

# Integration: Definitions and basic concepts

## Introduction to Engineering Mathematics

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# Indefinite integrals

# Indefinite integrals as antiderivatives

- **Antiderivative:** the opposite (inverse operation) of a derivative.
- The **indefinite integral** is an antiderivative.

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In these examples,  $C$  is an **arbitrary constant**.

# Anatomy of an indefinite integral

The diagram illustrates the components of the indefinite integral expression  $\int f(x) dx$ . Three orange lines with labels point to specific parts of the expression:

- Integral sign (curly 'S'):** Points to the large integral symbol  $\int$ .
- Differential (identifies the variable):** Points to the  $dx$  term, which is underlined in the original image.
- Integrand (the function to integrate):** Points to the  $f(x)$  term, which is underlined in the original image.

The full expression is  $\int f(x) dx$ .

# Examples

Polynomials:

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$  for  $n \neq -1$
- $\int \frac{dx}{x} = \ln |x| + C$

Trigonometric functions:

- $\int \sin x dx = -\cos x + C$
  - $\int \cos x dx = \sin x + C$
- $\int \tan x dx = -\ln |\cos x| + C$   
*Not an obvious antiderivative; we will see how to derive this integral later.*



# Examples

Exponential/logarithmic functions:

- $\int e^x dx = e^x + C$
- $\int \frac{dx}{x} = \ln |x| + C$

Inverse trigonometric functions:

- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$
- $\int \frac{dx}{1+x^2} = \arctan(x) + C$

# Examples

Special cases:

- $\int \sec^2 x \, dx = \tan x + C$
- $\int \csc^2 x \, dx = -\cot x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \csc x \cot x \, dx = -\csc x + C$

# Definite integrals

# The definite integral

Definite integral: integral with “bounds”

$$\int_a^b f(x) dx = F(b) - F(a)$$

*Upper bound*

*Lower bound*

How to compute:

- 1 Find primitive function  $F(x)$ :  $\int f(x) dx = F(x) + C$
- 2 Substitute bounds into  $F(x)$ :

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b.$$

# Examples

Compute:

- $\int_0^{\pi/2} \sin x \, dx$
- $\int_{-1}^2 (x^2 + 2x - 1) \, dx$

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- Linearity:

$$\int_a^b (Af(x) + Bg(x))dx = A \int_a^b f(x)dx + B \int_a^b g(x)dx$$



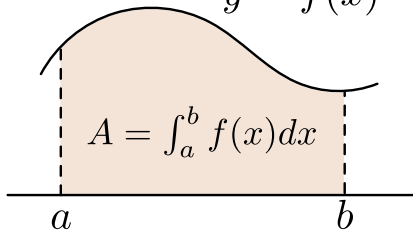
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- Linearity:  
$$\int_a^b (Af(x) + Bg(x))dx = A \int_a^b f(x)dx + B \int_a^b g(x)dx$$
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

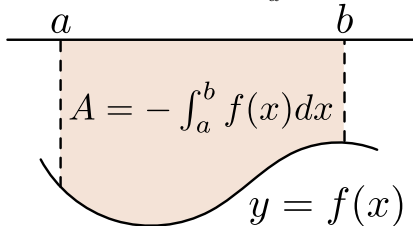
## Definite integrals correspond to signed areas

If  $f(x) \geq 0$ :  $A = \int_a^b f(x) dx$ .

$$y = f(x)$$



If  $f(x) \leq 0$ :  $A = -\int_a^b f(x) dx$ .



If  $f(x)$  changes sign: break up into parts above/below  $x$ -axis.

## Examples

Find the area...

- Between the graph of  $y = \sin x$  and the  $x$ -axis, from 0 to  $2\pi$ .
- Below the graph of  $y = 3x - x^2$  and above the  $x$ -axis.

## Properties (continued)

- For an even function,  $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$

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- For an odd function,  $\int_{-a}^a f(x) = 0$ .
- “King's rule”:  $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ .

## FREE Wi-Fi

$$\int_{-2}^2 \left( x^3 \cos \frac{x}{2} + \frac{1}{2} \right) \sqrt{4 - x^2} \, dx$$



The Wi-Fi password is the first 10 digits of the answer.

## Continuity and integration

- So far, we have silently assumed  $f(x)$  is continuous on  $[a, b]$  to define the integral  $\int_a^b f(x)dx$ .
- The integral can also be defined if  $f(x)$  is **piecewise continuous**.



## Examples

Compute the following integrals:

- $\int \left( 10x^4 - \frac{2}{\sqrt{1-x^2}} \right) dx$
- $\int_0^1 x^{5/2}(1-x) dx$
- $\int \frac{x^2}{1+x^2} dx$

## Examples

Show that  $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}.$

*Difficult, uses King's rule.*

# Fundamental theorem of calculus

## Integrals and derivatives are each other's inverse

- If  $G(x) = \int_1^x \ln y \, dy$ , find  $G'(x)$ .
- If  $H(x) = \int_1^{x^2} \ln y \, dy$ , find  $H'(x)$ .