The Laplace Transform

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Generalizing the Fourier Transform

The CTFT expresses a time-domain signal as a linear combination of **complex sinusoids** of the form $e^{j\omega t}$. In the generalization of the CTFT to the Laplace transform, the complex sinusoids become **complex exponentials** of the form e^{st} where s can have any complex value. Replacing the complex sinusoids with complex exponentials leads to this definition of the Laplace transform.

$$\mathcal{L}\left(\mathrm{x}(t)\right) = \mathrm{X}(s) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-st}\, dt$$

$$\mathrm{x}(t) \xleftrightarrow{\mathcal{L}} \mathrm{X}(s)$$

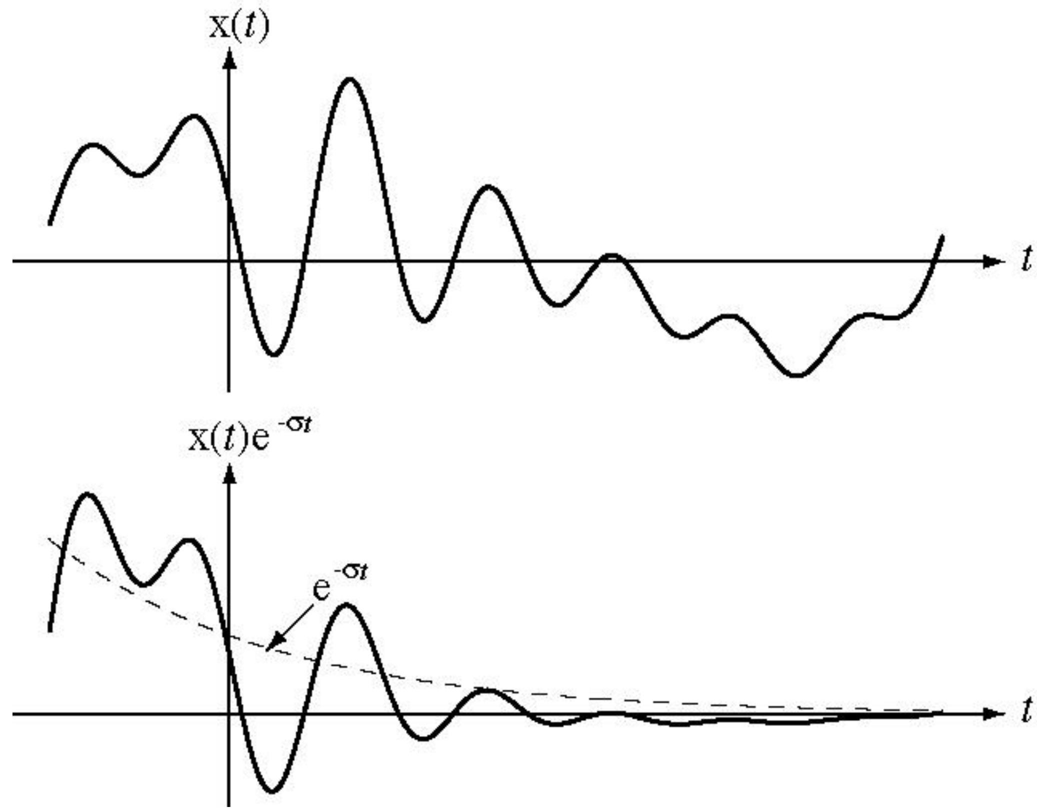
Generalizing the Fourier Transform

The variable s is viewed as a generalization of the variable ω of the form $s = \sigma + j\omega$. Then, when σ, the real part of s, is zero, the Laplace transform reduces to the CTFT. Using $s = \sigma + j\omega$ the Laplace transform is

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t}\, dt$$

$$= \mathcal{F}\left[x(t) e^{-\sigma t}\right]$$

which is the Fourier transform of $x(t)e^{-\sigma t}$



Generalizing the Fourier Transform

The extra factor e^{-st} is sometimes called a **convergence factor** because, when chosen properly, it makes the integral converge for some signals for which it would not otherwise converge. For example, strictly speaking, the signal $A\,\mathrm{u}(t)$ does not have a CTFT because the integral does not converge. But if it is multiplied by the convergence factor, and the real part of s is chosen appropriately, the CTFT integral will converge.

$$\int_{-\infty}^{\infty} A\,\mathrm{u}(t)\,e^{-j\omega t}\,dt = A\int_{0}^{\infty} e^{-j\omega t}\,dt \leftarrow \text{ Does not converge}$$

$$\int_{-\infty}^{\infty} A e^{-\sigma t}\,\mathrm{u}(t)\,e^{-j\omega t}\,dt = A\int_{0}^{\infty} e^{-(\sigma + j\omega)t}\,dt \leftarrow \text{ Converges (if } \sigma > 0)$$

Complex Exponential Excitation

If a continuous-time LTI system is excited by a complex exponential $\mathrm{x}(t) = Ae^{st}$, where A and s can each be any complex number, the system response is also a complex exponential of the same functional form except multiplied by a complex constant. The response is the convolution of the excitation with the impulse response and that is

$$\mathrm{y}(t) = \int_{-\infty}^{\infty} \mathrm{h}(\tau)\mathrm{x}(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \mathrm{h}(\tau) A e^{s(t-\tau)}\,d\tau = \underbrace{Ae^{st}}_{\mathrm{x}(t)} \int_{-\infty}^{\infty} \mathrm{h}(\tau) e^{-s\tau}\,d\tau$$

The quantity $\mathrm{H}(s) = \int_{-\infty}^{\infty} \mathrm{h}(\tau) e^{-s\tau}\,d\tau$ is called the **Laplace transform** of $\mathrm{h}(t)$.

Complex Exponential Excitation

Let $\mathrm{x}(t) = \underbrace{(6 + j3)}_{A} e^{\overbrace{(3-j2)}^{s}t} = (6.708\angle 0.4637)e^{(3-j2)t}$

and let $\mathrm{h}(t) = e^{-4t}\,\mathrm{u}(t)$. Then $\mathrm{H}(s) = \dfrac{1}{s+4}$, $\sigma > -4$ and,

in this case, $s = 3 - j2 = \sigma + j\omega$ with $\sigma = 3 > -4$ and $\omega = -2$.

$$\mathrm{y}(t) = \mathrm{x}(t)\mathrm{H}(s) = \frac{6 + j3}{3 - j2 + 4} e^{(3-j2)t} = (0.6793\angle 0.742)e^{(3-j2)t}.$$

The response is the same functional form as the excitation but multiplied by a different complex constant. This only happens when the excitation is a complex exponential and that is what makes complex exponentials unique.

Pierre-Simon Laplace

3/23/1749 - 3/2/1827

The Transfer Function

Let $\mathrm{x}(t)$ be the excitation and let $\mathrm{y}(t)$ be the response of a system with impulse response $\mathrm{h}(t)$. The Laplace transform of $\mathrm{y}(t)$ is

$$\mathrm{Y}(s) = \int_{-\infty}^{\infty} \mathrm{y}(t) e^{-st} dt = \int_{-\infty}^{\infty} \left[\mathrm{h}(t) * \mathrm{x}(t) \right] e^{-st} dt$$

$$\mathrm{Y}(s) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathrm{h}(\tau) \mathrm{x}(t-\tau) d\tau \right) e^{-st} dt$$

$$\mathrm{Y}(s) = \int_{-\infty}^{\infty} \mathrm{h}(\tau) d\tau \int_{-\infty}^{\infty} \mathrm{x}(t-\tau) e^{-st} dt$$

The Transfer Function

Let $\mathrm{x}(t) = \mathrm{u}(t)$ and let $\mathrm{h}(t) = e^{-4t}\,\mathrm{u}(t)$. Find $\mathrm{y}(t)$.

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(t)\,\mathrm{h}(t - t)\,dt = \int_{-\infty}^{\infty} \mathrm{u}(t)\,e^{-4(t-t)}\,\mathrm{u}(t - t)\,dt$$

$$\mathrm{y}(t) = \begin{cases} \int_0^t e^{-4(t-t)}\,dt = e^{-4t}\int_0^t e^{4t}\,dt = e^{-4t}\dfrac{e^{4t} - 1}{4} = \dfrac{1 - e^{-4t}}{4} & , \ t > 0 \\ 0 & , \ t < 0 \end{cases}$$

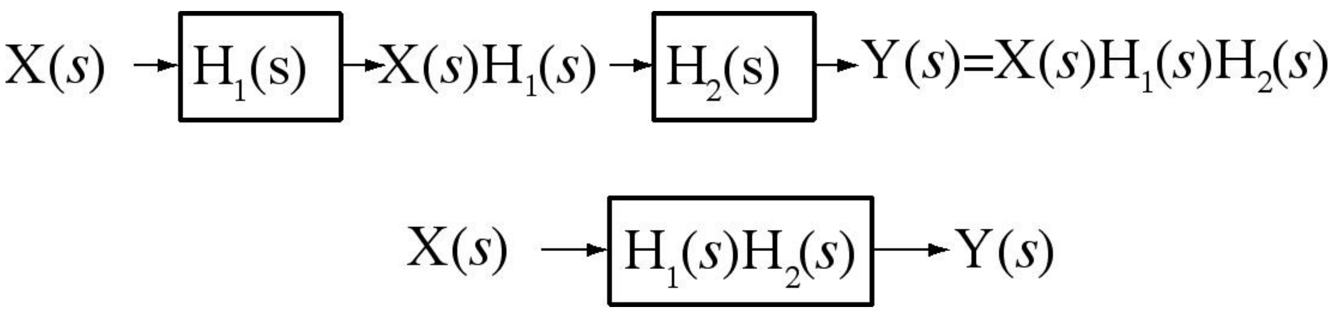
$$\mathrm{y}(t) = (1/4)\left(1 - e^{-4t}\right)\mathrm{u}(t)$$

$$\mathrm{X}(s) = 1/s \ , \ \mathrm{H}(s) = \frac{1}{s+4} \Rightarrow \mathrm{Y}(s) = \frac{1}{s} \times \frac{1}{s+4} = \frac{1/4}{s} - \frac{1/4}{s+4}$$

$$\mathrm{x}(t) = (1/4)\left(1 - e^{-4t}\right)\mathrm{u}(t)$$

Cascade-Connected Systems

If two systems are cascade connected the transfer function of the overall system is the product of the transfer functions of the two individual systems.



Direct Form II Realization

A very common form of transfer function is a ratio of two polynomials in s,

$$\mathrm{H}(s) = \frac{\mathrm{Y}(s)}{\mathrm{X}(s)} = \frac{\sum_{k=0}^{N} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

Direct Form II Realization

The transfer function can be conceived as the product of two transfer functions,

$$\mathrm{H}_1(s) = \frac{\mathrm{Y}_1(s)}{\mathrm{X}(s)} = \frac{1}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

and

$$\mathrm{H}_2(s) = \frac{\mathrm{Y}(s)}{\mathrm{Y}_1(s)} = b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0$$

$$\mathrm{X}(s) \rightarrow \boxed{\mathrm{H}_1(s) = \frac{1}{a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0}} \rightarrow \mathrm{Y}_1(s) \rightarrow \boxed{\mathrm{H}_2(s) = b_N s^N + b_{N-1} s^{N-1} + \ldots + b_1 s + b_0} \rightarrow \mathrm{Y}(s)$$

Direct Form II Realization

From

$$\mathrm{H}_1(s) = \frac{\mathrm{Y}_1(s)}{\mathrm{X}(s)} = \frac{1}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

we get

$$\mathrm{X}(s) = \left[a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0\right] \mathrm{Y}_1(s)$$

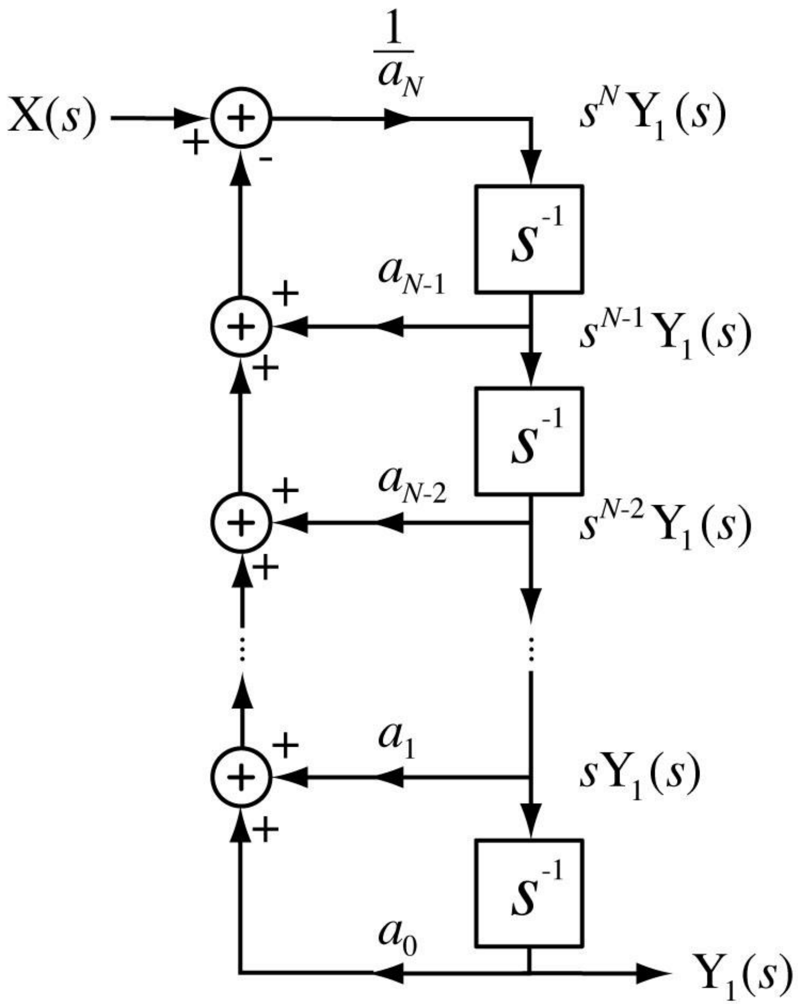
or

$$\mathrm{X}(s) = a_N s^N \, \mathrm{Y}_1(s) + a_{N-1} s^{N-1} \, \mathrm{Y}_1(s) + \cdots + a_1 s \, \mathrm{Y}_1(s) + a_0 \, \mathrm{Y}_1(s)$$

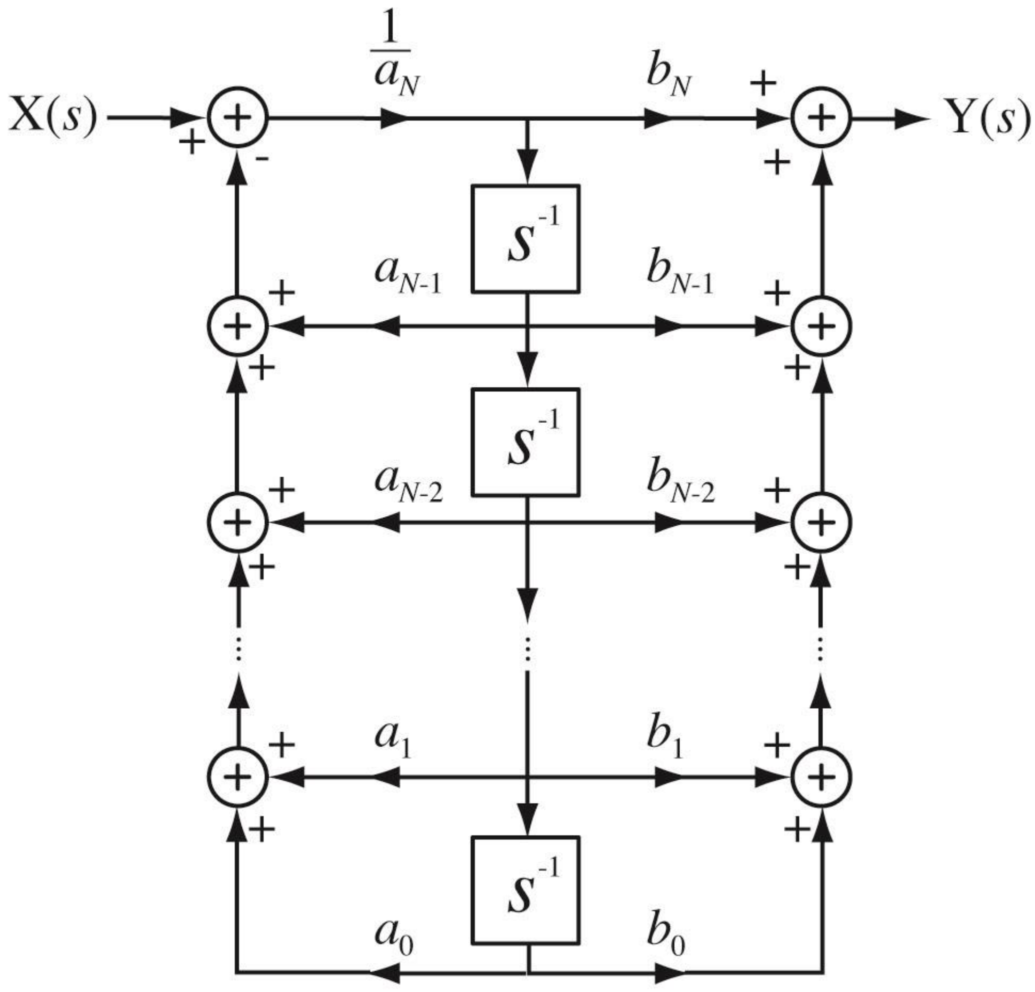
Rearranging

$$s^N \, \mathrm{Y}_1(s) = \frac{1}{a_N} \left\{ \mathrm{X}(s) - \left[a_{N-1} s^{N-1} \, \mathrm{Y}_1(s) + \cdots + a_1 s \, \mathrm{Y}_1(s) + a_0 \, \mathrm{Y}_1(s) \right] \right\}$$

Direct Form II Realization



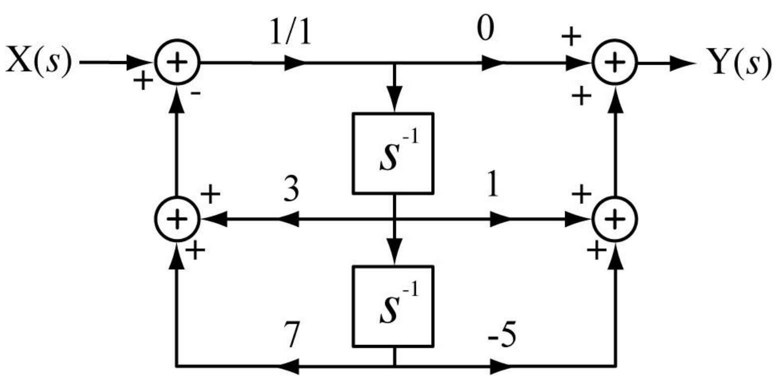
Direct Form II Realization



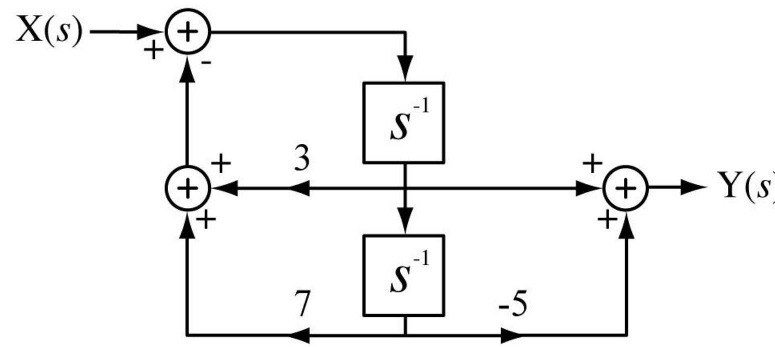
Direct Form II Realization

A system is defined by $\mathrm{y}''(t)+3\mathrm{y}'(t)+7\mathrm{y}(t)=\mathrm{x}'(t)-5\mathrm{x}(t)$.

$$\mathrm{H}(s)=\frac{s-5}{s^2+3s+7}$$



Simplified



Inverse Laplace Transform

There is an inversion integral

$$\mathrm{y}(t) = \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} \mathrm{Y}(s) e^{st} ds \quad , \quad s = \sigma + j\omega$$

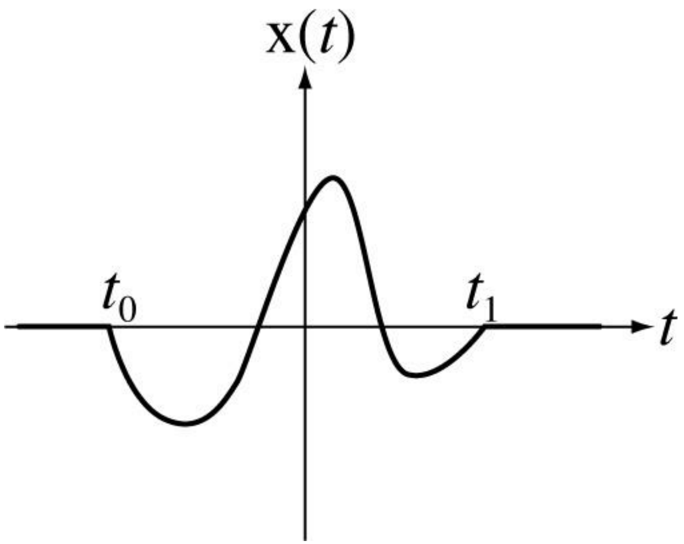
for finding $\mathrm{y}(t)$ from $\mathrm{Y}(s)$, but it is rarely used in practice.

Usually inverse Laplace transforms are found by using tables of standard functions and the properties of the Laplace transform.

Existence of the Laplace Transform

Time Limited Signals

If $x(t) = 0$ for $t < t_0$ and $t > t_1$ it is a **time limited** signal. If $x(t)$ is also bounded for all t, the Laplace transform integral converges and the Laplace transform exists for all s.



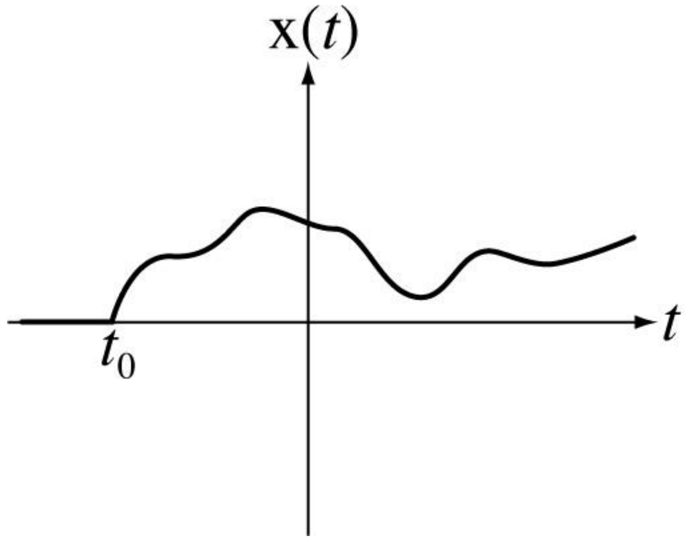
Existence of the Laplace Transform

$$\text{Let } \mathrm{x}(t) = \mathrm{rect}(t) = \mathrm{u}(t + 1/2) - \mathrm{u}(t - 1/2).$$

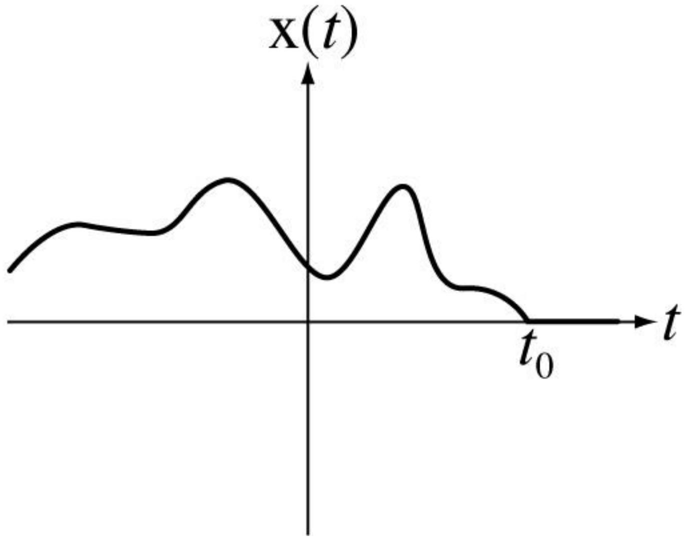
$$\mathrm{X}(s) = \int_{-\infty}^{\infty} \mathrm{rect}(t) e^{-st}\, dt = \int_{-1/2}^{1/2} e^{-st}\, dt = \frac{e^{-s/2} - e^{s/2}}{-s} = \frac{e^{s/2} - e^{-s/2}}{s}\ , \text{All } s$$

Existence of the Laplace Transform

Right- and Left-Sided Signals



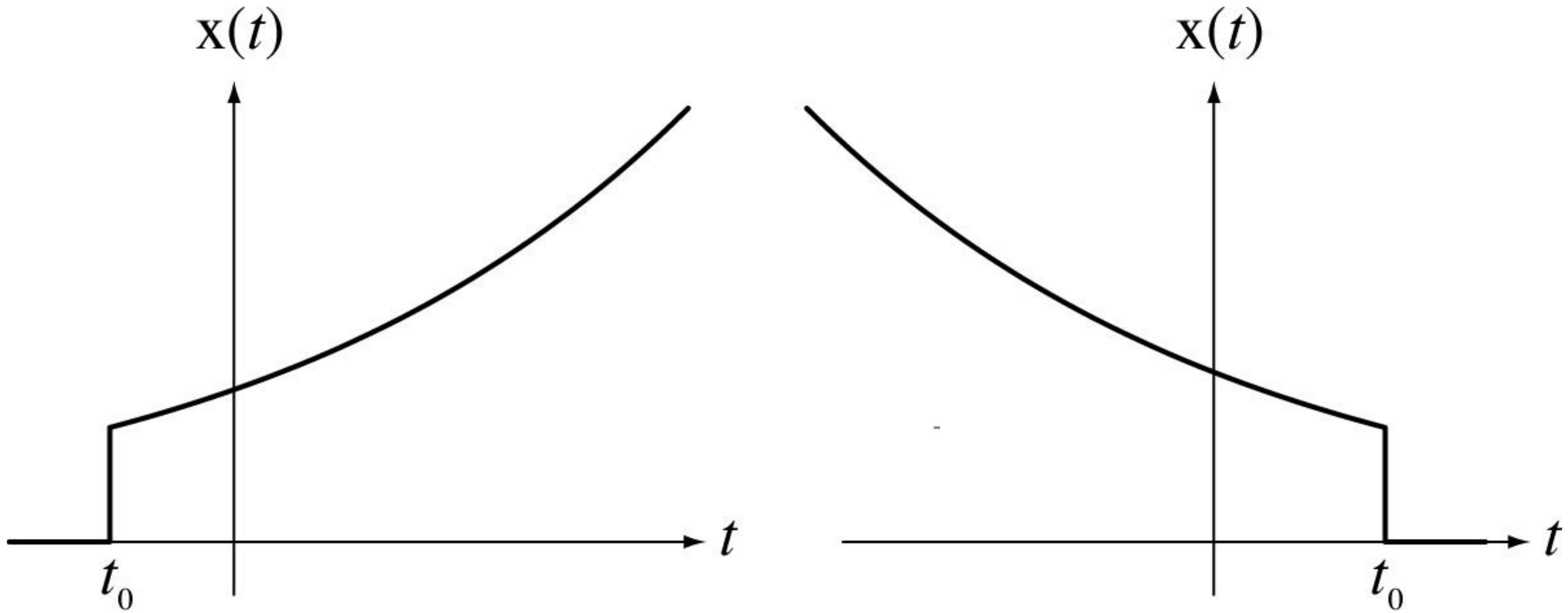
Right-Sided



Left-Sided

Existence of the Laplace Transform

Right- and Left-Sided Exponentials



Right-Sided

$$x(t) = e^{\alpha t}\,u(t - t_0)\ ,\ \alpha \in \mathbb{R}$$

Left-Sided

$$x(t) = e^{\beta t}\,u(t_0 - t)\ ,\ \beta \in \mathbb{R}$$

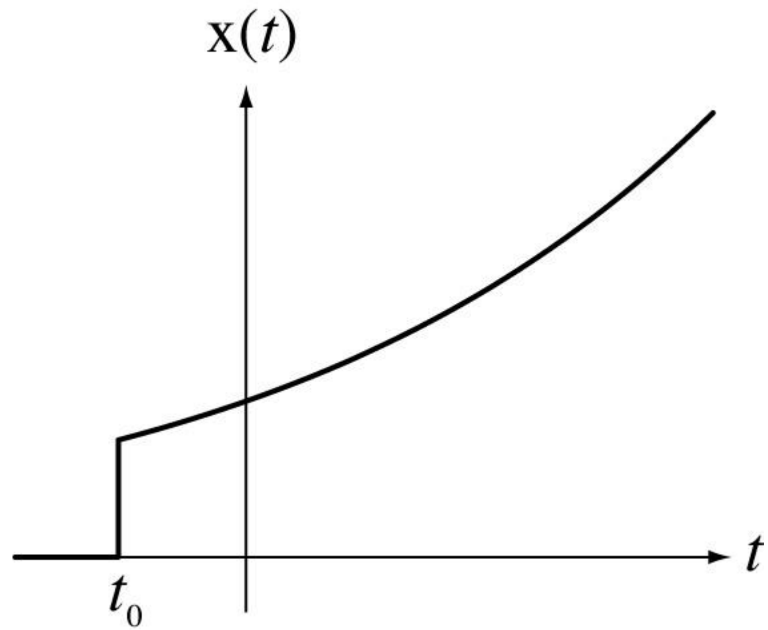
Existence of the Laplace Transform

Right-Sided Exponential

$$x(t) = e^{\alpha t}\,u(t - t_0)\ ,\ \ \alpha \in \mathbb{R}$$

$$X(s) = \int_{t_0}^{\infty} e^{\alpha t} e^{-st}\,dt = \int_{t_0}^{\infty} e^{(\alpha-\sigma)t} e^{-j\omega t}\,dt$$

If $\mathrm{Re}(s) = \sigma > \alpha$ the asymptotic behavior of $e^{(\alpha-\sigma)t} e^{-j\omega t}$ as $t \to \infty$ is to approach zero and the Laplace transform integral converges.



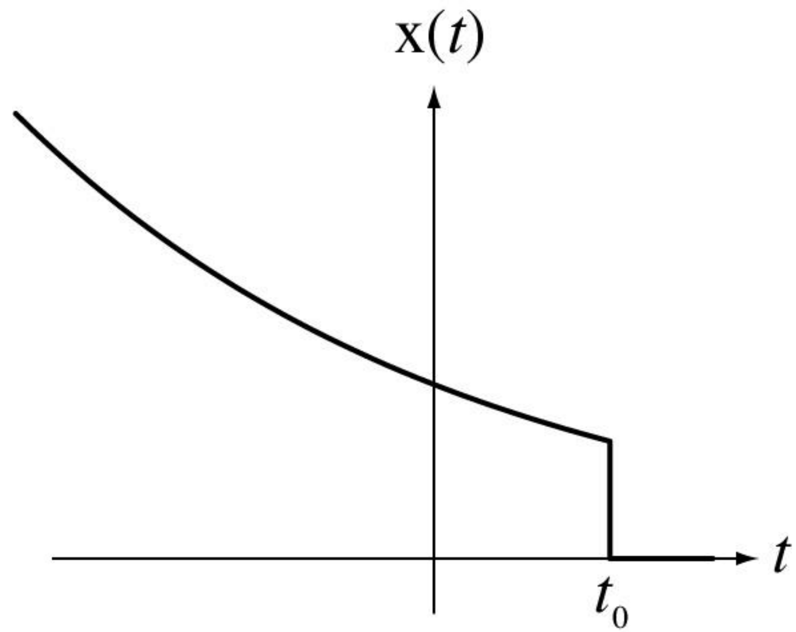
Existence of the Laplace Transform

Left-Sided Exponential

$$\mathrm{x}(t) = e^{\beta t}\,\mathrm{u}(t_0 - t)\ ,\ \beta \in \mathbb{R}$$

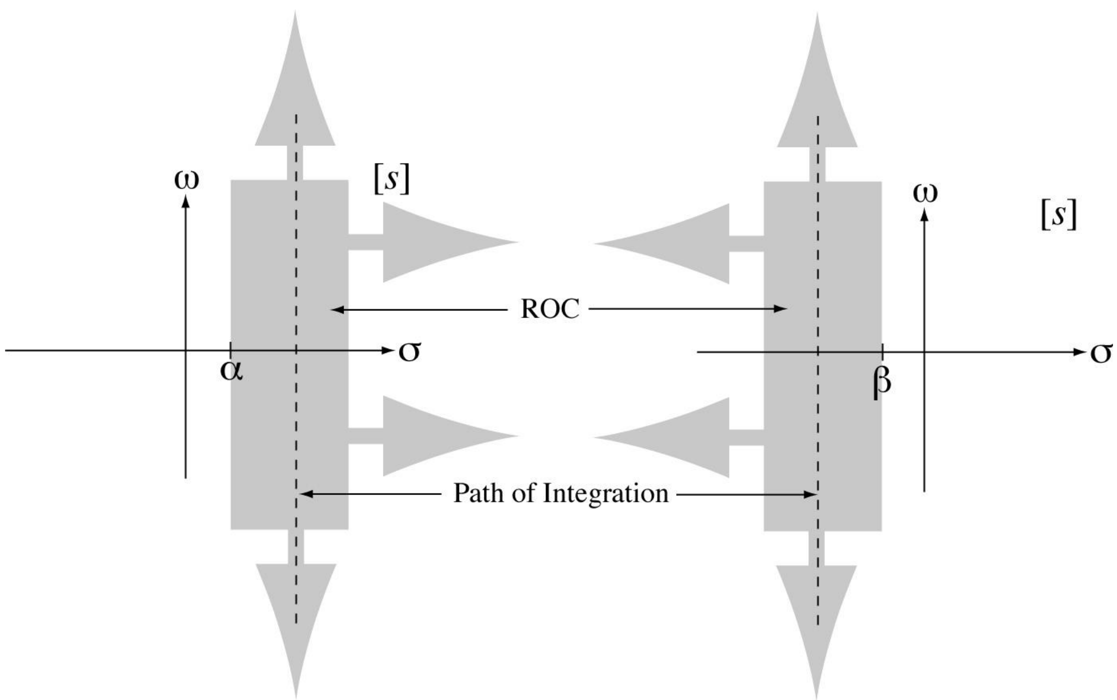
$$\mathrm{X}(s) = \int_{-\infty}^{t_0} e^{\beta t} e^{-st}\,dt = \int_{-\infty}^{t_0} e^{(\beta - \sigma)t} e^{-j\omega t}\,dt$$

If $\sigma < \beta$ the asymptotic behavior of $e^{(\beta-\sigma)t}e^{-j\omega t}$ as $t \to -\infty$ is to approach zero and the Laplace transform integral converges.



Existence of the Laplace Transform

The two conditions $s > a$ and $s < b$ define the **region of convergence** (**ROC**) for the Laplace transform of right- and left-sided signals.



Existence of the Laplace Transform

Any right-sided signal that grows no faster than an exponential in positive time and any left-sided signal that grows no faster than an exponential in negative time has a Laplace transform.

If $x(t) = x_r(t) + x_l(t)$ where $x_r(t)$ is the right-sided part and $x_l(t)$ is the left-sided part and if $|x_r(t)| < K_r e^{\alpha t}$ and $|x_l(t)| < K_l e^{\beta t}$ and α and β are as small as possible, then the Laplace-transform integral converges and the Laplace transform exists for $\alpha < \sigma < \beta$. Therefore if $\alpha < \beta$ the ROC is the region $\alpha < \beta$. If $\alpha > \beta$, there is no ROC and the Laplace transform does not exist.

Laplace Transform Pairs

The Laplace transform of $g_1(t) = Ae^{\alpha t}\,u(t)$ is

$$G_1(s) = \int_{-\infty}^{\infty} Ae^{\alpha t}\,u(t)\,e^{-st}\,dt = A\int_{0}^{\infty} e^{-(s-\alpha)t}\,dt = A\int_{0}^{\infty} e^{(\alpha-\sigma)t}\,e^{-j\omega t}\,dt = \frac{A}{s-\alpha}$$

This function has a **pole** at $s = \alpha$ and the ROC is the region to the right of that point. The Laplace transform of $g_2(t) = Ae^{\beta t}\,u(-t)$ is

$$G_2(s) = \int_{-\infty}^{\infty} Ae^{\beta t}\,u(-t)\,e^{-st}\,dt = A\int_{-\infty}^{0} e^{(\beta-s)t}\,dt = A\int_{-\infty}^{0} e^{(\beta-\sigma)t}\,e^{-j\omega t}\,dt = -\frac{A}{s-\beta}$$

This function has a pole at $s = \beta$ and the ROC is the region to the left of that point.

Region of Convergence

The following two Laplace transform pairs illustrate the importance of the region of convergence.

$$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha} \;,\; \sigma > -\alpha$$

$$-e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha} \;,\; \sigma < -\alpha$$

The two time-domain functions are different but the algebraic expressions for their Laplace transforms are the same. Only the ROC's are different.

Region of Convergence

Some of the most common Laplace transform pairs

(There is more extensive table in the book.)

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1 \;,\; \text{All } \sigma$$

$$\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 1/s \;,\; \sigma > 0 \qquad -\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} 1/s \;,\; \sigma < 0$$

$$\mathrm{ramp}(t) = t\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 1/s^2 \;,\; \sigma > 0 \qquad \mathrm{ramp}(-t) = -t\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} 1/s^2 \;,\; \sigma < 0$$

$$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 1/(s+\alpha) \;,\; \sigma > -\alpha \qquad -e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} 1/(s+\alpha) \;,\; \sigma < -\alpha$$

$$e^{-\alpha t}\sin(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2} \;,\; \sigma > -\alpha \qquad -e^{-\alpha t}\sin(\omega_0 t)\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2} \;,\; \sigma < -\alpha$$

$$e^{-\alpha t}\cos(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2} \;,\; \sigma > -\alpha \qquad -e^{-\alpha t}\cos(\omega_0 t)\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2} \;,\; \sigma < -\alpha$$

Laplace Transform Example

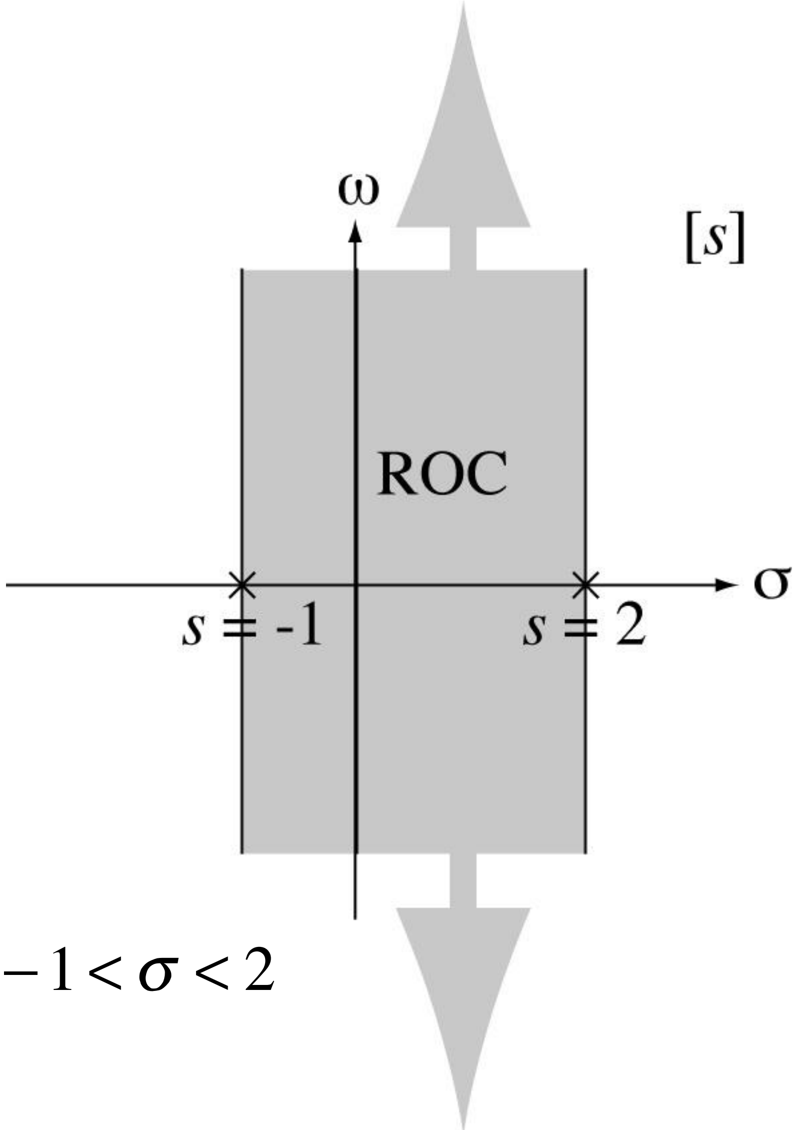
Find the Laplace transform of

$$x(t) = e^{-t}\,u(t) + e^{2t}\,u(-t)$$

$$e^{-t}\,u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} \;,\; \sigma > -1$$

$$e^{2t}\,u(-t) \xleftrightarrow{\mathcal{L}} -\frac{1}{s-2} \;,\; \sigma < 2$$

$$e^{-t}\,u(t) + e^{2t}\,u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} - \frac{1}{s-2} \;,\; -1 < \sigma < 2$$



Laplace Transform Example

Find the inverse Laplace transform of

$$\mathrm{X}(s) = \frac{4}{s+3} - \frac{10}{s-6} \;, \; -3 < \sigma < 6$$

The ROC tells us that $\dfrac{4}{s+3}$ must inverse transform into a right-sided signal and that $\dfrac{10}{s-6}$ must inverse transform into a left-sided signal.

$$\mathrm{x}(t) = 4e^{-3t}\,\mathrm{u}(t) + 10e^{6t}\,\mathrm{u}(-t)$$

Laplace Transform Example

Find the inverse Laplace transform of

$$X(s) = \frac{4}{s+3} - \frac{10}{s-6} \ , \ \sigma > 6$$

The ROC tells us that both terms must inverse transform into a right-sided signal.

$$x(t) = 4e^{-3t}\,u(t) - 10e^{6t}\,u(t)$$

Laplace Transform Example

Find the inverse Laplace transform of

$$\mathrm{X}(s) = \frac{4}{s+3} - \frac{10}{s-6} \ , \ s < -3$$

The ROC tells us that both terms must inverse transform into a left-sided signal.

$$\mathrm{x}(t) = -4e^{-3t}\,\mathrm{u}(-t) + 10e^{6t}\,\mathrm{u}(-t)$$

MATLAB System Objects

A MATLAB system object is a special kind of variable in MATLAB that contains all the information about a system. It can be created with the `tf` command whose syntax is

```
sys = tf(num,den)
```

where `num` is a vector of numerator coefficients of powers of s, `den` is a vector of denominator coefficients of powers of s, both in descending order and `sys` is the system object.

MATLAB System Objects

For example, the transfer function

$$H_1(s) = \frac{s^2 + 4}{s^5 + 4s^4 + 7s^3 + 15s^2 + 31s + 75}$$

can be created by the commands

```
»num = [1 0 4] ;  den = [1 4 7 15 31 75] ;

»H1 = tf(num,den) ;

»H1

Transfer function:

                 s^2 + 4

----------------------------------------

  s^5 + 4 s^4 + 7 s^3 + 15 s^2 + 31 s + 75
```

Partial-Fraction Expansion

The inverse Laplace transform can always be found (in principle at least) by using the inversion integral. But that is rare in engineering practice. The most common type of Laplace-transform expression is a ratio of polynomials in s,

$$\mathrm{G}(s)=\frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots a_1 s + a_0}$$

The denominator can be factored, putting it into the form,

$$\mathrm{G}(s)=\frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_1 s + b_0}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

Partial-Fraction Expansion

For now, assume that there are no repeated poles and that $N > M$, making the fraction **proper** in s. Then it is possible to write the expression in the **partial fraction** form,

$$\mathrm{G}(s) = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \cdots + \frac{K_N}{s-p_N}$$

where

$$\frac{b_M s^M + b_{M-1} s^{M-1} + \cdots b_1 s + b_0}{(s-p_1)(s-p_2)\cdots(s-p_N)} = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \cdots + \frac{K_N}{s-p_N}$$

The K's can be found be any convenient method.

Partial-Fraction Expansion

Multiply both sides by $s - p_1$

$$\left(s-p_1\right)\frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_1 s + b_0}{\left(s-p_1\right)\left(s-p_2\right)\cdots\left(s-p_N\right)} = \left[\begin{array}{c} K_1 + \left(s-p_1\right)\dfrac{K_2}{s-p_2} + \cdots \\ + \left(s-p_1\right)\dfrac{K_N}{s-p_N} \end{array}\right]$$

$$K_1 = \frac{b_M p_1^M + b_M p_1^{M-1} + \cdots + b_1 p_1 + b_0}{\left(p_1 - p_2\right)\cdots\left(p_1 - p_N\right)}$$

All the K's can be found by the same method and the inverse Laplace transform is then found by table look-up.

Partial-Fraction Expansion

$$\mathrm{H}(s) = \frac{10s}{(s+4)(s+9)} = \frac{K_1}{s+4} + \frac{K_2}{s+9} \quad , \quad s > -4$$

$$K_1 = \left[\cancel{(s+4)} \frac{10s}{\cancel{(s+4)}(s+9)} \right]_{s=-4} = \left[\frac{10s}{s+9} \right]_{s=-4} = \frac{-40}{5} = -8$$

$$K_2 = \left[\cancel{(s+9)} \frac{10s}{(s+4)\cancel{(s+9)}} \right]_{s=-9} = \left[\frac{10s}{s+4} \right]_{s=-9} = \frac{-90}{-5} = 18$$

$$\mathrm{H}(s) = \frac{-8}{s+4} + \frac{18}{s+9} = \frac{-8s - 72 + 18s + 72}{(s+4)(s+9)} = \frac{10s}{(s+4)(s+9)} \quad . \text{ Check.}$$

$$\downarrow\downarrow\downarrow\downarrow\downarrow$$

$$\mathrm{h}(t) = \left(-8e^{-4t} + 18e^{-9t}\right)\mathrm{u}(t)$$

Partial-Fraction Expansion

If the expression has a repeated pole of the form,

$$\mathrm{G}(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{(s - p_1)^2 (s - p_3) \cdots (s - p_N)}$$

the partial fraction expansion is of the form,

$$\mathrm{G}(s) = \frac{K_{12}}{(s - p_1)^2} + \frac{K_{11}}{s - p_1} + \frac{K_3}{s - p_3} + \cdots + \frac{K_N}{s - p_N}$$

and K_{12} can be found using the same method as before.

But K_{11} cannot be found using the same method.

Partial-Fraction Expansion

Instead K_{11} can be found by using the more general formula

$$K_{qk} = \frac{1}{(m-k)!} \frac{d^{m-k}}{ds^{m-k}} \left[\left(s - p_q\right)^m \mathrm{H}(s) \right]_{s \to p_q} \quad , \quad k = 1, 2, \cdots, m$$

where m is the order of the qth pole, which applies to repeated poles of any order.

If the expression is not a proper fraction in s the partial-fraction method will not work. But it is always possible to **synthetically divide** the numerator by the denominator until the remainder is a proper fraction and then apply partial-fraction expansion.

Partial-Fraction Expansion

$$\mathrm{H}(s)=\frac{10s}{(s+4)^2(s+9)}=\frac{K_{12}}{(s+4)^2}+\frac{K_{11}}{s+4}+\frac{K_2}{s+9}\quad,\ s>4$$

$\uparrow$

Repeated Pole

$$K_{12}=\left[\cancel{(s+4)^2}\frac{10s}{\cancel{(s+4)^2}(s+9)}\right]_{s=-4}=\frac{-40}{5}=-8$$

Using

$$K_{qk}=\frac{1}{(m-k)!}\frac{d^{m-k}}{ds^{m-k}}\left[(s-p_q)^m\mathrm{H}(s)\right]_{s\to p_q}\quad,\ k=1,2,\cdots,m$$

$$K_{11}=\frac{1}{(2-1)!}\frac{d^{2-1}}{ds^{2-1}}\left[(s+4)^2\mathrm{H}(s)\right]_{s\to-4}=\frac{d}{ds}\left[\frac{10s}{s+9}\right]_{s\to-4}$$

Partial-Fraction Expansion

$$K_{11} = \left[\frac{(s+9)10 - 10s}{(s+9)^2}\right]_{s=-4} = \frac{18}{5}$$

$$K_2 = -\frac{18}{5} \Rightarrow \mathrm{H}(s) = \frac{-8}{(s+4)^2} + \frac{18/5}{s+4} + \frac{-18/5}{s+9} \quad , \quad s > -4$$

$$\mathrm{H}(s) = \frac{-8s - 72 + \frac{18}{5}\left(s^2 + 13s + 36\right) - \frac{18}{5}\left(s^2 + 8s + 16\right)}{(s+4)^2 (s+9)} \quad , \quad s > -4$$

$$\mathrm{H}(s) = \frac{10s}{(s+4)^2 (s+9)} \quad , \quad s > -4$$

$$\mathrm{h}(t) = \left(-8te^{-4t} + \frac{18}{5}e^{-4t} - \frac{18}{5}e^{-9t}\right)\mathrm{u}(t)$$

Partial-Fraction Expansion

$$\mathrm{H}(s) = \frac{10s^2}{(s+4)(s+9)} \quad , \quad s > -4 \leftarrow \textbf{Improper in s}$$

$$\mathrm{H}(s) = \frac{10s^2}{s^2+13s+36} \quad , \quad s > -4$$

Synthetic Division $\rightarrow$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad\quad 10 \\ \hline s^2+13s+36 & 10s^2 \\ & \underline{10s^2+130s+360} \\ & \quad\; -130s-360 \end{array}$$

$$\mathrm{H}(s) = 10 - \frac{130s+360}{(s+4)(s+9)} = 10 - \left[\frac{-32}{s+4} + \frac{162}{s+9}\right] \quad , \quad s > -4$$

$$\mathrm{h}(t) = 10\delta(t) - \left[162e^{-9t} - 32e^{-4t}\right]\mathrm{u}(t)$$

Inverse Laplace Transform Example

Method 1

$$\mathrm{G}(s) = \frac{s}{(s-3)(s^2-4s+5)} \quad , \quad s < 2$$

$$\mathrm{G}(s) = \frac{s}{(s-3)(s-2+j)(s-2-j)} \quad , \quad s < 2$$

$$\mathrm{G}(s) = \frac{3/2}{s-3} - \frac{(3+j)/4}{s-2+j} - \frac{(3-j)/4}{s-2-j} \quad , \quad s < 2$$

$$\mathrm{g}(t) = \left(-\frac{3}{2}e^{3t} + \frac{3+j}{4}e^{(2-j)t} + \frac{3-j}{4}e^{(2+j)t} \right) \mathrm{u}(-t)$$

Inverse Laplace Transform Example

$$g(t) = \left(-\frac{3}{2}e^{3t} + \frac{3+j}{4}e^{(2-j)t} + \frac{3-j}{4}e^{(2+j)t} \right) u(-t)$$

This looks like a function of time that is complex-valued. But, with the use of some trigonometric identities it can be put into the form

$$g(t) = (3/2)\left\{ e^{2t}\left[\cos(t) + (1/3)\sin(t)\right] - e^{3t} \right\} u(-t)$$

which has only real values.

Inverse Laplace Transform Example

Method 2

$$\mathrm{G}(s) = \frac{s}{(s-3)\left(s^2-4s+5\right)} \quad , \quad \sigma < 2$$

$$\mathrm{G}(s) = \frac{s}{(s-3)(s-2+j)(s-2-j)} \quad , \quad \sigma < 2$$

$$\mathrm{G}(s) = \frac{3/2}{s-3} - \frac{(3+j)/4}{s-2+j} - \frac{(3-j)/4}{s-2-j} \quad , \quad \sigma < 2$$

Getting a common denominator and simplifying

$$\mathrm{G}(s) = \frac{3/2}{s-3} - \frac{1}{4}\frac{6s-10}{s^2-4s+5} = \frac{3/2}{s-3} - \frac{6}{4}\frac{s-5/3}{(s-2)^2+1} \quad , \quad \sigma < 2$$

Inverse Laplace Transform Example

$$\text{Method 2}$$

$$\mathrm{G}(s) = \frac{3/2}{s-3} - \frac{6}{4}\frac{s-5/3}{(s-2)^2+1} \quad , \quad s<2$$

The denominator of the second term has the form of the Laplace transform of a damped cosine or damped sine but the numerator is not yet in the correct form. But by adding and subtracting the correct expression from that term and factoring we can put it into the form

$$\mathrm{G}(s) = \frac{3/2}{s-3} - \frac{3}{2}\left[\frac{s-2}{(s-2)^2+1} + \frac{1/3}{(s-2)^2+1}\right] \quad , \quad s<2$$

Inverse Laplace Transform Example

$$\text{Method 2}$$

$$\mathrm{G}(s) = \frac{3/2}{s-3} - \frac{3}{2}\left[\frac{s-2}{(s-2)^2+1} + \frac{1/3}{(s-2)^2+1}\right] \ , \ \sigma < 2$$

This can now be directly inverse Laplace transformed into

$$\mathrm{g}(t) = (3/2)\left\{e^{2t}\left[\cos(t) + (1/3)\sin(t)\right] - e^{3t}\right\}\mathrm{u}(-t)$$

which is the same as the previous result.

Inverse Laplace Transform Example

Method 3

When we have a pair of poles p_2 and p_3 that are complex conjugates

we can convert the form $\mathrm{G}(s) = \dfrac{A}{s-3} + \dfrac{K_2}{s-p_2} + \dfrac{K_3}{s-p_3}$ into the

form $\mathrm{G}(s) = \dfrac{A}{s-3} + \dfrac{s(K_2+K_3) - K_3p_2 - K_2p_3}{s^2 - (p_1+p_2)s + p_1p_2} = \dfrac{A}{s-3} + \dfrac{Bs+C}{s^2 - (p_1+p_2)s + p_1p_2}$

In this example we can find the constants A, B and C by realizing that

$$\mathrm{G}(s) = \frac{s}{(s-3)(s^2-4s+5)} \circ \frac{A}{s-3} + \frac{Bs+C}{s^2-4s+5} \quad , \quad s < 2$$

is not just an equation, it is an **identity**. That means it must be an equality for any value of s.

Inverse Laplace Transform Example

Method 3

A can be found as before to be $3/2$. Letting $s = 0$, the identity becomes $0 \equiv -\frac{3/2}{3} + \frac{C}{5}$ and $C = 5/2$. Then, letting $s = 1$, and solving we get $B = -3/2$. Now

$$G(s) = \frac{3/2}{s-3} + \frac{(-3/2)s + 5/2}{s^2 - 4s + 5} \quad , \quad s < 2$$

or

$$G(s) = \frac{3/2}{s-3} - \frac{3}{2}\frac{s - 5/3}{s^2 - 4s + 5} \quad , \quad s < 2$$

This is the same as a result in Method 2 and the rest of the solution is also the same. The advantage of this method is that all the numbers are real.

Use of MATLAB in Partial Fraction Expansion

MATLAB has a function `residue` that can be very helpful in partial fraction expansion. Its syntax is `[r,p,k] = residue(b,a)` where `b` is a vector of coefficients of descending powers of s in the numerator of the expression and `a` is a vector of coefficients of descending powers of s in the denominator of the expression, `r` is a vector of residues, `p` is a vector of finite pole locations and `k` is a vector of so-called direct terms which result when the degree of the numerator is equal to or greater than the degree of the denominator. For our purposes, residues are simply the numerators in the partial-fraction expansion.

Laplace Transform Properties

Let $\mathrm{g}(t)$ and $\mathrm{h}(t)$ form the transform pairs, $\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)$

and $\mathrm{h}(t) \xleftrightarrow{\mathcal{L}} \mathrm{H}(s)$ with ROC's, ROC_G and ROC_H respectively.

Linearity $\alpha \mathrm{g}(t) + \beta \mathrm{h}(t) \xleftrightarrow{\mathcal{L}} \alpha \mathrm{G}(s) + \beta \mathrm{H}(s)$

$$\mathrm{ROC} \supseteq \mathrm{ROC}_\mathrm{G} \cap \mathrm{ROC}_\mathrm{H}$$

Time Shifting $\mathrm{g}(t - t_0) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s) e^{-st_0}$

$$\mathrm{ROC} = \mathrm{ROC}_\mathrm{G}$$

***s*-Domain Shift** $e^{s_0 t} \mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s - s_0)$

$\mathrm{ROC} = \mathrm{ROC}_\mathrm{G}$ shifted by s_0,

(s is in ROC if $s - s_0$ is in ROC_G)

Laplace Transform Properties

Time Scaling

$$g(at) \xleftrightarrow{\mathcal{L}} (1/|a|)\,G(s/a)$$

$$\text{ROC} = \text{ROC}_G \text{ scaled by } a$$

$$(s \text{ is in ROC if } s/a \text{ is in } \text{ROC}_G)$$

Time Differentiation

$$\frac{d}{dt}g(t) \xleftrightarrow{\mathcal{L}} sG(s)$$

$$\text{ROC} \supseteq \text{ROC}_G$$

***s*-Domain Differentiation**

$$-t\,g(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}G(s)$$

$$\text{ROC} = \text{ROC}_G$$

Laplace Transform Properties

Convolution in Time

$$\mathrm{g}(t) * \mathrm{h}(t) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)\mathrm{H}(s)$$

$$\mathrm{ROC} \supseteq \mathrm{ROC}_{\mathrm{G}} \cap \mathrm{ROC}_{\mathrm{H}}$$

Time Integration

$$\int_{-\infty}^{t} \mathrm{g}(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)/s$$

$$\mathrm{ROC} \supseteq \mathrm{ROC}_{\mathrm{G}} \cap (\sigma > 0)$$

If $\mathrm{g}(t) = 0$, $t < 0$ and there are no impulses or higher-order singularities at $t = 0$ then

Initial Value Theorem:

$$\mathrm{g}(0^{+}) = \lim_{s \to \infty} s\,\mathrm{G}(s)$$

Final Value Theorem:

$$\lim_{t \to \infty} \mathrm{g}(t) = \lim_{s \to 0} s\,\mathrm{G}(s) \text{ if } \lim_{t \to \infty} \mathrm{g}(t) \text{ exists}$$

Laplace Transform Properties

Final Value Theorem $\lim_{t\to\infty} \mathrm{g}(t) = \lim_{s\to 0} s\,\mathrm{G}(s)$

This theorem only applies if the limit $\lim_{t\to\infty} \mathrm{g}(t)$ actually exists.

It is possible for the limit $\lim_{s\to 0} s\,\mathrm{G}(s)$ to exist even though the limit $\lim_{t\to\infty} \mathrm{g}(t)$ does not exist. For example

$$\mathrm{x}(t) = \cos(\omega_0 t) \xleftrightarrow{\mathcal{L}} \mathrm{X}(s) = \frac{s}{s^2 + \omega_0^2}$$

$$\lim_{s\to 0} s\,\mathrm{X}(s) = \lim_{s\to 0} \frac{s^2}{s^2 + \omega_0^2} = 0$$

but $\lim_{t\to\infty} \cos(\omega_0 t)$ does not exist.

Laplace Transform Properties

Final Value Theorem

The final value theorem applies to a function $\mathrm{G}(s)$ if all the poles of $s\mathrm{G}(s)$ lie in the open left half of the s plane. Be sure to notice that <u>this does not say</u> that all the poles of $\mathrm{G}(s)$ must lie in the open left half of the s plane. $\mathrm{G}(s)$ could have a single pole at $s = 0$ and the final value theorem would still apply.

Use of Laplace Transform Properties

Find the Laplace transforms of $\mathrm{x}(t)=\mathrm{u}(t)-\mathrm{u}(t-a)$ and $\mathrm{x}(2t)=\mathrm{u}(2t)-\mathrm{u}(2t-a)$. From the table $\mathrm{u}(t)\xleftrightarrow{\mathcal{L}}1/s,\ \sigma>0.$

Then, using the time-shifting property $\mathrm{u}(t-a)\xleftrightarrow{\mathcal{L}}e^{-as}/s,\ \sigma>0.$

Using the linearity property $\mathrm{u}(t)-\mathrm{u}(t-a)\xleftrightarrow{\mathcal{L}}\left(1-e^{-as}\right)/s,\ \sigma>0.$

Using the time-scaling property

$$\mathrm{u}(2t)-\mathrm{u}(2t-a)\xleftrightarrow{\mathcal{L}}\frac{1}{2}\left[\frac{1-e^{-as}}{s}\right]_{s\to s/2}=\frac{1-e^{-as/2}}{s},\ \sigma>0$$

Use of Laplace Transform Properties

Use the s-domain differentiation property and

$$\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} 1/s \;, \sigma > 0$$

to find the inverse Laplace transform of $1/s^2$. The s-domain differentiation property is $-t\,\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}\left(\mathrm{G}(s)\right)$. Then

$-t\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}\left(\frac{1}{s}\right) = -\frac{1}{s^2}$. Then using the linearity property

$$t\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}.$$

The Unilateral Laplace Transform

In most practical signal and system analysis using the Laplace transform a modified form of the transform, called the **unilateral Laplace transform**, is used. The unilateral Laplace transform is defined by $G(s) = \int_{0^-}^{\infty} g(t) e^{-st}\, dt$. The only difference between this version and the previous definition is the change of the lower integration limit from $-\infty$ to 0^-. With this definition, all the Laplace transforms of causal functions are the same as before with the same ROC, the region of the s plane to the right of all the finite poles.

The Unilateral Laplace Transform

The unilateral Laplace transform integral excludes negative time. If a function has non-zero behavior in negative time its unilateral and bilateral transforms will be different. Also functions with the same positive time behavior but different negative time behavior will have the same unilateral Laplace transform. Therefore, to avoid ambiguity and confusion, the unilateral Laplace transform should only be used in analysis of causal signals and systems. This is a limitation but in most practical analysis this limitation is not significant and the unilateral Laplace transform actually has advantages.

The Unilateral Laplace Transform

The main advantage of the unilateral Laplace transform is that the ROC is simpler than for the bilateral Laplace transform and, in most practical analysis, involved consideration of the ROC is unnecessary. The inverse Laplace transform is unchanged. It is

$$g(t) = \frac{1}{j2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} G(s) e^{+st} ds$$

The Unilateral Laplace Transform

Some of the properties of the unilateral Laplace transform are different from the bilateral Laplace transform.

Time-Shifting $\quad \mathrm{g}(t-t_0) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)e^{-st_0} \;,\; t_0 > 0$

Time Scaling $\quad \mathrm{g}(at) \xleftrightarrow{\mathcal{L}} (1/|a|)\mathrm{G}(s/a) \;,\; a > 0$

First Time Derivative $\quad \frac{d}{dt}\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} s\,\mathrm{G}(s) - \mathrm{g}(0^-)$

*N*th Time Derivative $\quad \frac{d^N}{dt^N}(\mathrm{g}(t)) \xleftrightarrow{\mathcal{L}} s^N\,\mathrm{G}(s) - \sum_{n=1}^{N} s^{N-n}\left[\frac{d^{n-1}}{dt^{n-1}}(\mathrm{g}(t))\right]_{t=0^-}$

Time Integration $\quad \int_{0^-}^{t} \mathrm{g}(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)/s$

The Unilateral Laplace Transform

The time shifting property applies only for shifts to the right because a shift to the left could cause a signal to become non-causal. For the same reason scaling in time must only be done with positive scaling coefficients so that time is not reversed producing an anti-causal function. The derivative property must now take into account the initial value of the function at time $t = 0^-$ and the integral property applies only to functional behavior after time $t = 0$. Since the unilateral and bilateral Laplace transforms are the same for causal functions, the bilateral table of transform pairs can be used for causal functions.

The Unilateral Laplace Transform

The Laplace transform was developed for the solution of differential equations and the unilateral form is especially well suited for solving differential equations with initial conditions. For example,

$$\frac{d^2}{dt^2}\left[\mathrm{x}(t)\right]+7\frac{d}{dt}\left[\mathrm{x}(t)\right]+12\,\mathrm{x}(t)=0$$

with initial conditions $\mathrm{x}(0^-)=2$ and $\frac{d}{dt}\left(\mathrm{x}(t)\right)_{t=0^-} = -4.$

Laplace transforming both sides of the equation, using the new derivative property for unilateral Laplace transforms,

$$s^2\,\mathrm{X}(s) - s\,\mathrm{x}(0^-) - \frac{d}{dt}\left(\mathrm{x}(t)\right)_{t=0^-} + 7\left[s\,\mathrm{X}(s) - \mathrm{x}(0^-)\right]+12\,\mathrm{X}(s)=0$$

The Unilateral Laplace Transform

Solving for $\mathrm{X}(s)$

$$\mathrm{X}(s) = \frac{s\,\overbrace{\mathrm{x}(0^-)}^{=2} + 7\,\overbrace{\mathrm{x}(0^-)}^{=2} + \overbrace{\frac{d}{dt}(\mathrm{x}(t))_{t=0^-}}^{=-4}}{s^2 + 7s + 12}$$

or $\mathrm{X}(s) = \dfrac{2s+10}{s^2+7s+12} = \dfrac{4}{s+3} - \dfrac{2}{s+4}$. The inverse transform yields

$\mathrm{x}(t) = \left(4e^{-3t} - 2e^{-4t}\right)\mathrm{u}(t)$. This solution solves the differential equation with the given initial conditions.

Pole-Zero Diagrams and Frequency Response

If the transfer function of a stable system is $H(s)$, the frequency response is $H(j\omega)$. The most common type of transfer function is of the form,

$$\mathrm{H}(s) = A\frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

Therefore $H(j\omega)$ is

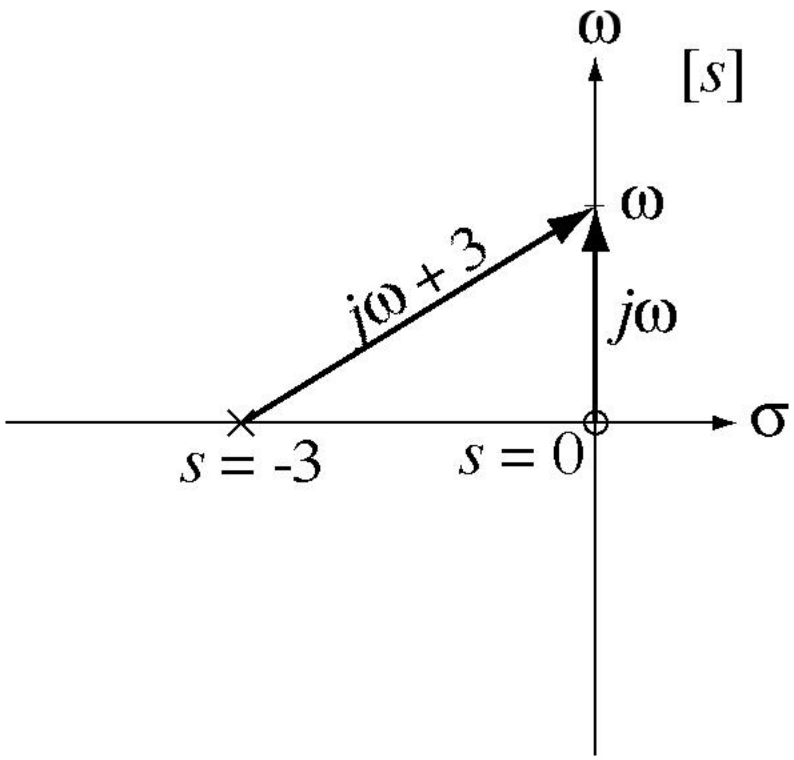
$$\mathrm{H}(j\omega) = A\frac{(j\omega-z_1)(j\omega-z_2)\cdots(j\omega-z_M)}{(j\omega-p_1)(j\omega-p_2)\cdots(j\omega-p_N)}$$

Pole-Zero Diagrams and Frequency Response

Let $\mathrm{H}(s) = \dfrac{3s}{s+3}$.

$$\mathrm{H}(j\omega) = 3\,\frac{j\omega}{j\omega + 3}$$

The numerator $j\omega$ and the denominator $j\omega + 3$ can be conceived as vectors in the s plane.



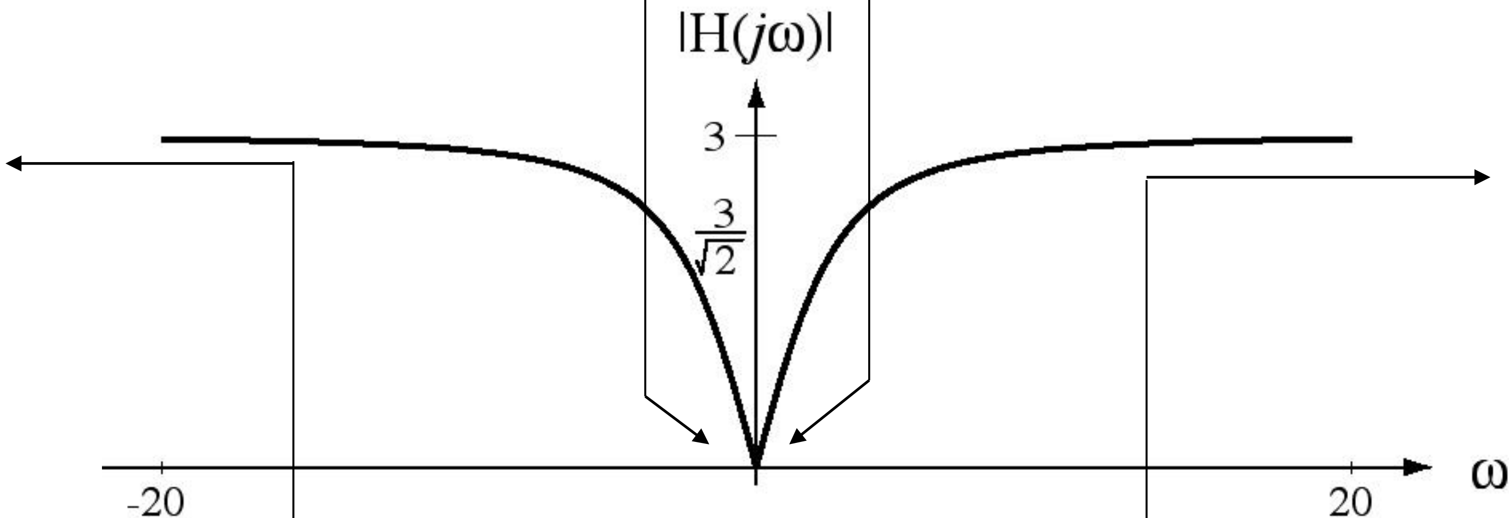
$$\left|\mathrm{H}(j\omega)\right| = 3\,\frac{\left|j\omega\right|}{\left|j\omega + 3\right|}$$

$$\measuredangle\,\mathrm{H}(j\omega) = \underbrace{\measuredangle 3}_{=0} + \measuredangle j\omega - \measuredangle(j\omega + 3)$$

Pole-Zero Diagrams and Frequency Response

$$\lim_{w\to 0^-} |H(jw)| = \lim_{w\to 0^-} 3\frac{|jw|}{|jw+3|} = 0$$

$$\lim_{w\to 0^+} |H(jw)| = \lim_{w\to 0^+} 3\frac{|jw|}{|jw+3|} = 0$$



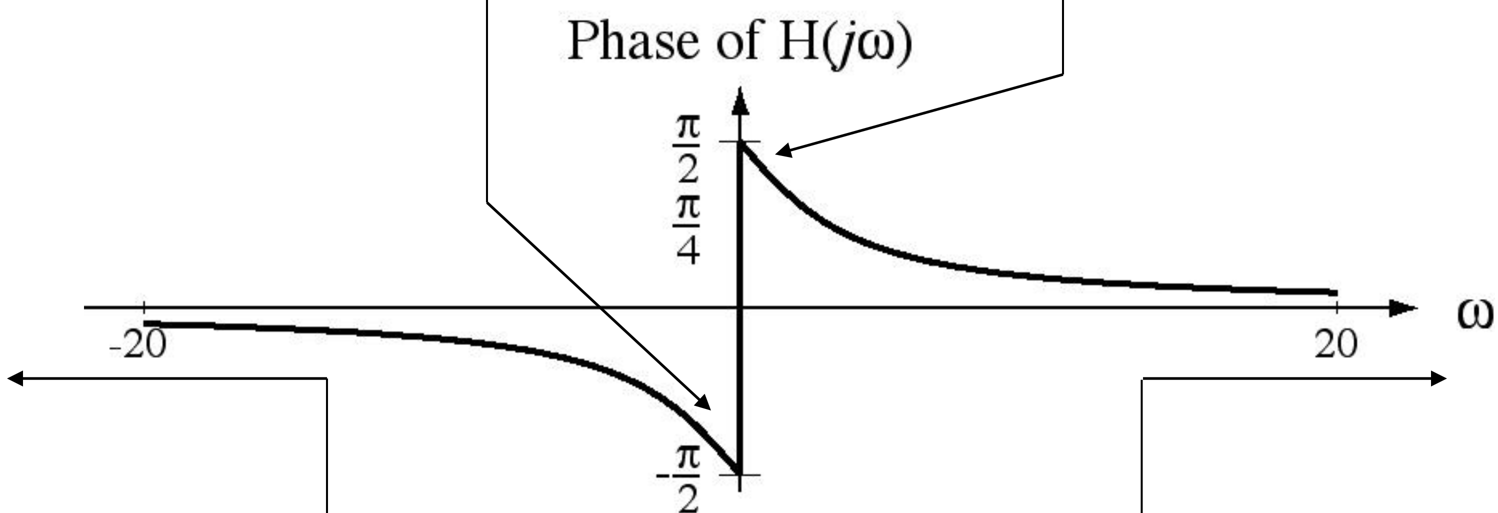
$$\lim_{w\to -\infty} |H(jw)| = \lim_{w\to -\infty} 3\frac{|jw|}{|jw+3|} = 3$$

$$\lim_{w\to +\infty} |H(jw)| = \lim_{w\to +\infty} 3\frac{|jw|}{|jw+3|} = 3$$

Pole-Zero Diagrams and Frequency Response

$$\lim_{w \to 0^-} \square\ \mathrm{H}(jw) = -\frac{p}{2} - 0 = -\frac{p}{2}$$

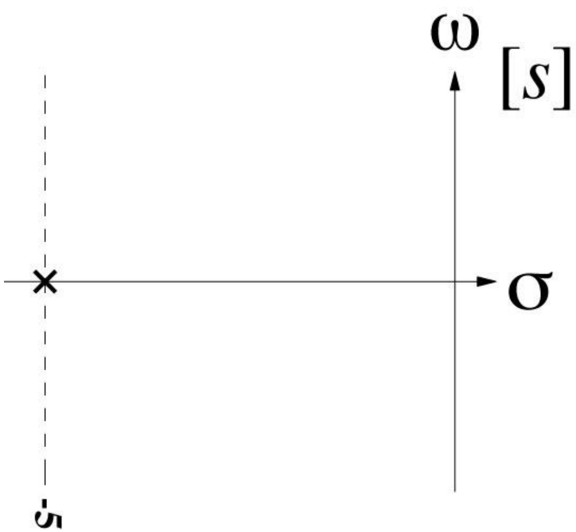
$$\lim_{w \to 0^+} \square\ \mathrm{H}(jw) = \frac{p}{2} - 0 = \frac{p}{2}$$

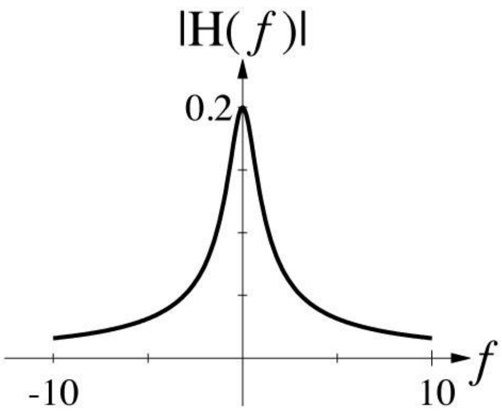


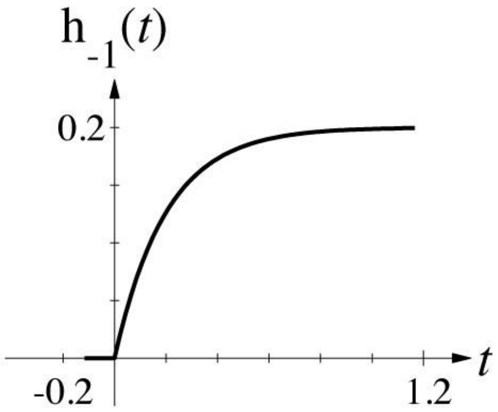
$$\lim_{w \to -\infty} \square\ \mathrm{H}(jw) = -\frac{p}{2} - \left(-\frac{p}{2}\right) = 0$$

$$\lim_{w \to +\infty} \square\ \mathrm{H}(jw) = \frac{p}{2} - \frac{p}{2} = 0$$

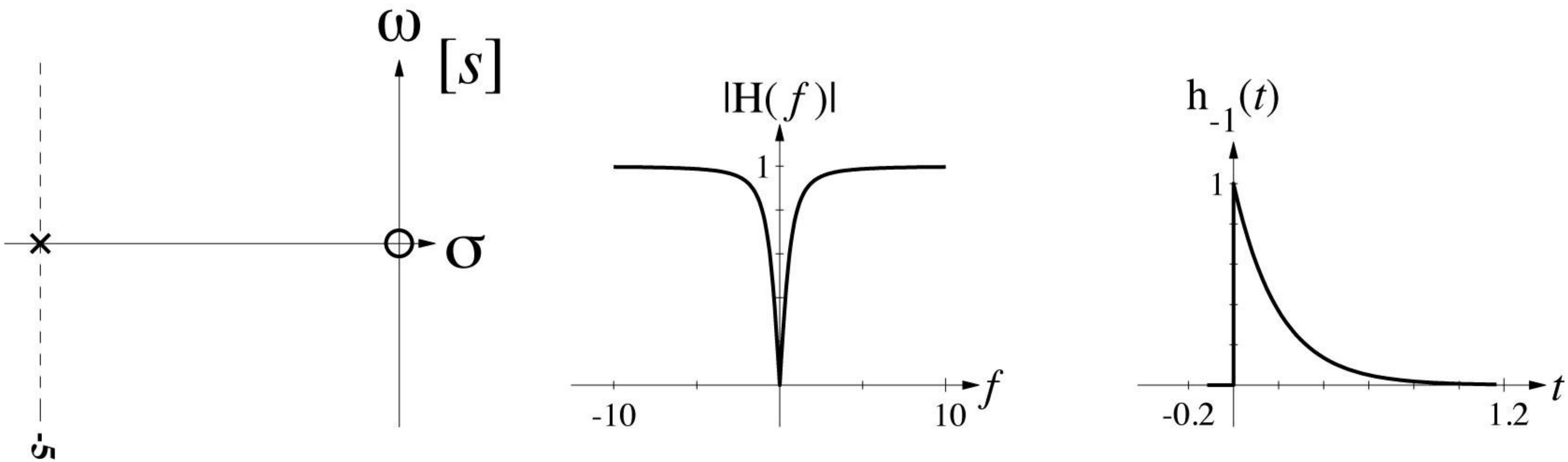
Pole-Zero Diagrams and Frequency Response



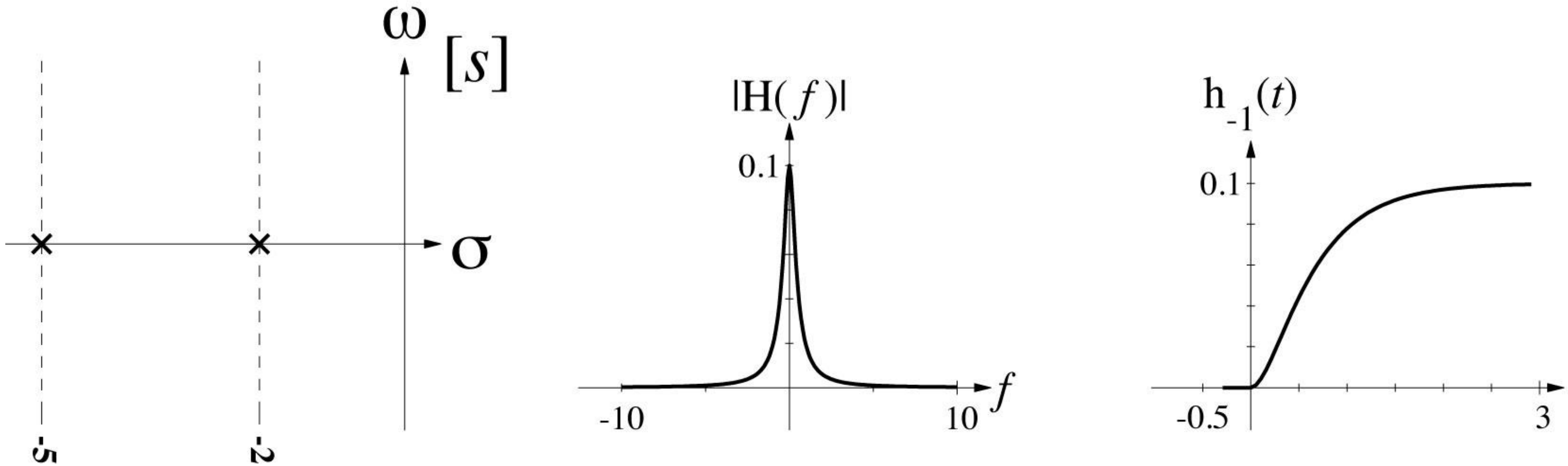




Pole-Zero Diagrams and Frequency Response



Pole-Zero Diagrams and Frequency Response



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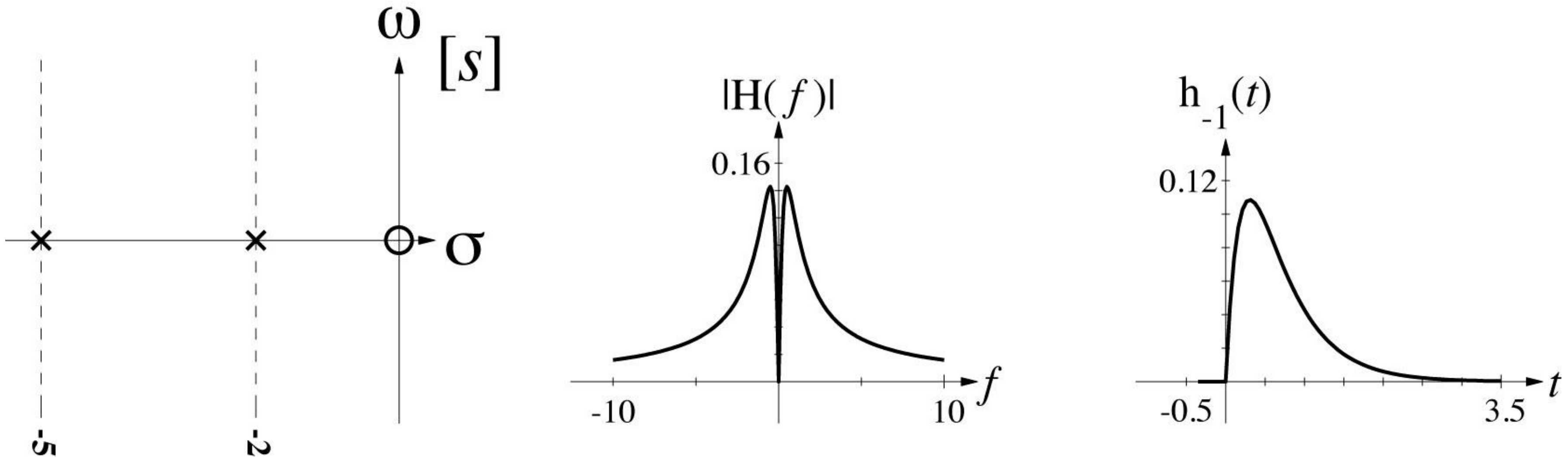
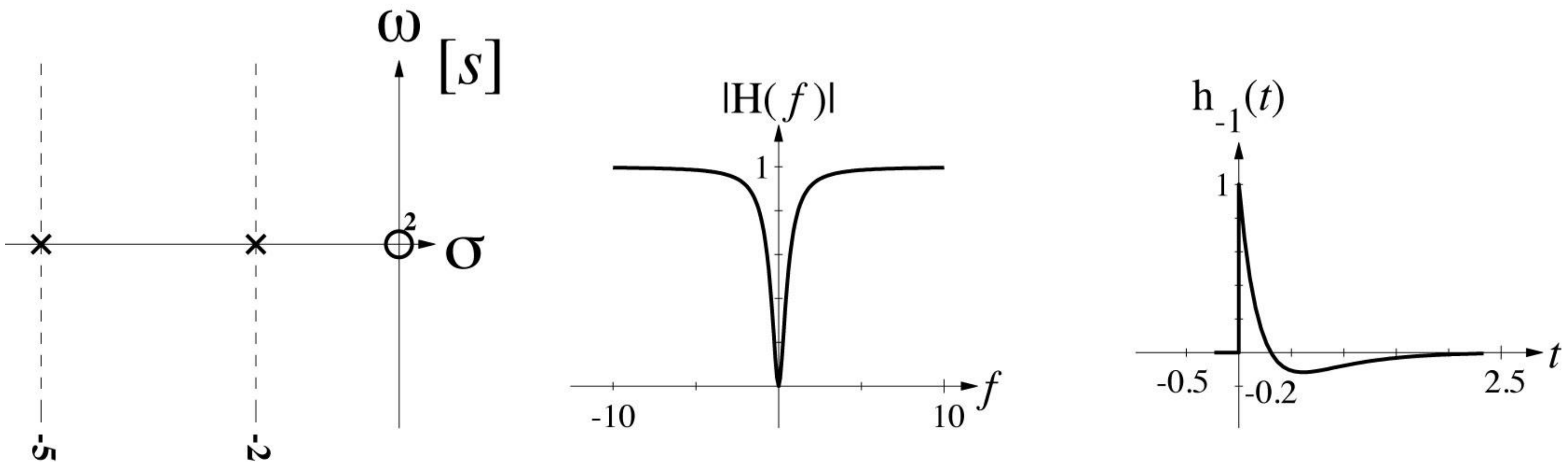
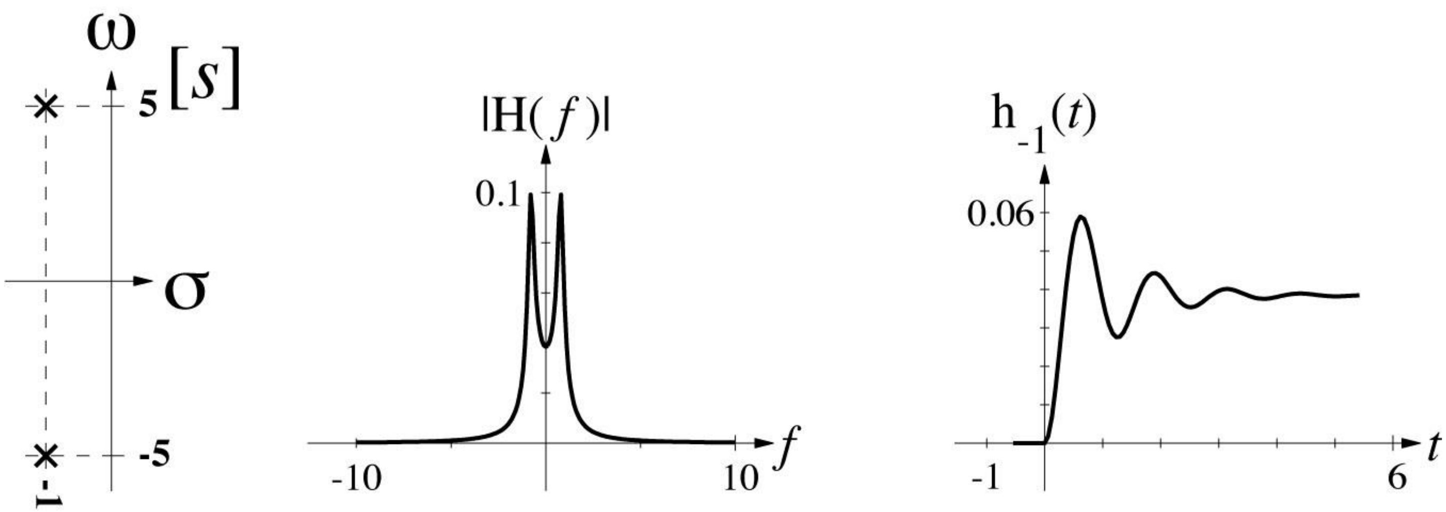


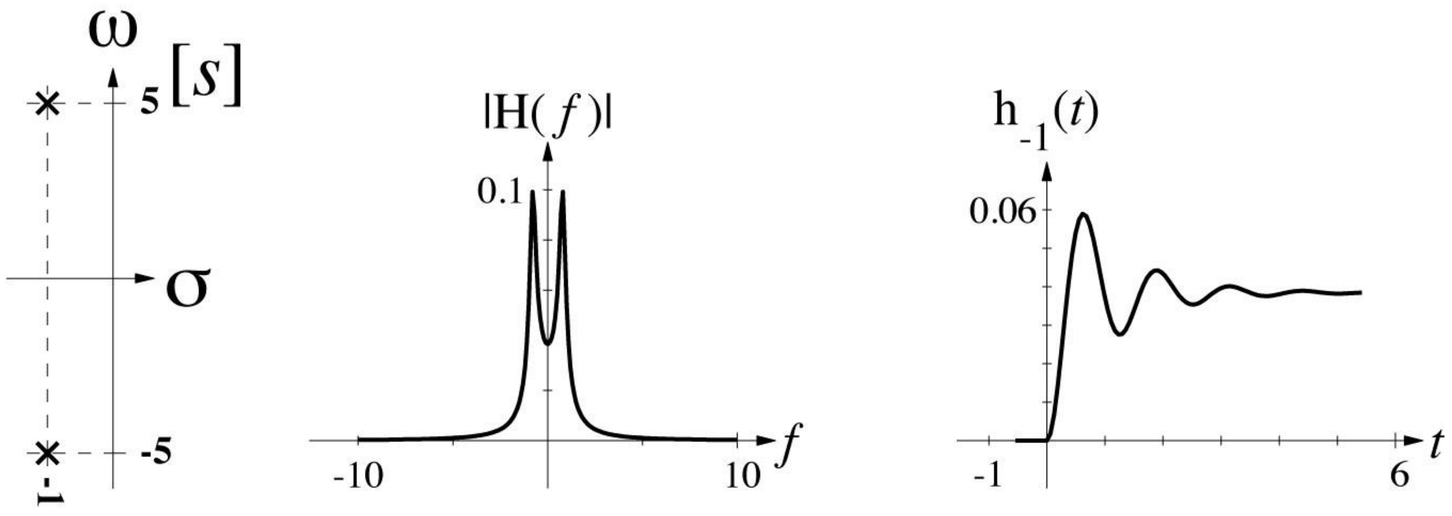
Pole-Zero Diagrams and Frequency Response



Pole-Zero Diagrams and Frequency Response



Pole-Zero Diagrams and Frequency Response



Pole-Zero Diagrams and Frequency Response