BLM3620 Digital Signal Processing*

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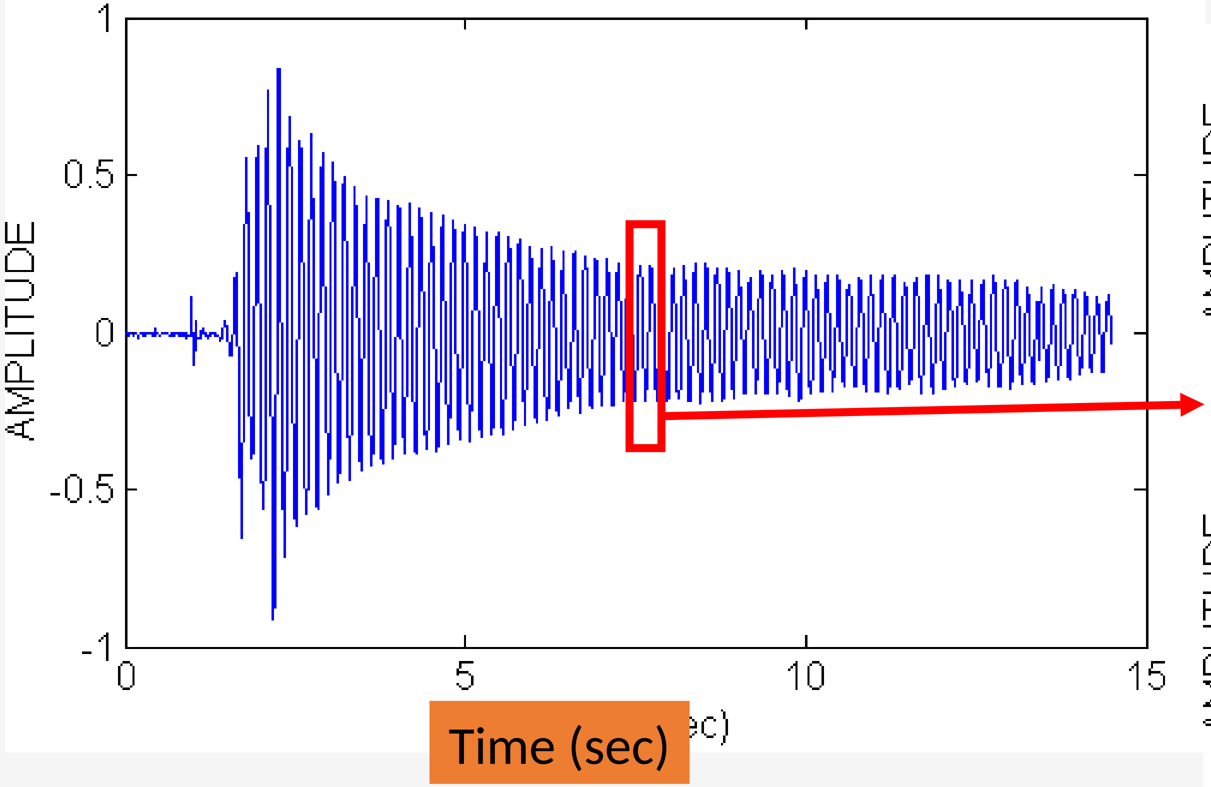
*Based on lecture notes from Ali Can Karaca & Ahmet Elbir

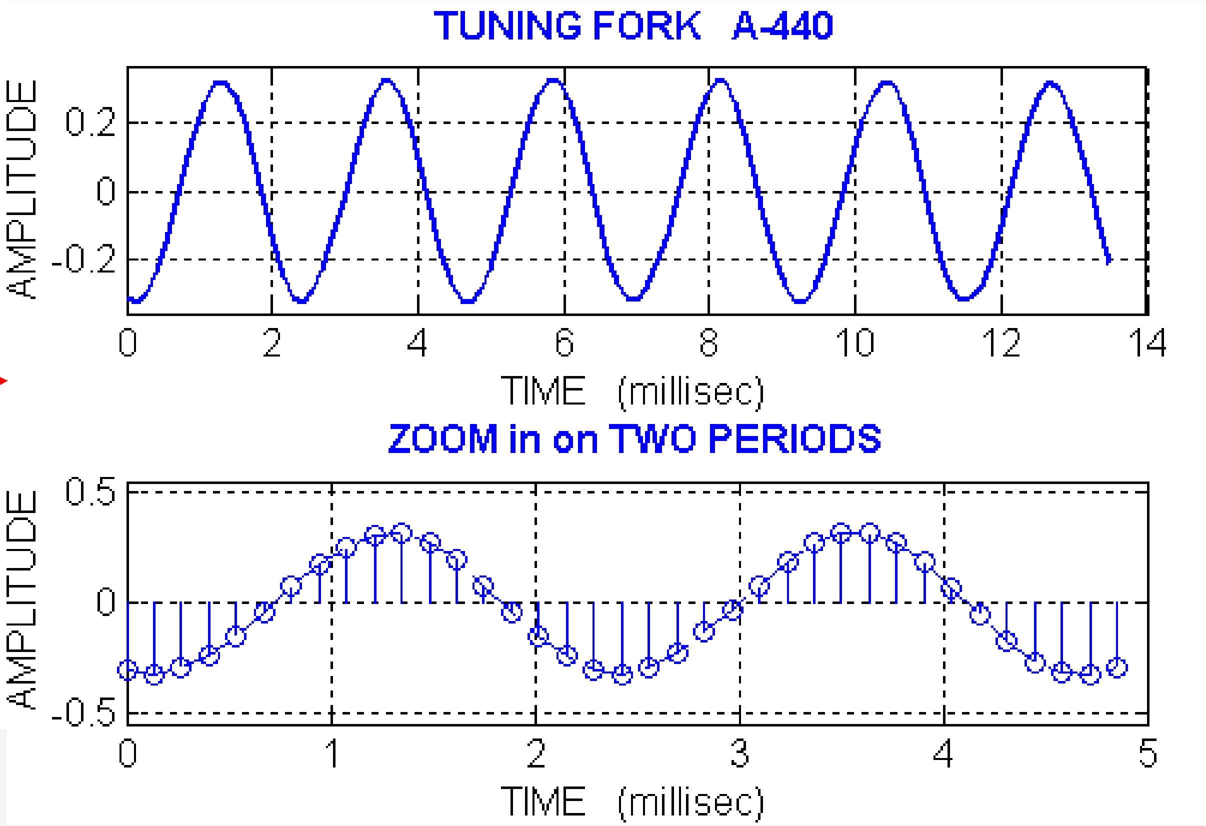
Lecture #2 – Sinusoids and Complex Exponentials

- Sinusoidal Signals
- Frequency, Period, Phase and Amplitude
- Complex Exponential Signals
- Phasor Addition
- MATLAB Applications

Recall: Tunning Fork

Sinusoids are important part of our world.





SINES and COSINES

- Always use the COSINE FORM

$$A\cos(2\pi(440)t+\varphi)$$

- Sine is a special case:

$$\sin(\omega t)=\cos(\omega t-\frac{\pi}{2})$$

Ref. DSP First lecture notes

Sinusoid Signal

$$A\cos(\omega t + \varphi)$$

- **FREQUENCY** ω
  - Radians/sec
  - Hertz (cycles/sec)

$$\omega = (2\pi) f$$

- **PERIOD** (in sec)

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

- **AMPLITUDE** A
  - Magnitude
- **PHASE** φ

Ref. DSP First lecture notes

Some Trigonometric Identities

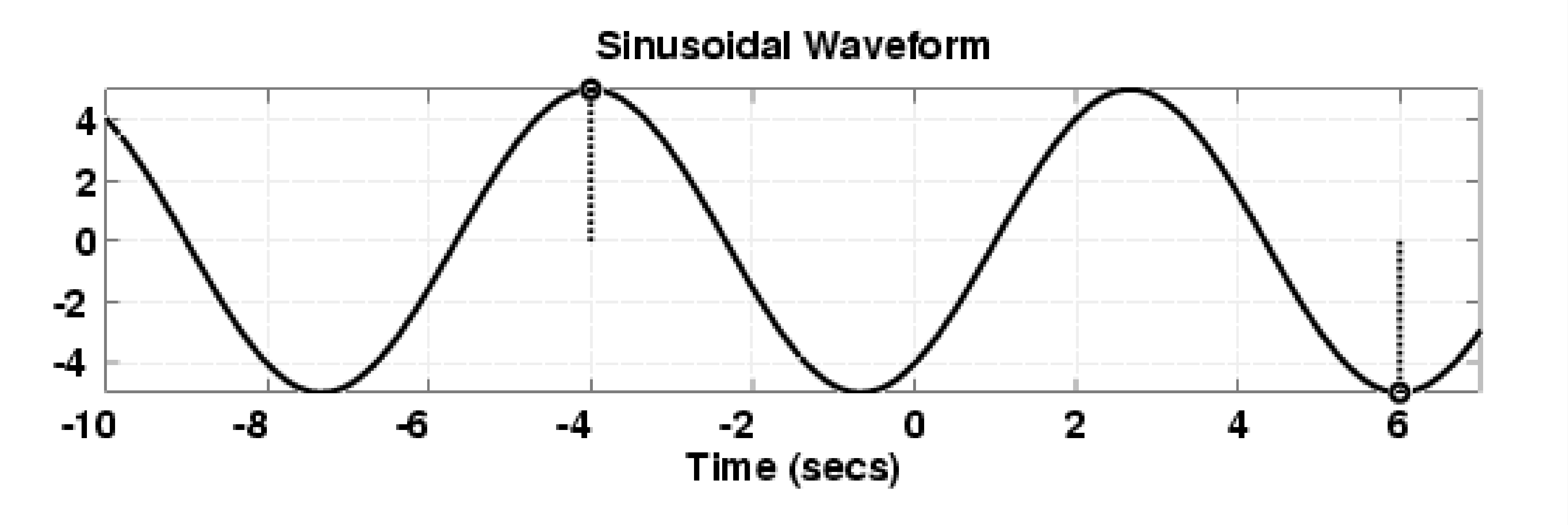
Number	Equation
1	$\sin^2\theta + \cos^2\theta = 1$
2	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
3	$\sin 2\theta = 2\sin\theta\cos\theta$
4	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
5	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

EXAMPLE of SINUSOID

- Given the Formula

$$5\cos(0.3\pi t + 1.2\pi)$$

- Make a plot



Ref. DSP First lecture notes

PLOT COSINE SIGNAL

$$5\cos(0.3\pi t + 1.2\pi)$$

- Formula defines A, ω, and ϕ

$$A = 5$$

$$\omega = 0.3\pi$$

$$\varphi = 1.2\pi$$

PLOTTING COSINE SIGNAL from the FORMULA

$$5\cos(0.3\pi t + 1.2\pi)$$

- Determine **period**:

$$T = 2\pi/\omega = 2\pi/0.3\pi = 20/3$$

- Determine a **peak** location by solving

$$(\omega t + \varphi) = 0 \quad \Rightarrow \quad (0.3\pi t + 1.2\pi) = 0$$

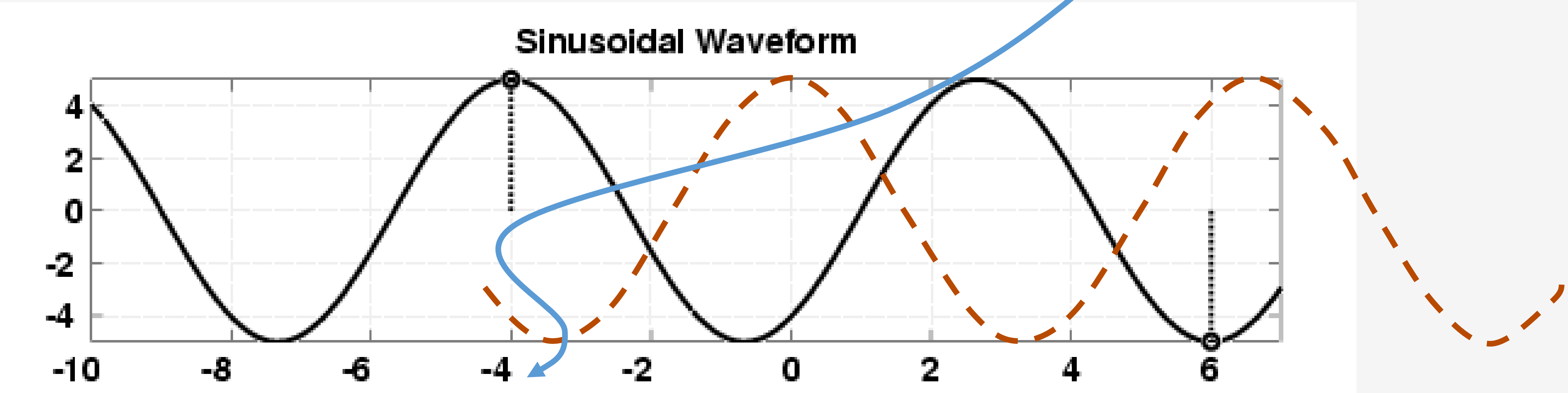
- **Zero** crossing is T/4 before or after
- Positive & Negative peaks spaced by T/2

Time-shifted Sinusoid

$$x(t) = 5\cos(0.3\pi t)$$

One peak at t = 0

$$x(t+4) = 5\cos(0.3\pi(t+4)) = 5\cos(0.3\pi(t-(-4)))$$



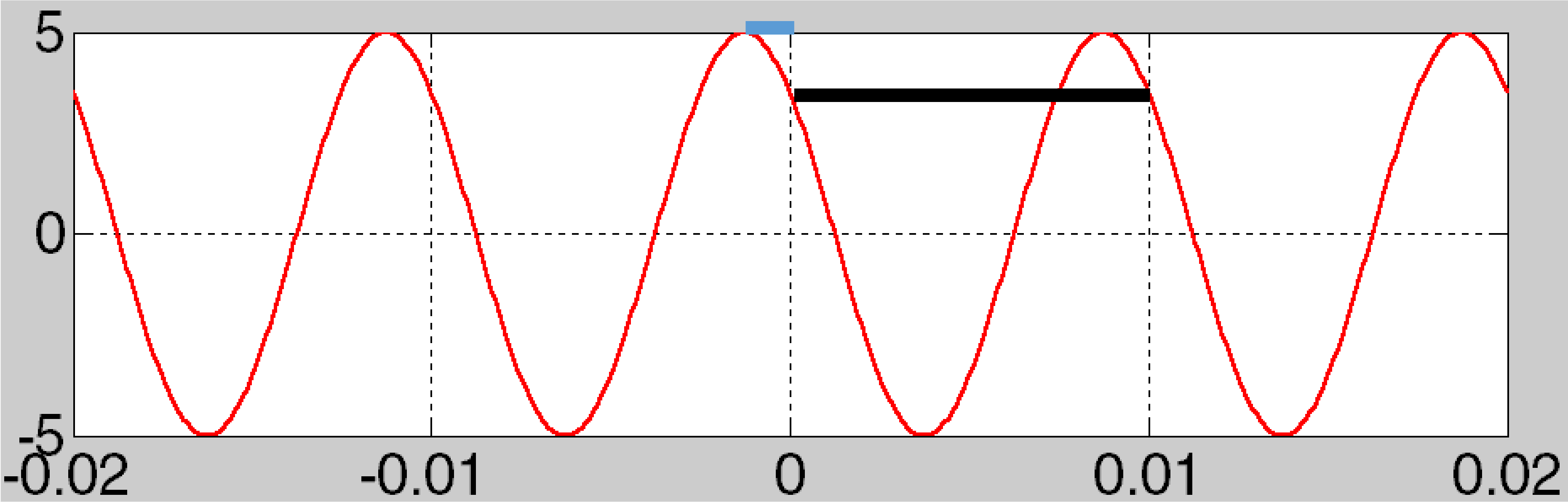
Peak shifts from t=0 to t = -4

How to determine Amplitude, Phase and Period from a plot

- Measure the period, T
  - Between peaks or zero crossings
  - Compute frequency: $\omega = 2\pi/T$
- Measure time of a peak: t_m
  - Compute phase: $\phi = -\omega\, t_m$
- Measure height of positive peak: A

3 steps

(A, ω, ϕ) from a PLOT



$$T = \frac{0.01\text{sec}}{1\,\text{period}} = \frac{1}{100} \longrightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{0.01} = 200\pi$$

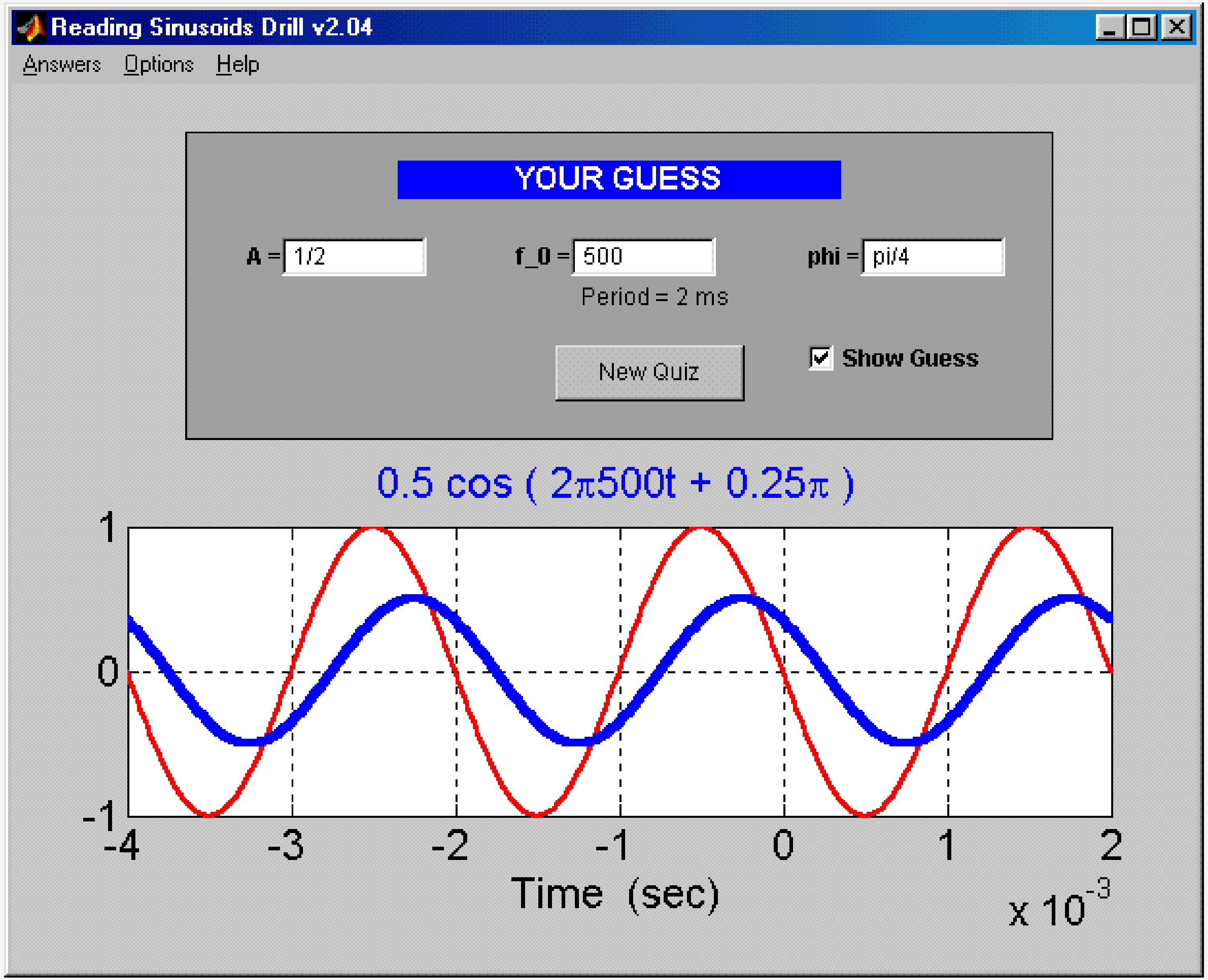
$$t_m = -0.00125\text{sec} \longrightarrow \varphi = -\omega t_m = -(200\pi)(t_m) = 0.25\pi$$

SINE DRILL (MATLAB GUI)

https://dspfirst.gatech.edu/matlab/#sindrill

SinDrill is a program that tests the users ability to determine basic parameters of a sinusoid.

After a plot of a sinusoid is displayed, the user must correctly guess its amplitude, frequency, and phase.



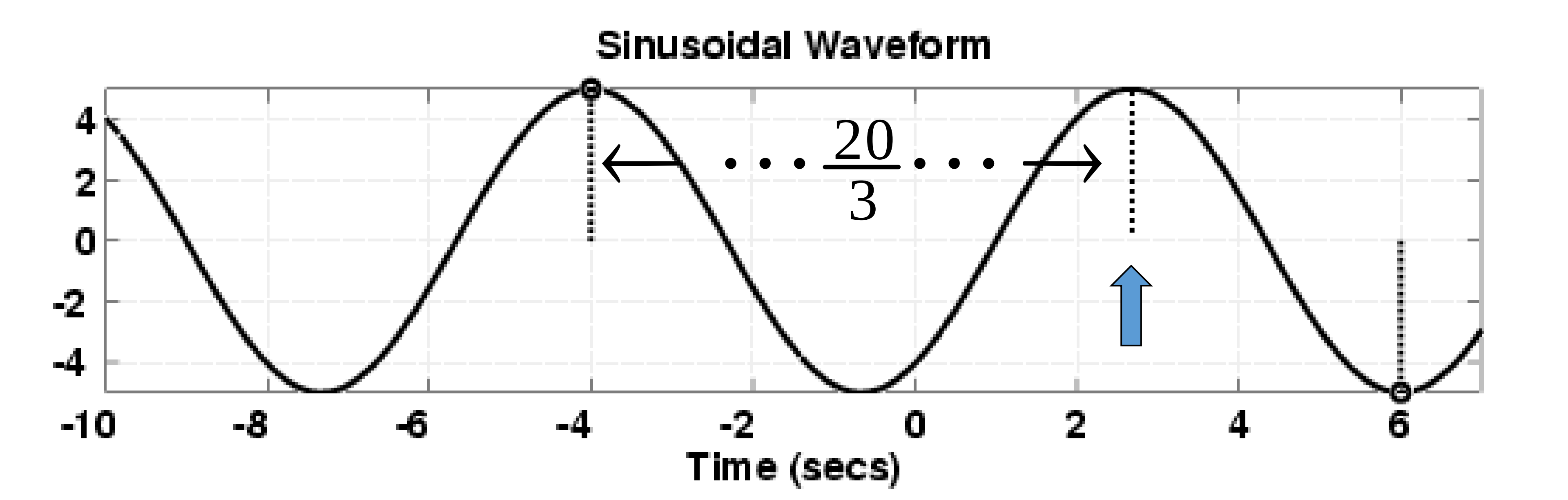
Phase is Ambiguous

The cosine signal is periodic

– Period is 2π

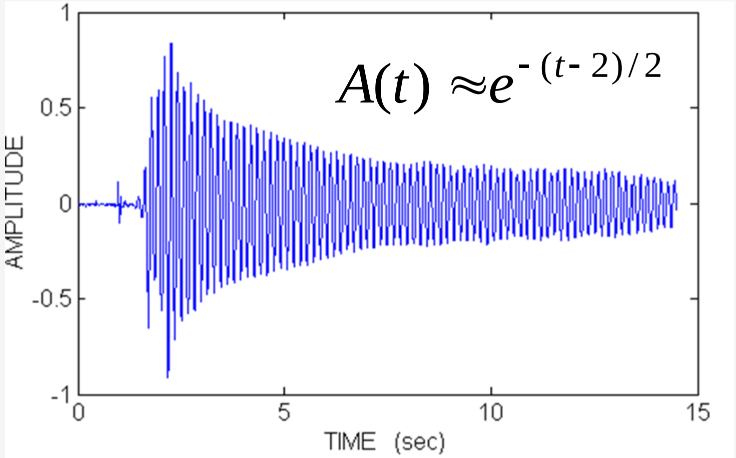
$$\boxed{A\cos(\omega t + \varphi + 2\pi) = A\cos(\omega t + \varphi)}$$

$$5\cos(0.3\pi t + 1.2\pi) = 5\cos(0.3\pi t - 0.8\pi)$$



Attenuaniton: Amplitude Varies with Time (Fade Out?)

$$x(t) = A\cos(\omega t + \varphi)$$ → $$A(t) = Ae^{-t/\alpha}$$




```
fs = 8000;
% define array tt for time
% time runs from -1s to +3.2s
% sampled at an interval of 1/fs
tt = 0: 1/fs : 3.2;
xx = 2.1 * cos(2*pi*440*tt + 0.4*pi);

soundsc (xx,fs)
```

```
fs = 8000;
tt = 0: 1/fs : 3.2;
yy = exp(-tt*1.2);% exponential decay
yy = xx.*yy;

soundsc(yy,fs)
```

$$x(t) = 2.1\cos(880\pi t + 0.4\pi)$$

$$y(t) = 2.1e^{-1.2t}\cos(880\pi t + 0.4\pi)$$

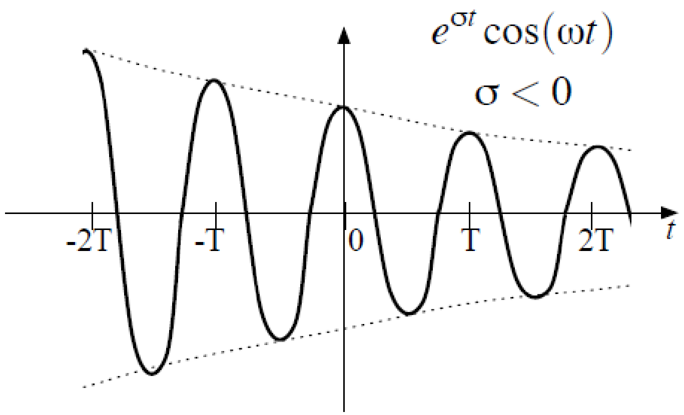
Growing Sinuzoid? (Exponential Sinuzoid)

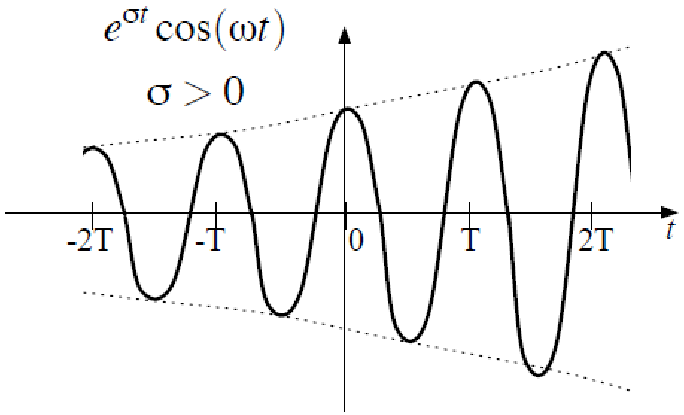
Damped or Growing Sinusoids

- A damped or growing sinusoid is given by

$$x(t) = e^{\sigma t} \cos(\omega t + \theta)$$

- Exponential growth ($\sigma > 0$) or decay ($\sigma < 0$), modulated by a sinusoid.

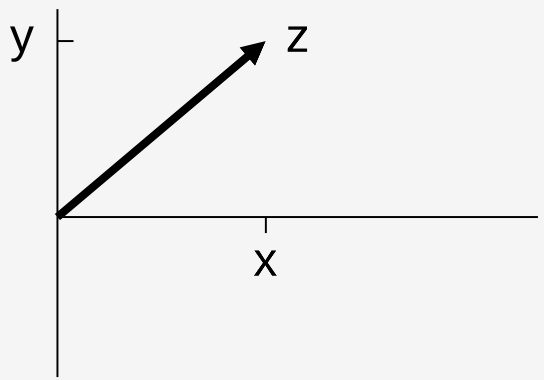




Remember: Complex Numbers

Cartesian Coordinate System

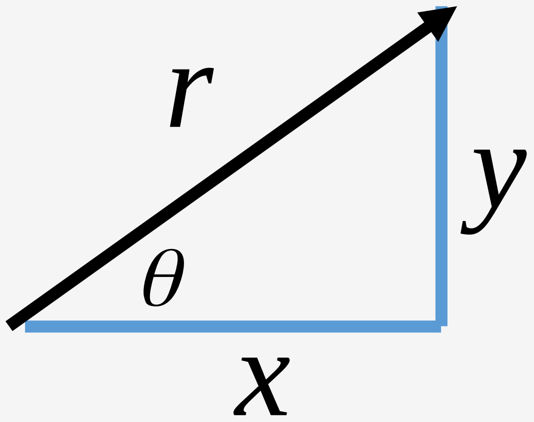
- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and Physics use $z = \boldsymbol{i}$
- Complex number: $z = x + \boldsymbol{j}\, y$



Polar Coordinate System

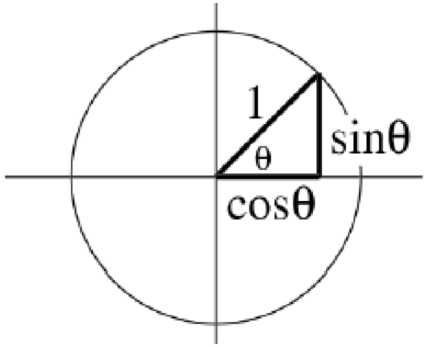
$$r^2 = x^2 + y^2 \qquad x = r\cos\theta$$

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right) \qquad y = r\sin\theta$$



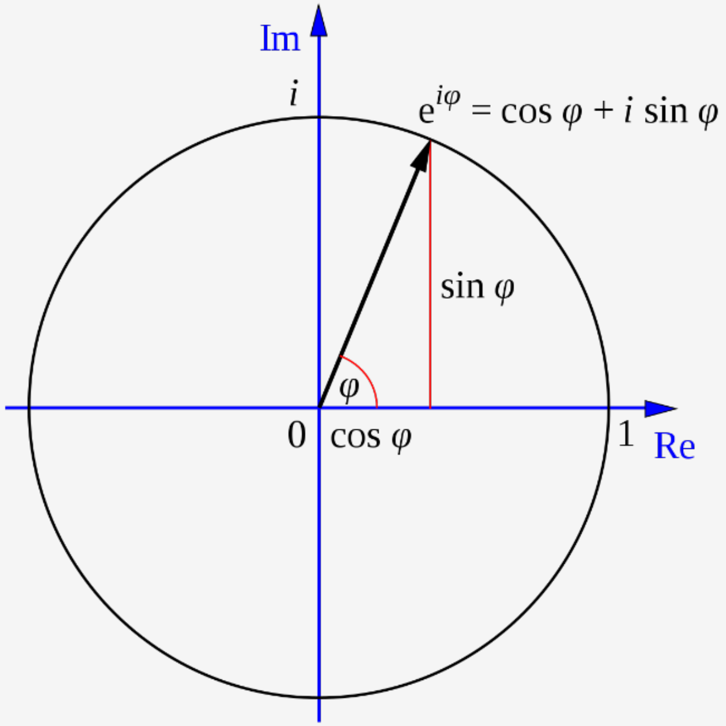
Euler’s Formula (Important!!)

- **Complex Exponential**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



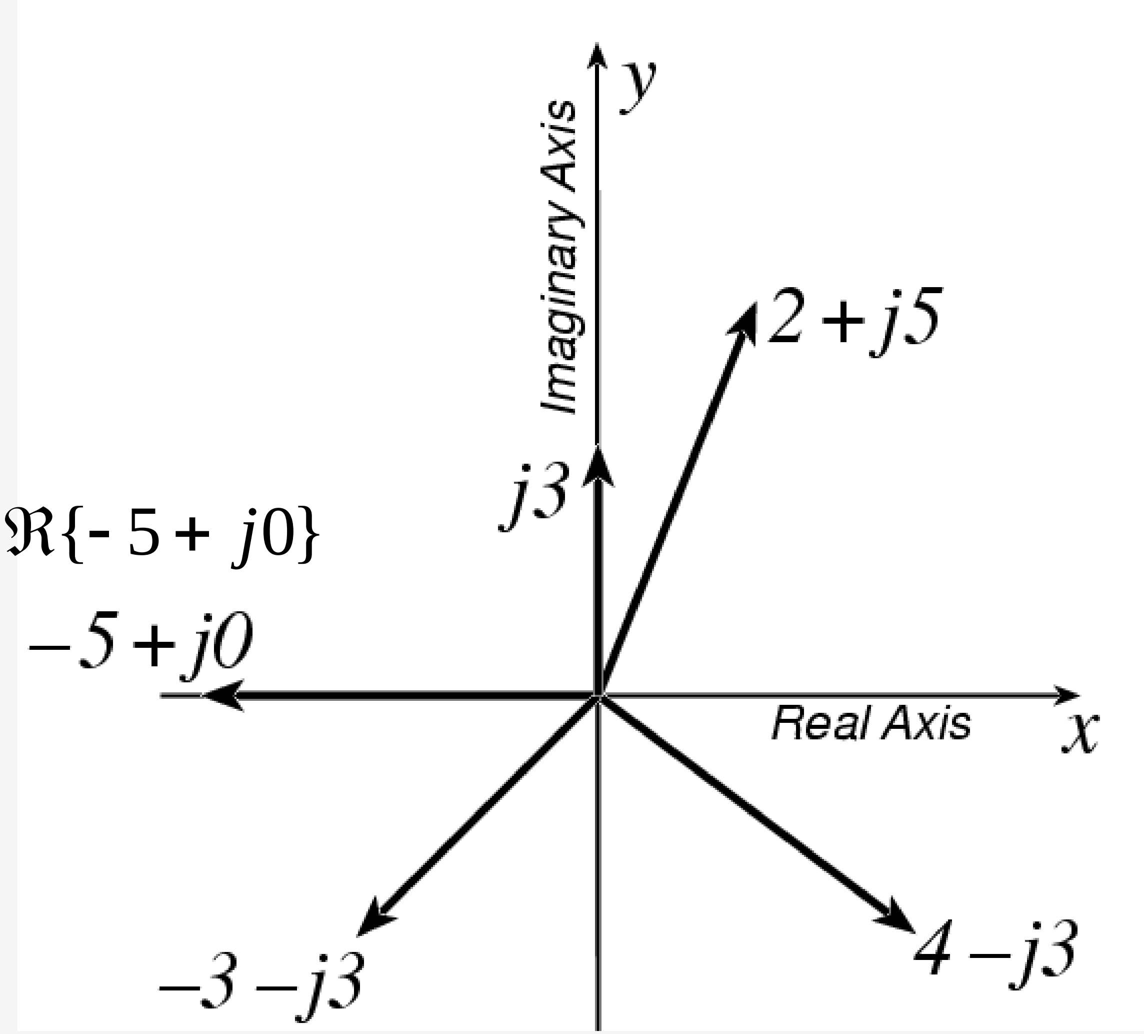
$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$



Remember: Complex Numbers

$$-5 = \Re\{-5 + j0\}$$



Complex addition?

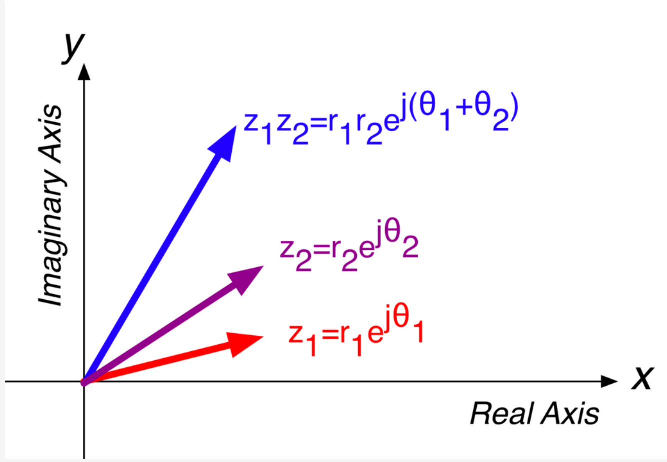
Complex multiplication?

Real part:

$$x = \Re\{z\}$$

Imaginary part:

$$y = \Im\{z\}$$



Zdrill tool

https://dspfirst.gatech.edu/matlab/#zdrill

Euler's Formula (Important!!)

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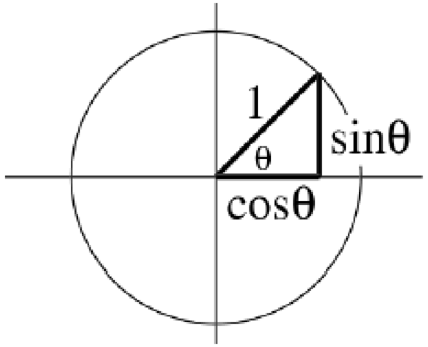

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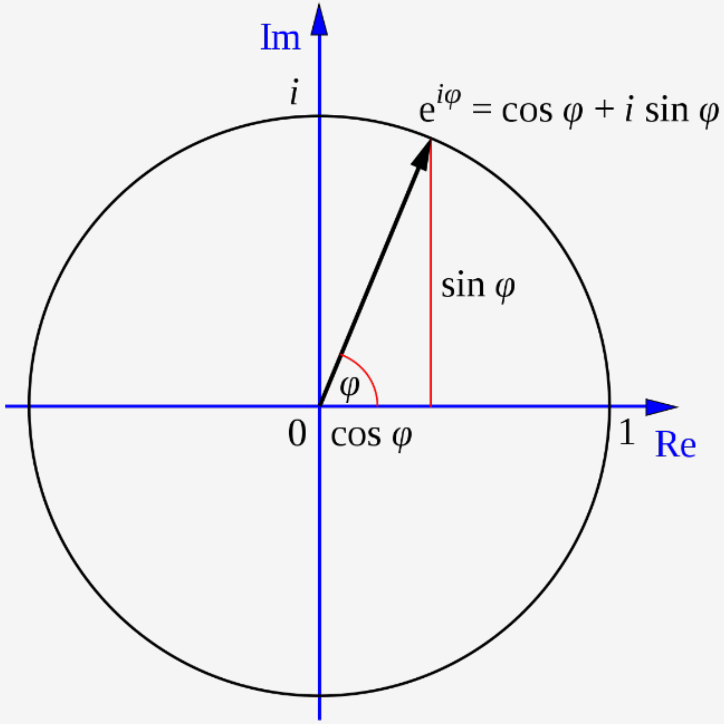
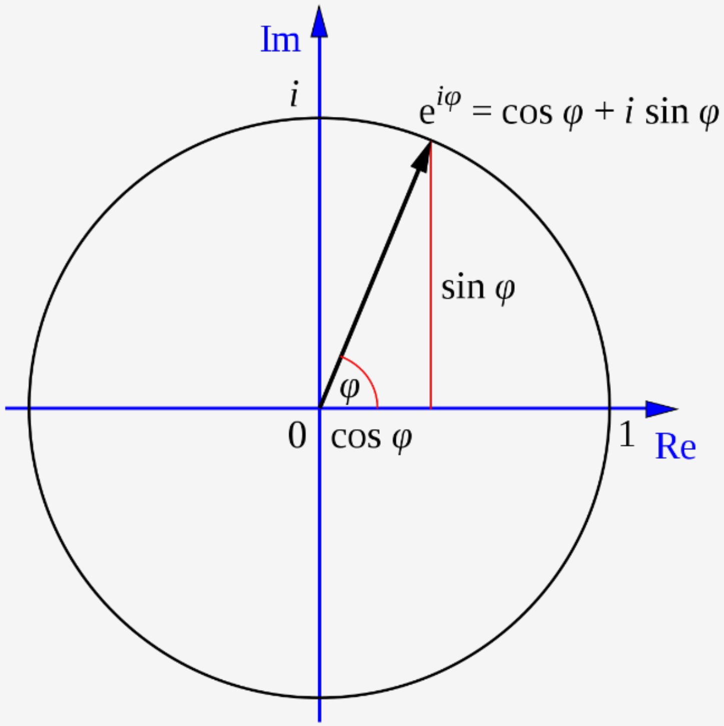


What happens if we write variable instead of Theta?

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

Euler’s Formula (Important!!)



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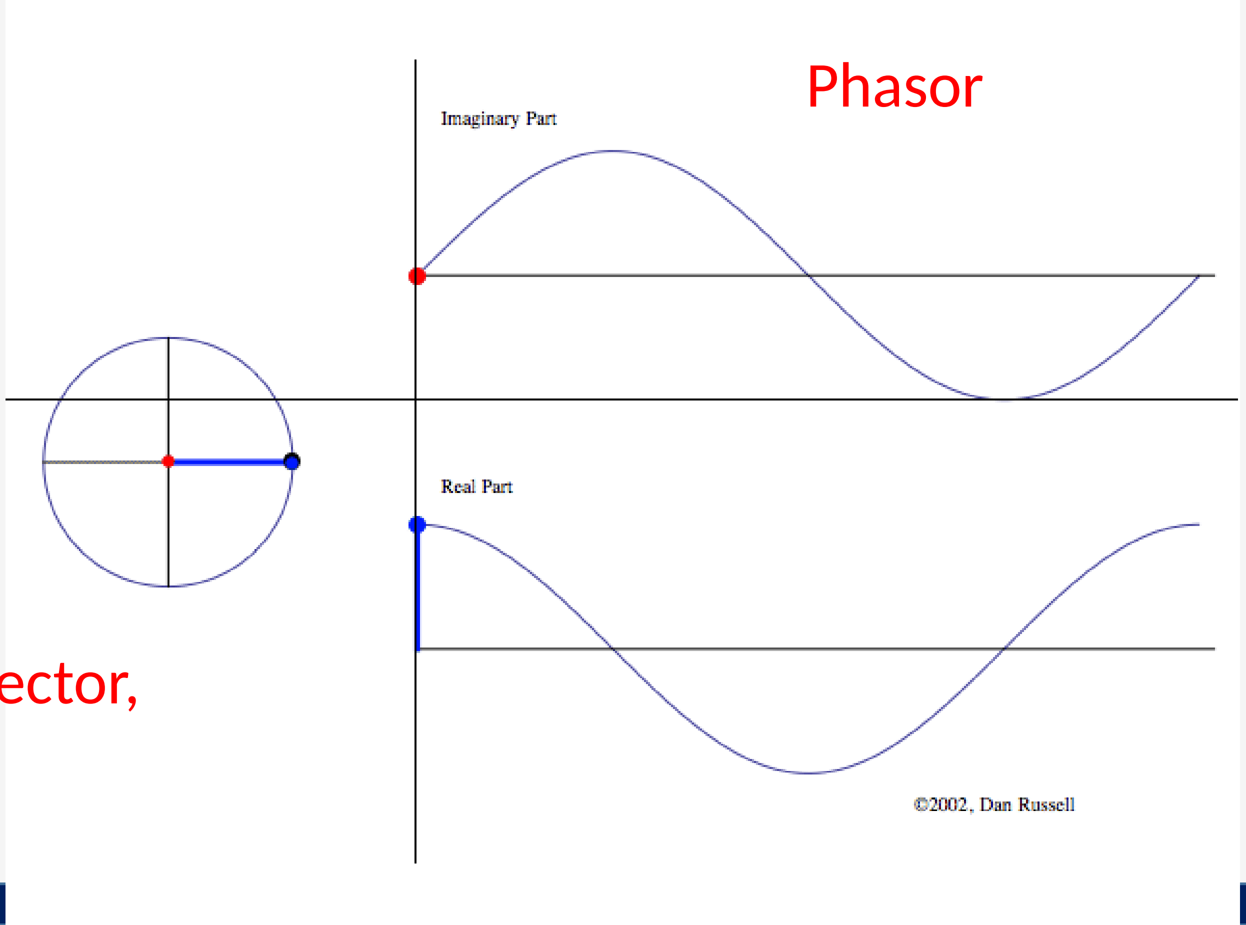


Complex Exponential includes a rotating vector,

= complex summation of sinuzoids

Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

INVERSE Euler's Formula

- Solve Euler's formula for cosine (or sine)

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

Phasor Form of A Cosine

$$A\cos(\omega t+\varphi)=\Re\{(Ae^{j\varphi})e^{j\omega t}\}$$

Complex Amplitude: Constant

Varies with time

- Find the COMPLEX AMPLITUDE for:

$$x(t)=\sqrt{3}\cos(77\pi t+0.5\pi)$$

- Use EULER's FORMULA:

$$x(t)=\Re\{\sqrt{3}e^{j(77\pi t+0.5\pi)}\}$$

$$=\Re\{\sqrt{3}e^{j0.5\pi}e^{j77\pi t}\}$$

$$X=\sqrt{3}e^{j0.5\pi}$$

POP QUIZ

- Determine the 60-Hz sinusoid whose COMPLEX AMPLITUDE is: $X = \sqrt{3} + j3$

- Convert **X** to **<u>POLAR</u>**:

$$x(t) = \Re\{\boxed{(\sqrt{3} + j3)}e^{j(120\pi t)}\}$$

$$= \Re\{\boxed{\sqrt{12}\, e^{j\pi/3}}e^{j120\pi t}\}$$

$$\Rightarrow \quad x(t) = \sqrt{12}\cos(120\pi t + \pi/3)$$

Want to Add Sinusoids with same frequency

Adding sinusoids of common frequency results in sinusoid with **SAME** frequency

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \varphi_k)$$

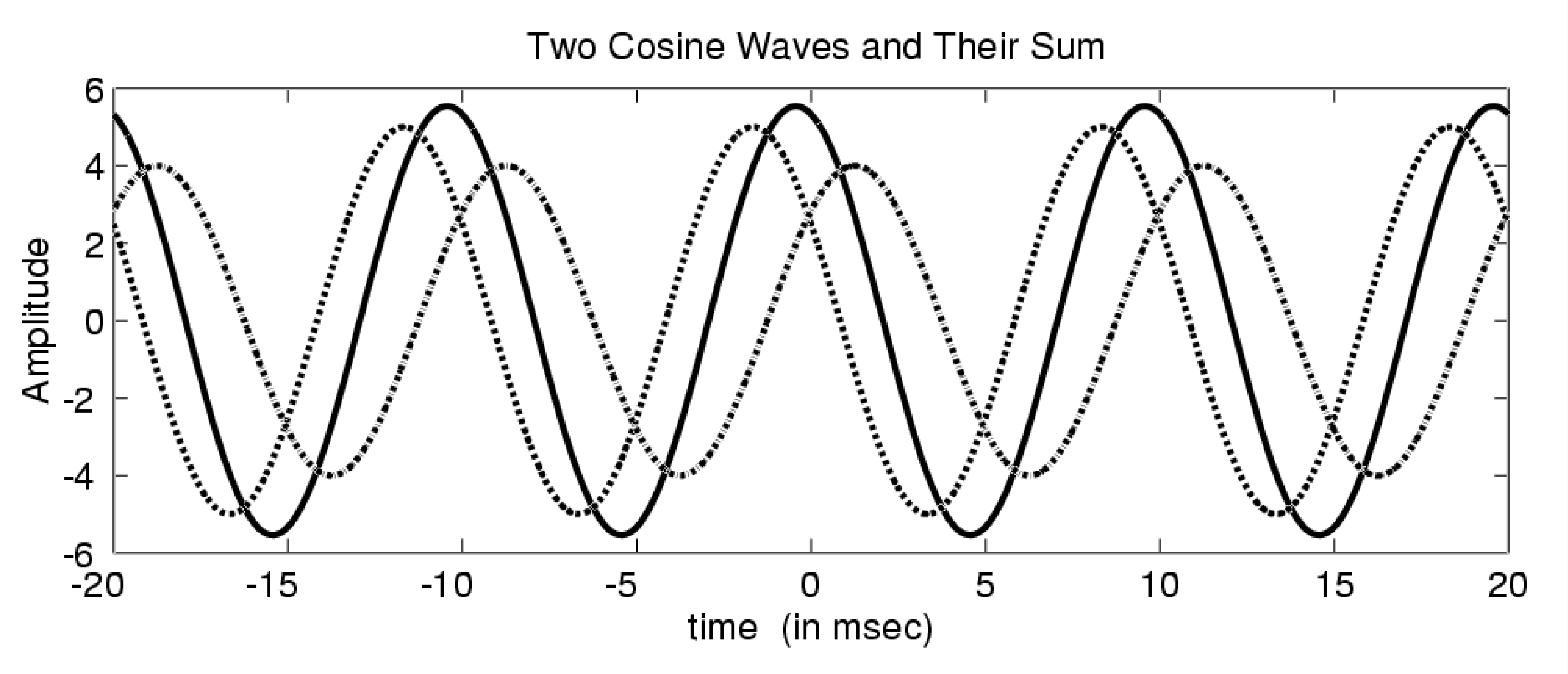
$$= A\cos(\omega_0 t + \varphi)$$

Get the new complex amplitude by complex addition

$$\sum_{k=1}^{N} A_k e^{j\varphi_k} = Ae^{j\varphi}$$

Want to Add Sinusoids with same frequency

Adding sinusoids of common frequency results in sinusoid with **SAME** frequency



Want to Add Sinusoids with same frequency

- ADD THESE 2 SINUSOIDS:

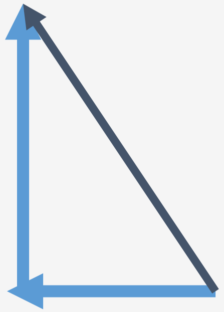
$$x_1(t) = \cos(77\pi t - \pi)$$

$$x_2(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

- COMPLEX (PHASOR) ADDITION:

$$1e^{-j\pi} + \sqrt{3}e^{j0.5\pi}$$

$$\sqrt{3}e^{j\pi/2} = j\sqrt{3}$$



$$e^{-j\pi} = -1$$

$$-1 + j\sqrt{3} = 2e^{j2\pi/3}$$

$$\boxed{x_3(t) = 2\cos(77\pi t + \tfrac{2\pi}{3})}$$

Phasor Addition

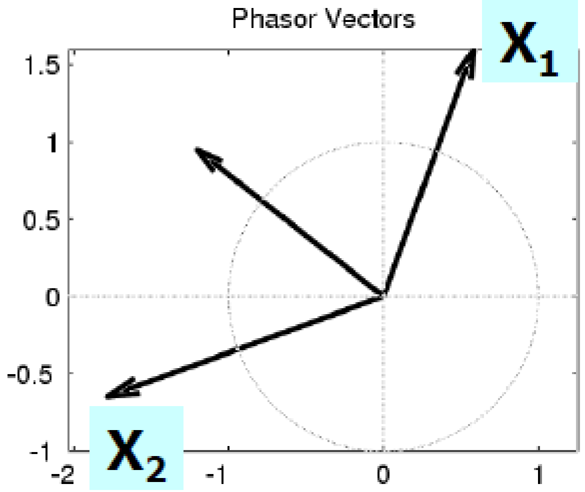
$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$

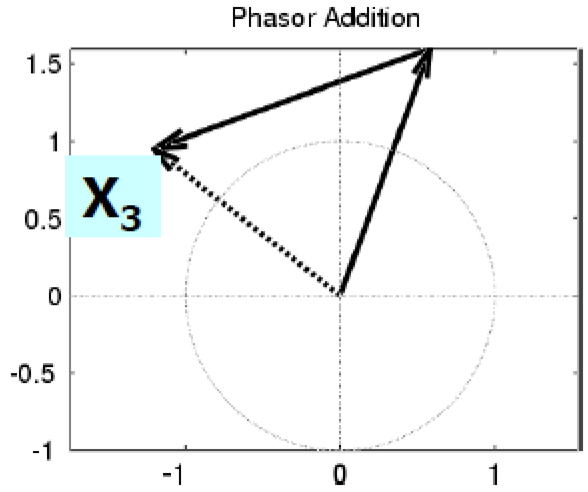
$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$

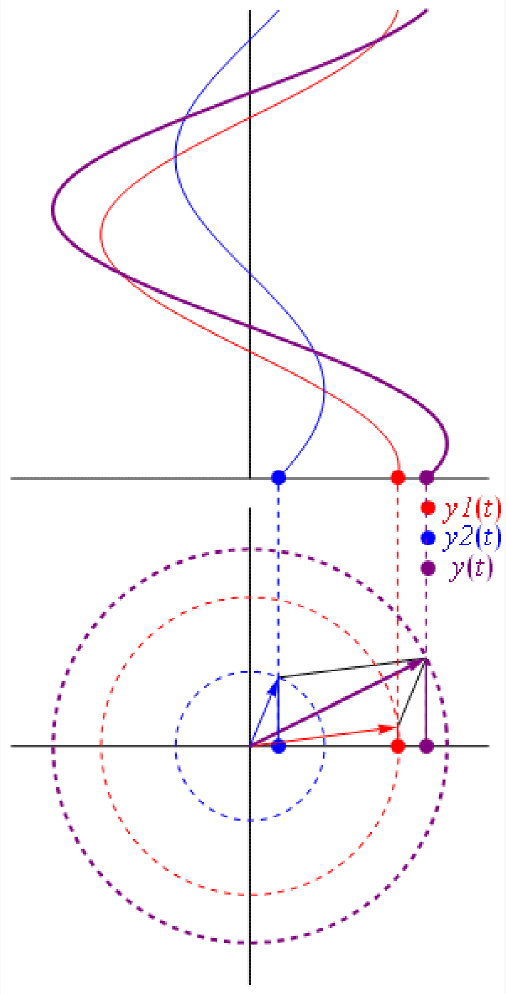
$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$

VECTOR (PHASOR) ADD





Sum of Phasors and Fourier Series

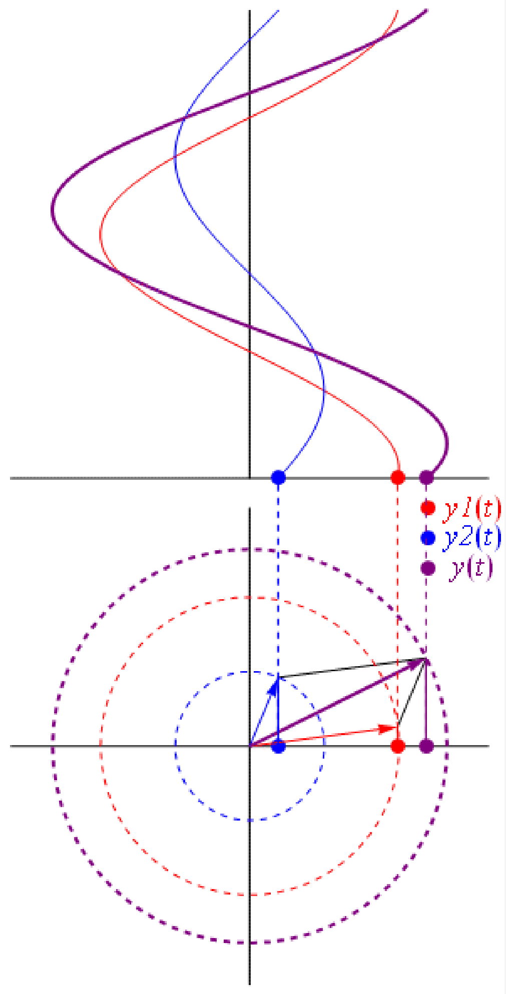


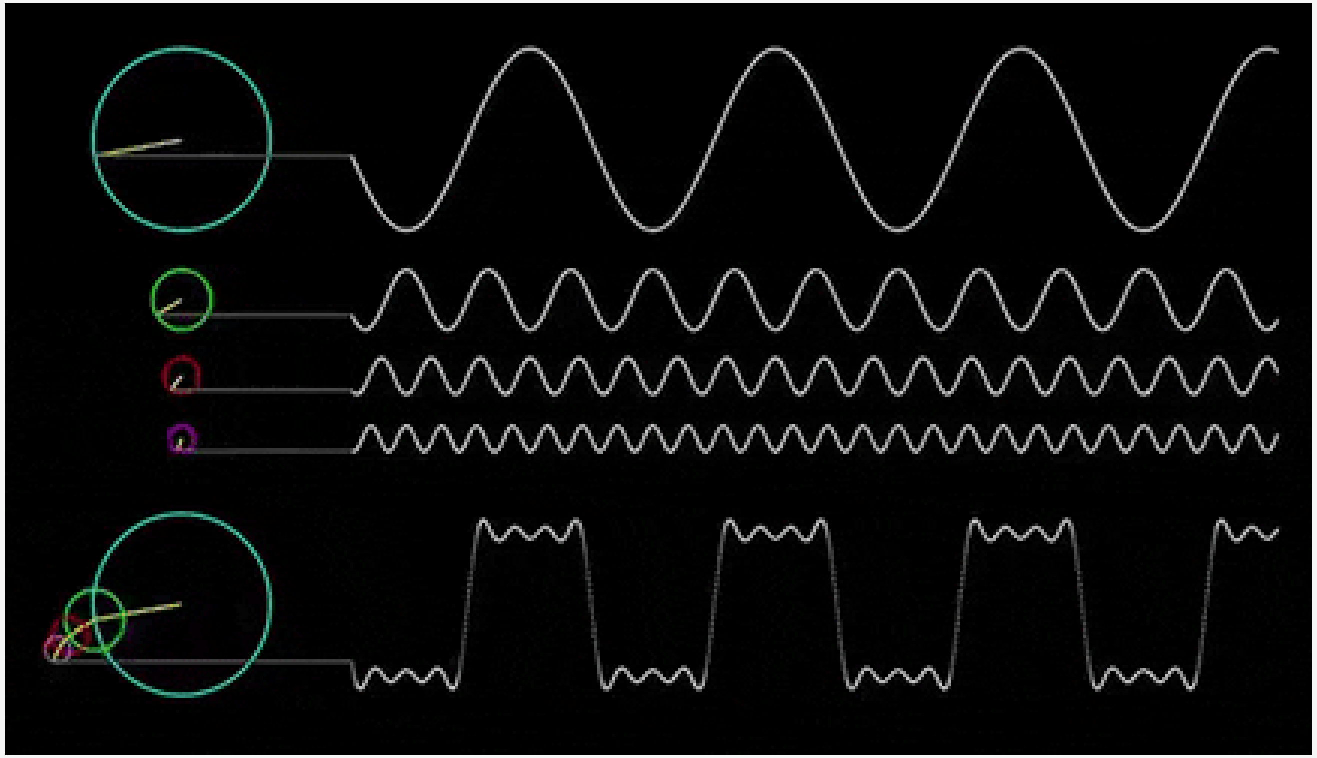
Plotting A Complex Exponential in MATLAB

```
%% Plot signal
tt = 0: 1/10000 : 3.2;
xx = 2.1*exp(2*pi*10*tt*1j);
xx2 = 0.5*exp(2*pi*10*tt*1j);

figure(1); plot (tt,real(xx)); xlim([0 0.01]);
figure(2); plot (tt,imag(xx)); xlim([0 0.01]);

%% Simulate Phasor
close all;
figure(1);

for i = 1:length(tt)

   x = real(xx(i));    y = imag(xx(i));

   plot([0 x],[0 y]);
   xlim([-4 4]);    ylim([-4 4]);    drawnow;

end
```

```
%% Simulate sum of Phasor-2
close all;
figure(1);

for i = 1:length(tt)

   x = real(xx(i));
   y = imag(xx(i));
   x2 = real(xx2(i));
   y2 = imag(xx2(i));

   plot([0 x],[0 y],'r'); hold on;
   plot([x x+x2],[y y+y2],'b');
   plot([0 x+x2],[0 y+y2],'k');
   xlim([-4 4]);    ylim([-4 4]);
   drawnow; hold off;

end
```

Sum of Phasors and Fourier Series

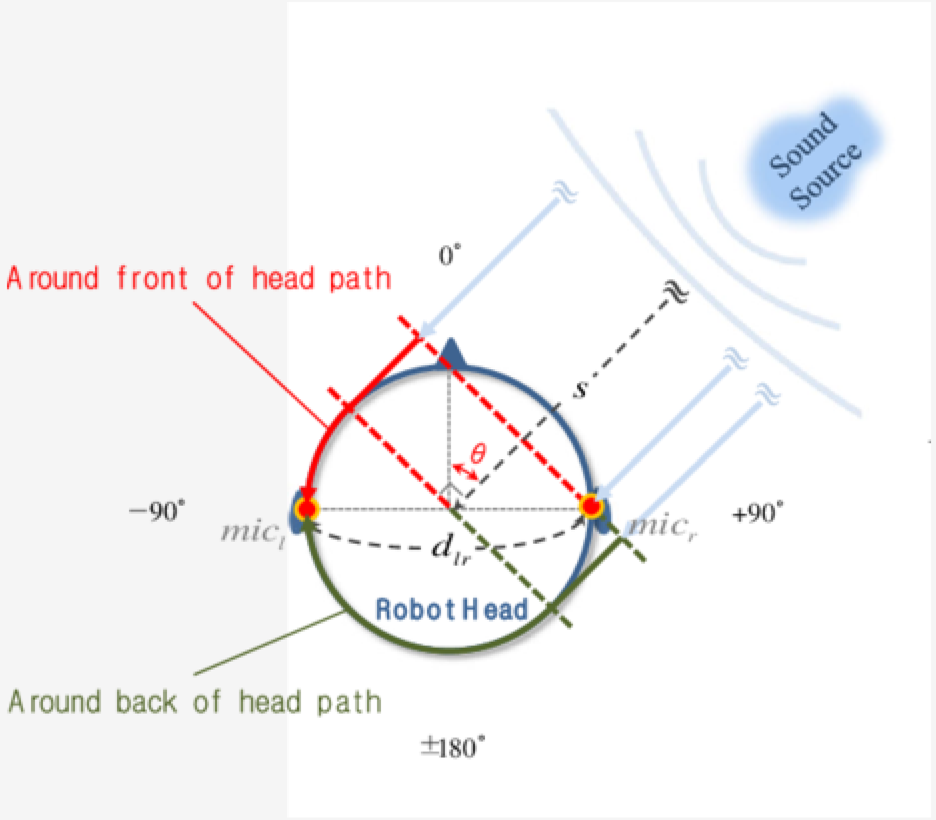


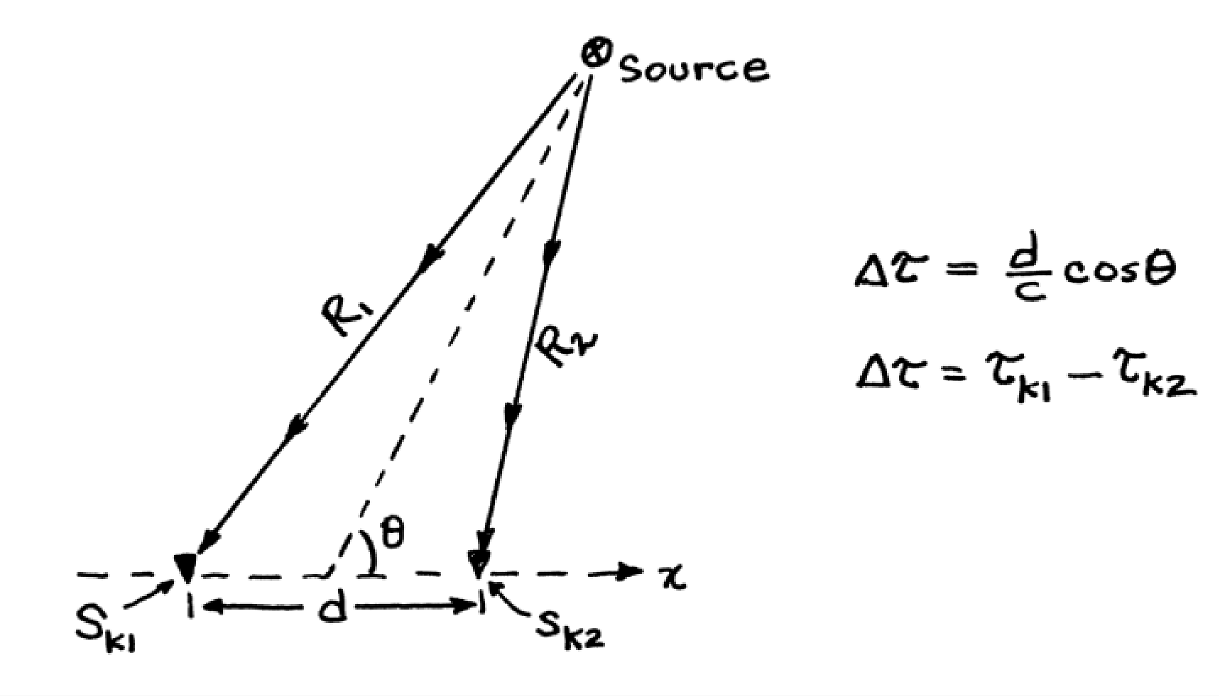


$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Demo Link:https://dspfirst.gatech.edu/chapters/02sines/demos/phasors/index.html

Where Can We Use Phase Info: Binaural Sound Localization





$$\text{Sensor } S_{k_1}: \quad r_{k_1}(t) = s(t - \tau_{k_1})$$
$$\text{Sensor } S_{k_2}: \quad r_{k_2}(t) = s(t - \tau_{k_2})$$

Exercise - 1

Define $x(t)$ as

$$x(t) = 7\cos(100\pi t - 3\pi/4) + 3\cos(100\pi(t + 0.005))$$

(a) Use phasor addition to express $x(t)$ in the form $x(t) = A\cos(\omega_0 t + \phi)$ by finding the numerical values of A and ϕ, as well as ω_0.

$$x(t) = 7\cos(100\pi t - 3\pi/4) + 3\cos(100\pi t + \pi/2)$$

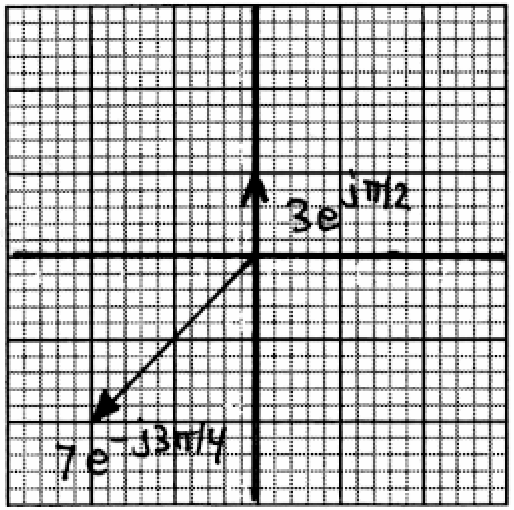
$$= \mathrm{Re}\left\{7e^{-j3\pi/4}e^{j100\pi t} + 3e^{j\pi/2}e^{j100\pi t}\right\}$$

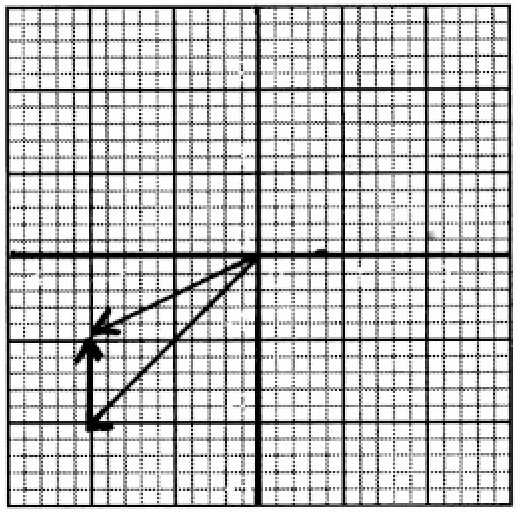
$$= \mathrm{Re}\left\{\underbrace{\left(7e^{-j3\pi/4} + 3e^{j\pi/2}\right)}_{5.3199e^{-j0.8806\pi}}e^{j100\pi t}\right\}$$

$$= \mathrm{Re}\left\{5.3199e^{-j0.8806\pi} \cdot e^{j100\pi t}\right\}$$

$$= 5.3199\cos(100\pi t - 0.8806\pi)$$

(b) Make two complex plane plots to illustrate how complex amplitudes (phasors) were used to solve part (a). On the first plot, show the two complex amplitudes being added; on the second plot, show your solution as a vector and the addition of the two complex amplitudes as vectors (head-to-tail).





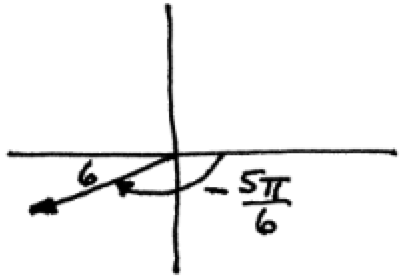
Exercise - 2

Simplify the following complex-valued expressions. In each case reduce the answers to a **simple** numerical form. Let

$$V = -3 + j3\sqrt{3}.$$

(a) Express jV in polar form. In addition plot jV as a vector.

$$jV = -3j - 3\sqrt{3}$$
$$= 6e^{-j\frac{5\pi}{6}}$$



(d) Express $\Re e\{j^3 V e^{j15t}\}$ in the standard "cosine" form.

$$\mathcal{R}e\{j^3 V e^{j15t}\} = \mathcal{R}e\{e^{-j\frac{\pi}{2}} \cdot 6e^{j\frac{2\pi}{3}} e^{j15t}\} = \mathcal{R}e\{6e^{j\frac{\pi}{6}} e^{j15t}\}$$
$$= \boxed{6\cos\left(15t + \frac{\pi}{6}\right)}$$

Exercise - 3

The phase of a sinusoid can be related to time shift: $x(t) = A\cos(2\pi f_\circ t + \phi) = A\cos(2\pi f_\circ (t - t_1))$
In the following parts, assume that the period of the sinusoidal wave is $T = 20$ sec.

(a) "When $t_1 = 5$ sec, the value of the phase is $\phi = 3\pi/2$."
Explain whether this is TRUE or FALSE.

$\varphi = -2\pi(t_1/T)$

$t_1 = 5 \Rightarrow \varphi = -2\pi(5/20) = -\pi/2$

$\boxed{\text{TRUE}}$

BUT YOU CAN ADD 2π, SO $\varphi = -\pi/2 + 2\pi = 3\pi/2$

(b) "When $t_1 = -5$ sec, the value of the phase is $\phi = \pi/4$."
Explain whether this is TRUE or FALSE.

$\varphi = -2\pi(-5/20) = +\pi/2$

$\boxed{\text{FALSE}}$

$\pi/2 - \pi/4 = \pi/4$ IS NOT MULTIPLE of 2π

Sample Q

P-2.10 Define $x(t)$ as

$$x(t) = 2\sin(\omega_0 t + \pi/4) + \cos(\omega_0 t)$$

(a) Express $x(t)$ in the form $x(t) = A\cos(\omega_0 t + \phi)$.

(b) Find a complex-valued signal $z(t)$ such that $x(t) = \Re e\{z(t)\}$.

P-2.7 Simplify the following expressions:

(a) $3e^{j\pi/3} + 4e^{-j\pi/6}$

(b) $\left(\sqrt{3} - j3\right)^{10}$

(c) $\left(\sqrt{3} - j3\right)^{-1}$

(d) $\left(\sqrt{3} - j3\right)^{1/3}$

(e) $\Re\left\{je^{-j\pi/3}\right\}$

Give the answers in *both* Cartesian form $(x + jy)$ and polar form $(re^{j\theta})$.

P-2.11 Define $x(t)$ as

$$x(t) = 5\cos(\omega t) + 5\cos(\omega t + 120^\circ) + 5\cos(\omega t - 120^\circ)$$

Simplify $x(t)$ into the standard sinusoidal form: $x(t) = A\cos(\omega t + \phi)$. Use phasors to do the algebra, but also provide a plot of the vectors representing each of the three phasors.