CHAPTER

42

LAPLACE TRANSFORM

42.1 INTRODUCTION

Laplace transforms help in solving the differential equations with boundary values without finding the general solution and the values of the arbitrary constants.

42.2 LAPLACE TRANSFORM

Definition. Let $f(t)$ be function defined for all positive values of t, then

$$F(s)=\int_0^\infty e^{-st} f(t)\,dt$$

provided the integral exists, is called the **Laplace Transform** of $f(t)$. It is denoted as

$$\boxed{L[f(t)]=F(s)=\int_0^\infty e^{-st} f(t)\,dt}$$

42.3 IMPORTANT FORMULAE

1. $L(1)=\frac{1}{s}$

2. $L\left(t^n\right)=\frac{n!}{s^{n+1}}$, when $n=0,\ 1,\ 2,\ 3...$

3. $L\left(e^{at}\right)=\frac{1}{s-a}$ $\quad (s>a)$

4. $L(\cosh at)=\frac{s}{s^2-a^2}$ $\quad \left(s^2>a^2\right)$

5. $L(\sinh at)=\frac{a}{s^2-a^2}$ $\quad (s^2>a^2)$

6. $L(\sin at)=\frac{a}{s^2+a^2}$ $\quad (s>0)$

7. $L(\cos at)=\frac{s}{s^2+a^2}$ $\quad (s>0)$

1. $\boxed{L(1)=\frac{1}{s}}$

Proof. $L(1)=\int_0^\infty 1\cdot e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_0^\infty=-\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty=-\frac{1}{s}[0-1]=\frac{1}{s}$

Hence $L(1)=\frac{1}{s}$ **Proved.**

2. $\boxed{L\left(t^n\right)=\frac{n!}{s^{n+1}}}$ where n and s are positive.

Proof. $L(t^n)=\int_0^\infty e^{-st}t^n dt$

Putting $st=x$ or $t=\frac{x}{s}$ or $dt=\frac{dx}{s}$

Thus, we have $\quad \mathrm{L}(t^n) = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} \quad \Rightarrow \quad \mathrm{L}(t^n) = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}.\,x^n\,dx$

$$\Rightarrow L(t^n) = \frac{\overline{|n+1}}{s^{n+1}} \quad \Rightarrow \mathrm{L}\left(t^n\right) = \frac{n!}{s^{n+1}} \qquad \left[\begin{array}{l} \overline{|n+1} = \int_0^\infty e^{-x}.x^n\,dx \\ \text{and} \quad \overline{|n+1} = n! \end{array}\right]$$

Proved.

3. $\boxed{\mathbf{L}\left(e^{at}\right) = \frac{1}{s-a}}$, where $s > a$

Proof. $\mathrm{L}\,(e^{at}) = \int_0^\infty e^{-st}.e^{at}dt = \int_0^\infty e^{-st+at}.dt$

$$= \int_0^\infty e^{(-s+a)t}.dt = \int_0^\infty e^{-(s-a)t}.dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$$

Proved.

4. $\boxed{\mathbf{L}(\cosh at) = \frac{s}{s^2-a^2}}$

Proof. $\mathrm{L}(\cosh at) = \mathrm{L}\left[\frac{e^{at}+e^{-at}}{2}\right] \qquad \left(\therefore \cosh at = \frac{e^{at}+e^{-at}}{2}\right)$

$$= \frac{1}{2}\mathrm{L}\left(e^{at}\right)+\frac{1}{2}\mathrm{L}\left(e^{-at}\right) = \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right] \qquad \left[\mathrm{L}\left(e^{at}\right) = \frac{1}{s-a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

Proved.

5. $\boxed{\mathbf{L}(\sinh at) = \frac{a}{s^2-a^2}}$

Proof. L (sinh at) $= \mathrm{L}\left[\frac{1}{2}\left(e^{at}-e^{-at}\right)\right]$

$$= \frac{1}{2}[L(e^{at})-L(e^{-at})] = \frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

Proved.

6. $\boxed{\mathbf{L}(\sin at) = \frac{a}{s^2+a^2}}$

Proof. L (sin at) $= \mathrm{L}\left[\frac{e^{iat}-e^{-iat}}{2i}\right] \qquad \left[\because \sin at = \frac{e^{iat}-e^{-iat}}{2i}\right]$

$$= \frac{1}{2i}\left[L\left(e^{iat}-e^{-iat}\right)\right] = \frac{1}{2i}\left[L\left(e^{iat}\right)-L\left(e^{-iat}\right)\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia}-\frac{1}{s+ia}\right] = \frac{1}{2i}\,\frac{s+ia-s+ia}{s^2+a^2} = \frac{1}{2i}\,\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

Proved.

7. $\boxed{\mathbf{L}\,(\cos at) = \frac{s}{s^2+a^2}}$

Proof. $\qquad \mathrm{L}\,(\cos at) = \mathrm{L}\left(\frac{e^{iat}+e^{-iat}}{2}\right) \qquad \left[\because \cos at = \frac{e^{iat}+e^{-iat}}{2}\right]$

$$=\frac{1}{2}[\mathrm{L}(e^{iat}+e^{-iat})]=\frac{1}{2}[\mathrm{L}(e^{iat})+\mathrm{L}(e^{-iat})]\ =\frac{1}{2}\left[\frac{1}{s-ia}+\frac{1}{s+ia}\right]=\frac{1}{2}\frac{s+ia+s-ia}{s^2+a^2}$$

$$=\frac{s}{s^2+a^2}$$ **Proved.**

Example 1. *Find the Laplace transform of f(t) defined as*

$$f(t)=\begin{cases}\frac{t}{k}, & when\ \ 0<t<k\\ 1, & when\ \ \ t>k\end{cases}$$

Solution. $L\ [f(t)]\ =\ \int_0^k \frac{t}{k}e^{-st}dt+\int_k^\infty 1.e^{-st}dt=\frac{1}{k}\left[\left(t\frac{e^{-st}}{-s}\right)_0^k-\int_0^k\frac{e^{-st}}{-s}\,dt\right]+\left[\frac{e^{-st}}{-s}\right]_k^\infty$

$$=\frac{1}{k}\left[\frac{ke^{-ks}}{-s}-\left(\frac{e^{-st}}{s^2}\right)_0^k\right]+\frac{e^{-ks}}{s}=\frac{1}{k}\left[\frac{ke^{-ks}}{-s}-\frac{e^{-sk}}{s^2}+\frac{1}{s^2}\right]+\frac{e^{-ks}}{s}$$

$$=-\frac{e^{-sk}}{s}-\frac{1}{k}\frac{e^{-ks}}{s^2}+\frac{1}{k}\frac{1}{s^2}+\frac{e^{-ks}}{s}=\frac{1}{ks^2}[-e^{-ks}+1]$$ **Ans.**

Example 2. *Find the Laplace transform of the function f(t)* = $\begin{cases}t-1, & 1<t<2\\ 3-t, & 2<t<3\end{cases}$.

(*U.P., II Semester, 2009*)

Solution. The given function is periodic with period 3.

$$\begin{aligned}L\ [f(t)]\ &=\ \int_1^3 f(t)\,e^{-st}dt\\ &=\ \left[\int_1^2 f(t)\,e^{-st}dt+\int_2^3 f(t)\,e^{-st}dt\right]\\ &=\ \left[\int_1^2 (t-1)\,e^{-st}dt+\int_2^3 (3-t)\,e^{-st}dt\right]\\ &=\ \left[\left\{(t-1)\frac{e^{-st}}{-s}-\frac{e^{-st}}{(-s)^2}\right\}_1^2+\left\{(3-t)\frac{e^{-st}}{-s}+\frac{e^{-st}}{(-s)^2}\right\}_2^3\right]\\ &=\ \left[\left\{\frac{e^{-2s}}{-s}-\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s^2}\right\}+\left\{\frac{e^{-3s}}{s^2}-\frac{e^{-2s}}{-s}-\frac{e^{-2s}}{s^2}\right\}\right]\\ &=\ \left[-\frac{e^{-2s}}{s}-\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s^2}+\frac{e^{-3s}}{s^2}+\frac{e^{-2s}}{s}-\frac{e^{-2s}}{s^2}\right]\\ &=\ \left[\frac{1}{s^2}(-e^{-2s}+e^{-s}+e^{-3s}-e^{-2s})\right]\ =\ \frac{1}{s^2}[e^{-s}-2e^{-2s}+e^{-3s}]\end{aligned}$$ **Ans.**

Example 3. *Find the Laplace transform of* $F(t)$ = $\begin{cases}1, & 0\le t<1\\ t, & 1\le t<2\\ t^2, & 2\le t<\infty\end{cases}$ (*Q. Bank U.P. 2001*)

Solution. Here, we have $F(t)=\begin{cases}1, & 0\le t<1\\ t, & 1\le t<2\\ t^2, & 2\le t<\infty\end{cases}$

$$L[F(t)]\ =\ \int_0^\infty e^{-st}\cdot F(t)dt=\int_0^1 e^{-st}dt+\int_1^2 t\,e^{-st}dt+\int_2^\infty t^2e^{-st}dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^1 + \left(t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right)_1^2 + \left(t^2\frac{e^{-st}}{-s}\right)_2^\infty - \int_2^\infty 2t\cdot\frac{e^{-st}}{-s}dt$$

$$= \left(\frac{1-e^{-s}}{s}\right) + \left(\frac{-2}{s}e^{-2s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right) + \frac{4}{s}e^{-2s} + \frac{2}{s}\int_2^\infty t\, e^{-st}dt$$

$$= \frac{1}{s} + \frac{2}{s}e^{-2s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2}{s}\left[\left(t\frac{e^{-st}}{-s}\right)_2^\infty - \int_2^\infty 1\cdot\frac{e^{-st}}{-s}dt\right]$$

$$= \frac{1}{s} + \frac{2}{s}e^{-2s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2}{s}\left[\frac{2}{s}e^{-2s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_2^\infty\right]$$

$$= \frac{1}{s} + \frac{2}{s}e^{-2s} + \frac{e^{-s}}{s^2} + \frac{3}{s^2}e^{-2s} + \frac{2}{s^3}e^{-2s}.$$ **Ans.**

Example 4. *Find the Laplace transform of* $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$

(U.P, II Semester, June 2007)

Solution. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^2 t^2 e^{-st}dt + \int_2^3 (t-1)e^{-st}dt + \int_3^\infty 7e^{-st}dt$

$$\left[\int I\ II = I\ II_1 - I'\ II_{11} + I''\ II_{111} - \ldots\right]$$

$$= \left[t^2\left(\frac{e^{-st}}{(-s)}\right) - 2t\frac{e^{-st}}{(-s)^2} + 2\frac{e^{-st}}{(-s)^3}\right]_0^2 + \left[(t-1)\left(\frac{e^{-st}}{(-s)}\right) - \frac{e^{-st}}{(-s)^2}\right]_2^3 + 7\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= \left[-4\left(\frac{e^{-2s}}{s}\right) - 4\left(\frac{e^{-2s}}{s^2}\right) - 2\left(\frac{e^{-2s}}{s^3}\right) + \frac{2}{s^3}\right] + \left[2\left(\frac{e^{-3s}}{-s}\right) - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2}\right] + 7\left(0 + \frac{e^{-3s}}{s}\right)$$

$$= \frac{2}{s^3} + e^{-2s}\left[-\frac{4}{s} - \frac{4}{s^2} - \frac{2}{s^3}\right] + e^{-3s}\left[-\frac{2}{s} - \frac{1}{s^2}\right] + e^{-2s}\left[\frac{1}{s} + \frac{1}{s^2}\right] + e^{-3s}\left[\frac{7}{s}\right]$$

$$= \frac{2}{s^3} + e^{-2s}\left[-\frac{4}{s} - \frac{4}{s^2} - \frac{2}{s^3} + \frac{1}{s} + \frac{1}{s^2}\right] + e^{-3s}\left[-\frac{2}{s} - \frac{1}{s^2} + \frac{7}{s}\right]$$

$$= \frac{2}{s^3} + e^{-2s}\left[-\frac{3}{s} - \frac{3}{s^2} - \frac{2}{s^3}\right] + e^{-3s}\left[\frac{5}{s} - \frac{1}{s^2}\right] = \frac{2}{s^3} - \frac{e^{-2s}}{s^3}\left(2 + 3s + 3s^2\right) + \frac{e^{-3s}}{s^2}(5s - 1)$$ **Ans.**

Example 5. *Find the Laplace transform of (1 + sin 2t).*

Solution. Laplace transform of $(1 + \sin 2t)$

$$= \int_0^\infty e^{-st}(1 + \sin 2t)\,dt = \int_0^\infty e^{-st}\left(1 + \frac{e^{2it} - e^{-2it}}{2i}\right)dt$$

$$= \frac{1}{2i}\int_0^\infty [2ie^{-st} + e^{(-s+2i)t} - e^{(-s-2i)t}]dt = \frac{1}{2i}\left[\frac{2ie^{-st}}{-s} + \frac{e^{(-s+2i)t}}{-s+2i} - \frac{e^{(-s-2i)t}}{-s-2i}\right]_0^\infty$$

$$= \frac{1}{2i}\left[\left(0 + \frac{2i}{s}\right) + \frac{1}{-s+2i}(0-1) - \frac{1}{-s-2i}(0-1)\right]$$

$$= \frac{1}{2i}\left[\frac{2i}{s} + \frac{1}{s-2i} - \frac{1}{s+2i}\right] = \frac{1}{2}\left[\frac{2}{s} + \frac{4}{s^2+4}\right] = \frac{1}{s} + \frac{2}{s^2+4}$$ **Ans.**

Alternate Method

$$L\,(1 + \sin 2t) = L\,(1) + L \sin 2t = \frac{1}{s} + \frac{2}{s^2+4}$$ **Ans.**

42.4 PROPERTIES OF LAPLACE TRANSFORM

(1) $L[af_1(t)+bf_2(t)] = a\,L[f_1(t)] + b\,L[f_2(t)]$

Proof. $L[af_1(t)+bf_2(t)] = \int_0^\infty e^{-st}[af_1(t)+bf_2(t)]\,dt$

$$= a\int_0^\infty e^{-st} f_1(t)\,dt + b\int_0^\infty e^{-st} f_2(t)\,dt$$

$$= a\,L\,[f_1(t)] + b\,L[f_2(t)]$$ **Proved.**

42.5 CHANGE OF SCALE PROPERTY

If $L\{f(t)\} = F(s)$ then $\boxed{L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)}$

Proof. $L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt = \int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\frac{du}{a}$ $\left[\text{Put } at = u \Rightarrow dt = \frac{du}{a}\right]$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\,du = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)\,dt$$

$$= \frac{1}{a}\int_0^\infty e^{-St} f(t)\,dt = \frac{1}{a}L\,\{f(t)\} = \frac{1}{a}F(S)$$ $\left[\text{Put } S = \frac{s}{a}\right]$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$ **Proved.**

Example 6. *If* $L\,\{J_0(\sqrt{t})\} = \frac{e^{-\frac{1}{4s}}}{s}$, *find* $L\,\{J_0(2\sqrt{t})\}$.

Solution. Here, we have

$$L\,\{J_0(\sqrt{t})\} = \frac{e^{-\frac{1}{4s}}}{s}$$

By change of scale property,

$$L\,\{J_0(\sqrt{4t})\} = \frac{1}{4}\cdot\left\{\frac{e^{-\frac{1}{4(s/4)}}}{(s/4)}\right\}$$

$$\Rightarrow \quad L\,\{J_0(2\sqrt{t})\} = \frac{1}{s}e^{-1/s}$$ **Ans.**

(2) First Shifting Theorem. If $L\,\{f(t)\} = F(s)$, then

$$\boxed{\mathbf{L[e^{at}f(t)] = F(s-a)}}$$

Proof. $L[e^{at}f(t)] = \int_0^\infty e^{-st}.e^{at} f(t)dt = \int_0^\infty e^{-(s-a)t} f(t)dt$

$$= \int_0^\infty e^{-rt} f(t)\,dt$$ where $r = s - a$

$$= F\,(r) = F\,(s-a)$$ **Proved.**

With the help of this property, we can have the following important results :

1. $L(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$

2. $L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2-b^2}$ **3.** $L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2-b^2}$

4. $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ **5.** $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$

42.6 HEAVISIDE'S SHIFTING THEOREM (Second Translation Property)

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & 0 < t < a \end{cases}$ then prove that

$$\boxed{L\{g(t)\} = e^{-as}F(s)}$$ *(U.P. II Semester, Summer 2006)*

Proof. $L\{g(t)\} = \int_0^\infty e^{-st} g(t)dt$

$= \int_0^a e^{-st} g(t)dt + \int_a^\infty e^{-st} g(t)dt$ [$g(t) = 0$, when $0 < t < a$]

$= 0 + \int_a^\infty e^{-st}.f(t-a)dt = \int_a^\infty e^{-st} f(t-a)dt$ [Put $t - a = u \Rightarrow dt = du$]

$= \int_0^\infty e^{-s(u+a)} f(u)\,du = e^{-sa}\int_0^\infty e^{-su} f(u)du = e^{-as}\int_0^\infty e^{-st} f(t)\,dt$ **Proved.**

$$\boxed{\mathbf{L\{g(t)\} = e^{-as}F(s)}}$$

Example 7. *Find the Laplace transform of* $\cos^2 t$.

Solution. We know that $\cos 2t = 2\cos^2 t - 1$

$$\cos^2 t = \frac{1}{2}[\cos 2t + 1]$$

$$L(\cos^2 t) = L\left[\frac{1}{2}(\cos 2t + 1)\right] = \frac{1}{2}[L(\cos 2t) + L(1)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(2)^2} + \frac{1}{s}\right] = \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right]$$ **Ans.**

Example 8. *If* $L(\cos^2 t) = \dfrac{s^2+2}{s(s^2+4)}$, *find* $L(\cos^2 at)$. *(U.P., II Semester, Summer 2006)*

Solution. We have, $L(\cos^2 t) = \dfrac{s^2+2}{s(s^2+4)}$

By change of scale property, we have

$$L(\cos^2 at) = \frac{1}{a}.\frac{\left(\frac{s}{a}\right)^2 + 2}{\frac{s}{a}\left[\left(\frac{s}{a}\right)^2 + 4\right]} = \frac{1}{a}\left[\frac{s^2+2a^2}{\frac{s}{a}(s^2+4a^2)}\right] = \frac{s^2+2a^2}{s(s^2+4a^2)}$$ **Ans.**

Example 9. *Find the Laplace transform of* $t^{-\frac{1}{2}}$.

Solution. We know that $L(t^n) = \dfrac{\overline{|n+1}}{s^{n+1}}$

Put $n = -\dfrac{1}{2}$, $L\left(t^{-1/2}\right) = \dfrac{\overline{\left|-\frac{1}{2}+1\right.}}{s^{-1/2+1}} = \dfrac{\overline{\left|\frac{1}{2}\right.}}{\sqrt{s}} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$, where $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$ **Ans.**

Example 10. *Find the Laplace transform of 2 sin 2t cos 4t.*

Solution. We have

$$f(t) = 2\sin 2t\cos 4t = \sin\frac{2t+4t}{2} + \sin\frac{2t-4t}{2} = \sin 3t - \sin t$$

$$L\,f(t) = L(\sin 3t) - L(\sin t) = \frac{3}{s^2+9} - \frac{1}{s^2+1}$$ **Ans.**

Example 11. *Find the Laplace transform of 4 sin³ t.*

Solution. We have

$$f(t) = 4\sin^3 t = 3\sin t - \sin 3t \qquad [\sin 3t = 3\sin t - 4\sin^3 t]$$

$$L\,f(t) = 3L\sin t - L\sin 3t = \frac{3}{s^2+1} - \frac{3}{s^2+9}$$ **Ans.**

Example 12. *Find the Laplace transform of 4 cosh 2t sin 4t*

Solution. We have

$$f(t) = 4\cosh 2t\sin 4t = 4\left(\frac{e^{2t}+e^{-2t}}{2}\right)\left(\frac{e^{4it}-e^{-4it}}{2i}\right)$$

$$= \frac{1}{i}\left[e^{(2+4i)t} - e^{(2-4i)t} + e^{(-2+4i)t} - e^{(-2-4i)t}\right]$$

$$L[f(t)] = -i\left[L(e^{(2+4i)t}) - L(e^{(2-4i)t}) + L(e^{(-2+4i)t}) - L(e^{(-2-4i)t})\right]$$

$$= -i\left[\frac{1}{s-2-4i} - \frac{1}{s-2+4i} + \frac{1}{s+2-4i} - \frac{1}{s+2+4i}\right]$$

$$= -i\left[\left(\frac{1}{s-2-4i} - \frac{1}{s+2+4i}\right) - \left(\frac{1}{s-2+4i} - \frac{1}{s+2-4i}\right)\right]$$

$$= -i\left[\frac{4+8i}{s^2-(2+4i)^2} - \frac{4-8i}{s^2-(2-4i)^2}\right]$$ **Ans.**

EXERCISE 42.1

Find the Laplace transforms of the following:

1. $t + t^2 + t^3$ **Ans.** $\frac{1}{s^2}+\frac{2}{s^3}+\frac{6}{s^4}$

2. $\sin t\cos t$ **Ans.** $\frac{1}{s^2+4}$

3. $t^{7/2}e^{5t}$ (*M.D.U. Dec. 2009*) **Ans.** $\frac{105\sqrt{\pi}}{16(s-5)^{9/2}}$

4. $\sin^3 2t$ **Ans.** $\frac{48}{(s^2+4)(s^2+36)}$

5. $e^{-t}\cos^2 t$ **Ans.** $\frac{1}{2s+2}+\frac{s+1}{2s^2+4s+10}$

6. $\sin 2t\cos 3t$ **Ans.** $\frac{2(s^2-5)}{(s^2+1)(s^2+25)}$

7. $\sin 2t\sin 3t$ **Ans.** $\frac{12s}{(s^2+1)(s^2+25)}$

8. $\cos at\sinh at$ **Ans.** $\frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}\right]$

9. $\sinh^3 t$ **Ans.** $\frac{6}{(s^2-1)(s^2-9)}$

10. $\cos t\cos 2t$ **Ans.** $\frac{s(s^2+5)}{(s^2+1)(s^2+9)}$

11. $\cosh at\ \sin at$ **Ans.** $\frac{a(s^2+2a^2)}{s^4+4a^4}$

12. $f(t)=\begin{cases}\cos\left(t-\dfrac{2\pi}{3}\right), & t>\dfrac{2\pi}{3}\\ 0, & t<\dfrac{2\pi}{3}\end{cases}$ **Ans.** $e^{\frac{-2\pi s}{3}}\cdot\frac{s}{s^2+1}$

42.7 EXISTENCE THEOREM

According to this theorem $\int_0^{\infty} e^{-st}f(t)\,dt$ exists if $\int_0^{\lambda} e^{-st}f(t)dt$ can actually be evaluated and its limit as $\lambda \to \infty$ exists.

Otherwise we may use the following theorem:

If $f(t)$ is continuous and $\lim\limits_{t\to\infty}\left[e^{-at}f(t)\right]$ is finite, then Laplace transform of $f(t)$ *i.e.* $\int_0^{\infty} e^{-st}f(t)\,dt$ exists for $s > a$.

It should however, be kept in mind that the above foresaid conditions are sufficient but not necessary.

For example; $L\left(\frac{1}{\sqrt{t}}\right)$ exists though $\frac{1}{\sqrt{t}}$ is infinite at $t = 0$. Similarly a function $f(t)$ for which $\lim\limits_{t\to\infty}\left[e^{-at}f(t)\right]$ is finite and having a finite discontinuity will have a Laplace transform of $s > a$.

42.8 LAPLACE TRANSFORM OF THE DERIVATIVE OF *f(t)*

$$\boldsymbol{L[f'(t)] = sL[f(t)] - f(0)} \qquad \text{where} \qquad L[f(t)] = F(s).$$

Proof. $L[f'(t)] = \int_0^{\infty} e^{-st}f'(t)dt$

Integrating by parts, we get

$$\begin{aligned} L[f'(t)] &= \left[e^{-st}f(t)\right]_0^{\infty} - \int_0^{\infty}(-se^{-st})f(t)\,dt \\ &= -f(0) + s\int_0^{\infty} e^{-st}f(t)dt \qquad (e^{-st}f(t)=0, \text{ when } t=\infty) \\ &= -f(0) + sL[f(t)] \end{aligned}$$

$$\boxed{\boldsymbol{L[f'(t)] = s\,L[f(t)] - f(0).}}$$ **Proved.**

Note. Roughly, Laplace transform of **derivative** of $f(t)$ corresponds to **multiplication** of the Laplace transform of $f(t)$ by s.

42.9 LAPLACE TRANSFORM OF DERIVATIVE OF ORDER *n* (*M.D.U. Dec. 2009*)

$$L[f^n(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \ldots - f^{n-1}(0)$$

Proof. We have already proved in Article 42.8 that

$$L[f'(t)] = sL[f(t)] - f(0) \qquad \ldots(1)$$

Replacing $f(t)$ by $f'(t)$ and $f'(t)$ by $f''(t)$ in (1), we get

$$L[f''(t)] = sL[f'(t)] - f'(0) \qquad \ldots(2)$$

Putting the value of $L[f'(t)]$ from (1) in (2), we have

$$L[f''(t)] = s\,[sL[f(t)] - f(0)] - f'(0)$$

$\Rightarrow \qquad L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$

Similarly, $\qquad L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - s f'(0) - f''(0)$

$$L[f^{iv}(t)] = s^4 L[f(t)] - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$$

$$\boxed{\boldsymbol{L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) + \ldots\ldots - f^{n-1}(0)}}$$

Example 13. *Given* $L\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{s^{3/2}}$, *show that* $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$. *(U.P., II Semester, 2005)*

Solution. Let $F(t) = 2\sqrt{\frac{t}{\pi}} \quad \Rightarrow \quad F'(t) = \frac{1}{\sqrt{\pi t}}$. Also $F(0) = 0$

$\therefore \qquad L\{F'(t)\} = s\,L\{F(t)\} - F(0) = s.\ \frac{1}{s^{3/2}} - 0$

$\therefore \qquad L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$. **Proved.**

Example 14. *Find the Laplace transform of* $\sin\sqrt{t}$ *; Hence find* $L\left(\frac{\cos\sqrt{t}}{2\sqrt{t}}\right)$

Solution. $\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \ldots\ldots$

$$= t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \ldots\ldots \qquad \left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\right]$$

$\therefore \qquad L(\sin\sqrt{t}) = L\left(t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \ldots\ldots\right) = \frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{3!s^{5/2}} + \frac{\Gamma 7/2}{5!s^{7/2}} - \ldots\ldots$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{2^2 s}\right) + \frac{1}{2!}\left(\frac{1}{2^2 s}\right)^2 - \ldots..\right\}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{2^2 s}\right) + \frac{1}{2!}\left(\frac{1}{2^2 s}\right)^2 - \ldots\ldots\right\} \qquad \left[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right]$$

$\Rightarrow \qquad L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-(1/4s)}$

Now, $L\left[\frac{d}{dt}(\sin\sqrt{t})\right] = s\,L(\sin\sqrt{t}) - 0 \qquad \left[\because F(0) = 0 \text{ and } L\left[\frac{d}{dt}[F(t)]\right] = sF(s)\right]$

$$L\left(\frac{\cos\sqrt{t}}{2\sqrt{t}}\right) = \frac{\sqrt{\pi}}{2\sqrt{s}} e^{-\left(\frac{1}{4s}\right)} \quad \Rightarrow \quad L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\sqrt{\pi}}{\sqrt{5}} e^{-1/45}$$ **Ans.**

42.10 LAPLACE TRANSFORM OF INTEGRAL OF *f(t)*

$$\boxed{\boldsymbol{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} F(s)}} \qquad \text{where} \quad L[f(t)] = F(s)$$

Proof. Let $\phi(t) = \int_0^t f(t)\,dt$ and $\phi(0) = 0$ then $\phi'(t) = f(t)$

We know the formula of Laplace transforms of $\phi'(t)$ *i.e.*

$$L[\phi'(t)] = s\,L[\phi(t)] - \phi(0)$$

$\Rightarrow \qquad L[\phi'(t)] = s\,L[\phi(t)] \qquad [\phi(0) = 0]$

$$\Rightarrow \qquad L[\phi(t)] = \frac{1}{s} L[\phi'(t)]$$

Putting the values of $\phi(t)$ and $\phi'(t)$, we get

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)] \Rightarrow \boxed{\mathbf{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} F(s)}$$ **Proved.**

Note. (1) Laplace transform of **Integral** of $f(t)$ corresponds to the division of the Laplace transform of $f(t)$ by s.

(2) $$\int_0^t f(t)\; dt = L^{-1}\left[\frac{1}{s} F(s)\right]$$

42.11 LAPLACE TRANSFORM OF *t.f(t)* (Multiplication by *t*)

If $L[f(t)] = F(s)$, then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)].$$ (*U.P., II Semester, Summer 2005*)

Proof. $$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt \qquad \text{...(1)}$$

Differentiating (1) w.r.t. 's', we get

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right] = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right) f(t)\,dt$$

$$= \int_0^\infty \left(-t\,e^{-st}\right).f(t)\,dt = \int_0^\infty e^{-st}\left[-t.f(t)\right] dt$$

$$= L[-t\,f(t)] \Rightarrow L[t\,f(t)] = (-1)^1 \frac{d}{ds}[F(s)]$$

Similarly, $L[t^2 f(t)] = (-1)^2 \dfrac{d^2}{ds^2}[F(s)]$

$$L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3}[F(s)]$$

$$\boxed{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]}$$ **Proved.**

42.12 INITIAL AND FINAL VALUE THEOREMS

(*a*) Initial Value Theorem. $L\{f(t)\} = F(s)$

$$\Rightarrow \qquad \lim_{t\to 0} f(t) = \operatorname*{Lim}_{s\to\infty}[sF(s)],$$ provided the limit exists.

Proof. $$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$$

$$\Rightarrow \qquad \int_0^\infty e^{-st} f'(t)\,dt = s\,F(s) - f(0)$$

$$\Rightarrow \qquad \operatorname*{Lim}_{s\to\infty} \int_0^\infty e^{-st} f'(t)\,dt = \operatorname*{Lim}_{s\to\infty}[sF(s) - f(0)]$$

$$\Rightarrow \qquad \operatorname*{Lim}_{s\to\infty}[s\,F(s)] = f(0) + \int_0^\infty \left(\operatorname*{Lim}_{s\to\infty} e^{-st}\right) f'(t)\,dt$$

$$= f(0) + \int_0^{\infty} (0)\, f'(t)\, dt \qquad (\because \lim_{s\to\infty} e^{-st} = 0)$$

$$= f(0) + 0 = f(0) = \lim_{t\to 0} f(t)$$

(*b*) Final Value Theorem. $L\{f(t)\} = F(s)$

$\Rightarrow$ $\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$, provided the limits exist.

Proof. $L\{f'(t)\} = sL\{f(t)\} - f(0) \quad \Rightarrow \int_0^{\infty} e^{-st} f'(t)\, dt = sF(s) - f(0)$

$$\Rightarrow \quad \lim_{s\to 0}\int_0^{\infty} e^{-st} f'(t)\, dt = \lim_{s\to 0}[sF(s) - f(0)]$$

$$\Rightarrow \quad \lim_{s\to 0}[sF(s) - f(0)] = \lim_{s\to 0}\int_0^{\infty} e^{-st} f'(t)\, dt$$

$$\Rightarrow \quad \lim_{s\to 0}[sF(s)] - f(0) = \lim_{s\to 0}\int_0^{\infty} e^{-st} f'(t)\, dt$$

$$\Rightarrow \quad \lim_{s\to 0}[sF(s)] - f(0) = \int_0^{\infty} \lim_{s\to 0} e^{-st} f'(t)\, dt = \int_0^{\infty} (1)\, f'(t)\, dt \qquad \left[\because \lim_{s\to 0} e^{-st} = 1\right]$$

$$\Rightarrow \quad \boxed{\lim_{s\to 0}[sF(s)] = \lim_{t\to\infty} f(t)}$$

Example 15. *If* $L\{F(t)\} = \dfrac{1}{s(s+\beta)}$ *then, find* $\lim_{t\to\infty} F(t)$

Solution. By final-value theorem,

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} sL\{F(t)\} = \lim_{s\to 0} \frac{s}{s(s+\beta)} = \lim_{s\to 0} \frac{1}{(s+\beta)} = \frac{1}{\beta} \quad \textbf{Ans.}$$

42.13. EXPONENTIAL INTEGRAL FUNCTION $\int_t^{\infty} \left(\frac{e^{-x}}{x}\right) dx$

Let $\quad f(t) = \int_t^{\infty} \frac{e^{-x}}{x}\, dx$

$\Rightarrow \quad f'(t) = -\dfrac{e^{-t}}{t} \quad \Rightarrow \quad tf'(t) = -e^{-t}$ [Here – ve sign appears due to lower limit]

Taking Laplace Transform of $tf'(t)$, we get $L\{t\, f'(t)\} = L\{-e^{-t}\} = -L\{e^{-t}\}$

$$\Rightarrow \quad -\frac{d}{ds}[sF(s) - f(0)] = -\frac{1}{s+1}$$

$$\Rightarrow \quad \frac{d}{ds}[s\, F(s)] = \frac{1}{s+1} \qquad \left[\because f(0) = \text{constant} \therefore \frac{d}{ds} f(0) = 0\right]$$

Integrating both the sides, we get

$$s\, F(s) = \log(s+1) + C \qquad \text{...(1)}$$

Now, by final value theorem, we have

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t) \qquad \text{...(2)}$$

Hence, $\lim_{s\to 0}[sF(s)] = \lim_{s\to 0}[\log(s+1) + C] = 0 + C = C$ ...(3)

Also, $$\lim_{t\to\infty}[f(t)] = \lim_{t\to\infty}\int_t^{\infty}\left(\frac{e^{-x}}{x}\right) dx = 0 \qquad \text{...(4)}$$

Putting the values of $\lim_{s\to 0}[s\,F(s)]$ and $\lim_{t\to\infty}[f(t)]$ from (3) and (4) in (2), we get
$C = 0$.

Hence from (1), $sF(s) = \log(s+1) \Rightarrow F(s) = \left\{\frac{\log(s+1)}{s}\right\}$

$$\Rightarrow \quad \boxed{L\int_t^\infty \left(\frac{e^{-x}}{x}\right)dx = \left[\frac{\log(s+1)}{s}\right]}$$

Example 16. *Find the Laplace Transform of t sin at.*

Solution. $L(t\sin at) = L\left(t\frac{e^{iat}-e^{-iat}}{2i}\right) = \frac{1}{2i}[L(t.e^{iat}) - L(te^{-iat})]$

$$= \frac{1}{2i}\left[-\frac{d}{ds}\frac{1}{s-ia} + \frac{d}{ds}\frac{1}{s+ia}\right] = \frac{1}{2i}\left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right] = \frac{1}{2i}\left[\frac{(s+ia)^2-(s-ia)^2}{(s-ia)^2(s+ia)^2}\right]$$

$$= \frac{1}{2i}\frac{(s^2+2ias-a^2)-(s^2-2ias-a^2)}{(s^2+a^2)^2} = \frac{1}{2i}\frac{4ias}{(s^2+a^2)^2} = \frac{2as}{(s^2+a^2)^2}$$ **Ans.**

Example 17. *Find the Laplace transform of t sinh at.*

Solution. $L(\sinh at) = \frac{a}{s^2-a^2}$

$$L[t\sinh at] = -\frac{d}{ds}\left(\frac{a}{s^2-a^2}\right) = \frac{2as}{(s^2-a^2)^2}$$ **Ans.**

Example 18. *Find the Laplace transform of t^2 cos at*

Solution. $L(\cos at) = \frac{s}{s^2+a^2}$

$$L(t^2\cos at) = (-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+a^2}\right] = \frac{d}{ds}\frac{(s^2+a^2).1-s(2s)}{(s^2+a^2)^2} = \frac{d}{ds}\frac{a^2-s^2}{(s^2+a^2)^2}$$

$$= \frac{(s^2+a^2)^2(-2s)-(a^2-s^2).2(s^2+a^2)(2s)}{(s^2+a^2)^4} = \frac{(s^2+a^2)(-2s)-(a^2-s^2)4s}{(s^2+a^2)^3}$$

$$= \frac{-2s^3-2a^2s-4a^2s+4s^3}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$ **Ans.**

Example 19. *Obtain the Laplace transform of $t^2e^t.\sin 4t$.*

(Uttarakhand II Sem., Summer 2010, U.P. II Semester, Summer 2002)

Solution. $L(\sin 4t) = \frac{4}{s^2+16}$,

$$L(e^t\sin 4t) = \frac{4}{(s-1)^2+16}$$

$$L(te^t\sin 4t) = -\frac{d}{ds}\left(\frac{4}{s^2-2s+17}\right) = \frac{4(2s-2)}{(s^2-2s+17)^2}$$

$$L\left(t^2e^t\sin 4t\right)=-\frac{d}{ds}\left(\frac{4(2s-2)}{\left(s^2-2s+17\right)^2}\right)=-4\frac{\left(s^2-2s+17\right)^2 2-(2s-2)2\left(s^2-2s+17\right)(2s-2)}{\left(s^2-2s+17\right)^4}$$

$$=-\,4\,\frac{(s^2-2s+17)\,2-2(2s-2)^2}{(s^2-2s+17)^3}=\frac{-4\left(2s^2-4s+34-8s^2+16s-8\right)}{\left(s^2-2s+17\right)^3}$$

$$=\frac{-4\left(-6s^2+12s+26\right)}{\left(s^2-2s+17\right)^3}=\frac{8\left[3s^2-6s-13\right]}{(s^2-2s+17)^3}$$ **Ans.**

Example 20. *Find the Laplace transform of the function*

$$f(t)=te^{-t}\sin 2t$$ *(U.P. II Semester, Summer 2002)*

Solution. $L[\sin 2t]=\dfrac{2}{s^2+4}$

$$L\left[e^{-t}\sin 2t\right]=\frac{2}{(s+1)^2+4}=F(s)\quad\text{(say)}$$

$$L\left(te^{-t}\sin 2t\right)=-F'(s)=-\frac{d}{ds}\left[\frac{2}{(s+1)^2+4}\right]=\frac{2\cdot 2(s+1)}{\left[(s+1)^2+4\right]^2}=\frac{4(s+1)}{\left[(s+1)^2+4\right]^2}$$ **Ans.**

EXERCISE 42.2

Find the Laplace transforms of the following :

1. $t_e at$ **Ans.** $\dfrac{1}{(s-a)^2}$

2. $t\cosh at$ **Ans.** $\dfrac{s^2+a^2}{\left(s^2-a^2\right)^2}$

3. $t\cos t$ **Ans.** $\dfrac{s^2-1}{\left(s^2+1\right)^2}$

4. $t\cosh t$ **Ans.** $\dfrac{s^2+1}{\left(s^2-1\right)^2}$

5. $t^2\sin t$ **Ans.** $\dfrac{2\left(3s^2-1\right)}{\left(s^2+1\right)^3}$

6. t^3e^{-3t} **Ans.** $\dfrac{6}{(s+3)^4}$

7. $t\sin^2 3t$ **Ans.** $\dfrac{1}{2}\left[\dfrac{1}{s^2}-\dfrac{s^2-36}{\left(s^2+36\right)^2}\right]$

8. $t\,e^{at}\sin at$ **Ans.** $\dfrac{2a(s-a)}{\left(s^2-2as+2a^2\right)^2}$

9. $te^{-t}\cosh t$ **Ans.** $\dfrac{s^2+2s+2}{\left(s^2+2s\right)^2}$

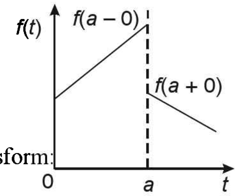
10. $t^2e^{-2t}\cos t$ **Ans.** $\dfrac{2(s^3+6s^2+9s+2)}{(s^2+4s+5)^3}$

11. $\int_0^t e^{-2t}t\sin^3 t\,dt$ **Ans.** $\dfrac{3\,(s+2)}{2s}\left[\dfrac{1}{\left[(s+2)^2+9\right]^2}-\dfrac{1}{\left[(s+2)^2+1\right]^2}\right]$

12. If $f(t)$ is continuous, except for an ordinary discontinuity at $t=a$, $(a>0)$ as given in the figure, then show that

$$L\left[f'(t)\right]=s\left[f(t)\right]-f(0)-e^{-as}\left[f(a+0)-f(a-0)\right]$$

(U.P. II Semester 2003)



13. Pick the correct statement for final value theorem of Laplace transform:

(*i*) $\underset{t \to 0}{\text{Lt}} f(t) = \underset{s \to \infty}{\text{Lt}} s\,F(s)$ (*ii*) $\underset{t \to \infty}{\text{Lt}} f(t) = \underset{s \to 0}{\text{Lt}} s\,F(s)$

(*U.P. II Semester 2010*) **Ans.** (*ii*)

42.14 LAPLACE TRANSFORM OF $\frac{1}{t}f(t)$ (Division by *t*)

If $L[f(t)] = F(s)$, then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty f(s)\,ds$ (*U.P. II Semester Summer, 2007, 2005*)

Proof. We know that $L[f(t)] = F(s)$ or $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ ...(1)

Integrating (1) w.r.t. '*s*', we have

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds = \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty \frac{-f(t)}{t}\left[e^{-st}\right]_s^\infty dt = \int_0^\infty \frac{-f(t)}{t}\left[0 - e^{-st}\right] dt = \int_0^\infty e^{-st}\left\{\frac{1}{t}.f(t)\right\} dt = L\left[\frac{1}{t}f(t)\right]$$

$$\Rightarrow \quad \boxed{L\left[\frac{1}{t}f(t)\right] = \int_0^\infty F(s)\,ds}$$ **Proved.**

Cor. $\mathrm{L}^{-1}\int_s^\infty F(s)\,ds = \frac{1}{t}\,f(t)$

Example 21. *Find the Laplace transform of* $\frac{\sin 2t}{t}$

Solution. $\mathrm{L}(\sin 2t) = \frac{2}{s^2+4}$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2+4}\,ds = 2.\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty = \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$= \cot^{-1}\frac{s}{2}$$ **Ans.**

Example 22. *Find the Laplace transform of* $f(t) = \int_0^t \frac{\sin at}{t}\,dt$

(*M.D.U., Dec. 2009, U.P., II Semester, Summer 2005*)

Solution. $L(\sin at) = \frac{a}{s^2+a^2}$

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

Hence, $L\left[\int_0^t \frac{\sin at}{t}\,dt\right] = \frac{1}{s}\cot^{-1}\frac{s}{a}$ **Ans.**

Example 23. *Find the Laplace transform of :*

$\frac{\cos at - \cos bt}{t}$ (*Uttarakhand, II Semester, June 2007, U.P., II Semester, 2004*)

Solution. Here, $f(t) = \frac{\cos at - \cos bt}{t}$

We know that, $L(\cos at - \cos bt) = L\,(\cos at) - L\,(\cos bt) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$$L\left(\frac{\cos at-\cos bt}{t}\right)=\int_s^{\infty}\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds=\left[\frac{1}{2}\log\left(s^2+a^2\right)-\frac{1}{2}\log\left(s^2+b^2\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^{\infty}=\frac{1}{2}\left[\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^{\infty}=\frac{1}{2}\log 1-\frac{1}{2}\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}=0-\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}\quad [\log 1=0]$$

$$=\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$ **Ans.**

Example 24. *If* $f(t)=\dfrac{e^{at}-\cos bt}{t}$, *find the Laplace transform of* $f(t)$.

(U.P. II Semester, Summer 2003)

Solution. $f(t)=\dfrac{e^{at}-\cos bt}{t}=\dfrac{e^{at}}{t}-\dfrac{\cos bt}{t}$

We know that, $L\left(e^{at}-\cos bt\right)=\left(\dfrac{1}{s-a}-\dfrac{s}{s^2+b^2}\right)$

$$\therefore\quad L\left(\frac{e^{at}-\cos bt}{t}\right)=\int_s^{\infty}\left(\frac{1}{s-a}-\frac{s}{s^2+b^2}\right)ds=\left[\log(s-a)-\frac{1}{2}\log\left(s^2+b^2\right)\right]_s^{\infty}$$

$$=\left[\frac{2\log(s-a)-\log\left(s^2+b^2\right)}{2}\right]_s^{\infty}=\frac{1}{2}\left[\log(s-a)^2-\log\left(s^2+b^2\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\log\frac{(s-a)^2}{s^2+b^2}\right]_s^{\infty}=\frac{1}{2}\left[\log\left\{\frac{\left(1-\frac{a}{s}\right)^2}{1+\frac{b^2}{s^2}}\right\}\right]_s^{\infty}$$

$$=\frac{1}{2}\left[0-\log\frac{\left(1-\frac{a}{s}\right)^2}{\left(1+\frac{b^2}{s^2}\right)}\right]=\frac{1}{2}\left[\log\frac{s^2+b^2}{(s-a)^2}\right]$$ **Ans.**

Example 25. *Find the Laplace transform of* $\dfrac{1-\cos t}{t^2}$.

Solution. L (1 – cos t) = L (1) – L (cos t) = $\dfrac{1}{s}-\dfrac{s}{s^2+1}$

$$L\left[\frac{1-\cos t}{t}\right]=\int_s^{\infty}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds=\left[\log s-\frac{1}{2}\log\left(s^2+1\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\log s^2-\log\left(s^2+1\right)\right]_s^{\infty}=\frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\log\frac{1}{\left(1+\frac{1}{s^2}\right)}\right]_s^{\infty}=\frac{1}{2}\left[0-\log\frac{s^2}{s^2+1}\right]=-\frac{1}{2}\log\frac{s^2}{s^2+1}$$

Again, $L\left[\dfrac{1-\cos t}{t^2}\right] = -\dfrac{1}{2}\int_s^{\infty} \log\dfrac{s^2}{s^2+1}\,ds = -\dfrac{1}{2}\int_s^{\infty}\left(\log\dfrac{s^2}{s^2+1}.1\right)ds$

Integrating by parts, we have,

$$= -\frac{1}{2}\left[\log\frac{s^2}{s^2+1}.s - \int \frac{s^2+1}{s^2}\frac{(s^2+1)2s - s^2(2s)}{(s^2+1)^2}.s\,ds\right]_s^{\infty}$$

$$= -\frac{1}{2}\left[s\log\frac{s^2}{s^2+1} - 2\int\frac{1}{s^2+1}ds\right]_s^{\infty} = -\frac{1}{2}\left[s\log\frac{s^2}{s^2+1} - 2\tan^{-1}s\right]_s^{\infty}$$

$$= -\frac{1}{2}\left[0-2\left(\frac{\pi}{2}\right) - s\log\frac{s^2}{s^2+1} + 2\tan^{-1}s\right] = -\frac{1}{2}\left[-\pi - s\log\frac{s^2}{s^2+1} + 2\tan^{-1}s\right]$$

$$= \frac{\pi}{2} + \frac{s}{2}\log\frac{s^2}{s^2+1} - \tan^{-1}s = \left(\frac{\pi}{2} - \tan^{-1}s\right) + \frac{s}{2}\log\frac{s^2}{s^2+1} = \cot^{-1}s + \frac{s}{2}\log\frac{s^2}{s^2+1}$$ **Ans.**

Example 26. *Evaluate* $L\left[e^{-4t}\dfrac{\sin 3t}{t}\right]$

Solution. $L\,[\sin 3t] = \dfrac{3}{s^2+3^2}$

$$\Rightarrow \quad L\left[\frac{\sin 3t}{t}\right] = \int_s^{\infty}\frac{3}{s^2+9}ds = \left[\frac{3}{3}\tan^{-1}\frac{s}{3}\right]_s^{\infty} = \left[\tan^{-1}\frac{s}{3}\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\frac{s}{3} = \cot^{-1}\frac{s}{3}$$

$$L\left[e^{-4t}\frac{\sin 3t}{t}\right] = \cot^{-1}\frac{s+4}{3} = \tan^{-1}\frac{3}{s+4}$$ **Ans.**

EXERCISE 42.3

Find Laplace transform of the following:

1. $\dfrac{1}{t}(1-e^t)$ **Ans.** $\log\dfrac{s-1}{s}$
2. $\dfrac{1}{t}\left(e^{-at} - e^{-bt}\right)$ **Ans.** $\log\dfrac{s+b}{s+a}$
3. $\dfrac{1}{t}(1-\cos at)$ **Ans.** $-\dfrac{1}{2}\log\dfrac{s^2}{s^2+a^2}$
4. $\dfrac{1}{t}\sin^2 t$ **Ans.** $\dfrac{1}{4}\log\dfrac{s^2+4}{s^2}$
5. $\dfrac{1}{t}\sinh t$ **Ans.** $-\dfrac{1}{2}\log\dfrac{s-1}{s+1}$
6. $\dfrac{1}{t}\left(e^{-t}\sin t\right)$ **Ans.** $\cot^{-1}(s+1)$
7. $\dfrac{1}{t}(1-\cos t)$ **Ans.** $\dfrac{1}{2}\,[\log\left(s^2+1\right) - \log s^{\partial}]$
8. $\displaystyle\int_0^{\infty}\frac{1}{t}e^{-2t}\sin t\,dt$ **Ans.** $\dfrac{1}{s}\cot^{-1}(s+2)$
9. $\displaystyle\int_0^{\infty}\frac{e^{-t}-e^{-3t}}{t}dt$ **Ans.** $\log 3$

42.15 LAPLACE TRANSFORM OF ERROR FUNCTION

Example 27. *Find* $L\left\{erf\sqrt{t}\right\}$ *and hence prove that*

$$L\left\{t.erf\,2\sqrt{t}\right\} = \frac{3s+8}{s^2(s+4)^{3/2}}$$ *(U.P. II Semester, Summer 2001)*

Solution. We know that $erf\sqrt{t} = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^{\sqrt{t}} e^{-x^2}dx$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right) dx = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} \ldots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left[\sqrt{t} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{10} - \frac{t^{7/2}}{42} + \ldots\right]$$

$$L\left\{erf\sqrt{t}\right\} = \frac{2}{\sqrt{\pi}} \left[\frac{\left|\frac{3}{2}\right.}{s^{3/2}} - \frac{\left|\frac{5}{2}\right.}{3s^{5/2}} + \frac{\left|\frac{7}{2}\right.}{10s^{7/2}} - \frac{\left|\frac{9}{2}\right.}{42s^{9/2}} + \ldots\right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\frac{1}{2}\left|\frac{1}{2}\right.}{s^{3/2}} - \frac{\frac{3}{2}.\frac{1}{2}\left|\frac{1}{2}\right.}{3s^{5/2}} + \frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\left|\frac{1}{2}\right.}{10s^{7/2}} - \frac{\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\left|\frac{1}{2}\right.}{42s^{9/2}} + \ldots\right] \qquad \left[\because \left|\frac{1}{2}\right. = \sqrt{\pi}\right]$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2}\frac{1}{s^{5/2}} + \frac{1.3}{2.4}\frac{1}{s^{7/2}} - \frac{1}{2}.\frac{3}{4}.\frac{5}{6}\frac{1}{s^{9/2}} + \ldots$$

$$= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2}.\frac{1}{s} + \frac{1}{2}.\frac{3}{4}.\frac{1}{s^2} - \frac{1}{2}.\frac{3}{4}.\frac{5}{6}\frac{1}{s^3} + \ldots.\right]$$

$$= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2}.\frac{1}{s} + \frac{\left(-\frac{1}{2}\right)\left\{-\frac{3}{2}\right\}}{2!}\frac{1}{s^2} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\frac{1}{s^3} + \ldots\right]$$

$$= \frac{1}{s^{3/2}} \left[1 + \frac{1}{s}\right]^{-\frac{1}{2}} = \frac{1}{s^{3/2}} \left[\frac{s}{s+1}\right]^{\frac{1}{2}} = \frac{1}{s\sqrt{(s+1)}}$$ **Ans.**

Now, $L\left\{erf\left(2\sqrt{t}\right)\right\} = L\left\{erf\sqrt{4t}\right\} = \frac{1}{4}\frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} = \frac{2}{s\sqrt{s+4}}$

$$L\left\{t.erf\left(2\sqrt{t}\right)\right\} = -\frac{d}{ds}\frac{2}{\sqrt{s^3+4s^2}} = -2\left(-\frac{1}{2}\right)\left[s^3+4s^2\right]^{-\frac{3}{2}}\left(3s^2+8s\right)$$

$$= \frac{3s^2+8s}{\left(s^3+4s^2\right)^{3/2}} = \frac{s(3s+8)}{s^3(s+4)^{3/2}} = \frac{3s+8}{s^2(s+4)^{3/2}}$$ **Proved.**

42.16 COMPLEMENTARY ERROR FUNCTION

This function is defined by

$$erf_c\left(\sqrt{t}\right) = 1 - erf\left(\sqrt{t}\right) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$$

Now, $L\left\{erf_c\left(\sqrt{t}\right)\right\} = L\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx\right\} = L(1) - \frac{2}{\sqrt{\pi}}L\left\{\int_0^{\sqrt{t}} e^{-x^2}\,dx\right\} = \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$

$$= \frac{\sqrt{(s+1)}-1}{s\sqrt{(s+1)}} = \frac{\left\{\sqrt{(s+1)}-1\right\}\left\{\sqrt{(s+1)}+1\right\}}{s\sqrt{(1+s)}\left\{\sqrt{(s+1)}+1\right\}}$$

$$= \frac{s+1-1}{s\sqrt{s+1}\,(\sqrt{s+1}+1)} = \frac{1}{\sqrt{(s+1)}\left\{\sqrt{(s+1)}+1\right\}}$$

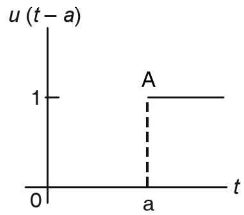
$$\boxed{\therefore L\left[erf_c\,(\sqrt{t})\right] = \frac{1}{\sqrt{s+1}\left\{\sqrt{s+1}+1\right\}}}$$ **Ans.**

42.17 UNIT STEP FUNCTION

With the help of unit step functions, we can find the inverse transform of functions, which cannot be determined with previous methods.

The unit step function $u(t - a)$ is defined as follows:

$$\boxed{u(t-a) = \begin{cases} \mathbf{0, \text{ when } } t < a \\ \mathbf{1, \text{ when } } t \geq a \end{cases}}$$ where $a \geq 0$



42.18 LAPLACE TRANSFORM OF UNIT FUNCTION

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

Proof. $L\left[u(t-a)\right] = \int_0^\infty e^{-st}u(t-a)\,dt = \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1\,dt = 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$

$$\boxed{L[u(t-a)] = \frac{e^{-as}}{s}}$$ **Proved.**

Example 28. *Express the following function in terms of unit step functions and find its Laplace transform:*

$$f(t) = \begin{cases} 8, & t < 2 \\ 6, & t \geq 2 \end{cases}$$

Solution. $f(t) = \begin{cases} 8+0, & t<2 \\ 8-2, & t \geq 2 \end{cases} = 8 + \begin{cases} 0, & t<2 \\ -2, & t\geq 2 \end{cases} = 8 + (-2)\begin{cases} 0, & t<2 \\ 1, & t \geq 2 \end{cases} = 8 - 2u(t-2)$

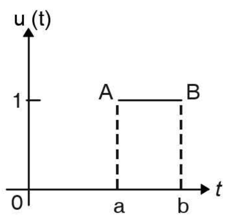
$$L\{f(t)\} = 8L(1) - 2\,L\,u(t-2) = \frac{8}{s} - 2\frac{e^{-2s}}{s}$$ **Ans.**

Example 29. *Draw the graph of $u(t - a) - u(t - b)$.*

Solution. As in Art 42.17 the graph of $u(t–a)$ is a straight line parallel to t-axis from A to ∞.

Similarly, the graph of $u(t-b)$ is a straight line parallel to t-axis from B to ∞.

Hence, the graph of $u(t - a) - u(t-b)$ is AB.



42.19 SECOND SHIFTING THEOREM

If $L[f(t)] = F(s)$, then $\boxed{L[f(t-a).\,u(t-a)] = e^{-as}F(s)}$

Proof. $L[f(t-a).u(t-a)] = \int_0^\infty e^{-st}[f(t-a).u(t-a)]dt$

$$= \int_0^a e^{-st}f(t-a).0\,dt + \int_a^\infty e^{-st}f(t-a)(1)dt = \int_a^\infty e^{-st}f(t-a)\,dt$$

$$= \int_0^\infty e^{-s(u+a)}f(u)\,du, \qquad \text{where } u = t-a$$

$$= e^{-sa}\int_0^\infty e^{-su}.f(u)\,du = e^{-sa}F(s)$$ **Proved.**

Example 30. *Express the following function in terms of unit step function and find its Laplace transform:*

$$f(t) = \begin{cases} E, & a < t < b \\ 0, & t \geq b \end{cases}$$

Solution. $f(t) = E\begin{cases} 1, & a < t < b \\ 0, & t \geq b \end{cases}$ $\quad [L[f(t-a).\,u(t-a)] = e^{-as}F(s)]$

$$L\{f(t)\} = E\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right]$$ **Ans.**

Example 31. *Express the following function in terms of unit step function :*

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$$

and find its Laplace transform. (*U.P.; II Semester, 2009*)

Solution. $f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$

$$= (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

$$= (t-1)u(t-1) - (t-1)u(t-2) + (3-t)u(t-2) + (t-3)u(t-3)$$

$$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$$

$$= e^{-s}L(t) - 2e^{-2s}L(t) - e^{-3s}L(t)$$

$[L[f(t-a).\,u(t-a)] = e^{-as}F(s)]$

$$L[f(t)] = \frac{e^{-s}}{s^2} - 2\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$ **Ans.**

Example 32. *Find L{F(t)} if*

$$F(t) = \begin{cases} \sin\left(t - \frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3} \end{cases}$$

Solution. $L\{F(t)\} = e^{-s\frac{\pi}{3}}L(\sin t)$ $\quad \left[\because a = \frac{\pi}{3}\right]$

$$= e^{-s\frac{\pi}{3}} \cdot \frac{1}{s^2+1}$$ (Using second shifting property) **Ans.**

42.20 THEOREM. $L[f(t)u(t-a)] = e^{-as}L[f(t+a)]$

Proof. $L[f(t).u(t-a)] = \int_0^\infty e^{-st}[f(t).u(t-a)]dt$

$$= \int_0^a e^{-st}[f(t).u(t-a)]dt + \int_a^\infty e^{-st}.[f(t).u(t-a)]dt = 0 + \int_a^\infty e^{-st}.f(t)(1)dt$$

$$= \int_a^\infty e^{-s(y+a)}.f(y+a)dy = e^{-as}\int_a^\infty e^{-sy}.f(y+a)\,dy \qquad (t-a=y)$$

$$= e^{-as}\int_a^\infty e^{-st}f(t+a)dt = e^{-as}L[f(t+a)]$$ **Proved.**

Example 33. *Find the Laplace transform of $t^2 u(t-3)$.*

Solution. $t^2.u(t-3) = \left[(t-3)^2 + 6(t-3) + 9\right]u(t-3)$

$$= (t-3)^2.u(t-3) + 6(t-3).u(t-3) + 9u(t-3)$$

$$L[t^2.u(t-3)] = L[(t-3)^2.u(t-3)] + 6L[(t-3).u(t-3)] + 9L[u(t-3)]$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Aliter. $L[t^2u(t-3)] = e^{-3s}L(t+3)^2 = e^{-3s}L\left[t^2+6t+9\right] = e^{-3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right]$ **Ans.**

Example 34. *Find the Laplace transform of $e^{-2t}u_\pi(t)$ where*

$$u_\pi(t) = \begin{cases} 0; & t<\pi \\ 1; & t>\pi \end{cases}$$

Solution. $u_\pi(t) = \begin{cases} 0; & t<\pi \\ 1; & t>\pi \end{cases}$

$$u_\pi(t) = u(t-\pi)$$

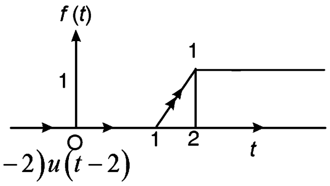
$$L[u_\pi(t)] = L[u(t-\pi)] = \frac{e^{-\pi s}}{s}$$

$$L\left[e^{-2t}u_\pi(t)\right] = \frac{e^{-\pi(s+2)}}{s+2}$$ **Ans.**

Example 35. *Express the following function in terms of unit step function and find its Laplace transform* $f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & 2<t \end{cases}$ *(U.P. II Semester, Summer 2002)*

Solution. The above function shown in the figure is expressed in algebraic form

$$f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & 2<t \end{cases} \qquad \ldots(1)$$



$$f(t) = (t-1)[u(t-1)-u(t-2)] + u(t-2)$$

$$= (t-1)u(t-1) - u(t-2)\{t-1-1\} = (t-1)u(t-1) - (t-2)u(t-2)$$

$$Lf(t) = L(t-1)u(t-1) - L(t-2)u(t-2) = \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$$ **Ans.**

Example 36. *Represent f(t) = sin 2t, 2π < t < 4π and f(t) = 0 otherwise, in terms of unit step function and then find its Laplace transform.*

Solution. $f(t)=\begin{cases}\sin 2t, & 2\pi<t<4\pi\\ 0, & \text{otherwise}\end{cases}$

$$f(t) = \sin 2t\left[u(t-2\pi)-u(t-4\pi)\right]$$

$$L[f(t)] = L[\sin 2t.\,u(t-2\pi)]-L[\sin 2t\,.\,u(t-4\pi)]$$

$$= e^{-2\pi s}L\sin 2(t+2\pi)-e^{-4\pi s}L\sin 2(t+4\pi)$$

$$= e^{-2\pi s}L\sin 2t-e^{-4\pi s}L\sin(2t)$$

$$= e^{-2\pi s}\frac{2}{s^2+4}-e^{-4\pi s}\frac{2}{s^2+4} = \left(e^{-2\pi s}-e^{-4\pi s}\right)\frac{2}{s^2+4}$$ **Ans.**

Example 37. *A function f (t) obeys the equation f (t) + 2* $\int_0^t f(t)\,dt = \cosh 2t$

Find the Laplace transform of f (t). (*U.P. II Semester Summer* 2006)

Solution. We have, $f(t)+2\int_0^t f(t)\ dt = \cosh 2t$

Taking Laplace transformation of both the sides, we get

$$L\{f(t)\}+2L\int_0^t f(t)\,dt = L(\cosh 2t) \quad\Rightarrow\quad F(s)+2\,.\,\frac{1}{s}F(s)=\frac{s}{s^2-4}$$

$$\Rightarrow\quad F(s)\left\{1+\frac{2}{s}\right\}=\frac{s}{s^2-4} \quad\Rightarrow\quad F(s)\left\{\frac{s+2}{s}\right\}=\frac{s}{s^2-4}$$

$$\Rightarrow\quad F(s)=\left(\frac{s}{s^2-4}\right)\left(\frac{s}{s+2}\right) \quad\Rightarrow\quad F(s)=\frac{s^2}{(s^2-4)(s+2)}$$ **Ans.**

EXERCISE 42.4

Find the Laplace transform of the following:

1. $f(t)=\begin{cases}t-1, & 1<t<2\\ 0, & \text{otherwise}\end{cases}$ **Ans.** $\frac{e^{-s}-e^{-2s}}{s^2}-\frac{e^{-2s}}{s}$
2. $e^t u(t-1)$ **Ans.** $\frac{e^{-(s-1)}}{s-1}$
3. $\frac{1-e^{2t}}{t}+tu(t)+\cosh t.\cos t$ **Ans.** $\log\frac{s-2}{s}+\frac{1}{s^2}+\frac{s^3}{s^4+4}$
4. $t^2u(t-2)$ **Ans.** $\frac{e^{-2s}}{s^3}\left(4s^2+4s+2\right)$
5. $\sin t\ u\ (t-4)$ **Ans.** $\frac{e^{-4s}}{s^2+1}[\cos 4+s\sin 4]$
6. $f(t)=K(t-2)[u(t-2)-u(t-3)]$ **Ans.** $\frac{K}{s^2}\left[e^{-2s}-(s+1)e^{-3s}\right]$
7. $f(t)=K\frac{\sin\pi t}{T}[u(t-2T)-u(t-3T)]$ **Ans.** $\frac{K\pi T}{s^2T^2+\pi^2}\left(e^{-2sT}-e^{-3sT}\right)$

Express the following in terms of unit step functions and obtain Laplace transforms.

8. $f(t)=\begin{cases}t, & 0<t<2\\ 0, & 2<t\end{cases}$ **Ans.** $u(t)-u(t-2),\frac{1-(2s+1)e^{-2s}}{s^2}$
9. $f(t)=\begin{cases}\sin t, & 0<t<\pi\\ t, & t>\pi\end{cases}$ **Ans.** $\frac{1+e^{-\pi s}}{s^2+1}+\frac{e^{-\pi s}(\pi s+1)}{s^2}$

10. $f(t)=\begin{cases}4, & 0<t<1\\ -2, & 0<t<3\\ 5, & t>3\end{cases}$ **Ans.** $\dfrac{4-6e^{-s}+7e^{-3s}}{s}$

42.21. PERIODIC FUNCTIONS

Let $f(t)$ be a periodic function with period T, then

$$\boxed{L[f(t)]=\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}}$$

Proof. $L[f(t)]=\int_0^\infty e^{-st}f(t)\,dt = \int_0^T e^{-st}f(t)\,dt+\int_T^{2T} e^{-st}f(t)\,dt+\int_{2T}^{3T} e^{-st}f(t)\,dt+\ldots$

Substituting $t = u + T$ in second integral and $t = u + 2\,T$ in third integral, and so on.

$$L[f(t)] = \int_0^T e^{-st}f(t)\,dt+\int_0^T e^{-s(u+T)}f(u+T)\,du+\int_0^T e^{-s(u+2T)}f(u+2T)\,du+\ldots$$

$$= \int_0^T e^{-st}f(t)\,dt+e^{-sT}\int_0^T e^{-su}f(u)\,du+e^{-2sT}\int_0^T e^{-su}f(u)\,du+\ldots$$

$$[f(u)=f(u+T)=f(u+2T)=f(u+3T)=\ldots.]$$

$$= \int_0^T e^{-st}f(t)\,dt+e^{-sT}\int_0^T e^{-st}f(t)\,dt+e^{-2sT}\int_0^T e^{-st}f(t)\,dt+\ldots$$

$$= \left[1+e^{-sT}+e^{-2sT}+e^{-3sT}+\ldots\right]\int_0^T e^{-st}f(t)\,dt \qquad \left[1+a+a^2+a^3+\ldots=\frac{1}{1-a}\right]$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt.$$

Proved.

Example 38. *Find the Laplace transform of the waveform*

$$f(t)=\left(\frac{2t}{3}\right),\ 0\le t\le 3.$$

Solution. $L[f(t)]=\dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st}f(t)\,dt$

$$L\left[\frac{2t}{3}\right]=\frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\left(\frac{2}{3}t\right)dt=\frac{1}{1-e^{-3s}}\frac{2}{3}\left[\frac{te^{-st}}{-s}-(1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$=\frac{2}{3}\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s}-\frac{e^{-3s}}{s^2}+\frac{1}{s^2}\right]=\frac{2}{3}\cdot\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s}+\frac{1-e^{-3s}}{s^2}\right]$$

$$=\frac{2e^{-3s}}{-s\left(1-e^{-3s}\right)}+\frac{2}{3s^2}$$

Ans.

Example 39. *Draw the graph and find the Laplace transform of the triangular wave function of period 2 C given by*

$$f(t)=\begin{cases} t, & 0<t\le C\\ 2C-t, & C<t<2C\end{cases}$$

(Uttarakhand, II Semester, June 2007)

Solution. Period = $2C = T$

Laplace transform of periodic function $f(t)$

$$L\{f(t)\}=\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

$$L\{f(t)\}=\frac{1}{1-e^{-2Cs}}\int_0^{2C} e^{-st}f(t)\,dt \qquad (T=2c)$$

On putting the values of $f(t)$, we get

$$L\left[f(t)\right]=\frac{1}{1-e^{-2Cs}}\left[\int_0^C e^{-st}t\,dt+\int_C^{2C}e^{-st}(2C-t)\,dt\right]$$

$$=\frac{1}{1-e^{-2Cs}}\left[\left\{\frac{te^{-st}}{-s}-1.\frac{e^{-st}}{(-s)^2}\right\}_0^C+\left\{(2C-t)\frac{e^{-st}}{(-s)}-(-1)\frac{e^{-st}}{(-s)^2}\right\}_C^{2C}\right]$$

$$=\frac{1}{1-e^{-2Cs}}\left[\left\{\frac{C.e^{-Cs}}{-s}-\frac{e^{-Cs}}{(-s)^2}-0+\frac{1}{s^2}\right\}+\left\{(2C-2C)\frac{e^{-2Cs}}{(-s)}+\frac{e^{-2Cs}}{s^2}-\left((2C-C)\frac{e^{-Cs}}{-s}+\frac{e^{-Cs}}{s^2}\right)\right\}\right]$$

$$=\frac{1}{1-e^{-2Cs}}\left\{-\frac{Ce^{-Cs}}{s}-\frac{e^{-Cs}}{s^2}+\frac{1}{s^2}+\frac{e^{-2Cs}}{s^2}+\frac{Ce^{-Cs}}{s}-\frac{e^{-Cs}}{s^2}\right\}$$

$$=\frac{1}{1-e^{-2Cs}}\left\{\frac{1}{s^2}(1-2e^{-Cs}+e^{-2Cs})\right\}=\frac{\left(1-e^{-Cs}\right)^2}{s^2\left(1+e^{-Cs}\right)\left(1-e^{-Cs}\right)}=\frac{1-e^{-Cs}}{s^2\left(1+e^{-Cs}\right)}$$ **Ans.**

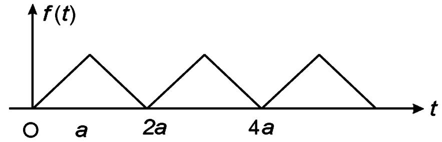
Example 40. *Draw the graph of the periodic function*

$$f(t)=\begin{cases} t, & 0<t<\pi \\ \pi-t, & \pi<t<2\pi \end{cases}$$

and find its Laplace transform. (*U.P. Second Semester, 2003*)

Solution. Period $=2\pi=T$

Laplace transform of Periodic functions



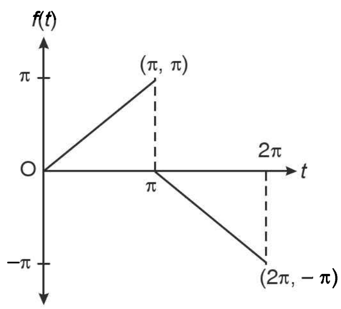
$$L\{f(t)\}=\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

$$=\frac{1}{1-e^{-2\pi s}}\int_0^{2\pi}e^{-st}f(t)\,dt=\frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi}e^{-st}t\,dt+\int_{\pi}^{2\pi}e^{-st}(\pi-t)\,dt\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{\frac{t.e^{-st}}{-s}-1.\frac{e^{-st}}{(-s)^2}\right\}_0^{\pi}+\left\{(\pi-t)\frac{e^{-st}}{(-s)}-(-1)\frac{e^{-st}}{(-s)^2}\right\}_{\pi}^{2\pi}$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\left\{\frac{\pi.e^{-\pi s}}{-s}-\frac{e^{-\pi s}}{(-s)^2}-0+\frac{1}{s^2}\right\}+\left\{(\pi-2\pi)\frac{e^{-2\pi s}}{-s}+\frac{e^{-2\pi s}}{s^2}-\left((\pi-\pi)\frac{e^{-\pi s}}{-s}+\frac{e^{-\pi s}}{s^2}\right)\right\}\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{-\frac{\pi e^{-\pi s}}{s}-\frac{e^{-\pi s}}{s^2}+\frac{1}{s^2}+\pi\frac{e^{-2\pi s}}{s}+\frac{e^{-2\pi s}}{s^2}-0-\frac{e^{-\pi s}}{s^2}\right\}$$



$$=\frac{1}{1-e^{-2\pi s}}\left\{-\frac{\pi}{s}e^{-\pi s}+\frac{\pi}{s}e^{-2\pi s}+\frac{1}{s^2}-\frac{1}{s^2}.e^{-\pi s}+\frac{1}{s^2}e^{-2\pi s}-\frac{e^{-\pi s}}{s^2}\right\}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{\pi}{s}\left(e^{-2\pi s}-e^{-\pi s}\right)+\frac{1}{s^2}\left(1+e^{-2\pi s}-2e^{-\pi s}\right)\right] = \frac{-\pi s e^{-\pi s}\left(1-e^{-\pi s}\right)+\left(1-e^{-\pi s}\right)^2}{s^2\left(1+e^{-\pi s}\right)\left(1-e^{-\pi s}\right)}$$

$$= \frac{-\pi s e^{-\pi s}+1-e^{-\pi s}}{s^2\left(1+e^{-\pi s}\right)}$$

Ans.

Example 41. *Find the Laplace transform of the function (Half wave rectifier)*

$$f(t) = \begin{cases} \sin \omega t & \text{for } 0 < t < \dfrac{\pi}{\omega} \\ 0 & \text{for } \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega}. \end{cases}$$

(U.P. II Semester, 2010, Summer 2002)

Solution. $L[f(t)] = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt \qquad \left[\begin{array}{l} f(t) \text{ is a periodic function} \\ T = \dfrac{2\pi}{\omega} \end{array}\right.$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\sin \omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\times 0 \times dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\pi/\omega} e^{-st}\sin\ \omega t\ dt \qquad \left[\int e^{ax}\sin bx\,dx = e^{ax}\frac{(a\sin bx - b\cos bx)}{a^2+b^2}\right]$$

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-st}(-s\sin\omega t - \omega\cos\omega t)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{\omega e^{-\frac{\pi}{\omega}s}+\omega}{s^2+\omega^2}\right] = \frac{\omega\left[1+e^{-\frac{\pi s}{\omega}}\right]}{\left(s^2+\omega^2\right)\left[1-e^{-\frac{2\pi s}{\omega}}\right]} = \frac{\omega\left[1+e^{-\frac{\pi}{\omega}s}\right]}{(s^2+\omega^2)\left(1-e^{-\frac{\pi}{\omega}s}\right)\left(1+e^{-\frac{\pi}{\omega}s}\right)}$$

$$= \frac{\omega}{\left(s^2+\omega^2\right)\left[1-e^{-\frac{\pi s}{\omega}}\right]}$$

Ans.

Example 42. *Find the Laplace Transform of the Periodic function (saw tooth wave)*

$$f(t) = \frac{kt}{T} \text{ for } 0 < t < T, \qquad f(t+T) = f(t)$$

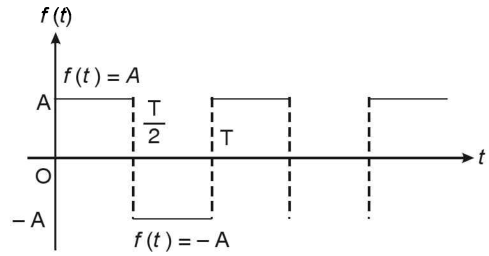
Solution. $L[f(t)] = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st} f(t)\,dt = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st}\frac{kt}{T}dt$

$$= \frac{1}{1-e^{-sT}}\frac{k}{T}\int_0^T e^{-st}.t\,dt = \frac{k}{T\left(1-e^{-sT}\right)}\left[t\frac{e^{-st}}{-s} - \int 1.\frac{e^{-st}}{-s}dt\right]_0^T \quad \text{Integrating by parts}$$

$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^T = \frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{Te^{-sT}}{-s} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{Te^{-sT}}{-s}+\frac{1}{s^2}\left(1-e^{-sT}\right)\right] = -\frac{ke^{-sT}}{s\left(1-e^{-sT}\right)}+\frac{k}{Ts^2}$$ **Ans.**

Example 43. *Obtain Laplace transform of rectangular wave given by*



Solution. We know that Laplace transform of a periodic function *i.e.*,

$$L f(t) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} = \frac{\int_0^{\frac{T}{2}} e^{-st} A\,dt + \int_{\frac{T}{2}}^{T} e^{-st}(-A)\,dt}{1-e^{-sT}}$$

$$= A\frac{\left[\frac{e^{-st}}{-s}\right]_0^{\frac{T}{2}} - \left[\frac{e^{-st}}{-s}\right]_{\frac{T}{2}}^{T}}{1-e^{-sT}} = \frac{A}{1-e^{-sT}}\left[-\frac{e^{-\frac{sT}{2}}}{s}+\frac{1}{s}+\frac{e^{-sT}}{s}-\frac{e^{-\frac{sT}{2}}}{s}\right]$$

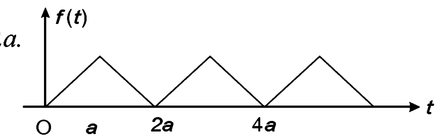
$$= \frac{A}{s\left(1-e^{-sT}\right)}\left[1-2e^{-\frac{sT}{2}}+e^{-sT}\right] = \frac{A}{s\left(1-e^{-sT}\right)}\left[1-e^{-\frac{sT}{2}}\right]^2$$

$$= \frac{A\left[1-e^{-\frac{sT}{2}}\right]^2}{s\left(1+e^{-\frac{sT}{2}}\right)\left(1-e^{-\frac{sT}{2}}\right)} = \frac{A}{s}\frac{\left(1-e^{-\frac{sT}{2}}\right)}{\left(1+e^{-\frac{sT}{2}}\right)} = \frac{A\left(e^{\frac{sT}{4}}-e^{-\frac{sT}{4}}\right)}{s\left(e^{\frac{sT}{4}}+e^{-\frac{sT}{4}}\right)} = \frac{A}{s}\tanh\frac{sT}{4}$$ **Ans.**

Example 44. *Draw the graph of the following periodic function and find its Laplace transform:*

$$f(t) = \begin{cases} t & \text{for } 0 < t \le a \\ 2a-t & \text{for } a < t < 2a \end{cases}$$ *(U.P. II Semester, Summer 2002)*

Solution. The given function is periodic with period $2a$.



$$\therefore \qquad L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} f(t)\,e^{-st}\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a f(t)e^{-st}dt + \int_a^{2a} f(t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a t\,e^{-st}dt + \int_a^{2a}(2a-t)\,e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left\{t\frac{e^{-st}}{-s}-\frac{e^{-st}}{(-s)^2}\right\}_0^a + \left\{(2a-t)\frac{e^{-st}}{-s}+\frac{e^{-st}}{(-s)^2}\right\}_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[-\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right] = \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2}+\frac{e^{-2as}}{s^2}-2\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{1}{\left(1-e^{-2as}\right)}\left(1+e^{-2as}-2e^{-as}\right) = \frac{1}{s^2}\frac{\left(1-e^{-as}\right)^2}{\left(1+e^{-as}\right)\left(1-e^{-as}\right)} = \frac{1}{s^2}\left[\frac{1-e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{1}{s^2}\left[\frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{e^{\frac{as}{2}}+e^{-\frac{as}{2}}}\right] = \frac{1}{s^2}\tanh\frac{as}{2}$$ **Ans.**

Example 45. *A periodic square wave function f(t), in terms of unit step functions, is written as*

$$f(t) = k\left[u_0(t)-2u_a(t)+2u_{2a}(t)-2u_{3a}(t)+\ldots\right]$$

Show that the Laplace transform of f (t) is given by

$$L\left[f(t)\right] = \frac{k}{s}\tanh\left(\frac{as}{2}\right)$$

Solution.
$$f(t) = k\left[u_0(t)-2u_a(t)+2u_{2a}(t)-2u_{3a}(t)+\ldots\right]$$

$$f(t) = k\,[u\,(t-0) - 2u\,(t-a) + 2u\,(t-2a) - 2u\,(t-3a) + \ldots\ldots.]$$

$$L\,[f(t)] = k\,[L\,u\,(t-0) - 2\,L\,u\,(t-a) + 2\,L\,u\,(t-2a) - 2\,L\,u\,(t-3a) + \ldots..]$$

$$= k\left[\frac{1}{s}-2\frac{e^{-as}}{s}+2\frac{e^{-2as}}{s}-2\frac{e^{-3as}}{s}+\ldots\right] = \frac{k}{s}\left[1-2e^{-as}+2e^{-2as}-2e^{-3as}+\ldots\right]$$

$$= \frac{k}{s}\left[1-2(e^{-as}-e^{-2as}+e^{-3as}-\ldots)\right] = \frac{k}{s}\left[1-2\frac{e^{-as}}{1+e^{-as}}\right] = \frac{k}{s}\left[\frac{1+e^{-as}-2e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{k}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right] = \frac{k}{s}\left[\frac{e^{\frac{as}{2}}-e^{\frac{-as}{2}}}{e^{\frac{as}{2}}+e^{-\frac{as}{2}}}\right] = \frac{k}{s}\tanh\frac{as}{2}$$ **Ans.**

EXERCISE 42.5

1. Find the Laplace transform of the periodic function

$f(t) = e^t$ for $0 < t < 2\pi$ **Ans.** $\dfrac{e^{2(1-s)\pi}-1}{(1-s)\left(1-e^{-2\pi s}\right)}$

2. Obtain Laplace transform of full wave rectified sine wave given by

$f(t) = \sin\omega t, \quad 0 < t < \dfrac{\pi}{\omega}$ **Ans.** $\dfrac{\omega}{\left(s^2+\omega^2\right)}\coth\dfrac{\pi s}{2\omega}$

3. Find the Laplace transform of the staircase function

$f(t) = kn,\ np < t < (n+1)p,\ n = 0, 1, 2, 3$ **Ans.** $\dfrac{ke^{ps}}{s\left(1-e^{-ps}\right)}$

Find Laplace transform of the following:

4. $f(t) = t^2,\ 0 < t < 2,\ f(t+2) = f(t)$ **Ans.** $\dfrac{2-e^{-2s}-4se^{-2s}-4s^2e^{-2s}}{s^3\left(1-e^{-2s}\right)}$

5. $f(t) = \begin{cases} 1, & 0 \le t \le \dfrac{a}{2} \\ -1, & \dfrac{a}{2} \le t < a \end{cases}$ *(U.P. II Semester, 2004)* **Ans.** $\dfrac{1}{s}\tanh\dfrac{as}{4}$

6. $f(t)=\begin{cases}\cos\omega t, & 0<t<\dfrac{\pi}{\omega}\\ 0, & \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{\omega}\end{cases}$ **Ans.** $\dfrac{s}{\left(s^2+\omega^2\right)\left(1-e^{-\frac{\pi s}{\omega}}\right)}$

7. $f(t)=\begin{cases}t, & 0<t<1\\ 0, & 1<t<2\end{cases}\quad f(t+2)=f(t)$ **Ans.** $\dfrac{1-e^{-s}(s+1)}{s^2\left(1-e^{-2s}\right)}$

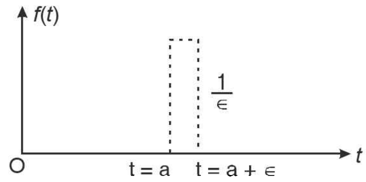
8. $f(t)=\begin{cases}\dfrac{2t}{T}, & 0\le t\le\dfrac{T}{2}\\ \dfrac{2}{T}(T-t), & \dfrac{T}{2}\le t\le T\end{cases}\quad f(t+T)=f(t)$ **Ans.** $\dfrac{2}{Ts^2}\tanh\dfrac{sT}{4}-\dfrac{1}{s\left(e^{\frac{sT}{2}}+1\right)}$

42.22 IMPULSE FUNCTION

When a large force acts for a short time, then the product of the force and the time is called impulse in applied mechanics. The unit impulse function is the limiting function.

$$\delta(t-1)=\begin{cases}\dfrac{1}{\varepsilon}, & a<t<a+\varepsilon\\ 0, & \text{otherwise}\end{cases}$$

The value of the function (height of the strip in the figure) becomes infinite as $\varepsilon\to 0$ and the area of the rectangle is unity.



(1) The Unit Impulse function is defined as follows:

$$\delta(t-a)=\begin{cases}\infty & \text{for } t=a\\ 0 & \text{for } t\ne a\end{cases}$$

and $\int_0^\infty \delta(t-a).dt=1$ [Area of strip = 1]

(2) Laplace Transform of Unit Impulse function

$$\int_0^\infty f(t)\ \delta(t-a)\,dt=\int_a^{a+\varepsilon} f(t).\frac{1}{\varepsilon}dt$$

$\left\{\begin{array}{l}\text{Mean value Theorem}\\ \int_a^b f(t)\,dt=(b-a)f(\eta)\end{array}\right.$

$$=(a+\varepsilon-a)f(\eta).\frac{1}{\varepsilon}$$

where $a<\eta<a+\varepsilon$

$$=f(\eta)$$

Property I. $\boxed{\int_0^\infty f(t)\delta(t-a)\,dt=f(a)}$ as $\varepsilon\to 0$

Note. If $f(t)=e^{-st}$ and $L[\delta(t-a)]=e^{-as}$

Example 46. *Evaluate* $\int_{-\infty}^{\infty} e^{-5t}\delta(t-2)\,dt$.

Solution. $\int_{-\infty}^{\infty} e^{-5t}\delta(t-2)\,dt=e^{-5\times 2}=e^{-10}$ **Ans.**

Property II: $\boxed{\int_{-\infty}^{\infty} f(t)\delta'(t-a)\,dt=-f'(a)}$

Proof. $\int_{-\infty}^{\infty} f(t)\delta'(t-a)\,dt=\left[f(t).\delta(t-a)\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty} f'(t)\delta(t-a)\,dt$

$$=0-0-f'(a)=-f'(a)$$

Example 47. *Find the Laplace transform of* $t^3\delta(t-4)$

Solution. $Lt^3\delta(t-4) = \int_0^\infty e^{-st}t^3\delta(t-4)\,dt \;=\; 4^3 e^{-4s}$ **Ans.**

EXERCISE 42.6

Evaluate the following:

1. $\int_0^\infty e^{-3t}\delta(t-4)\,dt$ **Ans.** e^{-12}
2. $\int_{-\infty}^\infty \sin 2t\,\delta\left(t-\frac{\pi}{4}\right)dt$ **Ans.** 1
3. $\int_{-\infty}^\infty e^{-3t}\delta'(t-2)\,dt$ **Ans.** $3e^{-6}$

Find Laplace transform of

4. $\dfrac{\delta(t-4)}{t}$ **Ans.** $\dfrac{e^{-4s}}{4}$
5. $\cos t \log t\, \delta\,(t-\pi)$ **Ans.** $-e^{-\pi s}\log\pi$
6. $e^{-4t}\delta(t-3)$ **Ans.** $e^{-3(s+4)}$

42.23 CONVOLUTION THEOREM

If $\quad L[f_1(t)] = F_1(s)$ and $L[f_2(t)] = F_2(s)$

then $\quad L\left\{\int_0^t f_1(x)\,f_2(t-x)\,dx\right\} = F_1(s)\cdot F_2(s)$

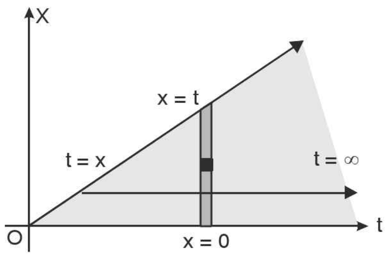
or $\quad \boxed{L^{-1}(F_1(s).F_2(s)) = \int_0^t f_1(x)f_2(t-x)dx}$

Proof. We have

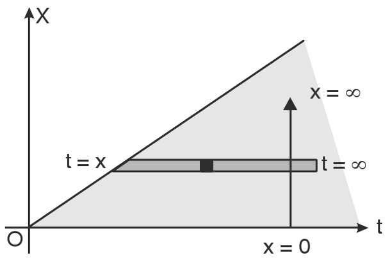
$$L\left\{\int_0^\infty f_1(x)\,f_2(t-x)\,dx\right\} = \int_0^\infty e^{-st}\left[\int_0^t f_1(x)\,f_2(t-x)\,dx\right]dt \qquad \text{(By Definition)}$$

where the double integral is taken over the infinite region in the first quadrant lying between the lines $x = 0$ and $x = t$.

Here first we are integrating w.r.t. "x", within limits $x = 0$ and $x = t$, and then we will integrate w.r.t. "t" with limits $t = 0$ and $t = \infty$.



On changing the order of integration first we integrate w.r.t. "t" with limits $t = x$ and $t = \infty$ and then w.r.t. "x" with limits $x = 0$ and $x = \infty$.



On changing the order of integration, the integral becomes

$$\int_0^\infty dx\left[\int_x^\infty e^{-st} f_1\,(x).\,f_2\,(t-x)\,dt\right]$$

$$=\int_0^\infty dx\left[\int_x^\infty e^{-s(t-x+x)} f_1.\,(x).\,f_2\,(t-x)\,dt\right]=\int_0^\infty dx\left[\int_x^\infty e^{-s(t-x)}.\,e^{-sx} f_1\,(x)\,.\,f_2\,(t-x)\,dt\right]$$

$$=\int_0^\infty e^{-sx}\,f_1\,(x)\,dx\left[\int_x^\infty e^{-s(t-x)}\,f_2\,(t-x)\,dt\right]=\int_0^\infty e^{-sx}\,f_1\,(x)\,dx\left[\int_x^\infty e^{-sz}\,f_2\,(z)\,dz\right]$$

[Put $t-x=z \Rightarrow dt=dz$]

$$=\int_0^\infty e^{-sx} f_1(x)\,dx\int_0^\infty e^{-sz} f_2(z)\,dz,$$ Lower limit $x-x=z \Rightarrow z=0$]

$$=\int_0^\infty e^{-sx} f_1(x)\,F_2(s)\,dx=\left[\int_0^\infty e^{-sx} f_1(x)\,dx\right]F_2(s)\;=\;F_1(s)\,F_2(s)$$ **Proved.**

Example 48. *Find the Laplace transform of* $\int_0^t e^x\,.\sin\,(t-x)\,dx$

Solution. By Convolution Theorem

$$L\int_0^t f_1\,(x)\;f_2(t-x)\;dx=F_1\,(s)\,.\,F_2\,(s)$$

$$\Rightarrow\; L\int_0^t e^x.\sin\,(t-x)\;dx\;=\;L\,(e^x)\,.\,L\sin t\;=\frac{1}{s-1}\,\frac{1}{s^2+1}\;=\frac{1}{(s-1)\,(s^2+1)}$$ **Ans.**

Note. Convolution Theorem is generally used to find Inverse Laplace transform of the product of two functions, discussed in the next chapter.

42.24 LAPLACE TRANSFORM OF BESSEL FUNCTIONS J_0 (*x*) and J_1 (*x*)

We know that

$$J_n(x)=\frac{x^2}{2^n\,\Gamma\,(n+1)}\left[1-\frac{x^2}{2\,.\,(2n+2)}+\frac{x^4}{2.4\,(2n+2)\,(2n+4)}-.....\right]$$

$$J_0(t)=\left[1-\frac{t^2}{2^2}+\frac{t^4}{2^2.4^2}-\frac{t^6}{2^2.4^2.6^2}+...\right]$$

Taking Laplace transforms of both sides, we have

$$LJ_0(t)=\frac{1}{s}-\frac{1}{2^2}.\frac{2!}{s^3}+\frac{1}{2^2.4^2}\frac{4!}{s^5}-\frac{1}{2^2.4^2.6^2}\frac{6!}{s^7}+...$$

$$=\frac{1}{s}\left[1-\frac{1}{2}\left(\frac{1}{s^2}\right)+\frac{1.3}{2.4}\left(\frac{1}{s^4}\right)-\frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right)+...\right]$$

$$=\frac{1}{s}\left[1+\left(-\frac{1}{2}\right)\left(\frac{1}{s^2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{1}{s^2}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{1}{s^2}\right)^3+...\right]$$

$$=\frac{1}{s}\left[1+\frac{1}{s^2}\right]^{-\frac{1}{2}}$$ (By Binomial theorem)

$$=\frac{1}{s}\left[\frac{s^2+1}{s^2}\right]^{-\frac{1}{2}}\;=\frac{1}{s}\left[\frac{s^2}{s^2+1}\right]^{\frac{1}{2}}\;=\frac{1}{\sqrt{s^2+1}}$$ ... (1) **Ans.**

We know that $\quad Lf\,(at)=\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

$$LJ_0(at) = \frac{1}{a}\frac{1}{\sqrt{\frac{s^2}{a^2}+1}} = \frac{1}{\sqrt{s^2+a^2}} \qquad \text{[From (1)]}$$

$$LJ_1(x) = -LJ_0'(x) = -[sLJ_0(x) - J_0(0)] = -\left[s.\frac{1}{\sqrt{s^2+1}} - 1\right] = 1 - \frac{s}{\sqrt{s^2+1}} \qquad \textbf{Ans.}$$

EXERCISE 42.7

Find the Laplace transform of the following:

1. $e^{ax} J_0(bx)$ **Ans.** $\dfrac{1}{\sqrt{s^2+2as+a^2+b^2}}$
2. $x J_0(ax)$ **Ans.** $\dfrac{s}{(s^2+a^2)^{3/2}}$
3. $x J_1(x)$ **Ans.** $\dfrac{1}{(s^2+1)^{3/2}}$

42.25 EVALUATION OF INTEGRALS

We can evaluate number of integrals having lower limit 0 and upper limit ∞ by the help of Laplace transform.

Example 49. *Evaluate* $\int_0^\infty te^{-3t}\sin t\, dt$

Solution. $\int_0^\infty te^{-3t}\sin t\, dt = \int_0^\infty te^{-st}\sin\ t\ dt \qquad (s = 3)$

$$= L(t\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

Putting $s = 3$, we get $= \dfrac{2\times 3}{(3^2+1)^2} = \dfrac{6}{100} = \dfrac{3}{50}$ **Ans.**

Example 50. *Evaluate* $\int_0^\infty \frac{e^{-t}\sin t}{t}dt$ and $\int_0^\infty \frac{\sin t}{t}dt$ (*U.P., II Semester, 2009*)

Solution. $\int_0^\infty \frac{e^{-t}\sin t}{t}dt = \int_0^\infty e^{-st}\frac{\sin t}{t}dt \qquad (s = 1)$

$$= L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}ds = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s \qquad ...(1)$$

$$= \frac{\pi}{2} - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \qquad (s = 1) \quad \textbf{Ans.}$$

On putting $s = 0$ in (1), we get $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2}$ **Ans.**

Example 51. *Evaluate* $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}dt$

Solution. $\int_0^\infty e^{-st}(e^{-at} - e^{-bt})dt = L(e^{-at} - e^{-bt}) = L(e^{-at}) - L(e^{-bt}) = \left(\frac{1}{s+a} - \frac{1}{s+b}\right)$

$$\int_0^\infty e^{-st}\left(\frac{e^{-at} - e^{-bt}}{t}\right)dt = L\frac{1}{t}(e^{-at} - e^{-bt}) = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds$$

$$= [\log(s+a) - \log(s+b)]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty = \left[\log \frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]_s^\infty = \left[\log 1 - \log \frac{s+a}{s+b}\right] = \log \frac{s+b}{s+a}$$

Putting s = 0 in above, we get $\int_0^\infty \left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \log\left(\frac{b}{a}\right)$ **Ans.**

Example 52. *Show that* $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0$

Solution. L $\{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3} L\{\sin t\}$

$$\Rightarrow \quad \int_0^\infty e^{-st} t^3 \sin t \, dt = \frac{-d^3}{ds^3} \frac{1}{s^2+1}$$

$$\left[\begin{array}{l}\text{This is G.P.}\\ \text{Sum} = \dfrac{a}{1-r}\end{array}\right]$$

$$= -\frac{d^2}{ds^2}\left[-\frac{2s}{(s^2+1)^2}\right] = \frac{d}{ds}\left[\frac{(s^2+1)^2(2) - 2s\,[2(s^2+1)]\,(2s)}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{2(s^2+1) - 8s^2}{(s^2+1)^3}\right] = \frac{d}{ds}\left[\frac{-6s^2+2}{(s^2+1)^3}\right] = \frac{(s^2+1)^3(-12s) - (-6s^2+2)3(s^2+1)^2(2s)}{(s^2+1)^6}$$

$$= \frac{(s^2+1)(-12s) - (-6s^2+2)\,6s}{(s^2+1)^4} = \frac{-12s^3 - 12s + 36s^3 - 12s}{(s^2+1)^4}$$

$$\int_0^\infty e^{-st} t^3 \sin t \, dt = \frac{24s^3 - 24s}{(s^2+1)^4} = \frac{24s\,(s^2-1)}{(s^2+1)^4} \qquad \ldots (1)$$

Putting $s = 1$ in (1), we get $\int_0^\infty e^{-t} t^3 \sin t \, dt = 0$ **Ans.**

Example 53. *Evaluate* $\int_0^\infty t^2 e^{3t} \sin^2 t \, dt.$

Solution. We have, $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$

$$\Rightarrow \quad L\{\sin^2 t\} = \frac{1}{2}[L\{1\} - L\{\cos 2t\}] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\Rightarrow \quad L\left[t^2 \sin^2 t\right] = (-1)^2 \cdot \frac{d^2}{ds^2}\left[\frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}\right]$$

$$\Rightarrow \quad \int_0^\infty e^{-st} . t^2 \sin^2 t \, dt = \frac{1}{2}\frac{d}{ds}\left[\frac{d}{ds}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}\right] = \frac{1}{2}\frac{d}{ds}\left[-\frac{1}{s^2} - \frac{(s^2+4)(1) - s\,(2s)}{(s^2+4)^2}\right]$$

$$= \frac{1}{2}\frac{d}{ds}\left[-\frac{1}{s^2} - \frac{-s^2+4}{(s^2+4)^2}\right] = \frac{1}{2}\left[\frac{2}{s^3} - \frac{(s^2+4)^2(-2s) - (-s^2+4)\,2\,(s^2+4)\,(2s)}{(s^2+4)^4}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3} - \frac{(s^2+4)(-2s) - (-s^2+4)\,4s}{(s^2+4)^3}\right] \qquad \ldots (1)$$

Putting the value of $s = -3$ in (1), we get

$$\int_0^\infty e^{3t} t^2 \sin^2 t \, dt = \frac{1}{2}\left[\frac{2}{-27} - \frac{(13)6 - (-5)(-12)}{(9+4)^3}\right]$$

$$= -\frac{1}{27} - \frac{9}{(13)^3} = \frac{-2197 - 243}{59319} = \frac{-2440}{59319}$$ **Ans.**

EXERCISE 42.8

Evaluate the following by using Laplace Transform:

1. $\int_0^\infty t e^{-4t} \sin t \, dt$ **Ans.** $\frac{8}{289}$
2. $\int_0^\infty \frac{e^{-2t} \sinh t \sin t}{t} dt$ **Ans.** $\frac{1}{2}\tan^{-1}\frac{1}{2}$
3. $\int_0^\infty \frac{\sin^2 t}{t^2} dt$ **Ans.** $i\frac{5}{2}$
4. $\int_0^\infty \frac{e^{-t} - e^{-4t}}{t} dt$ **Ans.** log 4

42.26 FORMULATION OF LAPLACE TRANSFORM

S.No.	$f(t)$	$F(s)$
1.	e^{at}	$\frac{1}{s-a}$
2.	t^n	$\frac{\lfloor n+1}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$
3.	$\sin at$	$\frac{a}{s^2+a^2}$
4.	$\cos at$	$\frac{s}{s^2+a^2}$
5.	$\sinh at$	$\frac{a}{s^2-a^2}$
6.	$\cosh at$	$\frac{s}{s^2-a^2}$
7.	$u(t-a)$	$\frac{e^{-as}}{s}$
8.	$\delta(t-a)$	e^{-as}
9.	$e^{bt}\sin at$	$\frac{a}{(s-b)^2+a^2}$
10.	$e^{bt}\cos at$	$\frac{s-b}{(s-b)^2+a^2}$
11.	$\frac{t}{2a}\sin at$	$\frac{s}{(s^2+a^2)^2}$
12.	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
13.	$\frac{1}{2a^3}(\sin at - at\cos at)$	$\frac{1}{(s^2+a^2)^2}$
14.	$\frac{1}{2a}(\sin at + at\cos at)$	$\frac{s^2}{(s^2+a^2)^2}$

42.27 PROPERTIES OF LAPLACE TRANSFORM

S.No.	*Property*	*f(t)*	*F (s)*
1.	Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$, $a > 0$
2.	Derivative	$\frac{df(t)}{dt}$ $\frac{d^2 f(t)}{dt^2}$ $\frac{d^3 f(t)}{dt^3}$	$s\,F(s) - f(0)$, $s > 0$ $s^2 F(s) - sf(0) - f'(0)$, $s > 0$ $s^3F(s) - s^2f(0) - s\,f'(0) - f''(0)$, $s > 0$
3.	Integral	$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s)$, $s > 0$
4.	Initial Value	$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$
5.	Final Value	$\lim_{t\to\infty} f(t)$	$\lim_{s\to\infty} sF(s)$
6.	First shifting	$e^{-at}f(t)$	$F(s+a)$
7.	Second shifting	$f(t)\,u(t-a)$	$e^{-a}\,L\,f(t+a)$
8.	Multiplication by t	$t\,f(t)$	$-\frac{d}{ds}F(s)$
9.	Multiplication by t^n	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$
10.	Division by t	$\frac{1}{t}f(t)$	$\int_s^\infty F(s)\,ds$
11.	Periodic function	$f(t)$	$\frac{\int_0^T e^{-st}f(t)}{1-e^{-st}}$ $f(t+T) = f(t)$
12.	Convolution	$f(t) * g(t)$	$F(s)\,G(s)$

OBJECTIVE TYPE QUESTIONS

Choose the correct alternative :

1. Laplace transform of t^3e^{-3t} is :

(*i*) $\frac{7}{(s+4)^3}$ (*ii*) $\frac{s}{(s+3)^3}$ (*iii*) $\frac{6}{(s+3)^4}$ (*iv*) $\frac{2}{(s+6)^3}$ **Ans.** (*iii*)

(*R.G.P.V., Bhopal, II Semester, Feb. 2006*)

2. Laplace transform of $e^{-2t} \sin 4t$ is :

(*i*) $\dfrac{2}{s^2+4s+20}$ (*ii*) $\dfrac{s-2}{s^2+4s+20}$ (*iii*) $\dfrac{s-4}{s^2+4s+20}$ (*iv*) $\dfrac{4}{s^2+4s+20}$ **Ans.** (*iv*)

(*R.G.P.V., Bhopal, II Semester, June 2007*)

3. If $\{F(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t F(x)\,dx\right\}$ is :

(*i*) $\int_0^s \bar{f}(s)\,ds$ (*ii*) $\int_0^s \frac{1}{s}\bar{f}(s)\,ds$ (*iii*) $\frac{1}{s}\bar{f}(s)$ (*iv*) $s\,\bar{f}(s)$ **Ans.** (*iii*)

(*R.G.P.V., Bhopal, II Semester, June 2006*)

4. If $L\{F(t)\} = \bar{f}(s)$, then $L\{t\,F(t)\}$ is :

(*i*) $\bar{f}'(s)$ (*ii*) $-\bar{f}'(s)$ (*iii*) $\bar{f}'(s)+\bar{f}(s)$ (*iv*) $s\bar{f}'(s)+\bar{f}(s)$ **Ans.** (*ii*)

(*R.G.P.V., Bhopal, II Semester, June 2006*)

5. Laplace transform of $\dfrac{\cos at - \cos bt}{t}$ is

(*i*) $\log\dfrac{s^2+b^2}{s^2+a^2}$ (*ii*) $\dfrac{1}{2}\log\dfrac{s^2+a^2}{s^2+b^2}$ (*iii*) $\dfrac{1}{2}\log\dfrac{s^2+b^2}{s^2+a^2}$ (*iv*) $\log\dfrac{s+b}{s+a}$ **Ans.** (*iii*)

(*R.G.P.V., Bhopal, II Semester, Feb. 2006, 2005*)

6. The Laplace transform of the function

$$F(t) = \begin{cases} 1, & 0 \le t < 2 \\ -1, & 2 \le t < 4 \end{cases}, \; f(t+4) = f(t) \text{ is given as,}$$

(*i*) $\dfrac{1-e^{-2s}}{s\left(1+e^{-2s}\right)}$ (*ii*) $\dfrac{1+e^{-2s}}{s\left(1+e^{-2s}\right)}$ (*iii*) 0 (*iv*) $\dfrac{s+1}{s-1}$ **Ans.** (*i*)

(*U.P., II Semester, 2009*)

Fill in the blank for each of the following question:

7. The Laplace transform of

$\int_0^t \int_0^t \int_0^t \cos au\,du\,du\,du$ is given as ...... [*U.P.T.U. (SUM) 2009*] **Ans.** $\dfrac{1}{s^2(s^2+a^2)}$