Limits and Continuity (1/2)

Introduction to Engineering Mathematics

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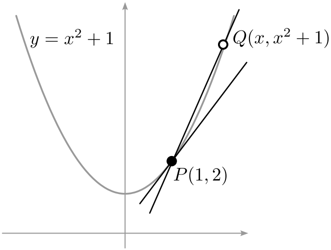
Contents

Motivation

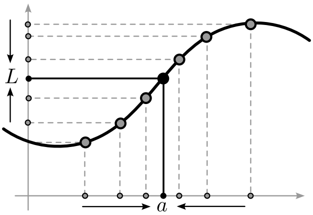
Compute the **tangent line** to the parabola $y = x^2 + 1$ at the point $P(1, 2)$.

- Take another point Q on the parabola
- Compute the line through P and Q
- Let Q "move towards" P

What is the slope of the line when Q gets "infinitely close" to P?



Definition of limit



- We write:

$$\lim_{x \to a} f(x) = L$$

- We say: *"The limit of $f(x)$ as x goes to a is L".*
- We mean:
    - $f(x)$ is defined for all x near a (possibly not a itself)
    - As x gets closer to a, $f(x)$ gets closer and closer to L

Examples

Compute $\lim_{x \to 2} f(x)$, with $f(x)$ given by

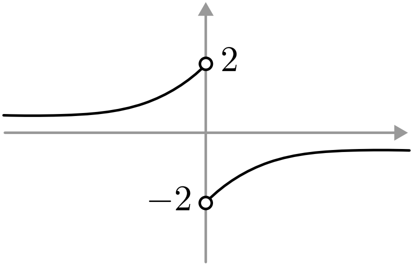
- $f(x) = x - 1$ when $x \neq 2$
- $f(x) = 2$ otherwise.

Examples

1. $\lim\limits_{x \to -2} \dfrac{x^2 + x - 2}{x^2 + 5x + 6}$
2. $\lim\limits_{x \to 0} \dfrac{x}{\sqrt{3 + x} - \sqrt{3 - x}}$

One-sided limits

Sometimes the limit is different depending on which side you approach a.



$$\lim_{x \to 0} f(x) = ?$$

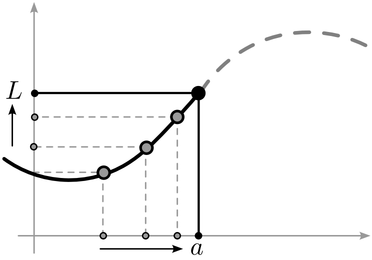
- $+2$ when $x \to 0$ from left
- -2 when $x \to 0$ from right

Definition

Left Limit:

$$\lim_{x \to a-} f(x) = L$$

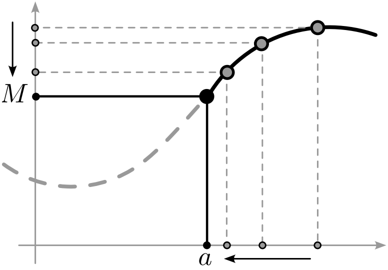
- $f(x)$ defined to the **left** of a
- $f(x) \to L$ as $x \to a$ from the **left**



Right Limit:

$$\lim_{x \to a+} f(x) = M$$

- $f(x)$ defined to the **right** of a
- $f(x) \to M$ as $x \to a$ from the **right**



Relation with two-sided limit

The two-sided limit exists,

$$\lim_{x \to a} f(x) = L,$$

if and only if both one-sided limits exist and are equal:

$$\lim_{x \to a+} f(x) = L \quad \text{and} \quad \lim_{x \to a-} f(x) = L$$

Examples

1. $\lim_{x \to 0+} \sqrt{x}$
2. $\lim_{x \to 0} \frac{|x|}{x}$
3. $\lim_{x \to 2} \frac{|x-2|}{x^2+x-6}$
4. $\lim_{x \to \pi} f(x)$, where $f(x) = \sin x$ when $x < \pi$ and $f(x) = \sqrt{x - \pi}$ otherwise.

Limit laws

1. $\lim_{x \to a} x = a$
2. $\lim_{x \to a} c = c$
3. $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
4. $\lim_{x \to a}(c(f(x))) = c \lim_{x \to a} f(x)$
5. $\lim_{x \to a}(f(x)g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$
7. $\lim_{x \to a} \left(f(x)^{m/n}\right) = \left(\lim_{x \to a} f(x)\right)^{m/n} = L^{m/n}$,
    - If n even: L must be positive
    - If $m < 0$: L must be different from 0.

Limit laws

8. If $f(x) = g(x)$ for all $x \neq a$ close to a, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.
9. If $f(x) \leq g(x)$ for all $x \neq a$ close to a, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

These laws also hold for one-sided limits, with appropriate modifications.

Limits of polynomials and rational functions

- Polynomial: $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$$\lim_{x \to a} P(x) = P(a).$$

- Rational function: $F(x) = \frac{P(x)}{Q(x)}$, with $P(x)$, $Q(x)$ polynomials

$$\lim_{x \to a} F(x) = \frac{\lim_{x \to a} P(x)}{\lim_{x \to a} Q(x)} = \frac{P(a)}{Q(a)} \quad \text{if } Q(a) \neq 0.$$

The squeeze theorem

Lets you compute the limit of a difficult function $g(x)$ “squeezed” between two simple functions $f(x)$ and $h(x)$.

Suppose

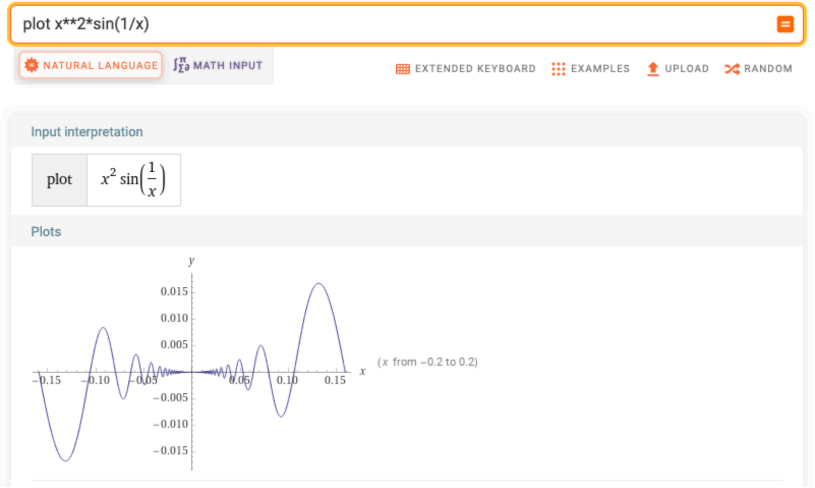
1. $f(x) \leq g(x) \leq h(x)$ for some x near a
2. $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.

Then

$$\lim_{x \to a} g(x) = L.$$

Example

Compute $\lim_{t \to 0} t^2 \sin\left(\frac{1}{t}\right)$.



Source: Wolfram Alpha