

# Theory of equations (2/2): Polynomial equations

## Introduction to Engineering Mathematics

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- Remainder theorems
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# Overview

- Every polynomial of degree  $N$  has  $N$  roots
  - Some of these roots may be *complex* (e.g.  $x^2 + 1$ )
  - Some of these roots may be *the same* (e.g.  $x^2 + 2x + 1$ )
- Roots correspond to factors of the polynomial
- There is no algorithm for finding all roots of a polynomial
- If a real polynomial has a complex root  $z$ , then the complex conjugate  $\bar{z}$  is also a root (e.g.  $x^3 - x^2 + x - 1$ )

## Recall

Polynomial of degree  $n$ :

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- The number  $n$  is called the **degree** of  $P(x)$ .
- A **root** or **zero** is a number  $\alpha$  such that  $P(\alpha) = 0$ .
- Roots can be real ( $\alpha \in \mathbb{R}$ ) or complex ( $\alpha \in \mathbb{C}$ ).
- A **factor** is a polynomial  $F(x)$  such that  $P(x) = F(x)Q(x)$  for some other polynomial  $Q(x)$ .
  - Linear factor:  $F(x) = x - \alpha$
  - Quadratic factor:  $F(x) = Ax^2 + Bx + C$

## Remainder theorem (special case)

If  $P(x)$  is a polynomial, then  $P(h)$  is the remainder of  $P(x)$  divided by  $x - h$ .

## Corollary

Note: “Corollary” means “consequence”.

If  $P(x)$  is a polynomial with zero  $\alpha \in \mathbb{C}$  (in other words,  $P(\alpha) = 0$ ), then  $x - \alpha$  is a factor of  $P(x)$ :

$$P(x) = (x - \alpha)Q(x).$$

## Example

Find all the factors of  $P(x) = 2x^3 + 3x^2 - 1$ .

## Remainder theorem (general version)

If  $P(x)$  is a polynomial with *distinct* zeros  $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{C}$ , then  $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)$  is a factor of  $P(x)$ :

$$P(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)Q(x).$$

Notes:

- There are at most  $n$  distinct zeros, where  $n$  is the degree of  $P(x)$  (see later).



## Examples

Find a polynomial of degree 4 with roots  $\pm i$ ,  $\pm 2$ , and such that  $P(3) = 25$ .

## Examples

Find a polynomial of degree 4 with roots 0 and  $-2$ , and where the root  $-2$  has multiplicity 3.

## How many roots can a polynomial have?

**Theorem:** A polynomial  $P(x) \neq 0$  cannot have more than  $n$  distinct roots, where  $n = \deg P(x)$ .

**Proof:** Assume that there are  $m$  distinct roots  $\alpha_1, \dots, \alpha_m$ , with  $m > \deg P(x)$ . Then by the remainder theorem,

$$P(x) = (x - \alpha_1) \cdots (x - \alpha_m) Q(x).$$

The left-hand side has degree  $n$ , whereas the right-hand side has degree at least  $m > n$ . This is a contradiction.

# Relation between roots and coefficients

Define the **symmetric polynomials**:

- $S_1 = a_1 + \cdots + a_n$
- $S_2 = a_1a_2 + a_1a_3 + \cdots + a_1a_n + a_2a_3 + \cdots + a_{n-1}a_n$
- $S_3 = a_1a_2a_3 + \cdots + a_{n-2}a_{n-1}a_n$
- ...
- $S_n = a_1a_2 \cdots a_n$

Then:

$$(x - a_1)(x - a_2) \cdots (x - a_n) = x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \cdots + (-1)^n S_n. \quad (1)$$

## Example

Given  $P(x) = x^3 + 2x^2 - 3x - 1$  with roots  $\alpha$ ,  $\beta$ , and  $\gamma$ , find the value of  $\alpha^2 + \beta^2 + \gamma^2$ .

# The fundamental theorem of algebra

**Theorem:** Each polynomial has *at least one* root (which may be complex).

**Proof:** Difficult.

**Consequence:** Each polynomial of degree  $n$  has exactly  $n$  roots (which may be same).

## How to find roots?

- Degree 2: formula for quadratic equation
- Degree 3, 4: formulas exist, but they are very complicated
- Degree 5 and up: **no general formula exists**

In general, proceed via trial and error, or numerically.

## Example

Factorize  $P(x) = x^4 + 2x^3 + 2x^2 + 2x + 1$ .



## Complex conjugates theorem

**Theorem:** If  $P(x)$  is a polynomial with real coefficients, then complex roots appear in *conjugates*.

In other words, if  $z = \alpha + i\beta$  is a root with multiplicity  $p$ , then  $\bar{z} = \alpha - i\beta$  is also a root with multiplicity  $p$ .

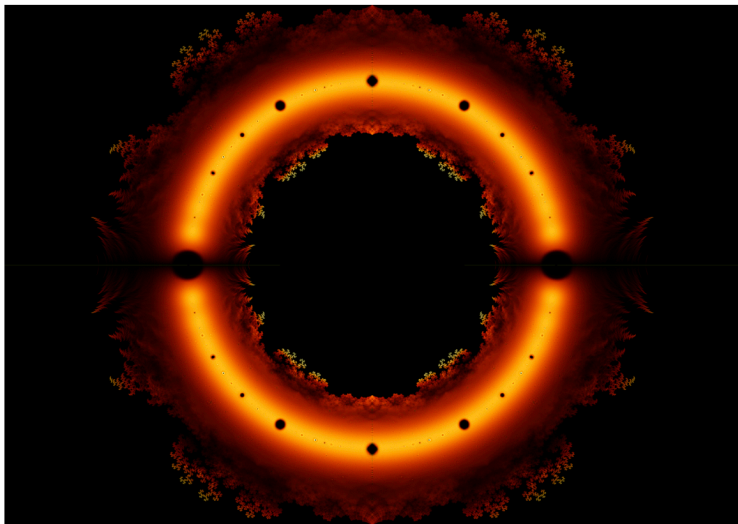


Figure 1. Roots of all polynomials of degree 23 whose coefficients are  $\pm 1$ . The brightness shows the number of roots per pixel.

Source: *The beauty of roots*, <https://arxiv.org/pdf/2310.00326>.