

Limits and Continuity (1/2)

Introduction to Engineering Mathematics

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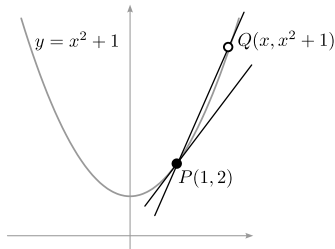
(Two-sided) limits

Motivation

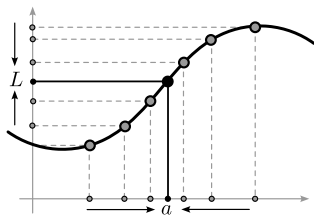
Compute the **tangent line** to the parabola $y = x^2 + 1$ at the point $P(1, 2)$.

- Take another point Q on the parabola
- Compute the line through P and Q
- Let Q “move towards” P

What is the slope of the line when Q gets “infinitely close” to P ?



Definition of limit



- We write:

$$\lim_{x \rightarrow a} f(x) = L$$

- We say: “The limit of $f(x)$ as x goes to a is L ”.
- We mean:
 - $f(x)$ is defined for all x near a (possibly not a itself)
 - As x gets closer to a , $f(x)$ gets closer and closer to L

Examples

Compute $\lim_{x \rightarrow 2} f(x)$, with $f(x)$ given by

- $f(x) = x - 1$ when $x \neq 2$
- $f(x) = 2$ otherwise.

Examples

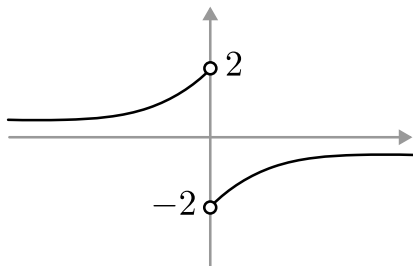
$$\textcircled{1} \quad \lim_{x \rightarrow -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}$$

$$\textcircled{2} \quad \lim_{x \rightarrow 0} \frac{x}{\sqrt{3+x} - \sqrt{3-x}}$$

One-sided limits

One-sided limits

Sometimes the limit is different depending on which side you approach a .



$$\lim_{x \rightarrow 0} f(x) = ?$$

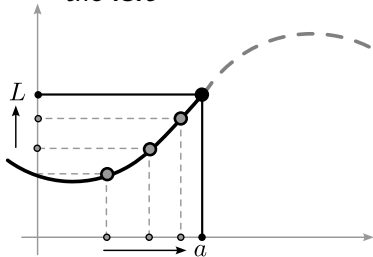
- $+2$ when $x \rightarrow 0$ from left
- -2 when $x \rightarrow 0$ from right

Definition

Left Limit:

$$\lim_{x \rightarrow a^-} f(x) = L$$

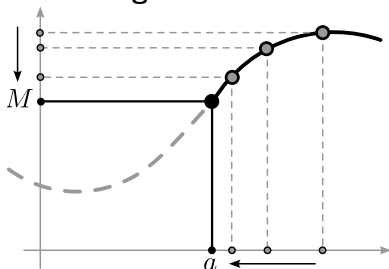
- $f(x)$ defined to the **left** of a
- $f(x) \rightarrow L$ as $x \rightarrow a$ from the **left**



Right Limit:

$$\lim_{x \rightarrow a^+} f(x) = M$$

- $f(x)$ defined to the **right** of a
- $f(x) \rightarrow M$ as $x \rightarrow a$ from the **right**



Relation with two-sided limit

The two-sided limit exists,

$$\lim_{x \rightarrow a} f(x) = L,$$

if and only if both one-sided limits exist and are equal:

$$\lim_{x \rightarrow a+} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a-} f(x) = L$$

Examples

① $\lim_{x \rightarrow 0^+} \sqrt{x}$

② $\lim_{x \rightarrow 0} \frac{|x|}{x}$

③ $\lim_{x \rightarrow 2} \frac{|x - 2|}{x^2 + x - 6}$

④ $\lim_{x \rightarrow \pi} f(x)$, where $f(x) = \sin x$ when $x < \pi$ and $f(x) = \sqrt{x - \pi}$ otherwise.

Limit laws and theorems

Limit laws

- ① $\lim_{x \rightarrow a} x = a$
- ② $\lim_{x \rightarrow a} c = c$
- ③ $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- ④ $\lim_{x \rightarrow a} (c(f(x))) = c \lim_{x \rightarrow a} f(x)$
- ⑤ $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- ⑥ $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ if $\lim_{x \rightarrow a} g(x) \neq 0$
- ⑦ $\lim_{x \rightarrow a} (f(x)^{m/n}) = (\lim_{x \rightarrow a} f(x))^{m/n} = L^{m/n},$
 - If n even: L must be positive
 - If $m < 0$: L must be different from 0.

Limit laws

- ⑧ If $f(x) = g(x)$ for all $x \neq a$ close to a , then
$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x).$$
- ⑨ If $f(x) \leq g(x)$ for all $x \neq a$ close to a , then
$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

These laws also hold for one-sided limits, with appropriate modifications.

Limits of polynomials and rational functions

- Polynomial: $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

$$\lim_{x \rightarrow a} P(x) = P(a).$$

- Rational function: $F(x) = \frac{P(x)}{Q(x)}$, with $P(x)$, $Q(x)$ polynomials

$$\lim_{x \rightarrow a} F(x) = \frac{\lim_{x \rightarrow a} P(x)}{\lim_{x \rightarrow a} Q(x)} = \frac{P(a)}{Q(a)} \quad \text{if } Q(a) \neq 0.$$

The squeeze theorem

Lets you compute the limit of a difficult function $g(x)$ “squeezed” between two simple functions $f(x)$ and $h(x)$.

Suppose

- ① $f(x) \leq g(x) \leq h(x)$ for some x near a
- ② $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$.

Then

$$\lim_{x \rightarrow a} g(x) = L.$$

Example

Compute $\lim_{t \rightarrow 0} t^2 \sin \left(\frac{1}{t} \right)$.

plot $x^2 \sin(1/x)$



NATURAL LANGUAGE



MATH INPUT



EXTENDED KEYBOARD



EXAMPLES



UPLOAD



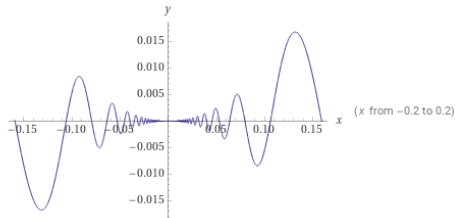
RANDOM

Input interpretation

plot

$$x^2 \sin\left(\frac{1}{x}\right)$$

Plots



Source: Wolfram Alpha