# BLM3620 Digital Signal Processing*

Erkan Uslu

*euslu@yildiz.edu.tr*

Yıldız Technical University – Computer Engineering

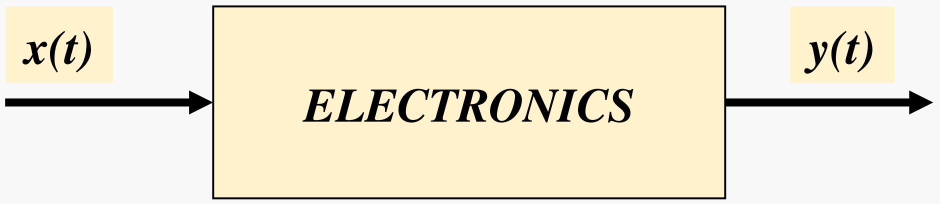
*Based on lecture notes from Ali Can Karaca & Ahmet Elbir

# Lecture #4 – Sampling and Aliasing

- Sampling
- Principal of Aliasing
- Spectrum of a Discrete-Time Signal
- Over-Sampling & Under-Sampling
- Stroboscopic effect

# Remember: Analog & Digital Systems

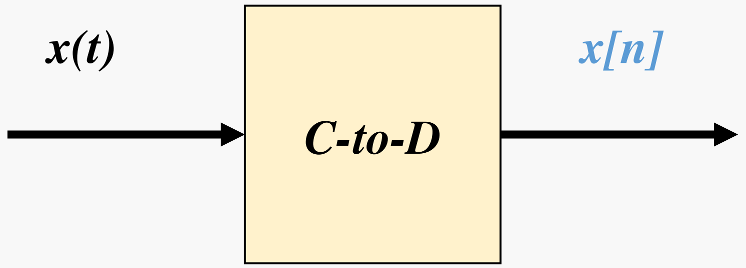
- ANALOG/ELECTRONIC:
  - Circuits: resistors, capacitors, op-amps




- DIGITAL/MICROPROCESSOR
  - Convert x(t) to numbers stored in memory

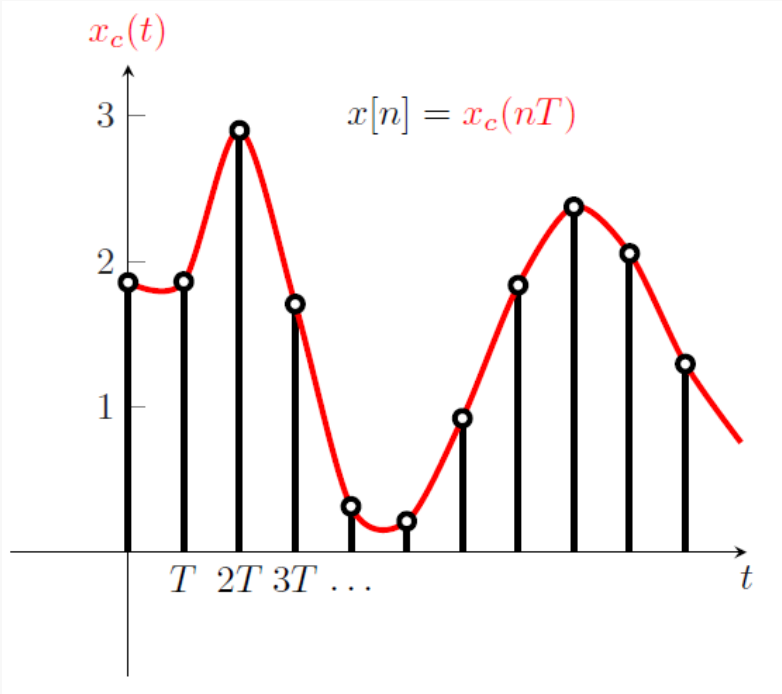
# Sampling of Analog Signals

- SAMPLING PROCESS
  - Convert x(t) to numbers x[n]
  - “n” is an integer index; x[n] is a sequence of values
  - Think of “n” as the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$

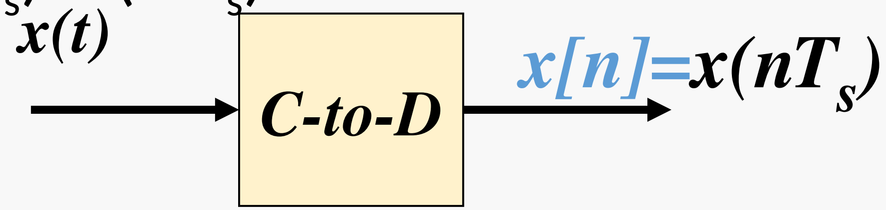


# Sampling of Analog Signals

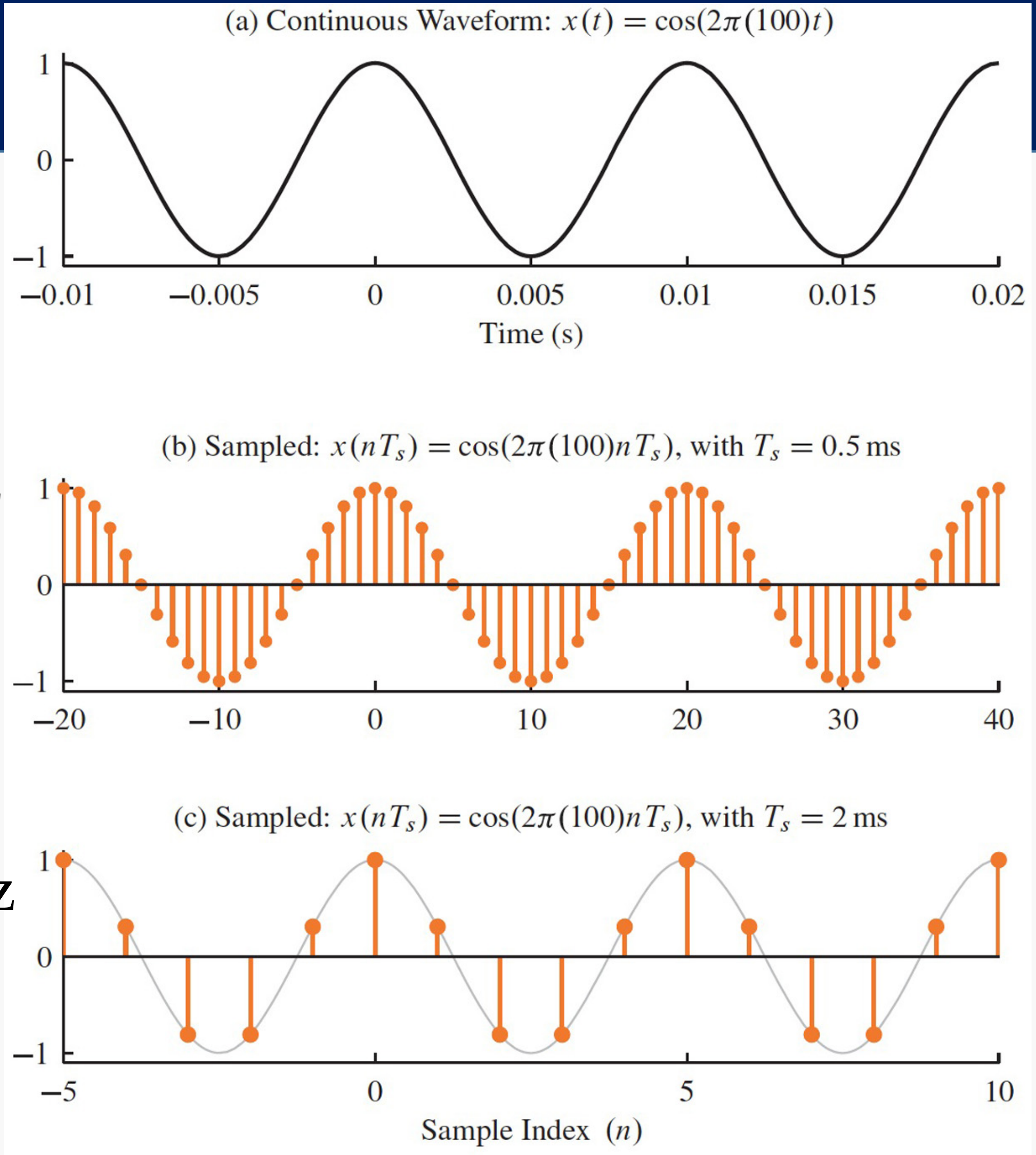
- SAMPLING RATE ($f_s$)
  - $f_s = 1/T_s$
    - NUMBER of SAMPLES PER SECOND
  - $T_s$ = 125 microsec $\Rightarrow$ $f_s$ = 8000 samples/sec
    - UNITS of $f_s$ ARE HERTZ: 8000 Hz




- UNIFORM SAMPLING at $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s)=x(n/f_s)$



$f = 100\text{Hz}$




$f_s = 2\,\text{kHz}$

$f_s = 500\text{Hz}$

Which one provides the most accurate representation of x(t)?

# Sampling Theorem

- HOW OFTEN DO WE NEED TO SAMPLE?
  - DEPENDS on FREQUENCY of SINUSOID
  - ANSWERED by SHANNON/NYQUIST Theorem
  - ALSO DEPENDS on “**RECONSTRUCTION**”

*Shannon Sampling Theorem*

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

# Nyquist Rate

- "**Nyquist Rate**" Sampling
  - $f_s$ > **TWICE** the HIGHEST Frequency in x(t)
  - "Sampling above the Nyquist rate"

- **BANDLIMITED SIGNALS**
  - DEF: HIGHEST FREQUENCY COMPONENT in SPECTRUM of x(t) is finite
  - NON-BANDLIMITED EXAMPLE
    - TRIANGLE WAVE is NOT BANDLIMITED

# Discrete-Time Sinusoid

- Change x(t) into x[n] **DERIVATION**

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi)$$

$$x[n] = A\cos((\omega T_s)n + \varphi)$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

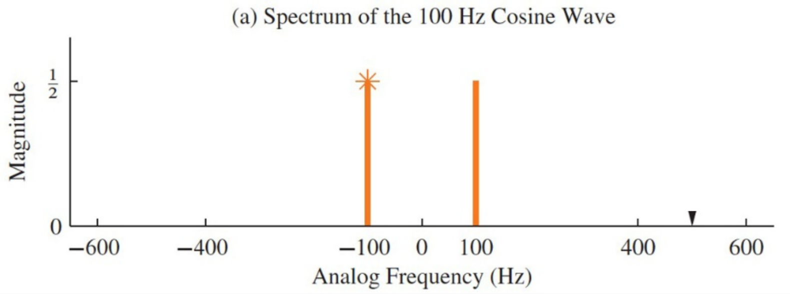
***DEFINE DIGITAL FREQUENCY***

# New Notion: Digital Frequency!

- $\hat{\omega}$ VARIES from **0** to **2π**, as *f* varies from 0 to the sampling frequency
- UNITS are radians, **not** rad/sec
  - – DIGITAL FREQUENCY is NORMALIZED

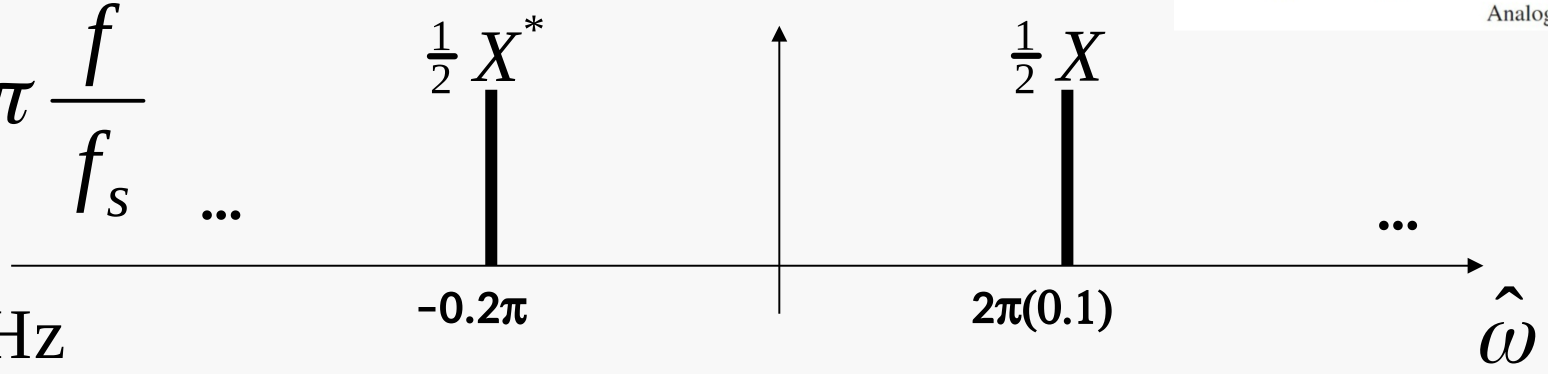
$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}$$

# Spectrum of Digital Signal

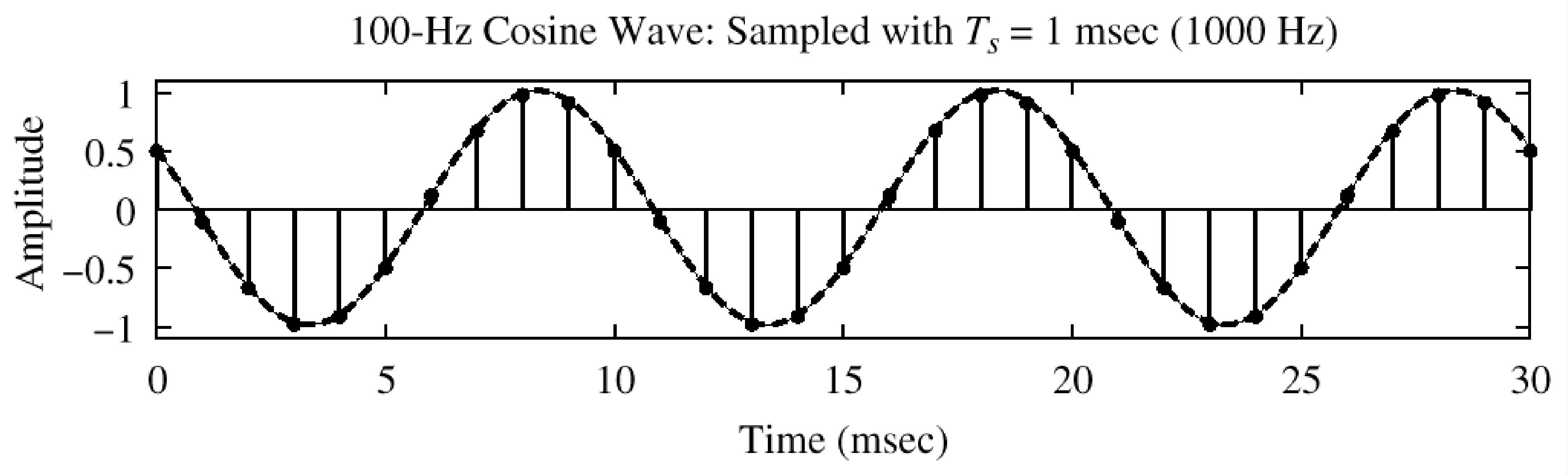



$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$



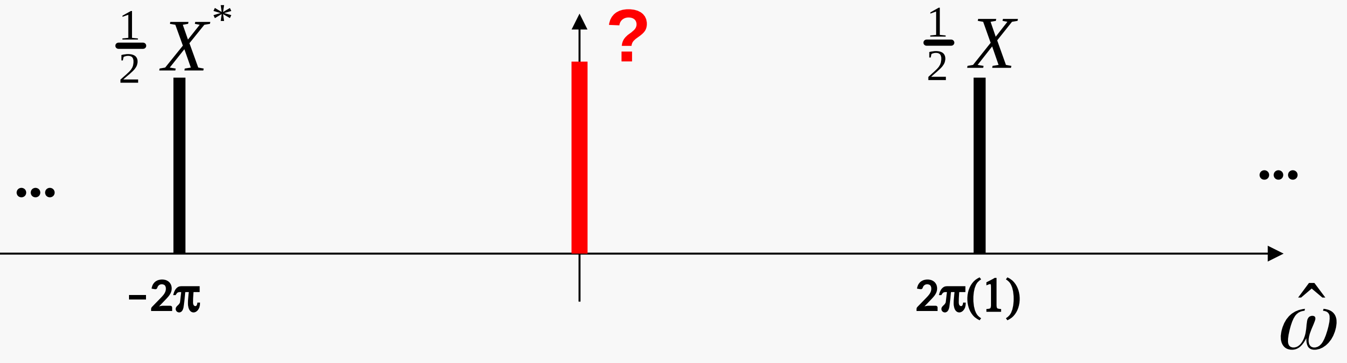

$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$



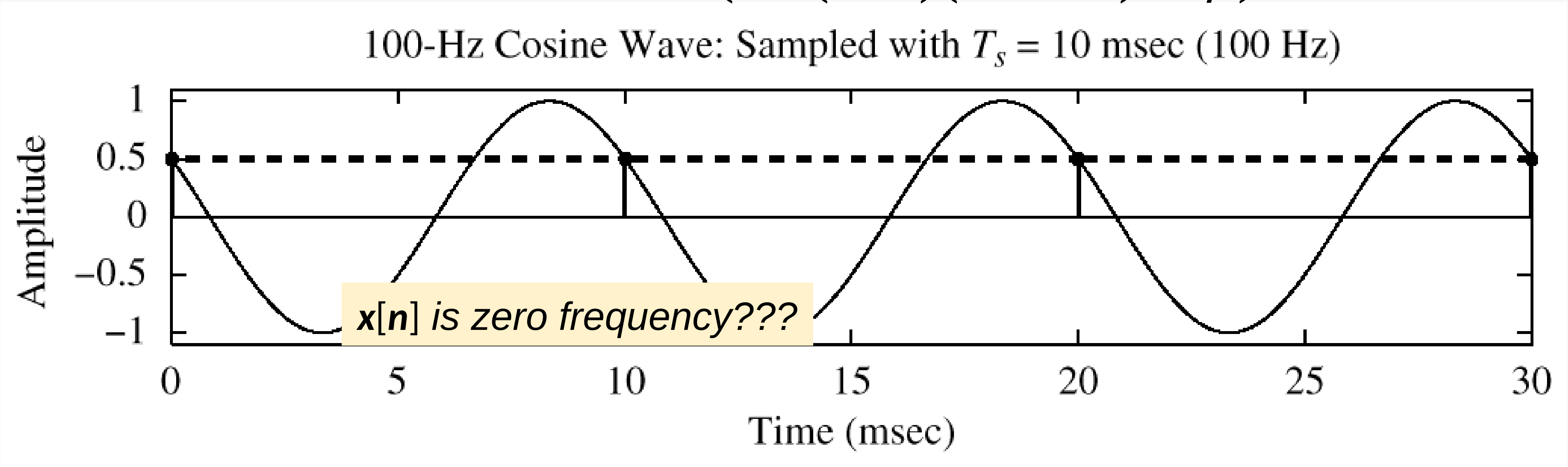
# Spectrum of Digital Signal

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 100\text{ Hz}$$




$$x[n] = A\cos(2\pi(100)(n/100) + \varphi)$$



# The Rest of the Story

- Spectrum of x[n] has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**

- SPECTRUM is PERIODIC with period = **2π**
  - Because

$$A\cos(\hat{\omega}n+\varphi)=A\cos((\hat{\omega}+2\pi\ell)n+\varphi)$$

# Example for clarification

- Other Frequencies give the same

$$\hat{\omega}$$

$$x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_1[n] = \cos(400\pi \tfrac{n}{1000}) = \cos(0.4\pi n)$$

$$x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_2[n] = \cos(2400\pi \tfrac{n}{1000}) = \cos(2.4\pi n)$$

$$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$$

$$\Rightarrow \; x_2[n] = x_1[n] \qquad\qquad 2400\pi - 400\pi = 2\pi(1000)$$

**GIVEN x[n], we CAN'T KNOW whether it came from a sinusoid at $f_o$ or ($f_o + f_s$) or ($f_o$ + $2f_s$) ...**

# Digital Frequency Repeats for each Every 2pi

- Other Frequencies give the same $\hat{\omega}$

$$\text{If} \quad x(t) = A\cos(2\pi(\underline{f + \ell f_s})t + \varphi) \qquad t \leftarrow \frac{n}{f_s}$$

$$\text{and we want :} \quad x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\text{then :} \hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$$

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

Normalized Frequency

# DIGITAL FREQ AGAIN $\hat{\omega}$

**Normalized Radian Frequency**

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

**ALIASING**

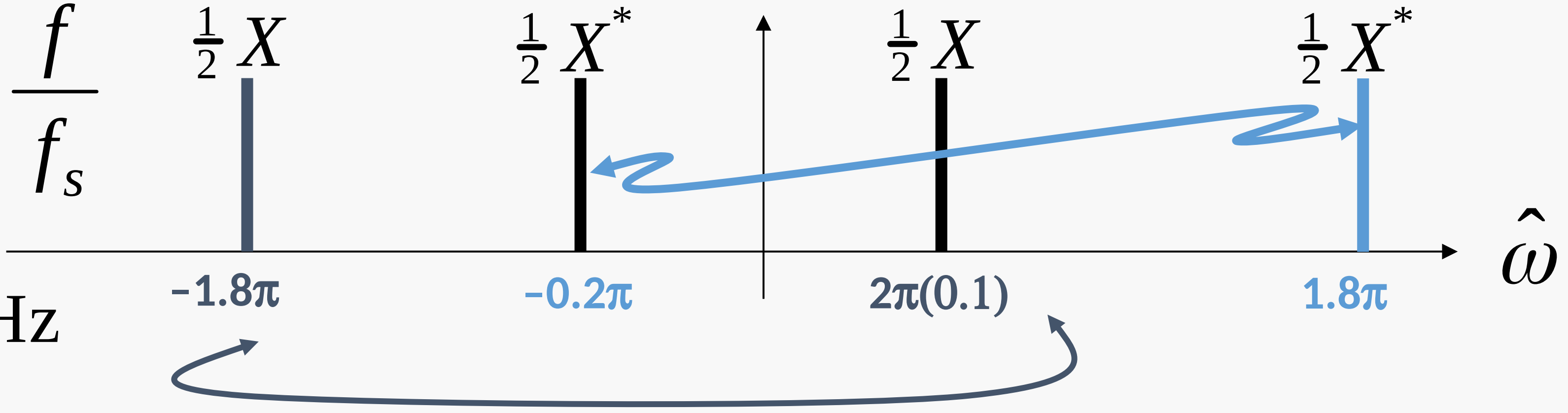
$$\hat{\omega} = \omega T_s = -\frac{2\pi f}{f_s} + 2\pi \ell$$

**FOLDED ALIAS**

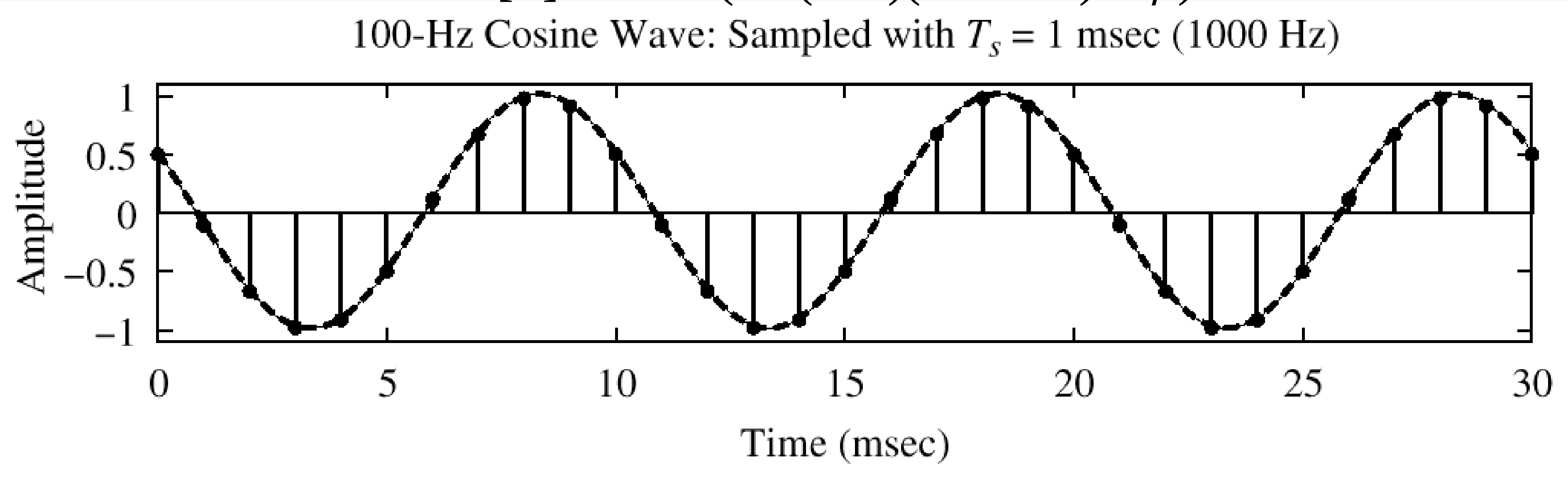
# Example Spectrum-1

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$




$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$




Principal alias :

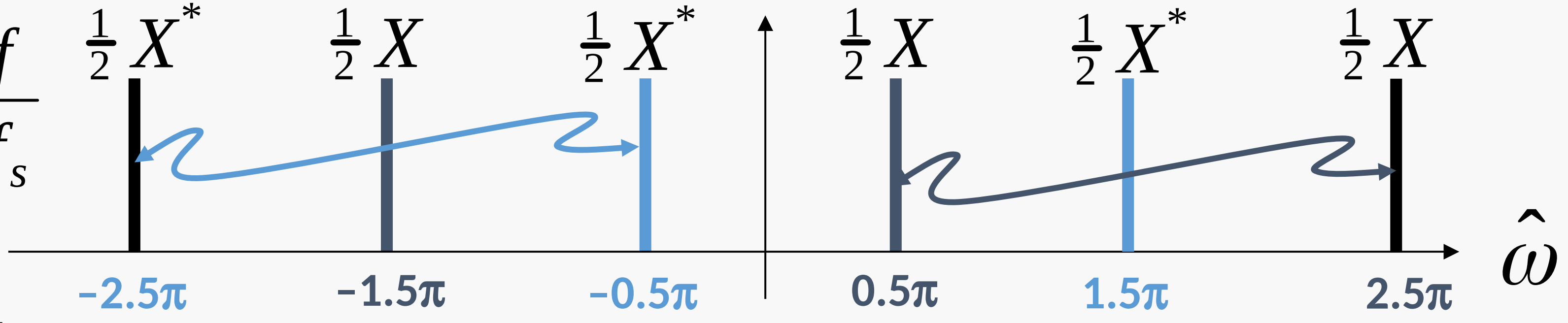
$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.1(1000) = 100\,\text{Hz}$$

$$x(t) = A\cos(2\pi 100 t + \varphi)$$

# Aliasing Example Spectrum-2

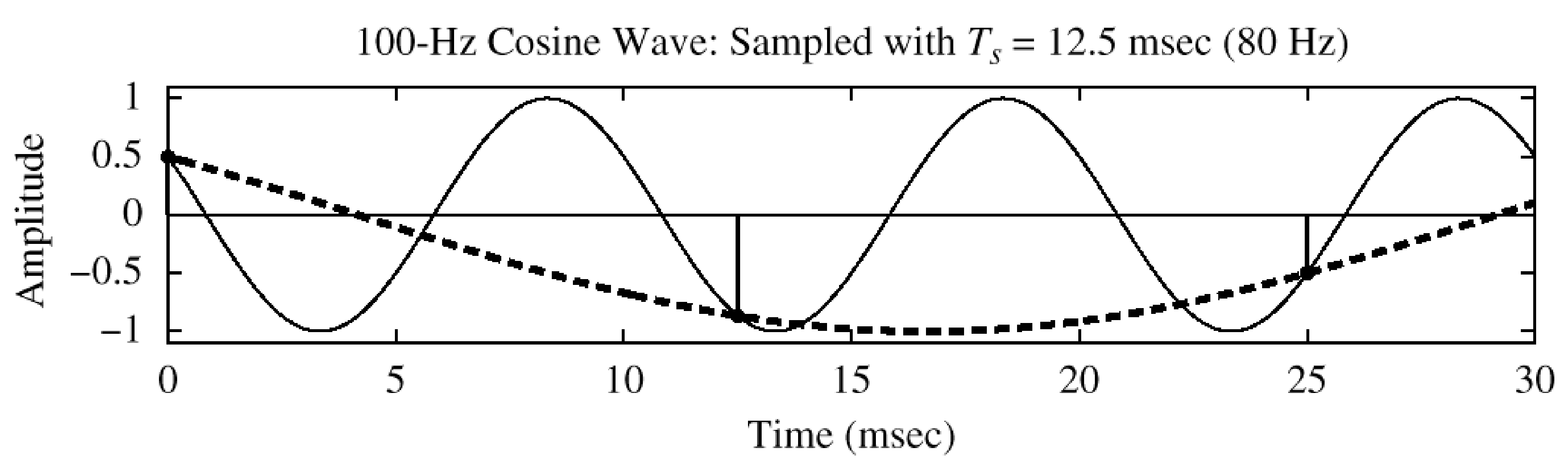
Principal alias is always between $-\pi \le \hat{\omega} \le \pi$

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$




$f_s = 80\,\text{Hz}$

$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$




Principal alias :

$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.25(80) = 20\,\text{Hz}$$

$$x(t) = A\cos(2\pi 20 t + \varphi)$$

# From the book (the same example, more clear)

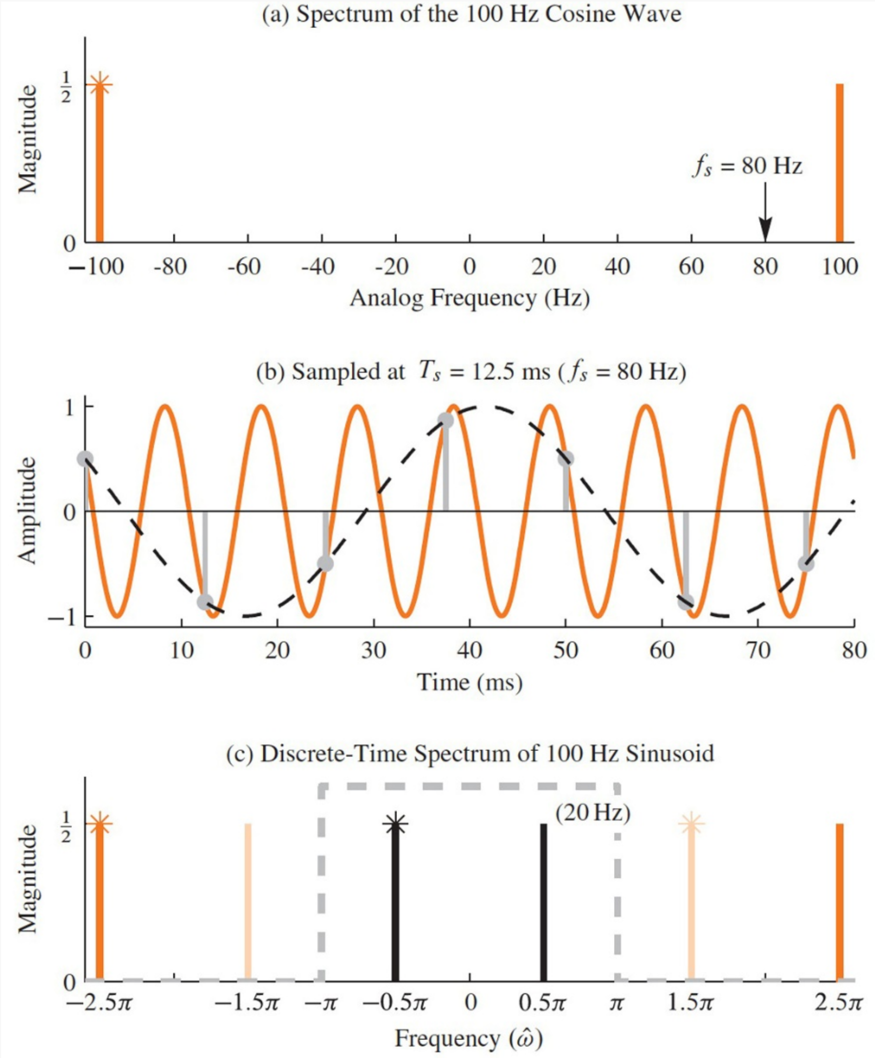



**Figure 4-9** *Under-sampling* a 100 Hz sinusoid at $f_s$ = 80 samples/s.

(a) Continuous-time spectrum;

(b) time-domain plot, showing the samples $x[n]$ as gray dots, the original signal $x(t)$ as a continuous **orange** line, and the reconstructed signal $y(t)$ as a dashed black line, which is a 20 Hz sinusoid passing through the same sample points; and

(c) discrete-time spectrum plot, showing the positive and negative frequency components of the original sinusoid at = $\pm 2.5\pi$ rad, along with two sets of alias components.

# FOLDING (a type of ALIASING)

- EXAMPLE: 3 different x(t); same x[n]

$$f_s = 1000 \qquad \hat{\omega} = 2\pi \frac{100}{1000} = 2\pi(0.1)$$

$$\cos(2\pi(100)t) \rightarrow \cos[2\pi(0.1)n]$$

$$\cos(2\pi(1100)t) \rightarrow \cos[2\pi(1.1)n] = \cos[2\pi(0.1)n]$$

$$\cos(2\pi(900)t) \rightarrow \cos[2\pi(0.9)n]$$

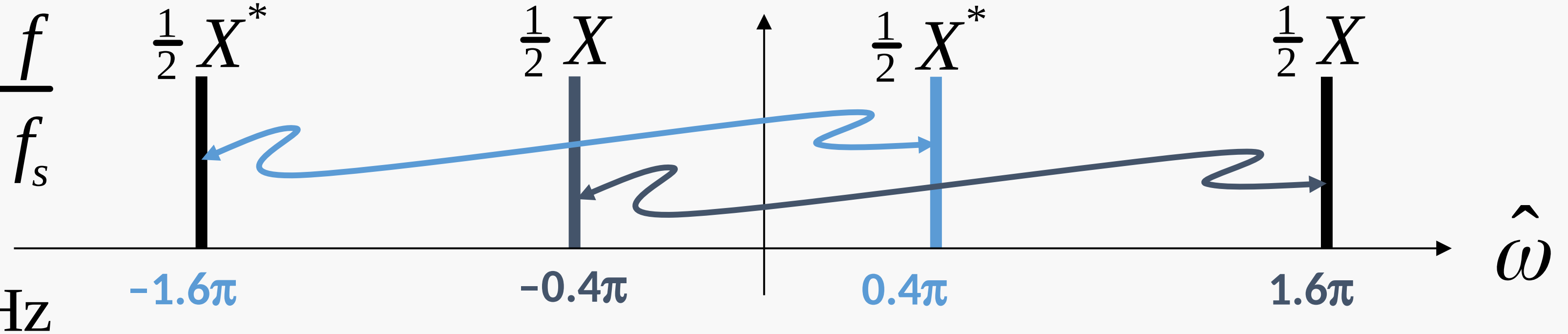
$$= \cos[2\pi(0.9)n - 2\pi n] = \cos[2\pi(-0.1)n] = \cos[2\pi(0.1)n]$$

- 900 Hz "folds" to 100 Hz when $f_s$=1kHz

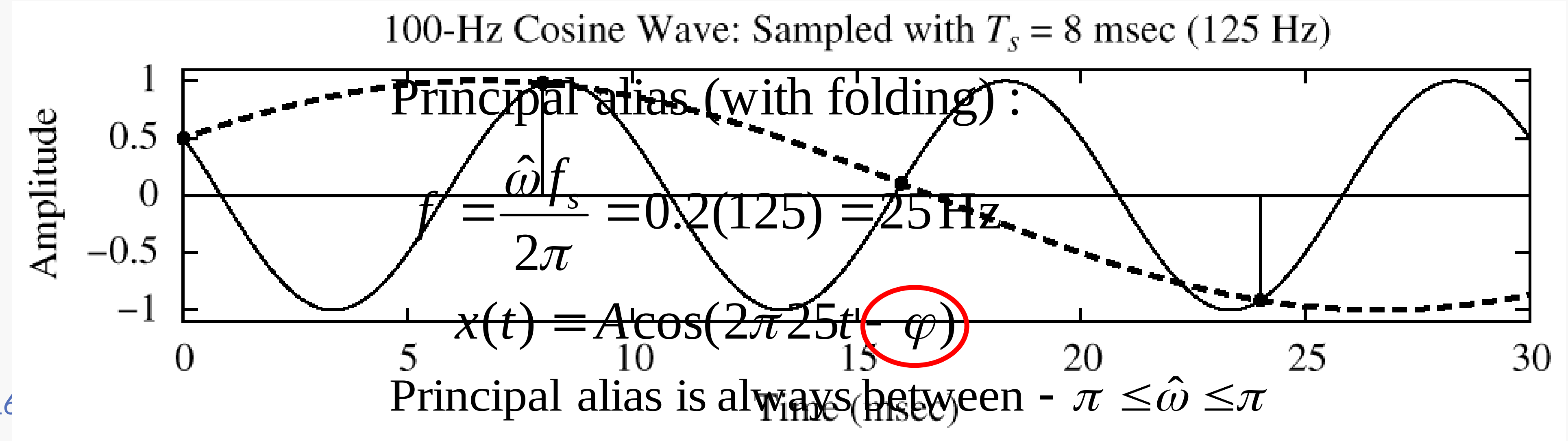
# Example Folding Case

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 125\text{Hz}$$




$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$




Principal alias (with folding) :

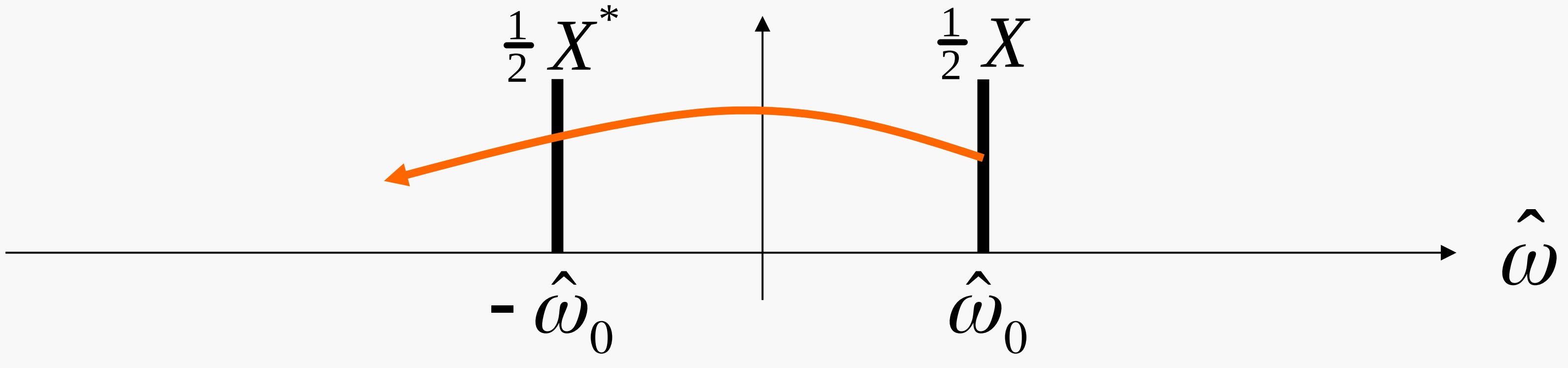
$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.2(125) = 25\text{ Hz}$$

$$x(t) = A\cos(2\pi 25t - \varphi)$$

Principal alias is always between $-\pi \le \hat{\omega} \le \pi$

# SPECTRUM Explanation of SAMPLING THEOREM

- How do we prevent aliasing?
- Guarantee original signal is principal alias:



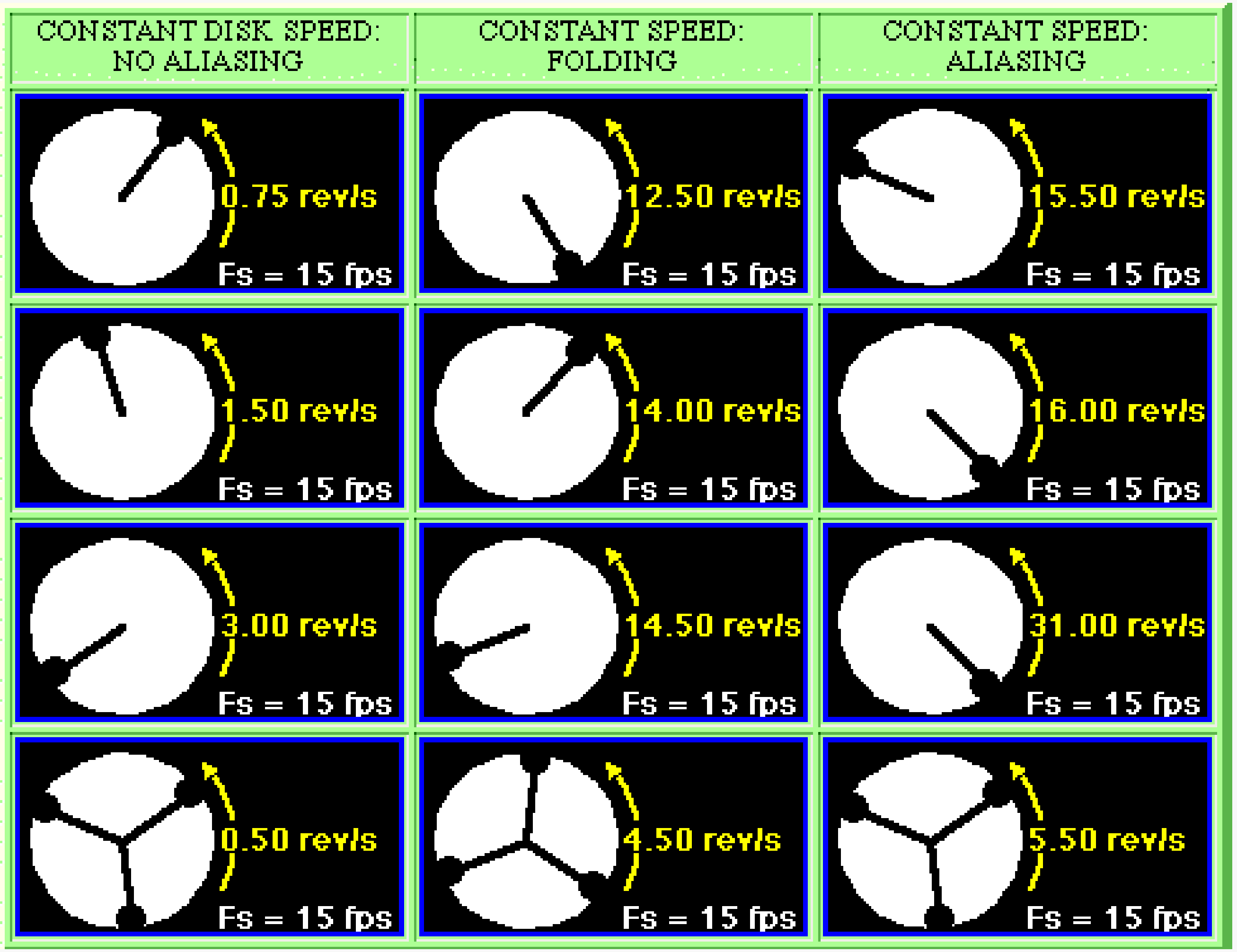

$$\hat{\omega}_0 - 2\pi < -\hat{\omega}_0 \;\Rightarrow\; \hat{\omega}_0 < \pi$$

$$\hat{\omega}_0 = \frac{2\pi f_0}{f_s} < \pi \;\Rightarrow\; f_0 < \frac{f_s}{2}$$

# Be Careful:

https://www.youtube.com/watch?v=qgvuQGY946g

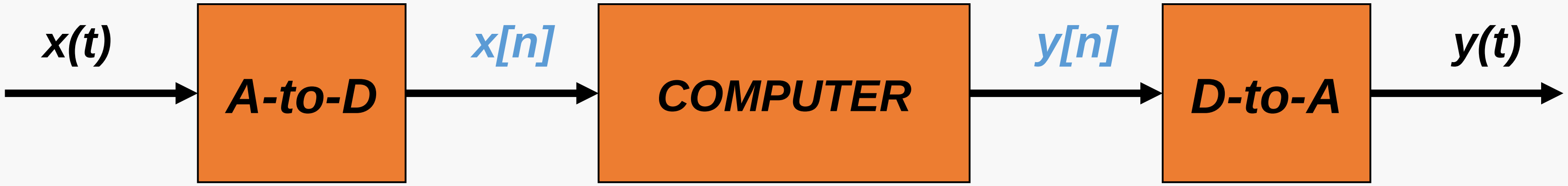
# Strobe Demo




https://dspfirst.gatech.edu/chapters/04samplin/demos/strobe/index.html

https://dspfirst.gatech.edu/chapters/04samplin/demos/synstrob/index.html

# Digital to Analog Reconstruction

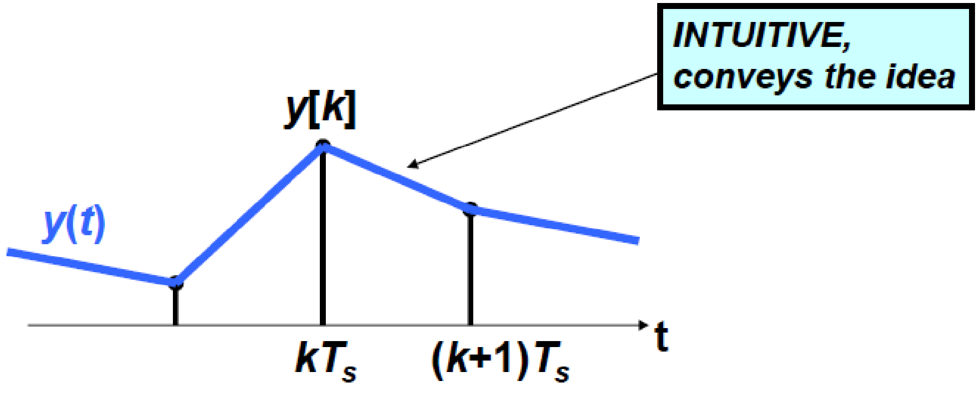



- Create continuous y(t) from y[n]
  - **IDEAL D-to-A:**
    - If you have formula for y[n]
  - Invert sampling ($t=nT_s$) by $n=f_s t$
  - $y[n] = A\cos(0.2\pi n+\phi)$ with $f_s = 8000$ Hz
  - $y(t) = A\cos(0.2\pi(8000t)+\phi) = A\cos(2\pi(800)t+\phi)$

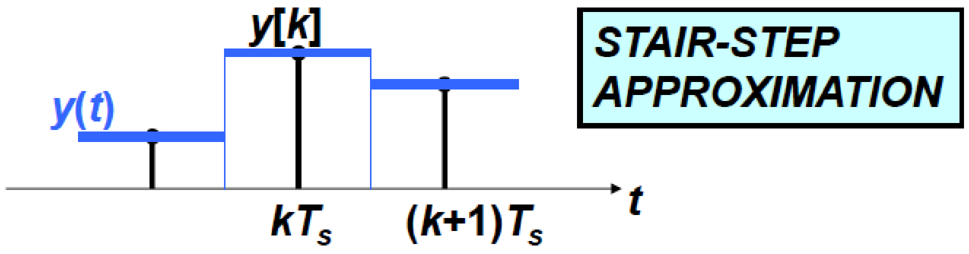
# Reconstruction

- RECONSTRUCT THE SMOOTHEST ONE
  - THE LOWEST FREQ, if $y[n]$ = sinusoid

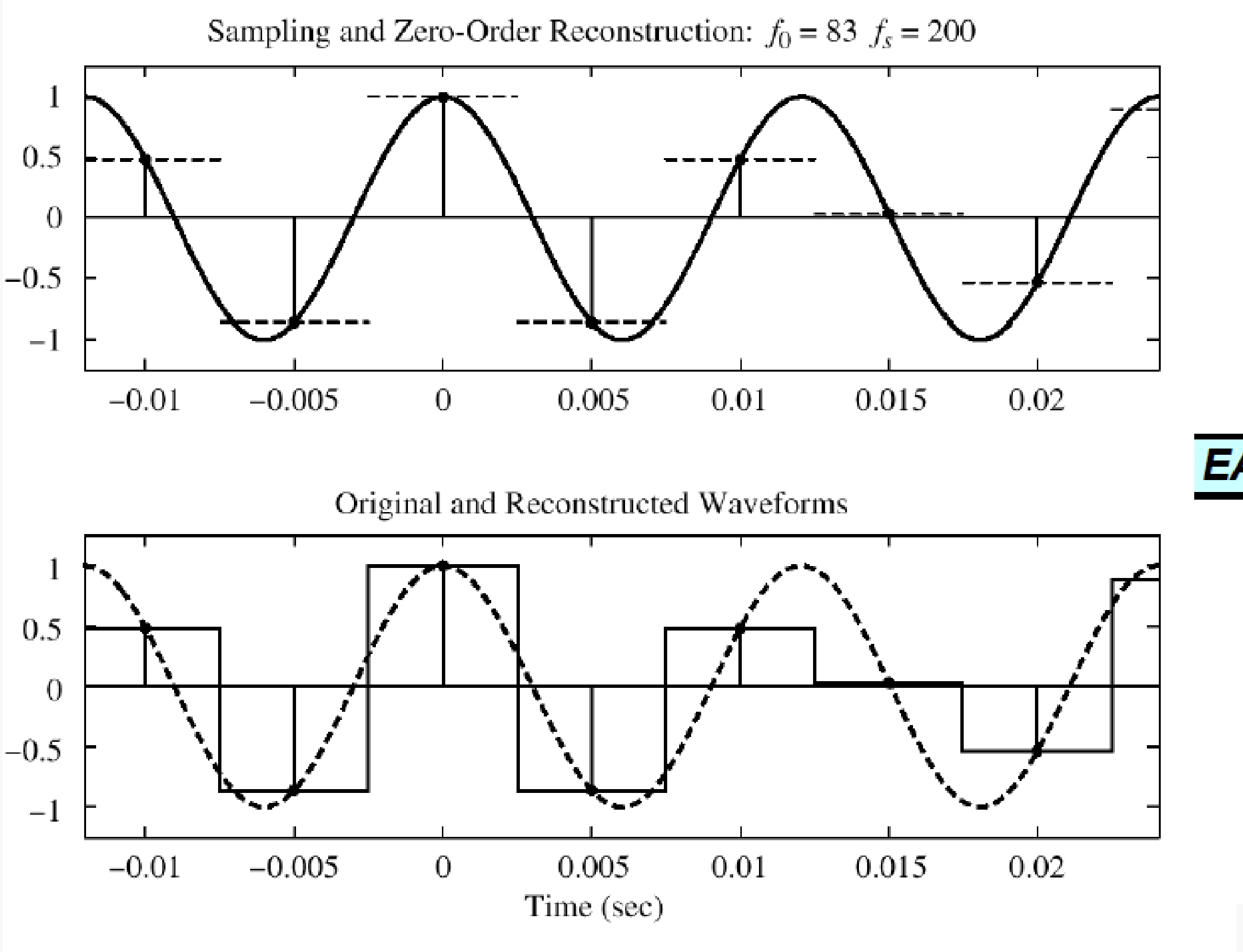
- CONVERT STREAM of NUMBERS to $x(t)$
- "CONNECT THE DOTS"
- INTERPOLATION



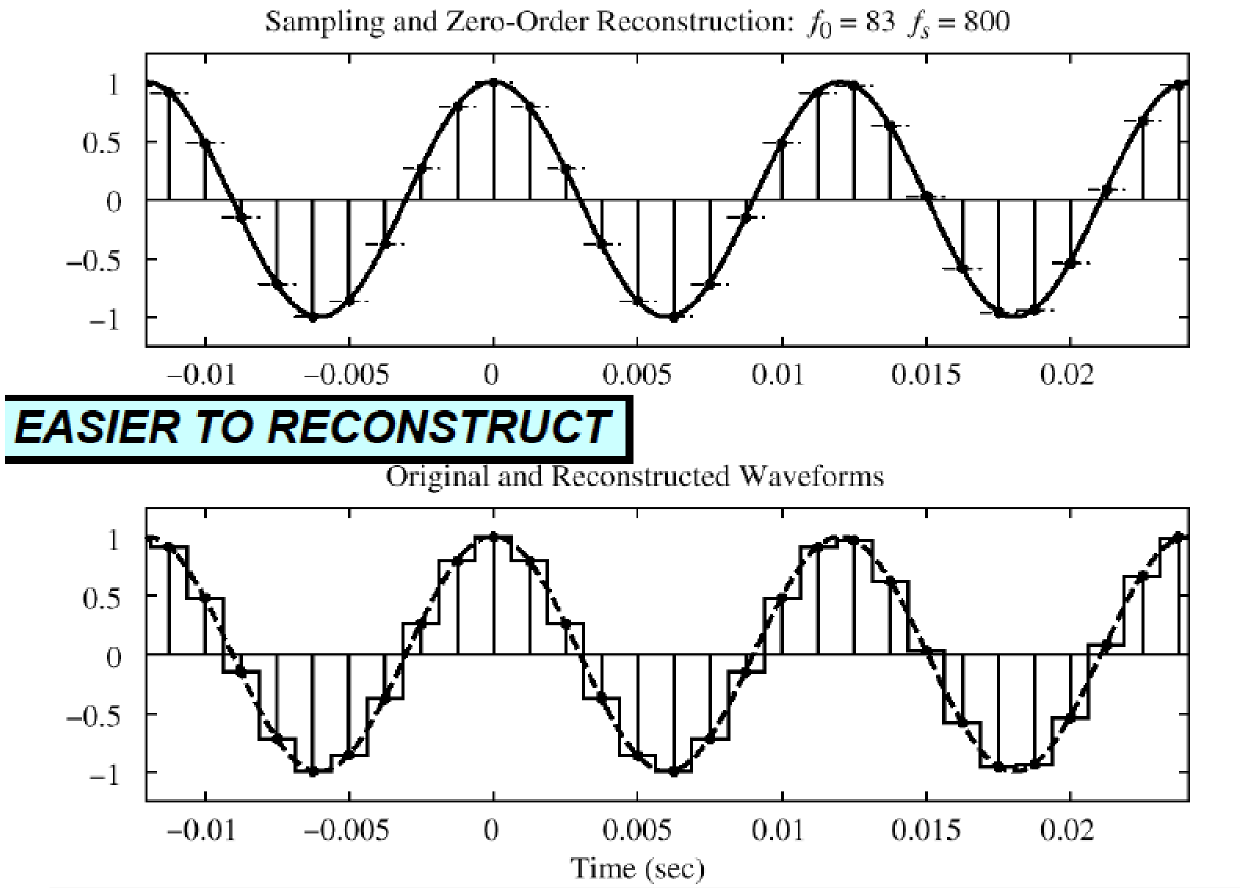

- CONVERT $y[n]$ to $y(t)$
  - $y[k]$ should be the value of $y(t)$ at $t = kT_s$
  - Make $y(t)$ equal to $y[k]$ for
    - $kT_s - 0.5T_s < t < kT_s + 0.5T_s$



# STAIR-STEP APPROXIMATION



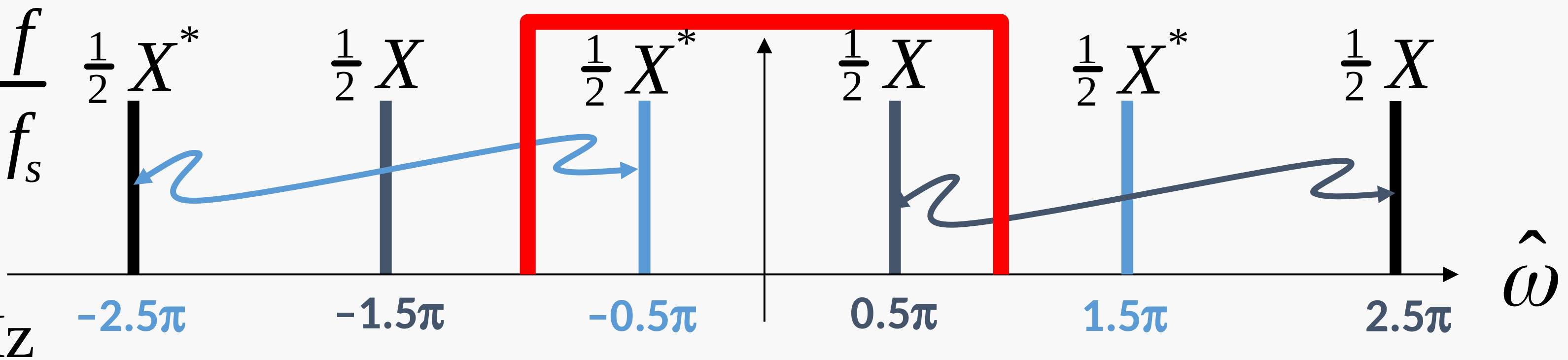

## OVER-SAMPLING CASE



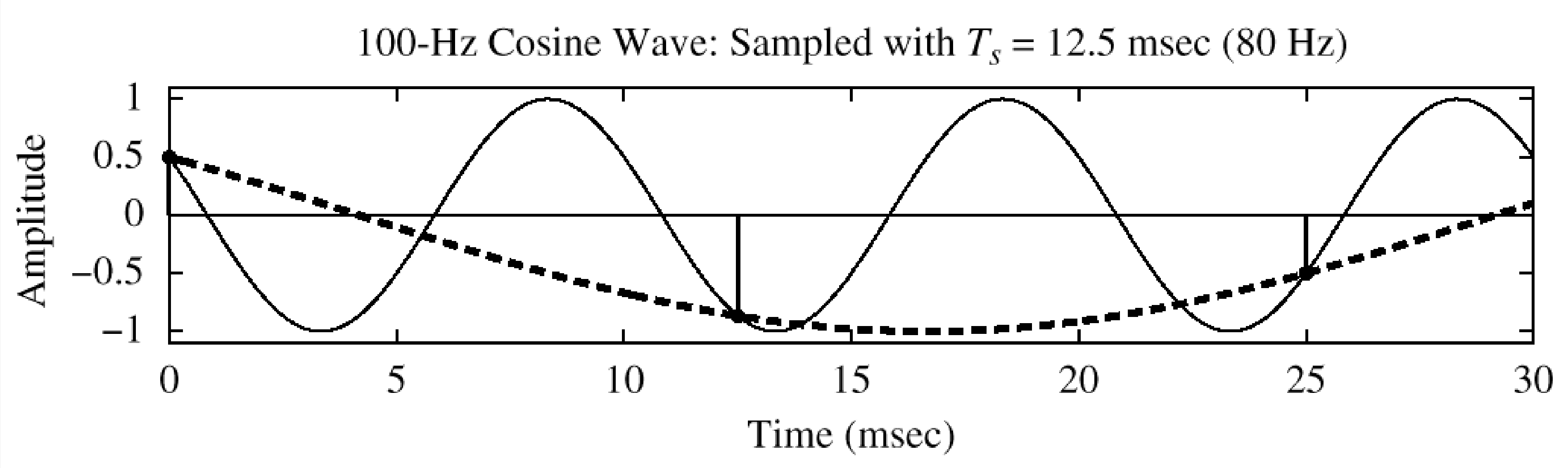
# SPECTRUM (ALIASING CASE)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

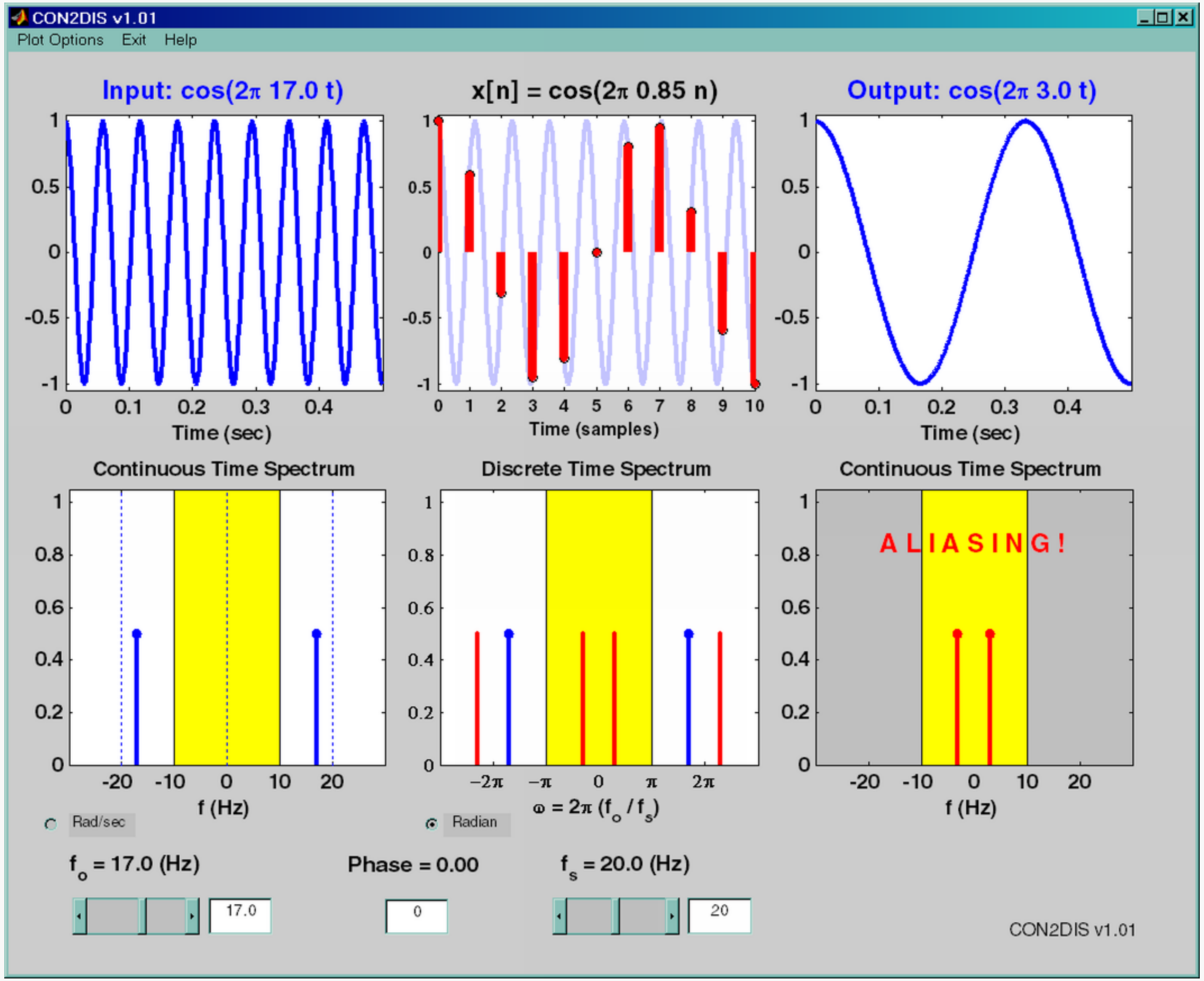
$$f_s = 80\text{Hz}$$




$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$

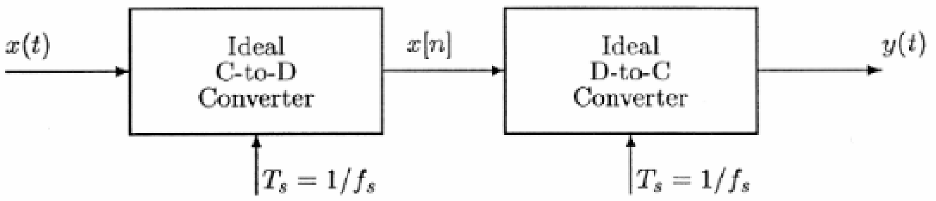


# Let’s Analyze with MATLAB




https://dspfirst.gatech.edu/matlab/#con2dis

# Example - 1




Suppose that the $\boxed{\text{output}}$ of the D-to-C converter in the system above is found to be

$$y(t) = 2 + 10\cos(2\pi(150)t + \pi/3)$$

when the sampling rate is $f_s = 1/T_s = 400$ samples/second.

(a) Give an equation for $x[n]$ in terms of cosine functions. **Write your answer on the line below.**

With no aliasing, going through a D/C converter is the inverse of going through the C/D. Therefore, we can get x[n] by passing y(t) through a C/D converter.

**Answer:** $x[n] = 2 + 10\cos(2\pi(150)n/400 + \pi/3)$

$= 2 + 10\cos(3n\pi/4 + \pi/3)$

(b) Determine two *different* input signals $x(t) = x_1(t)$ and $x(t) = x_2(t)$ that could have produced the given output of the D-to-C converter. **All of the frequencies in your answers must be positive and less than 400 Hz. Write your answers for both inputs on the lines below.**

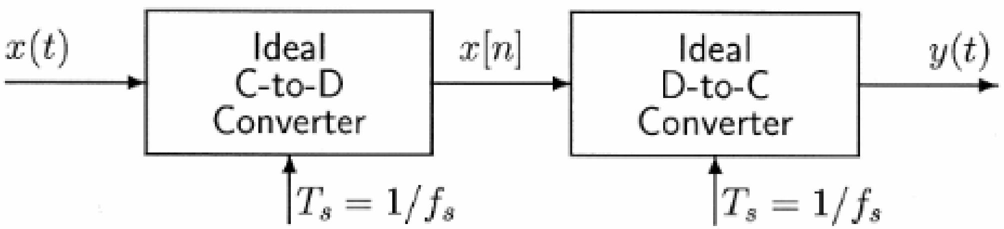
**Answer:** $x_1(t) = 2 + 10\cos(2\pi(150)t + \pi/3)$

(no aliasing)

**Answer:** $x_2(t) = 2 + 10\cos(2\pi(250)t - \pi/3)$

We have folding in the second case:

# Example-2




Suppose that the continuous-time input $x(t)$ to the above system is given as

$$x(t) = \cos(14000\pi t) + \cos(2000\pi t) + \cos(1000\pi t).$$

$f_{max} = 7{,}000$ Hz

(a) What sampling rate is required such that no aliasing occurs for $x(t)$? $\boxed{f_s = 14{,}000 \text{ Hz}}$

(c) Given that $x(t) = \cos(25000\pi t)$ and $f_s = 10000$ samples/second, write a simplified expression for the output $y(t)$ in terms of cosine functions.

$$\boxed{y(t) = \cos(5000\pi t)}$$

NOTE THAT ALIASING OCCURS.

$$x[n] = \cos\left(\frac{2500\pi n}{10000}\right) = \cos(0.5\pi n)$$

$$y(t) = \cos(0.5\pi t \, 10000) = \cos(5000\pi t)$$

# Example-3

**PROBLEM:**

The "spectrum" diagram gives the frequency content of a signal.

(a) Draw a sketch of the spectrum of $x(t)$ which is "cosine-times-sine"

$$x(t) = \cos(50\pi t)\ \sin(700\pi t)$$

Label the frequencies and complex amplitudes of each component.

(b) Determine the minimum sampling rate that can be used to sample $x(t)$ without any aliasing.

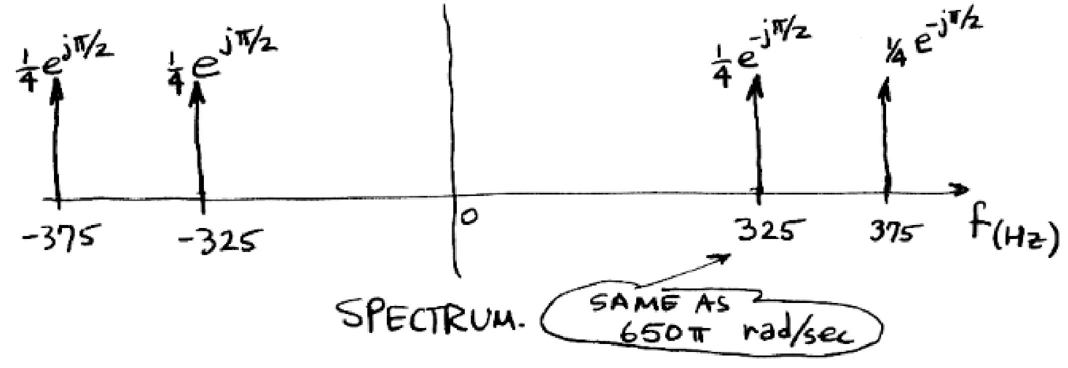
$x(t) = \cos(50\pi t)\sin(700\pi t)$

(a) $x(t) = \left(\frac{1}{2}e^{j50\pi t} + \frac{1}{2}e^{-j50\pi t}\right)\left(\frac{1}{2j}e^{j700\pi t} - \frac{1}{2j}e^{-j700\pi t}\right)$

$= \frac{1}{4j}e^{j750\pi t} + \frac{1}{4j}e^{j650\pi t} - \frac{1}{4j}e^{-j650\pi t} - \frac{1}{4j}e^{-j750\pi t}$

SAME AS $\frac{1}{4}e^{-j\pi/2}$

SAME AS $\frac{1}{4}e^{+j\pi/2}$



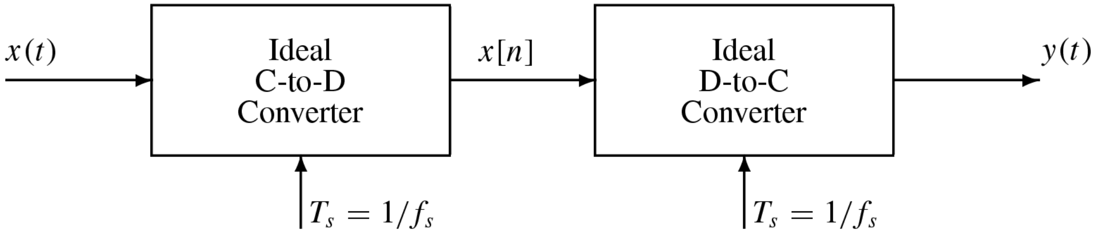

(b) Sampling Thm says sample at a rate greater than $\boxed{\text{two}}$ $\boxed{\text{times}}$ the highest freq.

HIGHEST FREQ = 375 Hz

$\Rightarrow f_s \geq 750$ Hz.

# Example-4

**PROBLEM:**
Consider the following system.




Suppose that the output of the C-to-D converter is

$$x[n] = 5 + 8\cos(0.4\pi n) + 4\cos(0.8\pi n + \pi/3)$$

when the sampling rate is $f_s = 1/T_s = 2000$ samples/second. Determine the output $y(t)$ of the ideal D-to-C converter.

$$x[n] = 5 + 8\cos(0.4\pi n) + 4\cos(0.8\pi n + \pi/3)$$


$F_s = 2000$

For discrete to continuous, we replace "n" with $F_s t$

$$y(t) = x[n]\Big|_{n=F_s t}$$

$$= 5 + 8\cos(0.4\pi(2000)t) + 4\cos(0.8\pi(2000)t + \pi/3)$$

$$= 5 + 8\cos(2\pi(400)t) + 4\cos(2\pi(800)t + \pi/3)$$