Theory of equations (1/2): Working with polynomials

Introduction to Engineering Mathematics

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Overview

- Polynomial long division
- Synthetic division of polynomials

Reminder: what is a polynomial?

Polynomial of degree n:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

with

- x: the unknown
- $a_0, \dots, a_n$: the coefficients (in $\mathbb{C}$ or $\mathbb{R}$)

Reminder: Euclidian division of numbers

For every two integers p and d, we can write the fraction p/d as

$$\frac{p}{d} = q + \frac{r}{d},$$

with q the **quotient** and r the **remainder**. In other words

$$p = qd + r.$$

The numbers q and r can be found by *long division*.

Properties:

- The remainder r is always smaller than d.
- The quotient q and remainder r are unique.

Example

If we divide 20 by 7, we get

$$\frac{20}{7} = 2 + \frac{6}{7},$$

or $20 = 2 \times 7 + 6$.

Therefore:

- Quotient: $q = 20$
- Remainder: $r = 6$.

Euclidian division of polynomials

For every two polynomials $P(x)$ and $D(x)$, we can write the rational function $P(x)/D(x)$ as

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)},$$

with $P(x)$ the **quotient** and $R(x)$ the **remainder**. In other words

$$P(x) = Q(x)D(x) + R(x).$$

The polynomials $Q(x)$ and $R(x)$ can be found by *long division*.

Properties:

- The degree of $R(x)$ is always smaller than the degree of $D(x)$.
- The quotient $Q(x)$ and remainder $R(x)$ are unique.

Example

If we divide $x^2 + 1$ by $x - 1$, we get:

$$\begin{aligned}
\frac{x^2+1}{x-1} &= \frac{x^2-1+2}{x-1} \\
&= \frac{x^2-1}{x-1} + \frac{2}{x-1} \\
&= x + 1 + \frac{2}{x-1}
\end{aligned}$$

Therefore, the quotient is $Q(x) = x + 1$ and the remainder is $R(x) = 2$.

Polynomial long division

If $P(x) = 3x^4 - x^3 + 2x^2 - 2x - 1$ and $D(x) = x + 2$, find the quotient and the remainder after dividing $P(x)$ by $D(x)$.

Algorithm:

1. Divide leading term by leading term and write down result
2. Multiply result by divisor and transfer the result to the left
3. Subtract
4. Drop next term
5. Repeat steps 1-4.

Example

Divide $P(x) = 3x^4 - x^3 + 2x^2 - 2x - 1$ by $D(x) = x + 2$.

Synthetic division

Algorithm:

1. Write down coefficients of $P(x)$ on top
2. Write down coefficients of $-D(x)$ (except the first) on left
3. Lower first coefficient
4. Multiply and put result back in the table
5. Add coefficients in next column
6. Repeat step 4-5 until done
7. Determine quotient and remainder

Example

If $P(x) = x^3 - 12x^2 - 42$ and $D(x) = x - 3$, find the quotient and the remainder after dividing $P(x)$ by $D(x)$.

Special case

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1.$$

Resources

Step-by-step walkthroughs of both algorithms (from the 2021-22 version of this course):

- Polynomial long division: https://youtu.be/RyRqUg5oycE?t=499
- Synthetic division: https://youtu.be/NqQeMfGEzk4

These videos can be found on Ufora as well.