Trigonometry (2/5): Formulas Introduction to Engineering Mathematics

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Overview

1 Reflection identities

2 Shift formulas

3 Addition/subtraction formulas

4 Double/half angle formulas

5 Formulas for product to sum and sum to product

Notes for this chapter

- You should learn all of these formulas by heart.
- The best way to learn these formulas is by doing lots of practice problems.
- Often, knowing the derivation of a formula will also help you remember it.
- Follow along with the lectures to fill in missing steps.

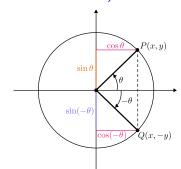
Reflection identities

Reflection across the x-axis (even/odd identities)

Formulas:

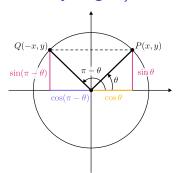
$$cos(-\theta) = cos \theta
sin(-\theta) = -sin \theta$$

Example: Compute $\sin(-\pi/4)$



Reflection across the y-axis (complementary angles)

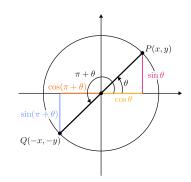
$$\cos(\pi - \theta) = -\cos\theta$$
$$\sin(\pi - \theta) = -\sin\theta$$



Shift formulas

Shift formulas (by π)

$$\begin{aligned} \cos(\pi + \theta) &= -\cos \theta \\ \sin(\pi + \theta) &= -\sin \theta \\ \tan(\pi + \theta) &= -\tan \theta \end{aligned}$$



Shift formulas (by $\pi/2$)

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$
$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

- Proof skipped.
- ullet Very important formulas to turn \sin into \cos and vice versa.

Addition/subtraction formulas

Addition/subtraction formulas (for \sin and \cos)

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$
$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$
$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

- Proof skipped.
- ullet Note the signs for $\cos!$
- Every formula on the previous slides can be derived from these formulas.

Example

Compute $\sin\left(\frac{7\pi}{12}\right)$.

Addition/subtraction formulas (for tan)

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$
$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

Double/half angle formulas

Double-angle formulas $(2\theta \to \theta)$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

- Follows from the addition/subtraction formulas.
- ullet The double-angle formula for \cos can also be written as

$$cos(2\theta) = cos^{2} \theta - sin^{2} \theta$$
$$= 2 cos^{2} \theta - 1$$
$$= 1 - 2 sin^{2} \theta$$

Example

Prove that
$$\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$$
.

Half-angle formulas $(\theta \to 2\theta)$

Formulas:

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$
$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$
$$\tan^2 \theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

These formulas are useful to compute integrals of powers of \sin and \cos .

Example

Express $\cos^2 x \sin^2 x$ as a combination of sines and cosines, without any powers.

Formulas for product to sum and sum to product

Product-to-sum formulas

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$
$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$
$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$$
$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

Sum-to-product formulas

$$\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2}\right) \cos \left(\frac{\alpha - \beta}{2}\right)$$

$$\sin \alpha - \sin \beta = 2 \cos \left(\frac{\alpha + \beta}{2}\right) \sin \left(\frac{\alpha - \beta}{2}\right)$$

$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2}\right) \cos \left(\frac{\alpha - \beta}{2}\right)$$

$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2}\right) \sin \left(\frac{\alpha - \beta}{2}\right)$$

Saw this guy wearing an interesting hoodie at the library today

