

## Midterm Exam 1

BLM 1541: Probability and Statistics — Fall 2016

Print family (or last) name: \_\_\_\_\_

Print given (or first) name: \_\_\_\_\_

Print student number: \_\_\_\_\_ Group: \_\_\_\_\_ Order: \_\_\_\_\_

I have read and understand all of the instructions below, and I will obey the Academic Integrity Code of Yıldız Technical University.

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### Signature and Date

- This exam has 4 pages in total, numbered 1 to 4. Make sure your exam has all the pages.
- Note the number written on the upper right-hand corner of the first page. On the sign-up sheet being passed around, sign your name next to this number.
- This exam will be 1 hour and 25 minutes in length.
- This is a closed-book and closed-note exam. Electronic devices (e.g., cellphone, smart watch) are not allowed. Single page A4 note sheet is allowed. *Don't forget to attach your A4 note sheet into your solution; otherwise your solution will not be graded.*
- For all problems, follow these instructions:
  - Give only your answers in the spaces provided. I will only grade what you put in the answer space, and I will take off points for any scratch work in the answer space. Use the scratch-work area or the backs of the sheets to work out your answers before filling in the answer space.
  - PMF stands for probability mass function; CDF stands for cumulative distribution function;  $\text{var}(X)$  stands for the variance of the random variable  $X$ ;  $\text{cov}(X, Y)$  stands for the covariance between the random variables  $X$  and  $Y$ .
  - For any proofs, be sure to provide a step-by-step argument, with justifications for every step.

Problem	1	2	3	4	5	6	Total
	20	15	20	15	15	15	100
Points							

1. [20 points] Let us given two events  $A$  and  $B$ , then answer the following questions

- (a) If events  $A$  and  $B$  are not independent and the probabilities  $P(A \cup B) = 0.6$ ,  $P(\bar{A} \cup \bar{B}) = 0.7$  and  $P(A \cap \bar{B}) = 0.2$  are given, then find the probability of event  $A$ , i.e.,  $P(A)$ ?

*Hints: You can start from  $A = (A \cap B) \cup (A \cap \bar{B})$ .*

*Herein write your answer.*

Most efficient way:

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap \bar{B}) \\ P(A \cap B) &= 1 - P(\bar{A} \cup \bar{B}) \end{aligned}$$

Therefore we have  $P(A) = 0.3 + 0.2 = 0.5$

Other way:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.6 &= P(A) + P(B) - 0.3 \text{ so } P(A) + P(B) = 0.9 \end{aligned}$$

$P(A \cup B) = P(B) + P(A \cap \bar{B})$  so we have  $P(B) = P(A \cup B) - P(A \cap \bar{B}) = 0.6 - 0.2 = 0.4$

At the end we have  $P(A) + P(B) = 0.9$  then  $P(A) = 0.9 - P(B) = 0.9 - 0.4 = 0.5$

(b) If events  $A$  and  $B$  are independent, then show that  $P(A \cap \bar{B}) = P(A)P(\bar{B})$ .

*Herein write your answer.*

We can write  $P(A) = P(A \cap B) + P(A \cap \bar{B})$ , then we have  $P(A \cap \bar{B}) = P(A) - P(A \cap B)$ . Under the condition of that  $A$  and  $B$  are independent, then we have  $P(A \cap B) = P(A)P(B)$ . Accordingly,  $P(A \cap \bar{B})$  can be written as

$$\begin{aligned} P(A \cap \bar{B}) &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \end{aligned}$$

where substituting  $P(\bar{B}) = 1 - P(B)$  results in  $P(A \cap \bar{B}) = P(A)P(\bar{B})$ , which proves the equality.

2. [15 points] A computer manufacturer uses chips from three sources. Chips from sources  $A$ ,  $B$ , and  $C$  are defective with probabilities 0.005, 0.001, and 0.010, respectively. If a randomly selected chip is found to be defective, find the probability that the manufacturer was  $A$ . Assume that the proportions of chips from  $A$ ,  $B$ , and  $C$  are 0.5, 0.1, and 0.4, respectively.

*Herein write your answer.*

$$\begin{aligned} P(\text{chip defective}) &= P(\text{def.}|A)P(A) + P(\text{def.}|B)P(B) + P(\text{def.}|C)P(C) \\ &= 5 \times 10^{-3}P(A) + 10^{-3}P(B) + 10^{-2}P(C) \\ &= 5 \times 10^{-3} \times 0.5 + 10^{-3} \times 0.1 + 10^{-2} \times 0.4 = 6.6 \times 10^{-3} \end{aligned}$$

By means of using Bayes theorem, we have

$$P(A|\text{chip defective}) = \frac{P(\text{def.}|A)P(A)}{P(\text{def.})} = \frac{5 \times 10^{-3} \times 0.5}{6.6 \times 10^{-3}} = 0.3788$$

3. [20 points] If  $X$  and  $Y$  have a covariance of  $\text{cov}(X, Y)$ , we can transform them to a new pair of random variables whose covariance is zero. To do so we let

$$W = X,$$

$$Z = cX + Y,$$

where  $c$  is a constant real number, then answer the following questions. Express  $\text{cov}(W, Z)$  in terms of  $c$ ,  $\text{var}(X)$  and  $\text{cov}(X, Y)$ .

*Herein write your answer.*

Let say  $\mu_X = E[X]$  and  $\mu_Y = E[Y]$ . Using the equation of covariance, we can write

$$\begin{aligned} \text{Cov}(W, Z) &= E[(W - E[W])(Z - E[Z])] \\ &= E[(X - E[X])(cX + Y - E[cX + Y])] \\ &= E[(X - \mu_X)(cX + Y - c\mu_X - \mu_Y)] \\ &= E[cX^2 + XY - 2cX\mu_X - Y\mu_X + c\mu_X^2 - X\mu_Y + \mu_X\mu_Y] \\ &= E[cX^2] + E[XY] - 2c\mu_X^2 - \mu_Y\mu_X + c\mu_X^2 - \mu_X\mu_Y + \mu_X\mu_Y \\ &= E[cX^2] - c\mu_X^2 + \underbrace{E[XY] - \mu_Y\mu_X}_{\text{Cov}(X, Y)} \\ &= c \underbrace{(E[X^2] - \mu_X^2)}_{\text{var}(X)} + \underbrace{E[XY] - \mu_Y\mu_X}_{\text{Cov}(X, Y)} \\ &= c \text{var}(X) + \text{cov}(X, Y) \end{aligned}$$

4. [15 points] Assume that  $X$  and  $Y$  are two discrete random variables whose joint PMF is given as

$$P_{X,Y}(x, y) = \begin{cases} \frac{1}{6}, & x = 0, y = 0 \\ \frac{1}{3}, & x = 0, y = 1 \\ \frac{1}{3}, & x = 1, y = 0 \\ \frac{1}{6}, & x = 1, y = 1 \end{cases}$$

Then, find the PMF of random variable  $X$ , i.e.,  $P_X(x) = ?$

*Herein write your answer.*

We can find the probabilities of  $X$  by using Bayes theorem as follows

$$P(X = 0) = P(X = 0|Y = 0) + P(X = 0|Y = 1) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

$$P(X = 1) = P(X = 1|Y = 0) + P(X = 1|Y = 1) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Then we have the PMF of  $X$  as

$$\begin{aligned} P_X(x) &= \begin{cases} P(X = 0), & x = 0, \\ P(X = 1), & x = 1 \end{cases} \\ &= \begin{cases} 1/2, & x = 0, \\ 1/2, & x = 1 \end{cases} \end{aligned}$$

5. [15 points] A television storeowner figures that 70 percent of the customers entering his store buy a television. The rest is just browsing. 5 customers enter the store that day. What is the probability that our storeowner sells three or more televisions on that day?

*Herein write your answer.*

Let say  $X$  be a random variable representing the Number of televisions sold that day.  $P(X \geq 3) = ?$

Each customer is a Bernoulli trial with definitions:

*Success* → Customer buys a TV.

*Failure* → Customer does not buy a TV

Then the random variable  $X$  is a binomial random variable, i.e.,  $X \sim \text{Binomial}(n = 5, p = 0.7)$ . Then, we can compute the probability as

$$\begin{aligned} P(X \geq 3) &= 1 - P(X \leq 2) \\ &= 1 - F(2) \\ &= 1 - P(0) - P(1) - P(2) \\ &= 1 - 0.163 = 0.837 \end{aligned}$$

6. [15 points] Let  $X$  be a random variable whose PDF is given as

$$f(x) = \begin{cases} c(1 - x^2), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- a) What is the value of the constant  $c$ ?

*Herein write your answer.*

$$\begin{aligned} \int_{-1}^1 f(x)dx &= 1 \text{ therefore } \int_{-1}^1 c(1 - x^2)dx \\ &= c \left( x - \frac{x^3}{3} \Big|_{-1}^1 \right) = c \left( 1 - \frac{1}{3} - 1 + \frac{1}{3} \right) \\ &= c \left( 2 - \frac{2}{3} \right) = c \frac{4}{3} = 1 \text{ then we find that } c = \frac{3}{4} \end{aligned}$$

- b) What is the CDF of  $X$ ? In other words  $F_X(x) = P(X < x) = ?$

*Herein write your answer.*

$$\begin{aligned} F_X(x) &= P(X < x) = \int_{-\infty}^x f(u)du \\ &= \int_{-1}^x \frac{3}{4}(1 - u^2)du \\ &= \frac{3}{4} \left( u - \frac{u^3}{3} \Big|_{-1}^x \right) = \frac{3}{4} \left( x - \frac{x^3}{3} \right) - \frac{3}{4} \left( -1 + \frac{1}{3} \right) \\ &= -\frac{x^3}{4} + \frac{3}{4}x + \frac{1}{2} \end{aligned}$$