

# Limits and Continuity (1/2)

## Introduction to Engineering Mathematics

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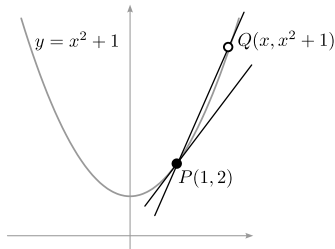
## (Two-sided) limits

## Motivation

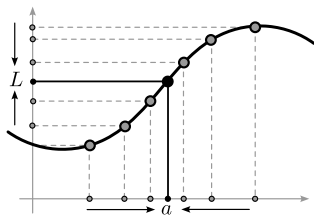
Compute the **tangent line** to the parabola  $y = x^2 + 1$  at the point  $P(1, 2)$ .

- Take another point  $Q$  on the parabola
- Compute the line through  $P$  and  $Q$
- Let  $Q$  “move towards”  $P$

What is the slope of the line when  $Q$  gets “infinitely close” to  $P$ ?



# Definition of limit



- We write:

$$\lim_{x \rightarrow a} f(x) = L$$

- We say: “The limit of  $f(x)$  as  $x$  goes to  $a$  is  $L$ ”.
- We mean:
  - $f(x)$  is defined for all  $x$  near  $a$  (possibly not  $a$  itself)
  - As  $x$  gets closer to  $a$ ,  $f(x)$  gets closer and closer to  $L$

## Examples

Compute  $\lim_{x \rightarrow 2} f(x)$ , with  $f(x)$  given by

- $f(x) = x - 1$  when  $x \neq 2$
- $f(x) = 2$  otherwise.

# Examples

$$\textcircled{1} \quad \lim_{x \rightarrow -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}$$

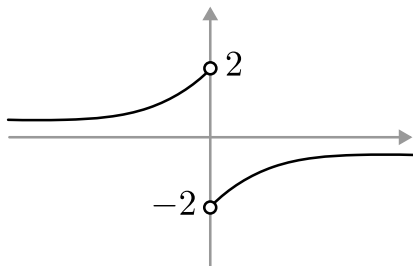
$$\textcircled{2} \quad \lim_{x \rightarrow 0} \frac{x}{\sqrt{3+x} - \sqrt{3-x}}$$

# One-sided limits



## One-sided limits

Sometimes the limit is different depending on which side you approach  $a$ .



$$\lim_{x \rightarrow 0} f(x) = ?$$

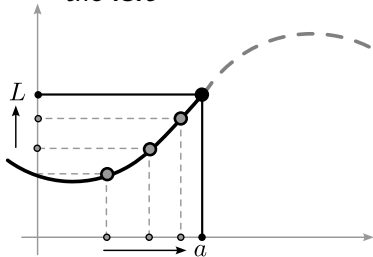
- $+2$  when  $x \rightarrow 0$  from left
- $-2$  when  $x \rightarrow 0$  from right

# Definition

**Left Limit:**

$$\lim_{x \rightarrow a^-} f(x) = L$$

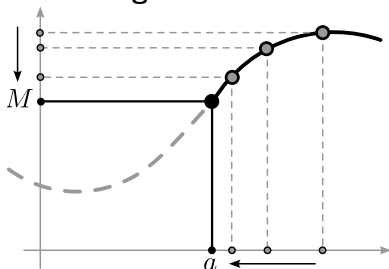
- $f(x)$  defined to the **left** of  $a$
- $f(x) \rightarrow L$  as  $x \rightarrow a$  from the **left**



**Right Limit:**

$$\lim_{x \rightarrow a^+} f(x) = M$$

- $f(x)$  defined to the **right** of  $a$
- $f(x) \rightarrow M$  as  $x \rightarrow a$  from the **right**



## Relation with two-sided limit

The two-sided limit exists,

$$\lim_{x \rightarrow a} f(x) = L,$$

if and only if both one-sided limits exist and are equal:

$$\lim_{x \rightarrow a+} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a-} f(x) = L$$

# Examples

①  $\lim_{x \rightarrow 0^+} \sqrt{x}$

②  $\lim_{x \rightarrow 0} \frac{|x|}{x}$

③  $\lim_{x \rightarrow 2} \frac{|x - 2|}{x^2 + x - 6}$

④  $\lim_{x \rightarrow \pi} f(x)$ , where  $f(x) = \sin x$  when  $x < \pi$  and  $f(x) = \sqrt{x - \pi}$  otherwise.

# Limit laws and theorems

# Limit laws

- ①  $\lim_{x \rightarrow a} x = a$
- ②  $\lim_{x \rightarrow a} c = c$
- ③  $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- ④  $\lim_{x \rightarrow a} (c(f(x))) = c \lim_{x \rightarrow a} f(x)$
- ⑤  $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- ⑥  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$
- ⑦  $\lim_{x \rightarrow a} (f(x)^{m/n}) = (\lim_{x \rightarrow a} f(x))^{m/n} = L^{m/n},$ 
  - If  $n$  even:  $L$  must be positive
  - If  $m < 0$ :  $L$  must be different from 0.

## Limit laws

- ⑧ If  $f(x) = g(x)$  for all  $x \neq a$  close to  $a$ , then
$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x).$$
- ⑨ If  $f(x) \leq g(x)$  for all  $x \neq a$  close to  $a$ , then
$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

These laws also hold for one-sided limits, with appropriate modifications.

# Limits of polynomials and rational functions

- Polynomial:  $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

$$\lim_{x \rightarrow a} P(x) = P(a).$$

- Rational function:  $F(x) = \frac{P(x)}{Q(x)}$ , with  $P(x)$ ,  $Q(x)$  polynomials

$$\lim_{x \rightarrow a} F(x) = \frac{\lim_{x \rightarrow a} P(x)}{\lim_{x \rightarrow a} Q(x)} = \frac{P(a)}{Q(a)} \quad \text{if } Q(a) \neq 0.$$



# The squeeze theorem

Lets you compute the limit of a difficult function  $g(x)$  “squeezed” between two simple functions  $f(x)$  and  $h(x)$ .

Suppose

- ①  $f(x) \leq g(x) \leq h(x)$  for some  $x$  near  $a$
- ②  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$ .

Then

$$\lim_{x \rightarrow a} g(x) = L.$$

## Example

Compute  $\lim_{t \rightarrow 0} t^2 \sin \left( \frac{1}{t} \right)$ .

plot  $x^2 \sin(1/x)$



NATURAL LANGUAGE



MATH INPUT



EXTENDED KEYBOARD



EXAMPLES



UPLOAD



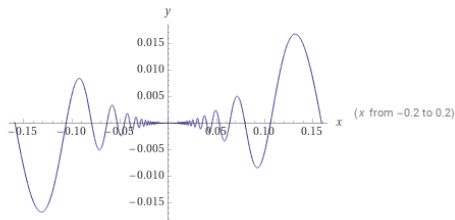
RANDOM

Input interpretation

plot

$$x^2 \sin\left(\frac{1}{x}\right)$$

Plots



Source: Wolfram Alpha