PROBLEM:

We have seen that musical tones can be modeled mathematically by sinusoidal signals. If you read music or play the piano you are aware of the fact that the piano keyboard is divided into octaves, with the tones in each octave being twice the frequency of the corresponding tones in the next lower octave. To calibrate the frequency scale, the reference tone is the A above middle-C, which is usually called A440 since its frequency is 440 Hz. Each octave contains 12 tones, and the ratio between the frequencies of successive tones is constant. Since middle C is 9 tones below A440, its frequency is approximately $(440)2^{-9/12} \approx 262$ Hz. The names of the tones (notes) of the octave starting with middle-C and ending with high-C are:

note name	C	$C^{\#}$	D	$E^{\flat}$	E	F	$F^{\#}$	G	$G^{\#}$	A	$B^{\flat}$	B	C
note number	40	41	42	43	44	45	46	47	48	49	50	51	52
frequency													

(a) Explain why the ratio of the frequencies of successive notes must be $2^{1/12}$.

(b) Make a table of the frequencies of the tones of the octave beginning with middle-C assuming that A above middle C (note #49) is tuned to 440 Hz.

(c) The above notes on a piano are numbered 40 through 52. If n denotes the note number, and f denotes the frequency of the corresponding tone, give a formula for the frequency of the tone as a function of the note number.

(d) A *chord* is a combination of musical notes sounded simultaneously. A *triad* is a three note chord. The E Minor chord is composed of the tones of E, G, B sounded simultaneously. From the set of corresponding frequencies determined in part (a), make a sketch of the essential features of the spectrum of the E-Minor chord assuming that each note is realized by a pure sinusoidal tone and that each note is equally loud. (You do not have to specify the complex amplitudes precisely.)

PROBLEM:

Suppose that a MATLAB function has been written to calculate a sum of discrete-time sinusoids, e.g., something similar to the `makecos( )` that was written for the lab. Here is the actual function:

```
function  xn = makedcos(omegahat,ZZ,Length)
xn = real( exp( j*(0:Length-1)'*omegahat(:)' ) * ZZ(:) );
```

If the following MATLAB command is used to make an output sound:

```
soundsc( makedcos(pi*(0.5:0.4:1.5),[-2i,1i,3-4i],1000000), 1000 )
```

(a) Draw a plot of the discrete-time spectrum (vs. $\hat{\omega}$) of the discrete-time signal defined by this MATLAB operation. Make sure that you include all the spectrum components in the $-\pi$ to $+\pi$ interval.

(b) Draw a plot of the continuous-time spectrum (vs. f in Hz) of the analog output signal defined by the `soundsc()` function.

PROBLEM:

A linear-FM "chirp" signal is one that sweeps in frequency from $\omega_1 = 2\pi f_1$ to $\omega_2 = 2\pi f_2$ as time goes from $t = 0$ to $t = T_2$. We can define the *instantaneous frequency* of the chirp as the derivative of the phase of the sinusoid:

$$x(t) = A\cos(\alpha t^2 + \beta t + \phi) \tag{1}$$

where the cosine function operates on a time-varying argument

$$\psi(t) = \alpha t^2 + \beta t + \phi$$

The derivative of the argument $\psi(t)$ is the *instantaneous frequency* which is also the audible frequency heard from the chirp *if the chirping frequency does not change too rapidly.*

$$\omega_i(t) = \frac{d}{dt}\psi(t) \qquad \text{radians/sec} \tag{2}$$

There are examples on the CD-ROM in the Chapter 3 demos.

(a) For the linear-FM "chirp" in (1), determine formulas for the beginning instantaneous frequency (ω_1) and the ending instantaneous frequency (ω_2) in terms of α, β and T_2. For this problem, assume that the starting time of the "chirp" is $t = 0$.

(b) For the "chirp" signal

$$x(t) = \Re e\left\{e^{j2\pi(29t^2 - 100t)}\right\}$$

derive a formula for the *instantaneous* frequency versus time. Should your answer for the frequency be a positive number?

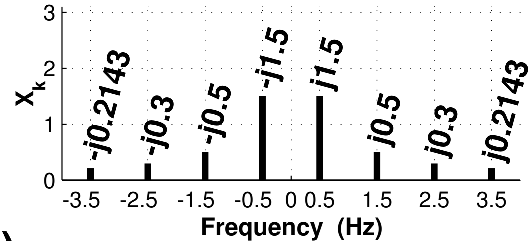
(c) For the signal in part (b), make a plot of the *instantaneous* frequency (in Hz) versus time over the range $0 \le t \le 1$ sec.

PROBLEM:

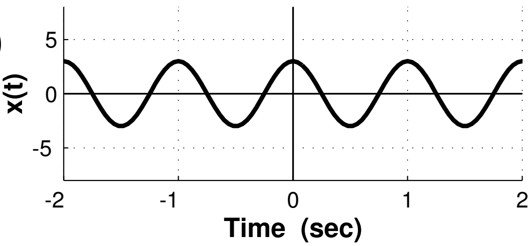
Several signals are plotted below along with their corresponding spectra. However, they are in a random order. For each of the signals (a)–(e), determine the correct complex spectrum (1)–(5). Write your answers in the following table:

(a)	**(b)**	**(c)**	**(d)**	**(e)**

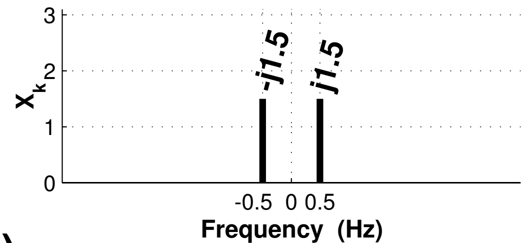
(1)



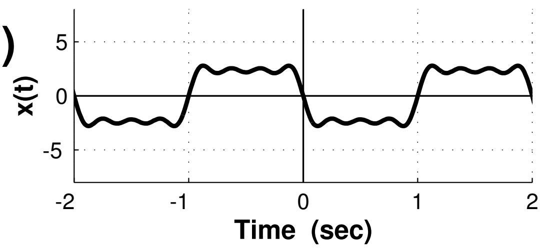
(a)



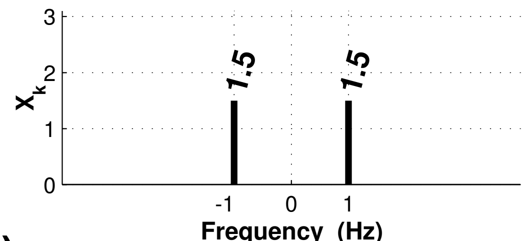
(2)



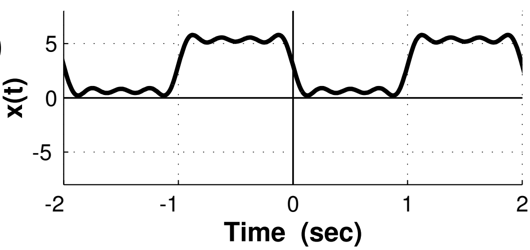
(b)



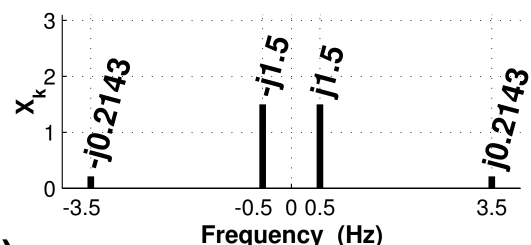
(3)



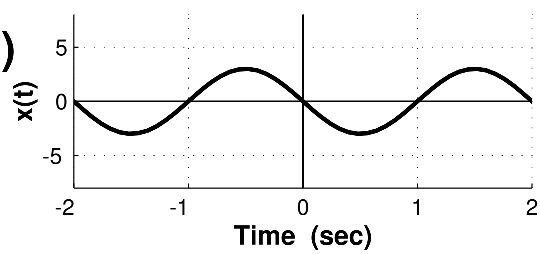
(c)



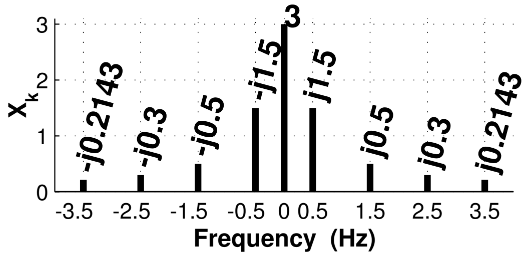
(4)



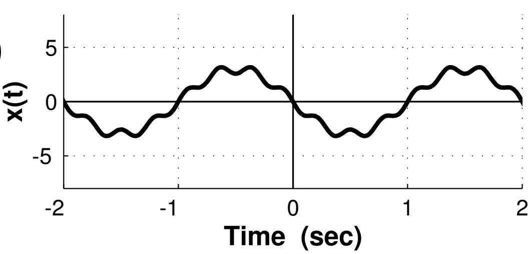
(d)



(5)



(e)

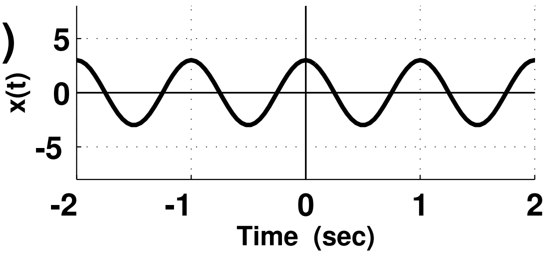


PROBLEM:

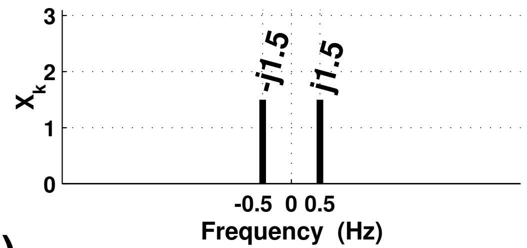
Several signals are plotted below along with their corresponding spectra. However, they are in a random order. For each of the signals (a)–(e), determine the correct complex spectrum (1)–(5). Write your answers in the following table:

(a)	(b)	(c)	(d)	(e)

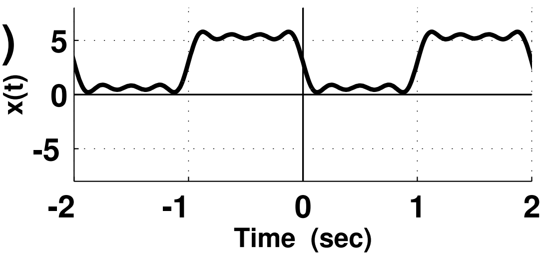
(a)



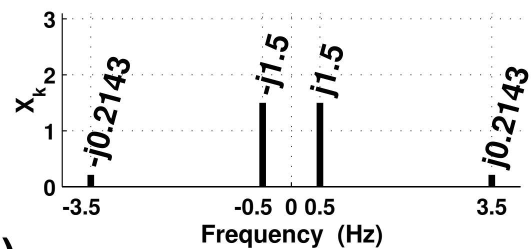
(1)



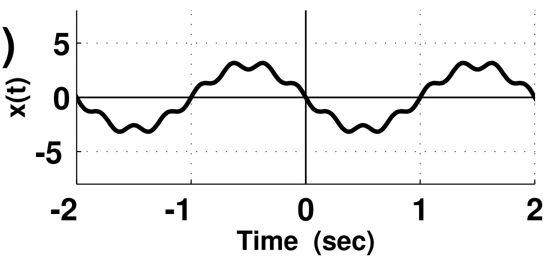
(b)



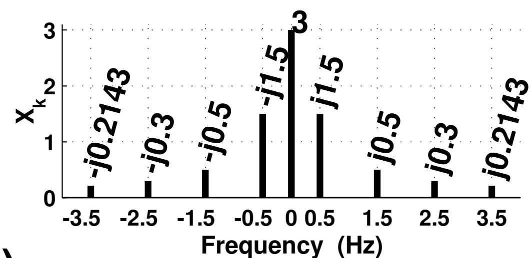
(2)



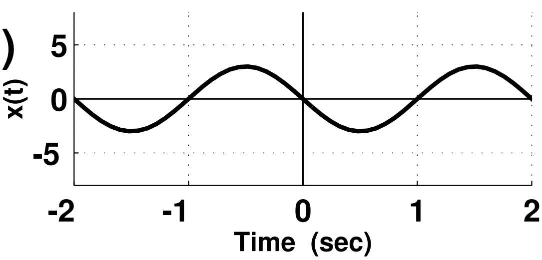
(c)



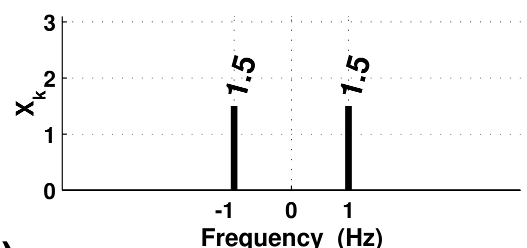
(3)



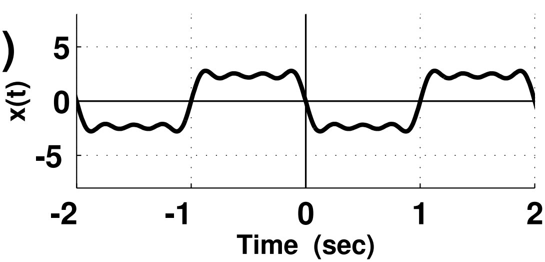
(d)



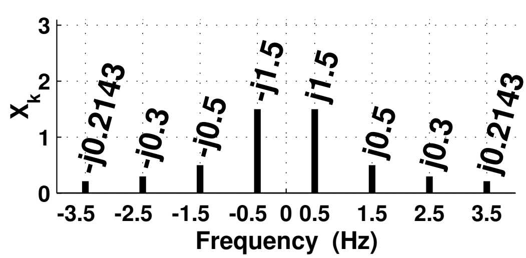
(4)



(e)

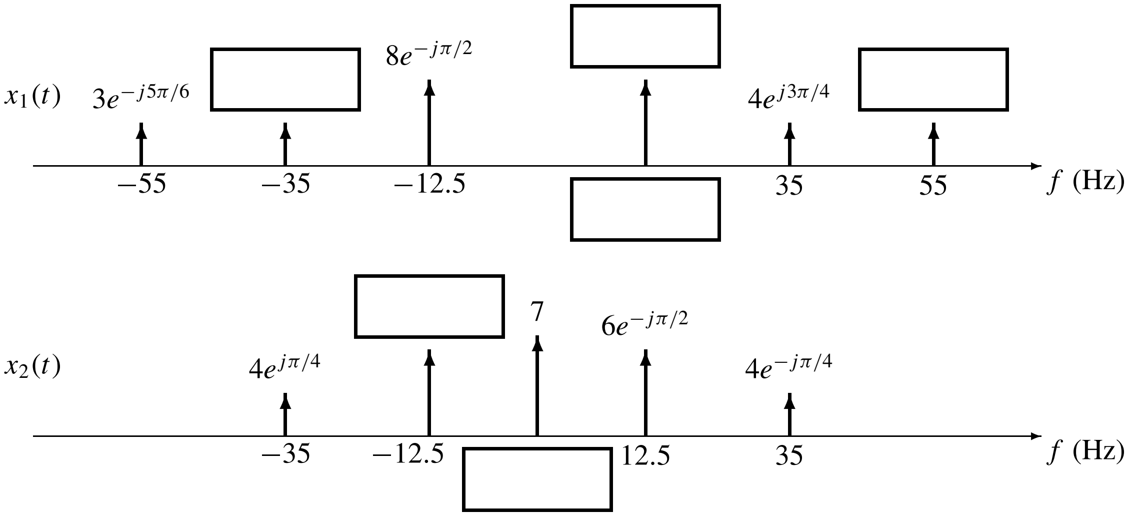


(5)



PROBLEM:

(a) The incomplete spectra for two *real* signals $x_1(t)$ and $x_2(t)$ are shown in the following figures. Fill in the empty boxes for the missing components.



(b) Write an equation for $x_2(t)$ in terms of cosine functions.

(c) Draw the spectrum representation for $x_3(t) = x_1(t) + x_2(t)$.

(d) Is $x_3(t) = x_1(t) + x_2(t)$ periodic? If so, what is the fundamental frequency?

PROBLEM:

Define $x(t)$ as

$$x(t) = 20 + 60\cos((2\pi/15)t - 2\pi/3)$$

(a) What is the fundamental period T_0 of $x(t)$?

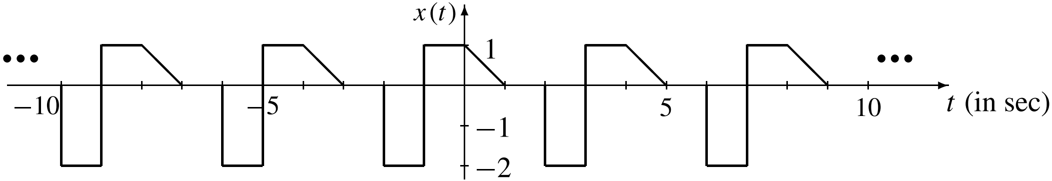
(b) What is the time shift t_m of $x(t)$?

(c) Draw a detailed plot of $x(t)$ over the domain $|t| \leq \frac{3}{2}T_0$. Label carefully and include the amplitude, t_m, and T_0.

(d) Define $y(t) = x(t - t_0)$. Find t_0 so that the signal $y(t)$ has its maximum value at $t = 0$. There are an infinite number of t_0s, so give the general form.

PROBLEM:

Suppose that a periodic signal $x(t)$ is defined as follows:



(a) Determine the $\boxed{\text{DC value}}$ of $x(t)$.

(b) Is $x(t)$ $\boxed{\text{bandlimited}}$? If so, give the maximum frequency. If not, explain why.

(c) Does $x(t)$ have a $\boxed{\text{fundamental frequency}}$? If so, give the frequency. If not, explain why.

(d) Determine the $\boxed{\text{instantaneous frequency } \omega_i(t)}$ at times $t = 1.5$ and $t = -0.6$ seconds? *This is easy, just think carefully.*

(e) Write the $\boxed{\text{Fourier integral}}$ expression for the coefficient a_3 in terms of the specific signal $x(t)$ defined above. *Set up all the specifics of the integral (e.g., limits of integration), but do not evaluate the integral. All parameters in the integral should have numeric values.*

PROBLEM:

For the chirp signal

$$x(t) = \Re e\{e^{j(100t^2+30t+77)}\}$$

Make a plot of the (instantaneous) frequency versus time over the range $0 \leq t \leq 1$ sec.

PROBLEM:

For the chirp signal

$$x(t) = \Re e\{e^{j\theta(t)}\} = \cos(2\pi(\alpha t^2 + \beta t + \phi))$$

(a) Determine values for α, β, and ϕ, so that the instantaneous frequency of $x(t)$ will start at 3800 Hz and end at 800 Hz over the time interval $0 \leq t \leq 3$ seconds.

(b) Make a plot of the (instantaneous) frequency versus time over the range $0 \leq t \leq 3$ sec.

PROBLEM:

Consider the signal

$$x(t) = 20\cos(\omega_1 t + \pi/3) + 10\cos(\omega_2 t)$$

(a] Sketch the (two-sided) spectrum of the signal for the case where $0 < \omega_1 < \omega_2$. Indicate the size of the complex phasors for each frequency.

(b] How should ω_1 and ω_2 be chosen so that $x(t)$ can be expressed in the form

$$x(t) = A\cos(300\pi t + \phi)$$

(You only need to find ω_1 and ω_2; you do not have to find A and ϕ.)

(c) If $0 < \omega_1 < \omega_2$, how should ω_1 and ω_2 be chosen so that $x(t)$ is periodic with period $T = .1$; i.e., so that $x(t + 0.1) = x(t)$ for all t?

PROBLEM:

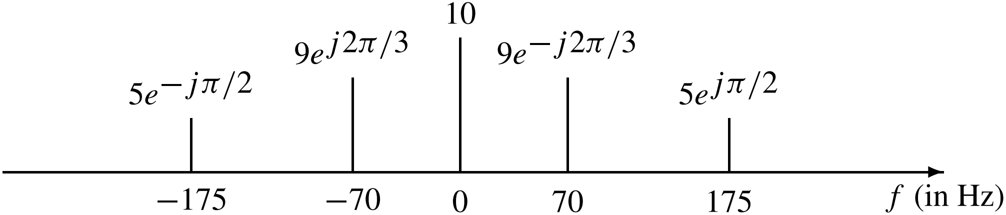
For the "chirp" signal

$$x(t) = \Re e\{e^{j(40t^2 + 27t + 13)}\}$$

Make a plot of the *instantaneous* frequency (in Hertz) versus time over the range $0 \le t \le 1$ sec.

PROBLEM:

A signal $x(t)$ has the two-sided spectrum representation shown below.



(a) Write an equation for $x(t)$.

(b) Is $x(t)$ a periodic signal? If so, what is its period?

(c) Prove that any real-valued sinusoid such as

$$y(t) = A\cos(\omega_0 t + \phi)$$

has a spectrum consisting of two components: one in negative frequency and one in positive frequency. Determine the complex amplitudes (Z_k) that go with each spectral component.
Hint: use the identity $\Re e\{z\} = \frac{1}{2}(z + z^*)$

PROBLEM:

Let $x(t) = \sin^3(27\pi t)$.

(a) Determine a formula for $x(t)$ as a sum of complex exponentials.

(b) What is the fundamental period for $x(t)$?

(c) Plot the *spectrum* for $x(t)$.

PROBLEM:

An amplitude modulated (AM) cosine wave is represented by the formula

$$x(t) = [A + \sin(\omega_0 t)]\sin(\omega_c t)$$

where $0 < \omega_0 \ll \omega_c$.

(a) Use *phasors* to show that $x(t)$ can be expressed in the form:

$$x(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) + A_3 \cos(\omega_3 t + \phi_3)$$

where $\omega_1 < \omega_2 < \omega_3$; i.e., find $A_1,\ A_2,\ A_3,\ \phi_1,\ \phi_2,\ \phi_3,\ \omega_1,\ \omega_2,\ \omega_3$ in terms of A, ω_0, and ω_c.

(b) Sketch the two-sided spectrum of this signal on a frequency axis. Be sure to label important features of the plot. Label your plot in terms of the unknown As and ωs.

PROBLEM:

An amplitude modulated (AM) cosine wave is represented by the formula

$$x(t) = [7 + 2\cos(\pi t + \pi/3)]\sin(11\pi t)$$

(a) Use *phasors* to show that $x(t)$ can be expressed in the form:

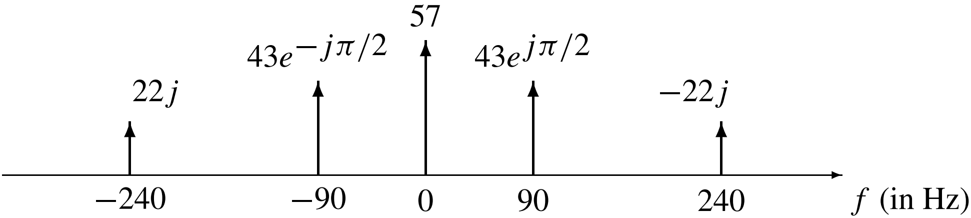
$$x(t) = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2) + A_3\cos(\omega_3 t + \phi_3)$$

where $\omega_1 < \omega_2 < \omega_3$; i.e., find $A_1,\ A_2,\ A_3,\ \phi_1,\ \phi_2,\ \phi_3,\ \omega_1,\ \omega_2,\ \omega_3$ in terms of A, ω_0, and ω_c.

(b) Sketch the two-sided spectrum of this signal on a frequency axis. Be sure to label important features of the plot. Label your plot in terms of the numerical values of the A_i's ϕ_i's and ω_i's.

PROBLEM:

A signal $x(t)$ has the two-sided spectrum representation shown below.



(a) Write an equation for $x(t)$.

(b) Is $x(t)$ a periodic signal? If so, what is its period?

PROBLEM:

The following MATLAB program makes a plot of a "cosine-times-cosine" signal:

```
tt = 0:0.01:2;
xx = cos(15*pi*tt) .* cos(2pi*tt);
plot(tt,xx)
```

(a) Make a sketch of the plot that will be done by MATLAB. Label the time axis carefully.

(b) The "spectrum" diagram gives the frequency content of a signal. Draw a sketch of the spectrum of the signal represented by `xx`. Label the frequencies and complex amplitudes of each component.

PROBLEM:

We have seen that musical tones can be modeled mathematically by sinusoidal signals. If you read music or play the piano you know well that the piano keyboard is divided into octaves, with the tones in each octave being twice the frequency of the corresponding tones in the next lower octave. To calibrate the frequency scale, the reference tone is the A above middle-C, which is usually called A440 since its frequency is 440 Hz. Each octave contains 12 tones, and the ratio between the frequencies of successive tones is constant. Thus, the ratio must be $2^{1/12}$. Since middle C is 9 tones below A440, its frequency is approximately $(440)2^{-9/12} \approx 262$ Hz. The names of the tones (notes) of the octave starting with middle-C and ending with high-C are:

$$C \quad C^{\sharp} \quad D \quad E^{\flat} \quad E \quad F \quad F^{\sharp} \quad G \quad G^{\sharp} \quad A(440) \quad B^{\flat} \quad B \quad C$$

(a) Make a table of the twelve frequencies of the tones of the octave beginning with middle-C assuming that A above middle C is tuned to 440 Hz.

(b) A chord in the C-major scale is composed of the tones of C E G sounded simultaneously. From the set of corresponding frequencies determined in part (a), make a sketch of the essential features of the spectrum of the chord assuming that each note is realized by a pure sinusoidal tone. (You do not have to specify the complex phasors precisely.)

PROBLEM:

It might be difficult to see why the derivative of the phase would be the instantaneous frequency. The following experiment provides a clue.

(a) Use the following parameters to define a "chirp" signal:

$$\begin{aligned}\omega_1 &= 2\pi(1) \text{ rad/sec} \\ \omega_2 &= 2\pi(9) \text{ rad/sec} \\ T_2 &= 2 \text{ sec}\end{aligned}$$

In other words, determine α and β in equation (**??**) to define $x(t)$ so that it sweeps the specified frequency range.

(b) Now make a plot of the signal synthesized in part (a). Pick a time sampling interval that is small enough so that the plot is very smooth. Put this plot in the middle panel of a 3×1 subplot, i.e., `subplot(3,1,2)`.

(c) It is difficult to verify whether or not this chirp signal will have the correct frequency content. However, the rest of this problem is devoted to an experiment that will demonstrate that the derivative of the phase is the "correct" definition of instantaneous frequency. First of all, make a plot of the instantaneous frequency $f_i(t)$ (in Hz) versus time.

(d) Now generate and plot a 4 Hz sinusoid. Put this plot in the upper panel of a 3×1 subplot, i.e., `subplot(3,1,1)`.

(e) Finally, generate and plot an 8 Hz sinusoid. Put this plot in the lower panel of a 3×1 subplot, i.e., `subplot(3,1,3)`.

(f) Compare the three signals and comment on the frequency content of the chirp. Concentrate on the frequency of the chirp in the time range $1.6 \leq t \leq 2$ sec. Which sinusoid matches the chirp in this time region? Compare the expected $f_i(t)$ in this region to 4 Hz and 8 Hz.

PROBLEM:

A signal composed of sinusoids is given by the equation

$$x(t) = 2\cos(6\pi t) + 3\cos(10\pi t - \pi/4)$$

(a) Sketch the spectrum of this signal indicating the complex amplitude of each frequency component. You do not have to make separate plots for real/imaginary parts or magnitude/phase. Just indicate the complex amplitude value at the appropriate frequency.

(b) Is $x(t)$ periodic? If so, what is the smallest period?

(c) Now consider a new signal $w(t) = x(t) - \cos(6t)$. Draw a carefully labelled sketch of the spectrum for $w(t)$. Explain why $w(t)$ is *not* periodic.

PROBLEM:

Consider a signal $x(t)$ such that

$$x(t) = 2\cos(\omega_1 t)\cos(\omega_2 t) = \cos[(\omega_2 + \omega_1)t] + \cos[(\omega_2 - \omega_1)t]$$

where $0 < \omega_1 < \omega_2$.

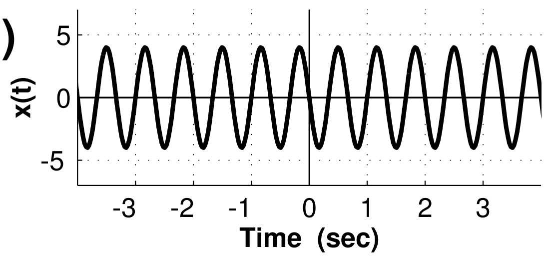
(a) Suppose that $\omega_1 = 25\pi$ and $\omega_2 = 60\pi$. Determine the (minimum) period of $x(t)$.

(b) Draw the spectrum for $x(t)$, using the parameters from part (a).

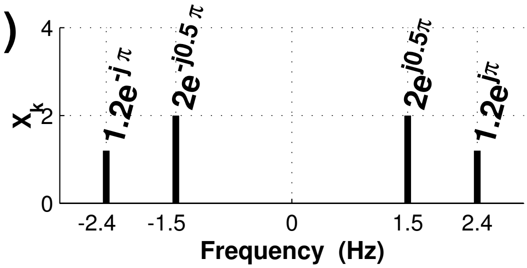
PROBLEM:

Several signals are plotted below along with their corresponding spectra. However, they are in a random order. For each of the signals (a)–(e), determine the correct spectrum (1)–(5). Explain your answers by deriving the formula for a time signal from each of the spectrum plots.

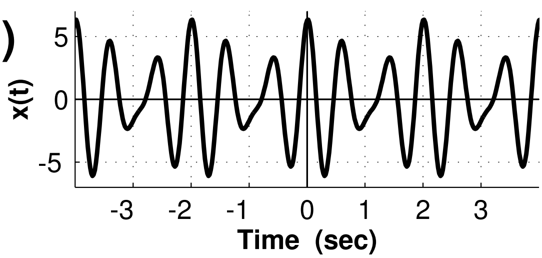
(a)



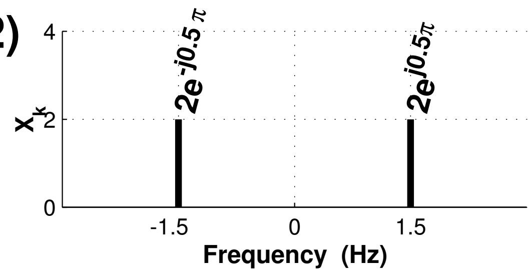
(1)



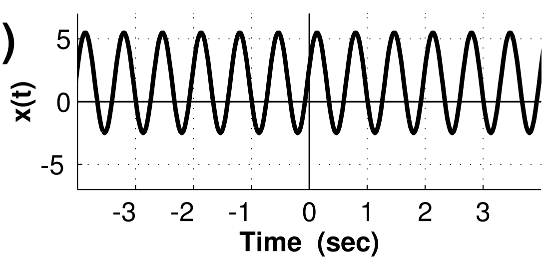
(b)



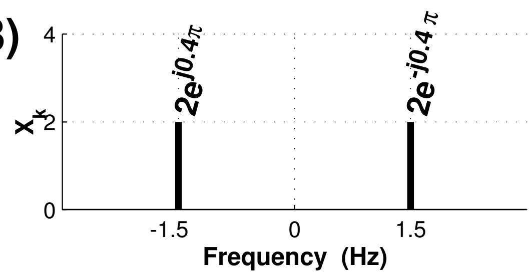
(2)



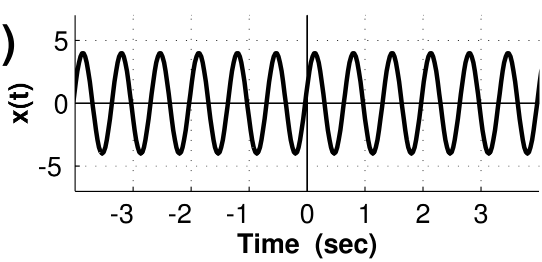
(c)



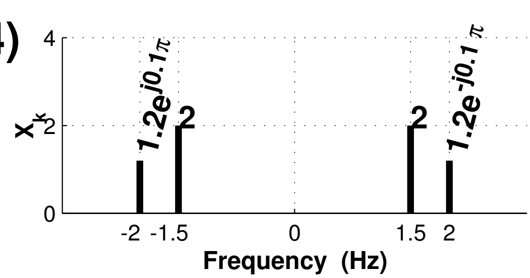
(3)



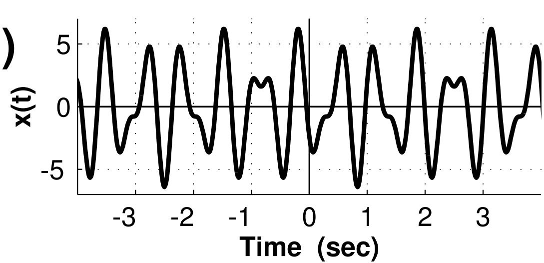
(d)



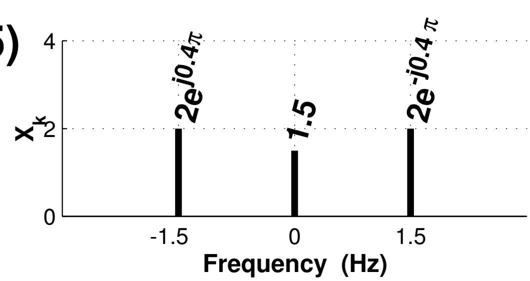
(4)



(e)



(5)



PROBLEM:

We have seen that musical tones can be modeled mathematically by sinusoidal signals. If you read music or play the piano you are aware of the fact that the piano keyboard is divided into octaves, with the tones in each octave being twice the frequency of the corresponding tones in the next lower octave. To calibrate the frequency scale, the reference tone is the A above middle-C, which is usually called A440 since its frequency is 440 Hz. Each octave contains 12 tones, and the ratio between the frequencies of successive tones is constant. Since middle C is 9 tones below A440, its frequency is approximately $(440)2^{-9/12} \approx 262$ Hz. The names of the tones (notes) of the octave starting with middle-C and ending with high-C are:

note name	C	$C^{\#}$	D	$E^{\flat}$	E	F	$F^{\#}$	G	$G^{\#}$	A	$B^{\flat}$	B	C
note number	40	41	42	43	44	45	46	47	48	49	50	51	52
frequency													

(a) Explain why the ratio of the frequencies of successive notes must be $2^{1/12}$.

(b) Make a table of the frequencies of the tones of the octave beginning with middle-C assuming that A above middle C (note #49) is tuned to 440 Hz.

(c) The above notes on a piano are numbered 40 through 52. If n denotes the note number, and f denotes the frequency of the corresponding tone, give a formula for the frequency of the tone as a function of the note number.

(d) A *chord* is a combination of musical notes sounded simultaneously. A *triad* is a three note chord. The C Minor chord is composed of the tones of $C \;\; E^{\flat} \;\; G$ sounded simultaneously. From the set of corresponding frequencies determined in part (a), make a sketch of the essential features of the spectrum of the C Minor chord assuming that each note is realized by a pure sinusoidal tone and that each note is equally loud. (You do not have to specify the complex amplitudes precisely.)

PROBLEM:

A linear-FM "chirp" signal is one that sweeps in frequency from $\omega_1 = 2\pi f_1$ to $\omega_2 = 2\pi f_2$ as time goes from $t = 0$ to $t = T_2$. We can define the *instantaneous frequency* of the chirp as the derivative of the phase of the sinusoid:

$$x(t) = A\cos(\alpha t^2 + \beta t + \phi) \tag{1}$$

where the cosine function operates on a time-varying argument

$$\psi(t) = \alpha t^2 + \beta t + \phi$$

The derivative of the argument $\psi(t)$ is the *instantaneous frequency* which is also the audible frequency heard from the chirp *if the chirping frequency does not change too rapidly.*

$$\omega_i(t) = \frac{d}{dt}\psi(t) \qquad \text{radians/sec} \tag{2}$$

There are examples on the CD-ROM in the Chapter 3 demos.

(a) For the linear-FM "chirp" in (1), determine formulas for the beginning instantaneous frequency (ω_1) and the ending instantaneous frequency (ω_2) in terms of α, β and T_2. For this problem, assume that the starting time of the "chirp" is $t = 0$.

(b) For the "chirp" signal

$$x(t) = \Re e\left\{e^{j2\pi(-33t^2 + 98t - 0.2)}\right\}$$

derive a formula for the *instantaneous* frequency versus time.

(c) For the signal in part (b), make a plot of the *instantaneous* frequency (in Hz) versus time over the range $0 \le t \le 1$ sec.