BLM2041 Signals and Systems

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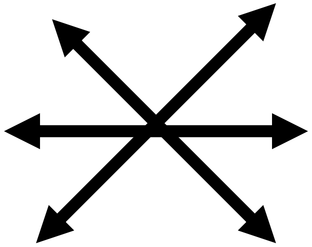
Spectrum Representation

Problem Solving Skills

- Math Formula
  - Sum of Cosines
  - Amp, Freq, Phase
- Recorded Signals
  - Speech
  - Music
  - No simple formula



- Plot & Sketches
  - $S(t)$ versus t
  - Spectrum
- MATLAB
  - Numerical
  - Computation
  - Plotting list of numbers

LECTURE OBJECTIVES

- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs

LECTURE OBJECTIVES

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

FREQUENCY can change vs. TIME

Chirps:

$$x(t) = \cos(\alpha t^2)$$

Introduce Spectrogram Visualization **(specgram.m)** **(plotspec.m)**

LECTURE OBJECTIVES

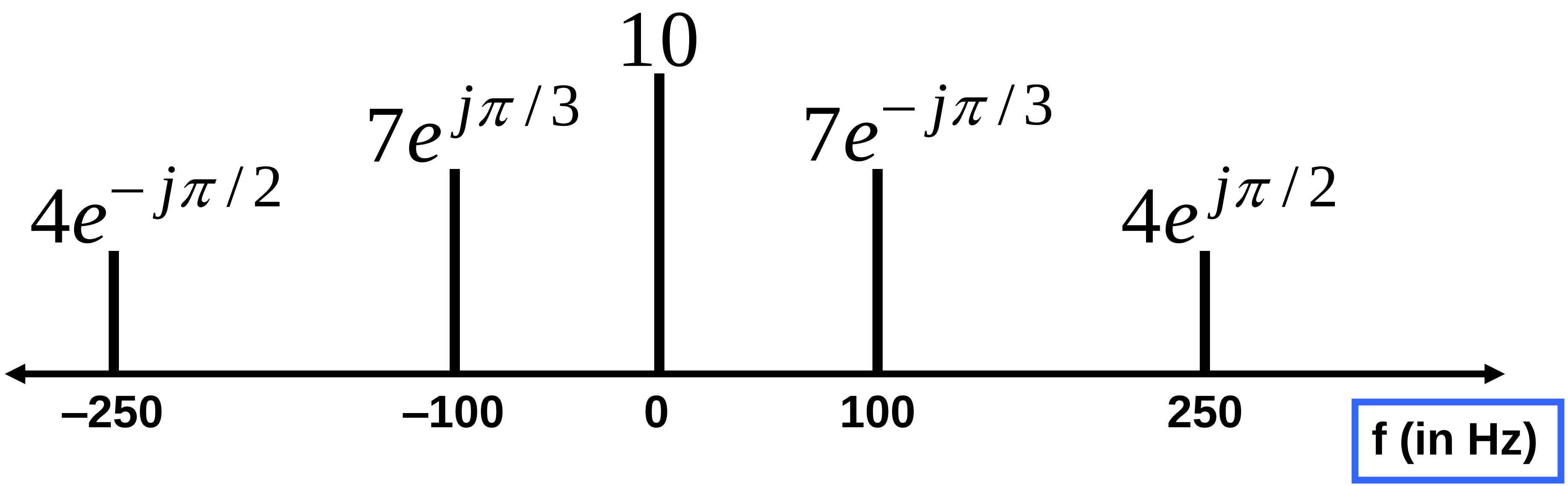
- Work with the Fourier Series Integral

$$\boxed{a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt}$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $\boldsymbol{x(t+T_0) = x(t)}$
- **SPECTRUM** from Fourier Series
  - a_k is Complex Amplitude for k-th Harmonic

FREQUENCY DIAGRAM

- Plot Complex Amplitude vs. Freq



Another FREQ. Diagram

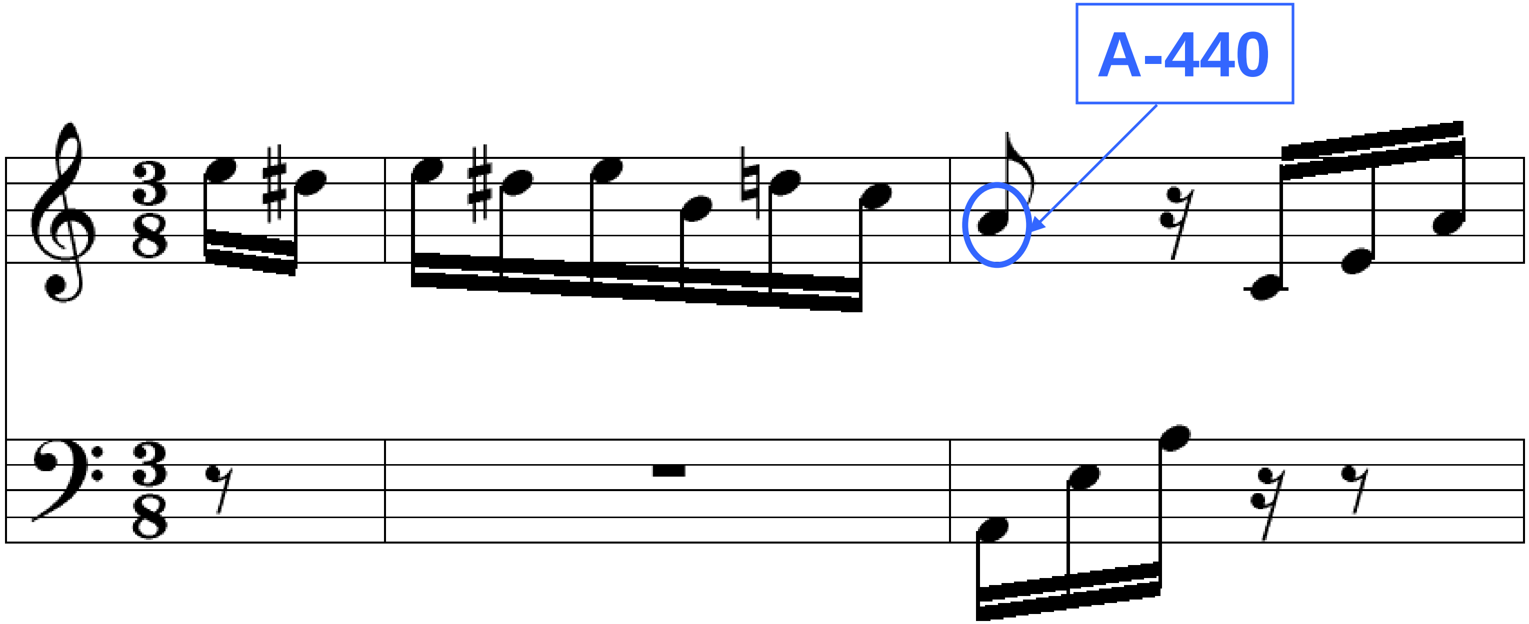


Figure 3.18 Sheet-music notation is a time–frequency diagram.

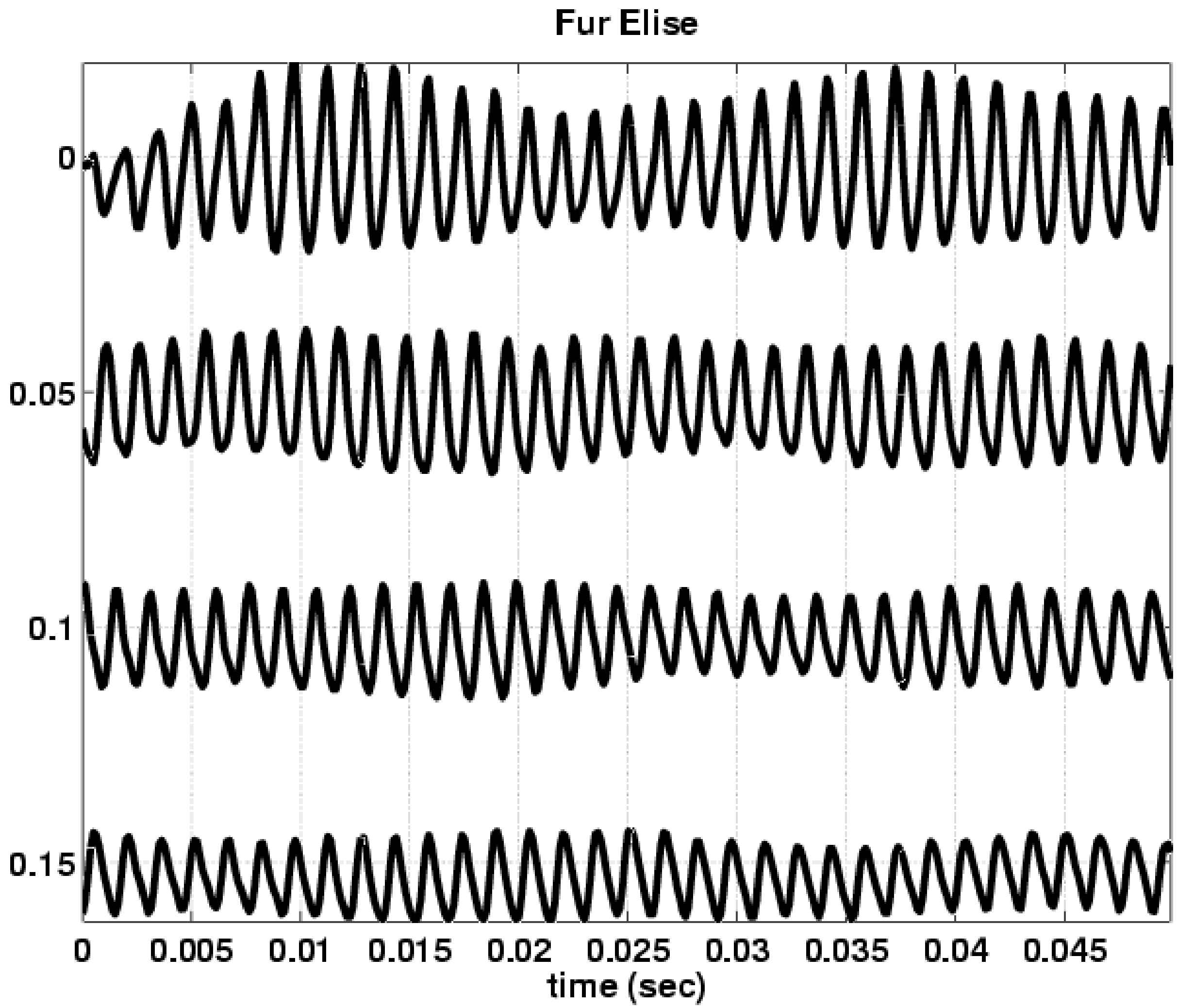
Frequency is the vertical axis

Time is the horizontal axis

MOTIVATION

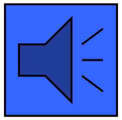
- Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?

Fur Elise WAVEFORM

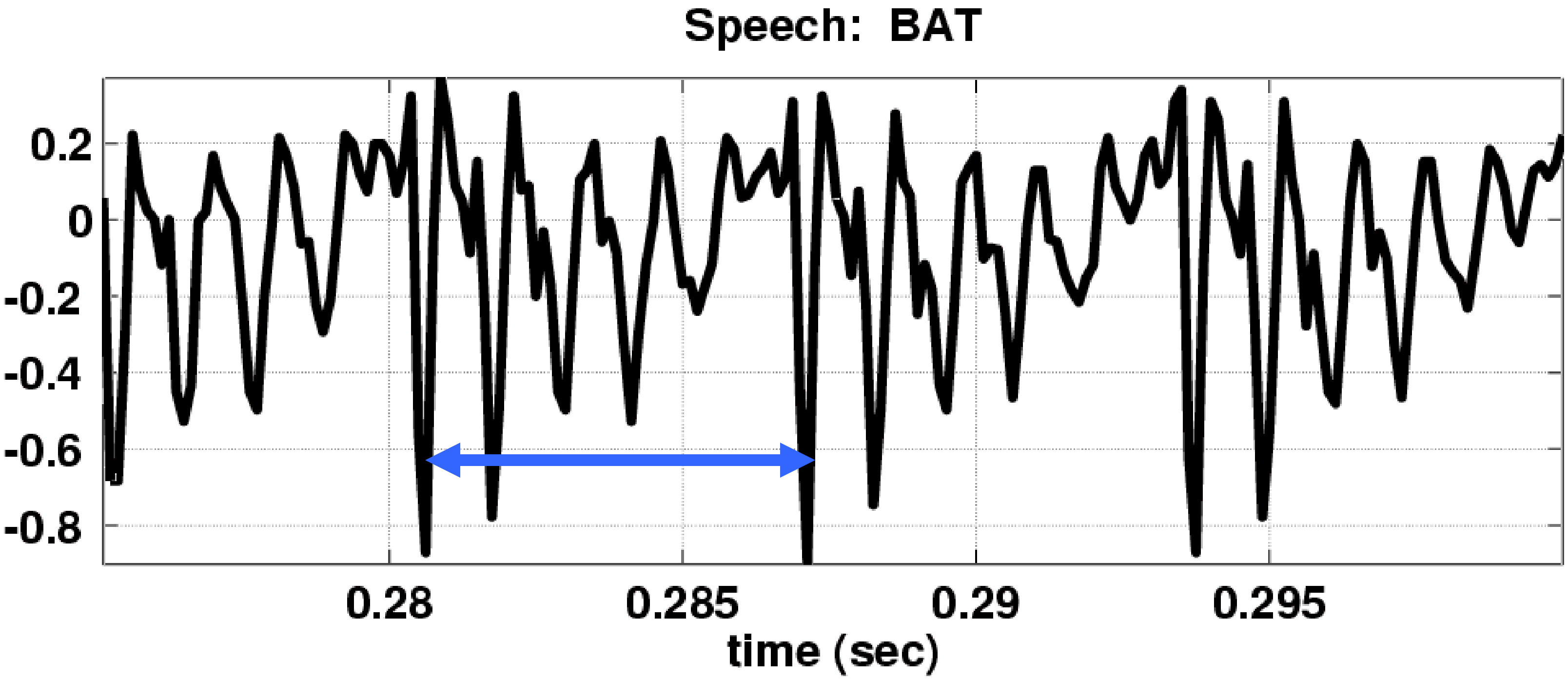


Beat Notes

Speech Signal: BAT



- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})}$$

INVERSE Euler's Formula

- Solve for cosine (or sine)

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big

NEGATIVE FREQUENCY

- Is negative frequency real?
- Doppler Radar provides an example
  - Police radar measures speed by using the Doppler shift principle
  - Let’s assume 400Hz ←→60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)
  - Think of a train whistle

SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

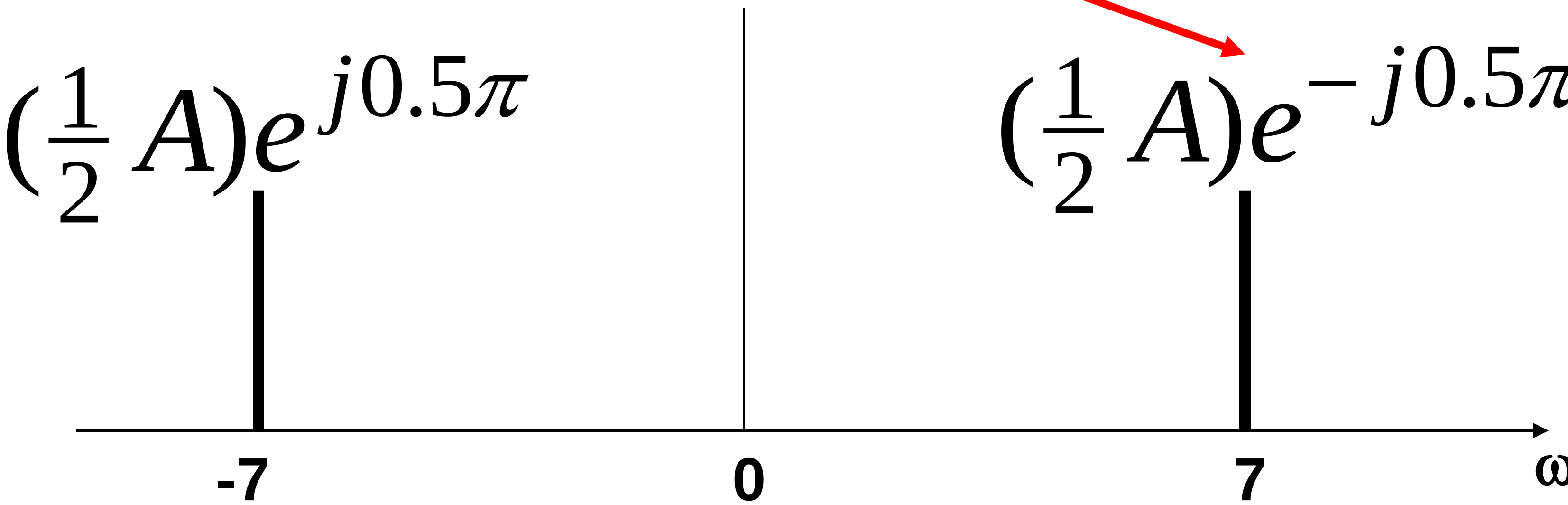
$$\boxed{\frac{-1}{j} = j = e^{j0.5\pi}}$$

  - Positive freq. has phase = -0.5π
  - Negative freq. has phase = +0.5π

GRAPHICAL SPECTRUM

EXAMPLE of SINE

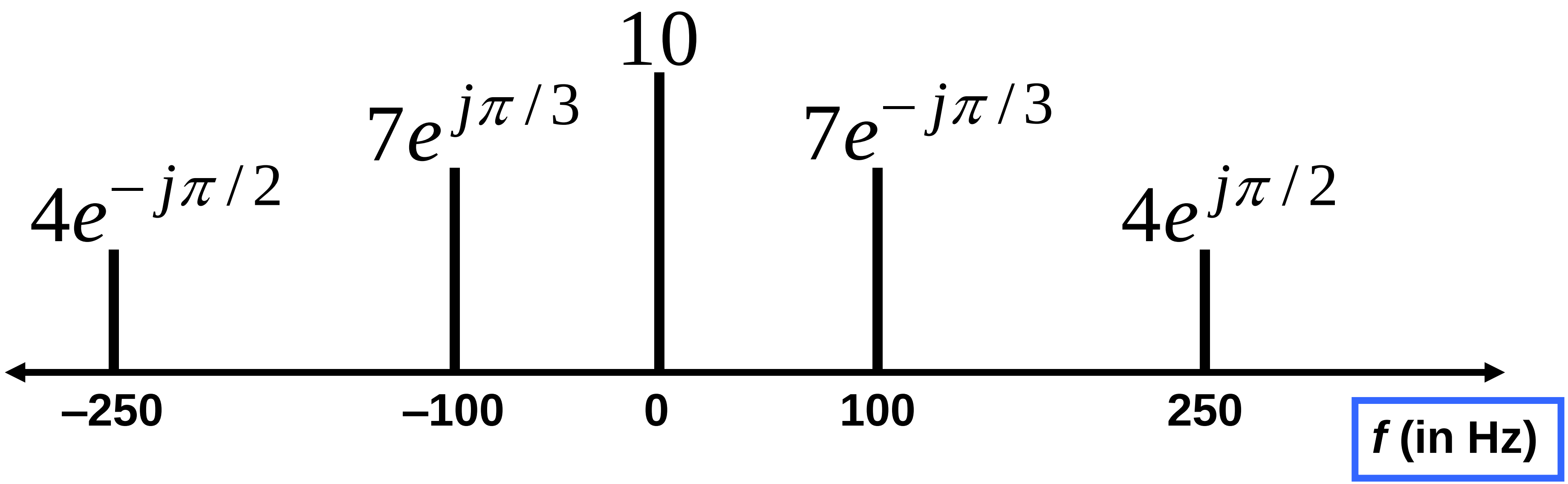
$$A\sin(7t) = \tfrac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \tfrac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are shown

SPECTRUM ---> SINUSOID

- Add the spectrum components:



What is the formula for the signal *x*(*t*)?

Gather (A, ω, ϕ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0** Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 **0**
  - 7 $-\pi/3$
  - 4 $+\pi/2$

Note the **conjugate phase**

DC is another name for zero-freq component

DC component always has $\phi=0$ or π (for real ***x(t)***)

Add Spectrum Components-1

- **Frequencies:**
  - **-250 Hz**
  - **-100 Hz**
  - **0 Hz**
  - **100 Hz**
  - **250 Hz**

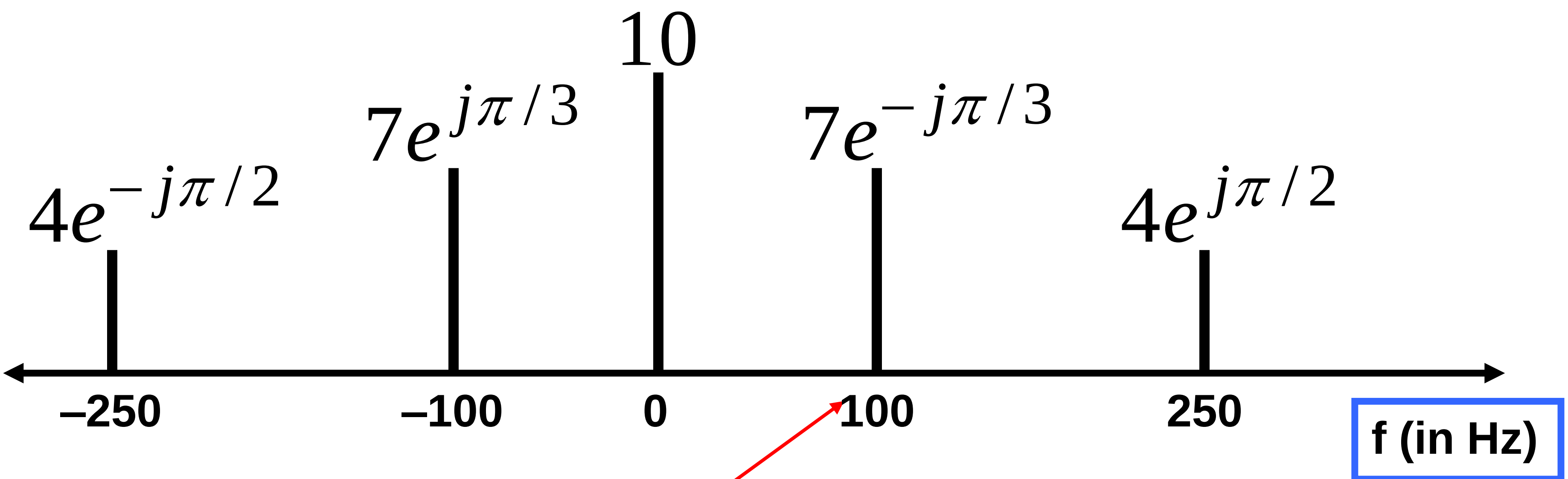
- **Amplitude & Phase**
  - **4** **-π/2**
  - **7** **+π/3**
  - **10** **0**
  - **7** **-π/3**
  - **4** **+π/2**

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Add Spectrum Components-2



$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Simplify Components

$$x(t) = 10 +$$
$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{-j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

FINAL ANSWER

$$\boxed{\begin{aligned} x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) \\ + 8\cos(2\pi(250)t + \pi/2) \end{aligned}}$$

<u>So, we get the general form:</u>

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$\Re e\{z\} = \tfrac{1}{2} z + \tfrac{1}{2} z^*$$

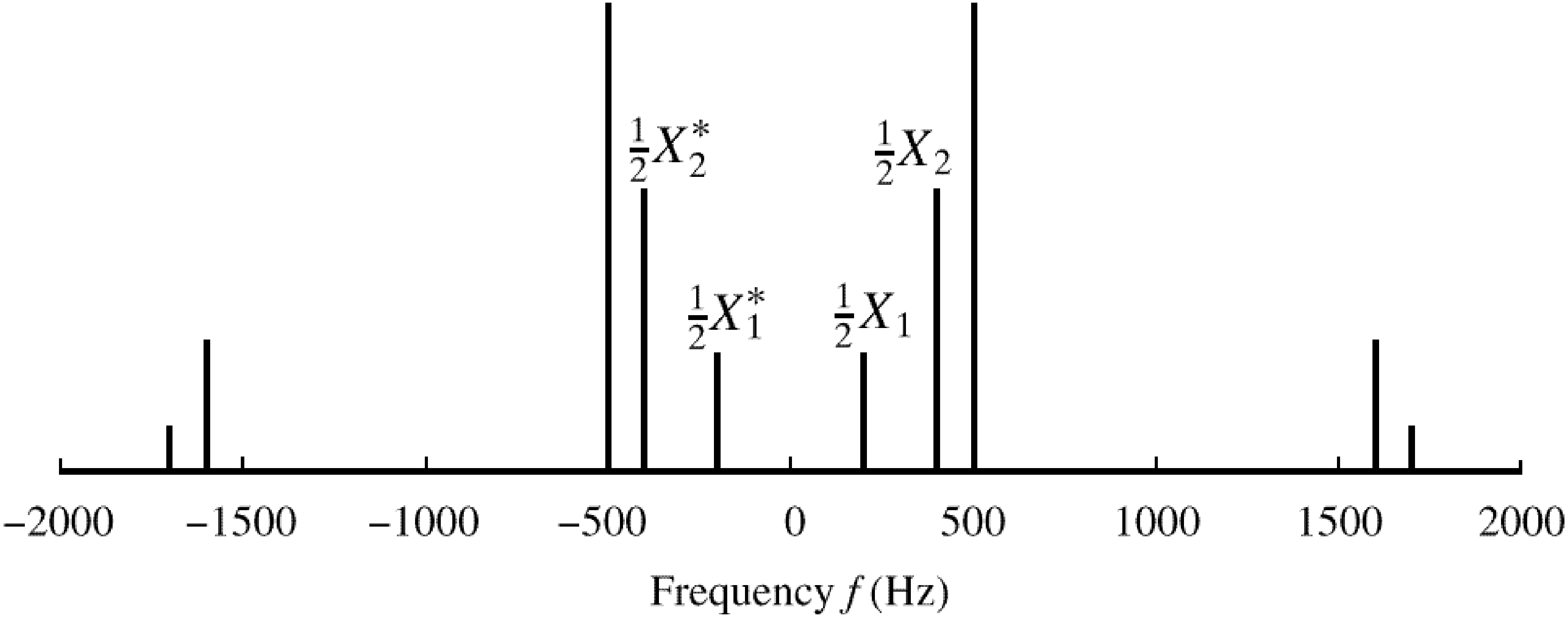
$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t}\right\}$$

Example: Synthetic Vowel

- Sum of 5 Frequency Components
  - Complex amplitudes for harmonic signal that approximates the vowel sound «ah»

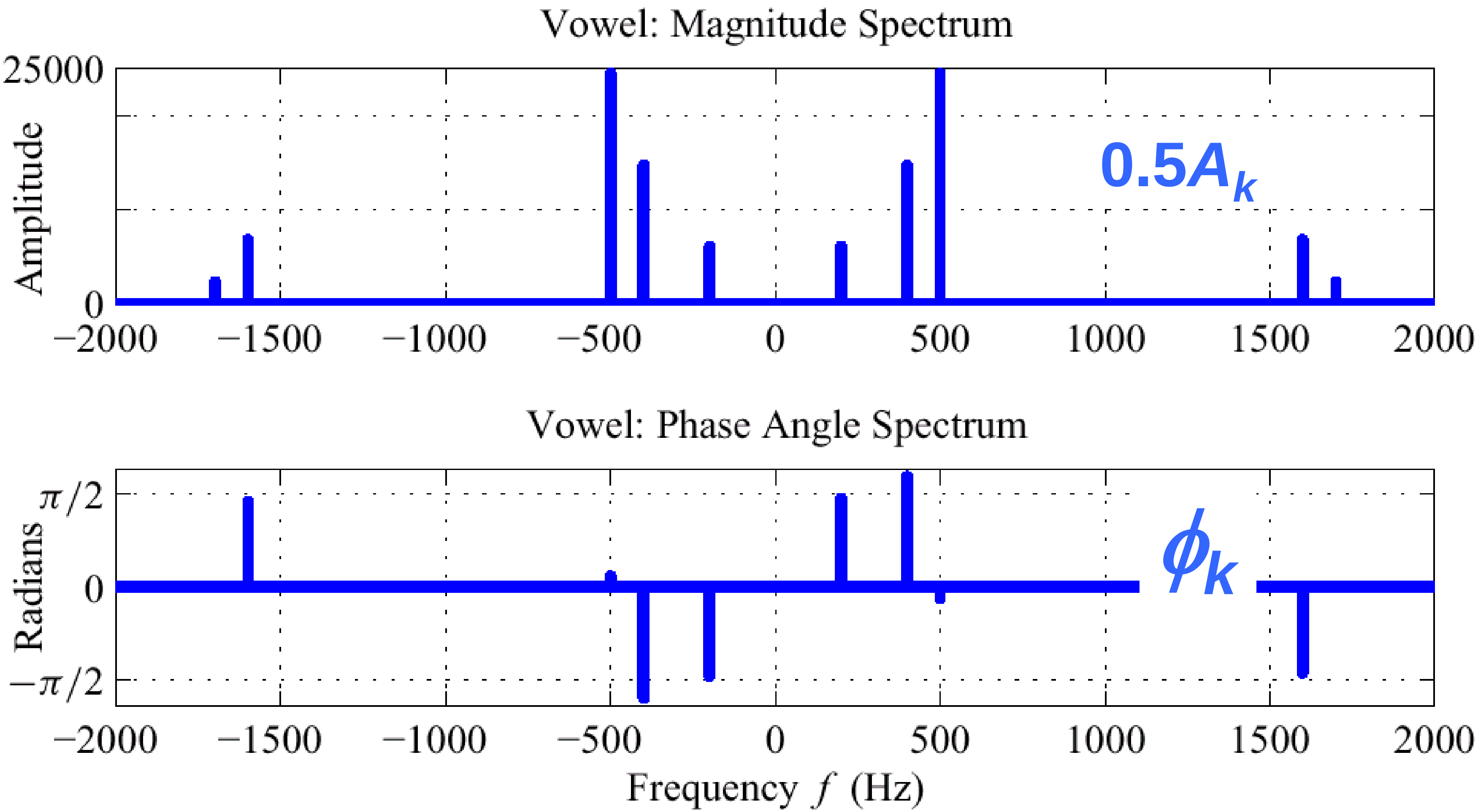
f_k **(Hz)**	X_k	**Mag**	**Phase (rad)**
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	-0.185
1600	$(1657 - j13520)$	13,621	-1.449
1700	$4723 + j0$	4723	0

SPECTRUM of VOWEL



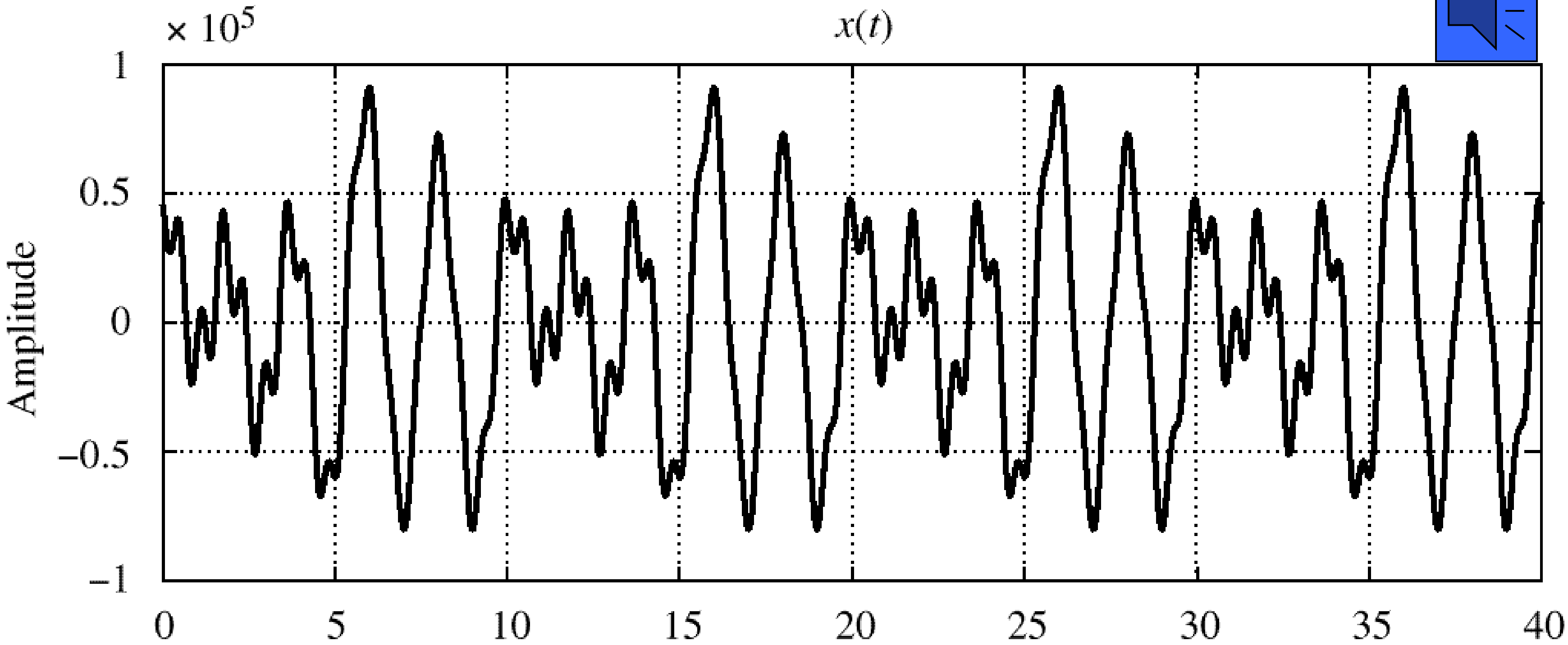
- Note: Spectrum has $0.5X_k$ (except X_{DC})
- Conjugates in negative frequency

SPECTRUM of VOWEL (Polar Format)



Vowel Waveform (sum of all 5 components)

- Sum of all of the signals in the prevous slides

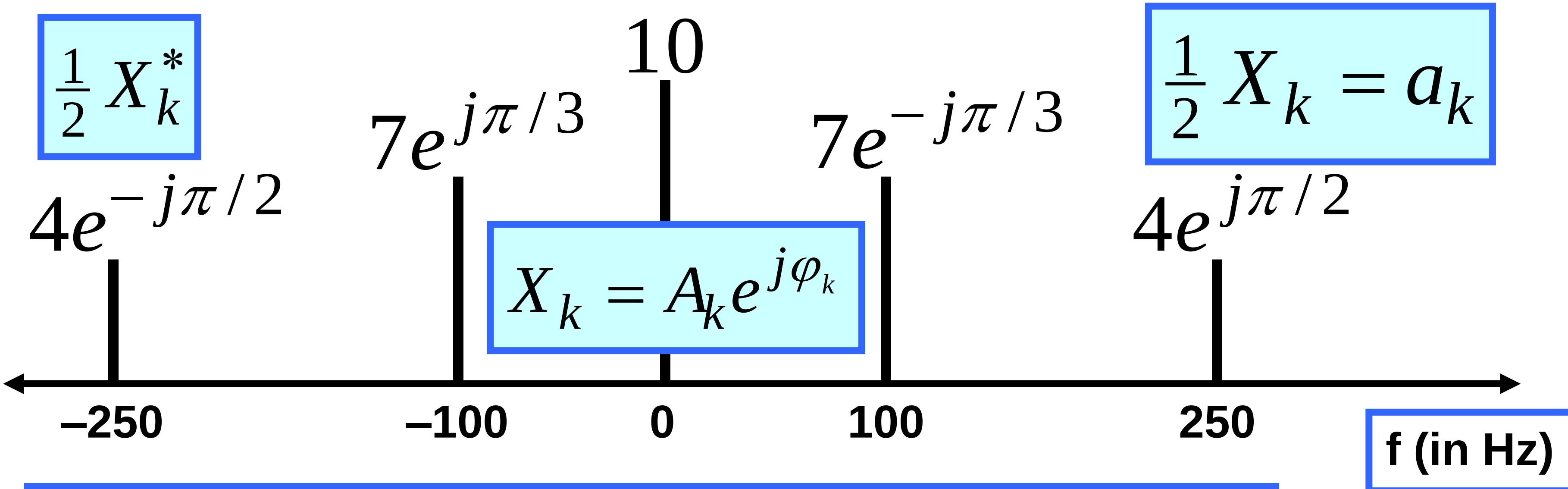


  - – Note that the period is 10 ms, which equals $1/f_0$

Periodic Signals, Harmonics & Time-Varying Sinusoids

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

SPECTRUM for PERIODIC ?

- Nearly Periodic in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec

PERIODIC SIGNALS

- Repeat every T secs
  - Definition

$$\boxed{x(t) = x(t+T)}$$

  - Example:

$$\boxed{x(t) = \cos^2(3t)}$$

$$T = ?$$

  - Speech can be "quasi-periodic"

$$T = \frac{2\pi}{3} \qquad T = \frac{\pi}{3}$$

Period of Complex Exponential

$$x(t) = e^{j\omega t}$$

$$x(t+T) = x(t) \, ?$$

Definition: Period is *T*

$$\cancel{e^{j\omega(t+T)}} = \cancel{e^{j\omega t}}$$

$$\boxed{e^{j2\pi k} = 1}$$

$$\Rightarrow e^{j\omega T} = 1 \quad \Rightarrow \omega T = 2\pi k$$

$$\omega = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right) k = \omega_0 k$$

k = integer

Harmonic Signal Spectrum

Periodic signal can only have : $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T}$$

$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

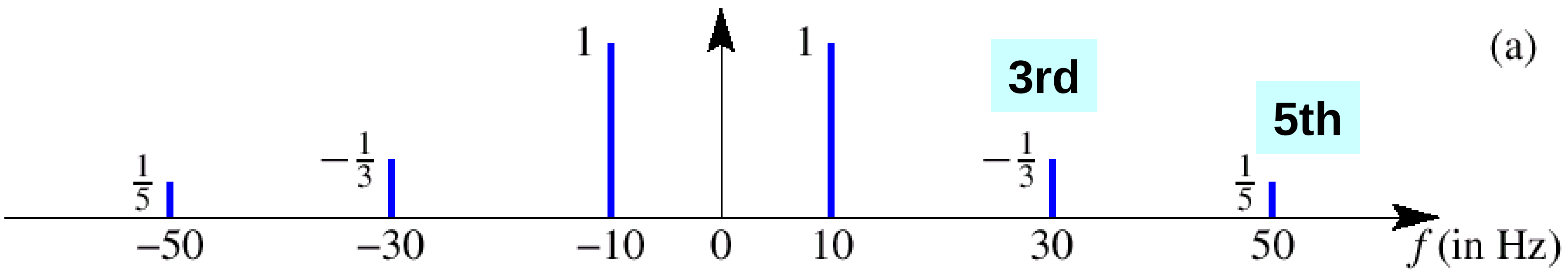
$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$$

$$f_0 = \frac{1}{T_0}$$

$f_0 =$ fundamental Frequency (largest)

$T_0 =$ fundamental Period (shortest)

Harmonic Signal (3 Freqs)

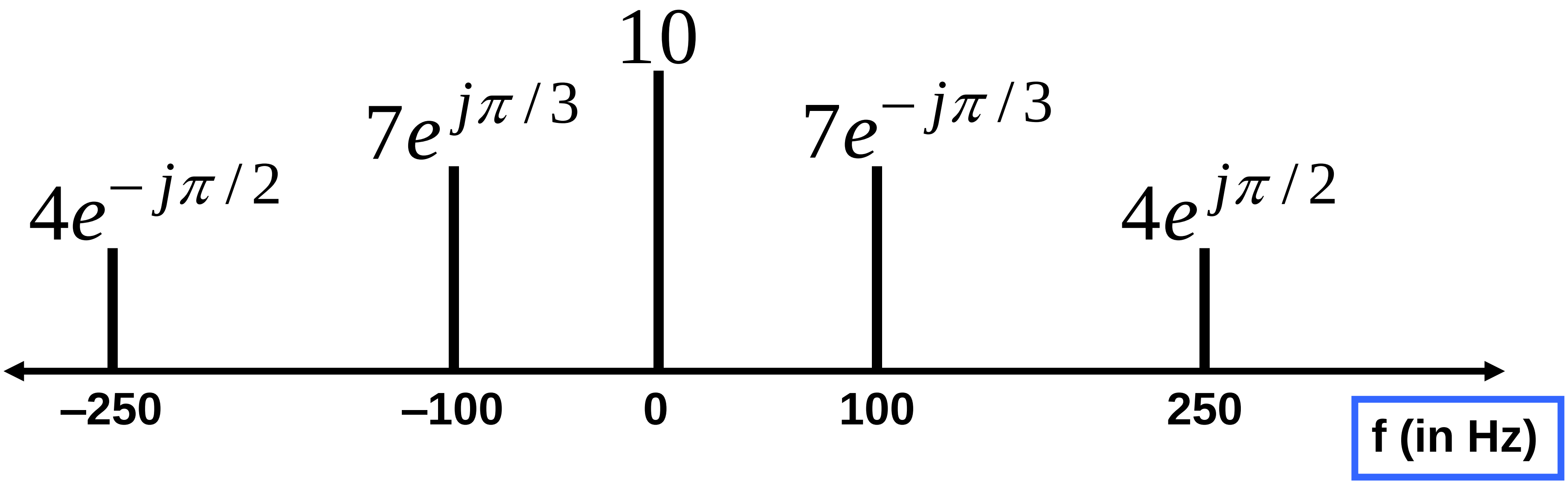


What is the fundamental frequency?

10 Hz

POP QUIZ: FUNDAMENTAL

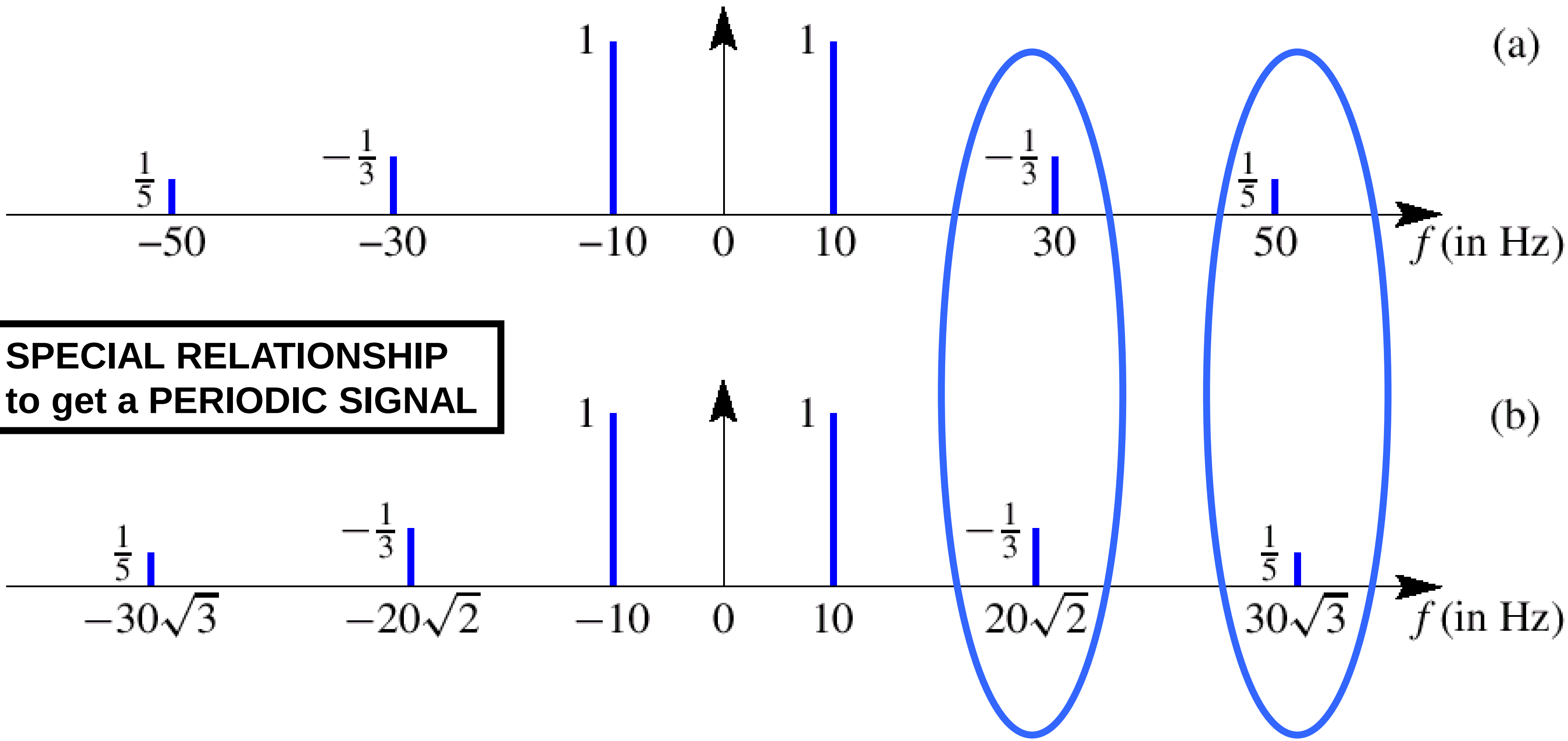
- Here's another spectrum:



What is the fundamental frequency?

100 Hz ? **50 Hz ?**

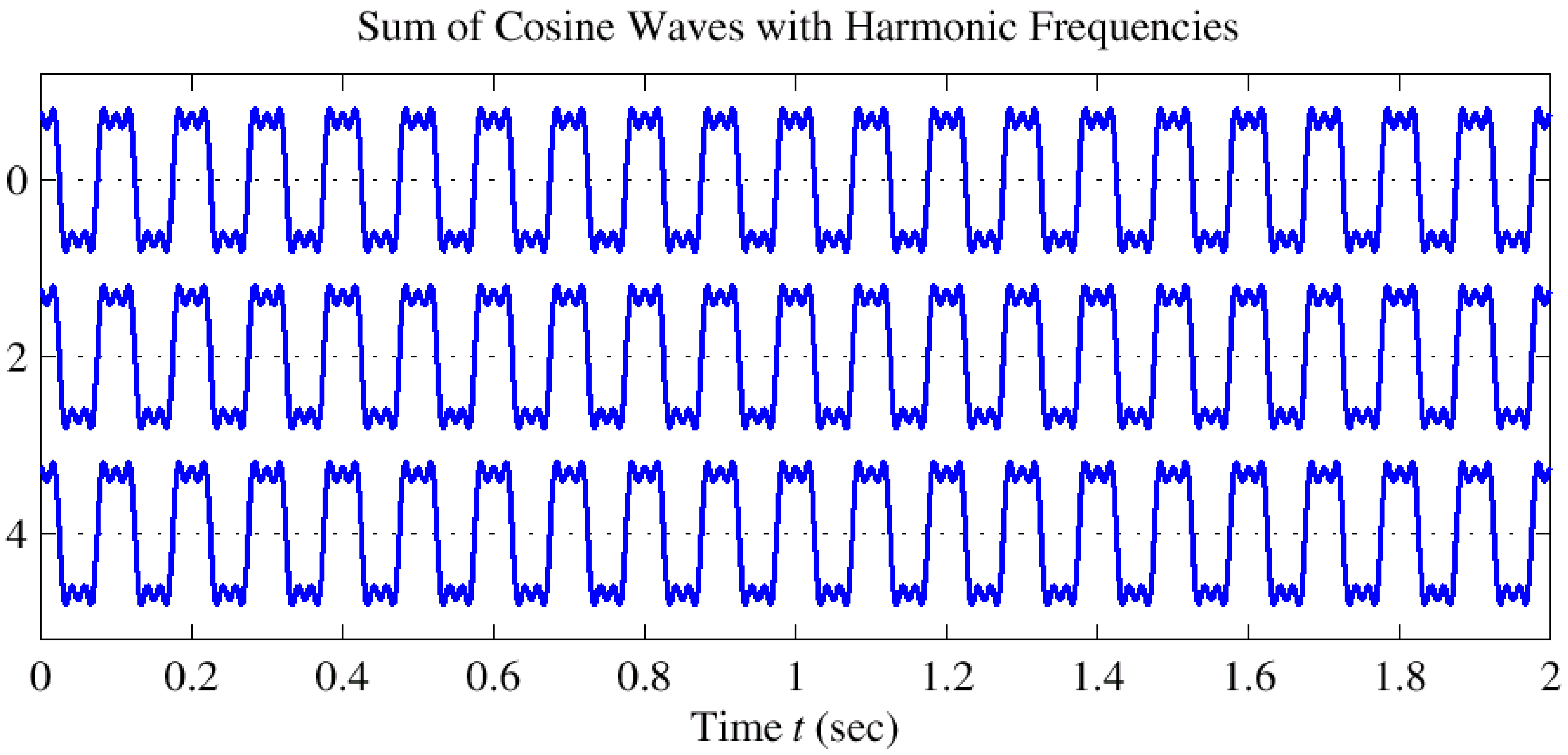
IRRATIONAL SPECTRUM



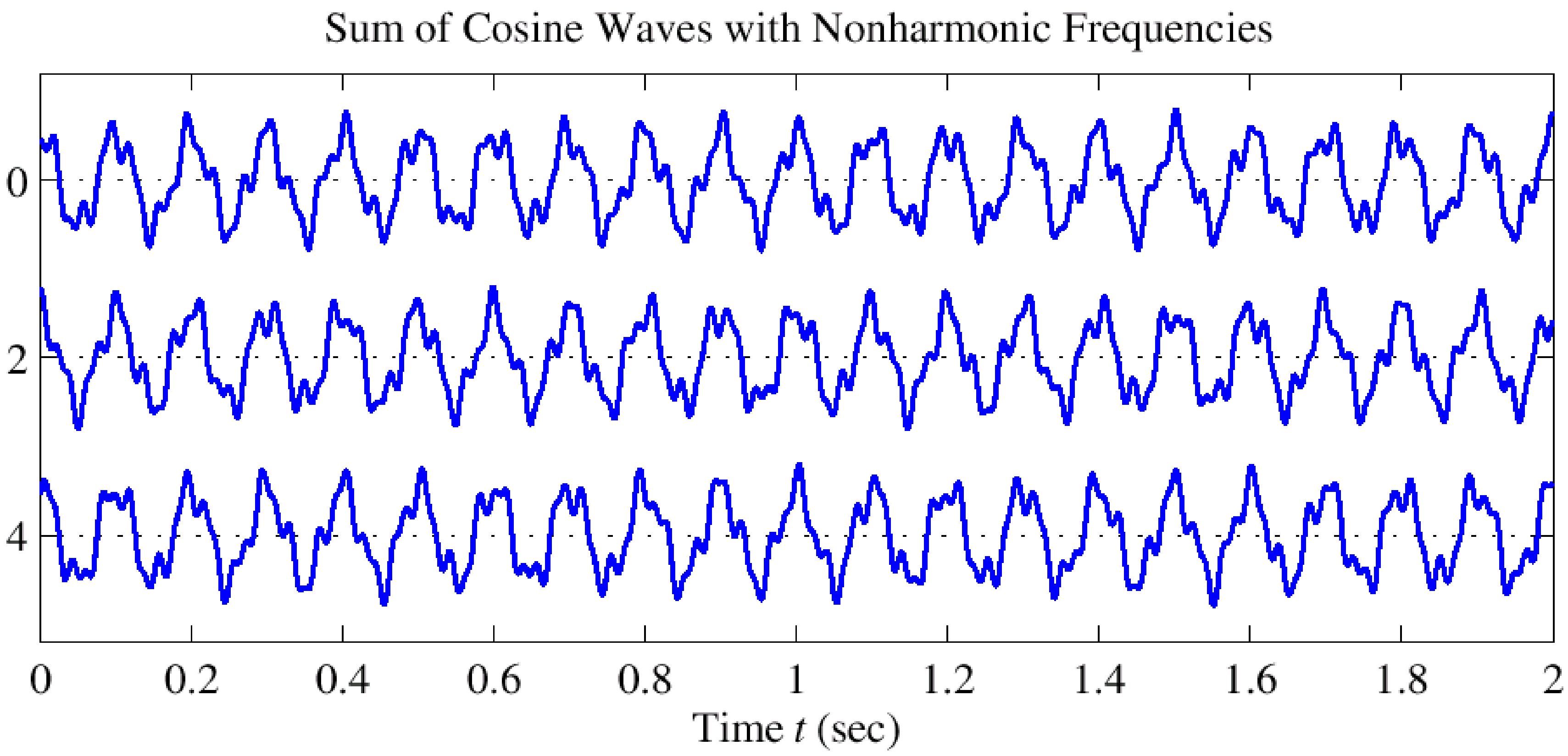
SPECIAL RELATIONSHIP to get a PERIODIC SIGNAL

Harmonic Signal (3 Freqs)

T=0.1



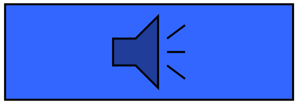
NON-Harmonic Signal

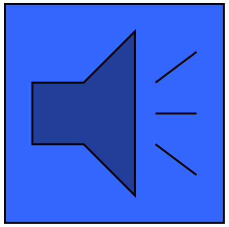


NOT PERIODIC

FREQUENCY ANALYSIS

- **<u>Now, a much HARDER problem</u>**
- Given a recording of a song, have the computer write the music





- Can a machine extract frequencies?
  - Yes, if we COMPUTE the spectrum for $x(t)$
    - During short intervals

Time-Varying FREQUENCIES Diagram

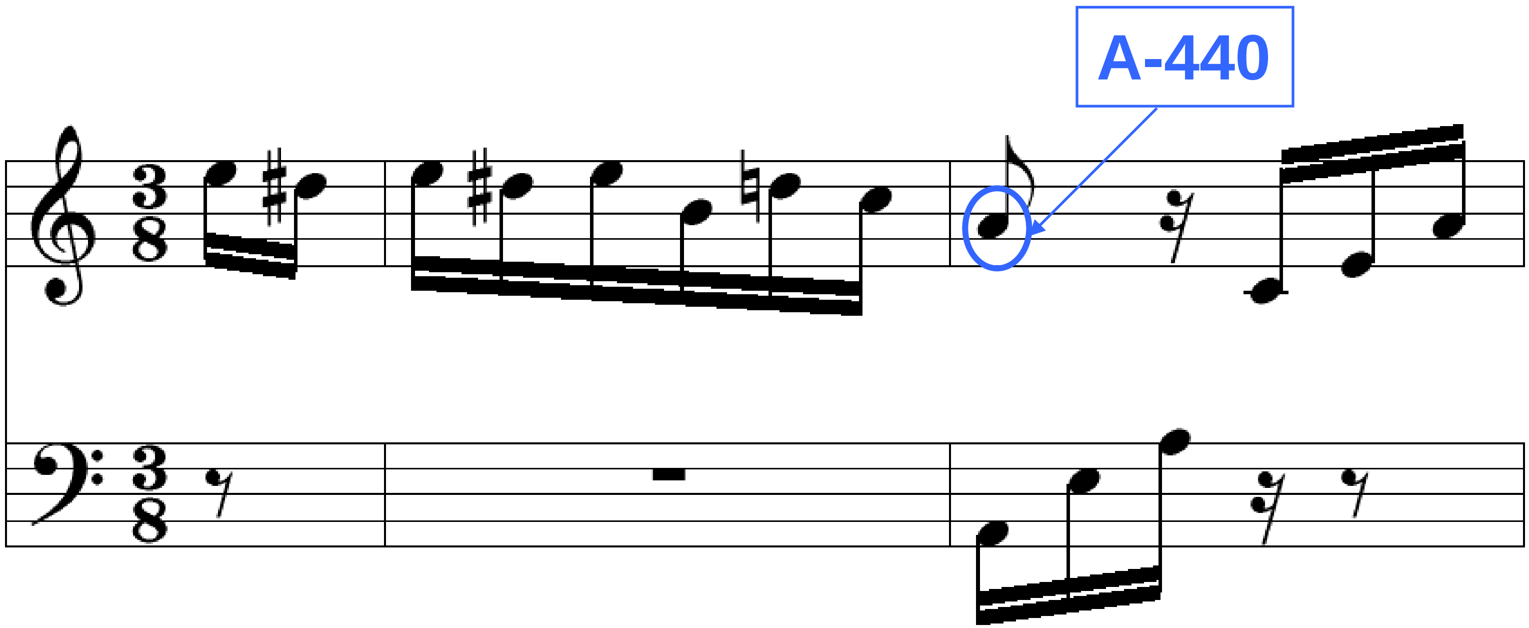


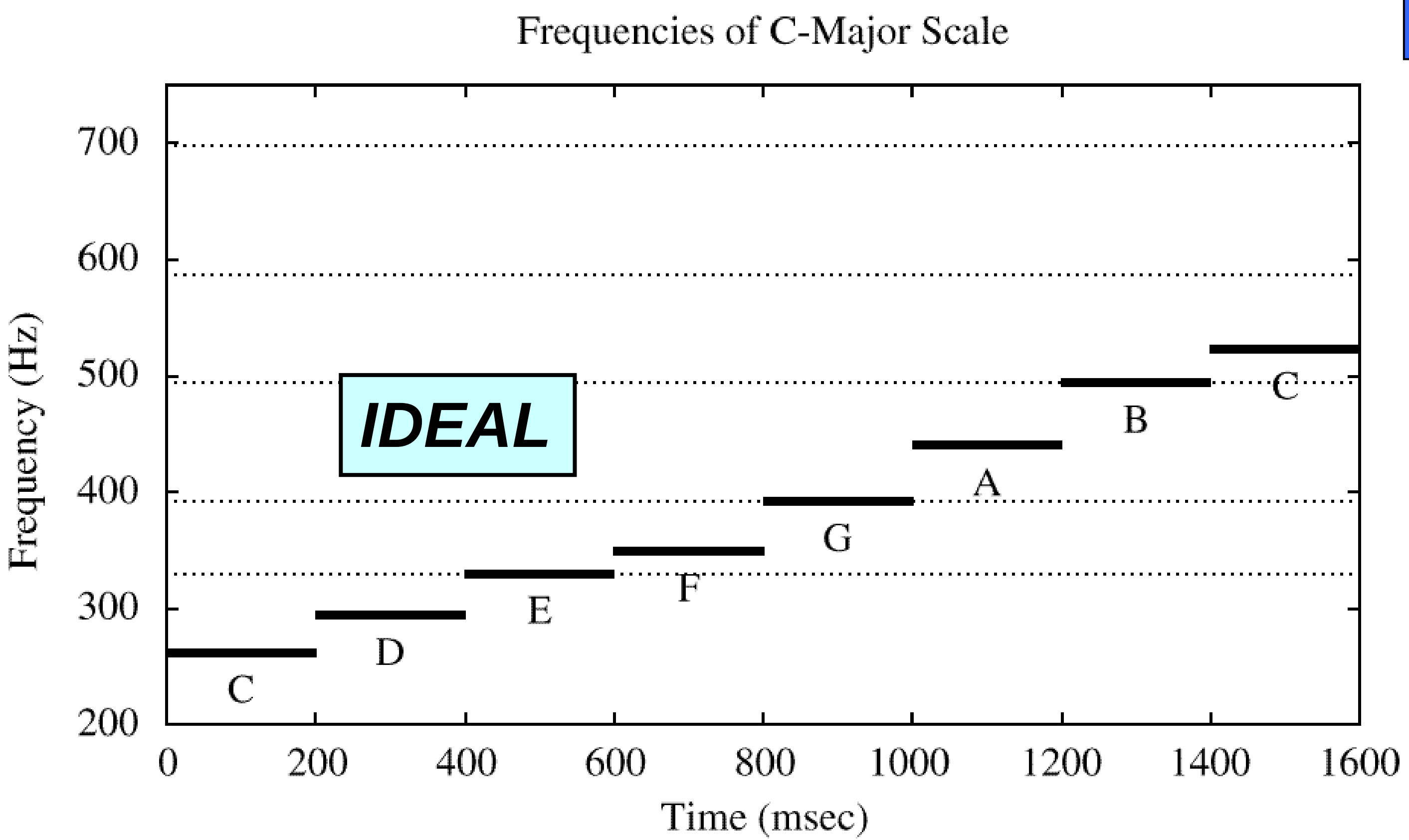
Figure 3.18 Sheet-music notation is a time–frequency diagram.

Frequency is the vertical axis

Time is the horizontal axis

SIMPLE TEST SIGNAL

- C-major SCALE: stepped frequencies



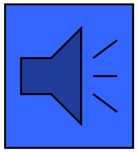
  - Frequency is constant for each note

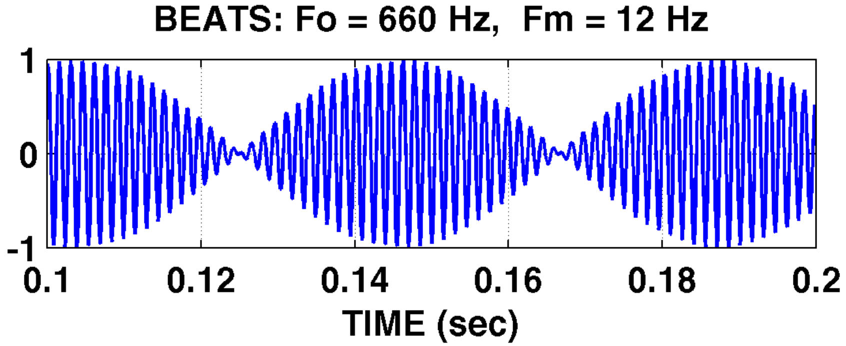
SPECTROGRAM

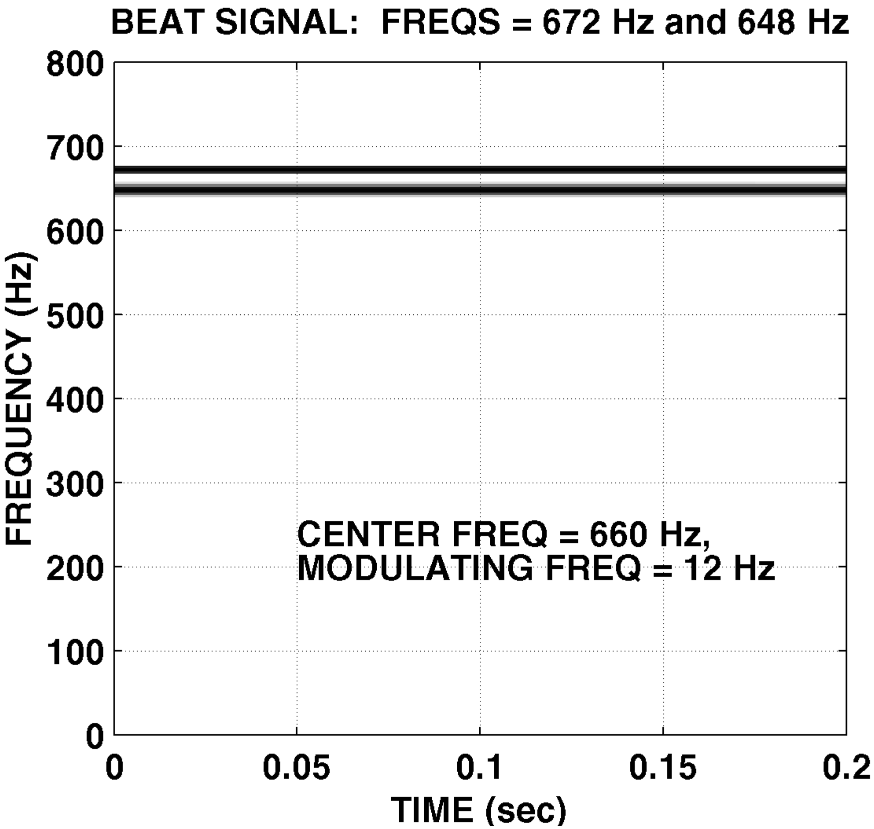
- SPECTROGRAM Tool
  - MATLAB function is `specgram.m`
  - SP-First has `plotspec.m` & `spectgr.m`
- **ANALYSIS** program
  - Takes x(t) as input &
  - Produces spectrum values X_k
  - Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (Fast Fourier Transform)

SPECTROGRAM EXAMPLE

- Two **Constant** Frequencies: Beats







$$\cos(2\pi(660)t)\sin(2\pi(12)t)$$

AM Radio Signal

- Same as BEAT Notes

$$\boxed{\cos(2\pi(660)t)\sin(2\pi(12)t)}$$

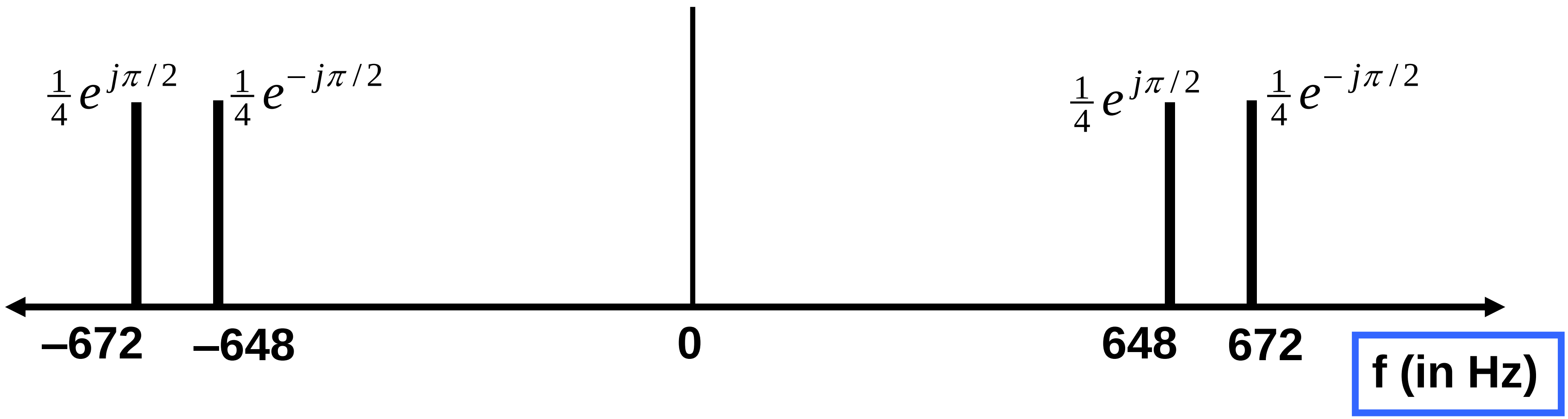
$$\frac{1}{2}\left(e^{j2\pi(660)t}+e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t}-e^{-j2\pi(12)t}\right)$$

$$\frac{1}{4j}\left(e^{j2\pi(672)t}-e^{-j2\pi(672)t}-e^{j2\pi(648)t}+e^{-j2\pi(648)t}\right)$$

$$\boxed{\tfrac{1}{2}\cos(2\pi(672)t-\tfrac{\pi}{2})+\tfrac{1}{2}\cos(2\pi(648)t+\tfrac{\pi}{2})}$$

SPECTRUM of AM (Beat)

- 4 complex exponentials in AM:



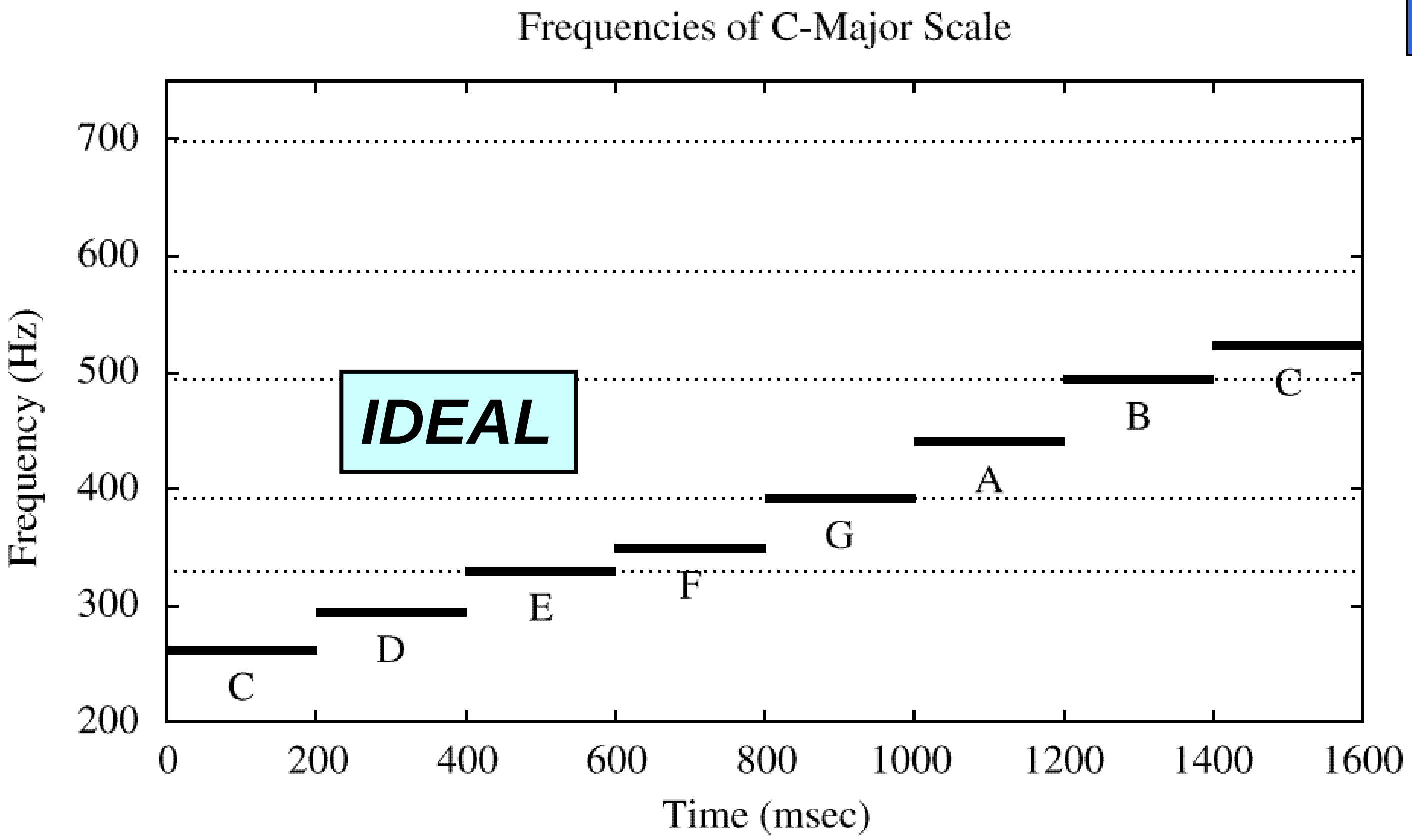
What is the fundamental frequency?

648 Hz ?

24 Hz ?

STEPPED FREQUENCIES

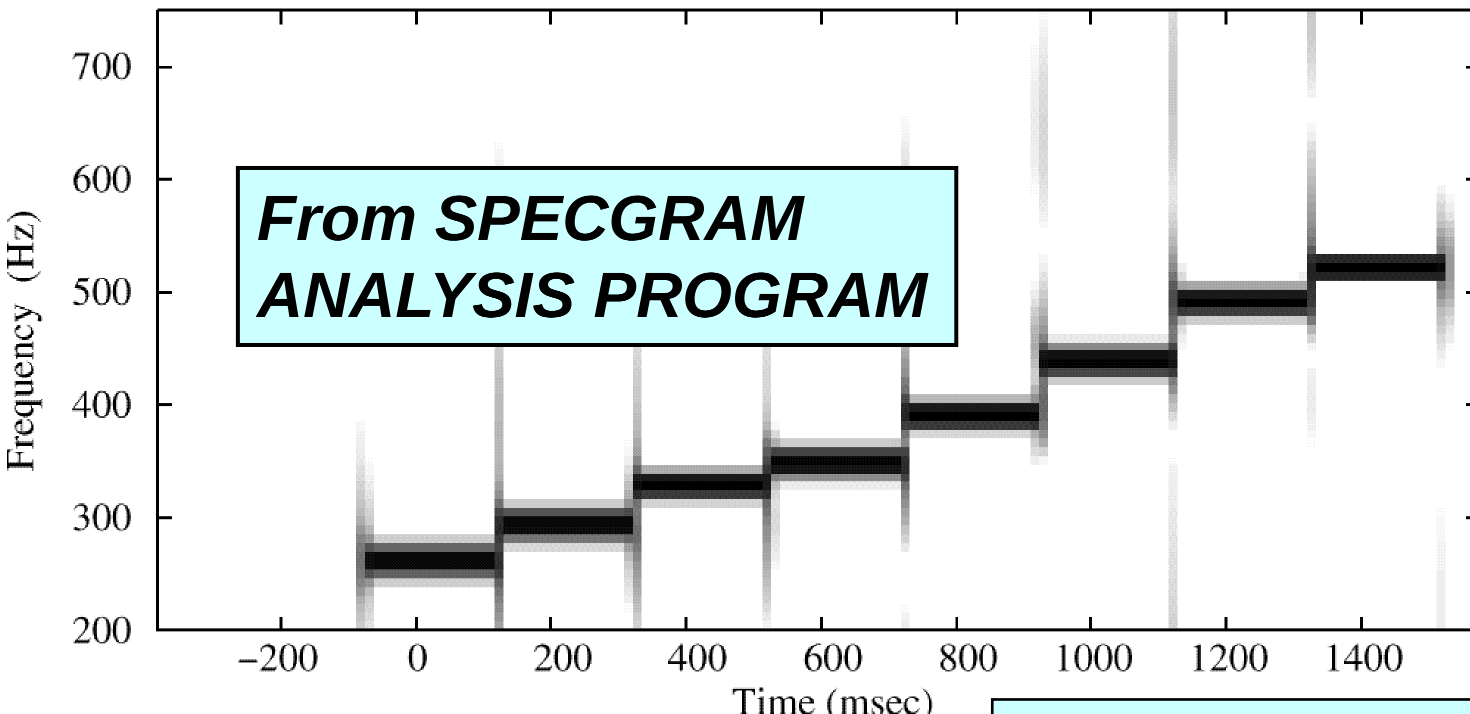
- C-major SCALE: stepped frequencies



  – Frequency is constant for each note

SPECTROGRAM of C-Scale

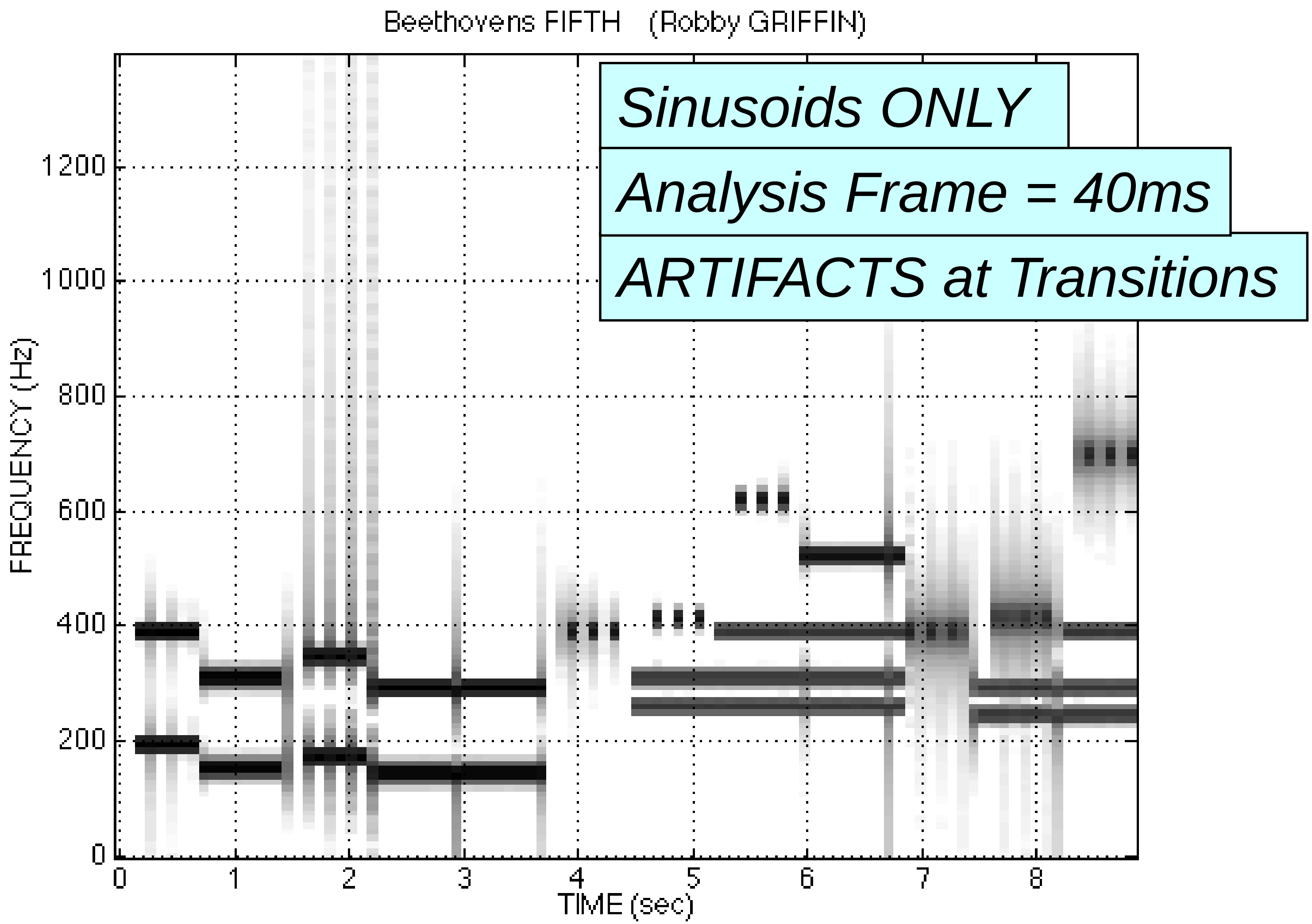
Sinusoids ONLY



From SPECGRAM ANALYSIS PROGRAM

ARTIFACTS at Transitions

Spectrogram of LAB SONG



Time-Varying Frequency

- Frequency can change vs. time
  - Continuously, not stepped
- **FREQUENCY MODULATION (FM)**

$$\boxed{x(t) = \cos(2\pi f_c t + v(t))}$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

New Signal: Linear FM

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC

$$\boxed{x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)}$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define "instantaneous frequency"

INSTANTANEOUS FREQ

- Definition

$$\boxed{\begin{aligned} &x(t) = A\cos(\psi(t)) \\ &\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) \end{aligned}}$$

Derivative of the "Angle"

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$

$$\psi(t) = 2\pi f_0 t + \varphi$$

Makes sense

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = 2\pi f_0$$

INSTANTANEOUS FREQ of the Chirp

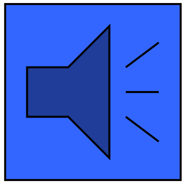
- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

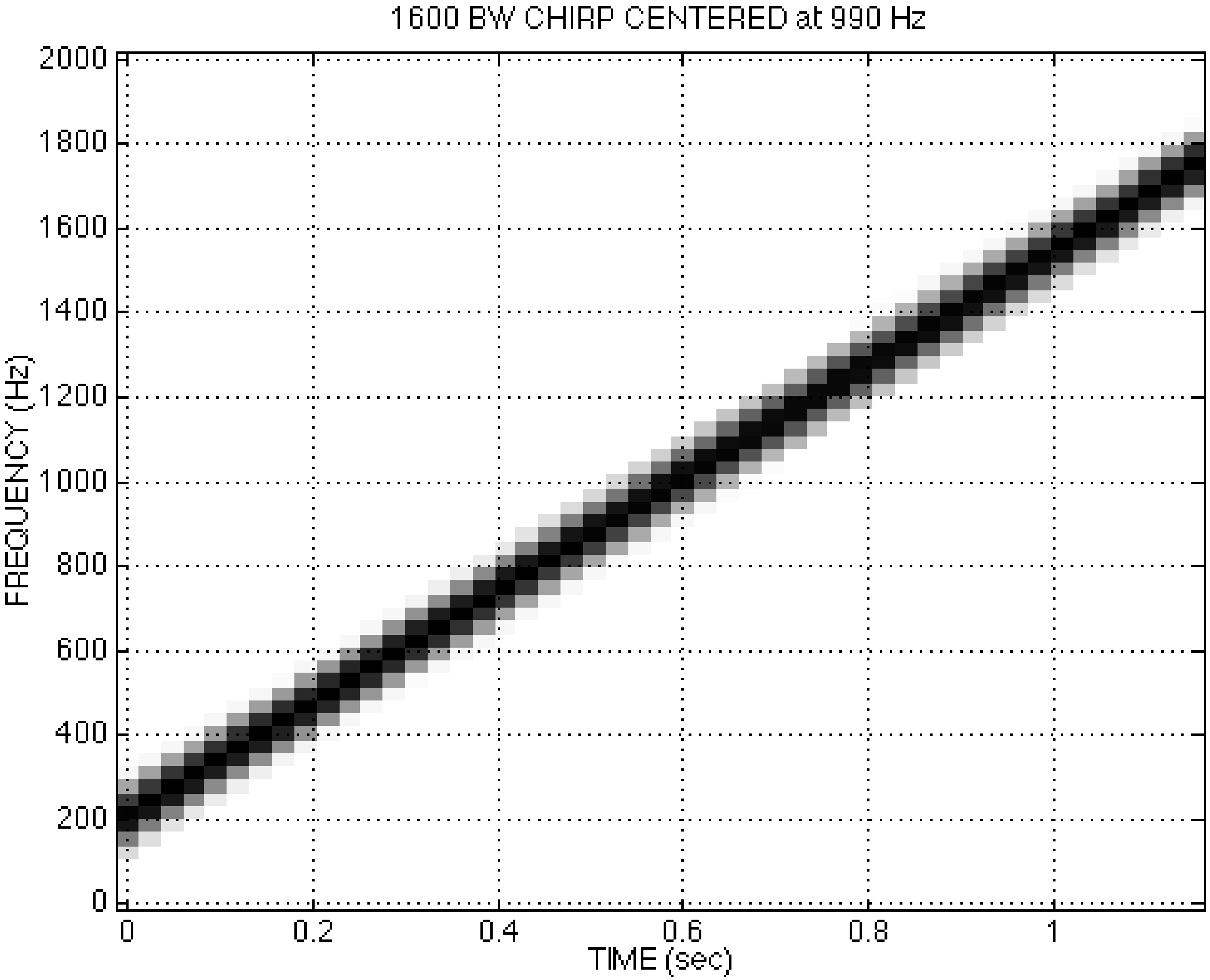
$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$

$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

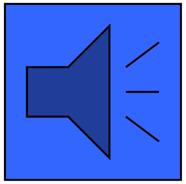
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\alpha t + \beta$$

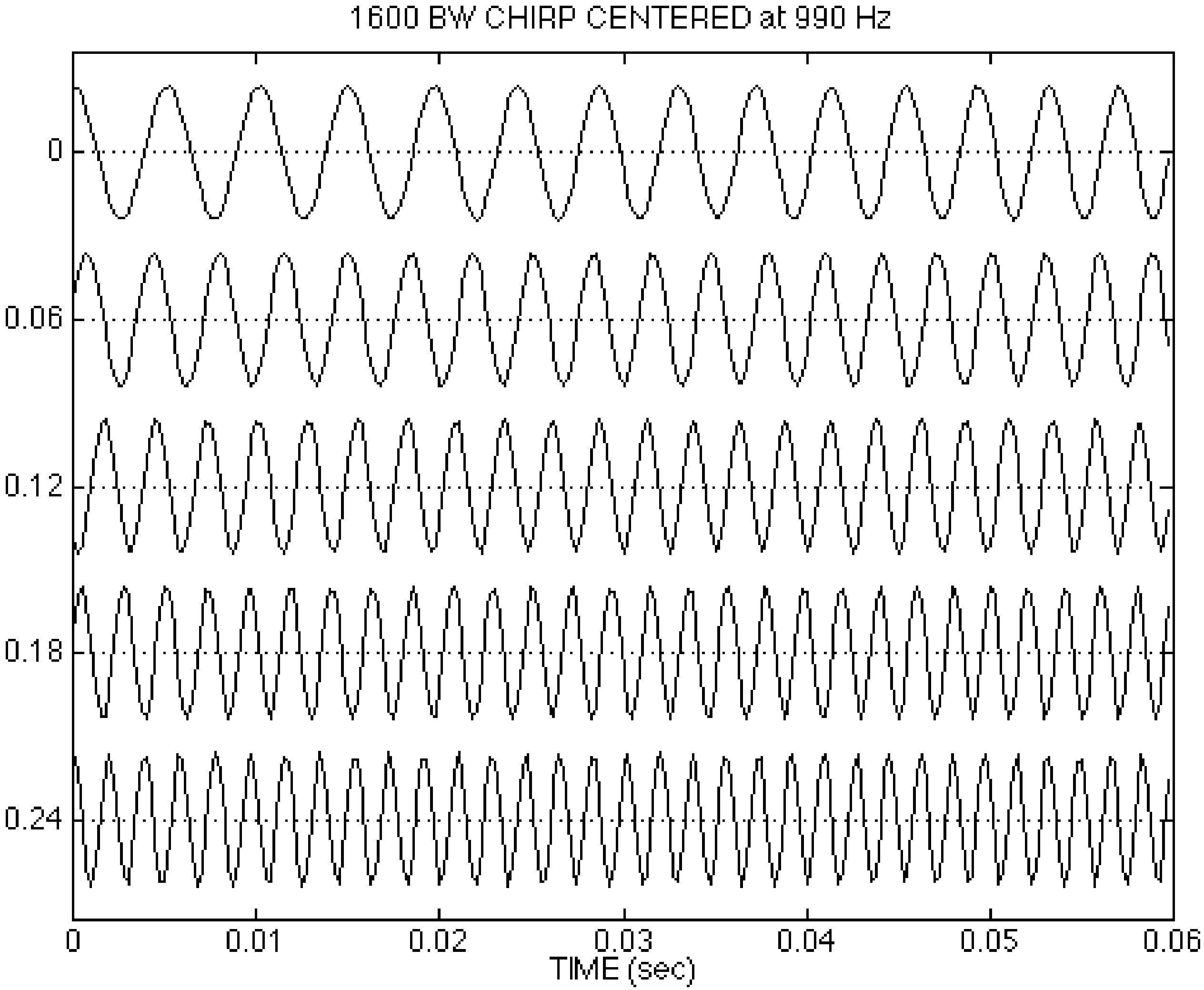
CHIRP SPECTROGRAM





CHIRP WAVEFORM





OTHER CHIRPS

$\psi(t)$ can be anything:

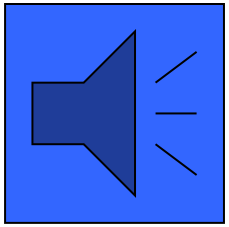
$$x(t) = A\cos(\alpha\cos(\beta t) + \varphi)$$

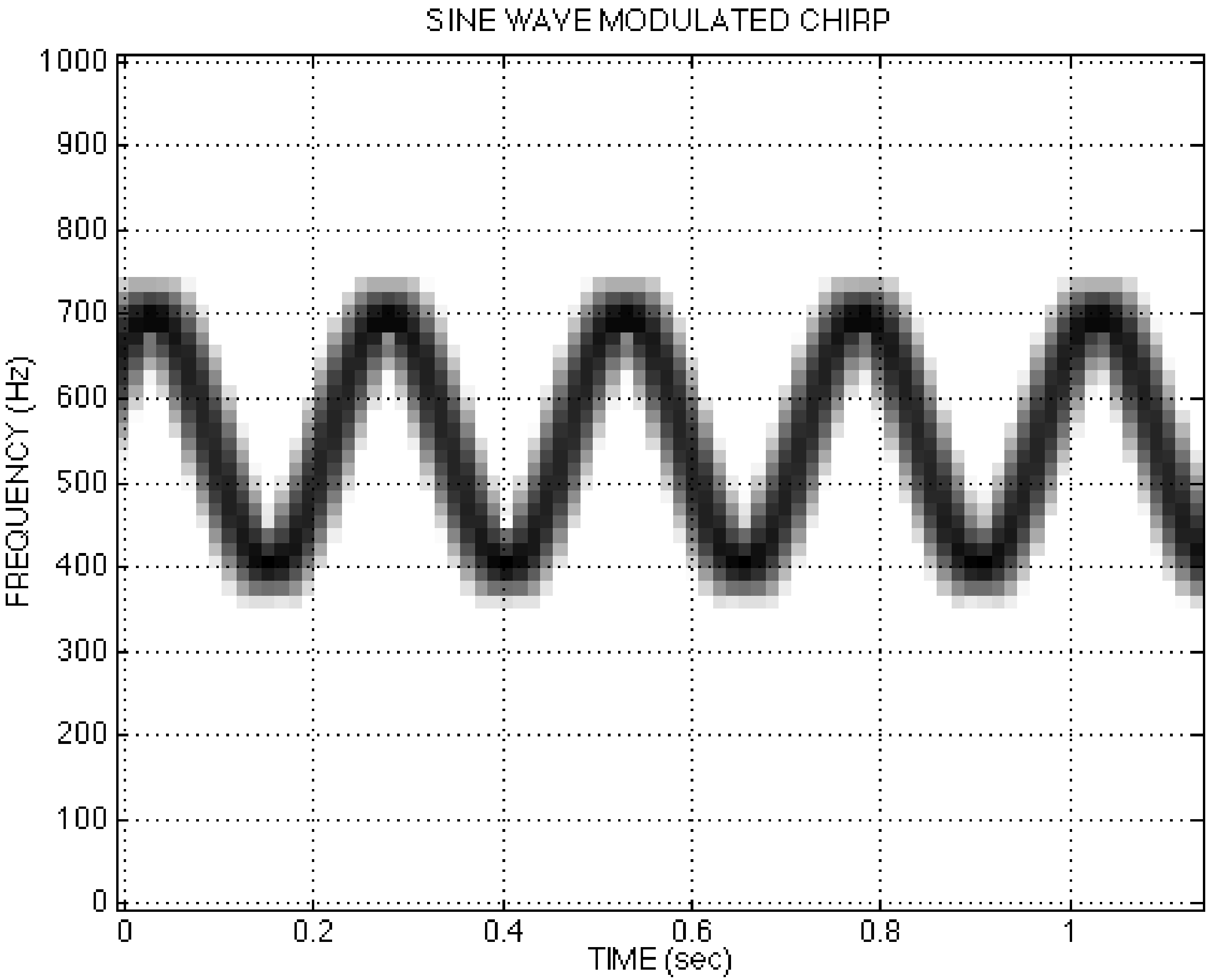
$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = -\alpha\beta\sin(\beta t)$$

$\psi(t)$ could be speech or music:

– FM radio broadcast

SINE-WAVE FREQUENCY MODULATION (FM)





Fourier Series Coefficients

HISTORY

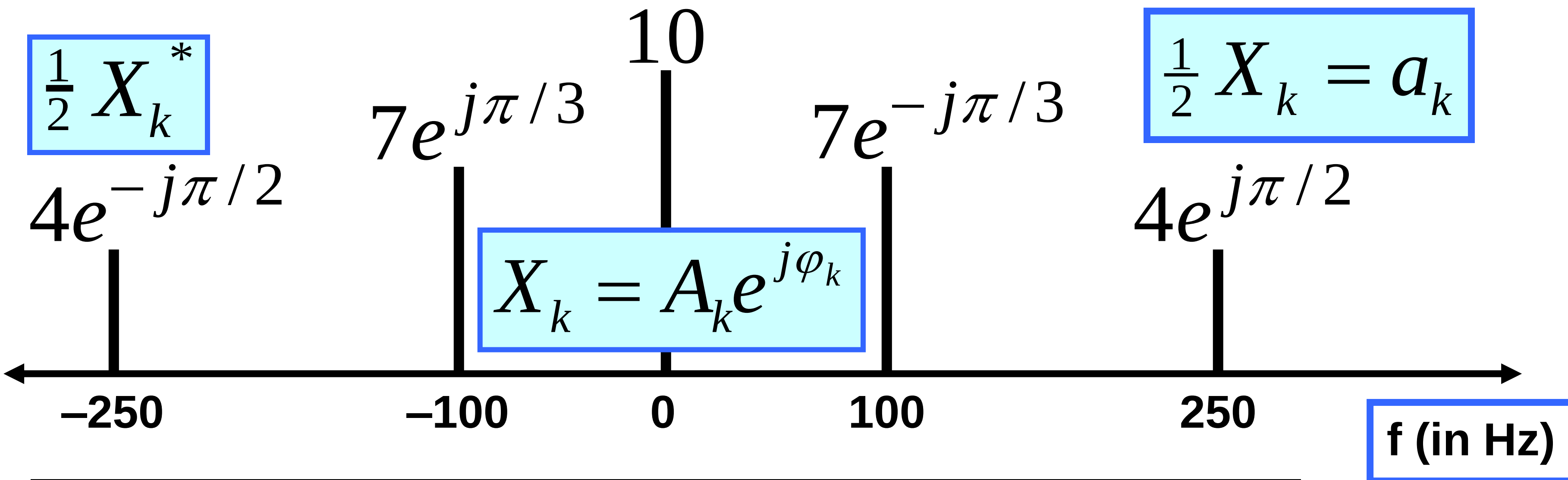
- Jean Baptiste Joseph Fourier (1768-1830)

  - Napoleonic era
  - Studied the mathematical theory of heat conduction
  - Established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric funcions.

- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = \boxed{a_0} + \sum_{k=1}^{N} \left\{ \boxed{a_k} e^{j2\pi f_k t} + \boxed{a_k^*} e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$\boxed{2\pi\left(f_0\right) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}}$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

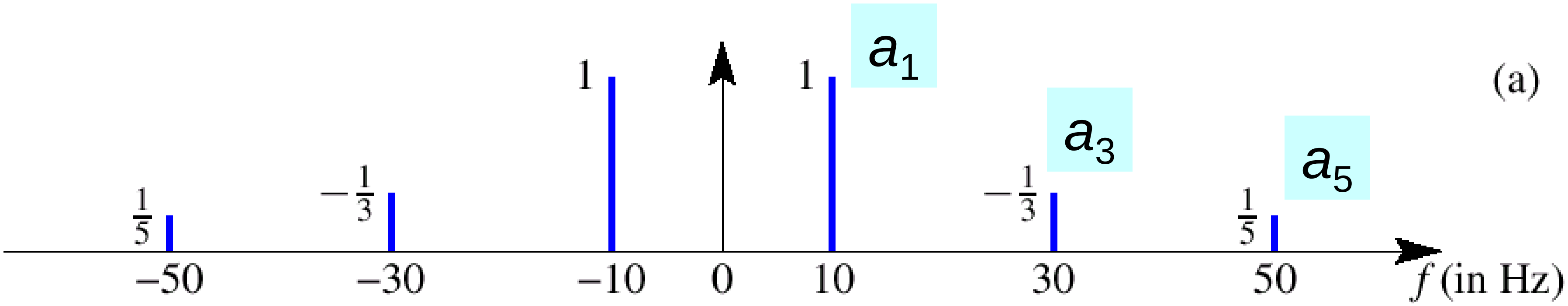
$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

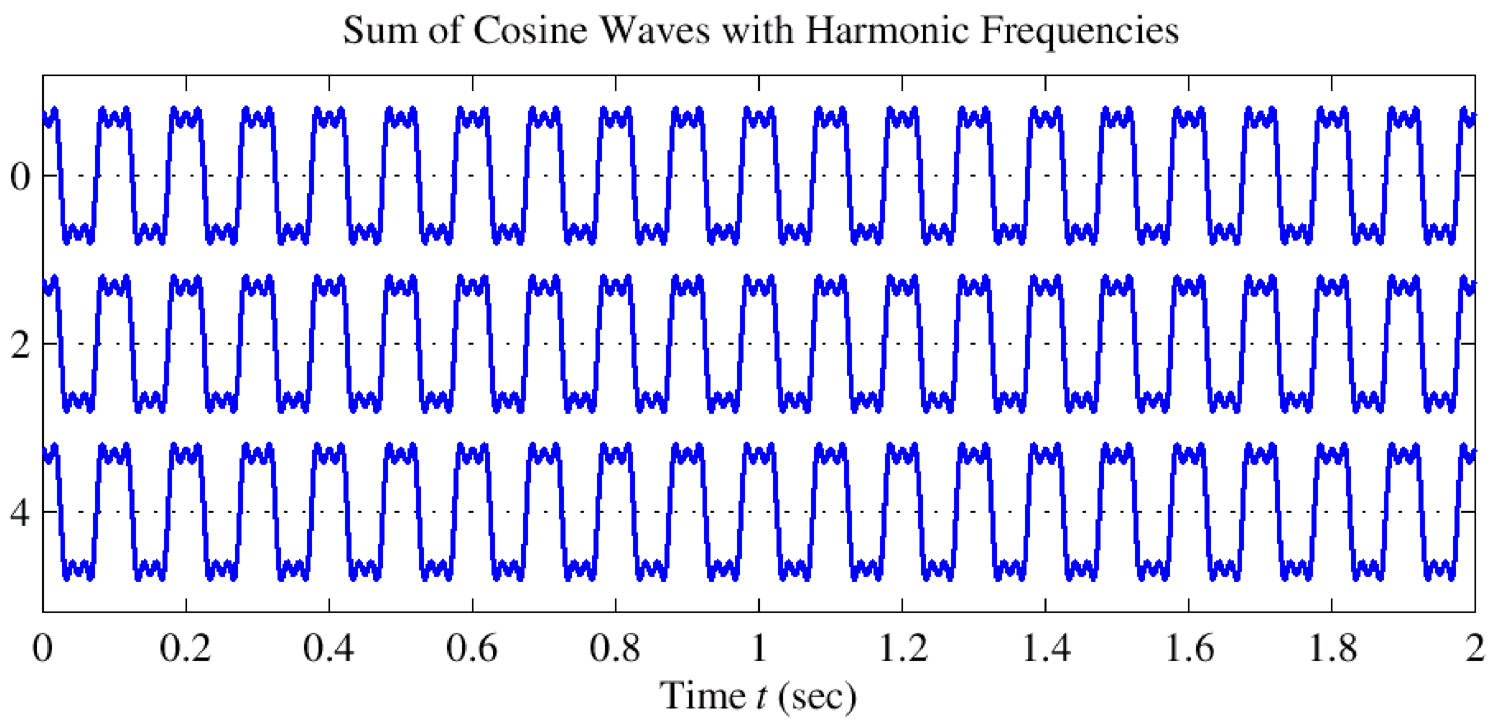
$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

Harmonic Signal (3 Freqs)



T = 0.1



SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - **Easy**
  - Given (ω_k, A_k, ϕ_k) create $x(t)$
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good

- ANALYSIS
  - **Hard**
  - Given $x(t)$, extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$\boxed{a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}dt}$$

INTEGRAL Property of *exp(j)*

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}\,dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}\left(e^{-j2\pi m} - 1\right)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}\,dt = 0$$

$$m \neq 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of *exp*(*j*)

- PRODUCT of *exp*(+*j*) and *exp*(-*j*)

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt} dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t} dt$$

Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\tfrac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \tfrac{1}{T_0}\int_0^{T_0} \left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \right) e^{-j(2\pi/T_0)\ell t} dt$$

$$\tfrac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \sum_{k=-\infty}^{\infty} a_k \left(\tfrac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)kt} e^{-j(2\pi/T_0)\ell t} dt \right) = a_\ell$$

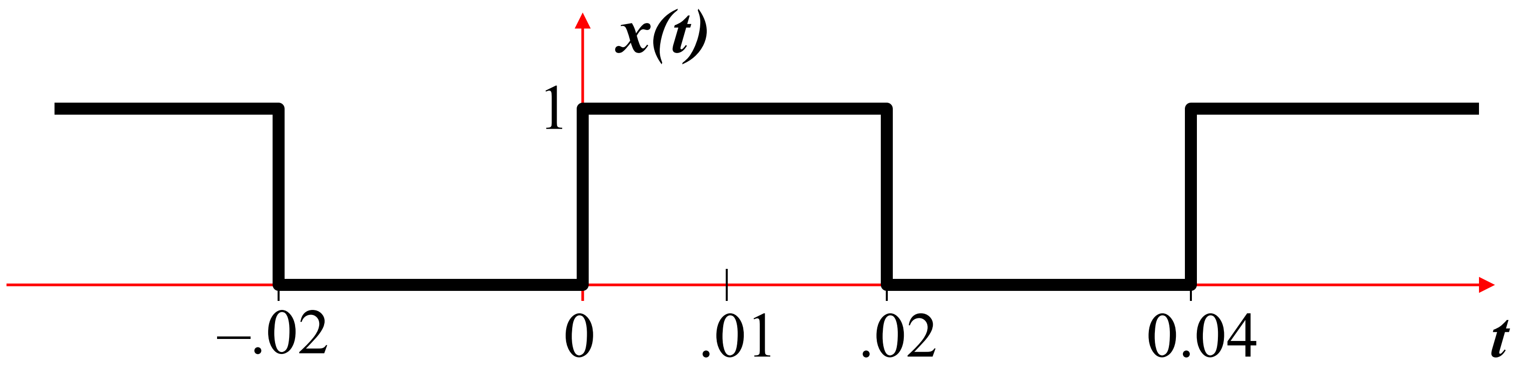
$$\Rightarrow a_k = \tfrac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Integral is zero except for $k = \ell$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04\,\text{sec.}$



FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j(2\pi/.04)kt}dt = \frac{1}{.04(-j2\pi k/.04)}e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient: a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt \qquad (k=0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}(\text{Area})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} 1\,dt = \frac{1}{.04}(.02-0) = \tfrac{1}{2}$$

Fourier Coefficients a_k

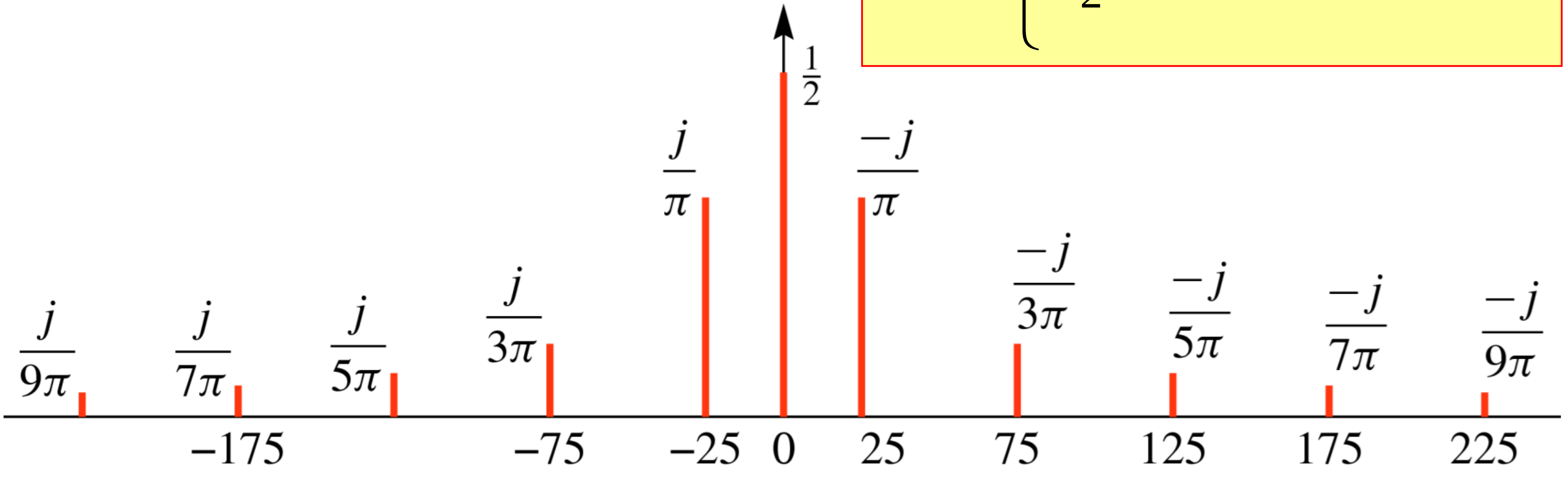
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Integral

- HOW do you determine a_k from $x(t)$?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Fundamental Frequency $f_0 = 1/T_0$

$a_{-k} = a_k^*$ when $x(t)$ is real

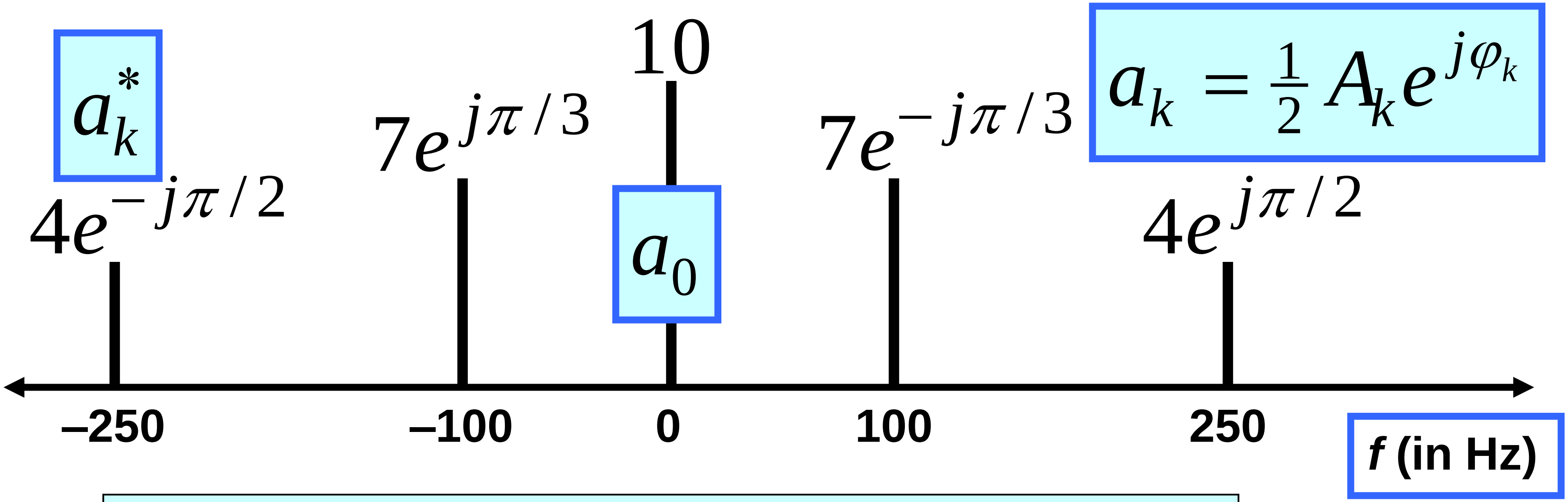
$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

(DC component)

Fourier Series & Spectrum

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k \, e^{j2\pi f_k t} + a_k^* \, e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

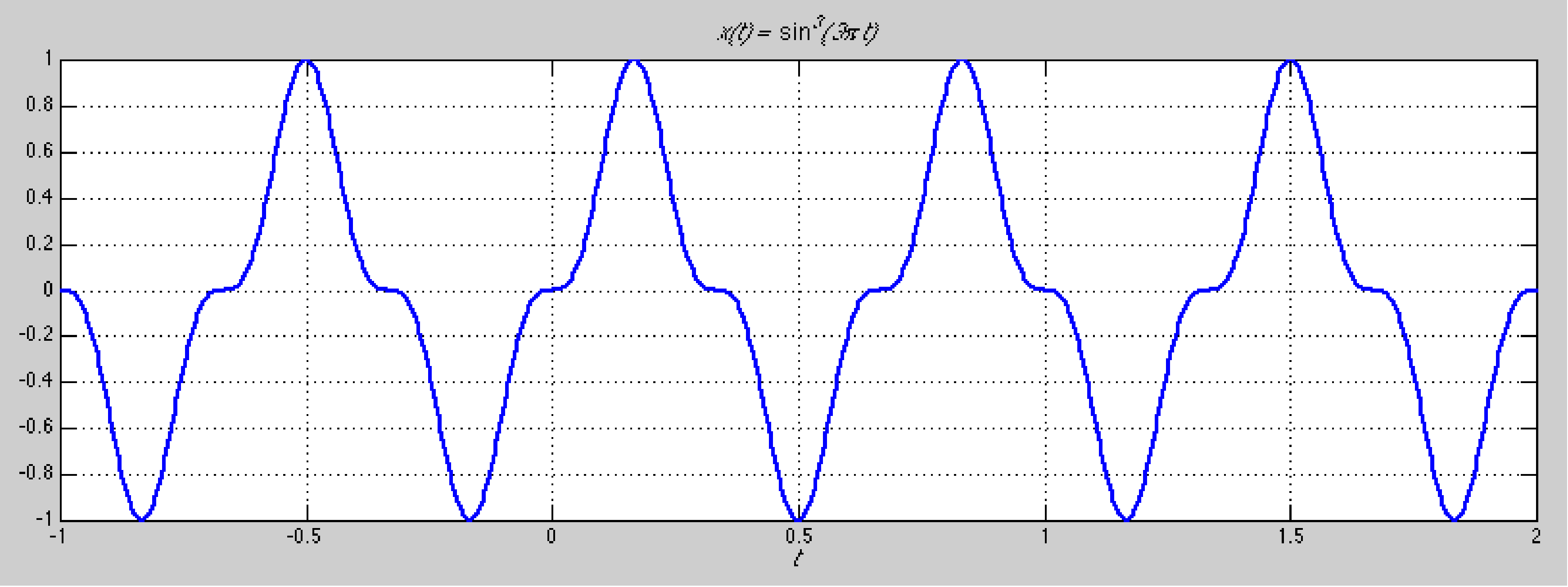
$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$\boxed{2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}}$$

Example

$$x(t) = \sin^3(3\pi t)$$



$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

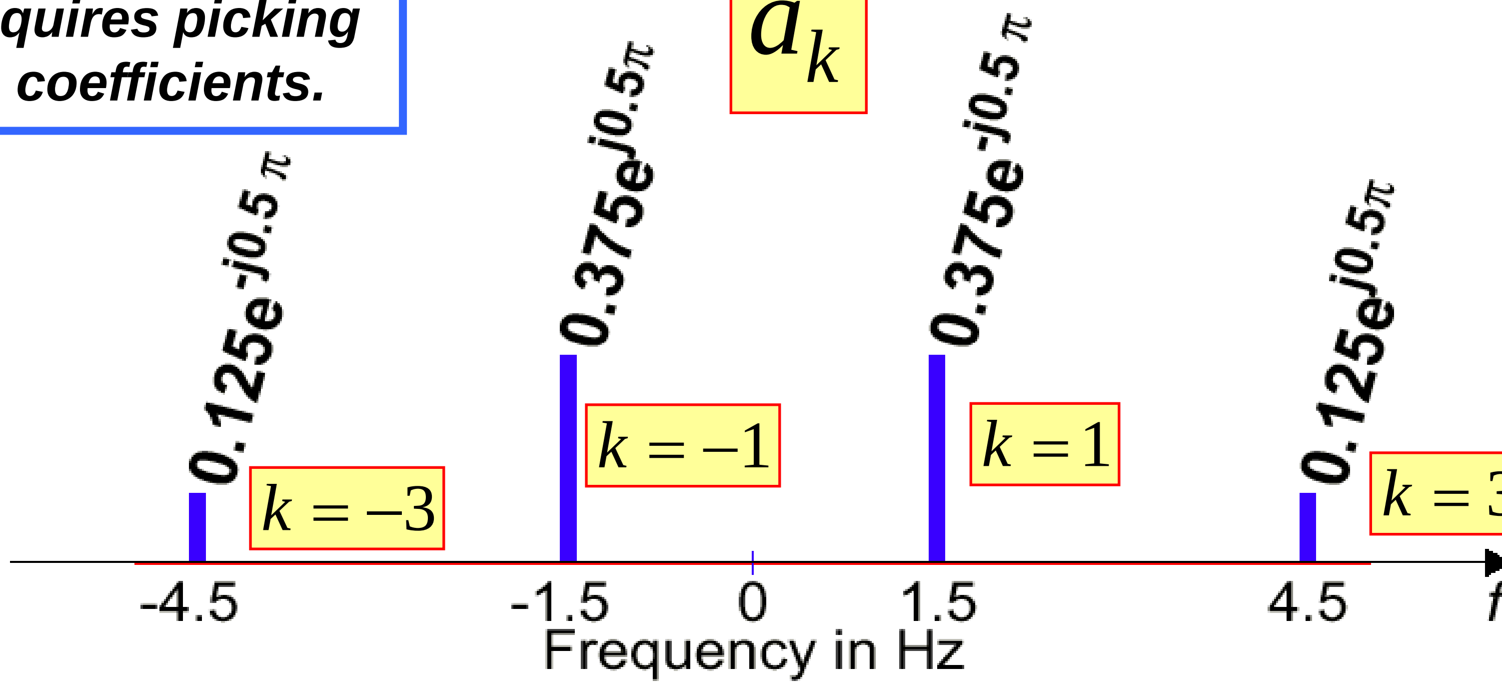
Example

$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.

a_k



STRATEGY: $x(t) \rightarrow a_k$

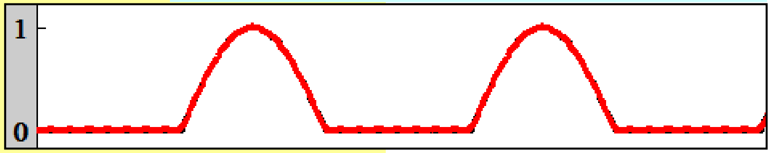
- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$\boxed{a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}\,dt}$$

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$a_k = \frac{1}{T_0}\int_0^{T_0/2} \sin(\tfrac{2\pi}{T_0}t)\, e^{-j(2\pi/T_0)kt}dt$$

$$= \frac{1}{T_0}\int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}dt$$

$$= \frac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t}dt - \frac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t}dt$$

$$= \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

$$= \frac{1}{4\pi(k-1)}\left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1\right) - \frac{1}{4\pi(k+1)}\left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1\right)$$

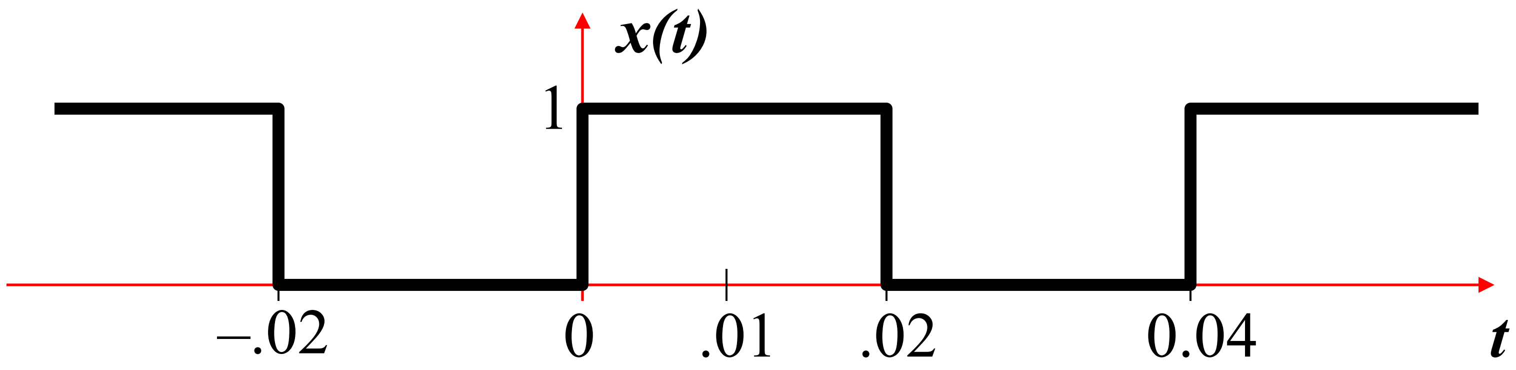
$$= \frac{1}{4\pi(k-1)}\left(e^{-j\pi(k-1)} - 1\right) - \frac{1}{4\pi(k+1)}\left(e^{-j\pi(k+1)} - 1\right)$$

$$= \left(\frac{k+1-(k-1)}{4\pi(k^2-1)}\right)\left(-(-1)^k - 1\right) = \begin{cases} 0 & k \text{ odd} \\ \boxed{\pm\frac{1}{j4}} & k = \pm 1 \\ \frac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases}$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2} T_0 \\ 0 & \tfrac{1}{2} T_0 \le t < T_0 \end{cases}$$

$$\text{for } T_0 = 0.04 \text{ sec.}$$



Fourier Coefficients a_k

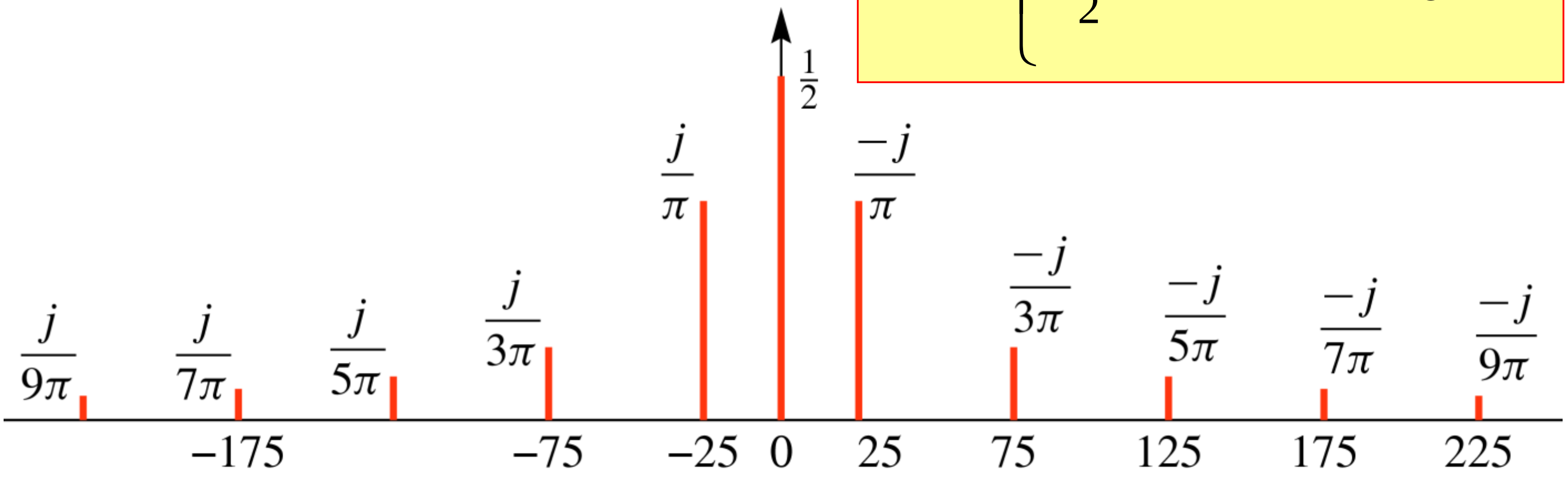
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Synthesis

- HOW do you **APPROXIMATE** $x(t)$?

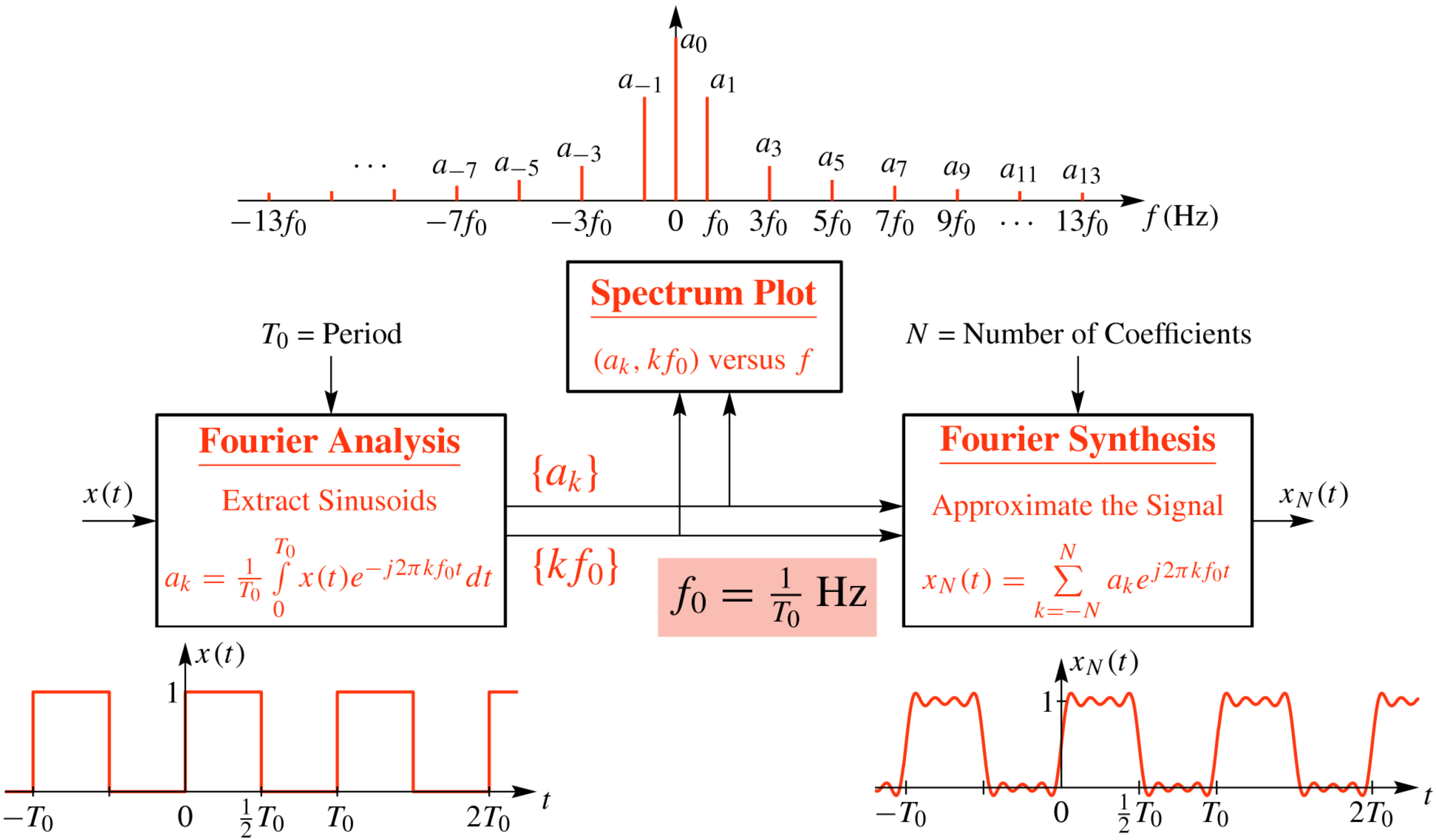
$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

- Use FINITE number of coefficients

$$\boxed{x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}}$$

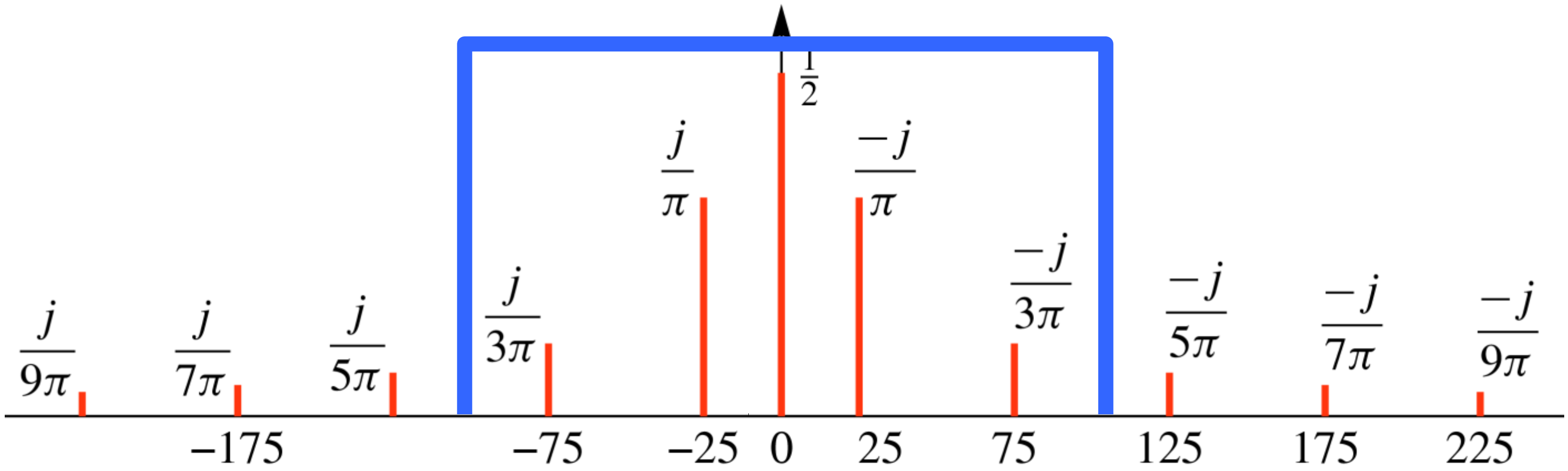
$$\boxed{a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}}$$

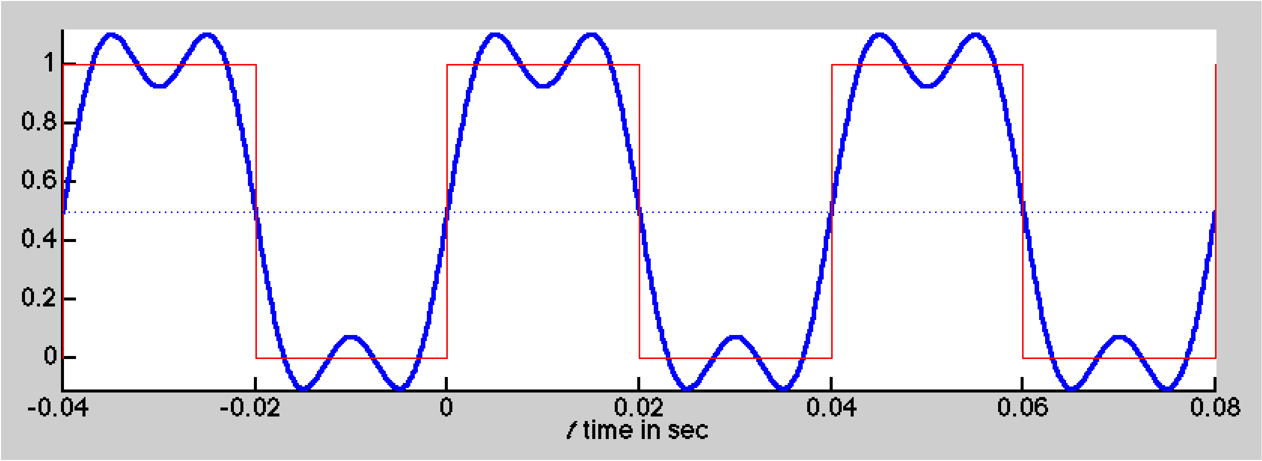
Fourier Series Synthesis



Synthesis: 1st & 3rd Harmonics

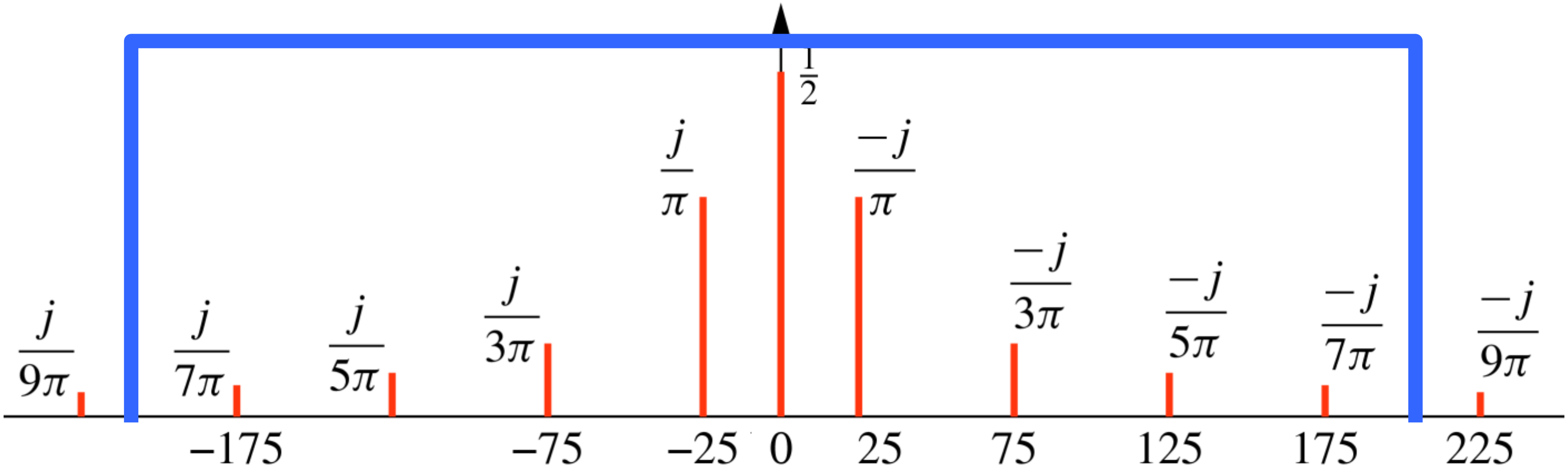
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

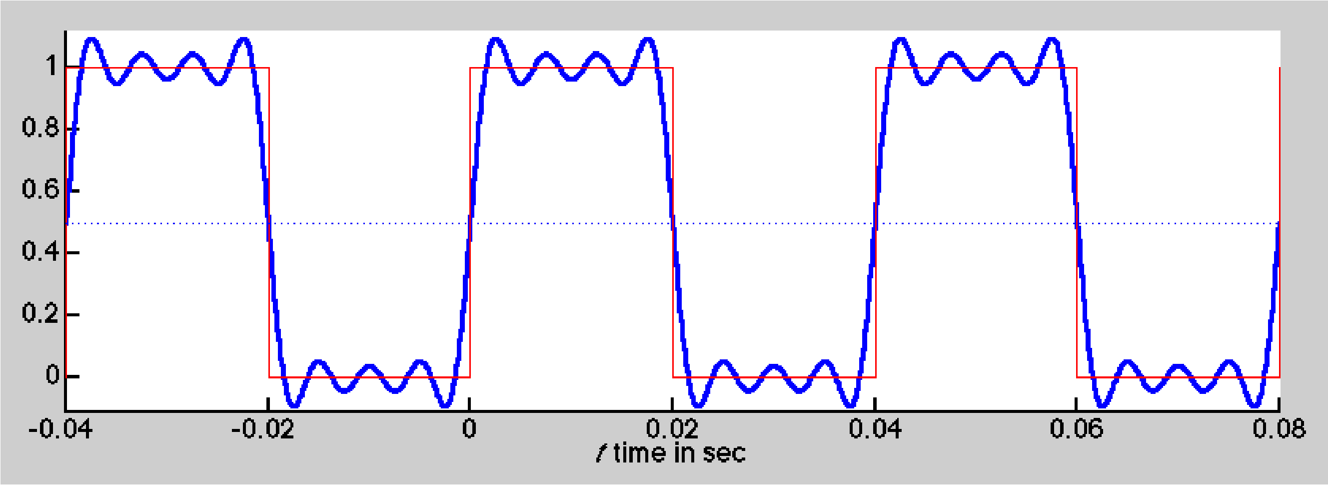




Synthesis: up to 7th Harmonic

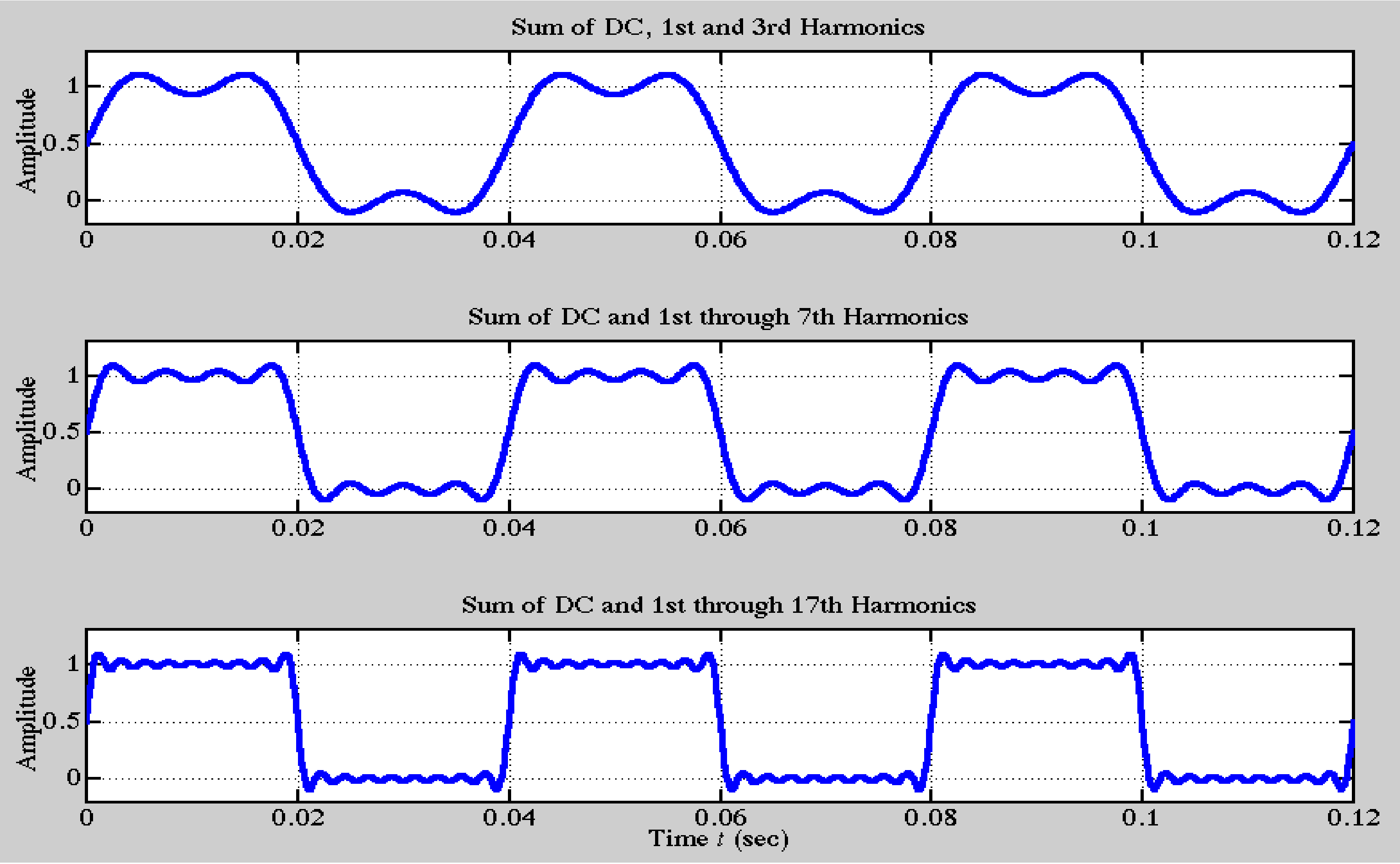
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$





Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



Gibbs' Phenomenon

- Convergence at DISCONTINUITY of $x(t)$
  - There is always an overshoot
  - 9% for the Square Wave case