# Integration: Partial Fractions

## Introduction to Engineering Mathematics

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# What are partial fractions?

Every rational function can be written as a sum of "simple" partial fractions. For example

$$\frac{x+2}{x^3-x} = \frac{-2}{x} + \frac{3}{2(x-1)} + \frac{1}{2(x+1)}.$$

In this lecture, we will find a recipe for the coefficients and terms in the partial fraction expansion.

# Why are partial fractions useful?

The advantage is that the partial fractions are *much* easier to integrate:

$$\begin{aligned}\int \frac{x+2}{x^3-x}dx &= -2\int \frac{dx}{x} + \frac{3}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} \\ &= -2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

# How to find the partial fraction expansion

# Goal

To integrate **any rational function**: determine

$$\int \frac{P(x)}{Q(x)} dx = ???$$

where $P(x)$ and $Q(x)$ are polynomials.

## Step 1: Divide if necessary

If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, do a polynomial division:

$$\int \frac{P(x)}{Q(x)}dx = \int S(x)dx + \int \frac{R(x)}{Q(x)}dx,$$

with $S(x)$ the quotient and $R(x)$ the remainder.

- Recall: $\deg R(x) < \deg Q(x)$.
- From now on, we will suppose that this division has already been done, so that $\deg P(x) < \deg Q(x)$.

# Special case 1: Linear denominator

If $Q(x)$ is a linear polynomial, i.e. $Q(x) = Ax + B$, then our integral takes the form

$$\int \frac{P(x)}{Q(x)} dx = \int \frac{K}{Ax + B} dx$$
$$= \cdots$$

# Special case 2: Quadratic denominator

If $Q(x)$ is a quadratic polynomial, then several cases are possible. After completing the square, we can have one of the following forms:

If $\deg P(x) = 0$:

- $\displaystyle\int \frac{dx}{x^2 - a^2}$
- $\displaystyle\int \frac{dx}{x^2 + a^2}$

If $\deg P(x) = 1$:

- $\displaystyle\int \frac{x\,dx}{x^2 - a^2}$
- $\displaystyle\int \frac{x\,dx}{x^2 + a^2}$

# Step 2: Find the roots of the denominator

Do a factorization of the denominator $Q(x)$ into factors with **real** coefficients.

You will find:

- Some linear factors $(x - \alpha)$, with $\alpha$ roots of $Q(x)$
- Some quadratic factors $(Ax^2 + Bx + C)$ that cannot be further reduced.

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**Important:** Do not split into complex factors. For example:

$$x^3 + x = x(x^2 + 1).$$

Stop here, don't factor into $x(x + i)(x - i)$.

# Case 2.1: Distinct roots

Assume that $Q(x) = (x - \alpha_1) \cdots (x - \alpha_k)$, with all $\alpha_i$ **distinct** and **real**.

Then the partial fraction expansion becomes

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \cdots + \frac{A_k}{x - \alpha_k},$$

where the coefficients $A_1, \ldots, A_k$ can be determined by adding the terms together and comparing with the left-hand side.

# Example

Find $\displaystyle\int \frac{x+2}{x^3-x}dx$

# Case 2.2: Irreducible quadratic factors

- For each quadratic factor, put a *linear term* in the numerator of the partial fraction.
- Deal with the linear factors as before.

# Example

$$\begin{aligned}\frac{x+2}{x^3+x} &= \frac{x+2}{x(x^2+1)} \\ &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\ &= \cdots\end{aligned}$$

Therefore $\int \frac{x+2}{x^3+x} dx = \cdots$

# Case 2.3: Repeated linear factors

- If $Q(x)$ has a repeated factor $(x-\alpha)^p$, then add $p$ terms to the partial fraction expansion:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_p}{(x-\alpha)^p} + \text{[other PFE]}$$

- Deal with other linear and quadratic factors as before.

# Example

Determine the partial fraction expansion of $\dfrac{1}{x^2(x-1)^3}$.

# Example

Find $\displaystyle\int \frac{dx}{x^3 - 5x^2 + 8x - 4}$.

# Summary

1. If $\deg P(x) \geq \deg Q(x)$, do polynomial division.
2. Factor the denominator $Q(x)$ and write partial fractions for each root:
   - Distinct roots (roots with multiplicity 1):
   $$\mathsf{PF} = \frac{A}{x-\alpha}.$$
   - Irreducible quadratic factors:
   $$\mathsf{PF} = \frac{Ax+B}{x^2+\cdots}$$
   - Root with multiplicity $p$:
   $$\frac{A_1}{x-\alpha} + \cdots + \frac{A_p}{(x-\alpha)^p}.$$
3. Find the coefficients in the partial fraction expansion by solving a system of equations.
4. Integrate the partial fraction expansion.