Theory of equations (2/2): Polynomial equations

Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

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Overview

- Every polynomial of degree N has N roots
  - Some of these roots may be *complex* (e.g. $x^2 + 1$)
  - Some of these roots may be *the same* (e.g. $x^2 + 2x + 1$)
- Roots correspond to factors of the polynomial
- There is no algorithm for finding all roots of a polynomial
- If a real polynomial has a complex root z, then the complex conjugate $\bar{z}$ is also a root (e.g. $x^3 - x^2 + x - 1$)

Roots and factors of polynomials

Reminder

Polynomial of degree n:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- The number n is called the **degree** of $P(x)$.
- A **root** or **zero** is a number α such that $P(\alpha) = 0$.
- Roots can be real ($\alpha \in \mathbb{R}$) or complex ($\alpha \in \mathbb{C}$).
- A **factor** is a polynomial $F(x)$ such that $P(x) = F(x)Q(x)$ for some other polynomial $Q(x)$.
  - Linear factor: $F(x) = x - \alpha$
  - Quadratic factor: $F(x) = Ax^2 + Bx + C$

Theorems involving polynomials

Remainder theorem

Remainder theorem (special case)

If $P(x)$ is a polynomial, then $P(h)$ is the remainder of $P(x)$ divided by $x - h$.

Corollary

Note: "Corollary" means "consequence".

If $P(x)$ is a polynomial with zero $\alpha \in \mathbb{C}$ (in other words, $P(\alpha) = 0$), then $x - \alpha$ is a factor of $P(x)$:

$$P(x) = (x - \alpha)Q(x).$$

Example

Find all the factors of $P(x) = 2x^3 + 3x^2 - 1$.

Remainder theorem (general version)

If $P(x)$ is a polynomial with *distinct* zeros $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{C}$, then $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$ is a factor of $P(x)$:

$$P(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)Q(x).$$

Notes:

- There are at most n distinct zeros, where n is the degree of $P(x)$ (see later).

Examples

Find a polynomial of degree 4 with roots $\pm i$, ± 2, and such that $P(3) = 25$.

Examples

Find a polynomial of degree 4 with roots 0 and -2, and where the root -2 has multiplicity 3.

How many roots can a polynomial have?

Theorem: A polynomial $P(x) \neq 0$ cannot have more than n distinct roots, where $n = \deg P(x)$.

Proof: Assume that there are m distinct roots $\alpha_1, \dots, \alpha_m$, with $m > \deg P(x)$. Then by the remainder theorem,

$$P(x) = (x - \alpha_1) \cdots (x - \alpha_m) Q(x).$$

The left-hand side has degree n, whereas the right-hand side has degree at least $m > n$. This is a contradiction.

Relation between roots and coefficients

Define the **symmetric polynomials**:

- $S_1 = a_1 + \cdots + a_n$
- $S_2 = a_1a_2 + a_1a_3 + \cdots + a_1a_n + a_2a_3 + \cdots + a_{n-1}a_n$
- $S_3 = a_1a_2a_3 + \cdots + a_{n-2}a_{n-1}a_n$
- ...
- $S_n = a_1a_2 \cdots a_n$

Then:

$$(x - a_1)(x - a_2)\cdots(x - a_n) = \\ x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \cdots + (-1)^n S_n. \quad (1)$$

Example

Given $P(x) = x^3 + 2x^2 - 3x - 1$ with roots α, β, and γ, find the value of $\alpha^2 + \beta^2 + \gamma^2$.

Fundamental theorem

The fundamental theorem of algebra

Theorem: Each polynomial has *at least one* root (which may be complex).

Proof: Difficult.

Consequence: Each polynomial of degree n has exactly n roots (which may be same).

How to find roots?

- Degree 2: formula for quadratic equation
- Degree 3, 4: formulas exist, but they are very complicated
- Degree 5 and up: **no general formula exists**

In general, proceed via trial and error, or numerically.

Example

Factorize $P(x) = x^4 + 2x^3 + 2x^2 + 2x + 1$.

Complex conjugates

Complex conjugates theorem

Theorem: If $P(x)$ is a polynomial with real coefficients, then complex roots appear in *conjugates*.

In other words, if $z = \alpha + i\beta$ is a root with multiplicity p, then $\bar{z} = \alpha - i\beta$ is also a root with multiplicity p.

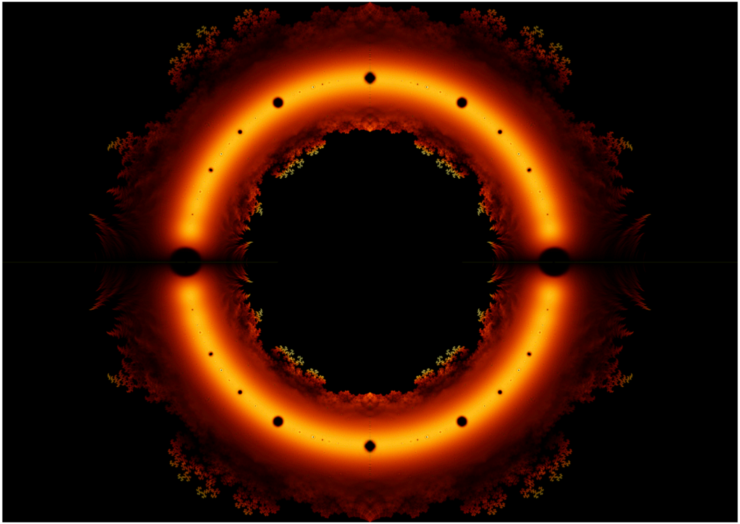

Figure 1. Roots of all polynomials of degree 23 whose coefficients are ± 1. The brightness shows the number of roots per pixel.

Source: *The beauty of roots*, https://arxiv.org/pdf/2310.00326.