

Integration: Definitions and basic concepts

Introduction to Engineering Mathematics

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Indefinite integrals

Indefinite integrals as antiderivatives

- **Antiderivative:** the opposite (inverse operation) of a derivative.
- The **indefinite integral** is an antiderivative.

Examples:

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In these examples, C is an **arbitrary constant**.

Anatomy of an indefinite integral

The diagram illustrates the components of an indefinite integral $\int f(x) dx$. It features three labels with orange lines pointing to specific parts of the expression:

- Integral sign (curly 'S')* points to the large integral symbol \int .
- Differential (identifies the variable)* points to the dx term.
- Integrand (the function to integrate)* points to the $f(x)$ term.

$$\int f(x) dx$$

Examples

Polynomials:

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
- $\int \frac{dx}{x} = \ln |x| + C$

Trigonometric functions:

- $\int \sin x dx = -\cos x + C$
 - $\int \cos x dx = \sin x + C$
- $\int \tan x dx = -\ln |\cos x| + C$
Not an obvious antiderivative; we will see how to derive this integral later.

Examples

Exponential/logarithmic functions:

- $\int e^x dx = e^x + C$
- $\int \frac{dx}{x} = \ln |x| + C$

Inverse trigonometric functions:

- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$
- $\int \frac{dx}{1+x^2} = \arctan(x) + C$

Examples

Special cases:

- $\int \sec^2 x \, dx = \tan x + C$
- $\int \csc^2 x \, dx = -\cot x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \csc x \cot x \, dx = -\csc x + C$

Definite integrals

The definite integral

Definite integral: integral with “bounds”

$$\int_a^b f(x) dx = F(b) - F(a)$$

Upper bound

Lower bound

How to compute:

- 1 Find primitive function $F(x)$: $\int f(x) dx = F(x) + C$
- 2 Substitute bounds into $F(x)$:

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b.$$

Examples

Compute:

- $\int_0^{\pi/2} \sin x \, dx$
- $\int_{-1}^2 (x^2 + 2x - 1) \, dx$

Properties of the definite integral

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- Linearity:

$$\int_a^b (Af(x) + Bg(x))dx = A \int_a^b f(x)dx + B \int_a^b g(x)dx$$

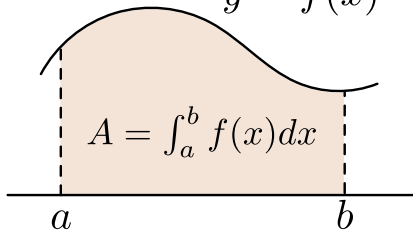
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- Linearity:
$$\int_a^b (Af(x) + Bg(x))dx = A \int_a^b f(x)dx + B \int_a^b g(x)dx$$
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

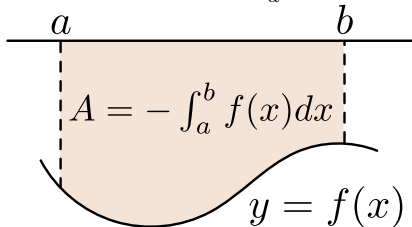
Definite integrals correspond to signed areas

If $f(x) \geq 0$: $A = \int_a^b f(x) dx$.

$$y = f(x)$$



If $f(x) \leq 0$: $A = -\int_a^b f(x) dx$.



If $f(x)$ changes sign: break up into parts above/below x -axis.

Examples

Find the area...

- Between the graph of $y = \sin x$ and the x -axis, from 0 to 2π .
- Below the graph of $y = 3x - x^2$ and above the x -axis.

Properties (continued)

- For an even function, $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$

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- For an odd function, $\int_{-a}^a f(x) = 0$.
- “King's rule”: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

FREE Wi-Fi

$$\int_{-2}^2 \left(x^3 \cos \frac{x}{2} + \frac{1}{2} \right) \sqrt{4 - x^2} \, dx$$



The Wi-Fi password is the first 10 digits of the answer.

Continuity and integration

- So far, we have silently assumed $f(x)$ is continuous on $[a, b]$ to define the integral $\int_a^b f(x)dx$.
- The integral can also be defined if $f(x)$ is **piecewise continuous**.

Examples

Compute the following integrals:

- $\int \left(10x^4 - \frac{2}{\sqrt{1-x^2}} \right) dx$
- $\int_0^1 x^{5/2}(1-x) dx$
- $\int \frac{x^2}{1+x^2} dx$

Examples

Show that $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}.$

Difficult, uses King's rule.

Fundamental theorem of calculus

Integrals and derivatives are each other's inverse

- If $G(x) = \int_1^x \ln y \, dy$, find $G'(x)$.
- If $H(x) = \int_1^{x^2} \ln y \, dy$, find $H'(x)$.