On the Ptolemy-Alhazen problem: source at infinite-distance

filename: cal20230511.tex

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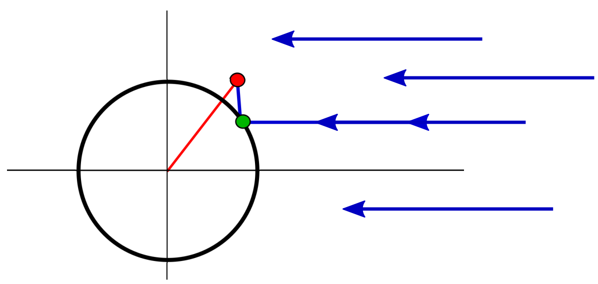
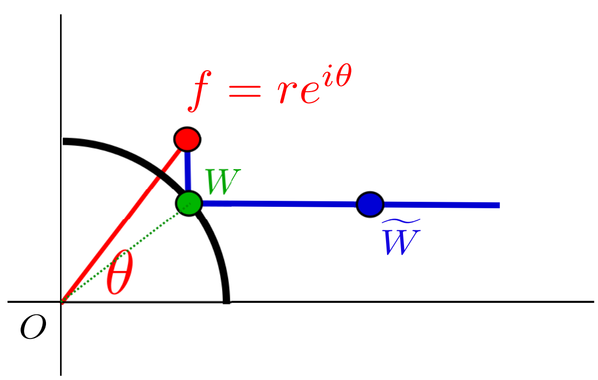
Assume that light rays reach the unit circle from the right side, parallel to the real axis.



Let $F(f), f = re^{i\theta}$ be the observation point, where we can assume $r > 1$ and $0 \le \theta \le \frac{\pi}{2}$ from symmetry.

Let $W(w)$ be the reflection point and $\widetilde{W}(\widetilde{w})$ be $\widetilde{w} = w+1$. Note that W and $\widetilde{W}$ are points on a line parallel to the real axis.



Claim

The refrection point w is given as a solution of the equation

$$re^{-i\theta}w^4 - w^3 + w - re^{i\theta} = 0. \quad (1)$$

Proof. From the reflection property, the following holds

$$\angle(O, W, \widetilde{W}) = \angle(F, W, O).$$

Therefore,

$$\begin{aligned}
&\arg \frac{\widetilde{w} - w}{0 - w} = \arg \frac{0 - w}{f - w} \\
&\arg \frac{\widetilde{w} - w}{-w} - \arg \frac{-w}{f - w} = 0 \\
&\arg \frac{w - \widetilde{w}}{w} \cdot \frac{w - f}{w} = 0 \\
&\arg \frac{(w - (1 + w))(w - f)}{w^2} = 0 \\
&\arg \frac{f - w}{w^2} = 0.
\end{aligned}$$

The last equality implies that $\dfrac{f - w}{w^2}$ is a real number. Since this complex conjugate is also real, we have

$$\frac{f - w}{w^2} = \frac{\overline{f} - \overline{w}}{\overline{w}^2}.$$

Hence, we have

$$\overline{w}^2 (re^{i\theta} - w) = w^2 (re^{-i\theta} - \overline{w}).$$

Since w is a point on the unit circle and satisfies $w\overline{w} = 1$, the above equation can be written as

$$re^{-i\theta} w^4 - w^3 + w - re^{i\theta} = 0.$$

□

Remark. Equation (1) has four roots. Under the assumptions here, we can choose the one that can be written in $e^{i\varphi}$ $(0 \leq \varphi \leq \frac{\pi}{2})$.

Point w is the point of tangency of the unit circle to the parabola whose directrix is perpendicular to the real axis and whose focus is F.