DSP First, 2/e

Lecture 24
Time-Domain Response for IIR Systems

READING ASSIGNMENTS

- This Lecture:
  - Chapter 10, Sects. 10-9, 10-10, & 10-11
    - Partial Fraction Expansion

LECTURE OBJECTIVES

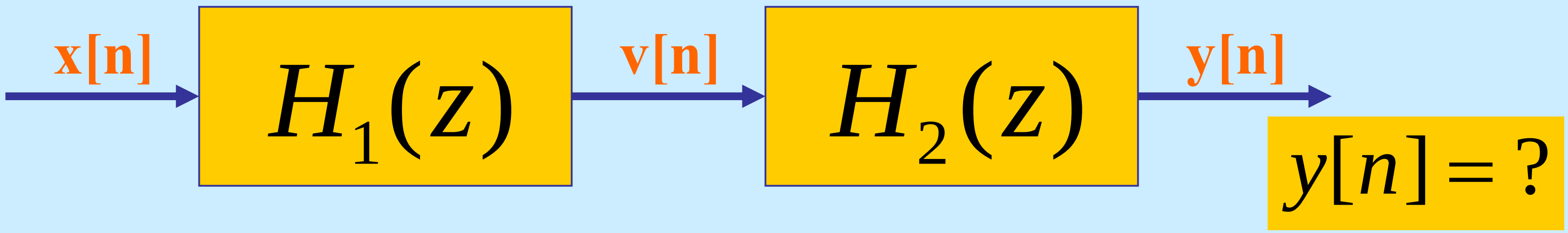
- Calculate output from Input
  - Transient and Steady State Responses
  - Z-Transform method with Partial Fraction Expansion
- SECOND-ORDER IIR FILTERS
  - TWO FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

- H(z) can have COMPLEX POLES & ZEROS

CASCADE: Pole-Zero Cancellation

- Multiply the z-transforms



$$v[n] = x[n] + 0.5x[n-1] - 0.5x[n-2]$$

$$H_2(z) = 1 - z^{-1}$$

$$x[n] = u[n] + (0.5)^n u[n]$$

What is Frequency Response?

- **<u>Sinusoid-in gives sinusoid-out</u>**
  - True for LTI systems
  - Seems to require an infinite-length sinusoid

$$x[n] = \cos(\hat{\omega}_0 n) \quad \text{for} \quad \boxed{-\infty < n < \infty}$$

  - But, real-world sinusoids start at n=0

$$x[n] = \cos(\hat{\omega}_0 n) u[n] = \begin{cases} \cos(\hat{\omega}_0 n) & n \geq 0 \\ 0 & n < 0 \end{cases}$$

  - With z-transforms, we can solve this one-sided problem

THREE significant INPUTS

- Given: $\longrightarrow \boxed{H(z) = \dfrac{5}{1+0.8z^{-1}}} \longrightarrow$
- Find the output, y[n]
  - For 3 cases:

$$x[n] = \cos(0.2\pi n)$$

$$x[n] = u[n]$$

$$x[n] = \cos(0.2\pi n)u[n]$$

SINUSOID ANSWER

- Given: $H(z) = \dfrac{5}{1 + 0.8z^{-1}}$
- The input: $x[n] = \cos(0.2\pi n)$
- Then y[n] $y[n] = M\cos(0.2\pi n + \psi)$

$$H(e^{j0.2\pi}) = \frac{5}{1 + 0.8e^{-j0.2\pi}} = 2.919e^{j0.089\pi}$$

Step Response: u[n]→U(z)

SHORT TABLE OF z-TRANSFORMS

	$x[n]$	$\Longleftrightarrow$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n - n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\dfrac{1}{1 - az^{-1}}$

Step Response: x[n] is u[n]

$$Y(z) = H(z)X(z) = \left(\frac{5}{1+0.8z^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right)$$

Product

Partial Fraction Expansion

$$Y(z) = \frac{A}{1+0.8z^{-1}} + \frac{B}{1-z^{-1}} = \frac{(A+B)+(0.8B-A)z^{-1}}{(1+0.8z^{-1})(1-z^{-1})}$$

$$\Rightarrow \quad (A+B) = 5 \quad \text{and} \quad (0.8B-A) = 0$$

$$Y(z) = \frac{A}{1+0.8z^{-1}} + \frac{B}{1-z^{-1}}$$

Need Sum of Terms

Step Response

$$\boxed{Y(z) = \frac{20/9}{1+0.8z^{-1}} + \frac{25/9}{1-z^{-1}}}$$

Do the INVERSE z-Transform of Y(z)

$$y[n] = \frac{20}{9}(-0.8)^n u[n] + \frac{25}{9}u[n]$$

$$y[n] \rightarrow \frac{25}{9} \quad \text{as} \quad n \rightarrow \infty$$

What is DC value of Frequency Response?

SINUSOID Starting at n=0

- Given: $$H(z) = \frac{5}{1+0.8z^{-1}}$$

- The input: $$x[n] = \cos(0.2\pi n)\,u[n]$$

- Then y[n] $$\begin{aligned} y[n] &= \Re\{h[n] * x[n]\} \\ &= \Re\{5(-0.8)^n u[n] * e^{j0.2\pi n} u[n]\} \end{aligned}$$

$$y[n] = A\cos(0.2\pi n + \varphi) + B(-0.8)^n u[n]$$

SINUSOID Starting at n=0

$$y[n] = \Re\{5(-0.8)^n u[n] * e^{j0.2\pi n} u[n]\}$$

$$Y(z) = H(z)X(z) = \frac{5}{1+0.8z^{-1}} \frac{1}{1-e^{j0.2\pi}z^{-1}}$$

$$Y(z) = \frac{\frac{5}{1+1.25e^{j0.2\pi}}}{1+0.8z^{-1}} + \frac{\frac{5}{1+0.8e^{-j0.2\pi}}}{1-e^{j0.2\pi}z^{-1}}$$

$$= \frac{2.19-j0.8}{1+0.8z^{-1}} + \frac{2.81+j0.8}{1-e^{j0.2\pi}z^{-1}}$$

$$H(e^{j0.2\pi})$$

SINUSOID Starting at n=0

$$Y(z) = \frac{\frac{5}{1+1.25e^{j0.2\pi}}}{1+0.8z^{-1}} + \frac{\frac{5}{1+0.8e^{-j0.2\pi}}}{1-e^{j0.2\pi}z^{-1}}$$

$$= \frac{2.19 - j0.8}{1+0.8z^{-1}} + \frac{2.81 + j0.8}{1-e^{j0.2\pi}z^{-1}}$$

$H(e^{j0.2\pi})$

$$y[n] = \Re\{(2.18 - j0.8)(-0.8)^n u[n] + 2.92e^{j0.28}e^{j0.2\pi n}u[n]\}$$

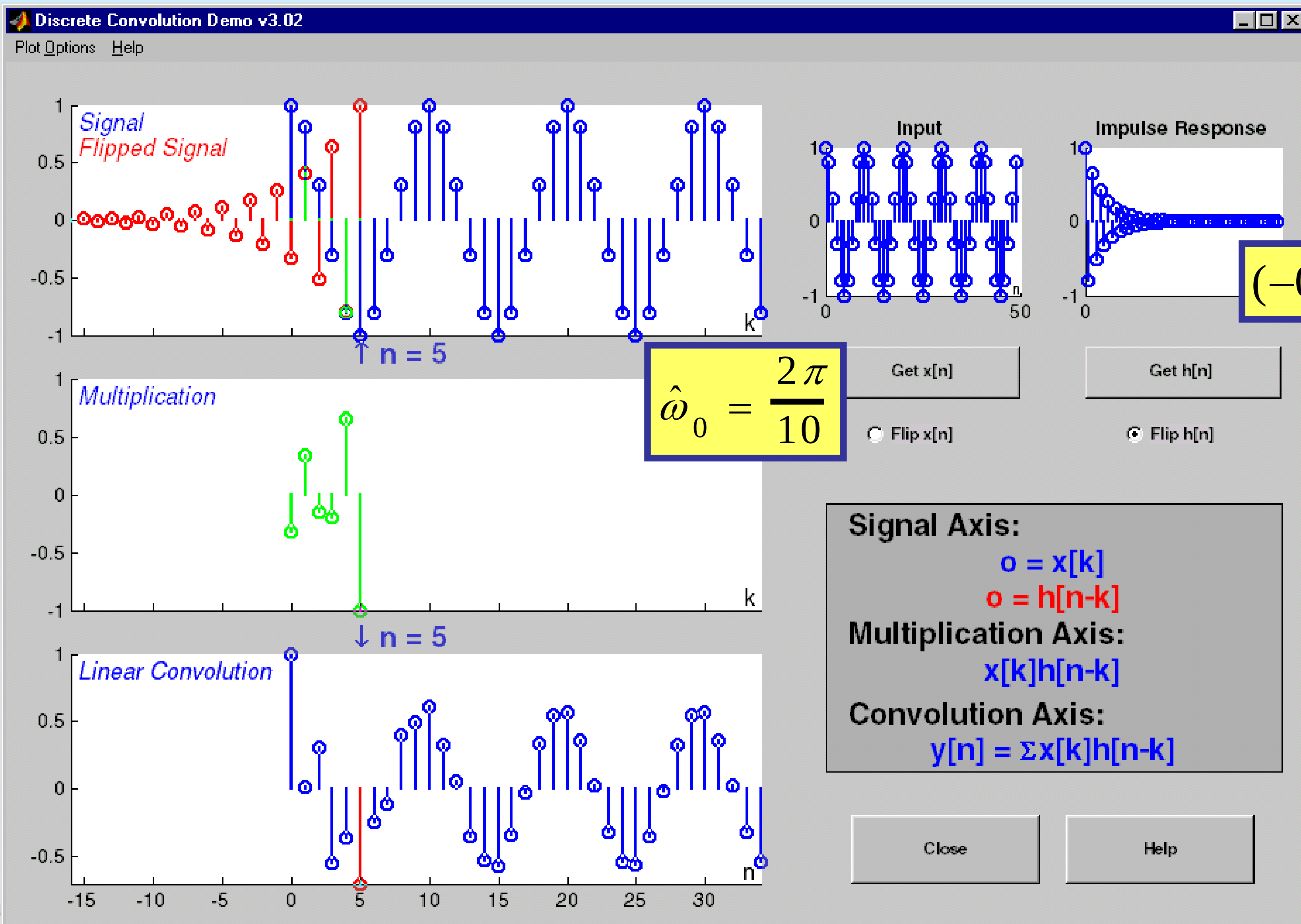
$$y[n] = 2.18(-0.8)^n u[n] + 2.92\cos(0.2\pi n + 0.28)u[n]$$

Transient **Steady-State**

BONUS QUESTION

- Given: $$H(z) = \frac{5}{1+0.8z^{-1}}$$
- The input is $x[n] = 4\cos(\pi n - 0.5\pi)$
- Then find y[n] $y[n] = ?$

Transient & Steady State



$$\hat{\omega}_0 = \frac{2\pi}{10}$$

$$(-0.8)^n$$

CALCULATE the RESPONSE

$$x[n] = e^{j\hat{\omega}_0 n}u[n]$$

$H(z)$

$$X(z) = \frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}$$

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}\right)$$

Use the Z-Transform Method
And PARTIAL FRACTIONS

GENERAL INVERSE Z

PROCEDURE FOR INVERSE z-TRANSFORMATION ($M < N$)

1. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form $(1 - p_k z^{-1})$ for $k = 1, 2, \ldots, N$.
2. Make a partial fraction expansion of $H(z)$ into a sum of terms of the form

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \quad \text{where} \quad A_k = H(z)(1 - p_k z^{-1})\Big|_{z=p_k}$$

3. Write down the answer as

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

$(\text{pole})^n$

SPLIT Y(z) to INVERT

- Need SUM of Terms:

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}\right)$$

$$= \frac{b_0}{(1 - a_1 z^{-1})(1 - e^{j\hat{\omega}_0} z^{-1})}$$

$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

INVERT Y(z) to y[n]

- Use the Z-Transform Table

$$Y(z) = \frac{\left(\dfrac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\dfrac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

$$y[n] = \left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n] + \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]$$

TWO PARTS of y[n]

- **TRANSIENT**
  - Acts Like (pole)^n
  - Dies out ?
    - IF $|a_1|<1$

$$\boxed{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n]}$$

- **STEADY-STATE**
  - Depends on the input
  - e.g., Sinusoidal

$$\boxed{\left(\frac{b_0}{1 - a_1 e^{j\hat{\omega}_0}}\right)e^{j\hat{\omega}_0 n} u[n]}$$

STEADY STATE HAPPENS

- When Transient dies out
- In the Limit as "n" approaches infinity
- Can use Frequency Response to get Magnitude & Phase for sinusoid

$$y_{ss}[n] \rightarrow \left(\frac{b_0}{1 - a_1 e^{j\hat{\omega}_0}} \right) e^{j\hat{\omega}_0 n} = H(e^{j\hat{\omega}_0}) e^{j\hat{\omega}_0 n}$$

NUMERICAL EXAMPLE

Example 8.12 If $b_0 = 5$, $a_1 = -0.8$, and $\hat{\omega}_0 = 2\pi/10$, the transient component is

$$y_t[n] = \left(\frac{-4}{-0.8 - e^{j0.2\pi}}\right)(-0.8)^n u[n] = 2.3351 e^{-j0.3502}(-0.8)^n u[n]$$

$$= 2.1933(-0.8)^n u[n] - j0.8012(-0.8)^n u[n]$$

Similarly, the steady-state component is

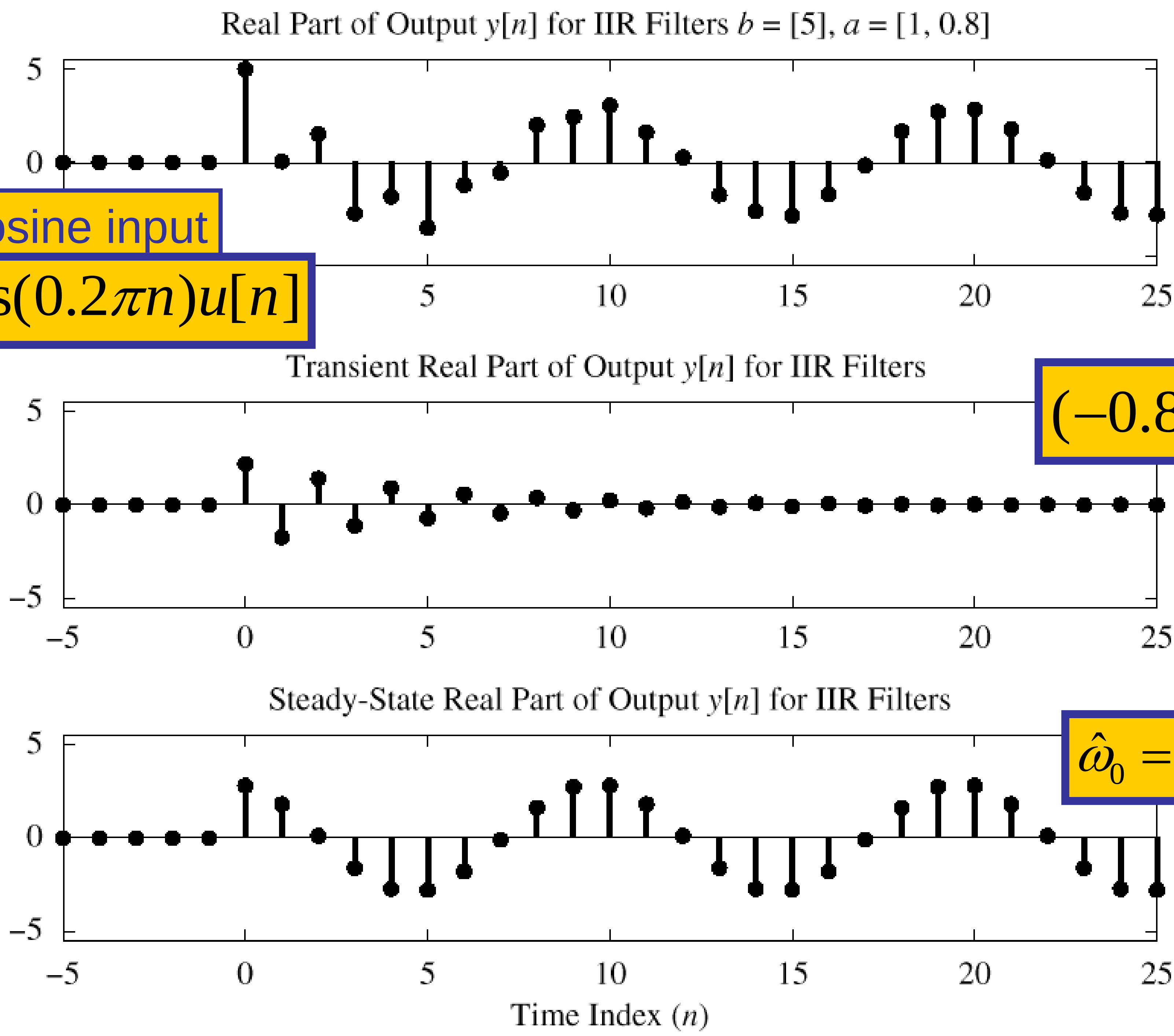
$$y_{ss}[n] = \left(\frac{5}{1 + 0.8e^{-j0.2\pi}}\right) e^{j0.2\pi n} u[n] = 2.9188 e^{j0.2781} e^{j0.2\pi n} u[n]$$

$$= 2.9188 \cos\left(\tfrac{2\pi}{10} n + 0.2781\right) u[n] + j\, 2.9188 \sin\left(\tfrac{2\pi}{10} n + 0.2781\right) u[n]$$

$\boxed{0.089\pi = 0.2781}$

SINUSOID starting at n=0

- We'll look at an example in MATLAB
  - $x[n]=\cos(0.2\pi n)u[n]$
  - Pole at –0.8, so a^n is $(-0.8)^n$
- There are two components:
  - TRANSIENT
    - Start-up region just after n=0; $\mathbf{(-0.8)^n}$
  - STEADY-STATE
    - Eventually, y[n] looks sinusoidal.
    - **Magnitude & Phase from Frequency Response**



Cosine input

$\cos(0.2\pi n)u[n]$

$(-0.8)^n$

$\hat{\omega}_0 = 0.2\pi$

STABILITY

- When Does the TRANSIENT DIE OUT ?

STEADY-STATE RESPONSE AND STABILITY

A stable system is one that does not "blow up." This intuitive statement can be formalized by saying that the output of a stable system can always be bounded ($|y[n]| < M_y$) whenever the input is bounded ($|x[n]| < M_x$).[3]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$h[n] = b_0 a_1^n u[n]$$

need $|a_1| < 1$

Stability

- Nec. & suff. condition: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$$h[n] = b(a)^n u[n] \Leftrightarrow H(z) = \frac{b}{1 - a z^{-1}}$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |b||a|^n < \infty \text{ if } |a| < 1 \Rightarrow$$

Pole at z=a must be Inside unit circle

STABILITY CONDITION

- ALL POLES INSIDE the UNIT CIRCLE
- UNSTABLE EXAMPLE:

POLE @ z=1.1

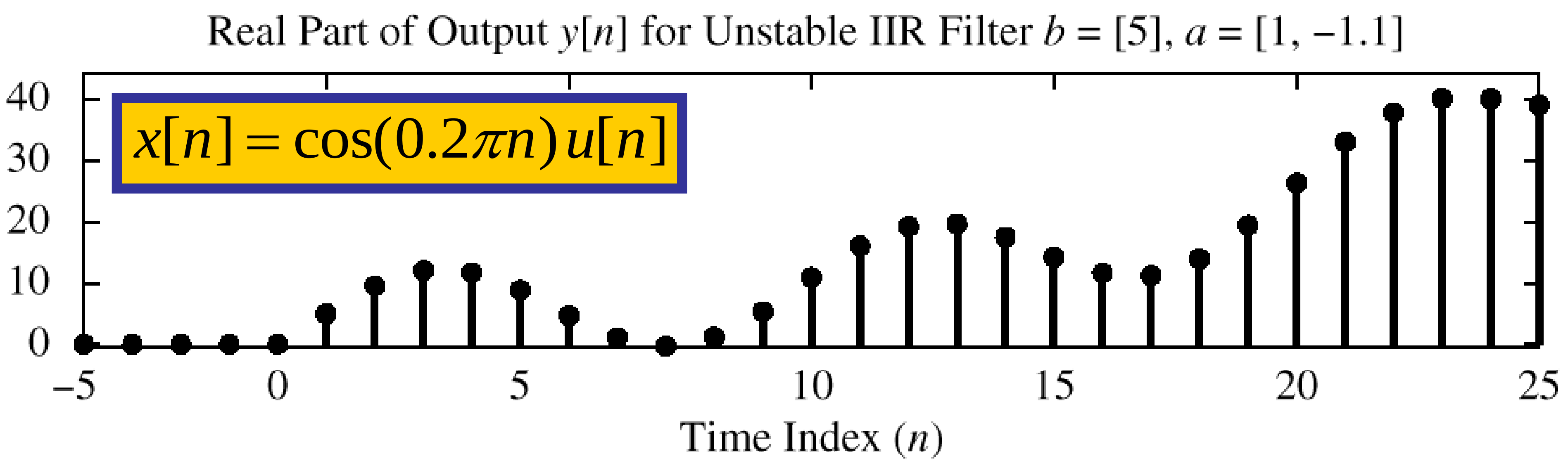


Figure 8.15 Illustration of an unstable IIR system. Pole is at $z = 1.1$.

SECOND-ORDER FILTERS

- Two FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2]$$
$$+ b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$\boxed{H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}}$$

MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z^1 + b_2}{z^2 - a_1 z^1 - a_2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

TWO COMPLEX POLES

- Find Impulse Response ?
  - Can OSCILLATE vs. n
  - "RESONANCE"

$$\boxed{(p_k)^n = (re^{j\theta})^n = r^n e^{jn\theta}}$$

- Find **FREQUENCY RESPONSE**
  - Depends on Pole Location
  - Close to the Unit Circle?
    - Make **BANDPASS FILTER**

$$\boxed{\begin{array}{l}\text{pole is @ } re^{j\theta} \\ r \to 1?\end{array}}$$

Inverse z-Transform?

- SECOND-ORDER IIR FILTERS

$$H(z) = \frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

- H(z) can have COMPLEX POLES & ZEROS

$$H(z) = \frac{0.5}{1 - 0.9e^{j\pi/3}z^{-1}} + \frac{0.5}{1 - 0.9e^{-j\pi/3}z^{-1}}$$

$$\boxed{2 \text{ Poles} : z = 0.9e^{\pm j\pi/3}}$$

2nd ORDER z-Transform

$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n] = (0.9)^n \tfrac{1}{2}(e^{j\pi n/3} + e^{-j\pi n/3}) u[n]$$

$$H(z) = \frac{0.5}{1 - 0.9e^{j\pi/3} z^{-1}} + \frac{0.5}{1 - 0.9e^{-j\pi/3} z^{-1}}$$

$$H(z) = \frac{1 - 0.9\cos(\pi/3) z^{-1}}{(1 - 0.9e^{j\pi/3} z^{-1})(1 - 0.9e^{-j\pi/3} z^{-1})}$$

$$H(z) = \frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

2nd ORDER Z-transform PAIRS

GENERAL ENTRY for z-Transform TABLE

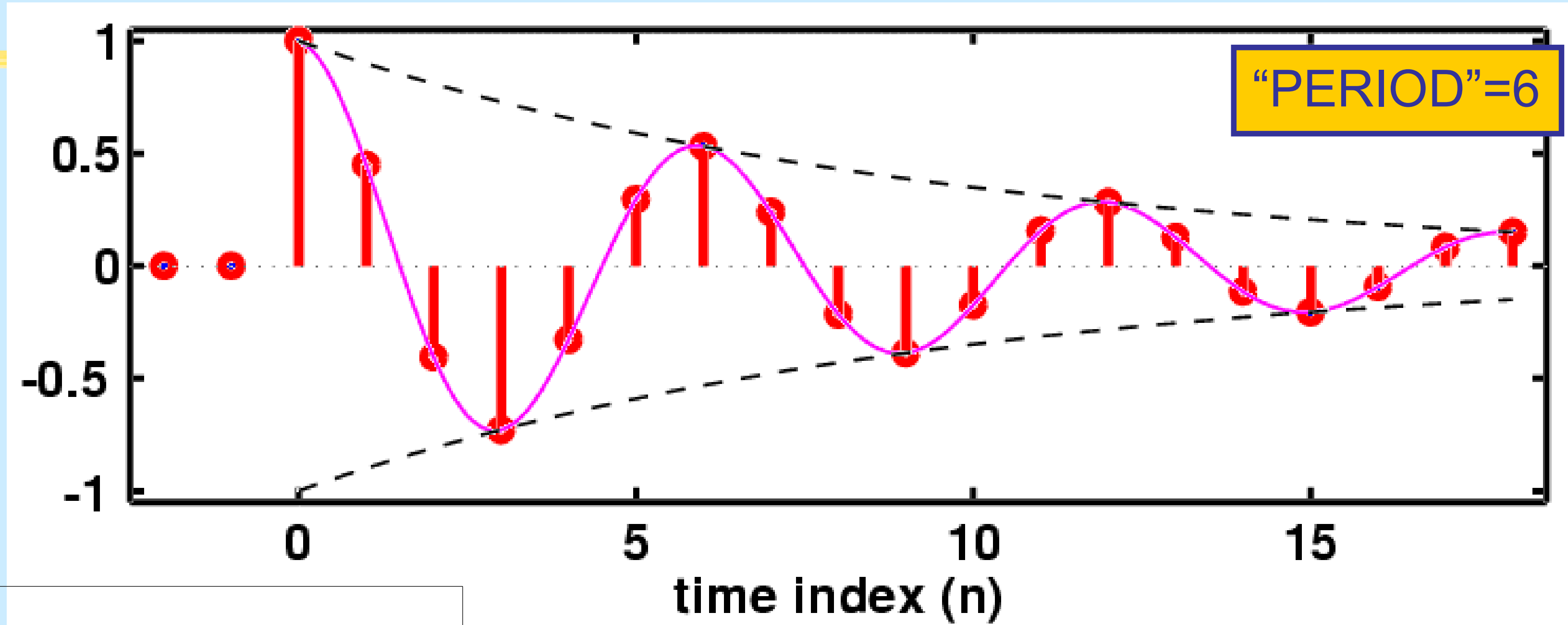
$$h[n] = r^n \cos(\theta n) u[n]$$

$$H(z) = \frac{1 - r\cos(\theta) z^{-1}}{1 - 2r\cos(\theta) z^{-1} + r^2 z^{-2}}$$

$$h[n] = A r^n \cos(\theta n + \varphi) u[n]$$

$$H(z) = \frac{\cos(\varphi) - r\cos(\theta - \varphi) z^{-1}}{1 - 2r\cos(\theta) z^{-1} + r^2 z^{-2}}$$

h[n]: Decays & Oscillates



$$h[n] = (0.9)^n \cos\left(\tfrac{\pi}{3} n\right) u[n]$$

$$H(z) = \frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

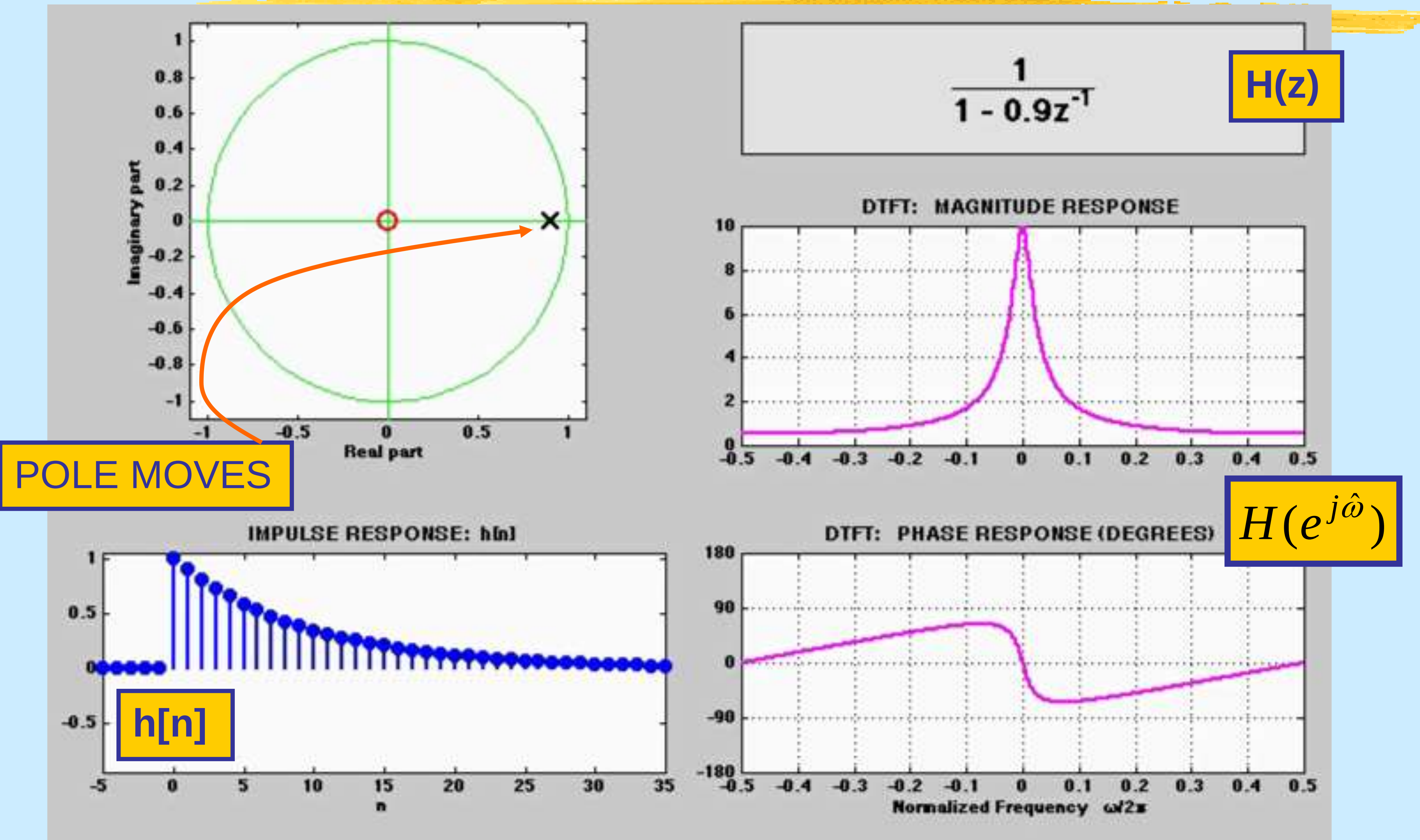
2nd ORDER EX: n-Domain

$$H(z) = \frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

$$y[n] = 0.9y[n-1] - 0.81y[n-2] + x[n] - 0.45x[n-1]$$

```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
nn = -2:19;
hh = filter( bb, aa, (nn==0) );
HH = freqz( bb, aa, [-pi,pi/100:pi] );
```

3 DOMAINS MOVIE: IIR



7 IIR MOVIES @ WEBSITE

- `http://dspfirst.gatech.edu/chapters/08feedbac/demos/3_domain/index.html`
- 3 DOMAINS MOVIES: <u>IIR</u> Filters
  - One pole moving and a zero at the origin
  - One pole and one zero; both moving
  - Two complex-conjugate poles moving radially
  - Two complex-conjugate poles moving in angle
  - Movement of a zero in a two-pole Filter
  - Radial Movement of Two out of Four Poles
  - Angular Movement of Two out of Four Poles