Integration: Definitions and basic concepts

Introduction to Engineering Mathematics

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Contents

Indefinite integrals as antiderivatives

- **Antiderivative**: the opposite (inverse operation) of a derivative.
- The **indefinite integral** is an antiderivative.

Examples:

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In these examples, C is an **arbitrary constant**.

Anatomy of an indefinite integral

Differential
(identifies the variable)

Integral sign
(curly 'S')

$$\int f(x)\,dx$$

Integrand
(the function to integrate)

Examples

Polynomials:

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
- $\int \frac{dx}{x} = \ln|x| + C$

Trigonometric functions:

- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$

$$\int \tan x \, dx = -\ln|\cos x| + C$$

Not an obvious antiderivative; we will see how to derive this integral later.

Examples

Exponential/logarithmic functions:

- $\displaystyle\int e^x dx = e^x + C$
- $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

Inverse trigonometric functions:

- $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$
- $\displaystyle\int \frac{dx}{1+x^2} = \arctan(x) + C$

Examples

Special cases:

- $\displaystyle\int \sec^2 x\,dx = \tan x + C$
- $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
- $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
- $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

The definite integral

Definite integral: integral with "bounds"

Upper bound

$$\int_a^b f(x)dx = F(b) - F(a)$$

Lower bound

How to compute:

1. Find primitive function $F(x)$: $\displaystyle\int f(x)\,dx = F(x) + C$
2. Substitute bounds into $F(x)$:
   $\displaystyle\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$.

Examples

Compute:

- $\displaystyle\int_0^{\pi/2} \sin x \, dx$
- $\displaystyle\int_{-1}^{2} (x^2 + 2x - 1) \, dx$

Properties of the definite integral

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- Linearity:
  $$\int_a^b (Af(x) + Bg(x))dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx$$

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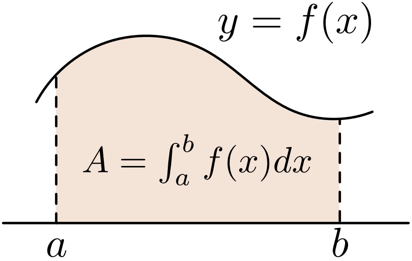
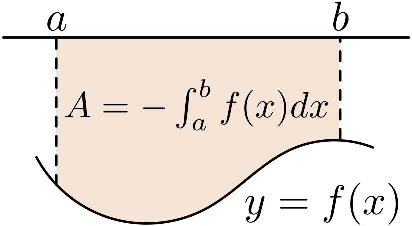
- $\displaystyle\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Definite integrals correspond to signed areas

If $f(x) \geq 0$: $A = \int_a^b f(x)\,dx$.



If $f(x) \leq 0$: $A = -\int_a^b f(x)\,dx$.



If $f(x)$ changes sign: break up into parts above/below x-axis.

Examples

Find the area...

- Between the graph of $y = \sin x$ and the x-axis, from 0 to 2π.
- Below the graph of $y = 3x - x^2$ and above the x-axis.

Properties (continued)

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- For an odd function, $\int_{-a}^{a} f(x) = 0.$
- "King's rule": $\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx.$


FREE Wi-Fi
$$\int_{-2}^{2}\left(x^3\cos\frac{x}{2}+\frac{1}{2}\right)\sqrt{4-x^2}\,dx$$
The Wi-Fi password is the first 10 digits of the answer.

Continuity and integration

- So far, we have silently assumed $f(x)$ is continuous on $[a, b]$ to define the integral $\int_a^b f(x)dx$.
- The integral can also be defined if $f(x)$ is **piecewise continuous**.

Examples

Compute the following integrals:

- $\displaystyle\int \left(10x^4 - \frac{2}{\sqrt{1-x^2}}\right)\, dx$
- $\displaystyle\int_0^1 x^{5/2}(1-x)\, dx$
- $\displaystyle\int \frac{x^2}{1+x^2}\, dx$

Examples

Show that $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$.

Difficult, uses King's rule.

Integrals and derivatives are each other's inverse

- If $G(x) = \int_1^x \ln y \, dy$, find $G'(x)$.
- If $H(x) = \int_1^{x^2} \ln y \, dy$, find $H'(x)$.