MA 2621 Probability for Applications
E1-Term Summer 2023 - Homework 04

Due: F 06/16 by 11.59 p.m.

Show all work as described in class. Partial credits will be given. Submit your solutions in a pdf format to canvas. Please write your name.

1. Consider the following joint probability distribution of X and Y.

X	$Y=0$	1	2
0	0.10	0.04	0.02
1	0.08	0.20	0.12
2	0	0.14	0.30

(a) What is $P(X=1, Y=2)$?

(b) Compute $P(X \leq 1, Y \geq 1)$.

(c) Find the marginal distribution of X and Y.

(d) Compute $P(X=2|Y=1)$.

(e) Find $E(X)$ and $E(Y)$.

(f) Are X and Y independent? Explain!

(g) Let $Z = 2X + Y$. Find the PMF of Z.

(h) Find the expected value of Z, $E(Z)$.

a).

$$P\,(X = 1,\, Y = 2) = .12$$

b).

$$P(X = 0,\, Y = 1),\, P(X = 0,\, Y = 2),\, P(X = 1,\, Y = 1),\, P(X = 1,\, Y = 2)$$? −1

c).

$$P(X = 0) = 0.10 + 0.04 + 0.02 = 0.16$$
$$P(X = 1) = 0.08 + 0.20 + 0.12 = 0.40$$
$$P(X = 2) = 0 + 0.14 + 0.30 = 0.44$$
$$P(Y = 0) = 0.10 + 0.08 + 0 = 0.18$$
$$P(Y = 1) = 0.04 + 0.20 + 0.14 = 0.38$$
$$P(Y = 2) = 0.02 + 0.12 + 0.30 = 0.44$$

d).

$$P(X = 2|Y = 1) = 0.14 / 0.38 = 0.3684$$

e).

$$E(X) = 0 * (0.16) + 1 * (0.40) + 2 * (0.44) = 0 + 0.40 + 0.88 = 1.28$$

$$E(Y) = 0 * (0.18) + 1 * (0.38) + 2 * (0.44) = 0 + 0.38 + 0.88 = 1.26$$

f).

$The\ joint\ probability\ P(X = x,\ Y = y)\ equals\ the\ product\ of\ the\ marginal\ probabilities\ P(X = x)P(Y = y)$

$$P(X = 1,\ Y = 1) = 0.20\ (from\ the\ table)$$

$$P(X = 1) = 0.40\ (from\ the\ marginal\ distribution\ of\ X)$$

$$P(Y = 1) = 0.38\ (from\ the\ marginal\ distribution\ of\ Y)$$

$$Since\ this\ is\ not\ true\ (0.20 \neq 0.152),\ X\ and\ Y\ are\ not\ independent$$

g).

$$We\ have\ Z = 2X + Y \qquad Z\ can\ take\ values\ of\ 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$ 6

$$P(Z = 0) = P(X = 0,\ Y = 0) = 0.10\ (from\ the\ table)$$

$$P(Z = 1) = P(X = 0,\ Y = 1) = 0.04\ (from\ the\ table)$$

$$P(Z = 2) = P(X = 0,\ Y = 2) + P(X = 1,\ Y = 0) = 0.02 + 0.08 = 0.10$$

−1

$$P(Z = 3) = P(X = 1,\ Y = 1) + P(X = 2,\ Y = 0) = 0.20 + 0 = 0.20$$

$$P(Z = 4) = P(X = 1,\ Y = 2) + P(X = 2,\ Y = 1) = 0.12 + 0.14 = 0.26$$

$$P(Z = 5) = P(X = 2,\ Y = 2) = 0.30\ (from\ the\ table)$$

h).

$$E(Z) = \sum[z * P(Z = z)]\ for\ all\ z$$

$$= 0P(Z = 0) + 1P(Z = 1) + 2P(Z = 2) + 3P(Z = 3) + 4P(Z = 4) + 5P(Z = 5)$$

$$= 00.10 + 10.04 + 20.10 + 30.20 + 40.26 + 50.30$$

$$= 0 + 0.04 + 0.20 + 0.60 + 1.04 + 1.50$$

$$= 3.38.$$

−1

2. Let X be a continuous random variable with PDF,

$$f(x) = \begin{cases} kx^2 & ; 0 < x < 4 \\ 0 & ; \text{otherwise} \end{cases}$$

(a) Find k.

(b) Find the expected value of X, $E(X)$.

(c) Find variance of X, $Var(X)$.

(d) Find Cumulative Distribution Function (CDF) of X.

(e) Find the values, $P(X > 2)$ and $P(1 < X < 3)$.

a).

$$\int_{-\infty}^{\infty} f(x)dx = 1 \qquad \int_{0}^{4} kx^2 = 1$$

$$k\left(\frac{4^3}{3}\right) - k\left(\frac{0^3}{3}\right) = 1 \qquad \frac{64k}{3} = 1 \qquad k = \frac{3}{64}$$

b).

$$\int_{-\infty}^{\infty} x(f(x))dx \qquad \int_{0}^{4} \left(\frac{3}{64}\right)x^3 dx$$

$$E(X) = \left(\frac{3}{64}\right)\left(\frac{4^4}{4}\right) - \left(\frac{3}{64}\right)\left(\frac{0^4}{4}\right) = 3$$

c).

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{0}^{4} \left(\frac{3}{64}\right)x^4 dx$$

$$\frac{3}{64}\left(\frac{4^4}{5-0}\right)$$

$$E[X^2] = 48.6$$

−1

$$48.6 - 3^2$$

$$39.6$$

d).

$$F(x) = \int_{0}^{x} f(t)dt \qquad F(x) = \int_{0}^{x} \left(\frac{3}{64}\right)t^2 dt$$

$$\left(\frac{3}{64}\right)\left(\frac{1}{3}t^3\right) \qquad \left(\frac{1}{64}\right)t^3$$

$= (\frac{1}{64})x^3 \; for \; 0 \leq x \leq 4$

$x < 0$

-1

$x > 4$

e).

$P(X > 2) = \int_2^4 (\frac{3}{64})x^2 dx \quad = (\frac{3}{64})(\frac{1}{3}x^3) \quad = \frac{1}{64}(4^3 - 2^3) \quad = \frac{1}{64}(56) \quad = 0.875$

$P(1 < X < 2) = \int_1^3 (\frac{3}{64})x^2 dx \quad = (\frac{3}{64})(\frac{1}{3}x^3) \quad = \frac{1}{64}(3^3 - 1^3) \quad = \frac{1}{64}(26) \quad = 0.40635$

3. Suppose X has the probability density

$$f(x) = \begin{cases} ax + b & ; 0 \leq x \leq 1 \\ 0 & ; \text{otherwise} \end{cases}$$

and $E(X^2) = \frac{1}{6}$. Find the values of a and b.

$\int_0^1 f(x)dx = 1 \quad \int_0^1 (ax + b)dx = 1 \quad \frac{a}{2} + b = 1 \text{ --> } (1)$

-1

$E[X^2] = \int_0^1 x^2 f(x)dx \quad \int_0^1 (ax^3 + bx^2)dx \quad \frac{a}{4}(x^4)\frac{b}{3}(x^3) \quad \frac{a}{4} + \frac{b}{3} = \frac{1}{6} \text{ --> } (2)$

$\frac{(2-2b)}{4} + \frac{b}{3} = \frac{1}{6} \quad b = \frac{1}{2} \; and \; a = 1$

4. Suppose you order a pizza from your favorite pizzeria at 7:00 pm, knowing that the time it takes for your pizza to be ready is uniformly distributed between 7:00 pm and 7:30 pm. What is the probability that you will have to wait longer than 10 minutes for your pizza?

Here, a = 0 minutes and b = 30 minutes. So, the pdf is 1/(30 − 0) = 1/30.

$P(X = 10) = \int_{10}^{30} f(x)dx \quad \int_{10}^{30} (\frac{1}{30})dx \quad (\frac{1}{30})(30 - 10) \quad \frac{2}{3}$

So, there's a 2/3 chance you'll have to wait more than 10 minutes for your pizza.

5. The total duration of baseball games in the major league in the 2011 season is uniformly distributed between 447 hours and 521 hours inclusive.
   (a) Find the mean and the standard deviation of the duration of baseball games.
   (b) What is the probability that the duration of games for a team for the 2011 season is between 480 and 500 hours?

a).

$$Mean\ \mu = (a + b) / 2$$

$$Standard\ deviation\ \sigma = \sqrt{\frac{(b-a)^2}{12}}$$

-1

$$Here,\ a = 447\ hours\ and\ b = 521\ hours$$

$$Standard\ deviation\ \sigma = \sqrt{\frac{(521-447)^2}{12}} \quad \approx 19.61\ Hours$$

b).

$$P(480 \leq X \leq 500) = \int_{480}^{500} f(x)dx \quad \int_{480}^{500} (\frac{1}{74})dx \quad (\frac{1}{74})(500 - 480) \quad \frac{20}{74} \quad \approx 0.27027$$

So, there's approximately a 27.03% chance that the duration of games for a team for the 2011 season is between 480 and 500 hours.

6. The length of time for an individual to be served at a cafeteria is a random variable having an exponential distribution with a mean of 2 minutes.
   (a) What is the probability that a person is served in less than 3 minutes?
   (b) Find the expected time and the standard deviation of time for one individual to be served at a cafeteria.

$$P(X = x) = \lambda\ e^{(-\lambda x)},\ for\ x \geq 0$$

a).

$$1/\mu = 1/2 = 0.5$$

$$P(X < 3) = 1 - e^{(-\lambda x)} \quad = 1 - e^{(-0.5 * 3)} \quad = 1 - e^{(-1.5)} \quad \approx 0.7768698$$

So, the probability that a person is served in less than 3 minutes is approximately 0.78.

b).

For an exponential distribution, both the expected value and the standard deviation are equal to the reciprocal of the rate parameter λ. Therefore, the expected time for one individual to be served at a cafeteria is 2 minutes (as given). The standard deviation is also 2 minutes.

7. The time till failure of an electronic component has an Exponential distribution and it is known that 10% of components have failed by 1000 hours.
   (a) What is the probability that a component is still working after 5000 hours?
   (b) Find the mean and standard deviation of the time till failure.

a).

$0.1 = 1 - e^{(-\lambda * 1000)} \qquad -\lambda * 1000 = ln(0.9) \qquad \lambda = -ln(0.9)/1000 \approx 0.00010536$

$$P(X > 5000) = e^{(-\lambda * 5000)} = e^{(-0.00010536 * 5000)} \approx 0.60653$$

$$So, \ the\ probability\ is\ approximately\ 0.61$$

b).

$$\mu = 1/\lambda = 1/0.00010536 \approx 9486.75\ hours$$

So, the mean time till failure is approximately 9486.75 hours and the standard deviation is also 9486.75 hours.