DSP First, 2/e

Lecture 22
IIR Filters: Feedback and H(z)

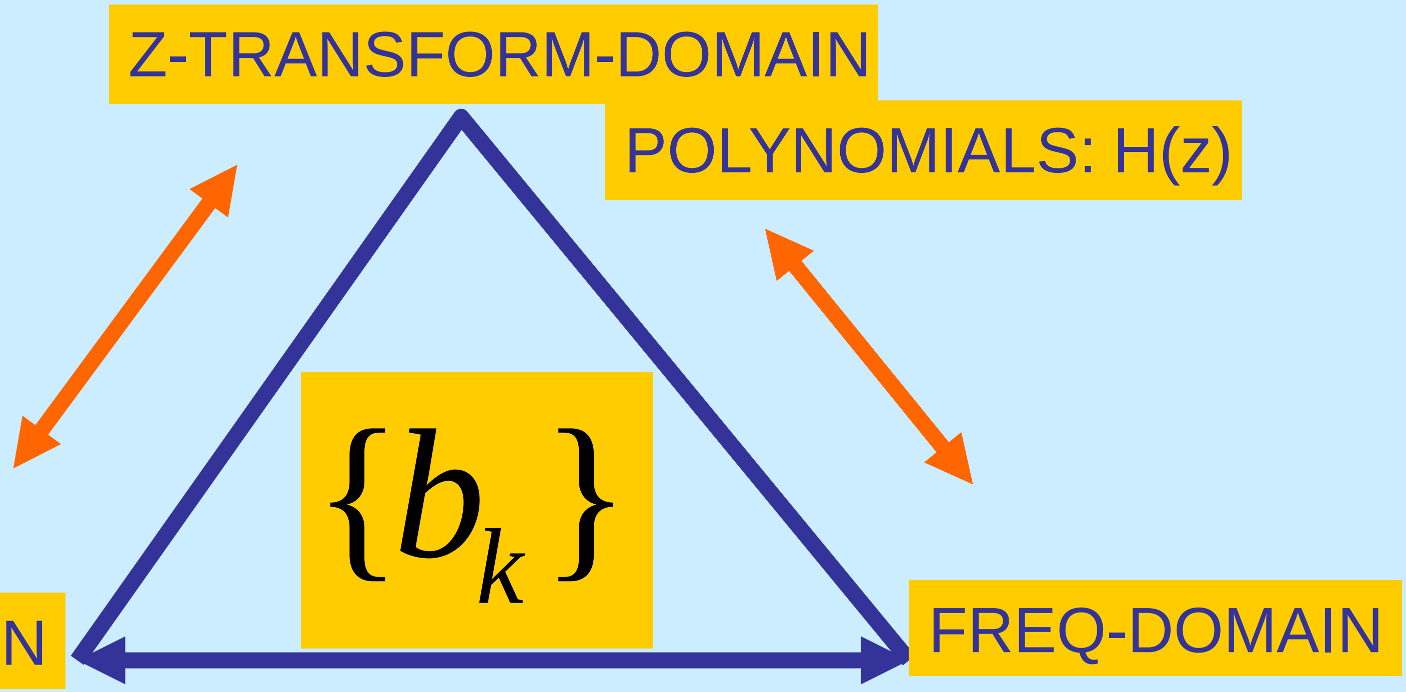
READING ASSIGNMENTS

- This Lecture:
  - Chapter 10, Sects. 10-1, 10-2, & 10-3
- Other Reading:
  - Optional: Ch. 10, Sect 10-4
    - FILTER STRUCTURES

LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
  - Define **IIR** DIGITAL Filters
    - Filters with **FEEDBACK**
    - use PREVIOUS OUTPUTS
  - Show how to compute the output y[n]
  - Derive Impulse Response h[n]
  - Derive *z*-transform: h[n] ←→ H(z)

THREE DOMAINS



$$\{b_k\}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

Quick Review: Delay by n_d

Difference Equation $y[n] = x[n - n_d]$

IMPULSE RESPONSE $h[n] = \delta[n - n_d]$

SYSTEM FUNCTION $H(z) = z^{-n_d}$

Frequency Response $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_d}$

Quick Review: L-pt Averager

Difference Equation

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} x[n-k]$$

IMPULSE RESPONSE

$$h[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} \delta[n-k]$$

SYSTEM FUNCTION

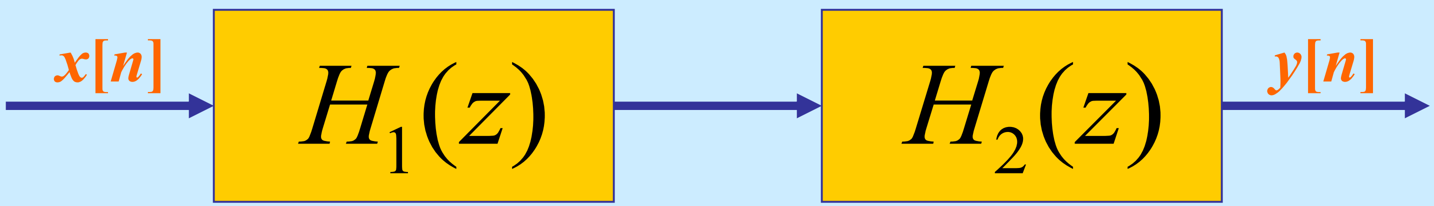
$$H(z) = \sum_{n=0}^{L-1} \tfrac{1}{L} z^{-n}$$

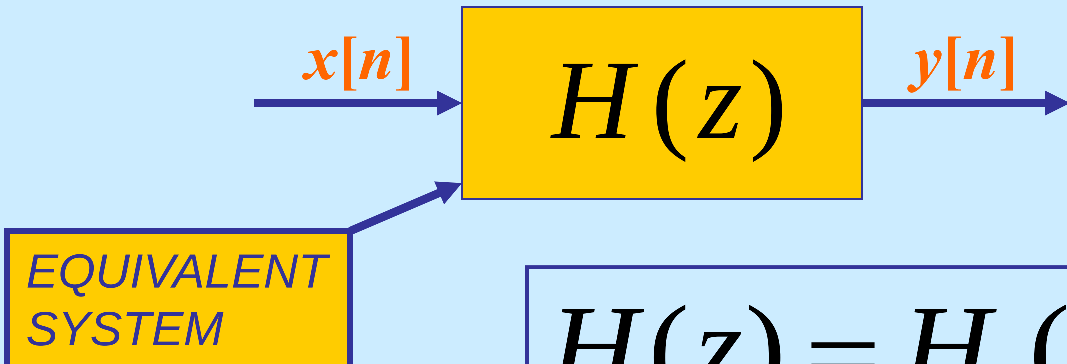
Frequency RESPONSE

$$H(e^{j\hat{\omega}}) = \frac{1}{L} e^{-j\frac{L-1}{2}\hat{\omega}} \frac{\sin(\frac{L}{2}\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})}$$

Recall: CASCADE Equivalent

- Multiply the System Functions

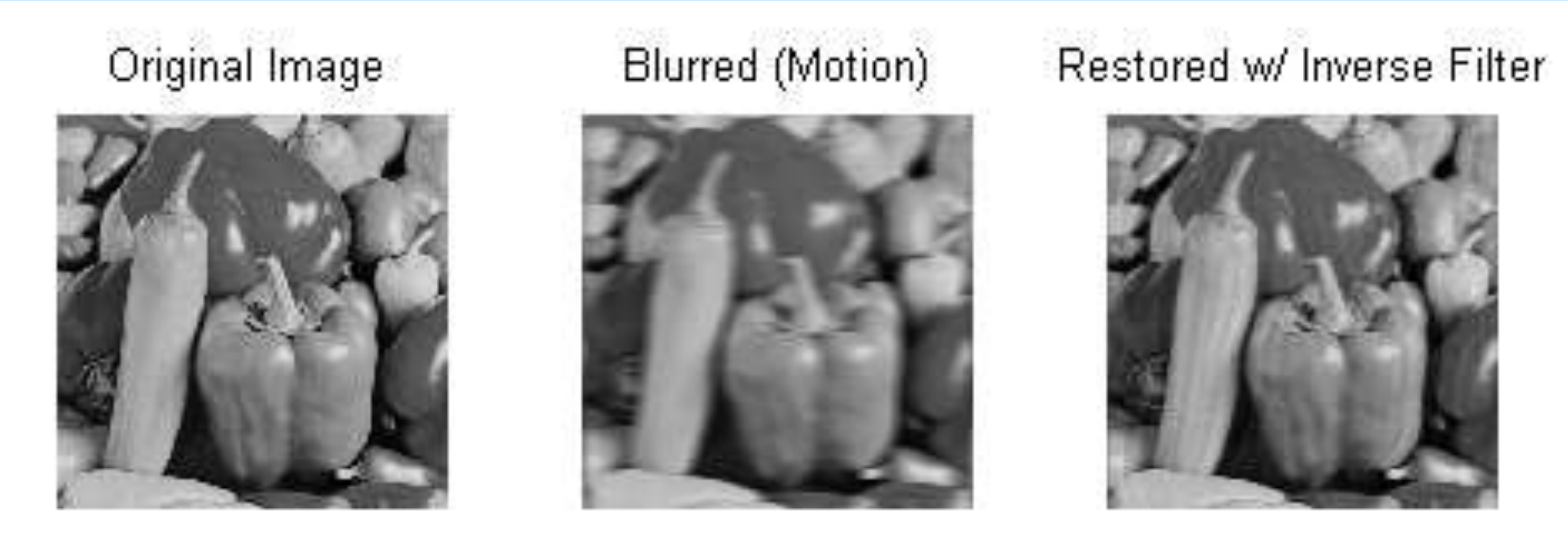




$$\boxed{H(z) = H_1(z)H_2(z)}$$

Motivation: DEconvolution

- Ex: Remove optical blur in postprocessing?



Deconvolution Filter

- System to remove optical blur in postprocessing
- Given $h_1[n]$, can we find $h_2[n]$ to make $y[n]$ equal to $s[n]$?

$$s[n] \longrightarrow \boxed{h_1[n]} \xrightarrow{x[n]} \boxed{h_2[n]} \xrightarrow{y[n]}$$

$$x[n] = s[n] * h_1[n]$$

$$y[n] = x[n] * h_2[n] = s[n] * h_1[n] * h_2[n]$$

$$\Rightarrow h_1[n] * h_2[n] = \delta[n]$$

Deconvolution in Z-DOMAIN

$$x[n] = s[n] - as[n-1] \Rightarrow h_1[n] = \delta[n] - a\delta[n-1]$$

- Hard to solve for $h_2[n]$ in convolution sum
- z-domain? $Y(z) = H_2(z)H_1(z)S(z) = H(z)S(z)$

$$s[n] \longrightarrow \boxed{H_1(z)} \xrightarrow{x[n]} \boxed{H_2(z)} \longrightarrow y[n]$$

$$H(z) = 1 = H_2(z)H_1(z)$$
$$\Rightarrow H_2(z) = 1/H_1(z)$$

Not FIR

$$H_1(z) = 1 - az^{-1}$$
$$\Rightarrow H_2(z) = \frac{1}{1 - az^{-1}}$$

IIR FILTERS

- IIR = **infinite impulse response**; the impulse response h[n] has infinite length
- **FIR**: is a weighted sum of inputs, so the current output value does **not** involve previous output values, only the input values
- **IIR**: the current output value involves **previous output values** (**feedback**) as well as input values

First Order IIR – ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]}$$

$a_0 = 1$

PREVIOUS

FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY: NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$\boxed{y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]}$$

SIGN CHANGE

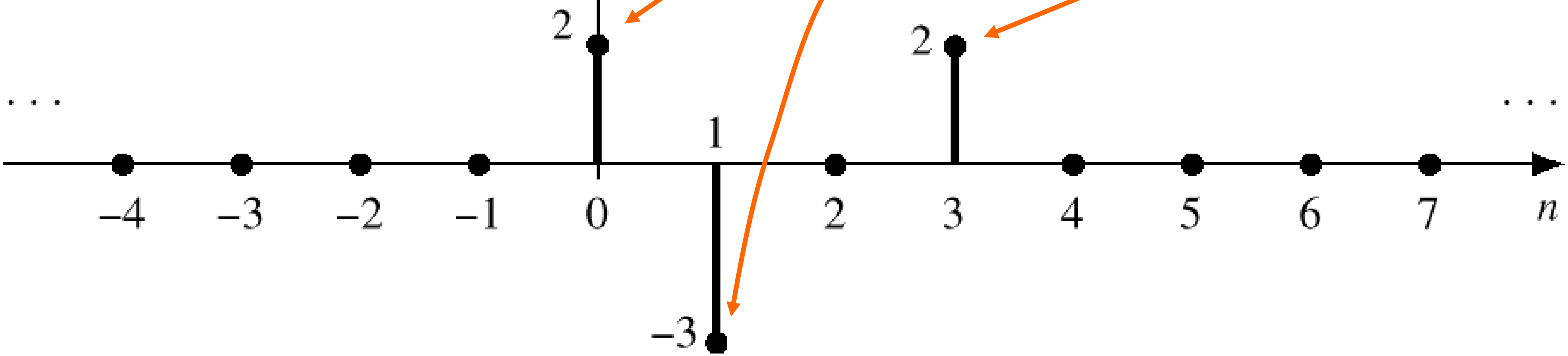
- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

$$\boxed{y[n] - 0.8y[n-1] = 3x[n] - 2x[n-1]}$$

COMPUTE OUTPUT

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$



COMPUTE y[n]

- FEEDBACK DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$

- NEED y[-1] to get started

$$\boxed{y[0] = 0.8y[-1] + 5x[0]}$$

AT REST CONDITION

- y[n] = 0, for n<0
- BECAUSE x[n] = 0, for n<0

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied*.
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- SAME with MORE FEEDBACK TERMS

$$\boxed{y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]}$$

COMPUTE MORE y[n]

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

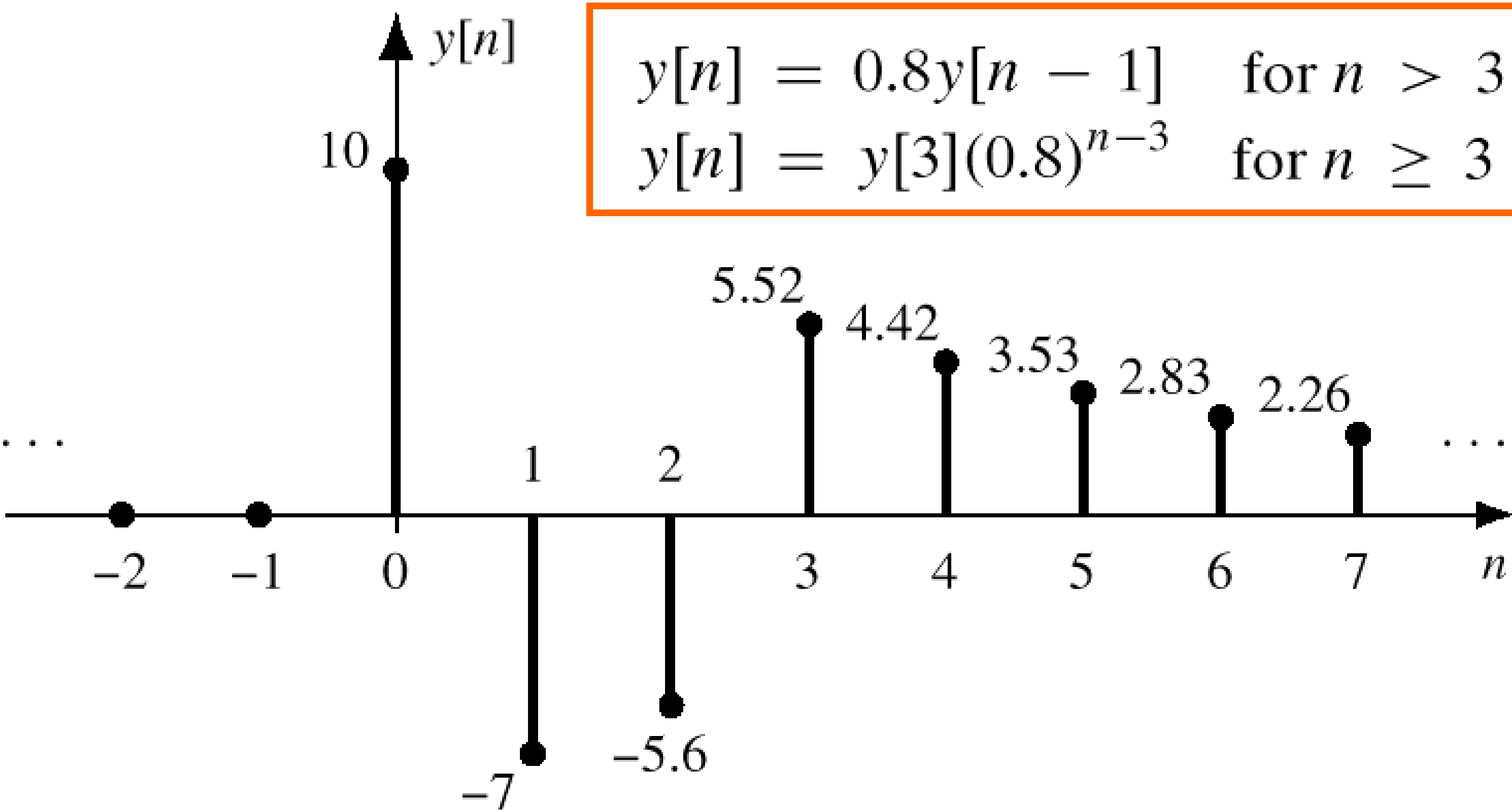
$$y[6] = 0.8y[5] + 5x[6] = 0.8(\underline{3.5328}) + 5(0) = 2.8262$$

Continues @ $(0.8)^{n-3}$

No more input

PLOT y[n]   (infinite length)

$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$
$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$



IMPULSE RESPONSE

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] \Rightarrow h[n] = a_1 h[n-1] + b_0 \delta[n]}$$

n	$n < 0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \qquad h[n] = b_0(a_1)^n u[n]$$

$$\boxed{u[n] = 1, \text{for } n \geq 0}$$

IMPULSE RESPONSE

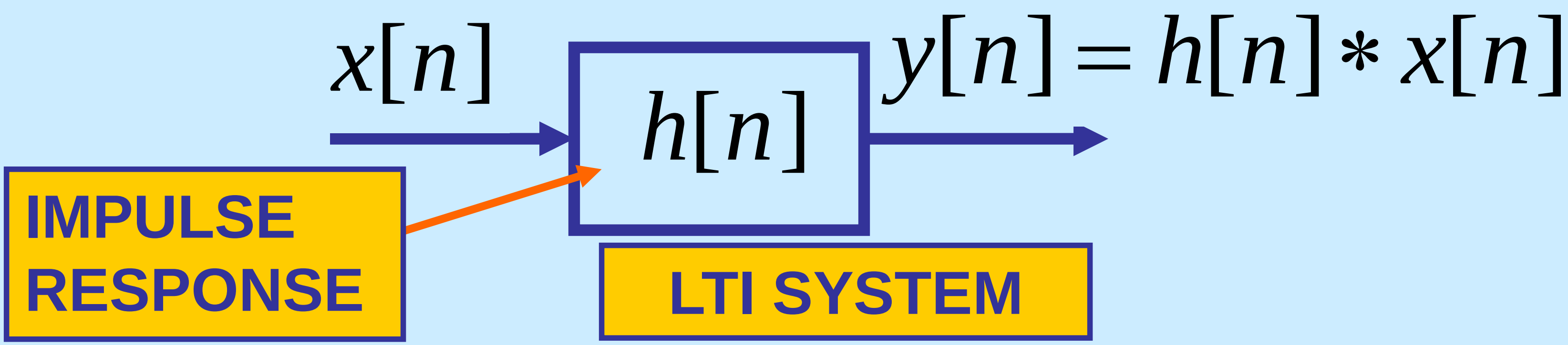
- DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 3x[n]}$$

- Find h[n]

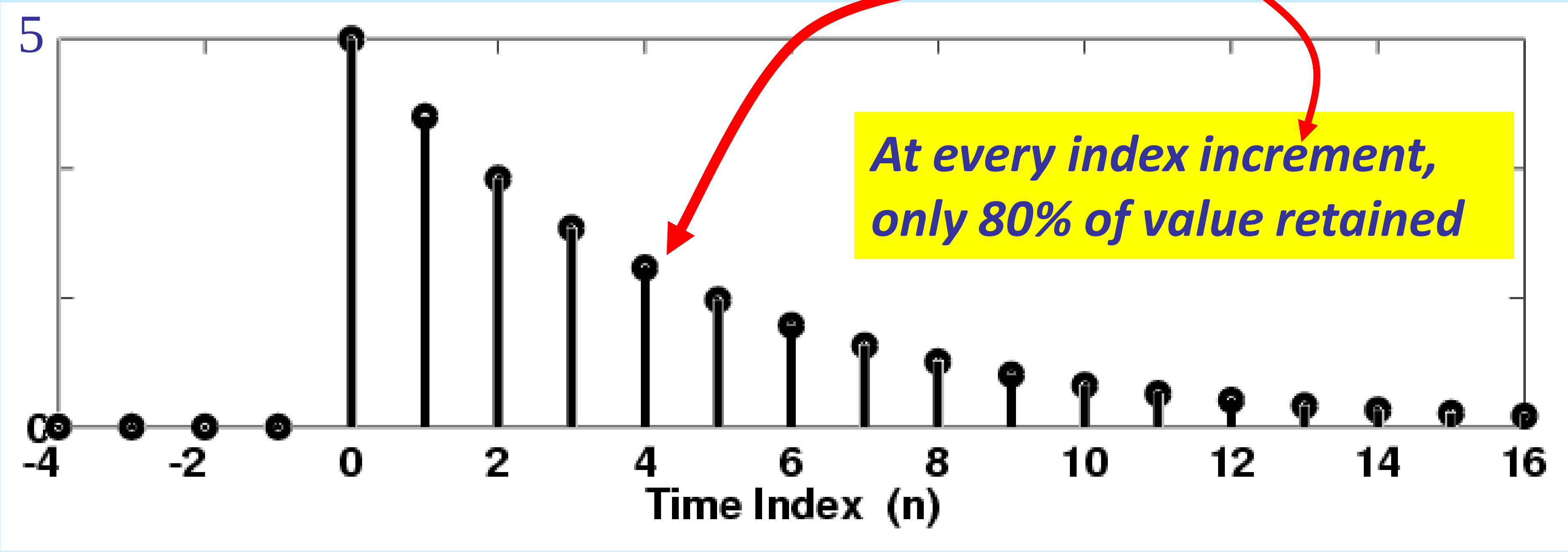
$$\boxed{h[n] = 3(0.8)^n u[n]}$$

- <u>**CONVOLUTION**</u> in TIME-DOMAIN



PLOT IMPULSE RESPONSE

$$h[n] = b_0(a_1)^n u[n] = 5(0.8)^n u[n]$$



Infinite-Length Signal: h[n]

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION in IIR

$$H(z) = \sum_{n=-\infty}^{\infty} b_0 (a_1)^n u[n] z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n}$$

Derivation of H(z)

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

H(z) = z-Transform{ h[n] }

- **FIRST-ORDER** IIR FILTER:

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n]}$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

$$\boxed{H(z) = \frac{b_0}{1 - a_1 z^{-1}}}$$

The impulse response is ***infinitely*** *long.*
But, the filter is specified by only a few coefficients –
The ***order*** *is* ***finite***.

Find H(z) from DE via ALGEBRA

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n]}$$

$$\mathfrak{Z}\{\bullet\} = z\text{-transform}$$

$$\mathfrak{Z}\{y[n]\} = a_1 \mathfrak{Z}\{y[n-1]\} + b_0 \mathfrak{Z}\{x[n]\}$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z)$$

$$Y(z) - a_1 z^{-1} Y(z) = Y(z)(1 - a_1 z^{-1}) = b_0 X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 - a_1 z^{-1}}$$

H(z) = z-Transform{ h[n] }

- ANOTHER FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]$$

z^{-1} is a shift

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

STEP RESPONSE: x[n]=u[n]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$u[n] = 1, \text{ for } n \geq 0$

n	$x[n]$	$y[n]$
$n < 0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
$\vdots$	$\vdots$	$\vdots$

DERIVE STEP RESPONSE

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \qquad \text{for } n \geq 0, \qquad \text{if } a_1 \neq 1$$

PLOT STEP RESPONSE

$$u[n] = 1, \quad \text{for } n \geq 0$$

$$y[n] = 0.8\,y[n-1] + 3u[n]$$

$$y[n] = 15\left(1 - 0.8^{n+1}\right)u[n]$$