BLM2041 Signals and Systems

Syllabus

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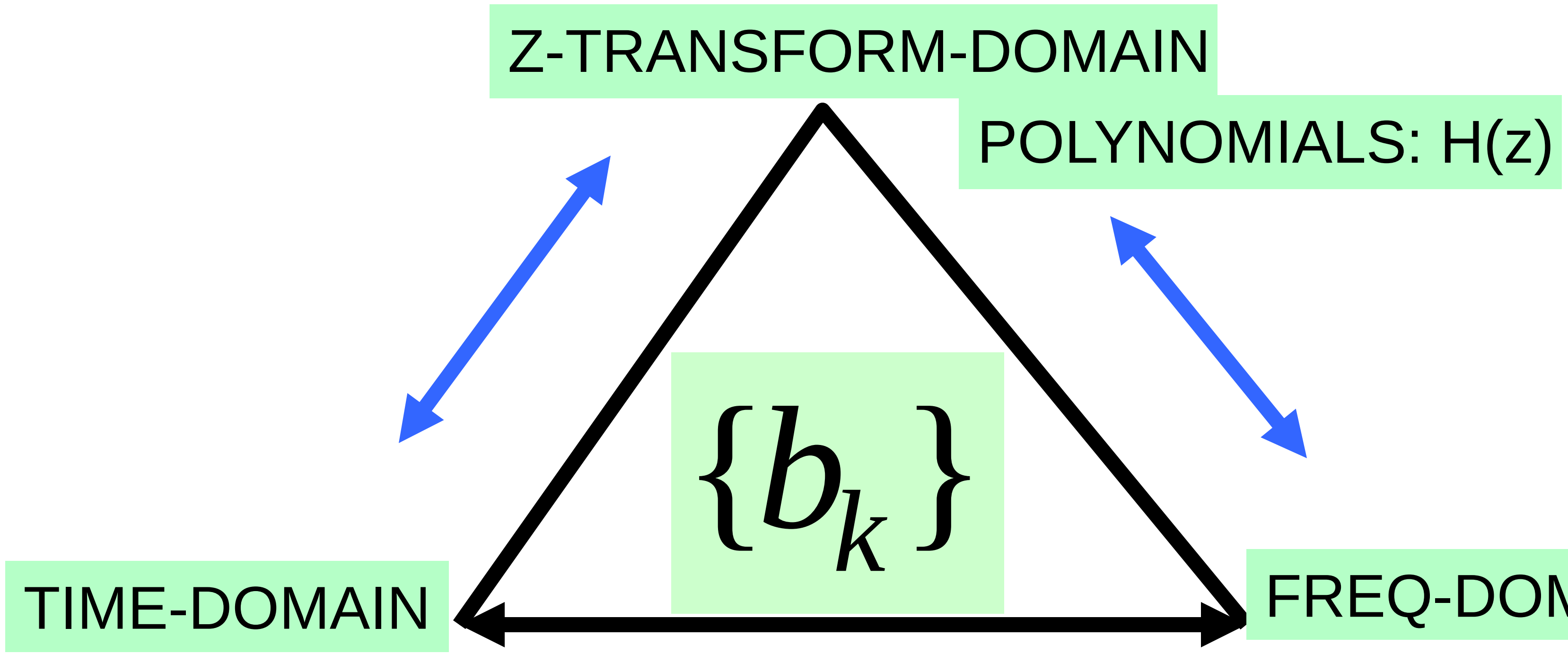
Z Transforms: Introduction

LECTURE OBJECTIVES

- INTRODUCE the Z-TRANSFORM
  - Give Mathematical Definition
  - Show how the H(z) POLYNOMIAL simplifies analysis
    - **CONVOLUTION** is SIMPLIFIED !
- Z-Transform can be applied to
  - FIR Filter: $h[n] \rightarrow H(z)$
  - Signals: $x[n] \rightarrow X(z)$

$$H(z) = \sum_{n} h[n] z^{-n}$$

THREE DOMAINS



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

Three main reasons for Z-Transform

- Offers compact and convenient notation for describing digital signals and systems
- Widely used by DSP designers, and in the DSP literature
- Pole-zero description of a processor is a great help in visualizing its stability and frequency response characteristic

TRANSFORM CONCEPT

- Move to a new domain where
  - OPERATIONS are EASIER & FAMILIAR
  - Use POLYNOMIALS

- TRANSFORM both ways
  - $x[n] \to X(z)$ (into the z domain)
  - $X(z) \to x[n]$ (back to the time domain)

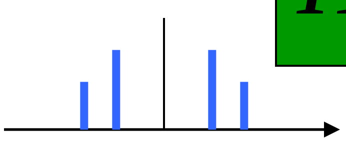
"TRANSFORM" EXAMPLE

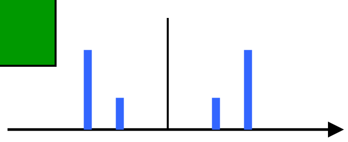
- Equivalent Representations

$x[n]$ → $h[n] = \delta[n] - \delta[n-1]$ → $y[n]$

$$H(e^{j\hat{\omega}}) = \sum_n h[n] e^{-j\hat{\omega}n}$$

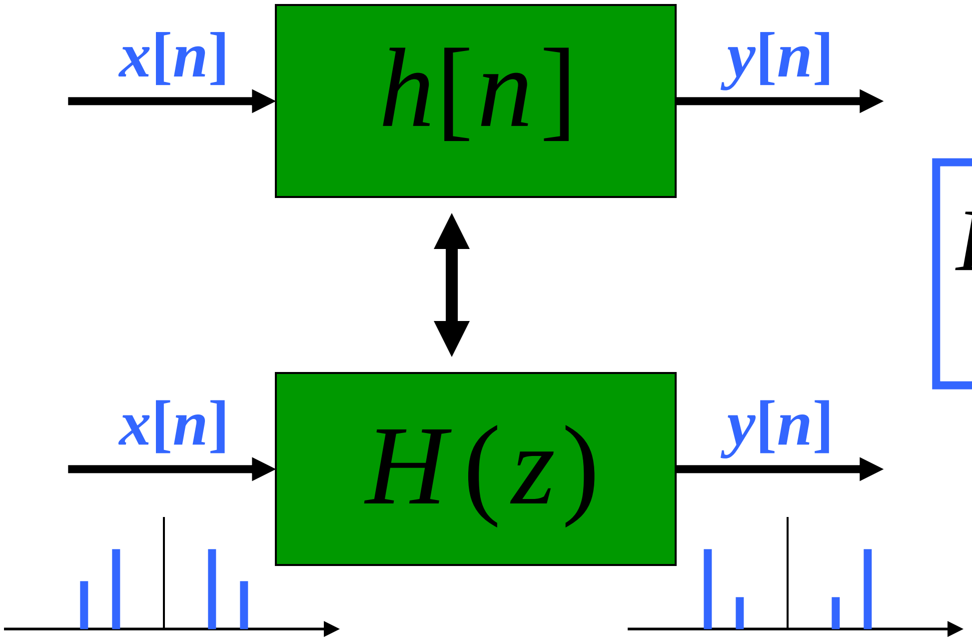
$x[n]$ → $H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}$ → $y[n]$





Z-TRANSFORM IDEA

- **<u>POLYNOMIAL</u>** REPRESENTATION



$$\boxed{H(z) = \sum_n h[n] z^{-n}}$$

Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM:

$$\boxed{H(z) = \sum_{n} h[n] z^{-n}}$$

APPLIES to Any SIGNAL

- EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$\begin{aligned} H(z) &= 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4} \\ &= 2 - 3z^{-2} + 2z^{-4} \\ &= 2 - 3(z^{-1})^2 + 2(z^{-1})^4 \end{aligned}$$

POLYNOMIAL in z^{-1}

Z-Transform EXAMPLE

- ANY SIGNAL has a z-Transform:

$$\boxed{X(z) = \sum_{n} x[n] z^{-n}}$$

Example 7.1

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ?$$

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Z-Transform EXAMPLE

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

EXPONENT GIVES TIME LOCATION

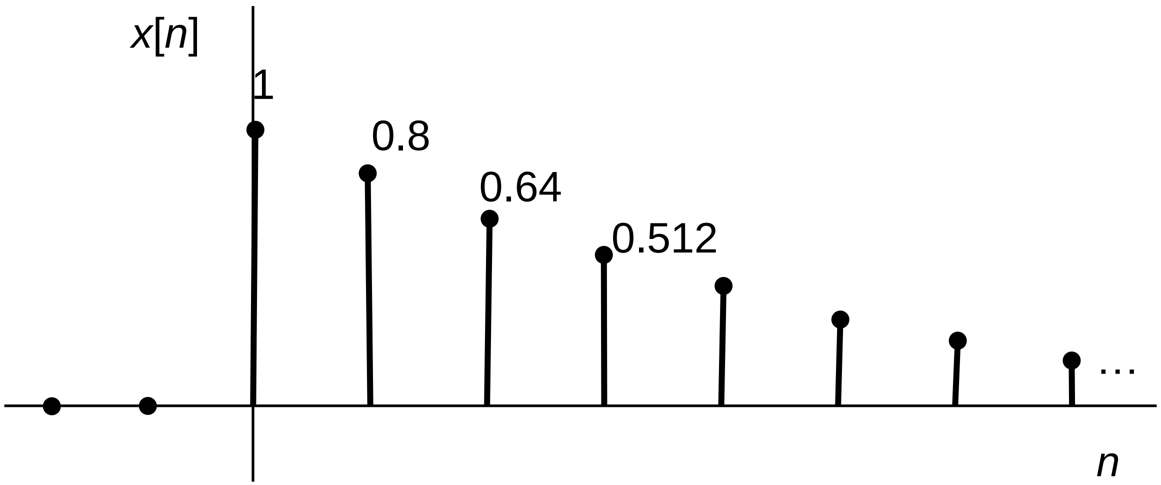
$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

$$x[n] = ?$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

Example

- Find the Z-Transform of the exponentially decaying signal shown in the following figure, expressing is as compact as possible.



Example

- The Z-Transform of the signal:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$= 1 + 0.8z^{-1} + 0.64z^{-2} + 0.512z^{-3} + \cdots$$

$$= 1 + (0.8z^{-1}) + (0.64z^{-1})^2 + (0.512z^{-1})^3 + \cdots$$

$$= \frac{1}{1 - 0.8z^{-1}} = \frac{z}{z - 0.8}$$

Example

- Find and sketch, the signal corresponding to the Z-Transform:

$$X(z) = \frac{1}{z+1.2}$$

Example

- Recasting X(z) as a power series in z^{-1}, we obtain:

$$X(z) = \frac{1}{(z+1.2)} = \frac{z^{-1}}{(1+1.2z^{-1})} = z^{-1}(1+1.2z^{-1})^{-1}$$

$$= z^{-1}\{1+(-1.2z^{-1})+(-1.2z^{-1})^2+(-1.2z^{-1})^3+\cdots\}$$

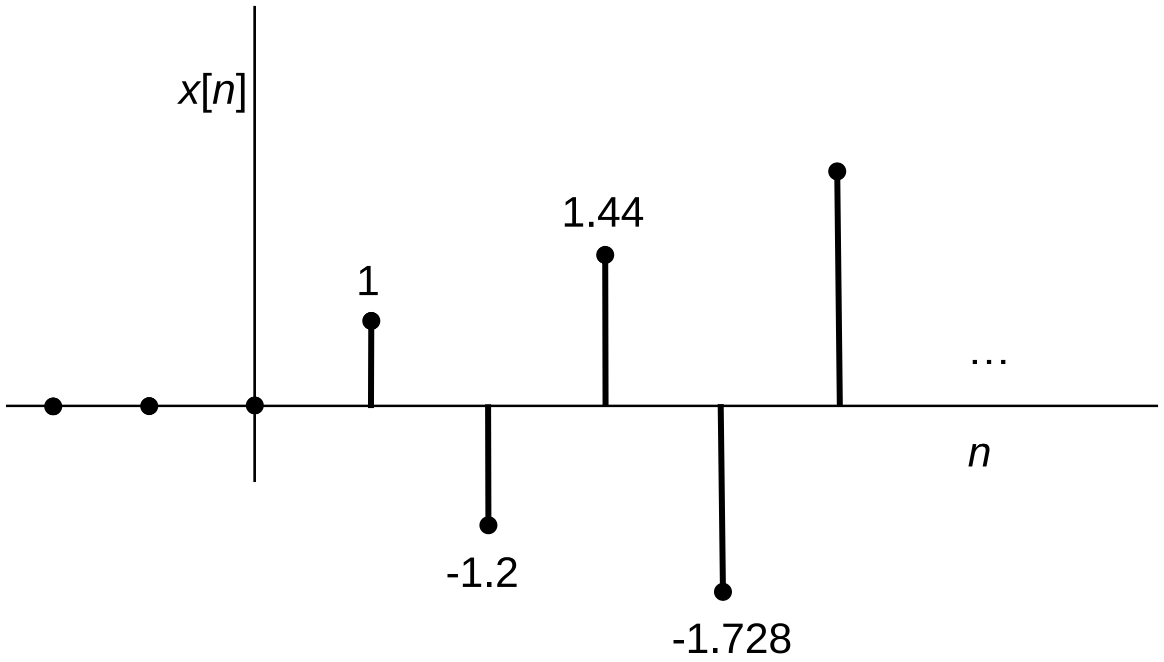
$$= z^{-1} - 1.2z^{-2} + 1.44z^{-3} - 1.728z^{-4} + \cdots$$

- Succesive values of $x[n]$, starting at n=0, are therefore:

0, 1, -1.2, 1.44, -1.728, $\cdots$

Example

- $x[n]$ is shown in the following figure:



Z-Transform of FIR Filter

- CALLED the **SYSTEM FUNCTION**
  - $h[n]$ is same as $\{b_k\}$

SYSTEM FUNCTION

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k] x[n-k]$$

FIR DIFFERENCE EQUATION **CONVOLUTION**

Z-Transform of FIR Filter

- Get $H(z)$ DIRECTLY from the $\{b_k\}$
- Example 7.3 in the book:

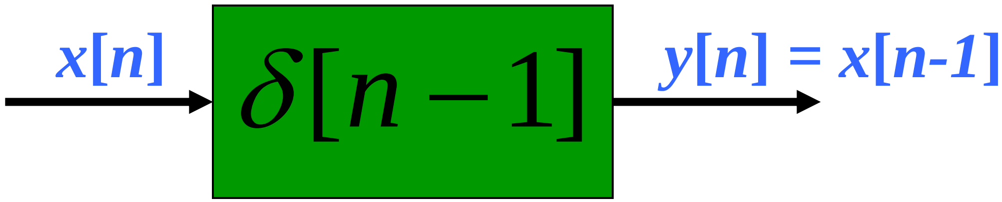
$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

$$\{b_k\} = \{6, -5, 1\}$$

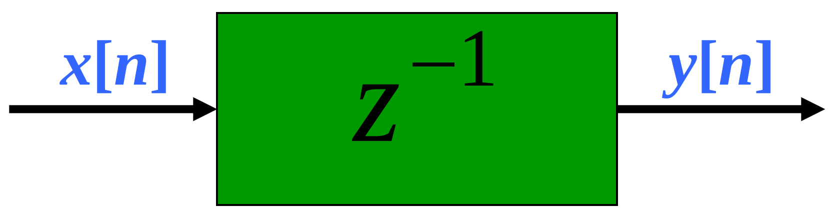
$$H(z) = \sum b_k z^{-1} = 6 - 5z^{-1} + z^{-2}$$

Ex. DELAY SYSTEM

- UNIT DELAY: find $h[n]$ and $H(z)$



$$H(z) = \sum \delta[n-1] z^{-n} = z^{-1}$$



DELAY EXAMPLE

- UNIT DELAY: find $y[n]$ via polynomials
  - x[n] = {3,1,4,1,5,9,0,0,0,...}

$$Y(z) = z^{-1} X(z)$$

$$Y(z) = z^{-1}(3 + z^{-1} + 4z^{-2} + z^{-3} + 5z^{-4} + 9z^{-5})$$

$$Y(z) = 0z^{0} + 3z^{-1} + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5} + 9z^{-6}$$

n	$n < 0$	0	1	2	3	4	5	6	$n > 6$
$y[n]$	0	0	3	1	4	1	5	9	0

DELAY PROPERTY

A delay of one sample multiplies the z-transform by z^{-1}.

$$x[n-1] \quad \Longleftrightarrow \quad z^{-1}X(z)$$

Time delay of n_0 *samples multiplies the z-transform by* z^{-n_0}

$$x[n-n_0] \quad \Longleftrightarrow \quad z^{-n_0}X(z)$$

GENERAL I/O PROBLEM

- Input is $x[n]$, find $y[n]$ (for FIR, $h[n]$)
- How to combine $X(z)$ and $H(z)$?

Example 7.5

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

$$\text{and} \quad h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

$$\text{and} \quad H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

FIR Filter = CONVOLUTION

```
x[n], X(z)      0    +1    -1    +1    -1
h[n], H(z)      1     2     3     4
              ------------------------------------------
                0    +1    -1    +1    -1
                      0    +2    -2    +2    -2
                            0    +3    -3    +3    -3
                                  0    +4    -4    +4    -4
              ------------------------------------------
 y[n], Y(z)     0    +1    +1    +2    +2    -3    +1    -4
```

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

CONVOLUTION

CONVOLUTION PROPERTY

- PROOF:

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

$$Y(z) = \sum_{k=0}^{M} h[k]\left(z^{-k}X(z)\right)$$

$$= \left(\sum_{k=0}^{M} h[k]z^{-k}\right)X(z) = H(z)X(z).$$

MULTIPLY Z-TRANSFORMS

CONVOLUTION EXAMPLE

- **MULTIPLY** the z-TRANSFORMS:

Example 7.5

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

and $$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

and $$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

MULTIPLY $\boldsymbol{H(z)X(z)}$

CONVOLUTION EXAMPLE

- Finite-Length input $x[n]$
- FIR Filter (L=4)

MULTIPLY Z-TRANSFORMS

$$Y(z) = H(z)X(z)$$

$$= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4})$$

$$= z^{-1} + (-1 + 2)z^{-2} + (1 - 2 + 3)z^{-3} + (-1 + 2 - 3 + 4)z^{-4}$$

$$+ \ (-2 + 3 - 4)z^{-5} + (-3 + 4)z^{-6} + (-4)z^{-7}$$

$$= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7}$$

$y[n]$ = ?

CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - Remember: $h_1[n] * h_2[n]$
  - How to combine $H_1(z)$ and $H_2(z)$?

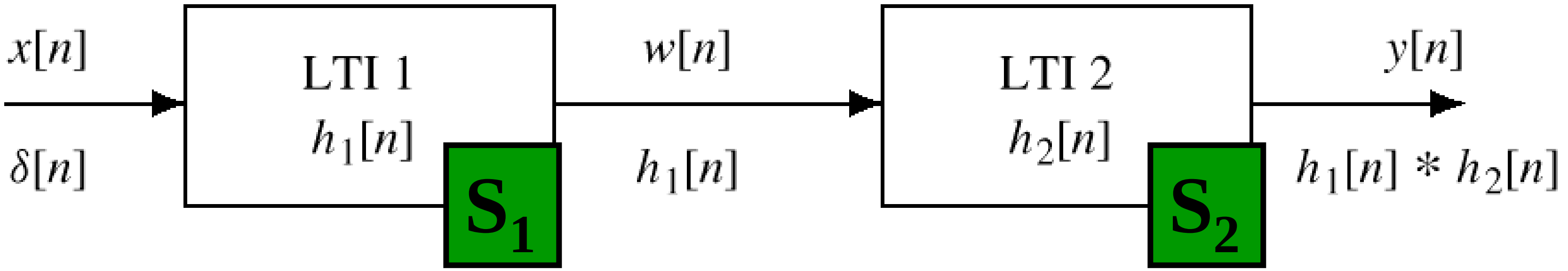
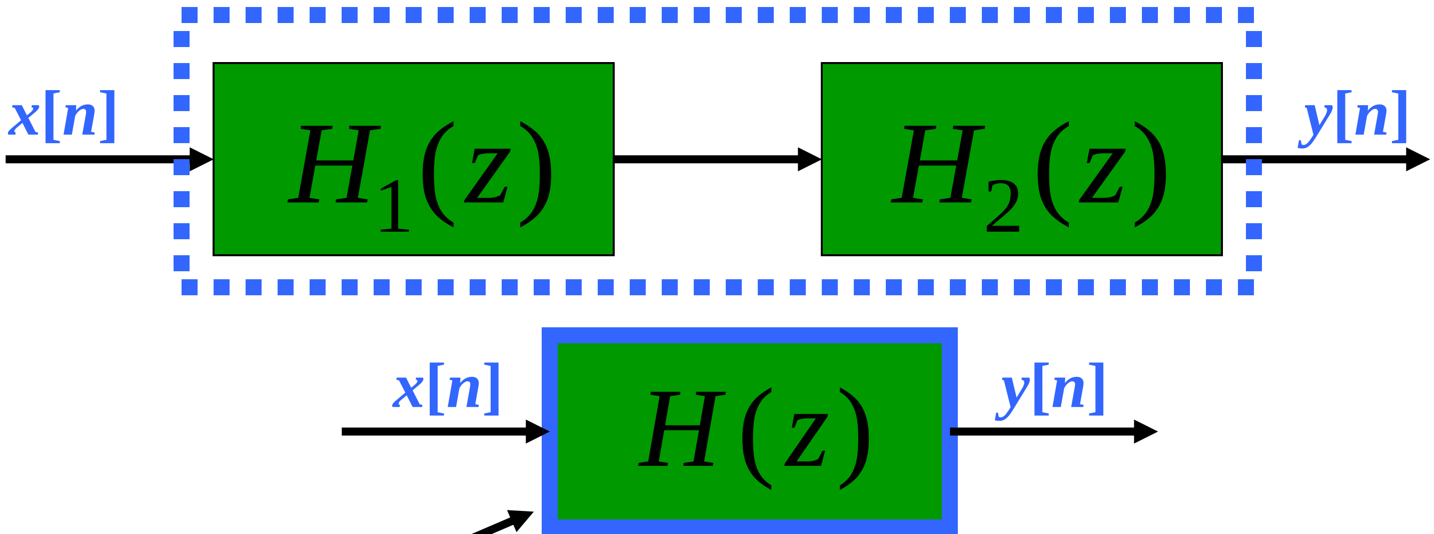


Figure 5.19 A Cascade of Two LTI Systems.

CASCADE EQUIVALENT

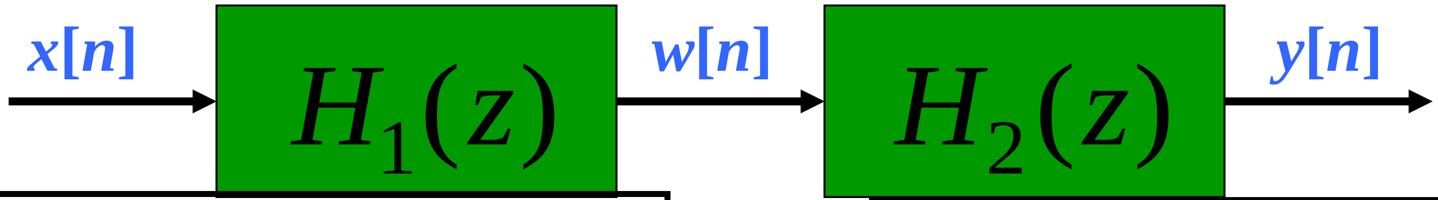
- Multiply the System Functions



EQUIVALENT SYSTEM

$$\boxed{H(z) = H_1(z)H_2(z)}$$

CASCADE EXAMPLE

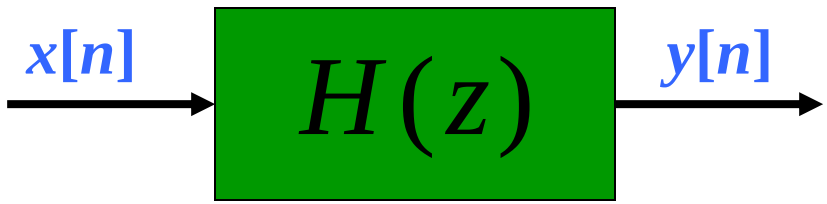


$$\boxed{w[n] = x[n] - x[n-1]}$$

$$\boxed{y[n] = w[n] + w[n-1]}$$

$$\boxed{H_1(z) = 1 - z^{-1}}$$

$$\boxed{H_2(z) = 1 + z^{-1}}$$



$$\boxed{H(z) = (1 - z^{-1})(1 + z^{-1}) = 1 - z^{-2}}$$

$$\boxed{y[n] = x[n] - x[n-2]}$$

Zeros of H(z) and the Frequency Domain

LECTURE OBJECTIVES

- ZEROS and POLES
- Relate H(z) to FREQUENCY RESPONSE

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE DOMAINS:
  - Show Relationship for FIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

DESIGN PROBLEM

- Example:
  - Design a Lowpass FIR filter
    - Find b_k
  - Reject completely 0.7π, 0.8π, and 0.9π
    - This is NULLING
  - Estimate the filter length needed to accomplish this task.
    - How many b_k ?

- Z POLYNOMIALS provide the TOOLS

Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM: $$\boxed{H(z)=\sum_{n} h[n]z^{-n}}$$
- EXAMPLE:

APPLIES to Any SIGNAL

$$\{h[n]\}=\{2,0,-3,0,2\}$$

$$H(z)=2z^{-0}+0z^{-1}-3z^{-2}+0z^{-3}+2z^{-4}$$
$$=2-3z^{-2}+2z^{-4}$$
$$=2-3(z^{-1})^{2}+2(z^{-1})^{4}$$

POLYNOMIAL in z^{-1}

CONVOLUTION PROPERTY

- Convolution in the n-domain
  - SAME AS
- Multiplication in the z-domain

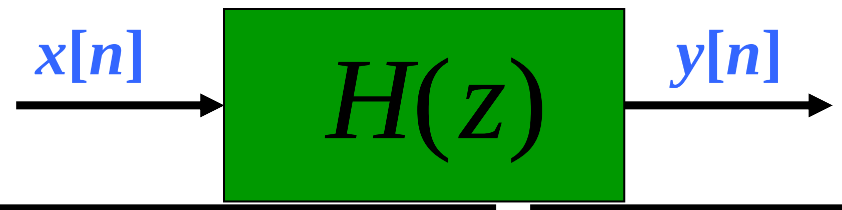
$$y[n] = h[n] * x[n] \quad \leftrightarrow \quad Y(z) = H(z)X(z)$$

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=0}^{M} h[k]x[n-k]$$

FIR Filter

MULTIPLY z-TRANSFORMS

CONVOLUTION EXAMPLE



$$x[n] = \delta[n-1] + 2\delta[n-2]$$

$$h[n] = \delta[n] - \delta[n-1]$$

$$y[n] = x[n] * h[n]$$

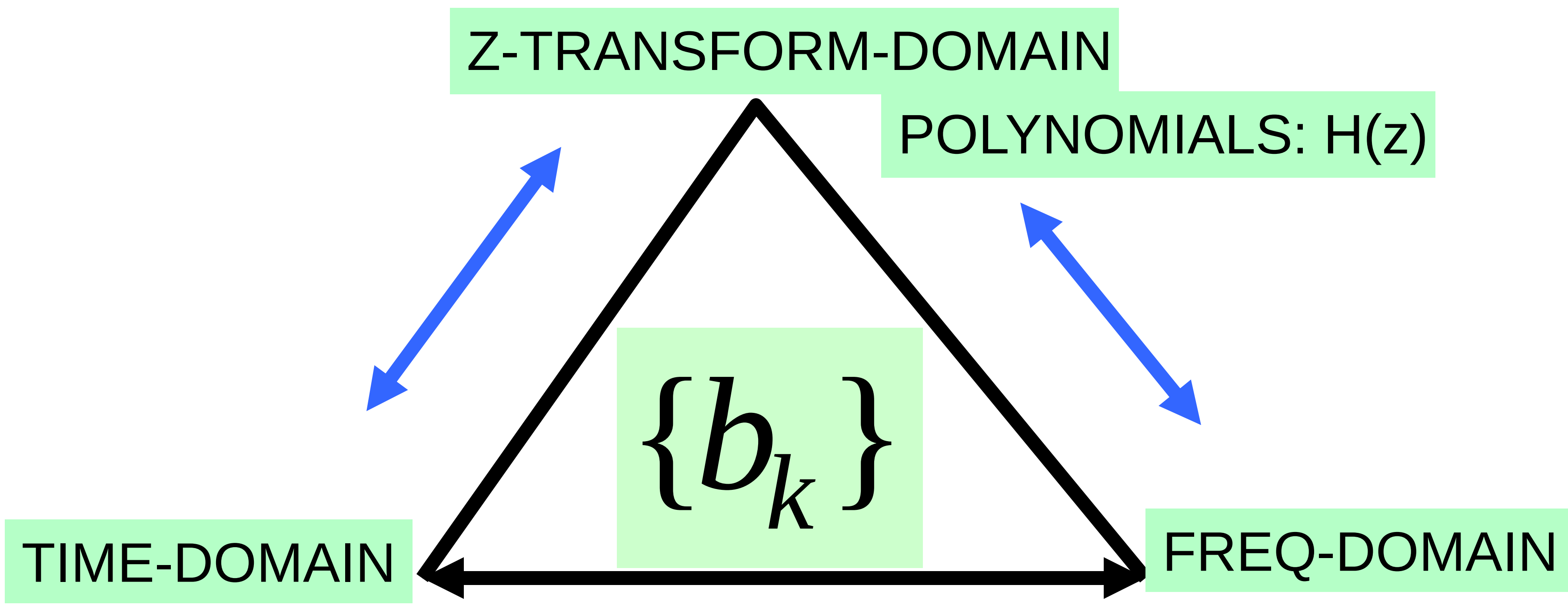
$$X(z) = z^{-1} + 2z^{-2}$$

$$H(z) = 1 - z^{-1}$$

$$Y(z) = (z^{-1} + 2z^{-2})(1 - z^{-1}) = z^{-1} + z^{-2} - 2z^{-3}$$

$$y[n] = \delta[n-1] + \delta[n-2] - 2\delta[n-3]$$

THREE DOMAINS



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

FREQUENCY RESPONSE ?

- Same Form:

$\hat{\omega}$ – Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k (e^{j\hat{\omega}})^{-k}$$

$$z = e^{j\hat{\omega}}$$

z – Domain

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

SAME COEFFICIENTS

ANOTHER ANALYSIS TOOL

- z-Transform POLYNOMIALS are EASY !
  - ROOTS, FACTORS, etc.
- **ZEROS and POLES: where is $H(z) = 0$?**
- The z-domain is COMPLEX
  - $H(z)$ is a COMPLEX-VALUED function of a COMPLEX VARIABLE z.

ZEROS of $H(z)$

- Find z, where $H(z)=0$

$$H(z) = 1 - \tfrac{1}{2} z^{-1}$$

$$1 - \tfrac{1}{2} z^{-1} = 0 \,?$$

$$z - \tfrac{1}{2} = 0$$

$$\boxed{\text{Zero at}: z = \tfrac{1}{2}}$$

ZEROS of $H(z)$

- Find z, where $H(z)$=0
  - Interesting when z is ON the unit circle.

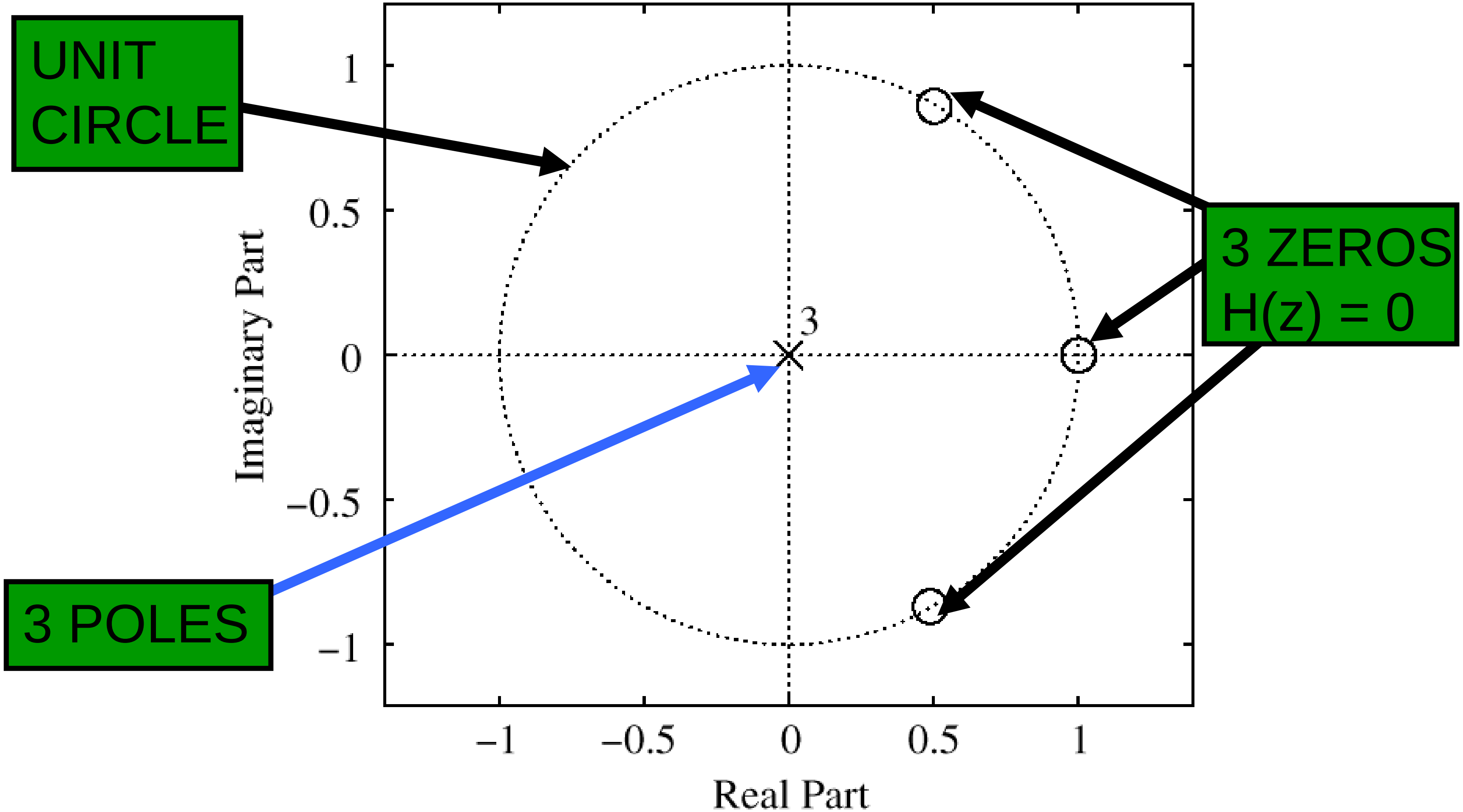
$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})$$

$$\text{Roots} : z = 1, \tfrac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$e^{\pm j\pi/3}$$

PLOT ZEROS in z-DOMAIN



POLES of H(z)

- Find *z*, where $H(z) \to \infty$
  - Not very interesting for the FIR case

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = \frac{z^3 - 2z^2 + 2z - 1}{z^3}$$

$$\boxed{\text{Three Poles at}: z = 0}$$

FREQ. RESPONSE from ZEROS

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

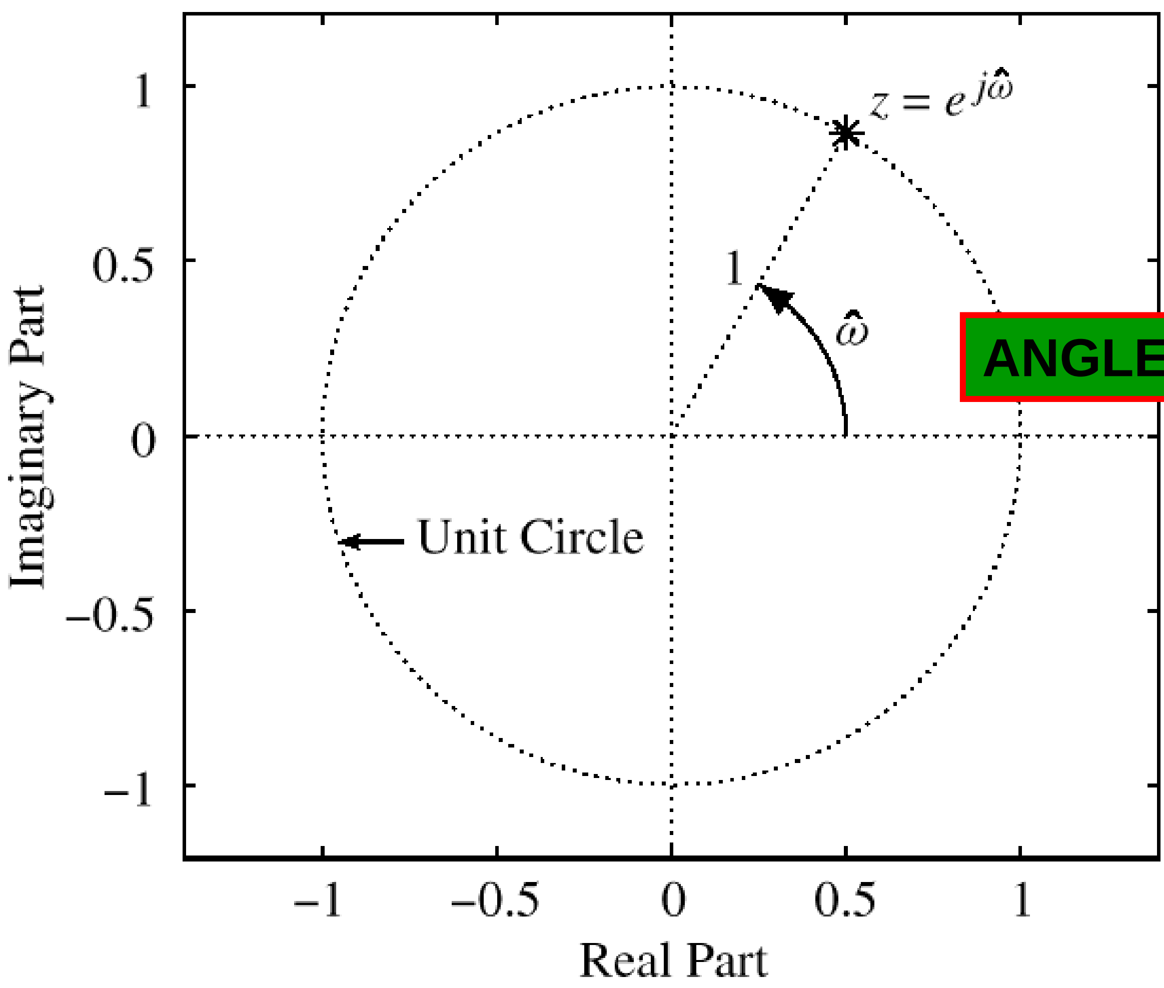
- Relate $H(z)$ to FREQUENCY RESPONSE
- EVALUATE $H(z)$ on the **UNIT CIRCLE**
  - ANGLE is same as FREQUENCY

$$z = e^{j\hat{\omega}} \quad \text{(as } \hat{\omega} \text{ varies)}$$

$$\text{defines a CIRCLE, radius} = 1$$

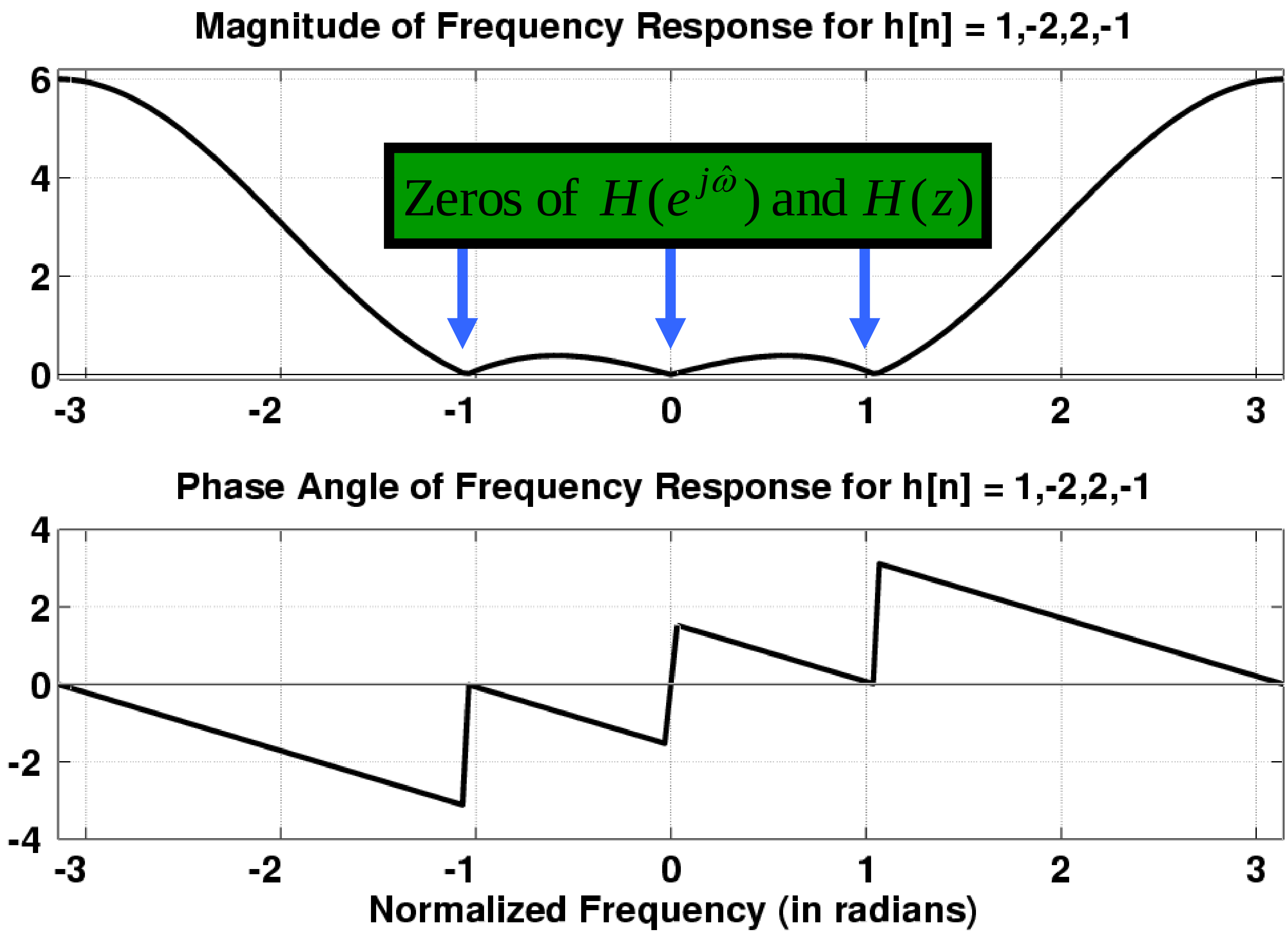
$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

The Complex z-Plane



ANGLE is FREQUENCY

FIR Frequency Response



NULLING PROPERTY of *H*(*z*)

- When $H(z)=0$ on the unit circle.
  - Find inputs $x[n]$ that give zero output

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

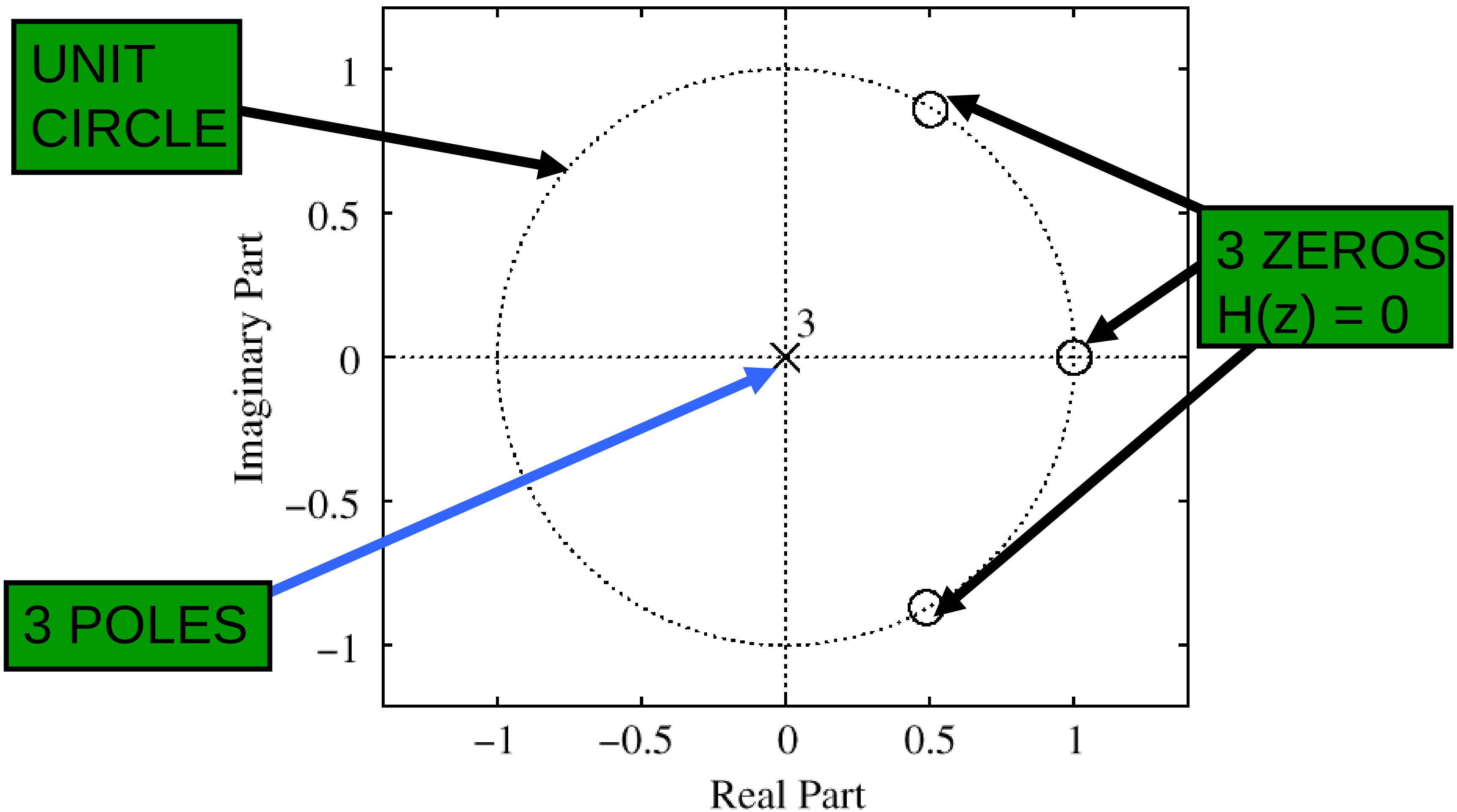
$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$

$$\boxed{H(e^{j\pi/3}) = ?}$$

$$x[n] = e^{j(\pi/3)n}$$

$$y[n] = H(e^{j(\pi/3)}) \cdot e^{j(\pi/3)n}$$

PLOT ZEROS in z-DOMAIN



NULLING PROPERTY of $H(z)$

- Evaluate $H(z)$ at the input "frequency"

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

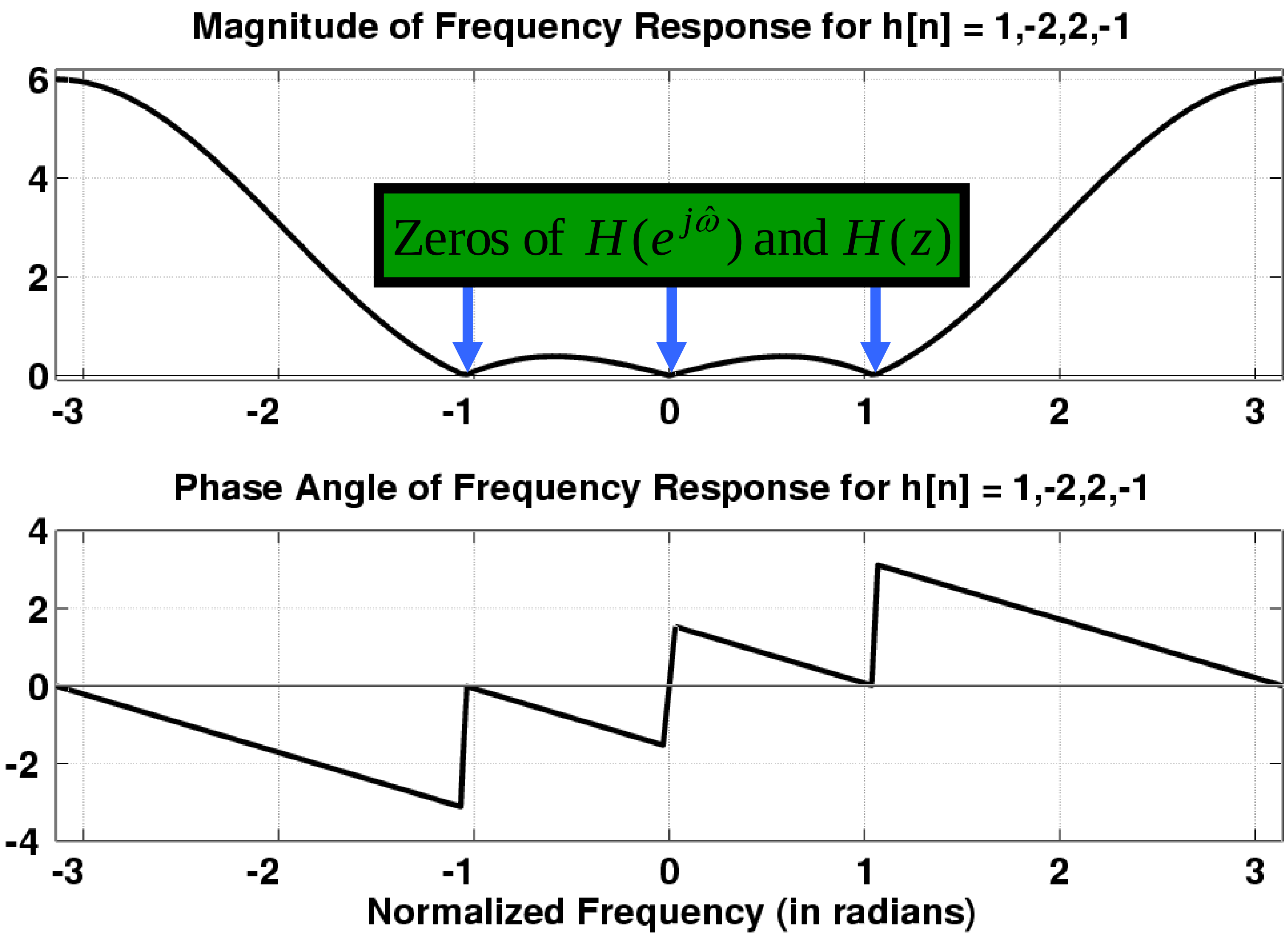
$$y[n] = H(e^{j\pi/3}) \cdot e^{j(\pi/3)n}$$

$$y[n] = (1 - 2e^{-j\pi/3} + 2e^{-j2\pi/3} - e^{-j3\pi/3}) \cdot e^{j(\pi/3)n}$$

$$(1 - 2(\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) + 2(-\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) - (-1))$$

$$y[n] = (1 - 1 + j\sqrt{3} - 1 - j\sqrt{3} + 1) \cdot e^{j(\pi/3)n} = 0$$

FIR Frequency Response



DESIGN PROBLEM

- Example:
  - Design a Lowpass FIR filter
    - Find b_k
  - Reject completely 0.7π, 0.8π, and 0.9π
  - Estimate the filter length needed to accomplish this task.
    - How many b_k

- Z POLYNOMIALS provide the TOOLS

IIR Filters: Feedback and H(z)

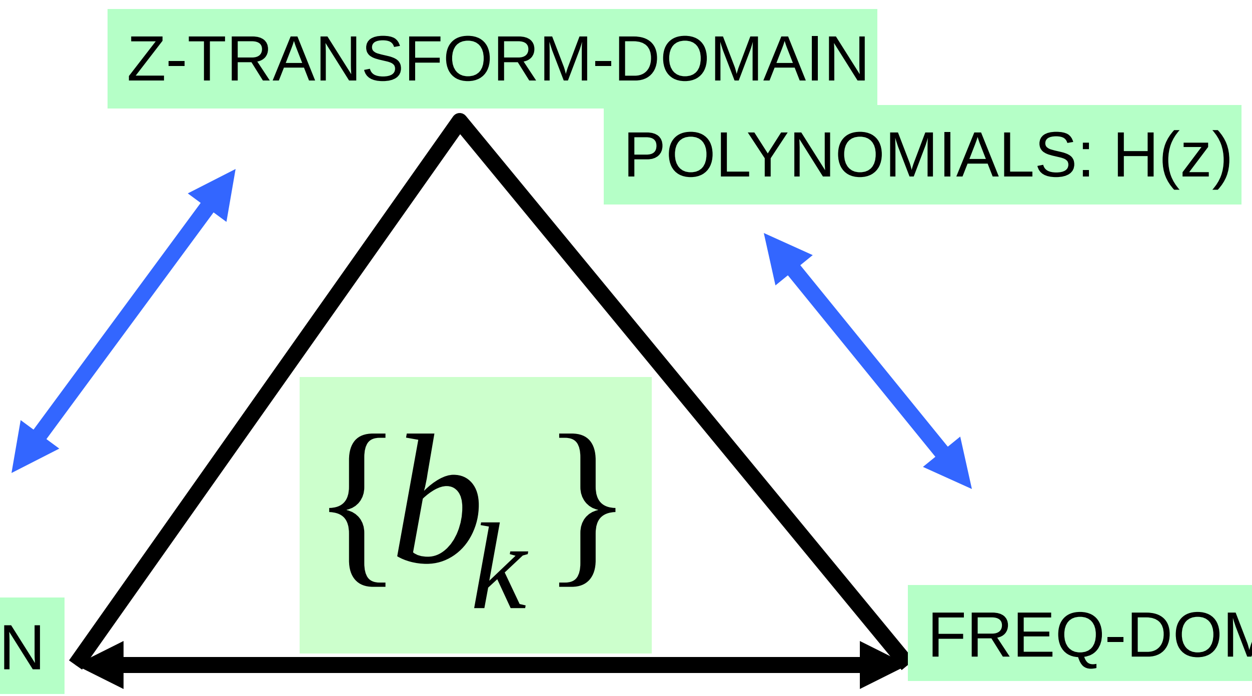
LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
  - Define **IIR** DIGITAL Filters
- Have **FEEDBACK**: use PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

– Show how to compute the output $y[n]$
  - FIRST-ORDER CASE (N=1)
  - Z-transform: Impulse Response $h[n]$ ←→ $H(z)$

THREE DOMAINS



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

Quick Review: Delay by n_d

$$y[n] = x[n - n_d]$$

IMPULSE RESPONSE

$$h[n] = \delta[n - n_d]$$

SYSTEM FUNCTION

$$H(z) = z^{-n_d}$$

FREQUENCY RESPONSE

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega} n_d}$$

LOGICAL THREAD

- FIND the IMPULSE RESPONSE, $h[n]$
  - INFINITELY LONG
  - **IIR** Filters

$$\boxed{H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}}$$

- EXPLOIT THREE DOMAINS:
  - Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]}$$

PREVIOUS

FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$\boxed{y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]}$$

FEEDBACK COEFFICIENT

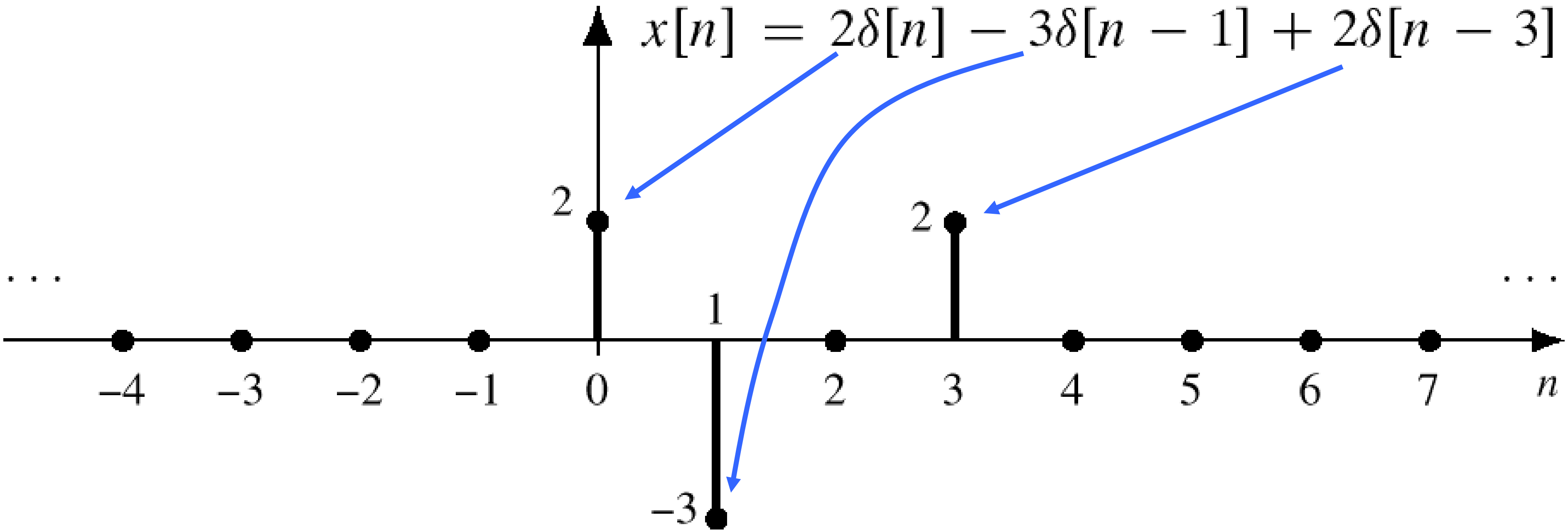
SIGN CHANGE

- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

COMPUTE OUTPUT

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$



COMPUTE $y[n]$

- FEEDBACK DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$

  - NEED y[-1] to get started

$$\boxed{y[0] = 0.8y[-1] + 5x[0]}$$

AT REST CONDITION

- $y[n] = 0$, for $n<0$
- BECAUSE $x[n] = 0$, for $n<0$

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied.*

2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- SAME with MORE FEEDBACK TERMS

$$\boxed{y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]}$$

COMPUTE MORE $y[n]$

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

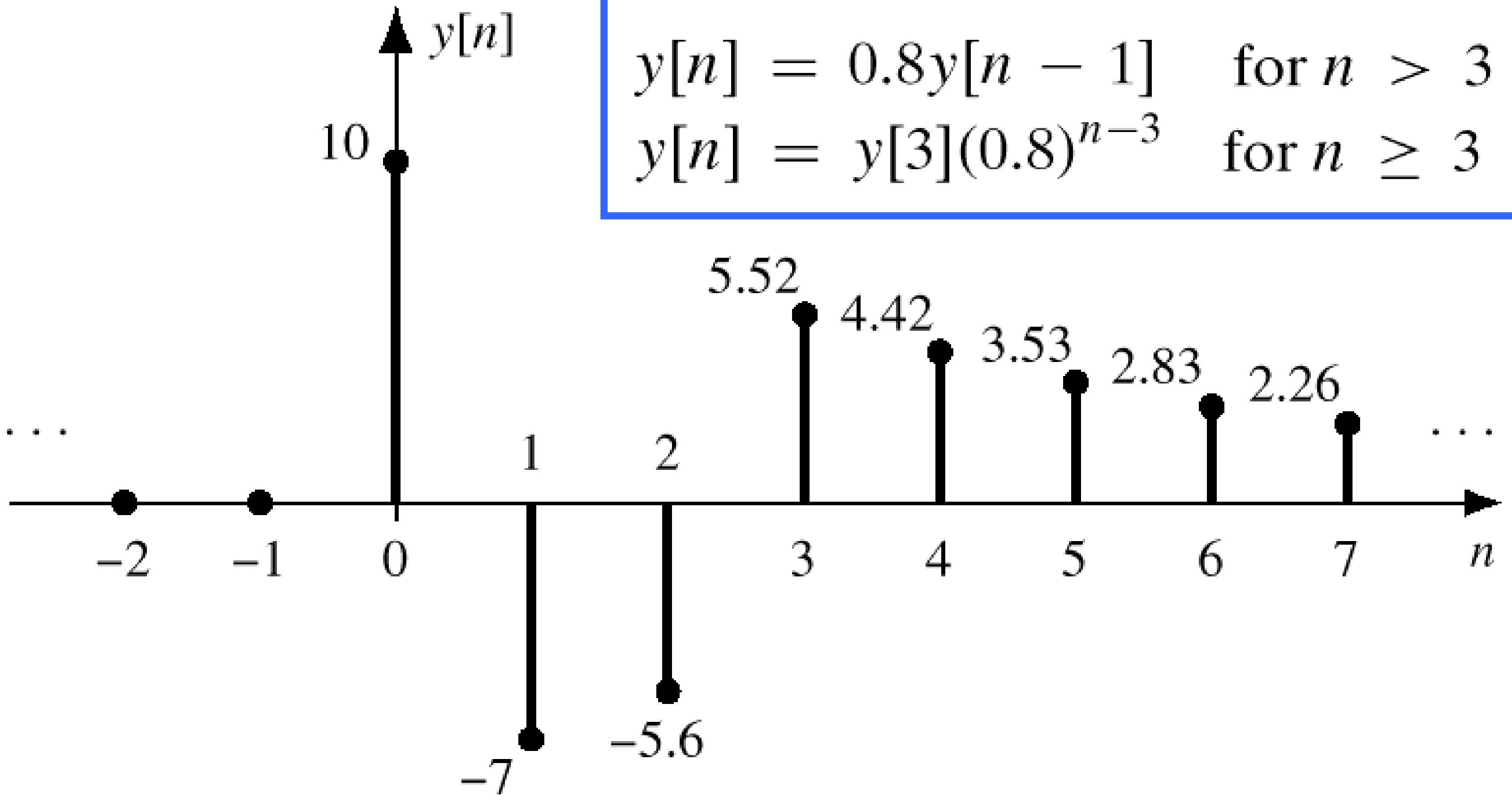
$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLOT $y[n]$



$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$
$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$

IMPULSE RESPONSE

$$\boxed{h[n] = a_1 h[n-1] + b_0 \delta[n]}$$

n	$n < 0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

$$\boxed{u[n] = 1, \quad \text{for } n \geq 0}$$

IMPULSE RESPONSE

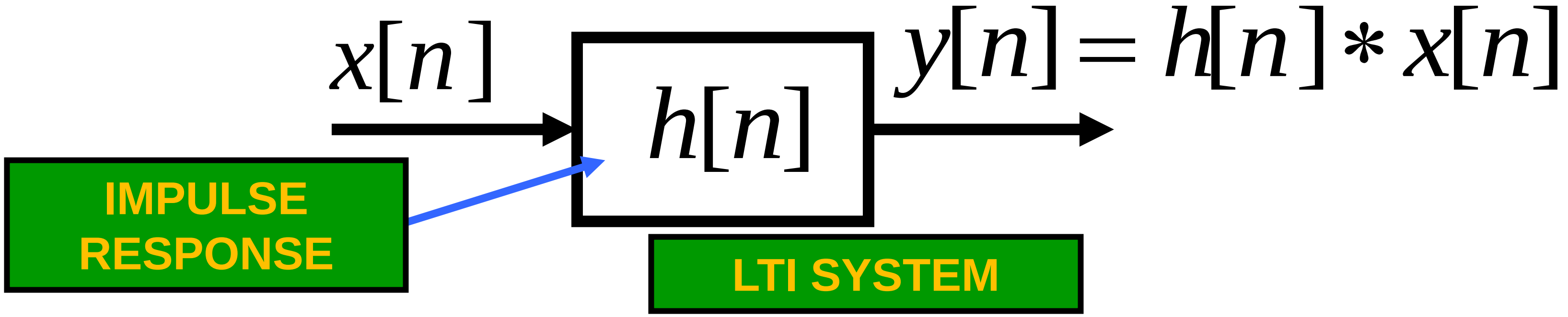
- DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 3x[n]}$$

- Find h[n]

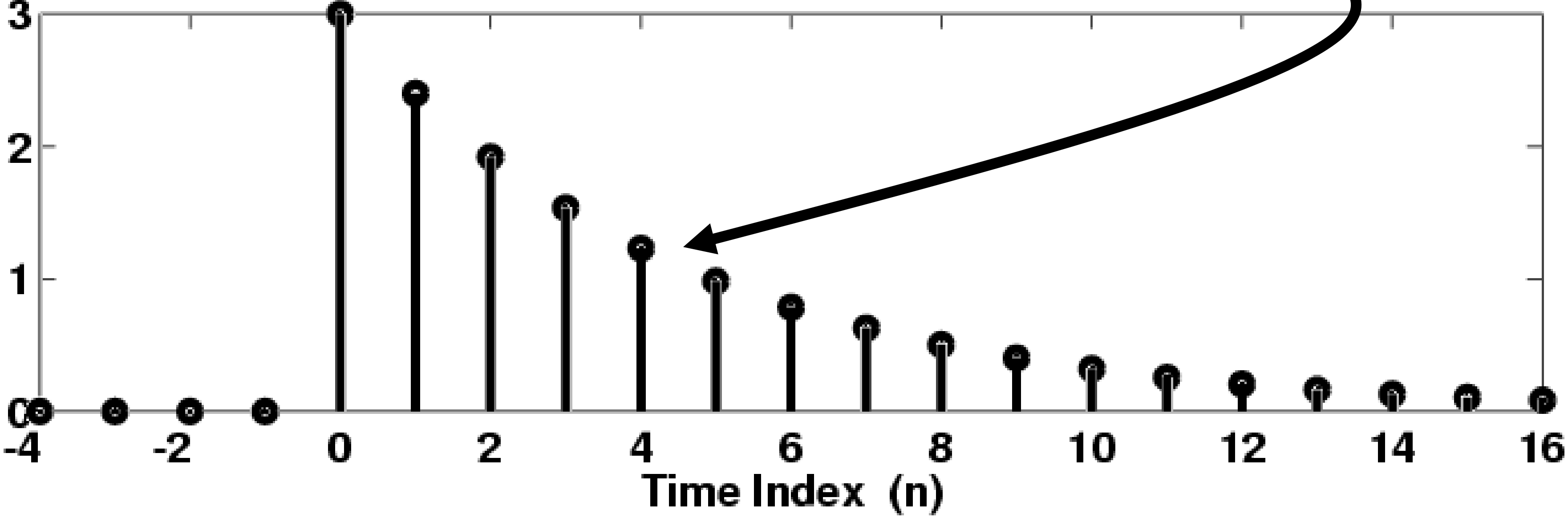
$$\boxed{h[n] = 3(0.8)^n u[n]}$$

- **CONVOLUTION** in TIME-DOMAIN



PLOT IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



Infinite-Length Signal: $h[n]$

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z) = \sum_{n=-\infty}^{\infty} b_0 (a_1)^n u[n] z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n}$$

Derivation of $H(z)$

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

$H(z)$ = z-Transform{ $h[n]$ }

- FIRST-ORDER IIR FILTER:

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n]}$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

$$\boxed{H(z) = \frac{b_0}{1 - a_1 z^{-1}}}$$

$H(z)$ = z-Transform{ $h[n]$ }

- ANOTHER FIRST-ORDER IIR FILTER:

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]}$$

$$\boxed{h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]}$$

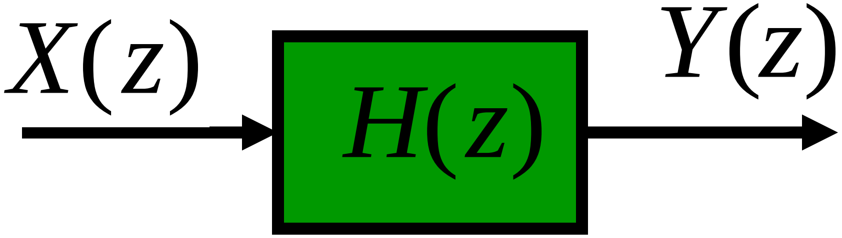
$\boxed{z^{-1} \text{ is a shift}}$

$$\boxed{H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}}$$

CONVOLUTION PROPERTY

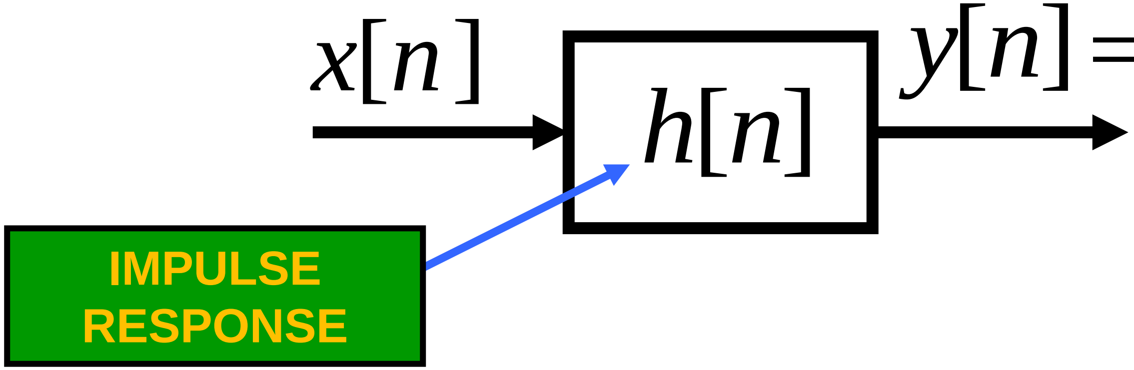
- MULTIPLICATION of z-TRANSFORMS

$$Y(z) = H(z)X(z)$$



- CONVOLUTION in TIME-DOMAIN

$$y[n] = h[n] * x[n]$$



STEP RESPONSE: $x[n]=u[n]$

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

n	$x[n]$	$y[n]$
$n < 0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
$\vdots$	1	$\vdots$

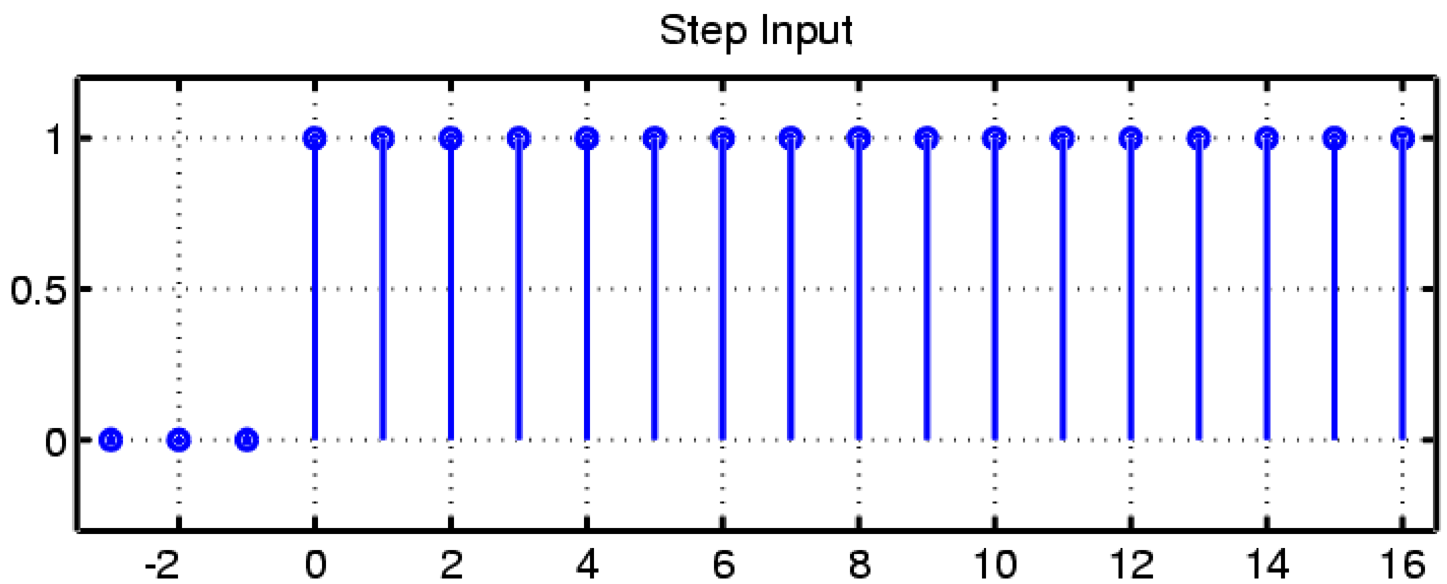
DERIVE STEP RESPONSE

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

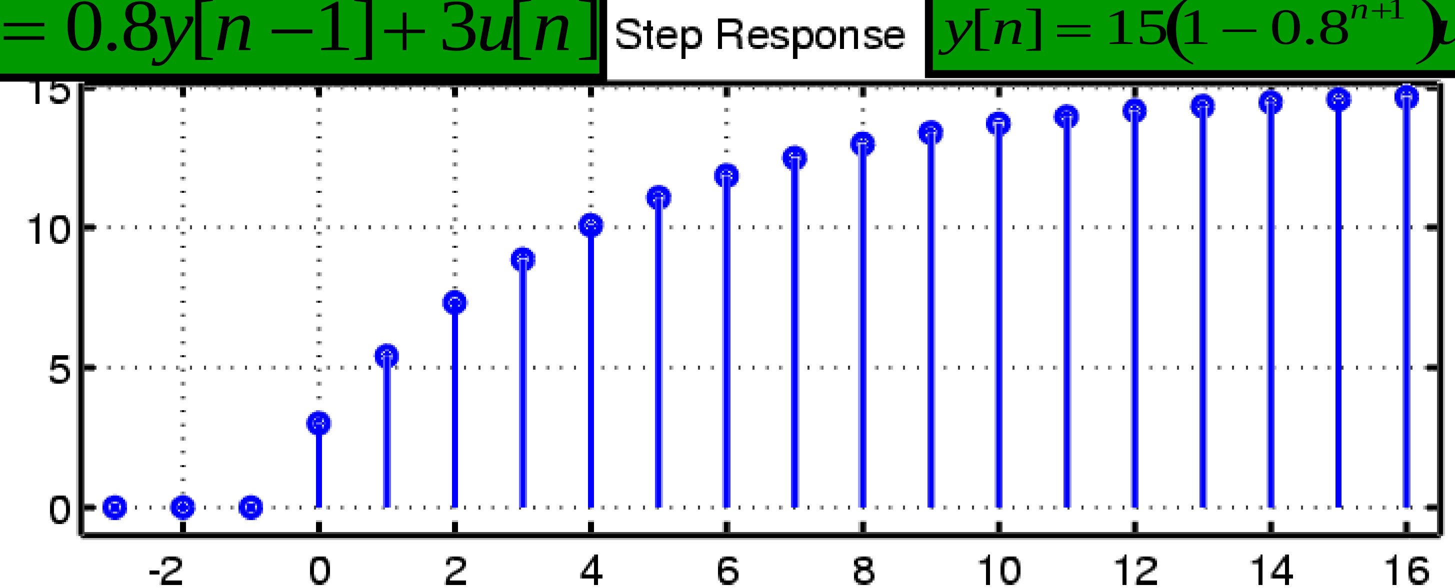
$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \qquad \text{for } n \geq 0, \qquad \text{if } a_1 \neq 1$$

PLOT STEP RESPONSE



$$y[n] = 0.8y[n-1] + 3u[n]$$

$$y[n] = 15\left(1 - 0.8^{n+1}\right)u[n]$$



IIR Filters: H(z) and Frequency Response

LECTURE OBJECTIVES

- SYSTEM FUNCTION: H(z)
- H(z) has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

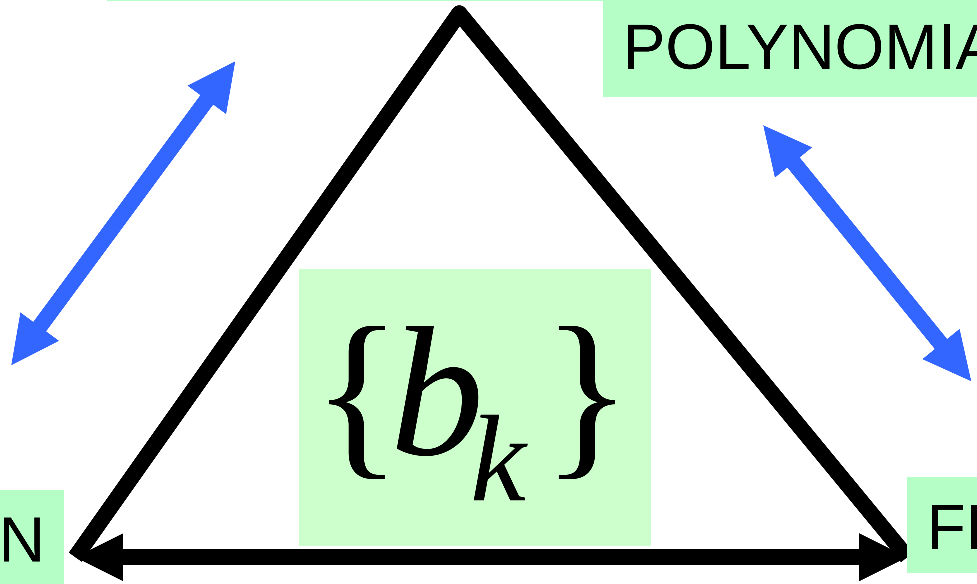
- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

THREE DOMAINS

Z-TRANSFORM-DOMAIN

POLYNOMIALS: H(z)



$$\{b_k\}$$

TIME-DOMAIN

FREQ-DOMAIN

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$H(z)$ = z-Transform{ $h[n]$ }

- FIRST-ORDER IIR FILTER:

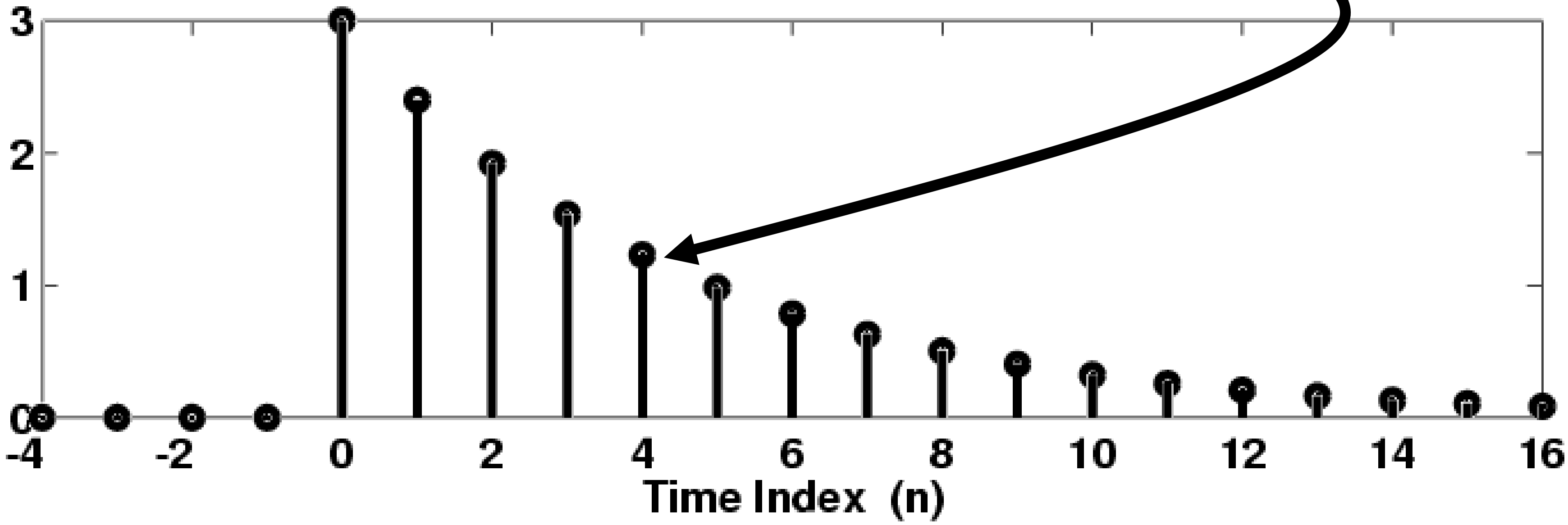
$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n]}$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

$$\boxed{H(z) = \frac{b_0}{1 - a_1 z^{-1}}}$$

Typical IMPULSE Response

$$h[n] = b_0(a_1)^n u[n] = 3(0.8)^n u[n]$$



First-Order Transform Pair

$$h[n] = ba^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - a z^{-1}}$$

- GEOMETRIC SEQUENCE:

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

DELAY PROPERTY of $X(z)$

- DELAY in TIME<-->Multiply $X(z)$ by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1}X(z)$$

Proof: $$\sum_{n=-\infty}^{\infty} x[n-1]z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+1)}$$

$$= z^{-1}\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell} = z^{-1}X(z)$$

Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION $H(z)$
  - Use **DELAY** PROPERTY

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]}$$

$$\boxed{Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)}$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

SYSTEM FUNCTION of IIR

- NOTE the FILTER COEFFICIENTS

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \boxed{\frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}} = \frac{B(z)}{A(z)}$$

SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]}$$

- **READ** the FILTER COEFFS:

H(z)

$$\boxed{Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8z^{-1}}\right) X(z)}$$

CONVOLUTION PROPERTY

- **MULTIPLICATION** of z-TRANSFORMS

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow \quad Y(z) = H(z)X(z)$$

- **CONVOLUTION** in TIME-DOMAIN

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow \quad y[n] = h[n] * x[n]$$

IMPULSE RESPONSE

POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

ZERO: H(z)=0

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

POLE: H(z) → inf

EXAMPLE: Poles & Zeros

- VALUE of $H(z)$ at POLES is **INFINITE**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

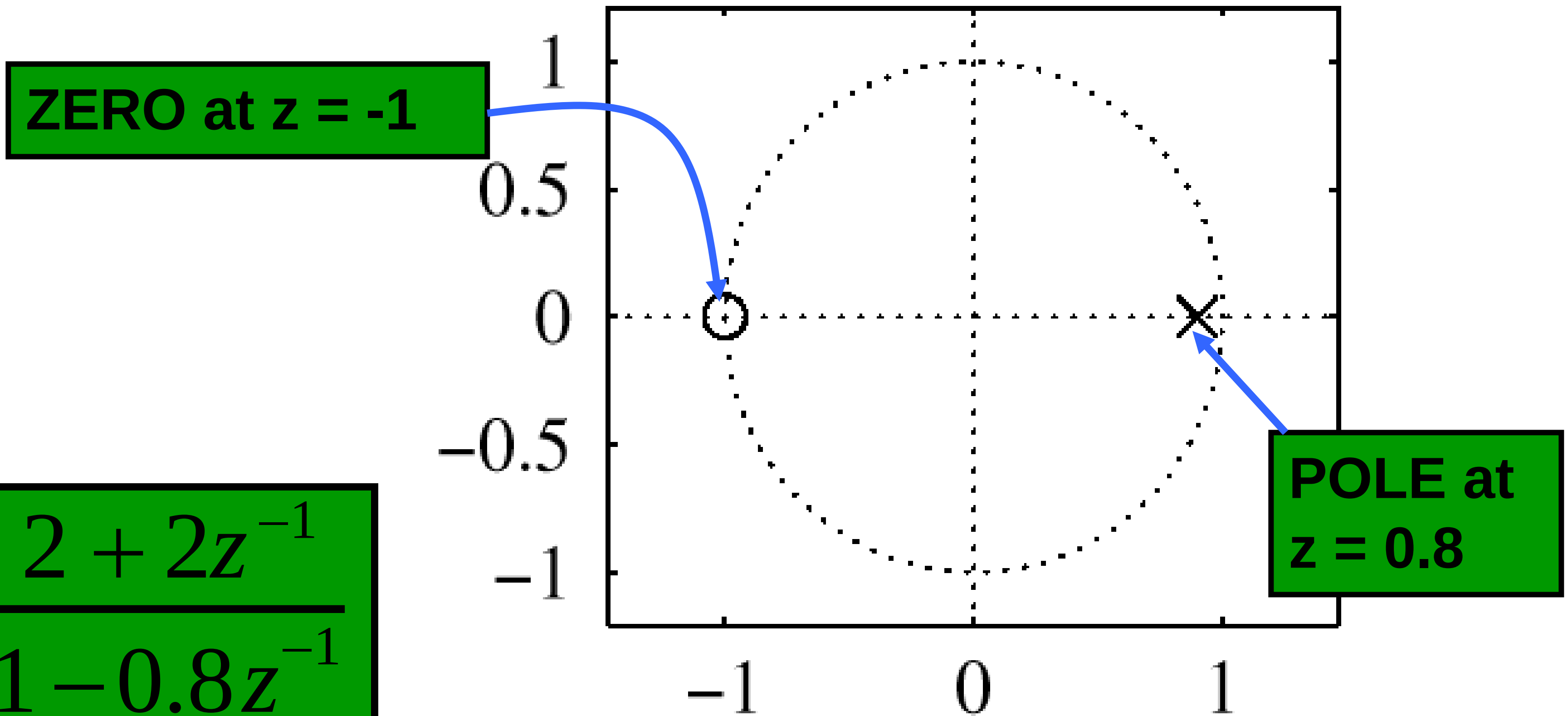
$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

ZERO at z= -1

$$H(z) = \frac{2 + 2(\frac{4}{5})^{-1}}{1 - 0.8(\frac{4}{5})^{-1}} = \frac{\frac{9}{2}}{0} \to \infty$$

POLE at z=0.8

POLE-ZERO PLOT



$$\frac{2+2z^{-1}}{1-0.8z^{-1}}$$

FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

FREQ. RESPONSE FORMULA

$$H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$

$$\left|H(e^{j\hat{\omega}})\right|^2 =$$

$$\left|\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{2+2e^{j\hat{\omega}}}{1-0.8e^{j\hat{\omega}}}$$

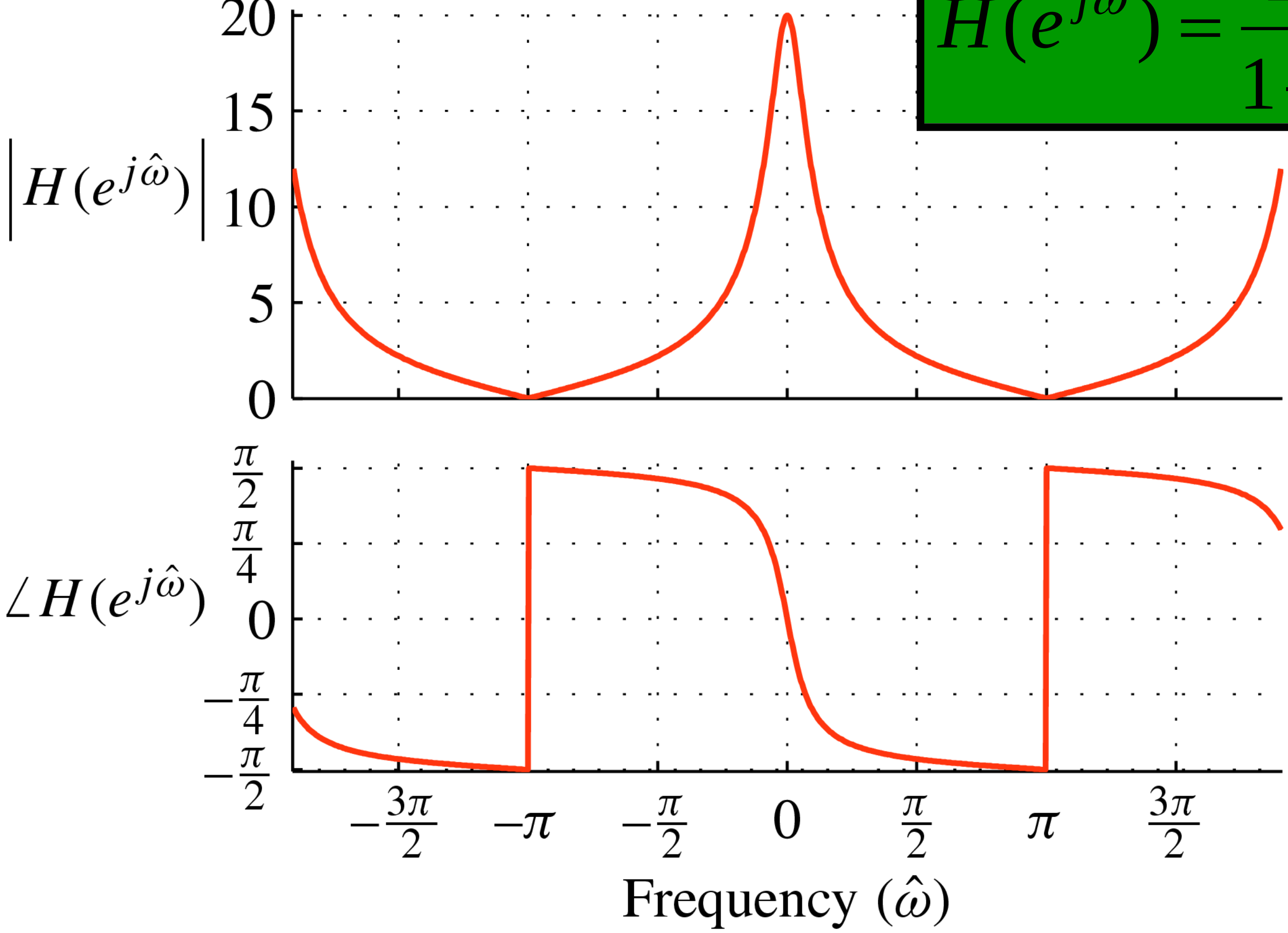
$$\frac{4+4+4e^{-j\hat{\omega}}+4e^{j\hat{\omega}}}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} =$$

$$\frac{8+8\cos\hat{\omega}}{1.64-1.6\cos\hat{\omega}}$$

$$@\,\hat{\omega}=0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{8+8}{0.04} = 400, \quad @\,\hat{\omega}=\pi\,?$$

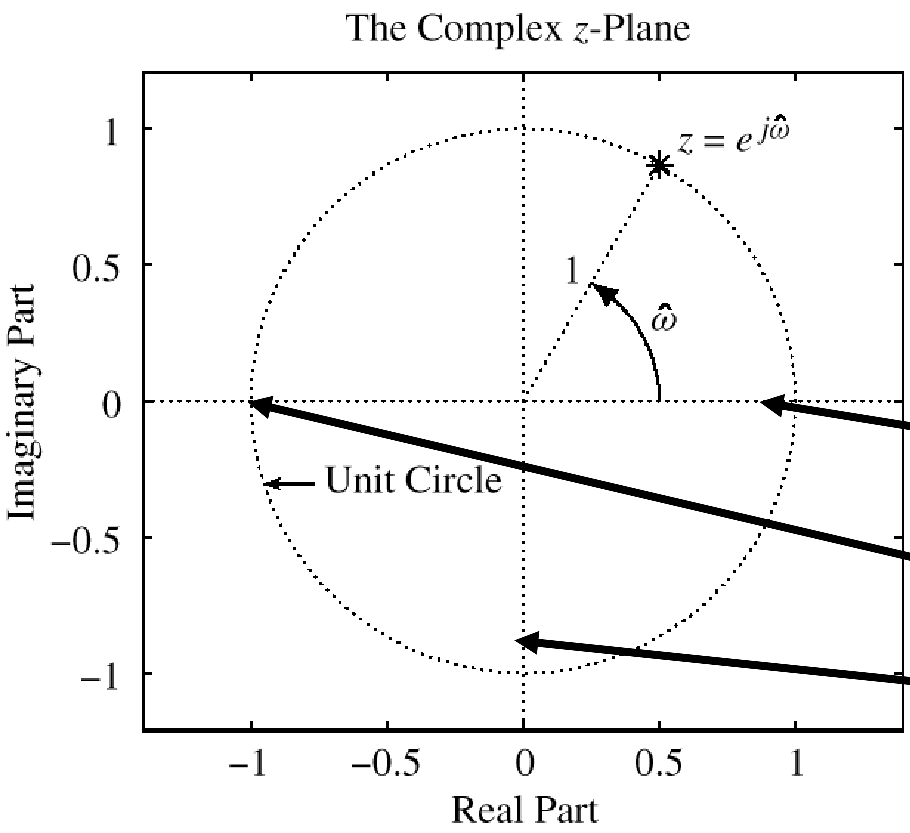
Frequency Response Plot

$$H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$



UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



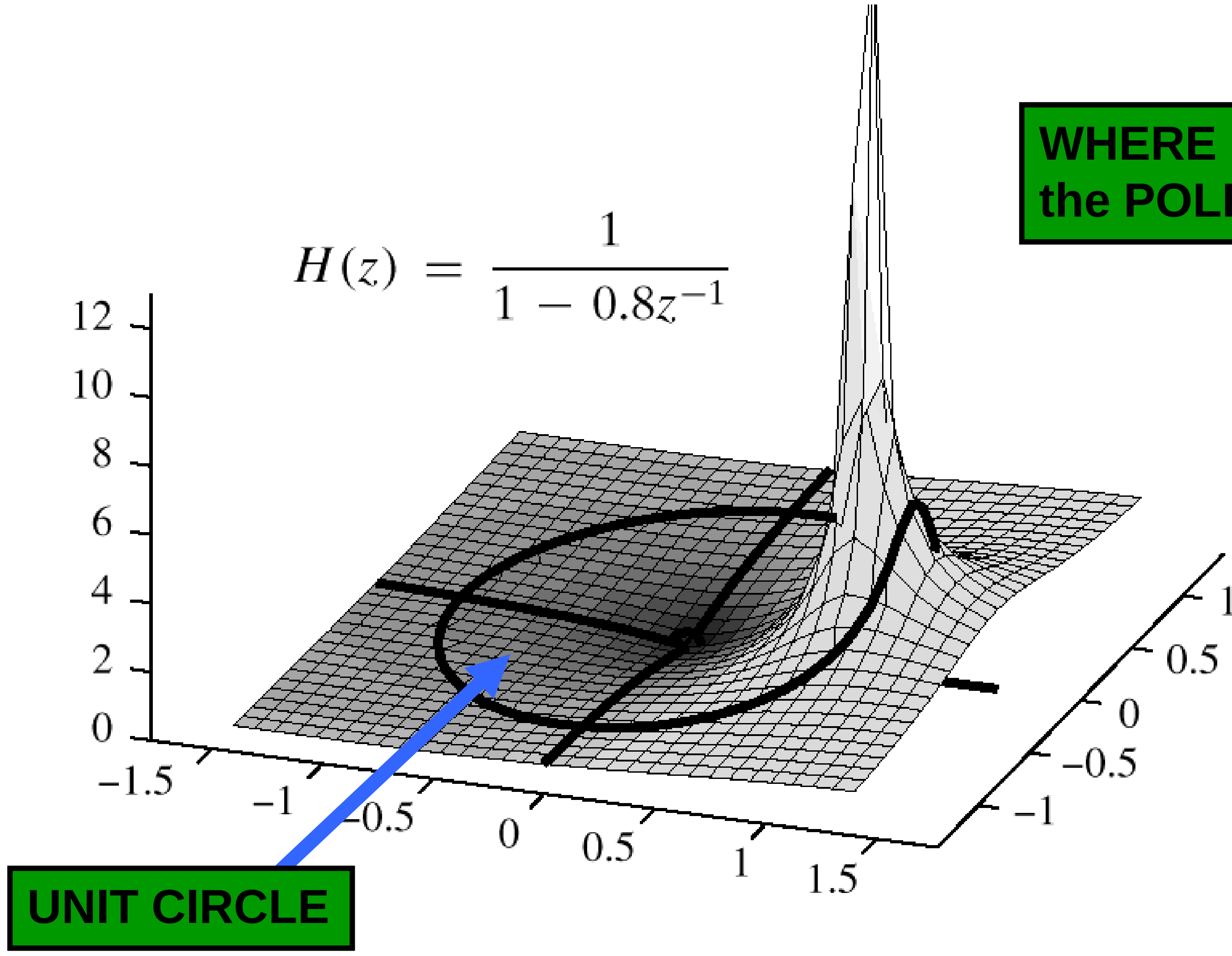
$$z = e^{j\hat{\omega}}$$

$$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$$

$$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$$

$$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\tfrac{1}{2}\pi$$

3-D VIEWPOINT: EVALUATE H(z) EVERYWHERE



SINUSOIDAL RESPONSE

- $x[n]$ = SINUSOID => $y[n]$ is SINUSOID
- Get MAGNITUDE & PHASE from $H(z)$

$$\boxed{\begin{array}{l} \text{if} \quad x[n] = e^{j\hat{\omega}n} \\[2ex] \text{then} \quad y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n} \\[2ex] \text{where} \quad H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} \end{array}}$$

POP QUIZ

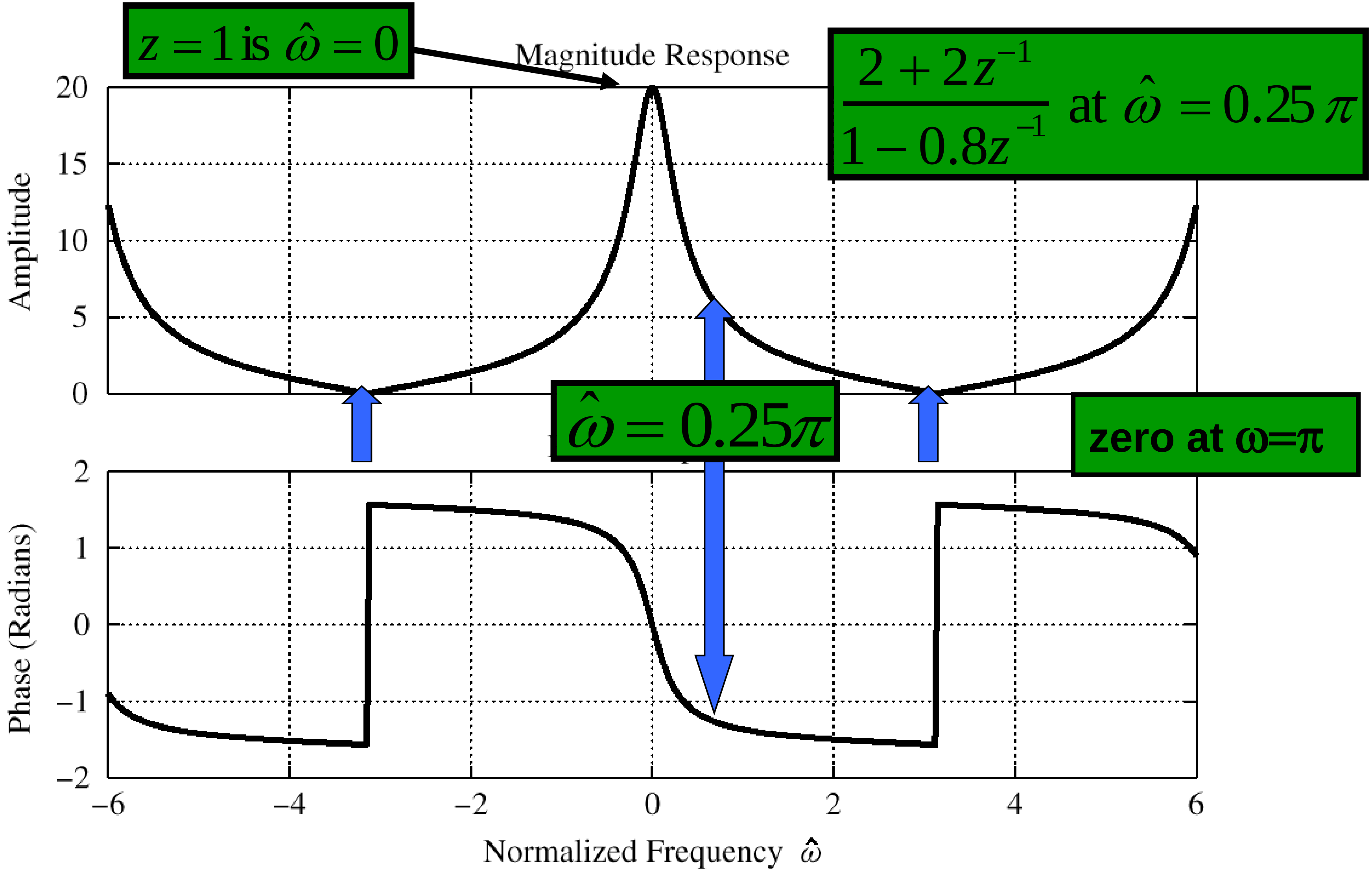
- Given: $\boxed{H(z) = \dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}}}$

- Find the Impulse Response, $h[n]$

- Find the output, $y[n]$
  - When

$$\boxed{x[n] = \cos(0.25\pi n)}$$

Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

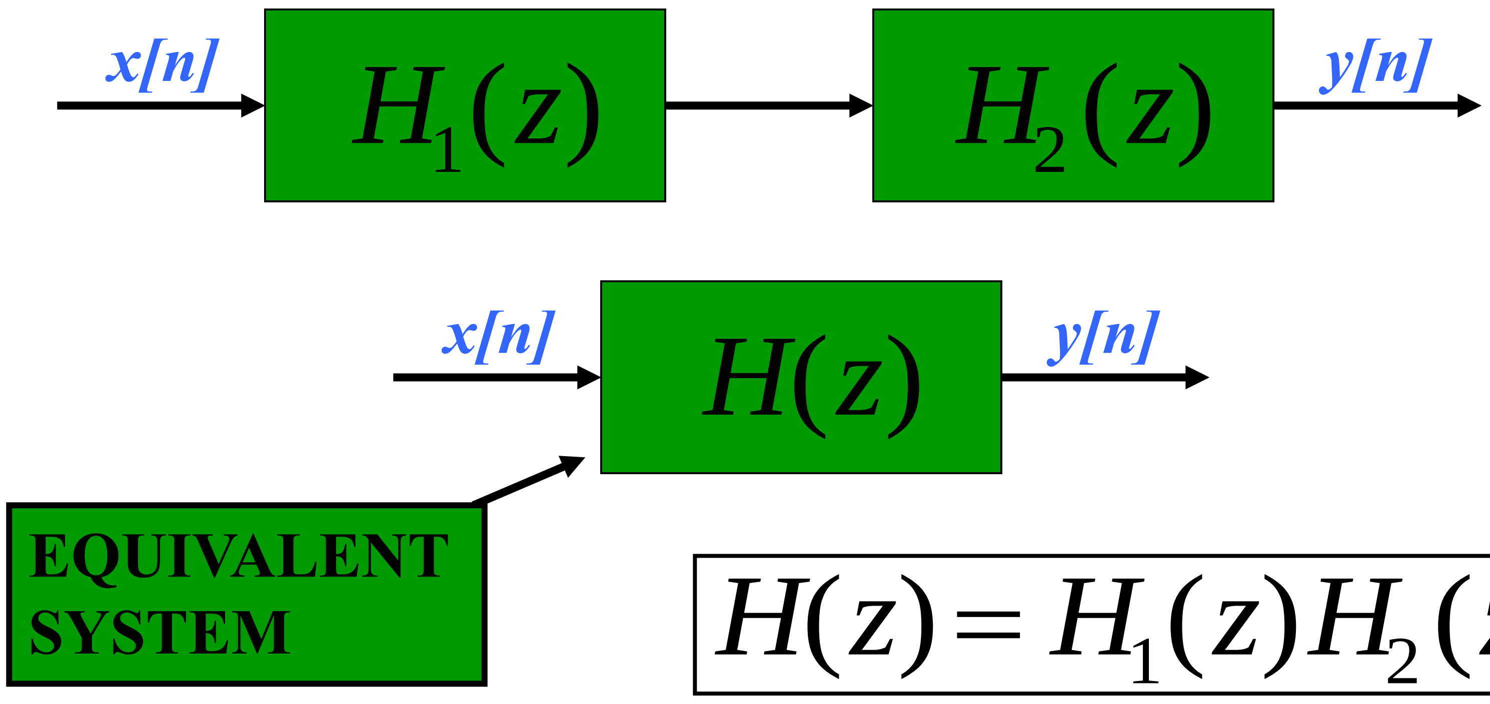
- Given: $$H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}$$
- Find output, $y[n]$, when $x[n]=\cos(0.25\pi n)$
  - Evaluate at $z=e^{j0.25\pi}$

$$H(z)=\frac{2+2(\frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2})}{1-0.8e^{-j0.25\pi}}=5.182e^{-j1.309}$$

$$y[n]=5.182\cos(0.25\pi n-0.417\pi)$$

CASCADE EQUIVALENT

- Multiply the System Functions



$$\boxed{H(z) = H_1(z)H_2(z)}$$