Limits and Continuity (1/2)Introduction to Engineering Mathematics

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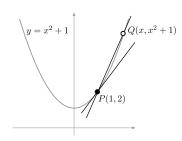
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Motivation

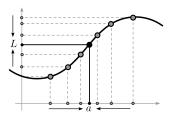
Compute the **tangent line** to the parabola $y=x^2+1$ at the point P(1,2).

- ullet Take another point Q on the parabola
- Compute the line through P and Q
- Let Q "move towards" P

What is the slope of the line when ${\cal Q}$ gets "infinitely close" to ${\cal P}$?



Definition of limit



• We write:

$$\lim_{x\to a} f(x) = L$$

- We say: "The limit of f(x) as x goes to a is L".
- We mean:
 - f(x) is defined for all x near a (possibly not a itself)
 - \bullet As x gets closer to $a,\,f(x)$ gets closer and closer to L

Examples

Compute $\lim_{x \to 2} f(x)$, with f(x) given by

- f(x) = x 1 when $x \neq 2$
- ullet f(x)=2 otherwise.

Examples

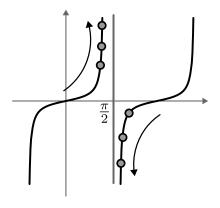
- $\begin{array}{l} \lim\limits_{x\to -2} \frac{x^2+x-2}{x^2+5x+6} \\ \mathbf{2} \lim\limits_{x\to 0} \frac{1}{\sqrt{3+x}-\sqrt{3-x}} \end{array}$

One-sided limits

Sometimes the limit exists on "one side" of the function.

$$\lim_{x \to \pi/2} \tan x = ?$$

- $+\infty$ when $x \to 0$ from right
- $-\infty$ when $x \to 0$ from left

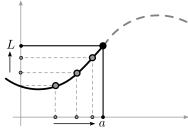


Definition

Left Limit:

$$\lim_{x\to a-}f(x)=L$$

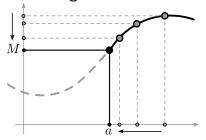
- f(x) defined to the **left** of a
- $f(x) \to L$ as $x \to a$ from the **left**



Right Limit:

$$\lim_{x\to a+} f(x) = M$$

- f(x) defined to the **right** of a
- $f(x) \to M$ as $x \to a$ from the **right**



Relation with two-sided limit

The two-sided limit exists,

$$\lim_{x \to a} f(x) = L,$$

if and only if both one-sided limits exist and are equal:

$$\lim_{x\to a+} f(x) = L \quad \text{and} \quad \lim_{x\to a-} f(x) = L$$

Examples

- $\lim_{x\to 0+} \sqrt{x}$
- $\lim_{x \to 0} \frac{|x|}{x}$
- $\lim_{x \to 2} \frac{|x-2|}{x^2 + x 6}$
- 4 $\lim_{x \to \pi} f(x)$, where $f(x) = \sin x$ when $x < \pi$ and $f(x) = \sqrt{x \pi}$ otherwise.

Limit laws

- $2 \lim_{r \to a} c = c$
- 3 $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- $4 \lim_{x \to a} (c(f(x))) = c \lim_{x \to a} f(x)$
- $\mathbf{5} \ \lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
- $\text{ 6 } \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)} \text{ if } \lim_{x\to a}g(x)\neq 0$
- - ullet If n even: L must be positive
 - If m < 0: L must be different from 0.

Limit laws

- $\textbf{ § If } f(x)=g(x) \text{ for all } x\neq a \text{ close to } a\text{, then } \lim_{x\to a}f(x)=\lim_{x\to a}g(x).$

These laws also hold for one-sided limits, with appropriate modifications.

Limits of polynomials and rational functions

• Polynomial: $P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ $\lim_{x \to a} P(x) = P(a).$

• Rational function: $F(x) = \frac{P(x)}{Q(x)}$, with P(x), Q(x) polynomials

$$\lim_{x \to a} F(x) = \frac{\lim_{x \to a} P(x)}{\lim_{x \to a} Q(x)} = \frac{P(a)}{Q(a)} \quad \text{if } Q(a) \neq 0.$$

The squeeze theorem

Lets you compute the limit of a difficult function g(x) "squeezed" between two simple functions f(x) and h(x).

Suppose

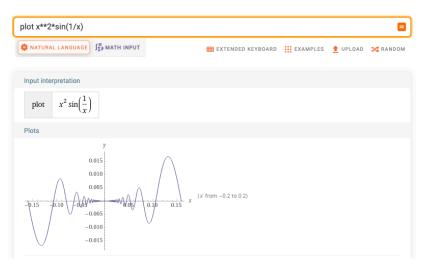
- $2 \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$

Then

$$\lim_{x \to a} g(x) = L.$$

Example

Compute
$$\lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right)$$
.



Source: Wolfram Alpha