

# Integration: Techniques

## Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

# Overview

- Integration by substitution
- Trigonometric integration
- Integration by parts
- Reduction formulae

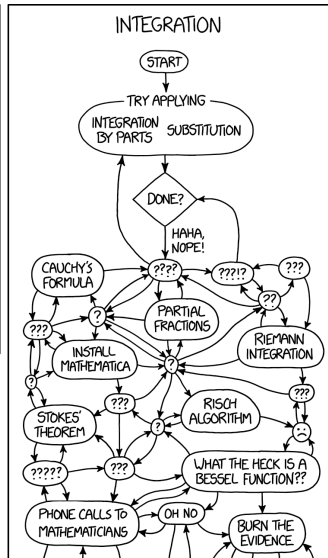
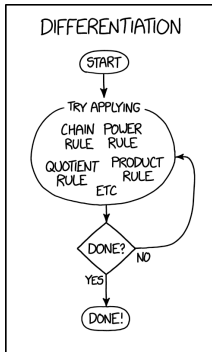


Figure 1: Source: <https://xkcd.com/2117/>

## Warning

*There are many more problems in these slides than we can cover in class. The solutions to all problems can be found at the end of this slide deck. You are encouraged to solve the problems that we can't cover in class and check your solutions with the key at the back.*

# Integration by substitution

**Idea:** *Find a function whose derivative also occurs in the integral.  
Replace this function by a new variable to get an easier integral.*

- Don't forget to change the differential!
- Substitute everything!

**Example:** compute  $\int x \sin(1 + x^2) dx$

## Substitution: definite integrals

Works the same as for indefinite integrals, but also adjust the boundaries.

Example:  $\int_0^2 (1+x)^5 dx$

## Examples

$$\int a^{bx} dx$$

## Examples

$$\int x^3 \cos(x^4 + 2) dx$$



## Examples

$$\int \frac{\sin(3 \ln x)}{x} dx$$

## Examples

$$\int x^5 \sqrt{1+x^2} \, dx$$

## Examples

$$\int e^x \sqrt{1 + e^x} dx$$

## Examples

$$\int \frac{dx}{x^2 + 4x + 5}$$

# Trigonometric integration

**Useful when:** integrand contains trigonometric functions.

**Examples:** compute

- $\int \tan x \, dx$
- $\int \cot x \, dx$

## Examples

$$\int \sec x \, dx$$

## Examples

$$\int \csc x \, dx$$

## Examples

$$\int \sin^3 x \cos^8 x \, dx$$



## Examples

$$\int \cos^5 x \, dx$$

## Examples

$$\int \cos^2 x \, dx$$

## Examples

$$\int \sin^2 x \, dx$$

## Examples

$$\int \sin^4 x \, dx$$

## Examples

$$\int \tan^2 x \, dx$$

## Examples

$$\int \sec^4 x \, dx$$

## Integration by parts

**Idea:** *Transform integral into a simpler integral (easier to solve).*

**Rule:** 
$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx$$

**Example:** compute  $\int \ln x dx$

## Examples

$$\int \sec^3 x \, dx$$



## Examples

$$\int e^{ax} \cos(bx) dx$$

## Reduction formula

**Idea:** For an integral  $I_n$  that depends on a parameter  $n$ , find a formula that relates  $I_n$  with  $I_{n-1}$  (or  $I_{n-2}$ , ...) Use this formula to find  $I_n$  recursively.

**Example:** Let  $I_n = \int x^n e^{-x} dx$ . Find a relation between  $I_n$  and  $I_{n-1}$ , and use this to determine  $I_0$ ,  $I_1$ , and  $I_2$ .

## Example

Let  $I_n = \int_0^{\pi/2} \cos^n x \, dx$ . Find a reduction formula for  $I_n$ .

# Solutions

- $\int x \sin(1 + x^2) dx = -\frac{1}{2} \cos(1 + x^2) + C$
- $\int_0^2 (1 + x)^5 dx = \frac{364}{3}$
- $\int a^{bx} dx = \frac{a^{bx}}{b \ln a} + C$
- $\int x^3 \cos(x^4 + 2) dx = \frac{1}{4} \sin(x^4 + 2) + C$
- $\int \frac{\sin(3 \ln x)}{x} dx = -\frac{1}{3} \cos(3 \ln x) + C$

- $\int x^5 \sqrt{1+x^2} \, dx = \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$
- $\int e^x \sqrt{1+e^x} \, dx = \frac{2}{3}(1+e^x)^{3/2} + C$
- $\int \frac{dx}{x^2+4x+5} = \tan^{-1}(x+2) + C$
- $\int \tan x \, dx = \ln |\sec x| + C$
- $\int \cot x \, dx = -\ln |\csc x| + C$

- $\int \sec x \, dx = \ln |\sec x + \tan x| + C$
- $\int \csc x \, dx = \ln |\csc x - \cot x| + C$
- $\int \sin^3 x \cos^8 x \, dx = \frac{1}{11} \cos^{11} x - \frac{1}{9} \cos^9 x + C$
- $\int \cos^5 x \, dx = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$

- $\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$
- $\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$
- $\int \sin^4 x \, dx = \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C$
- $\int \tan^2 x \, dx = \tan x - x + C$
- $\int \sec^4 x \, dx = \tan x + \frac{\tan^3 x}{3} + C$



- $\int \ln x \, dx = x(\ln x - 1) + C$
- $\int \sec^3 x \, dx = \frac{1}{2} (\ln |\sec x + \tan x| + \sec x \tan x) + C$
- $\int e^{ax} \cos(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos(bx) + b \sin(bx)) + C$

A GUIDE TO  
INTEGRATION BY PARTS:

GIVEN A PROBLEM OF THE FORM:

$$\int f(x)g(x)dx=?$$

CHOOSE VARIABLES  $u$  AND  $v$  SUCH THAT:

$$u = f(x)$$

$$dv = g(x)dx$$

NOW THE ORIGINAL EXPRESSION BECOMES:

$$\int u dv=?$$

WHICH DEFINITELY LOOKS EASIER.

ANYWAY, I GOTTA RUN.

BUT GOOD LUCK!

Figure 2: Source: <https://xkcd.com/1201/>