4.3.5 Probit regression

$$
\begin{aligned}
\Phi(a) &= \int_{-\infty}^{a} \mathcal{N}(\theta \mid 0,1)\mathrm{d}\theta \\
&= \frac{1}{2} + \int_{0}^{a} \mathcal{N}(\theta \mid 0,1) d\theta \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{0}^{a} \exp\left(-\theta^2/2\right) d\theta
\end{aligned}
$$

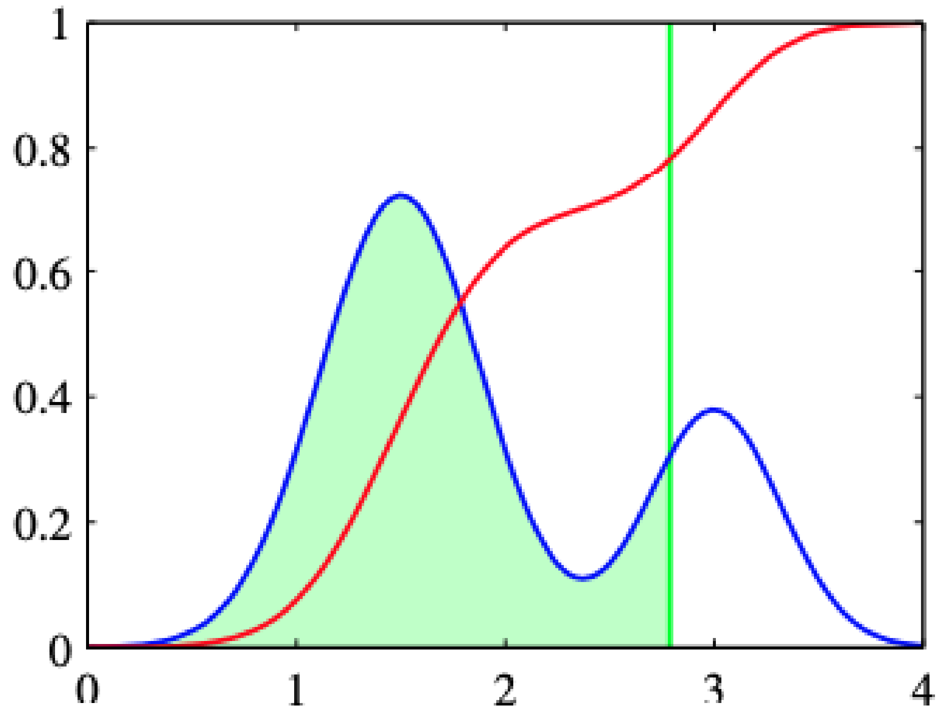
replace $\theta = \sqrt{2}\theta'$

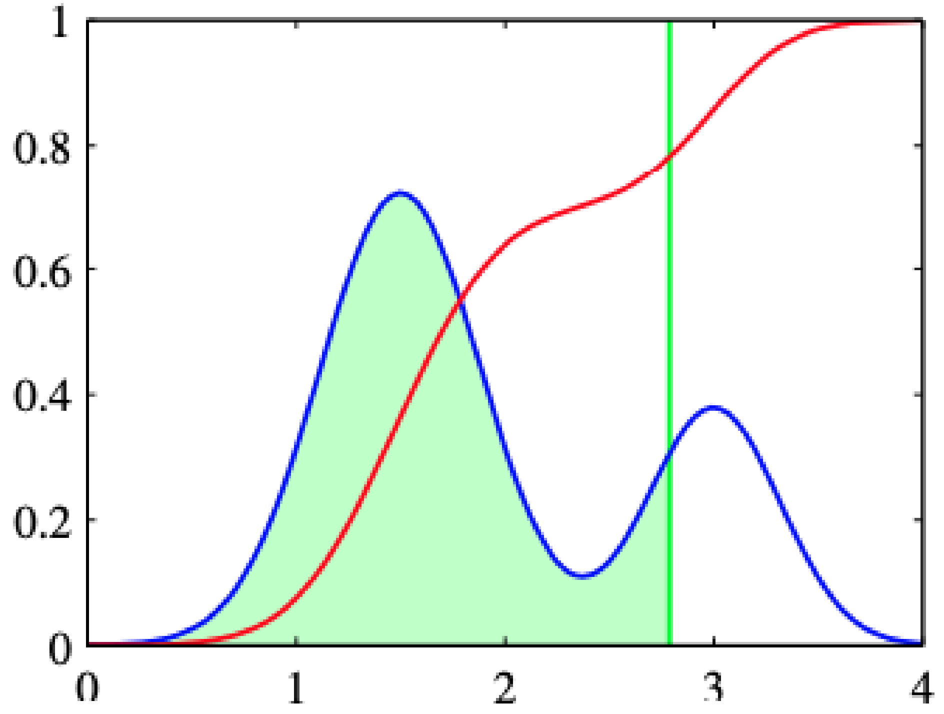
$$
\begin{aligned}
&= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_{0}^{a/\sqrt{2}} \exp\left(-\theta'^2\right) d\theta' \\
&= \frac{1}{2} \left\{ 1 + \mathrm{erf}\left(\frac{a}{\sqrt{2}}\right) \right\}
\end{aligned}
$$

$\mathrm{erf}(a) = \frac{2}{\sqrt{\pi}} \int_0^a \exp\left(-\theta^2\right) \mathrm{d}\theta$

4.3.5 Probit regression

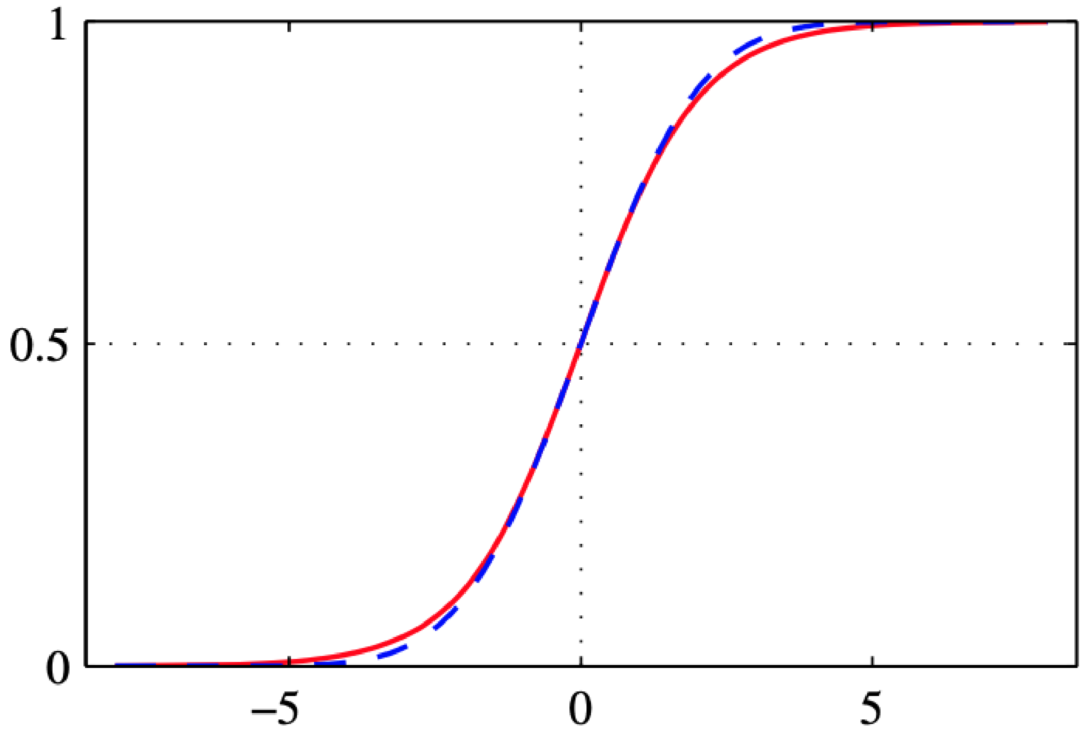
- aaa
- bbb




1
0.8
0.6
0.4
0.2
0
0
1
2
3
4

4.3.5 Probit regression

$\sigma(a) = \frac{1}{1+\exp(-a)}$



$$
\begin{aligned}
p(t \mid \mathbf{x}) &= (1-\epsilon)\sigma(\mathbf{x}) + \epsilon(1-\sigma(\mathbf{x})) \\
&= \epsilon + (1-2\epsilon)\sigma(\mathbf{x})
\end{aligned}
$$

4.3.6 Canonical link functions

$$p(t \mid \eta, s) = \frac{1}{s} h\left(\frac{t}{s}\right) g(\eta) \exp\left\{\frac{\eta t}{s}\right\}$$

$$y \equiv \mathbb{E}[t \mid \eta] = -s \frac{d}{d\eta} \ln g(\eta)$$

$$y = f\left(\mathbf{w}^{\mathrm{T}} \boldsymbol{\phi}\right)$$

$$\ln p(\mathbf{t} \mid \eta, s) = \sum_{n=1}^{N} \ln p\left(t_n \mid \eta, s\right) = \sum_{n=1}^{N}\left\{\ln g\left(\eta_n\right) + \frac{\eta_n t_n}{s}\right\} + \text{ const}$$

$$\begin{aligned}\nabla_{\mathbf{w}} \ln p(\mathbf{t} \mid \eta, s) &= \sum_{n=1}^{N} \left\{ \frac{d}{d\eta_n} \ln g\left(\eta_n\right) + \frac{t_n}{s} \right\} \frac{d\eta_n}{dy_n} \frac{dy_n}{da_n} \nabla a_n \\ &= \sum_{n=1}^{N} \frac{1}{s} \left\{ t_n - y_n \right\} \psi'\left(y_n\right) f'\left(a_n\right) \boldsymbol{\phi}_n\end{aligned}$$

$$f^{-1}(y) = \psi(y)$$

$$\nabla \ln E(\mathbf{w}) = \frac{1}{s} \sum_{n=1}^{N} \left\{ y_n - t_n \right\} \phi_n$$

$$p(z) = \frac{1}{Z}$$

$$\left.\frac{df(z)}{dz}\right|_{z=z_0} = 0$$

$$\ln f(z) \simeq \ln f\left(z_0\right) - \frac{1}{2} A \left(z - z_0\right)^2$$

$$A = -\left.\frac{d^2}{dz^2} \ln f(z)\right|_{z=z_0}$$

$$f(z) \simeq f\left(z_0\right) \exp\left\{-\frac{A}{2}\left(z - z_0\right)^2\right\}$$

$$q(z)=\left(\frac{A}{2\pi}\right)^{1/2}\exp\left\{-\frac{A}{2}\left(z-z_0\right)^2\right\}$$

$$\ln f(\mathbf{z})\simeq\ln f\left(\mathbf{z}_0\right)-\frac{1}{2}\left(\mathbf{z}-\mathbf{z}_0\right)^{\mathrm{T}}\mathbf{A}\left(\mathbf{z}-\mathbf{z}_0\right)$$

$$f(\mathbf{z})\simeq f\left(\mathbf{z}_0\right)\exp\left\{-\frac{1}{2}\left(\mathbf{z}-\mathbf{z}_0\right)^{\mathrm{T}}\mathbf{A}\left(\mathbf{z}-\mathbf{z}_0\right)\right\}$$

$$q(\mathbf{z})=\frac{|\mathbf{A}|^{1/2}}{(2\pi)^{M/2}}\exp\left\{-\frac{1}{2}\left(\mathbf{z}-\mathbf{z}_0\right)^{\mathrm{T}}\mathbf{A}\left(\mathbf{z}-\mathbf{z}_0\right)\right\}=\mathcal{N}\left(\mathbf{z}\mid\mathbf{z}_0,\mathbf{A}^{-1}\right)$$

4.4 The Laplace Approximation

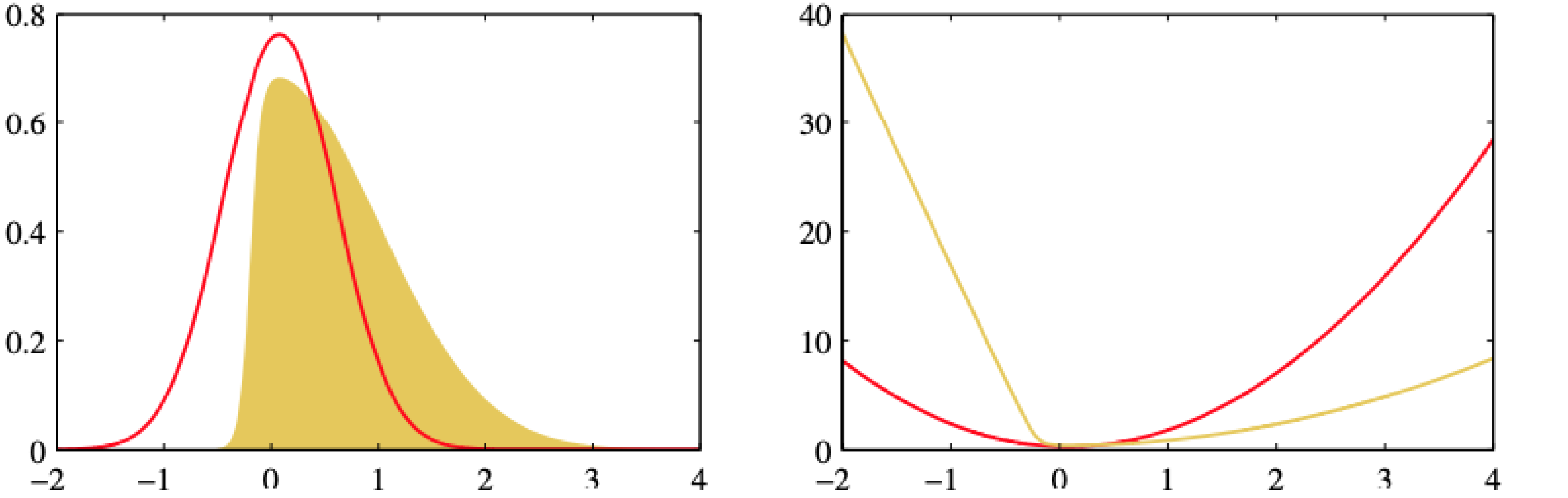


Figure 4.14 Illustration of the Laplace approximation applied to the distribution $p(z) \propto \exp(-z^2/2)\sigma(20z+4)$ where $\sigma(z)$ is the logistic sigmoid function defined by $\sigma(z) = (1+e^{-z})^{-1}$. The left plot shows the normalized distribution $p(z)$ in yellow, together with the Laplace approximation centred on the mode z_0 of $p(z)$ in red. The right plot shows the negative logarithms of the corresponding curves.

4.4.1 Model comparison and BIC

$$
\begin{aligned}
Z &= \int f(\mathbf{z}) \mathrm{d}\mathbf{z} \\
&\simeq f\left(\mathbf{z}_0\right) \int \exp\left\{-\frac{1}{2}\left(\mathbf{z}-\mathbf{z}_0\right)^{\mathrm{T}} \mathbf{A}\left(\mathbf{z}-\mathbf{z}_0\right)\right\} \mathrm{d}\mathbf{z} \\
&= f\left(\mathbf{z}_0\right) \frac{(2\pi)^{M/2}}{|\mathbf{A}|^{1/2}}
\end{aligned}
$$

$$
p(\mathcal{D}) = \int p(\mathcal{D} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta}
$$

$$
\ln p(\mathcal{D}) \simeq \ln p\left(\mathcal{D} \mid \boldsymbol{\theta}_{\mathrm{MAP}}\right) + \ln p\left(\boldsymbol{\theta}_{\mathrm{MAP}}\right) + \frac{M}{2} \ln(2\pi) - \frac{1}{2} \ln|\mathbf{A}|
$$

$$\mathbf{A} = -\nabla\nabla \ln p\left(\mathcal{D} \mid \boldsymbol{\theta}_{\mathrm{MAP}}\right) p\left(\boldsymbol{\theta}_{\mathrm{MAP}}\right) = -\nabla\nabla \ln p\left(\boldsymbol{\theta}_{\mathrm{MAP}} \mid \mathcal{D}\right)$$

$$\ln p(\mathcal{D}) \simeq \ln p\left(\mathcal{D} \mid \boldsymbol{\theta}_{\mathrm{MAP}}\right) - \frac{1}{2} M \ln N$$

Exercise 4.22

$$p(D)=\int p(D\mid\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}=\int f(\boldsymbol{\theta})d\boldsymbol{\theta}$$

$$=f\left(\boldsymbol{\theta}_{MAP}\right)\frac{(2\pi)^{M/2}}{|\mathbf{A}|^{1/2}}$$

$$=p\left(D\mid\boldsymbol{\theta}_{MAP}\right)p\left(\boldsymbol{\theta}_{MAP}\right)\frac{(2\pi)^{M/2}}{|\mathbf{A}|^{1/2}}$$

$$\ln p(D)=\ln p\left(D\mid\boldsymbol{\theta}_{MAP}\right)+\ln p\left(\boldsymbol{\theta}_{MAP}\right)+\frac{M}{2}\ln 2\pi-\frac{1}{2}\ln|\mathbf{A}|$$

Exercise 4.23

$$
\begin{aligned}
\ln p(D) &= \ln p\left(D \mid \boldsymbol{\theta}_{MAP}\right) + \ln p\left(\boldsymbol{\theta}_{MAP}\right) + \frac{M}{2}\ln 2\pi - \frac{1}{2}\ln|\mathbf{A}| \\
&= \ln p\left(D \mid \boldsymbol{\theta}_{MAP}\right) - \frac{M}{2}\ln 2\pi - \frac{1}{2}\ln|\mathbf{V}_0| - \frac{1}{2}\left(\boldsymbol{\theta}_{MAP} - \mathbf{m}\right)^T \mathbf{V}_0{}^{-1}\left(\boldsymbol{\theta}_{MAP} - \mathbf{m}\right) \\
&\quad + \frac{M}{2}\ln 2\pi - \frac{1}{2}\ln|\mathbf{A}| \\
&= \ln p\left(D \mid \boldsymbol{\theta}_{MAP}\right) - \frac{1}{2}\ln|\mathbf{V}_0| - \frac{1}{2}\left(\boldsymbol{\theta}_{MAP} - \mathbf{m}\right)^T \mathbf{V}_0{}^{-1}\left(\boldsymbol{\theta}_{MAP} - \mathbf{m}\right) - \frac{1}{2}\ln|\mathbf{A}|
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A} &= -\nabla\nabla \ln p\left(D \mid \boldsymbol{\theta}\right) p\left(\boldsymbol{\theta}\right)|_{\theta=\theta_{\boldsymbol{MAP}}} \\
&= -\nabla\nabla \ln p\left(D \mid \boldsymbol{\theta}\right)|_{\theta=\theta_{\boldsymbol{MAP}}} - \nabla\nabla \ln p\left(\boldsymbol{\theta}\right)|_{\theta=\theta_{\boldsymbol{MAP}}} \\
&= \mathbf{H} - \nabla\nabla\left\{-\frac{1}{2}\left(\boldsymbol{\theta} - \mathbf{m}\right)^T \mathbf{V}_{\mathbf{0}}^{-1}\left(\boldsymbol{\theta} - \mathbf{m}\right)\right\}|_{\theta=\theta_{\boldsymbol{MAP}}} \\
&= \mathbf{H} + \nabla\left\{\mathbf{V}_{\mathbf{0}}{}^{-1}\left(\boldsymbol{\theta} - \mathbf{m}\right)\right\}|_{\theta=\theta_{\boldsymbol{MAP}}} \\
&= \mathbf{H} + \mathbf{V}_{\mathbf{0}}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\ln\left\{|\mathbf{V}_0| \cdot \left|\mathbf{H} + \mathbf{V}_0^{-1}\right|\right\} \\
&\ln\left\{|\mathbf{V}_0\mathbf{H} + \mathbf{I}|\right\} \\
&\ln|\mathbf{V}_0| - \frac{1}{2}\ln|\mathbf{H}| \\
&\ln|\mathbf{H}| + \text{ const}
\end{aligned}
$$

$$
\mathbf{H} = \sum_{n=1}^{N} \mathbf{H_n} = N\hat{\mathbf{H}}
$$