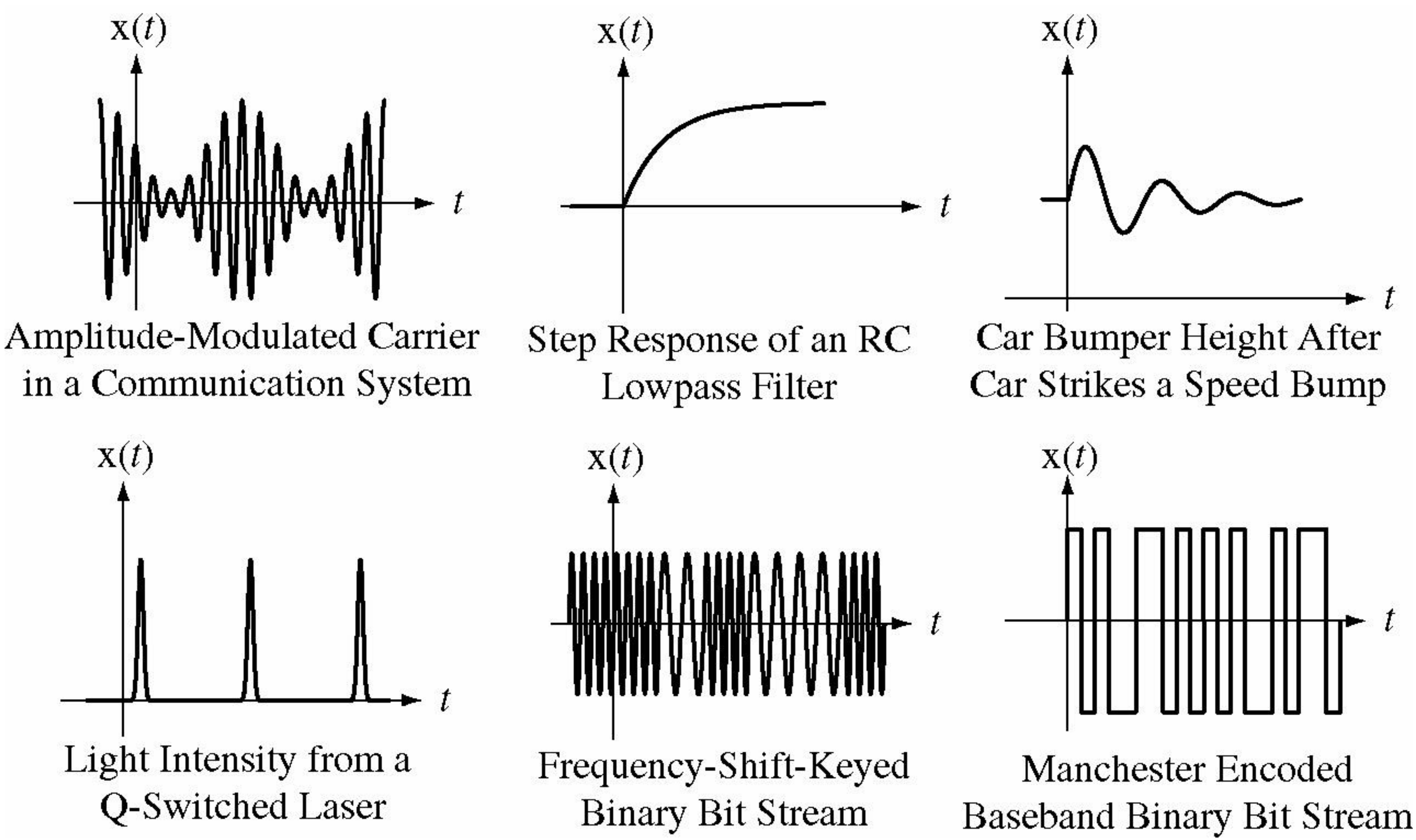
Mathematical Description of Continuous-Time Signals

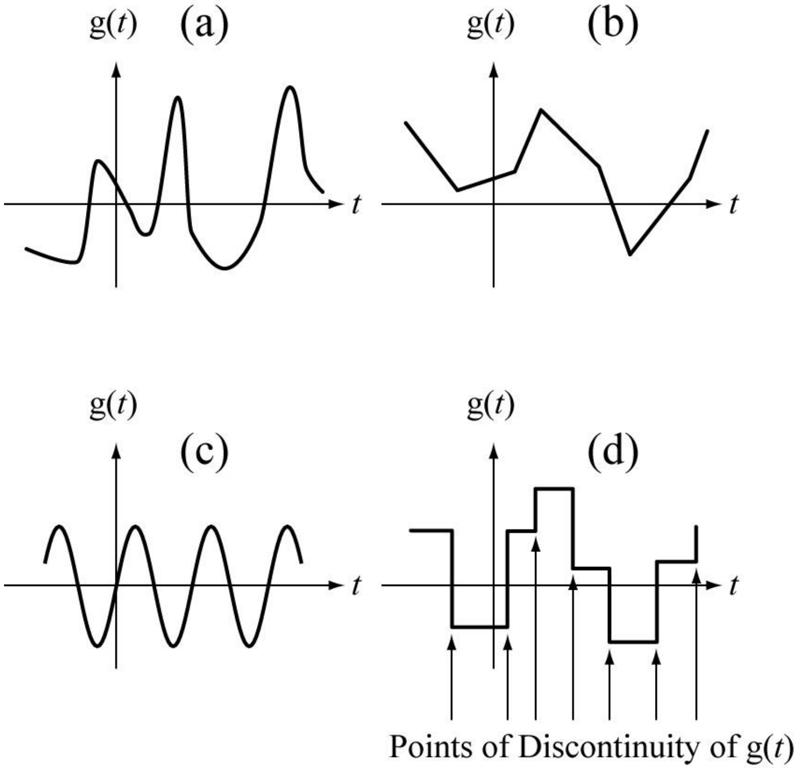
Edited by Dr. Robert Akl

Typical Continuous-Time Signals



Continuous vs Continuous-Time Signals

All continuous signals that are functions of time are **continuous-time** but not all continuous-time signals are continuous



Continuous-Time Sinusoids

$$g(t) = A\cos(2\pi t / T_0 + \theta) = A\cos(2\pi f_0 t + \theta) = A\cos(\omega_0 t + \theta)$$

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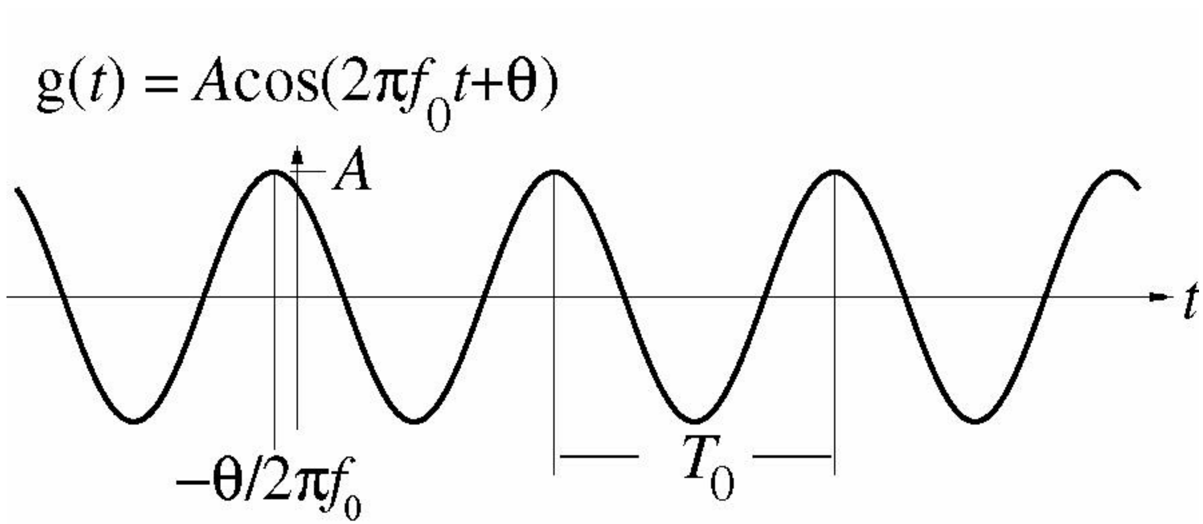
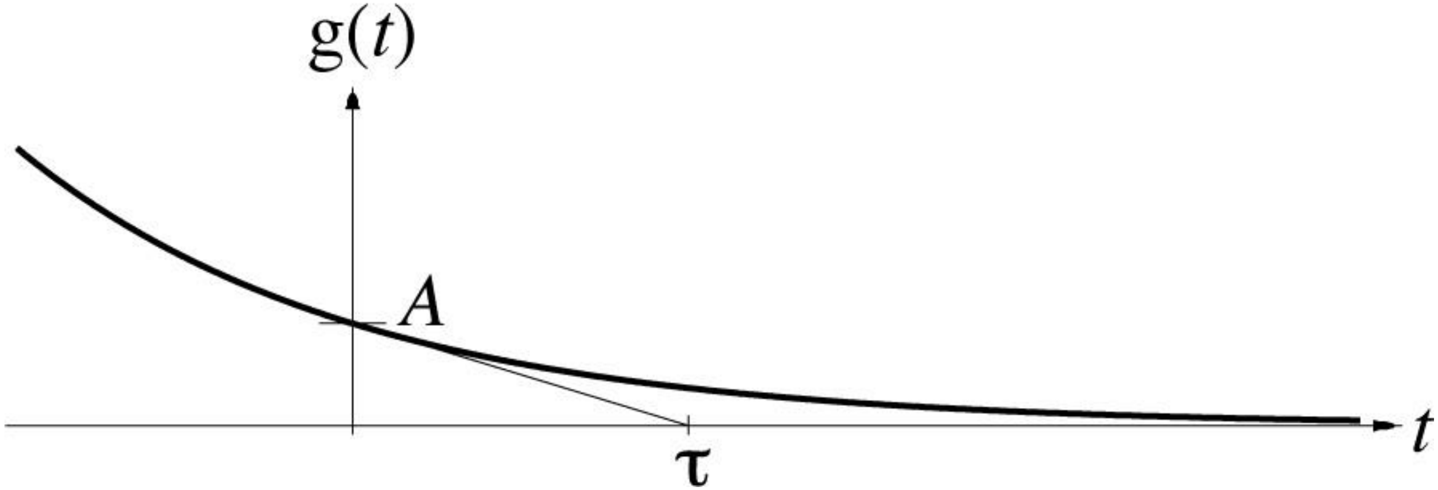
Amplitude	Period (s)	Phase Shift (radians)	Cyclic Frequency (Hz)	Radian Frequency (radians/s)



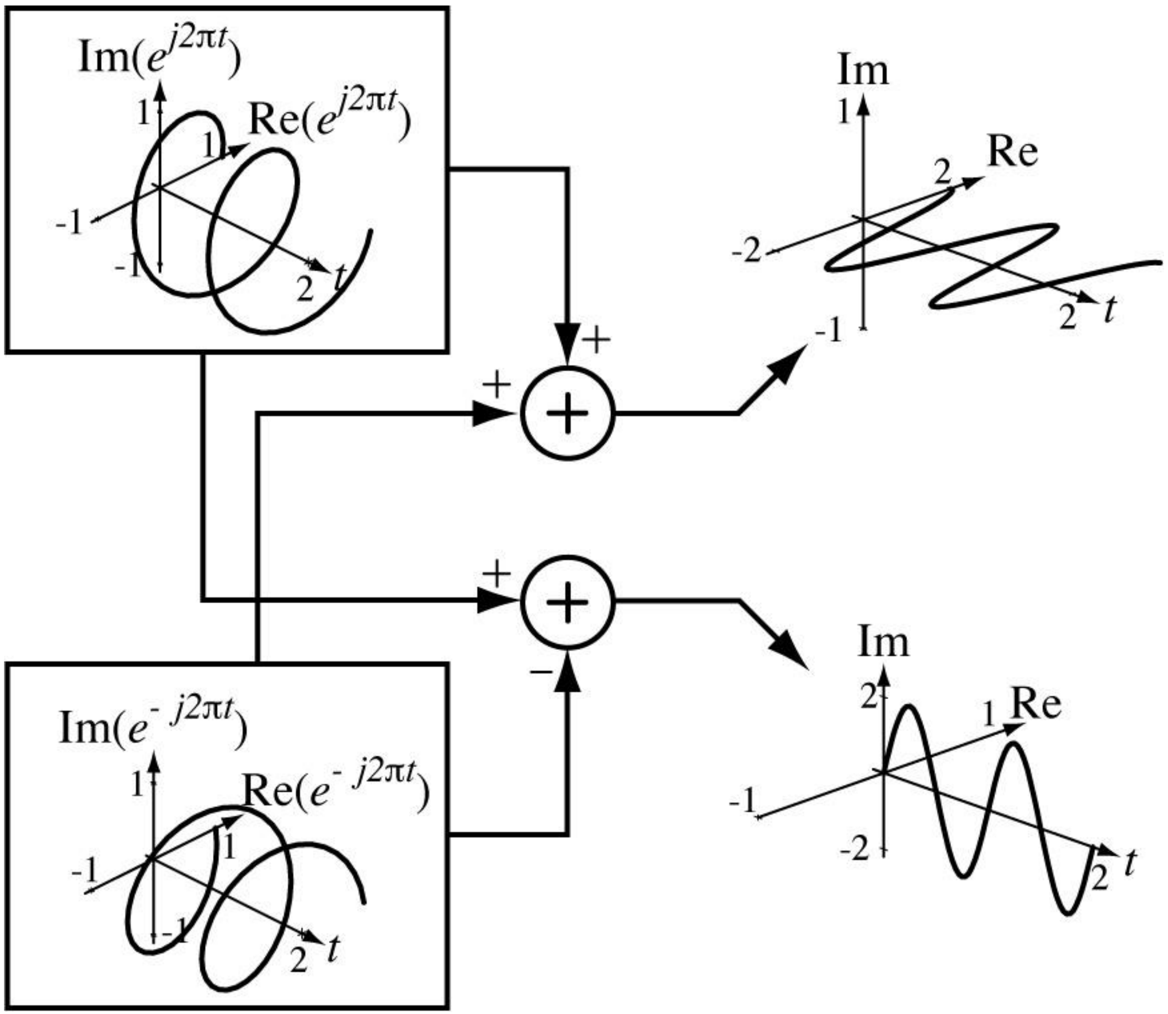
Continuous-Time Exponentials

$$g(t) = Ae^{-t/t}$$

Amplitude Time Constant (s)



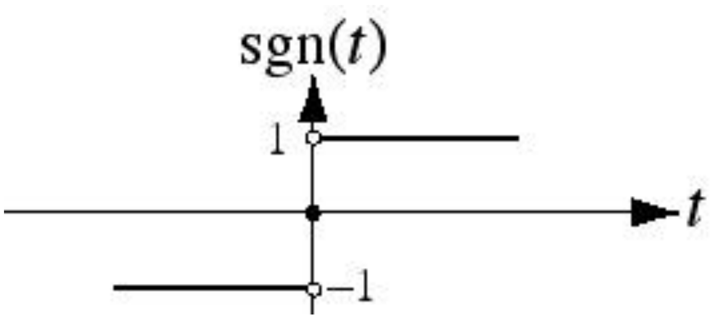
Complex Sinusoids



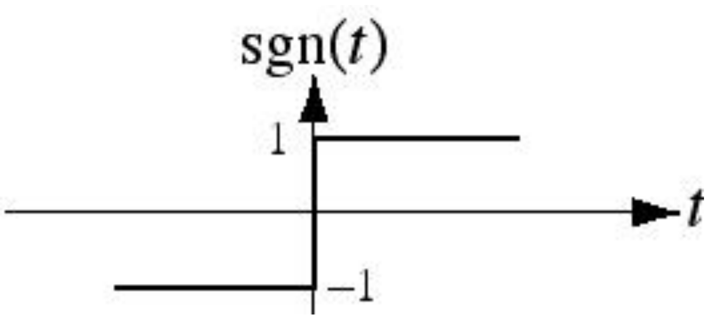
The Signum Function

$$\operatorname{sgn}(t) = \begin{cases} 1 & , \; t > 0 \\ 0 & , \; t = 0 \\ -1 & , \; t < 0 \end{cases}$$

Precise Graph



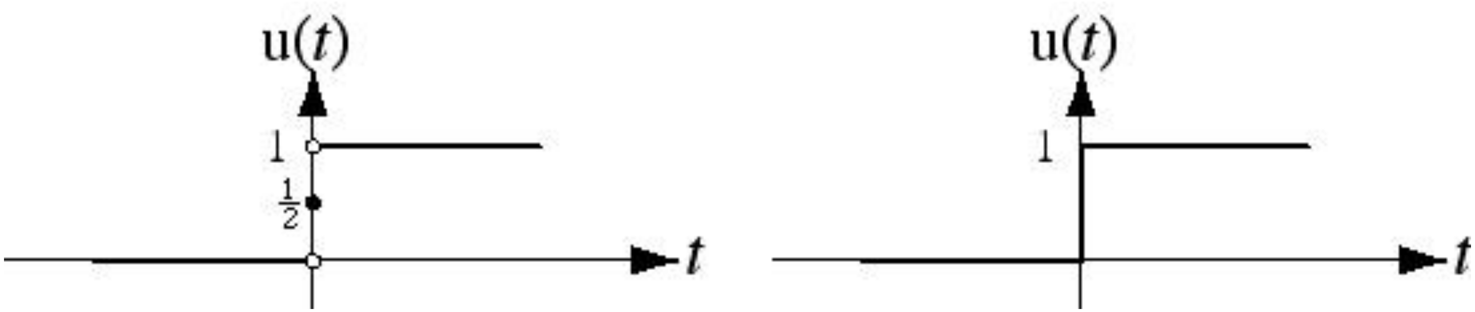
Commonly-Used Graph



The signum function, in a sense, returns an indication of the sign of its argument.

The Unit Step Function

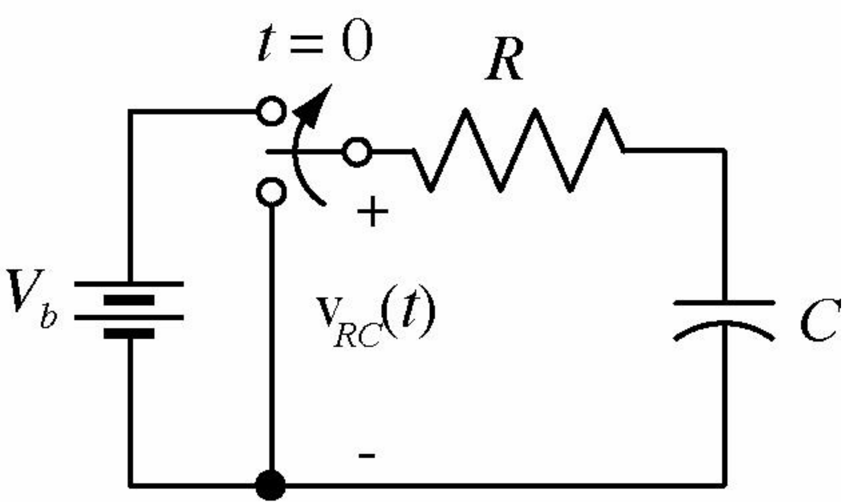
$$\mathrm{u}(t) = \begin{cases} 1 & , \ t > 0 \\ 1/2 & , \ t = 0 \\ 0 & , \ t < 0 \end{cases}$$



The product signal $\mathrm{g}(t)\mathrm{u}(t)$ can be thought of as the signal $\mathrm{g}(t)$ "turned on" at time $t = 0$.

The Unit Step Function

The unit step function can mathematically describe a signal that is zero up to some point in time and non-zero after that.

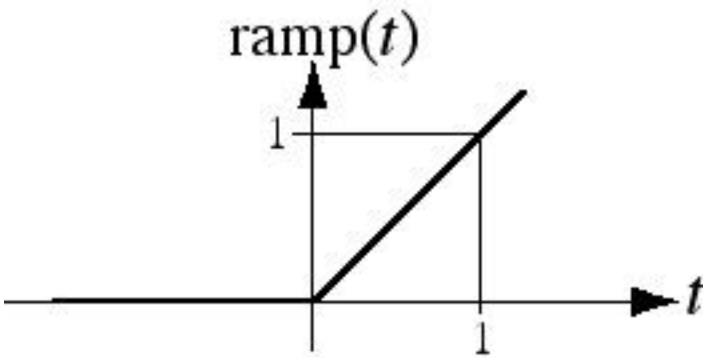


$$v_{RC}(t) = V_b \, u(t)$$

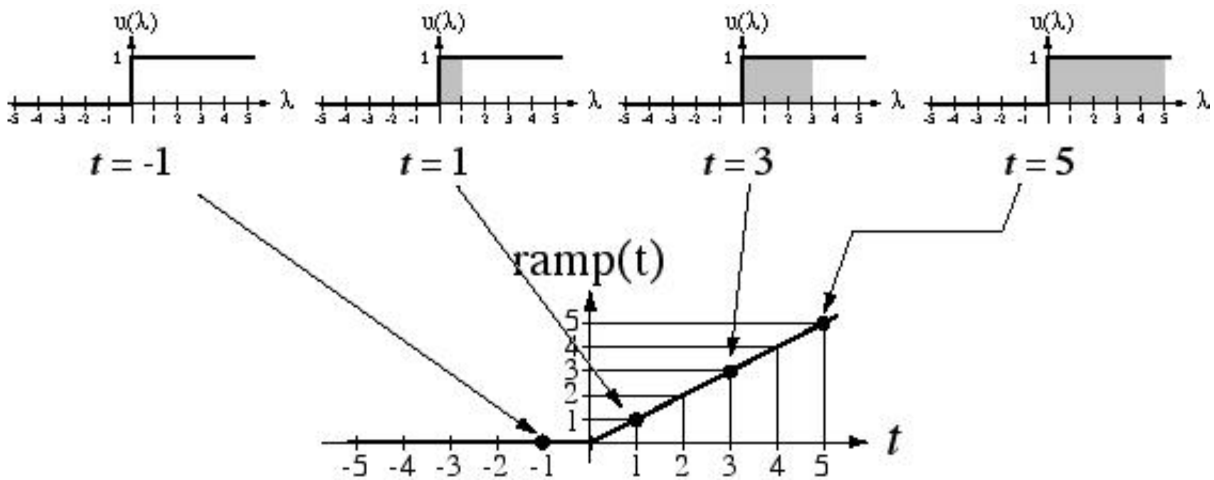
$$i(t) = (V_b / R) e^{-t/RC} u(t)$$

$$v_C(t) = V_b \left(1 - e^{-t/RC}\right) u(t)$$

The Unit Ramp Function

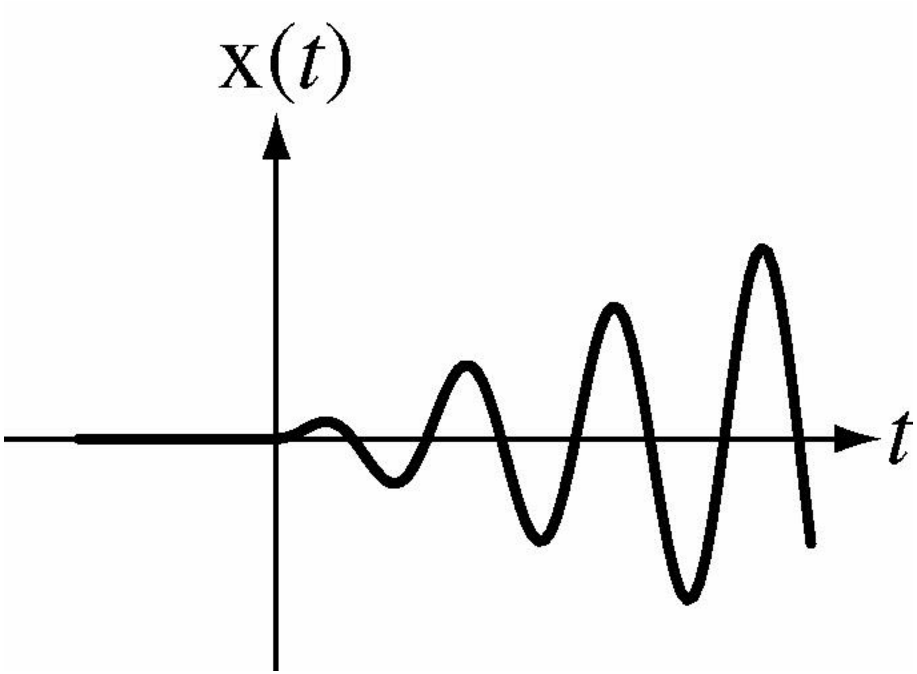


$$\mathrm{ramp}(t) = \begin{cases} t\ , & t > 0 \\ 0\ , & t \le 0 \end{cases} = \int_{-\infty}^{t} \mathrm{u}(\lambda)\, d\lambda = t\,\mathrm{u}(t)$$



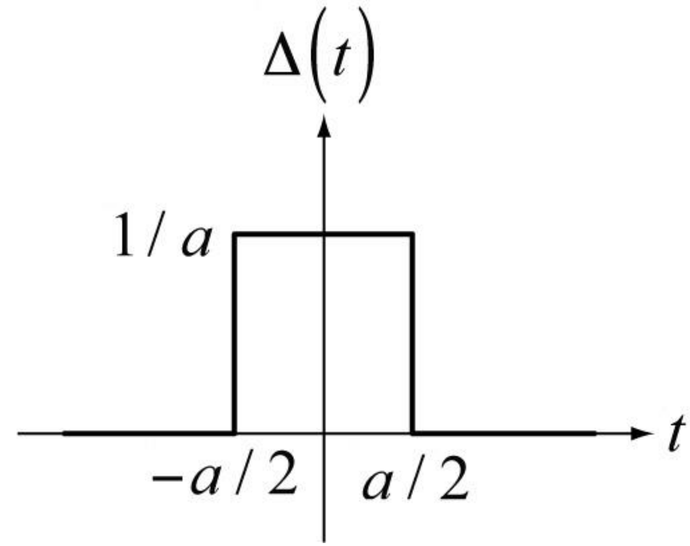
The Unit Ramp Function

Product of a sine wave and a ramp function.

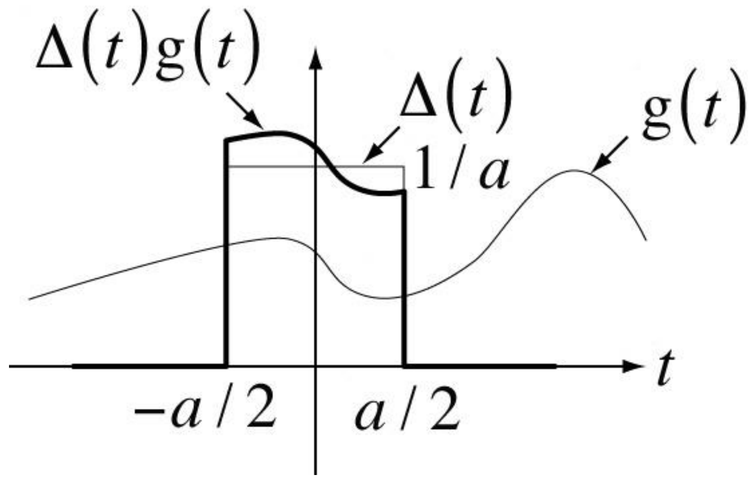


Introduction to the Impulse

Define a function $D(t) = \begin{cases} 1/a & , \ |t| < a/2 \\ 0 & , \ |t| > a/2 \end{cases}$



Let another function $g(t)$ be finite and continuous at $t = 0$.



Introduction to the Impulse

The area under the product of the two functions is

$$A = \frac{1}{a}\int_{-a/2}^{a/2} \mathrm{g}(t)\,dt$$

As the width of $\mathrm{D}(t)$ approaches zero,

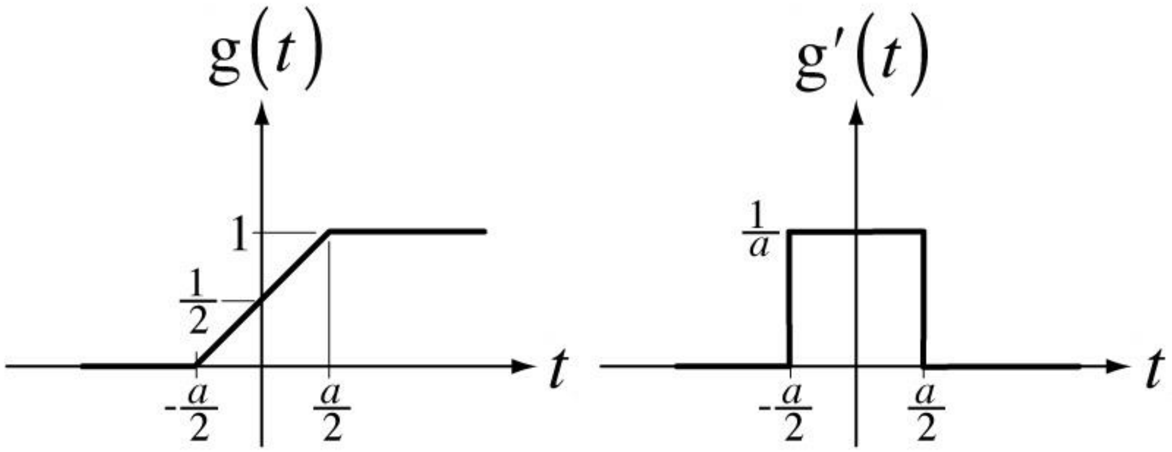
$$\lim_{a\to 0} A = \mathrm{g}(0)\lim_{a\to 0}\frac{1}{a}\int_{-a/2}^{a/2} dt = \mathrm{g}(0)\lim_{a\to 0}\frac{1}{a}(a) = \mathrm{g}(0)$$

The continuous-time unit impulse is implicitly defined by

$$\mathrm{g}(0) = \int_{-\infty}^{\infty} d(t)\,\mathrm{g}(t)\,dt$$

The Unit Step and Unit Impulse

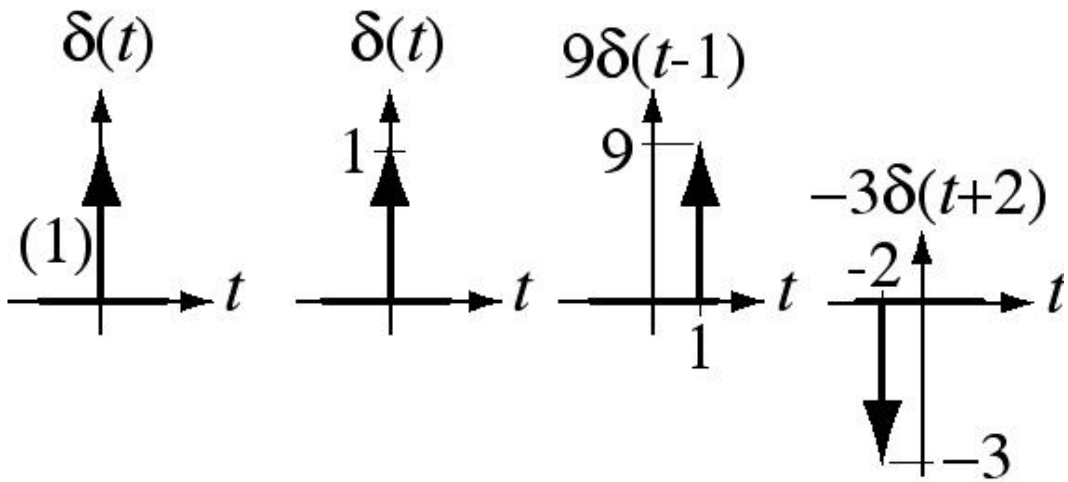
As a approaches zero, $\mathrm{g}(t)$ approaches a unit step and $\mathrm{g}'(t)$ approaches a unit impulse.



The unit step is the integral of the unit impulse and the unit impulse is the generalized derivative of the unit step.

Graphical Representation of the Impulse

The impulse is not a function in the ordinary sense because its value at the time of its occurrence is not defined. It is represented graphically by a vertical arrow. Its strength is either written beside it or is represented by its length.



Properties of the Impulse

The Sampling Property

$$\int_{-\infty}^{\infty} g(t)\,\delta(t - t_0)\,dt = g(t_0)$$

The sampling property "extracts" the value of a function at a point.

The Scaling Property

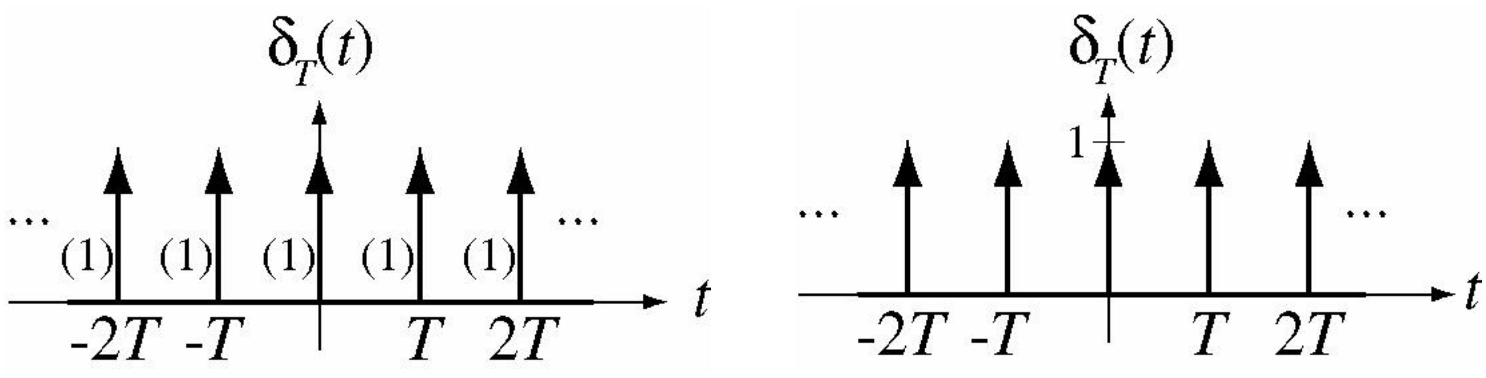
$$\delta(a(t - t_0)) = \frac{1}{|a|}\delta(t - t_0)$$

This property illustrates that the impulse is different from ordinary mathematical functions.

The Unit Periodic Impulse

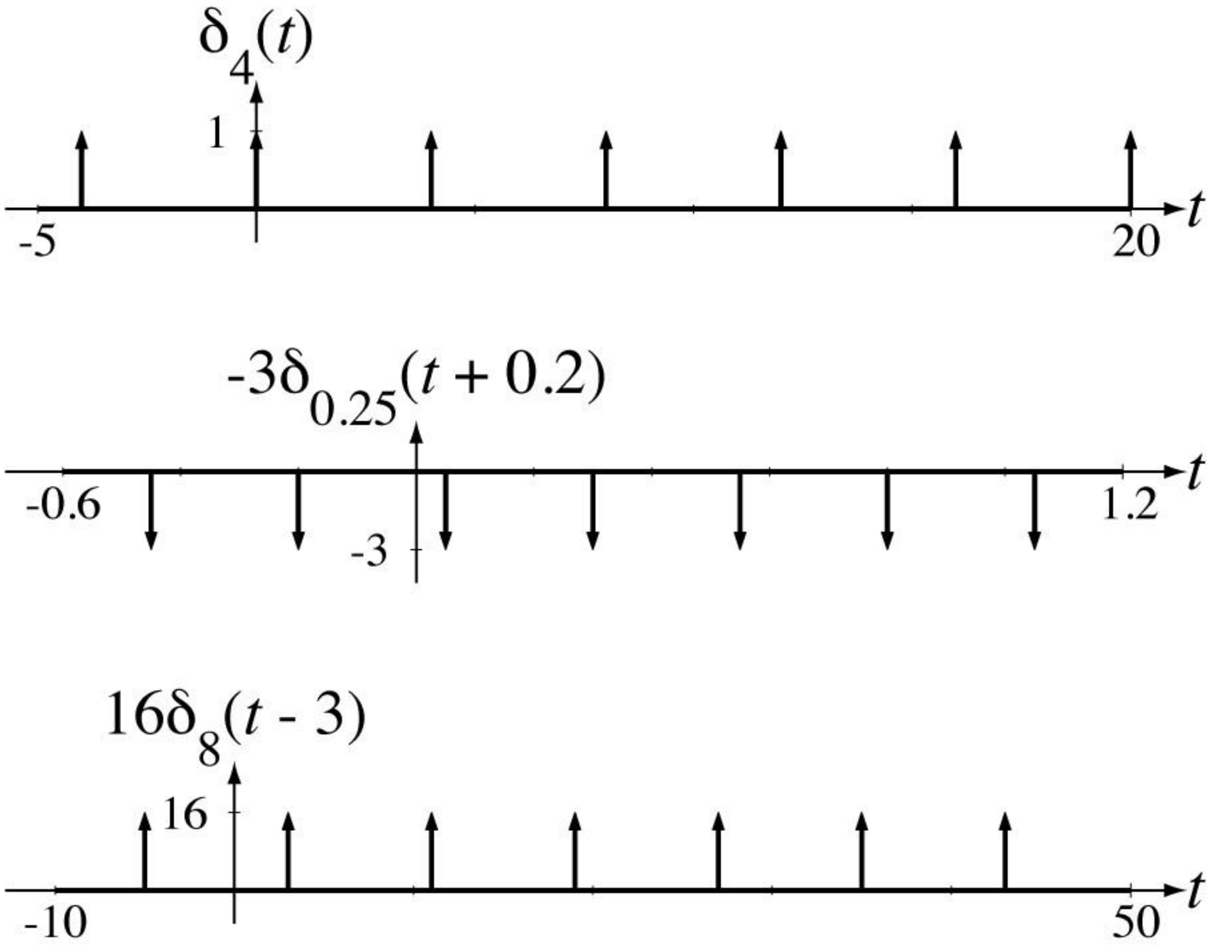
The unit periodic impulse is defined by

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \ , \ n \text{ an integer}$$



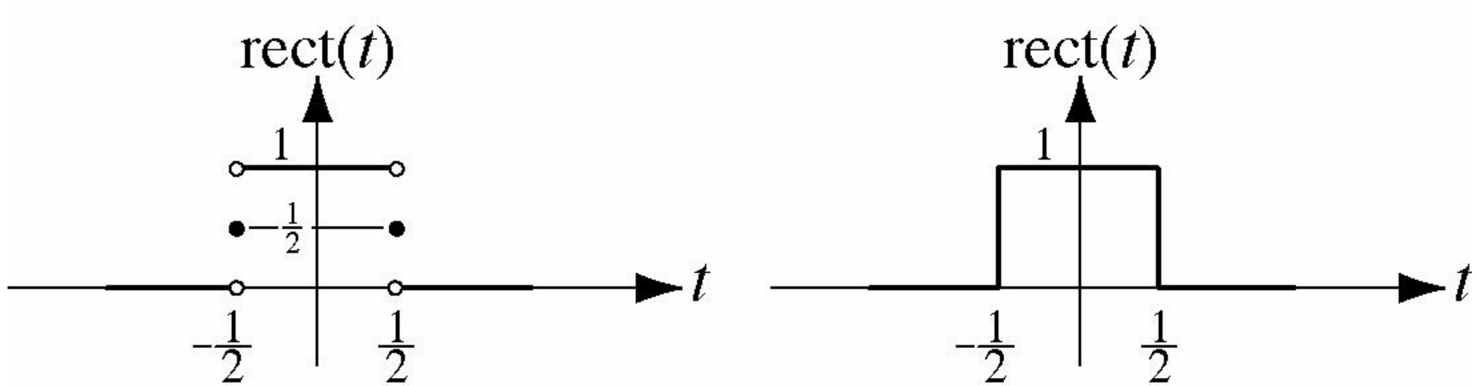
The periodic impulse is a sum of infinitely many uniformly-spaced impulses.

The Periodic Impulse



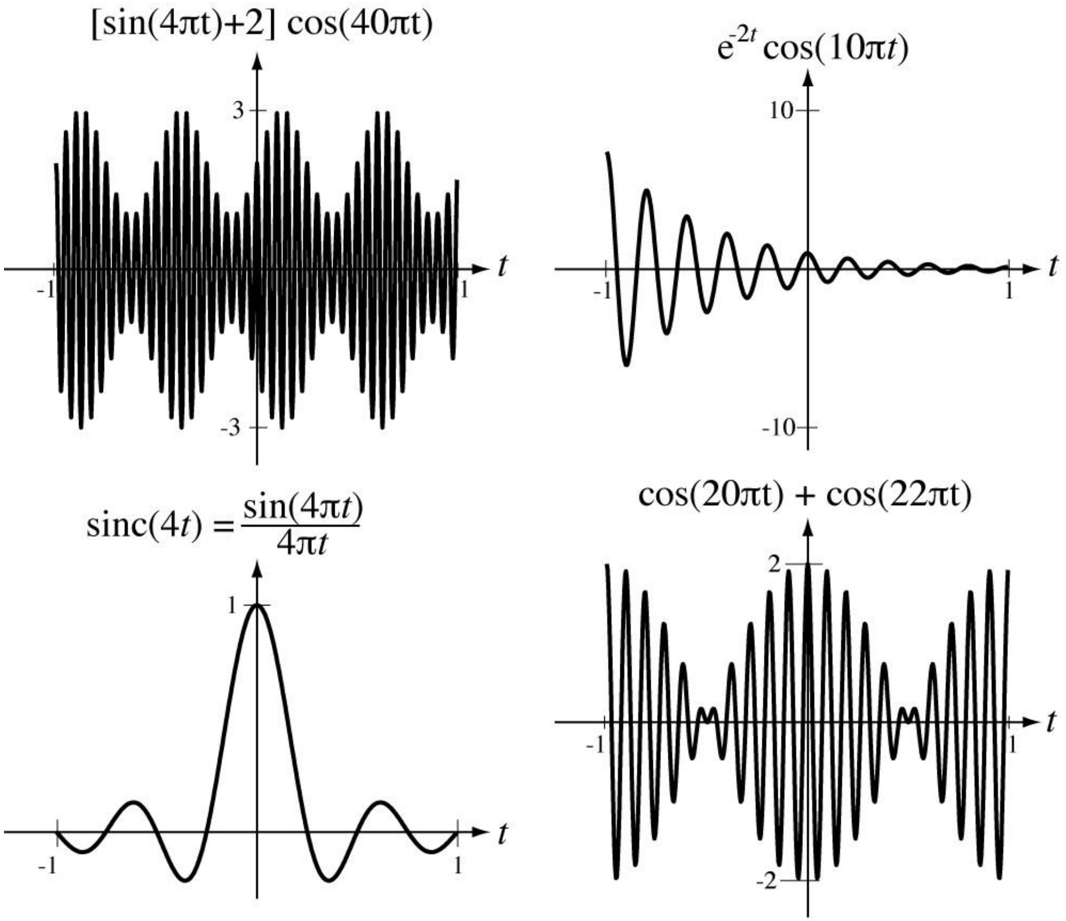
The Unit Rectangle Function

$$\mathrm{rect}(t) = \begin{cases} 1 \ , & |t| < 1/2 \\ 1/2 \ , & |t| = 1/2 \\ 0 \ , & |t| > 1/2 \end{cases} = \mathrm{u}(t + 1/2) - \mathrm{u}(t - 1/2)$$



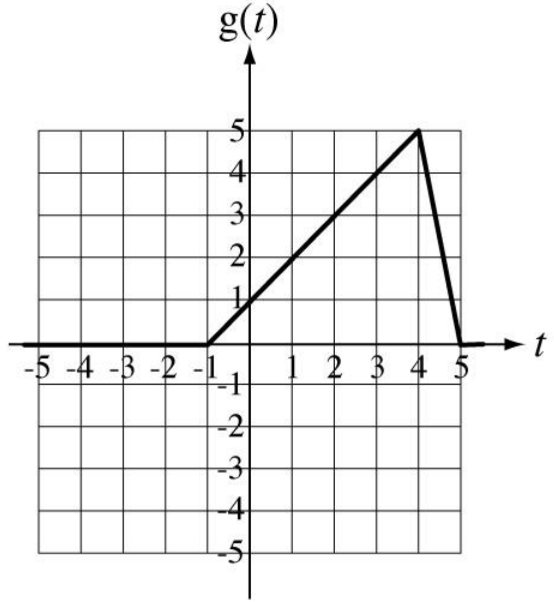
The product signal $\mathrm{g}(t)\mathrm{rect}(t)$ can be thought of as the signal $\mathrm{g}(t)$ "turned on" at time $t = -1/2$ and "turned back off" at time $t = +1/2$.

Combinations of Functions



Shifting and Scaling Functions

Let a function be defined graphically by

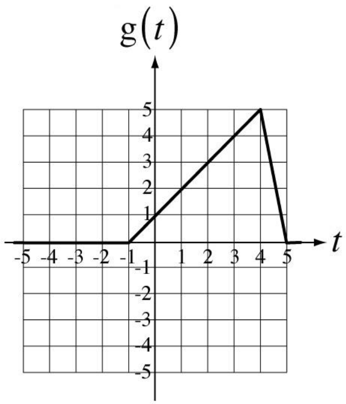


t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(t)$	0	0	0	0	0	1	2	3	4	5	0

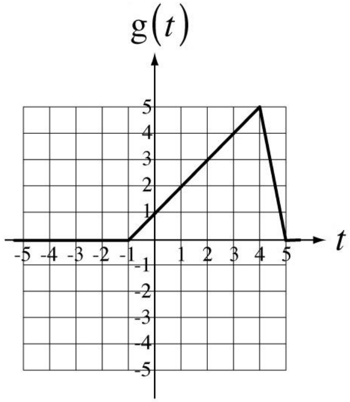
and let $g(t) = 0 \quad , \quad |t| > 5$

Shifting and Scaling Functions

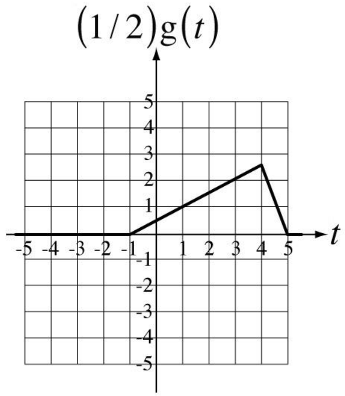
Amplitude Scaling, $\mathrm{g}(t) \to A\,\mathrm{g}(t)$



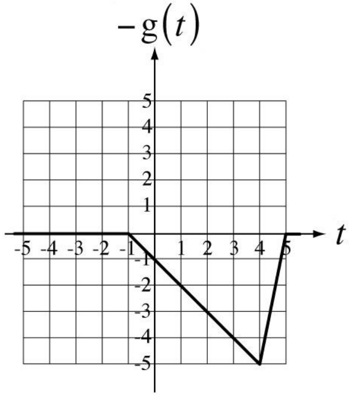
t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\mathrm{g}(t)$	0	0	0	0	0	1	2	3	4	5	0



t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\mathrm{g}(t)$	0	0	0	0	0	1	2	3	4	5	0

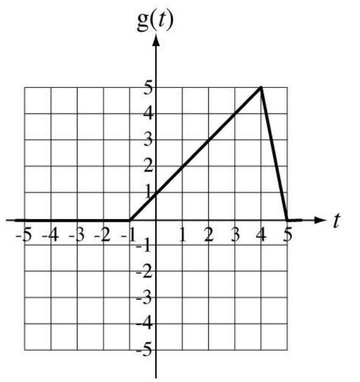


t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$(1/2)\mathrm{g}(t)$	0	0	0	0	0	$1/2$	1	$3/2$	2	$5/2$	0



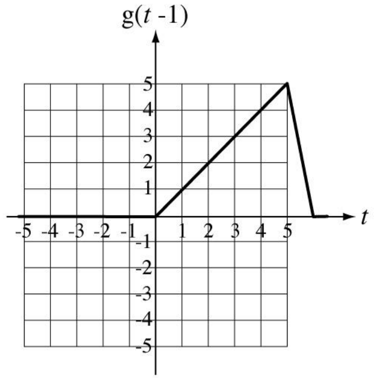
t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$-\mathrm{g}(t)$	0	0	0	0	0	-1	-2	-3	-4	-5	0

Shifting and Scaling Functions



t	−5	−4	−3	−2	−1	0	1	2	3	4	5
$g(t)$	0	0	0	0	0	1	2	3	4	5	0

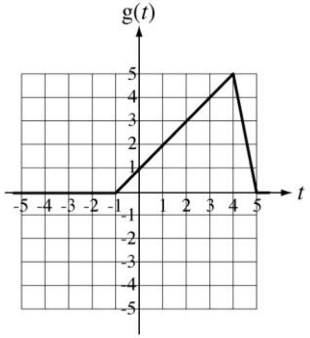
Time shifting, $t \rightarrow t - t_0$



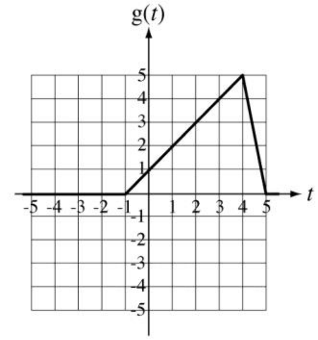
t	−5	−4	−3	−2	−1	0	1	2	3	4	5
$t-1$	−6	−5	−4	−3	−2	−1	0	1	2	3	4
$g(t-1)$	0	0	0	0	0	0	1	2	3	4	5

Shifting and Scaling Functions

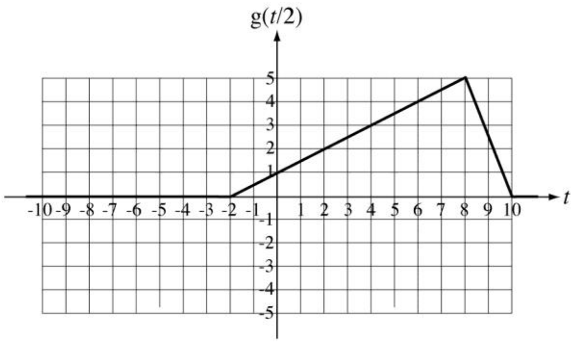
Time scaling, $t \to t/a$



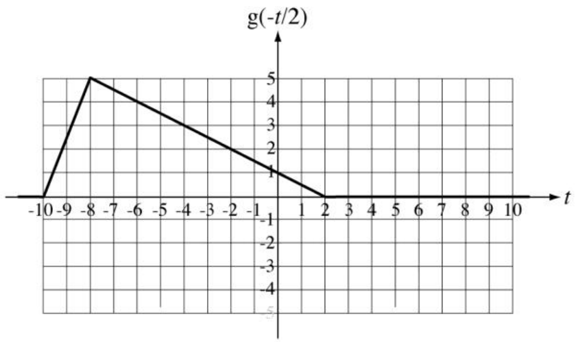
t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(t)$	0	0	0	0	0	1	2	3	4	5	0



t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(t)$	0	0	0	0	0	1	2	3	4	5	0



t	-10	-8	-6	-4	-2	0	2	4	6	8	10
$t/2$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(t/2)$	0	0	0	0	0	1	2	3	4	5	0



t	-10	-8	-6	-4	-2	0	2	4	6	8	10
$-t/2$	5	4	3	2	1	0	-1	-2	-3	-4	-5
$g(-t/2)$	0	5	4	3	2	1	0	0	0	0	0

Shifting and Scaling Functions

Multiple transformations $\quad \mathrm{g}(t) \to A\,\mathrm{g}\left(\dfrac{t - t_0}{a}\right)$

A multiple transformation can be done in steps

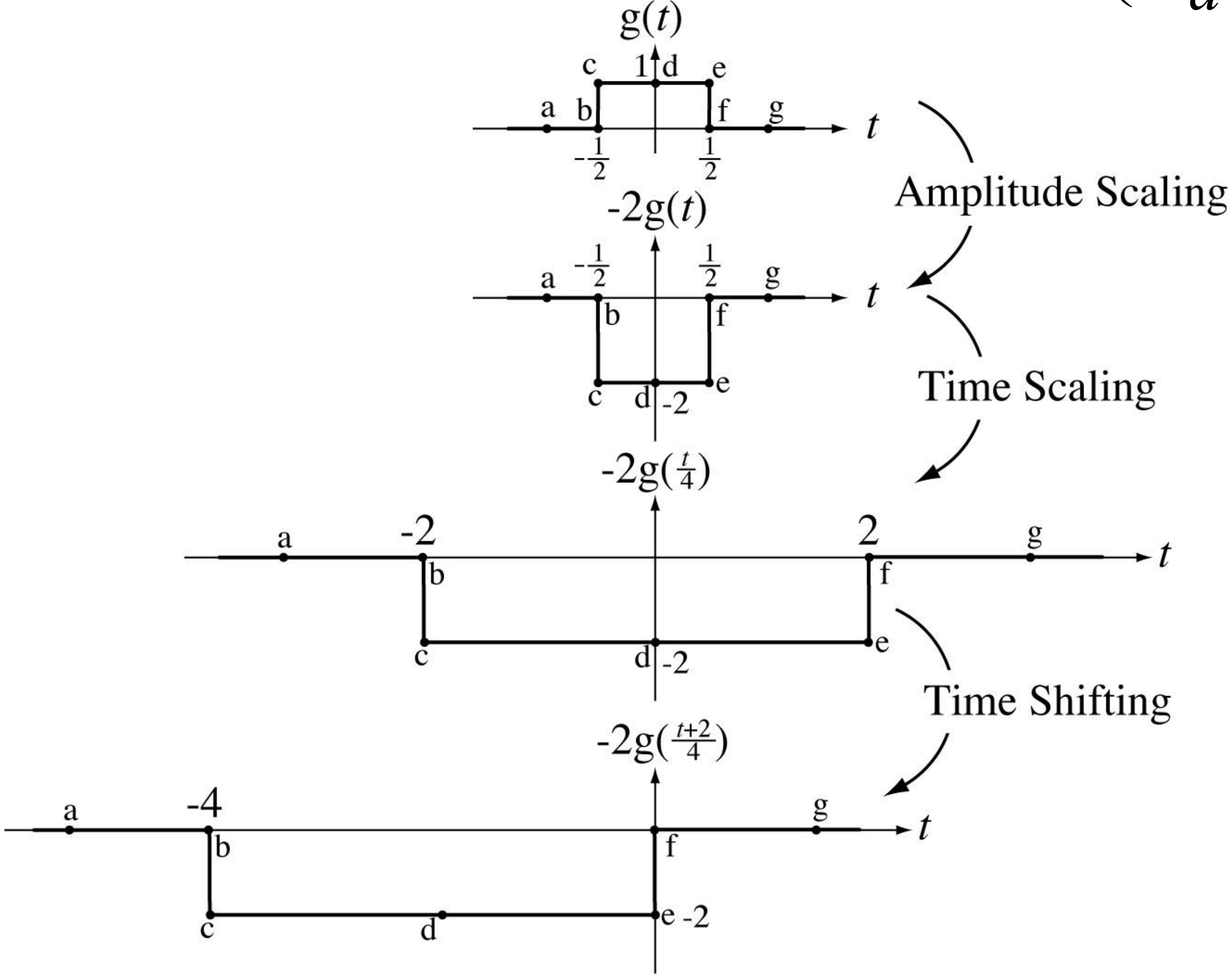
$$\mathrm{g}(t) \xrightarrow[\text{scaling, } A]{\text{amplitude}} A\,\mathrm{g}(t) \xrightarrow{t \to t/a} A\,\mathrm{g}\left(\frac{t}{a}\right) \xrightarrow{t \to t - t_0} A\,\mathrm{g}\left(\frac{t - t_0}{a}\right)$$

The sequence of the steps is significant

$$\mathrm{g}(t) \xrightarrow[\text{scaling, } A]{\text{amplitude}} A\,\mathrm{g}(t) \xrightarrow{t \to t - t_0} A\,\mathrm{g}(t - t_0) \xrightarrow{t \to t/a} A\,\mathrm{g}\left(\frac{t}{a} - t_0\right) \neq A\,\mathrm{g}\left(\frac{t - t_0}{a}\right)$$

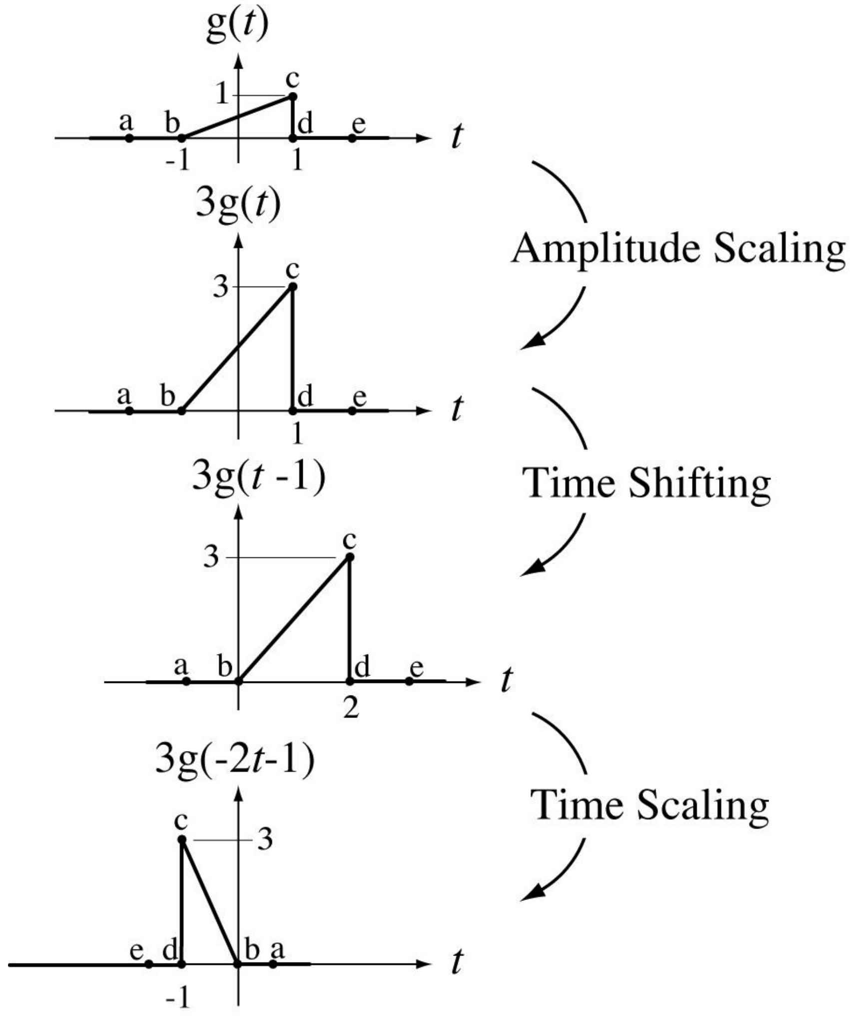
Shifting and Scaling Functions

Simultaneous scaling and shifting $\mathrm{g}(t) \to A\,\mathrm{g}\left(\dfrac{t - t_0}{a}\right)$

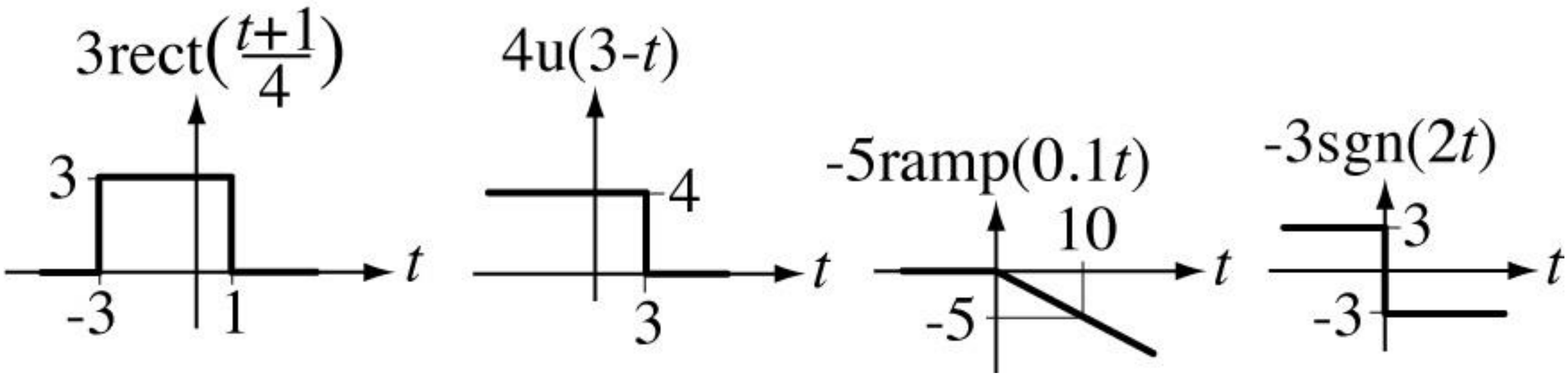


Shifting and Scaling Functions

Simultaneous scaling and shifting, $A\,\mathrm{g}\left(bt - t_0\right)$

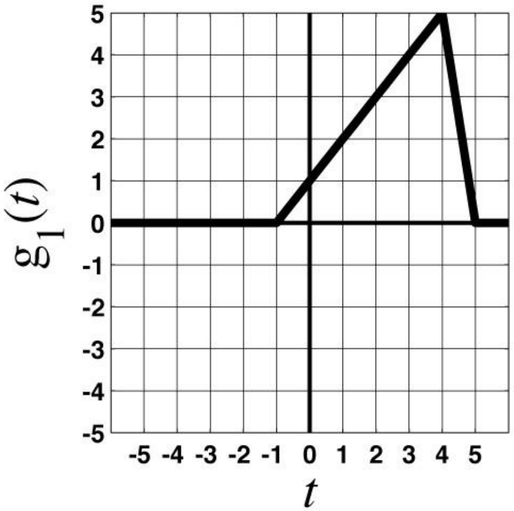


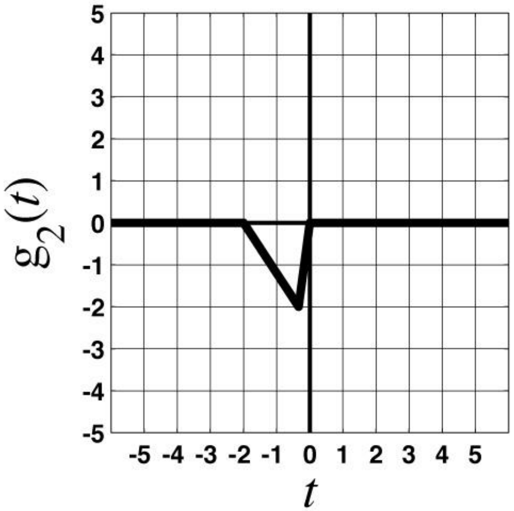
Shifting and Scaling Functions



Shifting and Scaling Functions

If $g_2(t) = A\,g_1((t - t_0)/w)$ what are A, t_0 and w?



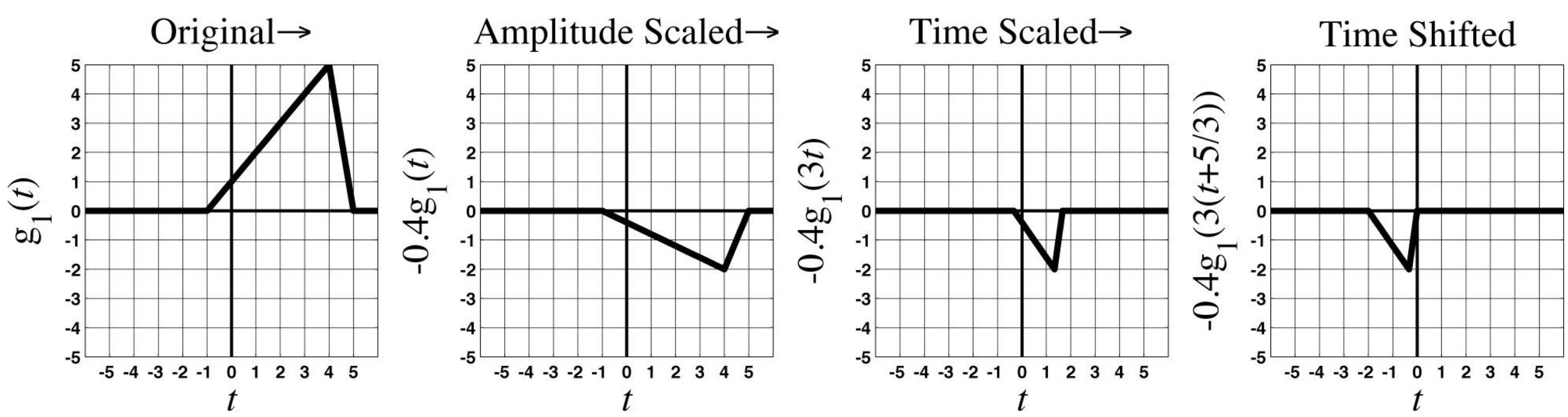


Shifting and Scaling Functions

$$\text{Height } +5 \rightarrow -2 \Rightarrow A = -0.4 \text{ , } \mathrm{g}_1(t) \rightarrow -0.4\,\mathrm{g}_1(t)$$

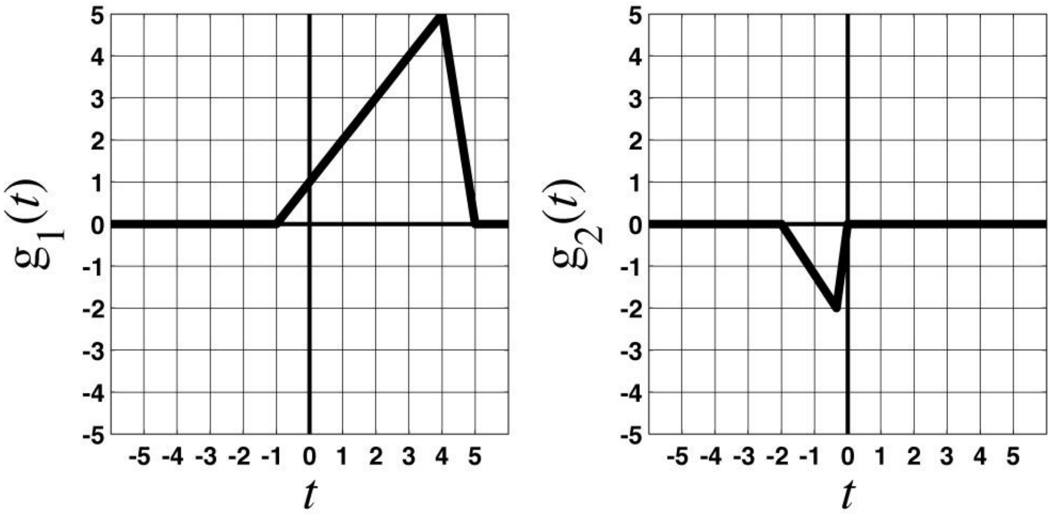
$$\text{Width } +6 \rightarrow +2 \Rightarrow w = 1/3 \Rightarrow -0.4\,\mathrm{g}_1(t) \rightarrow -0.4\,\mathrm{g}_1(3t)$$

$$\text{Shift left by } 5/3 \Rightarrow t_0 = -5/3 \Rightarrow -0.4\,\mathrm{g}_1(3t) \rightarrow -0.4\,\mathrm{g}_1(3(t+5/3))$$



Shifting and Scaling Functions

If $g_2(t) = A g_1(wt - t_0)$ what are A, t_0 and w?

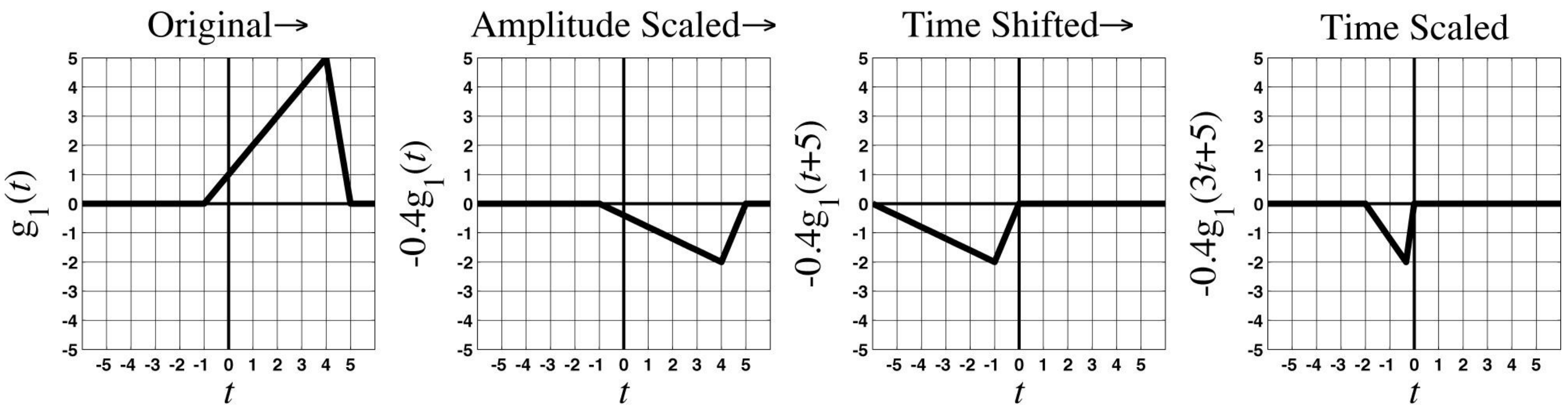


Shifting and Scaling Functions

$$\text{Height } +5 \rightarrow -2 \Rightarrow A = -0.4 \Rightarrow g_1(t) \rightarrow -0.4\,g_1(t)$$

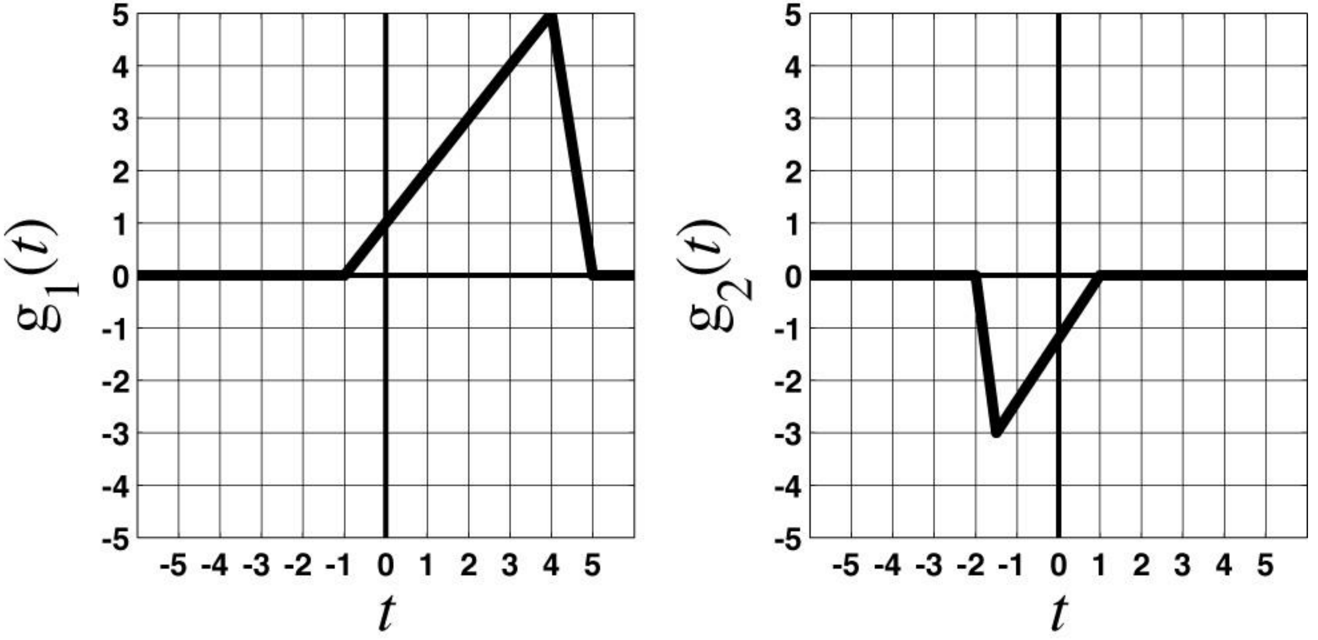
$$\text{Shift left } 5 \Rightarrow t_0 = -5 \Rightarrow -0.4\,g_1(t) \rightarrow -0.4\,g_1(t+5)$$

$$\text{Width } +6 \text{ to } +2 \Rightarrow w = 3 \Rightarrow -0.4\,g_1(t+5) \rightarrow -0.4\,g_1(3t+5)$$



Shifting and Scaling Functions

If $g_2(t) = A g_1(w(t - t_0))$ what are A, t_0 and w?

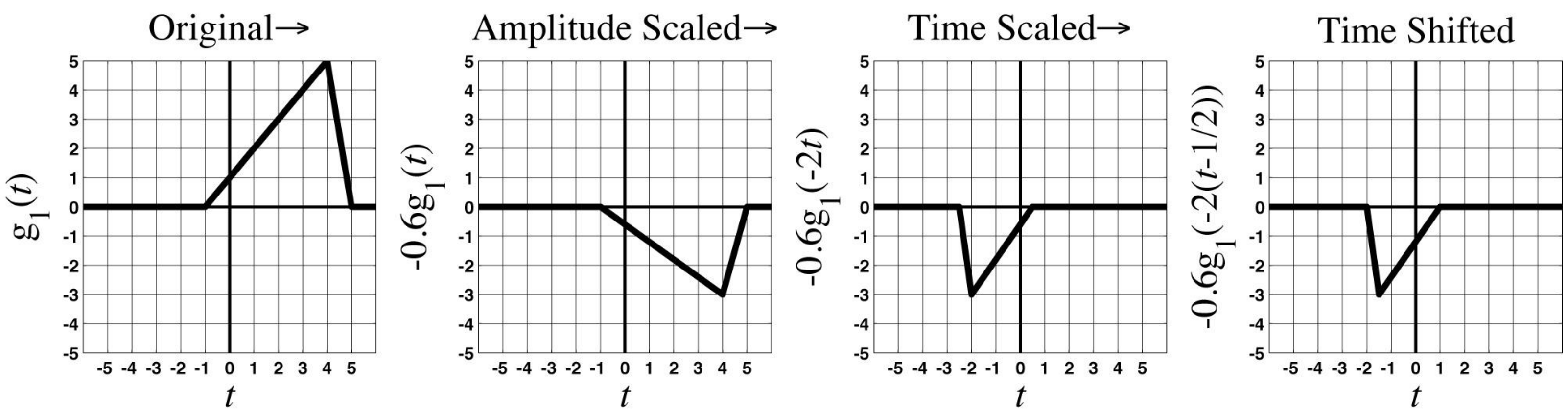


Shifting and Scaling Functions

$$\text{Height } +5 \rightarrow -3 \Rightarrow A = -0.6 \Rightarrow g_1(t) \rightarrow -0.6\, g_1(t)$$

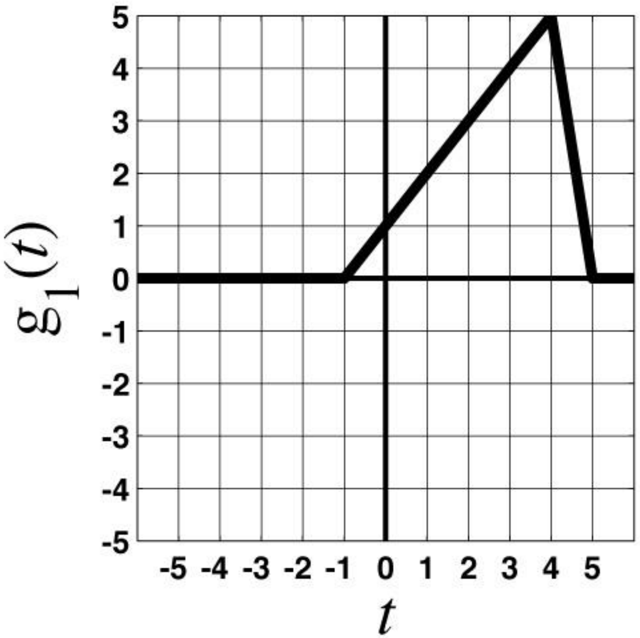
$$\text{Width } +6 \rightarrow -3 \Rightarrow w = -2 \Rightarrow -0.6\, g_1(t) \rightarrow -0.6\, g_1(-2t)$$

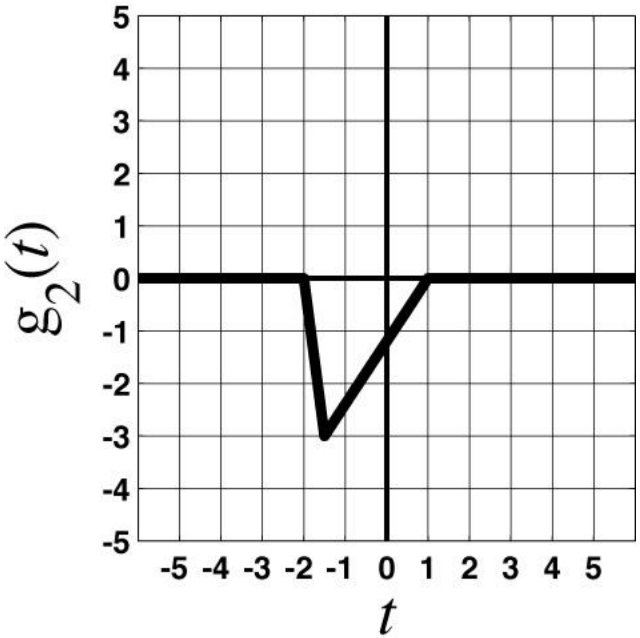
$$\text{Shift Right } 1/2 \Rightarrow t_0 = 1/2 \Rightarrow -0.6\, g_1(-2t) \rightarrow -0.6\, g_1(-2(t - 1/2))$$



Shifting and Scaling Functions

If $g_2(t) = A g_1(t/w - t_0)$ what are A, t_0 and w?



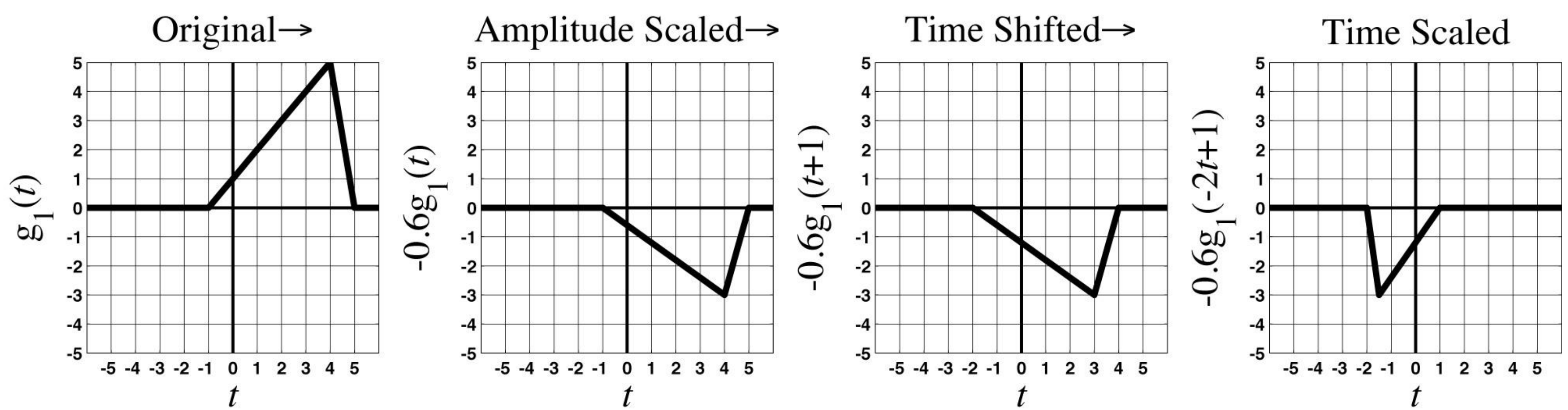


Shifting and Scaling Functions

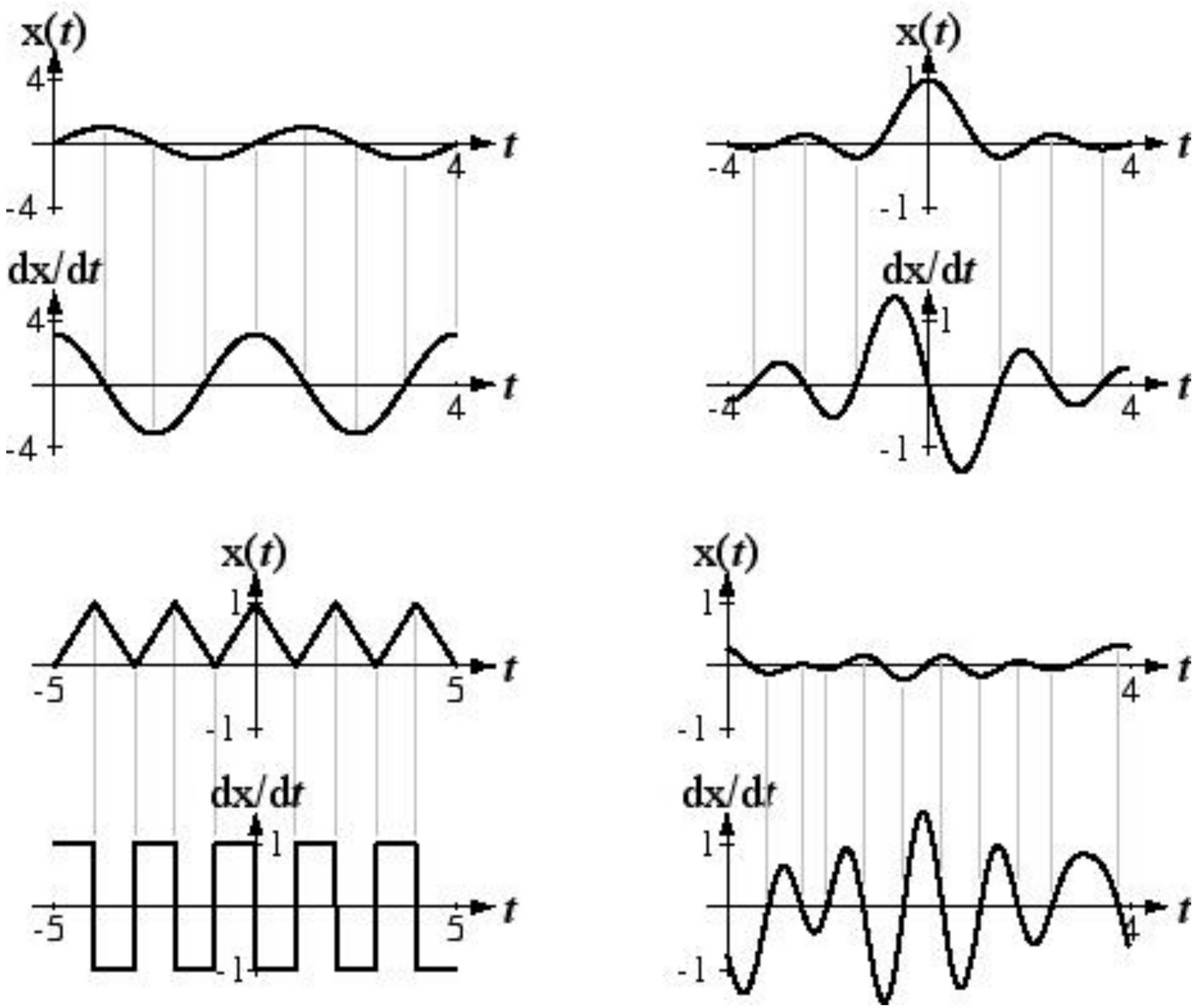
$$\text{Height } +5 \rightarrow -3 \Rightarrow A = -0.6 \Rightarrow g_1(t) \rightarrow -0.6\, g_1(t)$$

$$\text{Shift Left } 1 \Rightarrow t_0 = -1 \Rightarrow -0.6\, g_1(t) \rightarrow -0.6\, g_1(t+1)$$

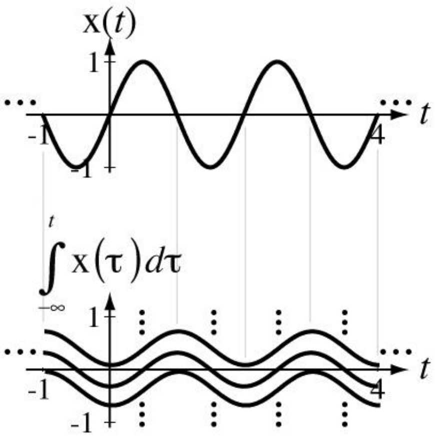
$$\text{Width } +6 \rightarrow -3 \Rightarrow w = -1/2 \Rightarrow -0.6\, g_1(t+1) \rightarrow -0.6\, g_1(-2t+1)$$

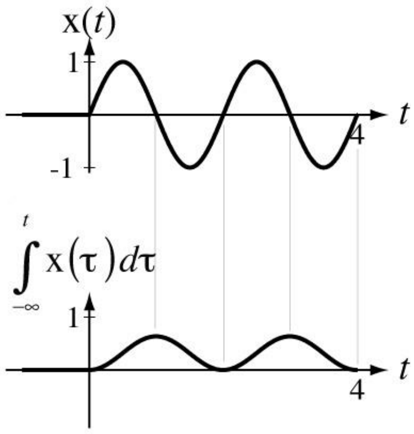


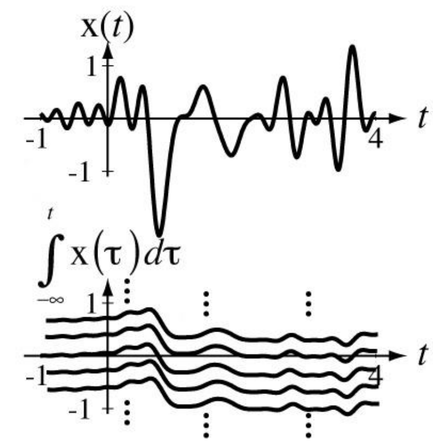
Differentiation

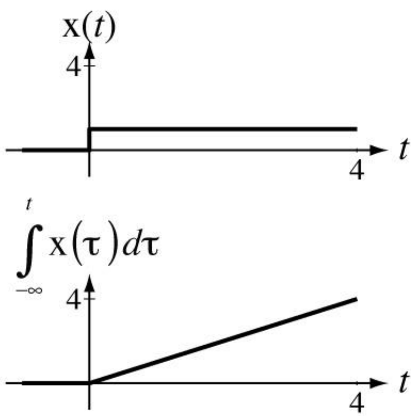


Integration



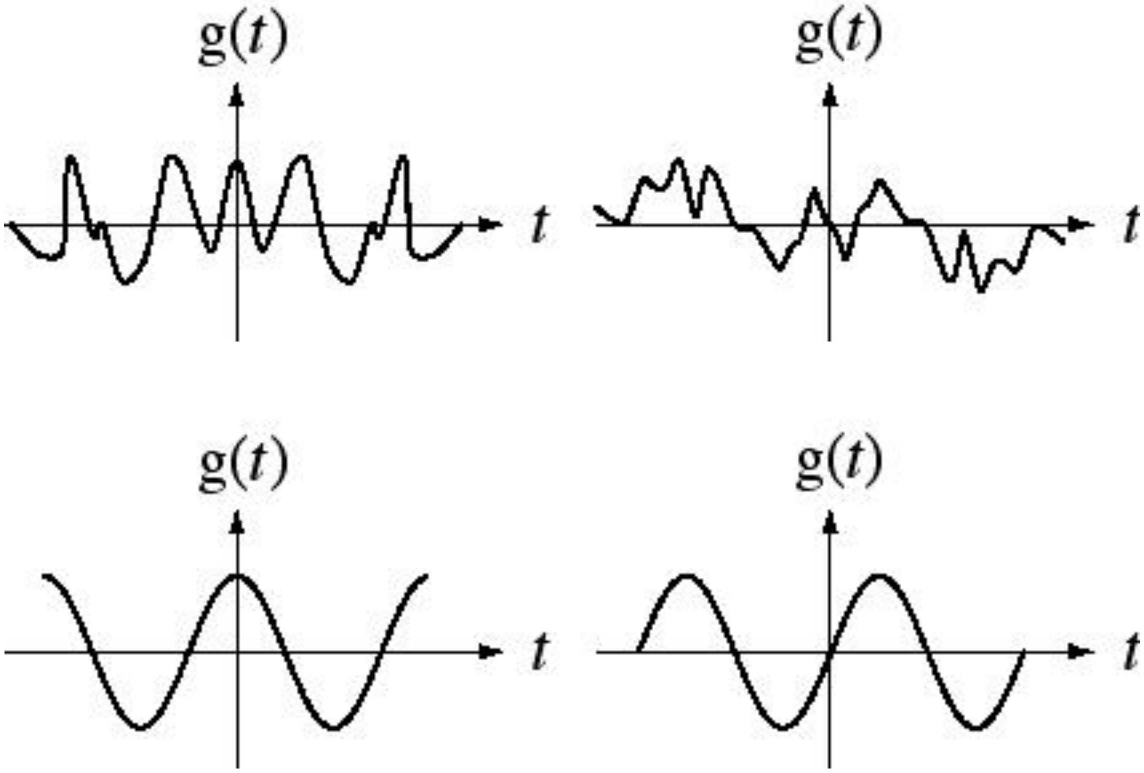






Even and Odd Signals

Even Functions	Odd Functions
$\mathrm{g}(t) = \mathrm{g}(-t)$	$\mathrm{g}(t) = -\mathrm{g}(-t)$



Even and Odd Parts of Functions

The **even part** of a function is $g_e(t) = \dfrac{g(t) + g(-t)}{2}$.

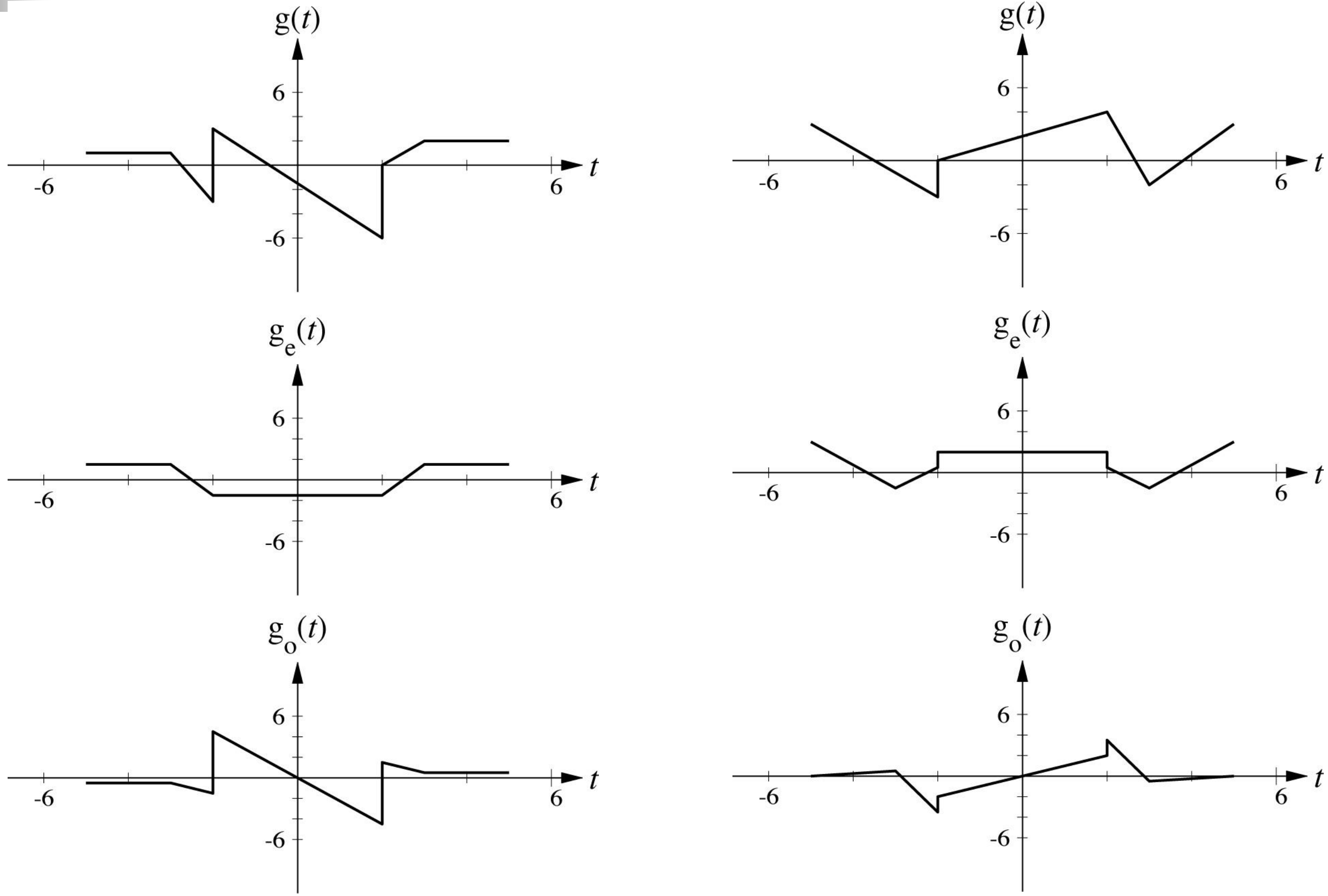
The **odd part** of a function is $g_o(t) = \dfrac{g(t) - g(-t)}{2}$.

A function whose even part is zero is odd and a function whose odd part is zero is even.

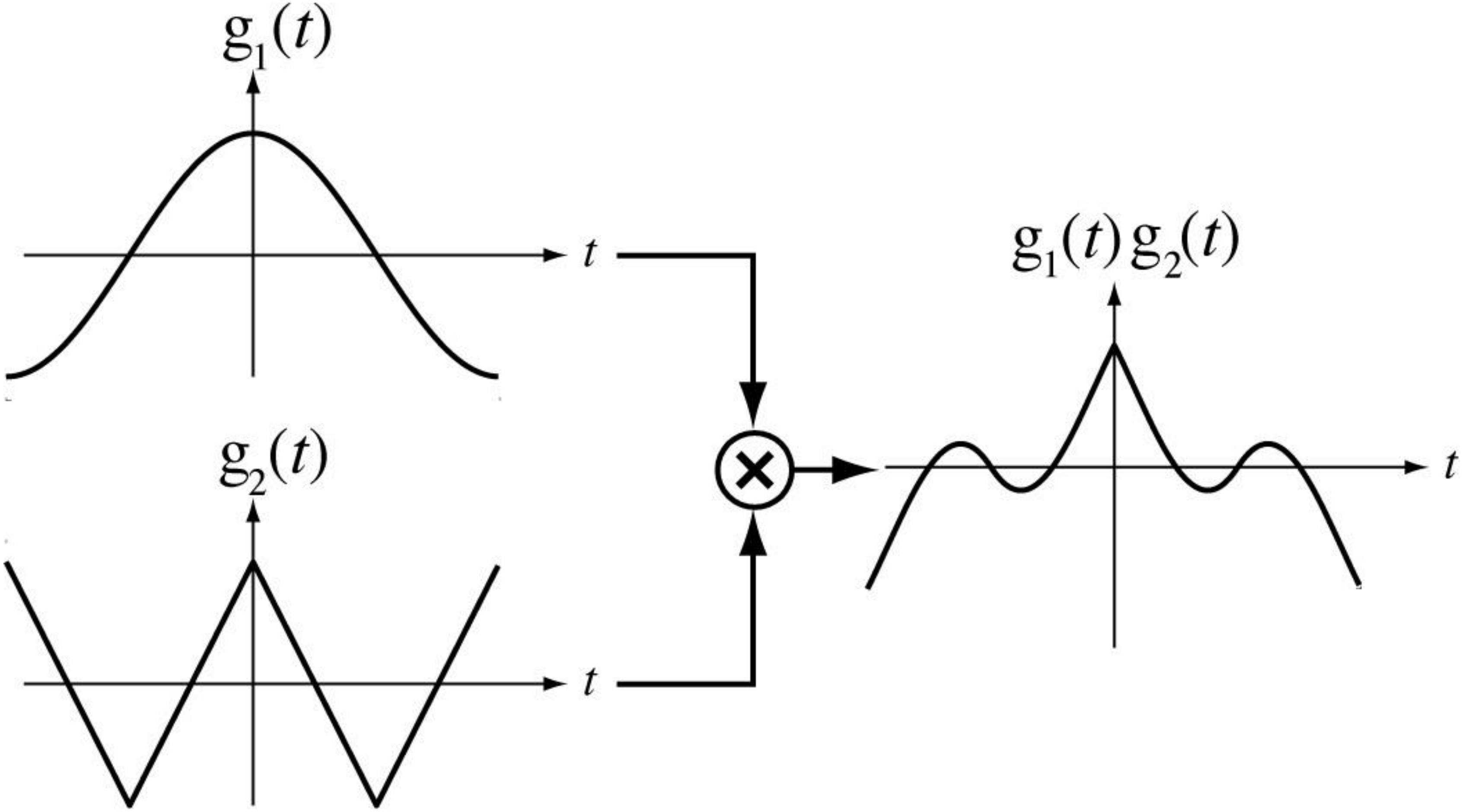
The derivative of an even function is odd and the derivative of an odd function is even.

The integral of an even function is an odd function, <u>plus a constant</u>, and the integral of an odd function is even.

Even and Odd Parts of Functions

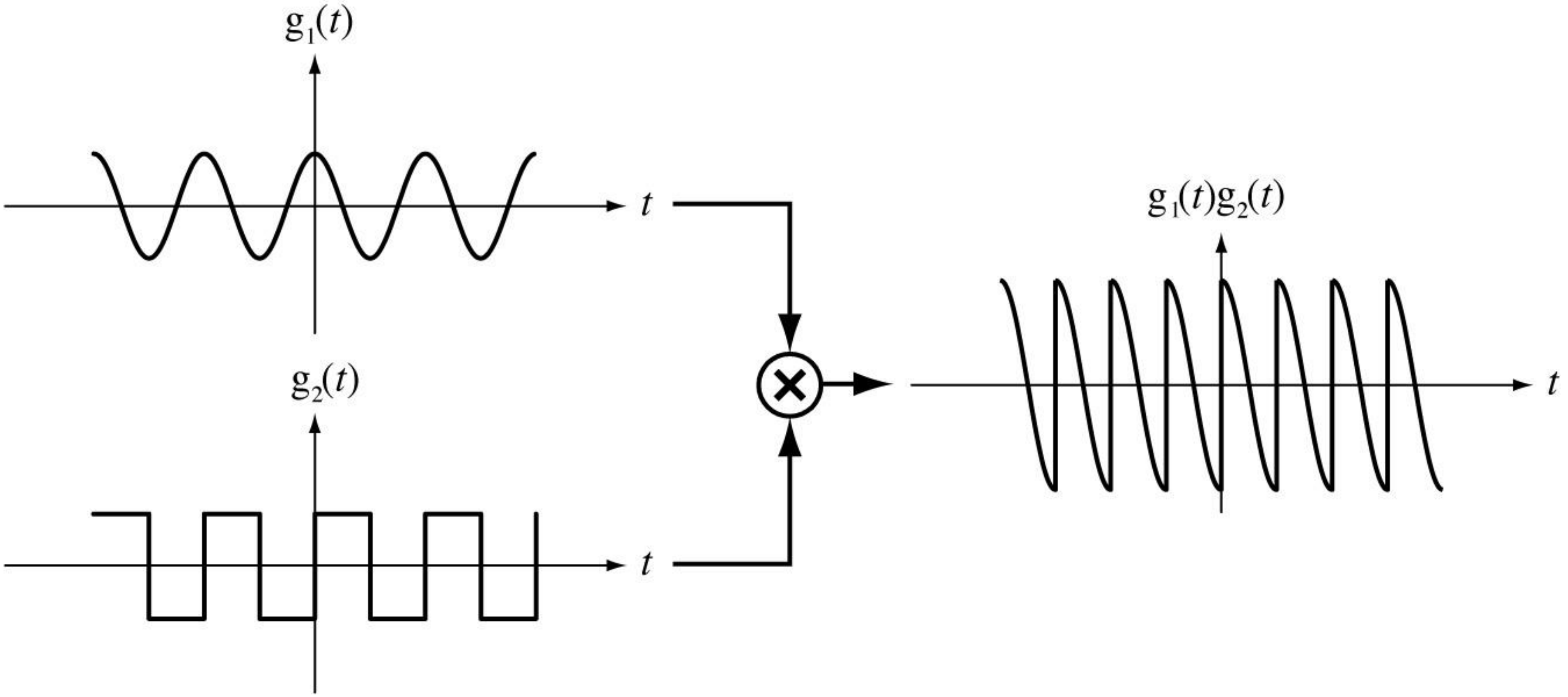


Products of Even and Odd Functions



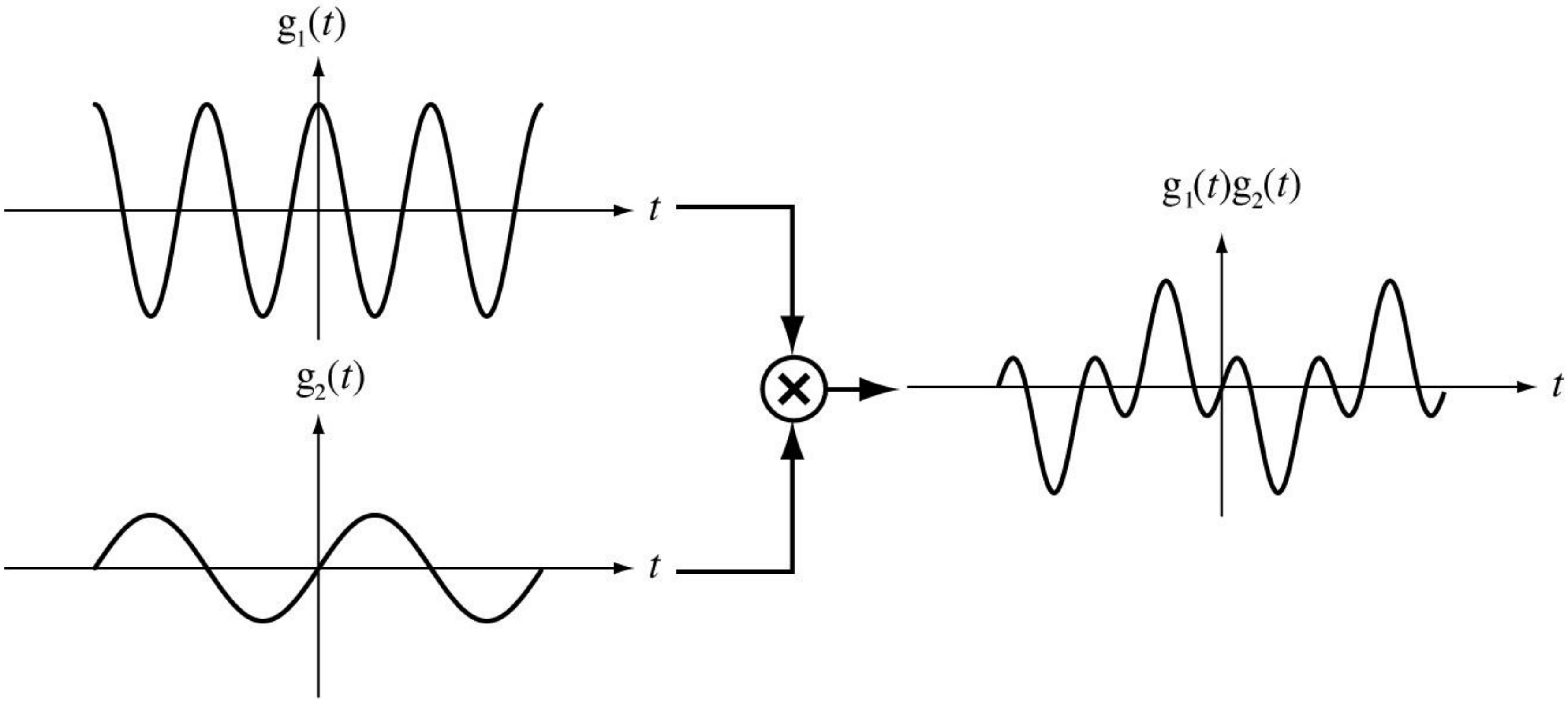
Products of Even and Odd Functions

An Even Function and an Odd Function

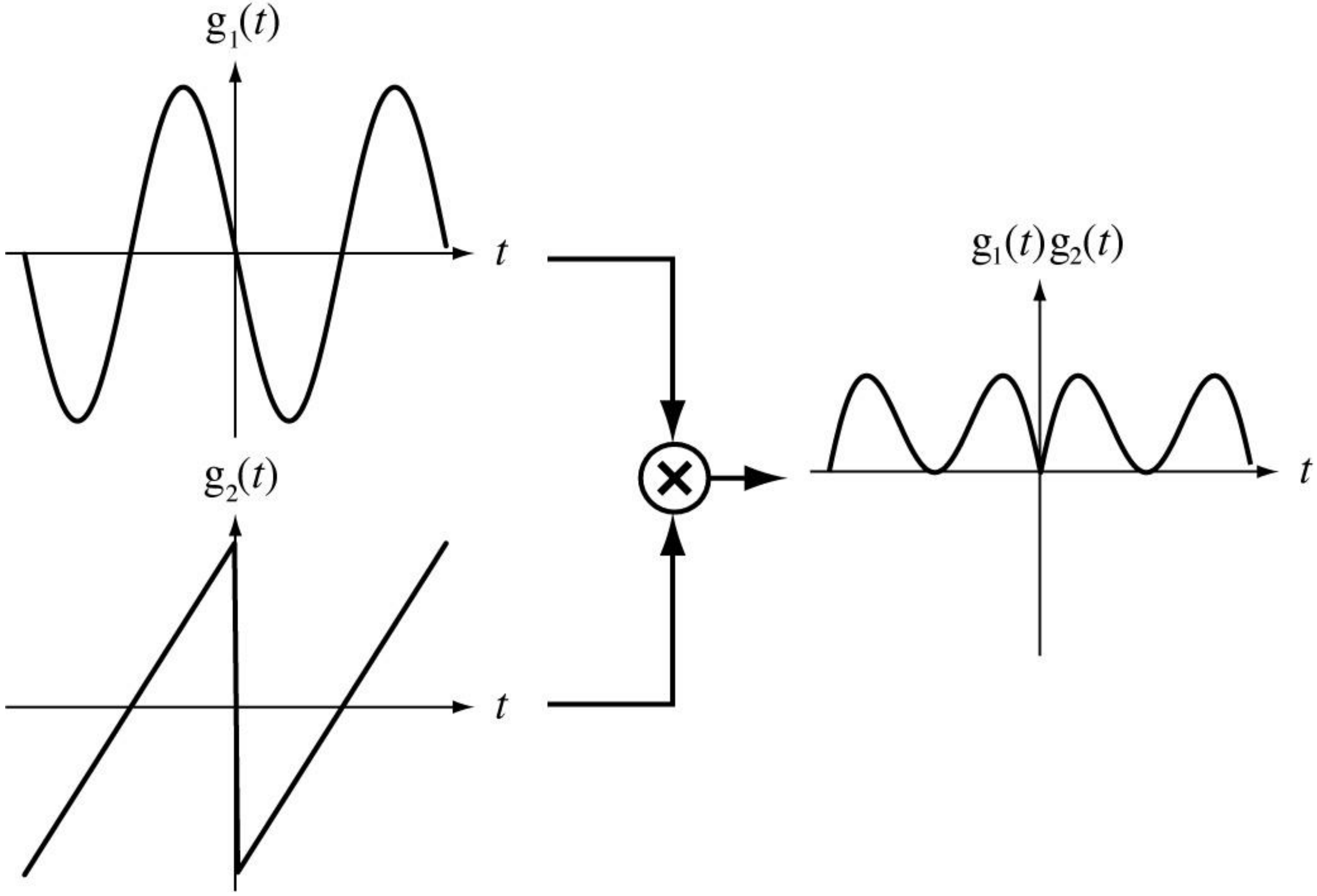


Products of Even and Odd Functions

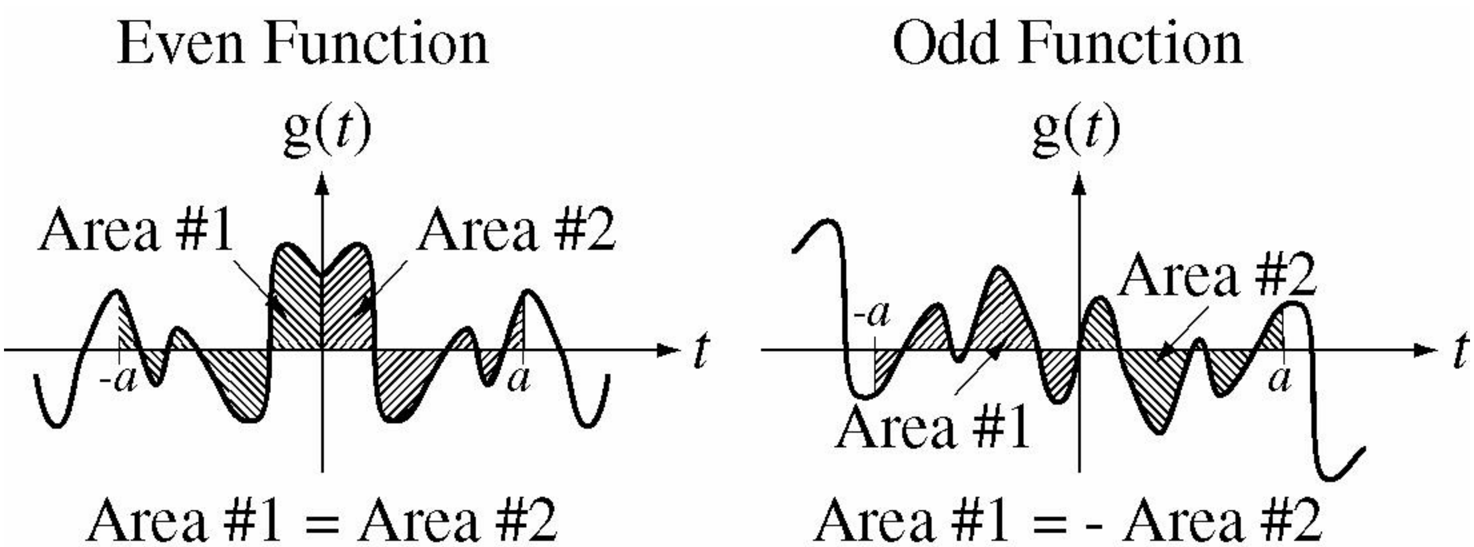
An Even Function and an Odd Function



Products of Even and Odd Functions



Integrals of Even and Odd Functions



$$\int_{-a}^{a} g(t)\,dt = 2\int_{0}^{a} g(t)\,dt \qquad \int_{-a}^{a} g(t)\,dt = 0$$

Integrals of Even and Odd Functions

Evaluate the integral

$$I = \int_{-10}^{10} 4\,\mathrm{rect}(t/8)e^{j2\pi t/16}dt$$

$$I = 4\int_{-4}^{4}\left[\underbrace{\cos(\pi t/8)}_{\text{even}} + j\underbrace{\sin(\pi t/8)}_{\text{odd}}\right]dt = 8\int_{0}^{4}\cos(\pi t/8)dt + \underbrace{j8\int_{-4}^{4}\sin(\pi t/8)dt}_{=0}$$

$$I = 8\left[\frac{\sin(\pi t/8)}{\pi/8}\right]_0^4 = \frac{64}{\pi}[1-0] = \frac{64}{\pi} \cong 20.372$$

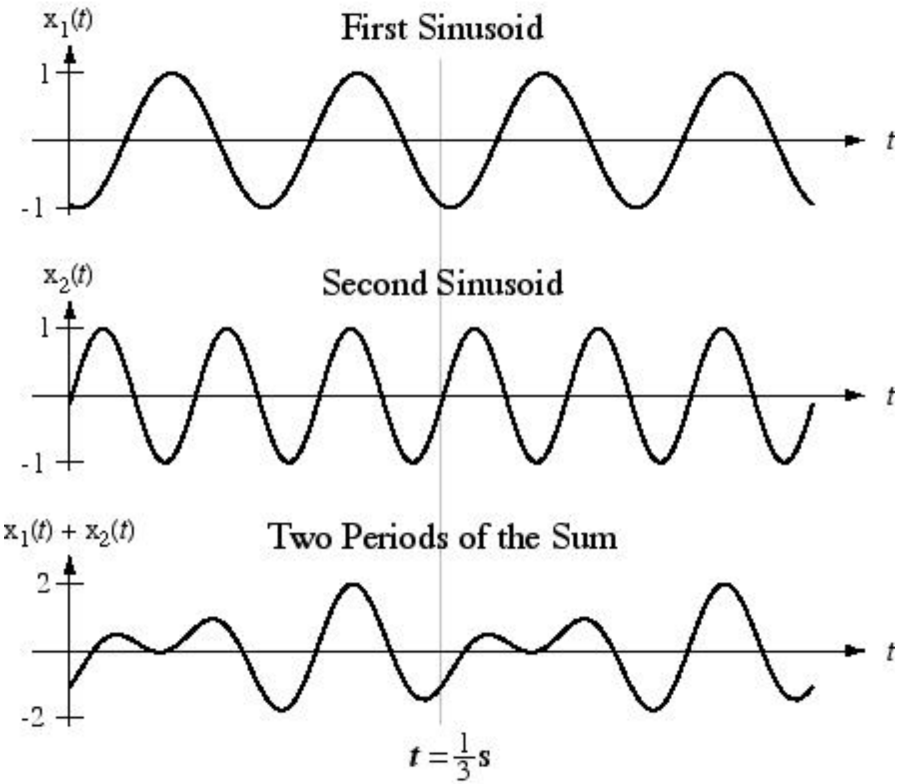
Periodic Signals

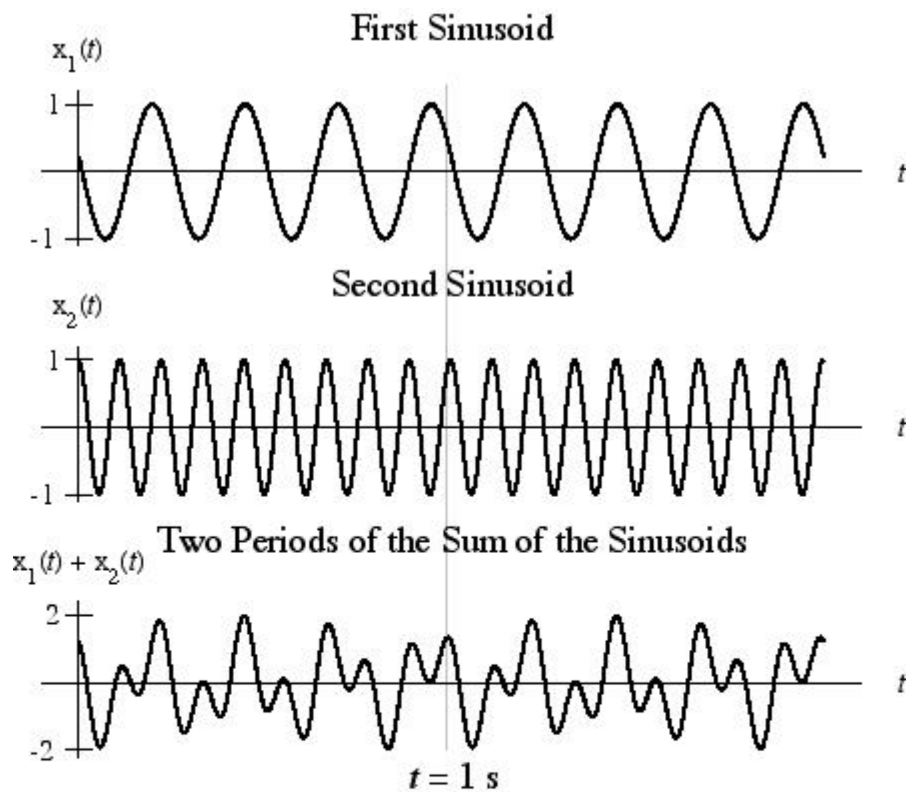
If a function $\mathrm{g}(t)$ is **periodic**, $\mathrm{g}(t) = \mathrm{g}(t + nT)$ where n is any integer and T is a **period** of the function. The minimum positive value of T for which $\mathrm{g}(t) = \mathrm{g}(t + T)$ is called the **fundamental period** T_0 of the function. The reciprocal of the fundamental period is the **fundamental frequency** $f_0 = 1 / T_0$.

A function that is not periodic is **aperiodic**.

Sums of Periodic Functions

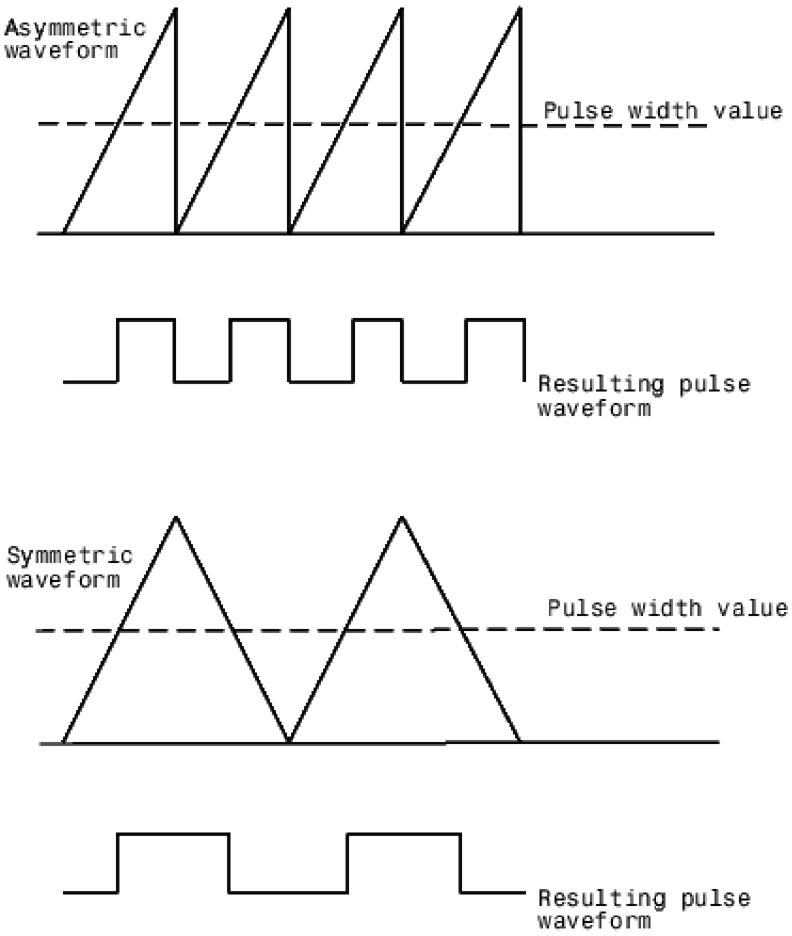
The period of the sum of periodic functions is the **least common multiple** of the periods of the individual functions summed. If the least common multiple is infinite, the sum function is aperiodic.





ADC Waveforms

Examples of waveforms which may appear in analog-to-digital converters. They can be described by a periodic repetition of a ramp returned to zero by a negative step or by a periodic repetition of a triangle-shaped function.

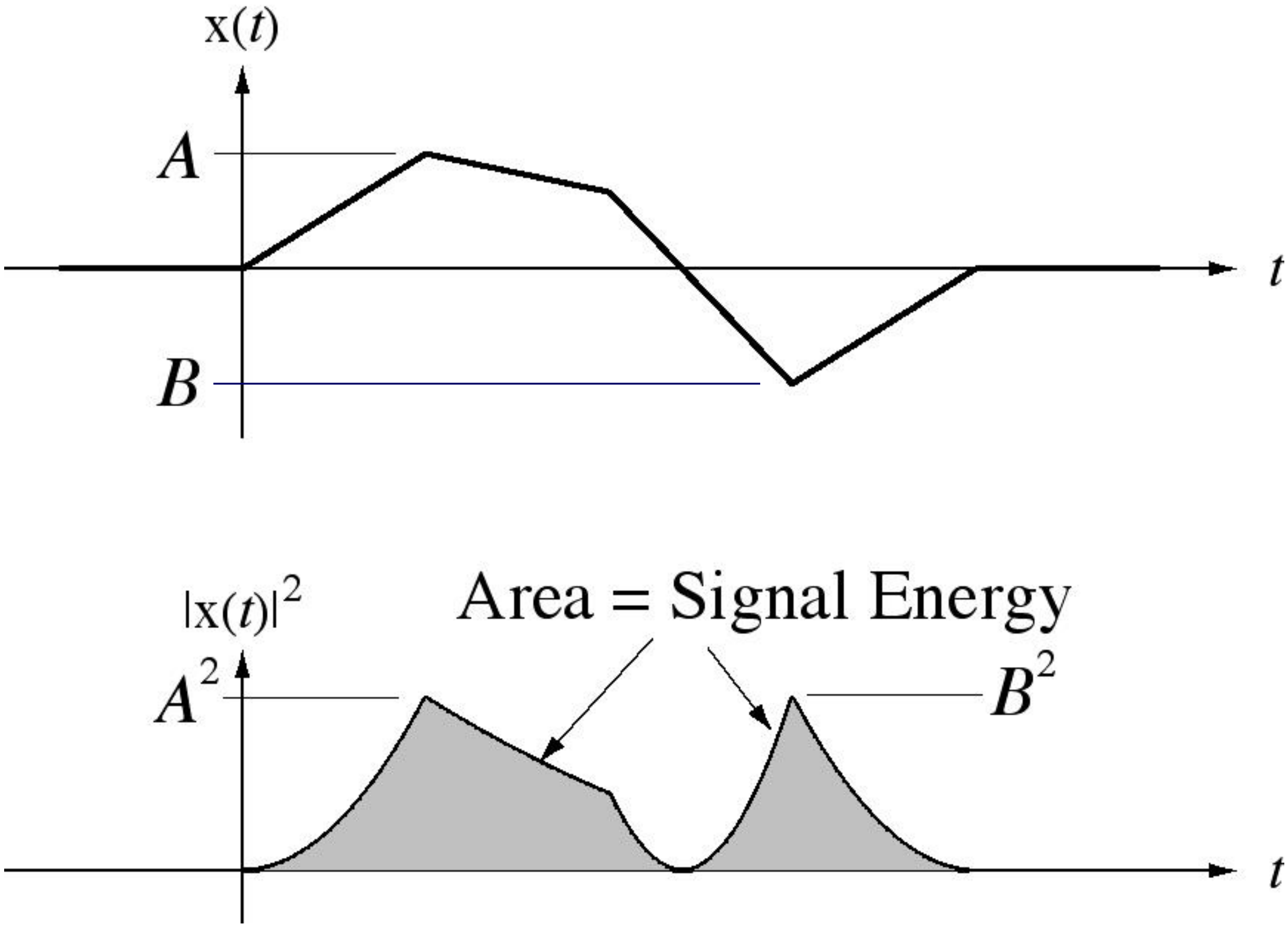


Signal Energy and Power

The signal energy of a signal $\mathrm{x}(t)$ is

$$E_{\mathrm{x}} = \int_{-\infty}^{\infty} \left|\mathrm{x}(t)\right|^2 dt$$

Signal Energy and Power



Signal Energy and Power

Find the signal energy of $x(t) = \left[2\,\mathrm{rect}(t/2) - 4\,\mathrm{rect}\left(\frac{t+1}{4}\right)\right]\mathrm{u}(t+2)$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} \left|\left[2\,\mathrm{rect}(t/2) - 4\,\mathrm{rect}\left(\frac{t+1}{4}\right)\right]\mathrm{u}(t+2)\right|^2 dt$$

$$E_x = \int_{-2}^{\infty} \left[2\,\mathrm{rect}(t/2) - 4\,\mathrm{rect}\left(\frac{t+1}{4}\right)\right]^2 dt$$

$$E_x = \int_{-2}^{\infty} \left[4\,\mathrm{rect}^2(t/2) + 16\,\mathrm{rect}^2\left(\frac{t+1}{4}\right) - 16\,\mathrm{rect}(t/2)\,\mathrm{rect}\left(\frac{t+1}{4}\right)\right] dt$$

$$E_x = 4\int_{-2}^{\infty} \mathrm{rect}(t/2)\,dt + 16\int_{-2}^{\infty} \mathrm{rect}\left(\frac{t+1}{4}\right) dt - 16\int_{-2}^{\infty} \mathrm{rect}(t/2)\,\mathrm{rect}\left(\frac{t+1}{4}\right) dt$$

$$E_x = 4\int_{-1}^{1} dt + 16\int_{-2}^{1} dt - 16\int_{-1}^{1} dt = 8 + 48 - 32 = 24$$

Signal Energy and Power

Some signals have infinite signal energy. In that case It is more convenient to deal with average signal power. The average signal power of a signal $\mathrm{x}(t)$ is

$$P_{\mathrm{x}} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\mathrm{x}(t)\right|^2 dt$$

For a periodic signal $\mathrm{x}(t)$ the average signal power is

$$P_{\mathrm{x}} = \frac{1}{T} \int_{T} \left|\mathrm{x}(t)\right|^2 dt$$

where T is any period of the signal.

Signal Energy and Power

A signal with finite signal energy is called an **energy signal**.

A signal with infinite signal energy and finite average signal power is called a **power signal**.

Signal Energy and Power

Find the average signal power of a signal $\mathrm{x}(t)$ with fundamental period 12, one period of which is described by

$$\mathrm{x}(t) = \mathrm{ramp}(-t/5) \quad , \quad -4 < t < 8$$

$$P_{\mathrm{x}} = \frac{1}{T}\int_{T} |\mathrm{x}(t)|^2\, dt = \frac{1}{12}\int_{-4}^{8} |\mathrm{ramp}(-t/5)|^2\, dt = \frac{1}{12}\int_{-4}^{0} (-t/5)^2\, dt$$

$$P_{\mathrm{x}} = \frac{1}{12}\int_{-4}^{0} \frac{t^2}{25}\, dt = \frac{1}{300}\left[t^3/3\right]_{-4}^{0} = \frac{0-(-64/3)}{300} = \frac{16}{225} @ 0.0711$$