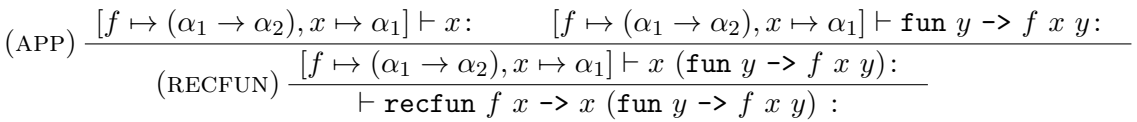
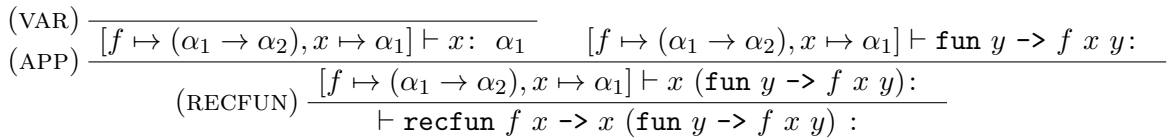
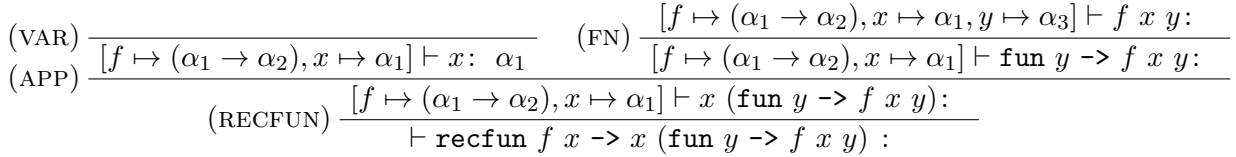
Aufgabe 9-4.

$$\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :$$

$$(\textsc{recfun})\ \frac{[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y) :}{\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :}$$

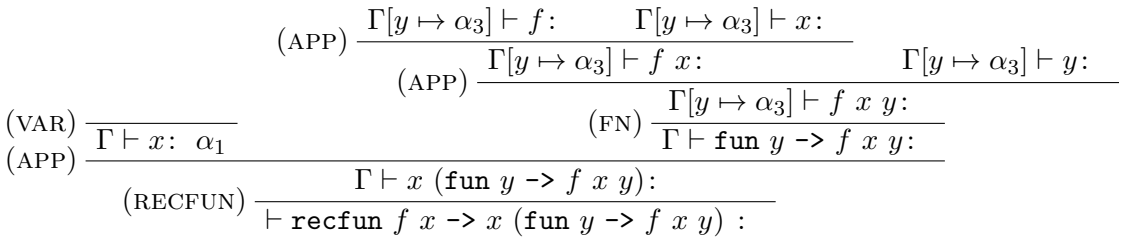

(APP)
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash x :$
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y :$
(RECFUN)
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y) :$
$\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :$

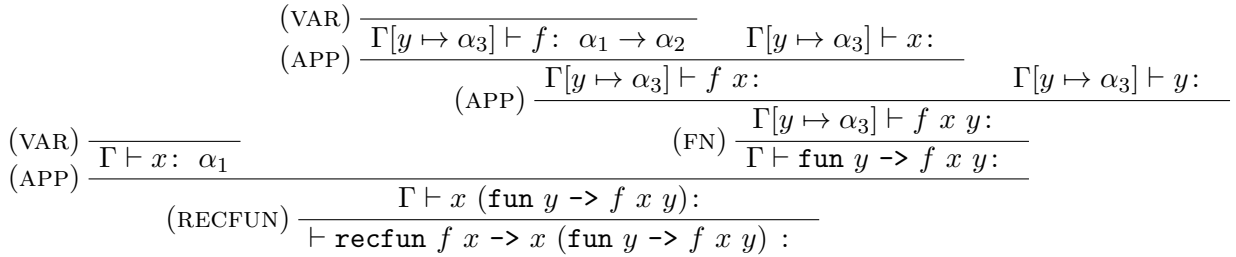

(VAR)
(APP)
(RECFUN)
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash x : \ \alpha_1$
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y :$
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y) :$
$\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :$

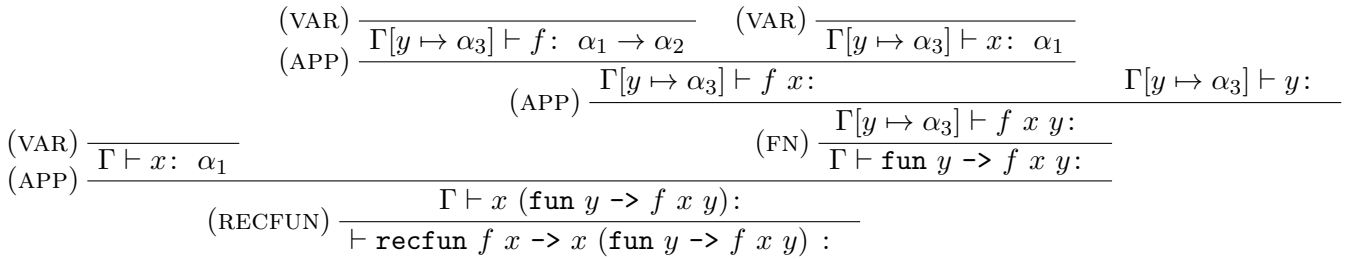

(VAR)
(APP)
(FN)
(RECFUN)
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash x : \alpha_1$
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1, y \mapsto \alpha_3] \vdash f\ x\ y :$
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y :$
$[f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1] \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y) :$
$\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :$

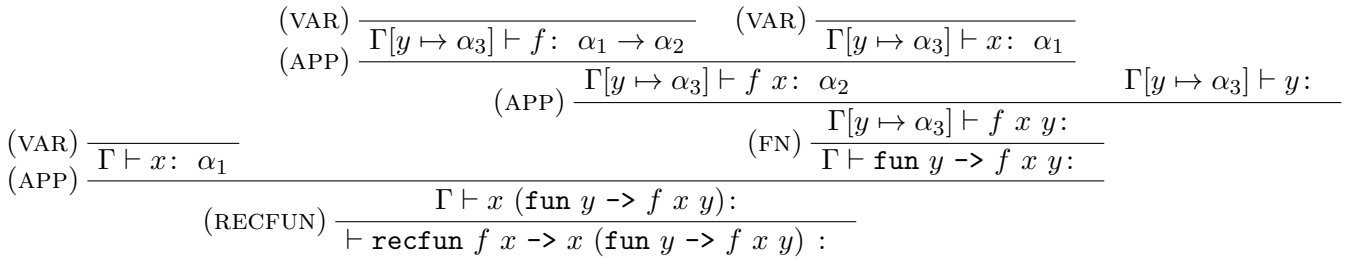
Abkürzung:
$\Gamma := [f \mapsto (\alpha_1 \to \alpha_2), x \mapsto \alpha_1]$

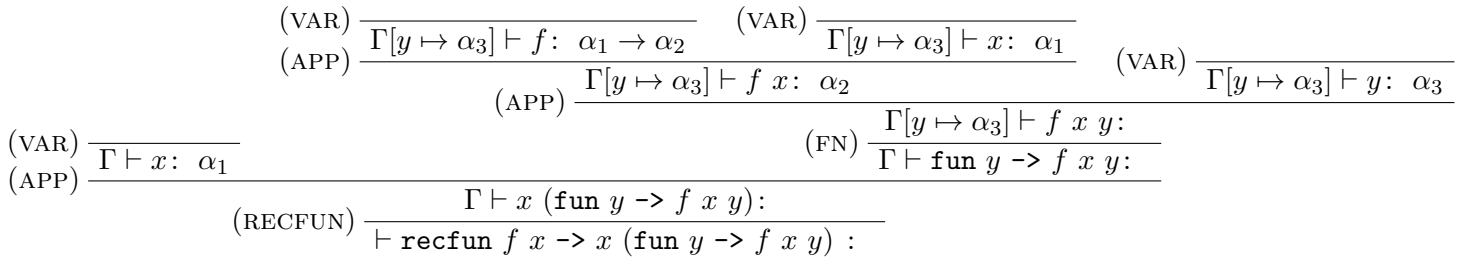
$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x\colon\ \alpha_1}\,(\text{VAR}) \qquad \dfrac{\dfrac{\Gamma[y \mapsto \alpha_3] \vdash f\ x\colon \qquad \Gamma[y \mapsto \alpha_3] \vdash y\colon}{\Gamma[y \mapsto \alpha_3] \vdash f\ x\ y\colon}\,(\text{APP})}{\Gamma \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y\colon}\,(\text{FN})}{\Gamma \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\colon}\,(\text{APP})}{\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :}\,(\text{RECFUN})$$


(APP)
(APP)
(VAR)
(APP)
(FN)
(RECFUN)


(VAR)
(APP)
(APP)
(VAR)
(APP)
(FN)
(RECFUN)


(VAR) $\Gamma[y \mapsto \alpha_3] \vdash f : \alpha_1 \to \alpha_2$
(VAR) $\Gamma[y \mapsto \alpha_3] \vdash x : \alpha_1$
(APP) $\Gamma[y \mapsto \alpha_3] \vdash f\ x :$
$\Gamma[y \mapsto \alpha_3] \vdash y :$
(APP) $\Gamma[y \mapsto \alpha_3] \vdash f\ x\ y :$
(VAR) $\Gamma \vdash x : \alpha_1$
(FN) $\Gamma \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y :$
(APP) $\Gamma \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y) :$
(RECFUN) $\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :$

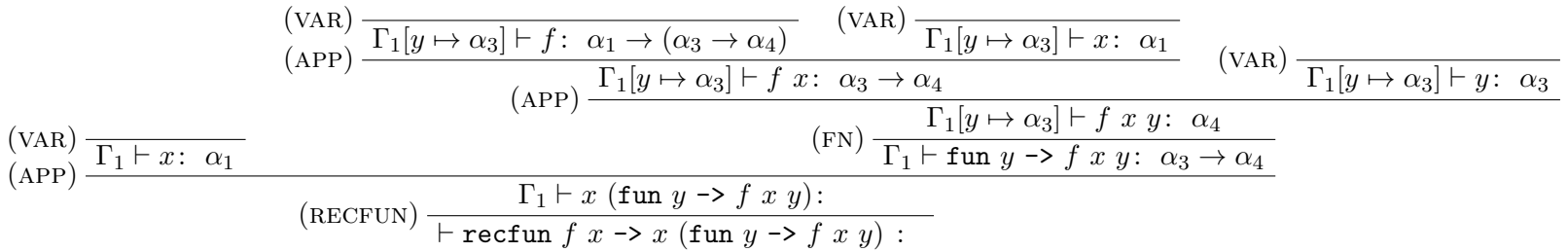

(VAR) $\Gamma[y \mapsto \alpha_3] \vdash f\colon\ \alpha_1 \to \alpha_2$
(VAR) $\Gamma[y \mapsto \alpha_3] \vdash x\colon\ \alpha_1$
(APP) $\Gamma[y \mapsto \alpha_3] \vdash f\ x\colon\ \alpha_2$
$\Gamma[y \mapsto \alpha_3] \vdash y\colon$
(VAR) $\Gamma \vdash x\colon\ \alpha_1$
(FN) $\Gamma[y \mapsto \alpha_3] \vdash f\ x\ y\colon$
(APP) $\Gamma \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y\colon$
(RECFUN) $\Gamma \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\colon$
$\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :$


(VAR) Γ[y ↦ α3] ⊢ f : α1 → α2
(VAR) Γ[y ↦ α3] ⊢ x : α1
(APP) Γ[y ↦ α3] ⊢ f x : α2
(VAR) Γ[y ↦ α3] ⊢ y : α3
(FN) Γ[y ↦ α3] ⊢ f x y :
(VAR) Γ ⊢ x : α1
(APP) Γ ⊢ x (fun y -> f x y):
(RECFUN) ⊢ recfun f x -> x (fun y -> f x y) :

Substitution: $[\alpha_2 \mapsto (\alpha_3 \to \alpha_4)]$
Abkürzung:
$\Gamma_1 := [f \mapsto (\alpha_1 \to (\alpha_3 \to \alpha_4)), x \mapsto \alpha_1]$

$$\dfrac{\dfrac{}{\Gamma_1 \vdash x\colon\ \alpha_1}\,(\textsc{var}) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma_1[y \mapsto \alpha_3] \vdash f\colon\ \alpha_1 \to (\alpha_3 \to \alpha_4)}\,(\textsc{var}) \quad \dfrac{}{\Gamma_1[y \mapsto \alpha_3] \vdash x\colon\ \alpha_1}\,(\textsc{var})}{\Gamma_1[y \mapsto \alpha_3] \vdash f\ x\colon\ \alpha_3 \to \alpha_4}\,(\textsc{app}) \quad \dfrac{}{\Gamma_1[y \mapsto \alpha_3] \vdash y\colon\ \alpha_3}\,(\textsc{var})}{\Gamma_1[y \mapsto \alpha_3] \vdash f\ x\ y\colon\ \alpha_4}\,(\textsc{app})}{\Gamma_1 \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y\colon}\,(\textsc{fn})}{\Gamma_1 \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\colon}\,(\textsc{app})$$

$$\dfrac{\Gamma_1 \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\colon}{\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :}\,(\textsc{recfun})$$


(VAR)
(APP)
(VAR)
(APP)
(VAR)
(APP)
(FN)
(RECFUN)
$\Gamma_1[y \mapsto \alpha_3] \vdash f\colon\ \alpha_1 \to (\alpha_3 \to \alpha_4)$
$\Gamma_1[y \mapsto \alpha_3] \vdash x\colon\ \alpha_1$
$\Gamma_1[y \mapsto \alpha_3] \vdash y\colon\ \alpha_3$
$\Gamma_1[y \mapsto \alpha_3] \vdash f\ x\colon\ \alpha_3 \to \alpha_4$
$\Gamma_1 \vdash x\colon\ \alpha_1$
$\Gamma_1[y \mapsto \alpha_3] \vdash f\ x\ y\colon\ \alpha_4$
$\Gamma_1 \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y\colon\ \alpha_3 \to \alpha_4$
$\Gamma_1 \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\colon$
$\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :$

Substitution: $[\alpha_1 \mapsto (\tau \to \alpha_5)]$
Abkürzungen:
$\Gamma_2 := [f \mapsto ((\tau \to \alpha_5) \to \tau), x \mapsto (\tau \to \alpha_5)]$,
$\tau := (\alpha_3 \to \alpha_4)$

$$\dfrac{\dfrac{}{\Gamma_2 \vdash x\colon\ \tau \to \alpha_5}\,(\textsc{var}) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma_2[y \mapsto \alpha_3] \vdash f\colon\ ((\tau \to \alpha_5) \to \tau)}\,(\textsc{var}) \qquad \dfrac{}{\Gamma_2[y \mapsto \alpha_3] \vdash x\colon\ \tau \to \alpha_5}\,(\textsc{var})}{\Gamma_2[y \mapsto \alpha_3] \vdash f\ x\colon\ \tau}\,(\textsc{app}) \qquad \dfrac{}{\Gamma_2[y \mapsto \alpha_3] \vdash y\colon\ \alpha_3}\,(\textsc{var})}{\Gamma_2[y \mapsto \alpha_3] \vdash f\ x\ y\colon\ \alpha_4}\,(\textsc{app}) \qquad \dfrac{\Gamma_2[y \mapsto \alpha_3] \vdash f\ x\ y\colon\ \alpha_4}{\Gamma_2 \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y\colon\ \tau}\,(\textsc{fn})}{\Gamma_2 \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\colon\ \alpha_5}\,(\textsc{app})$$

$$\dfrac{\Gamma_2 \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\colon\ \alpha_5}{\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :}\,(\textsc{recfun})$$

Substitution: $[\alpha_5 \mapsto \tau]$
Abkürzung:
$\Gamma_3 := [f \mapsto ((\tau \to \tau) \to \tau), x \mapsto (\tau \to \tau)]$
$\tau := (\alpha_3 \to \alpha_4)$

$$
\dfrac{
\dfrac{}{\Gamma_2 \vdash x\colon\ \tau \to \tau}\,(\text{VAR})
\qquad
\dfrac{
\dfrac{
\dfrac{}{\Gamma_2[y \mapsto \alpha_3] \vdash f\colon\ ((\tau \to \tau) \to \tau)}\,(\text{VAR})
\qquad
\dfrac{}{\Gamma_2[y \mapsto \alpha_3] \vdash x\colon\ \tau \to \tau}\,(\text{VAR})
}{\Gamma_2[y \mapsto \alpha_3] \vdash f\ x\colon\ \tau}\,(\text{APP})
\qquad
\dfrac{}{\Gamma_2[y \mapsto \alpha_3] \vdash y\colon\ \alpha_3}\,(\text{VAR})
}{\Gamma_2[y \mapsto \alpha_3] \vdash f\ x\ y\colon\ \alpha_4}\,(\text{APP})
}{\Gamma_2 \vdash \texttt{fun}\ y\ \texttt{->}\ f\ x\ y\colon\ \tau}\,(\text{FN})
}{\Gamma_2 \vdash x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\colon\ \tau}\,(\text{APP})
}{\vdash \texttt{recfun}\ f\ x\ \texttt{->}\ x\ (\texttt{fun}\ y\ \texttt{->}\ f\ x\ y)\ :\ (\tau \to \tau) \to \tau}\,(\text{RECFUN})
$$