# Further Transformations

Graphics 1 CMP-5010B


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# Content

- Inverse Rotations
- Scaling, Shearing, and Reflection
- Homogeneous Coordinates

# Inverse Rotations

$$R^{-1}$$

# Inverse Rotations

We commonly need to compute the inverse of a rotation, for example, in the hierarchical transformations in character animation skeletons.

# Inverse Rotations

$$v' = Rv$$
$$v = R^{-1}v'$$

Where $R$ is the rotation matrix and $v$ is a vertex.

# Properties of Rotation Matrices

Rotation matrices are **square**.

# Properties of Rotation Matrices

The **determinant** of a rotation matrix is 1.

- because: $\cos^2 \alpha + \sin^2 \alpha = 1$
- hint: think about the radius in the unit circle

# Properties of Rotation Matrices

Rotation matrices are **orthonormal**.

- column vectors are orthogonal
- column vectors are unit
- hint: think about the radius in the unit circle
- exercise: plot the column vectors

# Properties of Rotation Matrices

$$R^T R = I, \quad RR^T = I$$

Where $I$ is the **identity** matrix.

# Properties of Rotation Matrices

We can use all these properties to **test** if a matrix *is* a rotation matrix.

# Inverse Rotation Matrices

Therefore the *inverse* of a rotation matrix **is** the *transpose* of the rotation matrix.

$$R^{-1} = R^T$$

# Inverse Rotation Matrices

Therefore the *inverse* of a rotation matrix **is** the *transpose* of the rotation matrix.

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}^{T} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

# Scaling, Shearing and Reflection

for *Affine* transformations

# Scaling

We can separate scaling to uniform scaling and non-uniform scaling.

# Uniform Scaling

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

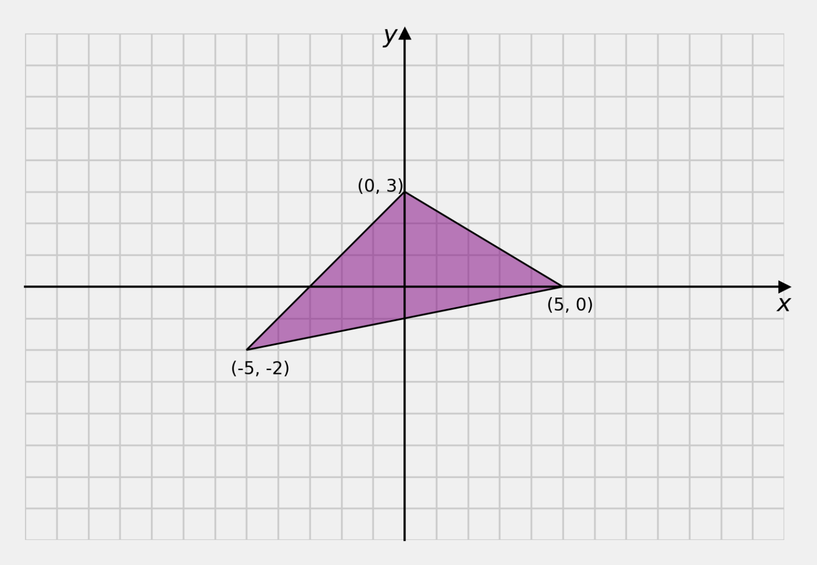


Figure 1: model to scale

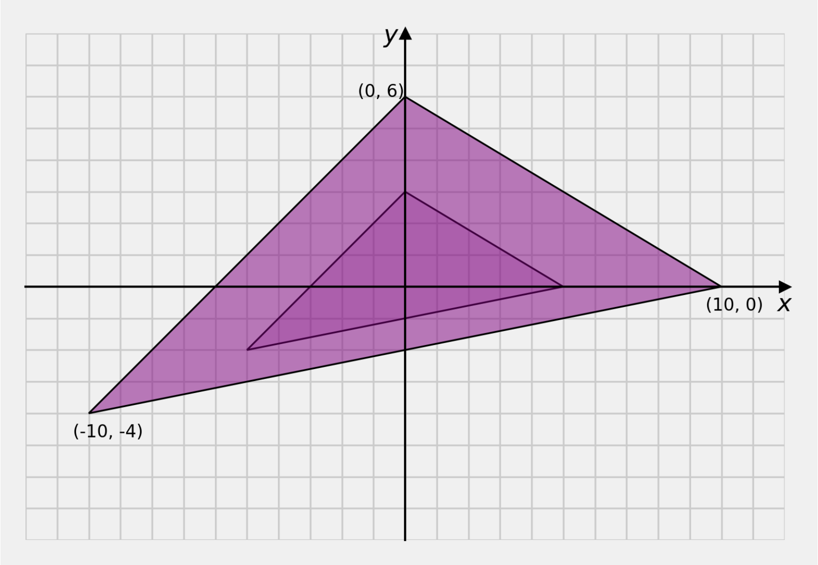


Figure 2: uniform scaling, $s = 2$

# Uniform Scaling

In the example, notice that all vertices are scaled equally by 2.

# Non-Uniform Scaling

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

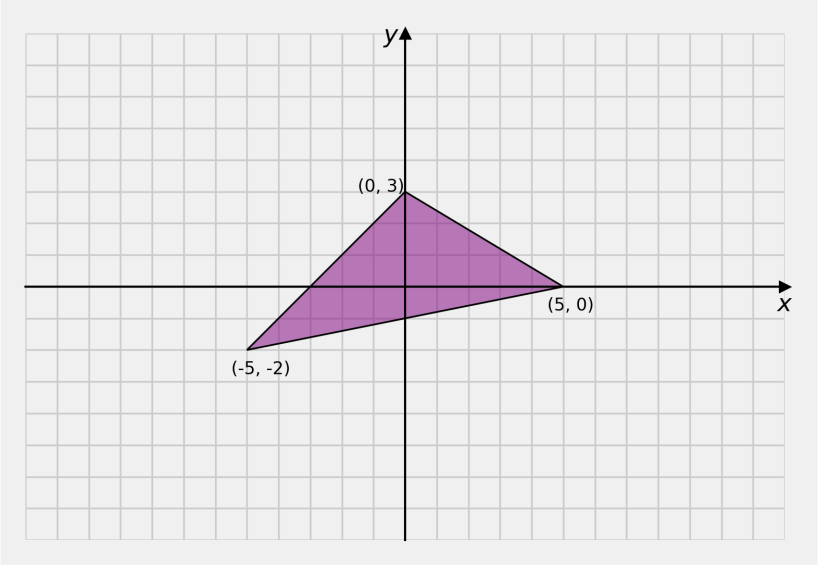


Figure 3: model to scale

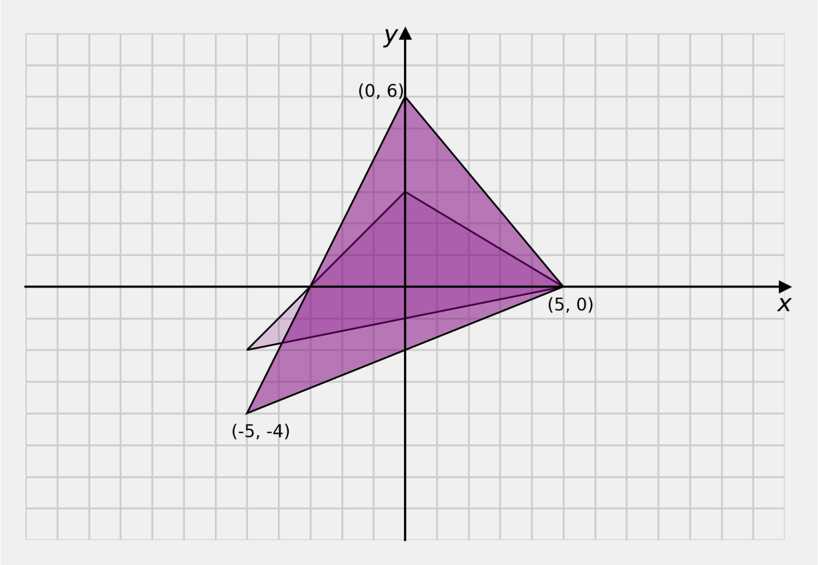


Figure 4: non-uniform scaling, $s_x = 1, s_y = 2$

# Non-Uniform Scaling

In the non-uniform example, notice that all vertices are scaled in the $y$ direction by 2 and in the $x$ direction by 1, so there is no change in $x$.

# Non-Uniform Scaling

*Uniform* scaling is a special case of non-uniform scaling where:

$$s_x = s_y$$

# Shearing

Shearing is an operation that moves vertices parallel to an axis, scaled by the distance from that axis.

# Shearing

To shear parallel to the $x$ axis:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

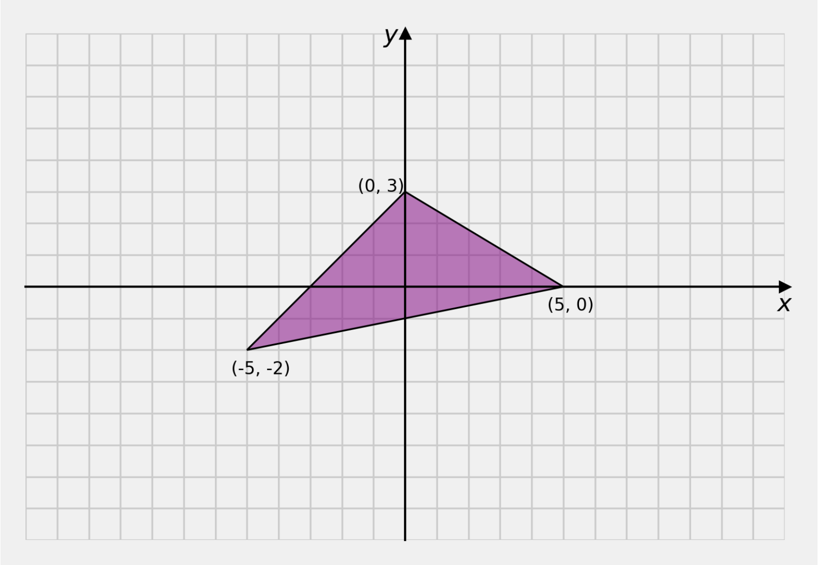


Figure 5: model to shear

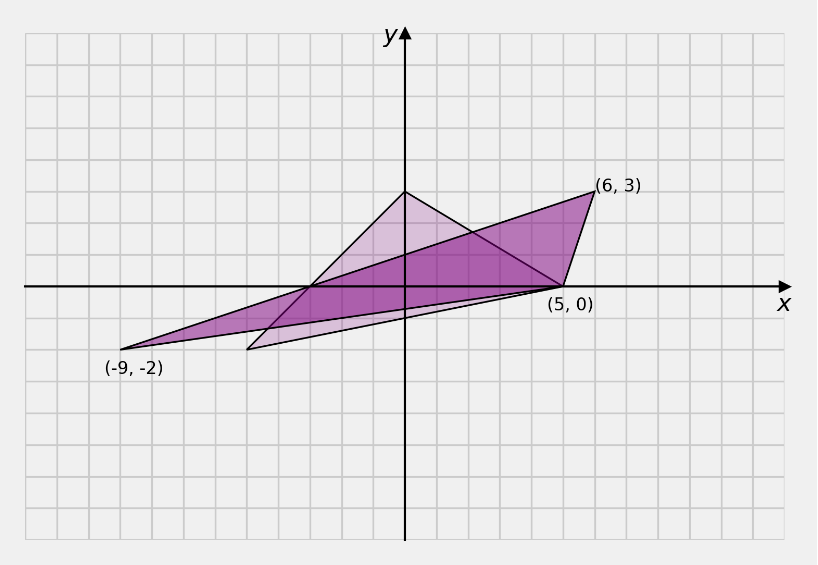


Figure 6: shearing parallel x, $\lambda = 2$

To shear parallel to the $y$ axis:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

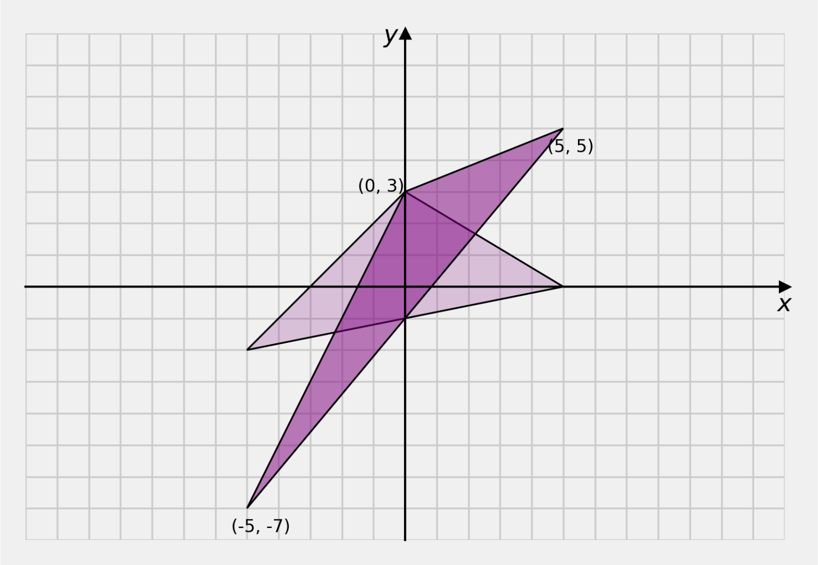


Figure 7: shearing parallel y, $\lambda = 1$

# Reflection

Reflection is an operation that imposes *symmetry* across an axis.

# Reflection

To reflect across the $y$ axis:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

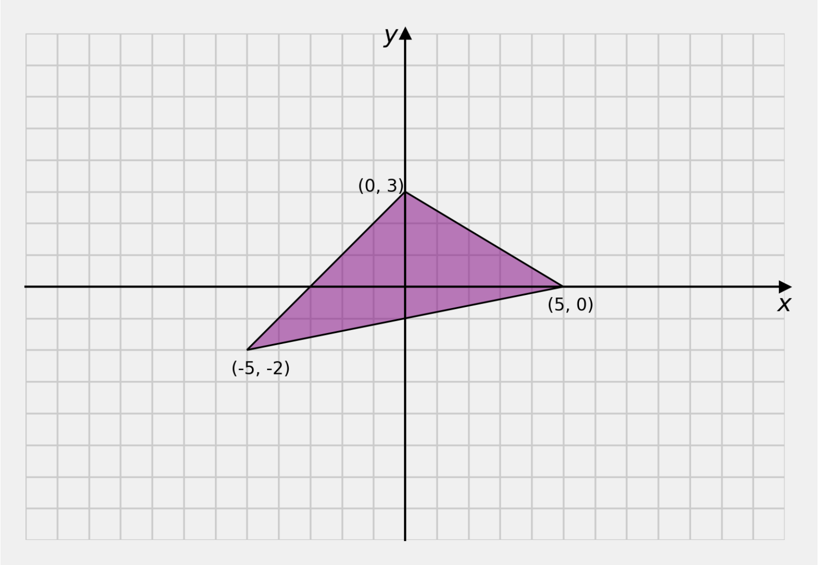


Figure 8: model to reflect

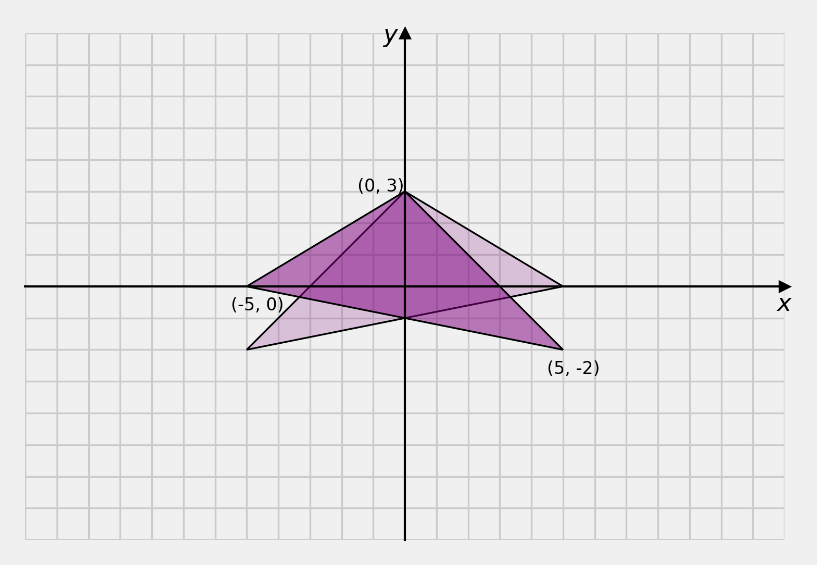


Figure 9: reflection across $y$ axis

# Reflection

To reflect across the $x$ axis:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

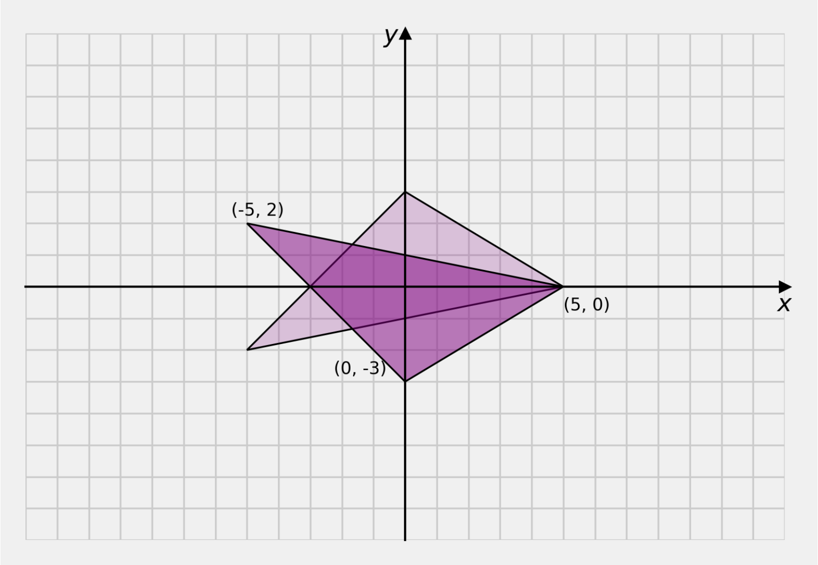


Figure 10: reflection across $x$ axis

# Reflection

To reflect across the $x = y$ axis:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

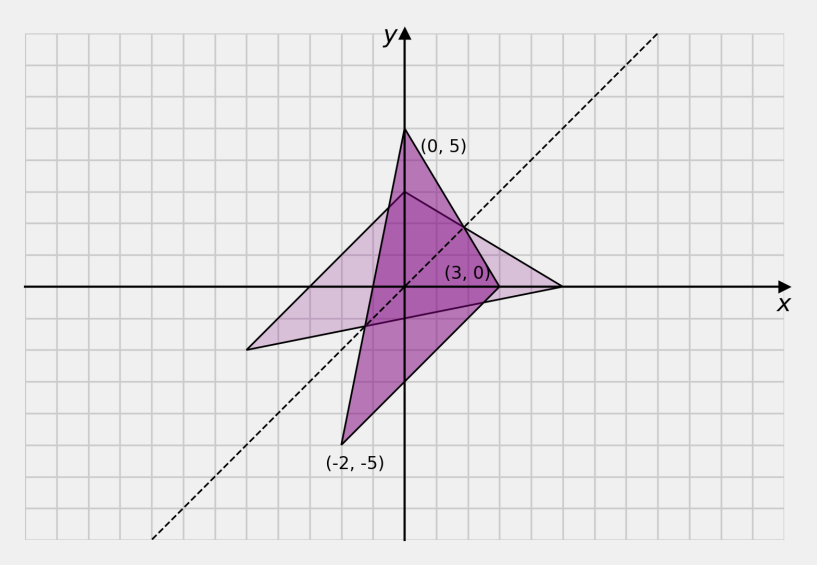


Figure 11: reflection across $x = y$ axis

# Homogeneous Coordinates

*adjective:* "of the same kind; alike."

# 2D transformation problem

- We have so far, explored a number of elementary transformations in 2D.
- For ease of implementation, it would be better if *translation* could also be done using matrix multiplication.
- Solution: **Homogeneous** Coordinates.

# Homogeneous Coordinates

- Define a new set of coordinates one dimension higher.
- For 2D, $\mathbb{R}^2 \to \mathbb{R}^3$
- We add a third coordinate $w$.

# Homogeneous Coordinates

The homogeneous coordinates relate to our 2D coordinates as follows:

$$x_h = \frac{x}{w}\ ,\ y_h = \frac{y}{w}\ ,\ w$$

# Homogeneous Coordinates

Thus: $x = wx_h$, $y = wy_h$.

- $w$ functions as a *scaling factor*.
- we can set $w$ to 1, so $x = x_h$, $y = y_h$
- How do we use 3D homogeneous coordinates to represent 2D transformations?

# Homogeneous Coordinates

For a general transformation operation, we extend the matrix multiplication we have seen so far, to include the *w* coordinate:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Homogeneous Rotation

For our homogeneous 3 x 3 transformation matrix, **rotation** is now:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Homogeneous Rotation

Remains a *true* rotation matrix.

- **All** the properties of a rotation matrix are preserved.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Homogeneous Scaling

For our homogeneous 3 x 3 transformation matrix, **scaling** is now:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Homogeneous Coordinates

For our homogeneous 3 x 3 transformation matrix, generally, the 2 x 2 matrix of the 2D elementary operations occupies the top left corner:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Homogeneous Translation

How do we fit a translation into our 3 x 3 matrix?

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Summary

- Inverse Rotations
- Scaling, Shearing, and Reflection
- Homogeneous Coordinates

Reading:

- Hearn, D. et al. (2004). Computer Graphics with OpenGL.
- Strang, Gilbert, et al. (1993) Introduction to linear algebra.