BLM2041 Signals and Systems

The Instructors:

Prof. Dr. Nizamettin Aydın

naydin@yildiz.edu.tr

Asist. Prof. Dr. Ferkan Yilmaz

ferkan@yildiz.edu.tr

Fourier Transform

LECTURE OBJECTIVES

- Review
  - Frequency Response
  - Fourier Series
- Definition of Fourier transform

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$

  - Relation to Fourier Series
- Examples of Fourier transform pairs
- Basic properties of Fourier transforms
  - **Convolution** property
  - **Multiplication** property

WHY use the Fourier transform?

- Manipulate the **Frequency Spectrum**
- Analog Communication Systems
  - AM: Amplitude Modulation; FM
  - What are the **Building Blocks** ?
    - **Abstract Layer**, not implementation
- Ideal Filters
  - mostly BPFs
- Frequency Shifters
  - aka Modulators, Mixers or Multipliers: $x(t) \times p(t)$

Everything = Sum of Sinusoids

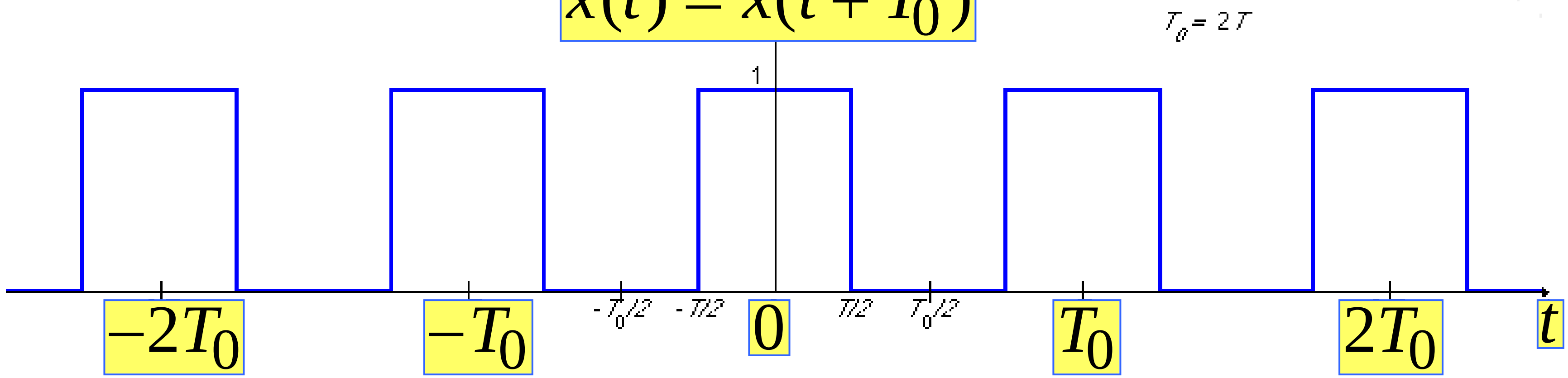
- One Square Pulse = Sum of Sinusoids
  - ???????????

- Finite Length
- Not Periodic

- Limit of Square Wave as Period → infinity
  - Intuitive Argument

Fourier Series: Periodic $x(t)$

$$x(t) = x(t + T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

Fourier Synthesis

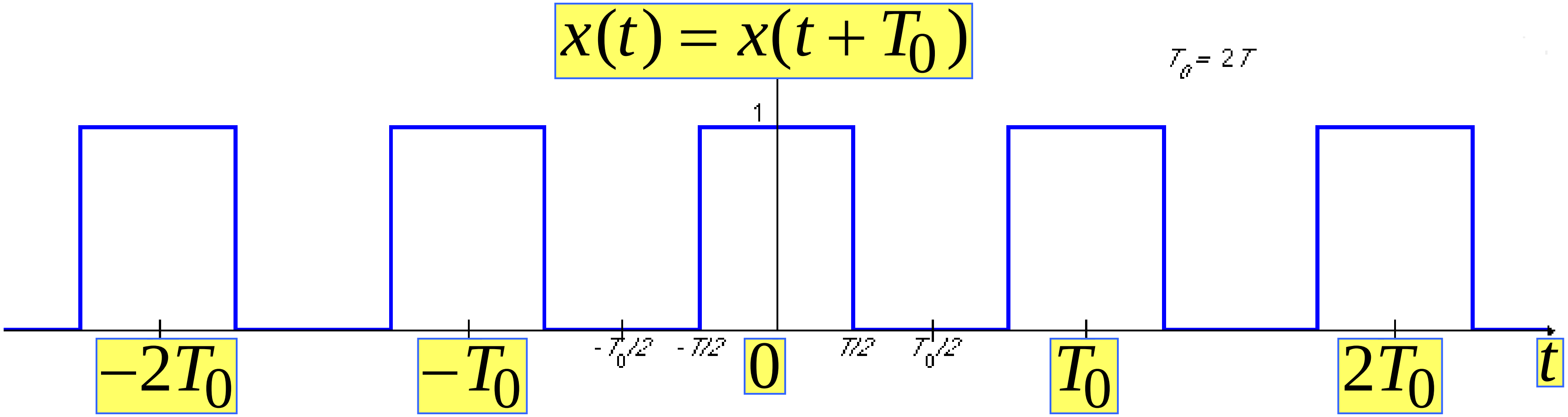
Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 kt} dt$$

Fourier Analysis

Square Wave Signal

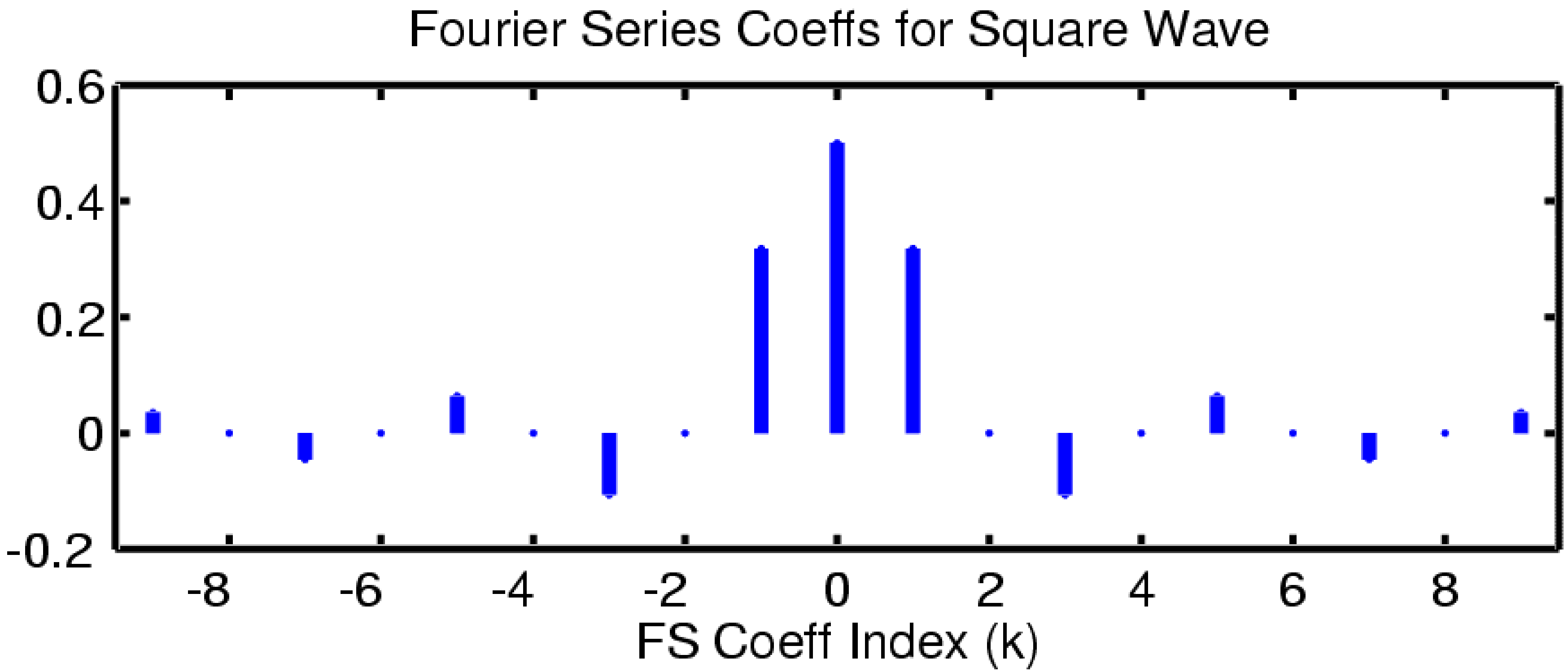


$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t} dt$$

$$a_k = \left. \frac{e^{-j\omega_0 kt}}{-j\omega_0 k T_0} \right|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k / 2)}{\pi k}$$

Spectrum from Fourier Series

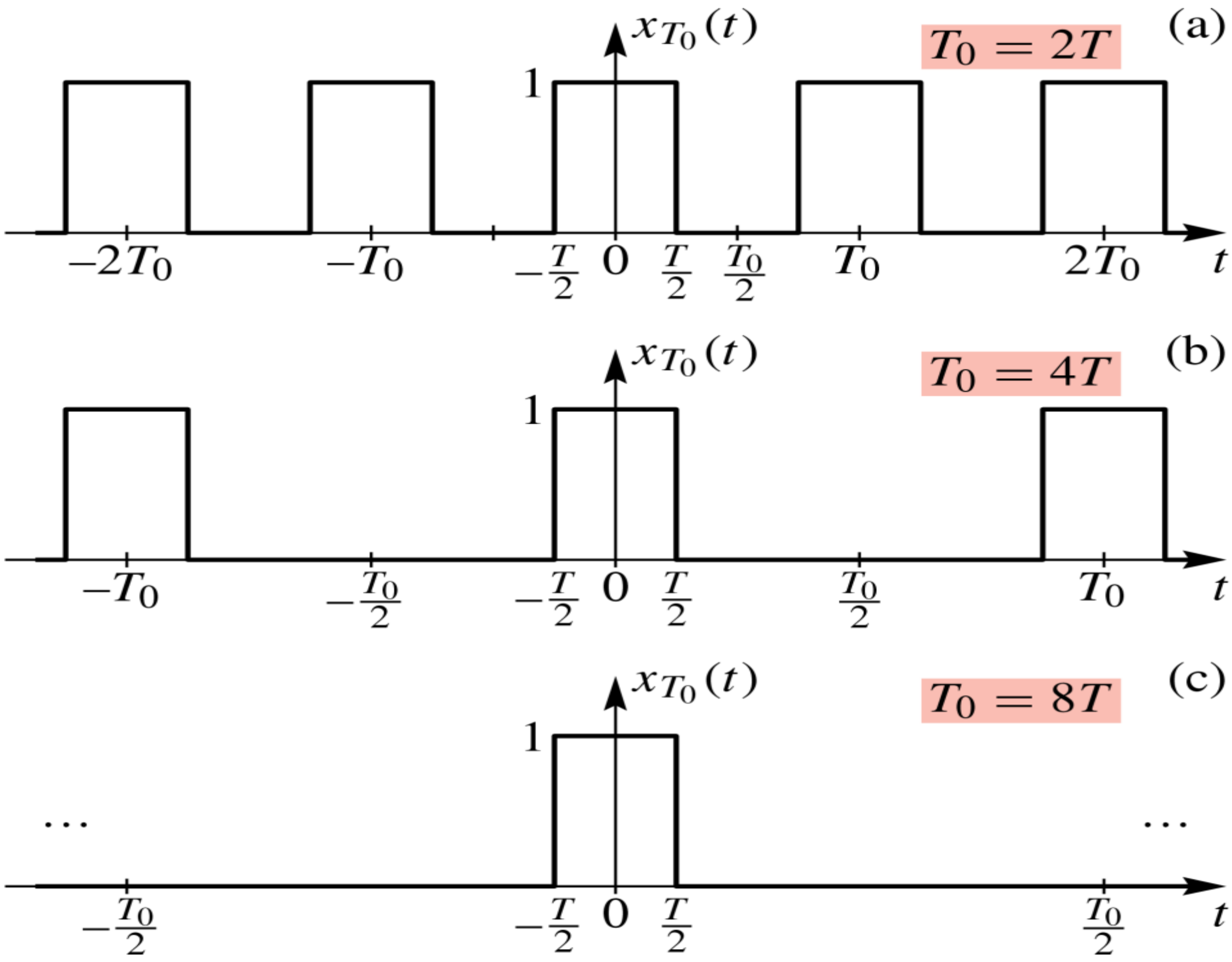
$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$



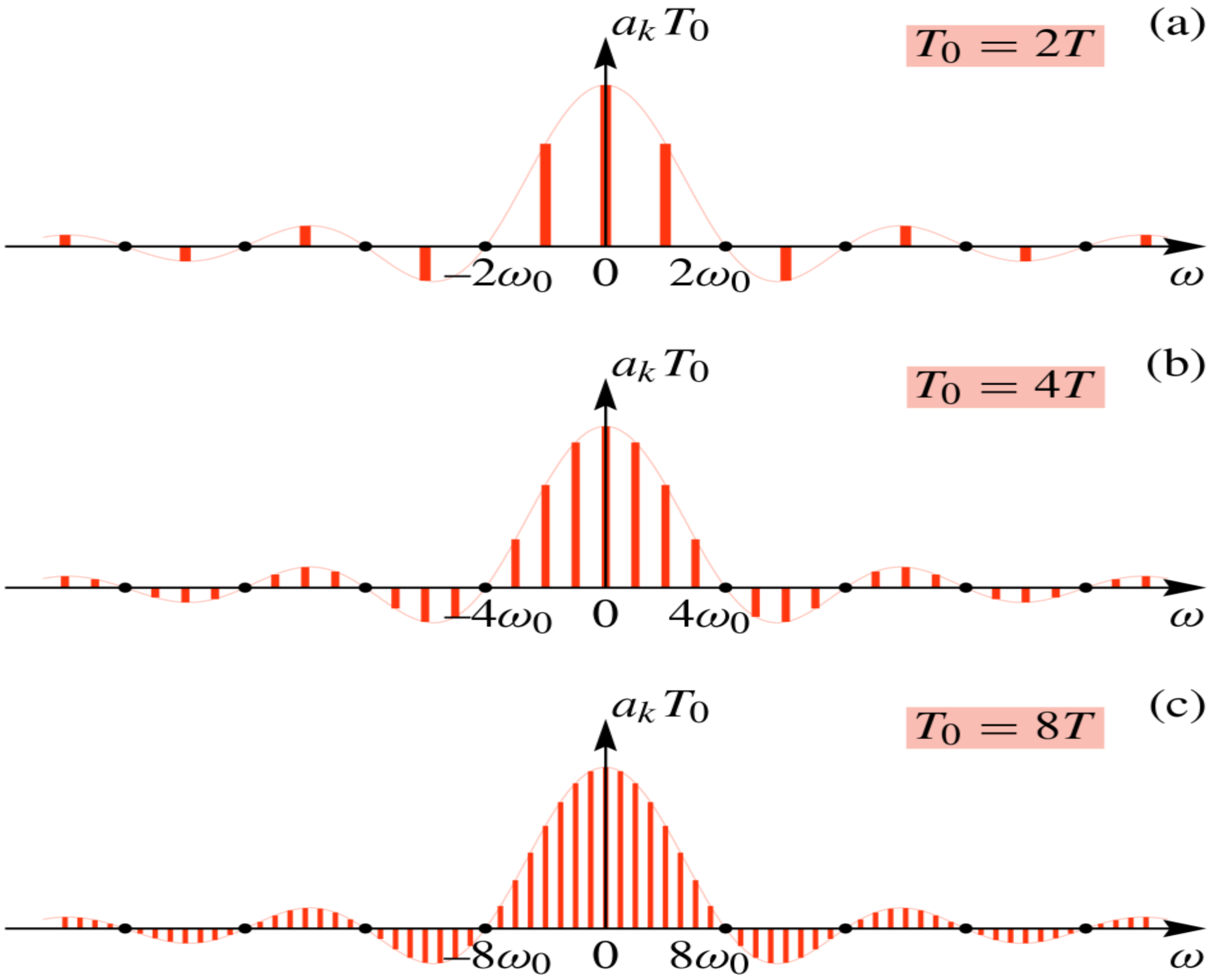
What if $x(t)$ is not periodic?

- Sum of Sinusoids?
  - Non-harmonically related sinusoids
  - Would not be periodic,
    - but would probably be non-zero for all t.
- Fourier transform
  - gives a “sum” (actually an integral) that involves ALL frequencies
  - can represent signals that are identically zero for negative t. !!!!!!!!!

Limiting Behavior of FS



Limiting Behavior of Spectrum



FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 k t} \left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega$$

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} k = \omega$$

$$\lim_{T_0 \to \infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Fourier Synthesis
(Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Analysis
(Forward Transform)

$$\text{Time - Domain} \Leftrightarrow \text{Frequency - Domain}$$

$$x(t) \Leftrightarrow X(j\omega)$$

Example 1

$$\boxed{x(t) = e^{-at}u(t)}$$

$$X(j\omega) = \int_0^\infty e^{-at}\, e^{-j\omega t}\, dt = \int_0^\infty e^{-(a+j\omega)t}\, dt$$

$$\boxed{a > 0}$$

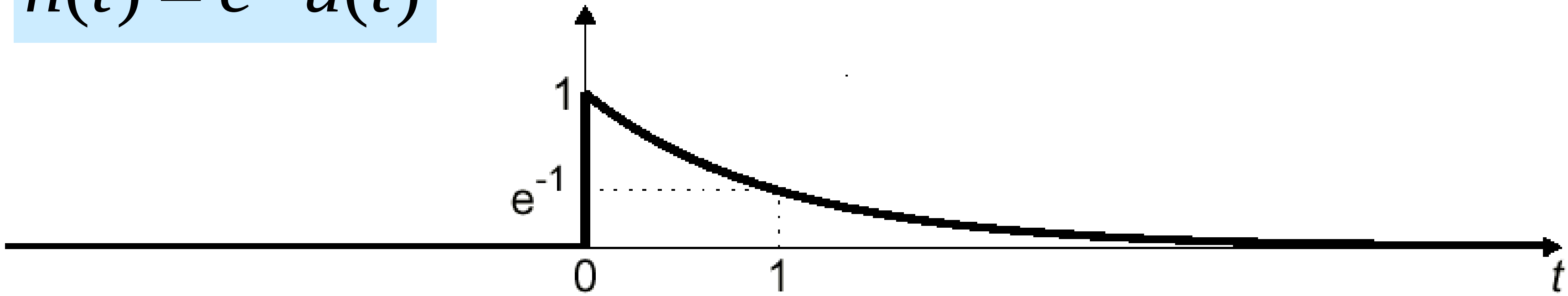
$$X(j\omega) = -\left.\frac{e^{-at}e^{-j\omega t}}{a+j\omega}\right|_0^\infty = \frac{1}{a+j\omega}$$

$$\boxed{X(j\omega) = \frac{1}{a+j\omega}}$$

Example 1 - Frequency Response

- Fourier Transform of $h(t)$ is
  - the Frequency Response

$$h(t) = e^{-t}u(t)$$

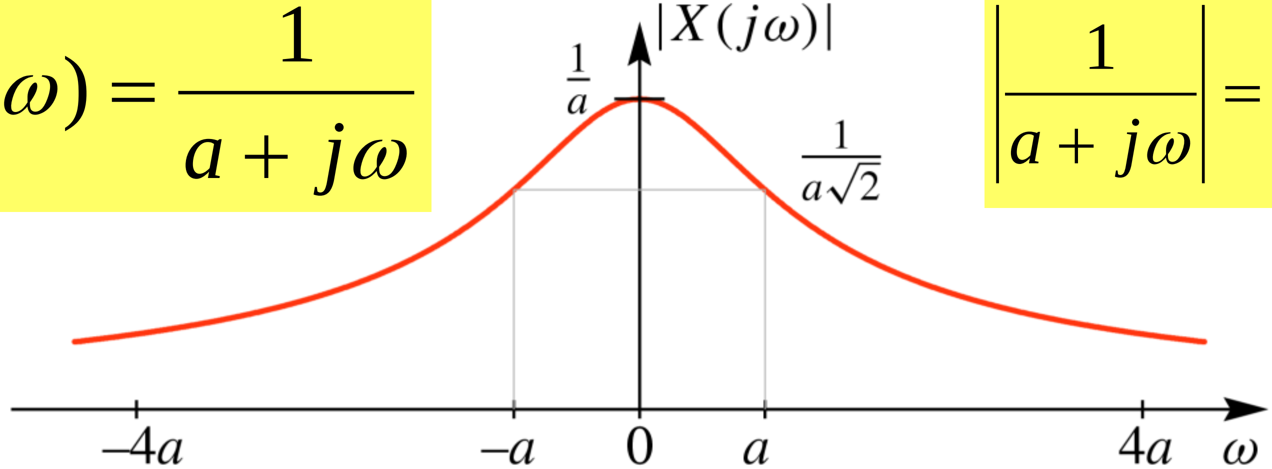


$$\boxed{h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}}$$

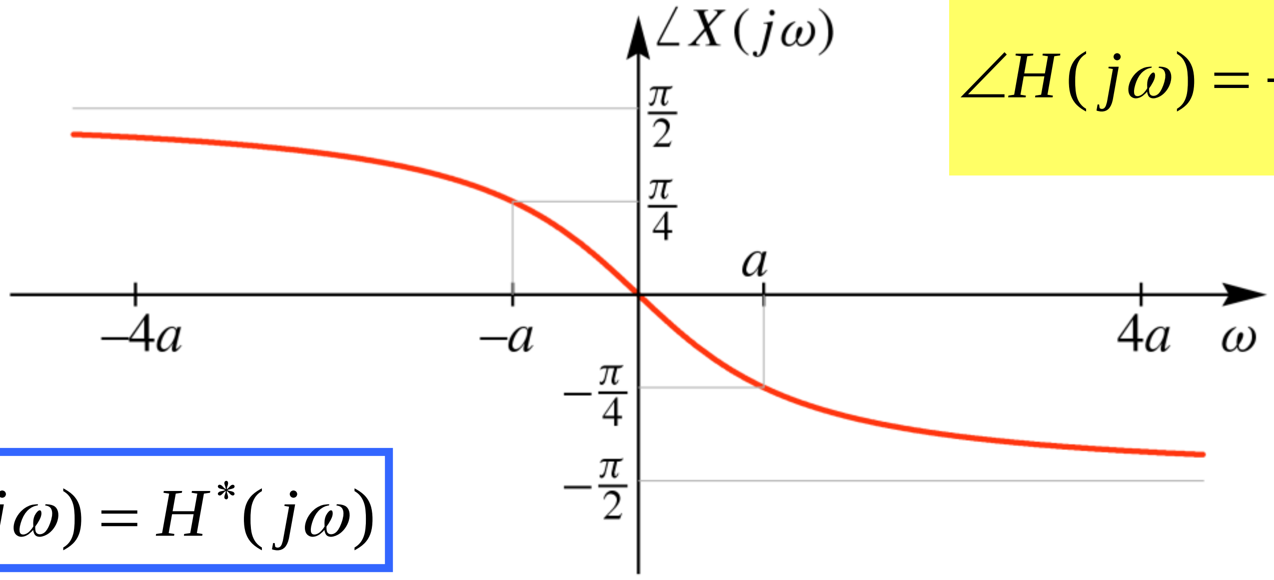
Example 1 - Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a + j\omega}$$

$$\left|\frac{1}{a + j\omega}\right| = \left|\frac{1}{\sqrt{a^2 + \omega^2}}\right|$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



$$\boxed{H(-j\omega) = H^*(j\omega)}$$

Example 2

$$\boxed{x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}}$$

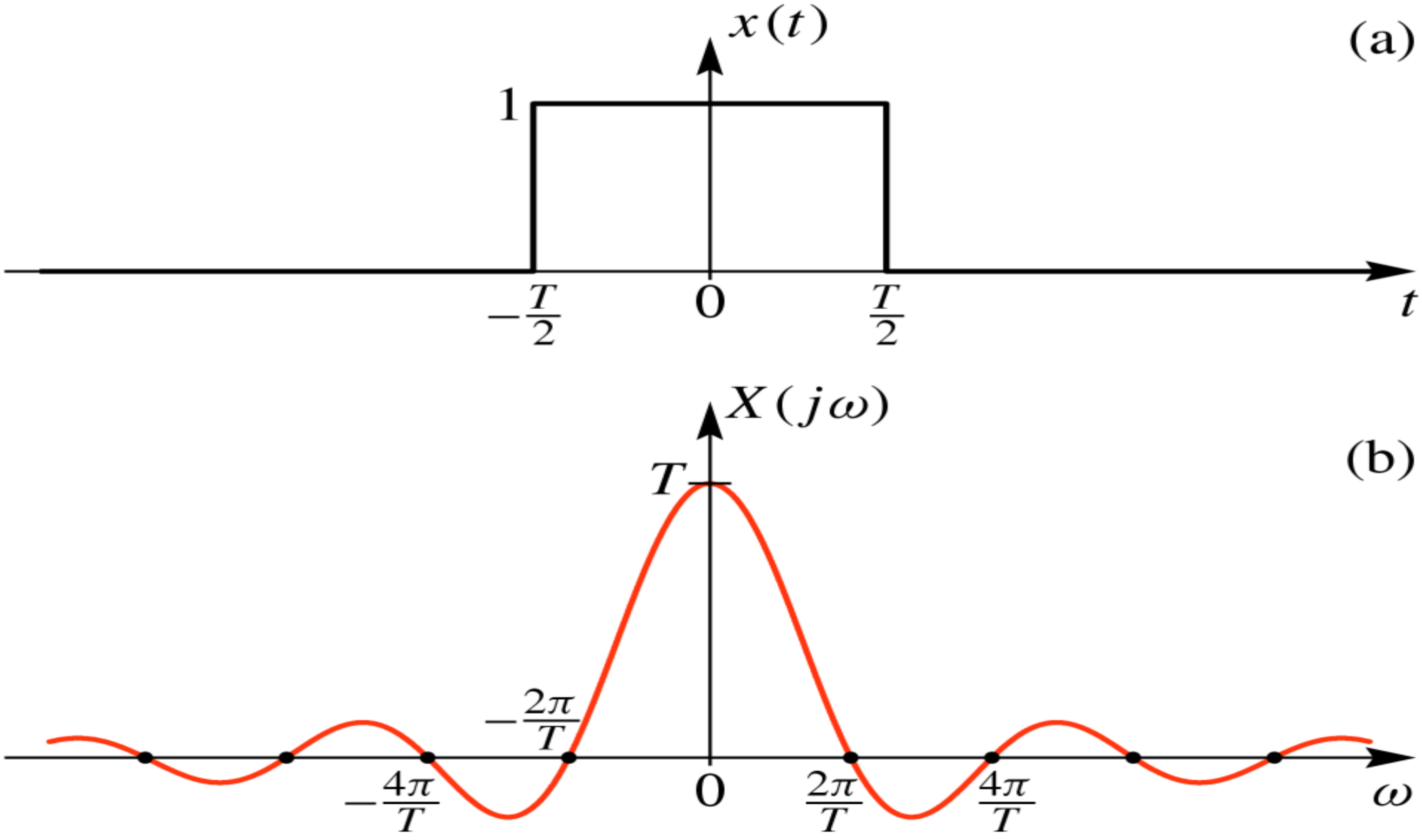
$$X(j\omega) = \int_{-T/2}^{T/2} (1) e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$\boxed{X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}}$$

Example 2

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$



Example 3

$$\boxed{X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}}$$

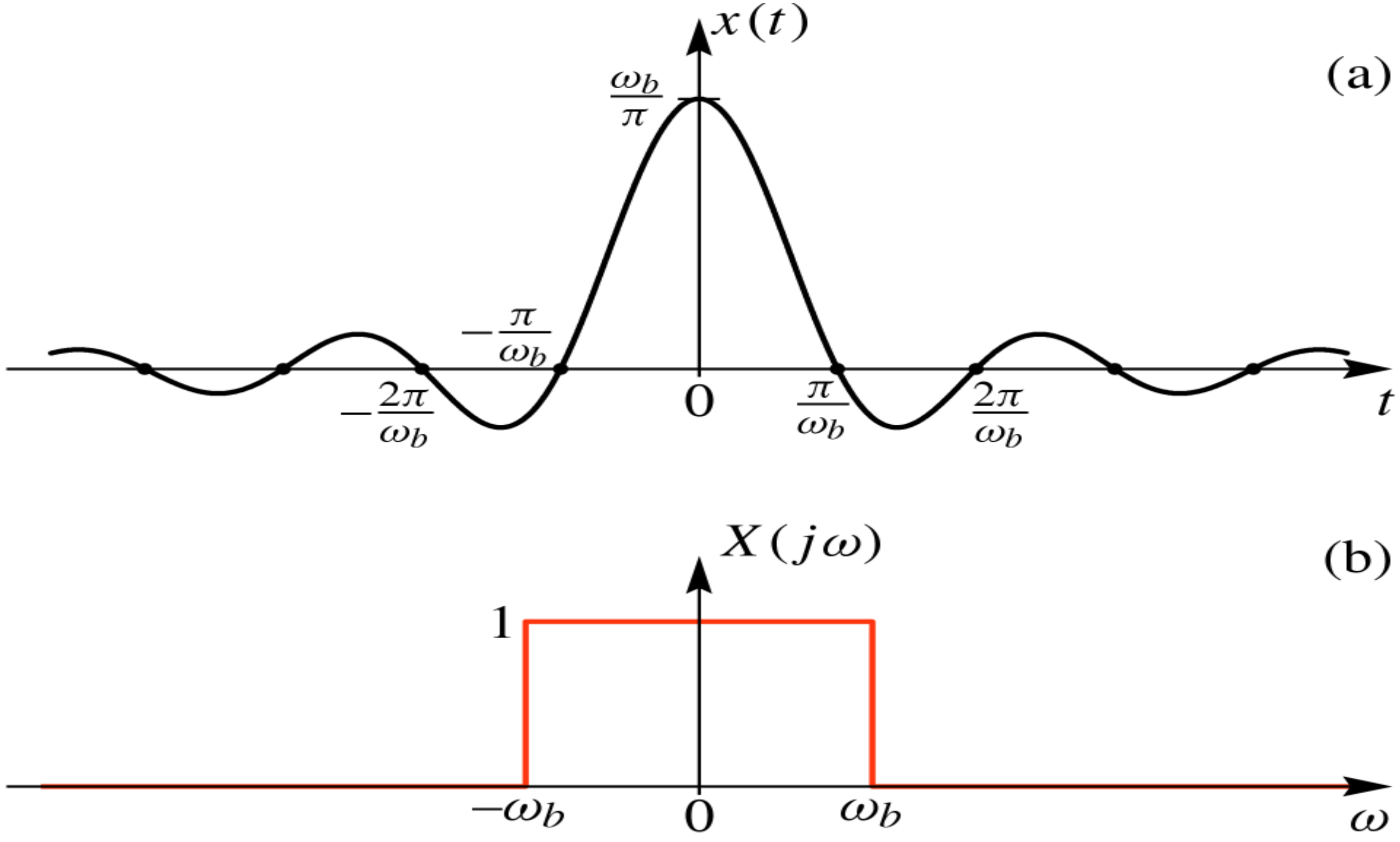
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_b}^{\omega_b} 1e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_b}^{\omega_b} = \frac{1}{2\pi}\frac{e^{j\omega_b t} - e^{-j\omega_b t}}{jt}$$

$$\boxed{x(t) = \frac{\sin(\omega_b t)}{\pi t}}$$

Example 3

$$x(t)=\frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega)=\begin{cases}1 & |\omega|<\omega_b \\ 0 & |\omega|>\omega_b\end{cases}$$



Example 4

$$\boxed{x(t) = \delta(t - t_0)}$$

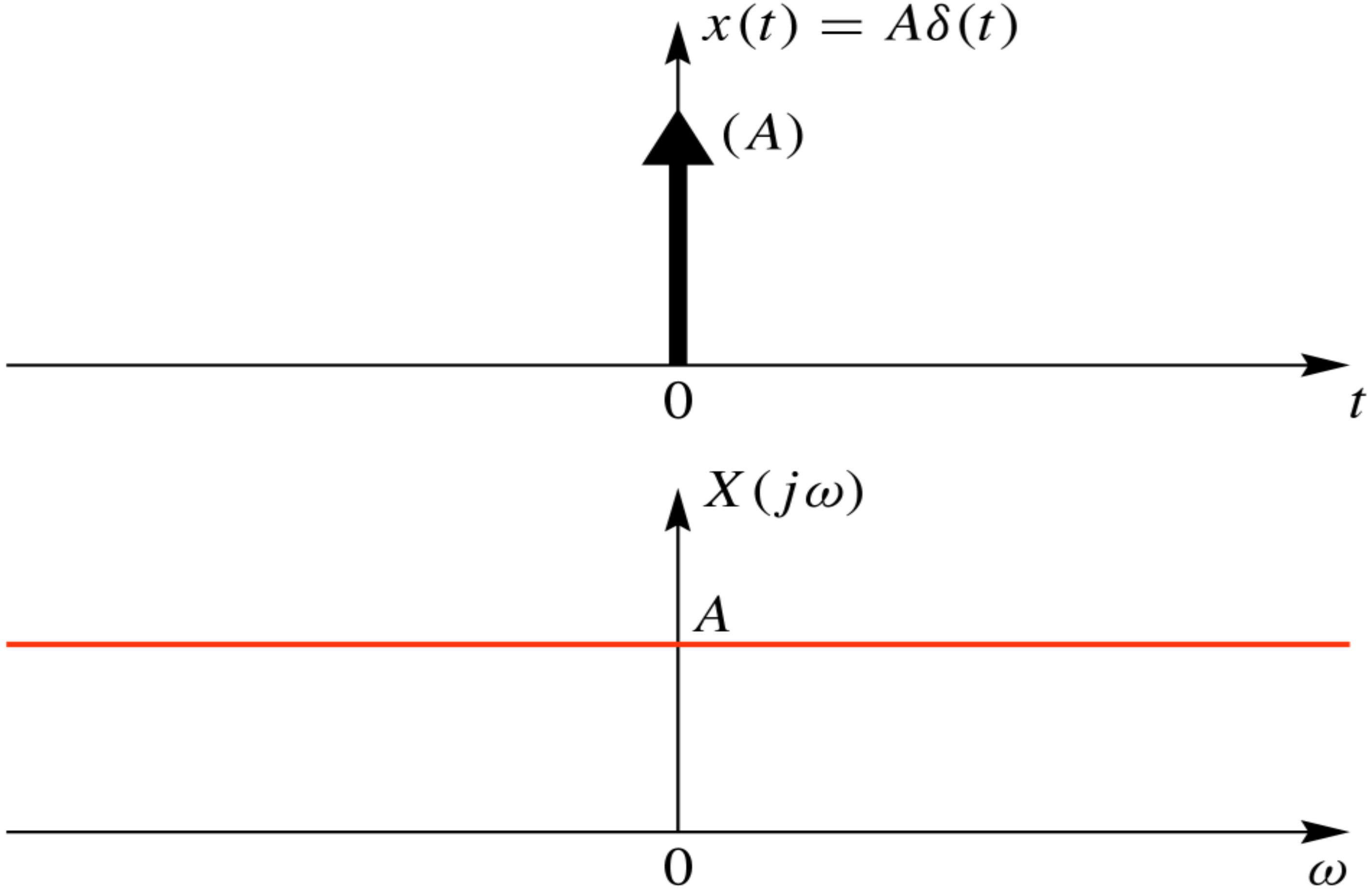
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}}$$

Shifting Property of the Impulse

Impulse function – Time and Frequency domains

$$\boxed{x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1}$$



Example 5

$$\boxed{X(j\omega) = 2\pi\delta(\omega - \omega_0)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

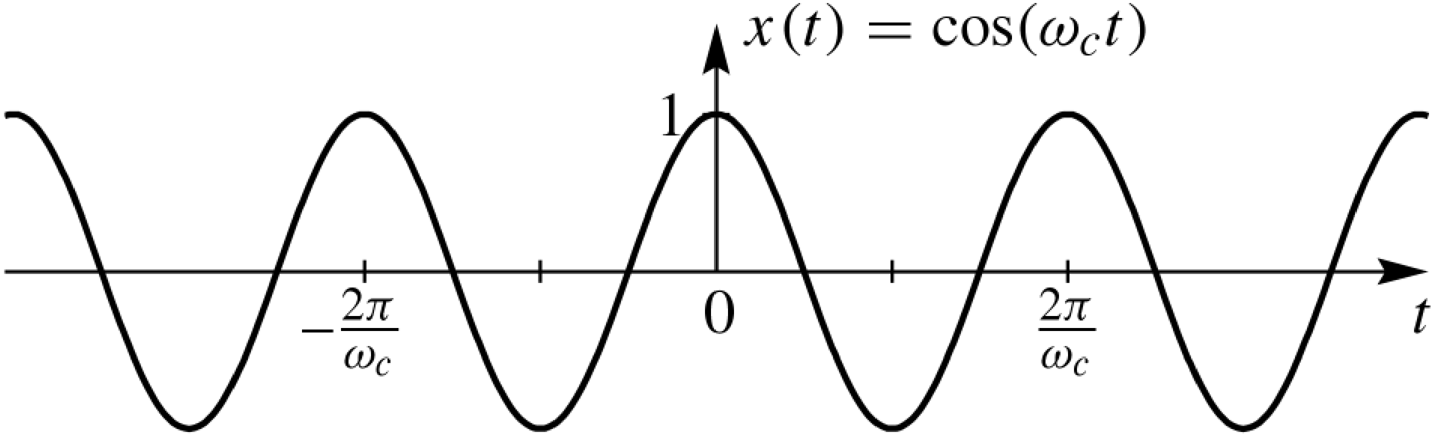
$$\boxed{x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)}$$

$$\boxed{x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)}$$

$$\boxed{\begin{aligned} &x(t) = \cos(\omega_0 t) \Leftrightarrow \\ &\qquad X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \end{aligned}}$$

Example 5

$$x(t) = \cos(\omega_c t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$



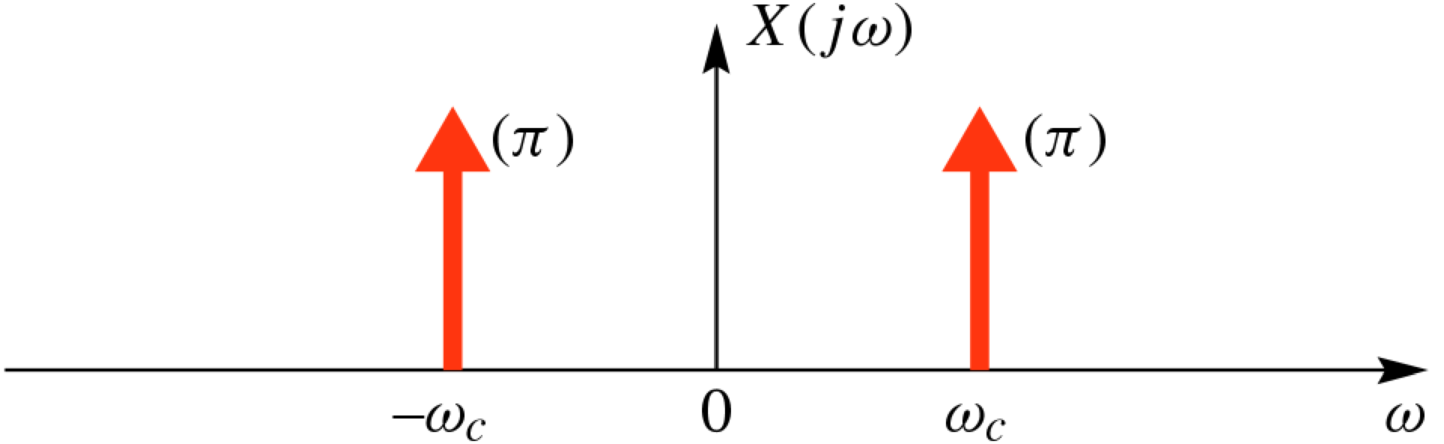


Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

Fourier Transform of a General Periodic Signal

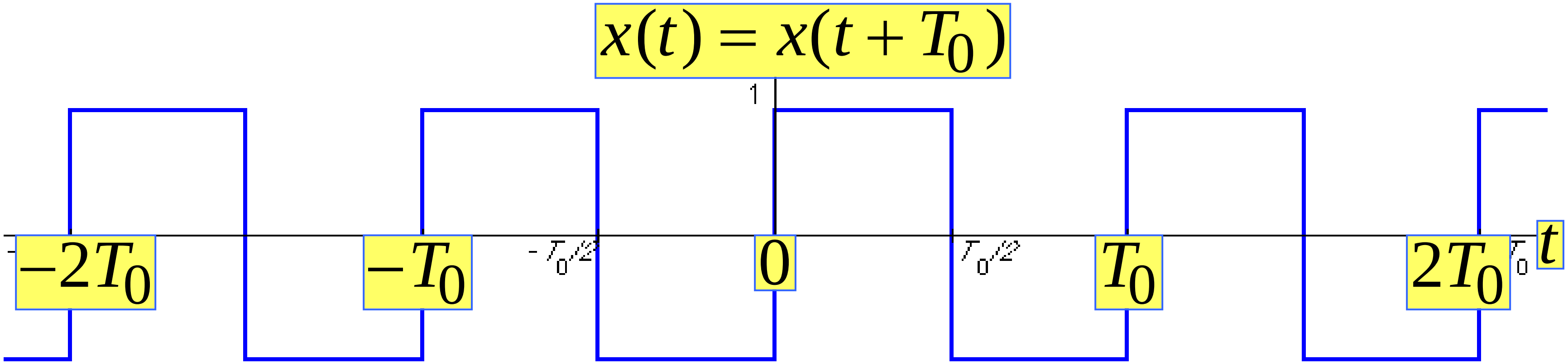
- If $x(t)$ is periodic with period T_0,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

Therefore, since $e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Square Wave Signal



$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

Square Wave Fourier Transform

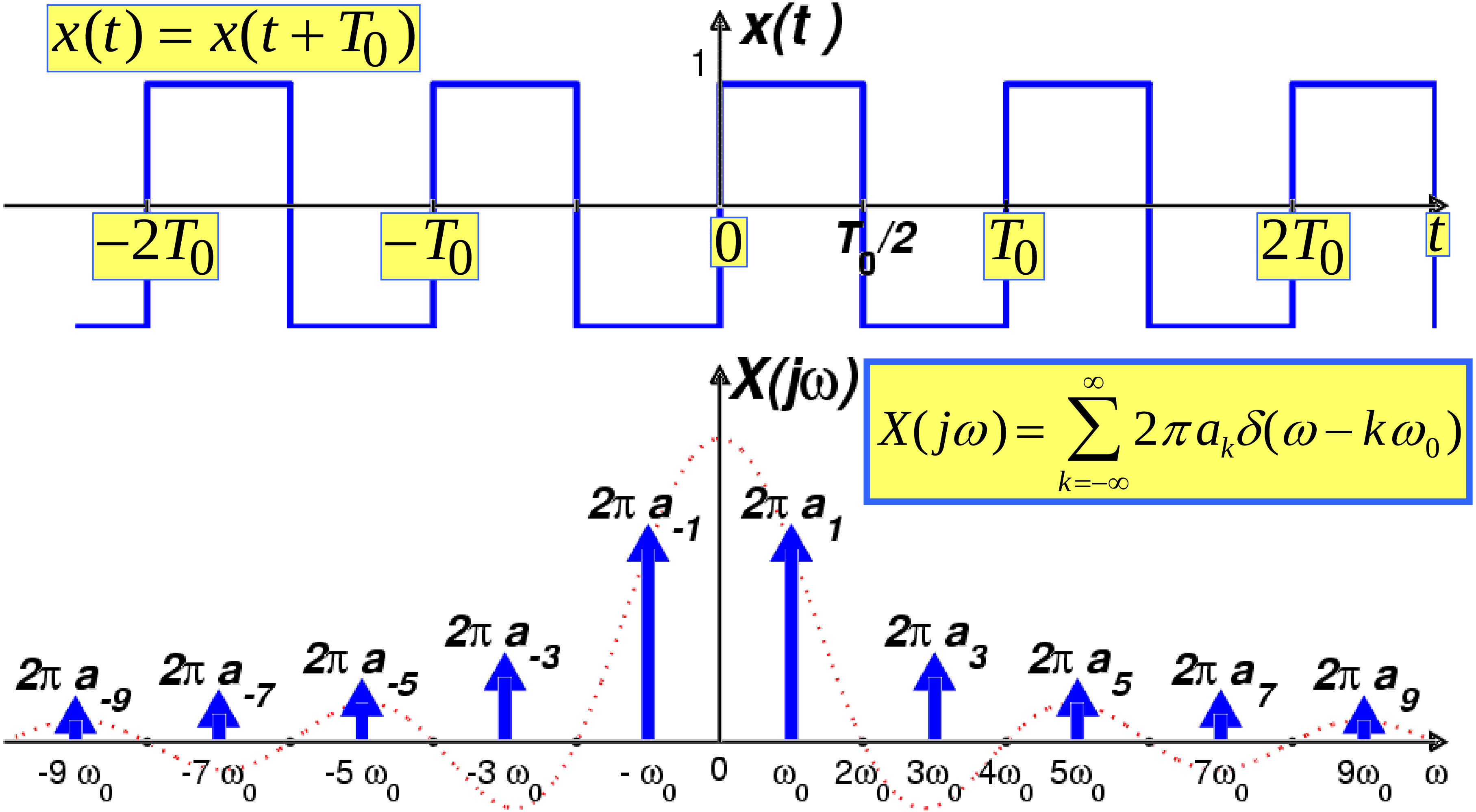


Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

Scaling Property

$$\boxed{x(at) \Leftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)}$$

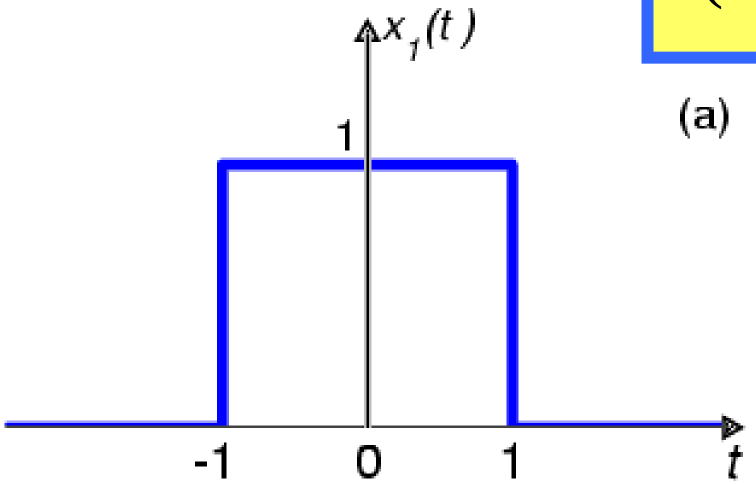
$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\lambda)e^{-j\omega(\lambda/a)}\frac{d\lambda}{|a|}$$

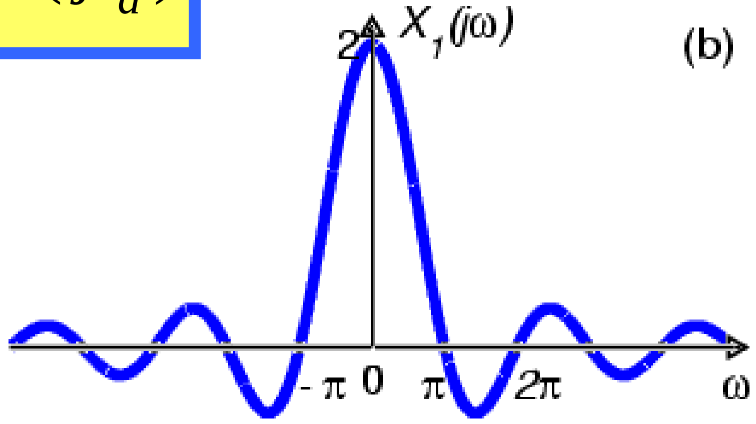
$$= \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

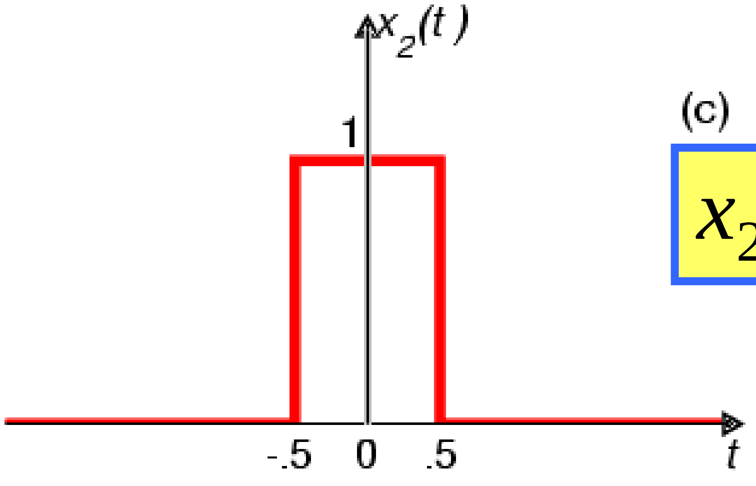
$x(2t)$ shrinks; $\frac{1}{2} X\left(j\frac{\omega}{2}\right)$ expands

Scaling Property

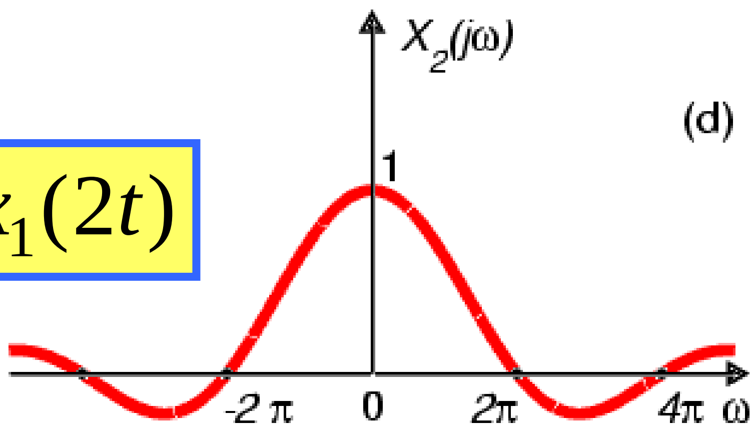
$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$







$$x_2(t) = x_1(2t)$$



Uncertainty Principle

- Try to make $x(t)$ shorter
  - Then $X(j\omega)$ will get wider
  - Narrow pulses have wide bandwidth

- Try to make $X(j\omega)$ narrower
  - Then $x(t)$ will have longer duration

- **Cannot simultaneously reduce time duration and bandwidth**

Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

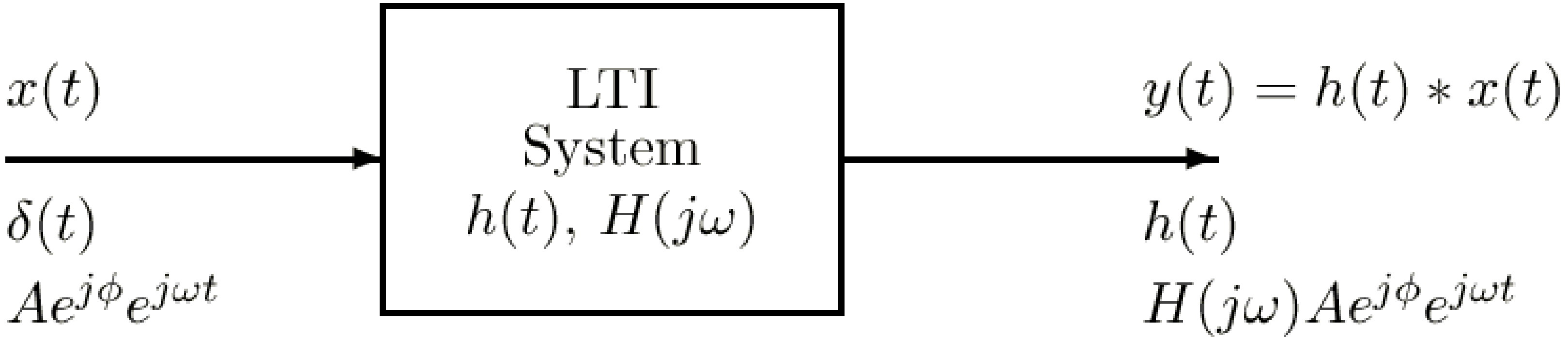
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property



- Convolution in the time-domain

$$\boxed{y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

corresponds to **MULTIPLICATION** in the frequency-domain

$$\boxed{Y(j\omega) = H(j\omega)X(j\omega)}$$

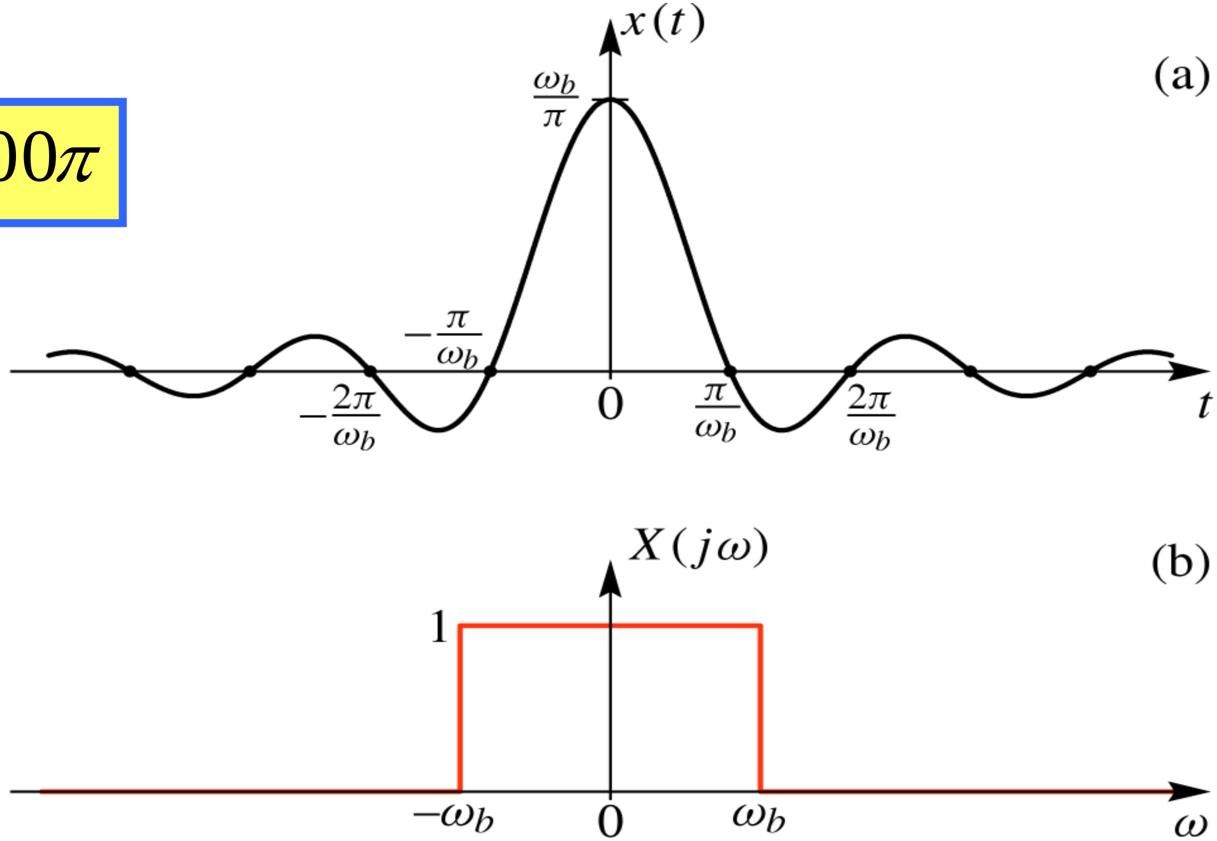
Convolution Example

- Bandlimited **Input** Signal
  - "sinc" function
- Ideal LPF (Lowpass Filter)
  - ***h*(*t*)** is a "sinc"
- **Output** is Bandlimited
  - Convolve "sincs"

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

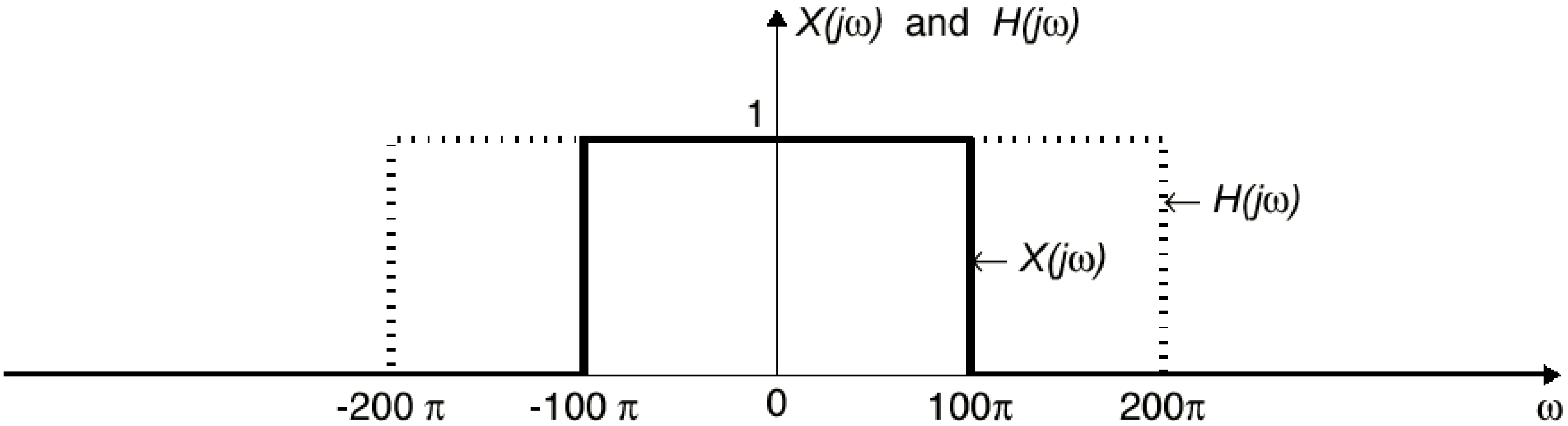
$$\omega_b = 100\pi$$



Convolution Example 1

$$\boxed{x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)}$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



Cosine Input to LTI System

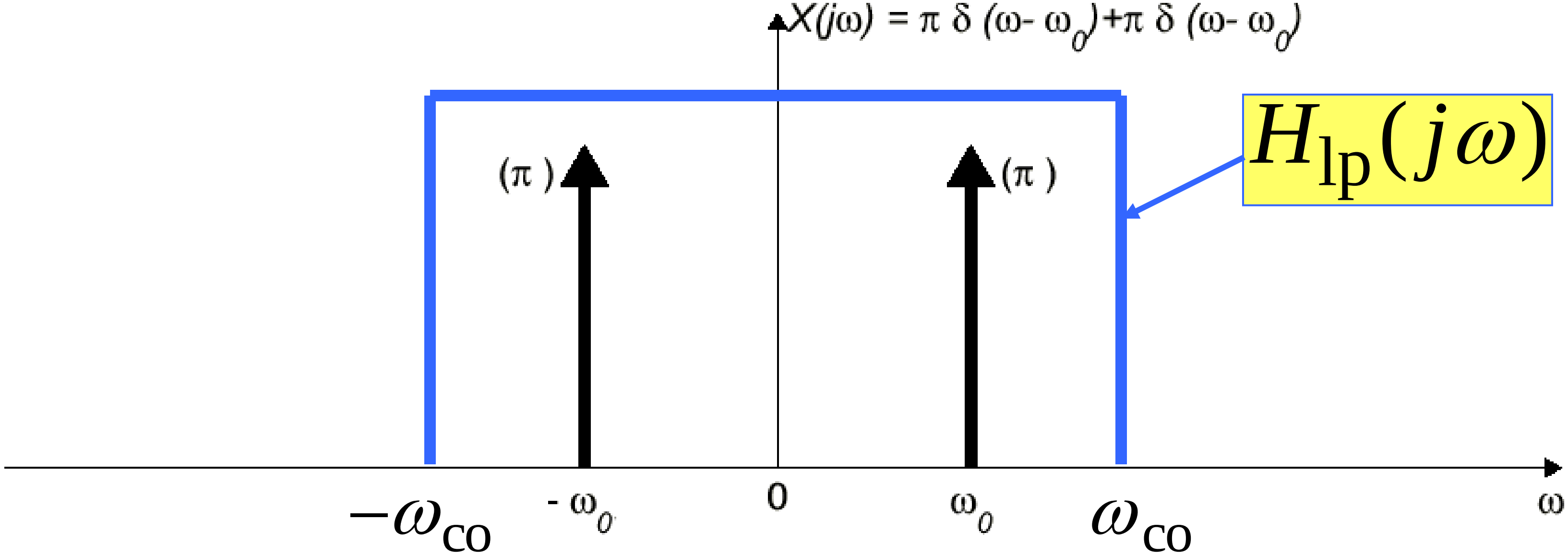
$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)]$$

$$= H(j\omega_0)\pi\delta(\omega-\omega_0) + H(-j\omega_0)\pi\delta(\omega+\omega_0)$$

$$\begin{aligned} y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

Ideal Lowpass Filter

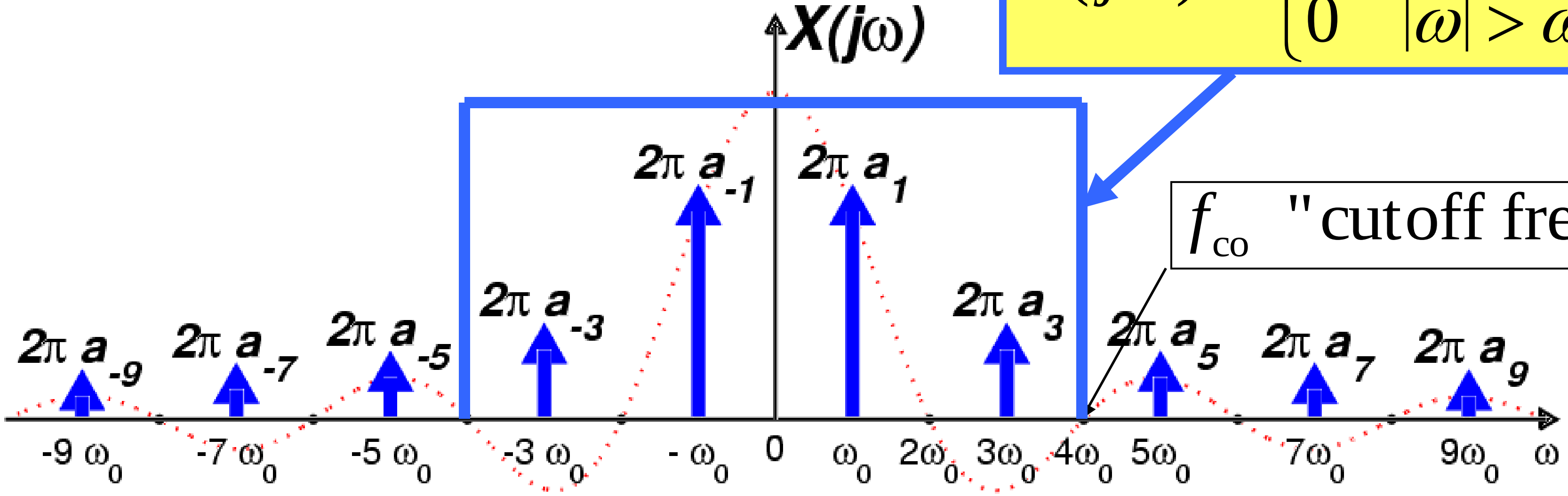


$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{\mathrm{co}}$$

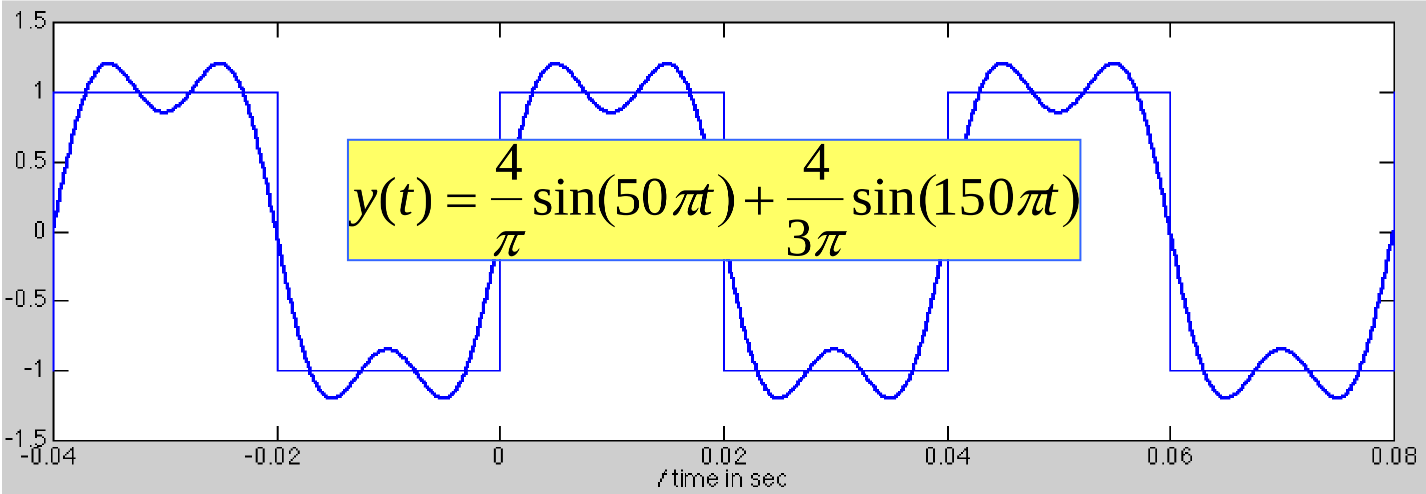
$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{\mathrm{co}}$$

Ideal Lowpass Filter

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

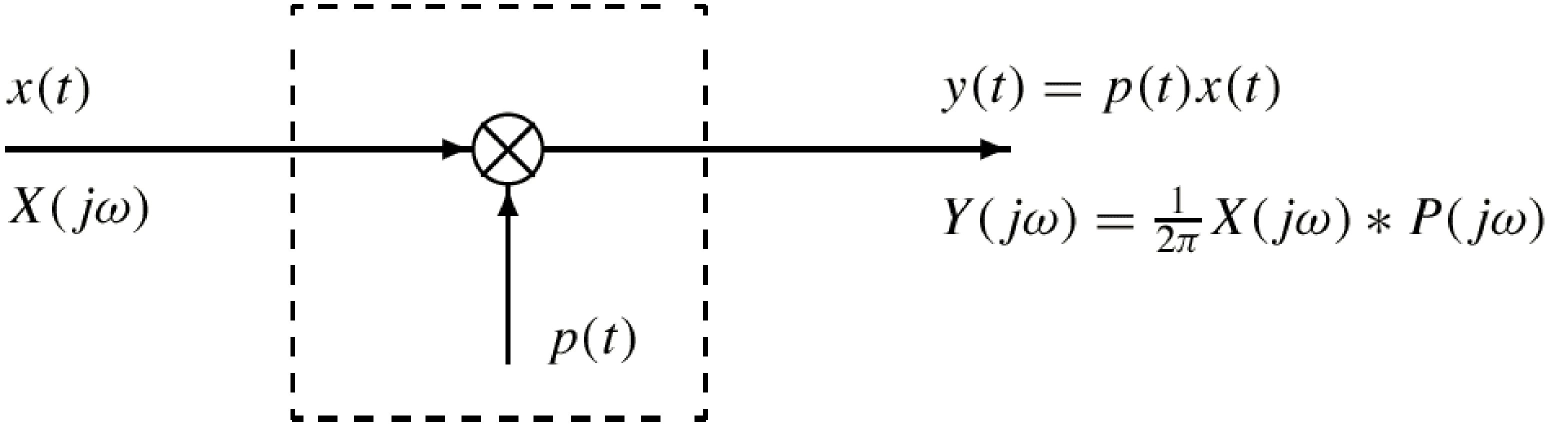


f_{co} "cutoff freq."



$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$

Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta$$

Frequency Shifting Property

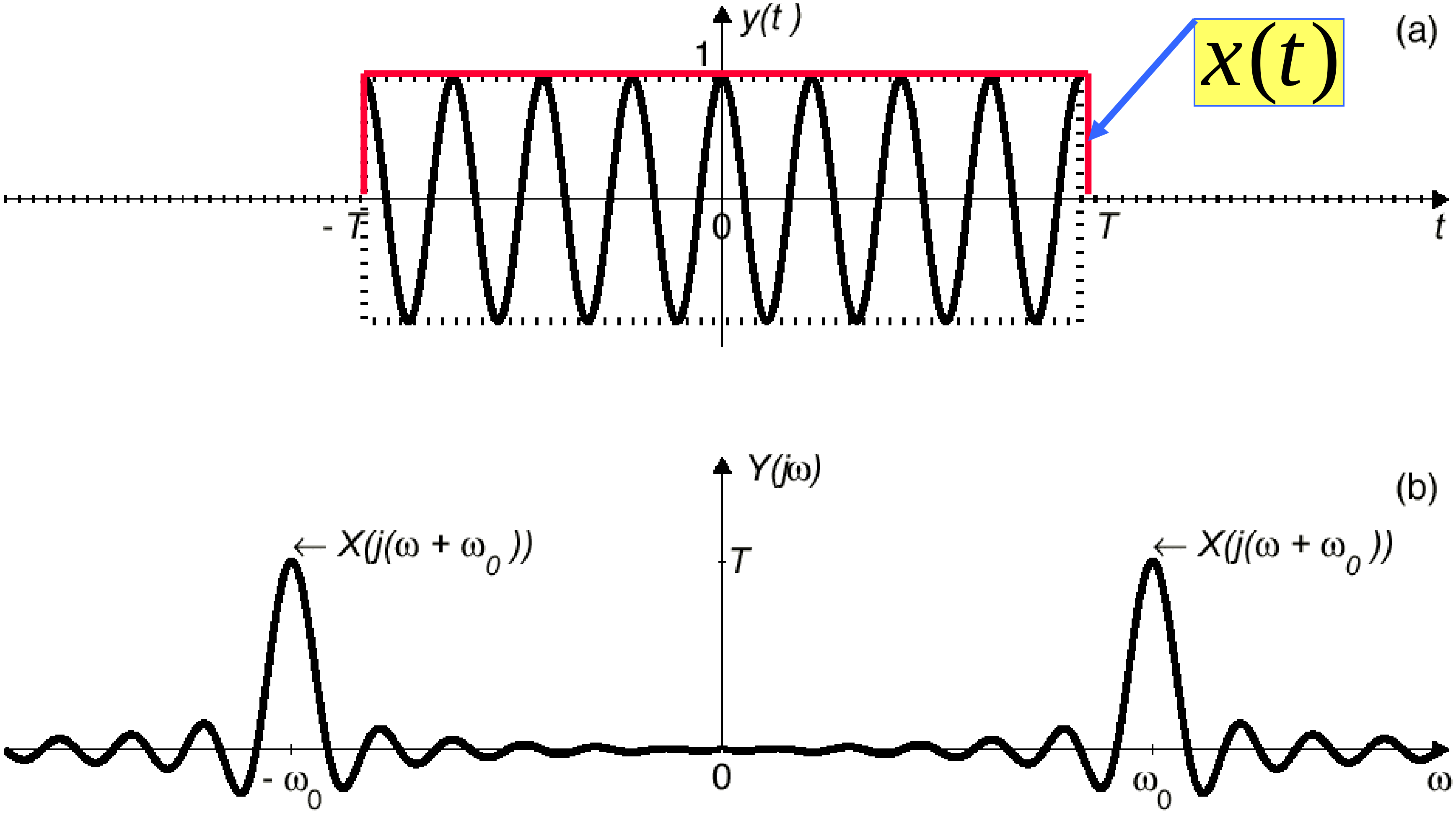
$$\boxed{x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))}$$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt$$

$$= X(j(\omega - \omega_0))$$

$$\boxed{y(t) = \frac{\sin 7t}{\pi t} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & elsewhere \end{cases}}$$

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$
$$Y(j\omega) = \frac{1}{2}X(j(\omega-\omega_0)) + \frac{1}{2}X(j(\omega+\omega_0))$$



Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)X(j\omega)e^{j\omega t}d\omega$$

Multiply by jω

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

$$\Leftrightarrow \frac{j\omega}{a + j\omega}$$