BLM2041 Signals and Systems

Syllabus

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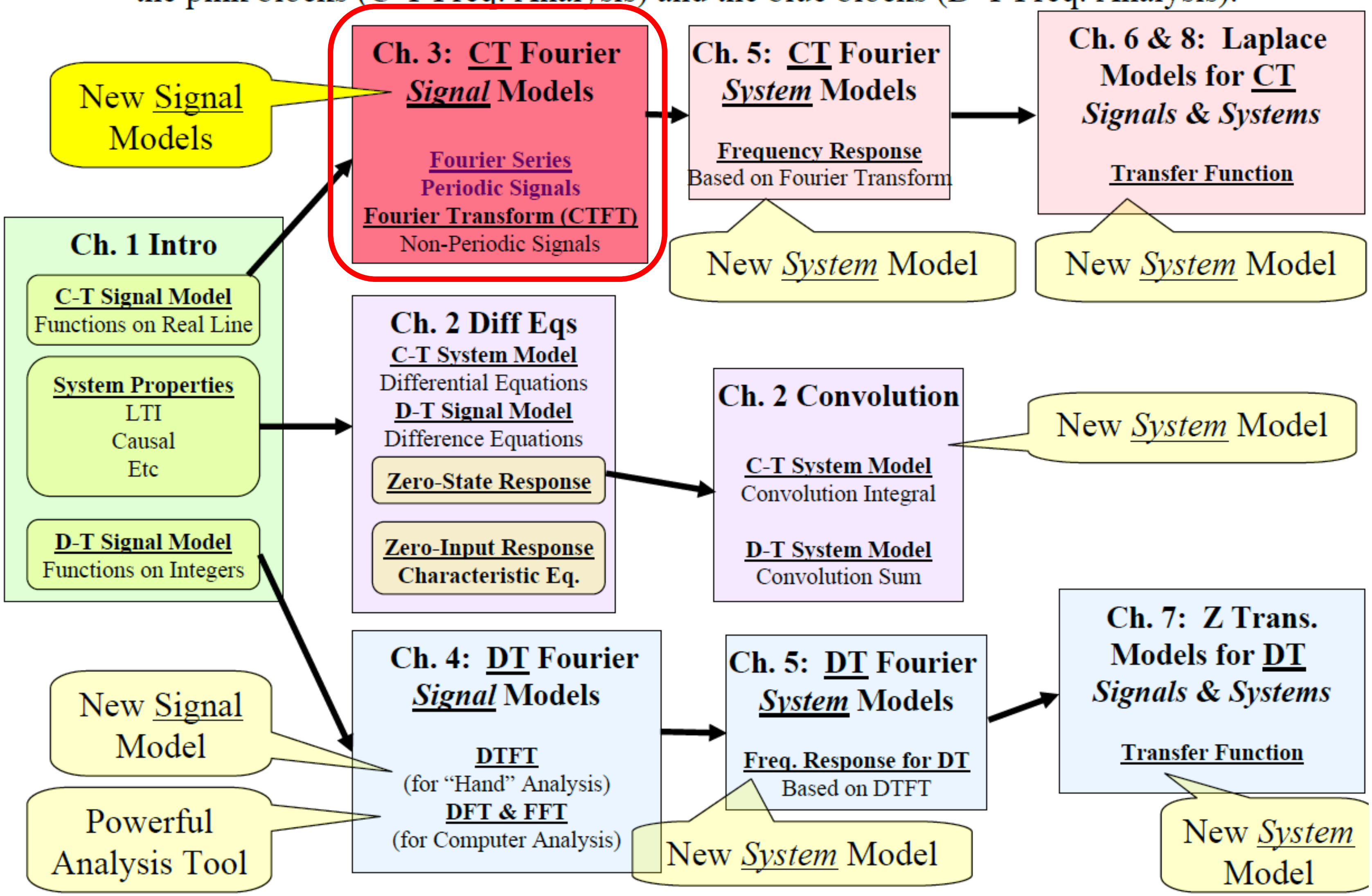
Where are we now?

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



LECTURE OBJECTIVES

- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs

LECTURE OBJECTIVES

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

FREQUENCY can change vs. TIME

Chirps:

$$\boxed{x(t) = \cos(\alpha t^2)}$$

Introduce Spectrogram Visualization **(specgram.m)** **(plotspec.m)**

LECTURE OBJECTIVES

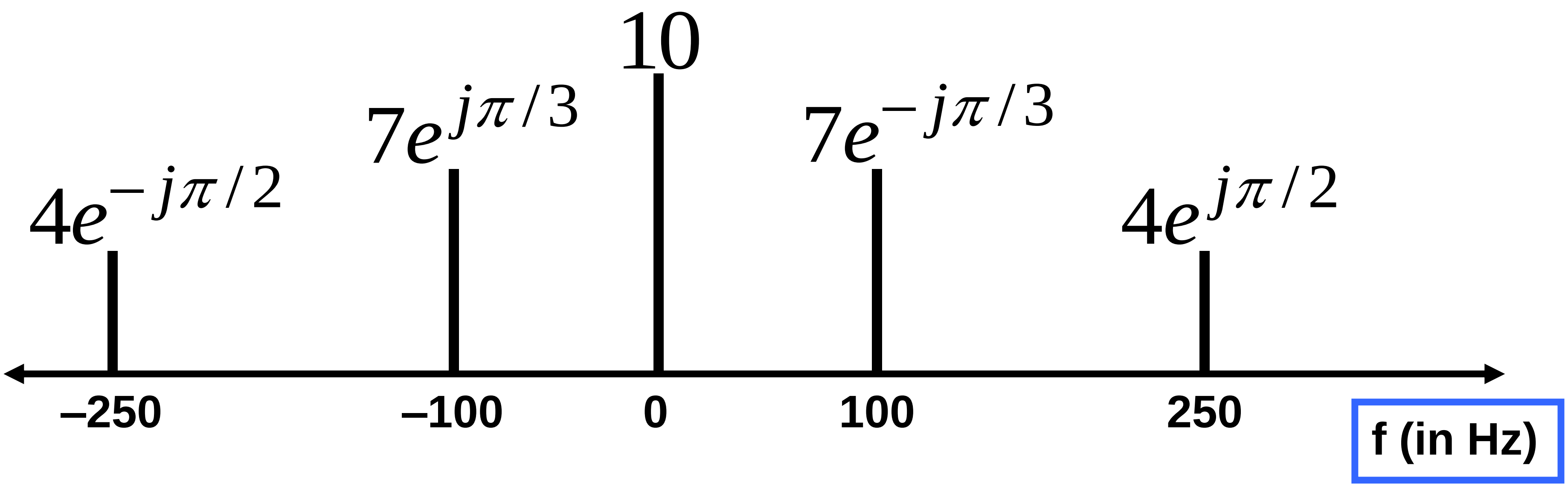
- Work with the Fourier Series Integral

$$\boxed{a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt}$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $\boldsymbol{x(t+T_0) = x(t)}$
- **SPECTRUM** from Fourier Series
  - a_k is Complex Amplitude for k-th Harmonic

FREQUENCY DIAGRAM

- Plot Complex Amplitude vs. Freq



Another FREQ. Diagram

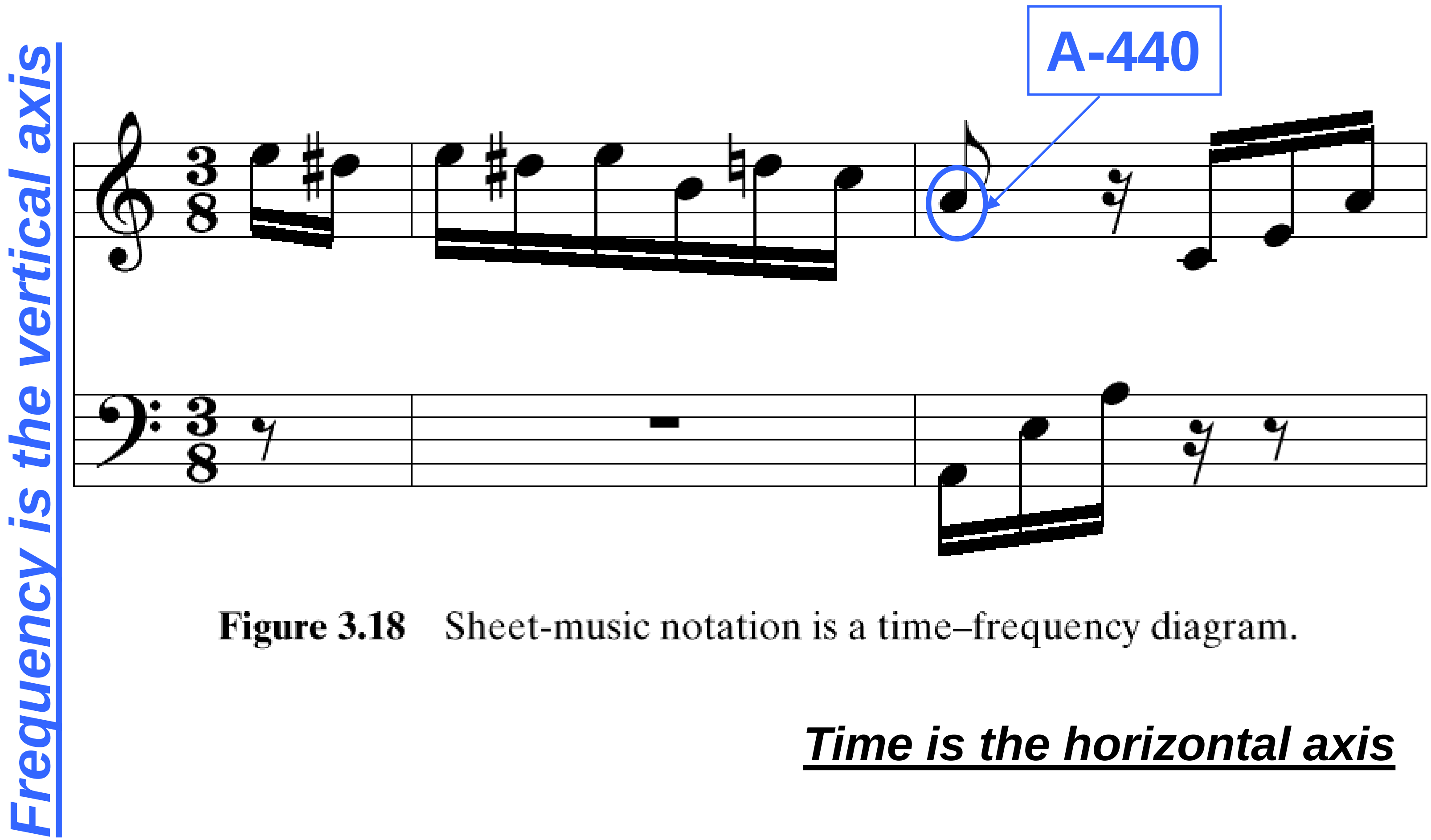


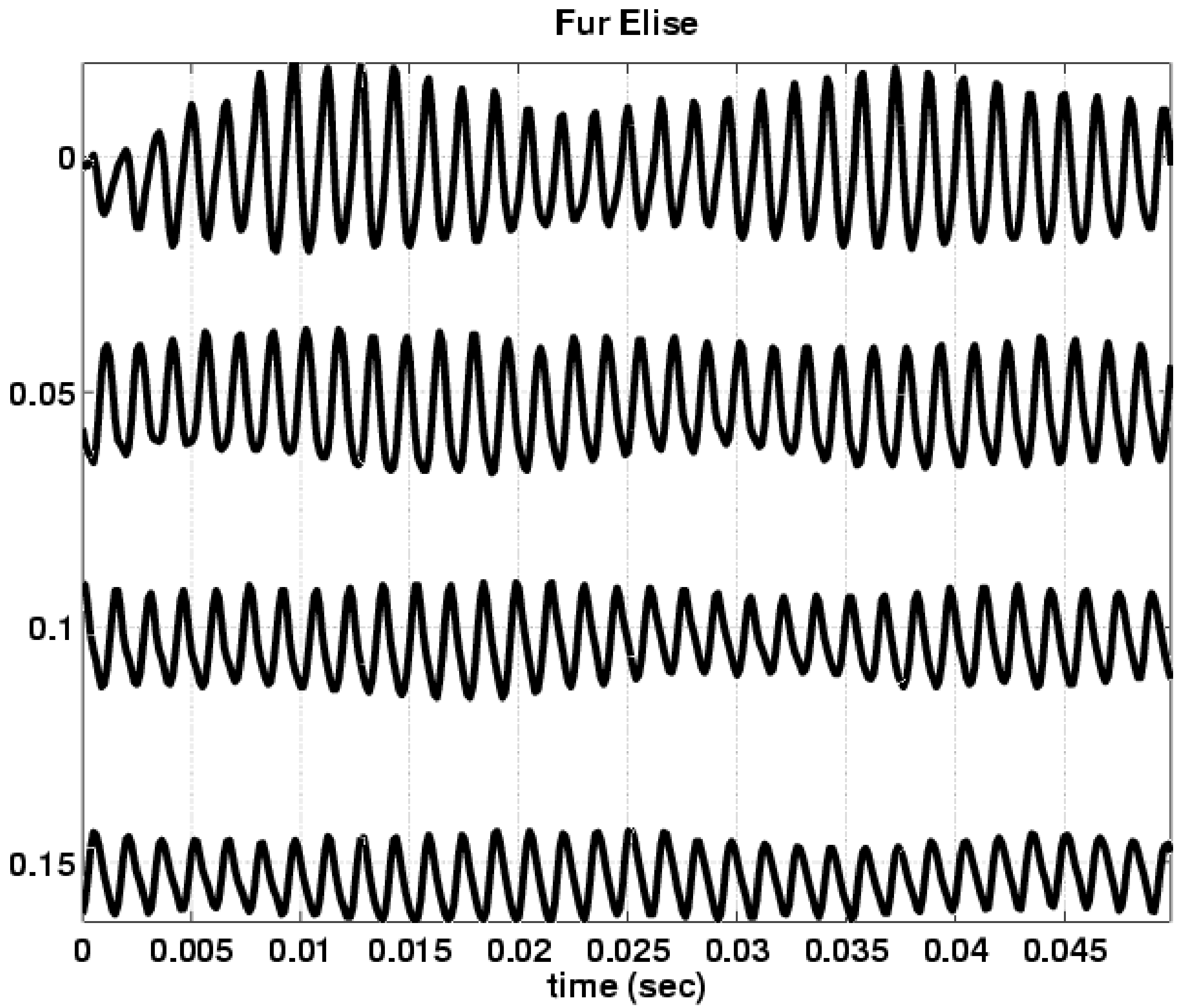
Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

MOTIVATION

- Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?

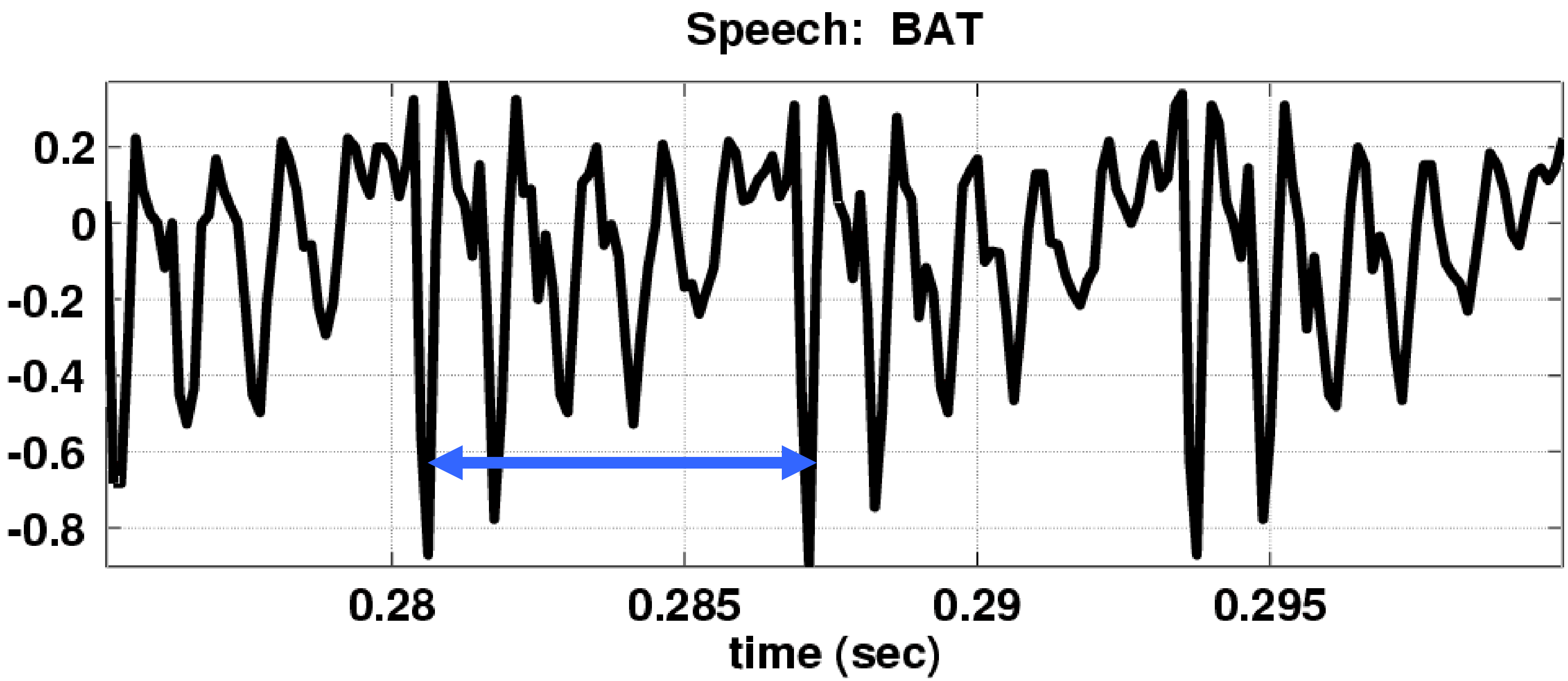
Fur Elise WAVEFORM



Beat Notes

Speech Signal: BAT

- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})}$$

INVERSE Euler's Formula

- Solve for cosine (or sine)

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big

NEGATIVE FREQUENCY

- Is negative frequency real?
- Doppler Radar provides an example
  - Police radar measures speed by using the Doppler shift principle
  - Let's assume 400Hz ←→60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)
  - Think of a train whistle

SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

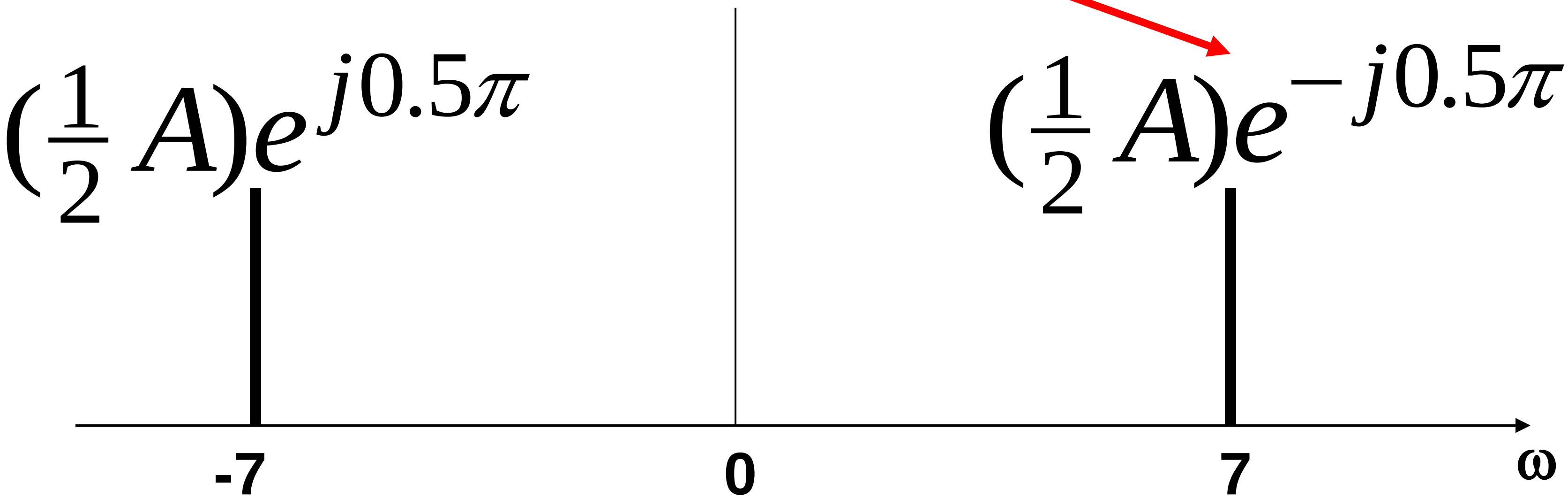
$$\boxed{\frac{-1}{j} = j = e^{j0.5\pi}}$$

 – Positive freq. has phase = -0.5π
 – Negative freq. has phase = +0.5π

GRAPHICAL SPECTRUM

EXAMPLE of SINE

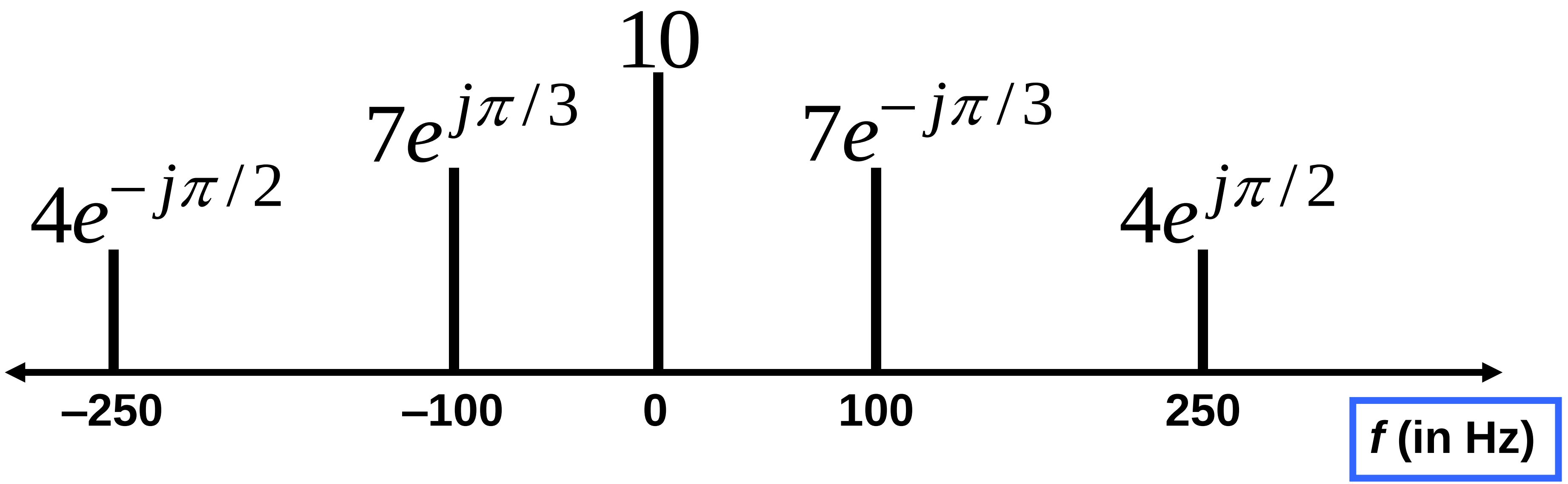
$$A\sin(7t) = \tfrac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \tfrac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are shown

SPECTRUM ---> SINUSOID

- Add the spectrum components:



What is the formula for the signal *x*(*t*)?

Gather (A,ω,ϕ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0** Hz
  - 100 Hz
  - 250 Hz

- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 **0**
  - 7 $-\pi/3$
  - 4 $+\pi/2$

Note the **conjugate phase**

DC is another name for zero-freq component

DC component always has $\phi=0$ or π (for real ***x(t)***)

Add Spectrum Components-1

- **Frequencies:**
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz

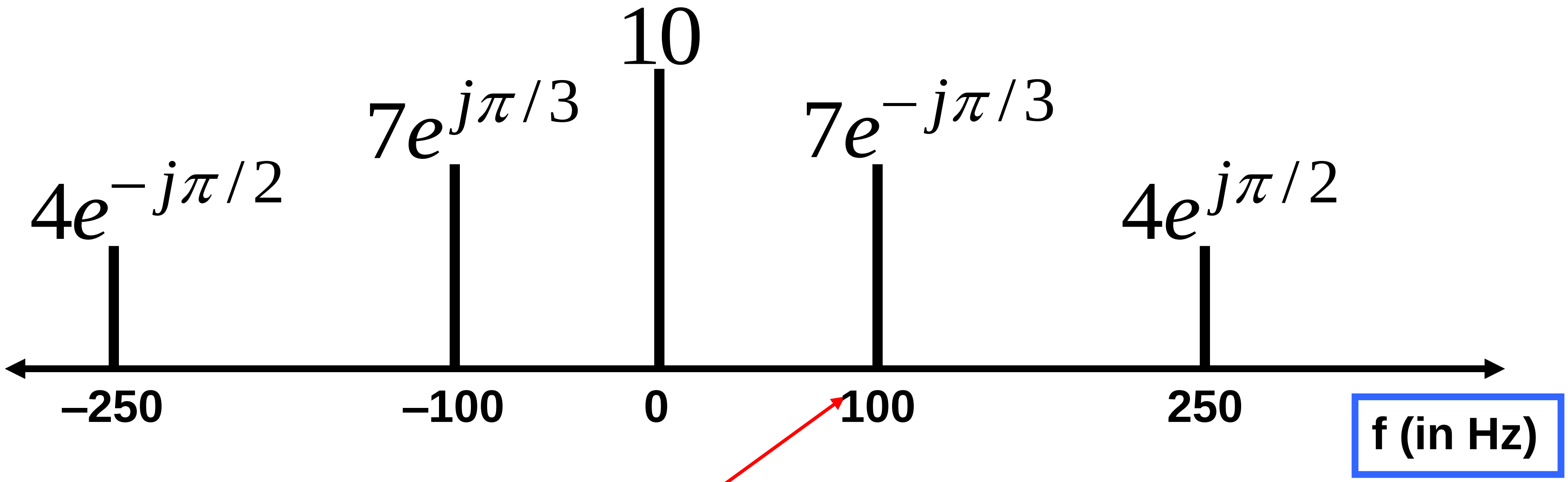
- **Amplitude & Phase**
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 0
  - 7 $-\pi/3$
  - 4 $+\pi/2$

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Add Spectrum Components-2



$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Simplify Components

$$x(t) = 10 +$$
$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$\boxed{A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{-j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}}$$

FINAL ANSWER

$$\boxed{\begin{aligned} x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) \\ + 8\cos(2\pi(250)t + \pi/2) \end{aligned}}$$

So, we get the general form:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$\Re e\{z\} = \tfrac{1}{2} z + \tfrac{1}{2} z^*$$

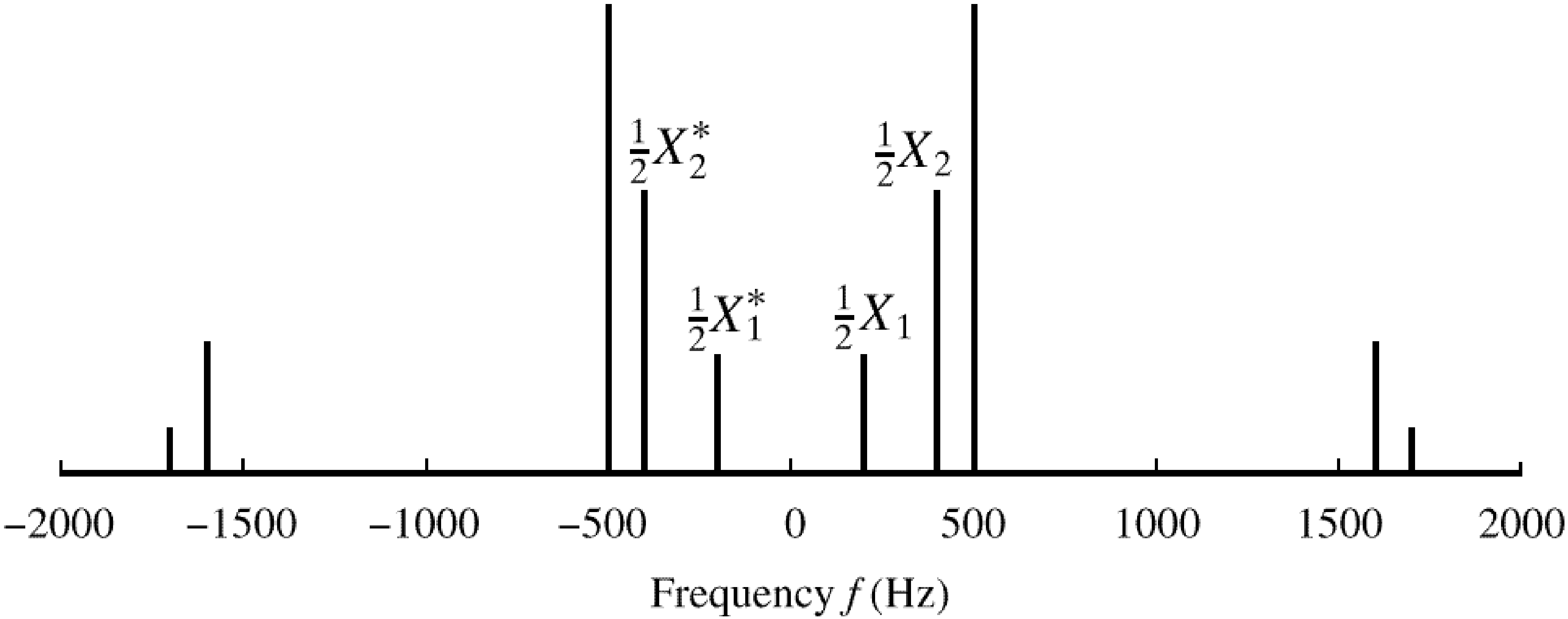
$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t}\right\}$$

Example: Synthetic Vowel

- Sum of 5 Frequency Components
  - Complex amplitudes for harmonic signal that approximates the vowel sound «ah»

f_k **(Hz)**	X_k	**Mag**	**Phase (rad)**
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	-0.185
1600	$(1657 - j13520)$	13,621	-1.449
1700	$4723 + j0$	4723	0

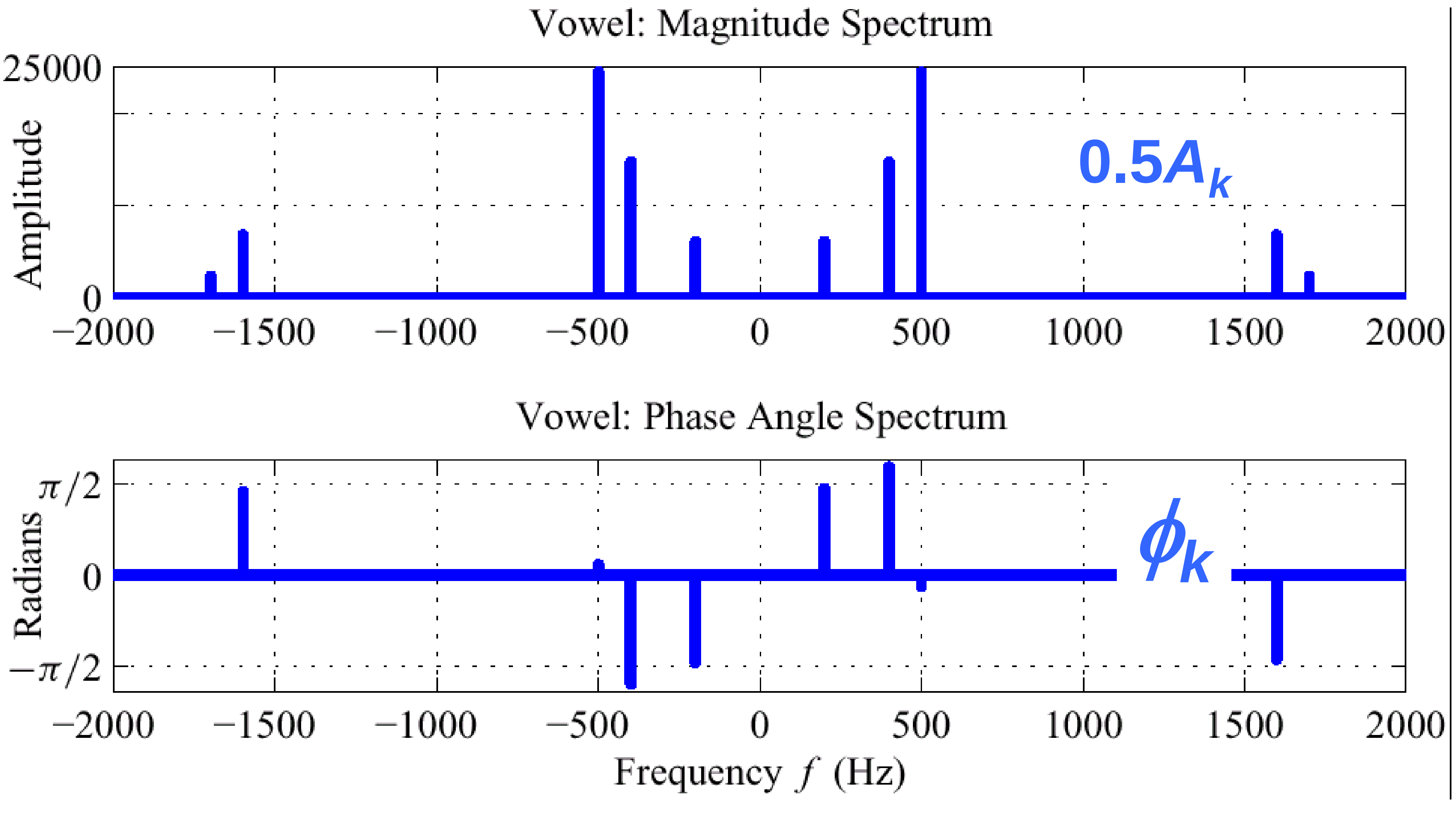
SPECTRUM of VOWEL



– Note: Spectrum has $0.5X_k$ (except X_{DC})

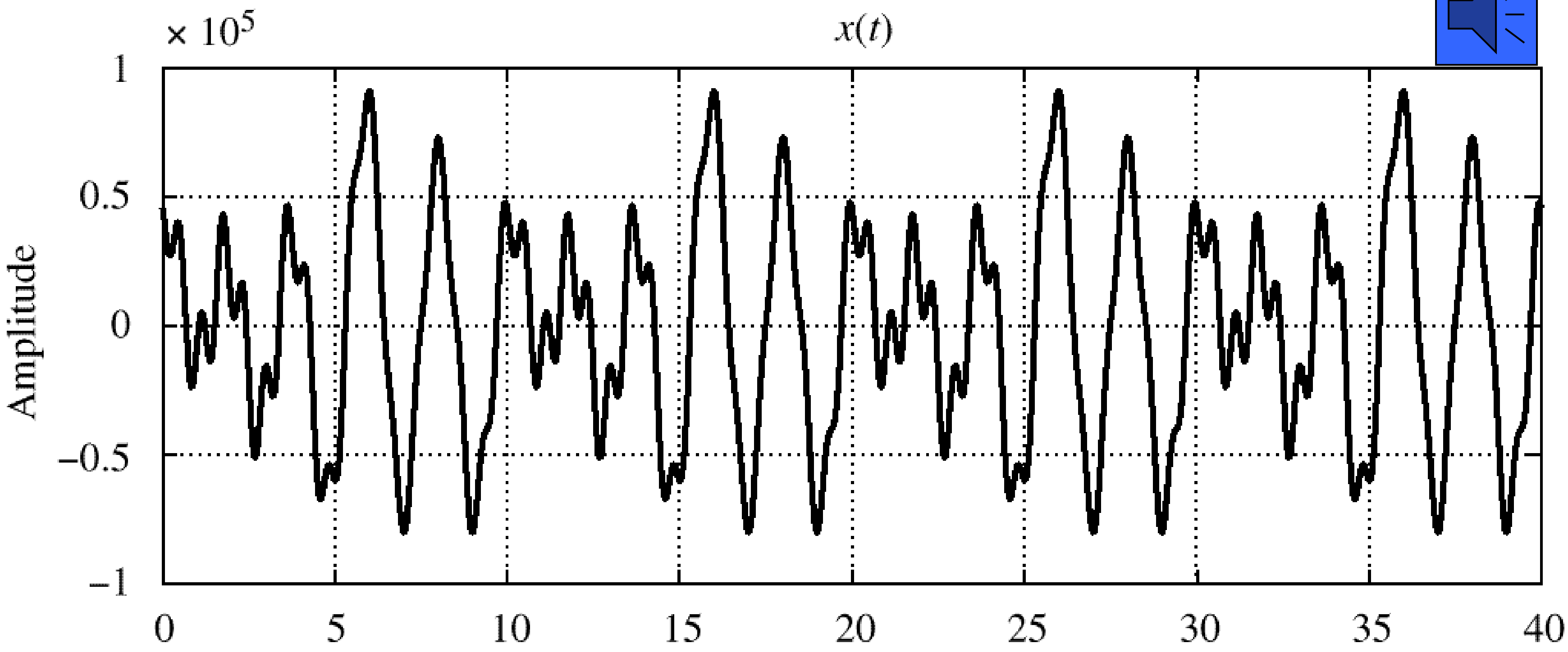
– Conjugates in negative frequency

SPECTRUM of VOWEL (Polar Format)



Vowel Waveform (sum of all 5 components)

- Sum of all of the signals in the prevous slides



  – Note that the period is 10 ms, which equals $1/f_0$

Fourier Series Motivation

"Fourier Series" allows us to write "virtually any" real-world <u>*PERIODIC*</u> signal as a sum of sinusoids with appropriate amplitudes and phases.

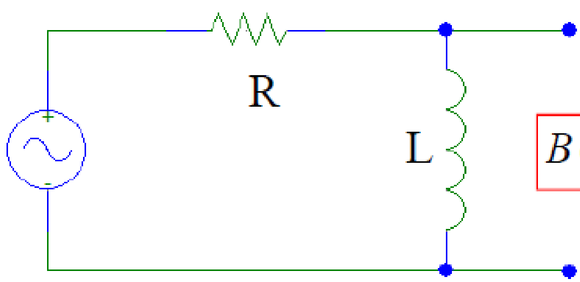
So… we can think of "building a periodic signal from sinusoidal building blocks".

Later we will extend that idea to also build many non-periodic signals from sinusoidal building blocks!

Thus, it is very common for engineers to think about "virtually any" signal as being made up of "sinusoidal components".

Q: Why all this attention to **<u>sinusoids</u>**?
A: Recall from Circuits… "sinusoidal analysis" of RLC circuits:
<u>Fundamental Result:</u> Sinusoid In $\Rightarrow$ Sinusoid Out
(Same Frequency, Different Amplitude & Phase)

$A\cos(\omega_0 t + \phi), \; -\infty < t < \infty$



$B\cos(\omega_0 t + \theta), \; -\infty < t < \infty$

Fourier Series Motivation

This "sinusoid in, sinusoid out" result holds for Constant-Coefficient, Linear Differential Equations as well as any LTI system. We'll only motivate this result for this Diff. Eq.: $$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = x(t)$$

If the input $x(t)$ is a sinusoid $A\cos(\omega_0 t + \phi),\ -\infty < t < \infty$

… then the solution $y(t)$ must be such that it and its derivatives can be combined to give the input sinusoid.

So… suppose the solution is $y(t) = B\cos(\omega_0 t + \theta),\ -\infty < t < \infty$

$$\omega_o^2 B\cos(\omega_o t + \theta) + a_1\omega_o B\sin(\omega_o t + \theta) + a_0 B\cos(\omega_o t + \theta) = A\cos(\omega_o t + \phi)$$

By slogging through lots of algebra and trig identities we can show this can be met with a proper choice of B and θ.

But it makes sense that to add up to a sinusoid we'd need all the terms on the left to be sinusoids of some sort!!!

So… we have reason to believe this:

Fundamental Result: Sinusoid In $\Rightarrow$ Sinusoid Out

(Same Frequency, Different Amplitude & Phase)

Fourier Series Motivation

Now… if our input is the linear combination of sinusoids:

$$x(t) = A_1 \cos\left(\omega_1 t + \phi_1\right) + A_2 \cos\left(\omega_2 t + \phi_2\right) + A_3 \cos\left(\omega_3 t + \phi_3\right) + \cdots, \quad -\infty < t < \infty$$

By linearity (i.e., superposition) we know that we can simply handle each term separately… and we know that each input sinusoid term gives an output sinusoid term:

$$y(t) = B_1 \cos\left(\omega_1 t + \theta_1\right) + B_2 \cos\left(\omega_2 t + \theta_2\right) + B_3 \cos\left(\omega_3 t + \theta_3\right) + \cdots, \quad -\infty < t < \infty$$

So… breaking a signal into sinusoidal parts makes the job of solving a Diff. Eq. EASIER!! (This was Fourier's big idea!!)

But…. What kind of signals can we use this trick on?

Or in other words…

What kinds of signals can we build by adding together sinusoids??!!!

What Can We Build with Sinusoids?

Let ω_0 be some given "fundamental" frequency

Q: What can I build from building blocks that looks like:

$$A_k \cos(\underbrace{k\omega_o} + \theta_k) \ ?$$

Only frequencies that are ***integer*** multiples of ω_o

Ex.: $\omega_o = 30$ rad/sec then consider 0, 30 60, 90, …

We can explore this by choosing a few different cases of values for the A_k and θ_k

On the next slide we limit ourselves to looking at three cases where we limit ourselves to having only three terms…

For this example let $\omega_0 = 2\pi$ rad/sec and look at a sum for $k = 1, 2, 3$:

$$x(t) = A_1 \cos\left(2\pi t + \phi_1\right) + A_2 \cos\left(2 \times 2\pi t + \phi_2\right) + A_3 \cos\left(3 \times 2\pi t + \phi_3\right)$$

What Can We Build with Sinusoids?

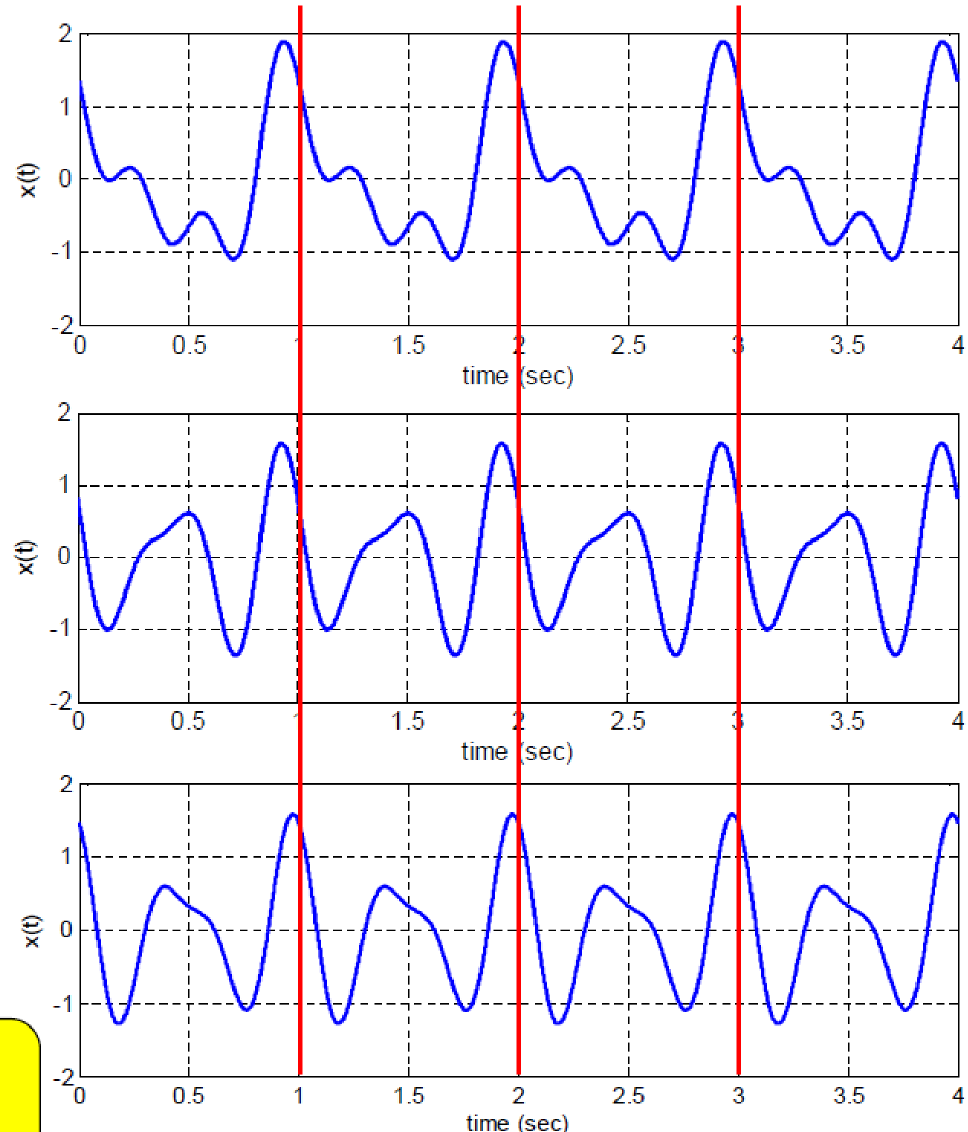
$$A_1 = 1.0 \qquad \theta_1 = 0$$
$$A_2 = 0.5 \qquad \theta_2 = \pi/4$$
$$A_3 = 0.5 \qquad \theta_3 = \pi/2$$

Equal

$$A_1 = 0.1 \qquad \theta_1 = 0$$
$$A_2 = 1.0 \qquad \theta_2 = \pi/4$$
$$A_3 = 0.5 \qquad \theta_3 = \pi/2$$

Equal

$$A_1 = 0.1 \qquad \theta_1 = 0$$
$$A_2 = 1.0 \qquad \theta_2 = \pi/7$$
$$A_3 = 0.5 \qquad \theta_3 = \pi/14$$



Note:
1. **All are periodic with period of 1s**
2. **All are "centered" vertically @ 0**

In one period: <u>Area</u> Above = <u>Area</u> Below

What Can We Build with Sinusoids?

Why do these all have period of 1 s???

$$x(t) = \underbrace{A_1 \cos(2\pi t + \phi_1)}_{\text{Repeats every 1 s}} + \underbrace{A_2 \cos(2 \times 2\pi t + \phi_2)}_{\text{Repeats every 1/2 s ... so it also repeats every 1 s}} + \underbrace{A_3 \cos(3 \times 2\pi t + \phi_3)}_{\text{Repeats every 1/3 s ... so it also repeats every 1 s}}$$

This motivates the following general statement:

A sum of sinusoids with frequencies that are integer multiples of some lowest "fundamental" frequency ω_o will give a periodic signal with period $T = 2\pi/\omega_o$ seconds.

So… we can now think about adding together any number of harmonically-related sinusoids… even infinitely many!

$$x(t) = \sum_{k=1}^{\infty} A_k \cos(k\omega_o t + \phi_k), \quad -\infty < t < \infty$$

i.e., all frequencies are an *integer* multiple of fund. freq. ω_o

What Can We Build with Sinusoids?

Why are these all centered vertically @ 0???

$$x(t) = \underbrace{A_1 \cos\left(2\pi t + \phi_1\right)}_{\text{Centered @ 0}} + \underbrace{A_2 \cos\left(2 \times 2\pi t + \phi_2\right)}_{\text{Centered @ 0}} + \underbrace{A_3 \cos\left(3 \times 2\pi t + \phi_3\right)}_{\text{Centered @ 0}}$$

This motivates the following general statement:

Unless we have a constant term added, a sum of sinusoids (with frequencies at ω_o, $2\omega_o$, $3\omega_o$, …) will be centered vertically at 0

So… we can now add a constant term

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(k\omega_o t + \phi_k\right), \quad -\infty < t < \infty$$

Note: for $k = 0$ we have $A_0\cos(0\times\omega_o t) = A_0$ so we can think of the constant term as a cosine with frequency = 0 and phase = 0

Fourier Series… A Way to Build a Periodic Signal

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(k\omega_o t + \phi_k\right), \; -\infty < t < \infty$$

This signal has Period $T = 2\pi/\omega_o$

Big Idea: We can think of (virtually) any real-world **periodic** signal as being made up of (possibly infinitely) many sinusoids whose frequencies are all an integer multiple of a fundamental frequency ω_o.

Once we set ω_o all we have to do is specify all the amplitudes (A_k) and phases (θ_k) and we get some periodic signal with period $T = 2\pi/\omega_o$.

But… if we are **GIVEN a periodic signal** how do we determine the correct:

- Fundamental Frequency ω_o (rad/sec)
- Amplitudes (A_k)
- Phases (θ_k)

Easy: $\omega_o = 2\pi/T$

Need to Learn How!!

Three Forms of Fourier Series

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(k\omega_o t + \phi_k\right)$$

"Amplitude & Phase" Form

The equation above is just one of three (totally equivalent!) different forms of the Fourier Series.

Each one contains the same information but presents it differently.

Which form you use in a particular setting depends….

- Partly on your preference
- Partly on what you are trying to do

Both of these come with experience…

We can easily find the other two by applying trig identities to the terms in the above form.

Convert to Complex Exponential Form

$$x(t) = A_0 + A_1 \cos\left(1\omega_o t + \phi_1\right) + A_2 \cos\left(2\omega_o t + \phi_2\right) + \cdots$$

"Amplitude & Phase" Form

Euler's Formula

$$\cos(\theta) = \tfrac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$$

$$x(t) = \underbrace{A_0}_{\triangleq c_0} + \left[\underbrace{\frac{A_1}{2} e^{j\phi_1}}_{\triangleq c_1} e^{j1\omega_o t} + \underbrace{\frac{A_1}{2} e^{-j\phi_1}}_{\triangleq c_{-1}} e^{j(-1)\omega_o t}\right] + \left[\underbrace{\frac{A_2}{2} e^{j\phi_2}}_{\triangleq c_2} e^{j2\omega_o t} + \underbrace{\frac{A_2}{2} e^{-j\phi_2}}_{\triangleq c_{-2}} e^{j(-2)\omega_o t}\right] + \cdots$$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

"Complex Exponential" Form

Convert to Sine-Cosine Form

$$x(t) = A_0 + A_1\cos\left(1\omega_o t + \phi_1\right) + A_2\cos\left(2\omega_o t + \phi_2\right) + \cdots$$

"Amplitude & Phase" Form

$\cos(\alpha+\beta)$ **Trig Identity**

$= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$$x(t) = \underbrace{A_0}_{\triangleq a_0} + \left[\underbrace{A_1\cos\left(\phi_1\right)}_{\triangleq a_1}\cos\left(1\omega_o t\right) - \underbrace{A_1\sin\left(\phi_1\right)}_{\triangleq b_1}\sin\left(1\omega_o t\right)\right]$$

$$+ \left[\underbrace{A_2\cos\left(\phi_2\right)}_{\triangleq a_2}\cos\left(2\omega_o t\right) - \underbrace{A_2\sin\left(\phi_2\right)}_{\triangleq b_2}\sin\left(2\omega_o t\right)\right] + \cdots$$

$$\boxed{x(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right]}$$

"Sine-Cosine" Form

Three (Equivalent) Forms of FS and Their Relationships

Best for "thinking about real-world ideas"

Trig Form: Amplitude & Phase

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

$$A_0 = c_0$$

$$\left.\begin{aligned} A_k &= 2|c_k| \\ \theta_k &= \angle c_k \end{aligned}\right\} k = 1, 2, 3, \ldots$$

$$c_0 = A_0$$

$$\left.\begin{aligned} c_k &= \tfrac{1}{2} A_k e^{j\theta_k} \\ c_{-k} &= \tfrac{1}{2} A_k e^{-j\theta_k} \end{aligned}\right\} k = 1, 2, 3, \ldots$$

Best for "doing math"
(c_k are like phasors!!)

Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$A_0 = a_0$$

$$A_k = \sqrt{a_k^2 + b_k^2}$$

$$\theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right)$$

$$a_0 = c_0$$

$$a_k = A_k \cos(\theta_k)$$

$$b_k = -A_k \sin(\theta_k)$$

$$c_0 = a_0$$

$$\left.\begin{aligned} c_k &= \tfrac{1}{2}(a_k - jb_k) \\ c_{-k} &= \tfrac{1}{2}(a_k + jb_k) \end{aligned}\right\} k = 1, 2, 3, \ldots$$

$$a_0 = c_0$$

$$a_k = 2\,\mathrm{Re}\{c_k\}, \quad k = 1, 2, 3, \ldots$$

$$b_k = -2\,\mathrm{Im}\{c_k\}, \quad k = 1, 2, 3, \ldots$$

Best for some "special scenarios"

Trig Form: Sine-Cosine

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)]$$

Example: Consider $\boxed{x(t) = \cos(t) + 0.5\cos(4t + \pi/3) + 0.25\cos(8t + \pi/2)}$

which is already in **Amp-Phase Form** of the Fourier Series with $\omega_0 = 1$:

$A_1 = 1$ $\quad$ $A_4 = 0.5$ $\quad$ $A_8 = 0.25$ $\quad$ (all other A_k are 0)

$\theta_1 = 0$ $\quad$ $\theta_4 = \pi/3$ $\quad$ $\theta_8 = \pi/2$

Using the conversion results on the previous slide we can re-write this in **Complex Exponential Form** of the FS as:

$c_0 = A_0$

$c_k = \frac{1}{2} A_k e^{j\theta_k}$

$c_{-k} = \frac{1}{2} A_k e^{-j\theta_k}$

$c_1 = 0.5$ $\quad$ $c_4 = 0.25e^{j\pi/3}$ $\quad$ $c_8 = 0.125e^{j\pi/2}$ $\quad$ (all other c_k are 0)

$c_{-1} = 0.5$ $\quad$ $c_{-4} = 0.25e^{-j\pi/3}$ $\quad$ $c_{-8} = 0.125e^{-j\pi/2}$

$$\boxed{x(t) = \left[0.5e^{jt} + 0.5e^{-jt}\right] + \left[0.25e^{j\pi/3}e^{j4t} + 0.25e^{-j\pi/3}e^{-j4t}\right] + \left[0.125e^{j\pi/2}e^{j8t} + 0.125e^{-j\pi/2}e^{-j8t}\right]}$$

Using the conversion results on the previous slide we can re-write this in **Sine-Cosine Form** of the FS as:

$a_1 = 1$ $\quad$ $a_4 = 0.25$ $\quad$ $a_8 = 0$ $\quad$ (all other a_k, b_k are 0)

$b_1 = 0$ $\quad$ $b_4 = 0.43$ $\quad$ $b_8 = 0.25$

$$\boxed{x(t) = [\cos(t)] + [0.25\cos(4t) - 0.43\sin(4t)] + [0.25\sin(8t)]}$$

$a_0 = c_0$

$a_k = A_k \cos(\theta_k)$

$b_k = -A_k \sin(\theta_k)$

Analytically Finding FS Coefficients

Q: How do we find the Exponential Form FS Coefficients?

A: Use this: (it can be proved but we won't do that here!)

$$\boxed{c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}\,dt}$$

Integrate over any complete period

Some books use only $t_0 = 0$.

where: T = fundamental period of $x(t)$ (in seconds)

ω_0 = fundamental frequency of $x(t)$ (in rad/second)

$= 2\pi/T$

t_0 = any time point (you pick t_0 to ease calculations)

$k \in$ all integers $(\ldots -3, -2, -1, 0, 1, 2, 3, \ldots)$

Looks like we have to do this integral infinitely many times!!! But…Usually you can do the integral in terms of arbitrary *k*!

Comment: Note that for $k = 0$ this gives

$$\boxed{c_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt}$$

c_0 is the "DC offset", which is the time-average over one period

Analytically Finding FS Coefficients

Q: How do we find the Sine-Cosine Form FS Coefficients?

A: Use these: (can be proved but we won't do that here!)

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$$

a_0 is the "DC offset", which is the time-average over one period

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos\left(k\omega_0 t\right)dt$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin\left(k\omega_0 t\right)dt$$

Integrate over any complete period

where: T = fundamental period of $x(t)$ (in seconds)

ω_0 = fundamental frequency of $x(t)$ (in rad/second)

$= 2\pi/T$

t_0 = any time point (you pick t_0 to ease calculations)

$k \in$ all integers

Analytically Finding FS Coefficients

Q: How do we find the Amplitude-Phase Form FS Coefficients?

A: No easy direct way! So convert from one of the other forms!

$$A_0 = a_0$$

$$A_k = \sqrt{a_k^2 + b_k^2}$$

$$\theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right)$$

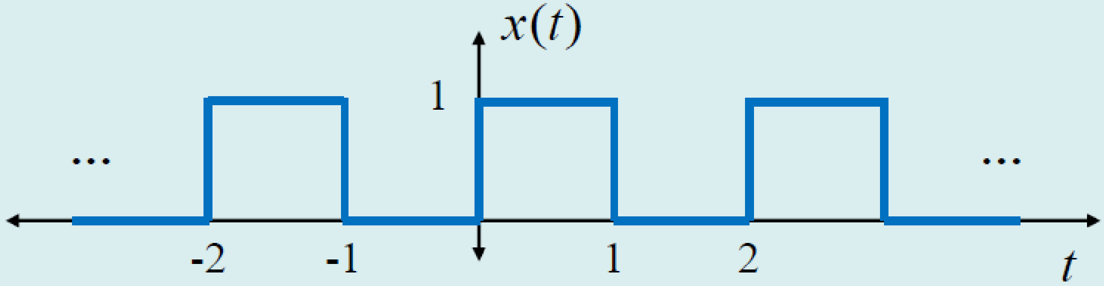
$$A_0 = c_0$$

$$\left.\begin{aligned} A_k &= 2|c_k| \\ \theta_k &= \angle c_k \end{aligned}\right\} k = 1, 2, 3, \ldots$$

- Recall… you can convert from any form into any other form using some simple equations!
- Thus… I tend to always find the c_k and then convert to other forms if needed.
- Why do I prefer to find the c_k?
  - Only one integral to actually do (although it is complex valued!)
  - Integrals involving exponential are usually easier than for sinusoids!

Analytically Finding FS Coefficients

Example: FS of Rectangular Pulse Train



$$T = 2 \;\Rightarrow\; \omega_0 = \frac{2\pi}{2} = \pi \text{ rad/sec}$$

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt$$

choose $t_0 = 0$

$$= \frac{1}{2}\int_0^2 x(t)e^{-jk\pi t}dt$$

$$= \frac{1}{2}\left[\int_0^1 1e^{-jk\pi t}dt + \int_1^2 0\times e^{-jk\pi t}dt\right]$$

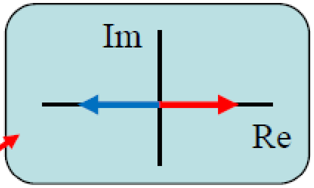
$$= \frac{1}{2}\int_0^1 e^{-jk\pi t}dt$$

$$= \frac{1}{2}\left[\frac{1}{-jk\pi}e^{-jk\pi t}\right]_0^1$$

Not valid for $k = 0$… so have to do that case separately!

$$= \frac{j}{2(k\pi)}\left[e^{-jk\pi} - 1\right]$$

$$e^{-jk\pi} = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd} \end{cases}$$



$$\boxed{c_k = \begin{cases} 0, & k \text{ even}, \neq 0 \\ \dfrac{-j}{k\pi}, & k \text{ odd} \end{cases}}$$

$$c_0 = \frac{1}{2}\int_0^1 1e^{-j0\pi t}dt = \frac{1}{2}\int_0^1 1dt$$

$$\boxed{c_0 = \frac{1}{2}}$$

DC Level (also called DC Offset)

Analytically Finding FS Coefficients

So… we've found the exponential FS to be:

$$x(t) = \cdots + \frac{-j}{-3\pi} e^{-j3\omega_o t} + \frac{-j}{-1\pi} e^{-j1\omega_o t} + \frac{1}{2} + \frac{-j}{1\pi} e^{j1\omega_o t} + \frac{-j}{3\pi} e^{j3\omega_o t} + \cdots$$

$$c_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \text{ even}, \neq 0 \\ \frac{-j}{k\pi}, & k \text{ odd} \end{cases}$$

$$a_0 = c_0$$

$$a_k = 2\,\mathrm{Re}\{c_k\},\ \ k = 1, 2, 3, \ldots$$

$$b_k = -2\,\mathrm{Im}\{c_k\},\ \ k = 1, 2, 3, \ldots$$

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$b_k = \begin{cases} 0, & k \text{ even} \\ \frac{2}{k\pi}, & k \text{ odd} \end{cases}$$

$$x(t) = \frac{1}{2} + \frac{2}{1\pi}\sin(1\omega_o t) + \frac{2}{3\pi}\sin(3\omega_o t) + \frac{2}{5\pi}\sin(5\omega_o t) + \cdots$$

Analytically Finding FS Coefficients

So… we've found the exponential FS to be:

$$x(t) = \cdots + \frac{-j}{-3\pi} e^{-j3\omega_o t} + \frac{-j}{-1\pi} e^{-j1\omega_o t} + \frac{1}{2} + \frac{-j}{1\pi} e^{j1\omega_o t} + \frac{-j}{3\pi} e^{j3\omega_o t} + \cdots$$

$$c_k = \begin{cases} \dfrac{1}{2}, & k = 0 \\ 0, & k \text{ even}, \neq 0 \\ \dfrac{-j}{k\pi}, & k \text{ odd} \end{cases}$$

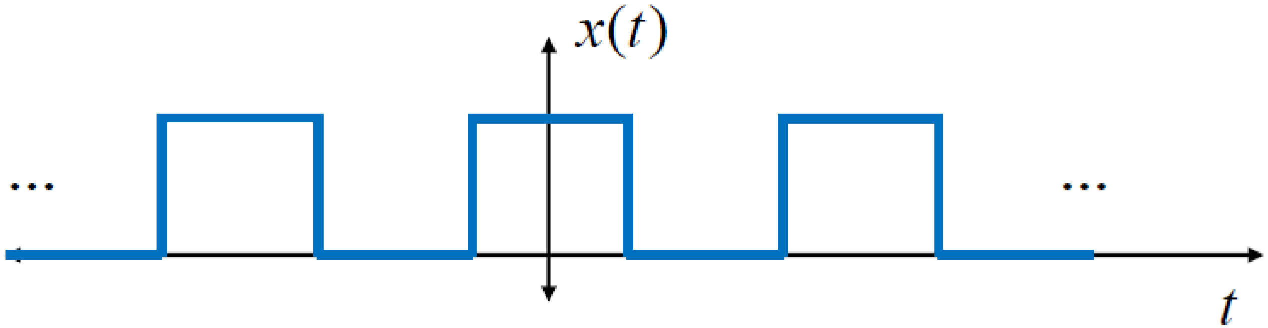
$$\left.\begin{aligned} A_0 &= c_0 \\ A_k &= 2|c_k| \\ \theta_k &= \angle c_k \end{aligned}\right\} k = 1, 2, 3, \ldots$$

$$A_k = \begin{cases} \dfrac{1}{2}, & k = 0 \\ 0, & k \text{ even} \\ \dfrac{2}{k\pi}, & k \text{ odd} \end{cases} \qquad \theta_k = \begin{cases} \text{N/A}, & k = 0 \\ \text{N/A}, & k \text{ even} \\ -\dfrac{\pi}{2}, & k \text{ odd} \end{cases}$$

$$x(t) = \tfrac{1}{2} + \tfrac{2}{1\pi}\cos(1\omega_o t - \pi/2) + \tfrac{2}{3\pi}\cos(3\omega_o t - \pi/2) + \tfrac{2}{5\pi}\cos(5\omega_o t - \pi/2) + \cdots$$

Symmetry "Tricks" for Finding FS Coefficients

Even Symmetry: $x(-t) = x(t)$ ("flipping" around $t = 0$ does nothing)



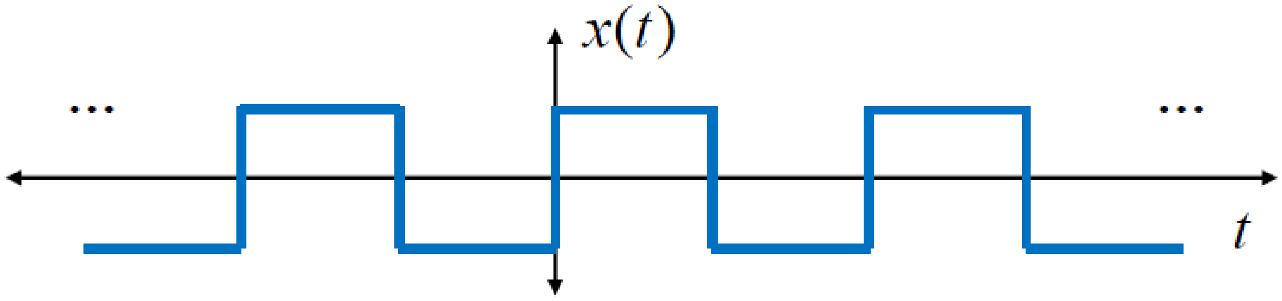
Noting that cosines have even symmetry and sines have odd symmetry it is not surprising that an even $x(t)$ needs only cosine components in the Sine-Cosine Form:

Sine-Cosine Form → $\boxed{b_k = 0}$ → ***Exp. Form*** $\boxed{c_k = \text{ Are Real}}$ → ***Amp.-Phase Form*** $\boxed{\theta_k = \pm\pi}$

$$c_0 = a_0$$
$$\left.\begin{array}{l} c_k = \tfrac{1}{2}(a_k - jb_k) \\ c_{-k} = \tfrac{1}{2}(a_k + jb_k) \end{array}\right\} k = 1, 2, 3, \ldots$$

$$A_0 = c_0$$
$$\left.\begin{array}{l} A_k = 2|c_k| \\ \theta_k = \angle c_k \end{array}\right\} k = 1, 2, 3, \ldots$$

Odd Symmetry: $x(-t) = -x(t)$ ("flipping" around $t = 0$ negates $x(t)$)



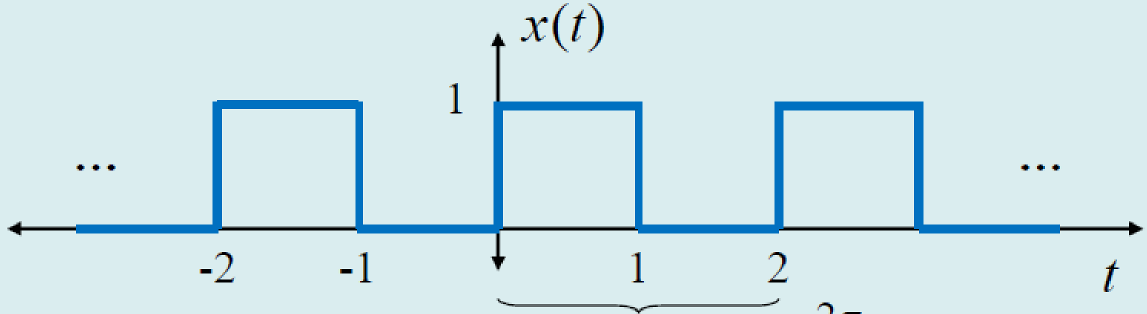
Noting that cosines have even symmetry and sines have odd symmetry it is not surprising that an ODD $x(t)$ needs only sine components in the Sine-Cosine Form:

Sine-Cosine Form	*Exp. Form*	*Amp.-Phase Form*
$\rightarrow$ $a_k = 0$ $\quad k \neq 0$	$\rightarrow$ $c_k =$ Are Imag $\quad k \neq 0$	$\rightarrow$ $\theta_k = \pm\pi/2$ $\quad k \neq 0$

$$c_0 = a_0$$
$$\left.\begin{aligned} c_k &= \tfrac{1}{2}(a_k - jb_k) \\ c_{-k} &= \tfrac{1}{2}(a_k + jb_k) \end{aligned}\right\} k = 1, 2, 3, \ldots$$

$$A_0 = c_0$$
$$\left.\begin{aligned} A_k &= 2|c_k| \\ \theta_k &= \angle c_k \end{aligned}\right\} k = 1, 2, 3, \ldots$$

Recall Example: FS of Rectangular Pulse Train



$$T = 2 \;\Rightarrow\; \omega_0 = \frac{2\pi}{2} = \pi \text{ rad/sec}$$

Sine-Cosine Form		*Exp. Form*		*Amp.-Phase Form*
$\underset{k\neq 0}{a_k = 0}$	→	$\underset{k\neq 0}{c_k = \text{ Are Imag}}$	→	$\underset{k\neq 0}{\theta_k = \pm\pi/2}$

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$b_k = \begin{cases} 0, & k \text{ even} \\ \frac{2}{k\pi}, & k \text{ odd} \end{cases}$$

$$c_k = \begin{cases} 0, & k \text{ even}, \neq 0 \\ \frac{-j}{k\pi}, & k \text{ odd} \end{cases}$$

$$\theta_k = \begin{cases} \text{N/A}, & k = 0 \\ \text{N/A}, & k \text{ even} \\ -\frac{\pi}{2}, & k \text{ odd} \end{cases}$$

Fourier Series Spectrum

Trig Form "Spectrum"… Is "Single Sided"

Best for "thinking about real-world ideas"

Trig Form: Amplitude & Phase

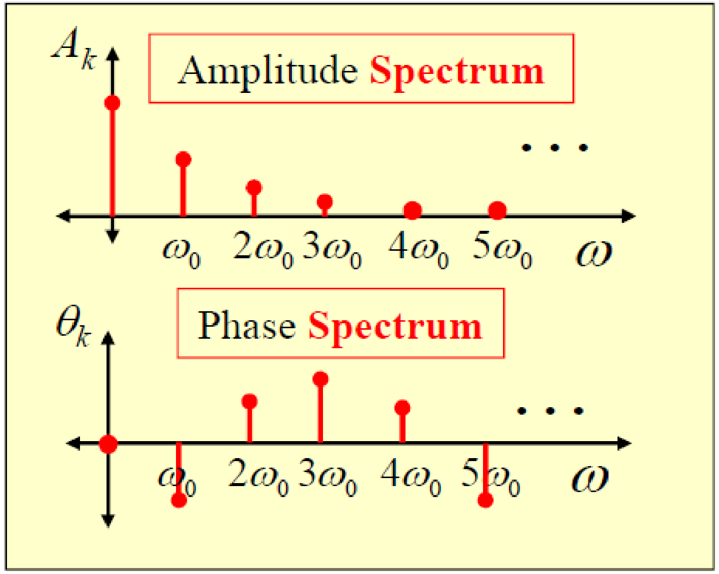
$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Need A_k and θ_k for $k = 0, 1, 2\ldots$

A_k = Amplitude
θ_k = Phase

So… to describe a signal via FS we specify:
"Amplitude & Phase @ Each Frequency"

A good way to "see" the FS coefficients is by plotting them vs. frequency:



For this form of FS:

- Do not need negative freqs
  ➔ "Single Sided" Spectrum

Fourier Series Spectrum

Exp Form "Spectrum"… Is "Double Sided"

Best for "doing math" (c_k are like phasors!!)

Exponential Form

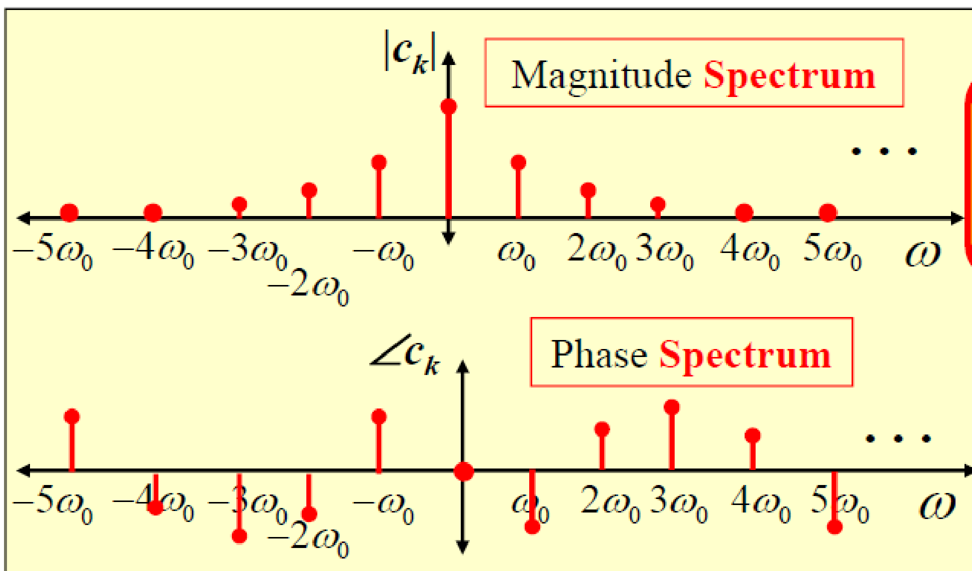
$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Need c_k (complex!)
for k = … -2, -1, 0, 1, 2…

$|c_k|$ = Magnitude
$\angle c_k$ = Phase

$$c_k e^{jk\omega_0 t} = \left[\,|c_k|\, e^{j\angle c_k}\right] e^{jk\omega_0 t}$$
$$= |c_k|\, e^{j(k\omega_0 t + \angle c_k)}$$

So… to describe a signal via FS we specify:
"Magnitude & Phase @ Each Frequency"



For this form of FS:
- Do need negative freqs
  → "Double Sided" Spectrum

Fourier Series Spectrum

Spectrum Characteristics

Trig Form: Amplitude & Phase

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Exponential Form

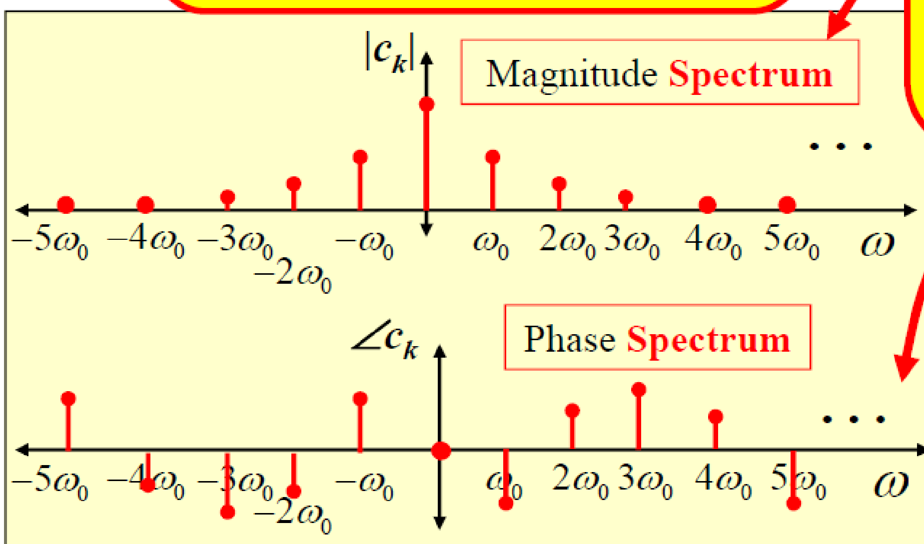
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

For Trig Form of FS Spectrum:

- "Single Sided" Spectrum
- $A_k \geq 0$ for $k > 0$
  - A_0: positive or negative
- θ_k is in radians $\theta_0 = 0$

For Exp Form of FS Spectrum:

- "Double Sided" Spectrum
- $|c_k| \geq 0$ for all k
  - *Even Symmetry for Magn.*
- $\angle c_k$ is in radians
- $\angle c_0 = 0$ or $\pm\pi$
- $\angle c_k = -\angle c_{-k}$
  - *Odd Symmetry for Phase*



$$\left.\begin{aligned} c_k &= \tfrac{1}{2} A_k e^{j\theta_k} \\ c_{-k} &= \tfrac{1}{2} A_k e^{-j\theta_k} \end{aligned}\right\} k = 1, 2, 3, \ldots$$

Fourier Series Spectrum

Parseval's Theorem

We saw earlier how to compute the average power of a periodic signal if we are given its time-domain model:

$$\boxed{P = \frac{1}{T}\int_{t_0}^{T+t_0} x^2(t)\,dt}$$

Q: Can we compute the average power from the frequency domain model

$$\boxed{\{c_k\}, \quad k = 0, \pm 1, \pm 2, \ldots}$$

A: Parseval's Theorem says… Yes!

Parseval's theorem says that the avg. power can be computed this way:

$$\boxed{P = \sum_{k=-\infty}^{\infty} |c_k|^2}$$

c_k are the Exp. Form FS coefficients

$$\boxed{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2}$$

Left side is clearly finite for real-world signals…

Thus, the $|c_k|$ must decay fast enough as $k \to \pm\infty$

Fourier Series Spectrum

Interpreting Parseval's Theorem

$$\underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)dt}_{\text{"sum" of squares in time-domain model}} = \underbrace{\sum_{k=-\infty}^{\infty} |c_k|^2}_{\text{"sum" of squares in freq.-domain model}}$$

"sum" of squares in time-domain model = **"sum" of squares in freq.-domain model**

$x^2(t)$ = power at time t (includes effects of all frequencies)

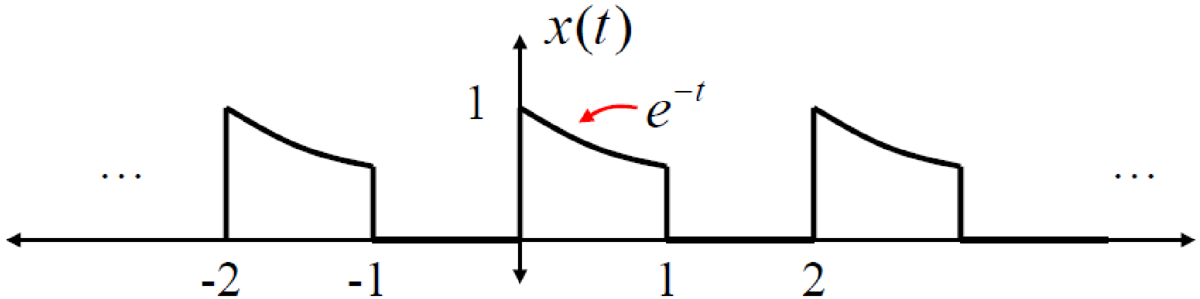
$|c_k|^2$ = power at frequency $k\omega_0$ (includes effects of all times)

We can find the power in the time domain by "adding up" all the "powers at each time"

We can find the power in the frequency domain by adding up all the "powers at each frequency"

Fourier Series Example

Example #1



choose

$$T = 2 \;\Rightarrow\; \omega_0 = \frac{2\pi}{2} = \pi \text{ rad/sec}$$

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{2}\int_0^2 x(t)e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\left[\int_0^1 e^{-t}e^{-jk\pi t}\,dt + \int_1^2 0\times e^{-jk\pi t}\,dt\right]$$

$$= \frac{1}{2}\int_0^1 e^{-(1+jk\pi)t}\,dt$$

$$= \frac{1}{2}\left[\frac{-1}{1+jk\pi}e^{-(1+jk\pi)t}\right]_0^1$$

$$= \frac{-1}{2(1+jk\pi)}\left[e^{-(1+jk\pi)} - 1\right]$$

$$= \frac{1-e^{-1}e^{jk\pi}}{2(1+jk\pi)}$$

$$\text{Note: } e^{-jk\pi} = \begin{cases} 1, & even\ \ k \\ -1, & odd\ \ k \end{cases}$$

$$\text{or equivalently } \; e^{-jk\pi} = (e^{-j\pi})^k = (-1)^k$$

So…

$$\boxed{c_k = \frac{1-e^{-1}(-1)^k}{2(1+jk\pi)}}$$

Now we can use Matlab to plot $|c_k|$ & $\angle c_k$

Spectrum