Integration: Inverse trigonometric substitution

Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

Overview

- Inverse trigonometric substitution
- Other substitutions
  - Higher order roots
  - $\tan \frac{\theta}{2}$-substitution

Inverse trigonometric substitution

What is inverse trigonometric substitution?

- Direct substitution: replace part of the integrand by a new variable
- Inverse substitution: **replace variable by new function**.

Example: $\int \sqrt{1-x^2}dx$ with the substitution $x = \cos\theta$

Why is this useful?

Inverse trigonometric substitution can solve integrals that have a power of $x^2 \pm a^2$ somewhere in the integrand.

Example:

- $\displaystyle\int \frac{dx}{(x^2-25)^3}$
- $\displaystyle\int \sqrt{1+x^2}dx$

Types of substitutions

The exact substitution depends on the integrand. We will distinguish the following three cases, depending on factors that occur in the integrand.

1. Factor $\sqrt{a^2 - x^2}$: use $x = a\sin\theta$
2. Factor $\sqrt{a^2 + x^2}$: use $x = a\tan\theta$
3. Factor $\sqrt{x^2 - a^2}$: use $x = a\sec\theta$

Don't memorize these substitutions: use your insight into trigonometry to think what substitution would simplify the integrand.

Remark: completing the square

Often the integrand will not exactly be in one of the three previous forms. In that case, **completing the square** may help:

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} = \int \frac{dt}{t^2+1} = \ldots$$

Case 1: $\sqrt{a^2 - x^2}$

- Substitution: $x = a \sin\theta$, with $\theta \in [-\pi/2, \pi/2]$
- Differential: $dx = a \cos\theta \, d\theta$

Example: $\displaystyle\int \frac{dx}{(5 - x^2)^{3/2}}$

Case 2: $\sqrt{a^2 + x^2}$

- Substitution: $x = a \tan\theta$, with $\theta \in (-\pi/2, \pi/2)$
- Differential: $dx = a \sec^2\theta \, d\theta$

Example: $\displaystyle\int \frac{dx}{\sqrt{1 + 4x^2}}$

Case 3: $\sqrt{x^2 - a^2}$

- Substitution: $x = a \sec\theta$, with $\theta \in (0, \pi)$ **(two cases)**
- Differential: $dx = a \tan\theta \sec\theta \, d\theta$

Example: $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$

Other substitutions

Higher roots

Useful when the integrand has a factor $\sqrt[n]{ax+b}$.

- Substitution: $ax + b = u^n$
- Differential: $a\,dx = nu^{n-1}\,du$

Example: $\displaystyle\int_{-1/3}^{2} \frac{x\,dx}{\sqrt[3]{3x+2}}$

Example

Sometimes this rule can be applied to integrands that are somewhat more general than what's shown on the previous slide.

Example: $\displaystyle\int \frac{dx}{x^{1/2}(1+x^{1/3})} = \ldots$

(Substitute $x^{1/6} = u$)

$\tan \frac{\theta}{2}$-substitution

Useful when the integrand is a rational function of $\sin\theta$ and $\cos\theta$.

- Substitution: $t = \tan\frac{\theta}{2}$, so that $\theta = 2\tan^{-1}(t)$
- Differential:

$$d\theta = \frac{2dt}{1+t^2}$$

- t-formulas:

$$\sin\theta = \frac{2t}{1+t^2} \quad \text{and} \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$

Example

$$\int \frac{d\theta}{2+\cos\theta} = \ldots$$