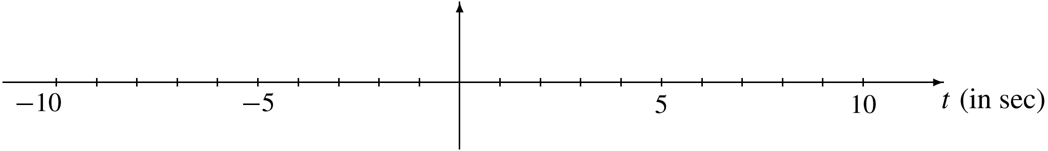
PROBLEM:

Suppose that a periodic signal is defined (over one period) as: $x(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 7 \\ 0 & \text{for } 7 < t < 8 \end{cases}$

(a) Assume that the period of $x(t)$ is 8 s. Draw a plot of $x(t)$ over the range $-10 \leq t \leq 10$ s.



(b) Determine the DC value of $x(t)$.

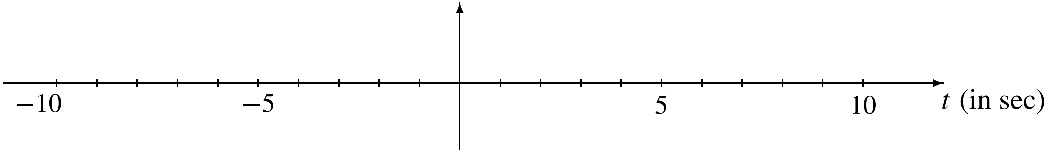
(c) Write the Fourier integral expression for the coefficient a_3 in terms of the specific signal $x(t)$ defined above. *Set up all the specifics of the integral (e.g., limits of integration), but do not evaluate the integral. All parameters in the integral should have numeric values.*

(d) Evaluate the following integral: $\displaystyle\int_0^9 e^{-j2\pi(15)t/10}\,dt$ Simplify your answer and express it in **polar form.**

PROBLEM:

Suppose that a periodic signal is defined (over one period) as: $x(t) = \begin{cases} t & \text{for } 0 \le t \le 1 \\ 0 & \text{for } 1 < t < 8 \end{cases}$

(a) Assume that the period of $x(t)$ is 8 s. Draw a plot of $x(t)$ over the range $-10 \le t \le 10$ s.



(b) Determine the DC value of $x(t)$.

(c) Write the Fourier integral expression for the coefficient a_4 in terms of the specific signal $x(t)$ defined above. *Set up all the specifics of the integral (e.g., limits of integration), but do not evaluate the integral. All parameters in the integral should have numeric values.*

(d) Evaluate the following integral: $\displaystyle\int_0^5 e^{-j2\pi(3)t/10} dt$ Simplify your answer and express it in **polar form.**

PROBLEM:

Consider the signal

$$x(t) = 5 + 5\cos(3000\pi t - \pi/6) + 4\cos(8000\pi t + \pi/2).$$

This signal has three sinusoidal components (including the "DC" component, whose frequency is equal to zero).

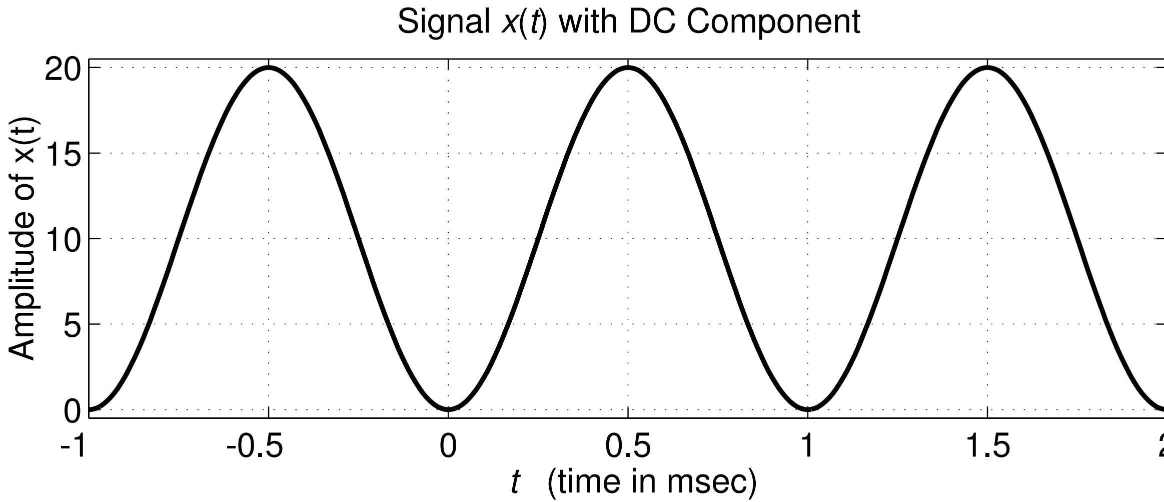
(a) Express the signal $x(t)$ as a sum of five complex exponential components using the *inverse Euler* relationship

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

(b) Which positive and negative frequencies (in Hz) are present in this signal?

(c) For each frequency identified in part (b), identify the complex amplitude of the corresponding complex exponential component. Give your answer as a table containing frequency in one column and complex amplitude in a second column.

PROBLEM:



The above signal $x(t)$ consists of a DC component plus a cosine signal. The terminology *DC component* means a component that is constant versus time.

(a) What is the frequency of the DC component? What is the frequency of the cosine component?

(b) Write an equation for the signal $x(t)$. You should be able to determine numerical values for all the amplitudes, frequencies, and phases in your equation by inspection of the above graph.

(c) Expand the equation obtained in the previous part into a sum of positive and negative frequency complex exponential signals.

(d) Plot the two-sided spectrum of the signal $x(t)$. Show the complex amplitudes for each positive and negative frequency contained in $x(t)$.

PROBLEM:

Try working this problem after you have worked Problem 3.1. It should be easy.

Consider the signal

$$x(t) = 20[\sin(1000\pi t)]^2.$$

(a) Using the inverse Euler relation for the sine function, express $x(t)$ as a sum of complex exponential signals with positive and negative frequencies.

(b) Use your result in part (a) to express $x(t)$ in the form $x(t) = A_0 + A_1 \cos(\omega_0 t)$.

(c) Determine the period T_0 of $x(t)$ and sketch its waveform over the interval $-T_0 \leq t \leq 2T_0$. Carefully label the graph.

(d) Plot the spectrum of $x(t)$.

PROBLEM:

A signal composed of sinusoids is given by the equation

$$x(t) = 4\cos(50\pi t - \pi/4) - 2\cos(150\pi t).$$

(a) Sketch the spectrum of this signal indicating the complex amplitude of each frequency component. You do not have to make separate plots for real/imaginary parts or magnitude/phase. Just indicate the complex amplitude value at the appropriate frequency.

(b) Is $x(t)$ periodic? If so, what is the period? Which harmonics are present?

(c) Now consider a new signal $y(t) = x(t) + 6\cos(60\pi t + \pi/3)$. How is the spectrum changed? Is $y(t)$ periodic? If so, what is the period of $y(t)$?

(d) Finally, consider another new signal $w(t) = x(t) + \cos(50t)$. How is the spectrum changed? Is $w(t)$ periodic? If so, what is the period of $w(t)$? If not, why not?

PROBLEM:

We have seen that musical tones can be modeled mathematically by sinusoidal signals. If you read music or play the piano you know well that the piano keyboard is divided into octaves, with the tones in each octave being twice the frequency of the corresponding tones in the next lower octave. To calibrate the frequency scale, the reference tone is the A above middle-C, which is usually called A440 since its frequency is 440 Hz. Each octave contains 12 tones, and the ratio between the frequencies of successive tones is constant. Thus, the ratio must be $2^{1/12}$. Since middle C is 9 tones below A440, its frequency is approximately $(440)2^{-9/12} \approx 262$ Hz. The names of the tones (notes) of the octave starting with middle-C and ending with high-C are:

note name	C	$C^\sharp$	D	$E^\flat$	E	F	$F^\sharp$	G	$G^\sharp$	$A(440)$	$B^\flat$	B	C
note number	40	41	42	43	44	45	46	47	48	49	50	51	52
frequency										440			

(a) Make a table of the frequencies of the tones of the octave beginning with middle-C assuming that A above middle C is tuned to 440 Hz.

(b) The above notes on a piano are numbered 40 through 52. If n denotes the note number, and f denotes the frequency of the corresponding tone, give a formula for the frequency of the tone as a function of the note number.

(c) A *chord* is a combination of musical notes sounded simultaneously. A *triad* is a three note chord. The D Major chord is composed of the tones of $D \quad F^\sharp \quad A$ sounded simultaneously. From the set of corresponding frequencies determined in part (a), make a sketch of the essential features of the spectrum of the D Major chord assuming that each note is realized by a pure sinusoidal tone. (You do not have to specify the complex phasors precisely.)

PROBLEM:

A linear-FM "chirp" signal is one that sweeps in frequency from $\omega_1 = 2\pi f_1$ to $\omega_2 = 2\pi f_2$ as time goes from $t = 0$ to $t = T_2$. We can define the *instantaneous frequency* of the chirp as the derivative of the phase of the sinusoid:

$$x(t) = A\cos(\alpha t^2 + \beta t + \phi) \tag{1}$$

where the cosine function operates on a time-varying argument

$$\psi(t) = \alpha t^2 + \beta t + \phi$$

The derivative of the argument $\psi(t)$ is the *instantaneous frequency* which is also the audible frequency heard from the chirp *if the chirping frequency does not change too rapidly.*

$$\omega_i(t) = \frac{d}{dt}\psi(t) \qquad \text{radians/sec} \tag{2}$$

There are examples on the CD-ROM in the Chapter 3 demos.

(a) For the linear-FM "chirp" in (1), determine formulas for the beginning instantaneous frequency (ω_1) and the ending instantaneous frequency (ω_2) in terms of α, β and T_2. For this problem, assume that the starting time of the "chirp" is $t = 0$.

(b) For the "chirp" signal

$$x(t) = \Re e\left\{e^{j2\pi(30t^2 - 30t)}\right\}$$

derive a formula for the *instantaneous* frequency versus time. Should your answer for the frequency be a positive number?

(c) For the signal in part (b), make a plot of the *instantaneous* frequency (in Hz) versus time over the range $0 \le t \le 1$ sec.

PROBLEM:

Let $x(t)$ be the signal

$$x(t) = [10 + 5\cos(2000\pi t + \pi/5)]\cos(10000\pi t).$$

(a) Use Euler's relation to expand $x(t)$ as a sum of complex exponential signals and show that it can be expressed in the Fourier series form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

(b) Determine the fundamental frequency ω_0 of this signal.

(c) What is the "DC value" of this signal?

(d) Determine all of the non-zero coefficients a_k of this signal and plot the spectrum of this signal. **Note carefully that you should be able to do this without evaluating any integrals.**

PROBLEM:

A periodic signal $x(t)$ is described over one period $0 \le t \le T_0$ by the equation

$$x(t) = \begin{cases} \dfrac{2t}{T_0} & 0 \le t < T_0/2 \\ 0 & T_0/2 \le t \le T_0. \end{cases}$$

We have seen that such a periodic signal can be represented by the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{where} \quad a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t}\, dt$$

(a) Sketch the periodic function $x(t)$ for $-T_0 < t < 2T_0$.

(b) Determine a_0, the D.C. coefficient for the Fourier series.

(c) Set up the Fourier $\boxed{\textit{analysis}}$ integral for determining a_k for $k \neq 0$. (Insert proper limits and integrand.) What integration technique from calculus could you apply to aid in evaluating this integral?

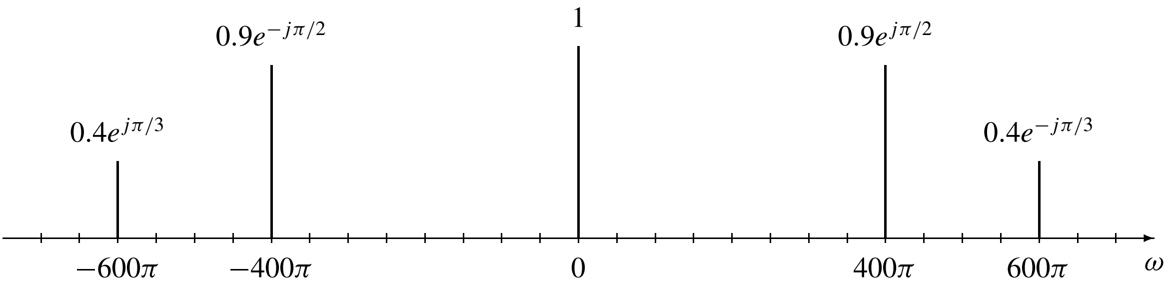
You do not have to evaluate the integral in part (c). If you are curious, the answer is:

$$a_k = \begin{cases} \dfrac{(1 + j\pi k)e^{-j\pi k} - 1}{2\pi^2 k^2} = \dfrac{(1 + j\pi k)(-1)^k - 1}{2\pi^2 k^2} & k \neq 0 \\ \text{your value of } a_0 \text{ found in (b)} & k = 0 \end{cases}$$

Note: A similar problem is worked out in great detail in Section 33.5.3 of the Fourier Series Notes on WebCT.

PROBLEM:

The spectrum of a signal $x(t)$ is shown in the following figure:



Note carefully that the frequency axis is radian frequency (ω) *not* cyclic frequency (f).

(a) Write an equation for $x(t)$ in terms of cosine functions.

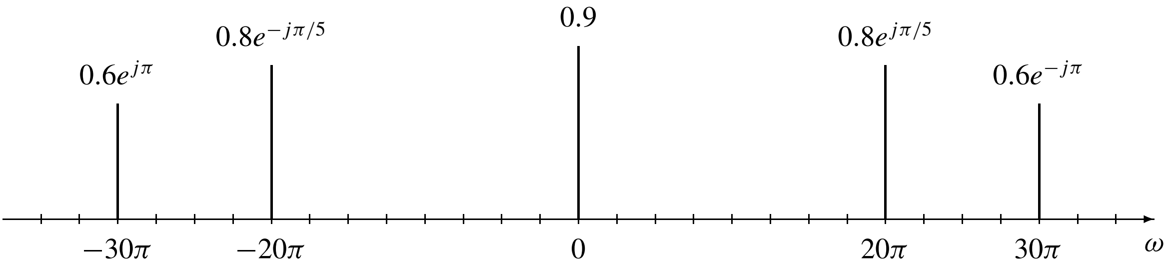
(b) Is $x(t)$ periodic? **You must explain this answer. Why or why not?**
If it is periodic, what is the fundamental frequency ω_0 and corresponding period T_0 of $x(t)$?

(c) A new signal is defined as $y(t) = \cos(\beta t + \pi) + x(t)$. Choose the radian frequency β so that the fundamental frequency of $y(t)$ is *half* the fundamental frequency of $x(t)$. *Note: There may be more than one possible solution.*

(d) Using the frequency β found in (c), modify the spectrum plot above so that it becomes the spectrum of $y(t)$. *Label the complex amplitude as well as the frequency.*

PROBLEM:

The spectrum of a signal $x(t)$ is shown in the following figure:



Note carefully that the frequency axis is radian frequency (ω) *not* cyclic frequency (f).

(a) Write an equation for $x(t)$ in terms of cosine functions.

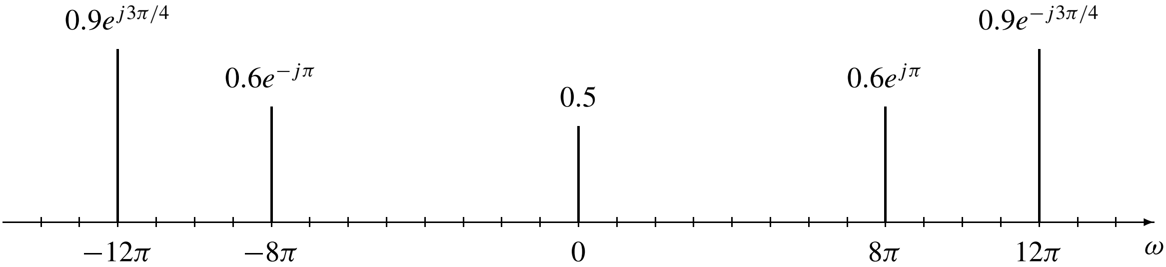
(b) Is $x(t)$ periodic? **You must explain this answer. Why or why not?**
If it is periodic, what is the fundamental frequency ω_0 and corresponding period T_0 of $x(t)$?

(c) A new signal is defined as $y(t) = \cos(\beta t - \pi/4) + x(t)$. Choose the radian frequency β so that the fundamental frequency of $y(t)$ is *half* the fundamental frequency of $x(t)$. *Note: There may be more than one possible solution.*

(d) Using the frequency β found in (c), modify the spectrum plot above so that it becomes the spectrum of $y(t)$. *Label the complex amplitude as well as the frequency.*

PROBLEM:

The spectrum of a signal $x(t)$ is shown in the following figure:



Note carefully that the frequency axis is radian frequency (ω) *not* cyclic frequency (f).

(a) Write an equation for $x(t)$ in terms of cosine functions.

(b) Is $x(t)$ periodic? **You must explain this answer. Why or why not?**
If it is periodic, what is the fundamental frequency ω_0 and corresponding period T_0 of $x(t)$?

(c) A new signal is defined as $y(t) = \cos(\beta t + \pi/4) + x(t)$. Choose the radian frequency β so that the fundamental frequency of $y(t)$ is *half* the fundamental frequency of $x(t)$. *Note: There may be more than one possible solution.*

(d) Using the frequency β found in (c), modify the spectrum plot above so that it becomes the spectrum of $y(t)$. *Label the complex amplitude as well as the frequency.*

PROBLEM:

A signal $x(t)$ is periodic with period $T_0 = 8$. Therefore it can be represented as a Fourier series of the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/8)kt}.$$

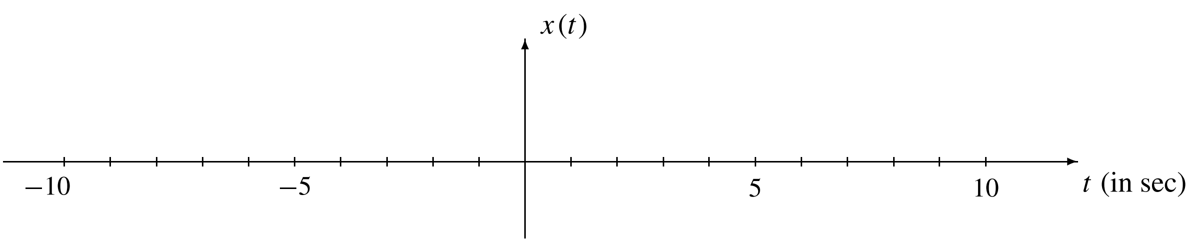
It is known that the Fourier series coefficients for this representation of a particular signal $x(t)$ are given by the integral

$$a_k = \frac{1}{8}\int_{-4}^{0} (4+t)e^{-j(2\pi/8)kt}\,dt. \qquad (1)$$

NOTE: Parts (c) and (d) can be worked independently of parts (a) and (b).

(a) In the expression for a_k in Equation (1) above, the integral and its limits define the signal $x(t)$. Determine an equation for $x(t)$ that is valid over one period.

(b) Using your result from part (a), draw a plot of $x(t)$ over the range $-10 \le t \le 10$ seconds. Label it carefully.



(c) Which value of k in Equation (1) gives the DC (or average) value of $x(t)$? $\boxed{k = \quad}$

(d) Determine the DC value of $x(t)$.

PROBLEM:

A signal $x(t)$ is periodic with period $T_0 = 10$. Therefore it can be represented as a Fourier series of the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/10)kt}.$$

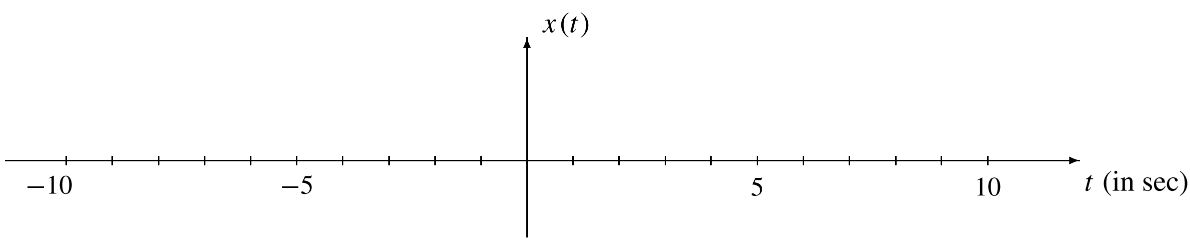
It is known that the Fourier series coefficients for this representation of a particular signal $x(t)$ are given by the integral

$$a_k = \frac{1}{10} \int_0^5 (t) e^{-j(2\pi/10)kt} dt. \qquad (1)$$

NOTE: Parts (c) and (d) can be worked independently of parts (a) and (b).

(a) In the expression for a_k in Equation (1) above, the integral and its limits define the signal $x(t)$. Determine an equation for $x(t)$ that is valid over one period.

(b) Using your result from part (a), draw a plot of $x(t)$ over the range $-10 \leq t \leq 10$ seconds. Label it carefully.



(c) Which value of k in Equation (1) gives the DC (or average) value of $x(t)$? $\boxed{k = \quad}$

(d) Determine the DC value of $x(t)$.

PROBLEM:

A signal $x(t)$ is periodic with period $T_0 = 8$. Therefore it can be represented as a Fourier series of the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/8)kt}.$$

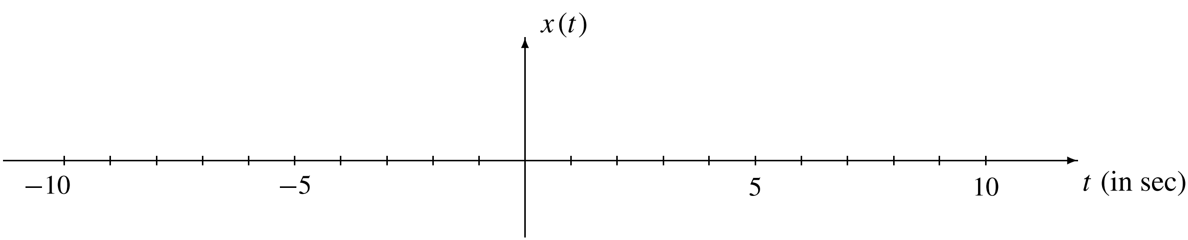
It is known that the Fourier series coefficients for this representation of a particular signal $x(t)$ are given by the integral

$$a_k = \frac{1}{8}\int_0^5 (5-t)e^{-j(2\pi/8)kt}\,dt. \qquad (1)$$

NOTE: Parts (c) and (d) can be worked independently of parts (a) and (b).

(a) In the expression for a_k in Equation (1) above, the integral and its limits define the signal $x(t)$. Determine an equation for $x(t)$ that is valid over one period.

(b) Using your result from part (a), draw a plot of $x(t)$ over the range $-10 \le t \le 10$ seconds. Label it carefully.



(c) Which value of k in Equation (1) gives the DC (or average) value of $x(t)$? $\boxed{k = \quad}$

(d) Determine the DC value of $x(t)$.

PROBLEM:

(a) Let $w(t) = 3\cos(200\pi t + 3\pi/4) + 2\cos(200\pi t - \pi/4) = A\cos(\omega_0 t + \phi)$.
Determine A, ω_0, and ϕ.

$A =$

$\omega_0 =$

$\phi =$

(b) A periodic signal $x(t)$ is given by

$$x(t) = -1 + \cos(100\pi t + \theta) + 2\cos(150\pi t + \psi).$$

Determine the period T_0 of this signal.

$T_0 =$

(c) If the Fourier series coefficients of the signal $x(t)$ in part (b) are $a_0 = -1$, $a_2 = 0.5e^{j\pi/6}$, $a_{-2} = 0.5e^{-j\pi/6}$, $a_3 = e^{-j\pi/3}$, and $a_{-3} = e^{j\pi/3}$, determine θ and ψ for the signal $x(t)$.

$\theta =$

$\psi =$

PROBLEM:

(a) Let $w(t) = \cos(100\pi t + \pi/4) + 2\cos(100\pi t - \pi/4) = A\cos(\omega_0 t + \phi)$.
Determine A, ω_0, and ϕ.

$A =$

$\omega_0 =$

$\phi =$

(b) A periodic signal $x(t)$ is given by

$$x(t) = 2 + 2\cos(1000\pi t + \theta) + \cos(1500\pi t + \psi).$$

Determine the period T_0 of this signal.

$T_0 =$

(c) If the Fourier series coefficients of the signal $x(t)$ in part (b) are $a_0 = 2$, $a_2 = e^{j\pi/2}$, $a_{-2} = e^{-j\pi/2}$, $a_3 = 0.5e^{-j\pi/6}$, and $a_{-3} = 0.5e^{j\pi/6}$, determine θ and ψ for the signal $x(t)$.

$\theta =$

$\psi =$

PROBLEM:

(a) Let $w(t) = 3\cos(200\pi t + 3\pi/4) + 2\cos(200\pi t - \pi/4) = A\cos(\omega_0 t + \phi)$.
Determine A, ω_0, and ϕ.

$A =$
$\omega_0 =$
$\phi =$

(b) A periodic signal $x(t)$ is given by

$$x(t) = 5 + 4\cos(100\pi t + \theta) + 2\cos(150\pi t + \psi).$$

Determine the period T_0 of this signal.

$T_0 =$

(c) If the Fourier series coefficients of the signal $x(t)$ in part (b) are $a_0 = 5$, $a_2 = 2e^{j\pi/4}$, $a_{-2} = 2e^{-j\pi/4}$, $a_3 = e^{-j\pi/2}$, and $a_{-3} = e^{j\pi/2}$, determine θ and ψ for the signal $x(t)$.

$\theta =$
$\psi =$

PROBLEM:

The two-sided spectrum of a signal $x(t)$ is given in the following table:

frequency (ω)	complex phasor
-200π	$1 + \sqrt{3}j$
-100π	X_{-1}
0	5
ω_1	$2e^{-j\pi 6}$
200π	X_2

(a) If $x(t)$ is a real signal, what are X_1, X_{-2}, and ω_1?

(b) Plot the spectrum of this signal as a graph.

(c) Write an expression for $x(t)$ involving only real numbers and cosine functions.

PROBLEM:

Consider the signal

$$x(t) = [10\cos(3000\pi t - \pi/6)]^2.$$

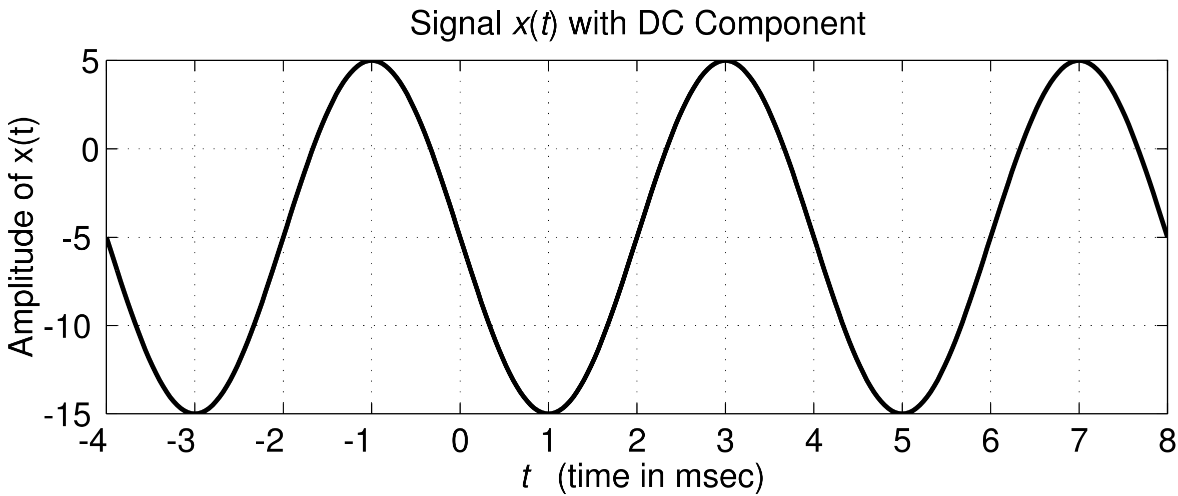
(a) Express the signal $x(t)$ as a sum of complex exponential components. Do this by using the relationship $\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$ to convert the cosine term into complex exponentials and then expand the squared expression.

(b) What frequencies (in Hz) are present in this signal?

(c) For each frequency identified in part (b), give the complex amplitude of the corresponding complex exponential component. Make a table of your analysis with frequency in one column and complex amplitude in a second column.

(d) Use your spectrum information to express $x(t)$ in the form $x(t) = A + B\cos(\omega_0 t + \phi)$.

PROBLEM:



The above signal $x(t)$ consists of a DC component plus a cosine signal. The terminology *DC component* means a component that is constant versus time.

(a) What is the frequency of the DC component? What is the frequency of the cosine component?

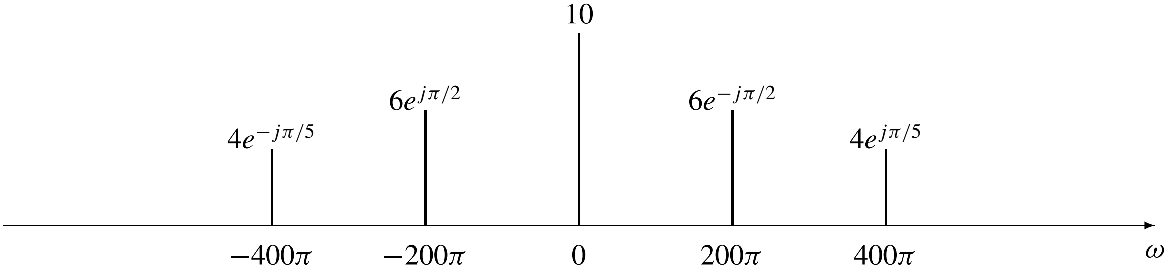
(b) Write an equation for the signal $x(t)$. You should be able to determine numerical values for all the amplitudes, frequencies, and phases in your equation by inspection of the above graph.

(c) Expand the equation obtained in the previous part into a sum of positive and negative frequency complex exponential signals.

(d) Plot the two-sided spectrum of the signal $x(t)$. Show the complex amplitudes for each positive and negative frequency contained in $x(t)$.

PROBLEM:

A real signal $x(t)$ has the following two-sided spectrum:

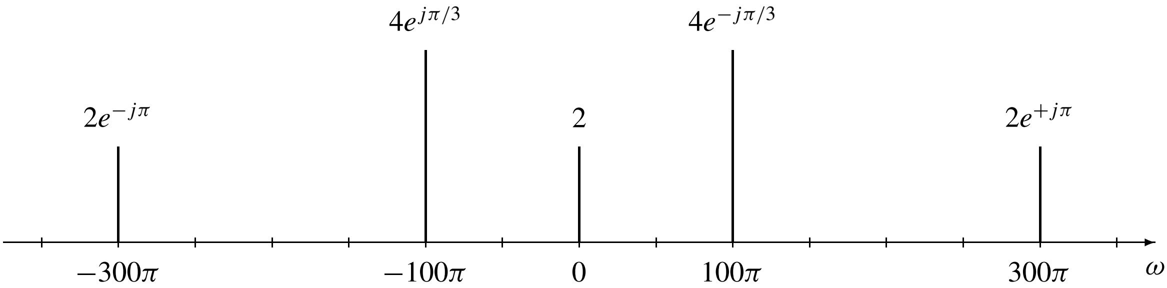


(a) Write an equation for $x(t)$ as a sum of cosines.

(b) Plot the spectrum of the signal $y(t) = 2x(t) + 10\cos(250\pi(t - 0.002))$.

PROBLEM:

The spectrum of a signal $x(t)$ is shown in the following figure:



(a) Write an equation for $x(t)$ in terms of cosine functions.

(b) Is $x(t)$ periodic? **You must explain this answer. Why or why not?**
If it is periodic, what is the fundamental frequency and corresponding period of $x(t)$?

(c) A new signal is defined as $y(t) = \cos(\alpha t + \pi) + x(t)$. It is known that $y(t)$ is periodic with period $T_0 = 0.04$ sec. Determine **two** positive values for the frequency α that will satisfy this condition.

(d) Using either of the frequencies α found in (c), modify the spectrum plot above so that it becomes the spectrum of $y(t)$.

PROBLEM:

We have seen that musical tones can be modeled mathematically by sinusoidal signals. If you read music or play the piano you know well that the piano keyboard is divided into octaves, with the tones in each octave being twice the frequency of the corresponding tones in the next lower octave. To calibrate the frequency scale, the reference tone is the A above middle-C, which is usually called A440 since its frequency is 440 Hz. Each octave contains 12 tones, and the ratio between the frequencies of successive tones is constant. Thus, the ratio must be $2^{1/12}$. Since middle C is 9 tones below A440, its frequency is approximately $(440)2^{-9/12} \approx 262$ Hz. The names of the tones (notes) of the octave starting with middle-C and ending with high-C are:

note name	C	$C^\sharp$	D	$E^\flat$	E	F	$F^\sharp$	G	$G^\sharp$	$A(440)$	$B^\flat$	B	C
note number	40	41	42	43	44	45	46	47	48	49	50	51	52
frequency										440			

(a) Make a table of the frequencies of the tones of the octave beginning with middle-C assuming that A above middle C is tuned to 440 Hz.

(b) The above notes on a piano are numbered 40 through 52. If n denotes the note number, and f denotes the frequency of the corresponding tone, give a formula for the frequency of the tone as a function of the note number.

(c) A *chord* is a combination of musical notes sounded simultaneously. A *triad* is a three note chord. The C Minor chord is composed of the tones of $C \quad E^\flat \quad G$ sounded simultaneously. From the set of corresponding frequencies determined in part (a), make a sketch of the essential features of the spectrum of the C Minor chord assuming that each note is realized by a pure sinusoidal tone. (You do not have to specify the complex phasors precisely.)