Derivatives (1/2)

Introduction to Engineering Mathematics

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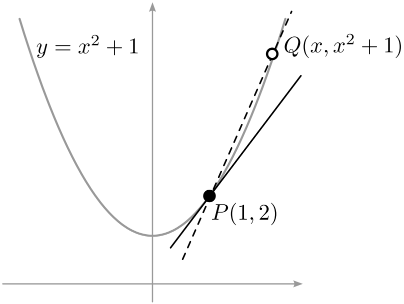
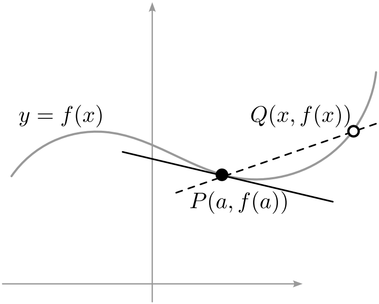
Motivation: tangent lines

In the previous class, we managed to compute the slope of the tangent line to the parabola $y = x^2$ by computing the limit

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

How would we do this for an **arbitrary curve** $y = f(x)$?





Derivative of a function

We define the **derivative of** $f(x)$ **at** $x = a$ ("f prime") as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{(if the limit exists).}$$

If we put $h = x - a$, we can rewrite this as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This is sometimes easier to compute.

If $f'(a)$ exists, we say that $f(x)$ is **differentiable** at $x = a$.

Examples

Compute the derivative of $f(x) = 3x^2 + 7x - 5$ at $x = 1$.

Examples

Compute the derivative of $f(x) = |x|$ at $x = a$.

The derivative as a function

By letting a vary in $f'(a)$, we obtain a function given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(if the limit exists).}$$

There are many notations for the derivative function: for $y = f(x)$,

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx} f(x).$$

all mean the same thing.

Example

Show that $\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$.

Properties of differentiable functions

We say that $y = f(x)$ is **differentiable on an interval** $[a, b]$ if

- $f'(x)$ exists for each $x \in (a, b)$
- At $x = a$, the *right derivative* exists:

$$f'(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}.$$

- At $x = b$, the *left derivative* exists:

$$f'(b) = \lim_{h \to 0-} \frac{f(b+h) - f(b)}{h}.$$

Link between differentiability and continuity

- If $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.
- The converse is not necessarily true!

Example: Show that $y = |x|$ is differentiable for all $x \neq 0$.

Higher-order derivatives

Now that we can take the derivative of a function, we can take the derivative of the derivative, and so on...

Notation	Name
$f'(x)$	1st-order derivative
$f''(x) = \frac{d}{dx} f'(x)$	2nd-order derivative
...	...
$f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x)$	nth-order derivative

Derivative of a polynomial: basic rules

1. $$\frac{d}{dx}c = 0$$

2. $$\frac{dx^n}{dx} = nx^{n-1} \quad \text{for } n \neq 0$$

Derivative of a polynomial: basic rules

3. $$\frac{d}{dx}(cf(x)) = c\frac{df}{dx}$$

4. $$\frac{d}{dx}(f(x)+g(x)) = \frac{df}{dx} + \frac{dg}{dx}$$

Examples

Compute the derivative of $f(x) = 3x^2 + 7x - 5$.

Derivative of the exponential/logarithm

$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

For a different base:

$$\frac{d}{dx}a^x = a^x \ln a$$
$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

- The last two rules will follow from the chain rule
- **Do not confuse the rules for a^x and x^n!**

Product and quotient rules

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$
$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Examples: Compute

- $\frac{d}{dx}(x^2e^x)$
- $\frac{d}{dx}\frac{x+1}{x+3}$

Derivatives of trigonometric functions

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$$

Examples

Using the rules from the previous slide, show that

$$\frac{d}{dx}\csc x = -\cot x \csc x$$
$$\frac{d}{dx}\sec x = \tan x \sec x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$

Example solution:

$$\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x} = \frac{1' \cdot \sin x - \sin'(x) \cdot 1}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}.$$

(This will be easier once we cover the **chain rule**)

Derivatives of inverse trigonometric functions

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

These rules can be derived by means of *implicit differentation*, which we will cover later.

The chain rule

- Useful for composite functions $F = f \circ g$ ("f after g")
- If $f(x)$ and $g(x)$ are differentiable, then the composite function $F(x)$ is also differentiable and

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Example

Compute $\dfrac{d}{dx}e^{\sqrt{\cos x}}$.

Example

Compute $\dfrac{d}{dx}x^{\sin(x)}$.