

Complex Numbers

Introduction to Engineering Mathematics

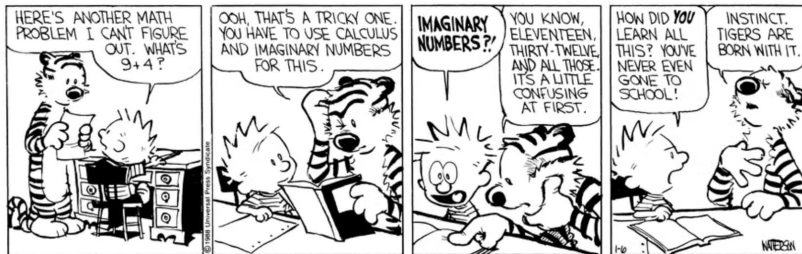
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What are complex numbers?

What are complex numbers?



Motivation: working with “impossible” numbers

At first, this expression looks like complete nonsense:

$$\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$$

But is it really?

Benefit: solving quadratic equations

Find x so that $x^2 - 2x + 2 = 0$.

- As $D = -4 < 0$, there are **no real solutions**
- If we set $i = \sqrt{-1}$, then we find **two solutions**.

OK to calculate with i , as long as we remember

$$i^2 = -1.$$

Number systems

Symbol	Elements	Used for
\mathbb{N}	0, 1, 2, ...	Counting
\mathbb{Z}	..., -1, 0, 1, 2, ...	Adding/subtracting
\mathbb{Q}	Fractions n/m	Dividing
\mathbb{R}	\mathbb{Q} and irrational numbers: e , π , ...	Limits
\mathbb{C}	$a + ib$, with $a, b \in \mathbb{R}$	Solving equations

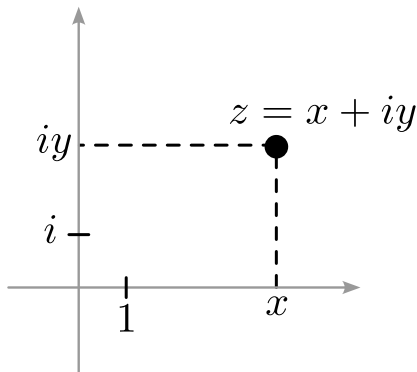
Cartesian and polar representation

Graphical representation of complex numbers

If $z = x + iy$ is a complex number, then

- $\operatorname{Re}(z) = x$ (the real part)
- $\operatorname{Im}(z) = y$ (the imaginary part)

are both real numbers and (x, y) determines a point in the plane.



Argand plane:

- x -axis: Real axis
- y -axis: Imaginary axis

Example

In the complex plane, find the location of:

- $z_1 = 1$
- $z_2 = 2 + 3i$
- $z_3 = -2i$
- $S = \{z \in \mathbb{C} : \operatorname{Re}(z) \leq 1\}$

Modulus and argument

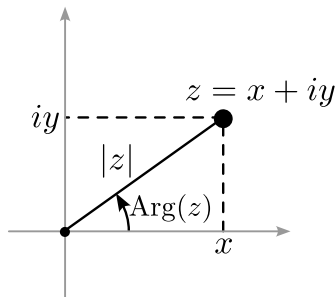
- **Modulus** (absolute value) of z : distance to the origin.

$$|z| = d(z, O) = \sqrt{x^2 + y^2}.$$

- **Argument** of z : angle with positive x -axis.

$$\text{Arg}(z) = \theta \in (-\pi, \pi] \text{ if}$$

$$\tan \theta = \frac{y}{x}.$$



Finding the argument (continued)

If z is in quadrant 1 or 4, then

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Otherwise (z in quadrant 2 or 3)

$$\theta = \pi + \tan^{-1}\left(\frac{y}{x}\right).$$

Polar representation of complex numbers

If r is the modulus and θ the argument of $z = x + iy$, then

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

This gives us the **polar representation of z** :

$$\begin{aligned} z &= x + iy \\ &= \boxed{r(\cos \theta + i \sin \theta)} \end{aligned}$$

Example

Find the polar representation of

- $z_1 = i$
- $z_2 = 1 + i$
- $z_3 = -\sqrt{3} - i$

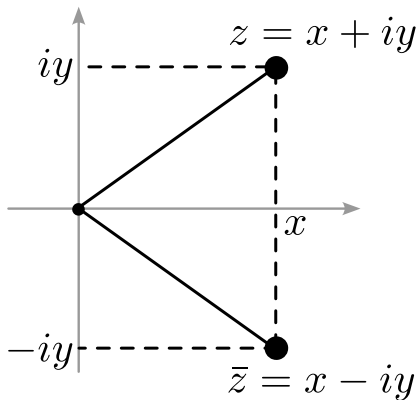
Complex conjugate

If $z = x + iy$, then the **complex conjugate** \bar{z} is given by

$$\bar{z} = x - iy.$$

Properties:

- ① $\operatorname{Re}(\bar{z}) = \operatorname{Re}(z)$
- ② $\operatorname{Im}(\bar{z}) = -\operatorname{Im}(z)$
- ③ $|\bar{z}| = |z|$
- ④ $\operatorname{Arg}(\bar{z}) = -\operatorname{Arg}(z)$



Calculating with complex numbers

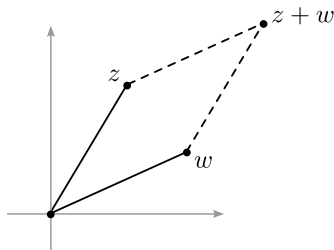
Adding and subtracting complex numbers

Complex numbers can be added/subtracted component-wise: if $z = x + iy$ and $w = a + ib$, then

$$\begin{aligned}z \pm w &= (x + iy) \pm (a + ib) \\ &= (x \pm a) + i(y \pm b)\end{aligned}$$

This has a nice geometric interpretation via the **parallelogram rule**:

- Draw a parallelogram with sides z and w
- $z + w$ is at the end of the diagonal



Multiplying complex numbers

If $z = x + iy$ and $w = a + ib$, then (using $i^2 = -1$)

$$\begin{aligned}zw &= (x + iy) \cdot (a + ib) \\ &= (xa - yb) + i(ya + xb)\end{aligned}$$

Properties:

- ① $z\bar{z} = |z|^2$
- ② $\overline{zw} = \bar{z} \cdot \bar{w}$

Product of complex numbers in polar form

Write

$$w = r(\cos \theta + i \sin \theta)$$

$$z = s(\cos \phi + i \sin \phi)$$

Then we get the following nice form for the complex product:

$$wz = \underbrace{rs}_{|wz|} (\underbrace{\cos(\theta + \phi)}_{\text{Arg}(wz)} + i \sin(\theta + \phi))$$

In particular, we get

- $|wz| = rs = |w||z|$
- $\text{Arg}(wz) = \theta + \phi = \text{Arg}(w) + \text{Arg}(z)$

De Moivre's theorem

From the product rule, we get

$$|z_1 z_2 \cdots z_n| = |z_1| |z_2| \cdots |z_n|$$
$$\text{Arg}(z_1 z_2 \cdots z_n) = \text{Arg}(z_1) + \cdots + \text{Arg}(z_n).$$

Substituting $z_1 = \dots = z_n = \cos \theta + i \sin \theta$ gives us **De Moivre's theorem**:

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta).$$

Division of complex numbers

We put

$$\frac{z}{w} = \frac{x + iy}{a + ib}.$$

- How can we make sense of this complex number?
- Multiply by the conjugate:

$$\frac{z}{w} = \frac{x + iy}{a + ib} \frac{a - ib}{a - ib} = \frac{ax + by}{a^2 + b^2} + i \frac{ay - bx}{a^2 + b^2}.$$

Properties:

$$|z/w| = |z|/|w| \quad \text{and} \quad \text{Arg}(z/w) = \text{Arg}(z) - \text{Arg}(w).$$

Useful properties of complex numbers

- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $\overline{\bar{z}} = z$
- $zw = 0$ iff $z = 0$ or $w = 0$

Property: to take the conjugate of a complicated expression, it suffices to take the conjugate of every term.

Example: Given $z = i\frac{Z-1}{Z+1}$, compute \bar{z} .

Examples

- Find the modulus and argument of $z = (3 + 5i)(4 - 2i)$.
- Simplify the complex number $z = \frac{7+3i}{4i}$.

Examples

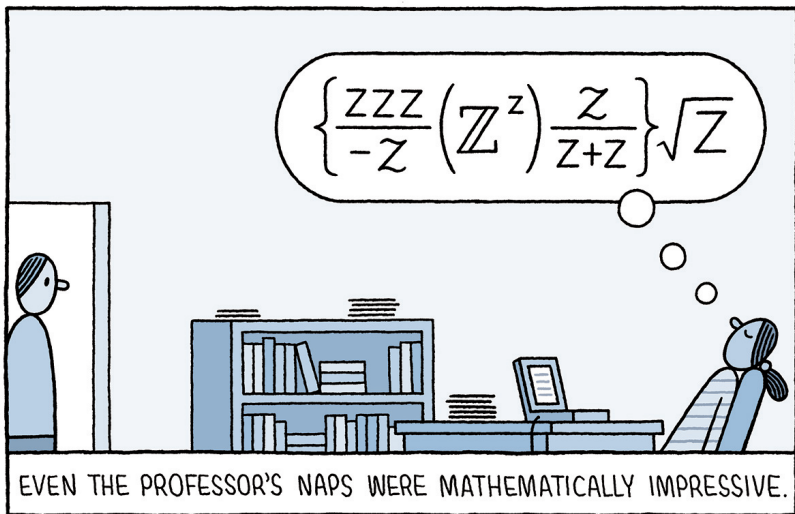
Simplify the following complex numbers as much as possible:

- $z = \frac{1+i}{1-i}$
- $z = i^{2022}$

Caveat

Keep in mind that, for complex numbers,

$$\sqrt{ab} \neq \sqrt{a}\sqrt{b}.$$



TOM GAULD for NEW SCIENTIST

Source: Tom Gauld