CHAPTER

25

Complex Numbers

25.1 INTRODUCTION

We have learnt the complex numbers in the previous class. Here we will review the complex number. In this chapter we will learn how to add, subtract, multiply and divide complex numbers.

25.2 COMPLEX NUMBERS

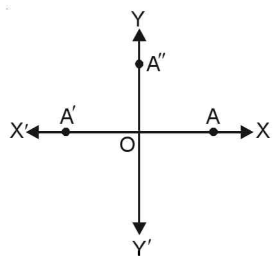
A number of the form $a + i\,b$ is called a complex number when a and b are real numbers and $i = \sqrt{-1}$. We call 'a' the real part and 'b' the imaginary part of the complex number $a + ib$. If $a = 0$ the number $i\,b$ is said to be purely imaginary, if $b = 0$ the number a is real.

A complex number $x + iy$ is denoted by z.

25.3 GEOMETRICAL REPRESENTATION OF IMAGINARY NUMBERS

Let OA be positive numbers which is represented by x and OA' by $-x$.

And $$-x = (i)^2\, x = i\,(ix) \text{ is on } OX'.$$



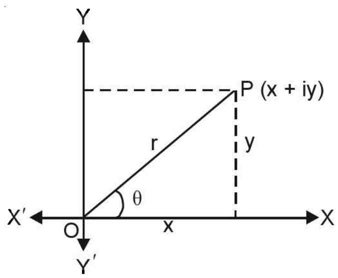
It means that the multiplication of the real number x by i twice amounts to the rotation of OA through two right angles to reach OA'.

Thus, it means that multiplication of x by i is equivalent to the rotation of x through one right angle to reach OA''.

Hence, y-axis is known as imaginary axis.

Multiplication by i rotates its direction through a right angle.

25.4 ARGAND DIAGRAM



Mathematician Argand represented a complex number in a diagram known as Argand diagram. A complex number $x + iy$ can be represented by a point P whose co-ordinate are (x, y). The axis of x is called the real axis and the axis of y the imaginary axis. The distance OP is the **modulus** and the angle, OP makes with the x-axis, is the **argument** of $x + iy$.

25.5 EQUAL COMPLEX NUMBERS

If two complex numbers $a + ib$ and $c + id$ are equal, prove that

$$a = c \quad \text{and} \quad b = d$$

Solution. We have, $$a + ib = c + id \;\Rightarrow\; a - c = i(d - b)$$

$$(a - c)^2 = -(d - b)^2 \;\Rightarrow\; (a - c)^2 + (d - b)^2 = 0$$

Here sum of two positive numbers is zero. This is only possible if each number is zero.

i.e., $(a - c)^2 = 0 \;\Rightarrow\; a = c$ and $(d - b)^2 = 0 \;\Rightarrow\; b = d$ **Ans.**

25.6 ADDITION OF COMPLEX NUMBERS

Let $a + ib$ and $c + id$ be two complex numbers, then

$$(a + ib) + (c + id) = (a + c) + i\,(b + d)$$

Procedure. In addition of complex numbers we add real parts with real parts and imaginary parts with imaginary parts.

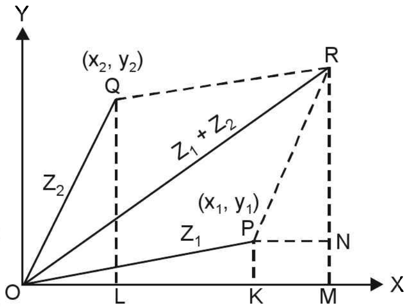
Example 1. *Add the following complex numbers:*

$$z_1 = 2 + \frac{3}{2}i, \quad z_2 = -5 + \frac{7}{4}i, \quad z_3 = \frac{5}{4} - \frac{8}{3}i, \quad z_4 = \frac{-11}{2} - i$$

Solution.
$$z_1 + z_2 + z_3 + z_4 = \left(2 + \frac{3}{2}i\right) + \left(-5 + \frac{7}{4}i\right) + \left(\frac{5}{4} - \frac{8}{3}i\right) + \left(-\frac{11}{2} - i\right)$$
$$= \left(2 - 5 + \frac{5}{4} - \frac{11}{2}\right) + \left(\frac{3}{2} + \frac{7}{4} - \frac{8}{3} - 1\right)i$$
$$= \left(\frac{8 - 20 + 5 - 22}{4}\right) + \left(\frac{18 + 21 - 32 - 12}{12}\right)i$$
$$= -\frac{29}{4} - \frac{5}{12}i$$
Ans.

25.7 ADDITION OF COMPLEX NUMBERS BY GEOMETRY

Let $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ be two complex numbers represented by the points P and Q on the Argand diagram.



Complete the parallelogram $OPRQ$.

Draw PK, RM, QL, perpendiculars on OX.

Also draw $PN \perp$ to RM.

$$OM = OK + KM = OK + OL = x_1 + x_2$$

and
$$RM = MN + NR = KP + LQ = y_1 + y_2$$

$\therefore$ The co-ordinates of R are $(x_1 + x_2, y_1 + y_2)$ and it represents the complex number.

$$(x_1 + x_2) + i\,(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2)$$

Thus the sum of two complex numbers is represented by the extremity of the diagonal of the parallelogram formed by OP (z_1) and OQ (z_2) as adjacent sides.

$$|z_1 + z_2| = OR \quad \text{and} \quad \text{amp}\,(z_1 + z_2) = \angle ROM.$$

25.8 SUBTRACTION

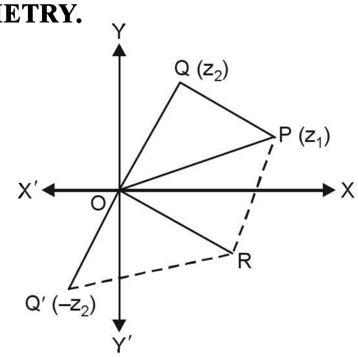
$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

Procedure. In subtraction of complex numbers we subtract real parts from real parts and imaginary parts from imaginary parts.

Example 2. *Subtract* $z_1 = \frac{3}{4} - \frac{7}{3}i$ *from* $z_2 = \frac{-5}{3} + \frac{11}{5}i$.

Solution.
$$z_2 - z_1 = \left(\frac{-5}{3} + \frac{11}{5}i\right) - \left(\frac{3}{4} - \frac{7}{3}i\right) = \left(\frac{-5}{3} - \frac{3}{4}\right) + \left(\frac{11}{5} + \frac{7}{3}\right)i$$
$$= \left(\frac{-20 - 9}{12}\right) + \left(\frac{33 + 35}{15}\right)i = \frac{-29}{12} + \frac{68}{15}i$$
Ans.

SUBTRACTION OF COMPLEX NUMBERS BY GEOMETRY.



Let P and Q represent two complex numbers

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2.$$

Then $$z_1 - z_2 = z_1 + (-z_2)$$

$z_1 - z_2$ means the addition of z_1 and $-z_2$.

$-z_2$ is represented by OQ' formed by producing OQ to OQ' such that $OQ = OQ'$.

Complete the parallelogram $OPRQ'$, then the sum of z_1 and $-z_2$ represented by OR.

25.9 POWERS OF *i*

Some time we need various powers of i.

We know that $i = \sqrt{-1}$.

On squaring both sides, we get

$$i^2 = -1$$

Multiplying by i both sides, we get

$$i^3 = -i$$

Again,
$$i^4 = (i^3)\,(i) = (-i)\,(i) = -(i^2) = -(-1) = 1$$
$$i^5 = (i^4)\,(i) = (1)\,(i) = i$$
$$i^6 = (i^4)\,(i^2) = (1)\,(-1) = -1$$
$$i^7 = (i^4)\,(i^3) = 1(-i) = -i$$
$$i^8 = (i^4)\,(i^4) = (1)\,(1) = 1.$$

Example 3. *Simplify the following:* (*a*) i^{49}, (*b*) i^{103}.

Solution. (*a*) We divide 49 by 4 and we get

$$49 = 4 \times 12 + 1$$
$$i^{49} = i^{4\times 12+1} = (i^4)^{12}\,(i^1) = (1)^{12}\,(i) = i$$

(*b*) we divide 103 by 4, we get

$$103 = 4 \times 25 + 3$$
$$i^{103} = i^{4\times 25+3} = (i^4)^{25}\,(i^3) = (1)^{25}\,(-i) = -i$$ **Ans.**

25.10 MULTIPLICATION

$$(a + ib) \times (c + id) = ac - bd + i(ad + bc)$$

Proof.
$$(a + ib) \times (c + id) = ac + iad + ibc + i^2bd$$
$$= ac + i(ad + bc) + (-1)bd \qquad [\because i^2 = -1]$$
$$= (ac - bd) + (ad + bc)i$$

Example 4. *Multiply 3 + 4i by 7 – 3i.*

Solution. Let $z_1 = 3 + 4i$ and $z_2 = 7 - 3i$

$$z_1.z_2 = (3 + 4i).(7 - 3i)$$
$$= 21 - 9i + 28i - 12i^2$$
$$= 21 - 9i + 28i - 12(-1) \qquad [\because i^2 = -1]$$
$$= 21 - 9i + 28i + 12$$
$$= 33 + 19i$$ **Ans.**

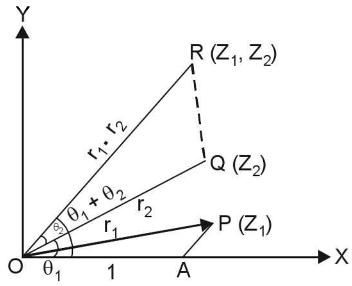
Multiplication of complex numbers (Polar form) :

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$x_1 = r_1\cos\theta_1, \quad y_1 = r_1\sin\theta_1$$
$$x_2 = r_2\cos\theta_2, \quad y_2 = r_2\sin\theta_2$$
$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad |z_1| = r_1$$
$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2) \qquad |z_2| = r_2$$
$$z_1 . z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$
$$= r_1r_1[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)], \qquad |z_1z_2| = r_1r_2$$

The modulus of the product of two complex numbers is the product of their moduli and the argument of the product is the sum of their arguments.

Graphical method



Let P, Q represent the complex numbers.

$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Cut off $OA = 1$ along x-axis. Construct $\Delta\, ORQ$ on OQ similar to $\Delta\, OAP$.

So that $$\frac{OR}{OP} = \frac{OQ}{OA} \Rightarrow \frac{OR}{OP} = \frac{OQ}{1} \Rightarrow OR = OP\,.\,OQ = r_1r_2$$

$$\angle XOR = \angle AOQ + \angle QOR = \theta_2 + \theta_1$$

Hence the product of two complex numbers z_1, z_2 is represented by the point R, such that

(*i*) $|z_1 . z_2| = |z_1|.|z_2|$ (*ii*) $\text{Arg}(z_1 . z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

25.11 *i* (IOTA) AS AN OPERATOR

Multipliction of a complex number by i.

Let $$z = x + iy = r(\cos\theta + i\sin\theta)$$
$$i = 0 + i\,.\,1 = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$
$$i\,.\,z = r(\cos\theta + i\sin\theta)\,.\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$
$$= r\left[\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right]$$

Hence a complex number multiplied by i results :

The rotation of the complex number by $\frac{\pi}{2}$ in anticlockwise direction without change in magnitude.

25.12 CONJUGATE OF A COMPLEX NUMBER

Two complex numbers which differ only in the sign of imaginary parts are called conjugate of each other.

A pair of complex number $a + ib$ and $a - ib$ are said to be conjugate of each other.

Theorem. Show that the sum and product of a complex number and its conjugate complex are both real.

Proof. Let $x + iy$ be a complex number and $x - iy$ its conjugate complex.

$$\text{Sum} = (x + iy) + (x - iy) = 2x \qquad \text{(Real)}$$

$$\text{Product} = (x + iy).(x - iy) = x^2 + y^2. \qquad \text{(Real)}$$

Proved.

Note. Let a complex number be z. Then the conjugate complex number is denoted by $\bar{z}$.

Example 5. *Find out the conjugate of a complex number $7 + 6i$.*

Solution. Let $z = 7 + 6i$

To find conjugate complex number of $7 + 6i$ we change the sign of imaginary number.

Conjugate of $z = \bar{z} = 7 - 6i$ **Ans.**

25.13 DIVISION

To divide a complex number $a + ib$ by $c + id$, we write it as $\dfrac{a + ib}{c + id}$.

To simplify further, we multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{a + ib}{c + id} = \frac{(a + ib)}{(c + id)} \times \frac{(c - id)}{(c - id)} = \frac{ac - iad + ibc - i^2 bd}{(c)^2 - (id)^2}$$

$$= \frac{ac - i(ad - bc) + bd}{c^2 - d^2 i^2} \qquad [\because i^2 = -1]$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i$$

Example 6. *Divide $1 + i$ by $3 + 4i$.*

Solution.
$$\frac{1 + i}{3 + 4i} = \frac{1 + i}{3 + 4i} \times \frac{3 - 4i}{3 - 4i}$$

$$= \frac{3 - 4i + 3i - 4i^2}{9 - 16i^2}$$

$$= \frac{3 - i + 4}{9 + 16} = \frac{7}{25} - \frac{1}{25} i$$

Ans.

DIVISION (By Algebra)

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{r_2(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)}$$

$$= \frac{r_1[(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \sin \theta_2 \cos \theta_1)]}{r_2(\cos^2 \theta_2 + \sin^2 \theta_2)}$$

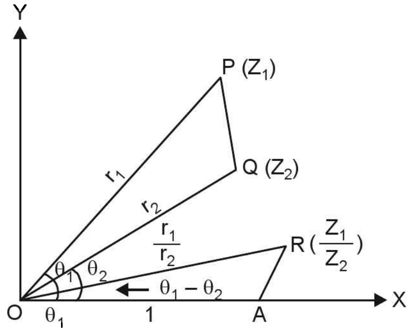
$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\right]$$

The modulus of the quotient of two complex numbers is the quotient of their moduli, and the argument of the quotient is the difference of their arguments.

25.14 DIVISION OF COMPLEX NUMBERS BY GEOMETRY

Let P and Q represent the complex numbers.

$$z_1 = x_1 + i\,y_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = x_2 + i\,y_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$



Cut off $OA = 1$, construct $\Delta\, OAR$ on OA similar to $\Delta\, OQP$.

So that $\dfrac{OR}{OA} = \dfrac{OP}{OQ} \Rightarrow \dfrac{OR}{1} = \dfrac{OP}{OQ}$

$$OR = \frac{OP}{OQ} = \frac{r_1}{r_2}$$

$$\angle AOR = \angle QOP = \angle AOP - \angle AOQ = \theta_1 - \theta_2$$

$\therefore$ R represents the number $\dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Hence the complex number $\dfrac{z_1}{z_2}$ is represented by the point R.

(*i*) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (*ii*) $\text{Arg.}\left(\dfrac{z_1}{z_2}\right) = \text{Arg.}(z_1) - \text{Arg.}(z_2)$.

Example 7. *Express* $\dfrac{(6+i)\,.\,(2-i)}{(4+3i)\,.\,(1-2i)}$ *in the form of* $a + ib$.

Solution. $\dfrac{(6+i)\,.\,(2-i)}{(4+3i)\,.\,(1-2i)} = \dfrac{12+1+i(2-6)}{4+6+i(3-8)} = \dfrac{13-4i}{10-5i}$

$$= \frac{(13-4i)(10+5i)}{(10-5i)(10+5i)} = \frac{150+25i}{100+25} = \frac{6+i}{5} = \frac{6}{5} + \frac{1}{5}i. \quad \textbf{Ans.}$$

Example 8. *If* $a = \cos\theta + i\sin\theta$, *prove that* $1 + a + a^2 = (1 + 2\cos\theta)(\cos\theta + i\sin\theta)$.

Solution. Here we have $a = \cos\theta + i\sin\theta$

$$\begin{aligned}
1 + a + a^2 &= 1 + (\cos\theta + i\sin\theta) + (\cos\theta + i\sin\theta)^2\\
&= 1 + \cos\theta + i\sin\theta + \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta\\
&= (\cos\theta + i\sin\theta) + (1 - \sin^2\theta) + \cos^2\theta + 2i\sin\theta\cos\theta\\
&= (\cos\theta + i\sin\theta) + \cos^2\theta + \cos^2\theta + 2i\sin\theta\cos\theta\\
&= (\cos\theta + i\sin\theta) + 2\cos^2\theta + 2i\sin\theta\cos\theta\\
&= (\cos\theta + i\sin\theta) + 2\cos\theta\,(\cos\theta + i\sin\theta)\\
&= (\cos\theta + i\sin\theta)\,(1 + 2\cos\theta)
\end{aligned}$$

Proved.

Example 9. *Solve for* θ *such that the expression* $\dfrac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ *is imaginary.*

Solution. $\dfrac{3+2i\sin\theta}{1-2i\sin\theta} = \dfrac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)} = \dfrac{3-4\sin^2\theta+8i\sin\theta}{1+4\sin^2\theta}$

If $3 - 4\sin^2\theta = 0$ then $\dfrac{3-4\sin^2\theta+8i\sin\theta}{1+4\sin^2\theta}$ = purely imaginary.

$$\sin^2\theta = \frac{3}{4} \quad \text{or} \quad \sin\theta = \frac{\sqrt{3}}{2} \quad \text{or} \quad \theta = \frac{\pi}{3} \quad \textbf{Ans.}$$

Example 10. *If* $a^2 + b^2 + c^2 = 1$ *and* $b + ic = (1 + a)z$, *prove that* $\dfrac{a+ib}{1+c} = \dfrac{1+iz}{1-iz}$.

Solution. Here, we have $b + ic = (1 + a)z \Rightarrow z = \dfrac{b+ic}{1+a}$

$$\frac{1+iz}{1-iz} = \frac{1+i\dfrac{b+ic}{1+a}}{1-i\dfrac{b+ic}{1+a}} = \frac{1+a+ib-c}{1+a-ib+c}$$

$$= \frac{[(1+a+ib)-c]}{(1+a+c-ib)} \times \frac{(1+a+ib+c)}{(1+a+c+ib)} = \frac{(1+a+ib)^2-c^2}{(1+a+c)^2+b^2}$$

$$= \frac{1+a^2-b^2+2a+2ib+2iab-c^2}{1+a^2+c^2+2a+2c+2ac+b^2} = \frac{1+a^2-b^2-c^2+2a+2ib+2iab}{1+(a^2+b^2+c^2)+2a+2c+2ac}$$

Putting the value of $a^2+b^2+c^2=1$ in the above, we get

$$= \frac{1+a^2-(1-a^2)+2a+2ib+2iab}{1+1+2a+2c+2ac} = \frac{2(a^2+a+ib+iab)}{2(1+a+c+ac)} = \frac{2(1+a)(a+ib)}{2(1+a)(1+c)} = \frac{a+ib}{1+c}$$

Proved.

Example 11. *If $z = \cos\theta + i\sin\theta$, prove that*

(*a*) $\dfrac{2}{1+z} = 1 - i\tan\dfrac{\theta}{2}$ (*b*) $\dfrac{1+z}{1-z} = i\cot\dfrac{\theta}{2}$

Solution. Here, we have $z = \cos\theta + i\sin\theta$

(*a*) $$\frac{2}{1+z} = \frac{2}{1+(\cos\theta+i\sin\theta)} = \frac{2}{(1+\cos\theta)+i\sin\theta} \times \frac{(1+\cos\theta)-i\sin\theta}{(1+\cos\theta)-i\sin\theta}$$

$$= \frac{2[(1+\cos\theta)-i\sin\theta]}{(1+\cos\theta)^2+\sin^2\theta}$$

$$= \frac{2[(1+\cos\theta)-i\sin\theta]}{2(1+\cos\theta)} = 1 - \frac{i\sin\theta}{1+\cos\theta}$$

$$= 1 - i\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)} = 1 - i\tan\left(\frac{\theta}{2}\right)$$

Proved.

$$\begin{aligned}&(1+\cos\theta)^2+\sin^2\theta\\ &= 1+\cos^2\theta+2\cos\theta+\sin^2\theta\\ &= 1+(\sin^2\theta+\cos^2\theta)+2\cos\theta\\ &= 1+1+2\cos\theta\\ &= 2+2\cos\theta\\ &= 2(1+\cos\theta)\end{aligned}$$

(*b*) $$\frac{1+z}{1-z} = \frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta} = \frac{2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \cdot \left(\frac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}}\right) = \cot\frac{\theta}{2}\left(\frac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}}\right)$$

Now, $$\frac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}} = \left(\frac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}}\right)\left(\frac{\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}}\right)$$

$$= \frac{\cos\frac{\theta}{2}\sin\frac{\theta}{2}+i\cos^2\frac{\theta}{2}+i\sin^2\frac{\theta}{2}-\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}} = \frac{i\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)}{\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)} = i$$

Thus, $$\frac{1+z}{1-z} = i\cot\frac{\theta}{2}$$

Proved.

Example 12. *If $x = \cos\theta + i\sin\theta$, $y = \cos\phi + i\sin\phi$, prove that*

$$\frac{x-y}{x+y} = i\tan\left(\frac{\theta-\phi}{2}\right) \qquad (M.U.\ 2008)$$

Solution. We have,

$$\frac{x-y}{x+y} = \frac{(\cos\theta + i\sin\theta) - (\cos\phi + i\sin\phi)}{(\cos\theta + i\sin\theta) + (\cos\phi + i\sin\phi)}$$

$$= \frac{(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)}{(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)}$$

$$= \frac{\left[-2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) + 2i\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right]}{\left[2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) + 2i\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)\right]}$$

$$= \frac{2i\sin\left(\frac{\theta-\phi}{2}\right)\left[\cos\left(\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\theta+\phi}{2}\right)\right]}{2\cos\left(\frac{\theta-\phi}{2}\right)\left[\cos\left(\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\theta+\phi}{2}\right)\right]} = i\tan\left(\frac{\theta-\phi}{2}\right)$$ **Proved.**

EXERCISE 25.1

1. If $z = 1 + i$, find (*i*) z^2 (*ii*) $\frac{1}{z}$ and plot them on the Argand diagram. **And.** (*i*) $2i$, (*ii*) $\frac{1}{2} - \frac{i}{2}$

Express the following in the form $a + ib$, where a and b are real (2 – 4):

2. $\frac{2-3i}{4-i}$ **Ans.** $\frac{11}{17} - \frac{10}{17}i$

3. $\frac{(3+4i)(2+i)}{1+i}$ **Ans.** $\frac{13}{2} + \frac{9}{2}i$

4. $\frac{(1+2i)^3}{(1+i)(2-i)}$ **Ans.** $-\frac{7}{2} + \frac{1}{2}i$

5. The points A, B, C represent the complex numbers z_1, z_2, z_3 respectively, and G is the centroid of the triangle ABC, if $4z_1 + z_2 + z_3 = 0$, show that the origin is the mid-point of AG.

6. $ABCD$ is a parallelogram on the Argand plane. The affixes of A, B, C are $8 + 5i$, $-7 - 5i$, $-5 + 5i$, respectively. Find the affix of D. **Ans.** $10 + 15i$

7. If z_1, z_2, z_3 are three complex numbers and

$$a_1 = z_1 + z_2 + z_3$$
$$b_1 = z_1 + \omega z_2 + \omega^2 z_3$$
$$c_1 = z_1 + \omega^2 z_2 + \omega z_3$$

show that $|a_1|^2 + |b_1|^2 + |c_1|^2 = 3\{|z_1|^2 + |z_2|^2 + |z_3|^2\}$

where ω, ω^2 are cube roots of unity.

8. Find the complex conjugate of $\frac{2+3i}{1-i}$. **Ans.** $-\frac{1}{2} - \frac{5}{2}i$

9. If $x + iy = \frac{1}{a+ib}$, prove that $(x^2 + y^2)(a^2 + b^2) = 1$

10. Find the value of $x^2 - 6x + 13$, when $x = 3 + 2i$. **Ans.** 0

11. If $\alpha - i\beta = \frac{1}{a-ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$. (*M.U. 2008*)

12. If $\frac{1}{\alpha + i\beta} + \frac{1}{a+ib} = 1$, where α, β, a, b are real, express b in terms of α, β.

Ans. $\frac{-\beta}{\alpha^2 + \beta^2 - 2\alpha + 1}$

13. If $(x + iy)^{1/3} = a + ib$, then show that $4(a^2 - b^2) = \dfrac{x}{a} + \dfrac{y}{b}$.

14. If $(x + iy)^3 = u + iv$, then show that $\dfrac{u}{x} + \dfrac{v}{y} = 4(x^2 - y^2)$.

15. Find the values of x and y, if $\dfrac{(1+i)x - 2i}{3+i} + \dfrac{(2-3i)y + i}{3-i} = i$. **Ans.** $x = 3$ and $y = -1$

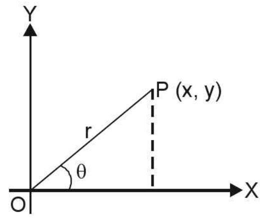
16. If $a + ib = \dfrac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$.

25.15 MODULUS AND ARGUMENT

Let $x + iy$ be a complex number.

Putting $x = r\cos\theta$ and $y = r\sin\theta$ so that $r = \sqrt{x^2 + y^2}$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$



the positive value of the root being taken.

Then r called the *modulus* or absolute value of the complex number $x + iy$ and is denoted by $|x + iy|$.

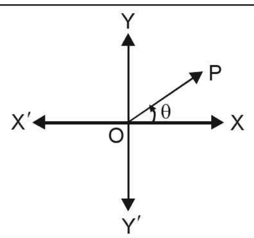
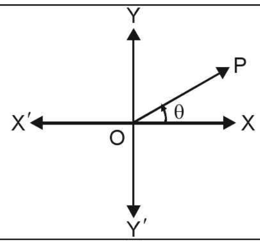
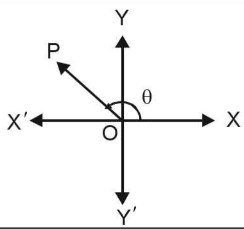
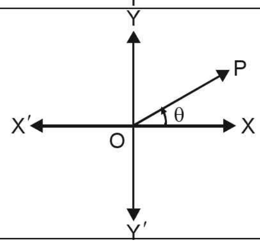
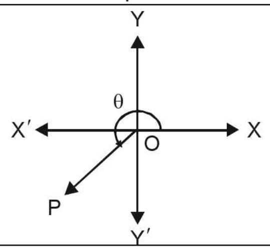
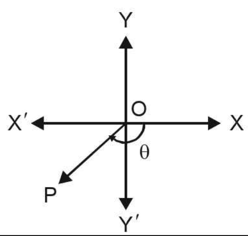
The angle θ is called the *argument* or *amplitude* of the complex number $x + iy$ and is denoted by arg. $(x + iy)$.

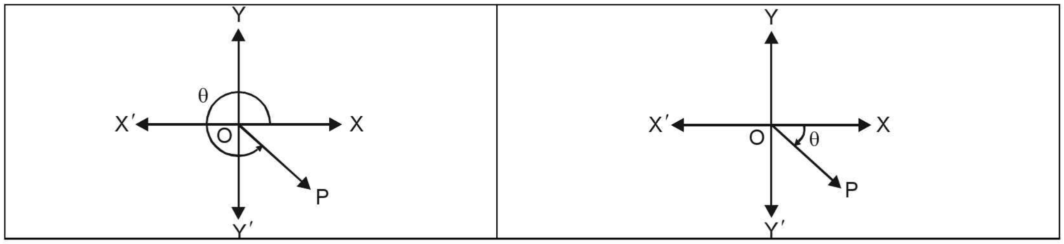
It is clear that θ will have infinite number of values differing by multiples of 2π. The values of θ lying in the range $-\pi < \theta \le \pi$ $[(0 < \theta < \pi)$ or $(-\pi < \theta < 0)]$ is called the *principal value* of the argument.

The principal value of θ is written either between 0 and π or between 0 and $-\pi$.

A complex number $x + iy$ is denoted by a single letter z. The number $x - iy$ (conjugate) is denoted by $\bar{z}$. The complex number in polar form is $r(\cos\theta + i\sin\theta)$.

Modulus of z is denoted by $|z|$ and $|z|^2 = x^2 + y^2$.

<table>
<tr><th>Angle θ</th><th>Principal value of θ</th></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
<tr><td></td><td></td></tr>
</table>



For example (*i*) the principal value of 240° is –120°.
(*ii*) the principal value of 330° is –30°.

Example 13. *Find the modulus and principal argument of the complex number*

$$\frac{1+2i}{1-(1-i)^2}.$$

Solution. $$\frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-(1-1-2i)} = \frac{1+2i}{1+2i} = 1 = 1+0i$$

$$\therefore \quad \left|\frac{1+2i}{1-(1-i)^2}\right| = |1+0i| = \sqrt{1^2} = 1$$ **Ans.**

Principal argument of $\dfrac{1+2i}{1-(1-i)^2}$ = Principal argument of $1+0i$

$$= \tan^{-1}\frac{0}{1} = \tan^{-1} 0 = 0°.$$

Hence modulus = 1 and principal argument = 0°. **Ans.**

Example 14. *Find the modulus and principal argument of the complex number :*

$$1 + \cos\alpha + i\sin\alpha. \qquad \left(0 < \alpha < \frac{\pi}{2}\right)$$

Solution. Let $(1+\cos\alpha) + i\sin\alpha = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts, we get

$$1+\cos\alpha = r\cos\theta \quad ...(1)$$

And $$\sin\alpha = r\sin\theta \quad ...(2)$$

Squaring and adding (1) and (2), we get

$$r^2(\cos^2\theta + \sin^2\theta) = (1+\cos\alpha)^2 + (\sin\alpha)^2$$

$$\Rightarrow \quad r^2 = 1 + \cos^2\alpha + 2\cos\alpha + \sin^2\alpha = 1 + 2\cos\alpha + 1$$

$$= 2(1+\cos\alpha) = 2\left(1 + 2\cos^2\frac{\alpha}{2} - 1\right) = 4\cos^2\frac{\alpha}{2}$$

$$\Rightarrow \quad r = 2\cos\frac{\alpha}{2}$$

From (1), we have, $$\cos\theta = \frac{1+\cos\alpha}{r} = \frac{1+2\cos^2\frac{\alpha}{2} - 1}{2\cos\frac{\alpha}{2}} = \cos\frac{\alpha}{2} \quad ...(3)$$

From (2), we have, $$\sin\theta = \frac{\sin\alpha}{r} = \frac{\sin\alpha}{2\cos\frac{\alpha}{2}} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos\frac{\alpha}{2}} = \sin\frac{\alpha}{2} \quad ...(4)$$

Argument = $$\tan^{-1}\frac{\sin\alpha}{1+\cos\alpha} = \tan^{-1}\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1+2\cos^2\frac{\alpha}{2}-1} = \tan^{-1}\tan\frac{\alpha}{2} = \frac{\alpha}{2}$$

General value of argument $= 2\pi k + \dfrac{\alpha}{2}$

$\theta = \dfrac{\alpha}{2}$ satisfied both equations, (1) and (2),

Arg $(1 + \cos\alpha + i\sin\alpha) = \dfrac{\alpha}{2}$ and modulus of $(1 + \cos\alpha + i\sin\alpha) = r = 2\cos\dfrac{\alpha}{2}$ **Ans.**

EXERCISE 25.2

Find the modulus and principal argument of the following complex numbers:

1. $-\sqrt{3} - i$ **Ans.** $2, -\dfrac{5\pi}{6}$

2. $\dfrac{(1+i)^2}{1-i}$ **Ans.** $\sqrt{2}, \dfrac{3\pi}{4}$

3. $\sqrt{\left(\dfrac{1+i}{1-i}\right)}$ **Ans.** $1, \dfrac{\pi}{4}$

4. $\tan\alpha - i$ **Ans.** $\sec\alpha, -\left(\dfrac{\pi}{2} - \alpha\right)$

5. $1 - \cos\alpha + i\sin\alpha$ **Ans.** $2\sin\dfrac{\alpha}{2}, \dfrac{\pi - \alpha}{2}$

6. $(4 + 2i)(-3 + \sqrt{2}i)$ **Ans.** $2\sqrt{55}, \tan^{-1}\left(\dfrac{3 - 2\sqrt{2}}{6 + \sqrt{2}}\right)$

Find the modulus of the following complex numbers :

7. $(\overline{7 + i^2}) + (6 - i) - (4 - 3i^3)$ **Ans.** $4\sqrt{5}$

8. $(\overline{5 - 6i}) - (5 + 6i) + (8 - i)$ **Ans.** $\sqrt{185}$

9. $(8 - i^3) - (7i^2 + 5) + (\overline{9 - i})$ **Ans.** $\sqrt{365}$

10. $(5 + 6i^{11}) + (8i^3 + i^5) + (i^2 - i^4)$ **Ans.** $\sqrt{178}$

11. If arg. $(z + 2i) = \dfrac{\pi}{4}$ and arg. $(z - 2i) = \dfrac{3\pi}{4}$, find z. **Ans.** $z = 2$

Example 15. *If* $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$ *show that*

$$\frac{1}{2i}\left(\frac{z_1}{z_2} - \frac{z_2}{z_1}\right) = \sin(\alpha - \beta)$$

Solution. We have

$$\frac{z_1}{z_2} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} \times \frac{\cos\beta - i\sin\beta}{\cos\beta - i\sin\beta}$$

$$= \frac{(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{\cos^2\beta + \sin^2\beta}$$

$$= \cos(\alpha - \beta) + i\sin(\alpha - \beta) \qquad \text{...(1)}$$

$$\frac{z_2}{z_1} = \frac{1}{\cos(\alpha - \beta) + i\sin(\alpha - \beta)} \times \frac{\cos(\alpha - \beta) - i\sin(\alpha - \beta)}{\cos(\alpha - \beta) - i\sin(\alpha - \beta)}$$

$$= \frac{\cos(\alpha - \beta) - i\sin(\alpha - \beta)}{\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)} = \cos(\alpha - \beta) - i\sin(\alpha - \beta) \qquad \text{...(2)}$$

Subtracting (2) from (1), we get

$$\frac{z_1}{z_2} - \frac{z_2}{z_1} = 2i\sin(\alpha - \beta)$$ **Proved.**

Example 16. *If* z_1 *and* z_2 *are any two complex numbers, prove that*

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$$

Solution. Let

$$z_1 = x_1 + iy_1$$
$$z_2 = x_2 + iy_2$$
$$|z_1 + z_2|^2 = |(x_1 + iy_1) + (x_2 + iy_2)|^2$$

$$= |(x_1 + x_2) + i(y_1 + y_2)|^2$$

$$= (x_1 + x_2)^2 + (y_1 + y_2)^2 \qquad ...(1)$$

Similarly $\qquad |z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \qquad ...(2)$

and $\qquad |z_1|^2 = x_1^2 + y_1^2 \qquad ...(3)$

$$|z_2|^2 = x_2^2 + y_2^2 \qquad ...(4)$$

$$\text{L.H.S.} = |z_1 + z_2|^2 + |z_1 - z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2$$

[Using (1) and (2)]

$$= x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 + x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2$$

$$= 2[x_1^2 + x_2^2 + y_1^2 + y_2^2] = 2[(x_1^2 + y_1^2) + (x_2^2 + y_2^2)] \qquad ...(5)$$

$$= 2[|z_1|^2 + |z_2|^2] = \text{R.H.S.}$$

Proved.

Example 17. *If z_1 and z_2 are two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, prove that*

$$\arg. z_1 - \arg. z_2 = \frac{\pi}{2}$$

(M.U. 2002, 2007)

Solution. Let

$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

Given that $\qquad |z_1 + z_2| = |z_1 - z_2|$

$$\Rightarrow \quad |(x_1 + iy_1) + (x_2 + iy_2)| = |(x_1 + iy_1) - (x_2 + iy_2)|$$

$$\Rightarrow \quad |(x_1 + x_2) + i(y_1 + y_2)| = |(x_1 - x_2) + (y_1 - y_2)i|$$

$$\Rightarrow \quad (x_1 + x_2)^2 + (y_1 + y_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$\Rightarrow x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 = x_1^2 + x_2^2 - 2x_1x_2 + y_1^2 + y_2^2 - 2y_1y_2$$

$$\Rightarrow \quad 2x_1x_2 + 2y_1y_2 = -2x_1x_2 - 2y_1y_2$$

$$\Rightarrow \quad 4x_1x_2 + 4y_1y_2 = 0$$

$$\Rightarrow \quad x_1x_2 + y_1y_2 = 0 \qquad ...(1)$$

Now, $\qquad \arg. z_1 - \arg. z_2 = \tan^{-1}\left(\frac{y_1}{x_1}\right) - \tan^{-1}\left(\frac{y_2}{x_2}\right)$

$$= \tan^{-1}\left[\frac{\frac{y_1}{x_1} - \frac{y_2}{x_2}}{1 + \left(\frac{y_1}{x_1}\right)\left(\frac{y_2}{x_2}\right)}\right] = \tan^{-1}\left(\frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2}\right)$$

$$= \tan^{-1}\left(\frac{x_2y_1 - x_1y_2}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2} \qquad \text{[Using (1)]}$$

$$\arg. z_1 - \arg. z_2 = \frac{\pi}{2}$$

Proved.

Example 18. *Find the complex number z if arg $(z + 1) = \frac{\pi}{6}$ and arg $(z - 1) = \frac{2\pi}{3}$.*

(M.U. 2009, 2000, 01, 02, 03)

Solution. Let $\qquad z = x + iy \qquad ...(1)$

$\therefore \qquad z + 1 = (x + 1) + iy$

We also given that

$$\text{Arg } (z + 1) = \tan^{-1}\left(\frac{y}{x + 1}\right) = \frac{\pi}{6}$$

$$\therefore \quad \frac{y}{x+1} = \tan 30° = \frac{1}{\sqrt{3}}$$

$$\therefore \quad \sqrt{3}y = x + 1 \qquad \ldots(2)$$

Now $\quad z - 1 = (x-1) + iy \qquad$ [From (1)]

and $\quad \tan^{-1}\left(\frac{y}{x-1}\right) = \frac{2\pi}{3} \Rightarrow \frac{y}{x-1} = \tan 120°$

$$\Rightarrow \quad \frac{y}{x-1} = -\cot 30° = -\sqrt{3}$$

$$\therefore \quad -y = \sqrt{3}x - \sqrt{3}$$

$$\Rightarrow \quad -\sqrt{3}y = 3x - 3 \qquad \ldots(3)$$

Adding (2) and (3), we get

$$0 = 4x - 2 \quad \Rightarrow \quad 4x = 2 \quad \Rightarrow \quad x = \frac{1}{2}$$

Putting $x = \frac{1}{2}$ in (2), we get

$$\sqrt{3}y = \frac{1}{2} + 1 \Rightarrow \sqrt{3}y = \frac{3}{2} \Rightarrow y = \frac{\sqrt{3}}{2}$$

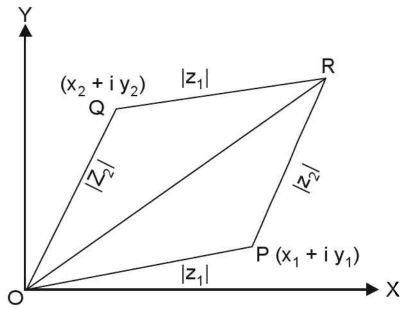
Putting the values of x and y in (1), we get

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad \textbf{Ans.}$$

Example 19. *Prove that*

(*i*) $|z_1 + z_2| \le |z_1| + |z_2|$ $\qquad$ (*ii*) $|z_1 - z_2| \ge |z_1| - |z_2|$

Solution. (*a*) (By Geometry) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be the two complex numbers shown in the figure



$$|z_1| = OP, \quad |z_2| = OQ$$

(*i*) Since in a traingle any side is less than the sum of the other two.

In Δ OPR, $\quad OR < OP + PR$, $OR < OP + OQ$

$\Rightarrow \quad |z_1 + z_2| < |z_1| + |z_2|$

$\quad OR = OP + PR \quad$ if $\quad O, P, R$ are collinear.

or $\quad |z_1 + z_2| = |z_1| + |z_2|$

(*ii*) Again, any side of a triangle is greater than the difference between the other two, we have

In Δ OPR

$OR > OP - PR, \quad \Rightarrow \quad OR > OP - OQ$

$$|z_1 - z_2| > |z_1| - |z_2| \qquad \textbf{Proved.}$$

(*b*) By Algebra. $\quad z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$

(*i*) $\quad |z_1 + z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$

$$= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(x_1x_2 + y_1y_2)$$

$$= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1x_2 + y_1y_2)^2}$$

$$= |z_1|^2 + |z_2|^2 + 2\sqrt{x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2}$$

$$[\because (x_1y_2 - x_2y_1)^2 \ge 0 \text{ or } x_1^2y_2^2 + x_2^2y_1^2 \ge 2x_1x_2y_1y_2]$$

$$|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2\sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2}$$
$$\le |z_1|^2 + |z_2|^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$
$$\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$
$$\le \{|z_1| + |z_2|\}^2$$
$$|z_1 + z_2| \le |z_1| + |z_2|$$

(*ii*) $$|z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2|$$
$$|z_1| - |z_2| \le |z_1 - z_2|$$
$$|z_1 - z_2| \ge |z_1| - |z_2|$$ **Proved.**

EXERCISE 25.3

1. If $z = x + iy$, prove that $\left(\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right) = 2\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$.

2. If $z = a\cos\theta + ia\sin\theta$, prove that $\left(\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right) = 2\cos 2\theta$.

3. Prove that $\left|\frac{z-1}{\bar{z}-1}\right| = 1$.

4. Let $z_1 = 2 - i$, $z_2 = -2 + i$, find

(*i*) $\text{Re}\left[\frac{z_1 z_2}{\bar{z}_1}\right]$ (*ii*) $\text{Im}\left[\frac{1}{z_1 \bar{z}_2}\right]$ **Ans.** (*i*) $-\frac{2}{5}$, (*ii*) 0

5. If $|z| = 1$, prove that $\frac{z-1}{z+1}$ $(z \ne 1)$ is a pure imaginary number, what will you conclude, if $z = 1$? **Ans.** If $z = 1$, $\frac{z-1}{z+1} = 0$, which is purely real.

25.16 POLAR FORM

Polar form of a complex number as we have discussed above

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$
$$\Rightarrow \quad x + iy = r(\cos\theta + i\sin\theta)$$
$$= r\,e^{i\theta} \quad \text{(Exponential form)} \qquad (e^{i\theta} = \cos\theta + i\sin\theta)$$

Procedure. To convert $x + iy$ into polar.

We write $$x = r\cos\theta$$
$$y = r\sin\theta$$

On solving these equations, we get the value of θ which satisfy both the equations and $r = \sqrt{x^2 + y^2}$.

25.17 TYPES OF COMPLEX NUMBERS

1. Cartesian form : $x + iy$ **2.** Polar form : $r(\cos\theta + i\sin\theta)$

3. Exponential form : $re^{i\theta}$

Example 20. *Express in polar form* : $1 - \sqrt{2} + i$

Solution. Let $(1 - \sqrt{2}) + i = r(\cos\theta + i\sin\theta)$

$$\therefore \quad 1 - \sqrt{2} = r\cos\theta \qquad ...(1)$$
$$1 = r\sin\theta \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$r^2(\cos^2\theta + \sin^2\theta) = (1 - \sqrt{2})^2 + 1^2$$

$$\Rightarrow \qquad r^2 = 1 - 2\sqrt{2} + 2 + 1$$

$$\Rightarrow \qquad r = \sqrt{4 - 2\sqrt{2}}$$

Putting the value of r in (1) and (2), we get

$$\cos\theta = \frac{1 - \sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{4 - 2\sqrt{2}}}$$

Hence, the polar form is $\sqrt{4 - 2\sqrt{2}}\left\{\dfrac{1 - \sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} + i\dfrac{1}{\sqrt{4 - 2\sqrt{2}}}\right\}$ **Ans.**

Example 21. *Find the smallest positive integer n for which*

$$\left(\frac{1+i}{1-i}\right)^n = 1. \qquad \textit{(Nagpur University, Winter 2004)}$$

Solution. $$\left[\frac{1+i}{1-i}\right]^n = 1$$

$$\left[\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right]^n = 1 \quad \Rightarrow \quad \left(\frac{1 - 1 + 2i}{1+1}\right)^n = 1$$

$$(i)^n = 1 = (i)^4 \quad \Rightarrow \quad n = 4$$ **Ans.**

Example 22. *If* $i^{\alpha + i\beta} = \alpha + i\beta$, *prove that* $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$

(Nagpur University, Summer 2003)

Solution. We have, $\alpha + i\beta = i^{(\alpha + i\beta)} = e^{\log i^{\alpha + i\beta}}$

$$\alpha + i\beta = e^{(\alpha + i\beta)\log i} = e^{(\alpha + i\beta)(\log i + 2n\pi i)}$$

$$= e^{(\alpha + i\beta)[\log(\cos \pi/2 + i\sin \pi/2) + 2n\pi i]}$$

$$\Rightarrow \qquad \alpha + i\beta = e^{(\alpha + i\beta)[\log e^{i\pi/2} + 2n\pi i]} = e^{(\alpha + i\beta)[i\pi/2 + 2n\pi i]}$$

$$= e^{i\alpha(\pi/2 + 2n\pi) - \beta(\pi/2 + 2n\pi)} = e^{-\beta\pi(2n + 1/2)} \times e^{\pi\alpha(2n + 1/2)i}$$

$$\Rightarrow \qquad \alpha + i\beta = e^{-\pi\beta(4n+1)/2}\left[\cos\left[\pi\alpha\frac{(4n+1)}{2}\right] + i\sin\left[\pi\alpha\frac{(4n+1)}{2}\right]\right]$$

Equating real and imaginary parts, we get

$$\alpha = e^{-\pi\beta(4n+1)/2}.\cos\left\{\frac{1}{2}\pi\alpha\,(4n+1)\right\} \qquad ...(1)$$

and

$$\beta = e^{-\pi\beta(4n+1)/2}.\sin\left\{\frac{1}{2}\pi\alpha\,(4n+1)\right\} \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$\alpha^2 + \beta^2 = e^{-\pi\beta(4n+1)}.\left[\cos^2\left\{\frac{1}{2}\pi\alpha\,(4n+1)\right\} + \sin^2\left\{\frac{1}{2}\pi\alpha\,(4n+1)\right\}\right]$$

$$\therefore \qquad \alpha^2 + \beta^2 = e^{-\pi\beta(4n+1)}$$

Hence the result. **Proved.**

EXERCISE 25.4

Express the following complex numbers into polar form :

1. $\dfrac{1+i}{1-i}$ **Ans.** $\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$

2. $\dfrac{-35+5i}{4\sqrt{2} + 3\sqrt{2}i}$ **Ans.** $5\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

3. $\dfrac{3(-4 - \sqrt{3} + 4\sqrt{3}i - i)}{8 + 2i}$ **Ans.** $3\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

4. $\dfrac{2 + 6\sqrt{3}i}{5 + \sqrt{3}i}$ **Ans.** $2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

5. $\dfrac{2+3i}{3-7i}$ **Ans.** $r = \sqrt{754}$, $\theta = \tan^{-1}\left(-\dfrac{23}{15}\right)$ **6.** $\left(\dfrac{4-5i}{2+3i}\right)\cdot\left(\dfrac{3+2i}{7+i}\right)$ **Ans.** 0.905, $\theta = \tan^{-1}(-7.2)$

7. $\dfrac{(2+5i)(-3+i)}{(1-2i)^2}$ **Ans.** $\dfrac{\sqrt{290}}{5}$, $\tan^{-1}\left(-\dfrac{1}{17}\right)$ **8.** $\dfrac{1+7i}{(2-i)^2}$ **Ans.** $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

9. $\dfrac{1+3i}{1-2i}$ **Ans.** $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$ **10.** $\dfrac{i-1}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$ **Ans.** $\sqrt{2}\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$

25.18 SQUARE ROOT OF A COMPLEX NUMBER

Let $a + ib$ be a complex number and its square root is $x + iy$.

i.e., $\sqrt{a+ib} = x + iy$ ...(1)

where x and $y \in R$.

Squaring both sides of (1), we get

$$a + ib = (x+iy)^2$$

$\Rightarrow$ $a + ib = x^2 + i^2y^2 + i\,2xy$

$\Rightarrow$ $a + ib = (x^2 - y^2) + i\,2xy$ $[\because i^2 = -1]$ ...(2)

Equating real and imaginary parts of (2), we get

$x^2 - y^2 = a$ ...(3)

and $2xy = b$ ...(4)

Also, we know that

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$\Rightarrow$ $(x^2+y^2)^2 = a^2 + b^2$ [Using (3) and (4)]

$\Rightarrow$ $x^2 + y^2 = \sqrt{a^2+b^2}$ ...(5)

Adding (3) and (5), we get

$$2x^2 = a + \sqrt{a^2+b^2} \quad \Rightarrow \quad x = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$$

Example 23. *Find the square root of the complex number 5 + 12i.*

Solution. Let $\sqrt{5+12i} = x + iy$ ...(1)

Squaring both sides of (1), we get $5 + 12i = (x+iy)^2 = (x^2-y^2) + i\,2xy$ ...(2)

Equating real and imaginary parts of (2), we get

$x^2 - y^2 = 5$ ...(3)

and $2xy = 12$ ...(4)

Now, $x^2 + y^2 = \sqrt{(x^2-y^2)^2 + 4x^2y^2} = \sqrt{(5)^2 + (12)^2}$

$$= \sqrt{25+144} = \sqrt{169} = 13$$

$\Rightarrow$ $x^2 + y^2 = 13$ ...(5)

Adding (3) and (5), we get $2x^2 = 5 + 13 = 18 \Rightarrow x = \sqrt{\dfrac{18}{2}} = \sqrt{9} = \pm 3$

Subtracting (3) from (5), we get $2y^2 = 13 - 5 = 8 \Rightarrow y = \sqrt{\dfrac{8}{2}} = \sqrt{4} = \pm 2$

Since, xy is positive, so x and y are of same sign. Hence, $x = \pm 3$, $y = \pm 2$

$\therefore$ $\sqrt{5+12i} = \pm 3 \pm 2i$ *i.e.* $(3+2i)$ or $-(3+2i)$ **Ans.**

Example 24. *Find the square root of* $-4 - 3i$.

Solution. Let $\sqrt{-4-3i} = x + iy$ ...(1)

Squaring both sides of (1), we get

$-4 - 3i = (x+iy)^2 = (x^2-y^2) + i\,2xy$ ...(2)

Equating real and imaginary parts of (2), we get

$$x^2 - y^2 = -4 \qquad ...(3)$$

And $$2xy = -3 \qquad ...(4)$$

Now, $$x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{16 + 9} = \sqrt{25} = \pm 5$$

$\Rightarrow$ $$x^2 + y^2 = 5 \qquad ...(5) \quad (\because x^2 + y^2 \geq 0)$$

Adding (3) and (5), we get

$$2x^2 = 5 - 4 = 1 \Rightarrow x = \sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$$

Subtracting (3) from (5), we get

$$2y^2 = 5 + 4 = 9 \Rightarrow y = \sqrt{\frac{9}{2}} = \pm\frac{3}{\sqrt{2}}$$

Since, xy is negative, so x and y will be of different signs. Hence, $x = \pm\frac{1}{\sqrt{2}}, y = \mp\frac{3}{\sqrt{2}}$

$\therefore$ $$\sqrt{-4 - 3i} = \pm\left(\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}}i\right)$$ **Ans.**

Example 25. *Prove that if the sum and product of two complex numbers are real then the two numbers must be either real or conjugate.* (*M.U. 2008*)

Solution. Let z_1 and z_2 be the two complex numbers.

We are given that $z_1 + z_2 = a$ (real)

and $z_1.z_2 = b$ (real)

If sum and product of the roots of a quadratic equation are given. Then the equation becomes

x^2 – (sum of the roots) x + product of the roots = 0

$$x^2 - ax + b = 0$$

$$\text{Root} = x = \frac{a \pm \sqrt{a^2 - 4b}}{2}$$

Case I. If $a^2 > 4b$ Then both the roots are real

Case II. If $a^2 < 4b$

Then $$\text{one root} = \frac{a}{2} + i\frac{\sqrt{4b - a^2}}{2}$$

$$\text{Second root} = \frac{a}{2} - i\frac{\sqrt{4b - a^2}}{2}$$

These roots are conjugate to each other. **Proved.**

EXERCISE 25.5

Find the square root of the following :

1. $1 + i$ **Ans.** $\left\{\pm\sqrt{\frac{\sqrt{2}+1}{2}} \pm \sqrt{\frac{\sqrt{2}-1}{2}}i\right\}$

2. $1 - i$ **Ans.** $\left\{\pm\sqrt{\frac{\sqrt{2}+1}{2}} \mp \sqrt{\frac{\sqrt{2}-1}{2}}i\right\}$

3. i **Ans.** $\left\{\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i\right\}$

4. $15 - 8i$ **Ans.** $1 - 4i, -1 + 4i$

5. $-2 + 2\sqrt{3}i$ **Ans.** $\pm(1 + \sqrt{3}i)$

6. $3 + 4\sqrt{7}i$ **Ans.** $\pm(\sqrt{7} + 2i)$

7. $\frac{2 + 3i}{5 - 4i} + \frac{2 - 3i}{5 + 4i}$ **Ans.** $\pm\frac{2}{\sqrt{41}}i$

8. $x^2 - 1 + i\,2x$ **Ans.** $\pm(x + i)$

9. $3 - 4i$ **Ans.** $\pm(2 - i)$

25.19 EXPONENTIAL AND CIRCULAR FUNCTIONS OF COMPLEX VARIABLES

Proof. $\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots \qquad \ldots(1)$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \qquad \ldots(2)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \qquad \ldots(3)$$

From (2) and (3), we have

$$\cos z + i\sin z = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)$$

$$= 1 + \frac{(iz)^1}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \ldots = e^{iz}$$

Therefore, $\cos z + i\sin z = e^{iz}$ ...(4)

Similarly, $\cos z - i\sin z = e^{-iz}$ ...(5)

From (4) and (5), we have

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \ldots(6)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \ldots(7)$$

25.20 DE MOIVRE'S THEOREM (By Exponential Function)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof. We know that $e^{i\theta} = \cos\theta + i\sin\theta$

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n$$

$$(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n$$ **Proved.**

If n is a fraction, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)$"

25.21 DE MOIVRE'S THEOREM (BY INDUCTION)

Statement: For any rational number n the value or one of the values of

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof. Case I. Let n be a non-negative integer. By actual multiplication,

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$$

$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \qquad \ldots(1)$$

Similarly we can prove that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3) = \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$$

Continuing in this way, we can prove that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$

Putting $\theta_1 = \theta_2 = \theta_3 = \ldots \theta_n = \theta$, we get

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

Case II. Let n be a negative integer, say $n = -m$ where m is a positive integer. Then,

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{(\cos m\theta + i\sin m\theta)} \qquad \text{[By case I]}$$

$$= \frac{1}{(\cos m\theta + i \sin m\theta)} \cdot \frac{(\cos m\theta - i \sin m\theta)}{(\cos m\theta - i \sin m\theta)} = \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$= \cos m\,\theta - i \sin m\,\theta$ $\qquad [\because \cos^2 m\,\theta + \sin^2 m\,\theta = 1]$

$= \cos(-m\,\theta) + i \sin(-m\,\theta) = \cos n\,\theta + i \sin n\,\theta$

Hence, the theorem is true for negative integers also.

Case III. Let n be a proper fraction $\frac{p}{q}$ where p and q are integers. Without loss of generality we can select q to be positive integer, p may be a positive or negative integer.

Since q is a positive integer

Now, $\left(\cos\frac{\theta}{q} + i \sin\frac{\theta}{q}\right)^q = \cos q\,.\frac{\theta}{q} + i \sin q\,.\frac{\theta}{q}$ $\qquad$ [By case I]

$= \cos\theta + i \sin\theta$

Taking the q th root of both sides, we get

$$(\cos\theta + i \sin\theta)^{\frac{1}{q}} = \cos\frac{\theta}{q} + i \sin\frac{\theta}{q}$$

Raising both sides to the power p,

$$(\cos\theta + i \sin\theta)^{\frac{p}{q}} = \left(\cos\frac{\theta}{q} + i \sin\frac{\theta}{q}\right)^p = \cos p.\frac{\theta}{q} + i \sin p.\frac{\theta}{q} \qquad \text{[By case I and II]}$$

Hence, one of the values of $(\cos\theta + \text{i} \sin\theta)^n$ is $\cos n\,\theta + i \sin n\,\theta$ when n is a proper fraction.

Thus, the theorem is true for all rational values of n.

Example 26. *Express* $\frac{(\cos\theta + i \sin\theta)^8}{(\sin\theta + i \cos\theta)^4}$ *in the form* $(x + iy)$.

Solution. $\frac{(\cos\theta + i \sin\theta)^8}{(\sin\theta + i \cos\theta)^4} = \frac{(\cos\theta + i \sin\theta)^8}{(i)^4\left(\cos\theta + \frac{1}{i}\sin\theta\right)^4}$

$$= \frac{(\cos\theta + i \sin\theta)^8}{(\cos\theta - i \sin\theta)^4} = \frac{(\cos\theta + i \sin\theta)^8}{[\cos(-\theta) + i \sin(-\theta)]^4}$$

$$= \frac{(\cos\theta + i \sin\theta)^8}{[(\cos\theta + i \sin\theta)^{-1}]^4} = \frac{(\cos\theta + i \sin\theta)^8}{(\cos\theta + i \sin\theta)^{-4}} = (\cos\theta + i \sin\theta)^{12}$$

$= \cos 12\,\theta + i \sin 12\,\theta$ **Ans.**

Example 27. *Prove that* $(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n = 2^{n+1} \cos^n\frac{\theta}{2} \cos\frac{n\,\theta}{2}$ *where n is an integer.*

Solution. L.H.S. $= (1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n$

$$= \left[1 + 2\cos^2\frac{\theta}{2} - 1 + 2\,i \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n + \left[1 + 2\cos^2\frac{\theta}{2} - 1 - i\,2 \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n$$

$$= \left[2\cos^2\frac{\theta}{2} + 2\,i \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n + \left[2\cos^2\frac{\theta}{2} - 2\,i \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n$$

$$= \left(2\cos\frac{\theta}{2}\right)^n\left[\cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\right]^n + \left(2\cos\frac{\theta}{2}\right)^n\left[\cos\frac{\theta}{2} - i \sin\frac{\theta}{2}\right]^n$$

$$= 2^n \cos^n\frac{\theta}{2}\left[\cos\frac{n\,\theta}{2} + i \sin\frac{n\,\theta}{2}\right] + 2^n \cos^n\frac{\theta}{2}\left[\cos\frac{n\,\theta}{2} - i \sin\frac{n\,\theta}{2}\right]$$

$$= 2^n \cos^n \frac{\theta}{2}\left[\cos \frac{n\theta}{2} + i \sin \frac{n\theta}{2} + \cos \frac{n\theta}{2} - i \sin \frac{n\theta}{2}\right]$$

$$= 2^n \cos^n \frac{\theta}{2}\left(2 \cos \frac{n\theta}{2}\right) = 2^{n+1} \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2} = \text{R.H.S.}$$ **Proved.**

Example 28. *Evaluate* $\left(\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n$ *(M.U. 2001, 2004, 2005)*

Solution. We know that,

$$1 = \sin^2\alpha + \cos^2\alpha$$

$$\Rightarrow \quad 1 = \sin^2\alpha - i^2\cos^2\alpha$$

$$\Rightarrow \quad 1 = (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha) \qquad ...(1)$$

Adding $\sin\alpha + i\cos\alpha$ both sides of (1), we get

$$1 + \sin\alpha + i\cos\alpha = (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha) + (\sin\alpha + i\cos\alpha)$$

$$= (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha + 1)$$

$$\therefore \quad \frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha} = \sin\alpha + i\cos\alpha$$

$$= \cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right) \qquad ...(2)$$

$$\Rightarrow \quad \left(\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n = \left\{\cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right)\right\}^n$$

$$= \cos n\left(\frac{\pi}{2}-\alpha\right) + i\sin n\left(\frac{\pi}{2}-\alpha\right)$$ **Ans.**

Example 29. *If* $2\cos\theta = x + \frac{1}{x}$ *and* $2\cos\phi = y + \frac{1}{y}$, *then prove that*

$x^p . y^q + \frac{1}{x^p . y^q} = 2\cos(p\theta + q\phi)$. *(Nagpur University, Summer 2000)*

Solution. We have,

$$x + \frac{1}{x} = 2\cos\theta \Rightarrow x^2 - 2x\cos\theta + 1 = 0$$

$$\Rightarrow \quad x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta}$$

Putting i^2 for -1 and considering the positive sign, we get

$$x = \cos\theta + i\sin\theta \text{ and similarly, } y = \cos\phi + i\sin\phi$$

Now, $\quad x^p = (\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta$

and $\quad y^q = (\cos\phi + i\sin\phi)^q = \cos q\phi + i\sin q\phi$

(by De-Moivre's theorem)

$$\therefore \quad x^p . y^q = (\cos p\theta + i\sin p\theta)(\cos q\phi + i\sin q\phi)$$

$$= \cos(p\theta + q\phi) + i\sin(p\theta + q\phi) \qquad ...(1)$$

Also $\quad \frac{1}{x^p . y^q} = [\cos(p\theta + q\phi) + i\sin(p\theta + q\phi)]^{-1}$

$$= \cos(p\theta + q\phi) - i\sin(p\theta + q\phi) \qquad ...(2)$$

Adding (1) and (2), we get

$$\therefore \quad x^p . y^q + \frac{1}{x^p . y^q} = 2\cos(p\theta + q\phi)$$ **Proved.**

Example 30. *Prove that the general value of* θ *which satisfies the equation*

$(\cos\theta + i\sin\theta).(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = 1$ is $\frac{4m\pi}{n(n+1)}$, *where m is any integer.*

Solution. $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ldots(\cos n\theta + i\sin n\theta) = 1$

$$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^2\ldots(\cos\theta + i\sin\theta)^n = 1$$

$$(\cos\theta + i\sin\theta)^{1+2\ldots+n} = 1$$

$$(\cos\theta + i\sin\theta)^{\frac{n(n+1)}{2}} = (\cos 2m\pi + i\sin 2m\pi)$$

$$\cos\frac{n(n+1)}{2}\theta + i\sin\frac{n(n+1)}{2}\theta = \cos 2m\pi + i\sin 2m\pi$$

$$\frac{n(n+1)}{2}\theta = 2m\pi \Rightarrow \theta = \frac{4m\pi}{n(n+1)}$$ **Proved.**

Example 31. *If* $(a_1 + ib_1)\,.\,(a_2 + ib_2)\ldots(a_n + ib_n) = A + iB$, *then prove that*

(*i*) $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$

(*ii*) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ \ldots\ (a_n^2 + b_n^2) = A^2 + B^2$

Solution. Let $a_1 = r_1\cos\alpha_1,\quad b_1 = r_1\sin\alpha_1$

$a_2 = r_2\cos\alpha_2,\quad b_2 = r_2\sin\alpha_2$

....

$a_n = r_n\cos\alpha_n,\quad b_n = r_n\sin\alpha_n$

$A = R\cos\theta,\quad B = R\sin\theta,$

$(a_1 + ib_1).(a_2 + ib_2)\ldots(a_n + ib_n) = A + iB$ (Given)

$r_1(\cos\alpha_1 + i\sin\alpha_1)\, r_2(\cos\alpha_2 + i\sin\alpha_2)\ldots r_n(\cos\alpha_n + i\sin\alpha_n) = R(\cos\theta + i\sin\theta)$

$r_1 r_2\ldots r_n[\cos(\alpha_1 + \alpha_2 + \ldots\alpha_n) + i\sin(\alpha_1 + \alpha_2 + \ldots\alpha_n)] = R(\cos\theta + i\sin\theta)$

$\therefore\quad r_1 r_2 \ldots\ldots\ldots\ldots r_n = R$

$\Rightarrow\quad (a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots(a_n^2 + b_n^2) = A^2 + B^2$

And $\alpha_1 + \alpha_2 + \ldots + \alpha_n = \theta$

$$\Rightarrow\quad \tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$$ **Proved.**

Example 32. *If* $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, *then prove that*

(*i*) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

(*ii*) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

(*iii*) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$

(*iv*) $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$ (*M.U. 2009*)

Solution. Here, we have

$$(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$$

$$(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma) = 0$$

$\therefore\quad a + b + c = 0$ say ...(1)

where, $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$

Also we can write

$$(\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma) = 0$$

$$\Rightarrow\quad (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1} = 0$$

$$\Rightarrow\quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

$$\Rightarrow \quad \frac{bc + ca + ab}{abc} = 0 \quad \Rightarrow \quad ab + bc + ca = 0 \qquad \ldots(2)$$

But $\quad (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$

$$0 = (a^2 + b^2 + c^2) + 0 \qquad \text{[From (1) and (2)]}$$

$$\Rightarrow \quad a^2 + b^2 + c^2 = 0$$

$$\Rightarrow \quad (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$$

$$\Rightarrow \quad (\cos 2\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin 2\beta) + (\cos 2\gamma + i\sin 2\gamma) = 0$$

$$\Rightarrow \quad (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$$

$$\Rightarrow \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0. \qquad \ldots(3)$$

$$\Rightarrow \quad 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 0$$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2} \qquad \ldots(4)$$

Further $\quad 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = \dfrac{3}{2}$

$$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2} \qquad \ldots(5)$$

Again consider $\quad ab + bc + ca = 0 \qquad$ [From (2)]

$$\Rightarrow \quad (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) + (\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma) + (\cos\gamma + i\sin\gamma)(\cos\alpha + i\sin\alpha) = 0$$

$$\Rightarrow \quad [\cos(\alpha + \beta) + i\sin(\alpha + \beta)] + [\cos(\beta + \gamma) + i\sin(\beta + \gamma)] + [\cos(\gamma + \alpha) + i\sin(\gamma + \alpha)] = 0$$

Equating real and imaginary parts, we get

$$\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$$

$$\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$$

Proved.

EXERCISE 25.6

1. If n is a positive integer show that $(a + ib)^n + (a - ib)^n = 2r^n \cos n\theta$ where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$. Hence deduce that $\left(1 + i\sqrt{3}\right)^8 + \left(1 - i\sqrt{3}\right)^8 = -2^8$.

2. If n be a positive integer, prove that $(1 + i)^n + (1 - i)^n = 2^{\frac{n+2}{2}} \cos\dfrac{n\pi}{4}$

3. Show that $(a + ib)^{m/n} + (a - ib)^{m/n} = 2(a^2 + b^2)^{\frac{m}{2n}} \cos\left(\dfrac{m}{n}\tan^{-1}\dfrac{b}{a}\right)$.

4. If $P = \cos\theta + i\sin\theta$, $q = \cos\phi + i\sin\phi$, show that

   (*i*) $\dfrac{P - q}{P + q} = i\tan\dfrac{\theta - \phi}{2}$ $\qquad$ (*ii*) $\dfrac{(P + q)(Pq - 1)}{(P - q)(Pq + 1)} = \dfrac{\sin 0 + \sin\phi}{\sin 0 - \sin\phi}$

5. If $x = \cos\theta + i\sin\theta$, show that (*i*) $x^m + \dfrac{1}{x^m} = 2\cos m\theta$ $\quad$ (*ii*) $x^m - \dfrac{1}{x^m} = 2i\sin m\theta$.

6. Prove that $\tanh(\log\sqrt{3}) = \dfrac{1}{2}$

7. Prove that $[\sin(\alpha + \theta) - e^{i\alpha}\sin\theta]^n = \sin^n\alpha\, e^{-in\theta}$

8. If $x + \dfrac{1}{x} = 2\cos\theta$, $y + \dfrac{1}{y} = 2\cos\phi$, $z + \dfrac{1}{z} = 2\cos\psi$, show that

   $$xyz + \frac{1}{xyz} = 2\cos(\theta + \phi + \psi)$$

25.22 ROOTS OF A COMPLEX NUMBER

We know that $\cos\theta + i\sin\theta = \cos(2m\pi + \theta) + i\sin(2m\pi + \theta)$, $m \in I$

$$[\cos\theta + i\sin\theta]^{1/n} = [\cos(2m\pi + \theta) + i\sin(2m\pi + \theta)]^{1/n}$$

$$= \cos\frac{(2m\pi + \theta)}{n} + i\sin\frac{(2m\pi + \theta)}{n}$$

Giving m the values 0, 1, 2, 3, ... $n - 1$ successively, we get the following n values of $(\cos\theta + i\sin\theta)^{1/n}$.

when $m = 0$, $\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$

When $m = 1$, $\cos\left(\frac{2\pi + \theta}{n}\right) + i\sin\left(\frac{2\pi + \theta}{n}\right)$

When $m = 2$, $\cos\left(\frac{4\pi + \theta}{n}\right) + i\sin\left(\frac{4\pi + \theta}{n}\right)$

When $m = n - 1$, $\cos\left(\frac{2(n-1)\pi + \theta}{n}\right) + i\sin\left(\frac{2(n-1)\pi + \theta}{n}\right)$

When $m = n$, $\cos\frac{2n\pi + \theta}{n} + i\sin\frac{2n\pi + \theta}{n} = \cos\left(2\pi + \frac{\theta}{n}\right) + i\sin\left(2\pi + \frac{\theta}{n}\right)$

$$= \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$$

which is the same as the value for $m = 0$. Thus, the values of $(\cos\theta + i\sin\theta)^{1/n}$ for $m = n$, $n + 1$, $n + 2$ etc., are the mere repetition of the first n values as obtained above.

Example 33. *Solve $x^4 + i = 0$.* (*M.U. 2008*)

Solution. Here, we have

$$x^4 = -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

$$x^4 = \cos\left(2n\pi + \frac{\pi}{2}\right) - i\sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$\Rightarrow \quad x = \left[\cos\left(2n\pi + \frac{\pi}{2}\right) - i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{\frac{1}{4}}$$

$$= \cos(4n + 1)\frac{\pi}{8} - i\sin(4n + 1)\frac{\pi}{8}$$

Putting $n = 0, 1, 2, 3$ we get the roots as

$$x_1 = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}, \quad x_2 = \cos\frac{5\pi}{8} - i\sin\frac{5\pi}{8}$$

$$x_3 = \cos\frac{9\pi}{8} - i\sin\frac{9\pi}{8}, \quad x_4 = \cos\frac{13\pi}{8} - i\sin\frac{13\pi}{8}$$ **Ans.**

Example 34. *Solve $x^5 = 1 + i$ and find the continued product of the roots.* (*M.U. 2005, 2004*)

Solution. $x^5 = 1 + i$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow x = 2^{\frac{1}{10}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{\frac{1}{5}}$$

$$\Rightarrow \quad x = 2^{\frac{1}{10}}\left[\cos\left(2k\pi + \frac{\pi}{4}\right).\frac{1}{5} + i\sin\left(2k\pi + \frac{\pi}{4}\right).\frac{1}{5}\right]$$

$$= 2^{\frac{1}{10}}\left[\cos(8k+1)\frac{\pi}{20} + i\sin(8k+1)\frac{\pi}{20}\right]$$

The roots are obtained by putting $k = 0, 1, 2, 3, 4...$

$$x_1 = 2^{\frac{1}{10}}\left[\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right], \qquad x_2 = 2^{\frac{1}{10}}\left[\cos\frac{9\pi}{20} + i\sin\frac{9\pi}{20}\right]$$

$$x_3 = 2^{\frac{1}{10}}\left[\cos\frac{17\pi}{20} + i\sin\frac{17\pi}{20}\right], \qquad x_4 = 2^{\frac{1}{10}}\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right]$$

$$x_5 = 2^{\frac{1}{10}}\left[\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right]$$

$$x_1 \,.\, x_2 \,.\, x_3 .\, x_4 .\, x_5 = \left(2^{\frac{1}{10}}\right)^5 \left(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right)\left(\cos\frac{9\pi}{20} + i\sin\frac{9\pi}{20}\right)\left(\cos\frac{17\pi}{20} + i\sin\frac{17\pi}{20}\right)$$

$$\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\left(\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right)$$

$$= 2^{\frac{1}{2}}\left[\cos\left(\frac{\pi}{20} + \frac{9\pi}{20} + \frac{17\pi}{20} + \frac{5\pi}{4} + \frac{33\pi}{20}\right) + i\sin\left(\frac{\pi}{20} + \frac{9\pi}{20} + \frac{17\pi}{20} + \frac{5\pi}{4} + \frac{33\pi}{20}\right)\right]$$

$$= \sqrt{2}\left[\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right] = \sqrt{2}\left[\cos\left(4\pi + \frac{\pi}{4}\right) + i\sin\left(4\pi + \frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \sqrt{2}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = 1 + i$$

Ans.

Example 35. *If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$ find them and show that $(1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4) = 5$.* (*M.U. 2007*)

Solution. Here, we have

$$x^5 - 1 = 0$$

$$\Rightarrow \quad x^5 = 1 = \cos 0 + i\sin 0$$

$$\Rightarrow \quad x^5 = \cos(2k\pi) + i\sin(2k\pi)$$

$$\Rightarrow \quad x = (\cos 2k\pi + i\sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$$

Putting $k = 0, 1, 2, 3, 4$, we get the five roots as below

$$x_0 = \cos 0 + i\sin 0, \quad x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$$x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \quad x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$$

$$x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$$

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$, we see that

$$x_2 = \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \alpha^2$$

Similarly, $\quad x_3 = \alpha^3$ and $x_4 = \alpha^4$

$\therefore$ The roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$

Hence $\quad x^5 - 1 = (x - 1)(x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)$

$$\Rightarrow \quad \frac{x^5 - 1}{x - 1} = (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)$$

On dividing $x^5 - 1$ by $x - 1$, we get

$$x^4 + x^3 + x^2 + x + 1 = (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)$$

$$\therefore (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4) = x^4 + x^3 + x^2 + x + 1$$

Putting $x = 1$, we get

$$(1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4) = 1 + 1 + 1 + 1 + 1 = 5.$$ **Proved.**

Example 36. *If ω is a cube root of unity, prove that*

$$(1 - \omega)^6 = -27$$ (*M.U. 2003*)

Solution. Let $x^3 = 1$

$$\Rightarrow \quad x = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3} = (\cos 2n\pi + i \sin 2n\pi)^{1/3}$$

$$= \cos\left(\frac{2n\pi}{3}\right) + i \sin\left(\frac{2n\pi}{3}\right)$$

Putting $n = 0, 1, 2$ the roots of unity are

$$x_0 = 1$$

$$x_1 = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = \omega \text{ (say)}$$

$$x_2 = \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = \left[\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right]^2 = \omega^2$$

Now,

$$1 + \omega + \omega^2 = 1 + \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} + \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}$$

$$= 1 + \cos\left(\pi - \frac{\pi}{3}\right) + i \sin\left(\pi - \frac{\pi}{3}\right) + \cos\left(\pi + \frac{\pi}{3}\right) + i \sin\left(\pi + \frac{\pi}{3}\right)$$

$$= 1 - \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} - \cos\frac{\pi}{3} - i \sin\frac{\pi}{3}$$

$$= 1 - 2\cos\frac{\pi}{3} = 1 - 2\left(\frac{1}{2}\right) = 0$$

$$\Rightarrow \quad 1 + \omega + \omega^2 = 0$$

$$\Rightarrow \quad 1 + \omega^2 = -\omega \qquad \text{...(1)}$$

Now,

$$(1 - \omega)^6 = [(1 - \omega)^2]^3 = [1 - 2\omega + \omega^2]^3 = [-\omega - 2\omega]^3$$

$$= (-3\omega)^3 = -27\omega^3 = -27$$ [Using (1)] **Proved.**

Example 37. *Use De Moivre's theorem to solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$.*

Solution. $x^4 - x^3 + x^2 - x + 1 = 0$

$$(x + 1)(x^4 - x^3 + x^2 - x + 1) = 0$$

$$x^5 + 1 = 0$$

$$x^5 = -1 = (\cos \pi + i \sin \pi) = \cos(2n\pi + \pi) + i \sin(2n\pi + \pi)$$

$$x = [\cos(2n + 1)\pi + i \sin(2n + 1)\pi]^{1/5}$$

$$= \cos\frac{(2n + 1)\pi}{5} + i \sin\frac{(2n + 1)\pi}{5}$$

When $n = 0, 1, 2, 3, 4$, the values are

$$\cos\frac{\pi}{5} + i \sin\frac{\pi}{5},\ \cos\frac{3\pi}{5} + i \sin\frac{3\pi}{5},\ \cos\pi + i \sin\pi,\ \cos\frac{7\pi}{5} + i \sin\frac{7\pi}{5},$$

$$\cos\frac{9\pi}{5} + i \sin\frac{9\pi}{5}.$$

$\cos \pi + i \sin \pi = -1$, which is rejected as it is corresponding to $x + 1 = 0$.
Hence, the required roots are

$$\cos\frac{\pi}{5} + i \sin \frac{\pi}{5}, \cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5}, \cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5}, \cos \frac{9\pi}{5} + i \sin \frac{9\pi}{5}. \qquad \textbf{Ans.}$$

EXERCISE 25.7

Find the values of:

1. $(1 + i)^{1/5}$. **Ans.** $2^{1/10}\left[\cos \frac{1}{5}\left(2n\pi + \frac{\pi}{4}\right) + i \sin \frac{1}{5}\left(2n\pi + \frac{\pi}{4}\right)\right]$, where $n = 0, 1, 2, 3, 4$

2. $(1 + \sqrt{-3})^{3/4}$ **Ans.** $(2)^{3/4}\left[\cos \frac{3}{4}\left(2n\pi + \frac{\pi}{3}\right) + i \sin \frac{3}{4}\left(2n\pi + \frac{\pi}{3}\right)\right]$, where $n = 0, 1, 2, 3$.

3. $(-i)^{1/6}$ **Ans.** $\cos (4n + 1) \frac{\pi}{12} - i \sin (4n + 1) \frac{\pi}{12}$, where $n = 0, 1, 2, 3, 4, 5$.

4. $(1 + i)^{2/3}$ **Ans.** $2^{1/3}\left[\cos\left(\frac{4n\pi}{3} + \frac{\pi}{6}\right) + i \sin\left(\frac{4n\pi}{3} + \frac{\pi}{6}\right)\right]$, where $n = 0, 1, 2$

5. Solve the equation with the help of De Moivre's theore $x^7 - 1 = 0$

   Ans. $\cos \frac{2n\pi}{7} + i \sin \frac{2n\pi}{7}$ where $n = 0, 1, 2, 3, 4, 5, 6$.

6. Find the roots of the equation $x^3 + 8 = 0$.

   Ans. $2\left[\cos\left(\frac{2n\pi + \pi}{3}\right) + i \sin\left(\frac{2n\pi + \pi}{3}\right)\right]$, where $n = 0, 1, 2$.

7. Use De-Moivre's theorem to solve $x^9 - x^5 + x^4 - 1 = 0$

   Ans. $\left[\cos (2n + 1) \frac{\pi}{5} + i \sin (2n + 1) \frac{\pi}{5}\right]$, where $n = 0, 1, 2, 3, 4$,

   and $\left[\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}\right]$, where $n = 0, 1, 2, 3$.

8. Show that the roots of $(x + 1)^6 + (x - 1)^6 = 0$ are given by

   $i \cot \frac{(2n + 1)}{12}$, $n = 0, 1, 2, 3, 4, 5$. Deduce $\tan^2 \frac{\pi}{12} + \tan^2 \frac{3\pi}{12} + \tan^2 \frac{5\pi}{12} = 15$.

9. Show that all the roots of $(x + 1)^7 = (x - 1)^7$ are given by $\pm i \cot \left(\frac{n\pi}{7}\right)$, where $n = 1, 2, 3$. Why $n \neq 0$.

25.23 CIRCULAR FUNCTIONS OF COMPLEX NUMBERS

We have already discussed circular functions in terms of exponential functions i.e, Euler's exponential form of circular functions:

$$\boxed{\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}, \quad \boxed{\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

If $\theta = z$, then $\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}}$ and $\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}$

25.24 HYPERBOLIC FUNCTIONS

(*i*) $\sinh x = \frac{e^x - e^{-x}}{2}$ (*ii*) $\cosh x = \frac{e^x + e^{-x}}{2}$ (*iii*) $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(iv) $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ (v) $\text{sech } x = \frac{2}{e^x + e^{-x}}$ (*vi*) $\text{cosech } x = \frac{2}{e^x - e^{-x}}$

(*vii*) $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$

(*viii*) $\cosh x - \sinh x = \dfrac{e^x + e^{-x}}{2} - \dfrac{e^x - e^{-x}}{2} = e^{-x}$

(*ix*) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx.$

25.25 RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS

$\sin ix = i \sinh x$	$\sinh ix = i \sin x$
$\cos ix = \cosh x$	$\cosh ix = \cos x$
$\tan ix = i \tanh x$	$\tanh ix = i \tan x$

25.26 FORMULAE OF HYPERBOLIC FUNCTIONS

A. (1) $\cosh^2 x - \sinh^2 x = 1,$ (2) $\text{sech}^2\, x = 1 - \tanh^2 x,$

(3) $\text{cosech}^2\, x = \coth^2 x - 1$

B. (1) $\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(2) $\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(3) $\tanh (x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

C. (1) $\sinh 2x = 2 \sinh x \cosh x$ (2) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(3) $\cosh 2x = 2\cosh^2 x - 1$ (4) $\cosh 2x = 1 + 2\sinh^2 x$

(5) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

D. (1) $\sinh x + \sinh y = 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

(2) $\sinh x - \sinh y = 2 \cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

(3) $\cosh x + \cosh y = 2 \cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

(4) $\cosh x - \cosh y = 2 \sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

Note: For proof, put $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Example 38. *Prove that*

$$(\cosh x - \sinh x)^n = \cosh nx - \sinh nx. \qquad \text{(M.U. 2001, 2002)}$$

Solution. L.H.S. $= (\cosh x - \sinh x)^n$

$$= \left[\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right]^n = \left[\frac{2e^{-x}}{2}\right]^n = (e^{-x})^n = e^{-nx} \qquad \text{...(1)}$$

R.H.S. $= \cosh nx - \sinh nx$

$$= \left(\frac{e^{nx} + e^{-nx}}{2} - \frac{e^{nx} + e^{-nx}}{2}\right) = \frac{2e^{-nx}}{2} = e^{-nx} \qquad \text{...(2)}$$

From (1) and (2), we have

L.H.S. = R.H.S. **Proved.**

Example 39. *If* $x = 2 \sin \alpha \cosh \beta$, $y = 2 \cosh \alpha \sinh \beta$, *show that*

$$\text{cosec}\,(\alpha - i\beta) + \text{cosec}\,(\alpha + i\beta) = \frac{4x}{x^2 + y^2}$$

Solution. We know that $\text{cosec}\,(\alpha + i\beta) = \dfrac{1}{\sin(\alpha + i\beta)} = \dfrac{1}{\sin \alpha \cos i\beta + \cos \alpha \sin i\beta}$

$$= \frac{1}{\sin\alpha\cosh\beta + i\cos\alpha\sinh\beta} = \frac{1}{\frac{x}{2} + i\frac{y}{2}} = \frac{2}{x+iy} \quad \text{... (1) (Given)}$$

$$\text{cosec}\,(\alpha - i\beta) = \frac{2}{x-iy} \quad \text{... (2)}$$

Adding (1) and (2), we get

$$\text{cosec}\,(\alpha - i\beta) + \text{cosec}\,(\alpha + i\beta) = \frac{2}{x-iy} + \frac{2}{x+iy} = \frac{4x}{x^2+y^2}$$ **Proved.**

Example 40. *If tan (x + iy) = i, where x and y are real, prove that x is indeterminate and y is infinite.*

Solution. $\tan(x+iy) = i \Rightarrow \tan(x-iy) = -i$

$$\tan 2x = \tan(\overline{x+iy} + \overline{x-iy}) = \frac{\tan(x+iy) + \tan(x+iy)}{1 - \tan(x+iy)\tan(x-iy)}$$

$$= \frac{i-i}{1-i(-i)} = \frac{i-i}{1-1} = \frac{0}{0}, \text{ which is indeterminate.}$$

Also $$\tan 2iy = \tan[(x+iy)-(x-iy)] = \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{i-(-i)}{1+i(-i)} = \frac{2i}{1+1} = i$$

$i \tanh 2y = i \quad \Rightarrow \quad \tanh 2y = 1 \quad \Rightarrow \quad 2y = \tanh^{-1}(1) = \frac{1}{2}\log\frac{1+1}{1-1} = \infty$

$\therefore$ y is infinite. **Proved.**

Example 41. *If tan* $(\theta + i\phi) = \cos\alpha + i\sin\alpha$, *prove that:*

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4} \text{ and } \phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$ (*Nagpur University, Summer 2002, Winter 2001*)

Solution. We have, $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$

$\therefore \quad \tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

But $\quad \tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)} = \frac{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{1-1} = \infty = \tan\frac{\pi}{2}$$

$\therefore \quad 2\theta = \frac{\pi}{2}$ or for general values,

$$2\theta = n\pi + \frac{\pi}{2} \Rightarrow \theta = \frac{n\pi}{2} + \frac{\pi}{4}$$

Again, $$\tan(2i\phi) = \tan[(\theta + i\phi) - (\theta - i\phi)] = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$= \frac{\cos\alpha + i\sin\alpha - (\cos\alpha - i\sin\alpha)}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2i\sin\alpha}{1 + \cos^2\alpha + \sin^2\alpha} = \frac{2i\sin\alpha}{1+1} = \frac{2i\sin\alpha}{2} = i\sin\alpha$$

$\Rightarrow \quad i\tanh 2\phi = i\sin\alpha \quad (\because \tan ix = i\tanh x)$

$\Rightarrow \quad \tanh 2\phi = \sin\alpha$

i.e., $$\frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$$

$$\therefore \quad \frac{e^{2\phi} - e^{-2\phi} + e^{2\phi} + e^{-2\phi}}{(e^{2\phi} + e^{-2\phi}) - (e^{2\phi} - e^{-2\phi})} = \frac{1+\sin\alpha}{1-\sin\alpha} \qquad \text{(Componendo and dividendo)}$$

i.e.
$$\frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{1-\cos\left(\frac{\pi}{2}+\alpha\right)}{1+\cos\left(\frac{\pi}{2}+\alpha\right)} \qquad \because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$

$$\Rightarrow \quad e^{4\phi} = \frac{2\sin^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}$$

$$\Rightarrow \quad e^{4\phi} = \tan^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \quad \Rightarrow \quad e^{2\phi} = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

Hence,
$$2\phi = \log_e \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \quad \Rightarrow \quad \phi = \frac{1}{2}\log_e \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$ **Proved.**

Example 42. *If* $u = \log_e \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, *prove that*

$$\tanh\frac{u}{2} = \tan\frac{\theta}{2}$$

Solution. Here, we have,

$$u = \log_e \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \quad \Rightarrow \quad e^u = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\Rightarrow \quad e^u = \frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{\theta}{2}} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} \qquad \text{...(1)}$$

$$\Rightarrow \quad e^{-u} = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}} \qquad \text{... (2)}$$

By componendo and dividendo on (1), we have

$$\frac{e^u+1}{e^u-1} = \frac{2}{2\tan\frac{\theta}{2}} \quad \therefore \quad \frac{e^u-1}{e^u+1} = \tan\frac{\theta}{2} \qquad \text{...(3)}$$

Now,
$$\tanh\frac{u}{2} = \frac{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}{e^{\frac{u}{2}} + e^{-\frac{u}{2}}} = \frac{e^{\frac{u}{2}} - e^{-\frac{u}{2}}}{e^{\frac{u}{2}} + e^{-\frac{u}{2}}}\cdot\frac{e^{\frac{u}{2}}}{e^{\frac{u}{2}}}$$

$$\Rightarrow \quad \tanh\frac{u}{2} = \frac{e^u-1}{e^u+1}$$

$$\Rightarrow \quad \tanh\frac{u}{2} = \tan\frac{\theta}{2} \qquad \text{[Using (3) and (4)]}$$ **Proved.**

Example 43. *If* $\cosh x = \sec\theta$, *prove that:*

(i) $$\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$$

(ii) $$\tanh\frac{\pi}{2} = \tan\frac{\theta}{2}$$ (*M.U.* 2003, 2005)

Solution. (*i*) Let

$$\tan^{-1} e^{-x} = \alpha$$

$$\Rightarrow \quad e^{-x} = \tan\alpha \quad \text{and} \quad \alpha = \tan^{-1}(e^{-x}) \qquad \text{...(1)}$$

$$\Rightarrow \quad e^{x} = \cot\alpha \qquad \text{...(2)}$$

Now,

$$\sec\theta = \cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \text{...(3)} \quad \text{(Given)}$$

Putting the values of e^{-x} and e^{x} from (1) and (2) in (3), we get

$$\sec\theta = \frac{\cot\alpha + \tan\alpha}{2}$$

$$\therefore \quad 2\sec\theta = \cot\alpha + \tan\alpha = \frac{\cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{\cos\alpha} = \frac{\cos^2\alpha + \sin^2\alpha}{\sin\alpha\cos\alpha}$$

$$= \frac{2}{2\sin\alpha\cos\alpha} \qquad [\because \cos^2\alpha + \sin^2\alpha = 1]$$

$$= \frac{2}{\sin 2\alpha}$$

$$\therefore \quad \cos\theta = \sin 2\alpha$$

$$\Rightarrow \quad \cos\theta = \cos\left(\frac{\pi}{2} - 2\alpha\right)$$

$$\therefore \quad \theta = \frac{\pi}{2} - 2\alpha = \frac{\pi}{2} - 2\tan^{-1}(e^{-x}) \qquad \text{[From (1)]} \quad \textbf{Proved.}$$

(*ii*) We have,

$$\cosh x = \sec\theta \qquad \text{(Given)}$$

$$\Rightarrow \quad \frac{e^{x} + e^{-x}}{2} = \sec\theta \qquad \left[\because \cosh x = \frac{e^{x} + e^{-x}}{2}\right]$$

$$\therefore \quad e^{x} - 2\sec\theta + e^{-x} = 0$$

$$\therefore \quad (e^{x})^2 - 2e^{x}\sec\theta + 1 = 0$$

Solving the quadratic equation in e^{x}.

$$e^{x} = \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2}$$

$$\Rightarrow \quad e^{x} = \sec\theta \pm \sqrt{\sec^2\theta - 1}$$

$$\Rightarrow \quad e^{x} = \sec\theta \pm \tan\theta \qquad \text{...(4)}$$

Now,

$$\tanh\frac{x}{2} = \frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}} = \frac{e^{x} - 1}{e^{x} + 1} \qquad \text{...(5)}$$

Putting the value of e^{x} from (4) in (5), we get

$$\tanh\frac{x}{2} = \frac{\sec\theta + \tan\theta - 1}{\sec\theta + \tan\theta + 1} \qquad \text{[Using (1)]}$$

$$= \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \frac{(1 - \cos\theta) + \sin\theta}{(1 + \cos\theta) + \sin\theta}$$

$$= \frac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} \qquad \textbf{Proved.}$$

EXERCISE 25.8

1. If $\tan\left(\frac{\pi}{8}+i\alpha\right)=x+iy$, prove that $x^2+y^2+2x=1$.
2. If $\cot\left(\frac{\pi}{8}+i\alpha\right)=x+iy$, prove that $x^2+y^2-2x=1$.
3. Prove that if $(1+i\tan\alpha)^{1+i\tan\beta}$ can have real values, one of them is $(\sec\alpha)^{\sec^2\beta}$.
4. If $\frac{(1+i)^{x+iy}}{(1-i)^{x-iy}}=\alpha+i\beta$, prove that the value of $\tan^{-1}\frac{\beta}{\alpha}$ *is* $\frac{\pi x}{2}+y\log 2$.
5. If $\tanh x=\frac{1}{2}$, find the value of $\sinh 2x$. **Ans.** $\frac{4}{3}$
6. If $\sin\alpha\cosh\beta=\frac{x}{2}$, $\cos\alpha\sinh\beta=\frac{y}{2}$, show that
   (*i*) $\operatorname{cosec}(\alpha-i\beta)+\operatorname{cosec}(\alpha+i\beta)=\frac{4x}{x^2+y^2}$
   (*ii*) $\operatorname{cosec}(\alpha-i\beta)-\operatorname{cosec}(\alpha+i\beta)=\frac{4iy}{x^2+y^2}$
7. Show that $\tan\left(\frac{u+iv}{2}\right)=\frac{\sin u+i\sinh v}{\cos u+\cosh v}$
8. If $\cot(\alpha+i\beta)=x+iy$, prove that
   (*i*) $x^2+y^2-2x\cot 2\alpha=1$
   (*ii*) $x^2+y^2+2y\coth 2\beta+1=0$
9. If $\tan\frac{x}{2}=\tanh\frac{u}{2}$ prove that
   (*i*) $\sinh u=\tan x$
   (*ii*) $\cosh u=\sec x$.
10. Solve the following equation for real values of x.
    $17\cosh x+18\sinh x=1$ **Ans.** $-\log 5$

25.27 SEPARATION OF REAL AND IMAGINARY PARTS OF CIRCULAR FUNCTIONS

Example 44. *Separate the following into real and imaginary parts:*

(*i*) *sin* $(x+iy)$ (*ii*) *cos* $(x+iy)$ (*iii*) *tan* $(x+iy)$
(*iv*) *cot* $(x+iy)$ (*v*) *sec* $(x+iy)$ (*vi*) *cosec* $(x+iy)$.

Solution. (*i*) $\sin(x+iy)=\sin x\cos iy+\cos x\sin(iy)=\sin x\cosh y+i\cos x\sinh y$.

(*ii*) $\cos(x+iy)=\cos x\cos(iy)-\sin x\sin(iy)=\cos x\cosh y-i\sin x\sinh y$.

(*iii*)
$$\tan(x+iy)=\frac{\sin(x+iy)}{\cos(x+iy)}=\frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$
$$=\frac{\sin 2x+\sin(2iy)}{\cos 2x+\cos 2iy}=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$
$$\left\{\begin{array}{l}\because\quad 2\sin A.\cos B=\sin(A+B)+\sin(A-B)\\ \text{and } 2\cos A.\cos B=\cos(A+B)+\cos(A-B)\end{array}\right\}$$

(*iv*)
$$\cot(x+iy)=\frac{\cos(x+iy)}{\sin(x+iy)}=\frac{2\cos(x+iy)\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)}$$
$$=\frac{\sin 2x-\sin(2iy)}{\cos(2iy)-\cos 2x}=\frac{\sin 2x-i\sinh 2y}{\cosh 2y-\cos 2x}$$

(*v*)
$$\sec(x+iy)=\frac{1}{\cos(x+iy)}=\frac{2\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$
$$=\frac{2[\cos x\cos(iy)+\sin x\sin(iy)]}{\cos 2x+\cos(2iy)}=\frac{2[\cos x\cosh y+i\sin x\sinh y]}{\cos 2x+\cosh 2y}$$

(*vi*) cosec $(x + iy)$ $= \dfrac{1}{\sin(x+iy)} = \dfrac{2\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)} = \dfrac{2[\sin x\cos(iy)-\cos x\sin(iy)]}{\cos(2iy)-\cos 2x}$

$$= \frac{2[\sin x\cosh y - i\cos x\sinh y]}{\cosh 2y - \cos 2x}$$ **Ans.**

Example 45. *If tan* $(A + iB) = x + iy$, *prove that*

$\tan 2A = \dfrac{2x}{1-x^2-y^2}$ *and* $\tanh 2B = \dfrac{2x}{1+x^2+y^2}$ (*Nagpur University, Summer 2000*)

Solution. $\tan(A + iB) = x + iy$; $\tan(A - iB) = x - iy$

$$\tan 2A = \tan(A + iB + A - iB)$$

$$= \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$\tan 2A = \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-\left(x^2+y^2\right)} = \frac{2x}{1-x^2-y^2}$$

Again $$\tan 2iB = \tan(A + iB - A + iB) = \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$\tan 2iB = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{(2y)i}{1+x^2+y^2}$$

$$\tanh 2B = \frac{2y}{1+x^2+y^2}$$ $\tan ix = i\tanh x$ **Proved.**

Example 46. *If* $\sin(\alpha + i\beta) = x + iy$, *prove that*

(*a*) $\dfrac{x^2}{\cosh^2\beta}+\dfrac{y^2}{\sinh^2\beta}=1$ (*b*) $\dfrac{x^2}{\sin^2\alpha}-\dfrac{y^2}{\cos^2\alpha}=1$

Solution. (*a*) $x + iy = \sin(\alpha + i\beta) = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta$

Equating real and imaginary parts, we get

$$x = \sin\alpha\cosh\beta,\ y = \cos\alpha\sinh\beta$$

$$\sin\alpha = \frac{x}{\cosh\beta} \text{ and } \cos\alpha = \frac{y}{\sinh\beta}$$

Squaring and adding, $$\sin^2\alpha + \cos^2\alpha = \frac{x^2}{\cosh^2\beta}+\frac{y^2}{\sinh^2\beta}$$

$$\Rightarrow \quad 1 = \frac{x^2}{\cosh^2\beta}+\frac{y^2}{\sinh^2\beta}$$ **Proved.**

(*b*) Again $$\cosh\beta = \frac{x}{\sin\alpha} \text{ and } \sinh\beta = \frac{y}{\cos\alpha}$$

$$\cosh^2\beta - \sinh^2\beta = \frac{x^2}{\sin^2\alpha}-\frac{y^2}{\cos^2\alpha}$$

$$1 = \frac{x^2}{\sin^2\alpha}-\frac{y^2}{\cos^2\alpha}$$ **Proved.**

25.28 SEPARATION OF REAL AND IMAGINARY PARTS OF HYPERBOLIC FUNCTIONS

Example 47. *Separate the following into real and imaginary parts of hyperbolic functions.*

(a) sinh (x + iy) (b) cosh (x + iy) (c) tanh (x + iy)

Solution. (*a*) $\sinh(x + iy) = \sinh x\cosh(iy) + \cosh x\sinh(iy)$

$$= \sinh x\cos y + i\sin y\cosh x.$$ **Ans.**

(*b*) $\cosh(x+iy) = \cosh x \cosh(iy) - \sinh x \sinh iy = \cosh x \cos y - i \sinh x \sin y.$

Ans.

(*c*) $$\tanh(x+iy) = \frac{\sinh(x+iy)}{\cosh(x+iy)} = \frac{-i\sin i(x+iy)}{\cos i(x+iy)}$$

$$= \frac{-i\sin(ix-y)}{\cos(ix-y)} = \frac{-i\,2\sin(ix-y)\cos(ix+y)}{2\cos(ix-y)\cos(ix+y)} \qquad \text{(Note this step)}$$

$$= -i\frac{\sin 2ix - \sin 2y}{\cos 2ix + \cos 2y} = -i\frac{i\sinh 2x - \sin 2y}{\cosh 2x + \cos 2y} = \frac{\sinh 2x + i\sin 2y}{\cosh 2x + \cos 2y}$$

$$= \frac{\sinh 2x}{\cosh 2x + \cos 2y} + i\frac{\sin 2y}{\cosh 2x + \cos 2y}$$ **Ans.**

Example 48. *If tan (x + iy) = sin (u + iv), prove that*

$$\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$$

Solution. Now $\tan(x+iy) = \sin(u+iv)$ separating the real and imaginary parts of both sides, we have

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} + \frac{i\sinh 2y}{\cos 2x + \cosh 2y} = \sin u \cosh v + i\cos u \sinh v$$

Equating real and imaginary parts, we get

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = \sin u \cosh v \qquad ...(1)$$

and $$\frac{\sinh 2y}{\cos 2x + \cosh 2y} = \cos u \sinh v \qquad ...(2)$$

Dividing (1) by (2), we obtain

$$\frac{\sin 2x}{\sinh 2y} = \frac{\sin u \cosh v}{\cos u \sinh v}$$

$$\Rightarrow \quad \frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$$ **Proved.**

Example 49. *If sin (θ + iφ) = tan α + i sec α, show that cos 2θ cosh 2φ = 3*

Solution. $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$

$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \tan\alpha + i\sec\alpha$

Equating real and imaginary parts, we get

$$\sin\theta\cosh\phi = \tan\alpha \qquad ...(1)$$

$$\cos\theta\sinh\phi = \sec\alpha \qquad ...(2)$$

We know that

$$\sec^2\alpha - \tan^2\alpha = 1$$

$$\cos^2\theta\sinh^2\phi - \sin^2\theta\cosh^2\phi = 1 \qquad \text{[From (1) and (2)]}$$

$$\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\cosh 2\phi - 1}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{\cosh 2\phi + 1}{2}\right) = 1$$

$$[-1 + \cosh 2\phi - \cos 2\theta + \cos 2\theta\cosh 2\phi] - [\cosh 2\phi + 1 - \cos 2\theta\cosh 2\phi - \cos 2\theta] = 4$$

$$\Rightarrow \quad -2 + 2\cos 2\theta\cosh 2\phi = 4$$

$$\Rightarrow \quad 2\cos 2\theta\cosh 2\phi = 6 \quad \Rightarrow \quad \cos 2\theta\cosh 2\phi = 3$$ **Proved.**

Example 50. *If sinh (θ + iφ) = cos α + i sin α, prove that* $\sinh^4\theta = \cos^2\alpha = \cos^4\phi$.

Solution. $\sinh(\theta + i\phi) = \cos\alpha + i\sin\alpha$

$\sinh\theta\cos\phi + i\sin\phi\cosh\theta = \cos\alpha + i\sin\alpha$

Equating real and imaginary parts, we have

$$\sinh\theta\cos\phi = \cos\alpha \text{ and} \qquad ...(1)$$

$$\sin\phi\cosh\theta = \sin\alpha \qquad ...(2)$$

Let us eliminate ϕ from (1) and (2).

$$\cos\phi = \frac{\cos\alpha}{\sinh\theta} \text{ and } \sin\phi = \frac{\sin\alpha}{\cosh\theta}$$

Squaring and adding, we get

$$1 = \frac{\cos^2\alpha}{\sinh^2\theta} + \frac{\sin^2\alpha}{\cosh^2\theta} \Rightarrow \frac{\cos^2\alpha}{\sinh^2\theta} = 1 - \frac{\sin^2\alpha}{\cosh^2\theta}$$

$$\Rightarrow \quad \frac{\cos^2\alpha}{\sinh^2\theta} = 1 - \frac{1-\cos^2\alpha}{1+\sinh^2\theta} = \frac{1+\sinh^2\theta - 1 + \cos^2\alpha}{1+\sinh^2\theta}$$

$$\frac{\cos^2\alpha}{\sinh^2\theta} = \frac{\sinh^2\theta + \cos^2\alpha}{1+\sinh^2\theta}$$

$\sinh^4\theta + \sinh^2\theta\cos^2\alpha = \cos^2\alpha + \cos^2\alpha\sinh^2\theta$

$\Rightarrow \quad \sinh^4\theta = \cos^2\alpha$ **Proved.**

For second result, let us eliminate θ.

$$\sinh\theta = \frac{\cos\alpha}{\cos\phi} \quad \text{and} \quad \cosh\theta = \frac{\sin\alpha}{\sin\phi}$$

$$\cosh^2\theta - \sinh^2\theta = \frac{\sin^2\alpha}{\sin^2\phi} - \frac{\cos^2\alpha}{\cos^2\phi} \quad \Rightarrow \quad 1 = \frac{1-\cos^2\alpha}{1-\cos^2\phi} - \frac{\cos^2\alpha}{\cos^2\phi}$$

$$\Rightarrow \quad \frac{\cos^2\alpha}{\cos^2\phi} = \frac{1-\cos^2\alpha - 1 + \cos^2\phi}{1-\cos^2\phi}$$

$$\frac{\cos^2\alpha}{\cos^2\phi} = \frac{\cos^2\phi - \cos^2\alpha}{1-\cos^2\phi}$$

$\Rightarrow \quad \cos^4\phi - \cos^2\phi\cos^2\alpha = \cos^2\alpha - \cos^2\alpha\cos^2\phi$

$\Rightarrow \quad \cos^4\phi = \cos^2\alpha.$ **Proved.**

Example 51. *If $e^z = \sin(u+iv)$ and $z = x+iy$, prove that*

$$2e^{2x} = \cosh 2v - \cos 2u \qquad (M.U.\ 2006)$$

Solution. We have, $\quad e^z = \sin(u+iv)$

$\Rightarrow \quad e^{x+iy} = \sin(u+iv)$

$\Rightarrow \quad e^x . e^{iy} = \sin u\cos iv + \cos u\sin iv$

$\Rightarrow \quad e^x(\cos y + i\sin y) = \sin u\cosh v + i\cos u\sinh v$

Equating real and imaginary parts, we get

$$e^x\cos y = \sin u\cosh v$$

and $\quad e^x\sin y = \cos u\sinh v$

Squaring and adding, we get

$$e^{2x}(\cos^2 y + \sin^2 y) = \sin^2 u\cosh^2 v + \cos^2 u\sinh^2 v$$

$\Rightarrow \quad e^{2x} = (1-\cos^2 u)\cosh^2 v + \cos^2 u(\cosh^2 v - 1)$

$\Rightarrow \quad e^{2x} = \cosh^2 v - \cos^2 u$

$$\Rightarrow \quad e^{2x} = \frac{1}{2}(1+\cosh 2v) - \frac{1}{2}(1+\cos 2u)$$

$$\Rightarrow \quad e^{2x} = \frac{1}{2}(\cosh 2v - \cos 2u)$$

$\Rightarrow \quad 2e^{2x} = \cosh 2v - \cos 2u$ **Proved.**

Example 52. *If* $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, *prove that*

$$\cos^4\theta = \sin^2\alpha = \sinh^4\phi. \qquad (M.U.\ 2003,\ 2004)$$

Solution. Here, we have

$$\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \quad \sin\theta\cosh\phi + i\cos\theta\sinh\phi = \cos\alpha + i\sin\alpha$$

Equating real and imaginary parts, we get

$$\sin\theta\cosh\phi = \cos\alpha \quad \Rightarrow \quad \cosh\phi = \frac{\cos\alpha}{\sin\theta} \qquad ...(1)$$

and

$$\cos\theta\sinh\phi = \sin\alpha \quad \Rightarrow \quad \sinh\phi = \frac{\sin\alpha}{\cos\theta} \qquad ...(2)$$

But

$$\cosh^2\phi - \sinh^2\phi = 1$$

$$\Rightarrow \quad \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta} = 1 \qquad \text{[Using (1) and (2)]}$$

$$\Rightarrow \quad \cos^2\alpha.\cos^2\theta - \sin^2\alpha.\sin^2\theta = \sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad (1 - \sin^2\alpha)\cos^2\theta - \sin^2\alpha.\sin^2\theta = (1 - \cos^2\theta)\cos^2\theta$$

$$\Rightarrow \quad \cos^2\theta - \sin^2\alpha(\cos^2\theta + \sin^2\theta) = \cos^2\theta - \cos^4\theta$$

$$\Rightarrow \quad \sin^2\alpha = \cos^4\theta \qquad ...(3)$$

Again

$$\sin^2\theta + \cos^2\theta = 1$$

$$\therefore \quad \frac{\cos^2\alpha}{\cosh^2\phi} + \frac{\sin^2\alpha}{\sinh^2\phi} = 1 \qquad \text{[Using (1) and (2)]}$$

$$\Rightarrow \quad \cos^2\alpha.\sinh^2\phi + \sin^2\alpha\cosh^2\phi = \sinh^2\phi\cosh^2\phi$$

$$\Rightarrow \quad (1 - \sin^2\alpha)\sinh^2\phi + \sin^2\alpha(1 + \sinh^2\phi) = \sinh^2\phi(1 + \sinh^2\phi)$$

$$\Rightarrow \quad \sinh^2\phi - \sin^2\alpha\sinh^2\phi + \sin^2\alpha + \sin^2\alpha\sinh^2\phi = \sinh^2\phi + \sinh^4\phi$$

$$\Rightarrow \quad \sin^2\alpha = \sinh^4\phi. \qquad ...(4)$$

From (3) and (4), we have $\cos^4\theta = \sin^2\alpha = \sinh^4\phi$ **Proved.**

Example 53. *If* $\operatorname{cosec}\left(\frac{\pi}{4} + ix\right) = u + iv$, *prove that*

$$(u^2 + v^2)^2 = 2(u^2 - v^2) \qquad (M.U.\ 2009)$$

Solution. Here, we have

$$u + iv = \operatorname{cosec}\left(\frac{\pi}{4} + ix\right)$$

$$= \frac{1}{\sin\left(\frac{\pi}{4} + ix\right)} \Rightarrow = \frac{1}{\sin\frac{\pi}{4}\cos ix + \cos\frac{\pi}{4}\sin ix}$$

$$= \frac{1}{\frac{1}{\sqrt{2}}\cosh x + \frac{1}{\sqrt{2}} i\sinh x} = \frac{\sqrt{2}}{\cosh x + i\sinh x}$$

$$= \frac{\sqrt{2}(\cosh x - i\sinh x)}{\cosh^2 x + \sinh^2 x} = \frac{\sqrt{2}(\cosh x - i\sinh x)}{\cosh 2x}$$

Equating real and imaginary parts, we get $u = \dfrac{\sqrt{2}\cosh x}{\cos h\, 2x}$, $v = -\dfrac{\sqrt{2}\sinh x}{\cosh 2x}$

Squaring and adding, we get

$$u^2 + v^2 = \frac{2(\cosh^2 x + \sinh^2 x)}{\cosh^2 2x} = \frac{2\cosh 2x}{\cosh^2 2x}$$

$$\Rightarrow \qquad (u^2+v^2)^2 = \left(\frac{2}{\cosh 2x}\right)^2 = \frac{4}{\cosh^2 2x} \qquad \text{...(1)}$$

Also, $$u^2 - v^2 = \frac{2}{\cosh^2 2x}(\cosh^2 x - \sinh^2 x) = \frac{2}{\cosh^2 2x} \qquad \text{...(2)}$$

From (1) and (2), we have

$$(u^2+v^2)^2 = 2\,(u^2-v^2)$$ **Proved.**

Example 54. *Separate into real and imaginary parts* $\sqrt{i}^{\sqrt{i}}$. (*M.U. 2008*)

Solution. We have,

$$\sqrt{i} = i^{\frac{1}{2}} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\frac{1}{2}}$$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}$$

Also, $$\sqrt{i} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\frac{1}{2}} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$$

$$\therefore \qquad \left(\sqrt{i}\right)^{\sqrt{i}} = \left(e^{i\frac{\pi}{4}}\right)^{\left(\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)} = e^{i\frac{\pi}{4\sqrt{2}} - \frac{\pi}{4\sqrt{2}}}$$

$$= e^{-\frac{\pi}{4\sqrt{2}}}.e^{i\frac{\pi}{4\sqrt{2}}} = e^{-\frac{\pi}{4\sqrt{2}}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$$

$$\therefore \qquad \text{Real part} = e^{-\frac{\pi}{4\sqrt{2}}}\cos\left(\frac{\pi}{4\sqrt{2}}\right)$$

$$\text{Imaginary part} = e^{-\frac{\pi}{4\sqrt{2}}}\sin\left(\frac{\pi}{4\sqrt{2}}\right)$$ **Ans.**

EXERCISE 25.9

Separate into real and imaginary parts.

1. $\operatorname{sech}(x+iy)$ **Ans.** $\dfrac{2\cosh x\cos y - 2i\sinh x\sin y}{\cosh 2x + \cos 2y}$

2. $\coth i\,(x+iy)$ **Ans.** $\dfrac{-\sinh 2y - i\sin 2x}{\cosh 2x - \cos 2y}$

3. $\coth(x+iy)$ **Ans.** $\dfrac{\sinh 2x - i\sin 2y}{\cosh 2x - \cos 2y}$

4. If $\sin(\theta + i\phi) = p\,(\cos\alpha + i\sin\alpha)$, prove that
$$p^2 = \frac{1}{2}\left[\cosh 2\phi - \cos 2\theta\right],\ \tan\alpha = \tanh\phi\cot\theta$$

5. If $\sin(\alpha + i\beta) = x + iy$, prove that $x^2\operatorname{sech}^2\beta + y^2\operatorname{cosech}^2\beta = 1$
and $x^2\operatorname{cosec}^2\alpha - y^2\sec^2\alpha = 1$

6. If $\cos(\theta + i\phi) = r\,(\cos\alpha + i\sin\alpha)$, prove that $\theta = \dfrac{1}{2}\log\left[\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$

7. If $\tan\left(\dfrac{\pi}{6} + i\alpha\right) = x + iy$, prove that $x^2 + y^2 + \dfrac{2x}{\sqrt{3}} = 1$

8. If $\tan(A+B) = \alpha + i\beta$, show that $\dfrac{1-(\alpha^2+\beta^2)}{1+(\alpha^2+\beta^2)} = \dfrac{\cos 2A}{\cosh 2B}$

9. If $\dfrac{x+iy-c}{x+iy+c} = e^{u+iv}$, prove that

$$x = -\frac{c\sinh u}{\cosh u - \cos v}, \qquad y = \frac{c\sinh v}{\cosh u - \cos v}$$

Further, if $v = (2n+1)\dfrac{\pi}{2}$, prove that $x^2 + y^2 = c^2$ where n is an integer.

10. If $\dfrac{u-1}{u+1} = \sin(x+iy)$, where $u = \alpha + i\beta$ show that the argument of u is $\theta + \phi$ where

$\tan\theta = \dfrac{\cos x\sinh y}{1+\sin x\cosh y}$ and , $\tan\phi = \dfrac{\cos x\sinh y}{1-\sin x\sinh y}$

11. If $A + iB = C\tan(x+iy)$, prove that $\tan 2x = \dfrac{2CA}{C^2-A^2-B^2}$

12. If $\cosh(\alpha+i\beta) = x+iy$, prove that

(*a*) $\dfrac{x^2}{\cosh^2\alpha} + \dfrac{y^2}{\sinh^2\alpha} = 1$ (*b*) $\dfrac{x^2}{\cos^2\beta} - \dfrac{y^2}{\sin^2\beta} = 1$

13. If $\cos(\theta + i\phi) = R(\cos\alpha + i\sin\alpha)$, prove that $\phi = \dfrac{1}{2}\log_e\left[\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$

14. If $\cos(\alpha+i\beta)\cos(\gamma+i\delta) = 1$, prove that $\tanh^2\delta\cosh^2\beta = \sin^2\alpha$

15. If $\dfrac{u-1}{u+1} = \sin(x+iy)$, find u. **Ans.** $\tan^{-1}\dfrac{2\cos x\sinh y}{\cos^2 x - \sinh^2 y}$

25.29 LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

Example 55. *Define logarithm of a complex number.*

Solution. If z and w are two complex numbers and $z = e^w$ then $w = \log z$, and if $w = \log z$; then $z = e^w$

Here $\log z$ is a many valued function. General value of $\log z$ is defined by Log z, where Log $z = \log z + 2n\pi i$.

Example 56. *Separate log* $(x+iy)$ *into its real and imaginary parts.*

Solution. Let $x = r\cos\theta$ ...(1)

and $y = r\sin\theta$ ...(2)

Squaring and adding (1) and (2) we have $x^2 + y^2 = r^2$

$\therefore$ $r = \sqrt{x^2+y^2}$,

We have, $\tan\theta = \dfrac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ [Dividing (2) by (1)]

$$\therefore \quad \log(x+iy) = \log[r(\cos\theta + i\sin\theta)]$$
$$= [\log r + \log(\cos\theta + i\sin\theta)]$$
$$\log(x+iy) = \log r + \log[\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)]$$
$$= \log r + \log e^{i(2n\pi+\theta)} = \log r + i(2n\pi+\theta)$$
$$\text{Log}(x+iy) = \log\sqrt{x^2+y^2} + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$$

and $\log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\dfrac{y}{x}$ **Ans.**

Example 57. *Find the general value of Log* $(1+i)$ + *Log* $(1-i)$.

Solution. $1 + i = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right) = \sqrt{2}\,e^{i\frac{\pi}{4}}$

$$\log(1+i) = \log \sqrt{2}\,.\,e^{i\frac{\pi}{4}} = \log \sqrt{2} + i\frac{\pi}{4}$$

$$\text{Log}(1+i) = \log \sqrt{2} + i\frac{\pi}{4} + 2n\pi i = \log \sqrt{2} + \left(2n\pi + \frac{\pi}{4}\right)i$$

$$\text{Log}(1-i) = \log \sqrt{2} + \left(2n\pi - \frac{\pi}{4}\right)i$$

Hence, $\text{Log}(1+i) + \text{Log}(1-i) = \left[\log\sqrt{2} + \left(2n\pi + \frac{\pi}{4}\right)i\right] + \left[\log\sqrt{2} + \left(2n\pi - \frac{\pi}{4}\right)i\right]$

$$= 2\log\sqrt{2} + 4n\pi i = \log 2 + 4n\pi i$$ **Ans.**

Example 58. *Show that* $\log \dfrac{x+iy}{x-iy} = 2\, i \tan^{-1}\dfrac{y}{x}$. (*Nagpur University, Winter 2003*)

Solution. Let $\log(x+iy) = \log(r\cos\theta + ir\sin\theta) = \log r\, e^{i\theta}$

$$= \log r + i\,\theta \qquad \begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix}$$

Similarly, $\log(x - iy) = \log r - i\,\theta$

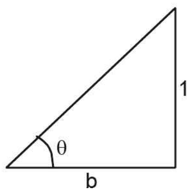
$$\log\frac{x+iy}{x-iy} = \log(x+iy) - \log(x-iy) = (\log r + i\theta) - (\log r - i\theta) = 2i\,\theta$$

$$= 2\,i\tan^{-1}\frac{y}{x}.$$ **Proved.**

Example 59. *Show that for real values of a and b*

$$e^{2ai\cot^{-1}b}\left[\frac{bi-1}{bi+1}\right]^{-a} = 1$$ (*M.U. 2008*)

Solution. Consider $\dfrac{bi-1}{bi+1} = \dfrac{bi+i^2}{bi-i^2} = \dfrac{b+i}{b-i}$



$$\Rightarrow \left(\frac{bi-1}{bi+1}\right)^{-a} = \left(\frac{b+i}{b-i}\right)^{-a}$$

$$\log\left[\frac{bi-1}{bi+1}\right]^{-a} = \log\left(\frac{b+i}{b-i}\right)^{-a} = -a\,[\log(b+i) - \log(b-i)]$$

$$= -a\left[\log\sqrt{b^2+1} + i\tan^{-1}\frac{1}{b} - \log\sqrt{b^2+1} + i\tan^{-1}\frac{1}{b}\right]$$

$$= -2ai\tan^{-1}\frac{1}{b}$$

$$\left[\begin{array}{l} \text{If } \cot\theta = b,\ \tan\theta = \dfrac{1}{b} \\ \text{Since } \cot^{-1}b = \tan^{-1}\left(\dfrac{1}{b}\right) \end{array}\right]$$

$$\left(\frac{bi-1}{bi+1}\right)^{-a} = e^{-2ai\tan^{-1}\left(\frac{1}{b}\right)}$$

$$e^{2ai\cot^{-1}b}\left(\frac{bi-1}{bi+1}\right)^{-a} = \left[e^{2ai\tan^{-1}\left(\frac{1}{b}\right)}\right]\cdot\left[e^{-2ai\tan^{-1}\left(\frac{1}{b}\right)}\right] = 1$$ **Proved.**

EXERCISE 25.10

1. Find the general value of Log i. **Ans.** $(4n+1)\dfrac{\pi i}{2}$
2. Express Log (-5) in terms of $a + ib$. **Ans.** $\log 5 + i(2n+1)\pi$
3. Find the value of z if

   (a) $\cos z = 2$. **Ans.** $z = 2n\pi \pm i\log\left(2+\sqrt{3}\right)$

   (b) $\cosh z = -1$. **Ans.** $z = (2n+1)\pi i$
4. Find the general and principal values of i^i **Ans.** $e^{-\left(2n\pi+\frac{\pi}{2}\right)}$, $e^{-\frac{\pi}{2}}$
5. If $i^{(\alpha + i\beta)} = x + iy$, prove that $x^2 + y^2 = e^{-(4m+1)\pi\theta}$.
6. Prove that $\log \dfrac{1}{1-e^{i\theta}} = \log\left(\dfrac{1}{2}\operatorname{cosec}\theta\right) + i\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)$
7. Show that $\log\sin(x+iy) = \dfrac{1}{2}\log\dfrac{\cosh 2y - \cos 2x}{2} + i\tan^{-1}(\cot x\tanh y)$.
8. Prove that $\tan\left[i\log\dfrac{a-ib}{a+ib}\right] = \dfrac{2ab}{a^2-b^2}$.
9. $\log\dfrac{\cos(x-iy)}{\cos(x+iy)} = 2i\tan^{-1}(\tan x\tanh y)$.
10. Separate $i^{(1+i)}$ into real and imaginary parts. **Ans.** $ie^{-\frac{\pi}{2}}$

25.30 INVERSE FUNCTIONS

If $\sin\theta = \dfrac{1}{2}$ then $\theta = \sin^{-1}\left(\dfrac{1}{2}\right)$, so here θ is called inverse sine of $\dfrac{1}{2}$.

Similarly, we can define inverse hyperbolic function sinh, cosh, tanh, etc. If $\cosh\theta = z$ then $\theta = \cosh^{-1} z$.

25.31 INVERSE HYPERBOLIC FUNCTIONS

Example 60. *Prove that* $\sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$ *(M.U. 2009)*

Solution. Let $\sinh^{-1} x = y \quad \Rightarrow \quad x = \sinh y$

$$x = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow \quad e^y - e^{-y} = 2x$$

$$\Rightarrow \quad e^{2y} - 2x\,e^y - 1 = 0$$

This is quadratic in e^y.

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

$$y = \log\left(x+\sqrt{x^2+1}\right) \qquad \text{(Taking positive sign only)}$$

$$\sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$$ **Proved.**

Example 61. *Prove that* $\cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)$ *(M.U. 2009)*

Solution. Let $y = \cosh^{-1} x \quad \Rightarrow \quad x = \cosh y$

$$x = \frac{e^y + e^{-y}}{2} \quad \Rightarrow \quad 2x = e^y + e^{-y}$$

$$\Rightarrow \quad e^{2y} - 2x\, e^y + 1 = 0 \qquad \text{(This is quadratic in } e^y\text{)}$$

$$\Rightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow \quad y = \log\,(x + \sqrt{x^2 - 1}) \qquad \text{(Taking positive sign only)}$$

$$\Rightarrow \quad \cosh^{-1} x = \log\,(x + \sqrt{x^2 - 1})$$ **Proved.**

Example 62. *Prove that* $\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$

Solution. Let $\tanh^{-1} x = y \quad \Rightarrow \quad x = \tanh y$

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

Applying componendo and dividendo, we obtain

$$\frac{1+x}{1-x} = \frac{e^y}{e^{-y}} = e^{2y}, \quad 2y = \log \frac{1+x}{1-x}$$

$$\Rightarrow \quad \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}.$$

Similarly, $\coth^{-1} x = \frac{1}{2} \log \frac{x+1}{x-1}$ **Proved.**

Example 63. *Prove that* $\text{sech}^{-1}\, x = \log \frac{1+\sqrt{1-x^2}}{x}$

Solution. Let $y = \text{sech}^{-1} x \quad \Rightarrow \quad x = \text{sech}\, y$

$$x = \frac{2}{e^y + e^{-y}} \quad \Rightarrow \quad x = \frac{2e^y}{e^{2y} + 1}$$

$$\Rightarrow \quad xe^{2y} - 2e^y + x = 0 \quad \Rightarrow \quad e^y = \frac{2 \pm \sqrt{4 - 4x^2}}{2x} = \frac{1 \pm \sqrt{1 - x^2}}{x}$$

We take only positive sign

$$e^y = \frac{1 + \sqrt{1-x^2}}{x} \quad \Rightarrow \quad y = \log \frac{1 + \sqrt{1-x^2}}{x}$$

$$\text{sech}^{-1} x = \log \frac{1 + \sqrt{1-x^2}}{x}$$

Similarly, $\text{cosech}^{-1} x = \log \frac{1 + \sqrt{1+x^2}}{x}$ **Proved.**

Example 64. *If* $x + iy = \cos(\alpha + i\beta)$ *or if* $\cos^{-1}(x + iy) = \alpha + i\beta$ *express* x *and* y *in terms of* α *and* β. *Hence show that* $\cos^2 \alpha$ *and* $\cosh^2 \beta$ *are the roots of the equation* $\lambda^2 - (x^2 + y^2 + 1)\,\lambda + x^2 = 0.$ (*M.U. 2002, 2004*)

Solution. Here, we have

$$\cos(\alpha + i\,\beta) = x + iy$$

$$\Rightarrow \quad \cos\alpha \cos i\,\beta - \sin\alpha \sin i\,\beta = x + iy$$

$$\Rightarrow \quad \cos\alpha \cosh\beta - i \sin\alpha \sinh\beta = x + iy$$

Equating real and imaginary parts, we get

$$\cos\alpha \cosh\beta = x \text{ and } \sin\alpha \sinh\beta = -y$$

We want to find the equation whose roots are $\cos^2\alpha$ and $\cosh^2\beta$.

Now, $x^2 + y^2 + 1 = \cos^2\alpha \cos h^2\beta + \sin^2\alpha \sinh^2\beta + 1$

$$= \cos^2\alpha \cosh^2\beta + (1 - \cos^2\alpha)(\cosh^2\beta - 1) + 1$$

$$= \cos^2 \alpha \cosh^2 \beta + \cosh^2 \beta - 1 - \cos^2 \alpha \cosh^2 \beta + \cos^2 \alpha + 1$$
$$= \cos^2 \alpha + \cosh^2 \beta$$

Sum of the roots $= \cos^2 \alpha + \cosh^2 \beta$
$$= x^2 + y^2 + 1$$

And product of the roots $= \cos^2 \alpha \cosh^2 \beta$
$$= x^2$$

Hence, the equation whose roots are $\cos^2 \alpha$, $\cosh^2 \beta$ is

$\lambda^2 - (x^2 + y^2 + 1)\lambda + x^2 = 0$ **Proved.**

Example 65. *Separate into real and imaginary part* $\cos^{-1}\left(\frac{3i}{4}\right)$ (*M.U. 2003*)

Solution. Let $\cos^{-1}\left(\frac{3i}{4}\right) = x + iy$

$$\Rightarrow \quad \frac{3i}{4} = \cos(x + iy)$$

$$\Rightarrow \quad \frac{3i}{4} = \cos x \cosh y - i \sin x \sinh y$$

Equating real and imaginary parts, we get

$$\therefore \quad \cos x \cosh y = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$$

$$\text{and} \quad -\sin x \sinh y = \frac{3}{4}$$

$$-1 \sinh y = \frac{3}{4} \qquad \sin x = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\therefore \quad \sinh y = -\frac{3}{4}$$

$$\Rightarrow \quad y = \log\left(\frac{-3}{4} + \sqrt{1 + \frac{9}{16}}\right) \Rightarrow y = \log\left(\frac{-3}{4} + \frac{5}{4}\right) = -\log 2 = \log\left(\frac{1}{2}\right)$$

$\therefore$ Real part $= \frac{\pi}{2}$ and imaginary Part $= -\log 2$ **Proved.**

25.32 SOME OTHER INVERSE FUNCTIONS

Example 66. *Separate* $\tan^{-1}(\cos\theta + i\sin\theta)$ *into real and imaginary parts.* (*M.U. 2009*)

Solution. Let $\tan^{-1}(\cos\theta + i\sin\theta) = x + iy$

$$\Rightarrow \quad \cos\theta + i\sin\theta = \tan(x + iy)$$

Similarly, $\cos\theta - i\sin\theta = \tan(x - iy)$

$$\tan 2x = \tan[(x + iy) + (x - iy)] = \frac{\tan(x+iy) + \tan(x-iy)}{1 - \tan(x+iy)\tan(x-iy)}$$

$$= \frac{(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)}{1 - (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)} = \frac{2\cos\theta}{1 - (\cos^2\theta + \sin^2\theta)}$$

$$= \frac{2\cos\theta}{1-1} = \frac{2\cos\theta}{0} = \infty = \tan\frac{\pi}{2}$$

$$\tan 2x = \tan\left(n\pi + \frac{\pi}{2}\right) \Rightarrow 2x = n\pi + \frac{\pi}{2} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{4}$$

Now, $\tan 2iy = \tan[(x + iy) - (x - iy)] = \dfrac{\tan(x+iy) - \tan(x-iy)}{1 + \tan(x+iy)\tan(x-iy)}$

$$= \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{1 + (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)} = \frac{2i\sin\theta}{1 + (\cos^2\theta + \sin^2\theta)} = \frac{2i\sin\theta}{1+1} = i\sin\theta$$

$$i\tanh 2y = i\sin\theta \quad \Rightarrow \quad \frac{e^{2y} - e^{-2y}}{e^{2y} + e^{-2y}} = \frac{\sin\theta}{1}$$

By componendo and dividendo, we have

$$\frac{2e^{2y}}{2e^{-2y}} = \frac{1 + \sin\theta}{1 - \sin\theta} \Rightarrow e^{4y} = \frac{1 + \cos\left(\frac{\pi}{2} - \theta\right)}{1 - \cos\left(\frac{\pi}{2} - \theta\right)} = \frac{1 + 2\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) - 1}{1 - \left[1 - 2\sin^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]}$$

$$= \frac{\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\sin^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} = \cot^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \Rightarrow e^{2y} = \cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\Rightarrow \quad 2y = \log\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \Rightarrow y = \frac{1}{2}\log\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\text{Imaginary part} = \frac{1}{2}\log\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\text{Real part} = \frac{n\pi}{2} + \frac{\pi}{4}$$

$$\tan^{-1}(\cos\theta + i\sin\theta) = \frac{n\pi}{2} + \frac{\pi}{4} + \frac{i}{2}\log\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$ **Ans.**

Example 67. *Separate* $\sin^{-1}(\alpha + i\beta)$ *into real and imaginary parts.*

Solution. Let $\sin^{-1}(\alpha + i\beta) = x + iy$

$$\alpha + i\beta = \sin(x + iy)$$

$$\Rightarrow \quad \alpha + i\beta = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i\cos x \sinh y$$

Equating real and imaginary parts, we have

$$\alpha = \sin x \cosh y \quad \text{...(1)}$$

and $$\beta = \cos x \sinh y \quad \text{....(2)}$$

We know that $\cosh^2 y - \sinh^2 y = 1$

$$\left(\frac{\alpha}{\sin x}\right)^2 - \left(\frac{\beta}{\cos x}\right)^2 = 1 \qquad \left[\begin{array}{l}\cosh y = \dfrac{\alpha}{\sin x} \\ \sinh y = \dfrac{\beta}{\cos x}\end{array}\right]$$

$$\Rightarrow \quad \alpha^2\cos^2 x - \beta^2\sin^2 x = \sin^2 x\cos^2 x$$

$$\Rightarrow \quad \alpha^2(1 - \sin^2 x) - \beta^2\sin^2 x = \sin^2 x(1 - \sin^2 x)$$

$$\Rightarrow \quad \sin^4 x - (\alpha^2 + \beta^2 + 1)\sin^2 x + \alpha^2 = 0$$

This is quadratic equation in $\sin^2 x$.

$$\sin^2 x = \frac{(\alpha^2 + \beta^2 + 1) \pm \sqrt{(\alpha^2 + \beta^2 + 1)^2 - 4\alpha^2}}{2}$$

$$\Rightarrow \quad \sin x = \sqrt{\frac{1}{2}\left[(\alpha^2 + \beta^2 + 1) \pm \sqrt{(\alpha^2 + \beta^2 + 1)^2 - 4\alpha^2}\right]}$$

$$\Rightarrow \quad x = \sin^{-1}\sqrt{\frac{1}{2}\left[(\alpha^2 + \beta^2 + 1) \pm \sqrt{(\alpha^2 + \beta^2 + 1)^2 - 4\alpha^2}\right]}$$

We know that $\sin^2 x + \cos^2 x = 1$

$$\left[\begin{array}{l}\text{From (1) and (2)}\\ \sin x = \dfrac{\alpha}{\cosh y}\\ \cos x = \dfrac{\beta}{\sinh y}\end{array}\right]$$

$$\Rightarrow \quad \left(\frac{\alpha}{\cosh y}\right)^2 + \left(\frac{\beta}{\sinh y}\right)^2 = 1$$

$$\Rightarrow \quad \alpha^2 \sinh^2 y + \beta^2 \cosh^2 y = \sinh^2 y \cosh^2 y$$

$$\Rightarrow \quad \alpha^2 \sinh^2 y + \beta^2 (1 + \sinh^2 y) = \sinh^2 y (1 + \sinh^2 y)$$

$$\Rightarrow \quad \sinh^4 y - (\alpha^2 + \beta^2 - 1) \sinh^2 y - \beta^2 = 0$$

This is quadratic equation in $\sinh^2 y$.

$$\sinh^2 y = \frac{(\alpha^2 + \beta^2 - 1) \pm \sqrt{(\alpha^2 + \beta^2 - 1)^2 + 4\beta^2}}{2}$$

$$\Rightarrow \quad \sinh y = \sqrt{\frac{1}{2}\left[(\alpha^2 + \beta^2 - 1) \pm \sqrt{(\alpha^2 + \beta^2 - 1)^2 + 4\beta^2}\right]}$$

$$\Rightarrow \quad y = \sinh^{-1} \sqrt{\frac{1}{2}\left[(\alpha^2 + \beta^2 - 1) \pm \sqrt{(\alpha^2 + \beta^2 - 1)^2 + 4\beta^2}\right]}$$

$$\text{Real part} = \sin^{-1} \sqrt{\frac{1}{2}\left[(\alpha^2 + \beta^2 + 1) \pm \sqrt{(\alpha^2 + \beta^2 + 1)^2 - 4\alpha^2}\right]}$$

$$\text{Imaginary part} = \sinh^{-1} \sqrt{\frac{1}{2}\left[(\alpha^2 + \beta^2 - 1) \pm \sqrt{(\alpha^2 + \beta^2 - 1)^2 + 4\beta^2}\right]}$$

Ans.

Example 68. *Separate $\tan^{-1} (a + i\, b)$ into real and imaginary parts.*

(*Nagpur University, Summer 2008, 2004*)

Solution. Let $\tan^{-1} (a + i\, b) = x + i\, y$ ...(1)

$$\therefore \quad \tan (x + i\, y) = a + i\, b$$

On both sides for i write $- i$ we get,

$$\therefore \quad \tan (x - i\, y) = a - i\, b$$

Now, $\tan 2x = \tan [(x + i\, y) + (x - i\, y)]$

$$= \frac{\tan(x + i\, y) + \tan(x - i\, y)}{1 - \tan(x + i\, y) \tan(x - i\, y)} = \frac{a + i\, b + a - i\, b}{1 - (a + i\, b)(a - i\, b)} = \frac{2a}{1 - a^2 - b^2}$$

$$2x = \tan^{-1}\left[\frac{2a}{1 - a^2 - b^2}\right] \quad \Rightarrow \quad x = \frac{1}{2} \tan^{-1}\left[\frac{2a}{1 - a^2 - b^2}\right] \qquad \text{...(2)}$$

and $\tan (2\, y\, i) = \tan [(x + i\, y) - (x - i\, y)]$

$$= \frac{\tan (x + i\, y) - \tan (x - i\, y)}{1 + \tan (x + i\, y) \tan (x - i\, y)} = \frac{a + b\,i - a + b\,i}{1 + (a + b\,i)(a - b\,i)}$$

$$i \tanh 2y = \frac{2\,b\,i}{1 + a^2 + b^2} \quad \text{so, } \tanh 2y = \frac{2\,b}{1 + a^2 + b^2}$$

$$2y = \tanh^{-1}\left[\frac{2\,b}{1 + a^2 + b^2}\right]$$

$$\text{so } y = \frac{1}{2} \tanh^{-1}\left[\frac{2\,b}{1 + a^2 + b^2}\right] \qquad \text{...(3)}$$

From (1), (2) and (3), we have

$$\tan^{-1}(a+ib) = \frac{1}{2}\tan^{-1}\left[\frac{2a}{1-a^2-b^2}\right] + \frac{i}{2}\tanh^{-1}\left[\frac{2b}{1+a^2+b^2}\right] \qquad \textbf{Ans.}$$

Example 69. *Show that* $\tan^{-1} i\left(\frac{x-a}{x+a}\right) = \frac{i}{2}\log\left(\frac{x}{a}\right)$. (*M.U. 2006, 2002*)

Solution. Let $\tan^{-1} i\left(\frac{x-a}{x+a}\right) = u + iv$ ...(1)

$\Rightarrow \quad \tan(u+iv) = i\left(\frac{x-a}{x+a}\right)$ and $\tan(u-iv) = -i\left(\frac{x-a}{x+a}\right)$

$$\tan 2u = \tan[(u+iv)+(u-iv)] = \frac{\tan(u+iv)+\tan(u-iv)}{1-\tan(u+iv)\tan(u-iv)}$$

$$= \frac{ix-ia-ix+ia}{x+a} = 0$$

$\therefore \quad \tan 2u = 0 \Rightarrow 2u = 0 \Rightarrow u = 0$

Putting the value of u in (1), we get

$\therefore \quad \tan^{-1} i\left(\frac{x-a}{x+a}\right) = iv \qquad \therefore \quad i\left(\frac{x-a}{x+a}\right) = \tan iv = i\tanh v$

$\therefore \quad \frac{x-a}{x+a} = \tanh v = \frac{e^v-e^{-v}}{e^v+e^{-v}}$

By Componendo and dividendo, we get

$$\frac{2x}{2a} = \frac{2e^v}{2e^{-v}} \quad\Rightarrow\quad \frac{x}{a} = e^{2v} \quad\Rightarrow\quad v = \frac{1}{2}\log\left(\frac{x}{a}\right)$$

$\therefore \tan^{-1} i\left(\frac{x-a}{x+a}\right) = u + iv = 0 + \frac{i}{2}\log\frac{x}{a} = \frac{i}{2}\log\left(\frac{x}{a}\right)$ **Proved.**

Example 70. *Prove that*

(*i*) $\cosh^{-1}\sqrt{1+x^2} = \sinh^{-1} x$ (*M.U. 2007*)

(*ii*) $\cosh^{-1}\sqrt{1+x^2} = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ (*M.U. 2002*)

Solution. (*i*) Let $\cosh^{-1}\sqrt{1+x^2} = y$ ...(1)

$\Rightarrow \quad \sqrt{1+x^2} = \cosh y$ ...(2)

On squaring both sides, we get

$$1 + x^2 = \cosh^2 y$$

$\therefore \quad x^2 = \cosh^2 y - 1$

$\Rightarrow \quad x^2 = \sinh^2 y$

$\Rightarrow \quad x = \sinh y$ ...(3)

$\Rightarrow \quad y = \sinh^{-1} x$

$\Rightarrow \quad \cosh^{-1}\sqrt{1+x^2} = \sinh^{-1} x$ [Using (1)] **Proved.**

(*ii*) Dividing (3) by (2), we get

$$\frac{\sinh y}{\cosh y} = \frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow \qquad \tanh y = \frac{x}{\sqrt{1+x^2}} \quad \Rightarrow \quad y = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \qquad \cosh^{-1}\sqrt{1+x^2} = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) \quad \text{[Using (1)]}$$ **Proved.**

EXERCISE 25.11

1. Prove that $\sin^{-1}(\operatorname{cosec}\theta) = \frac{\pi}{2} + i \log \cot \frac{\theta}{2}$.

2. If $\tan(\alpha + i\beta) = x + iy$, prove that
(a) $x^2 + y^2 + 2x \cot 2\alpha = 1$
(b) $x^2 + y^2 - 2y \coth 2\beta = -1$.

3. If $\tan(\theta + i\phi) = \sin(x + iy)$, then prove that
$\coth y \sinh 2\phi = \cot x \sin 2\theta$.

4. If $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$, show that.
(a) $x = \cos^{-1}\sqrt{\sin\theta}$
(b) $y = \log\left[\sqrt{\sin\theta} + \sqrt{1+\sin\theta}\right]$.

5. Separate into real and imaginary parts $\sin^{-1}(e^{i\theta})$

Ans. $\cos^{-1}\sqrt{\sin\theta} + i\log\left[\sqrt{\sin\theta} + \sqrt{1+\sin\theta}\right]$

6. Prove that

$$\tan^{-1}\left(\frac{\tan 2\theta + \tan 2\phi}{\tan 2\theta - \tan 2\phi}\right) + \tan^{-1}\left(\frac{\tan\theta - \tan\phi}{\tan\theta + \tan\phi}\right) = \tan^{-1}(\cot\theta \coth\phi)$$

7. Prove that $\tanh^{-1} x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$.

8. Prove that $\tanh^{-1}(\sin\theta) = \cosh^{-1}(\sec\theta)$

9. Prove that

$$\cosh^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right) = \log\left[\frac{\sqrt{b+a} + \sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a} - \sqrt{b-a}\tan\frac{x}{a}}\right]$$

10. Prove that $\tan^{-1}(e^{i\theta}) = \frac{n\pi}{2} + \frac{\pi}{4} = \log\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

11. If $\cosh^{-1}(x + iy) + \cosh^{-1}(x - iy) = \cosh^{-1} a$, prove that
$2(a - 1)x^2 + 2(a + 1)y^2 = a^2 - 1$.

12. Prove that : $\tanh^{-1}\cos\theta = \cosh^{-1}\operatorname{cosec}\theta$

13. Prove that : $\sinh^{-1}\tan\theta = \log(\sec\theta + \tan\theta)$

14. Prove that : $\sinh^{-1}\tan\theta = \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$

Separate into real and imaginary parts

15. $\cos^{-1} e^{i\theta}$ or $\cos^{-1}(\cos\theta + i\sin\theta)$ **Ans.** $\sin^{-1}\sqrt{\sin\theta} + i\log\left(\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right)$

16. If $\sinh^{-1}(x + iy) + \sinh^{-1}(x - iy) = \sinh^{-1} a$, prove that
$2(x^2 + y^2)\sqrt{a^2+1} = a^2 - 2x^2 - 2y^2$.