BLM3620 Digital Signal Processing*

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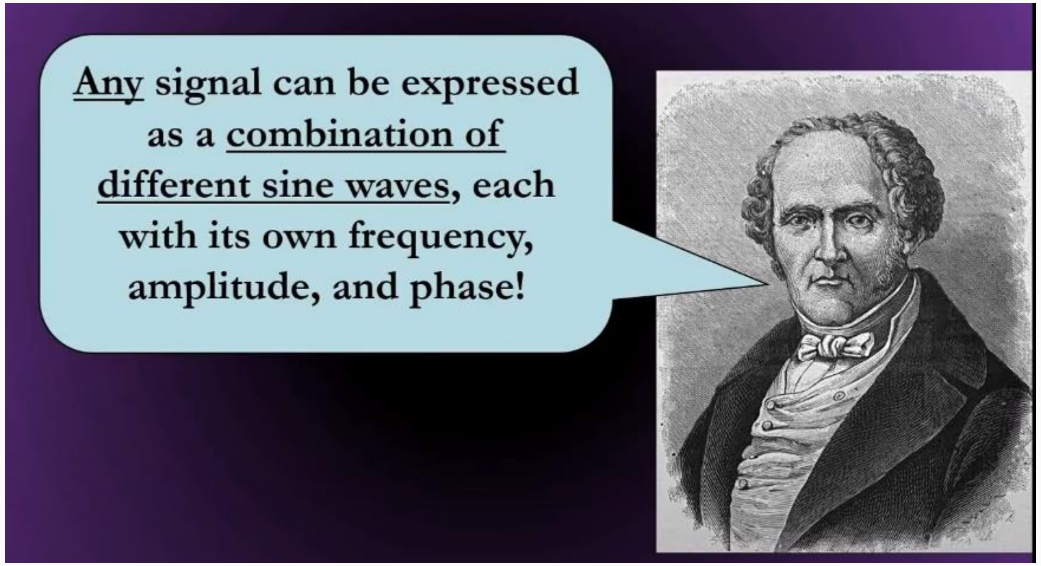
*Based on lecture notes from Ali Can Karaca & Ahmet Elbir

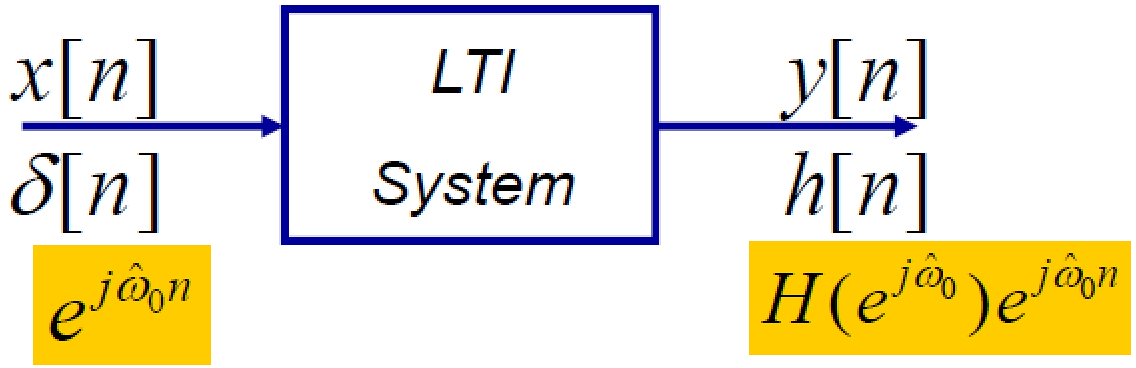
Lecture #9 – Discrete Time Fourier Transform and Properties

- Discrete Time Fourier Transform
- Examples
- Solution with Properties
- MATLAB Applications
- Exercises



Credit by Mike X Cohen, @Youtube

Recap: Frequency Response $H(e^{j\hat{\omega}})$

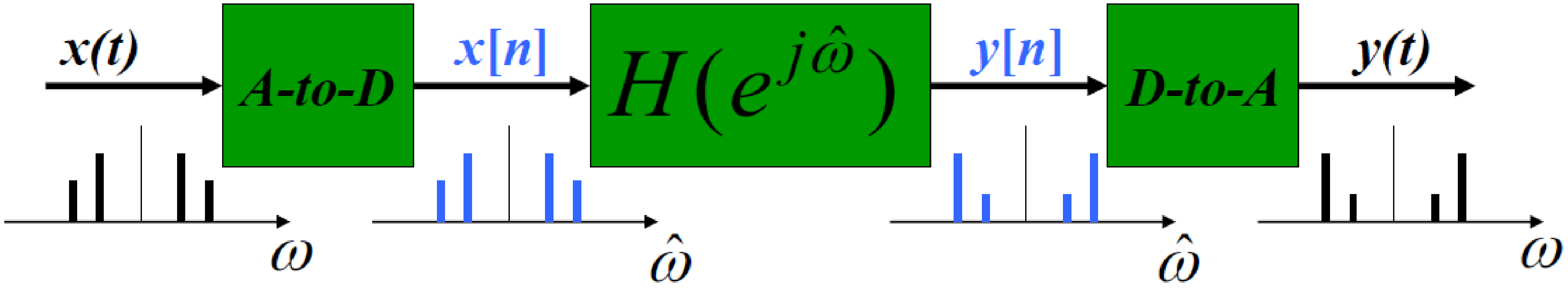


$$\boxed{H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n}}$$

$$\boxed{\text{Periodic}: H(e^{j(\hat{\omega}+2\pi)}) = H(e^{j\hat{\omega}})}$$

$$\boxed{y[n] = A \cdot \left|H(e^{j\hat{\omega}_0})\right| \cos(\hat{\omega}_0 n + \varphi + \angle H(e^{j\hat{\omega}_0}))}$$

Recap:Digital Filtering



- $\boxed{\omega}$ – SPECTRUM of x(t) (SUM of SINUSOIDS)
- $\boxed{\hat{\omega}}$ – SPECTRUM of x[n]
  - Is ALIASING a PROBLEM ?
- – SPECTRUM y[n] (FIR Gain or Nulls)
- $\boxed{\omega}$ – Then, OUTPUT y(t) = SUM of SINUSOIDS

Discrete Fourier Transform

- It is a Generalized version of Frequency Response

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n}}$$

- Definition of the **DTFT**:

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

- Forward DTFT

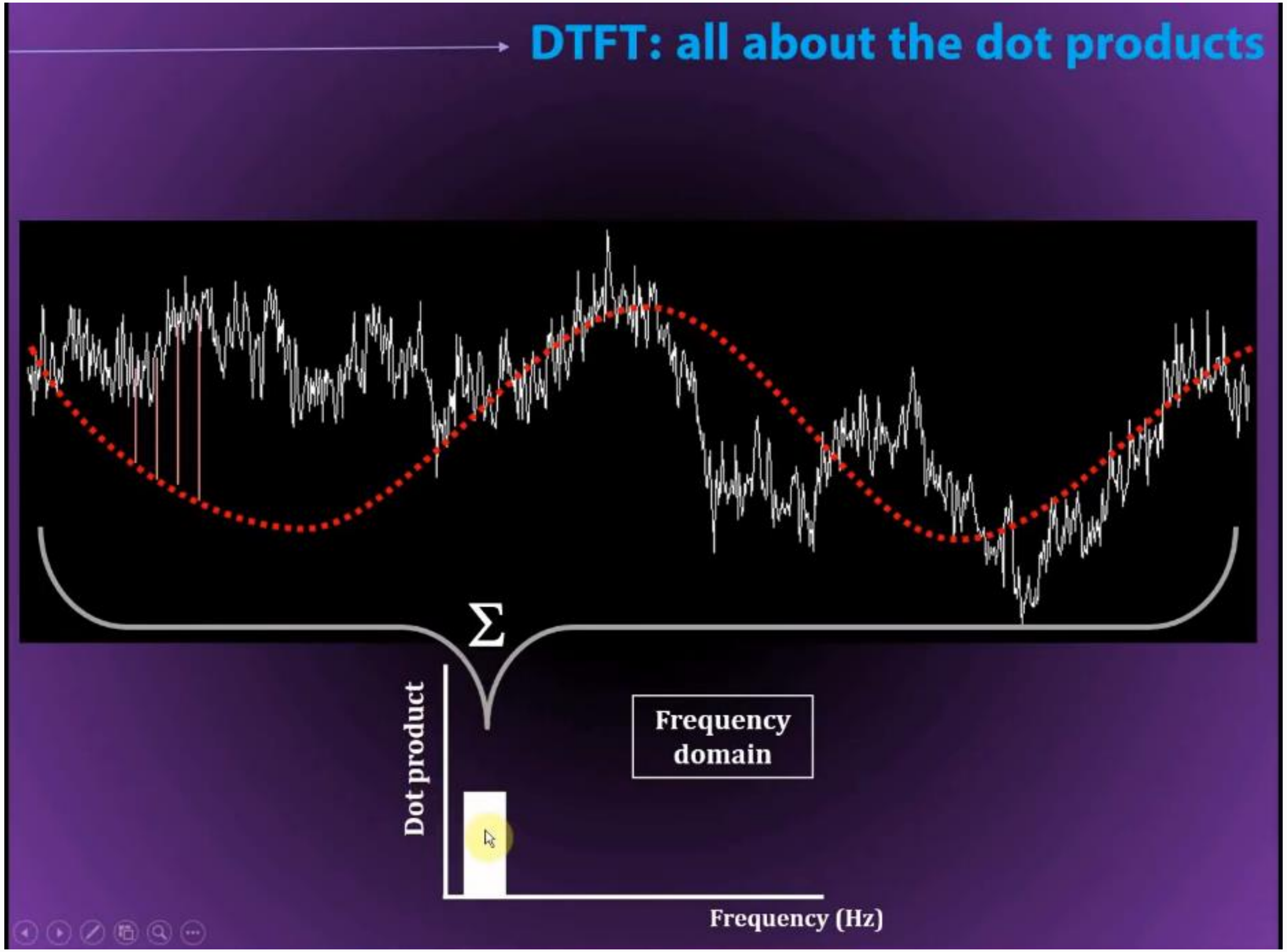
$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$$

- **Inverse DTFT**

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}$$

- Always periodic with a period of 2π

$$\boxed{X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})}$$



$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\hat{\omega}n}$$

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Periodicity of DTFT

- For any integer m: $$X(e^{j\hat{\omega}}) = X(e^{j(\hat{\omega}+2m\pi)})$$

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$$X(e^{j(\hat{\omega}+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\hat{\omega}+2m\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}e^{-j2\pi mn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

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Existence of DTFT

- Discrete-time Fourier transform (DTFT) exists – provided that the sequence is **absolutely-summable**

$$\left|X(e^{j\hat{\omega}})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}\right|$$

$$\leq \sum_{n=-\infty}^{\infty} \left|x[n]e^{-j\hat{\omega}n}\right| = \sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$$

- DTFT applies to discrete time sequences, *x*[*n*], regardless of length (if *x*[*n*] is **absolute summable**)

DTFT of a Single Sample

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{elsewhere} \end{cases} = \delta[n]$$

Unit Impulse function

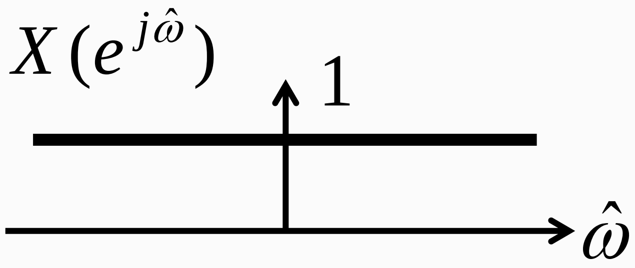
DTFT of a Single Sample

$$x[n]=\begin{cases}1, & n=0\\ 0, & \text{elsewhere}\end{cases}=\delta[n]$$

Unit Impulse function

$$X(e^{j\hat{\omega}})=\sum_{n=-\infty}^{\infty}\delta[n]e^{-j\hat{\omega}n}$$

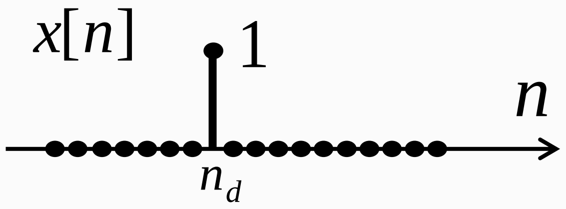
$$=\sum_{n=0}^{0}e^{-j\hat{\omega}n}=1$$



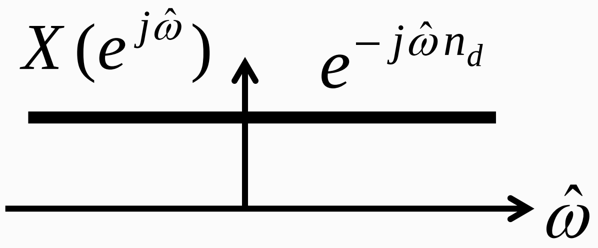
$$x[n]=\delta[n]\overset{\mathcal{F}}{\Leftrightarrow}X(e^{j\hat{\omega}})=1$$

Delayed Unit Impulse

$$x_d[n] = \delta[n - n_d] = \begin{cases} 1, & n = n_d \\ 0, & \text{elsewhere} \end{cases}$$



$$\boxed{X_d(e^{j\hat{\omega}}) = \sum_{n=n_d}^{n_d} e^{-j\hat{\omega} n} = e^{-j\hat{\omega} n_d}}$$



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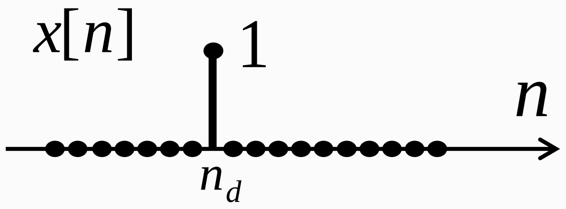

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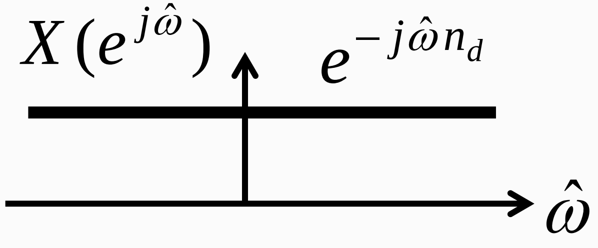


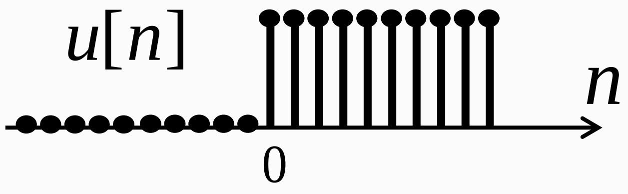
Generalizes to the <u>delay property</u>

$$x_d[n] = x[n - n_d] \Leftrightarrow$$

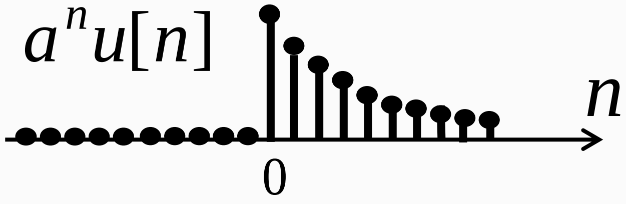
$$X_d(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}}) e^{-j\hat{\omega} n_d} = e^{-j\hat{\omega} n_d}$$

DTFT of Right-Sided Exponential

Unit Step Function: $u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$



$$\boxed{x[n] = a^n u[n], \quad |a| < 1}$$



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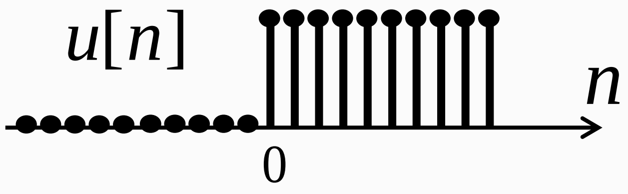

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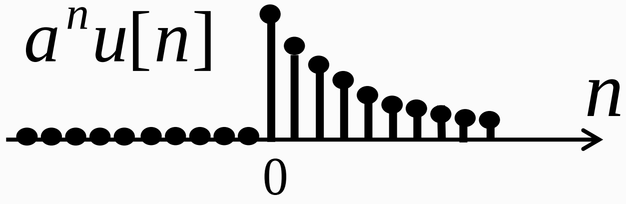

$$X(e^{j\hat{\omega}}) = \sum_{n=0}^{\infty} a^n e^{-j\hat{\omega}n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\hat{\omega}})^n = \frac{1}{1 - ae^{-j\hat{\omega}}} \quad \text{if } |a| < 1$$

Plotting: Magnitude and Angle Form

$$\boxed{x[n] = a^n u[n] \Leftrightarrow X(e^{j\hat{\omega}}) = \frac{1}{1 - ae^{-j\hat{\omega}}}}$$

$$X(e^{j\hat{\omega}}) = \left|X(e^{j\hat{\omega}})\right| e^{j\angle X(e^{j\hat{\omega}})}$$

$$\left|X(e^{j\hat{\omega}})\right|^2 = X(e^{j\hat{\omega}}) X^*(e^{j\hat{\omega}}) = \frac{1}{1 - ae^{-j\hat{\omega}}} \cdot \frac{1}{1 - ae^{j\hat{\omega}}}$$

$$\boxed{= \frac{1}{1 + a^2 - 2a\cos(\hat{\omega})}}$$

$$\boxed{\angle X(e^{j\hat{\omega}}) = \arctan\left(\frac{-a\sin(\hat{\omega})}{1 - a\cos(\hat{\omega})}\right)}$$

Magnitude and Angle Plots

EVEN Funct ion

$$\left|X(e^{j\hat{\omega}})\right| = \frac{1}{(1+a^2-2a\cos(\hat{\omega}))^{1/2}}$$

$$\left|X(e^{-j\hat{\omega}})\right| = \left|X(e^{j\hat{\omega}})\right|$$

ODD Function

$$\angle X(e^{j\hat{\omega}}) = \arctan\left(\frac{-a\sin(\hat{\omega})}{1-a\cos(\hat{\omega})}\right)$$

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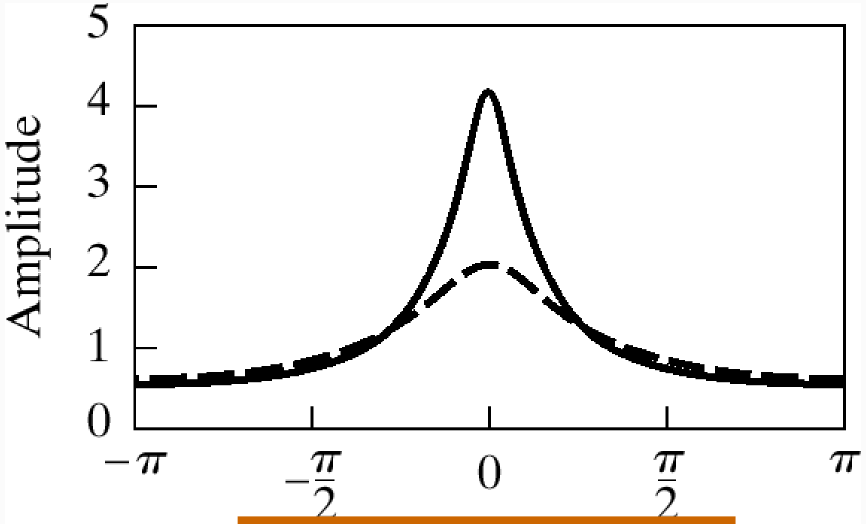

Radian Frequency ($\hat{\omega}$)

Magnitude and Angle Plots

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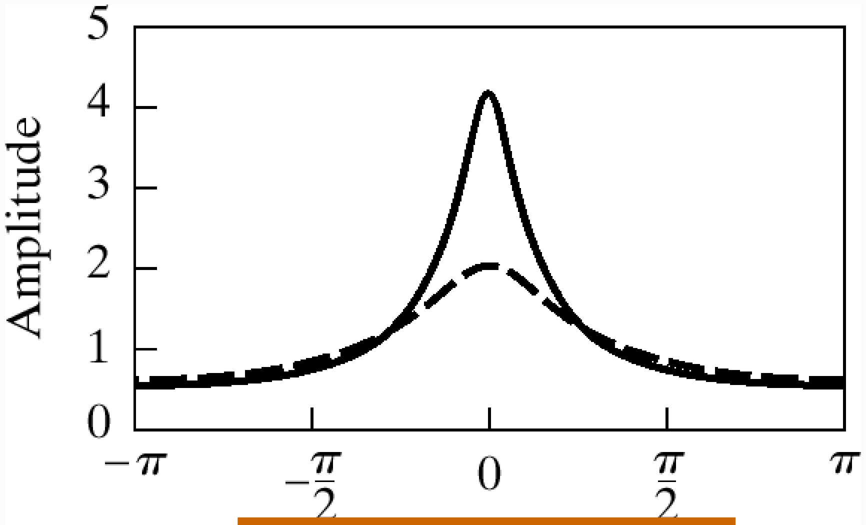

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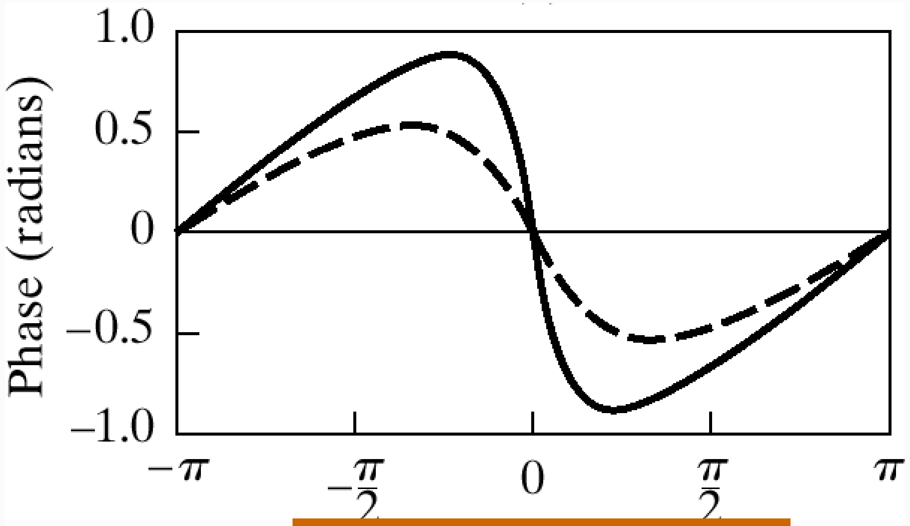

Radian Frequency ($\hat{\omega}$)

Inverse DTFT ?

$$\boxed{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}d\hat{\omega}}$$

$$X(e^{j\hat{\omega}}) = \frac{1}{1+0.3e^{-j\hat{\omega}}} \quad \Rightarrow x[n] = ?$$

$$x[n] = \int_{-\pi}^{\pi} \frac{1}{1+0.3e^{-j\hat{\omega}}} e^{j\hat{\omega}n} d\hat{\omega} \quad ??$$

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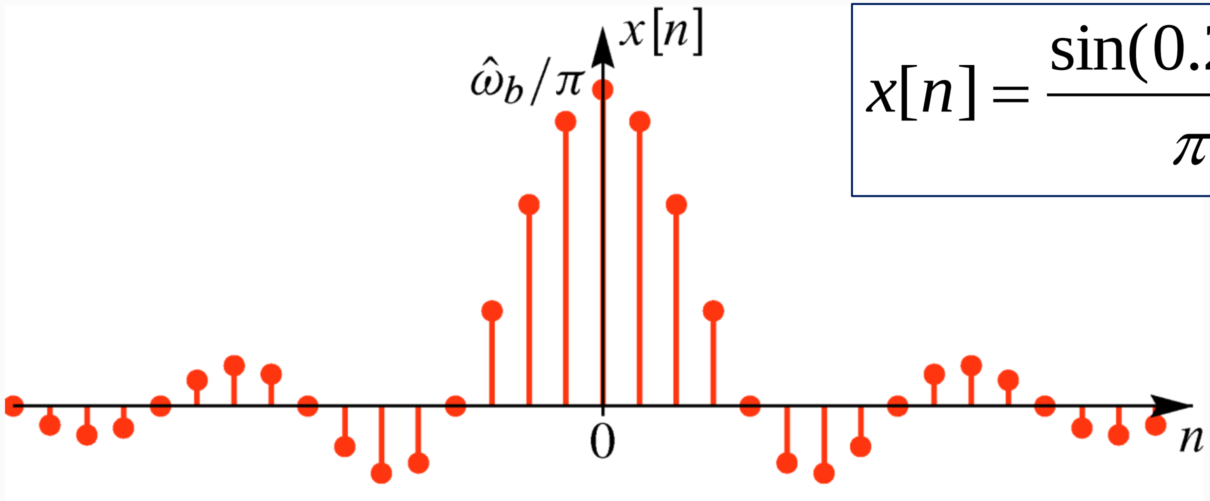
SINC Function:

- A “**sinc**” function or sequence



$$x[n] = \frac{\sin(0.25\pi\, n)}{\pi\, n}, \quad -\infty < n < \infty$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} \frac{\sin(0.25\pi n)}{\pi n} e^{-j\hat{\omega} n} = \quad ??$$

SINC Function from the inverse DTFT integral

Given a "**sinc**" function or sequence

$$x[n] = \frac{\sin(0.2\pi n)}{\pi n}, \quad -\infty < n < \infty$$

Consider an ideal band-limited signal:

$$X(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le 0.2\pi \\ 0, & 0.2\pi < |\hat{\omega}| \le \pi \end{cases}$$

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$$x[n] = \frac{1}{2\pi}\int_{-0.2\pi}^{0.2\pi} e^{j\hat{\omega}n}\, d\hat{\omega} = \left. \frac{e^{j\hat{\omega}n}}{2\pi\, jn} \right|_{-0.2\pi}^{0.2\pi}$$

$$= \frac{e^{j0.2\pi n} - e^{-j0.2\pi n}}{2\pi\, jn} = \frac{\sin(0.2\pi n)}{\pi n}$$

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Discrete-time Fourier Transform Pair

$$x[n] = \frac{1}{2\pi} \int_{-0.2\pi}^{0.2\pi} e^{j\hat{\omega}n} d\hat{\omega} = \left. \frac{e^{j\hat{\omega}n}}{2\pi\, jn} \right|_{-0.2\pi}^{0.2\pi}$$

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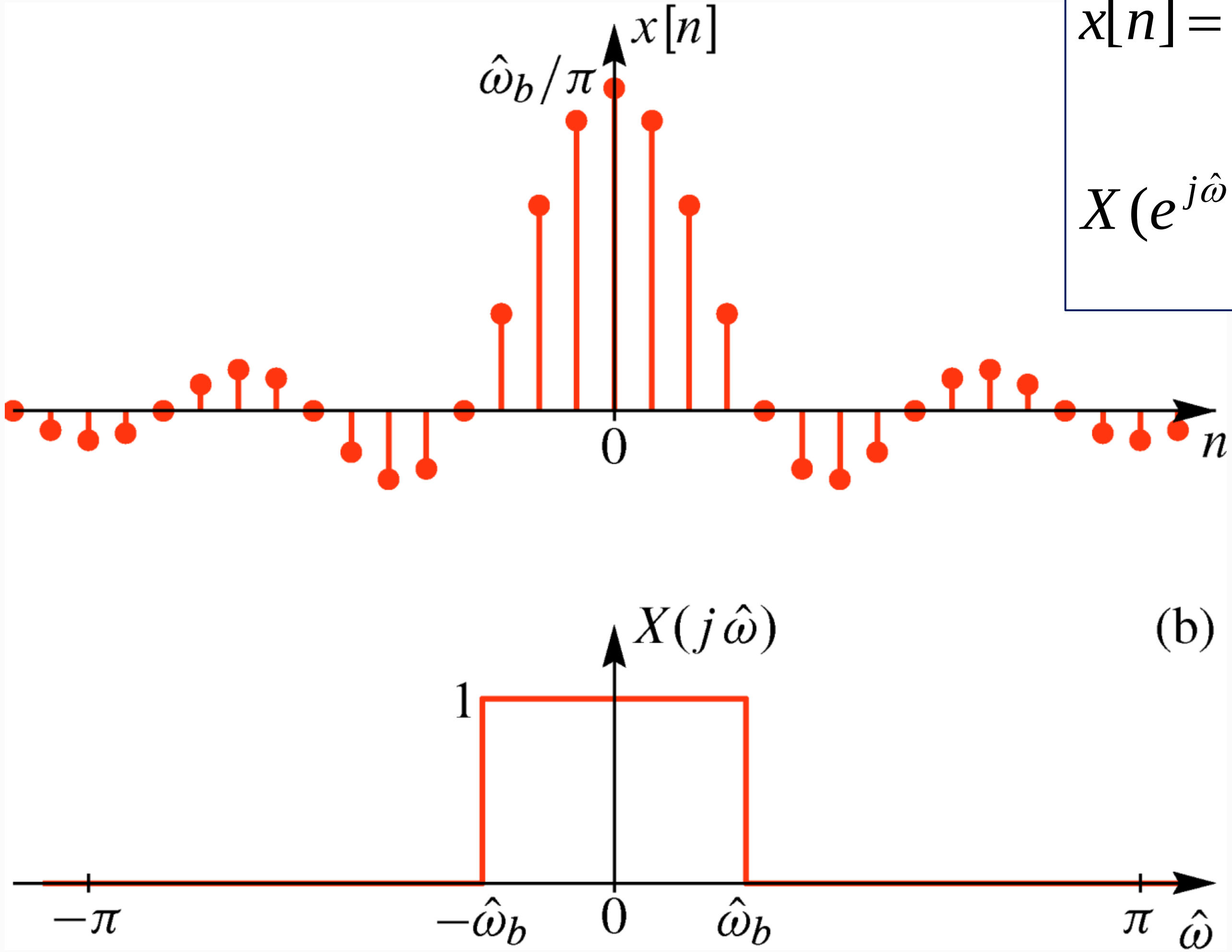
$$x[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

$$X(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \leq \hat{\omega}_b \\ 0, & \hat{\omega}_b < |\hat{\omega}| \leq \pi \end{cases}$$

SINC Function – Rectangle DTFT pair



$$x[n] = \frac{\sin(\hat{\omega}_b n)}{\pi n} \quad \stackrel{DTFT}{\Leftrightarrow}$$

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Summary of DTFT Pairs

$$x[n] = \delta[n - n_d] \Leftrightarrow X(e^{j\hat{\omega}}) = e^{-j\hat{\omega} n_d}$$

Delayed Impulse

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\hat{\omega}}) = \frac{1}{1 - ae^{-j\hat{\omega}}}$$

Right-sided Exponential

$$x[n] = \frac{\sin(\hat{\omega}_c n)}{\pi n} \Leftrightarrow X(e^{j\hat{\omega}}) = \begin{cases} 1 & |\hat{\omega}| \le \hat{\omega}_c \\ 0 & \hat{\omega}_c < |\hat{\omega}| < \pi \end{cases}$$

sinc function is **Bandlimited**

$$x[n] = \begin{cases} 1 & 0 \le n < L \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X(e^{j\hat{\omega}}) = \frac{\sin(\frac{1}{2} L\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} e^{-j\hat{\omega}(L-1)/2}$$

Using DTFT

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- To use it effectively you must
  - know **PAIRS**: the Fourier transforms of certain important signals
  - know ***properties*** and certain key ***theorems***
  - be able to combine time-domain and frequency domain methods appropriately

Property of DTFT

Table 7-2 Basic discrete-time Fourier transform properties.

Table of DTFT Properties

Property Name	*Time-Domain: $x[n]$*	*Frequency-Domain: $X(e^{j\hat{\omega}})$*
Periodic in $\hat{\omega}$		$X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\hat{\omega}}) + bX_2(e^{j\hat{\omega}})$
Conjugate Symmetry	$x[n]$ is real	$X(e^{-j\hat{\omega}}) = X^*(e^{j\hat{\omega}})$
Conjugation	$x^*[n]$	$X^*(e^{-j\hat{\omega}})$
Time-Reversal	$x[-n]$	$X(e^{-j\hat{\omega}})$
Delay	$x[n - n_d]$	$e^{-j\hat{\omega}n_d} X(e^{j\hat{\omega}})$
Frequency Shift	$x[n]e^{j\hat{\omega}_0 n}$	$X(e^{j(\hat{\omega}-\hat{\omega}_0)})$
Modulation	$x[n]\cos(\hat{\omega}_0 n)$	$\frac{1}{2}X(e^{j(\hat{\omega}-\hat{\omega}_0)}) + \frac{1}{2}X(e^{j(\hat{\omega}+\hat{\omega}_0)})$
Convolution	$x[n] * h[n]$	$X(e^{j\hat{\omega}})H(e^{j\hat{\omega}})$
Autocorrelation	$x[-n] * x[n]$	$\lvert X(e^{j\hat{\omega}})\rvert^2$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\hat{\omega}})\rvert^2 d\hat{\omega}$	

Properties of DTFT

- Linearity

$$x[n] = ax_1[n] + bx_2[n] \Leftrightarrow X(e^{j\hat{\omega}}) = aX_1(e^{j\hat{\omega}}) + bX_2(e^{j\hat{\omega}})$$

- Time-Delay ←→ phase shift

$$y[n] = x[n - n_d] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

- Frequency-Shift ←→ multiply by sinusoid

$$y[n] = e^{j\hat{\omega}_c n}x[n] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j(\hat{\omega}-\hat{\omega}_c)})$$

Example-1:

$x[n] = 0.2^n u[n-4] + 0.4^{n-1} u[n]$ -> Find the DTFT of this signal?

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Using Linearity and Shifting Properties:

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Using Linearity and Shifting Properties:

$$x[n] = 0.2^4 0.2^{n-4} u[n-4] + 0.4^{-1} 0.4^n u[n]$$

$$X(e^{j\Omega}) = \frac{0.2^4 e^{-j4\Omega}}{1 - 0.2e^{-j\Omega}} + \frac{2.5}{1 - 0.4e^{-j\Omega}}$$

Example-1:

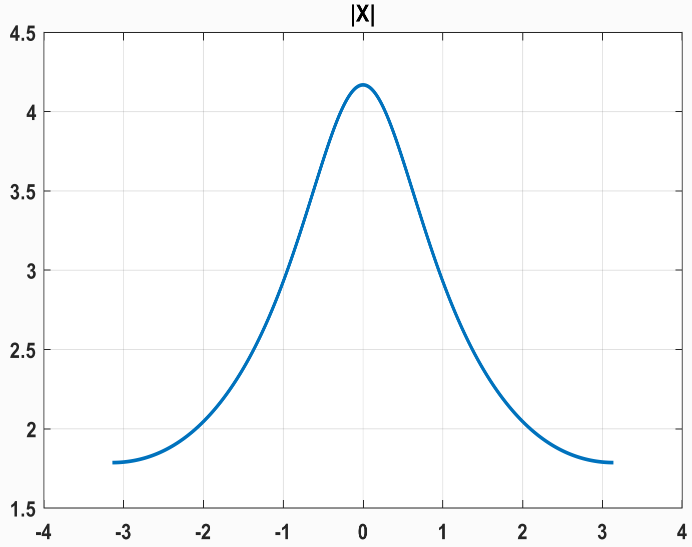
Let's plot the magnitude of DTFT

$$X(e^{j\Omega}) = \frac{0.2^4 e^{-j4\Omega}}{1 - 0.2e^{-j\Omega}} + \frac{2.5}{1 - 0.4e^{-j\Omega}}$$

```
clc; clear all;
what = -pi:0.01:pi;
N = 6;
firstPart = (0.2^4)*(exp(-j*4*omega))./(1-0.2*exp(-j*omega));
secondPart = (2.5)./(1-0.4*exp(-j*omega));

DTFT = firstPart + secondPart ;

figure(1);
plot(what,abs(DTFT));
title('|X|');
```



Example-2:

Örnek-2:

Given DT Fourier Transform $X\left(e^{j\hat{\omega}}\right) = \dfrac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})}$, find x[n] signal.

Example-2:

Örnek-2:

Given DT Fourier Transform $X\left(e^{j\hat{\omega}}\right) = \dfrac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})}$, find x[n] signal.

$$x[n] = a^n u[n] \overset{\text{DTFT}}{\Longleftrightarrow} X\left(e^{j\hat{\omega}}\right) = \frac{1}{1-ae^{-j\hat{\omega}}}, |a| < 1$$

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Örnek-2:

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$$X\left(e^{j\hat{\omega}}\right) = \frac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})} = \frac{A}{(1-0.3e^{-j\hat{\omega}})} + \frac{B}{(1-0.4e^{-j\hat{\omega}})}$$

Example-2:

Örnek-2:

Given DT Fourier Transform $X(e^{j\hat{\omega}}) = \dfrac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})}$, find x[n] signal.

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$$X(e^{j\hat{\omega}}) = \frac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})} = \frac{A}{(1-0.3e^{-j\hat{\omega}})} + \frac{B}{(1-0.4e^{-j\hat{\omega}})}$$

$$A = X(e^{j\hat{\omega}})\,(1-0.3e^{-j\hat{\omega}}) \xrightarrow{e^{j\hat{\omega}}=0.3} \frac{1}{(1-0.4/0.3)} = -3$$

Example-2:
Örnek-2:

Given DT Fourier Transform $X(e^{j\hat{\omega}}) = \dfrac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})}$, find x[n] signal.

$$x[n] = a^n u[n] \xleftrightarrow{\text{DTFT}} X(e^{j\hat{\omega}}) = \frac{1}{1-ae^{-j\hat{\omega}}}, |a| < 1$$

$$X(e^{j\hat{\omega}}) = \frac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})} = \frac{A}{(1-0.3e^{-j\hat{\omega}})} + \frac{B}{(1-0.4e^{-j\hat{\omega}})}$$

$$A = X(e^{j\hat{\omega}})\,(1-0.3e^{-j\hat{\omega}}) \xrightarrow{e^{j\hat{\omega}}=0.3} \frac{1}{(1-0.4/0.3)} = -3$$

$$B = X(e^{j\hat{\omega}})\,(1-0.4e^{-j\hat{\omega}}) \xrightarrow{e^{j\hat{\omega}}=0.4} \frac{1}{(1-0.3/0.4)} = 4$$

Example-2:
Örnek-2:

Given DT Fourier Transform $X(e^{j\hat{\omega}}) = \dfrac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})}$

$$X(e^{j\hat{\omega}}) = \frac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})} = \frac{-3}{(1-0.3e^{-j\hat{\omega}})} + \frac{4}{(1-0.4e^{-j\hat{\omega}})}$$

Example-2:
Örnek-2:

Given DT Fourier Transform $X\left(e^{j\hat{\omega}}\right) = \dfrac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})}$

$$X\left(e^{j\hat{\omega}}\right) = \frac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})} = \frac{-3}{(1-0.3e^{-j\hat{\omega}})} + \frac{4}{(1-0.4e^{-j\hat{\omega}})}$$

Using Linearity Property:

Example-2:
Örnek-2:

Given DT Fourier Transform $X(e^{j\hat{\omega}}) = \dfrac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})}$

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Using Linearity Property:

$$x[n] = -3 \times 0.3^n\, u[n] + 4 \times 0.4^n\, u[n]$$

Example-2:
Örnek-2:

Given DT Fourier Transform $X\left(e^{j\hat{\omega}}\right) = \dfrac{1}{(1-0.3e^{-j\hat{\omega}})(1-0.4e^{-j\hat{\omega}})}$

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Using Linearity Property:

$$x[n] = -3 \times 0.3^n\, u[n] + 4 \times 0.4^n\, u[n]$$

Example-3:

$$y[n] - 0.2y[n-1] = 0.5x[n] - 0.3x[n-1] + 0.1x[n-2]$$

Find frequency response and impulse response of the system given above.

$$y[n] = x[n-n_d] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

Example-3:

$$y[n] - 0.2y[n-1] = 0.5x[n] - 0.3x[n-1] + 0.1x[n-2]$$

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$$y[n] = x[n - n_d] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

Using linearity property:

$$Y\left(e^{j\widehat{\omega}}\right) - 0.2e^{-j\Omega}Y\left(e^{j\widehat{\omega}}\right) = 0.5X\left(e^{j\widehat{\omega}}\right) - 0.3e^{-j\widehat{\omega}}X\left(e^{j\widehat{\omega}}\right) + 0.1e^{-j2\widehat{\omega}}X\left(e^{j\widehat{\omega}}\right)$$

Example-3:

$$y[n] - 0.2y[n-1] = 0.5x[n] - 0.3x[n-1] + 0.1x[n-2]$$

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$Y\left(e^{j\widehat{\omega}}\right) = X\left(e^{j\widehat{\omega}}\right)H\left(e^{j\widehat{\omega}}\right)$ ->

Example-3:

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$Y\left(e^{j\widehat{\omega}}\right) = X\left(e^{j\widehat{\omega}}\right)H\left(e^{j\widehat{\omega}}\right)$ ->

$$H\left(e^{j\widehat{\omega}}\right) =$$

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Find frequency response and impulse response of the system given above.

Using shifting property:

$$y[n] = x[n-n_d] \Leftrightarrow Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})e^{-j\hat{\omega}n_d}$$

Using linearity property:

$$Y\left(e^{j\widehat{\omega}}\right) - 0.2e^{-j\Omega}Y\left(e^{j\widehat{\omega}}\right) = 0.5X\left(e^{j\widehat{\omega}}\right) - 0.3e^{-j\widehat{\omega}}X\left(e^{j\widehat{\omega}}\right) + 0.1e^{-j2\widehat{\omega}}X\left(e^{j\widehat{\omega}}\right)$$

$Y\left(e^{j\widehat{\omega}}\right) = X\left(e^{j\widehat{\omega}}\right)H\left(e^{j\widehat{\omega}}\right)$ ->

$$H\left(e^{j\widehat{\omega}}\right) = \frac{Y\left(e^{j\widehat{\omega}}\right)}{X\left(e^{j\widehat{\omega}}\right)} = \frac{0.5 - 0.3e^{-j\widehat{\omega}} + 0.1e^{-j2\widehat{\omega}}}{1 - 0.2e^{-j\widehat{\omega}}} = \frac{0.5}{1 - 0.2e^{-j\widehat{\omega}}} - \frac{0.3e^{-j\widehat{\omega}}}{1 - 0.2e^{-j\widehat{\omega}}} + \frac{0.1e^{-j2\widehat{\omega}}}{1 - 0.2e^{-j\widehat{\omega}}}$$

Example-3:

Example-3:

Impulse Response:

Example-3:

Impulse Response:

$$h[n] = 0.5 \times 0.2^{n}\, u[n] - 0.3 \times 0.2^{n-1}\, u[n-1] + 0.1 \times 0.2^{n-2}\, u[n-2]$$

Example-3:

$$H\left(e^{j\hat{\omega}}\right) =$$

Impulse Response:

$$\boxed{h[n] = 0.5 \times 0.2^{n}\, u[n] - 0.3 \times 0.2^{n-1}\, u[n-1] + 0.1 \times 0.2^{n-2}\, u[n-2]}$$

Example-3:

$$H(e^{j\hat{\omega}}) = \frac{Y(e^{j\hat{\omega}})}{X(e^{j\hat{\omega}})} = \frac{0.5 - 0.3e^{-j\hat{\omega}} + 0.1e^{-j2\hat{\omega}}}{1 - 0.2e^{-j\hat{\omega}}} = \frac{0.5}{1 - 0.2e^{-j\hat{\omega}}} - \frac{0.3e^{-j\hat{\omega}}}{1 - 0.2e^{-j\hat{\omega}}} + \frac{0.1e^{-j2\hat{\omega}}}{1 - 0.2e^{-j\hat{\omega}}}$$

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$$\boxed{h[n] = 0.5 \times 0.2^{n}\, u[n] - 0.3 \times 0.2^{n-1}\, u[n-1] + 0.1 \times 0.2^{n-2}\, u[n-2]}$$

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$$H(e^{j\hat{\omega}}) = \frac{Y(e^{j\hat{\omega}})}{X(e^{j\hat{\omega}})} = \frac{0.5 - 0.3e^{-j\hat{\omega}} + 0.1e^{-j2\hat{\omega}}}{1 - 0.2e^{-j\hat{\omega}}} = \frac{0.5}{1 - 0.2e^{-j\hat{\omega}}} - \frac{0.3e^{-j\hat{\omega}}}{1 - 0.2e^{-j\hat{\omega}}} + \frac{0.1e^{-j2\hat{\omega}}}{1 - 0.2e^{-j\hat{\omega}}}$$

$$x[n] = a^n u[n] \xLeftrightarrow{\text{DTFT}} X(e^{j\hat{\omega}}) = \frac{1}{1 - ae^{-j\hat{\omega}}}, |a| < 1$$

Impulse Response:

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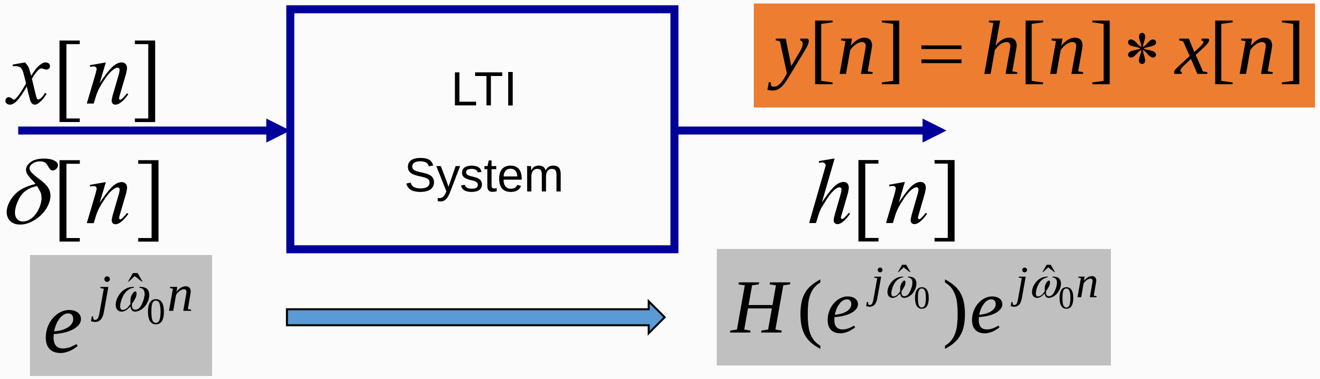
DTFT maps Convolution to Multiplication



DTFT maps Convolution to Multiplication

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \quad \Leftrightarrow \quad Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$$

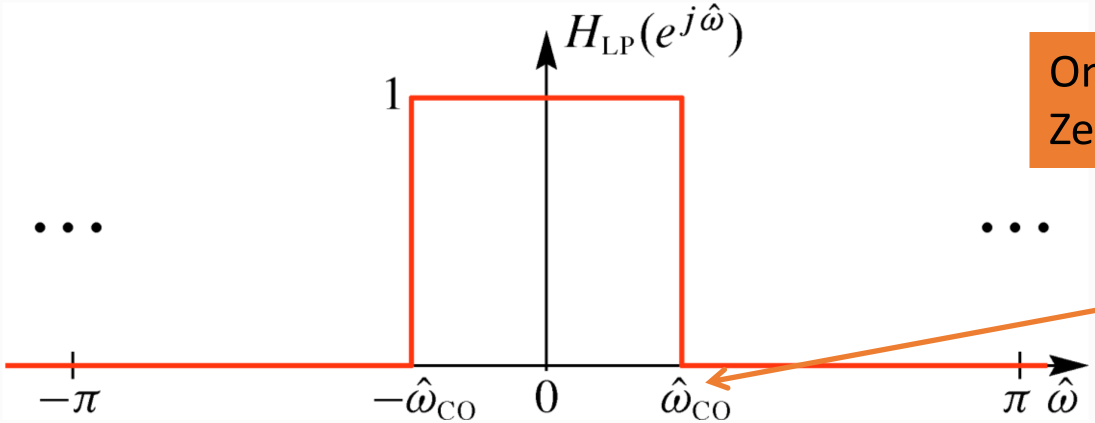
$x[n]$ → LTI System → $y[n] = h[n] * x[n]$

$\delta[n]$ → $h[n]$

$e^{j\hat{\omega}_0 n}$ → $H(e^{j\hat{\omega}_0})e^{j\hat{\omega}_0 n}$

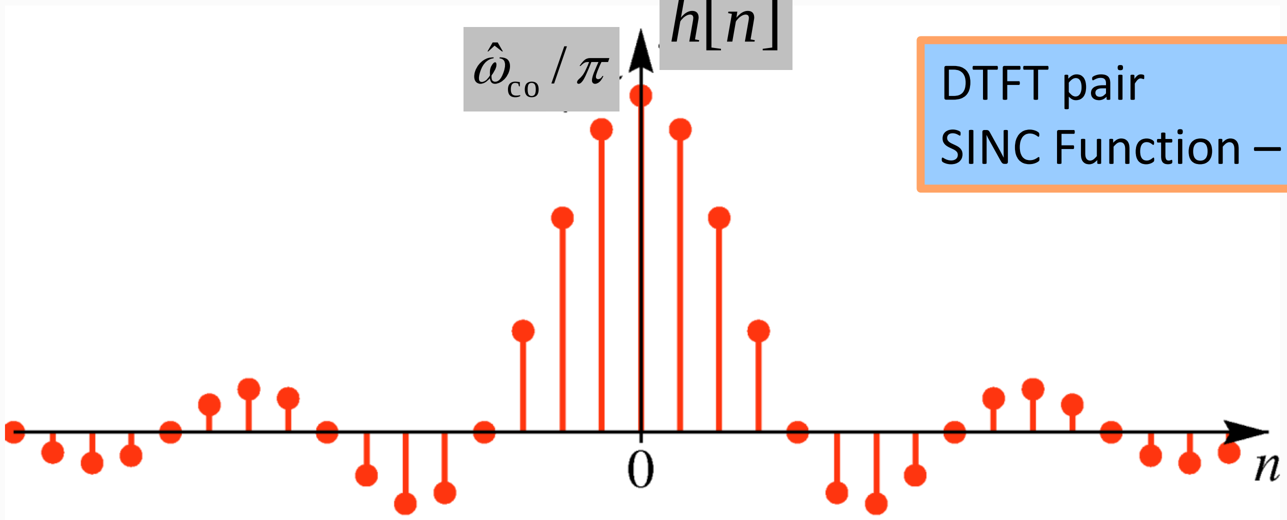
$X(e^{j\hat{\omega}_0})$ → $H(e^{j\hat{\omega}_0})X(e^{j\hat{\omega}_0})$

IDEAL LowPass Filter (LPF)



One in the passband
Zero in the stopband

$\hat{\omega}_{co}$ is cutoff frequency

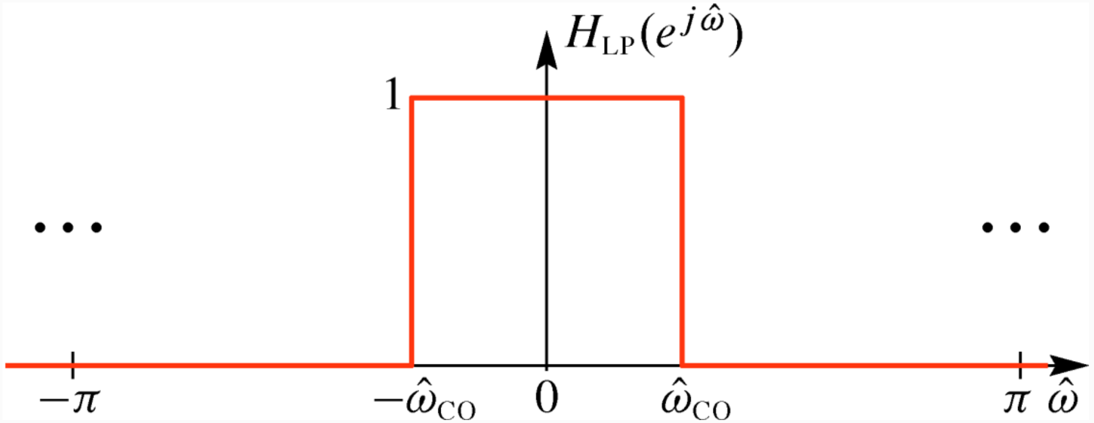


DTFT pair
SINC Function – Rectangle

$$h[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} \overset{DTFT}{\Leftrightarrow}$$

$$H(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_{co} \\ 0, & \hat{\omega}_{co} < |\hat{\omega}| \le \pi \end{cases}$$

Filtering with the IDEAL LPF



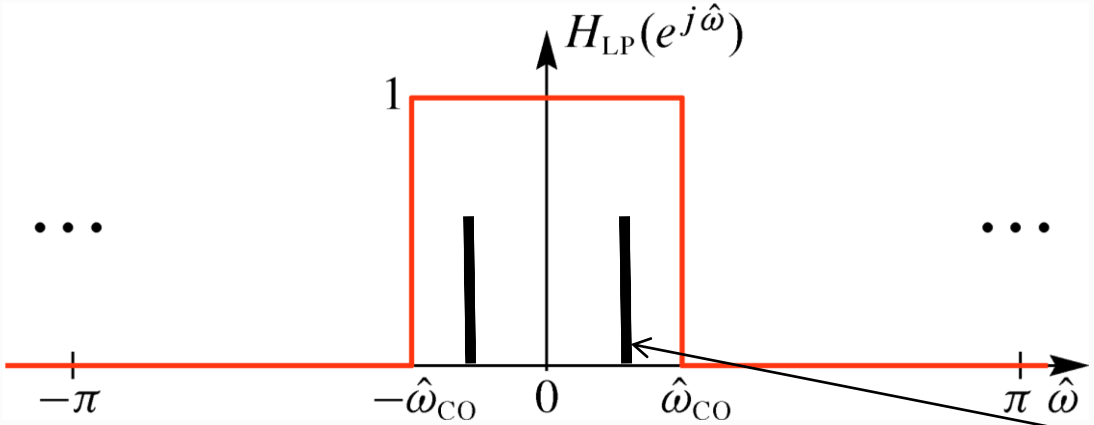
$$h[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} \quad \stackrel{DTFT}{\Leftrightarrow}$$

$$H(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \leq \hat{\omega}_{co} \\ 0, & \hat{\omega}_{co} < |\hat{\omega}| \leq \pi \end{cases}$$

Find the output when the input is a sinusoid:

$$\boxed{x[n] = \cos(\hat{\omega}_0 n)}$$

Filtering with the IDEAL LPF



$$h[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

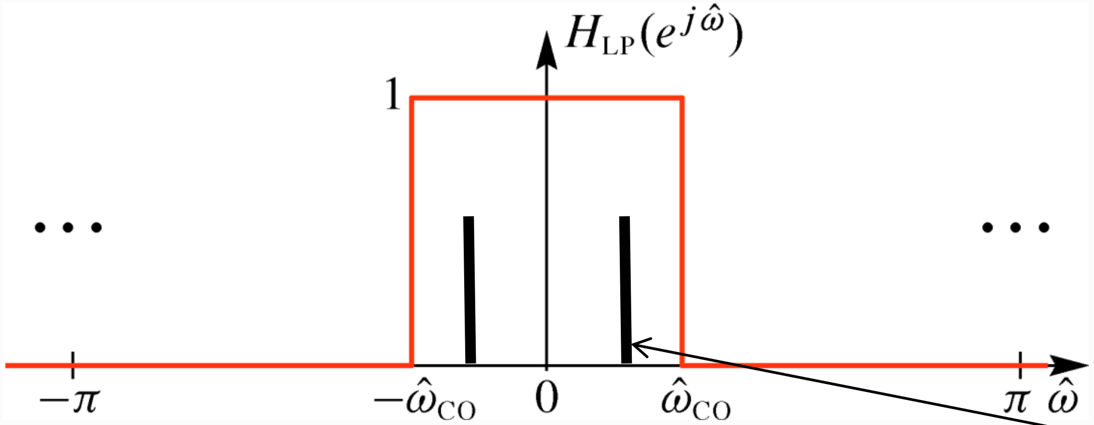
$$H(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_{co} \\ 0, & \hat{\omega}_{co} < |\hat{\omega}| \le \pi \end{cases}$$

$$x[n] = \cos(\hat{\omega}_0 n)$$

Find the output when the input is a sinusoid:

$$y[n] = h[n] * x[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} * \cos(\hat{\omega}_0 n) \quad ??$$

Filtering with the IDEAL LPF



$$h[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

$$H(e^{j\hat{\omega}}) = \begin{cases} 1, & |\hat{\omega}| \le \hat{\omega}_{co} \\ 0, & \hat{\omega}_{co} < |\hat{\omega}| \le \pi \end{cases}$$

$$x[n] = \cos(\hat{\omega}_0 n)$$

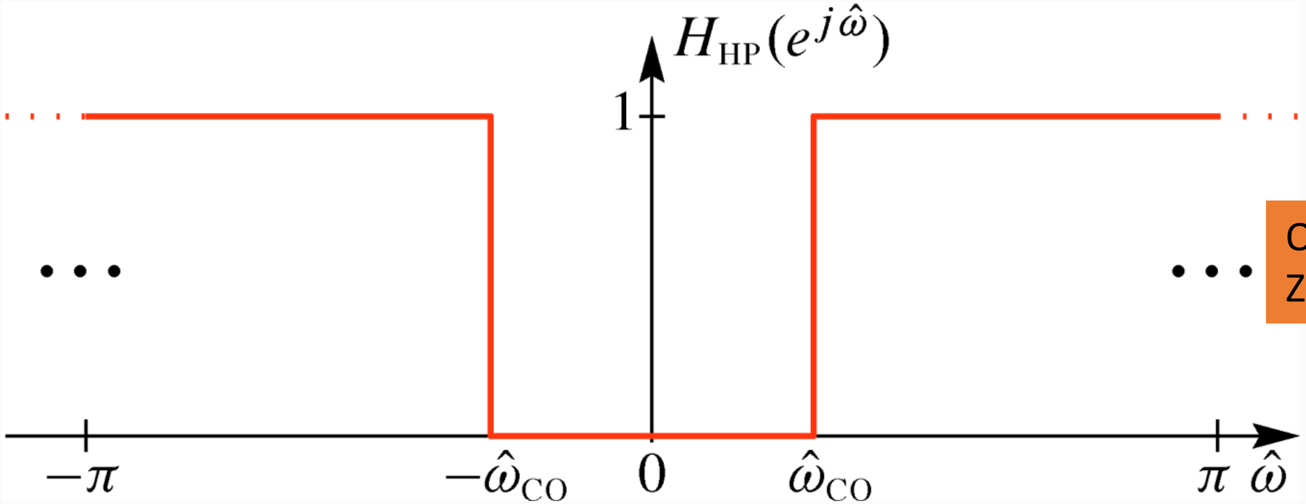
Find the output when the input is a sinusoid:

$$y[n] = h[n] * x[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n} * \cos(\hat{\omega}_0 n) \quad ??$$

Multiply the spectrum of the input times the DTFT of the filter to get

$$y[n] = \begin{cases} \cos(\hat{\omega}_0 n) & \hat{\omega}_0 \le \hat{\omega}_{co} \\ 0 & \hat{\omega}_0 > \hat{\omega}_{co} \end{cases}$$

IDEAL HighPass Filter (HPF)



One in the passband
Zero in the stopband

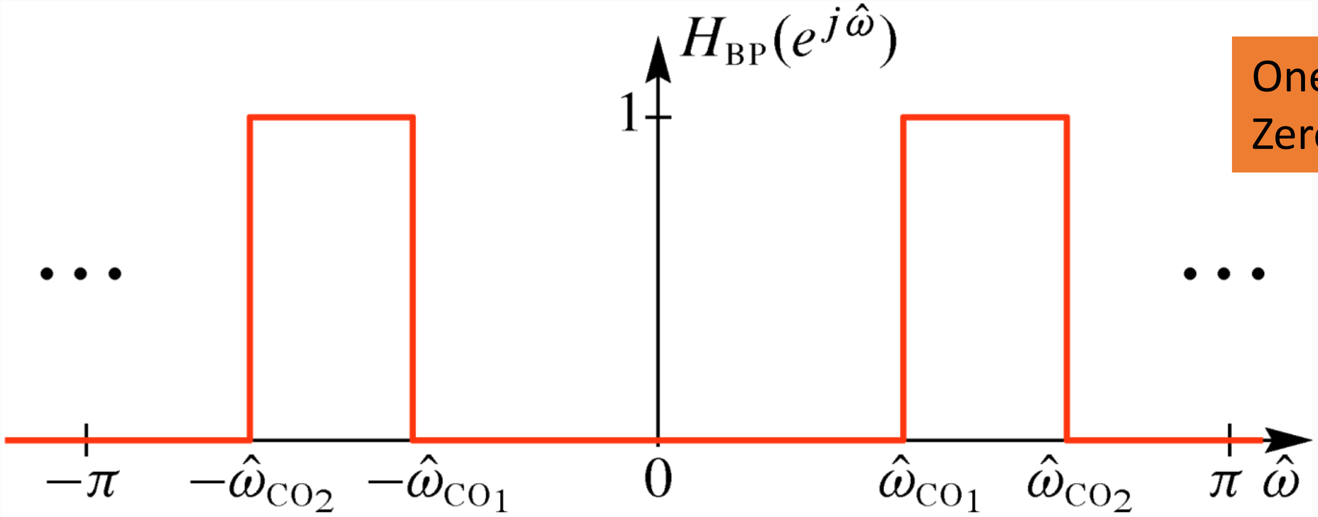
HPF is 1 minus LPF

Inverse DTFT of 1 is a delta

$$h_{\mathrm{HP}}[n] = \delta[n] - \frac{\sin(\hat{\omega}_{co} n)}{\pi n} \quad \overset{DTFT}{\Leftrightarrow}$$

$$H_{\mathrm{HP}}(e^{j\hat{\omega}}) = \begin{cases} 0, & |\hat{\omega}| \le \hat{\omega}_{co} \\ 1, & \hat{\omega}_{co} < |\hat{\omega}| \le \pi \end{cases}$$

IDEAL BandPass Filter (BPF)



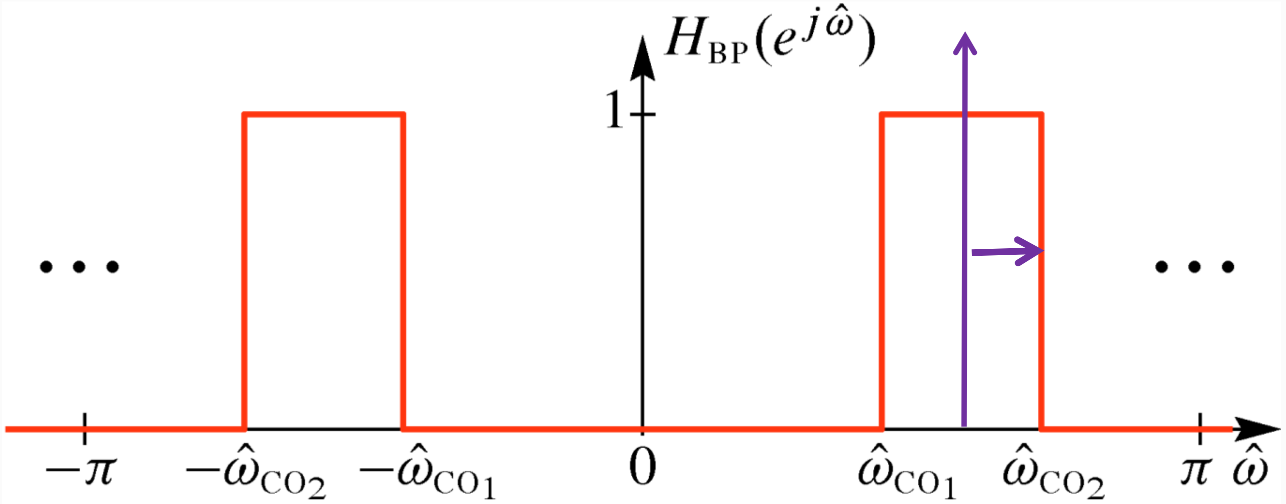
One in the passband
Zero in the 2 stopbands

BPF has two stopbands

Band Reject Filter has one stopband and two passbands.
It is one minus BPF

$$H_{BP}(e^{j\hat{\omega}}) = \begin{cases} 0 & |\hat{\omega}| \le \hat{\omega}_{co_1} \\ 1 & \hat{\omega}_{co_1} < |\hat{\omega}| \le \hat{\omega}_{co_2} \\ 0 & \hat{\omega}_{co_2} < |\hat{\omega}| \le \pi \end{cases}$$

Make IDEAL BPF from LPF



BPF is frequency shifted version of LPF

Frequency shifting **up and down** is done by cosine multiplication in the time domain

$$h_{BP}[n] = 2\cos(\hat{\omega}_{mid} n)\frac{\sin(\frac{1}{2}\hat{\omega}_{diff} n)}{\pi n}$$

$$\overset{DTFT}{\Leftrightarrow} H_{BP}(e^{j\hat{\omega}}) = \begin{cases} 0 & |\hat{\omega}| \le \hat{\omega}_{co_1} \\ 1 & \hat{\omega}_{co_1} < |\hat{\omega}| \le \hat{\omega}_{co_2} \\ 0 & \hat{\omega}_{co_2} < |\hat{\omega}| \le \pi \end{cases}$$

Exercise-1

EXERCISE 7.3: Use the linearity of the DTFT and (7.6) to determine the DTFT of the following sum of two right-sided exponential signals: $x[n] = (0.8)^n u[n] + 2(-0.5)^n u[n]$.

McClellan, Schafer, and Yoder, *DSP First, 2e*, ISBN 0-13-065562-7.

Exercise-1

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McClellan, Schafer, and Yoder, *DSP First, 2e*, ISBN 0-13-065562-7.

SOLUTION to EXERCISE 7.3: **DSP First 2e**

Equation (7.6) states in general

$$x[n] = a^n u[n] \longleftrightarrow X(e^{j\hat{\omega}}) = \frac{1}{1 - ae^{-j\hat{\omega}}}$$

Now if $x[n] = (0.8)^n u[n] + 2(-0.5)^n u[n]$ we can use (7.6) with $a = 0.8$ and -0.5 and the linearity property to write down by inspection, the result

$$X(e^{j\hat{\omega}}) = \frac{1}{1 - 0.8e^{-j\hat{\omega}}} + \frac{2}{1 + 0.5e^{-j\hat{\omega}}}$$

McClellan, Schafer, and Yoder, *DSP First, 2e*, ISBN 0-13-065562-7.
Prentice Hall, Upper Saddle River, NJ 07458. ©2016 Pearson Education, Inc.

Exercise-2

EXERCISE 7.10: Using the results of Exercise 7.9, show that the impulse response of the ideal HPF is

$$h_{\text{HP}}[n] = \delta[n] - \frac{\sin(\hat{\omega}_{\text{co}} n)}{\pi n}$$

McClellan, Schafer, and Yoder, *DSP First, 2e*, ISBN 0-13-065562-7.

Exercise-2

EXERCISE 7.10: Using the results of Exercise 7.9, show that the impulse response of the ideal HPF is

$$h_{\mathrm{HP}}[n] = \delta[n] - \frac{\sin(\hat{\omega}_{\mathrm{co}} n)}{\pi n}$$

SOLUTION to EXERCISE 7.10: **DSP First 2e**

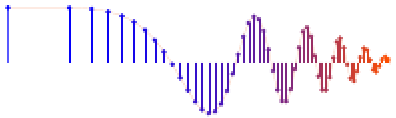
Given that $H_{\mathrm{HP}}(e^{j\hat{\omega}}) = 1 - H_{\mathrm{LP}}(e^{j\hat{\omega}})$ we can use linearity of the DTFT to find $h_{\mathrm{HP}}[n]$. We have the following DTFT pairs:

$$1 \quad \longleftrightarrow \quad \delta[n]$$

$$h_{\mathrm{HP}}[n] = \frac{\sin(\hat{\omega}_{co} n)}{\pi n)} \quad \longleftrightarrow \quad H_{\mathrm{LP}}(e^{j\hat{\omega}}) = \begin{cases} 1 & |\hat{\omega}| < \hat{\omega}_{co} \\ 0 & \hat{\omega}_{co} < |\hat{\omega}| \le \pi \end{cases}$$

Therefore,

$$h_{\mathrm{HP}}[n] = \delta[n] - \frac{\sin(\hat{\omega}_{\mathrm{co}} n)}{\pi n}$$



Exercise-3

Example 7-1: The following FIR filter

$$y[n] = 5x[n-1] - 4x[n-3] + 3x[n-5]$$

has a finite-length impulse response:

$$h[n] = 5\delta[n-1] - 4\delta[n-3] + 3\delta[n-5]$$

Each impulse in $h[n]$ is transformed using (**??**), and then combined according to the linearity property of the DTFT which gives

$$H(e^{j\hat{\omega}}) = 5e^{-j\hat{\omega}} - 4e^{-j3\hat{\omega}} + 3e^{-j5\hat{\omega}}$$

McClellan, Schafer, and Yoder, *DSP First, 2e*, ISBN 0-13-065562-7.