Limits and Continuity (2/2)

Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

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Infinite limits and limits at infinity

Infinite limits

We say that $f(x)$ has an **infinite limit** for $x \to a$ if the values of $f(x)$ become **arbitrarily large** when $x \to a$.

We note

$$\lim_{x\to a} f(x) = \infty \quad \text{or} \quad \lim_{x\to a} f(x) = -\infty,$$

depending on the sign of $f(x)$ near $x = a$.

Examples:

- $\displaystyle\lim_{x\to 0} \frac{1}{x^2}$
- $\displaystyle\lim_{x\to 0+} \ln x$

Vertical asymptotes

The line $x = a$ is a **vertical asymptote** of $f(x)$ if *at least one* of the following is true:

$$\lim_{x \to a+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a-} f(x) = \pm\infty.$$

Example: find the vertical asymptotes of

- $y = \ln x$
- $y = \tan x$

Example

Find all vertical asymptotes of the function

$$f(x) = \frac{|x-1|}{x^2 - 3x + 2}.$$

Hint: *Look for zeros of the denominator that are not zeros of the numerator.*

Example

Find all vertical asymptotes of the function

$$f(x) = \ln\left(\frac{x^2 - x}{x^2 - 4}\right).$$

Limits at infinity (intuition)

What happens to $f(x)$ when x becomes very large (positive/negative)?

Examples:

- $\lim\limits_{x \to \pm\infty} \dfrac{x^2 - 1}{x^2 + 1}$
- $\lim\limits_{x \to +\infty} e^{-x}$

Limits at infinity (definition)

We write:

$$\lim_{x \to +\infty} f(x) = L$$

$$\lim_{x \to -\infty} f(x) = M$$

We mean:

- $f(x)$ is defined on an interval $(a, +\infty)$
- $f(x)$ gets as close as we want to L by taking x large enough.

- $f(x)$ is defined on an interval $(-\infty, b)$
- $f(x)$ gets as close as we want to M by taking x large enough (in the negative direction).

Horizontal asymptotes

The line $y = L$ is a **horizontal asymptote** of $f(x)$ if *at least one* of the following is true:

$$\lim_{x \to +\infty} f(x) = L \quad \text{or} \lim_{x \to -\infty} f(x) = L.$$

Examples: Find the horizontal asymptotes of

- $y = \dfrac{x^2 - 1}{x^2 + 1}$
- $y = e^{-x}$
- $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Example

Find all horizontal asymptotes of the function

$$f(x) = \ln\left(\frac{x^2 - x}{x^2 - 4}\right).$$

Limit laws

1. $\lim_{x\to\infty} x = \infty$
2. $\lim_{x\to\infty} c = c$ (for c a constant)
3. $\lim_{x\to\infty}(f(x) + g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$
4. $\lim_{x\to\infty}(cf(x)) = c\lim_{x\to\infty} f(x)$
5. $\lim_{x\to\infty}(f(x)g(x)) = \lim_{x\to\infty} f(x)\lim_{x\to\infty} g(x)$
6. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$, if the denominator is not zero

Limit laws (continued)

7. $\lim_{x\to\infty} f(x)^{m/n} = \left(\lim_{x\to\infty} f(x)\right)^{m/n} = L^{m/n}$, if
   - $L \geq 0$ for n even
   - $L \neq 0$ for m negative
8. If $f(x) \leq g(x)$, then $\lim_{x\to\infty} f(x) \leq \lim_{x\to\infty} g(x)$
9. $\lim_{x\to\infty} 1/x^n = 0$, if $n > 0$
10. $\lim_{x\to\infty} g(f(x)) = \lim_{y\to c} g(y)$, where $c = \lim_{x\to\infty} f(x)$, if
    - c is constant (not equal to $\pm\infty$)
    - f is continuous (see later)

Examples

$$\lim_{x \to +\infty} \frac{2x^5 + 1}{x^5 + x^3 + 1}$$

Examples

$$\lim_{x \to -\infty} \frac{x^7}{\sqrt{x^{14} + 1}}$$

Examples

$$\lim_{x \to +\infty} \left(\sqrt{x^2 + 1} - x \right)$$

Examples

$$\lim_{x \to 0-} e^{1/x}$$

Examples

$$\lim_{x \to +\infty} \sin x$$

Examples

$$\lim_{x\to\pi/2-} e^{\tan x}$$

Infinite limits at infinity

Examples:

Summary of limit techniques

1. Try "just substituting" $\pm\infty$
2. Use common manipulations:
   - Simplify
   - Highest powers
   - Conjugate
   - ...

Special cases (to memorize):

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

Continuity

Motivation and definition

- Intuitively, continuity = "no jumps"
- Mathematically, $f(x)$ is **continuous** at an interior point $x = a$ of its domain if

$$\lim_{x \to a} f(x) = f(a).$$

Example: $f(x) = \dfrac{x^2 - x - 3}{x - 2}$ is not continuous at $x = 2$.

Examples

Define $f(x) = \frac{x^2-x-2}{x-2}$ when $x \neq 2$ and $f(x) = 1$ otherwise (i.e. $f(2) = 1$). Is $f(x)$ continuous at $x = 2$?

Left/right continuity

At $x = a$, $f(x)$ is ...

- **Right continuous** if $\lim_{x \to a+} f(x) = f(a)$
- **Left continuous** if $\lim_{x \to a-} f(x) = f(a)$.

Example: the Heaviside function is defined by $H(x) = 0$ when $x < 0$ and $H(x) = 1$ when $x \geq 0$. Is $H(x)$ right or left continuous at $x = 0$?

Continuity on an interval

The function $f(x)$ is continuous on an interval $[a, b]$ if ...

- $f(x)$ is continuous at every $x \in (a, b)$
- $f(x)$ is right-continuous in a
- $f(x)$ is left-continuous in b.

Example: Show that $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1, 1]$.

Making new continuous functions out of old ones

If $f(x), g(x)$ are continuous at $x = a$, and c is a constant, then the following are also continuous at $x = a$:

- $f + g$
- $f - g$
- cf
- fg
- f/g (if $g(a) \neq 0$)

Which functions are continuous?

Continuous **on their domain**:

- Polynomials
- Rational functions
- Trigonometric functions + inverse trigonometric functions
- Square roots, nth roots
- $\log$, $\exp$

In particular:

- Polynomials are continuous for all $x \in \mathbb{R}$
- Rational functions $P(x)/Q(x)$ are continuous for all x so that $Q(x) \neq 0$.

Composition of functions

If $f(x)$ is continuous and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(b) = f\left(\lim_{x\to a} g(x)\right).$$

Example: $\displaystyle\lim_{x\to 2} \sin\left(\frac{2-x}{4-x^2}\pi\right)$

Corollary

If g is continuous at $x = a$ and f is continuous at $y = g(a)$, then $f \circ g$ is continuous at $x = a$.

Examples: Where are the following functions continuous?

- $h(x) = \sin x^2$
- $h(x) = \ln(1 + \cos x)$