

# Trigonometry (2/5): Formulas

## Introduction to Engineering Mathematics

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# Overview

- ① Reflection identities
- ② Shift formulas
- ③ Addition/subtraction formulas
- ④ Double/half angle formulas
- ⑤ Formulas for product to sum and sum to product

## Notes for this chapter

- You should learn all of these formulas by heart.
- The best way to learn these formulas is by doing lots of practice problems.
- Often, knowing the derivation of a formula will also help you remember it.
- Follow along with the lectures to fill in missing steps.

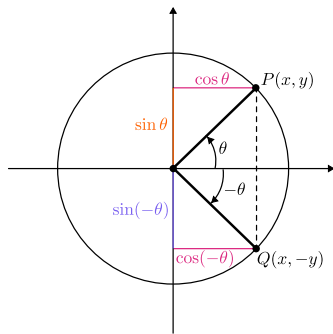
# Reflection identities

# Reflection across the $x$ -axis (even/odd identities)

Formulas:

$$\begin{aligned}\cos(-\theta) &= \cos \theta \\ \sin(-\theta) &= -\sin \theta\end{aligned}$$

Example: Compute  $\sin(-\pi/4)$

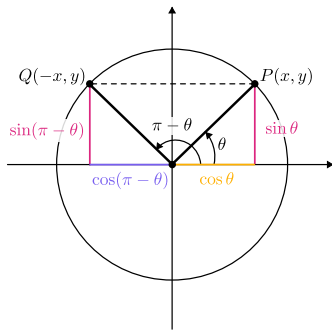


# Reflection across the $y$ -axis (complementary angles)

Formulas:

$$\cos(\pi - \theta) = -\cos \theta$$

$$\sin(\pi - \theta) = \sin \theta$$



## Shift formulas

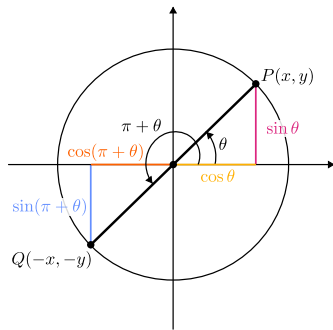
## Shift formulas (by $\pi$ )

Formulas:

$$\cos(\pi + \theta) = -\cos \theta$$

$$\sin(\pi + \theta) = -\sin \theta$$

$$\tan(\pi + \theta) = \tan \theta$$





## Shift formulas (by $\pi/2$ )

Formulas:

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta \\ \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \cot \theta\end{aligned}$$

- Proof skipped.
- **Very important formulas** to turn sin into cos and vice versa.

## Addition/subtraction formulas

# Addition/subtraction formulas (for sin and cos)

Formulas:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

- Proof skipped.
- Note the signs for cos!
- Every formula on the previous slides can be derived from these formulas.

## Example

Compute  $\sin\left(\frac{7\pi}{12}\right)$ .

## Addition/subtraction formulas (for tan)

Formulas:

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ \tan(\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}\end{aligned}$$

## Double/half angle formulas

## Double-angle formulas ( $2\theta \rightarrow \theta$ )

Formulas:

$$\begin{aligned}\sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ \tan(2\theta) &= \frac{2 \tan \theta}{1 - \tan^2 \theta}\end{aligned}$$

- Follows from the addition/subtraction formulas.
- The double-angle formula for  $\cos$  can also be written as

$$\begin{aligned}\cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta\end{aligned}$$

## Example

Prove that  $\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$ .



## Half-angle formulas ( $\theta \rightarrow 2\theta$ )

Formulas:

$$\begin{aligned}\cos^2 \theta &= \frac{1 + \cos(2\theta)}{2} \\ \sin^2 \theta &= \frac{1 - \cos(2\theta)}{2} \\ \tan^2 \theta &= \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}\end{aligned}$$

These formulas are useful to compute integrals of powers of  $\sin$  and  $\cos$ .

## Example

Express  $\cos^2 x \sin^2 x$  as a combination of sines and cosines, without any powers.

## Formulas for product to sum and sum to product

## Product-to-sum formulas

Formulas:

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$$

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

# Sum-to-product formulas

Formulas:

$$\sin \alpha + \sin \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right)$$

$$\sin \alpha - \sin \beta = 2 \cos \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha + \cos \beta = 2 \cos \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha - \cos \beta = -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right)$$