

# The binomial theorem

## Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

# Table of contents

- ① Pascal's triangle
- ② Binomial coefficients
- ③ The binomial theorem

# Pascal's triangle

# Pascal's triangle

Expand the following expressions and look at the coefficients.

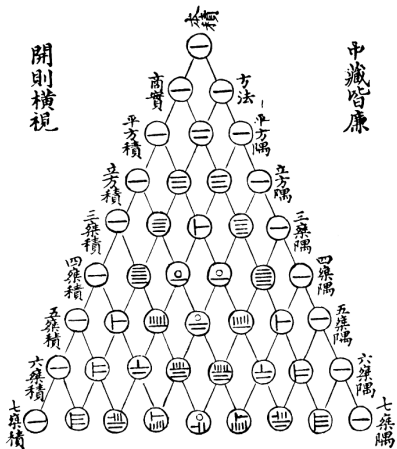
- $(a + b)^0 = 1$
- $(a + b)^1 = a + b$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

What do you notice?

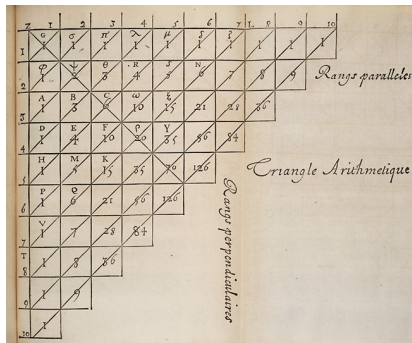
Based on this pattern, what is  $(a + b)^7$ ?

Would you be able to write down  $(a + b)^{27}$ ?

# 古法七乘方圖



七乘積	六乘積	五乘積	四乘積	三乘積	二乘積	一乘積	方積
-----	-----	-----	-----	-----	-----	-----	----



Blaise Pascal, 1665 CE

## Example

Use Pascal's triangle to expand  $\left(2x + \frac{1}{x}\right)^5$ .

# Binomial coefficients

## Binomial coefficients

- Factorial:  $n! = n(n-1)(n-2) \cdots 2 \cdot 1$ .
- Binomial coefficient (also called “n-choose-k”):

$$\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}.$$

- Measures the number of ways of choosing  $k$  objects from among  $n$  choices.



# Properties

For all  $n$  and  $k \leq n$ :

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$

$$\binom{n}{k} = \binom{n}{n-k}$$

# Rewriting Pascal's triangle using binomial coefficients

# The binomial theorem

# The binomial expansion

Putting everything we've learned together, we get

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n.$$

This can be written more compactly as

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$


## Example

Use the binomial expansion to expand  $(\sqrt{x} - 1)^7$ .

# Visual proof of the binomial expansion (optional)

$$(a+b)^1 = \underline{a} + \underline{b}$$

$$(a+b)^2 = a^2 + 2ab + b^2$$


$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$


$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
