CHAPTER

43

INVERSE LAPLACE TRANSFORMS

(SOLUTION OF DIFFERENTIAL EQUATIONS)

43.1 INVERSE LAPLACE TRANSFORMS

If $F(s)$ is the Laplace Transform of a function $f(t)$, then $f(t)$ is known as Inverse Laplace Transform. Now we will discuss how to find $f(t)$ when $F(s)$ is given.

If $L[f(t)] = F(s)$, then $L^{-1}[F(s)] = f(t)$, where L^{-1} is called the Inverse Laplace Transform operator.

From the application point of view, the Inverse Laplace Transform is very useful.

Inverse Laplace Transform is used in solving differential equations without finding the general solution and arbitrary constants.

43.2 IMPORTANT FORMULAE

1. $L^{-1}\left(\frac{1}{s}\right) = 1$

2. $L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}$

3. $L^{-1}\frac{1}{s-a} = e^{at}$

4. $L^{-1}\frac{s}{s^2-a^2} = \cosh at$ (*R.G.P.V., Bhopal, Dec. 2007*)

5. $L^{-1}\frac{1}{s^2-a^2} = \frac{1}{a}\sinh at$

6. $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$

7. $L^{-1}\frac{s}{s^2+a^2} = \cos at$

8. $L^{-1}F(s-a) = e^{at}f(t)$

9. $L^{-1}\frac{1}{(s-a)^2+b^2} = \frac{1}{b}e^{at}\sin bt$

10. $L^{-1}\frac{s-a}{(s-a)^2+b^2} = e^{at}\cos bt$

11. $L^{-1}\frac{1}{(s-a)^2-b^2} = \frac{1}{b}e^{at}\sinh bt$

12. $L^{-1}\frac{s-a}{(s-a)^2-b^2} = e^{at}\cosh bt$

13. $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}(\sin at - at\cos at)$

14. $L^{-1}\frac{s}{(s^2+a^2)^2} = \frac{1}{2a}t\sin at$

15. $L^{-1}\frac{s^2-a^2}{(s^2+a^2)^2} = t\cos at$

16. $L^{-1}(1) = s(t)$

17. $L^{-1}\frac{s^2}{(s^2+a^2)^2} = \frac{1}{2a}[\sin at + at\cos at]$

18. $L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(t)\,dt$

Example 1. *Show that* $\dfrac{1}{s^{1/2}} = L\left[\dfrac{1}{\sqrt{\pi t}}\right]$. (*U.P., II Semester, Summer 2005*)

Solution. We have to show that $\dfrac{1}{s^{1/2}} = L\left[\dfrac{1}{\sqrt{\pi t}}\right]$.

Now, $$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} = \frac{t^{n-1}}{\lceil n}$$

So $$L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{t^{\frac{1}{2}-1}}{\lceil 1/2} = \frac{t^{-1/2}}{\lceil 1/2} = \frac{t^{-1/2}}{\sqrt{\pi}}$$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{1}{\sqrt{\pi t}} \quad \Rightarrow \quad \frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$$ **Proved.**

Example 2. *Find the inverse Laplace Transform of the following:*

(*i*) $\dfrac{1}{s-2}$ (*ii*) $\dfrac{1}{s^2-9}$ (*iii*) $\dfrac{1}{s^2+25}$

(*iv*) $\dfrac{s}{s^2+9}$ (*v*) $\dfrac{1}{(s-2)^2+1}$ (*vi*) $\dfrac{s-1}{(s-1)^2+4}$

(*vii*) $\dfrac{1}{(s+3)^2-4}$ (*viii*) $\dfrac{s+2}{(s+2)^2-25}$

Solution.

(*i*) $L^{-1}\dfrac{1}{s-2} = e^{2t}$

(*ii*) $L^{-1}\dfrac{1}{s^2-9} = L^{-1}\dfrac{1}{3}\cdot\dfrac{3}{s^2-(3)^2} = \dfrac{1}{3}\sinh 3t$

(*iii*) $L^{-1}\dfrac{s}{s^2-16} = L^{-1}\dfrac{s}{s^2-(4)^2} = \cosh 4t$

(*iv*) $L^{-1}\dfrac{1}{s^2+25} = \dfrac{1}{5}\dfrac{5}{s^2+(5)^2} = \dfrac{1}{5}\sin 5t$

(*v*) $L^{-1}\dfrac{s}{s^2+9} = \dfrac{s}{s^2+(3)^2} = \cos 3t$

(*vi*) $L^{-1}\dfrac{1}{(s-2)^2+1} = e^{2t}\sin t$

(*vii*) $L^{-1}\dfrac{1}{(s+3)^2-4} = \dfrac{1}{2}\dfrac{2}{(s+3)^2-(2)^2} = \dfrac{1}{2}e^{-3t}\sinh 2t$

(*viii*) $L^{-1}\dfrac{s+2}{(s+2)^2-25} = L^{-1}\dfrac{(s+2)}{(s+2)^2-(5)^2} = e^{-2t}\cosh 5t$

Example 3. *Find* $L^{-1}\dfrac{s^2+2s+6}{s^3}$ (*M.D.U. 2010*)

Solution. Here, we have

$$L^{-1}\frac{s^2+2s+6}{s^3} = L^{-1}\left[\frac{1}{s}+\frac{2}{s^2}+\frac{6}{s^3}\right] = 1+\frac{2t}{1!}+\frac{6}{2!}t^2$$
$$= 1 + 2t + 3t^2$$ **Ans.**

Example 4. *Find Inverse Laplace Transform of*

(*a*) $\left\{\dfrac{6}{2s-3}-\dfrac{3+4s}{9s^2-16}+\dfrac{8-6s}{16s^2+9}\right\}$ (*b*) $\dfrac{2s-5}{9s^2-25}$

(*c*) $\dfrac{s-2}{6s^2+20}$ (*U.P. II. Semester Summer* 2001)

Solution.

(*a*) $$L^{-1}\left\{\frac{6}{2s-3}-\frac{3}{9s^2-16}-\frac{4s}{9s^2-16}+\frac{8}{16s^2+9}-\frac{6s}{16s^2+9}\right\}$$

$$= L^{-1}\left\{\frac{3}{s-\frac{3}{2}} - \frac{\frac{1}{3}}{s^2-\left(\frac{4}{3}\right)^2} - \frac{\frac{4}{9}s}{s^2-\left(\frac{4}{3}\right)^2} + \frac{\frac{1}{2}}{s^2+\left(\frac{3}{4}\right)^2} - \frac{\frac{3}{8}s}{s^2+\left(\frac{3}{4}\right)^2}\right\}$$

$$= L^{-1}\left\{\frac{3}{s-\frac{3}{2}} - \frac{1}{4}\frac{\frac{4}{3}}{s^2-\left(\frac{4}{3}\right)^2} - \frac{4}{9}\frac{s}{s^2-\left(\frac{4}{3}\right)^2} + \frac{2}{3}\frac{\frac{3}{4}}{s^2+\left(\frac{3}{4}\right)^2} - \frac{3}{8}\frac{s}{s^2+\left(\frac{3}{4}\right)^2}\right\}$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{4}\sinh\frac{4}{3}t - \frac{4}{9}\cosh\frac{4}{3}t + \frac{2}{3}\sin\frac{3}{4}t - \frac{3}{8}\cos\frac{3}{4}t$$ **Ans.**

(*b*) $$L^{-1}\frac{2s-5}{9s^2-25} = L^{-1}\left[\frac{2s}{9s^2-25} - \frac{5}{9s^2-25}\right] = L^{-1}\left[\frac{2s}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]} - \frac{5}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}\right]$$

$$= \frac{2}{9}\cosh\frac{5}{3}t - \frac{1}{3}L^{-1}\left(\frac{\frac{5}{3}}{s^2-\left(\frac{5}{3}\right)^2}\right) = \frac{2}{9}\cosh\frac{5t}{3} - \frac{1}{3}\sin\frac{5t}{3}$$ **Ans.**

(*c*) $$L^{-1}\frac{s-2}{6s^2+20} = L^{-1}\frac{s}{6s^2+20} - L^{-1}\frac{2}{6s^2+20} = \frac{1}{6}L^{-1}\frac{s}{s^2+\frac{10}{3}} - \frac{1}{3}L^{-1}\frac{1}{s^2+\frac{10}{3}}$$

$$= \frac{1}{6}L^{-1}\frac{s}{s^2+\frac{10}{3}} - \frac{1}{3}\times\sqrt{\frac{3}{10}}\,L^{-1}\frac{\sqrt{\frac{10}{3}}}{s^2+\frac{10}{3}} = \frac{1}{6}\cos\sqrt{\frac{10}{3}}\,t - \frac{1}{\sqrt{30}}\sin\sqrt{\frac{10}{3}}\,t$$ **Ans.**

Example 5. *Find the inverse Laplace transform of following function:*

$$\frac{14s+10}{49s^2+28s+13}.$$ [*U.P., II Semester, 2007*]

Solution. The given function can be written as

$$\frac{14s+10}{49s^2+28s+13} = \frac{14s+10}{(7s+2)^2+9} = \frac{14\left(s+\frac{2}{7}\right)+6}{49\left(s+\frac{2}{7}\right)^2+9}$$

$$\therefore \quad L^{-1}\left(\frac{14s+10}{49s^2+28s+13}\right) = L^{-1}\left[\frac{14\left(s+\frac{2}{7}\right)+6}{49\left(s+\frac{2}{7}\right)^2+9}\right] = e^{-\frac{2t}{7}}L^{-1}\left(\frac{14s+6}{49s^2+9}\right)$$

$$= e^{-\frac{2}{7}t}L^{-1}\frac{14}{49}\left(\frac{s+\frac{6}{14}}{s^2+\frac{9}{49}}\right) = e^{-\frac{2t}{7}}\left[\frac{14}{49}L^{-1}\left(\frac{s}{s^2+\frac{9}{49}}\right) + \left(\frac{14}{49}\right)\left(\frac{6}{14}\right)L^{-1}\left(\frac{1}{s^2+\frac{9}{49}}\right)\right]$$

$$= e^{-\frac{2t}{7}}\left[\frac{2}{7}\cos\frac{3}{7}t + \frac{6}{49}.\frac{7}{3}\sin\frac{3}{7}t\right]$$

$$= \frac{2}{7}e^{-\frac{2t}{7}}\left(\cos\frac{3}{7}t + \sin\frac{3}{7}t\right)$$ **Ans.**

EXERCISE 43.1

Find the Inverse Laplace Transform of the following:

1. $\dfrac{3s-8}{4s^2+25}$ **Ans.** $\dfrac{3}{4}\cos\dfrac{5t}{2}-\dfrac{4}{5}\sin\dfrac{5t}{2}$

2. $\dfrac{3\left(s^2-2\right)^2}{2s^5}$ **Ans.** $\dfrac{3}{2}-3t^2+\dfrac{1}{2}t^4$

3. $\dfrac{2s-5}{4s^2+25}+\dfrac{4s-18}{9-s^2}$ **Ans.** $\dfrac{1}{2}\left(\cos\dfrac{5t}{2}-\sin\dfrac{5t}{2}\right)-4\cosh 3t+6\sinh 3t$

4. $\dfrac{5s-10}{9s^2-16}$ **Ans.** $\dfrac{5}{9}\cosh\dfrac{4}{3}t-\dfrac{5}{6}\sinh\dfrac{4}{3}t$

5. $\dfrac{1}{4s}+\dfrac{16}{1-s^2}$ **Ans.** $\dfrac{1}{4}-16\sinh t$

6. $L^{-1}\left\{\dfrac{1}{s^n}\right\}$ exist only when the value of n is :

(*i*) Negative integer (*ii*) Positive integer
(*iii*) Zero (*iv*) None of these **Ans.** (*ii*) (*U.P. II Semester, 2010*)

43.3 MULTIPLICATION BY *S*

$$\boxed{L^{-1}\left[s\,F(s)\right]=\frac{d}{dt}f(t)+f(0)\,\delta(t)}$$

Example 6. *Find the Inverse Laplace Transform of* (*i*) $\dfrac{s}{s^2+1}$ (*ii*) $\dfrac{s}{4s^2-25}$ (*iii*) $\dfrac{3s}{2s+9}$

Solution.

(*i*) $L^{-1}\dfrac{1}{s^2+1}=\sin t$

$$L^{-1}\frac{s}{s^2+1}=\frac{d}{dt}(\sin t)+\sin(0)\,\delta(t)=\cos t \qquad \textbf{Ans.}$$

(*ii*) $$L^{-1}\frac{1}{4s^2-25}=\frac{1}{4}L^{-1}\frac{1}{s^2-\frac{25}{4}}=\frac{1}{4}\cdot\frac{2}{5}L^{-1}\frac{\frac{5}{2}}{s^2-\left(\frac{5}{2}\right)^2}=\frac{1}{10}\sinh\frac{5}{2}t$$

$$L^{-1}\frac{s}{4s^2-25}=\frac{1}{10}\frac{d}{dt}\sinh\frac{5}{2}t+\frac{1}{10}\sinh\frac{5}{2}(0)\,\delta(t)$$

$$=\frac{1}{10}\left(\frac{5}{2}\right)\cosh\frac{5}{2}t=\frac{1}{4}\cosh\frac{5}{2}t \qquad \textbf{Ans.}$$

(*iii*) $$L^{-1}\frac{3}{2s+9}=\frac{3}{2}L^{-1}\frac{1}{s+\frac{9}{2}}=\frac{3}{2}e^{-\frac{9}{2}t}$$

$$L^{-1}\frac{3s}{2s+9}=\frac{3}{2}\frac{d}{dt}\left(e^{-\frac{9}{2}t}\right)+\frac{3}{2}e^{-\frac{9}{2}(0)\delta(t)}=\frac{3}{2}\left(-\frac{9}{2}\right)e^{-\frac{9}{2}t}+\frac{3}{2}=-\frac{27}{4}e^{-\frac{9}{2}t}+\frac{3}{2} \qquad \textbf{Ans.}$$

EXERCISE 43.2

Find the Inverse Laplace Transform of the following:

1. $\dfrac{s}{s+5}$ **Ans.** $-5e^{-5t}$ 2. $\dfrac{2s}{3s+6}$ **Ans.** $-\dfrac{4}{3}e^{-2t}$

3. $\dfrac{s}{2s^2-1}$ **Ans.** $\dfrac{1}{2}\cosh\dfrac{t}{2}$ 4. $\dfrac{s^2}{s^2+a^2}$ **Ans.** $-a\sin at+1$ 5. $\dfrac{s^2+4}{s^2+9}$ **Ans.** $-\dfrac{5}{3}\sin 3t+1$

43.4 DIVISION BY s (MULTIPLICATION BY $\frac{1}{s}$)

$$\boxed{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t \left[L^{-1}\left[F(s)\right]\right] dt = \int_0^t f(t)\, dt}$$

Example 7. *Find the Inverse Laplace Transform of*

(*i*) $\frac{1}{s(s+a)}$ (*ii*) $\frac{1}{s(s^2+1)}$ (*iii*) $\frac{s^2+3}{s(s^2+9)}$

Solution.

(*i*) $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left(\frac{1}{s+a}\right) dt = \int_0^t e^{-at}\, dt = \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{e^{-at}}{-a} + \frac{1}{a} = \frac{1}{a}\left[1 - e^{-at}\right] \qquad \textbf{Ans.}$$

(*ii*) $L^{-1}\frac{1}{s^2+1} = \sin t$

$$L^{-1}\frac{1}{s}\left(\frac{1}{s^2+1}\right) = \int_0^t L^{-1}\left(\frac{1}{s^2+1}\right) dt = \int_0^t \sin t\, dt = \left[-\cos t\right]_0^t = -\cos t + 1 \qquad \textbf{Ans.}$$

(*iii*) $L^{-1}\frac{s^2+3}{s(s^2+9)} = L^{-1}\left[\frac{s^2+9-6}{s(s^2+9)}\right] = L^{-1}\left[\frac{1}{s} - \frac{6}{s(s^2+9)}\right]$

$$= 1 - 2\int_0^t \sin 3t\, dt = 1 + 2 \times \frac{1}{3}\left[\cos 3t\right]_0^t = 1 + \frac{2}{3}\cos 3t - \frac{2}{3}$$

$$= \frac{2}{3}\cos 3t + \frac{1}{3} = \frac{1}{3}\left[2\cos 3t + 1\right] \qquad \textbf{Ans.}$$

EXERCISE 43.3

Find the Inverse Laplace Transform of the following:

1. $\frac{1}{2s(s-3)}$ **Ans.** $\frac{1}{2}\left[\frac{e^{3t}}{3} - 1\right]$
2. $\frac{1}{s(s+2)}$ **Ans.** $\frac{1-e^{-2t}}{2}$
3. $\frac{1}{s(s^2-16)}$ **Ans.** $\frac{1}{16}\left[\cosh 4t - 1\right]$
4. $\frac{1}{s(s^2+a^2)}$ **Ans.** $\frac{1-\cos at}{a^2}$
5. $\frac{s^2+2}{s(s^2+4)}$ **Ans.** $\cos^2 t$
6. $\frac{1}{s^2(s+1)}$ **Ans.** $t - 1 + e^{-t}$
7. $\frac{1}{s^3(s^2+1)}$ **Ans.** $\frac{t^2}{2} + \cos t - 1$

43.5 FIRST SHIFTING PROPERTY

If $L^{-1} F(s) = f(t)$, then $\boxed{L^{-1} F(s+a) = e^{-at} L^{-1}\left[F(s)\right]}$

Example 8. *Find the Inverse Laplace Transform of*

(*i*) $\frac{1}{(s+2)^5}$ (*ii*) $\frac{s}{s^2+4s+13}$ (*iii*) $\frac{1}{9s^2+6s+1}$

Solution.

(*i*) $L^{-1}\dfrac{1}{s^5}=\dfrac{t^4}{4!}$

then $L^{-1}\dfrac{1}{(s+2)^5}=e^{-2t}.\dfrac{t^4}{4!}$ **Ans.**

(*ii*) $L^{-1}\left(\dfrac{s}{s^2+4s+13}\right)=L^{-1}\dfrac{s+2-2}{(s+2)^2+(3)^2}=L^{-1}\dfrac{s+2}{(s+2)^2+(3)^2}-L^{-1}\dfrac{2}{(s+2)^2+(3)^2}$

$= e^{-2t}L^{-1}\dfrac{s}{s^2+3^2}-e^{-2t}L^{-1}\dfrac{2}{3}\left(\dfrac{3}{s^2+3^2}\right) = e^{-2t}\cos 3t-\dfrac{2}{3}e^{-2t}\sin 3t$ **Ans.**

(*iii*) $L^{-1}\dfrac{1}{9s^2+6s+1}=L^{-1}\dfrac{1}{(3s+1)^2}=\dfrac{1}{9}L^{-1}\dfrac{1}{\left(s+\dfrac{1}{3}\right)^2}=\dfrac{1}{9}e^{-t/3}L^{-1}\dfrac{1}{s^2}$

$= \dfrac{1}{9}e^{-t/3}\,t=\dfrac{t\,e^{-t/3}}{9}$ **Ans.**

Example 9. *Find the Inverse Laplace Transform of* $\dfrac{s+1}{s^2-6s+25}$

(*U.P., II Semester 2010*)

Solution. $L^{-1}\left(\dfrac{s+1}{s^2-6s+25}\right)=L^{-1}\left[\dfrac{s+1}{(s-3)^2+(4)^2}\right]=L^{-1}\left[\dfrac{s-3+4}{(s-3)^2+(4)^2}\right]$

$=L^{-1}\left[\dfrac{s-3}{(s-3)^2+(4)^2}\right]+L^{-1}\left[\dfrac{4}{(s-3)^2+(4)^2}\right]$

$= e^{3t}\cos 4t+e^{3t}\sin\ 4t.$ **Ans.**

EXERCISE 43.4

Obtain the Inverse Laplace Transform of the following:

1. $\dfrac{s+8}{s^2+4s+5}$ **Ans.** $e^{-2t}(\cos t+6\sin t)$
2. $\dfrac{s}{(s+3)^2+4}$ **Ans.** $e^{-3t}(\cos 2t-1.5\sin 2t)$
3. $\dfrac{s}{(s+7)^4}$ **Ans.** $e^{-7t}\dfrac{t^2}{6}(3-7t)$
4. $\dfrac{s+2}{s^2-2s-8}$ **Ans.** $e^{-t}(\cosh 3t+\sinh 3t)$
5. $\dfrac{s}{s^2+6s+25}$ **Ans.** $e^{-3t}\left[\cos 4t-\dfrac{3}{4}\sin 4t\right]$
6. $\dfrac{1}{2(s-1)^2+32}$ **Ans.** $\dfrac{e^t}{8}\sin 4t$
7. $\dfrac{s-4}{4(s-3)^2+16}$ **Ans.** $\dfrac{1}{4}e^{3t}\cos 2t-\dfrac{1}{8}e^{3t}\sin 2t$

43.6 SECOND SHIFTING PROPERTY

$$\boxed{L^{-1}\left[e^{-as}F(s)\right]=f(t-a)\,u(t-a)}$$

Example 10. *Obtain Inverse Laplace Transform of*

(*i*) $\dfrac{e^{-\pi s}}{(s+3)}$ (*ii*) $\dfrac{e^{-s}}{(s+1)^3}$

Solution.

(*i*) $L^{-1}\dfrac{1}{s+3}=e^{-3t},\quad L^{-1}\dfrac{e^{-\pi s}}{s+3}=e^{-3(t-\pi)}\,u(t-\pi)$ **Ans.**

(*ii*) $L^{-1}\dfrac{1}{s^3}=\dfrac{t^2}{2!} \quad \Rightarrow \quad L^{-1}\dfrac{1}{(s+1)^3}=e^{-t}\dfrac{t^2}{2!}$

$$L^{-1}\frac{e^{-s}}{(s+1)^3}=e^{-(t-1)}\cdot\frac{(t-1)^2}{2!}\cdot u(t-1)$$ **Ans.**

Example 11. *Evaluate*

$$L^{-1}\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right]$$ (*U.P. II Semester, Summer 2002*)

Solution. $L^{-1}\left[\dfrac{e^{-s}-3e^{-3s}}{s^2}\right]=L^{-1}\left[\dfrac{e^{-s}}{s^2}-\dfrac{3e^{-3s}}{s^2}\right]$ ...(1)

We know that $L\,[u\,(t-\;a)]=\dfrac{e^{-as}}{s}$

and $\qquad L\,[(t-a)\,u(t-a)]=\dfrac{e^{-as}}{s^2}$

Using these results in (1), we get

$$\therefore \quad L^{-1}\left[\frac{e^{-s}-3e^{-3s}}{s^2}\right]=(t-1)\,u\,(t-1)-3(t-3)\,u\,(t-3)$$ **Ans.**

Example 12. *Find the Inverse Laplace Transform of*

$$\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$$

in terms of unit step functions.

Solution. $\qquad L^{-1}\dfrac{\pi}{s^2+\pi^2}=\sin\pi t$

$$L^{-1}\left[e^{-s}\frac{\pi}{s^2+\pi^2}\right]=\sin\pi(t-1)\,.\,u(t-1)=-\sin(\pi t)\,.\,u(t-1) \quad ...(1)$$

and $\qquad L^{-1}\dfrac{s}{s^2+\pi^2}=\cos\pi t$

$$L^{-1}\left[e^{-s/2}\frac{s}{s^2+\pi^2}\right]=\cos\pi\left(t-\frac{1}{2}\right).\,u\left(t-\frac{1}{2}\right)=\sin\pi t\,.\,u\left(t-\frac{1}{2}\right) \quad ...(2)$$

On adding (1) and (2), we get

$$L^{-1}\left[\frac{e^{-s/2}\,s+e^{-s}\,.\,\pi}{s^2+\pi^2}\right]=\sin(\pi t)\,.\,u\left(t-\frac{1}{2}\right)-\sin(\pi t)\,.\,u(t-1)$$

$$=\sin\pi t\left[u\left(t-\frac{1}{2}\right)-u\,(t-1)\right]$$ **Ans.**

Example 13. *Find the value of*

Solution $\qquad \dfrac{1}{(s^2+a^2)^2}=\dfrac{1}{s}-\dfrac{s}{(s^2+a^2)^2}=-\dfrac{1}{2s}\dfrac{d}{ds}\left(\dfrac{1}{s^2+a^2}\right)$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=L^{-1}\left\{-\frac{1}{2s}\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\}$$

$$=-\frac{1}{2s}\left\{-t\frac{1}{a}\sin at\right\}=-\frac{1}{2a}\frac{1}{s}\{t-\sin at\}=\frac{1}{2a}\int_0^t\sin at\,dt$$

$$= \frac{1}{2a}\left[t\left(\frac{-\cos at}{a}\right) - \int_0 \frac{\cos at}{a}\,dt\right]_0^t = \frac{1}{2a}\left[-\frac{t}{a}\cos at + \frac{\sin at}{a^2}\right]_0^t$$

$$= \frac{1}{2a^3}\left[-at\cos at + \sin at\right]$$ **Ans.**

EXERCISE 43.5

Obtain Inverse Laplace Transform of the following:

1. $\dfrac{e^{-s}}{(s+2)^3}$ **Ans.** $e^{-(t-2)}\dfrac{(t-2)^2}{2}u(t-2)$

2. $\dfrac{e^{-2s}}{(s+1)(s^2+2s+2)}$ **Ans.** $e^{-(t-2)}\,|\,1-\cos(t-2)\,|\,u(t-2)$

3. $\dfrac{e^{-s}}{\sqrt{s+1}}$ **Ans.** $\dfrac{e^{-(t-1)}}{\sqrt{\pi(t-1)}}\,.\,u(t-1)$

4. $\dfrac{e^{-\frac{\pi}{2}s}+e^{-\frac{3\pi}{2}s}}{s^2+1}$ **Ans.** $\cot t\left[u\left(t-\frac{3\pi}{2}\right)-u\left(t-\frac{\pi}{2}\right)\right]$

5. $\dfrac{e^{-4s}(s+2)}{s^2+4s+5}$ **Ans.** $e^{-2(t-u)}\cos(t-u)\,u(t-4)$

6. $\dfrac{e^{-as}}{s^2}$ **Ans.** $f(t) = t - a$, when $t > a$

   $= 0$, when $t < a$

7. $\dfrac{e^{-\pi s}}{s^2+1}$ **Ans.** $-\sin t.\ u(t-\pi)$

43.7 INVERSE LAPLACE TRANSFORMS OF DERIVATIVES

$$L^{-1}\left[\frac{d}{ds}F(s)\right] = -t\,L^{-1}[F(s)] = -t\,f(t) \quad \Rightarrow \quad \boxed{L^{-1}[F(s)] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]}$$

Example 14. *Find* $L^{-1}\left\{\log\dfrac{s+1}{s-1}\right\}$. (*Uttarakhand, II Semester, June 2010, 2009,* 2007)

Solution. $L^{-1}\left\{\log\left(\dfrac{s+1}{s-1}\right)\right\} = -\dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\log\left(\dfrac{s+1}{s-1}\right)\right]$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(s+1) - \frac{d}{ds}\log(s-1)\right] = -\frac{1}{t}L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= -\frac{1}{t}\left[e^{-t} - e^{t}\right] = \frac{1}{t}\left[e^{t} - e^{-t}\right]$$ **Ans.**

Example 15. *Find the Inverse Laplace Transform of F(s)* $= \log\dfrac{s+a}{s+b}$

(*U.P., II Semester, Summer* 2003)

Solution. $L^{-1}\log\left(\dfrac{s+a}{s+b}\right) = -\dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\log\dfrac{s+a}{s+b}\right]$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(s+a) - \frac{d}{ds}\log(s+b)\right] = -\frac{1}{t}L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$= -\frac{1}{t}\left[e^{-at} - e^{-bt}\right] = \frac{1}{t}\left(e^{-bt} - e^{-at}\right)$$ **Ans.**

Example 16. *Obtain the Inverse Laplace Transform of log* $\dfrac{s^2-1}{s^2}$.

Solution. $L^{-1}\left[\log\dfrac{s^2-1}{s^2}\right]=-\dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\log\dfrac{s^2-1}{s^2}\right]$

$$=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left\{\log\left(s^2-1\right)-2\log s\right\}\right]=-\frac{1}{t}L^{-1}\left[\frac{2s}{s^2-1}-\frac{2}{s}\right]=-\frac{1}{t}[\,2\cosh t-2\,]$$

$=\dfrac{2}{t}[\,1-\cosh t\,]$ **Ans.**

Example 17. *Find the function whose Laplace transform is*

log $\left(1+\dfrac{1}{s}\right)$. (*U.P., II Semester, June 2007*)

Solution. $L^{-1}\left[\log\left(1+\dfrac{1}{s}\right)\right]=\dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\log\left(\dfrac{s+1}{s}\right)\right]$

$$=-\frac{1}{t}L^{-1}\left[\left(\frac{s}{s+1}\right)\left(-\frac{1}{s^2}\right)\right]=-\frac{1}{t}L^{-1}\left[-\frac{1}{s\,(s+1)}\right]$$

$$=-\frac{1}{t}L^{-1}\left[\frac{1}{s+1}-\frac{1}{s}\right] \qquad \text{(Partial fraction)}$$

$$=-\frac{1}{t}[e^{-t}-1]=\frac{1}{t}[1-e^{-t}]$$ **Ans.**

Example 18. *Find the inverse Laplace transform of* $\tan^{-1}\left(\dfrac{2}{s^2}\right)$

Solution. Here, we have $L^{-1}\left[\tan^{-1}\left(\dfrac{2}{s^2}\right)\right]=-\dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\tan^{-1}\dfrac{2}{s^2}\right]$

$$=-\frac{1}{t}L^{-1}\left[\frac{1}{1+\frac{4}{s^4}}\left(-\frac{4}{s^3}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{s^4}{s^4+4}\left(-\frac{4}{s^3}\right)\right]=\frac{1}{t}L^{-1}\left[\frac{4s}{s^4+4}\right]$$

$$=\frac{4}{t}L^{-1}\left[\frac{s}{s^4+4}\right]=\frac{4}{t}L^{-1}\left[\frac{s}{(s^2+2s+2)\,(s^2-2s+2)}\right]$$

$$=\frac{4}{t}L^{-1}\left[-\frac{1}{4}\frac{1}{(s^2+2s+2)}+\frac{1}{4}\frac{1}{(s^2-2s+2)}\right] \qquad \left(\text{By partial fraction}\right)$$

$$=\frac{1}{t}L^{-1}\left[-\frac{1}{(s^2+2s+2)}+\frac{1}{(s^2-2s+2)}\right]=\frac{1}{t}L^{-1}\left[-\frac{1}{(s+1)^2+1}+\frac{1}{(s-1)^2+1}\right]$$

$$=\frac{1}{t}\left[-e^{-t}\sin t+e^{t}\sin t\right]=\frac{\sin t}{t}\left[e^{t}-e^{-t}\right]$$ **Ans.**

Example 19. *Find Inverse Laplace Transform of* $\tan^{-1}\dfrac{1}{s}$.

Solution. $L^{-1}\left(\tan^{-1}\dfrac{1}{s}\right)=-\dfrac{1}{t}L^{-1}\left[\dfrac{d}{ds}\tan^{-1}\dfrac{1}{s}\right]$ (*M.D.U., 2010*)

$$=-\frac{1}{t}L^{-1}\left[\frac{1}{1+\frac{1}{s^2}}\left(-\frac{1}{s^2}\right)\right]=\frac{1}{t}L^{-1}\left[\frac{1}{1+s^2}\right]=\frac{\sin t}{t}$$ **Ans.**

Example 20. *Find* $L^{-1}\left[\tan^{-1}(1+s)\right]$ (*M.D.U. 2010*)

Solution. $L^{-1}\left[\tan^{-1}(1+s)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\tan^{-1}(1+s)\right]$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{1+(s+1)^2}\right]=-\frac{1}{t}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -\frac{1}{t}e^{-t}\sin t$$ **Ans.**

Example 21. *Find the inverse Laplace transform of*

$$\cot^{-1}\left(\frac{s}{2}\right)$$ (*Q.Bank U.P. 2001*)

Solution.

Let $L^{-1}\left[\cot^{-1}\left(\frac{s}{2}\right)\right]=f(t) \quad \Rightarrow \quad L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right]=-t\,f(t)$

$$\Rightarrow \quad L^{-1}\left[\frac{-1}{1+\frac{s^2}{4}}\,\frac{1}{2}\right]=-t\,f(t) \;\Rightarrow\; L^{-1}\left[\frac{2}{s^2+2}\right]=t\,f(t)$$

$$\Rightarrow \quad \sin 2t = t\,f(t) \quad \Rightarrow \quad f(t)=\frac{1}{t}\sin 2t$$ **Ans.**

Example 22. *Obtain the Inverse Laplace Transform of* $\cot^{-1}\left(\frac{s+3}{2}\right)$

(*U. P., II Semester, Summer* 2002)

Solution. We know that $L^{-1}\left[F(s)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$

$$\therefore \quad L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(\frac{s+3}{2}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{-\frac{1}{2}}{1+\left(\frac{s+3}{2}\right)^2}\right\} = \frac{1}{2t}L^{-1}\left\{\frac{4}{4+(s+3)^2}\right\}$$

$$= \frac{1}{t}L^{-1}\left\{\frac{2}{2^2+(s+3)^2}\right\} = \frac{1}{t}e^{-3t}L^{-1}\left(\frac{2}{2^2+s^2}\right)$$

$$= \frac{e^{-3t}}{t}\sin 2t$$ **Ans.**

Example 23. *Find the inverse Laplace transform of* $\frac{2as}{(s^2+a^2)^2}$

Solution. $L^{-1}\left(\frac{a}{s^2+a^2}\right)=\sin at$

$$L^{-1}\left[\frac{d}{ds}\left\{\frac{a}{s^2+a^2}\right\}\right]=-t\sin at \quad \Rightarrow \quad L^{-1}\left\{\frac{-2as}{(s^2+a^2)^2}\right\} \quad =-t\sin at$$

$$\Rightarrow \qquad L^{-1}\left\{\frac{2as}{(s^2+a^2)^2}\right\} = t \sin at$$ **Ans.**

Example 24. *Find the inverse Laplace transform of* $\dfrac{s^2-a^2}{(s^2+a^2)^2}$

Solution. We know that

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$\therefore \qquad L^{-1}\left[\frac{d}{ds}\left\{\frac{a}{s^2+a^2}\right\}\right] = -t\cos at$$

$$\Rightarrow \quad L^{-1}\left\{\frac{(s^2+a^2).1-s(2s)}{(s^2+a^2)}\right\} = -t\cos at \quad \Rightarrow L^{-1}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = -t\cos at$$

$$\therefore \qquad L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$$ **Ans.**

EXERCISE 43.6

Obtain Inverse Laplace Transform of the following:

1. $\log\left(1+\dfrac{\omega^2}{s^2}\right)$ **Ans.** $-\dfrac{2}{t}\cos\omega t+2$ 2. $\log\left(1+\dfrac{1}{s^2}\right)$ **Ans.** $\dfrac{2}{t}[1-\cos\omega t]$

3. $\dfrac{s}{1+s^2+s^4}$ **Ans.** $\dfrac{2}{\sqrt{3}}\sin\dfrac{\sqrt{3}}{2}t\sinh\dfrac{t}{2}$

4. $\dfrac{s}{\left(s^2+a^2\right)^2}$ **Ans.** $\dfrac{t\sin at}{2a}$ 5. $s\log\dfrac{s}{\sqrt{s^2+1}}+\cot^{-1}s$ **Ans.** $\dfrac{1-\cos t}{t^2}$

6. $\dfrac{1}{2}\log\left\{\dfrac{s^2+b^2}{(s-a)^2}\right\}$ **Ans.** $\dfrac{e^{-at}-\cos bt}{t}$

43.8 INVERSE LAPLACE TRANSFORM OF INTEGRALS

$$L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t} = \frac{1}{t}L^{-1}[F(s)] \qquad \text{or} \qquad \boxed{L^{-1}[F(s)] = t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right]}$$

Example 25. *Obtain* $L^{-1}\dfrac{2s}{\left(s^2+1\right)^2}$

Solution. $L^{-1}\dfrac{2s}{\left(s^2+1\right)^2} = t\,L^{-1}\displaystyle\int_s^\infty\frac{2s\,ds}{\left(s^2+1\right)^2} = t\,L^{-1}\left[-\frac{1}{s^2+1}\right]_s^\infty = t\,L^{-1}\left[-0+\frac{1}{s^2+1}\right]$

$= t\sin t$ **Ans.**

43.9 PARTIAL FRACTIONS METHOD

Example 26. *Find the Inverse Laplace Transform of* $\dfrac{1}{s^2-5s+6}$.

Solution. Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{1}{s^2-5s+6}\right] = L^{-1}\left[\frac{1}{s-3}-\frac{1}{s-2}\right]$$

$$= L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right) = e^{3t}-e^{2t}$$ **Ans.**

Example 27. *Find the inverse Laplace transform of*

$$\frac{s^3}{s^4 - a^4} \qquad (Q.\ Bank\ U.P.\ 2001)$$

Solution. Here, we have

$$L^{-1}\left(\frac{s^3}{s^4 - a^4}\right) = L^{-1}\left[s\left\{\frac{s^2}{(s^2 - a^2)(s^2 + a^2)}\right\}\right] = L^{-1}\left[\frac{s}{2}\left(\frac{1}{s^2 - a^2} + \frac{1}{s^2 + a^2}\right)\right]$$

$$= \frac{1}{2} L^{-1}\left(\frac{s}{s^2 - a^2} + \frac{s}{s^2 + a^2}\right) \qquad \text{(By partial fractions)}$$

$$= \frac{1}{2}(\cosh at + \cos at) \qquad \textbf{Ans.}$$

Example 28. *Find the Inverse Laplace Transforms of* $\dfrac{s+4}{s(s-1)(s^2+4)}$.

Solution. Let us first resolve $\dfrac{s+4}{s(s-1)(s^2+4)}$ into partial fractions.

$$\frac{s+4}{s(s-1)(s^2+4)} \equiv \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4} \qquad \text{... (1)}$$

$$s + 4 \equiv A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)\,s(s-1)$$

Putting $s = 0$, we get $4 = -4A \quad \Rightarrow \quad A = -1$

Putting $s = 1$, we get $5 = B\,.\,1\,.\,(1+4) \qquad \Rightarrow B = 1$

Equating the coefficients of s^3 on both sides of (1), we have

$$0 = A + B + C \quad \Rightarrow \quad 0 = -1 + 1 + C \quad \Rightarrow \quad C = 0.$$

Equating the coefficients of s on both sides of (1), we get

$$1 = 4A + 4B - D \quad \Rightarrow \quad 1 = -4 + 4 - D \quad \Rightarrow \quad D = -1.$$

On putting the values of A, B, C, D in (1), we get

$$\frac{s+4}{s(s-1)(s^2+4)} = -\frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}$$

$$\therefore \quad L^{-1}\left[\frac{s+4}{s(s-1)(s^2+4)}\right] = L^{-1}\left[-\frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}\right]$$

$$= -L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right) = -1 + e^t - \frac{1}{2}\sin 2t. \qquad \textbf{Ans.}$$

Example 29. *Find the inverse Laplace transform of*

$$\frac{1}{s^4 + 4} \qquad [U.P.,\ II\ Semester,\ (SUM)\ 2007]$$

Solution. Here, we have

$$s^4 + 4 = (s^2 + 2)^2 - (2s)^2 = (s^2 - 2s + 2)(s^2 + 2s + 2)$$

$$\frac{1}{s^4 + 4} = \frac{1}{(s^2 - 2s + 2)(s^2 + 2s + 2)}$$

$$= \frac{1}{4s}\left[\frac{1}{s^2 - 2s + 2} - \frac{1}{s^2 + 2s + 2}\right] \qquad \begin{pmatrix}\text{By}\\ \text{partial}\\ \text{fractions}\end{pmatrix} \text{...(1)}$$

Now, $$L^{-1}\left(\frac{1}{s^2-2s+2}\right) = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] = e^t \sin t$$

and $$L^{-1}\left(\frac{1}{s^2+2s+2}\right) = L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t} \sin t$$

$\therefore$ $$\frac{1}{4}L^{-1}\left(\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right) = \frac{1}{4}(e^t-e^{-t})\sin t$$

Hence, $$L^{-1}\left[\frac{1}{4s}\left(\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right)\right] = \frac{1}{4}\int_0^t (e^t-e^{-t})\sin t\, dt$$

$\Rightarrow$ $$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{4}\left[\frac{e^t}{2}(\sin t-\cos t)-\frac{e^{-t}}{2}(-\sin t-\cos t)\right]$$

$$= \frac{1}{4}\left[\sin t\left(\frac{e^t+e^{-t}}{2}\right)-\cos t\left(\frac{e^t-e^{-t}}{2}\right)\right]$$

$\Rightarrow$ $$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{4}\left[\sin t\cosh t-\cos t\sinh t\right]$$ **Ans.**

Example 30. *Find the inverse Laplace transform of*

$$\frac{s}{s^4+4a^4}$$

Solution. $$s^4+4a^4 = (s^2+2a^2)^2-(2as)^2 = (s^2-2as+2a^2)(s^2+2as+2a^2)$$
$$= \{(s-a)^2+a^2\}\{(s+a)^2+a^2\}$$

$$\frac{s}{s^4+4a^4} = \frac{s}{\{(s-a)^2+a^2\}\{(s+a)^2+a^2\}}$$

$$= \frac{1}{4a}\left[\frac{1}{(s-a)^2+a^2}-\frac{1}{(s+a)^2+a^2}\right]$$ (By partial fraction)

$\therefore$ $$L^{-1}\left(\frac{s}{s^4+4a^4}\right) = \frac{1}{4a}\left[L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}-L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{a}e^{at}\sin at-e^{-at}\frac{1}{a}\sin at\right]$$

$$= \frac{1}{2a^2}\sin at\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{1}{2a^2}\sin at\sinh at.$$ **Ans.**

Example 31. *Find the Inverse Laplace Transform of* $\frac{e^{-cs}}{s^2(s+a)}$, $c > 0$.

(*U.P. II Semester, Summer* 2002)

Solution. We have,

$$L^{-1}\left[\frac{e^{-cs}}{s^2(s+a)}\right] = L^{-1}\left[-\frac{e^{-cs}}{a^2 s}+\frac{e^{-cs}}{as^2}+\frac{e^{-cs}}{a^2(s+a)}\right]$$ (By Partial fractions)

$$= L^{-1}\left[\left(\frac{-1}{a^2}\frac{e^{-cs}}{s}\right)+\left(\frac{1}{a}\right)\frac{e^{-cs}}{s^2}+\left(\frac{1}{a^2}\right)\frac{e^{-c(s+a)}}{e^{-ca}(s+a)}\right]$$

$$= -\frac{1}{a^2}u(t-c)+\frac{1}{a}(t-c)\,u(t-c)+\frac{1}{a^2e^{-ca}}e^{at}\,u(t-c)$$

$$= u\,(t-c)\left[\frac{-1}{a^2} + \frac{1}{a}(t-c) + \frac{1}{a^2}\,e^{a(c+t)}\right]$$, where $u(t-c)$ is unit step function. **Ans.**

Example 32. *Find the Inverse Laplace Transform of*

$$\frac{5s+3}{(s-1)(s^2+2s+5)}$$ (*U.P. II Semester Summer* 2005)

Solution. $L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

Let $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$$
$$5s+3 = s^2(A+B) + s(2A-B+C) + (5A-C)$$

Comparing the coefficients of s^2, s and constant, we get

$$A + B = 0 \quad \text{... (1)}$$
$$2A - B + C = 5 \quad \text{... (2)}$$
$$5A - C = 3 \quad \text{... (3)}$$

On adding equations (1) and (2), we have $3A + C = 5$ ... (4)

Adding equations (3) and (4), we get $8A = 8 \quad \Rightarrow \quad A = 1$

Putting A = 1 in (3), we get $C = 2$

Putting $A = 1$, $C = 2$ in (2), we get

$$B = -1$$

Thus $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5} = \frac{1}{s-1} - \frac{s-2}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+2^2} + \frac{3}{(s+1)^2+2^2}$$

$$L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{3}{(s+1)^2+2^2}\right\} - L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}$$

$$= e^t + 3e^{-t}\,L^{-1}\left\{\frac{1}{s^2+2^2}\right\} - e^{-t}\,L^{-1}\left\{\frac{s}{s^2+2^2}\right\}$$

$$= e^t + 3e^{-t}\,.\,\frac{1}{2}\sin 2t - e^{-t}\cos 2t$$ **Ans.**

Example 33. *Find the Inverse Laplace Transform of* $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$.

Solution. Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{a^2}{a^2-b^2}\cdot\frac{1}{s^2+a^2} - \frac{b^2}{a^2-b^2}\cdot\frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{a^2-b^2}\,L^{-1}\left[\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right] = \frac{1}{a^2-b^2}\left[a^2\left(\frac{1}{a}\sin at\right) - b^2\left(\frac{1}{b}\sin bt\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[\,a\sin at - b\sin bt\,\right]$$ **Ans.**

Note: This question is also solved by using the Convolution Theorem as an example 37.

EXERCISE 43.7

Find the Inverse Laplace Transforms of the following by partial fractions method:

1. $\dfrac{1}{s^2-7s+12}$ **Ans.** $e^{4t}-e^{3t}$

2. $\dfrac{s+2}{s^2-4s+13}$ **Ans.** $e^{2t}\cos 3t+\dfrac{4}{3}e^{2t}\sin 3t$

3. $\dfrac{3s+1}{(s-1)(s^2+1)}$ **Ans.** $e^t-2\cos t+\sin t$

4. $\dfrac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ **Ans.** $2e^{-t}+5e^{2t}-\dfrac{3}{2}e^{t/2}$

5. $\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$ **Ans.** $\dfrac{1}{2}e^t-e^{2t}+\dfrac{5}{2}e^{3t}$

6. $\dfrac{s-4}{(s-4)^2+9}$ **Ans.** $e^{4t}\cos 3t$

7. $\dfrac{16}{(s^2+2s+5)^2}$ **Ans.** $e^{-t}(\sin 2t-2t\cos 2t)$

8. $\dfrac{1}{(s+1)(s^2+2s+2)}$ **Ans.** $e^{-t}(1-\cos t)$

9. $\dfrac{1}{(s-2)(s^2+1)}$ **Ans.** $\dfrac{1}{5}e^{2t}-\dfrac{1}{5}\cos t-\dfrac{2}{5}\sin t$

10. $\dfrac{s^2-6s+7}{(s^2-4s+5)^2}$ **Ans.** $t\,e^{2t}\{\cos t-\sin t\}$

43.10 INVERSE LAPLACE TRANSFORM BY CONVOLUTION

$$L\left\{\int_0^t f_1(x)*f_2(t-x)\,dx\right\}=F_1(s).F_2(s) \text{ or } \boxed{\int_0^t f_1(x).f_2(t-x)\,dx=L^{-1}[F_1(s).F_2(s)]}$$

Example 34. *Use convolution theorem to evaluate:*

$$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$$ (*U.P., II Semester, 2010*)

Solution. $$\frac{s}{(s^2+4)^2}=\frac{1}{s^2+4}\cdot\frac{s}{s^2+4}$$

Let $$F_1(s)=\frac{1}{s^2+4} \text{ and } F_2(s)=\frac{s}{s^2+4}$$

and $$L^{-1}[F_1(s)]=L^{-1}\left(\frac{1}{s^2+4}\right)=\frac{1}{2}\sin 2t$$

and $$L^{-1}[F_2(s)]=L^{-1}\left(\frac{s}{s^2+4}\right)=\cos 2t$$

According to Convolution Theorem

$$L^{-1}[F_1(s).F_2(s)]=\int_0^t f_1(x).f_2(t-x)dx=\int_0^t\frac{1}{2}\sin 2x\cos 2(t-x)dx$$

$$=\frac{1}{4}\int_0^t[\sin(2x+2t-2x)+\sin(2x-2t+2x)]\,dx=\frac{1}{4}\int_0^t[\sin 2t+\sin(4x-2t)]dx$$

$$=\frac{1}{4}\left[x\sin 2t-\frac{1}{4}\cos(4x-2t)\right]_0^t=\frac{1}{4}\left[t\sin 2t-\frac{1}{4}\cos(4t-2t)+\frac{1}{4}\cos(-2t)\right]$$

$$=\frac{1}{4}\left[t\sin 2t-\frac{1}{4}\cos 2t+\frac{1}{4}\cos 2t\right]=\frac{1}{4}\sin 2t$$ **Ans.**

Example 35. *Use convolution theorem to find the inverse of the function* $\dfrac{1}{(s^2+a^2)^2}$.

Solution. We know that (*U.P., II Semester, 2009*)

$$L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at$$

Hence by convolution theorem

$$L^{-1}\frac{1}{(s^2+a^2)}\frac{1}{(s^2+a^2)}=\int_0^t\frac{1}{a}\sin ax\frac{1}{a}\sin a(t-x)\,dx$$

$$= \frac{1}{a^2}\int_0^t \frac{1}{2}[\cos(ax - at + ax) - \cos(ax + at - ax)]\,dx \quad \left\{\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]\right\}$$

$$= \frac{1}{2a^2}\int_0^t [\cos(2ax - at) - \cos a]\,dx = \frac{1}{2a^2}\left[\frac{1}{2a}\sin(2ax - at) - x\cos at\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin(2at - at) - t\cos at - \frac{1}{2a}\sin(-at)\right] = \frac{1}{2a^2}\left[\frac{1}{2a}\sin at - t\cos at + \frac{1}{2a}\sin at\right]$$

$$= \frac{1}{2a^2}\left[\frac{2}{2a}\sin at - t\cos at\right] = \frac{1}{2a^3}[\sin at - at\cos at]$$ **Ans.**

Example 36. *State convolution theorem and hence find*

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$$ (*Uttarakhand, II Semester, June 2007*)

Solution. Convolution Theorem (See Art 43.10 on page 1185).

Let $L\{f_1(t)\} = F_1(s)$ and Let $L\{f_2(t)\} = F_2(s)$

$$F_1(s) = \frac{1}{(s+2)^2} \quad \text{and} \quad F_2(s) = \frac{1}{s-2}$$

$$f_1(t) = L^{-1}\left[\frac{1}{(s+2)^2}\right] = t\,e^{-2t}$$

$$f_2(t) = L^{-1}\left[\frac{1}{(s-2)}\right] = e^{2t}$$

According to Convolution Theorem

$$L^{-1}[F_1(s).F_2(s)] = \int_0^t f_1(x) f_2(t-x)\,dx$$

$$L^{-1}\left[\frac{1}{(s+2)^2(s-2)}\right] = \int_0^t x\,e^{-2x}\cdot e^{2(t-x)}\,dx = \int_0^t x e^{2t-4x}\,dx$$

$$= \left[x\frac{e^{2t-4x}}{-4} - \int 1.\frac{e^{(2t-4x)}}{-4}\,dx\right]_0^t = \left[-\frac{x}{4}e^{2t-4x} + \frac{1}{4}\left\{\frac{e^{(2t-4x)}}{-4}\right\}\right]_0^t$$

$$= \frac{-t}{4}e^{2t-4t} - \frac{1}{16}e^{2t-4t} + \frac{1}{16}e^{2t} = \frac{-t}{4}e^{-2t} - \frac{1}{16}e^{-2t} + \frac{1}{16}e^{2t}$$

$$= \frac{e^{2t}}{16} - \frac{1}{16}e^{-2t}[4t+1]$$ **Ans.**

Example 37. *Using the Convolution Theorem find*

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}, \quad a \neq b.$$

(*M.D.U., 2009, U.P. II Semester Summer 2006, 2004*)

Solution. We have, $L(\cos at) = \dfrac{s}{s^2+a^2}$ and $L(\cos bt) = \dfrac{s}{s^2+b^2}$

Hence, by the convolution theorem

$$L\left\{\int_0^t \cos ax \cos b(t-x)\,dx\right\} = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Therefore,

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos ax \cos b(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t \{\cos(ax+bt-bx)+\cos(ax-bt+bx)\}\,dx$$

$$= \frac{1}{2}\int_0^t \cos\left[(a-b)x+bt\right]dx + \frac{1}{2}\int_0^t \cos\left[(a+b)x-bt\right]dx$$

$$= \left[\frac{\sin[(a-b)x+bt]}{2(a-b)}\right]_0^t + \left[\frac{\sin[(a+b)x-bt]}{2(a+b)}\right]_0^t = \frac{\sin at-\sin bt}{2(a-b)} + \frac{\sin at+\sin bt}{2(a+b)}$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$ **Ans.**

Example 38. *Evaluate* $L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$ (*U.P., II Semester, Summer* 2002)

Solution. We know that $L^{-1}\frac{s}{s^2+1} = \cos x$ and $L^{-1}\frac{2}{s^2+2^2} = \sin 2x$

$$L^{-1}\left(\frac{s}{(s^2+1)(s^2+4)}\right) = \frac{1}{2}L^{-1}\left[\left(\frac{s}{s^2+1}\right)\left(\frac{2}{s^2+4}\right)\right]$$

$$= \frac{1}{2}\int_0^t \sin 2x\cos(t-x)\,dx$$ [By Convolution Th.]

$$= \int_0^t \sin x\cos x\,\{\cos t\cos x+\sin t\sin x\}\,dx = \int_0^t\left[\sin x\cos^2 x\cos t+\sin^2 x\cos x\sin t\right]dx$$

$$= \left[-\frac{\cos^3 x}{3}\cos t+\frac{\sin^3 x}{3}\sin t\right]_0^t = -\frac{\cos^4 t}{3}+\frac{\sin^4 t}{3}+\frac{\cos t}{3} = \frac{1}{3}\left[\sin^4 t-\cos^4 t\right]+\frac{\cos t}{3}$$

$$= \frac{1}{3}\left(\sin^2 t+\cos^2 t\right)\left(\sin^2 t-\cos^2 t\right)+\frac{\cos t}{3} = \frac{1}{3}(\sin^2 t-\cos^2 t)+\frac{\cos t}{3} = -\frac{1}{3}\cos 2t+\frac{\cos t}{3}$$

$$= \frac{1}{3}(\cos t-\cos 2t)$$ **Ans.**

Example 39. *Obtain* $L^{-1}\frac{1}{s(s^2+a^2)}$.

Solution. $L^{-1}\frac{1}{s} = 1$ and $L^{-1}\frac{1}{s^2+a^2} = \frac{\sin at}{a}$.

$$L^{-1}\{F_1(s)\,.\,F_2(s)\} = \int_0^t f_1(t)\,f_2(t-x)\,dx$$ (Convolution Theorem)

Hence by the Convolution Theorem

$$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \frac{\sin a(t-x)}{a}\,dx = \left[\frac{-\cos(at-ax)}{-a^2}\right]_0^t$$

$$= \frac{1}{a^2}\left[1-\cos at\right]$$ **Ans.**

Example 40. *Using Convolution Theorem, prove that*

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \frac{t^2}{2}+\cos t-1$$ (*U.P., II Semester, Summer* 2005)

Solution. We know that,

$$L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2!}$$

$$L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

Using Convolution Theorem,

$$L^{-1}\left\{\frac{1}{s^3\left(s^2+1\right)}\right\} = \int_0^t \frac{(t-x)^2}{2!}\sin x\,dx$$

$$= \frac{1}{2}\int_0^t \left(t^2+x^2-2tx\right)\sin x\,dx = \frac{1}{2}\left[\left(t^2+x^2-2tx\right)(-\cos x) - \int (2x-2t)(-\cos x)\,dx\right]_0^t$$

$$= \frac{1}{2}\left[(t^2+x^2-2tx)(-\cos x) + 2\int (x-t)\cos x\,dx\right]_0^t$$

$$= \frac{1}{2}\left[(t^2+x^2-2tx)(-\cos x) + 2(x-t)\sin x + 2\cos x\right]_0^t$$

$$= \frac{1}{2}\left[(t^2+t^2-2t^2)(-\cos t) + 0 + 2\cos t + t^2\cos 0 - 2\cos 0\right]$$

$$= \frac{1}{2}[2\cos t + t^2 - 2] = \cos t + \frac{t^2}{2} - 1 = \frac{t^2}{2} + \cos t - 1 \qquad \textbf{Ans.}$$

EXERCISE 43.8

Obtain the Inverse Laplace Transform by convolution:

1. $\dfrac{s^2}{\left(s^2+a^2\right)^2}$ **Ans.** $\dfrac{1}{2}t\cos at + \dfrac{1}{2a}\sin at$

2. $\dfrac{1}{\left(s^2+1\right)^3}$ **Ans.** $\dfrac{1}{8}\left[\left(3-t^2\right)\sin t - 3t\cos t\right]$

3. $\dfrac{s}{\left(s^2+a^2\right)^2}$ **Ans.** $\dfrac{t\sin at}{2a}$

4. $\dfrac{1}{s^2\left(s^2-a^2\right)}$ **Ans.** $\dfrac{1}{a^3}[-at + \sinh at]$

5. $\dfrac{1}{(s+1)\left(s^2+1\right)}$ **Ans.** $\dfrac{1}{2}\left(\cos t - \sin t - e^{-t}\right)$

43.11 HEAVISIDE INVERSE FORMULA OF $\dfrac{F(s)}{G(s)}$

If $F(s)$ and G (s) be two polynomials in S. The degree of F(s) is less than that of G (s).
Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be n roots of the equation $G(s) = 0$

Inverse Laplace formula of $\dfrac{F(s)}{G(s)}$ is given by

$$L^{-1}\left\{\frac{F(s)}{G(s)}\right\} = \sum_{i=1}^{n}\frac{F(\alpha_i)}{G'(\alpha_i)}e^{\alpha_i t}$$

Example 41. *Find* $L^{-1}\left\{\dfrac{2s^2+5s-4}{s^3+s^2-2s}\right\}$.

Solution. Let $F(s) = 2s^2 + 5s - 4$

and $G(s) = s^3 + s^2 - 2s = s(s^2+s-2) = s(s+2)(s-1)$

$G'(s) = 3s^2 + 2s - 2$

$G(s) = 0$ has three roots, 0, 1, – 2

$\Rightarrow \qquad \alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = -2$

By Heaviside Inverse formula

$$L^{-1}\left\{\frac{F(s)}{G(s)}\right\} = \sum_{i=1}^{n}\frac{F(\alpha_i)}{G'(\alpha_i)}e^{\alpha_i t}$$

$$= \left\{ \frac{F(\alpha_1)}{G'(\alpha_1)} \right\} e^{t\alpha_1} + \frac{F(\alpha_2)}{G'(\alpha_2)} e^{t\alpha_2} + \frac{F(\alpha_3)}{G'(\alpha_3)} e^{t\alpha_3} = \frac{F(0)}{G'(0)} e^0 + \frac{F(1)}{G'(1)} e^t + \frac{F(-2)}{G'(-2)} e^{-2t}$$

$$= \frac{-4}{-2} e^0 + \frac{3}{3} e^t + \frac{(-6)}{(6)} e^{-2t} = 2 + e^t - e^{-2t} \qquad \textbf{Ans.}$$

Example 42. *Find* $L^{-1}\left[\dfrac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}\right]$ (*U.P. II Semester,* 2004)

Solution. Let
$F(s) = 2s^2 - 6s + 5$
$G(s) = s^3 - 6s^2 + 11s - 6 = (s - 1)(s - 2)(s - 3)$
$G(s) = 0$ has three roots, 1, 2, 3.
$\Rightarrow \quad \alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 3$
$G'(s) = 3s^2 - 12s + 11$

By Heaviside Inverse formula, we have $L^{-1}\left\{\dfrac{F(s)}{G(s)}\right\} = \displaystyle\sum_{i=1}^{n} \frac{F(\alpha_i)}{G'(\alpha_i)} e^{t\alpha_i}$

$$L^{-1}\left\{\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)} e^{t\alpha_1} + \frac{F(\alpha_2)}{G'(\alpha_2)} e^{t\alpha_2} + \frac{F(\alpha_3)}{G'(\alpha_3)} e^{t\alpha_3}$$

$$= \frac{F(1)}{G'(1)} e^t + \frac{F(2)}{G'(2)} e^{2t} + \frac{F(3)}{G'(3)} e^{3t} = \frac{(1)}{(2)} e^t + \frac{(1)}{(-1)} e^{2t} + \frac{(5)}{(2)} e^{3t} = \frac{1}{2} e^t - e^{2t} + \frac{5}{2} e^{3t} \quad \textbf{Ans.}$$

EXERCISE 43.9

Using Heaviside expansion formula, find the Inverse Laplace Transform of the following :

1. $\dfrac{s-1}{s^2+3s+2}$ **Ans.** $-2e^{-t} + 3e^{-2t}$

2. $\dfrac{s}{(s-1)(s-2)(s-3)}$ **Ans.** $\dfrac{1}{2}e^t - 2e^{2t} + \dfrac{3}{2}e^{3t}$

3. $\dfrac{2s+3}{(s-2)(s-3)(s-4)}$ **Ans.** $\dfrac{7}{2}e^{2t} - 9e^{3t} + \dfrac{11}{2}e^{4t}$

4. $\dfrac{11s^2 - 2s + 5}{2s^3 - 3s^2 - 3s + 2}$ **Ans.** $2e^{-2t} + 5 \cdot e^{2t} - \dfrac{3}{2}e^{\frac{t}{2}}$

43.12 SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace Transform method, without finding the general solution and the arbitrary constants. The method will be clear from the following examples:

Example 43. *Solve the following equation by Laplace transform*

$$y''' - 2y'' + 5y' = 0\ ;\ y = 0, \quad y' = 1 \text{ at } t = 0 \text{ and } y = 1 \text{ at } t = \frac{\pi}{8}.$$

(*Q. Bank U.P., II Semester 2001*)

Solution. Here, we have

$$y''' - 2y'' + 5y' = 0 \qquad ...(1)$$

Taking Laplace transform on both sides of (1), we get

$$L(y''') - 2L(y'') + 5L(y') = L(0)$$

$$\Rightarrow s^3\bar{y} - s^2 y(0) - sy'(0) - y''(0) - 2[s^2\bar{y} - sy(0) - y'(0)] + 5[s\bar{y} - y(0)] = 0$$

$$\Rightarrow (s^3 - 2s^2 + 5s)\,\bar{y} - s - k + 2 = 0 \qquad [\text{Let } y''(0) = k \text{ (say)}]$$

$$\Rightarrow \quad \bar{y} = \frac{(k-2) + s}{s(s^2 - 2s + 5)}$$

$$= \left(\frac{k-2}{5}\right)\left(\frac{1}{s} - \frac{s-2}{s^2 - 2s + 5}\right) + \frac{1}{s^2 - 2s + 5}$$

$$= \left(\frac{k-2}{5}\right)\frac{1}{s} - \left(\frac{k-2}{5}\right)\left\{\frac{(s-1)-1}{(s-1)^2+4}\right\} + \frac{1}{(s-1)^2+4}$$

$$\Rightarrow \quad \bar{y} = \left(\frac{k-2}{5}\right)\frac{1}{s} - \left(\frac{k-2}{5}\right)\left\{\frac{(s-1)}{(s-1)^2+4}\right\} + \left(\frac{k+3}{10}\right)\cdot\left\{\frac{2}{(s-1)^2+4}\right\} \quad ...(3)$$

Taking Inverse Laplace transform on both sides of (3), we get

$$y = \left(\frac{k-2}{5}\right) - \left(\frac{k-2}{5}\right)e^t \cos 2t + \left(\frac{k+3}{10}\right)e^t \sin 2t \quad ...(4)$$

Puttting $y\left(\frac{\pi}{8}\right) = 1$, we get $1 = \left(\frac{k-2}{5}\right) - \left(\frac{k-2}{5}\right)e^{\pi/8}\cdot\frac{1}{\sqrt{2}} + \left(\frac{k+3}{10}\right)e^{\pi/8}\cdot\frac{1}{\sqrt{2}}$ ...(5)

$\Rightarrow \quad k = 7$

On putting the value of k in (4), we get (on simplification)

Hence required solution is

$$y = 1 + e^t(\sin 2t - \cos 2t) \quad \textbf{Ans.}$$

Example 44. *Using Laplace transforms, find the solution of the initial value problem*

$$y'' - 4y' + 4y = 64 \sin 2t$$
$$y(0) = 0, \; y'(0) = 1.$$

Solution. Here, we have $y'' - 4y' + 4y = 64 \sin 2t$ ... (1)

$$y(0) = 0, \; y'(0) = 1.$$

Taking Laplace transform of both sides of (1), we have

$$[s^2\bar{y} - sy(0) - y'(0)] - 4[s\bar{y} - y(0)] + 4\bar{y} = \frac{64\times 2}{s^2+4} \quad ...(2)$$

On putting the values of $y(0)$ and y' (0) in (2), we get

$$s^2\bar{y} - 1 - 4s\bar{y} + 4\bar{y} = \frac{128}{s^2+4}$$

$$\Rightarrow \quad (s^2 - 4s + 4)\bar{y} = 1 + \frac{128}{s^2+4}, \quad \Rightarrow \quad (s-2)^2\bar{y} = 1 + \frac{128}{s^2+4}$$

$$\Rightarrow \quad \bar{y} = \frac{1}{(s-2)^2} + \frac{128}{(s-2)^2(s^2+4)} = \frac{1}{(s-2)^2} - \frac{8}{s-2} + \frac{16}{(s-2)^2} + \frac{8s}{s^2+4}$$

$$y = L^{-1}\left[-\frac{8}{s-2} + \frac{17}{(s-2)^2} + \frac{8s}{s^2+4}\right]$$

$$y = -8e^{2t} + 17t\,e^{2t} + 8\cos 2t \quad \textbf{Ans.}$$

Example 45. *Using Laplace transforms, find the solution of the initial value problem*

$$y'' + 9y = 6\cos 3t$$
$$y(0) = 2, \; y'(0) = 0$$

(*U. P. II Semester Summer* 2006)

Solution. We have, $y'' + 9y = 6\cos 3t$ ... (1)

$$y(0) = 2, \; y'(0) = 0$$

Taking Laplace transform of (1), we get

$$[s^2\bar{y} - sy(0) - y'(0)] + 9\bar{y} = 6\frac{s}{s^2+9} \quad ...(2)$$

Putting the values of $y(0)$ and $y'(0)$ in (2), we have

$$s^2\bar{y} - 2s + 9\bar{y} = \frac{6s}{s^2+9}$$

$$\Rightarrow \quad (s^2+9)\bar{y} = 2s + \frac{6s}{s^2+9} \quad \Rightarrow \quad \bar{y} = \frac{2s}{s^2+9} + \frac{6s}{(s^2+9)^2}$$

$$y = L^{-1}\frac{2s}{s^2+9} + L^{-1}\frac{6s}{(s^2+9)^2} = 2\cos 3t + L^{-1}\frac{d}{ds}\left[\frac{-3}{(s^2+9)}\right]$$

$$= 2\cos 3t - t\sin 3t \quad \textbf{Ans.}$$

Example 46. *Using Laplace transformation solve the following differential equation:*

$$\frac{d^2 x}{dt^2} + 9x = \cos 2t, \quad \text{if } x(0) = 1, \quad x\left(\frac{\pi}{2}\right) = -1$$ (*U. P. II Semester, Summer* 2002)

Solution. $\dfrac{d^2 x}{dt^2} + 9x = \cos 2t$ ... (1)

Taking Laplace transform of both the sides of (1), we get

$$L\frac{d^2 x}{dt^2} + 9\,L\,x = L\cos 2t$$

$$\Rightarrow \quad s^2\,\bar{x} - sx(0) - x'(0) + 9\,\bar{x} = \frac{s}{s^2+4} \qquad \text{... (2)}$$

On putting $x(0) = 1$ in (2), we get

$$s^2\,\bar{x} - s + 9\,\bar{x} - x'(0) = \frac{s}{s^2+4}$$

$$(s^2+9)\,\bar{x} = s + \frac{s}{s^2+4} + x'(0) = \frac{s\left(s^2+4\right)+s}{s^2+4} + x'(0) = \frac{s^3+5s}{s^2+4} + x'(0)$$

$$\Rightarrow \quad \bar{x} = \frac{(s^3+5s)}{(s^2+4)(s^2+9)} + \frac{x'(0)}{s^2+9} = \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\frac{s}{s^2+9} + \frac{x'(0)}{s^2+9}$$

Taking the Inverse Laplace Transform, we get

$$x(t) = \frac{1}{5}L^{-1}\frac{s}{s^2+4} + \frac{4}{5}L^{-1}\frac{s}{s^2+9} + L^{-1}\frac{x'(0)}{s^2+9}$$

$$x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{x'(0)\sin 3t}{3} \qquad \text{... (3)}$$

On putting $x\left(\dfrac{\pi}{2}\right) = -1$ in (3), we get

$$-1 = -\frac{1}{5} + 0 - \frac{x'(0)}{3} \qquad \Rightarrow \qquad x'(0) = \frac{12}{5}$$

On putting the value of $x'(0)$ in (3), we get

$$x = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{12}{5}\frac{\sin 3t}{3} = \frac{1}{5}[\cos 2t + 4\cos 3t + 4\sin 3t]$$ **Ans.**

Example 47. *Solve, using Laplace transform method*

$$y(0) = -2,\ y'(0) = 8; \quad y'' + 4y' + 4y = 6e^{-t}$$ (*U.P., II Semester, 2007*)

Solution. Here, we have

$$y'' + 4y' + 4y = 6e^{-t} \qquad \text{...(1)}$$

Taking Laplace transform on both sides of (1), we get

$$L(y'') + 4L(y') + 4L(y) = 6L\,(e^{-t})$$

$$\Rightarrow [s^2\bar{y} - sy(0) - y'(0)] + 4[s\,\bar{y} - y(0)] + 4\bar{y} = \frac{6}{s+1} \qquad \text{... (2)} \quad [\text{Here } \bar{y} = L(y)]$$

Putting the values of $y(0) = -2$ and $y'(0) = 8$ in (2), we get

$$(s^2 + 4s + 4)\,\bar{y} + 2s - 8 + 8 = \frac{6}{s+1}$$

$$\Rightarrow \qquad (s^2 + 4s + 4)\,\bar{y} + 2s = \frac{6}{s+1}$$

$$(s+2)^2\,\bar{y} + 2s = \frac{6}{s+1}$$

$$\bar{y} = \frac{6}{(s+1)(s+2)^2} - \frac{2s}{(s+2)^2} = 6\left[\frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}\right] - \frac{2\{(s+2)-2\}}{(s+2)^2}$$

$$= \frac{6}{s+1} - \frac{6}{s+2} - \frac{6}{(s+2)^2} - \frac{2}{s+2} + \frac{4}{(s+2)^2} = \frac{6}{s+1} - \frac{8}{s+2} - \frac{2}{(s+2)^2} \qquad ...(3)$$

Taking inverse Laplace transform on both sides of (3), we get

$$y = 6e^{-t} - 8e^{-2t} - 2t\, e^{-2t}$$ **Ans.**

Example 48. *Solve the following differential equation using Laplace transform*

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2 e^t$$

where $y(0) = 1, \left(\frac{dy}{dt}\right)_{t=0} = 0, \left(\frac{d^2y}{dt^2}\right)_{t=0} = -2$ *(U. P., II Semester, (SUM) 2008)*

Solution. Here we have equation

$$y''' - 3y'' + 3y' - y = t^2 e^t \qquad ... (1)$$

Taking Laplace transform on both side of equation (1), we get

$$L(y''') - 3L(y'') + 3L(y') - L(y) = L(t^2 e^t)$$

$$\Rightarrow [s^3\bar{y} - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2\bar{y} - sy(0) - y'(0)] + 3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

... (2) [where $\bar{y} = L(y)$]

Putting the values of $y(0)$, $y'(0)$ and y'' at $x = 0$ in (2), we get

$$\Rightarrow (s^3\bar{y} - s^2 + 2) - 3(s^2\bar{y} - s) + 3(s\bar{y} - 1) - \bar{y} = \frac{2}{(s-1)^3}$$

$$\Rightarrow (s^3 - 3s^2 + 3s - 1)\bar{y} - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$

$$\Rightarrow (s-1)^3\bar{y} = s^2 - 3s + 1 + \frac{2}{(s-1)^3}$$

$$\Rightarrow \bar{y} = \frac{(s-1)^2}{(s-1)^3} - \frac{s}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{1}{s-1} - \frac{(s-1)+1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6} \qquad ...(3)$$

Taking inverse Laplace transform on both sides of equation (3), we get

$$y = e^t - te^t - \frac{t^2}{2}e^t + \frac{t^5 e^t}{60} = \left(1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right)e^t$$ **Ans.**

Example 49. *Solve* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x}\sin x$ *where* $y(0) = 0, y'(0) = 1.$

(U.P., II Semester, 2004)

Solution. Here, we have $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x}\sin x$

Taking the Laplace Transform of both the sides, we get

$$[s^2\bar{y} - sy(0) - \bar{y}(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = L(e^{-x}\sin x)$$

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \frac{1}{(s+1)^2 + 1} \qquad ... (1)$$

On substituting the values of y (0) and y' (0) in (1), we get

$$(s^2\,\bar{y}-1)+2(s\,\bar{y})+5\bar{y}=\frac{1}{s^2+2s+2}$$

$$(s^2+2s+5)\,\bar{y}=1+\frac{1}{s^2+2s+2}=\frac{s^2+2s+3}{s^2+2s+2}$$

$$\bar{y}=\frac{s^2+2s+3}{\left(s^2+2s+5\right)\left(s^2+2s+2\right)}$$

On resolving the R.H.S. into partial fractions, we get

$$\bar{y}=\frac{2}{3}\,\frac{1}{s^2+2s+5}+\frac{1}{3}\,\frac{1}{s^2+2s+2}$$

On inversion, we obtain

$$y=\frac{2}{3}L^{-1}\,\frac{1}{s^2+2s+5}+\frac{1}{3}L^{-1}\,\frac{1}{s^2+2s+2}$$

$$\Rightarrow \quad y=\frac{1}{3}L^{-1}\,\frac{2}{(s+1)^2+(2)^2}+\frac{1}{3}L^{-1}\,\frac{1}{(s+1)^2+(1)^2}$$

$$\Rightarrow \quad y=\frac{1}{3}e^{-x}\sin 2x+\frac{1}{3}e^{-x}\sin x \quad \Rightarrow \quad y=\frac{1}{3}e^{-x}(\sin x+\sin 2x)$$ **Ans.**

Example 50. *Solve the equation by the transform method:*

$$\frac{dy}{dt}+2y+\int_0^t y\,dt=\sin t, \qquad y(0)=1$$

(*R. G. P. V. Bhopal, June* 2003)

Solution. Taking Laplace transform of the given equation, we get

$$[s\,\bar{y}-y(0)]+2\bar{y}+\frac{\bar{y}}{s}=\frac{1}{s^2+1} \qquad \ldots(1) \qquad \left[\because L\left\{\int_0^t y\,dt=\frac{\bar{y}}{s}\right\}\right]$$

Putting the values of y (0) = 1 in (1), we get

$$[s\,\bar{y}-1]+2\,\bar{y}+\frac{\bar{y}}{s}=\frac{1}{s^2+1}$$

$$\bar{y}\left(s+2+\frac{1}{s}\right)=1+\frac{1}{s^2+1} \qquad [\because y(0)=1]$$

$$\frac{1}{s}\,\bar{y}(s^2+2s+1)=\frac{s^2+1+1}{s^2+1}$$

$$\frac{1}{s}\,\bar{y}\,(s+1)^2=\frac{s^2+2}{s^2+1} \quad \Rightarrow \quad \bar{y}\,(s+1)^2=\frac{s^3+2s}{s^2+1}$$

$$\Rightarrow \quad \bar{y}=\frac{s^3+2s}{(s+1)^2\,(s^2+1)}=\frac{1}{s+1}-\frac{3}{2\,(s+1)^2}+\frac{1}{2\,(s^2+1)}$$ [By partial fractions]

Taking Inverse Laplace Transform, we have

$$y=e^{-t}-\frac{3}{2}t\,e^{-t}+\frac{1}{2}\sin t$$ **Ans.**

Example 51. *Solve the following differential equation using Laplace transform:*

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0, \qquad \textit{given } y(0)=2,\ y'(0)=0.$$

(*Uttarakhand, II Semester, June* 2007)

Solution. Taking Laplace transform on both sides of the given equation, we have

$$L[xy'']+L[y']+L[xy]=L(0)$$

$$\Rightarrow \quad -\frac{d}{ds}L(y'')+L(y')-\frac{d}{ds}L(y)=0$$

$$\Rightarrow \quad -\frac{d}{ds}\{s^2\bar{y} - sy(0) - y'(0)\} + \{s\bar{y} - y(0)\} - \frac{d\bar{y}}{ds} = 0 \qquad \dots (1)$$

Putting $y(0) = 2$ and $y'(0) = 0$ in (1), we get

$$-\frac{d}{ds}\{s^2\bar{y} - 2s - 0\} + \{s\bar{y} - 2\} - \frac{d\bar{y}}{ds} = 0$$

$$\Rightarrow \quad -s^2\frac{d\bar{y}}{ds} - (2s)\bar{y} + 2 + s\bar{y} - 2 - \frac{d\bar{y}}{ds} = 0$$

$$\Rightarrow \quad (s^2+1)\frac{d\bar{y}}{ds} + s\bar{y} = 0 \qquad \dots (2)$$

Separating the variables, we have

$$\frac{d\bar{y}}{\bar{y}} + \frac{s\,ds}{s^2+1} = 0 \qquad \dots (3)$$

On integrating, we have

$$\log\bar{y} + \frac{1}{2}\log(s^2+1) = \log C$$

$$\Rightarrow \quad \log\bar{y} = \log C - \log\sqrt{s^2+1} \quad \Rightarrow \quad \log\bar{y} = \log\frac{C}{\sqrt{s^2+1}}$$

$$\Rightarrow \quad \bar{y} = \frac{C}{\sqrt{s^2+1}} \qquad \dots (4)$$

Taking Inverse Laplace Transform, we get

$$y = L^{-1}\left[\frac{C}{\sqrt{s^2+1}}\right]$$

$$\Rightarrow \quad y = C\,J_0(x) \qquad \dots (5)\ [\text{ See Art. } 42.24]$$

At $x = 0$ $\quad y(0) = C\,J_0(0)$ ... (6)

Putting $y(0) = 2$ and $J_0(0) = 1$ in (6), we get

$$2 = C(1) \quad \Rightarrow \quad C = 2$$

On putting the value of C in (5), we get

$$y = 2\,J_0(x)$$ **Ans.**

Example 52. *Using Laplace transforms, find the solution of the initial value problem*

$$y'' + 9y = 9\,u(t-3), \qquad y(0) = y'(0) = 0$$

where $u(t-3)$ is the unit step functions.

Solution. We have, $\quad y'' + 9y = 9\,u(t-3)$ ... (1)

Taking Laplace transform of (1), we have

$$s^2\bar{y} - sy(0) - y'(0) + 9\bar{y} = 9\frac{e^{-3s}}{s} \qquad \dots (2)$$

Putting the values of $y(0) = 0$ and $y'(0) = 0$ in (2), we get

$$s^2\bar{y} + 9\bar{y} = 9\frac{e^{-3s}}{s}$$

$$(s^2+9)\bar{y} = 9\frac{e^{-3s}}{s}$$

$$\bar{y} = \frac{9e^{-3s}}{s(s^2+9)} \quad \Rightarrow \quad y = L^{-1}\frac{9e^{-3s}}{s(s^2+9)} \qquad \dots (3)$$

We know that $\quad L^{-1}\dfrac{3}{s^2+9} = \sin 3t$

$$\Rightarrow \qquad 3\,L^{-1}\frac{3}{s\,(s^2+9)} = 3\int_0^t \sin 3t\,dt = -[\cos 3t]_0^t = 1-\cos 3t \qquad \text{... (4)}$$

[Using second shifting theorem]

Using (4), we get the inverse of (3)

$$y = [\,1-\cos 3\,(t-3)\,]\,u\,(t-3)$$ **Ans.**

Example 53. *Solve, by the method of Laplace transform, the differential equation*

$$(D^2+n^2)\,x = a\sin(nt+\alpha),$$

$$x = Dx = 0 \text{ at } t = 0.$$ *(U.P., II Semester, Summer, 2010, 2002)*

Solution. Taking Laplace transform of the given differential equation, we get

$$[\,s^2\bar{x} - sx(0) - x'(0)\,] + n^2\bar{x} = a\,L\sin(nt+\alpha)$$

$$= a\,L\,[\,\sin nt\cos\alpha + \cos nt\sin\alpha\,] = a\left[\frac{n}{s^2+n^2}\cos\alpha + \frac{s}{s^2+n^2}\sin\alpha\right]$$

$$= \frac{a\,[n\cos\alpha + s\sin\alpha\,]}{s^2+n^2} \qquad \text{... (1)}$$

Putting $x(0) = x'(0) = 0$ in (1), we get

$$s^2\bar{x} + n^2\bar{x} = a\left[\frac{n\cos\alpha + s\sin\alpha}{s^2+n^2}\right] \quad\Rightarrow\quad (s^2+n^2)\,\bar{x} = \frac{a\,[\,n\cos\alpha + s\sin\alpha\,]}{s^2+n^2}$$

$$\Rightarrow \qquad \bar{x} = \frac{a\,n\cos\alpha + as\sin\alpha}{(s^2+n^2)^2}$$

$$\Rightarrow \qquad \bar{x} = a\cos\alpha\,.\,\frac{n}{(s^2+n^2)^2} + a\sin\alpha\,.\,\frac{s}{(s^2+n^2)^2} \qquad \text{... (2)}$$

Taking the Inverse Laplace Transform of (2), we get

$$x = a\cos\alpha\,L^{-1}\left[\frac{n}{(s^2+n^2)^2}\right] + (a\sin\alpha)\,L^{-1}\left[\frac{s}{(s^2+n^2)^2}\right] \qquad \text{... (3)}$$

Let us find out the inverse of the term on R.H.S.

$$\text{But} \quad L^{-1}\left\{\frac{2s}{(s^2+n^2)^2}\right\} = L^{-1}\frac{d}{ds}\frac{1}{s^2+n^2} = \frac{t}{n}\sin nt \qquad \left[-\frac{d}{ds}[\,F(s)\,] = t\,f(t)\right]$$

$$\Rightarrow \qquad L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} = \frac{t}{2n}\sin nt \qquad \text{... (4)}$$

$$\text{Again} \quad L^{-1}\frac{1}{s}\left\{\frac{s}{(s^2+n^2)^2}\right\} = \frac{1}{2n}\int_0^t t\sin nt\,dt \qquad \left[\frac{F(s)}{s} = \int_0^t f(t)\,dt\right]$$

$$L^{-1}\left[\frac{1}{(s^2+n^2)^2}\right] = \frac{1}{2n}\left[t\left(\frac{-\cos nt}{n}\right) + \frac{1}{n^2}\sin nt\right]_0^t$$

$$L^{-1}\left[\frac{n}{(s^2+n^2)^2}\right] = \frac{1}{2n^2}[\,-nt\cos nt + \sin nt\,] \qquad \text{... (5)}$$

Putting the values of $L^{-1}\left[\dfrac{s}{(s^2+n^2)^2}\right]$ from (4) and $L^{-1}\left[\dfrac{n}{(s^2+n^2)^2}\right]$ from (5) in (3), we get

$$x = (a\cos\alpha)\,\frac{1}{2n^2}[\,\sin nt - nt\cos nt\,] + (a\sin\alpha)\,\frac{t}{2n}\sin nt$$

$$= \frac{a}{2n^2}[\cos\alpha \sin nt - nt\cos\alpha\cos nt + nt\sin\alpha\sin nt]$$

$$= \frac{a}{2n^2}[\cos\alpha\sin nt - nt(\cos nt\cos\alpha - \sin nt\sin\alpha)] = \frac{a}{2n^2}[\sin nt\cos\alpha - nt\cos(nt+\alpha)]$$

Which is the required solution. **Ans.**

Example 54. *Solve* $t\,y'' + 2\,y' + t\,y = \cos t$, *if* $y(0) = 1,\ y'(0) = 0$.

Solution. We have, $t\,y'' + 2\,y' + t\,y = \cos t$

$$\Rightarrow \quad L\{t\,y''\} + L\{2y'\} + L\{t\,y\} = L\{\cos t\}$$

$$\Rightarrow \quad -\frac{d}{ds}L\{y''\} + 2L\{y'\} - \frac{d}{ds}L\{y\} = L\{\cos t\}$$

$$\Rightarrow \quad -\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] - \frac{d}{ds}\bar{y} = \frac{s}{s^2+1}$$

$$\Rightarrow \quad -\frac{d}{ds}[s^2\bar{y} - s] + 2s\bar{y} - 2 - \frac{d}{ds}\bar{y} = \frac{s}{s^2+1}$$

$$\Rightarrow \quad -2s\bar{y} - s^2\frac{d\bar{y}}{ds} + 1 + 2s\bar{y} - 2 - \frac{d\bar{y}}{ds} = \frac{s}{s^2+1}$$

$$\Rightarrow \quad (s^2+1)\frac{d\bar{y}}{ds} + 1 = \frac{-s}{s^2+1} \qquad \Rightarrow \qquad \frac{d\bar{y}}{ds} = \frac{-s}{(s^2+1)^2} - \frac{1}{s^2+1}$$

Taking Inverse Laplace Transform, we get

$$L^{-1}\frac{d}{ds}(\bar{y}) = L^{-1}\left[\frac{-s}{(s^2+1)^2} - \frac{1}{s^2+1}\right] \qquad \left[L^{-1}\frac{d}{ds}F(s) = (-1)^1 t^1 f(t)\right]$$

$$\Rightarrow \quad (-1)^1 t^1\, y = -\frac{1}{2}L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\} = -\frac{1}{2}t\sin t - \sin t$$

$$\Rightarrow \quad y = \frac{1}{2}\left(1 + \frac{2}{t}\right)\sin t$$ **Ans.**

Example 55. *Solve* $[t\,D^2 + (1-2t)\,D - 2]\,y = 0$, *where* $y(0) = 1,\ y'(0) = 2$.

(*M.D.U., 2010, U. P. II Semester, June* 2002)

Solution. Here, $t\,D^2 y + (1-2t)\,Dy - 2y = 0 \quad \Rightarrow \quad ty'' + y' - 2\,t\,y' - 2y = 0$

Taking Laplace transform of given differential equation, we get

$$L(t\,y'') + L(y') - 2\,L(t\,y') - 2\,L(y) = 0$$

$$\Rightarrow \quad -\frac{d}{ds}L\{y''\} + L\{y'\} + 2\frac{d}{ds}L(y') - 2L(y) = 0$$

$$\Rightarrow \quad -\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + [s\bar{y} - y(0)] + 2\frac{d}{ds}[s\bar{y} - y(0)] - 2\bar{y} = 0$$

Putting the values of $y(0)$ and $y'(0)$, we get

$$-\frac{d}{ds}(s^2\bar{y} - s - 2) + (s\bar{y} - 1) + 2\frac{d}{ds}(s\bar{y} - 1) - 2\bar{y} = 0 \qquad [\because y(0) = 1,\ y'(0) = 2]$$

$$\Rightarrow \quad -s^2\frac{d\bar{y}}{ds} - 2s\bar{y} + 1 + s\bar{y} - 1 + 2\left(s\frac{d\bar{y}}{ds} + \bar{y}\right) - 2\bar{y} = 0$$

$$\Rightarrow \quad -(s^2 - 2s)\frac{d\bar{y}}{ds} - s\bar{y} = 0$$

$$\Rightarrow \quad -\frac{d\bar{y}}{\bar{y}} - \frac{1}{s-2}ds = 0 \quad \textit{(Separating the variables)}$$

$$\Rightarrow \quad \int \frac{d\,\bar{y}}{\bar{y}} + \int \frac{ds}{s-2} = 0 \quad \Rightarrow \quad \log \bar{y} + \log(s-2) = \log C \quad \Rightarrow \quad \log \bar{y}\,(s-2) = \log C$$

$$\Rightarrow \quad \bar{y}\,(s-2) = C \quad \Rightarrow \quad \bar{y} = \frac{C}{s-2} \quad \Rightarrow \quad y = C\,L^{-1}\left\{\frac{1}{s-2}\right\} \quad \Rightarrow \quad y = C\,e^{2t} \qquad \ldots (1)$$

$$y\,(0) = C\,e^{0} \qquad \ldots (2)$$

Putting $y\,(0) = 1$ in (2), we get $1 = C\,e^{0}$ $\Rightarrow$ $C = 1$.

Putting $C = 1$ in (1), we get $y = e^{2t}$

This is the required solution. **Ans.**

Example 56. *Using Laplace transform, solve the following differential equation: -*

$$y'' + 2t\,y' - y = t$$

when $y\,(0) = 0$ *and* $y'\,(0) = 1$ (*U.P., II Semester, Summer* 2003)

Solution. We have, $y'' + 2t\,y' - y = t$ ... (1)

Taking Laplace transform of (1), we get

$$[\,s^2\,\bar{y} - sy\,(0) - y'(0)\,] - 2\frac{d}{ds}[\,s\,\bar{y} - y\,(0)\,] - \bar{y} = \frac{1}{s^2} \qquad \ldots (2)$$

On putting $y\,(0) = 0$ and $y'\,(0) = 1$ in (2), we get

$$(\,s^2\,\bar{y} - 1\,) - 2\frac{d}{ds}(\,s\,\bar{y} - 0\,) - \bar{y} = \frac{1}{s^2}$$

$$\Rightarrow \quad (\,s^2\,\bar{y} - 1\,) - 2\,\bar{y} - 2s\frac{d\,\bar{y}}{ds} - \bar{y} = \frac{1}{s^2} \quad \Rightarrow \quad -2s\frac{d\,\bar{y}}{ds} + (\,s^2 - 3\,)\,\bar{y} = \frac{1}{s^2} + 1 = \frac{1+s^2}{s^2}$$

$$\Rightarrow \quad \frac{d\,\bar{y}}{ds} - \frac{s^2-3}{2s}\,\bar{y} = \frac{1+s^2}{-2s^3} \quad \Rightarrow \quad \frac{d\,\bar{y}}{ds} - \left(\frac{s}{2} - \frac{3}{2s}\right)\bar{y} = -\frac{1}{2s^3} - \frac{1}{2s} \qquad \ldots (3)$$

Thus (3) is a linear differential equation.

$$\text{I.F.} = e^{\frac{1}{2}\int\left(\frac{3}{s} - s\right)ds} = e^{\frac{1}{2}\left(3\log s - \frac{s^2}{2}\right)} = e^{-\frac{s^2}{4}}\,.\,s^{\frac{3}{2}}$$

Solution of differential equation (3) is

$$\bar{y}\,e^{-\frac{s^2}{4}}\,.\,s^{\frac{3}{2}} = -\frac{1}{2}\int\left(\frac{1}{s^3} + \frac{1}{s}\right)s^{\frac{3}{2}}\,.\,e^{-\frac{s^2}{4}}\,ds = -\frac{1}{2}\int\left(\sqrt{s} + \frac{1}{s^{\frac{3}{2}}}\right)e^{-\frac{s^2}{4}}\,ds$$

Put $s^2 = 4\,z$ $\Rightarrow$ $s = 2\sqrt{z}$ so that $ds = \dfrac{dz}{\sqrt{z}}$

$$\bar{y}\,s^{\frac{3}{2}}\,.\,e^{-\frac{s^2}{4}} = -\frac{1}{2}\int\left(\sqrt{2}\,z^{\frac{1}{4}} + \frac{1}{2\sqrt{2}}\,z^{-\frac{3}{4}}\right)e^{-z}\,\frac{dz}{\sqrt{z}}$$

$$= -\frac{1}{\sqrt{2}}\int\left(z^{-\frac{1}{4}} + \frac{1}{4}z^{-\frac{5}{4}}\right)e^{-z}\,dz = -\frac{1}{\sqrt{2}}\int z^{-\frac{1}{4}}\,e^{-z}\,dz - \frac{1}{4\sqrt{2}}\int z^{-\frac{5}{4}}\,e^{-z}\,dz$$

$$= -\frac{1}{\sqrt{2}}\left[z^{-\frac{1}{4}}\,\frac{e^{-z}}{-1} + \int\left(-\frac{1}{4}\right)z^{-\frac{5}{4}}\,e^{-z}\,dz\right] - \frac{1}{4\sqrt{2}}\int z^{-\frac{5}{4}}\,e^{-z}\,dz + C$$

$$= \frac{1}{\sqrt{2}}\,e^{-z}\,z^{-\frac{1}{4}} + C = \frac{1}{\sqrt{2}}\,e^{-\frac{s^2}{4}}\left(\frac{s^2}{4}\right)^{-\frac{1}{4}} + C = \frac{1}{\sqrt{s}}\,e^{-\frac{s^2}{4}} + C \Rightarrow \bar{y} = \frac{1}{s^2} + C$$

(Particular case)

$$\Rightarrow \quad \bar{y} = \frac{1}{s^2} + c \quad \Rightarrow \quad y = L^{-1}\left(\frac{1}{s^2} + c\right) = t + c$$

$y = t$ (c must vanish if $\bar{y}$ is a transform since $\bar{y} \to 0$ as $s \to \infty$) **Ans.**

Example 57. *A particle moves in a line so that its displacement x from a fixed point O at any time t, is given by*

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+5x = 80 \sin 5t$$

Using Laplace transform, find its displacement at any time t if initially particle is at rest at x = 0 *(U.P., II Semester 2009)*

Solution. Here, we have

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+5x = 80 \sin 5t, \quad x(0)=0,\ x'(0)=0 \qquad ...(1)$$

Taking Laplace transform of both sides of (1), we get

$$[s^2\bar{x} - sx(0) - x'(0)] + 4[s\bar{x} - x(0)] + 5\bar{x} = L[80 \sin 5t]$$

$$[s^2\bar{x} - 0 - 0] + 4s\bar{x} + 5\bar{x} = 80\left(\frac{5}{s^2+25}\right), \qquad (s^2+4s+5)\,\bar{x} = \frac{400}{s^2+25}$$

$$\bar{x} = \left(\frac{1}{s^2+4s+5}\right)\left(\frac{400}{s^2+25}\right)$$

$$\bar{x} = \frac{2s+18}{s^2+4s+5} - \frac{2s+10}{s^2+25} \qquad \text{[By Partial fraction]}$$

$$= \frac{2(s+2)+14}{(s+2)^2+1} - \frac{2s}{s^2+(5)^2} - \frac{10}{s^2+(5)^2}$$

$$= \frac{2(s+2)}{(s+2)^2+1} + \frac{14}{(s+2)^2+1} - \frac{2s}{s^2+(5)^2} - \frac{10}{s^2+(5)^2}$$

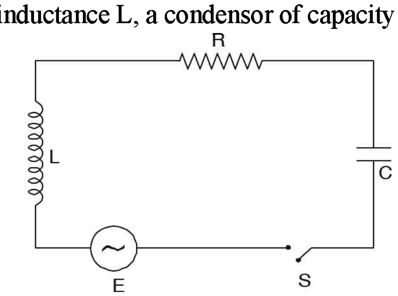
$$\Rightarrow \quad x = 2L^{-1}\frac{(s+2)}{(s+2)^2+1} + 14L^{-1}\frac{1}{(s+2)^2+1} - 2L^{-1}\frac{s}{s^2+(5)^2} - 2L^{-1}\frac{5}{s^2+(5)^2}$$

$$= 2e^{-2t}\cos t + 14e^{-2t}\sin t - 2\cos 5t - 2\sin 5t$$

$$= 2e^{-2t}(\cos t + 7\sin t) - 2(\cos 5t + \sin 5t)$$ **Ans.**

43.13 ELECTRIC CIRCUIT

Consider an electric circuit consisting of a resistance R, inductance L, a condensor of capacity C and electromotive power of voltage E in a series. A switch is also connected in the circuit. Here,



$$i = \frac{dq}{dt}$$

Voltage developed by Ri, $L\frac{di}{dt}$ and $\frac{q}{C}$

By Kirchhoff low $\boxed{L\frac{di}{dt} + Ri + \frac{q}{C} = E}$

Example 58. *A resistance R in series with inductance L is connected with e.m.f. E (t). The current i is given by*

$$L\frac{di}{dt} + Ri = E$$

If the switch is connected at t = 0 and disconnected at t = a, find the current i in terms of t.

(*U.P. II Semester, Summer* 2001)

Solution. Conditions under which current i flows are

$$E(t) = \begin{cases} E, & 0 < t < a \\ 0, & t > a \end{cases} \qquad [\, i = 0 \text{ at } t = 0]$$

Given equation is $L\dfrac{di}{dt} + Ri = E$ ... (1)

Taking Laplace transform of (1), we get

$$L[s\,\bar{i} - i(0)] + R\,\bar{i} = \int_0^\infty e^{-st}\, E\, dt$$

$$L\,s\,\bar{i} + R\,\bar{i} = \int_0^\infty e^{-st}\, E\, dt \qquad [\, i(0) = 0]$$

$$(Ls + R)\,\bar{i} = \int_0^\infty e^{-st}\,.\,E\, dt = \int_0^a e^{-st} E\, dt + \int_a^\infty e^{-st}\, E\, dt$$

$$= E\left[\frac{e^{-st}}{-s}\right]_0^a + 0 = \frac{E}{s}[1 - e^{-as}] = \frac{E}{s} - \frac{E}{s}e^{-as}$$

$$\Rightarrow \qquad \bar{i} = \frac{E}{s(Ls+R)} - \frac{Ee^{-as}}{s(Ls+R)}$$

Taking Inverse Laplace Transform, we obtain

$$i = \text{Inverse Lap.}\left[\frac{E}{s(Ls+R)}\right] - \text{Inverse Lap.}\left[\frac{E\,e^{-as}}{s(Ls+R)}\right] \qquad \ldots (2)$$

Now we have to find the value of $\text{Inverse Lap.}\left[\dfrac{E}{s(Ls+R)}\right]$

$$\text{Inverse Lap.}\left[\frac{E}{s(Ls+R)}\right] = \frac{E}{L}\,\text{Inverse Lap.}\left[\frac{1}{s\left(s + \frac{R}{L}\right)}\right]$$

$$= \frac{E}{L}\frac{L}{R}\,\text{Inverse Lap.}\left[\frac{1}{s} - \frac{1}{s + \frac{R}{L}}\right] = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right]$$

(*Resolving into partial fractions*)

and $$\text{Inverse Lap.}\left[\frac{E\,e^{-as}}{s(Ls+R)}\right] = \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right]u(t-a)$$

(*By the Second Shifting Theorem*)

On substituting the values of the inverse transforms in (2), we get

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right]u(t-a)$$

Hence $$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] \text{ for } 0 < t < a, \qquad [\, u(t-a) = 0]$$

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right] \qquad \text{for } t > a \quad [u(t-a) = 1]$$

$$= \frac{E}{R}\left[e^{-\frac{R}{L}(t-a)} - e^{-\frac{R}{L}t}\right] = \frac{E}{R}e^{-\frac{R}{L}t}\left[e^{\frac{Ra}{L}} - 1\right]$$

Ans.

Example 59. *Voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L and resistance R. Show that the current at time t is* $\dfrac{E}{R-aL}(e^{-at} - e^{-Rt/L})$. [*U.P., II Semester, (SUM) 2007*]

Solution. We know that

$$L\frac{dI}{dt} + RI = Ee^{-at}, \qquad ...(1)$$

where $I(0) = 0$

Taking Laplace transform of both sides of (1), we get

$$L[s\bar{I} - I(0)] + R\bar{I} = \frac{E}{s+a} \qquad ...(2)$$

Putting I (0) = 0 in (2), we get

$$(Ls+R)\bar{I} = \frac{E}{s+a}$$

$$\Rightarrow \qquad \bar{I} = \frac{E}{(s+a)(Ls+R)} = \frac{E}{R-aL}\left(\frac{1}{s+a} - \frac{1}{Ls+R}\right)$$

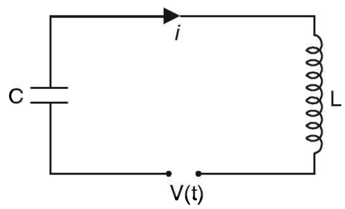
$$= \frac{E}{R-aL}\left(\frac{1}{s+a} - \frac{1}{s+R/L}\right) \qquad ...(3)$$

Taking the Inverse Laplace transform of both sides of (3), we get

$$I = \frac{E}{R-aL}L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+R/L}\right\} = \frac{E}{R-aL}[e^{-at} - e^{-Rt/L}] \qquad \textbf{Ans.}$$

Example 60. *Using the Laplace transform, find the current i (t) in the LC - circuit. Assuming L = 1 henry, C = 1 farad, zero initial current and charge on the capacitor, and*

$$v(t) = t, \text{ when } 0 < t < 1$$
$$= 0 \text{ otherwise.}$$



Solution. The differential equation for L and C circuit is given by

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E \qquad ... (1)$$

Putting $L = 1$, $C = 1$, $E = v(t)$ in (1), we get

$$\frac{d^2q}{dt^2} + q = v(t) \qquad ... (2)$$

Taking Laplace Transform of (2), we have

$$s^2\bar{q} - sq(0) - q'(0) + \bar{q} = \int_0^\infty v(t)\,e^{-st}\,dt$$

Substituting $q(0) = 0$, and $q'(0) = 0$, we get

$$s^2\bar{q} + \bar{q} = \int_0^1 te^{-st}\,dt + \int_1^\infty 0\,e^{-st}\,dt$$

$$(s^2+1)\bar{q} = \left[t\frac{e^{-st}}{-s}\right]_0^1 - \int_0^1 \frac{e^{-st}}{-s}\,dt = \frac{e^{-s}}{-s} - \left[\frac{e^{-st}}{s^2}\right]_0^1 = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

$$\bar{q} = \frac{1}{s^2+1}\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

$$\bar{q} = \frac{-e^{-s}}{s(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} + \frac{1}{s^2(s^2+1)}$$

Taking Inverse Laplace Transform, we get

$$q = \text{Inverse Lap.} \frac{-e^{-s}}{s(s^2+1)} - \text{Inverse Lap.} \frac{e^{-s}}{s^2(s^2+1)} + \text{Inverse Lap.} \frac{1}{s^2(s^2+1)} \quad \ldots (3)$$

We know that

$$\text{Inverse Lap.}\ [e^{-as} F(s)] = f(t-a)\, u(t-a)$$

$$\text{Inverse Lap.} \frac{1}{s(s^2+1)} = \int_0^t \sin t\, dt = [-\cos t]_0^t = 1 - \cos t \quad \ldots (4)$$

$$\text{Inverse Lap.} \frac{1}{s^2(s^2+1)} = \int_0^t (1-\cos t)\, dt = t - \sin t \quad \ldots (5)$$

In view of this, we have

$$\text{Inverse Lap.} \left[\frac{-e^{-s}}{s(s^2+1)}\right] = -[1-\cos(t-1)]\, u(t-1) \quad \text{[From (4)]}$$

$$\text{Inverse Lap.} \frac{e^{-s}}{s^2(s^2+1)} = [(t-1) - \sin(t-1)]\, u(t-1) \quad \text{[From (5)]}$$

Putting the above values in (3), we get

$$q = -[1-\cos(t-1)]\, u(t-1) - [(t-1) - \sin(t-1)]\, u(t-1) + t - \sin t \qquad \textbf{Ans.}$$

EXERCISE 43.10

Solve the following differential equations:

1. $\frac{d^2y}{dx^2} + y = 0$ where $y = 1$ and $\frac{dy}{dx} = -1$ at $x = 0$. **Ans.** $y = \cos x - \sin x$

2. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$,where $y = 2, \frac{dy}{dx} = -4$ at $x = 0$. **Ans.** $y = e^{-x}(2\cos 2x - \sin 2x)$

3. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$, given $y = \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 6$ at $x = 0$.
Ans. $y = e^x - 3e^{-x} + 2e^{-2x}$

4. $\frac{d^2y}{dx^2} + y = 3\cos 2x$, where $y = \frac{dy}{dx} = 0$ at $x = 0$. **Ans.** $y = \cos x - \cos 2x$.

5. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 1 - 2x$, given $y = 0, \frac{dy}{dx} = 4$ at $x = 0$. **Ans.** $y = e^x - e^{-2x} + x$

6. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4e^{2x}$, given $y = -3$, and $\frac{dy}{dx} = 5$ at $x = 0$
Ans. $y = -7e^x + 4e^{2x} + 4xe^{2x}$

7. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{2x}$, where y = 1, $\frac{dy}{dx} = -1$ at $x = 0$.
Ans. $y = 3 + 2x + \frac{1}{2}e^{3x} - 2e^{2x} - \frac{1}{2}e^x$.

8. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$, where y = 1, $\frac{dy}{dx} = 2, \frac{d^2y}{dx^2} = 2$ at $x = 0$.
Ans. $y = \frac{5}{3}e^x - e^{-x} + \frac{1}{2}e^{-2x}$

9. $(D^2 - D - 2)x = 20\sin 2t, x_0 = -1, x_1 = 2$ **Ans.** $x = 2e^{2t} - 4e^{-t} + \cos 2t - 3\sin 2t$

10. $(D^3 + D^2)\, x = 6t^2 + 4$, $x(0) = 0$, $x'(0) = 2$, $x''(0) = 0$
Ans. $x = \frac{1}{2}t^4 - 2t^3 + 8t^2 - 16t + 16 - 16e^{-t}$

11. $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = e^t$, where $x(0) = 2$, $\dfrac{dx}{dt} = -1$ at $t = 0$

Ans. $x = 2e^t - 3\,t\,e^t + \dfrac{1}{2}t^2e^t$

12. $y'' + 2y' + y = t\,e^{-t}$ if $y(0) = 1$, $y'(0) = -2$. **Ans.** $y = \left(1 - t + \dfrac{t^3}{6}\right)e^{-t}$

13. $\dfrac{d^2y}{dx^2} + y = x\cos 2x$, where $y = \dfrac{dy}{dx} = 0$ at $x = 0$. **Ans.** $y = \dfrac{4}{9}\sin 2x - \dfrac{5}{9}\sin x - \dfrac{x}{3}\cos 2x$

14. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = x^2e^{2x}$, where $y = 1$, $\dfrac{dy}{dx} = 0$, $\dfrac{d^2y}{dx^2} = -2$ at $x = 0$.

Ans. $y = e^{2x}(x^2 - 6x + 12) - e^x(15x^2 + 7x + 11)$.

15. $y'' + 4y' + 3y = t$, $t > 0$; given that $y(0) = 0$ and $y'(0) = 1$.

Ans. $y = -\dfrac{4}{9} + \dfrac{t}{6} + e^{-t} - \dfrac{5}{9}e^{-3t}$

43.14 SOLUTION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

Simultaneous differential equations can also be solved by Laplace Transform method.

Example 61. *Solve* $\dfrac{dx}{dt} + y = 0$ *and* $\dfrac{dy}{dt} - x = 0$ *under the condition* $x(0) = 1$, $y(0) = 0$.

Solution. We have,
$$x' + y = 0 \qquad \text{... (1)}$$
and
$$y' - x = 0 \qquad \text{... (2)}$$
Taking the Laplace transform of (1) and (2), we get
$$[s\,\bar{x} - x(0)] + \bar{y} = 0 \qquad \text{... (3)}$$
$$[s\,\bar{y} - y(0)] - \bar{x} = 0 \qquad \text{... (4)}$$
On substituting the values of $x(0)$ and $y(0)$ in (3) and (4), we get
$$s\,\bar{x} - 1 + \bar{y} = 0 \qquad \text{... (5)}$$
$$s\,\bar{y} - \bar{x} = 0 \qquad \text{... (6)}$$
Solving (5) and (6) for $\bar{x}$ and $\bar{y}$, we get
$$\bar{x} = \frac{s}{s^2+1}, \qquad \bar{y} = \frac{1}{s^2+1},$$
On inversion, we obtain
$$x = L^{-1}\left(\frac{s}{s^2+1}\right), \qquad y = L^{-1}\left(\frac{1}{s^2+1}\right),$$
$$x = \cos t, \qquad y = \sin t$$
Ans.

Example 62. *Solve the following simultaneous differential equations by Laplace transform*
$$3\frac{dx}{dt} - y = 2t, \quad \frac{dx}{dt} + \frac{dy}{dt} - y = 0$$
with the conditions $x(0) = y(0) = 0$. [*U.P., II Semester, (SUM) 2008*]

Solution. Here, we have
$$3\frac{dx}{dt} - y = 2t, \qquad \text{...(1)}$$
and
$$\frac{dx}{dt} + \frac{dy}{dt} - y = 0 \qquad \text{...(2)}$$
Taking Laplace transform on both sides of equation (1), we get
$$3L(x') - L(y) = L(2t)$$
$$3[s\bar{x} - x(0)] - \bar{y} = \frac{2}{s^2} \qquad [\text{where } L(x) = \bar{x} \text{ and } L(y) = \bar{y}]$$

$$3s\bar{x} - \bar{y} = \frac{2}{s^2} \qquad ...(3)$$

Again taking Laplace transform on both sides of equation (2), we get

$$L(x') + L(y') - L(y) = L\,(0)$$

$$\Rightarrow \quad [s\bar{x} - x(0)] + [s\bar{y} - y\,(0)] - \bar{y} = 0 \quad \Rightarrow \quad s\bar{x} + (s-1)\,\bar{y} = 0 \qquad ...(4)$$

Multiplying equation (4) by 3, we get

$$3s\bar{x} + 3(s-1)\,\bar{y} = 0 \qquad ...(5)$$

Subtracting equation (3) from (5), we get

$$(3s-2)\,\bar{y} = -\frac{2}{s^2}$$

$$\Rightarrow \quad \bar{y} = -\frac{2}{s^2\,(3s-2)} = \frac{1}{s^2} + \frac{3}{2s} - \frac{3}{2\left(s - \frac{2}{3}\right)}$$

Taking inverse Laplace transform on both sides, we get

$$y = t + \frac{3}{2} - \frac{3}{2}e^{\frac{2t}{3}} \qquad ...(6)$$

Substituting $\bar{y}$ in (3), we get

$$3s\bar{x} - \frac{1}{s^2} - \frac{3}{2s} + \frac{3}{2\left(s - \frac{2}{3}\right)} = \frac{2}{s^2} \quad \Rightarrow \quad 3s\bar{x} = \frac{3}{s^2} + \frac{3}{2s} - \frac{3}{2\left(s - \frac{2}{3}\right)}$$

$$\Rightarrow \quad \bar{x} = \frac{1}{s^3} + \frac{1}{2s^2} - \frac{1}{2s\left(s - \frac{2}{3}\right)} \quad \Rightarrow \quad \bar{x} = \frac{1}{s^3} + \frac{1}{2s^2} - \frac{3}{4}\left(\frac{1}{s - \frac{2}{3}} - \frac{1}{s}\right)$$

Taking inverse Laplace transform on both sides, we get

$$x = \frac{t^2}{2} + \frac{t}{2} - \frac{3}{4}e^{\frac{2t}{3}} + \frac{3}{4} \qquad ...(7)$$

Equation (6) and (7) when taken together, give the complete solution. **Ans.**

Example 63. *Solve the simultaneous equations:*

$$\frac{dx}{dt} - y = e^t,$$

$$\frac{dy}{dt} + x = \sin t, \text{ given } x\,(0) = 1,\ y\,(0) = 0,$$ (*U.P.,II Semester, Summer* 2006)

Solution. $\dfrac{dx}{dt} - y = e^t$ ... (1) $\qquad \dfrac{dy}{dt} + x = \sin t$ ... (2)

Taking Laplace transform of (1), we get

$$[\,s\,\bar{x} - x(0)\,] - \bar{y} = \frac{1}{s-1}$$

i.e.,

$$s\,\bar{x} - 1 - \bar{y} = \frac{1}{s-1} \qquad [\,\because\ x\,(0) = 1]$$

$$s\,\bar{x} - \bar{y} = 1 + \frac{1}{s-1}$$

$$s\,\bar{x} - \bar{y} = \frac{s}{s-1} \qquad ...\ (3)$$

Taking Laplace Transform of (2), we get

$$[\,s\,\bar{y} - y(0)\,] + \bar{x} = \frac{1}{s^2+1} \qquad [y\,(0) = 0)$$

$$\bar{x} + s\,\bar{y} = \frac{1}{s^2+1} \quad \text{... (4)}$$

Solving (3) and (4) for $\bar{x}$ and $\bar{y}$, we have

$$\bar{x} = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2} = \frac{1}{2}\left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + \frac{1}{(s^2+1)^2} \quad \text{... (5)}$$

and $$\bar{y} = \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)} = \frac{s}{(s^2+1)^2} - \frac{1}{2}\left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] \quad \text{... (6)}$$

Taking Inverse Laplace Transform of (5), we get

$$x = \frac{1}{2}L^{-1}\left[\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] + L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

$$= \frac{1}{2}[e^t + \cos t + \sin t] + \frac{1}{2}(\sin t - t\cos t) \quad \left[\because L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}(\sin at - at\cos at)\right]$$

$$= \frac{1}{2}[e^t + \cos t + 2\sin t - t\cos t]$$

Taking Inverse Laplace Transform of (6), we get

$$y = L^{-1}\left[\frac{s}{(s^2+1)^2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}t\sin t - \frac{1}{2}[e^t - \cos t + \sin t] \quad \left[\because L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}t\sin at\right]$$

$$= \frac{1}{2}[t\sin t - e^t + \cos t - \sin t]$$

Hence, $$x = \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$$

$$y = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t)$$

Ans.
[$i = 0$ at $t = 0$]

Example 64. *Use Laplace transform to solve:*

$$\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t$$

given that $x = 2$, $y = 0$ *at* $t = 0$. [*U.P., II Semester, 2004*]

Solution. Here, we have

$$\frac{dx}{dt} + y = \sin t \quad \text{...(1)}$$

$$\frac{dy}{dt} + x = \cos t \quad \text{...(2)}$$

Taking Laplace transform of (1) and (2), we get

$$s\bar{x} - x(0) + \bar{y} = \frac{1}{s^2+1} \Rightarrow s\bar{x} + \bar{y} = \frac{1}{s^2+1} + 2 \quad \text{...(3)}$$

and $$s\bar{y} - y(0) + \bar{x} = \frac{s}{s^2+1} \Rightarrow \bar{x} + s\bar{y} = \frac{s}{s^2+1} \quad \text{...(4)}$$

Solving (3) and (4) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{2s}{s^2-1} \text{ and } \bar{y} = \frac{1}{1+s^2} + \frac{2}{1-s^2}$$

$$\bar{x} = \frac{1}{s+1} + \frac{1}{s-1} \quad \text{...(5)}$$

and $$\bar{y} = \frac{1}{1+s^2} + \frac{1}{s+1} - \frac{1}{s-1} \qquad \text{... (6) (By partial fractions)}$$

Taking Inverse Laplace transform on both sides of (5) and (6), we get

$$x = e^{-t} + e^{t}$$

and $$y = \sin t + e^{-t} - e^{t}$$ **Ans.**

Example 65. *The co-ordinates (x, y) of a particle moving along a plane curve at any time t are given by*

$$\frac{dy}{dt} + 2x = \sin 2t, \quad \frac{dx}{dt} - 2y = \cos 2t; \ (t > 0)$$

It is given that at t = 0, x = 1 and y = 0. Show using transforms that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$. [*U.P. II Semester 2003*]

Solution. Here, we have

$$\left[\begin{array}{l} \dfrac{dy}{dt} + 2x = \sin 2t \\ \dfrac{dx}{dt} - 2y = \cos 2t \end{array}\right. \Rightarrow \left[\begin{array}{ll} 2x + Dy = \sin 2t & \text{...(1)} \\ Dx - 2y = \cos 2t, & \text{...(2)} \end{array}\right.$$

Taking Laplace transform of (1) on both sides, we get

$$2\bar{x} + s\bar{y} - y(0) = \frac{2}{s^2+4} \qquad \text{where } \bar{x} = L(x) \text{ and } \bar{y} = L(y)$$

$$2\bar{x} + s\bar{y} = \frac{2}{s^2+4} \qquad \text{...(3) } [\therefore y(0) = 0]$$

Again, taking Laplace transform of equation, (2) on both sides, we get

$$s\bar{x} - x(0) - 2\bar{y} = \frac{s}{s^2+4}, \text{ where } \bar{x} = L(x) \text{ and } \bar{y} = L(y)$$

$$\Rightarrow \qquad s\bar{x} - 2\bar{y} = \frac{s}{s^2+4} + 1 \qquad \text{...(4) } [\because x(0) = 0]$$

Multiplying equation (3) by 2 and equation (4) by s and then adding, we get

$$s\bar{x} + s^2\bar{x} = \frac{4}{s^2+4} + \frac{s^2}{s^2+4} + s$$

$$(4 + s^2)\,\bar{x} = 1 + s$$

$$\bar{x} = \frac{1+s}{4+s^2} = \frac{1}{s^2+4} + \frac{s}{s^2+4} \qquad \text{...(5)}$$

Taking Inverse Laplace transform of (5), we get

$$x = \frac{1}{2}\sin 2t + \cos 2t \qquad \text{...(6)}$$

Again, multiplying (3) by s and (4) by 2 then subtracting equation (6) from (3), we get

$$s^2\bar{y} + 4\bar{y} = \frac{2s}{s^2+4} - \frac{2s}{s^2+4} - 2$$

$$\Rightarrow \qquad \bar{y} = \frac{-2}{s^2+4} \qquad \text{...(7)}$$

Taking Inverse Laplace transform of (7), we get $y = -\sin 2t$

Now, $$4x^2 = 4\left[\frac{1}{4}\sin^2 2t + \cos^2 2t + \sin 2t \cos 2t\right]$$

$$5y^2 = 5\sin^2 2t$$

$$4xy = 4\left[\left(\frac{1}{2}\sin 2t + \cos 2t\right).(-\sin 2t)\right]$$
$$= -(2\sin^2 2t + 4\sin 2t\cos 2t)$$

$\therefore \quad 4x^2 + 5y^2 + 4xy = 4\sin^2 2t + 4\cos^2 2t = 4$ **Ans.**

Example 66. *Solve the following simultaneous differential equations by Laplace transform*

$$\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0; \quad \frac{dx}{dt} + 2y = e^{-t}$$

with conditions $x(0) = y(0) = 0.$ [*U.P., II Semester, 2008*]

Solution. Here, we have

$$\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0 \qquad ...(1)$$

and
$$\frac{dx}{dt} + 2y = e^{-t} \qquad ...(2)$$

Taking Laplace transform on both sides of equation (1), we get

$$L(x') + 4L(y') - L(y) = L(0)$$

$$\Rightarrow \quad s\bar{x} - x(0) + 4[s\bar{y} - y(0)] - \bar{y} = 0$$

$$\Rightarrow \quad s\bar{x} + (4s-1)\,\bar{y} = 0 \qquad ...(3)$$

Again, taking Laplace transform on both sides of equation (2), we get

$$L(x') + 2L(y) = L(e^{-t})$$

$$\Rightarrow \quad s\bar{x} - x(0) + 2\bar{y} = \frac{1}{s+1} \quad \Rightarrow \quad s\bar{x} + 2\bar{y} = \frac{1}{s+1} \qquad ...(4)$$

Subtracting (4) from (3), we get

$$(4s-3)\bar{y} = -\frac{1}{s+1}$$

$$\bar{y} = -\frac{1}{(s+1)(4s-3)} = -\frac{1}{7}\left(\frac{-1}{s+1} + \frac{1}{s-3/4}\right) = \frac{1}{7}\left(\frac{1}{s+1} - \frac{1}{s-3/4}\right) \qquad ...(5)$$

Taking inverse Laplace transform on both sides of (5), we get

$$y = \frac{1}{7}\left(e^{-t} - e^{\frac{3t}{4}}\right) \qquad ...(6)$$

Substituting $\bar{y}$ in (4), we get

$$s\bar{x} + \frac{2}{7}\left(\frac{1}{s+1} - \frac{1}{s-3/4}\right) = \frac{1}{s+1}$$

$$\Rightarrow \quad s\bar{x} = \frac{5}{7(s+1)} + \frac{2}{7(s-3/4)}$$

$$\Rightarrow \quad \bar{x} = \frac{5}{7s(s+1)} + \frac{2}{7s(s-3/4)} = \frac{5}{7}\left(\frac{1}{s} - \frac{1}{s+1}\right) + \frac{8}{21}\left(\frac{1}{s-3/4} - \frac{1}{s}\right)$$

$$= \frac{1}{3s} - \frac{5}{7(s+1)} + \frac{8}{21(s-3/4)} \qquad ...(7)$$

Taking Inverse Laplace transform on both sides of (7), we get

$$x = \frac{1}{3} - \frac{5}{7}e^{-t} + \frac{8}{21}e^{\frac{3t}{4}}$$ **Ans.**

Example 67. *Using Laplace Transformation, solve*

$$(D-2)\,x-(D+1)\,y=6\,e^{3t}$$

$$(2D-3)\,x+(D-3)\,y=6e^{3t} \qquad \ldots (1)$$

Given $x=3,\ y=0$ *when* $t=0.$ (*U.P., II Semester Summer* 2001)

Solution. Taking Laplace transformation of the given equations, we get

$$\begin{bmatrix} L\,Dx-2\,Lx-LDy-Ly=6\,Le^{3t} \\ 2L\,Dx-3Lx+LDy-3Ly=6\,Le^{3t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} s\,\bar{x}-x(0)-2\,\bar{x}-s\bar{y}+y(0)-\bar{y}=6\,\dfrac{1}{s-3} \\ 2s\,\bar{x}-2x(0)-3\,\bar{x}+s\,\bar{y}-y(0)-3\,\bar{y}=\dfrac{6}{s-3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (s-2)\,\bar{x}-(s+1)\,\bar{y}-3=\dfrac{6}{s-3} \\ (2s-3)\,\bar{x}+(s-3)\,\bar{y}-6=\dfrac{6}{s-3} \end{bmatrix} \Rightarrow \begin{bmatrix} (s-2)\,\bar{x}-(s+1)\,\bar{y}=\dfrac{3s-3}{s-3} \\ (2s-3)\,\bar{x}+(s-3)\,\bar{y}=\dfrac{6s-12}{s-3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (s-3)\,(s-2)\,\bar{x}-(s-3)\,(s+1)\,\bar{y}=3s-3 \\ (s+1)\,(2s-3)\,\bar{x}+(s+1)\,(s-3)\,\bar{y}=\dfrac{(s+1)\,(6s-12)}{s-3} \end{bmatrix} \text{ on adding, we get:}$$

$$(3s^2-6s+3)\,x=3\,(s-1)+\frac{6\,(s^2-s-2)}{s-3} \quad\Rightarrow\quad \bar{x}=\frac{3\,(s-1)}{3\,(s-1)^2}+\frac{6\,(s^2-s-2)}{3\,(s-1)^2\,(s-3)}$$

$$x=L^{-1}\left[\frac{1}{s-1}+\frac{2}{(s-1)^2}+\frac{2}{s-3}\right]=e^t+2t\,e^t+2e^{3t}$$

Putting the value of x in (1), we get

$$(D-2)\,(e^t+2te^t+2e^{3t})-(D+1)\,y=6e^{3t}$$

$$\Rightarrow \quad e^t+2te^t+2e^t+6e^{3t}-2e^t-4te^t-4e^{3t}-(D+1)\,y=6e^{3t}$$

$$\Rightarrow \quad (D+1)\,y=e^t-2te^t-4e^{3t} \qquad \ldots (2)$$

Taking Laplace transform of (2), we get

$$s\,\bar{y}-y(0)+\bar{y}=\frac{1}{s-1}-\frac{2}{(s-1)^2}-\frac{4}{s-3}$$

$$\Rightarrow \quad (s+1)\,\bar{y}=\frac{1}{s-1}-\frac{2}{(s-1)^2}-\frac{4}{s-3} \qquad [\because\ y(0)=0]$$

$$\Rightarrow \quad \bar{y}=\frac{1}{s^2-1}-\frac{2}{(s+1)\,(s-1)^2}-\frac{4}{(s+1)\,(s-3)}$$

$$=\frac{1}{s^2-1}-\frac{\frac{1}{2}}{s+1}+\frac{\frac{1}{2}}{s-1}-\frac{1}{(s-1)^2}+\frac{1}{s+1}-\frac{1}{s-3}$$

$$\Rightarrow \quad \bar{y}=\frac{1}{s^2-1}+\frac{\frac{1}{2}}{s+1}+\frac{\frac{1}{2}}{s-1}-\frac{1}{(s-1)^2}-\frac{1}{s-3}$$

$$\Rightarrow \quad y=L^{-1}\left[\frac{1}{s^2-1}+\frac{1}{2}\,\frac{1}{s+1}+\frac{1}{2}\,\frac{1}{s-1}-\frac{1}{s-3}-\frac{1}{(s-1)^2}\right]$$

$$\Rightarrow \qquad y = \sinh t + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} - e^{3t} - te^{t}$$

$$y = \sinh t + \cosh t - e^{3t} - te^{t}$$ **Ans.**

Example 68. *Solve the simultaneous equations*

$$(D^2 - 3)x - 4y = 0$$
$$x + (D^2 + 1)y = 0$$

for $t > 0$, given that $x = y = \frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 2$ at $t = 0$. [*U.P., II Semester, 2004*]

Solution. Here, we have

$$(D^2 - 3)x - 4y = 0 \qquad ...(1)$$
$$x + (D^2 + 1)y = 0 \qquad ...(2)$$

Taking Laplace transform of (1) and (2), we get

$$s^2\bar{x} - sx(0) - x'(0) - 3\bar{x} - 4\bar{y} = 0$$

i.e., $$(s^2 - 3)\bar{x} - 4\bar{y} = 2 \qquad ...(3)[\because x(0) = 0, x'(0) = 2]$$

and $$\bar{x} + s^2\bar{y} - sy(0) - y'(0) + \bar{y} = 0$$

i.e., $$\bar{x} + (s^2 + 1)\bar{y} = 0 \qquad ...(4)[\because y(0) = 0, y'(0) = 0]$$

Solving (3) and (4) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{2(s^2+1)}{(s^2-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2} \qquad ...(5)$$

and $$\bar{y} = -\frac{2}{(s^2-1)^2} = -\frac{1}{2}\left[\frac{1}{s+1} - \frac{1}{s-1} - \frac{1}{(s+1)^2} + \frac{1}{(s-1)^2}\right] \qquad ...(6)$$

Taking Inverse Laplace transform of both sides of (5) and (6), we get

$$x = L^{-1}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right] = te^{t} + te^{-t} = 2t\left(\frac{e^{t}+e^{-t}}{2}\right) = 2t\cosh t$$

and $$y = -\frac{1}{2}L^{-1}\left(\frac{1}{s+1} - \frac{1}{s-1} - \frac{1}{(s+1)^2} + \frac{1}{(s-1)^2}\right)$$

$$= -\frac{1}{2}(e^{-t} - e^{t} - te^{-t} + te^{t}) = \frac{e^{t}-e^{-t}}{2} - t\left(\frac{e^{t}-e^{-t}}{2}\right) = (1-t)\sinh t$$

Hence, $$x = 2t\cosh t,\ y = (1-t)\sinh t.$$ **Ans.**

EXERCISE 43.11

Solve the following:

1. $\frac{dx}{dt} + 4y = 0$, $\frac{dy}{dt} - 9x = 0$. Given $x = 2$ and $y = 1$ at $t = 0$.

Ans. $x = -\frac{2}{3}\sin 6t + 2\cos 6t,\ y = \cos 6t + 3\sin 6t$

2. $4\frac{dy}{dt} + \frac{dx}{dt} + 3y = 0$, $3\frac{dx}{dt} + 2x + \frac{dy}{dt} = 1$ under the condition $x = y = 0$ at $t = 0$.

Ans. $x = \frac{1}{2} - \frac{1}{5}e^{-t} - \frac{3}{10}e^{-\frac{6}{11}t},\ y = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-\frac{6}{11}t}$

3. $\dfrac{dx}{dt}+5x-2y=t, \dfrac{dy}{dt}+2x+y=0$ being given $x = y = 0$ when $t = 0$.

Ans. $x=-\dfrac{1}{27}(1+6t)\,e^{-3t}+\dfrac{1}{27}(1+3t),\ y=-\dfrac{2}{27}(2+3t)\,e^{-3t}-\dfrac{2t}{9}+\dfrac{4}{27}$

43.15 SOLUTION OF PARTIAL DIFFERENTIAL EQUATION BY LAPLACE TRANSFORM

Example 69. *Solve the differential equation using Laplace transform method:*

$$\frac{\partial y}{\partial t}=3\frac{\partial^2 y}{\partial t^2} \quad \textit{where} \quad y\left(\frac{\pi}{2},t\right)=0,\left(\frac{\partial y}{\partial x}\right)_{x=0}=0 \ \textit{and}\ y\,(x,\,0) = 30 \cos 5x.$$

(*U.P., II Semester Summer* 2005)

Solution. Given equation is

$$\frac{\partial y}{\partial t}=3\frac{\partial^2 y}{\partial t^2}$$

Taking Laplace transform of both sides, we get

$$sL\{y\}-y(x,0)=3\frac{d^2}{dx^2}L\{y\}, \qquad [\text{Let } L\{y\}=\bar{y}]$$

$$s\,\bar{y}-y(x,0)=3\frac{d^2\,\bar{y}}{dx^2}$$

$$3\frac{d^2\bar{y}}{dx^2}-s\,\bar{y}=-30\cos 5x$$

$$\left(D^2-\frac{s}{3}\right)\bar{y}=-10\cos 5x$$

A.E. is $\qquad m^2-\dfrac{s}{3}=0 \quad \Rightarrow \quad m=\pm\sqrt{\dfrac{s}{3}}$

$$C.F.=C_1\,e^{x\sqrt{s/3}}+C_2\,e^{-x\sqrt{s/3}}$$

$$P.I.=\frac{1}{\left(D^2-\frac{s}{3}\right)}(-10\cos 5x)$$

$$P.I.=\frac{30\cos 5x}{75+s}$$

Thus $\qquad \bar{y}=C_1\,e^{-x\sqrt{s/3}}+C_2\,e^{x\sqrt{s/3}}+\dfrac{30\cos 5x}{75+s} \qquad \ldots (1)$

$$\frac{\partial y}{\partial x}=0 \quad \text{when} \quad x=0$$

$\Rightarrow \qquad L\left\{\dfrac{\partial y}{\partial x}\right\}=0 \quad \text{at} \quad x=0$

$\Rightarrow \qquad \dfrac{d\bar{y}}{dx}=0 \quad \text{at} \quad x=0$

Again, $\qquad y\left(\dfrac{\pi}{2},t\right)=0 \Rightarrow L\left\{y\left(\dfrac{\pi}{2},t\right)\right\}=0 \Rightarrow \bar{y}\left(\dfrac{\pi}{2},s\right)=0$

$$\frac{d\,\bar{y}}{dx}=\sqrt{\frac{s}{3}}\left[Ae^{x\sqrt{s/3}}-Be^{-x\sqrt{s/3}}-\frac{150\sin 5x}{75+s}\right]$$

Putting $\qquad \dfrac{d\,\bar{y}}{dx}=0 \quad \text{at} \quad x=0$

$$0 = \sqrt{\frac{s}{3}}\,[A - B] \qquad \Rightarrow \qquad A = B$$

Equation (1) becomes,

$$\bar{y} = A\left[\, e^{x\sqrt{s/3}} + e^{-x\sqrt{s/3}} \,\right] + \frac{30\cos 5x}{75+s} \qquad \ldots (2)$$

Subjecting this to the condition

$$\bar{y}\left(\frac{\pi}{2}, s\right) = 0$$

$$0 = A\left[\, e^{\frac{\pi}{2}\sqrt{s/3}} + e^{-\frac{\pi}{2}\sqrt{s/3}} \,\right] + \frac{30\cos(5\pi/2)}{75+s}$$

$$24\cosh\left[\frac{\pi}{2}\sqrt{\frac{s}{3}}\right] = 0 \qquad \Rightarrow \quad A = 0$$

From equation (2), $\bar{y} = \dfrac{30\cos 5x}{75+s}$, taking inverse Laplace, we get

$$y = 30e^{-75t}\cos 5x$$ **Ans.**

EXERCISE 43.12

Solve the following differential equation using Laplace transform method:

1. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = \sin \pi x$ **Ans.** $u = \sin \pi x.\, e^{-p^2 t}$

2. $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = 4x - \dfrac{1}{2}x^2$ **Ans.** $u = \left(4x - \dfrac{x^2}{2}\right)e^{-p^2 t}$

3. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = \dfrac{1}{2}x(1-x)$

   Ans. $u = \dfrac{x}{2}(1-x)\cos pt + C_2 \sin pt\,(C_3 \cos px + C_4 \sin px)$

4. $16\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = x^2(5-x)$

   Ans. $u = x^2(5-x)\cos pt + C_4 \sin pt\left(C_1 \cos\dfrac{px}{4} + C_2 \sin\dfrac{px}{4}\right)$

5. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = \begin{cases} 2x, & \text{when } 0 \le x \le \dfrac{1}{2} \\ 2(1-x), & \text{when } \dfrac{1}{2} \le x \le 1 \end{cases}$