

## CS 2223 – Algorithms – HW: 4

1. Explain why the *ForwardElimination* algorithm on 210 of Levitin fails to provide a solution for:

$$x_1 + x_2 + x_3 = 6$$

$$x_1 + x_2 + 2x_3 = 9$$

$$x_1 + 2x_2 + 3x_3 = 14$$

Despite the fact that  $x = (1, 2, 3)$  can be easily verified as solution to the system.

How does the *BetterForwardElimination* algorithm on page 211 of Levitin remedy this?

Before getting into the solution directly, let's split this problem into two parts. One which explains the *ForwardElimination* and the second part which discusses the *BetterForwardElimination*.

The Forward Elimination method uses Gaussian elimination on a matrix A, which contains a system's coefficients, combined with a vector b that includes the system's right-hand side values. This technique aims to convert matrix A into an equivalent upper-triangular matrix where all elements beneath the primary diagonal are zero.

The algorithm requires an input matrix A[1..n, 1..n] and a column-vector b[1..n], and outputs an equivalent upper-triangular matrix in place of A, with corresponding right-hand side values in the (n+1)st column.

The algorithm operates in the following manner:

- For each row i ranging from 1 to n, the matrix A is augmented with the corresponding right-hand side value b[i] in the (n+1)st column, resulting in a new matrix with n rows and n+1 columns.
- For each row i ranging from 1 to n-1, the following steps are executed:
  - For each row j ranging from i+1 to n, the i-th row multiplied by a scaling factor is subtracted from the j-th row. The scaling factor is  $A[j, i] / A[i, i]$ .
  - For each column k ranging from i to n+1, the value of A[j, k] is updated to  $A[j, k] - A[j, i] * A[i, k] / A[i, i]$ . This step aims to eliminate the elements below the main diagonal in the i-th column.

The resulting matrix A is now an upper-triangular matrix with all elements below the main diagonal equal to zero.

The ForwardElimination algorithm provides a solution for the given system of equations because it does not account for the possibility of encountering a zero pivot element, which can cause division by zero and numerical instability. When using the ForwardElimination algorithm, you might encounter a situation where the algorithm fails to find the correct solution due to issues related to the selection of pivot elements.

The BetterForwardElimination algorithm remedies this issue by implementing Gaussian elimination with partial pivoting. Partial pivoting helps to avoid division by zero and numerical instability by selecting the row with the largest absolute value in the pivot column as the pivot row, and then swapping the current row with the pivot row. This process helps to maximize the absolute value of the pivot element, reducing the risk of encountering a zero pivot element and improving the numerical stability of the algorithm.

In the case of the given system of equations, the BetterForwardElimination algorithm correctly identifies  $x = (1, 2, 3)$  as the solution because of partial pivoting.

2. Explain in some detail why the *BetterForwardElimination* algorithm on page 211 of Levitin fails to provide a solution for

$$x_1 + x_2 + x_3 = 6$$

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_1 + 2x_2 + 3x_3 = 15$$

Despite the fact that  $x = (1, 2, 3)$  can be easily verified as a solution to the system

What can be done to remedy this shortcoming in the algorithm

This failure occurs because division by zero or numerical instability can arise when a pivot element is zero or close to zero. To remedy this shortcoming in the algorithm, we can use a modified version of the algorithm that implements full pivoting instead of partial pivoting. Full pivoting involves selecting the largest absolute value in the entire submatrix below and to the right of the pivot element as the pivot element. Then, we swap both the rows and columns to ensure that the pivot element is located in the diagonal position. This approach helps to avoid division by zero and improve numerical stability even further than partial pivoting, and it can provide a solution for a system of linear equations where partial pivoting fails.