Coordinate Geometry

Introduction to Engineering Mathematics

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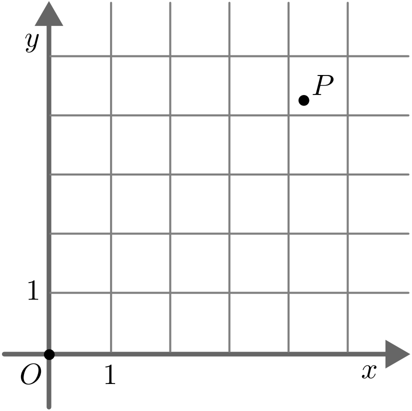
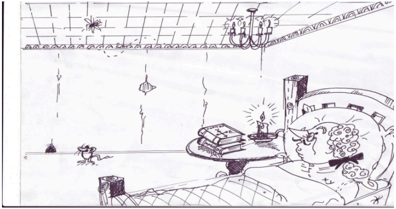
Foundations of coordinate geometry

What is coordinate geometry?

Studying geometry through coordinate calculations.

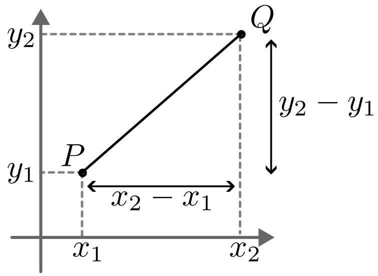




Example: distance between two points

Distance between P and Q:

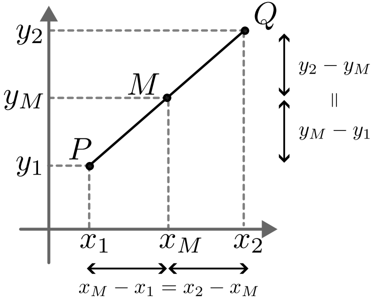
$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example: midpoint between two points

Coordinates of midpoint between P and Q:

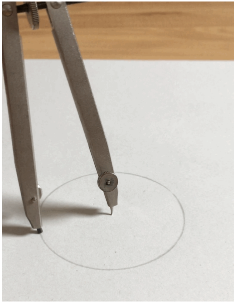
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

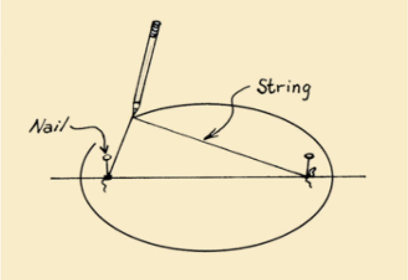


Locus of points

"Locus" = Set of points satisfying some condition.

- **Circle**: All points at given distance from a fixed center.
- **Ellipse**: All points for which the sum of distances to two fixed points (focal points) is constant
- **Parabola**: All points that are at equal distance from a fixed point and a given line (directrix)





Example

Find the locus of points for which the distance to the x-axis is equal to the distance to the point $(0, 1)$.

Circles

Circles

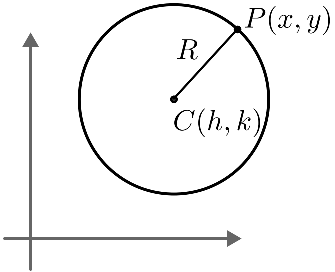
Locus of points $P(x,y)$ at distance R from center $C(h,k)$.

We have $d(P,C) = R$ so that

$$\sqrt{(x-h)^2 + (y-k)^2} = R,$$

and by squaring

$$\boxed{(x-h)^2 + (y-k)^2 = R^2}$$



Example

Find the equation of the circle that has the points $(1, 1)$ and $(7, 9)$ as end points of a diameter.

Example

Find the center and radius of the circle given by
$x^2 + y^2 - 6x + 2y + 8 = 0$.

Lines

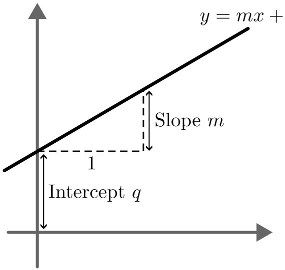
Lines

Line not parallel to the y-axis:

$$\boxed{y = mx + q}$$

with

- m: the **slope**
- q: the **intercept**



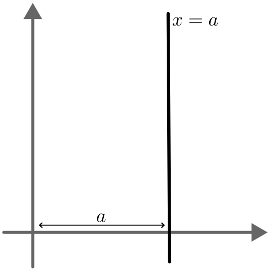
Lines

Line parallel to the y-axis:

$$\boxed{x = a}$$

with

- a: where the line intersects the x-axis



Finding the slope of a line

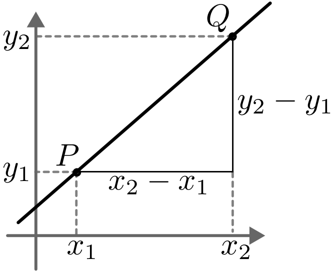
Take

- $\Delta x = x_2 - x_1$
- $\Delta y = y_2 - y_1$

Then

$$\boxed{m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}}$$

"Δx steps to the right, Δy steps up/down."



Example

Find the equation for the line through $(1, 5)$ and $(2, 7)$.

Properties

Equation for the line through (x_0, y_0) with slope m:

$$y - y_0 = m(x - x_0)$$

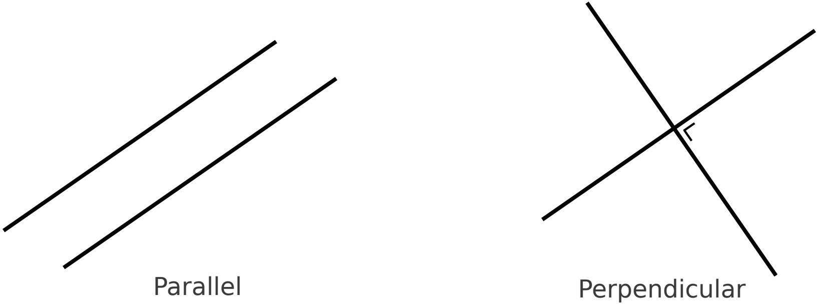
Equation for the line through the points (x_1, y_1) and (x_2, y_2):

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Parallel/perpendicular lines

Two lines are ...

- **parallel** if their slopes are the same: $\boxed{m_1 = m_2}$
- **perpendicular** if their slopes satisfy: $\boxed{m_1 m_2 = -1}$

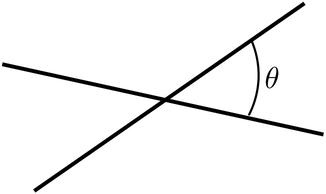


Angle between two lines

In general, the angle θ between two lines is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|,$$

where m_1 and m_2 are the slopes of the lines.



Example

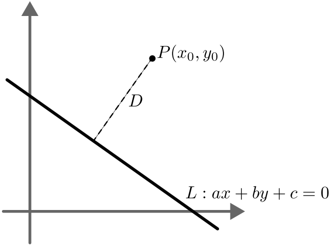
Given two lines $L_1 : x + 2y - 3 = 0$ and $L_2 : kx + y - 5 = 0$, for which value of k are L_1 and L_2 ...

1. Parallel?
2. Perpendicular?
3. At an angle of 45°?

Distance of a point to a line

Distance between point $P(x_0, y_0)$ and line $L : ax + by + c = 0$:

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$



Different representations of lines

1. Slope/intercept:
   - $y = mx + q$ (not parallel to y-axis)
   - $x = a$ (parallel)
2. Linear representation: $ax + by + c = 0$
3. Polar representation:
   - For line through the origin: $\tan\theta = m$
   - For line not through the origin:

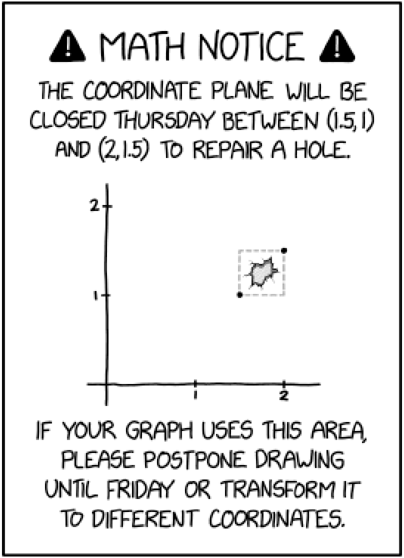
$$r = \frac{q}{\sin\theta - m\cos\theta}$$

Applications

Exercise

Find the equation of the common tangent line between two touching circles given by

$$\begin{aligned} C_1 &: x^2 + y^2 - 6x - 12y + 37 = 0 \\ C_2 &: x^2 + y^2 - 6y + 7 = 0. \end{aligned}$$



Source: xkcd 2735