

# Derivatives (1/2)

## Introduction to Engineering Mathematics

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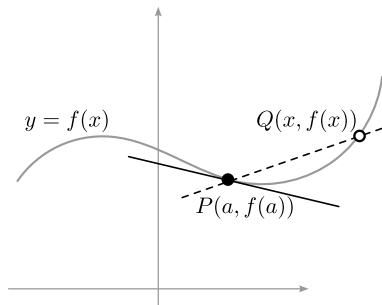
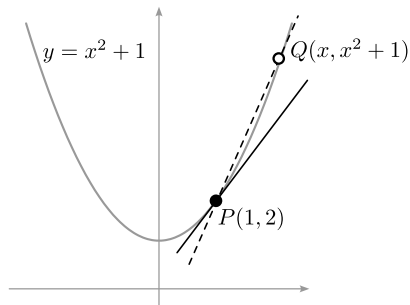
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## Motivation: tangent lines

In the previous class, we managed to compute the slope of the tangent line to the parabola  $y = x^2$  by computing the limit

$$m = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2.$$

How would we do this for an **arbitrary curve**  $y = f(x)$ ?



## Derivative of a function

We define the **derivative of**  $f(x)$  **at**  $x = a$  (“f prime”) as

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (\text{if the limit exists}).$$

If we put  $h = x - a$ , we can rewrite this as

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

This is sometimes easier to compute.

If  $f'(a)$  exists, we say that  $f(x)$  is **differentiable** at  $x = a$ .

## Examples

Compute the derivative of  $f(x) = 3x^2 + 7x - 5$  at  $x = 1$ .

## Examples

Compute the derivative of  $f(x) = |x|$  at  $x = a$ .

# The derivative as a function

By letting  $a$  vary in  $f'(a)$ , we obtain a function given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (\text{if the limit exists}).$$

There are many notations for the derivative function: for  $y = f(x)$ ,

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f(x).$$

all mean the same thing.

## Example

Show that  $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$ .



# Properties of differentiable functions

We say that  $y = f(x)$  is **differentiable on an interval**  $[a, b]$  if

- $f'(x)$  exists for each  $x \in (a, b)$
- At  $x = a$ , the *right derivative* exists:

$$f'(a) = \lim_{h \rightarrow 0+} \frac{f(a+h) - f(a)}{h}.$$

- At  $x = b$ , the *left derivative* exists:

$$f'(b) = \lim_{h \rightarrow 0-} \frac{f(b+h) - f(b)}{h}.$$

## Link between differentiability and continuity

- If  $f(x)$  is differentiable at  $x = a$ , then  $f(x)$  is also continuous at  $x = a$ .
- The converse is not necessarily true!

Example: Show that  $y = |x|$  is differentiable for all  $x \neq 0$ .

## Higher-order derivatives

Now that we can take the derivative of a function, we can take the derivative of the derivative, and so on...

| Notation                                 | Name                    |
|--|-------------------------|
| $f'(x)$                                  | 1st-order derivative    |
| $f''(x) = \frac{d}{dx} f'(x)$            | 2nd-order derivative    |
| ...                                      | ...                     |
| $f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x)$ | $n$ th-order derivative |

# Derivative of a polynomial: basic rules

1

$$\frac{d}{dx}c = 0$$

2

$$\frac{dx^n}{dx} = nx^{n-1} \quad \text{for } n \neq 0$$

## Derivative of a polynomial: basic rules

3

$$\frac{d}{dx}(cf(x)) = c \frac{df}{dx}$$

4

$$\frac{d}{dx}(f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx}$$

## Examples

Compute the derivative of  $f(x) = 3x^2 + 7x - 5$ .

# Derivative of the exponential/logarithm

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

For a different base:

$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

- The last two rules will follow from the chain rule
- **Do not confuse the rules for  $a^x$  and  $x^n$ !**

## Product and quotient rules

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$
$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Examples: Compute

- $\frac{d}{dx}(x^2 e^x)$
- $\frac{d}{dx} \frac{x+1}{x+3}$



# Derivatives of trigonometric functions

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$$

## Examples

Using the rules from the previous slide, show that

$$\frac{d}{dx} \csc x = -\cot x \csc x$$

$$\frac{d}{dx} \sec x = \tan x \sec x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

Example solution:

$$\frac{d}{dx} \csc x = \frac{d}{dx} \frac{1}{\sin x} = \frac{1' \cdot \sin x - \sin'(x) \cdot 1}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}.$$

(This will be easier once we cover the **chain rule**)

## Derivatives of inverse trigonometric functions

$$\begin{aligned}\frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \arccos x &= \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \arctan x &= \frac{1}{1+x^2}\end{aligned}$$

These rules can be derived by means of *implicit differentiation*, which we will cover later.

## The chain rule

- Useful for composite functions  $F = f \circ g$  (“ $f$  after  $g$ ”)
- If  $f(x)$  and  $g(x)$  are differentiable, then the composite function  $F(x)$  is also differentiable and

$$F'(x) = f'(g(x)) \cdot g'(x).$$

## Example

Compute  $\frac{d}{dx} e^{\sqrt{\cos x}}$ .

## Example

Compute  $\frac{d}{dx} x^{\sin(x)}$ .