The binomial theorem

Introduction to Engineering Mathematics

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Overview

- Pascal's triangle
- Binomial coefficients
- Binomial theorem

Pascal's triangle

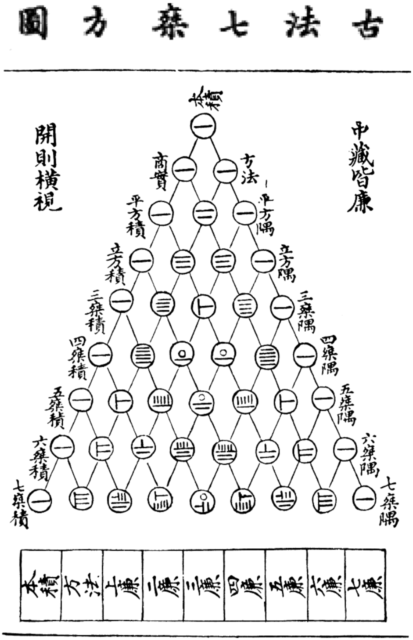
Expand the following expressions and look at the coefficients.

- $(a+b)^0 = 1$
- $(a+b)^1 = a+b$
- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

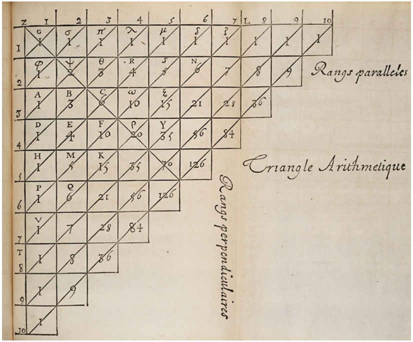
What do you notice?

Based on this pattern, what is $(a+b)^7$?

Would you be able to write down $(a+b)^{27}$?



Jian Xian ( ), 11th century CE



Blaise Pascal, 1665 CE

Example

Use Pascal's triangle to expand $\left(2x + \frac{1}{x}\right)^5$.

Binomial coefficients

- Factorial: $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.
- Binomial coefficient (also called "n-choose-k"):

$$\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}.$$

- Measures the number of ways of choosing k objects from among n choices.

Properties

For all n and $k \leq n$:

$$\binom{n}{0} = \binom{n}{n} = 1$$
$$\binom{n}{1} = \binom{n}{n-1} = n$$
$$\binom{n}{k} = \binom{n}{n-k}$$

Rewriting Pascal's triangle using binomial coefficients

The binomial expansion

Putting everything we've learned together, we get

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n.$$

This can be written more compactly as

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Example

Use the binomial expansion to expand $(\sqrt{x}-1)^7$.

Visual proof of the binomial expansion (optional)