The *z* Transform

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Generalizing the DTFT

The forward DTFT is defined by $\mathrm{X}\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty}\mathrm{x}[n]e^{-j\Omega n}$ in which Ω is discrete-time radian frequency, a real variable. The quantity $e^{j\Omega n}$ is then a complex sinusoid whose magnitude is always one and whose phase can range over all angles. It always lies on the unit circle in the complex plane. If we now replace $e^{j\Omega}$ with a variable z that can have any complex value we define the z transform $\mathrm{X}(z)=\sum_{n=-\infty}^{\infty}\mathrm{x}[n]z^{-n}$.

The DTFT expresses signals as linear combinations of complex <u>sinusoids</u>. The z transform expresses signals as linear combinations of complex <u>exponentials</u>.

Complex Exponential Excitation

Let the excitation of a discrete-time LTI system be a complex exponential of the form Az^n where z is, in general, complex and A is any constant. Using convolution, the response $\mathrm{y}[n]$ of an LTI system with impulse response $\mathrm{h}[n]$ to a complex exponential excitation $\mathrm{x}[n]$ is

$$\mathrm{y}[n] = \mathrm{h}[n] * Az^n = A\sum_{m=-\infty}^{\infty} \mathrm{h}[m] z^{n-m} = \underbrace{Az^n}_{=\mathrm{x}[n]} \sum_{m=-\infty}^{\infty} \mathrm{h}[m] z^{-m}$$

The response is the product of the excitation and the z transform of $\mathrm{h}[n]$ defined by $\mathrm{H}(z) = \sum_{m=-\infty}^{\infty} \mathrm{h}[n] z^{-n}$.

The Transfer Function

If an LTI system with impulse response $\mathrm{h}[n]$ is excited by a signal, $\mathrm{x}[n]$, the z transform $\mathrm{Y}(z)$ of the response $\mathrm{y}[n]$ is

$$\mathrm{Y}(z) = \sum_{n=-\infty}^{\infty} \mathrm{y}[n] z^{-n} = \sum_{n=-\infty}^{\infty} (\mathrm{h}[n] * \mathrm{x}[n]) z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathrm{h}[m]\mathrm{x}[n-m] z^{-n}$$

$$\mathrm{Y}(z) = \sum_{m=-\infty}^{\infty} \mathrm{h}[m] \sum_{n=-\infty}^{\infty} \mathrm{x}[n-m] z^{-n}$$

Let $q = n - m$. Then

$$\mathrm{Y}(z) = \sum_{m=-\infty}^{\infty} \mathrm{h}[m] \sum_{q=-\infty}^{\infty} \mathrm{x}[q] z^{-(q+m)} = \underbrace{\sum_{m=-\infty}^{\infty} \mathrm{h}[m] z^{-m}}_{=\mathrm{H}(z)} \underbrace{\sum_{q=-\infty}^{\infty} \mathrm{x}[q] z^{-q}}_{=\mathrm{X}(z)}$$

$$\mathrm{Y}(z) = \mathrm{H}(z)\mathrm{X}(z)$$

$\mathrm{H}(z)$ is the **transfer function**.

Systems Described by Difference Equations

The most common description of a discrete-time system is a difference equation of the general form

$$\sum_{k=0}^{N} a_k \, \mathrm{y}[n-k] = \sum_{k=0}^{M} b_k \, \mathrm{x}[n-k].$$

It was shown in Chapter 5 that the transfer function for a system of this type is

$$\mathrm{H}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

or

$$\mathrm{H}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = z^{N-M} \frac{b_0 z^{M} + b_1 z^{M-1} + \cdots + b_{M-1} z + b_M}{a_0 z^{N} + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}$$

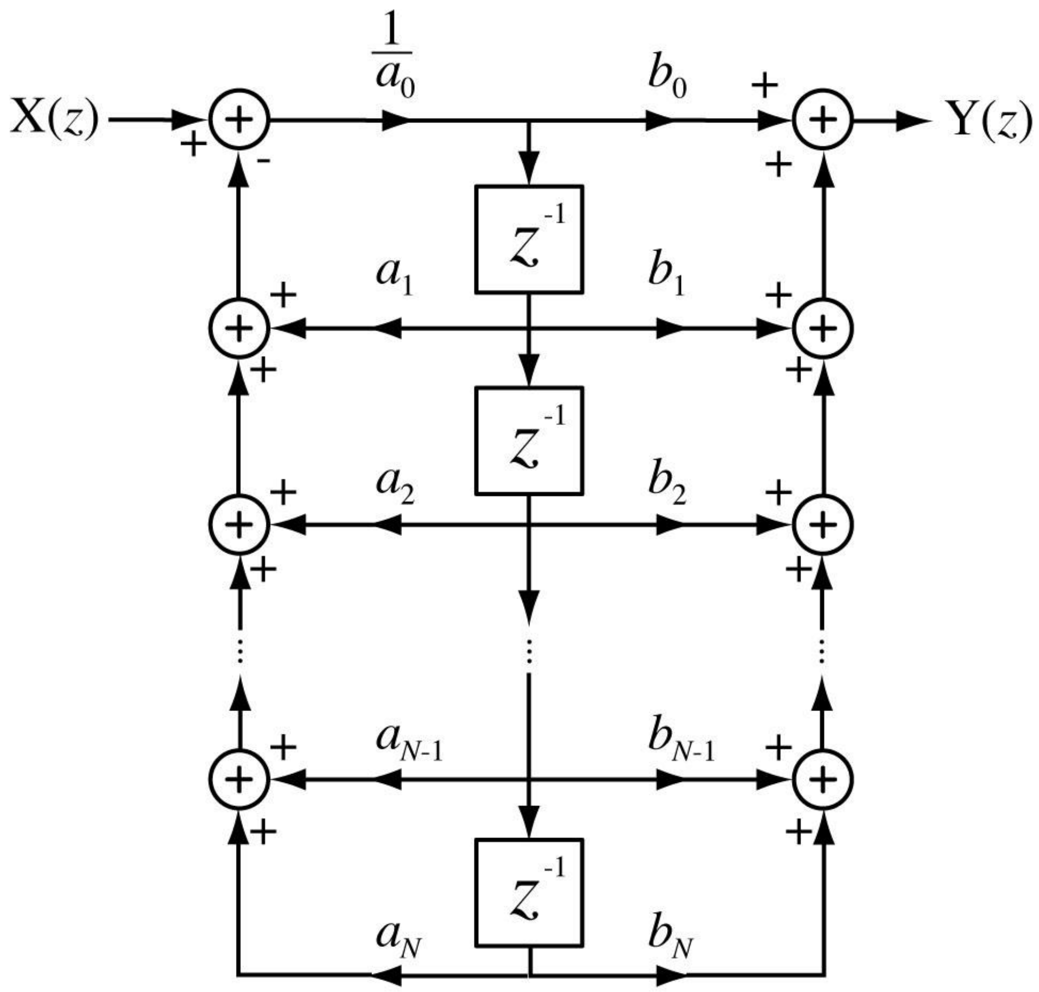
Direct Form II Realization

Direct Form II realization of a discrete-time system is similar in form to Direct Form II realization of continuous-time systems

A continuous-time system can be realized with integrators, summing junctions and multipliers

A discrete-time system can be realized with delays, summing junctions and multipliers

Direct Form II Realization



The Inverse z Transform

The inversion integral is

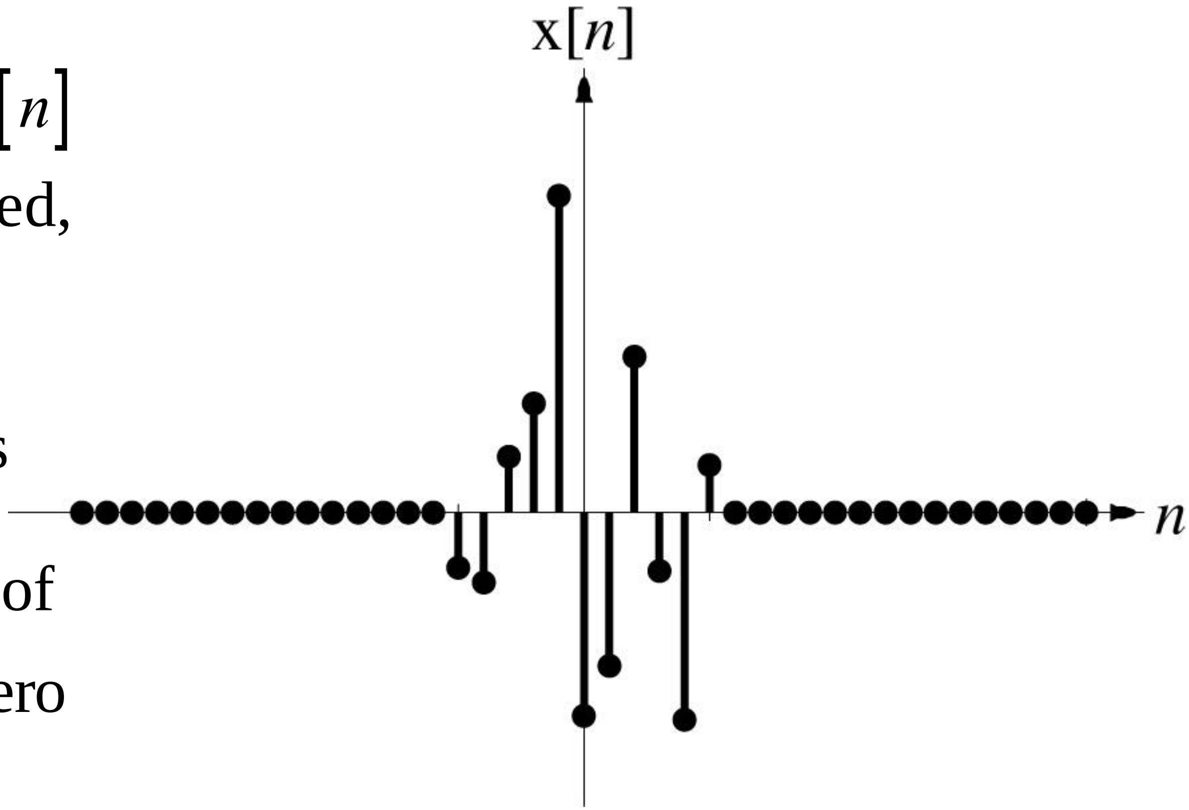
$$\mathrm{x}[n] = \frac{1}{j2\pi} \oint_{\mathrm{C}} \mathrm{X}(z) z^{n-1} dz.$$

This is a contour integral in the complex plane and is beyond the scope of this course. The notation $\mathrm{x}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{X}(z)$ indicates that $\mathrm{x}[n]$ and $\mathrm{X}(z)$ form a "z-transform pair".

Existence of the z Transform

Time Limited Signals

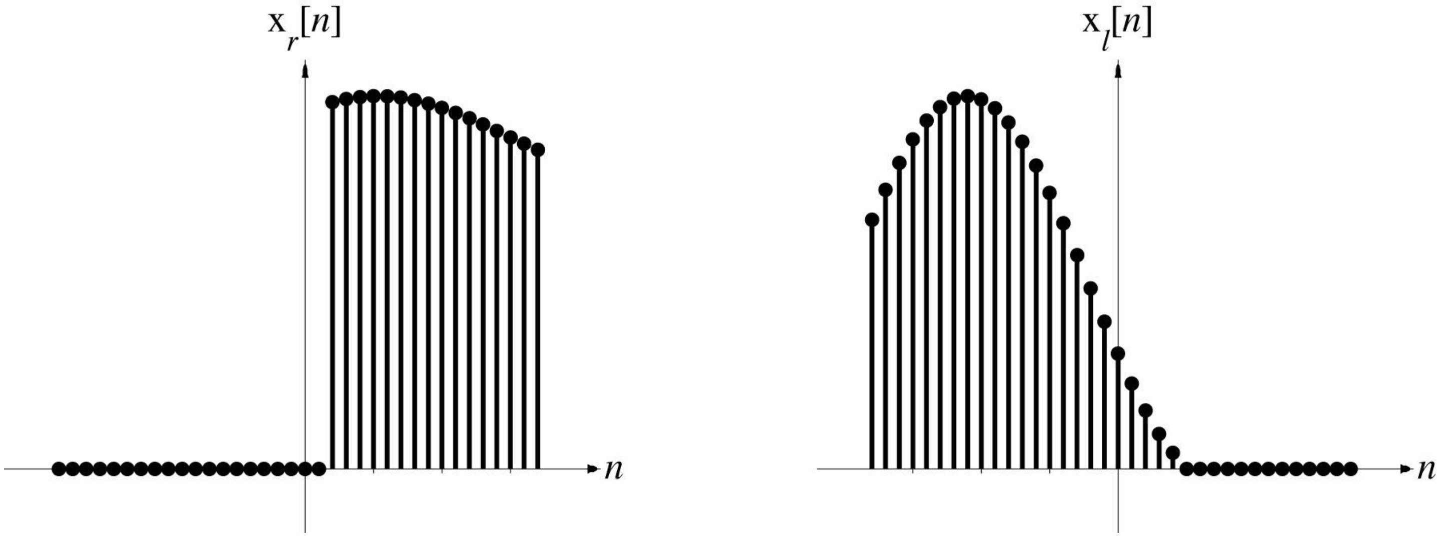
If a discrete-time signal $\mathrm{x}[n]$ is time limited and bounded, the z transformation summation $\sum_{n=-\infty}^{\infty} \mathrm{x}[n] z^{-n}$ is finite and the z transform of $\mathrm{x}[n]$ exists for any non-zero value of z.



Existence of the z Transform

Right- and Left-Sided Signals

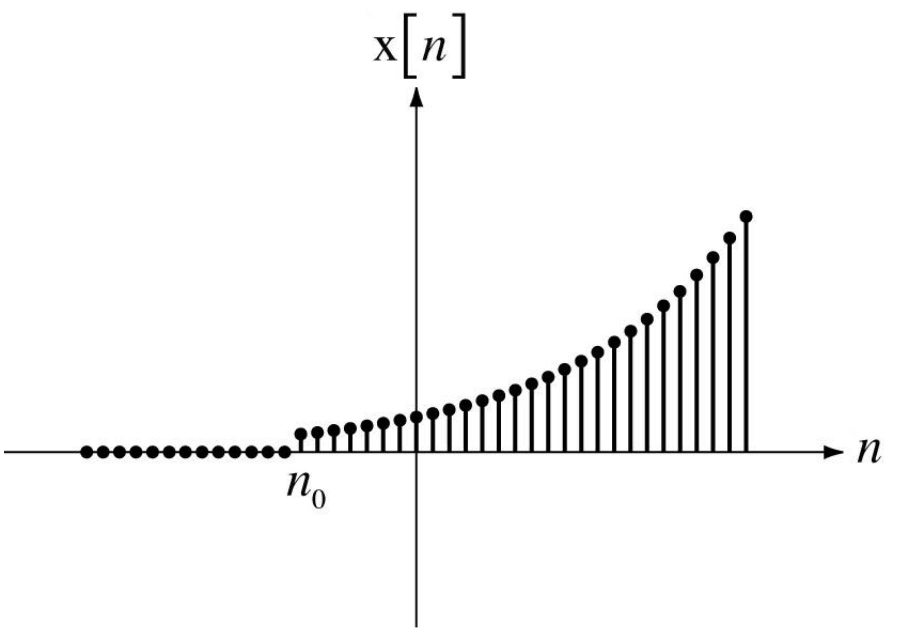
A right-sided signal $x_r[n]$ is one for which $x_r[n] = 0$ for any $n < n_0$ and a left-sided signal $x_l[n]$ is one for which $x_l[n] = 0$ for any $n > n_0$.

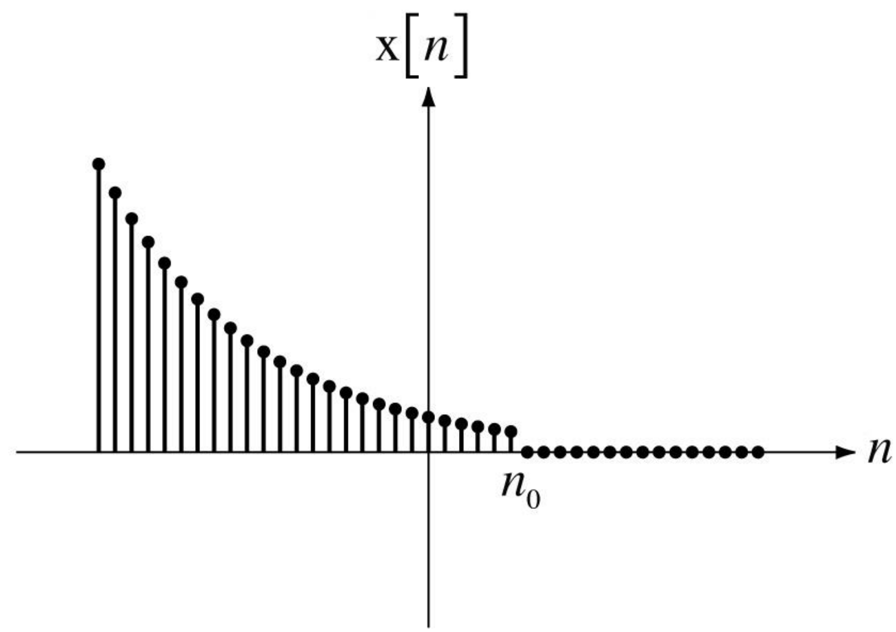


Existence of the z Transform

Right- and Left-Sided Exponentials

$$\mathrm{x}[n] = a^n \,\mathrm{u}[n - n_0] \;,\; a \in \square \qquad\qquad \mathrm{x}[n] = b^n \,\mathrm{u}[n_0 - n] \;,\; b \in \square$$



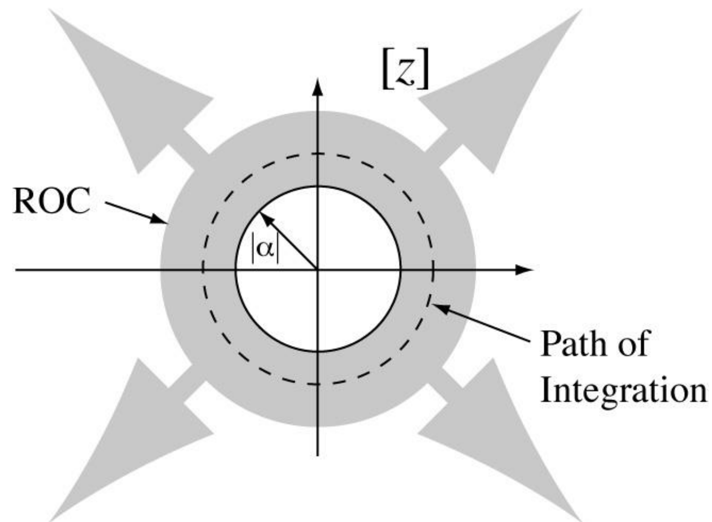


Existence of the z Transform

The z transform of $\mathrm{x}[n] = \alpha^n \mathrm{u}[n - n_0]$, $\alpha \in \mathbb{C}$ is

$$\mathrm{X}(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mathrm{u}[n - n_0] z^{-n} = \sum_{n=n_0}^{\infty} \left(\alpha z^{-1}\right)^n$$

<u>if the series converges</u> and it converges if $|z| > |\alpha|$. The path of integration of the inverse z transform must lie in the region of the z plane outside a circle of radius $|\alpha|$.

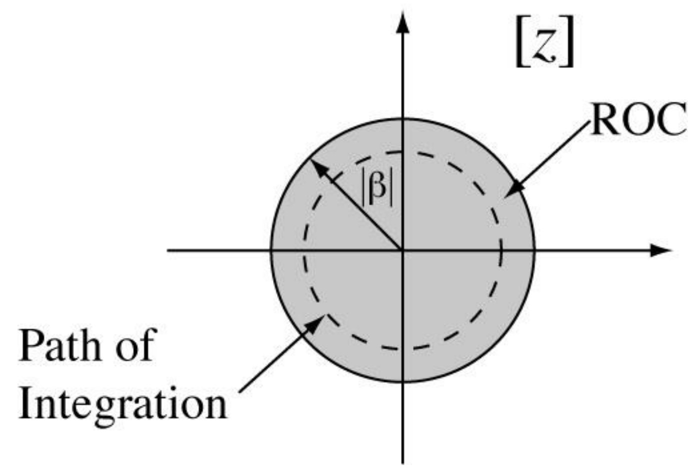


Existence of the z Transform

The z transform of $x[n] = \beta^n u[n_0 - n]$, $\beta \in \mathbb{C}$ is

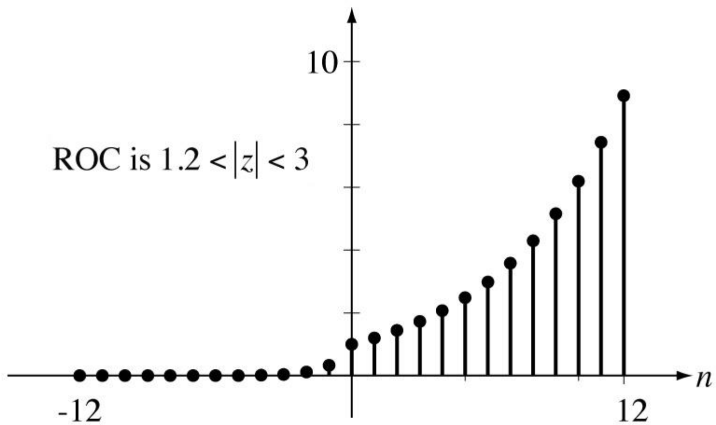
$$X(z) = \sum_{n=-\infty}^{n_0} \beta^n z^{-n} = \sum_{n=-\infty}^{n_0} \left(\beta z^{-1}\right)^n = \sum_{n=-n_0}^{\infty} \left(\beta^{-1} z\right)^n$$

if the series converges and it converges if $|z| < |\beta|$. The path of integration of the inverse z transform must lie in the region of the z plane inside a circle of radius $|\beta|$.

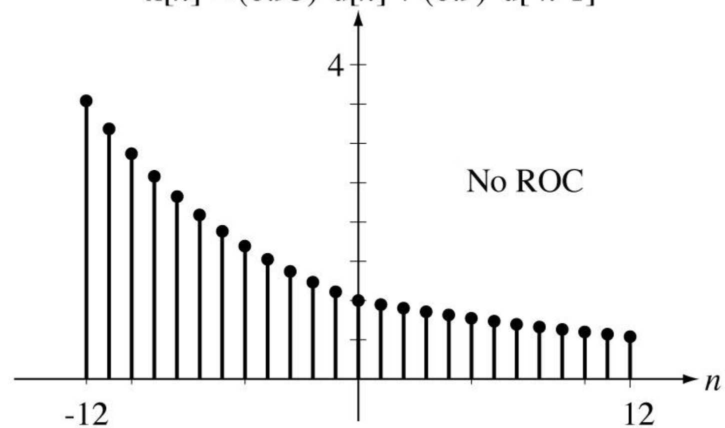


Existence of the z Transform

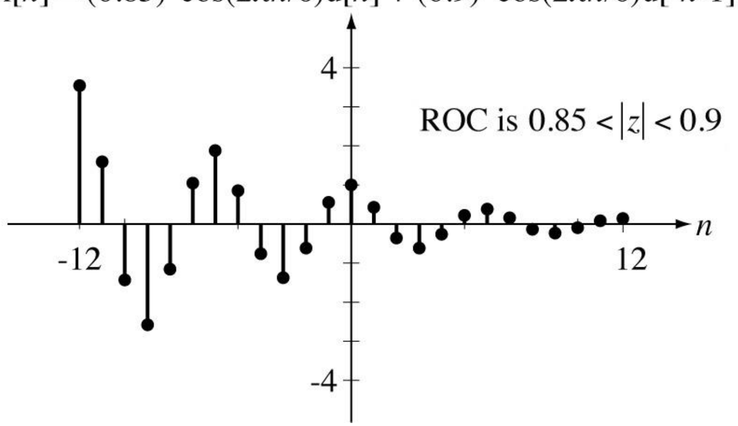
$x[n] = (1.2)^n u[n] + (3)^n u[-n-1]$



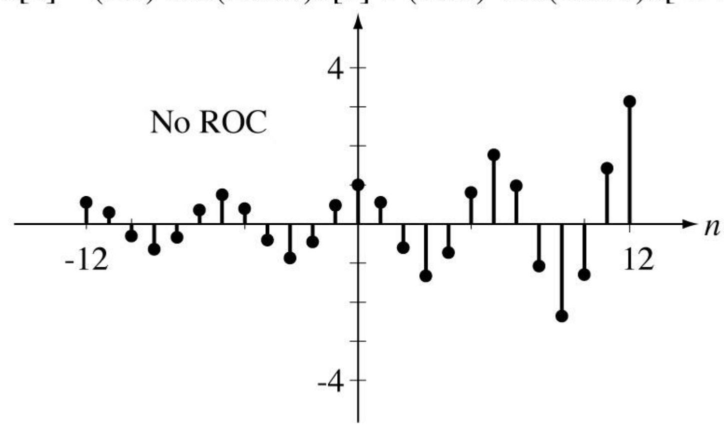
$x[n] = (0.95)^n u[n] + (0.9)^n u[-n-1]$



$x[n] = (0.85)^n \cos(2\pi n/6) u[n] + (0.9)^n \cos(2\pi n/6) u[-n-1]$



$x[n] = (1.1)^n \cos(2\pi n/6) u[n] + (1.05)^n \cos(2\pi n/6) u[-n-1]$



Some Common z Transform Pairs

$$\delta[n] \xleftrightarrow{\mathcal{Z}} 1 \quad , \text{ All } z$$

$$\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1} = \frac{1}{1-z^{-1}} \quad , \; |z| > 1 \qquad , \qquad -\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1} \quad , \; |z| < 1$$

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \; |z| > |\alpha| \qquad , \qquad -\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \; |z| < |\alpha|$$

$$n\,\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{(z-1)^2} = \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \quad , \; |z| > 1 \qquad , \qquad -n\,\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{(z-1)^2} = \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \quad , \; |z| < 1$$

$$n\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{\alpha z}{(z-\alpha)^2} = \frac{\alpha z^{-1}}{\left(1-\alpha z^{-1}\right)^2} \quad , \; |z| > |\alpha| \qquad , \qquad -n\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{\alpha z}{(z-\alpha)^2} = \frac{\alpha z^{-1}}{\left(1-\alpha z^{-1}\right)^2} \quad , \; |z| < |\alpha|$$

$$\sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z\sin(\Omega_0)}{z^2 - 2z\cos(\Omega_0) + 1} \quad , \; |z| > 1 \qquad , \qquad -\sin(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z\sin(\Omega_0)}{z^2 - 2z\cos(\Omega_0) + 1} \quad , \; |z| < 1$$

$$\cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z[z-\cos(\Omega_0)]}{z^2 - 2z\cos(\Omega_0) + 1} \quad , \; |z| > 1 \qquad , \qquad -\cos(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} = \frac{z[z-\cos(\Omega_0)]}{z^2 - 2z\cos(\Omega_0) + 1} \quad , \; |z| < 1$$

$$\alpha^n \sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z\alpha\sin(\Omega_0)}{z^2 - 2\alpha z\cos(\Omega_0) + \alpha^2} \quad , \; |z| > |\alpha| \quad , \quad -\alpha^n \sin(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z\alpha\sin(\Omega_0)}{z^2 - 2\alpha z\cos(\Omega_0) + \alpha^2} \quad , \; |z| < |\alpha|$$

$$\alpha^n \cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z[z-\alpha\cos(\Omega_0)]}{z^2 - 2\alpha z\cos(\Omega_0) + \alpha^2} \quad , \; |z| > |\alpha| \quad , \quad -\alpha^n \cos(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z[z-\alpha\cos(\Omega_0)]}{z^2 - 2\alpha z\cos(\Omega_0) + \alpha^2} \quad , \; |z| < |\alpha|$$

$$\alpha^{|n|} \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} - \frac{z}{z-\alpha^{-1}} \quad , \; |\alpha| < |z| < \left|\alpha^{-1}\right|$$

$$\mathrm{u}[n-n_0] - \mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1}\left(z^{-n_0} - z^{-n_1}\right) = \frac{z^{n_1-n_0-1} + z^{n_1-n_0-2} + \cdots + z + 1}{z^{n_1-1}} \quad , \; |z| > 0$$

z-Transform Properties

Given the z-transform pairs $\mathrm{g}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}(z)$ and $\mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{H}(z)$ with ROC's of $\mathrm{ROC_G}$ and $\mathrm{ROC_H}$ respectively the following properties apply to the z transform.

Linearity

$$\alpha\,\mathrm{g}[n] + \beta\,\mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \alpha\,\mathrm{G}(z) + \beta\,\mathrm{H}(z)$$

$$\mathrm{ROC} = \mathrm{ROC_G} \cap \mathrm{ROC_H}$$

Time Shifting

$$\mathrm{g}[n - n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0}\,\mathrm{G}(z)$$

$$\mathrm{ROC} = \mathrm{ROC_G} \text{ except perhaps } z = 0 \text{ or } z \to \infty$$

Change of Scale in z

$$\alpha^n\,\mathrm{g}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}(z/\alpha)$$

$$\mathrm{ROC} = |\alpha|\,\mathrm{ROC_G}$$

z-Transform Properties

Time Reversal

$$\mathrm{g}[-n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}\left(z^{-1}\right)$$

$$\mathrm{ROC} = 1/\mathrm{ROC}_\mathrm{G}$$

Time Expansion

$$\left\{ \begin{array}{ll} \mathrm{g}[n/k] & ,\ n/k \text{ and integer} \\ 0 & ,\ \text{otherwise} \end{array} \right\} \xleftrightarrow{\mathcal{Z}} \mathrm{G}\left(z^{k}\right)$$

$$\mathrm{ROC} = \left(\mathrm{ROC}_\mathrm{G}\right)^{1/k}$$

Conjugation

$$\mathrm{g}^*[n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}^*\left(z^*\right)$$

$$\mathrm{ROC} = \mathrm{ROC}_\mathrm{G}$$

***z*-Domain Differentiation**

$$-n\,\mathrm{g}[n] \xleftrightarrow{\mathcal{Z}} z\frac{d}{dz}\mathrm{G}(z)$$

$$\mathrm{ROC} = \mathrm{ROC}_\mathrm{G}$$

z-Transform Properties

Convolution $\quad \mathrm{g}[n] * \mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{H}(z)\mathrm{G}(z)$

First Backward Difference $\quad \mathrm{g}[n] - \mathrm{g}[n-1] \xleftrightarrow{\mathcal{Z}} \left(1 - z^{-1}\right)\mathrm{G}(z)$

$$\text{ROC} \supseteq \text{ROC}_{\mathrm{G}} \cap |z| > 0$$

Accumulation $\quad \sum_{m=-\infty}^{n} \mathrm{g}[m] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1}\mathrm{G}(z)$

$$\text{ROC} \supseteq \text{ROC}_{\mathrm{G}} \cap |z| > 1$$

Initial Value Theorem $\quad$ If $\mathrm{g}[n] = 0 \;, n < 0$ then $\mathrm{g}[0] = \lim_{z \to \infty} \mathrm{G}(z)$

Final Value Theorem $\quad$ If $\mathrm{g}[n] = 0 \;, n < 0, \; \lim_{n \to \infty} \mathrm{g}[n] = \lim_{z \to 1} (z-1)\mathrm{G}(z)$

if $\lim_{n \to \infty} \mathrm{g}[n]$ exists.

z-Transform Properties

For the final-value theorem to apply to a function $\mathrm{G}(z)$ all the finite poles of the function $(z-1)\mathrm{G}(z)$ must lie in the open interior of the unit circle of the z plane. Notice this does not say that all the poles of $\mathrm{G}(z)$ must lie in the open interior of the unit circle. $\mathrm{G}(z)$ could have a single pole at $z=1$ and the final-value theorem could still apply.

The Inverse z Transform

Synthetic Division

For rational z transforms of the form

$$\mathrm{H}(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

we can always find the inverse z transform by synthetic division. For example,

$$\mathrm{H}(z) = \frac{(z-1.2)(z+0.7)(z+0.4)}{(z-0.2)(z-0.8)(z+0.5)} \quad , \quad |z| > 0.8$$

$$\mathrm{H}(z) = \frac{z^3 - 0.1z^2 - 1.04z - 0.336}{z^3 - 0.5z^2 - 0.34z + 0.08} \quad , \quad |z| > 0.8$$

The Inverse z Transform

Synthetic Division

$$
\begin{array}{r|l}
 & 1+0.4z^{-1}+0.5z^{-2}\cdots \\
\cline{2-2}
z^3-0.5z^2-0.34z+0.08 & z^3-0.1z^2-1.04z-0.336 \\
 & \underline{z^3-0.5z^2-0.34z+0.08} \\
 & \qquad 0.4z^2-\ 0.7z-\ 0.256 \\
 & \qquad \underline{0.4z^2\ -0.2z-\ 0.136-0.032z^{-1}} \\
 & \qquad\qquad 0.5z-\ 0.12+0.032z^{-1} \\
 & \qquad\qquad \vdots \qquad \vdots \qquad \vdots
\end{array}
$$

The inverse z transform is

$$\delta[n]+0.4\delta[n-1]+0.5\delta[n-2]\cdots \xleftrightarrow{\mathcal{Z}} 1+0.4z^{-1}+0.5z^{-2}\cdots$$

The Inverse z Transform

Synthetic Division

We could have done the synthetic division this way.

$$
\begin{array}{r}
-4.2-30.85z-158.613z^2\cdots \\
0.08-0.34z-0.5z^2+z^3\overline{\big)-0.336-1.04z-0.1z^2+z^3} \\
\underline{-0.336+1.428z+2.1z^2-4.2z^3} \\
-2.468z-2.2z^2+5.2z^3 \\
\underline{-2.468z+10.489z^2+15.425z^3-30.85z^4} \\
-12.689z^2-10.225z^3+30.85z^4 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}
$$

$$-4.2\delta[n]-30.85\delta[n+1]-158.613\delta[n+2]\cdots\xleftrightarrow{\mathcal{Z}}-4.2-30.85z-158.613z^2\cdots$$

but with the restriction $|z|>0.8$ this second form does not converge and is therefore not the inverse z transform.

The Inverse z Transform

Synthetic Division

We can always find the inverse z transform of a rational function with synthetic division but the result is not in closed form. In most practical cases a closed-form solution is preferred.

Partial Fraction Expansion

Partial-fraction expansion works for inverse z transforms the same way it does for inverse Laplace transforms. But there is a situation that is quite common in inverse z transforms which deserves mention. It is very common to have z-domain functions in which the number of finite zeros equals the number of finite poles (making the expression improper in z) with at least one zero at $z = 0$.

$$\mathrm{H}(z) = \frac{z^{N-M}(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

Partial Fraction Expansion

Dividing both sides by z we get

$$\frac{H(z)}{z} = \frac{z^{N-M-1}(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

and the fraction on the right is now proper in z and can be expanded in partial fractions.

$$\frac{H(z)}{z} = \frac{K_1}{z-p_1} + \frac{K_2}{z-p_2} + \cdots + \frac{K_N}{z-p_N}$$

Then both sides can be multiplied by z and the inverse transform can be found.

$$H(z) = \frac{zK_1}{z-p_1} + \frac{zK_2}{z-p_2} + \cdots + \frac{zK_N}{z-p_N}$$

$$h[n] = K_1 p_1^n u[n] + K_2 p_2^n u[n] + \cdots + K_N p_N^n u[n]$$

z-Transform Properties

An LTI system has a transfer function

$$\mathrm{H}(z) = \frac{\mathrm{Y}(z)}{\mathrm{X}(z)} = \frac{z - 1/2}{z^2 - z + 2/9} \quad , \quad |z| > 2/3$$

Using the time-shifting property of the z transform draw a block diagram realization of the system.

$$\mathrm{Y}(z)\left(z^2 - z + 2/9\right) = \mathrm{X}(z)(z - 1/2)$$

$$z^2\,\mathrm{Y}(z) = z\,\mathrm{X}(z) - (1/2)\mathrm{X}(z) + z\,\mathrm{Y}(z) - (2/9)\mathrm{Y}(z)$$

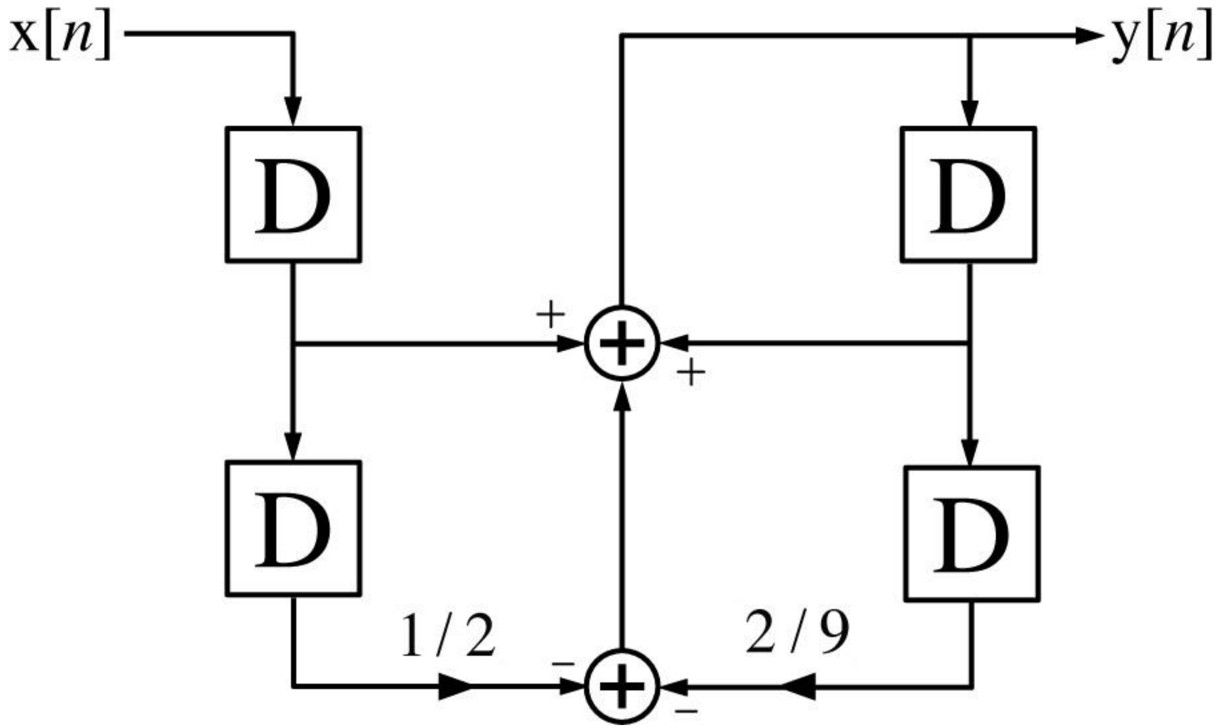
$$\mathrm{Y}(z) = z^{-1}\,\mathrm{X}(z) - (1/2)z^{-2}\,\mathrm{X}(z) + z^{-1}\,\mathrm{Y}(z) - (2/9)z^{-2}\,\mathrm{Y}(z)$$

z-Transform Properties

$$Y(z) = z^{-1}X(z) - (1/2)z^{-2}X(z) + z^{-1}Y(z) - (2/9)z^{-2}Y(z)$$

Using the time-shifting property

$$y[n] = x[n-1] - (1/2)x[n-2] + y[n-1] - (2/9)y[n-2]$$



z-Transform Properties

Let $g[n] \xleftrightarrow{\mathcal{Z}} G(z) = \dfrac{z-1}{\left(z-0.8e^{-j\pi/4}\right)\left(z-0.8e^{+j\pi/4}\right)}$. Draw a pole-zero diagram for $G(z)$ and for the z transform of $e^{j\pi n/8}g[n]$.

The poles of $G(z)$ are at $z = 0.8e^{\pm j\pi/4}$ and its single finite zero is at $z = 1$. Using the change of scale property

$$e^{j\pi n/8}g[n] \xleftrightarrow{\mathcal{Z}} G\left(ze^{-j\pi/8}\right) = \frac{ze^{-j\pi/8}-1}{\left(ze^{-j\pi/8}-0.8e^{-j\pi/4}\right)\left(ze^{-j\pi/8}-0.8e^{+j\pi/4}\right)}$$

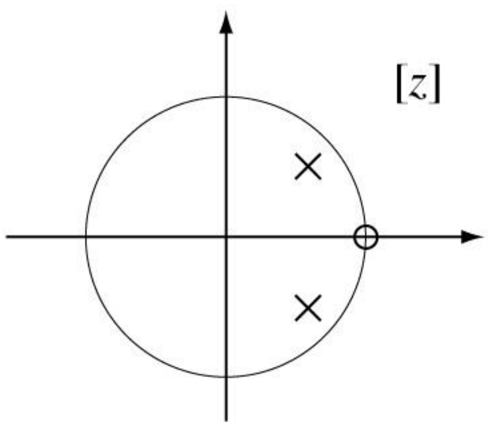
$$G\left(ze^{-j\pi/8}\right) = \frac{e^{-j\pi/8}\left(z-e^{j\pi/8}\right)}{e^{-j\pi/8}\left(z-0.8e^{-j\pi/8}\right)e^{-j\pi/8}\left(z-0.8e^{+j3\pi/8}\right)}$$

$$G\left(ze^{-j\pi/8}\right) = e^{j\pi/8}\frac{z-e^{j\pi/8}}{\left(z-0.8e^{-j\pi/8}\right)\left(z-0.8e^{+j3\pi/8}\right)}$$

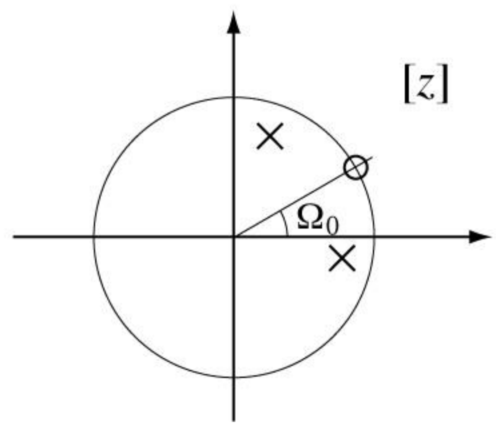
z-Transform Properties

$\mathrm{G}\left(ze^{-jp/8}\right)$ has poles at $z = 0.8e^{-jp/8}$ and $0.8e^{+j3p/8}$ and a zero at $z = e^{jp/8}$. All the finite zero and pole locations have been rotated in the z plane by $p/8$ radians.

Pole-zero Plot of $\mathrm{G}(z)$



Pole-zero Plot of $\mathrm{G}(ze^{-j\Omega_0})$



z-Transform Properties

Using the accumulation property and $\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \dfrac{z}{z-1} \quad , \quad |z|>1$

show that the z transform of $n\,\mathrm{u}[n]$ is $\dfrac{z}{(z-1)^2} \quad , \quad |z|>1$.

$$n\,\mathrm{u}[n] = \sum_{m=0}^{n} \mathrm{u}[m-1]$$

$$\mathrm{u}[n-1] \xleftrightarrow{\mathcal{Z}} z^{-1}\frac{z}{z-1} = \frac{1}{z-1} \quad , \quad |z|>1$$

$$n\,\mathrm{u}[n] = \sum_{m=0}^{n} \mathrm{u}[m-1] \xleftrightarrow{\mathcal{Z}} \left(\frac{z}{z-1}\right)\frac{1}{z-1} = \frac{z}{(z-1)^2} \quad , \quad |z|>1$$

Inverse z Transform Example

Find the inverse z transform of

$$\mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \; 0.5 < |z| < 2$$

Right-sided signals have ROC's that are outside a circle and left-sided signals have ROC's that are inside a circle. Using

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \; |z| > |\alpha|$$

$$-\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \; |z| < |\alpha|$$

We get

$$(0.5)^n \mathrm{u}[n] + (-2)^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \; 0.5 < |z| < 2$$

Inverse z Transform Example

Find the inverse z transform of

$$\mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| > 2$$

In this case, both signals are right sided. Then using

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| > |\alpha|$$

We get

$$\left[(0.5)^n - (-2)^n\right]\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| > 2$$

Inverse z Transform Example

Find the inverse z transform of

$$\mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| < 0.5$$

In this case, both signals are left sided. Then using

$$-\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| < |\alpha|$$

We get

$$-\left[(0.5)^n - (-2)^n\right]\mathrm{u}[-n-1] \xleftrightarrow{\mathcal{Z}} \mathrm{X}(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| < 0.5$$

The Unilateral z Transform

Just as it was convenient to define a unilateral Laplace transform it is convenient for analogous reasons to define a unilateral z transform

$$\mathrm{X}(z) = \sum_{n=0}^{\infty} \mathrm{x}[n] z^{-n}$$

Properties of the Unilateral z Transform

If two causal discrete-time signals form these transform pairs,

$\mathrm{g}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}(z)$ and $\mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{H}(z)$ then the following properties hold for the unilateral z transform.

Time Shifting

Delay: $$\mathrm{g}[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0}\,\mathrm{G}(z)\ ,\ n_0 \geq 0$$

Advance: $$\mathrm{g}[n+n_0] \xleftrightarrow{\mathcal{Z}} z^{n_0}\left(\mathrm{G}(z) - \sum_{m=0}^{n_0-1} \mathrm{g}[m] z^{-m}\right)\ ,\ \ n_0 > 0$$

Accumulation:

$$\sum_{m=0}^{n} \mathrm{g}[m] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1}\mathrm{G}(z)$$

Solving Difference Equations

The unilateral z transform is well suited to solving difference equations with initial conditions. For example,

$$\mathrm{y}[n+2] - \frac{3}{2}\mathrm{y}[n+1] + \frac{1}{2}\mathrm{y}[n] = (1/4)^n \quad , \quad \text{for } n \geq 0$$

$$\mathrm{y}[0] = 10 \quad \text{and} \quad \mathrm{y}[1] = 4$$

z transforming both sides,

$$z^2\left[\mathrm{Y}(z) - \mathrm{y}[0] - z^{-1}\mathrm{y}[1]\right] - \frac{3}{2}z\left[\mathrm{Y}(z) - \mathrm{y}[0]\right] + \frac{1}{2}\mathrm{Y}(z) = \frac{z}{z - 1/4}$$

the initial conditions are called for systematically.

Solving Difference Equations

Applying initial conditions and solving,

$$\mathrm{Y}(z) = z\left(\frac{16/3}{z - 1/4} + \frac{4}{z - 1/2} + \frac{2/3}{z - 1}\right)$$

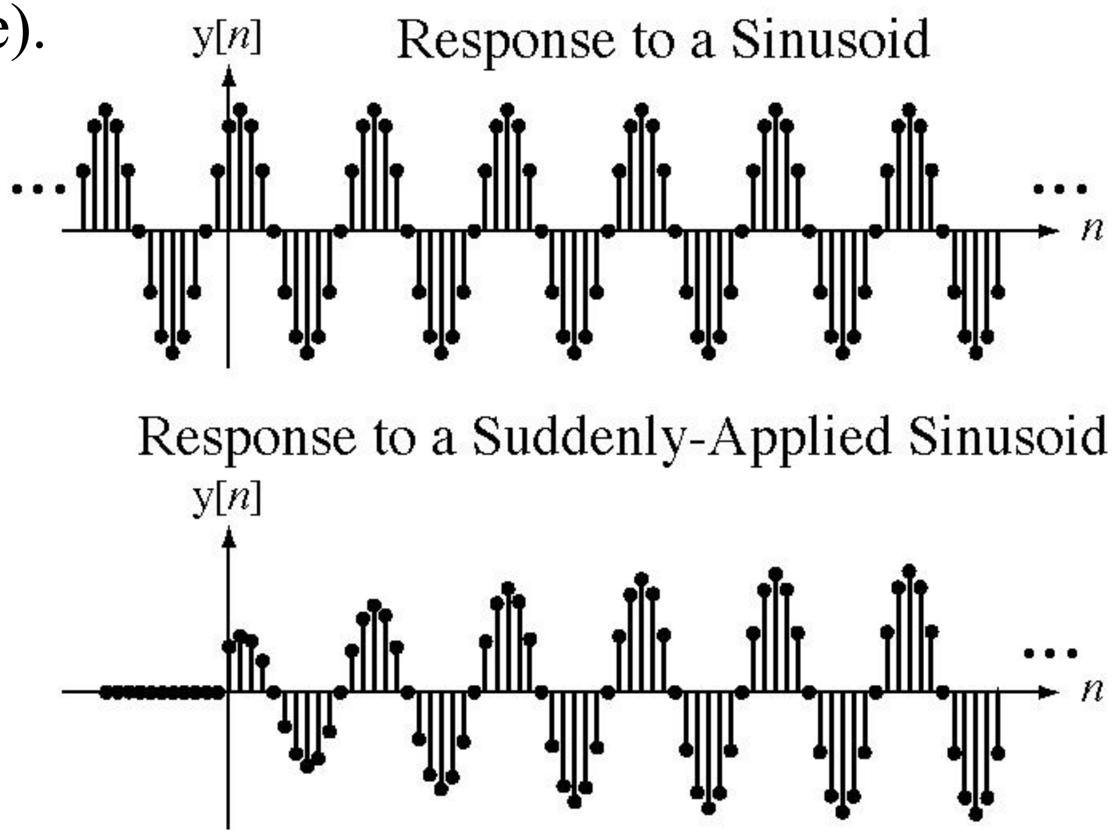
and

$$\mathrm{y}[n] = \left[\frac{16}{3}\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{2}\right)^n + \frac{2}{3}\right]\mathrm{u}[n]$$

This solution satisfies the difference equation and the initial conditions.

Pole-Zero Diagrams and Frequency Response

For a stable system, the response to a sinusoid applied at time $t = 0$ approaches the response to a true sinusoid (applied for all time).



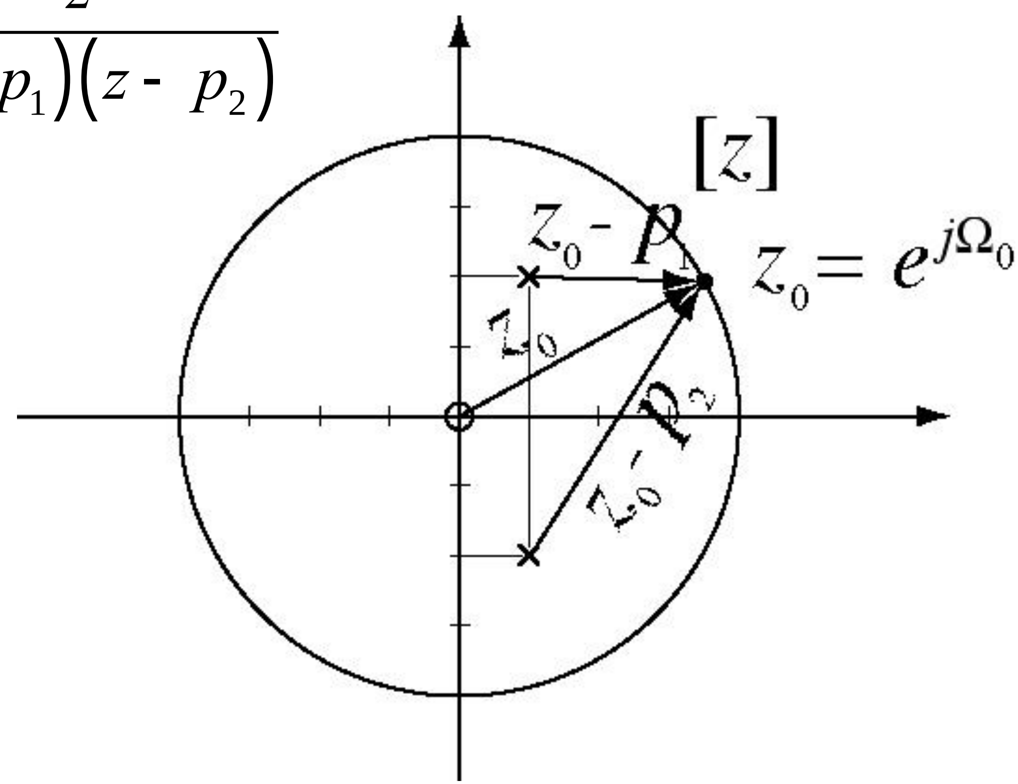
Pole-Zero Diagrams and Frequency Response

Let the transfer function of a system be

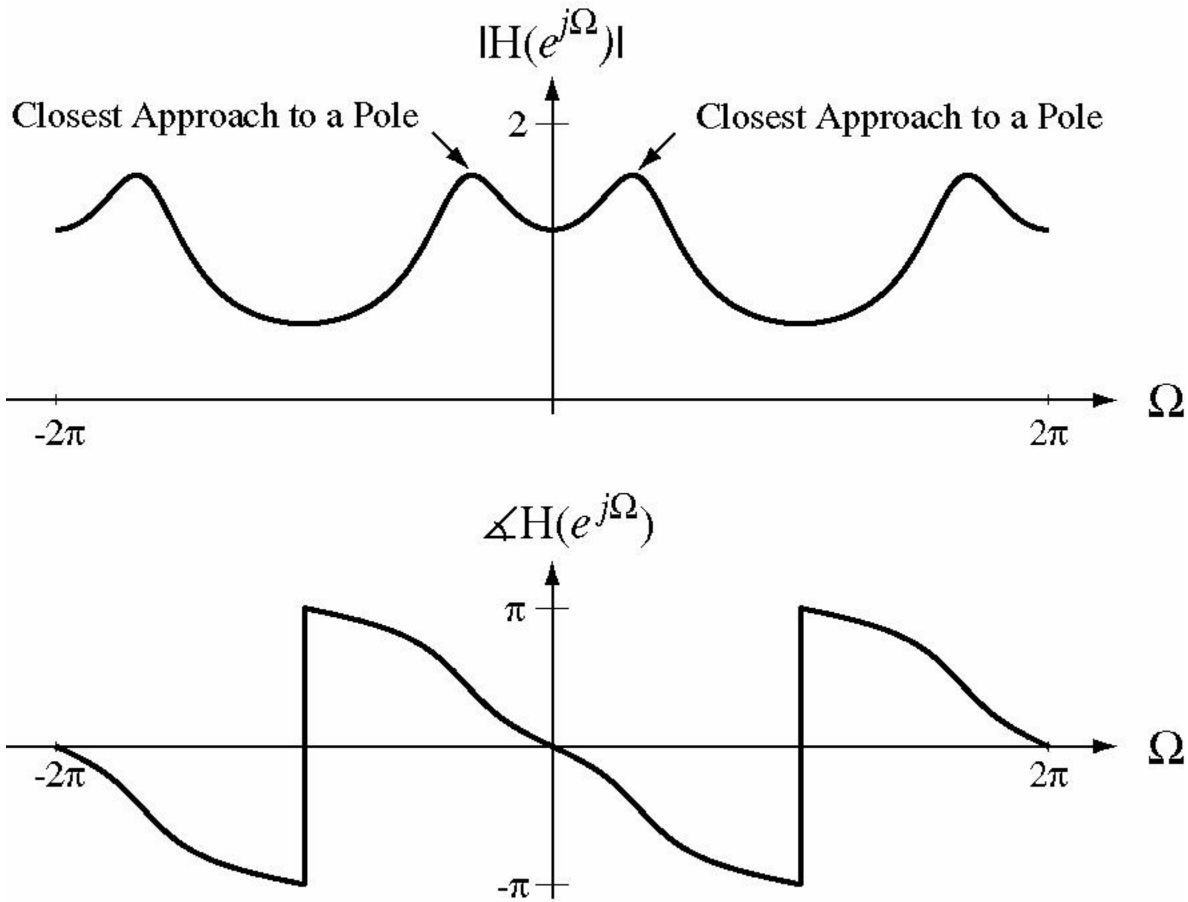
$$\mathrm{H}(z) = \frac{z}{z^2 - z/2 + 5/16} = \frac{z}{(z - p_1)(z - p_2)}$$

$$p_1 = \frac{1 + j2}{4} \;\; , \;\; p_2 = \frac{1 - j2}{4}$$

$$\left|\mathrm{H}\left(e^{jW}\right)\right| = \frac{\left|e^{jW}\right|}{\left|e^{jW} - p_1\right|\left|e^{jW} - p_2\right|}$$



Pole-Zero Diagrams and Frequency Response



Transform Method Comparison

A system with transfer function $\mathrm{H}(z) = \dfrac{z}{(z-0.3)(z+0.8)}\ ,\ |z| > 0.8$

is excited by a unit sequence. Find the total response.

Using z-transform methods,

$$\mathrm{Y}(z) = \mathrm{H}(z)\mathrm{X}(z) = \frac{z}{(z-0.3)(z+0.8)} \times \frac{z}{z-1}\ ,\quad |z| > 1$$

$$\mathrm{Y}(z) = \frac{z^2}{(z-0.3)(z+0.8)(z-1)} = -\frac{0.1169}{z-0.3} + \frac{0.3232}{z+0.8} + \frac{0.7937}{z-1}\ ,\quad |z| > 1$$

$$\mathrm{y}[n] = \left[-0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937\right]\mathrm{u}[n-1]$$

Transform Method Comparison

Using the DTFT,

$$\mathrm{H}\left(e^{j\Omega}\right)=\frac{e^{j\Omega}}{\left(e^{j\Omega}-0.3\right)\left(e^{j\Omega}+0.8\right)}$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\mathrm{H}\left(e^{j\Omega}\right)\mathrm{X}\left(e^{j\Omega}\right)=\frac{e^{j\Omega}}{\left(e^{j\Omega}-0.3\right)\left(e^{j\Omega}+0.8\right)}\times\underbrace{\left(\frac{1}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega)\right)}_{\text{DTFT of a Unit Sequence}}$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{e^{j2\Omega}}{\left(e^{j\Omega}-0.3\right)\left(e^{j\Omega}+0.8\right)\left(e^{j\Omega}-1\right)}+\pi\frac{e^{j\Omega}}{\left(e^{j\Omega}-0.3\right)\left(e^{j\Omega}+0.8\right)}\delta_{2\pi}(\Omega)$$

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{-0.1169}{e^{j\Omega}-0.3}+\frac{0.3232}{e^{j\Omega}+0.8}+\frac{0.7937}{e^{j\Omega}-1}+\frac{\pi}{(1-0.3)(1+0.8)}\delta_{2\pi}(\Omega)$$

Transform Method Comparison

Using the equivalence property of the impulse and the periodicity of both $\delta_{2\pi}(\Omega)$ and $e^{j\Omega}$

$$Y\left(e^{j\Omega}\right)=\frac{-0.1169e^{-j\Omega}}{1-0.3e^{-j\Omega}}+\frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}}+\frac{0.7937e^{-j\Omega}}{1-e^{-j\Omega}}+2.4933\delta_{2\pi}(\Omega)$$

Then, manipulating this expression into a form for which the inverse DTFT is direct

$$Y\left(e^{j\Omega}\right)=\frac{-0.1169e^{-j\Omega}}{1-0.3e^{-j\Omega}}+\frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}}+0.7937\left(\frac{e^{-j\Omega}}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega)\right)$$
$$\underbrace{-0.7937\pi\delta_{2\pi}(\Omega)+2.4933\delta_{2\pi}(\Omega)}_{=0}$$

Transform Method Comparison

$$\mathrm{Y}\left(e^{j\Omega}\right)=\frac{-0.1169e^{-j\Omega}}{1-0.3e^{-j\Omega}}+\frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}}+0.7937\left(\frac{e^{-j\Omega}}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega)\right)$$

Finding the inverse DTFT,

$$\mathrm{y}[n]=\left[-0.1169(0.3)^{n-1}+0.3232(-0.8)^{n-1}+0.7937\right]\mathrm{u}[n-1]$$

The result is the same as the result using the z transform, but the effort and the probability of error are considerably greater.

System Response to a Sinusoid

A system with transfer function

$$\mathrm{H}(z) = \frac{z}{z - 0.9} \quad , \quad |z| > 0.9$$

is excited by the sinusoid $\mathrm{x}[n] = \cos(2\pi n / 12)$. Find the response.

The z transform of a true sinusoid does not appear in the table of z transforms. The z transform of a causal sinusoid of the form $\mathrm{x}[n] = \cos(2\pi n / 12)\mathrm{u}[n]$ does appear. We can use the DTFT to find the response to the true sinusoid and the result is

$\mathrm{y}[n] = 1.995 \cos(2\pi n / 12 - 1.115)$.

System Response to a Sinusoid

Using the z transform we can find the response of the system to a causal sinusoid $\mathrm{x}[n] = \cos(2\pi n/12)\mathrm{u}[n]$ and the response is

$$\mathrm{y}[n] = 0.1217(0.9)^n \mathrm{u}[n] + 1.995\cos(2\pi n/12 - 1.115)\mathrm{u}[n]$$

Notice that the response consists of two parts, a transient response $0.1217(0.9)^n \mathrm{u}[n]$ and a forced response $1.995\cos(2\pi n/12 - 1.115)\mathrm{u}[n]$ that, except for the unit sequence factor, is exactly the same as the forced response we found using the DTFT.

System Response to a Sinusoid

This type of analysis is very common. We can generalize it to say that if a system has a transfer function $\mathrm{H}(z) = \frac{\mathrm{N}(z)}{\mathrm{D}(z)}$ that the response to a causal cosine excitation $\cos(\Omega_0 n)\mathrm{u}[n]$ is

$$\mathrm{y}[n] = \underbrace{\mathcal{Z}^{-1}\left(z\frac{\mathrm{N}_1(z)}{\mathrm{D}(z)}\right)}_{\text{Natural or Transient Response}} + \underbrace{\left|\mathrm{H}(p_1)\right|\cos\left(\Omega_0 n + \measuredangle \mathrm{H}(p_1)\right)\mathrm{u}[n]}_{\text{Forced Response}}$$

where $p_1 = e^{j\Omega_0}$. This consists of a natural or transient response and a forced response. If the system is stable the transient response dies away with time leaving only the forced response which, except for the $\mathrm{u}[n]$ factor is the same as the forced response to a true cosine. So we can use the z transform to find the response to a true sinusoid.