BLM3620 Digital Signal Processing*

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*Based on lecture notes from Ali Can Karaca & Ahmet Elbir

Lecture #14 – Filter Design Basics

- Relation of z-Transform and DTFT
- Examples
- Filter Design Using z-Plane
- MATLAB Example

Z-Transform EXAMPLE

$$H(z) = \sum_n h[n] z^{-n}$$

- ANY SIGNAL has a z-Transform:

$$X(z) = \sum_n x[n] z^{-n}$$

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Example 7.1

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

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$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

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FREQUENCY RESPONSE ?

- Same Form:

$\hat{\omega}$ – Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k (e^{j\hat{\omega}})^{-k}$$

$$z = e^{j\hat{\omega}}$$

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z – Domain

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

SAME COEFFICIENTS

SINUSOIDAL RESPONSE

- $x[n]$ = SINUSOID => $y[n]$ is SINUSOID
- Get MAGNITUDE & PHASE from $H(z)$

$$\boxed{\begin{array}{l} \text{if } \ x[n] = e^{j\hat{\omega}n} \\ \text{then } \ y[n] = H(e^{j\hat{\omega}})\, e^{j\hat{\omega}n} \\ \text{where } \ H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} \end{array}}$$

Exercise-1

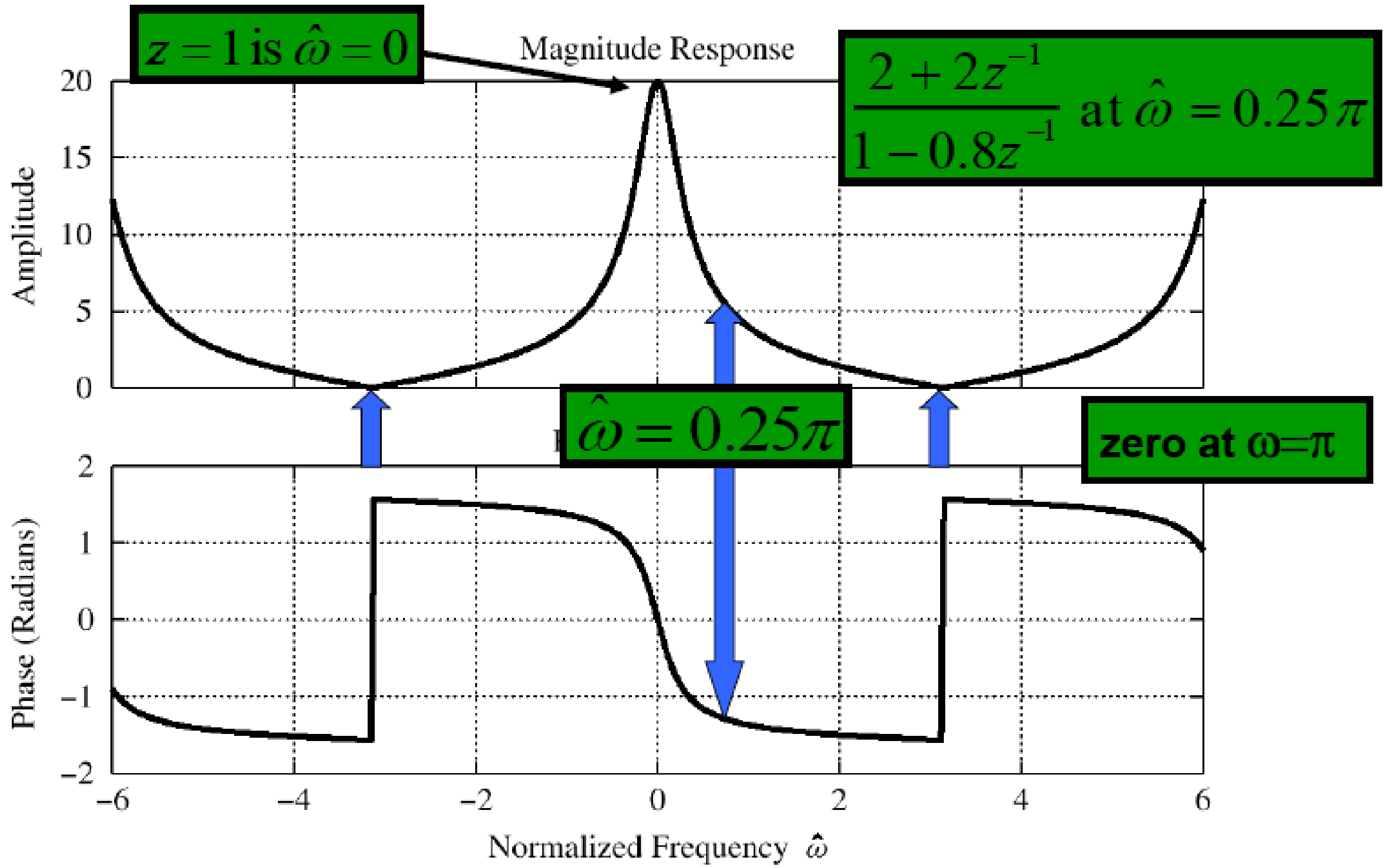
POP QUIZ

- Given: $$\boxed{H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}}$$
- Find the Impulse Response, $h[n]$
- Find the output, $y[n]$
  - When

$$\boxed{x[n]=\cos(0.25\pi n)}$$

Exercise-1

Evaluate FREQ. RESPONSE



Exercise-1

POP QUIZ: Eval Freq. Resp.

- Given: $$H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}$$

- Find output, $y[n]$, when
  - Evaluate at

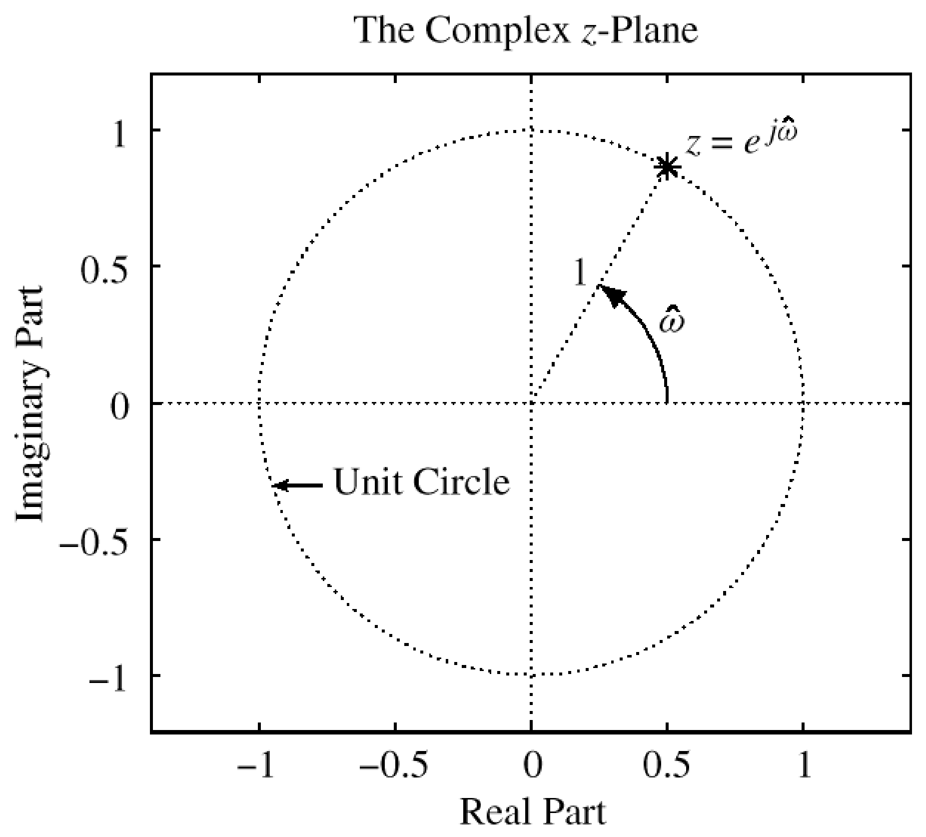
$$x[n]=\cos(0.25\pi n)$$

$$z=e^{j0.25\pi}$$

$$H(z)=\frac{2+2(\frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2})}{1-0.8e^{-j0.25\pi}}=5.182e^{-j1.309}$$

$$y[n]=5.182\cos(0.25\pi n-0.417\pi)$$

- MAPPING BETWEEN z and $\hat{\omega}$



$$\boxed{z = e^{j\hat{\omega}}}$$

- MAPPING BETWEEN

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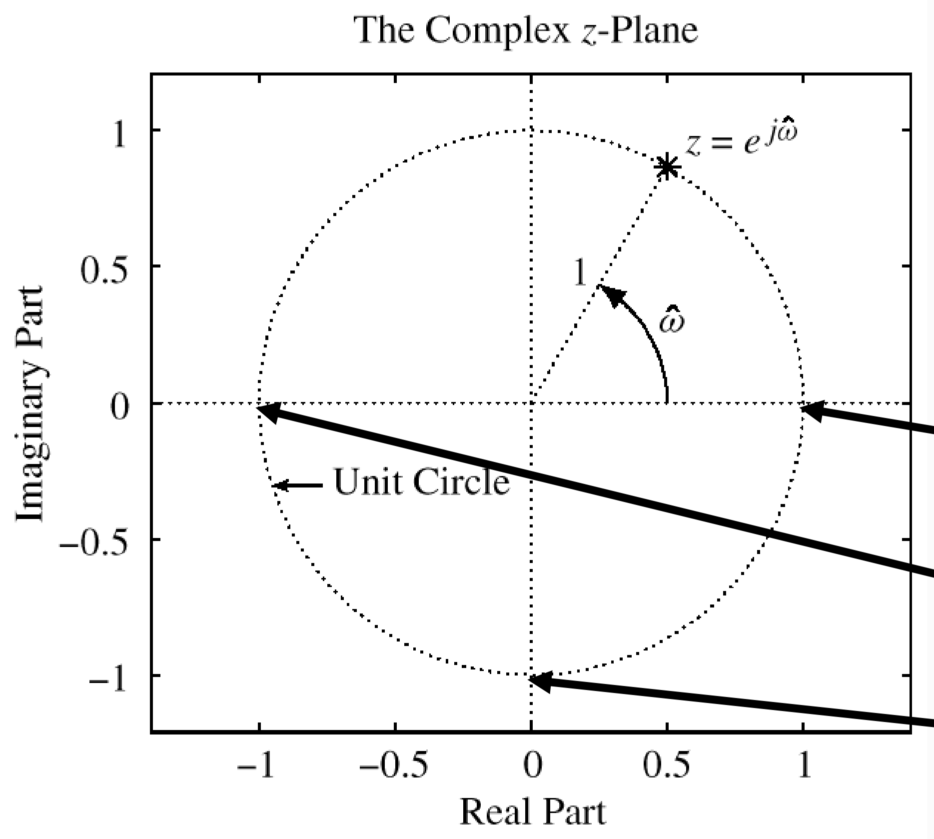


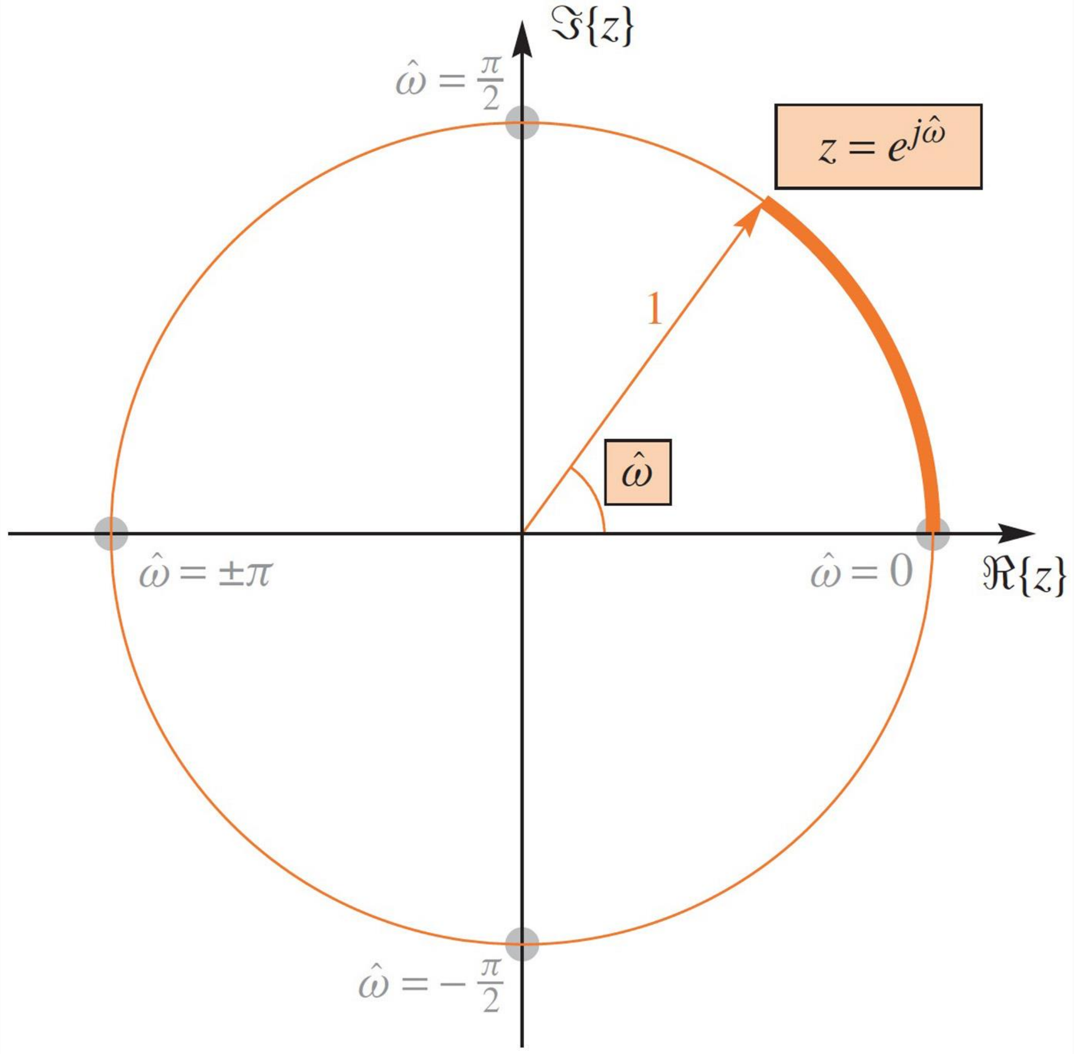
$$\boxed{z = e^{j\hat{\omega}}}$$

$$z = 1 \leftrightarrow \hat{\omega} = 0$$

$$z = -1 \leftrightarrow \hat{\omega} = \pm\pi$$

$$z = \pm j \leftrightarrow \hat{\omega} = \pm\tfrac{1}{2}\pi$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$



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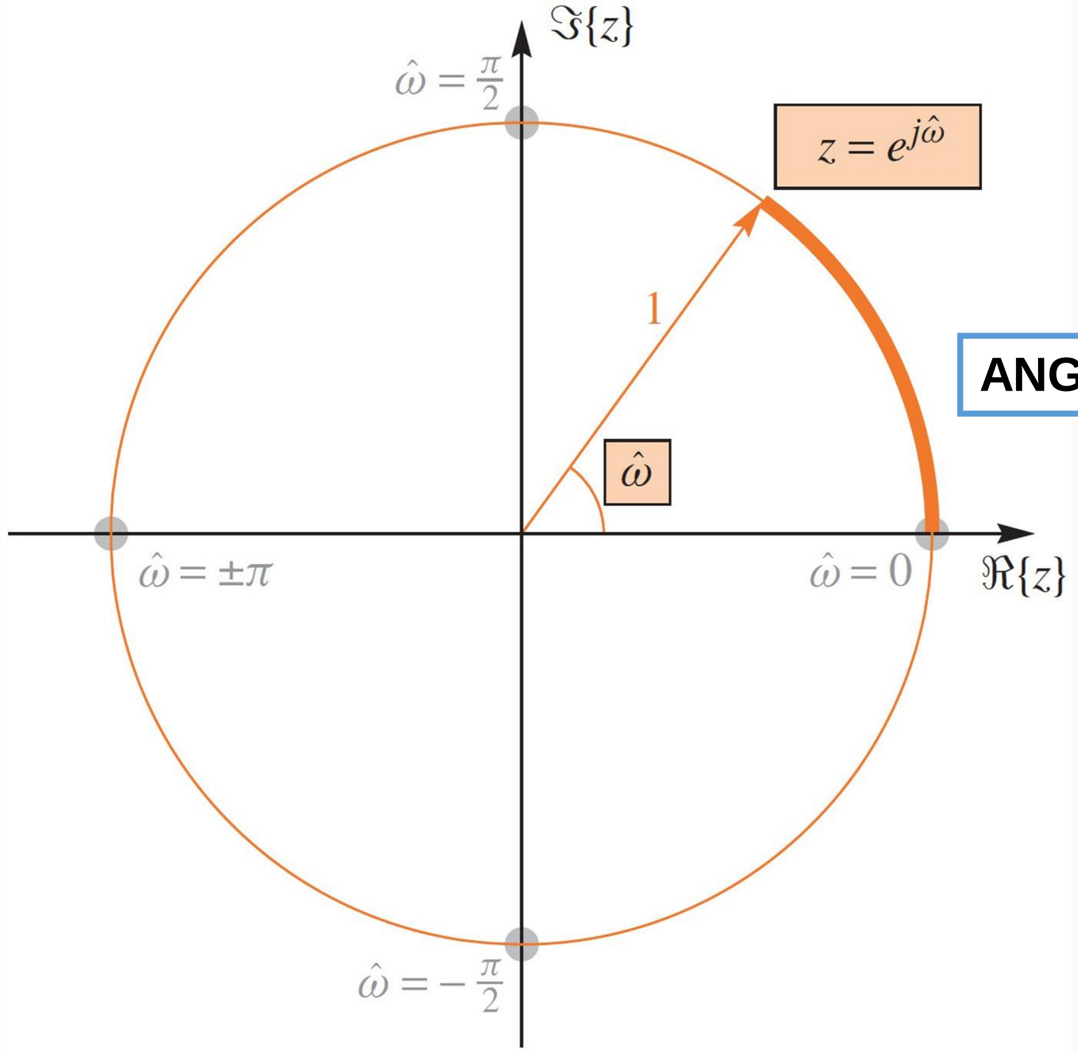
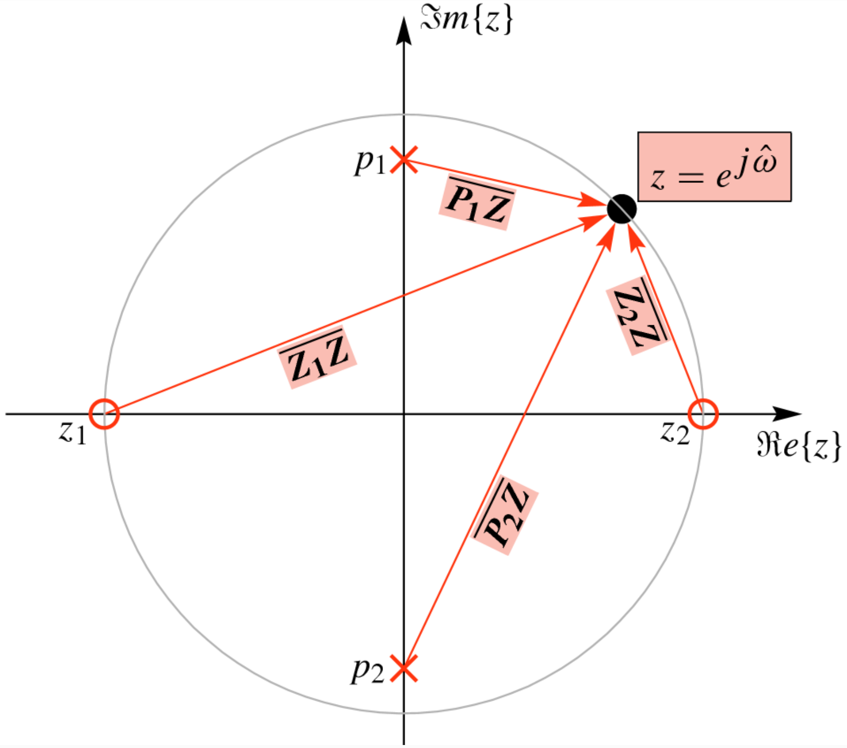


ANGLE is FREQUENCY

Frequency Response from poles and zeros

$$|H(e^{j\hat{\omega}})| = G \frac{|e^{j\hat{\omega}} - z_1|\,|e^{j\hat{\omega}} - z_2|}{|e^{j\hat{\omega}} - p_1|\,|e^{j\hat{\omega}} - p_2|}$$

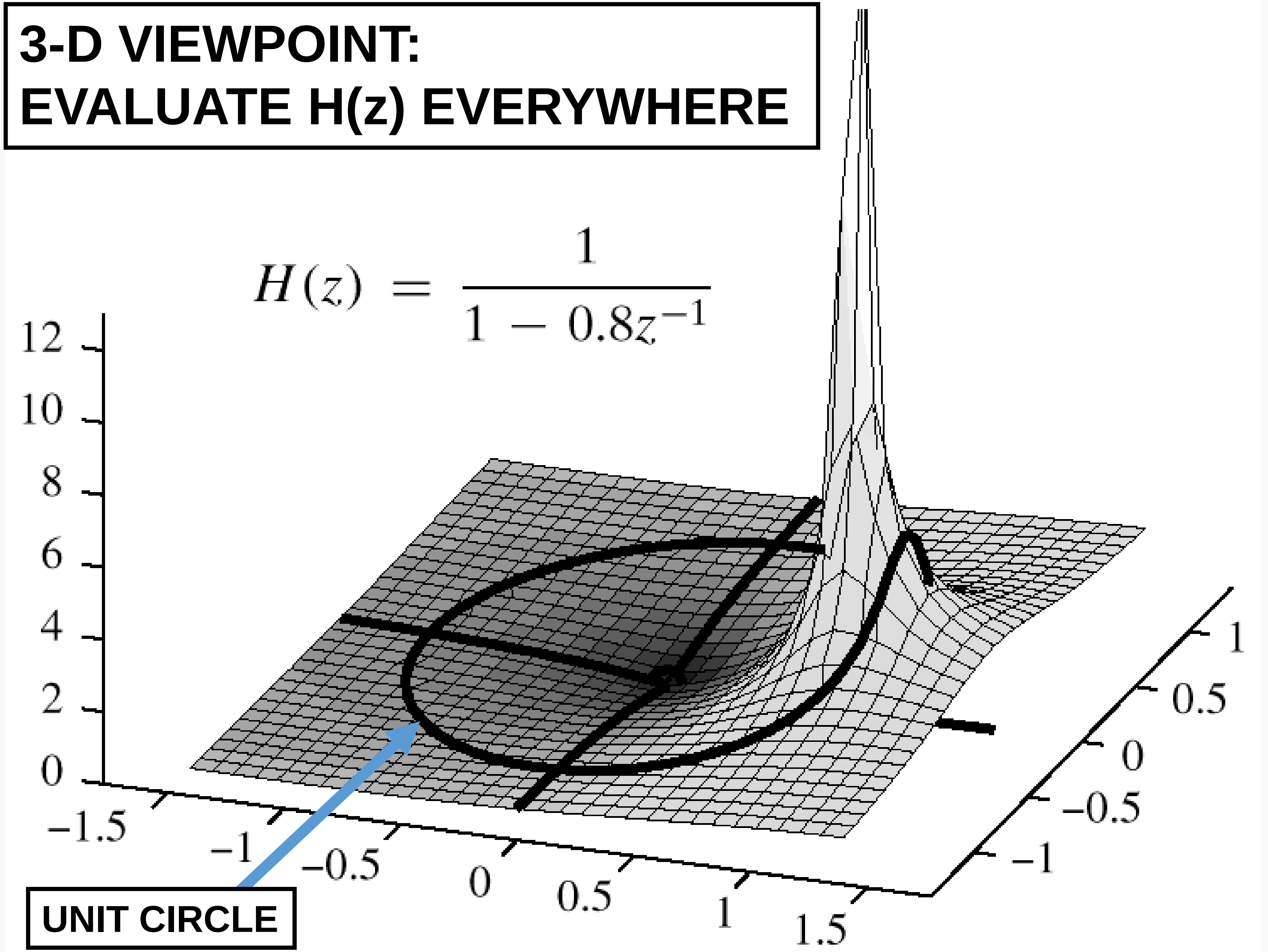
$$|H(e^{j\hat{\omega}})| = G \frac{\overline{Z_1 Z} \cdot \overline{Z_2 Z}}{\overline{P_1 Z} \cdot \overline{P_2 Z}}$$



$$H(z) = G \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

3-D VIEWPOINT: EVALUATE H(z) EVERYWHERE

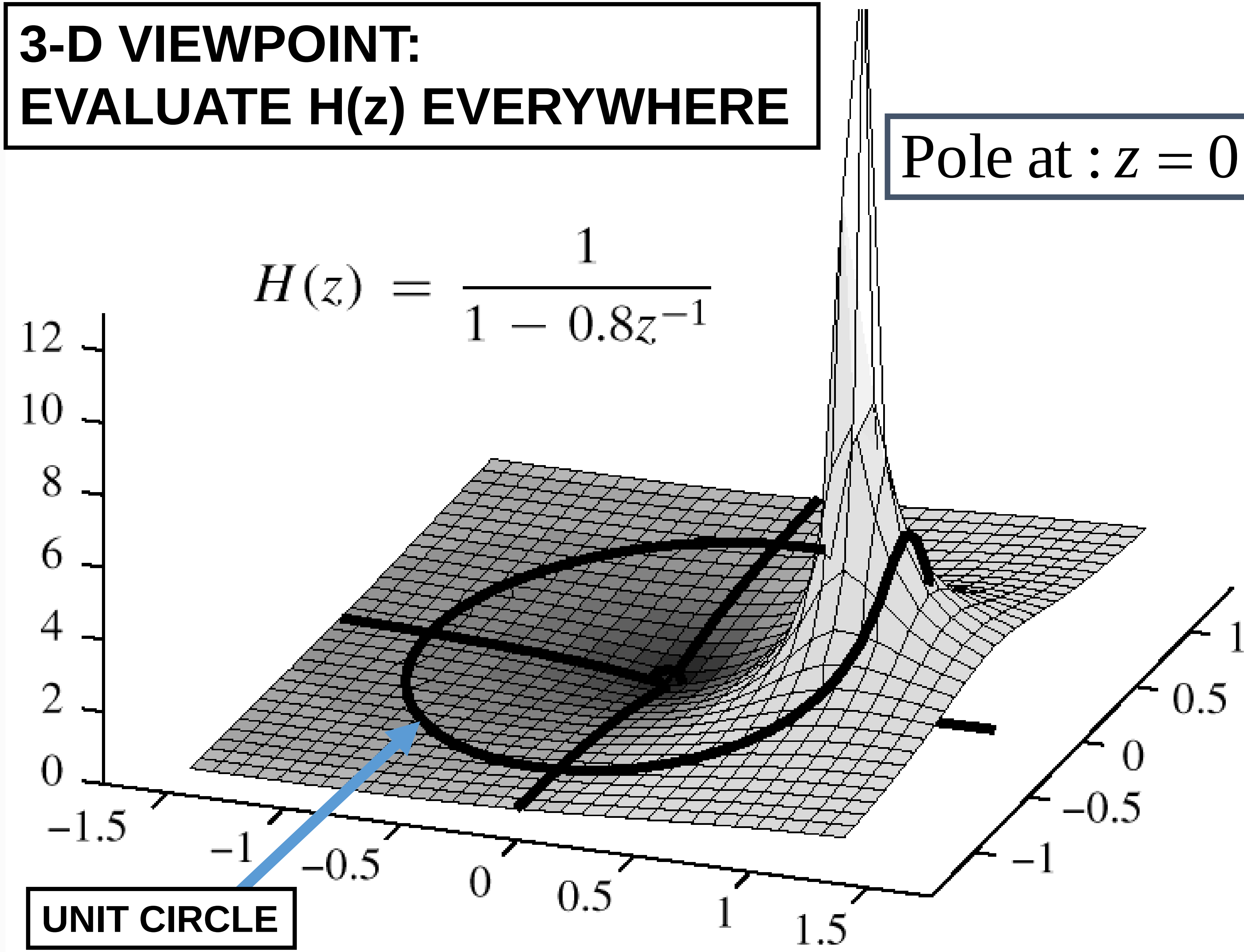
$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$



3-D VIEWPOINT:
EVALUATE H(z) EVERYWHERE

$$\text{Pole at}: z = 0.8$$

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$



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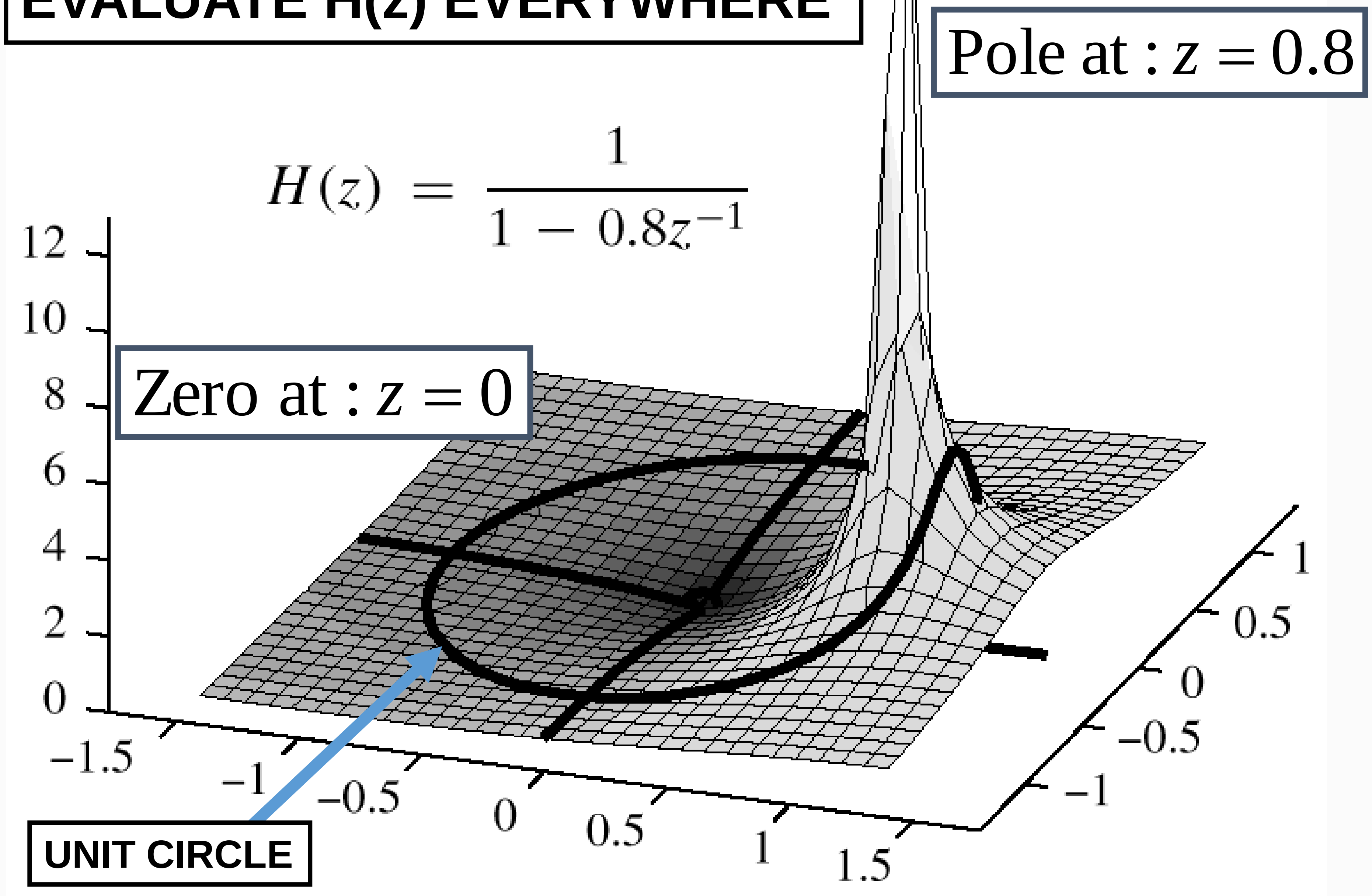
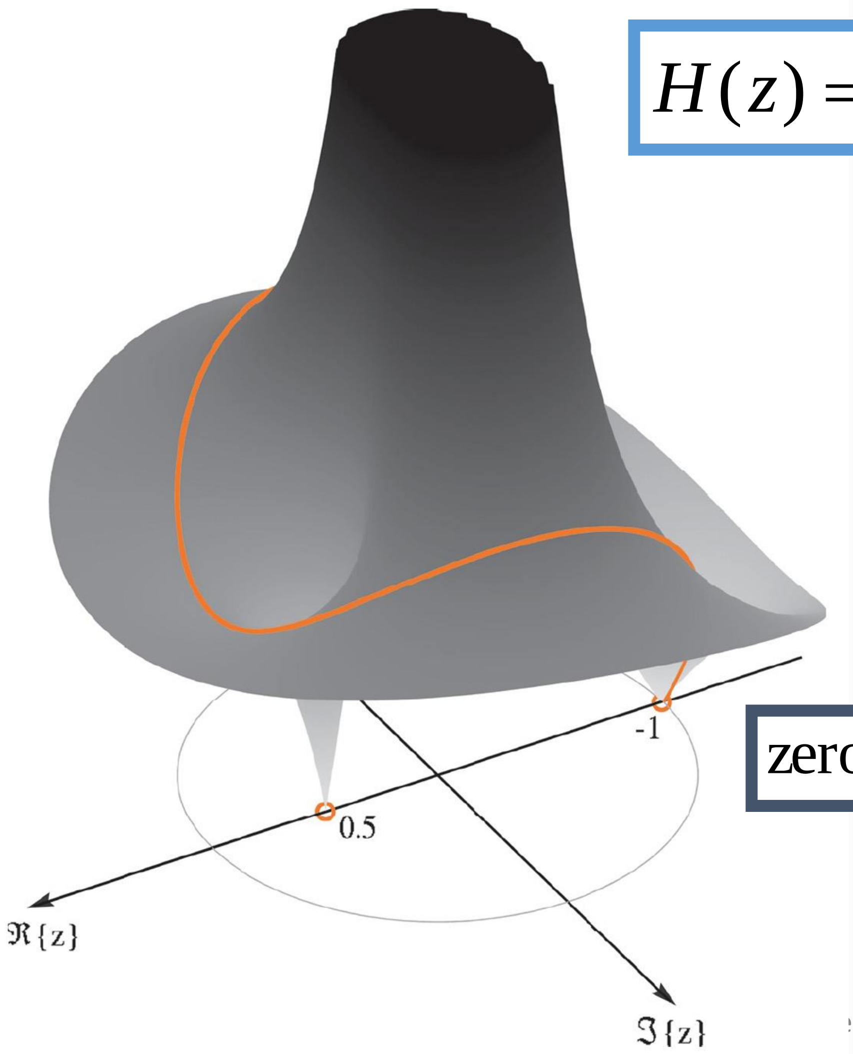


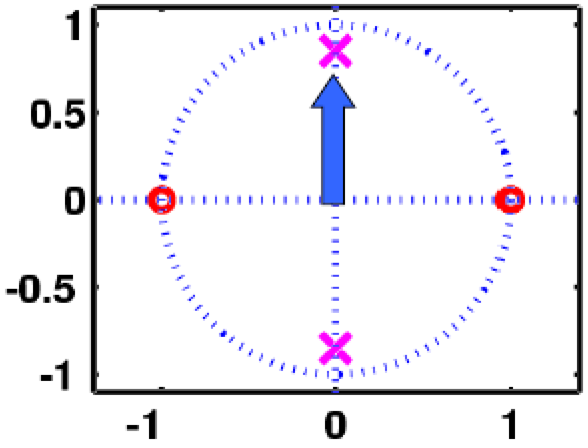
Evaluate H(z) on Unit Circle



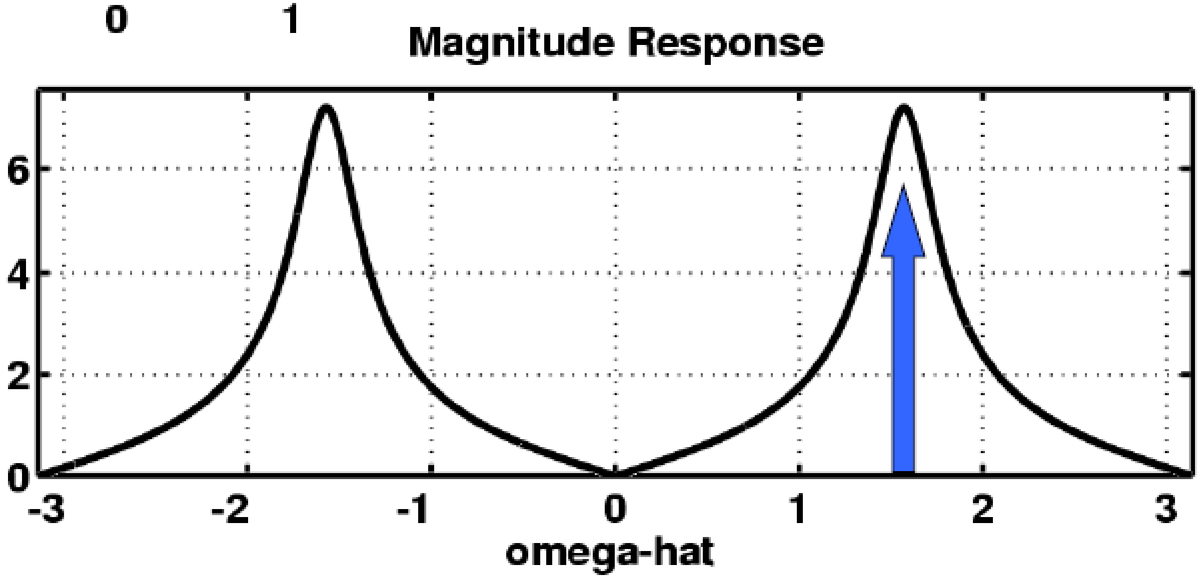
$$\boxed{H(z) = (1 + z^{-1})(1 - 0.5z^{-1})}$$

$$\boxed{\text{zeros at } z = -1 \text{ and } z = 0.5}$$

FREQUENCY RESPONSE from POLE-ZERO PLOT



$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-j2\hat{\omega}}}{1 + 0.7225e^{-j2\hat{\omega}}}$$



ZEROS of H(z) – example 2

- Find z, where H(z)=0
  - Interesting when z is ON the unit circle.

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

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$$\boxed{\text{Roots}: z = 1, \tfrac{1}{2} \pm j\frac{\sqrt{3}}{2}}$$

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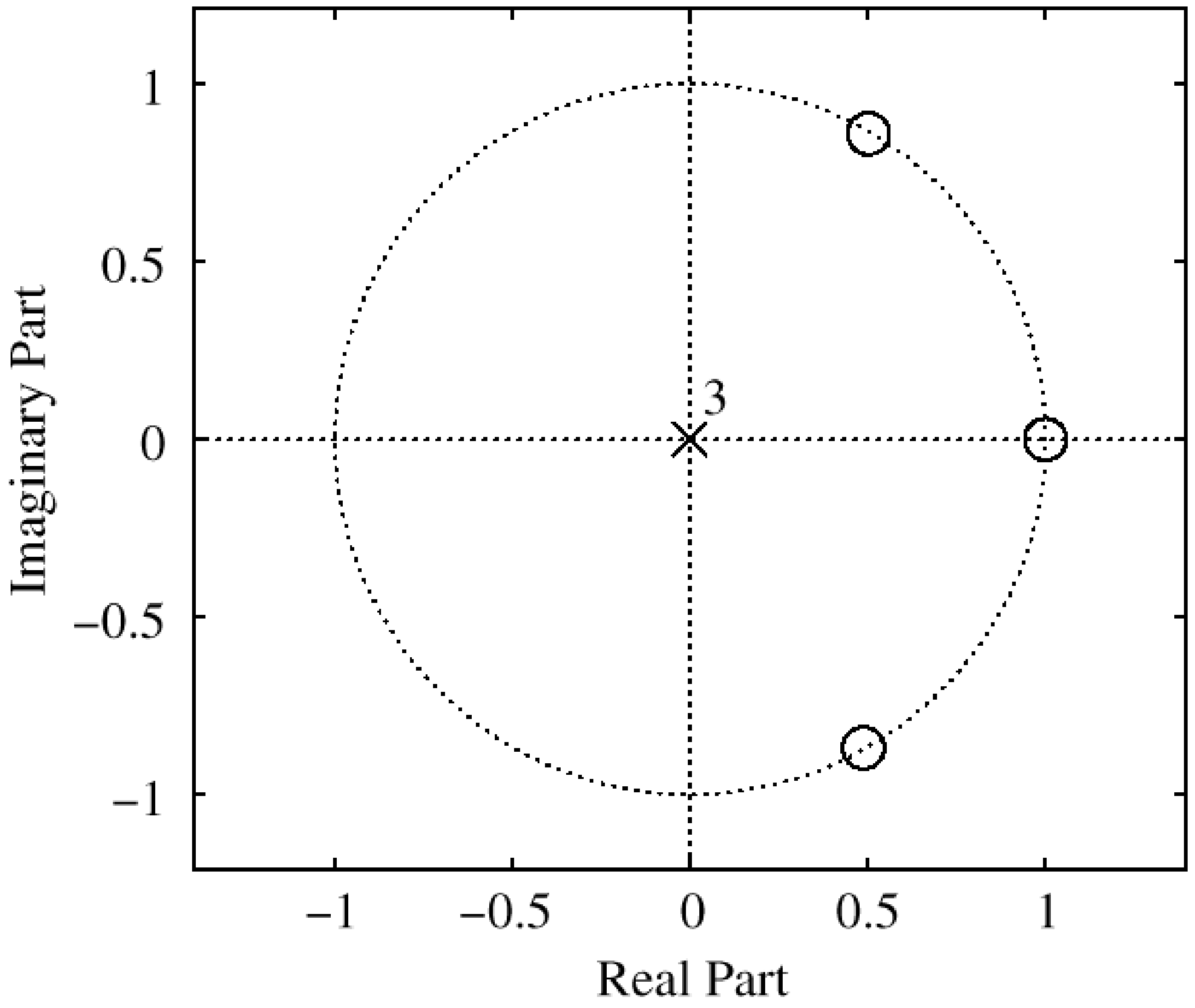
$$\boxed{\text{Roots}: z = 1, \tfrac{1}{2} \pm j\frac{\sqrt{3}}{2}} \rightarrow \boxed{e^{\pm j\pi/3}}$$

Recall: Roots occur in Conjugate pairs when coefficients are real

PLOT ZEROS in z-DOMAIN



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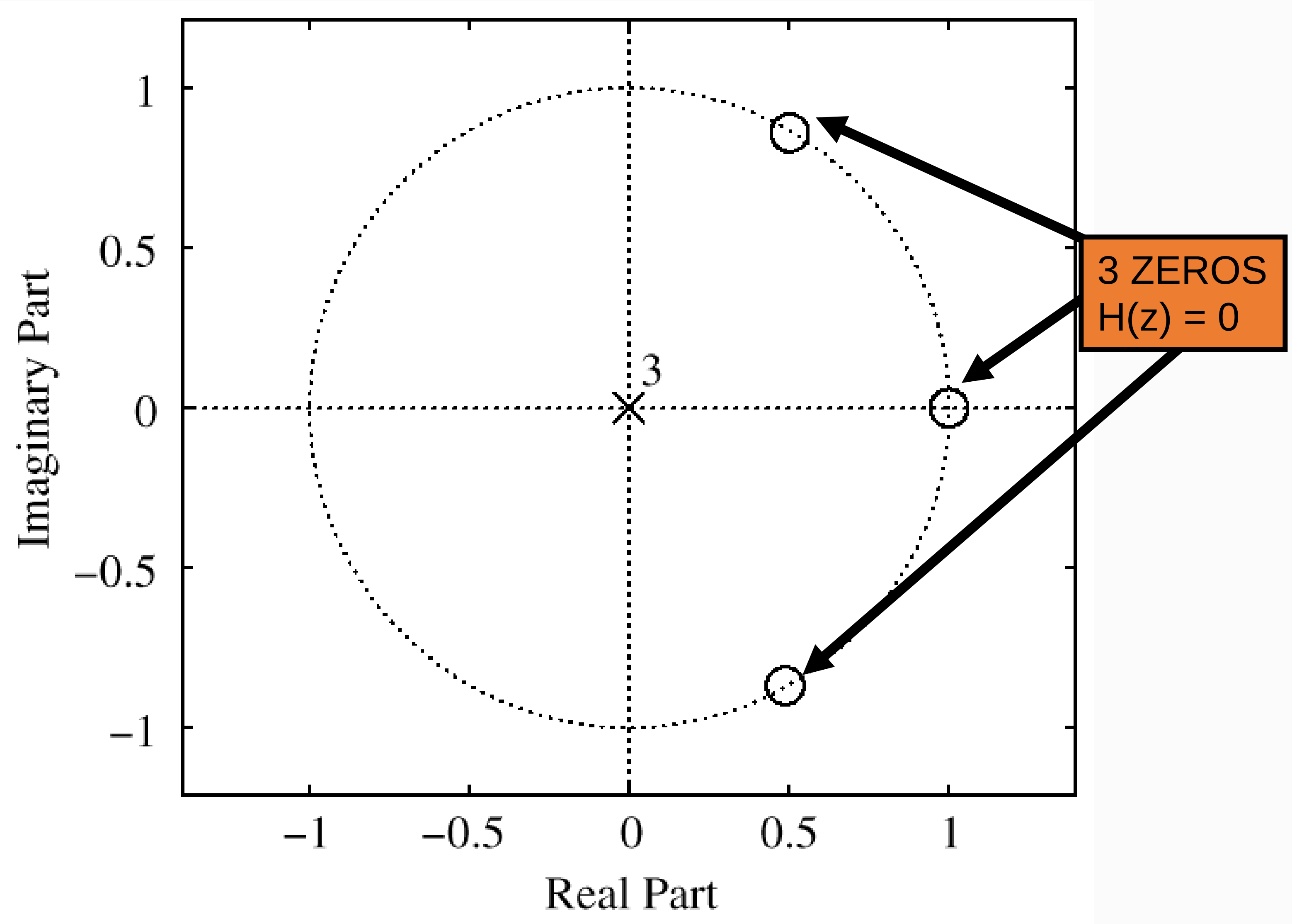

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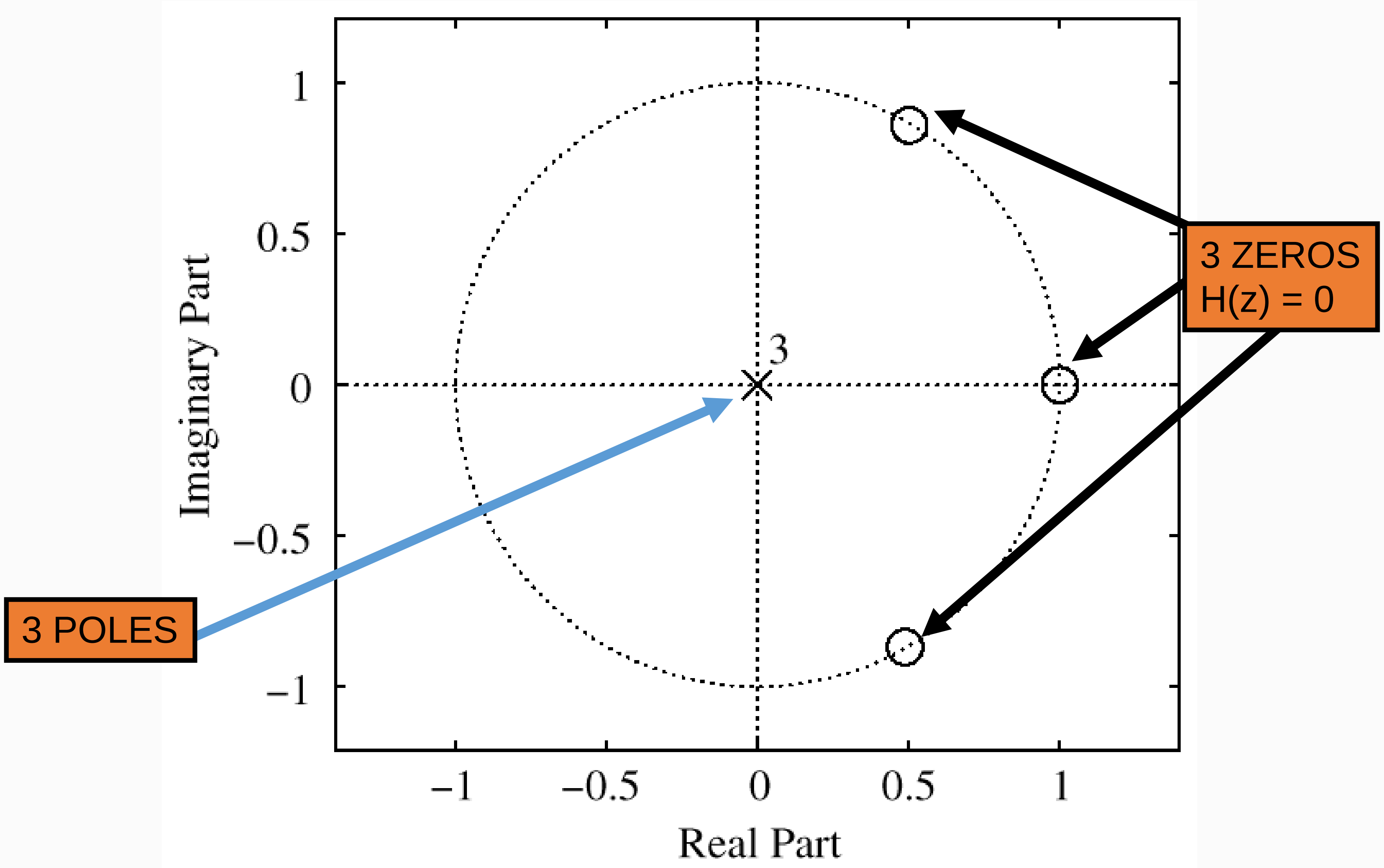

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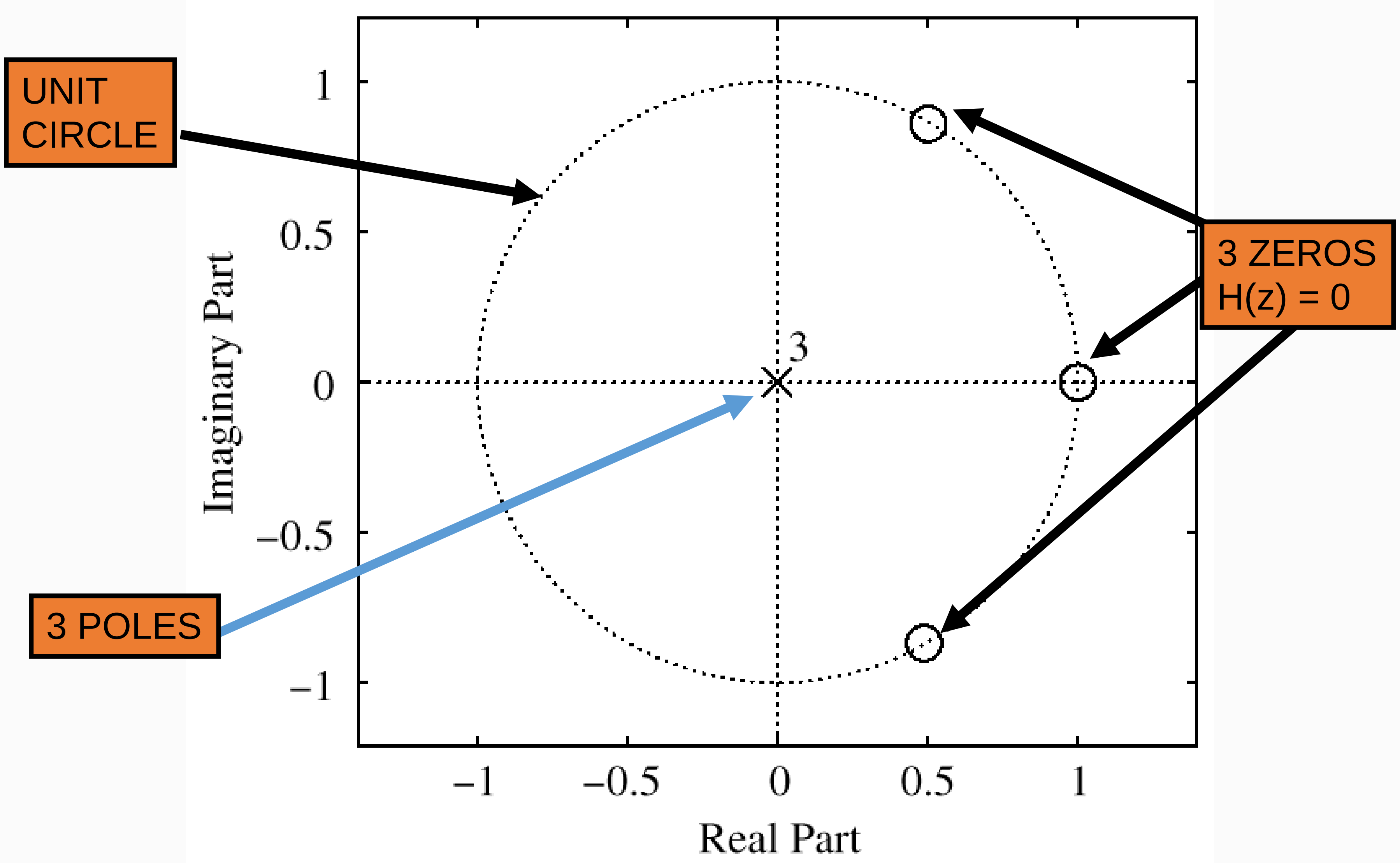

POLES of H(z)

- Find z, where
  - FIR only has poles at z=0

$$H(z) \to \infty$$

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POLES of H(z)

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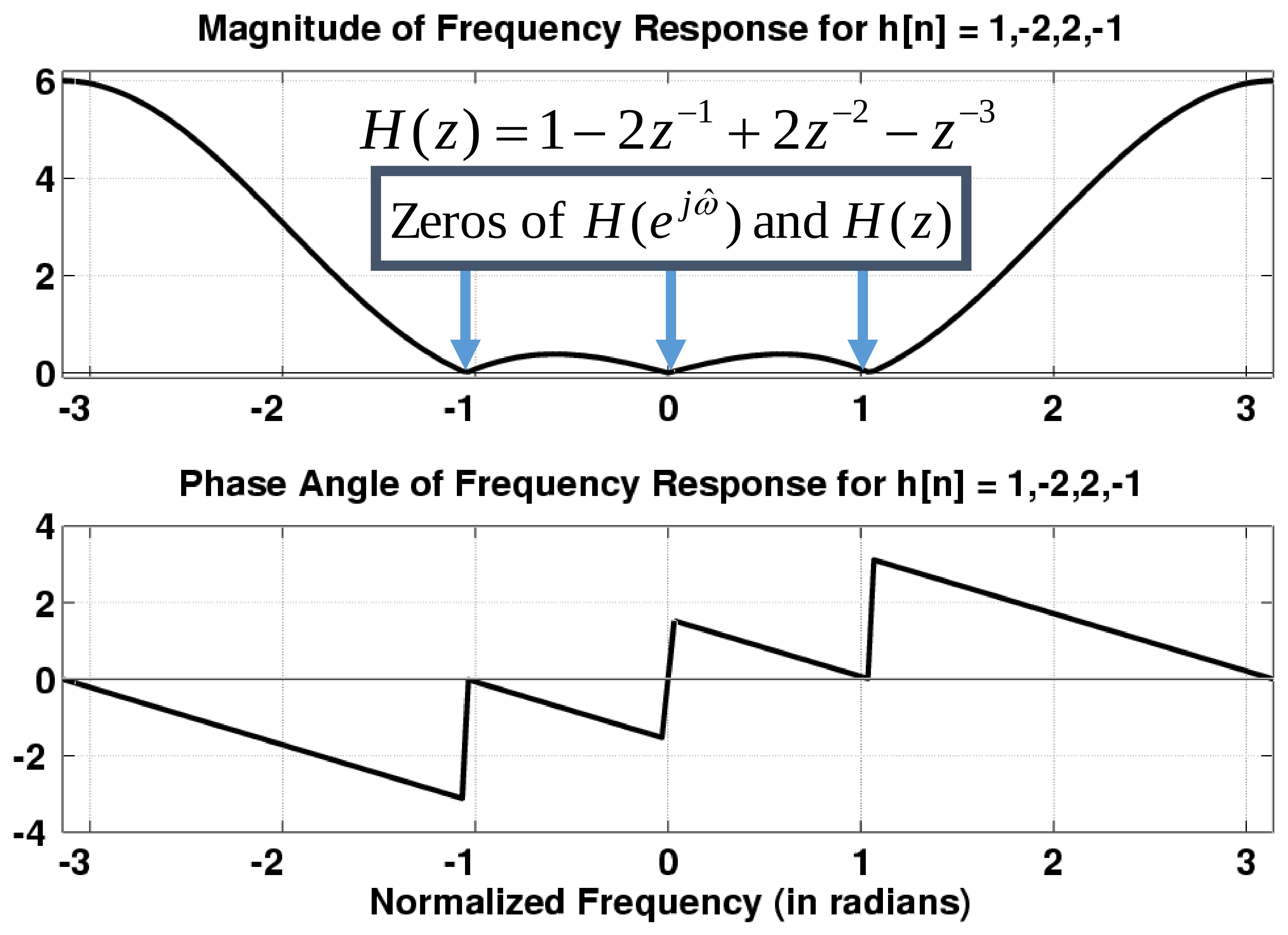
$$H(z) \to \infty$$

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

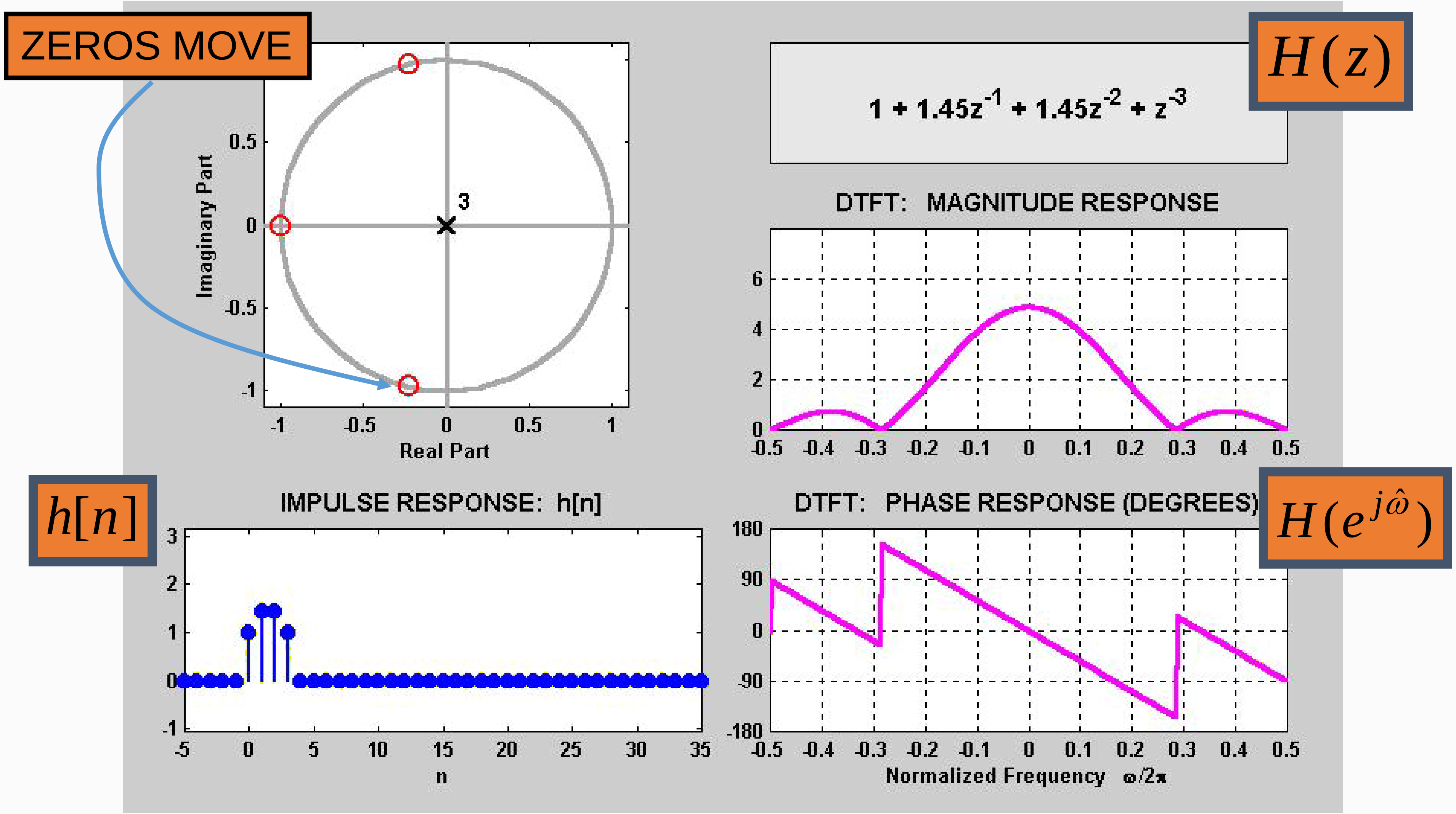
$$H(z) = \frac{z^3 - 2z^2 + 2z \;\; -1}{z^3}$$

$$\boxed{\text{Three Poles at} : z = 0}$$

FIR Frequency Response



3 DOMAINS MOVIE: FIR



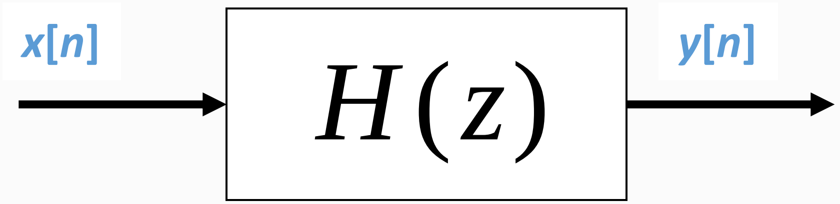
4 MOVIES @ WEBSITE

- http://dspfirst.gatech.edu/chapters/07ztrans/demos/3_domain/index.html
- 3 DOMAINS MOVIES: FIR Filters
  - Two zeros moving around UC and inside
  - Three zeros; one held fixed at $z=-1$
  - Ten zeros; 9 equally spaced around UC; one moving
  - Ten zeros; 8 equally spaced around UC; two moving

NULLING PROPERTY of H(z)

- When H(z)=0 on the unit circle.
  - Find inputs x[n] that give zero output

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$



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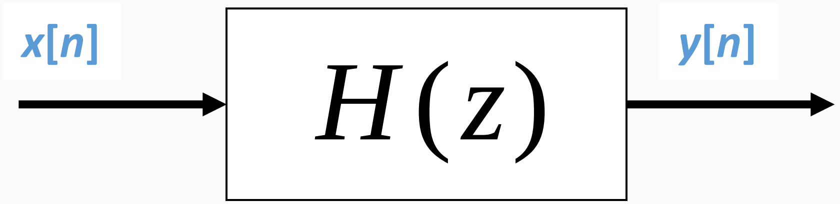
$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$



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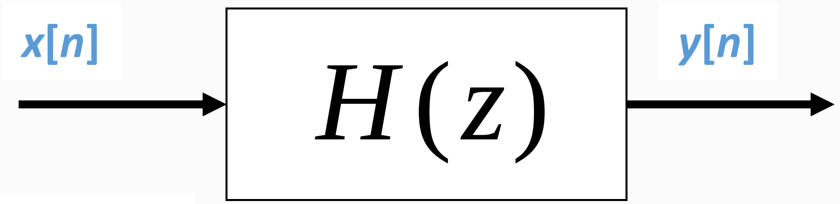

$$x[n] = e^{j(\pi/3)n}$$

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$$\xrightarrow{x[n]} \boxed{H(z)} \xrightarrow{y[n]}$$

$$x[n] = e^{j(\pi/3)n}$$

$$y[n] = H(e^{j(\pi/3)}) \cdot e^{j(\pi/3)n}$$

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$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$

$$\boxed{H(e^{j\pi/3}) = ?}$$

$$x[n] = e^{j(\pi/3)n}$$

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$$y[n] = H(e^{j\pi/3}) \cdot e^{j(\pi/3)n}$$

$$y[n] = (1 - 2e^{-j\pi/3} + 2e^{-j2\pi/3} - e^{-j3\pi/3}) \cdot e^{j(\pi/3)n}$$

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$$(1 - 2(\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) + 2(-\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) - (-1))$$

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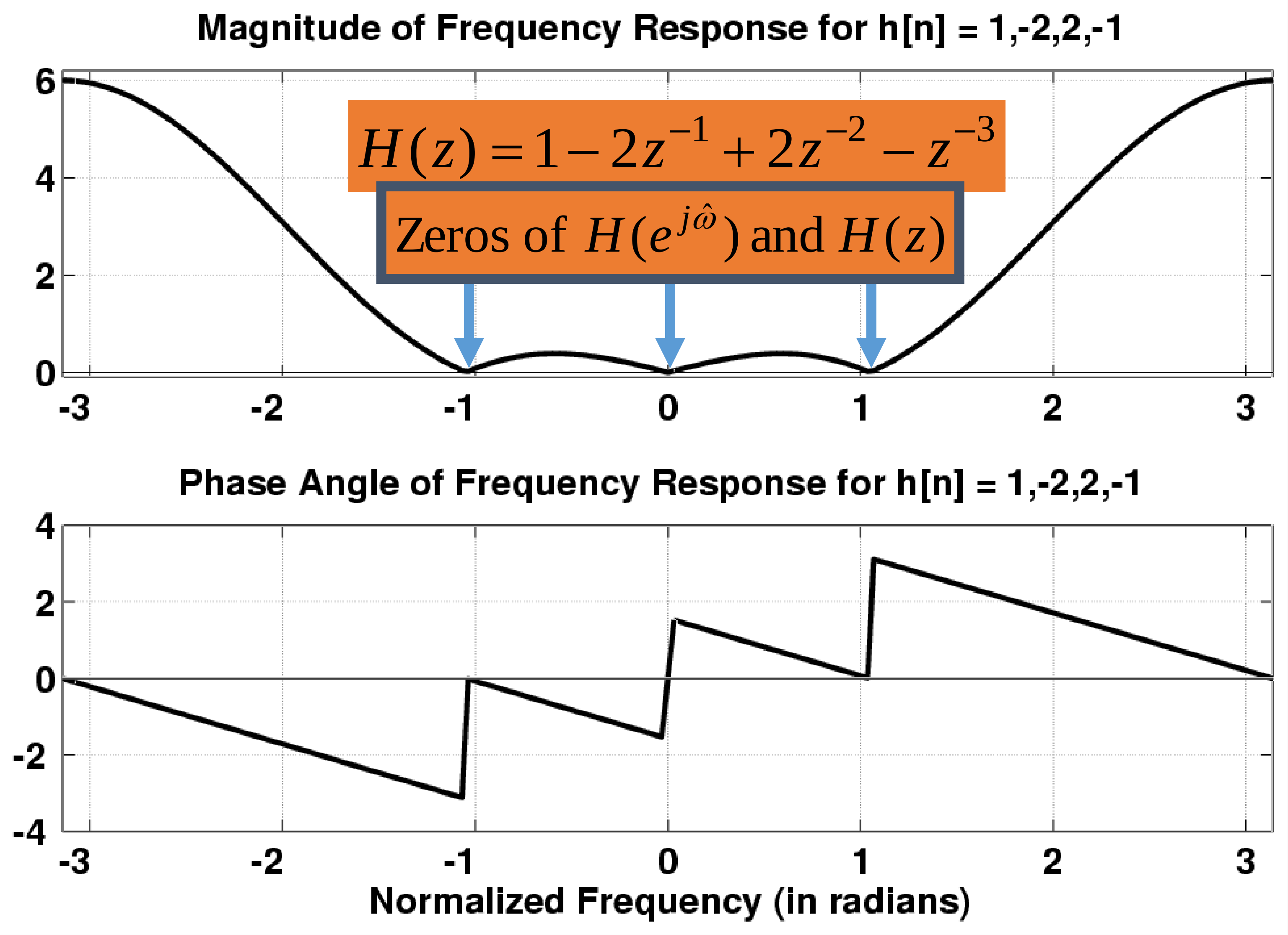
$$y[n] = (1 - 2e^{-j\pi/3} + 2e^{-j2\pi/3} - e^{-j3\pi/3}) \cdot e^{j(\pi/3)n}$$

$$(1 - 2(\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) + 2(-\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}) - (-1))$$

$$y[n] = (1 - 1 + j\sqrt{3} - 1 - j\sqrt{3} + 1) \cdot e^{j(\pi/3)n} = 0$$

FIR Frequency Response



DESIGN PROBLEM

- Example:
  - Design a Lowpass FIR filter (Find b_k)
  - Reject completely 0.7π, 0.8π, and 0.9π
  - Estimate the filter length needed to accomplish this task. How many b_k ?

- Z POLYNOMIALS provide the TOOLS

NULLING FILTER DESIGN

- PLACE ZEROS to make y[n] = 0

Need 6 ZEROS
where H(z) = 0

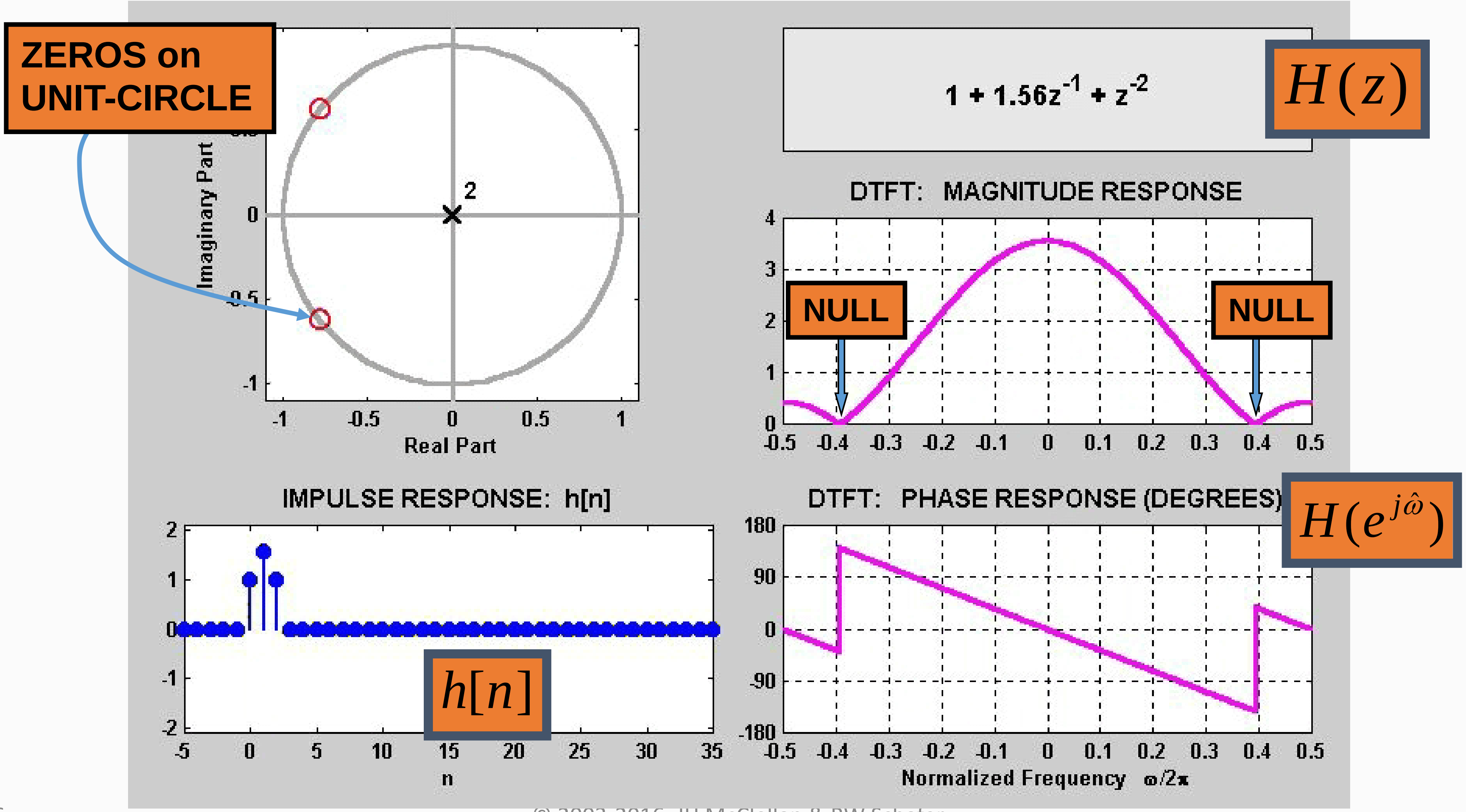
$$\boxed{H(z_k) = 0, \quad \text{for } z_k = e^{\pm j0.7\pi}, e^{\pm j0.8\pi}, e^{\pm j0.9\pi}}$$

- 6th order FIR has 7 filter coefficients

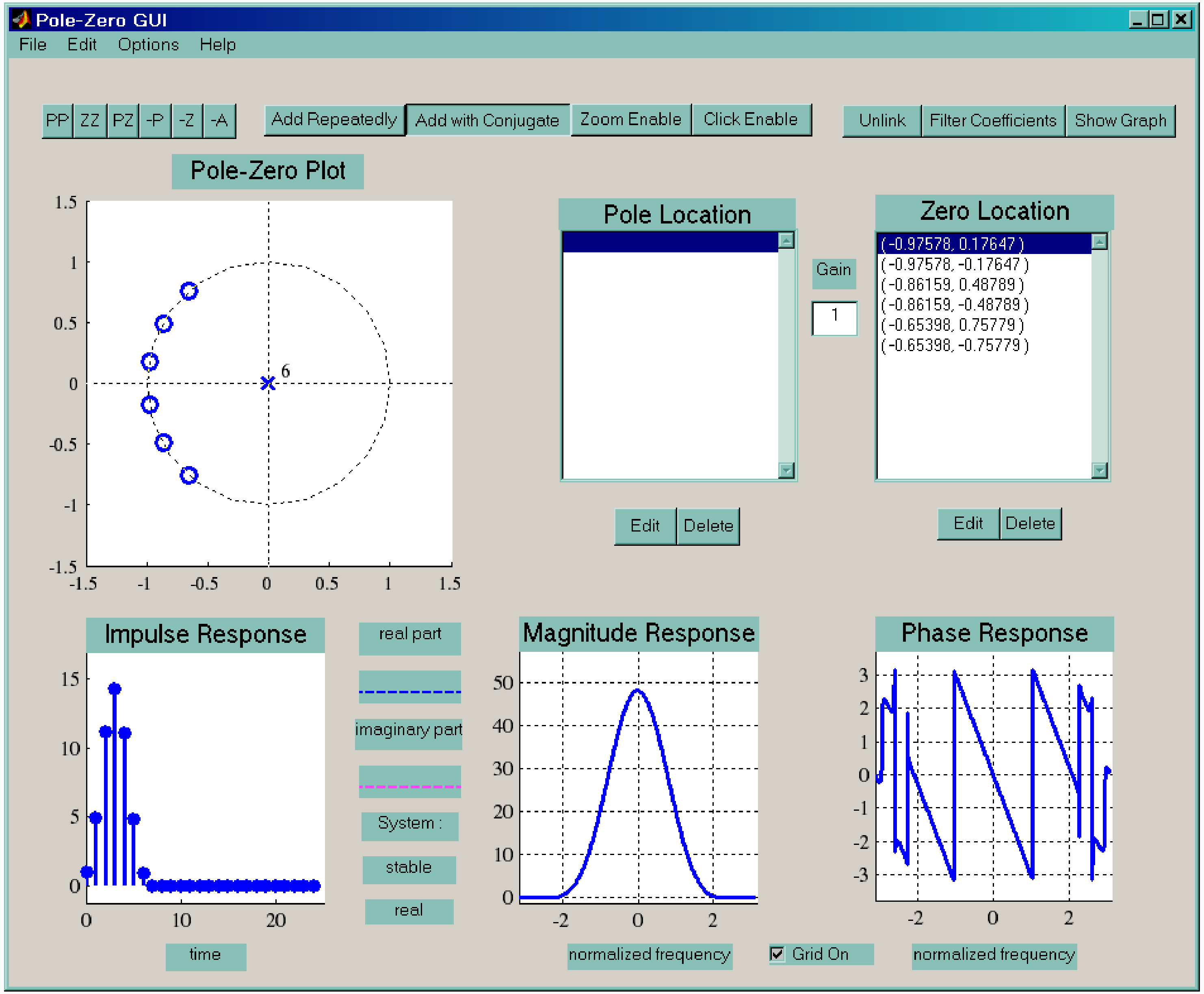
$$\boxed{x[n] = e^{j0.8\pi n} \quad \Rightarrow \quad y[n] = H(e^{j0.8\pi})e^{j0.8\pi n}}$$

$$\boxed{H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} + b_5 z^{-5} + b_6 z^{-6}}$$

3 DOMAINS MOVIE: FIR



PeZ Demo: Zero Placing



One zero, two zeros, …

We usually want filters with real coefficients

$$\boxed{H(z) = 1 - az^{-1} \quad \Rightarrow \quad H(z) = 0 \ @ \ z = a}$$

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If we want to block sinusoid with $\hat{\omega} = \pm 0.8\pi$

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If we want to block sinusoid with $\hat{\omega} = \pm 0.8\pi$

$$\begin{aligned}
&H(z_k) = 0 \quad \text{for } z_k = e^{\pm j0.8\pi} \\
&\Rightarrow H(z) = z^{-2}(z - e^{j0.8\pi})(z - e^{-j0.8\pi}) \\
&= z^{-2}(z^2 - z(e^{j0.8\pi} + e^{-j0.8\pi}) + 1) \\
&= 1 - 2(\cos 0.8\pi) z^{-1} + z^{-2} = 1 + 1.618 z^{-1} + z^{-2}
\end{aligned}$$

z^{-2} needed for causality

One zero, two zeros, …

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\end{aligned}$$

z^{-2} needed for causality

$$h[0] = 1, \quad h[1] = 1.618, \quad h[2] = 1$$

Block Multiple Frequencies

Want to totally block: $\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_m$

***H*(*z*)** must have zeros at: $z = e^{\pm j\hat{\omega}_1}, e^{\pm j\hat{\omega}_2}, \cdots, e^{\pm j\hat{\omega}_m}$

To block $\hat{\omega} = 0$ or π must have zero at ***z*** =1 or -1

So, the general form becomes:

$$H(z) = (1 - z^{-1})(1 + z^{-1}) \prod_{n=1}^{m} (1 - e^{j\hat{\omega}_n} z^{-1})(1 - e^{-j\hat{\omega}_n} z^{-1})$$

to block DC

to block $f_s/2$

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Want to totally block: $\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_m$

***H*(*z*)** must have zeros at: $z = e^{\pm j\hat{\omega}_1}, e^{\pm j\hat{\omega}_2}, \cdots, e^{\pm j\hat{\omega}_m}$

To block $\hat{\omega} = 0$ or π must have zero at ***z*** =1 or -1

So, the general form becomes:

$$H(z) = (1 - z^{-1})(1 + z^{-1}) \prod_{n=1}^{m} (1 - e^{j\hat{\omega}_n} z^{-1})(1 - e^{-j\hat{\omega}_n} z^{-1})$$

to block DC

to block $f_s/2$

On the other hand: Not much control over other frequencies

L-pt RUNNING SUM $H(z)$

L-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{L-1} z^{-k}$$

L-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1 - z^{-L}}{1 - z^{-1}} = \frac{z^{L} - 1}{z^{L-1}(z-1)}$$

L-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1-z^{-L}}{1-z^{-1}} = \frac{z^L - 1}{z^{L-1}(z-1)}$$

$$\boxed{z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{j2\pi k}}$$

L-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1 - z^{-L}}{1 - z^{-1}} = \frac{z^{L} - 1}{z^{L-1}(z-1)}$$

$$\boxed{z^{L} - 1 = 0 \quad \Rightarrow \quad z^{L} = 1 = e^{j2\pi k}}$$

$$\boxed{z = e^{j(2\pi/L)k} \quad \text{for } k = \; 1, 2, \ldots L-1}$$

ZEROS on
UNIT CIRCLE

L-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1 - z^{-L}}{1 - z^{-1}} = \frac{z^L - 1}{z^{L-1}(z-1)}$$

$$\boxed{z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{j2\pi k}}$$

$$\boxed{z = e^{j(2\pi/L)k} \quad \text{for } k = 1,2,\ldots L-1}$$

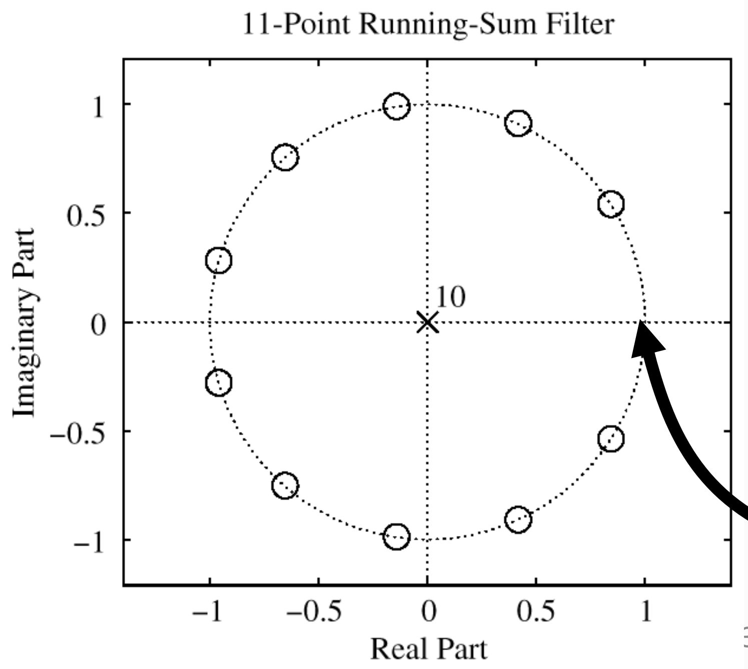
ZEROS on
UNIT CIRCLE

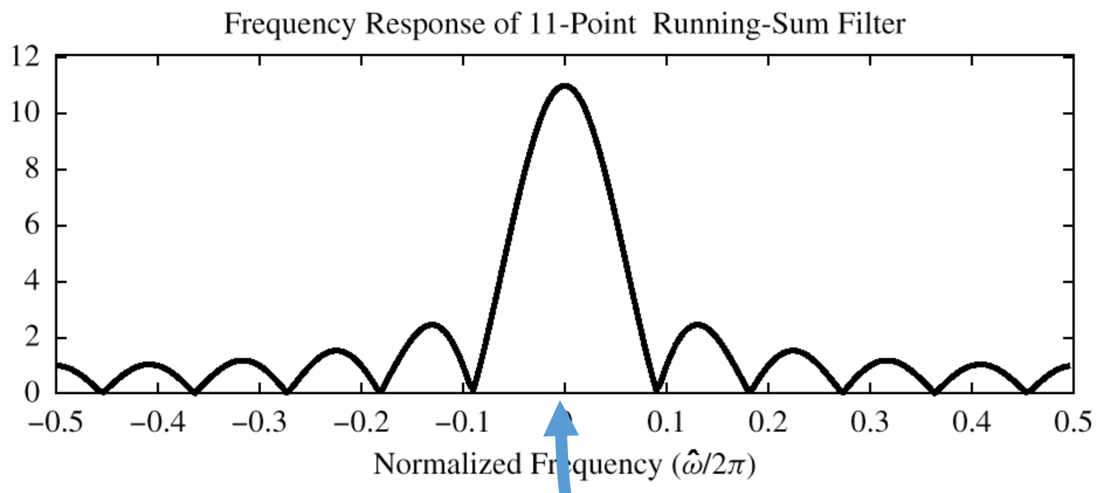
(z-1) in
denominator
cancels k=0 term

11-pt RUNNING SUM H(z)

$$H(z) = \sum_{k=0}^{10} z^{-k}$$

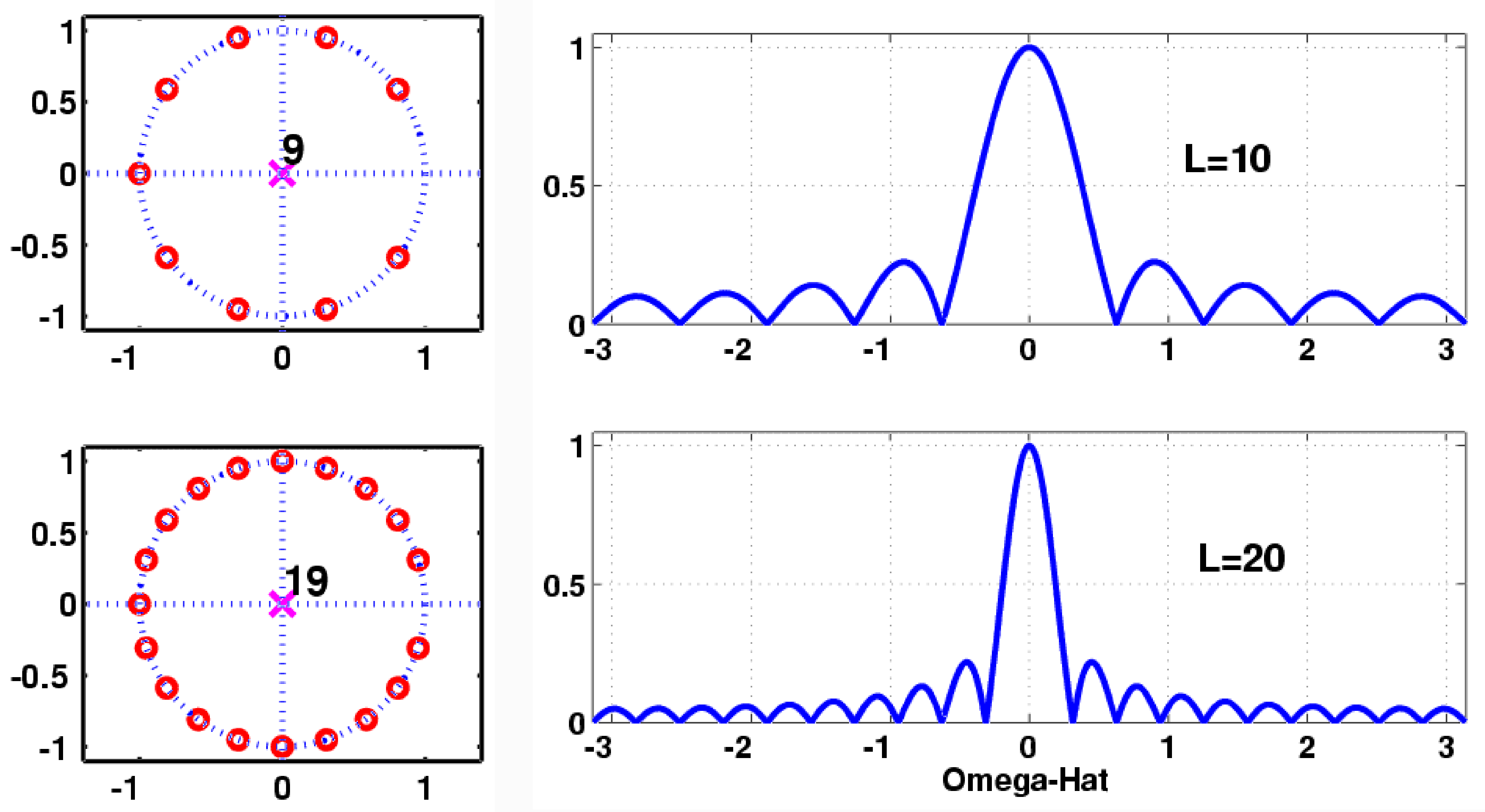
$$H(z) = (1 - e^{j2\pi/11} z^{-1})(1 - e^{j4\pi/11} z^{-1}) \cdots (1 - e^{j20\pi/11} z^{-1})$$



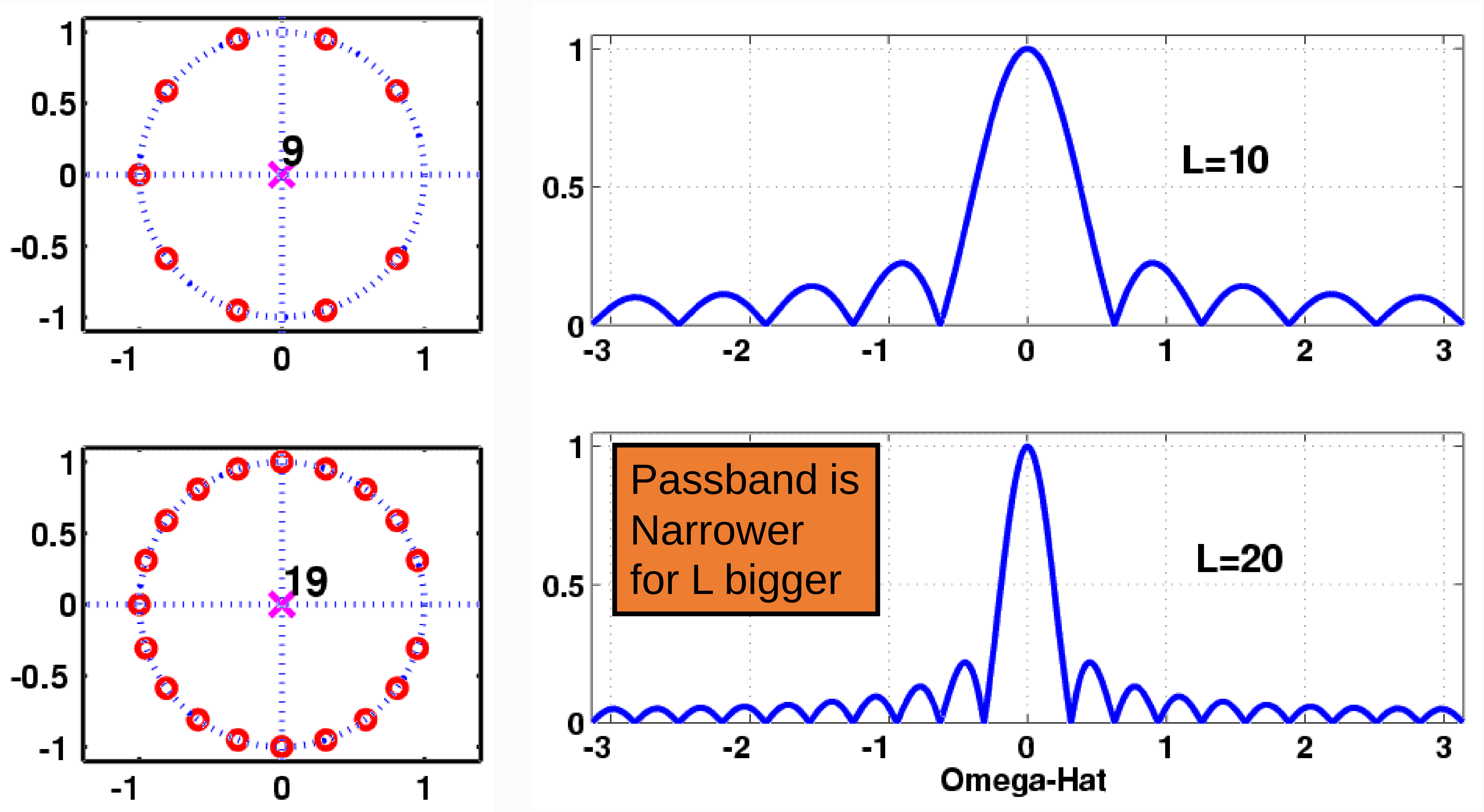


NO zero at z=1

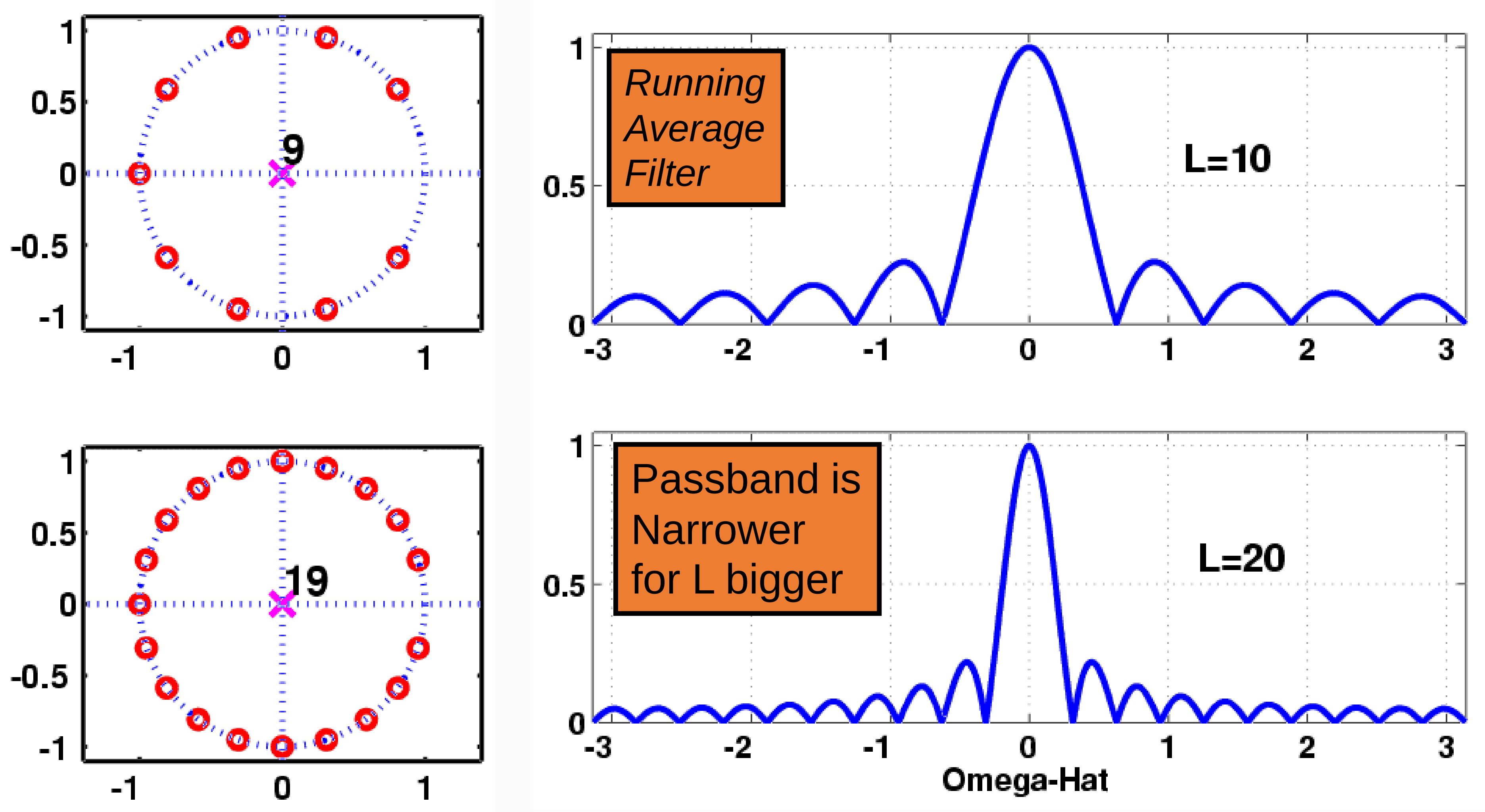
FILTER DESIGN: CHANGE *L*



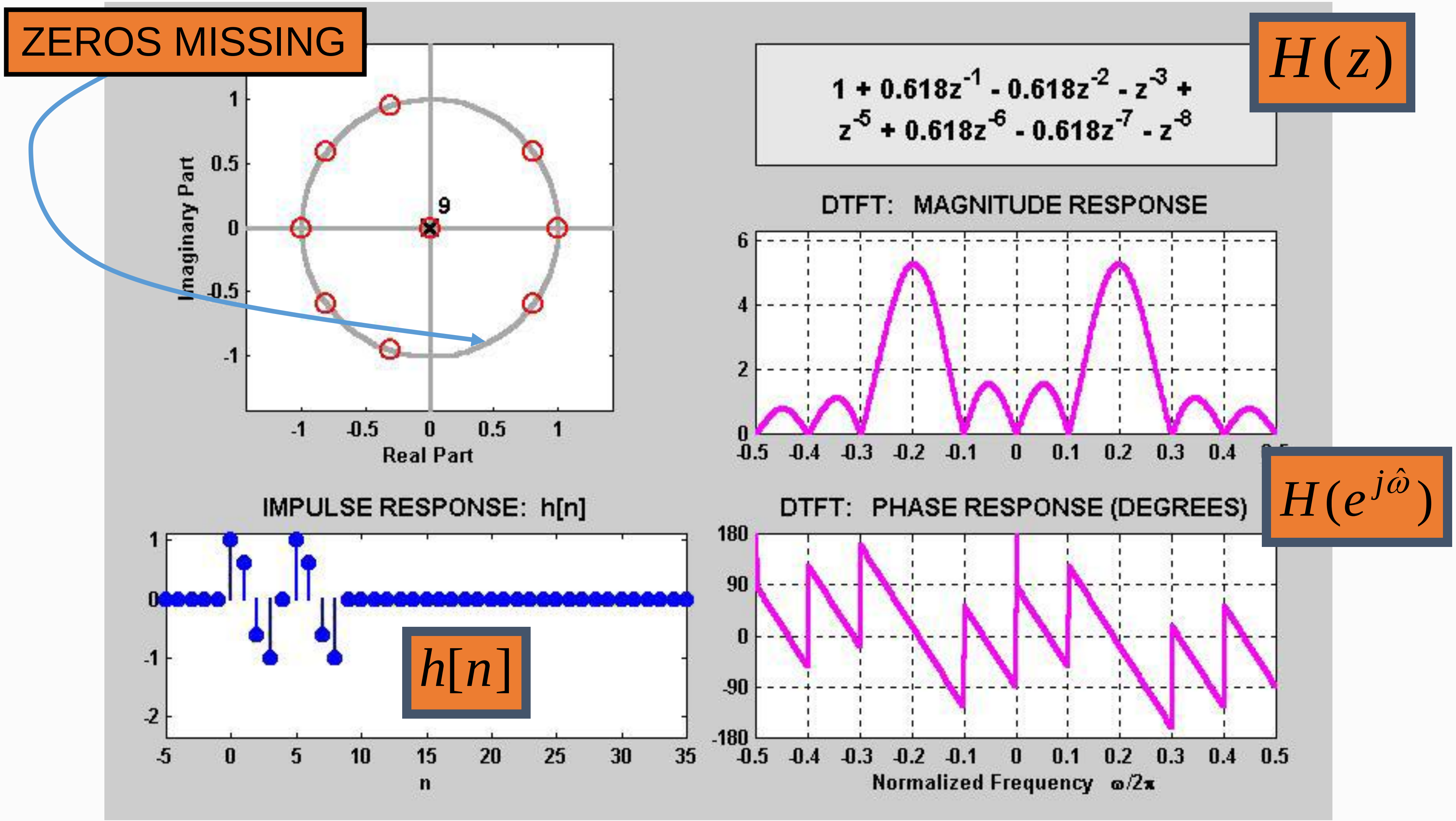
FILTER DESIGN: CHANGE *L*



FILTER DESIGN: CHANGE *L*



3 DOMAINS MOVIE: FIR BPF



Check Website & MATLAB Code

https://dspfirst.gatech.edu/chapters/07ztrans/demos/3_domain/index.html

```
%% load piano sounds
load labtest.mat;
sound(xx,fs);
figure(1); spectrogram(xx);

%% If I want to pass w = 0.2pi
load filter2.mat;
yy = filter( z_values, p_values, xx);
[h,w] = freqz( z_values, p_values, 'whole',120);
plot((w)/pi,abs(h));

sound(yy,fs);
figure(2); spectrogram(yy);
```