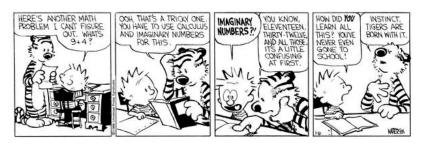
Complex Numbers Introduction to Engineering Mathematics

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Overview

- Motivation
- Number systems
- Graphical representation of complex numbers
- Modules, argument, complex conjugate
- Complex arithmetic



Motivation: working with "impossible" numbers

At first, this expression looks like complete nonsense:

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}$$

But is it really?

Benefit: solving quadratic equations

Find x so that $x^2 - 2x + 2 = 0$.

- As D = -4 < 0, there are **no real solutions**
- If we set $i = \sqrt{-1}$, then we find **two solutions**.

OK to calculate with i, as long as we remember

$$i^2 = -1.$$

Number systems

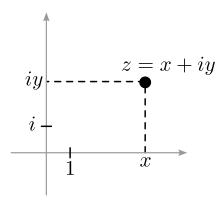
Symbol	Elements	Used for
N Z	0, 1, 2, , -1, 0, 1, 2,	Counting Adding/subtracting
\mathbb{Q}	Fractions n/m	Dividing
\mathbb{R}	$\mathbb Q$ and irrational numbers: e , π ,	Limits
\mathbb{C}	$a+ib$, with $a,b\in\mathbb{R}$	Solving equations

Graphical representation of complex numbers

If z = x + iy is a complex number, then

- Re(z) = x (the real part)
- Im(z) = y (the imaginary part)

are both real numbers and (x,y) determines a point in the plane.



Argand plane:

- x-axis: Real axis
- y-axis: Imaginary axis

Example

In the complex plane, find the location of:

- $z_1 = 1$
- $z_2 = 2 + 3i$
- $z_3 = -2i$
- $S = \{z \in \mathbb{C} : \operatorname{Re}(z) \le 1\}$

Modulus and argument

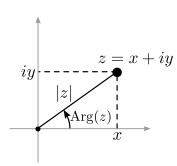
• **Modulus** (absolute value) of *z*: distance to the origin.

$$|z| = d(z, O) = \sqrt{x^2 + y^2}.$$

• **Argument** of *z*: angle with positive *x*-axis.

$$\operatorname{Arg}(z) = \theta \in (-\pi, \pi]$$
 if

$$\tan \theta = \frac{y}{x}.$$



Finding the argument (continued)

If z is in quadrant 1 or 4, then

$$\theta = \tan^1(\frac{y}{x}).$$

Otherwise (z in quadrant 2 or 3)

$$\theta = \pi + \tan^1(\frac{y}{x}).$$

Polar representation of complex numbers

If r is the modulus and θ the argument of z=x+iy, then

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

This gives us the **polar representation of** z:

$$z = x + iy$$
$$= r(\cos \theta + i \sin \theta)$$

Example

Find the polar representation of

- $z_1 = i$
- $\begin{array}{l} \bullet \ \, z_2 = 1 + i \\ \bullet \ \, z_3 = -\sqrt{3} i \end{array}$

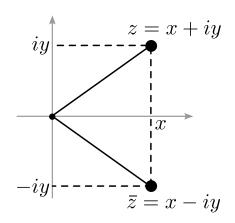
Complex conjugate

If z=x+iy, then the complex conjugate \bar{z} is given by

$$\bar{z} = x - iy$$
.

Properties:

- $2 \operatorname{Im}(\bar{z}) = -\operatorname{Im}(z)$
- 3 $|\bar{z}| = |z|$
- $4 \operatorname{Arg}(\bar{z}) = -\operatorname{Arg}(z)$



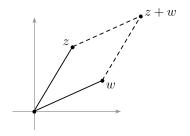
Adding and subtracting complex numbers

Complex numbers can be added/subtracted component-wise: if z=x+iy and w=a+ib, then

$$z \pm w = (x + iy) \pm (a + ib)$$
$$= (x \pm a) + i(y \pm b)$$

This has a nice geometric interpretation via the **parallellogram** rule:

- Draw a parallellogram with sides z and w
- ullet z+w is at the end of the diagonal



Multiplying complex numbers

If
$$z=x+iy$$
 and $w=a+ib$, then (using $i^2=-1$)
$$zw=(x+iy)\cdot(a+ib)$$

$$=(xa-yb)+i(ya+xb)$$

Properties:

- $\mathbf{1} \ z\bar{z} = |z|^2$

Product of complex numbers in polar form

Write

$$w = r(\cos \theta + i \sin \theta)$$
$$z = s(\cos \phi + i \sin \phi)$$

Then we get the following nice form for the complex product:

$$wz = \underbrace{rs}_{|wz|} \left(\cos(\underbrace{\theta + \phi}_{\text{Arg}(wz)}) + i \sin(\theta + \phi) \right)$$

In particular, we get

- |wz| = rs = |w||z|
- $\operatorname{Arg}(wz) = \theta + \phi = \operatorname{Arg}(w) + \operatorname{Arg}(z)$

De Moivre's theorem

From the product rule, we get

$$\begin{split} |z_1z_2\cdots z_n| &= |z_1||z_2|\cdots|z_n|\\ \operatorname{Arg}(z_1z_2\cdots z_n) &= \operatorname{Arg}(z_1) + \cdots + \operatorname{Arg}(z_n). \end{split}$$

Substituting $z_1 = ... = z_n = \cos\theta + i\sin\theta$ gives us **De Moivre's theorem**:

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta).$$

Division of complex numbers

We put

$$\frac{z}{w} = \frac{x + iy}{a + ib}.$$

- How can we make sense of this complex number?
- Multiply by the conjugate:

$$\frac{z}{w} = \frac{x+iy}{a+ib} \frac{a-ib}{a-ib} = \frac{ax+by}{a^2+b^2} + i\frac{ay-bx}{a^2+b^2}.$$

Properties:

$$|z/w| = |z|/|w| \quad \text{and} \quad \operatorname{Arg}(z/w) = \operatorname{Arg}(z) - \operatorname{Arg}(w).$$

Useful properties of complex numbers

- $\overline{z+w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $\bar{z} = z$
- zw = 0 iff z = 0 or w = 0

Property: to take the conjugate of a complicated expression, it suffices to take the conjugate of every term.

Example: Given $z = i \frac{Z-1}{Z+1}$, compute \bar{z} .

Examples

- Find the modulus and argument of z = (3+5i)(4-2i).
- Simplify the complex number $z=\frac{7+3i}{4i}.$

Examples

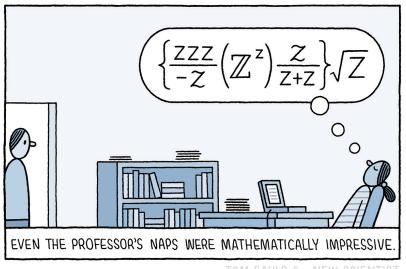
Simplify the following complex numbers as much as possible:

- $z = \frac{1+i}{1-i}$ $z = i^{2022}$

Caveat

Keep in mind that, for complex numbers,

$$\sqrt{ab} \neq \sqrt{a}\sqrt{b}$$
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TOM GAULD for NEW SCIENTIST

Source: Tom Gauld