# Complex Numbers

## Introduction to Engineering Mathematics

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# What are complex numbers?

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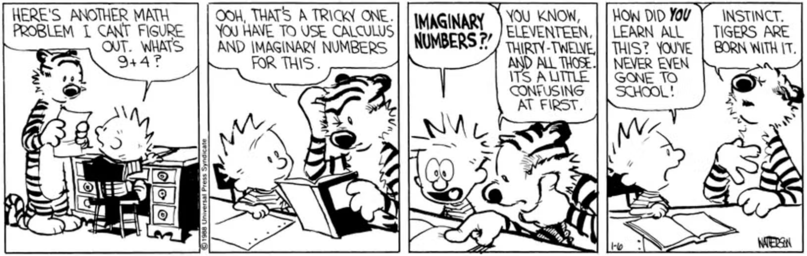


# Motivation: working with "impossible" numbers

At first, this expression looks like complete nonsense:

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}$$

But is it really?

# Benefit: solving quadratic equations

Find $x$ so that $x^2 - 2x + 2 = 0$.

- As $D = -4 < 0$, there are **no real solutions**
- If we set $i = \sqrt{-1}$, then we find **two solutions**.

OK to calculate with $i$, as long as we remember

$$\boxed{i^2 = -1.}$$

# Number systems

| Symbol | Elements | Used for |
|---|---|---|
| $\mathbb{N}$ | 0, 1, 2, … | Counting |
| $\mathbb{Z}$ | …, -1, 0, 1, 2, … | Adding/subtracting |
| $\mathbb{Q}$ | Fractions $n/m$ | Dividing |
| $\mathbb{R}$ | $\mathbb{Q}$ and irrational numbers: $e$, $\pi$, … | Limits |
| $\mathbb{C}$ | $a + ib$, with $a, b \in \mathbb{R}$ | Solving equations |

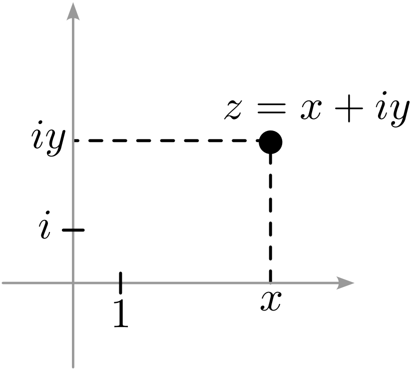
# Cartesian and polar representation

# Graphical representation of complex numbers

If $z = x + iy$ is a complex number, then

- $\mathrm{Re}(z) = x$ (the real part)
- $\mathrm{Im}(z) = y$ (the imaginary part)

are both real numbers and $(x, y)$ determines a point in the plane.




Argand plane:

- $x$-axis: Real axis
- $y$-axis: Imaginary axis

# Example

In the complex plane, find the location of:

- $z_1 = 1$
- $z_2 = 2 + 3i$
- $z_3 = -2i$
- $S = \{z \in \mathbb{C} : \mathrm{Re}(z) \leq 1\}$

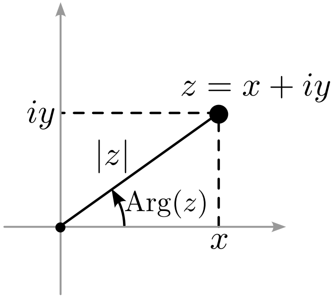
# Modulus and argument

- **Modulus** (absolute value) of $z$: distance to the origin.

$$|z| = d(z, O) = \sqrt{x^2 + y^2}.$$

- **Argument** of $z$: angle with positive $x$-axis.
  $\mathrm{Arg}(z) = \theta \in (-\pi, \pi]$ if

$$\tan\theta = \frac{y}{x}.$$



# Finding the argument (continued)

If $z$ is in quadrant 1 or 4, then

$$\theta = \tan^{-1}(\frac{y}{x}).$$

Otherwise ($z$ in quadrant 2 or 3)

$$\theta = \pi + \tan^{-1}(\frac{y}{x}).$$

# Polar representation of complex numbers

If $r$ is the modulus and $\theta$ the argument of $z = x + iy$, then

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

This gives us the **polar representation of** $z$:

$$\begin{aligned} z &= x + iy \\ &= \boxed{r(\cos\theta + i\sin\theta)} \end{aligned}$$

# Example

Find the polar representation of

- $z_1 = i$
- $z_2 = 1 + i$
- $z_3 = -\sqrt{3} - i$

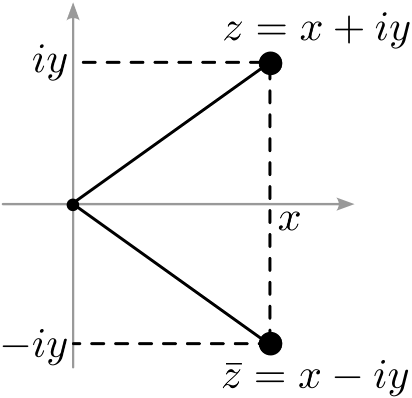
# Complex conjugate

If $z = x + iy$, then the **complex conjugate** $\bar{z}$ is given by

$$\bar{z} = x - iy.$$

Properties:

1. $\mathrm{Re}(\bar{z}) = \mathrm{Re}(z)$
2. $\mathrm{Im}(\bar{z}) = -\mathrm{Im}(z)$
3. $|\bar{z}| = |z|$
4. $\mathrm{Arg}(\bar{z}) = -\mathrm{Arg}(z)$



# Calculating with complex numbers

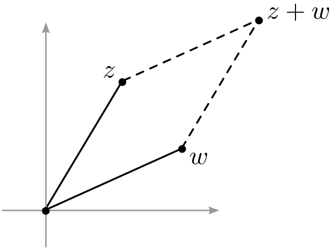
# Adding and subtracting complex numbers

Complex numbers can be added/subtracted component-wise: if $z = x + iy$ and $w = a + ib$, then

$$\begin{aligned} z \pm w &= (x + iy) \pm (a + ib) \\ &= (x \pm a) + i(y \pm b) \end{aligned}$$

This has a nice geometric interpretation via the **parallellogram rule**:

- Draw a parallellogram with sides $z$ and $w$
- $z + w$ is at the end of the diagonal



# Multiplying complex numbers

If $z = x + iy$ and $w = a + ib$, then (using $i^2 = -1$)

$$\begin{aligned} zw &= (x + iy) \cdot (a + ib) \\ &= (xa - yb) + i(ya + xb) \end{aligned}$$

Properties:

1. $z\bar{z} = |z|^2$
2. $\overline{zw} = \bar{z} \cdot \bar{w}$

# Product of complex numbers in polar form

Write

$$w = r(\cos\theta + i\sin\theta)$$
$$z = s(\cos\phi + i\sin\phi)$$

Then we get the following nice form for the complex product:

$$wz = \underbrace{rs}_{|wz|}\,(\cos(\underbrace{\theta+\phi}_{\mathrm{Arg}(wz)}) + i\sin(\theta+\phi))$$

In particular, we get

- $|wz| = rs = |w||z|$
- $\mathrm{Arg}(wz) = \theta + \phi = \mathrm{Arg}(w) + \mathrm{Arg}(z)$

# De Moivre's theorem

From the product rule, we get

$$
\begin{aligned}
|z_1 z_2 \cdots z_n| &= |z_1||z_2| \cdots |z_n| \\
\mathrm{Arg}(z_1 z_2 \cdots z_n) &= \mathrm{Arg}(z_1) + \cdots + \mathrm{Arg}(z_n).
\end{aligned}
$$

Substituting $z_1 = \ldots = z_n = \cos\theta + i\sin\theta$ gives us **De Moivre's theorem**:

$$
\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).}
$$

# Division of complex numbers

We put

$$\frac{z}{w} = \frac{x+iy}{a+ib}.$$

- How can we make sense of this complex number?
- Multiply by the conjugate:

$$\frac{z}{w} = \frac{x+iy}{a+ib}\frac{a-ib}{a-ib} = \frac{ax+by}{a^2+b^2} + i\frac{ay-bx}{a^2+b^2}.$$

Properties:

$$|z/w| = |z|/|w| \quad \text{and} \quad \mathrm{Arg}(z/w) = \mathrm{Arg}(z) - \mathrm{Arg}(w).$$

# Useful properties of complex numbers

- $\overline{z+w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $\bar{\bar{z}} = z$
- $zw = 0$ iff $z = 0$ or $w = 0$

Property: to take the conjugate of a complicated expression, it suffices to take the conjugate of every term.

Example: Given $z = i\dfrac{Z-1}{Z+1}$, compute $\bar{z}$.

# Examples

- Find the modulus and argument of $z = (3 + 5i)(4 - 2i)$.
- Simplify the complex number $z = \frac{7+3i}{4i}$.

# Examples

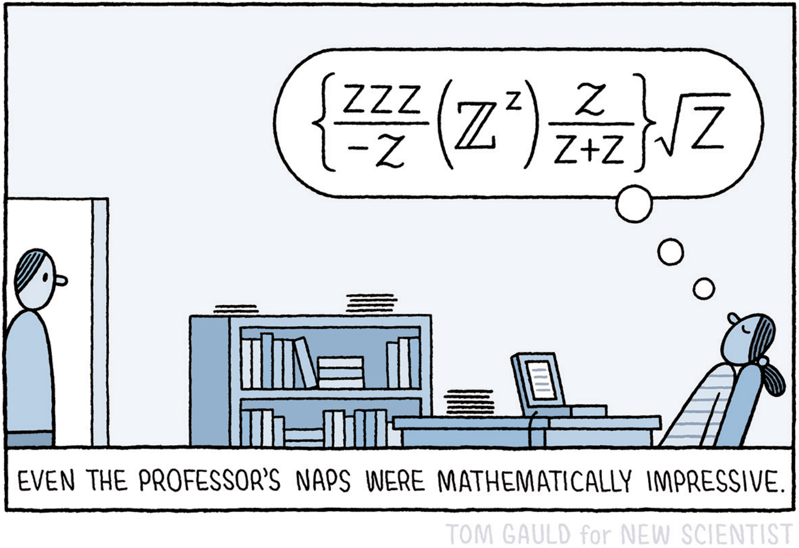
Simplify the following complex numbers as much as possible:

- $z = \frac{1+i}{1-i}$
- $z = i^{2022}$

# Caveat

Keep in mind that, for complex numbers,

$$\sqrt{ab} \neq \sqrt{a}\sqrt{b}.$$




Source: Tom Gauld