DSP First, 2/e

Lecture 23

Frequency Response, H(z), Poles and Zeros

for IIR and FIR Systems

READING ASSIGNMENTS

- This Lecture:
  - Chapter 9, Sects. 9-5 and 9-6
  - Chapter 10, Sects. 10-5 and 10-7

LECTURE OBJECTIVES

- ZEROS and POLES
- Relate H(z) to FREQUENCY RESPONSE

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- Four demos: PeZ, 3-Domain movies
  - Placing Poles and Zeros
- Bandpass Filters: IIR
- Nulling Filters: FIR   Notch Filters: IIR

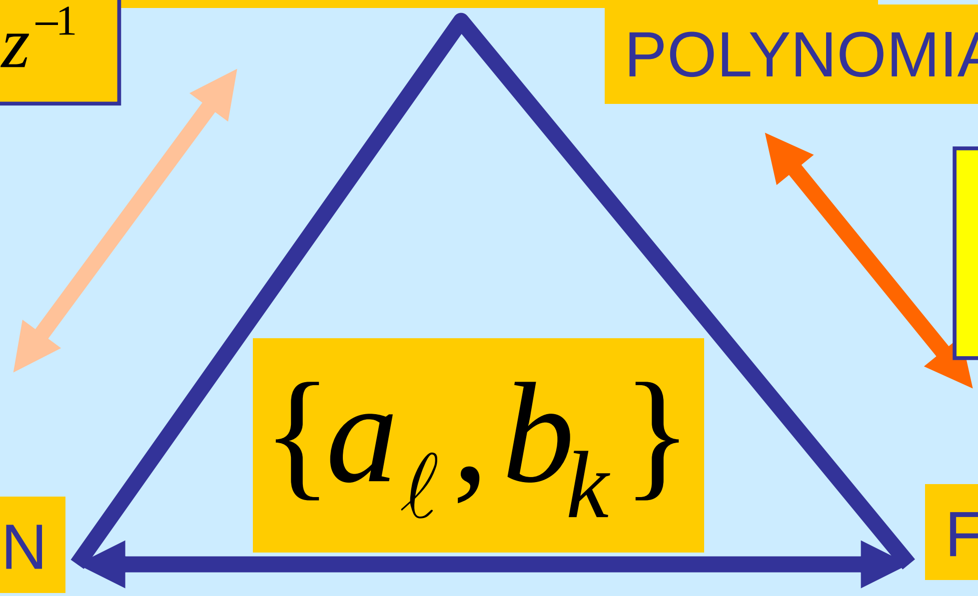
THREE DOMAINS: $H(e^{j\hat{\omega}})$

$$H(z)=\frac{b_0+b_1z^{-1}}{1-a_1z^{-1}}$$

Z-TRANSFORM-DOMAIN

POLYNOMIALS: H(z)

$$z=e^{j\hat{\omega}}$$



$$\{a_\ell, b_k\}$$

TIME-DOMAIN

FREQ-DOMAIN

$$y[n]=a_1y[n-1]+b_0x[n]+b_1x[n-1]$$

Impulse response, h[n]

$$H(e^{j\hat{\omega}})=\frac{b_0+b_1e^{-j\hat{\omega}}}{1-a_1e^{-j\hat{\omega}}}$$

Motivation: Filter Design

- Some tasks/analysis easier in one domain
  - Freq domain: system response to sinusoids
  - Time domain: calculate output to any signal
  - Z-domain: given specs, build a filter
- Can we design a filter that removes DC and sinusoids at frequency $\hat{\omega} = \pi / 3$?
- Z-domain reduces this to polynomial roots

H(z) = Rational Function

- First Order:

$$H(z)=\frac{b_0+b_1 z^{-1}}{1-a_1 z^{-1}}$$

- We can also study Second-Order Systems:

$$H(z)=\frac{b_0+b_1 z^{-1}+b_2 z^{-2}}{1-a_1 z^{-1}-a_2 z^{-2}}=\frac{B(z)}{A(z)}$$

- Numerator & Denominator Polynomials

POLES & ZEROS of H(z)

- Zeros of H(z), i.e., where is H(z)=0?
  - Look for Roots of Numerator Polynomial

$$H(z) = \frac{B(z)}{A(z)}, \text{ so } B(z_0) = 0 \Rightarrow H(z_0) = 0$$

$$\text{if } \; A(z_0) \neq 0$$

- Poles of H(z), i.e., where is H(z)=infinity?
  - Look for Roots of Denominator Polynomial

$$H(z) = \frac{B(z)}{A(z)}, \text{ so } A(z_0) = 0 \Rightarrow H(z_0) \rightarrow \infty$$

$$\text{if } \; B(z_0) \neq 0$$

Poles/Zeros of 1st-order H(z)

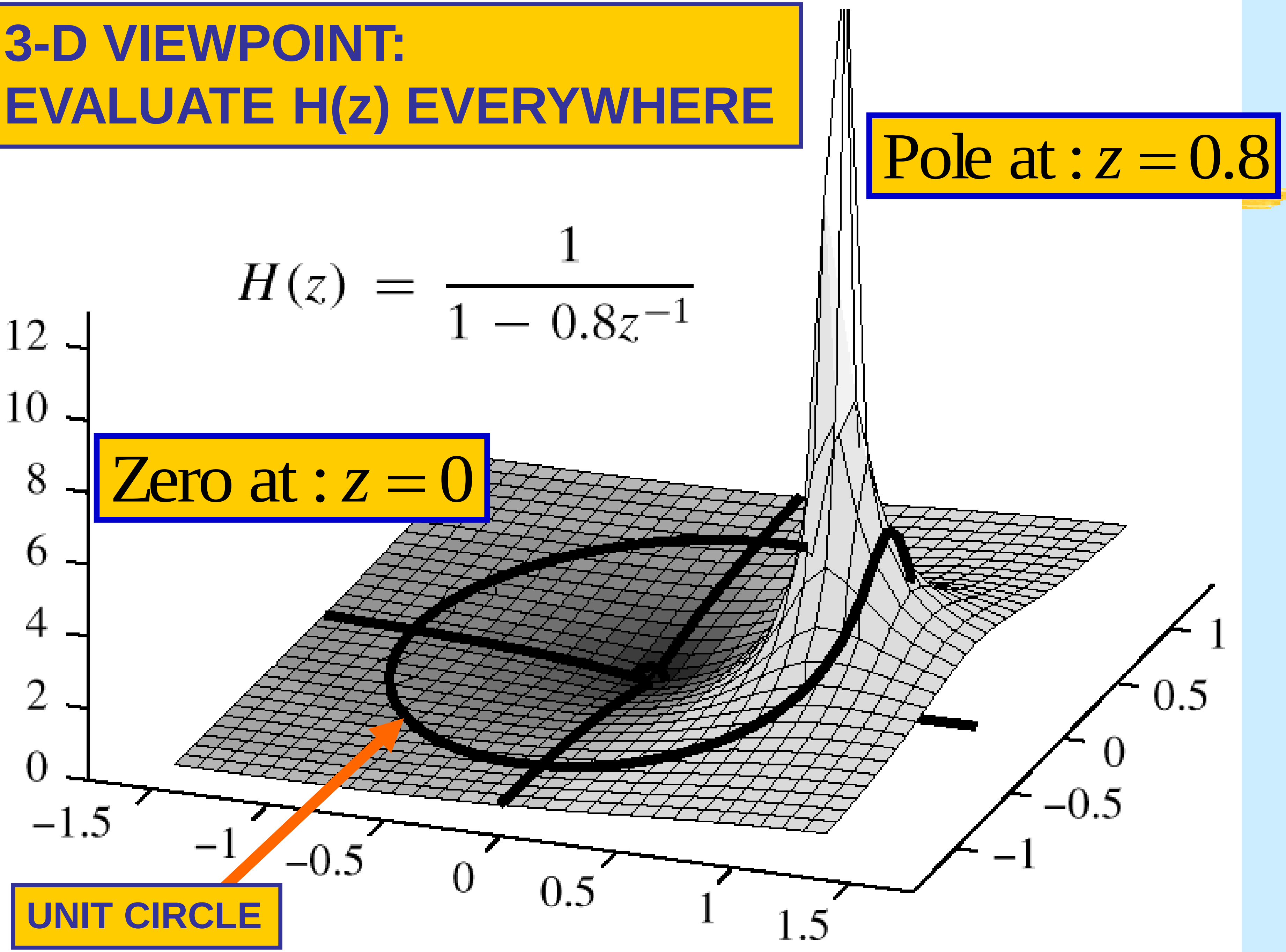
- Roots of Numerator & Denominator Polys:

$$\boxed{H(z) = \frac{1 + b_1 z^{-1}}{1 - 0.8z^{-1}}}$$

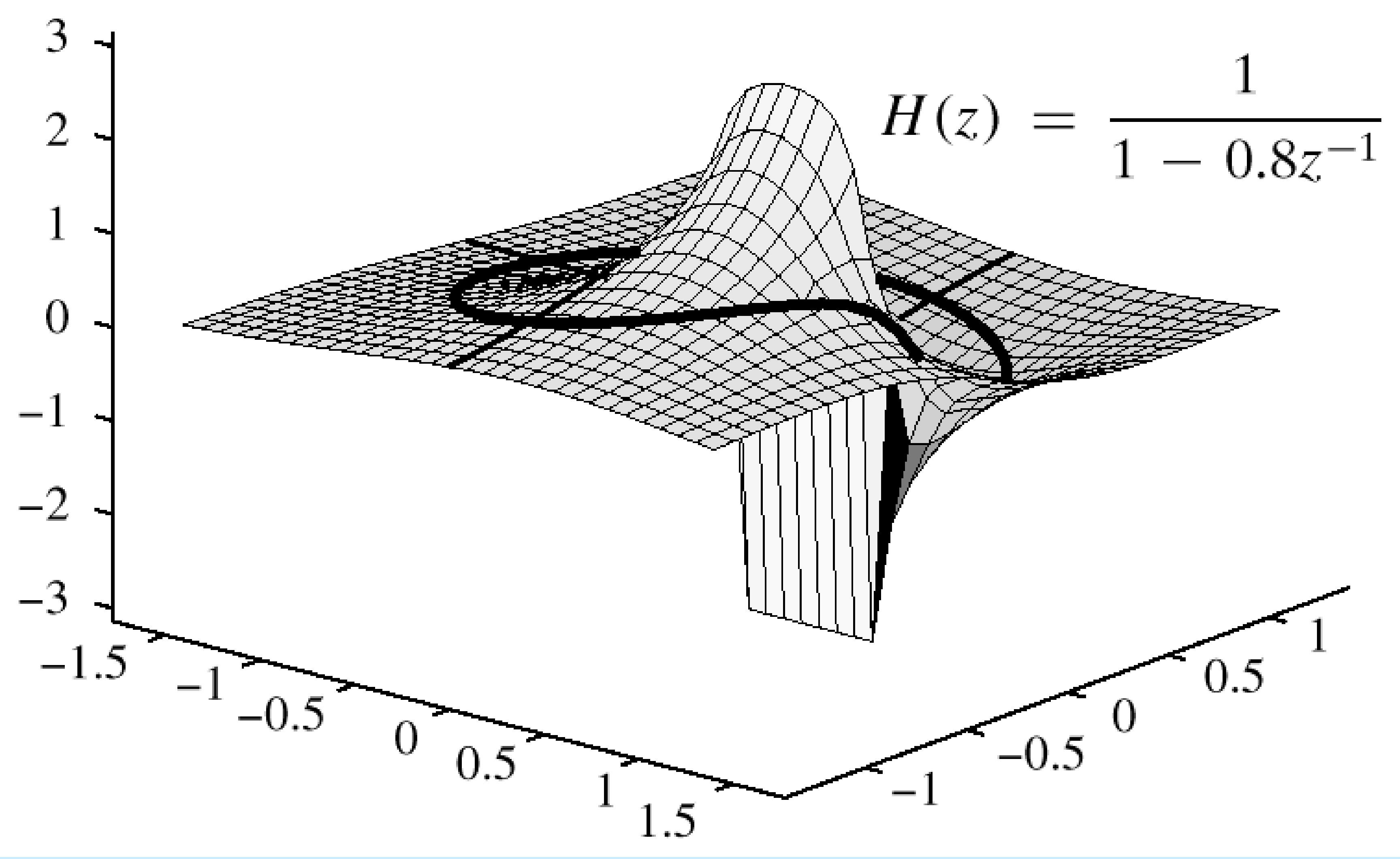
$$H(z) = \frac{z(1 + b_1 z^{-1})}{z(1 - 0.8z^{-1})} = \frac{z + b_1}{z - 0.8}$$

$$\boxed{\text{Pole at}: z = 0.8} \quad \boxed{\text{Zero at}: z = -b_1}$$

3-D VIEWPOINT:
EVALUATE H(z) EVERYWHERE



PHASE from 3-D PLOT



FREQ. RESPONSE from H(z)

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

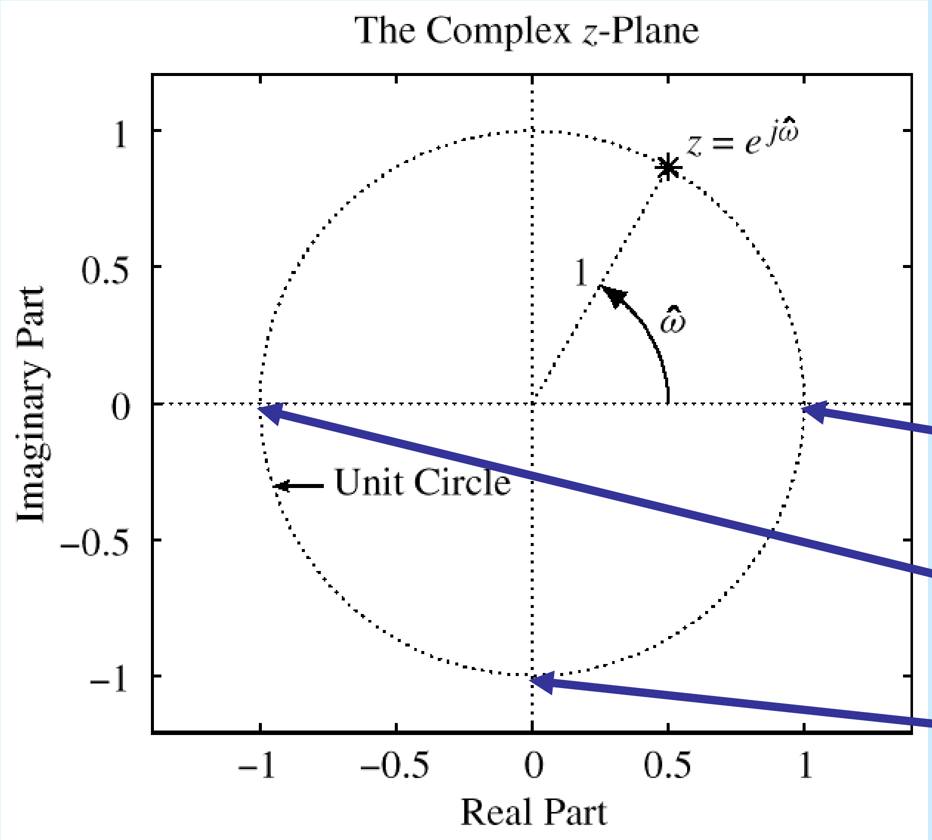
- Relate H(z) to FREQUENCY RESPONSE
- EVALUATE H(z) on the **UNIT CIRCLE**
  - ANGLE is same as FREQUENCY

$$z = e^{j\hat{\omega}} \qquad \text{(as } \hat{\omega} \text{ varies)}$$

$$\text{defines a CIRCLE, radius } = 1$$

UNIT CIRCLE: RECAP

- MAPPING BETWEEN z and $\hat{\omega}$



$$z = e^{j\hat{\omega}}$$

$$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$$

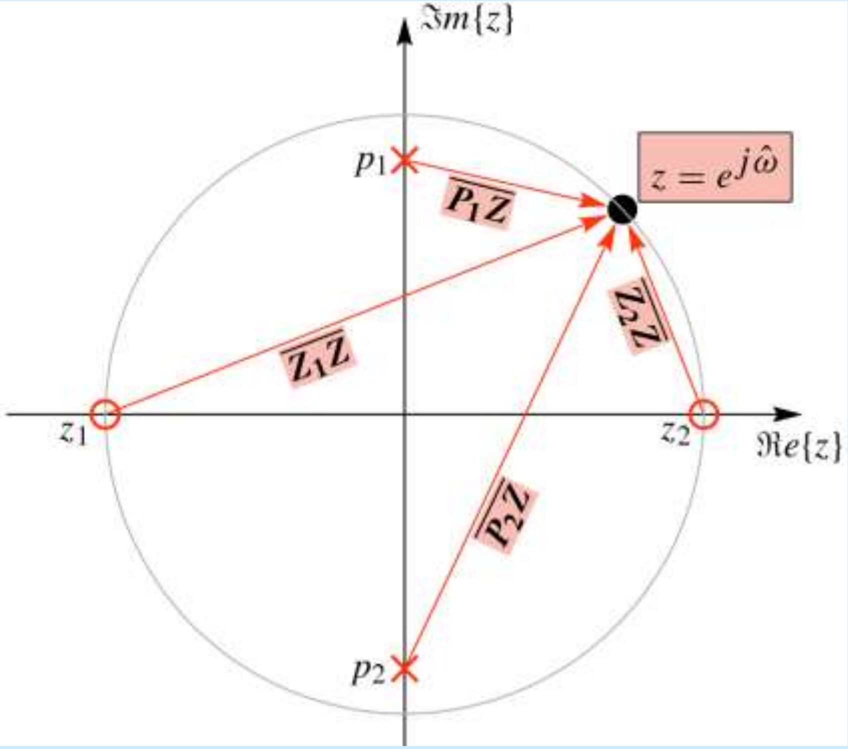
$$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$$

$$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\tfrac{1}{2}\pi$$

Frequency Response from poles and zeros

$$|H(e^{j\hat{\omega}})| = G\,\frac{|e^{j\hat{\omega}} - z_1|\,|e^{j\hat{\omega}} - z_2|}{|e^{j\hat{\omega}} - p_1|\,|e^{j\hat{\omega}} - p_2|}$$

$$|H(e^{j\hat{\omega}})| = G\,\frac{\overline{Z_1 Z} \cdot \overline{Z_2 Z}}{\overline{P_1 Z} \cdot \overline{P_2 Z}}$$



$$H(z) = G\,\frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

IIR H(z) example: two poles

- Poles just inside the unit circle (for stability)

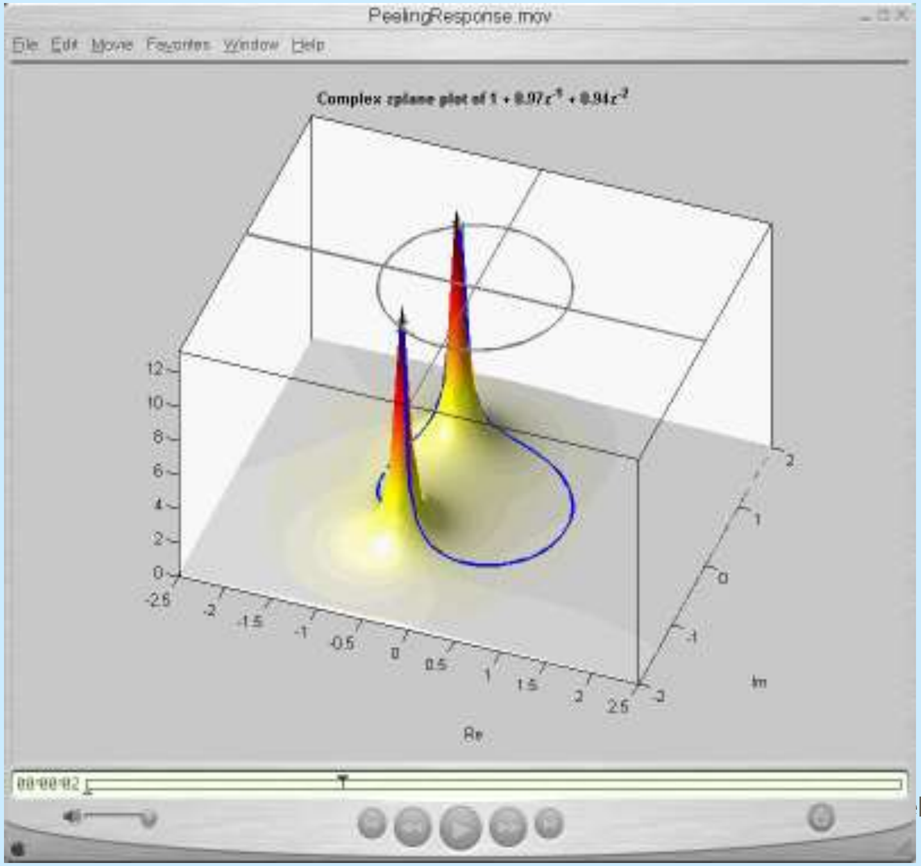
$$\boxed{H(z) = \frac{1}{1 + 0.97z^{-1} + 0.9409z^{-2}}}$$

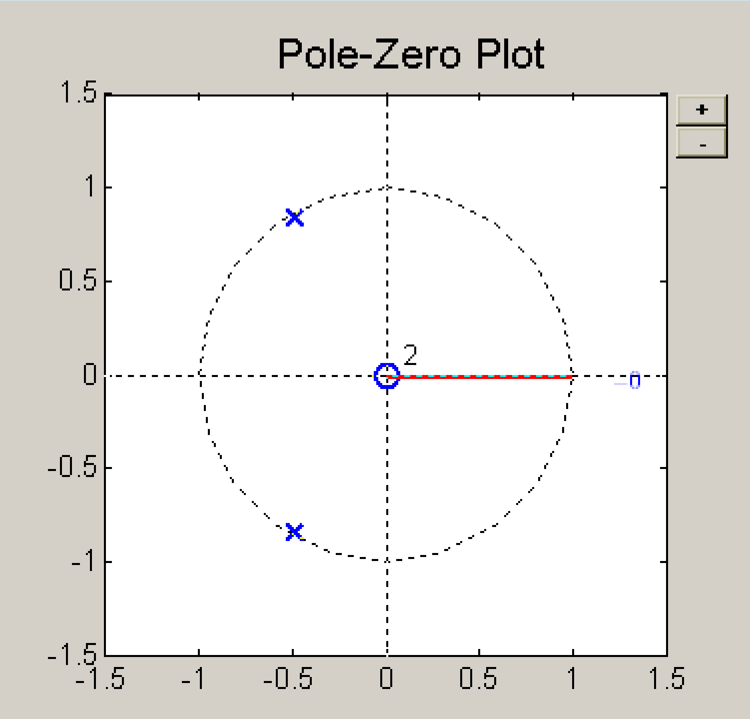
$\boxed{2 \text{ Poles}: z = 0.97e^{\pm j2\pi/3}}$ $\boxed{2 \text{ Zeros}: z = 0,0}$

- MATLAB: `roots( )` and `poly( )`
  - `roots( [1, 0.97, 0.9409] )`
  - `poly( 0.97*exp(j*2*pi*[1,-1]/3) )`

MOVIE for H(z) in 3-D

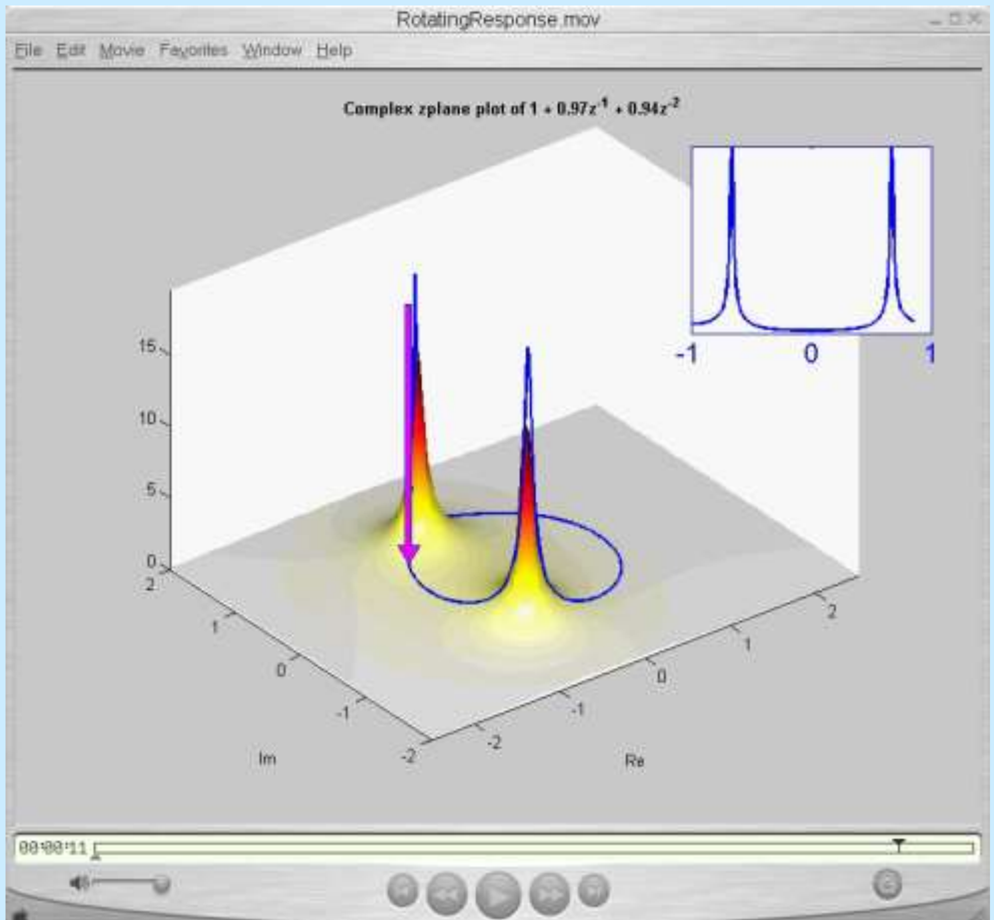
- POLES to H(z) to Frequency Reponse
  - TWO POLES SHOWN





Frequency Response from H(z)

Walking around the Unit Circle



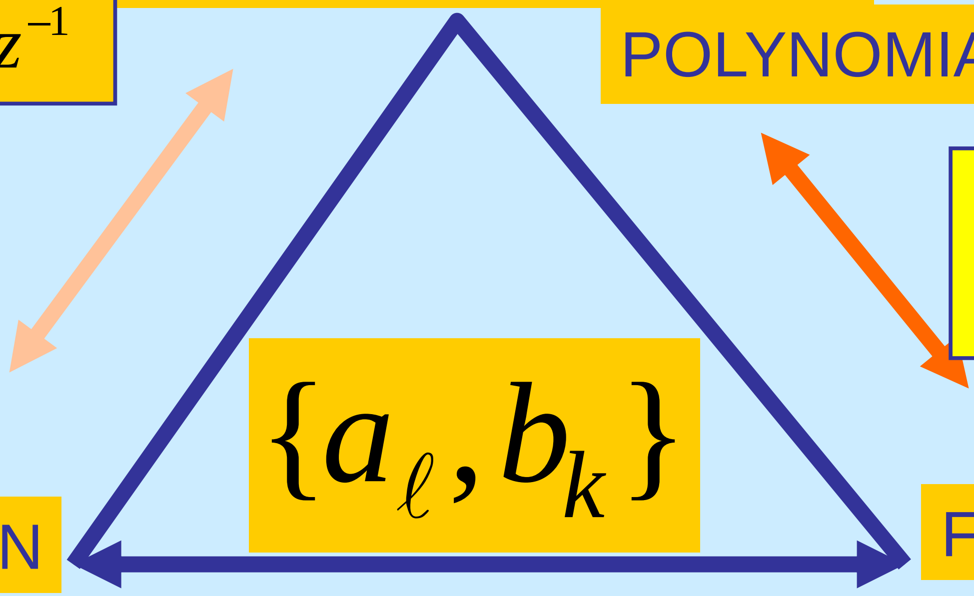
THREE DOMAINS: $H(e^{j\hat{\omega}})$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

Z-TRANSFORM-DOMAIN

POLYNOMIALS: H(z)

$$z = e^{j\hat{\omega}}$$



TIME-DOMAIN

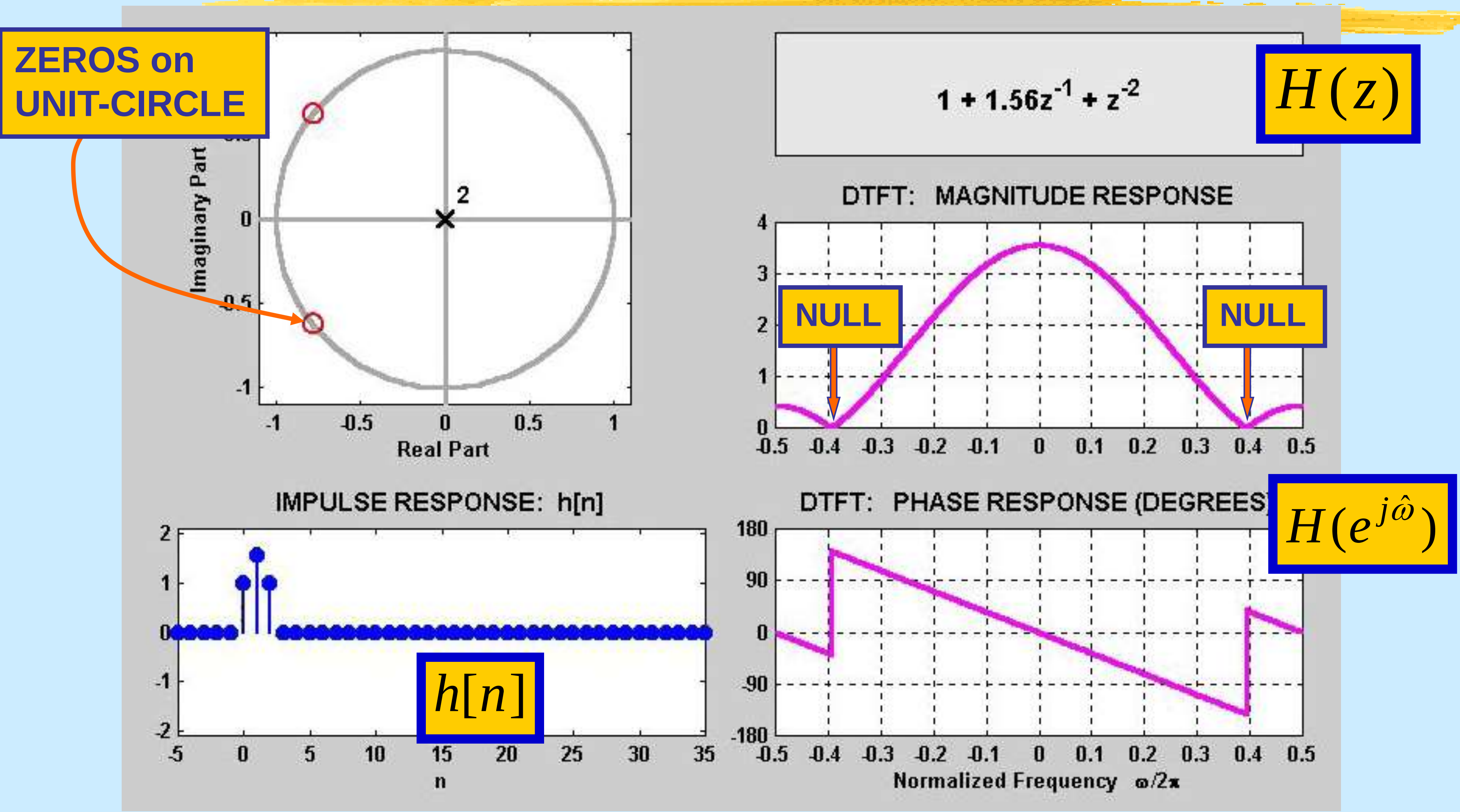
FREQ-DOMAIN

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

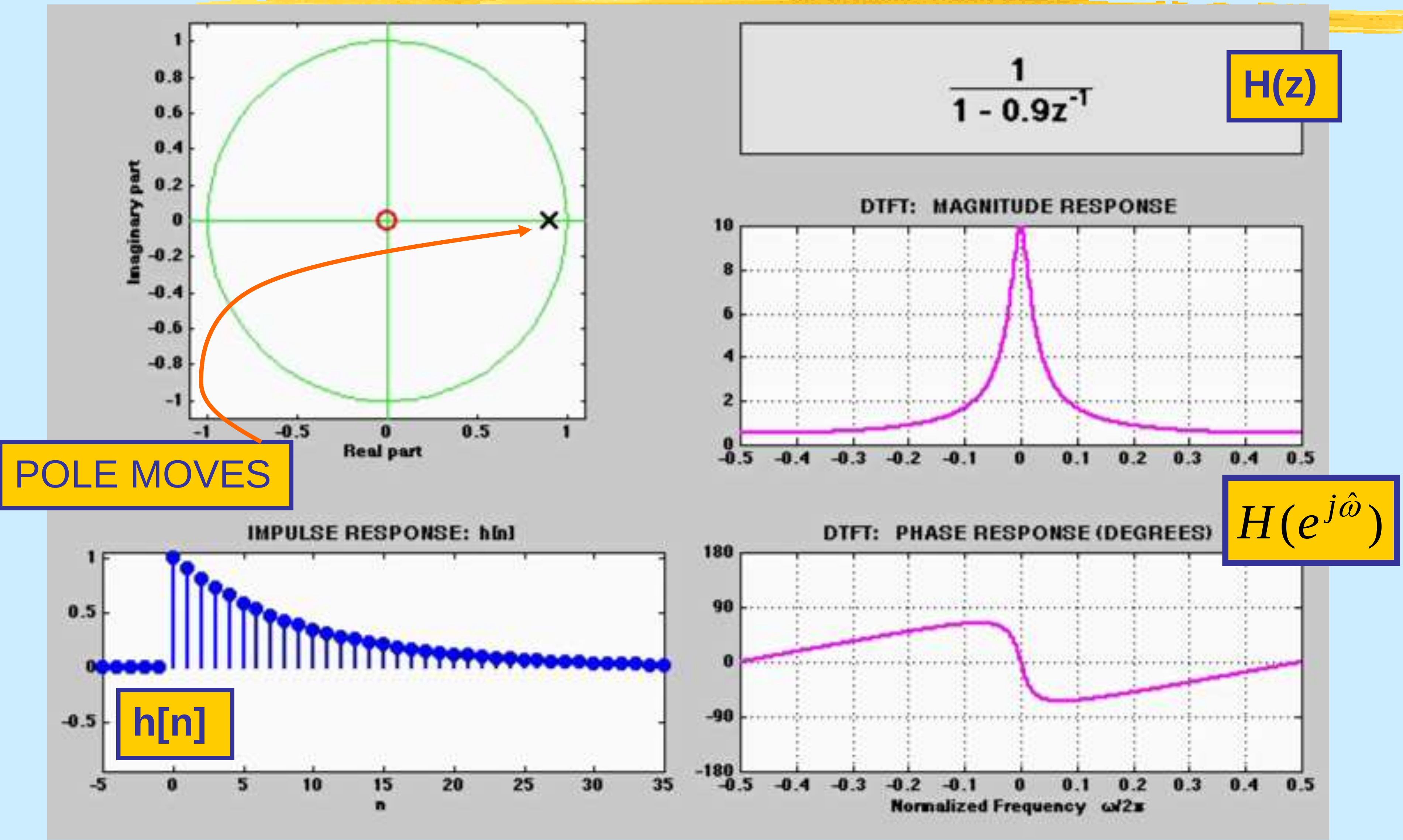
Impulse response, h[n]

$$H(e^{j\hat{\omega}}) = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

3 DOMAINS MOVIE: FIR



3 DOMAINS MOVIE: IIR



7 IIR MOVIES @ WEBSITE

- `http://dspfirst.gatech.edu/chapters/08feedbac/demos/3_domain/index.html`
- 3 DOMAINS MOVIES: IIR Filters
  - One pole moving and a zero at the origin
  - One pole and one zero; both moving
  - Two complex-conjugate poles moving radially
  - Two complex-conjugate poles moving in angle
  - Movement of a zero in a two-pole Filter
  - Radial Movement of Two out of Four Poles
  - Angular Movement of Two out of Four Poles

Reminder:
4 FIR MOVIES @ WEBSITE

- `http://dspfirst.gatech.edu/chapters/08feedbac/demos/3_domain/index.html`
- 3 DOMAINS MOVIES: FIR Filters
  - Two zeros moving around UC and inside
  - Three zeros; one held fixed at $z = -1$
  - Ten zeros; 9 equally spaced around UC; one moving
  - Ten zeros; 8 equally spaced around UC; two moving

Remove Interference

- Design a NOTCH filter (Find a_k and b_k)
  - To **Reject** completely 0.7π
    - This is NULLING
    - Zeros on UC

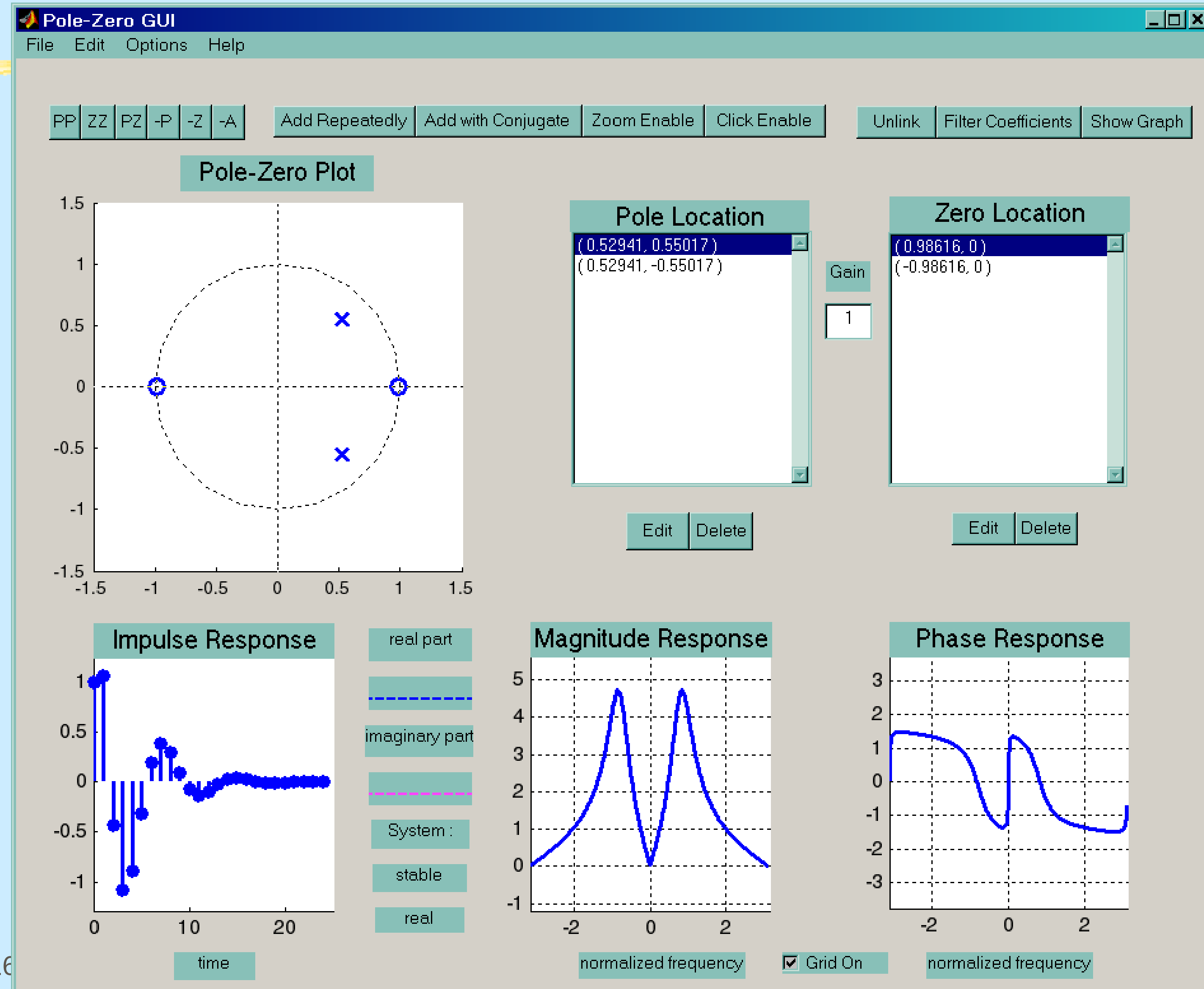
$$\boxed{2\ \text{Zeros}: z = e^{\pm j0.7\pi}}$$

  - Make the frequency response magnitude FLAT away from the notch.

$$\boxed{2\ \text{Poles}: z = 0.97e^{\pm j0.7\pi}}$$

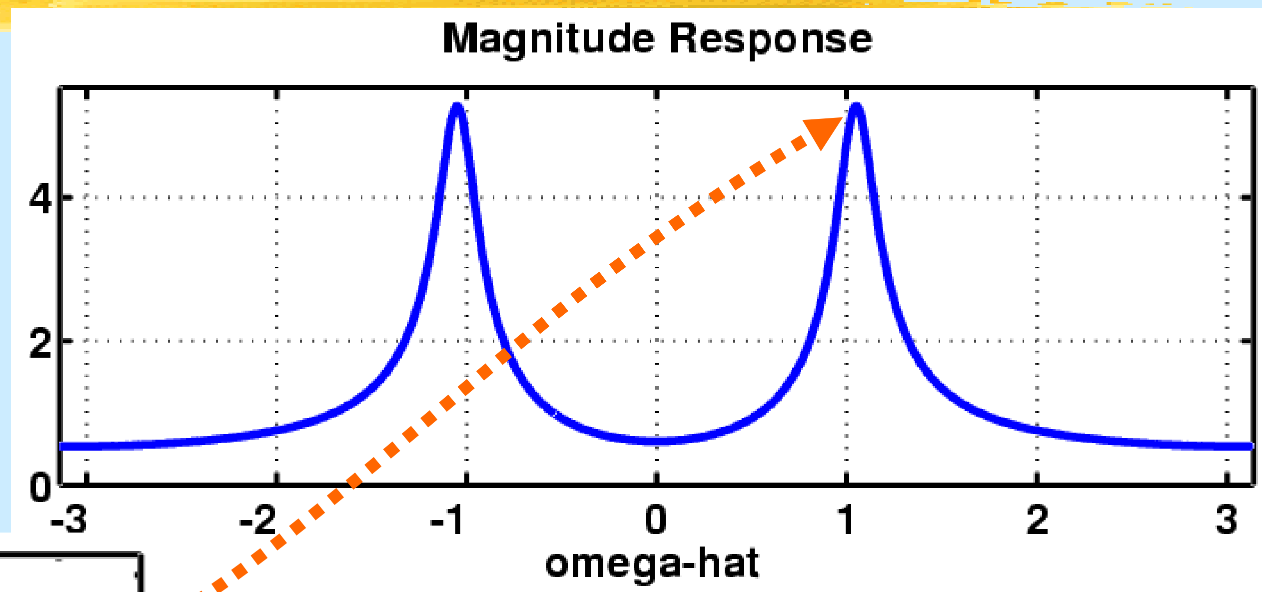
    - Use poles at the **same angle**
- Z-POLYNOMIALS provide the TOOLS
  - PEZDEMO GUI

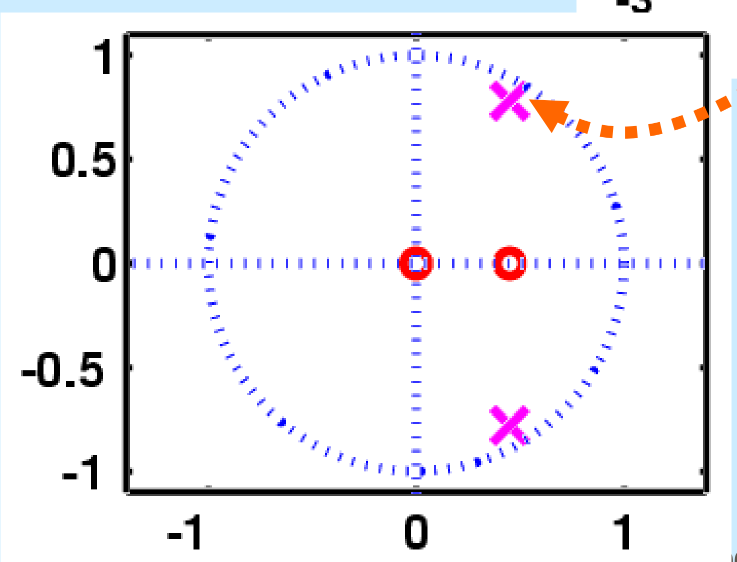
PeZ Demo: Pole-Zero Placing



Complex POLE-ZERO PLOT

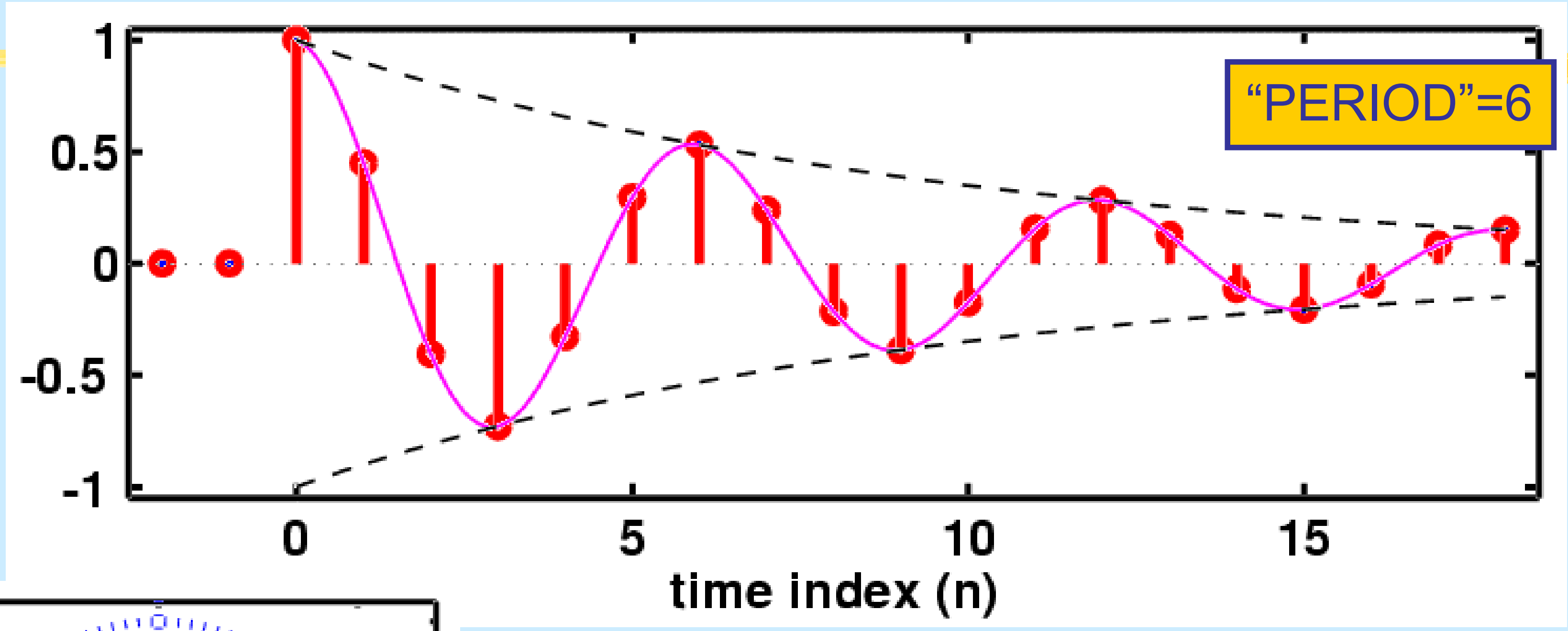
Where is the peak?

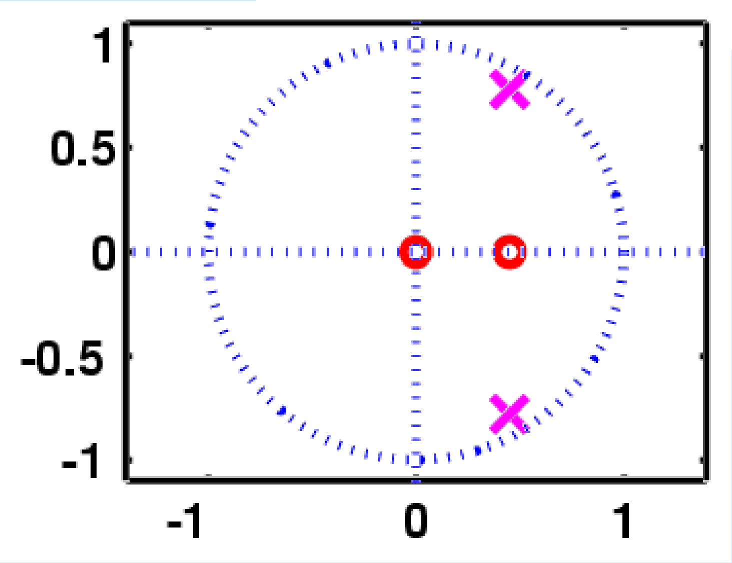




$$H(z) = \frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

h[n]: Decays & Oscillates

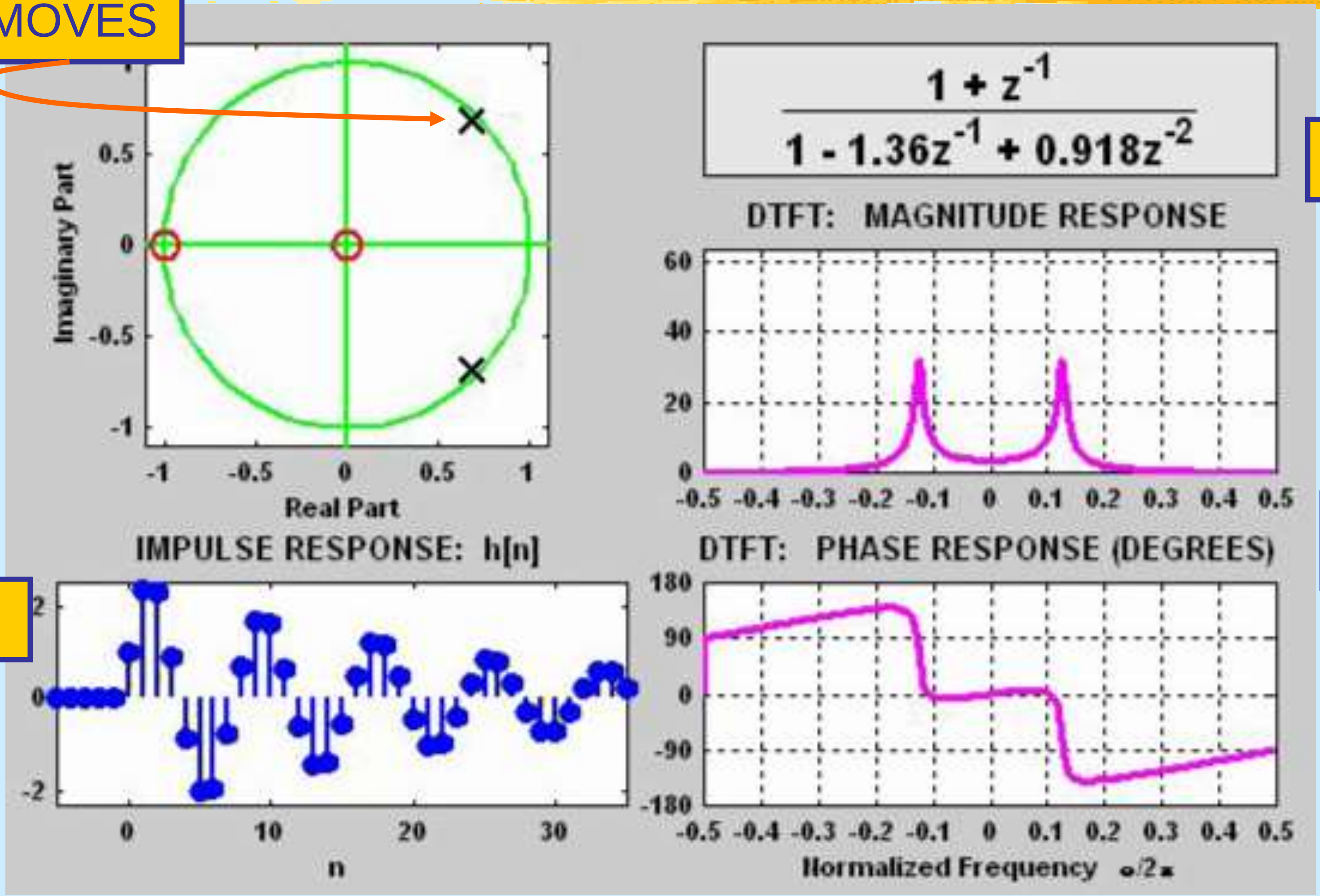




$$h[n] = (0.9)^n \cos(\pi n / 3) u[n]$$

3 DOMAINS MOVIE: IIR

POLE MOVES



H(z)

H(ω)

h[n]

SINUSOIDAL RESPONSE

- x[n] = SINUSOID => y[n] is SINUSOID
- Get MAGNITUDE & PHASE from H(z)

$$\boxed{\begin{aligned} &\text{if} \quad x[n] = e^{j\hat{\omega}n} \\ &\text{then} \quad y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n} \\ &\text{where} \quad H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} \end{aligned}}$$

POP QUIZ

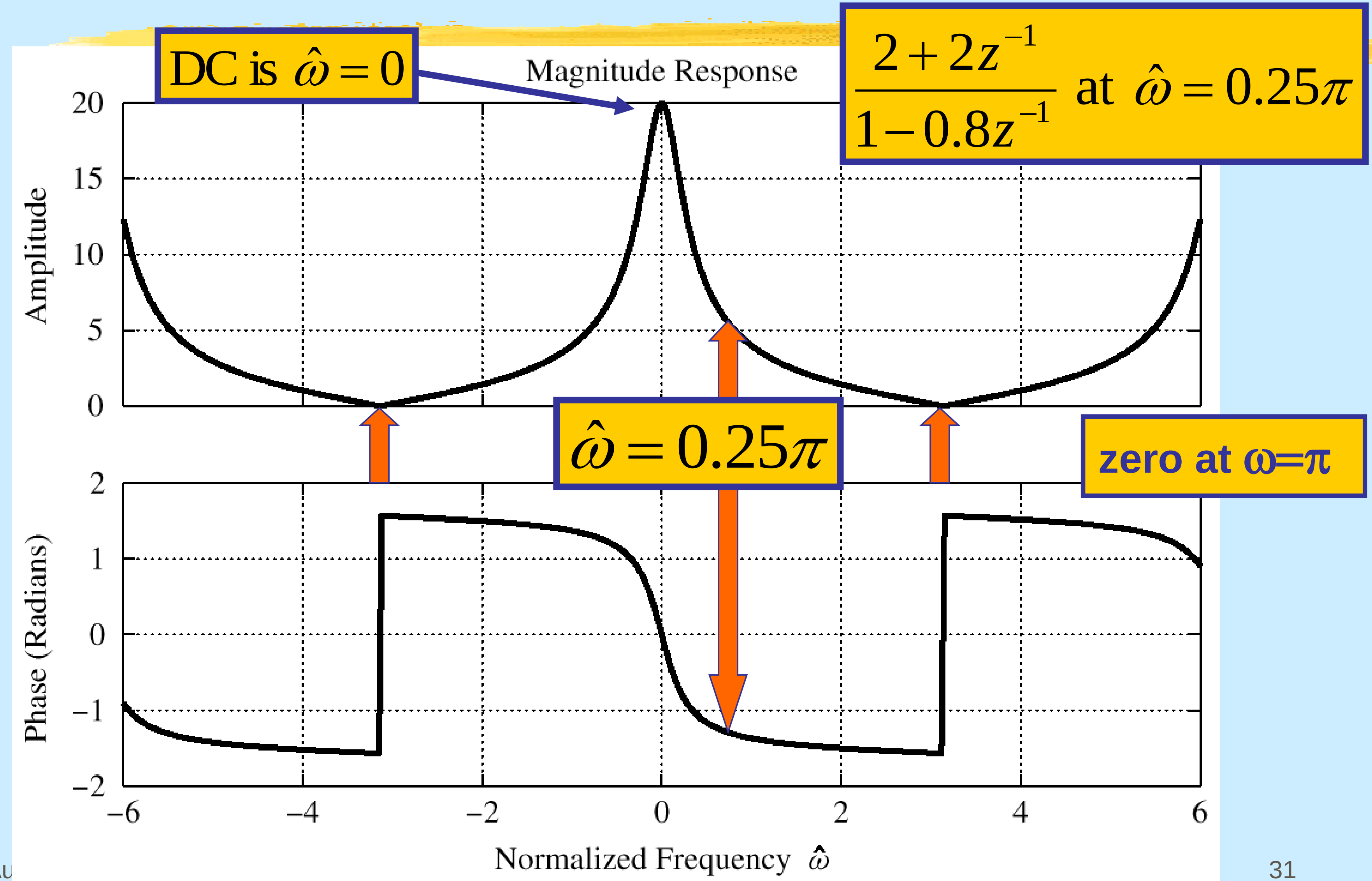
- Given:

$$\boxed{H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}}$$

- Find the Impulse Response, h[n]
- Find the output, y[n]
  - When

$$\boxed{x[n] = \cos(0.25\pi n)}$$

Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

- Given: $$H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}$$
- Find output, y[n], when $x[n]=\cos(0.25\pi n)$
  - Evaluate at $z=e^{j0.25\pi}$

$$H(z)=\frac{2+2(\frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2})}{1-0.8e^{-j0.25\pi}}=5.182e^{-j1.309}$$

$$y[n]=5.182\cos(0.25\pi n-0.417\pi)$$