Integration: Techniques

Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

Overview

- Integration by substitution
- Trigonometric integration
- Integration by parts
- Reduction formulae

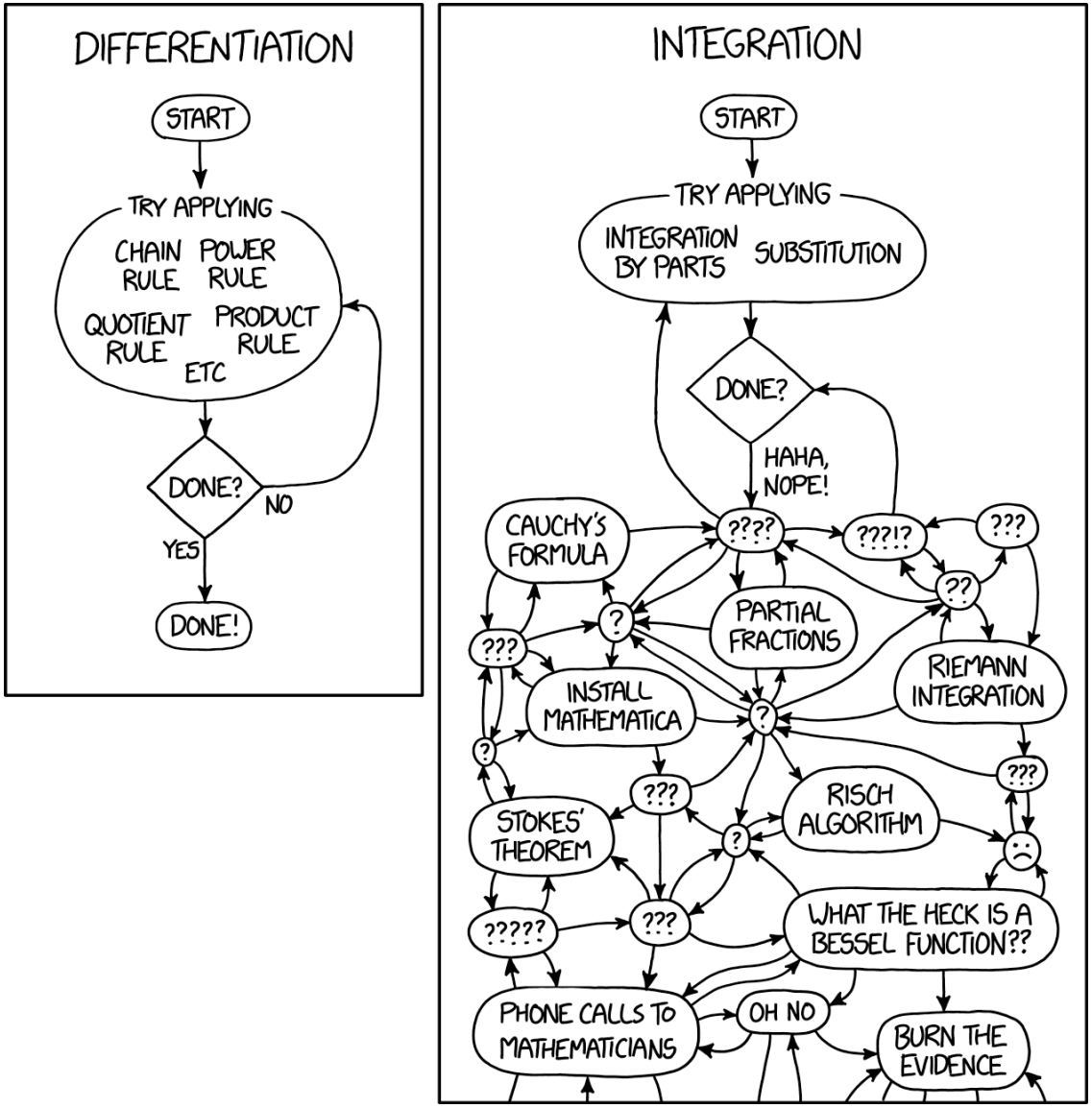

Figure 1: Source: https://xkcd.com/2117/

Warning

There are many more problems in these slides than we can cover in class. The solutions to all problems can be found at the end of this slide deck. You are encouraged to solve the problems that we can't cover in class and check your solutions with the key at the back.

Integration by substitution

Idea: *Find a function whose derivative also occurs in the integral. Replace this function by a new variable to get an easier integral.*

- Don't forget to change the differential!
- Substitute everything!

Example: compute $\int x \sin(1 + x^2)\, dx$

Substitution: definite integrals

Works the same as for indefinite integrals, but also adjust the boundaries.

Example: $\displaystyle\int_0^2 (1+x)^5\,dx$

Examples

$$\int a^{bx}\,dx$$

Examples

$$\int x^3 \cos(x^4 + 2)\, dx$$

Examples

$$\int \frac{\sin(3\ln x)}{x}\,dx$$

Examples

$$\int x^5 \sqrt{1+x^2}\, dx$$

Examples

$$\int e^x \sqrt{1+e^x}\, dx$$

Examples

$$\int \frac{dx}{x^2 + 4x + 5}$$

Trigonometric integration

Useful when: integrand contains trigonometric functions.

Examples: compute

- $\displaystyle\int \tan x \, dx$
- $\displaystyle\int \cot x \, dx$

Examples

$$\int \sec x \, dx$$

Examples

$$\int \csc x \, dx$$

Examples

$$\int \sin^3 x \cos^8 x \, dx$$

Examples

$$\int \cos^5 x \, dx$$

Examples

$$\int \cos^2 x \, dx$$

Examples

$$\int \sin^2 x \, dx$$

Examples

$$\int \sin^4 x \, dx$$

Examples

$$\int \tan^2 x \, dx$$

Examples

$$\int \sec^4 x \, dx$$

Integration by parts

Idea: *Transform integral into a simpler integral (easier to solve).*

Rule: $\displaystyle\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

Example: compute $\displaystyle\int \ln x\,dx$

Examples

$$\int \sec^3 x \, dx$$

Examples

$$\int e^{ax} \cos(bx)\, dx$$

Reduction formula

Idea: *For an integral I_n that depends on a parameter n, find a formula that relates I_n with I_{n-1} (or I_{n-2}, ...) Use this formula to find I_n recursively.*

Example: Let $I_n = \int x^n e^{-x}\, dx$. Find a relation between I_n and I_{n-1}, and use this to determine I_0, I_1, and I_2.

Example

Let $I_n = \int_0^{\pi/2} \cos^n x \, dx$. Find a reduction formula for I_n.

Solutions

- $\displaystyle\int x\sin(1+x^2)\,dx = -\frac{1}{2}\cos(1+x^2)+C$
- $\displaystyle\int_0^2 (1+x)^5\,dx = \frac{364}{3}$
- $\displaystyle\int a^{bx}\,dx = \frac{a^{bx}}{b\ln a}+C$
- $\displaystyle\int x^3\cos(x^4+2)\,dx = \frac{1}{4}\sin(x^4+2)+C$
- $\displaystyle\int \frac{\sin(3\ln x)}{x}\,dx = -\frac{1}{3}\cos(3\ln x)+C$

- $\displaystyle\int x^5\sqrt{1+x^2}\,dx =$
  $\displaystyle\frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$
- $\displaystyle\int e^x\sqrt{1+e^x}\,dx = \frac{2}{3}(1+e^x)^{3/2} + C$
- $\displaystyle\int \frac{dx}{x^2+4x+5} = \tan^{-1}(x+2) + C$
- $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$
- $\displaystyle\int \cot x\,dx = -\ln|\csc x| + C$

- $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$
- $\displaystyle\int \sin^3 x \cos^8 x\,dx = \frac{1}{11}\cos^{11} x - \frac{1}{9}\cos^9 x + C$
- $\displaystyle\int \cos^5 x\,dx = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

- $\displaystyle\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$
- $\displaystyle\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$
- $\displaystyle\int \sin^4 x\,dx = \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C$
- $\displaystyle\int \tan^2 x\,dx = \tan x - x + C$
- $\displaystyle\int \sec^4 x\,dx = \tan x + \frac{\tan^3 x}{3} + C$

- $\displaystyle\int \ln x \, dx = x(\ln x - 1) + C$
- $\displaystyle\int \sec^3 x \, dx = \frac{1}{2}\left(\ln|\sec x + \tan x| + \sec x \tan x\right) + C$
- $\displaystyle\int e^{ax} \cos(bx) \, dx = \frac{e^{ax}}{a^2 + b^2}\left(a\cos(bx) + b\sin(bx)\right) + C$

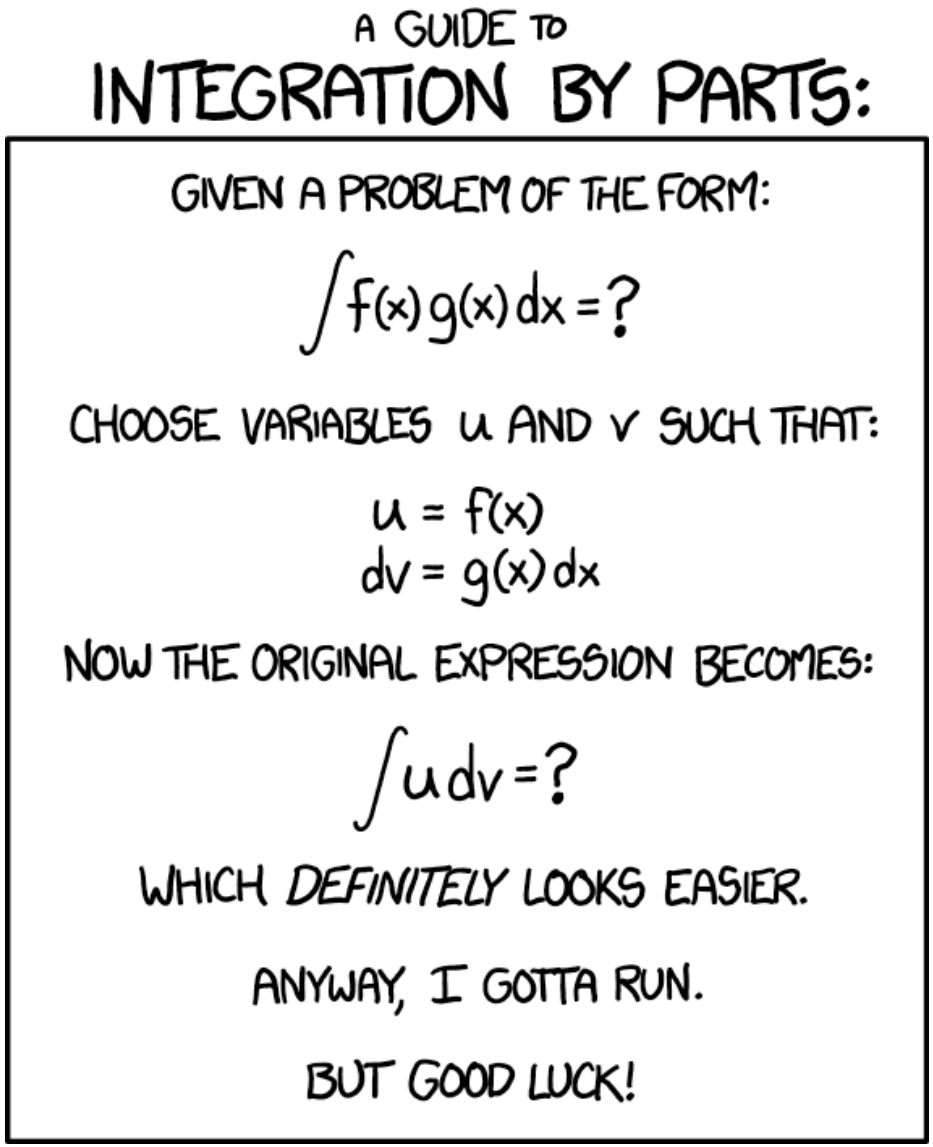

Figure 2: Source: https://xkcd.com/1201/