Introduction to Quantum Computing: Solutions

Contents

1 Qubits

1.1 Dirac/Braket notation

1. Give the matrix representation of $|00\rangle, |01\rangle, |10\rangle$, $|11\rangle$ and $|\psi\rangle = \frac{1+i}{3}|00\rangle - \frac{1-i}{3}|01\rangle + \frac{1+2i}{3}|10\rangle$.

$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \quad |\psi\rangle = \begin{bmatrix}\frac{1+i}{3}\\-\frac{1-i}{3}\\\frac{1+2i}{3}\\0\end{bmatrix}$$

2. Give the braket notation of the followings vectors:

   (a) $\begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$

   (b) $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   (c) $\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

   (d) $\begin{bmatrix}0\\0\\\frac{1}{2}\\-\frac{3+i\sqrt{2}}{4}\\0\\0\\\frac{1}{4}\\0\end{bmatrix} = \frac{1}{2}|010\rangle - \frac{3+i\sqrt{2}}{4}|011\rangle + \frac{1}{4}|110\rangle$

3. Show that $\langle 0|0\rangle = 1$ and $\langle 0|1\rangle = 0$.

$$\langle 0|0\rangle = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 1\times 1 + 0\times 0 = 1$$

$$\langle 0|1\rangle = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 1\times 0 + 0\times 1 = 0$$

4. Let $|\psi\rangle = \sum_{k=0}^{2^n-1}\alpha_k|k\rangle$ be a quantum state of n qubits living on the orthonormal basis $\{|k\rangle\}_{k=0,\dots,2^n-1}$. Show that the probability of measuring the basis state $|a\rangle$ is $|\langle a|\psi\rangle|^2 = |\alpha_a|^2$.

$$|\langle a|\psi\rangle|^2 = |\langle a|\sum_{k=0}^{2^n-1}\alpha_k|k\rangle|^2 = |\sum_{k=0}^{2^n-1}\alpha_k\langle a|k\rangle|^2$$

Since all the basis vectors $|k\rangle$ are orthogonal, we have $\forall a \neq k, \langle a|k\rangle = 0$ and $\langle a|a\rangle = 1$, thus $|\langle a|\psi\rangle|^2 = |\alpha_a|^2$.

5. Show that $|0\rangle\langle 0| = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

$$\begin{bmatrix}1 & 0\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

6. Given that $|0\rangle\langle 0| = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, give the braket notation of the 2×2 Identity matrix $\mathbb{I}_2$.

$$\mathbb{I}_2 = |0\rangle\langle 0| + |1\rangle\langle 1|$$

7. Develop the following tensor products:

   (a) $|0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) = |0\rangle \otimes \alpha|0\rangle + |0\rangle \otimes \beta|1\rangle = \alpha|00\rangle + \beta|01\rangle$

   (b) $(\alpha|0\rangle + \beta|1\rangle) \otimes (\phi|0\rangle + \lambda|1\rangle) = \alpha\phi|00\rangle + \alpha\lambda|01\rangle + \beta\phi|10\rangle + \beta\lambda|11\rangle$

8. Decompose the following tensor products as much as you can (if possible):

   (a) $|01\rangle = |0\rangle \otimes |1\rangle$

   (b) $\frac{i}{\sqrt{3}}(|101\rangle + |011\rangle - |111\rangle) = \frac{i}{\sqrt{3}}(|10\rangle + |01\rangle - |11\rangle) \otimes \frac{i}{\sqrt{3}}|1\rangle = \frac{i}{\sqrt{3}}(|10\rangle + (|0\rangle - |1\rangle) \otimes |1\rangle) \otimes |1\rangle$

   (c) $\frac{1}{\sqrt{2}}(|10\rangle + |11\rangle) = |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   (d) $\frac{i}{\sqrt{2}}(|00\rangle + |11\rangle)$

   (e) $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$

1.2 Quantum measurement

Let $|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$ with $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$ be an arbitrary quantum state.

1. What condition must verified by those complex numbers ? Give the following probabilities:
   They must verify: $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.

   (a) Probability of measuring $|10\rangle$: $|\gamma|^2$

   (b) Probability of measuring $|0\rangle$ on the first qubit (left one): $|\alpha|^2 + |\beta|^2$

   (c) Probability of measuring $|1\rangle$ on the second qubit (right one): $|\beta|^2 + |\delta|^2$

2. Depending on the outcome of the measurement, give the new state of $|\psi\rangle$:

   (a) *Outcome*: $|00\rangle \to |00\rangle$

   (b) *Outcome*: $|00\rangle + |01\rangle \to \frac{\alpha|00\rangle + \beta|01\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}}$

   (c) *Outcome*: $|1\rangle$ on the first qubit (left one) $\to \frac{\gamma|10\rangle + \delta|11\rangle}{\sqrt{|\gamma|^2 + |\delta|^2}}$

3. Let $|\phi\rangle = \frac{i}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Give the state of the system if we measure $|0\rangle$ on the leftmost qubit.

$$\to \frac{i(|00\rangle + |01\rangle)}{2\sqrt{2|\frac{i}{2}|^2}} = \frac{i(|00\rangle + |01\rangle)}{\sqrt{2}}$$

4. Let $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Give the state of the system if we measure $|a\rangle_{a=0,1}$ on one of the two qubits.

$$\text{If we measure } |0\rangle :\to |00\rangle$$
$$\text{If we measure } |1\rangle :\to |11\rangle$$

   As the state is highly entangled we only have to measure 1 qubit to know the state of both qubits.

2 Basic operations

Let $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ be an orthonormal basis of the space $\mathcal{H} \otimes \mathcal{H}$. Give the matrix representations of the following operations:

1.

$$
\begin{aligned}
|00\rangle &\mapsto |00\rangle \\
|01\rangle &\mapsto |01\rangle \\
|10\rangle &\mapsto |11\rangle \\
|11\rangle &\mapsto |10\rangle
\end{aligned}
$$

This operation is called CNOT.

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & X \end{bmatrix}
$$

2.

$$
\begin{aligned}
|00\rangle &\mapsto |00\rangle \\
|01\rangle &\mapsto |10\rangle \\
|10\rangle &\mapsto |01\rangle \\
|11\rangle &\mapsto |11\rangle
\end{aligned}
$$

This operation is called SWAP.

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

3.

$$
\begin{aligned}
|00\rangle &\mapsto |00\rangle \\
|01\rangle &\mapsto |01\rangle \\
|10\rangle &\mapsto |10\rangle \\
|11\rangle &\mapsto -|11\rangle
\end{aligned}
$$

This operation is called CZ.

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & Z \end{bmatrix}
$$

4.

$$
\begin{aligned}
|00\rangle &\mapsto i\,|01\rangle \\
|01\rangle &\mapsto -i\,|00\rangle \\
|10\rangle &\mapsto |10\rangle \\
|11\rangle &\mapsto |11\rangle
\end{aligned}
$$

$$
\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Y & 0 \\ 0 & \mathbb{I} \end{bmatrix}
$$

5. Give a unitary map that performs the following computation:

$$
|a\rangle \otimes |b\rangle \mapsto |a\rangle \otimes |\overline{b}\rangle
$$

where $\overline{b} = 1 - b$.

$$
\mathbb{I} \otimes X = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

6. Starting from the state $|00\rangle$, compute the Bell state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ using a Hadamard and a CNOT gate.

$$CNOT(H \otimes \mathbb{I})\,|00\rangle = CNOT\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Psi\rangle$$

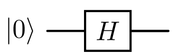
7. Starting from the Bell state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, compute the ground state $|00\rangle$ using a Hadamard and a CNOT gate.

$$(H \otimes \mathbb{I})^{\dagger} CNOT^{\dagger}\,|\Psi\rangle = (H \otimes \mathbb{I}) CNOT\,|\Psi\rangle = (H \otimes \mathbb{I})\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = |00\rangle$$

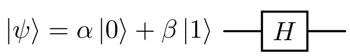
3 Quantum circuits

3.1 Getting started

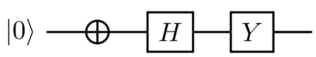
Compute the final outcome of the following quantum circuits. Indicate which circuits perform the same computation.

1. 

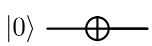
$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

2. 

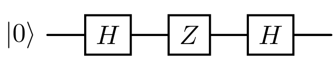
$$(\alpha + \beta)\,|0\rangle + (\alpha - \beta)\,|1\rangle$$

3. 

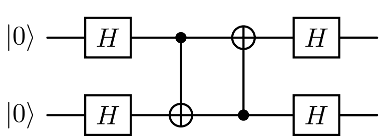
$$\frac{i}{\sqrt{2}}(|0\rangle + |1\rangle)$$

4. 

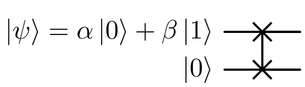
$$|1\rangle$$

5. 

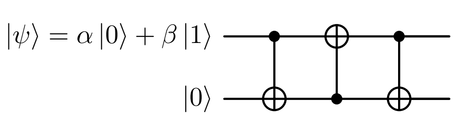
$$|1\rangle$$

6. 

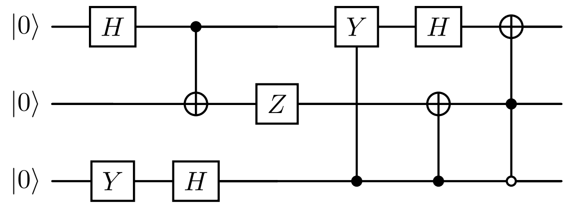
$$|00\rangle$$

7. 

$$|0\rangle \otimes (\alpha\,|0\rangle + \beta\,|1\rangle)$$

8. 

$$|0\rangle \otimes (\alpha\,|0\rangle + \beta\,|1\rangle)$$

9. 

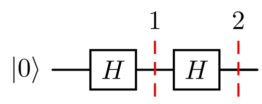
$$\frac{\sqrt{2}}{4}(|001\rangle + |011\rangle + |101\rangle - |111\rangle) + i\frac{\sqrt{2}}{4}(|000\rangle + |010\rangle + |100\rangle - |110\rangle)$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \quad X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad Y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \quad Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

3.2 Getting into it

3.2.1 Single qubit interference

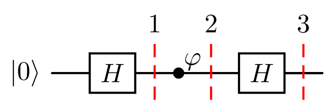
1. Compute the state of the qubit at each marked step.



   (a) (1): $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   (b) (2): $|0\rangle$

2. Same with an additional phase φ such that $P_\varphi = \begin{bmatrix}1 & 0\\ 0 & e^{i\varphi}\end{bmatrix}$:



1. (1): $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
2. (2): $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle)$
3. (3): $\frac{1}{2}(|0\rangle + |1\rangle + e^{i\varphi}(|0\rangle - |1\rangle)) = \frac{1}{2}(|0\rangle(1 + e^{i\varphi}) + |1\rangle(1 - e^{i\varphi}))$

3.2.2 Superdense coding

Classicaly, to transmit 2 bits of informations, one must send 2 of them. However, with quantum information it is possible to do it with only sending 1 qubit. This is called the Superdense coding.

1. Alice and Bob share an entangled state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$. She wants to communicate 2 bits of classical information to Bob by just physically sending 1 qubit. The system is composed of 2 qubits, therefore the possible bit strings $b_1 b_2$ to send are:

   (a) 00

   (b) 01

   (c) 10

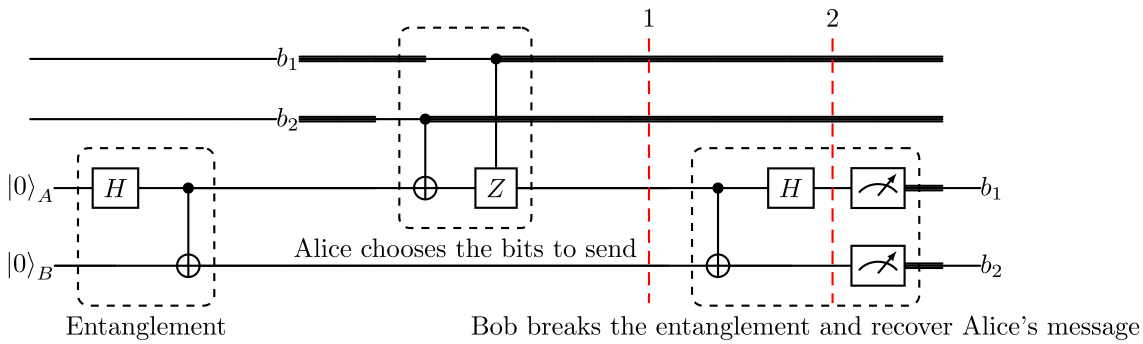
   (d) 11

   Depending on the information she wants to send, Alice has to perform some operations on her qubit:

(a) 00: She does nothing, i.e. apply the $\mathbb{I}$ gate
(b) 01: She apply the X gate
(c) 10: She apply the Z gate
(d) 11: She apply the X gate, then the Z gate

Once Alice did her operations, Bob breaks the entanglement and recover the message.

The protocol is summarized on the following quantum circuit:



Compute the state of the system at the marked steps for the four possibles bit strings b_1b_2 to send.

(a) 00:
   i. (1): $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
   ii. (2): $|00\rangle$

(b) 01:
   i. (1): $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$
   ii. (2): $|01\rangle$

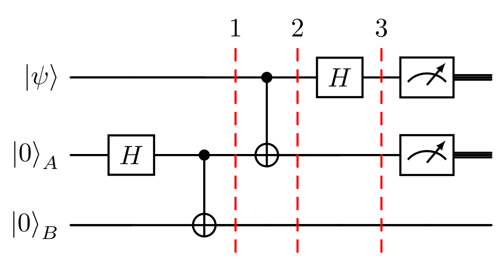
(c) 10:
   i. (1): $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$
   ii. (2): $|10\rangle$

(d) 11:
   i. (1): $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$
   ii. (2): $|11\rangle$

3.2.3 Quantum Teleportation

Alice and Bob are separated from each other by an unknown distance. Before splitting, they shared an entangled state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$. Charlie gives an unknown state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ to Alice and he wants her to send it to Bob. Alice is going to use the Quantum Teleportation protocol to do the job.

1. Compute the state of the system at each marked step.



(a) (1):

$$
\begin{aligned}
&= (\alpha |0\rangle + \beta |1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= \frac{\alpha}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\beta}{\sqrt{2}}(|100\rangle + |111\rangle)
\end{aligned}
$$

(b) (2):

$$
\begin{aligned}
&= \frac{\alpha}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\beta}{\sqrt{2}}(|110\rangle + |101\rangle) \\
&= \alpha |0\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \beta |1\rangle \otimes \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)
\end{aligned}
$$

(c) (3):

$$
\begin{aligned}
&= \frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\
&= \frac{\alpha}{2}(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \frac{\beta}{2}(|010\rangle + |001\rangle - |110\rangle - |101\rangle) \\
&= \frac{1}{2}|00\rangle \otimes (\alpha |0\rangle + \beta |1\rangle) + \frac{1}{2}|01\rangle \otimes (\beta |0\rangle + \alpha |1\rangle) + \frac{1}{2}|10\rangle \otimes (\alpha |0\rangle - \beta |1\rangle) + \frac{1}{2}|11\rangle \otimes (\beta |0\rangle - \alpha |1\rangle)
\end{aligned}
$$

2. After the last Hadamard gate, Alice performed a measurement on her qubits, making the quantum superposition collapse into four possible states. Depending on the outcome of those measurements, Bob has to perform some simple operation on his qubit to get $|\psi\rangle$. Give those operations for the four possible states.

   The four possible outcome states are:

   (a) $|00\rangle$: we have nothing to do, i.e. apply $\mathbb{I}$ on Bob's qubit

   $$(\mathbb{I} \otimes \mathbb{I}) |00\rangle \otimes \mathbb{I}(\alpha |0\rangle + \beta |1\rangle) = |00\rangle \otimes (\alpha |0\rangle + \beta |1\rangle) = |00\rangle \otimes |\psi\rangle$$

   (b) $|01\rangle$: the complex numbers α and β got flipped, thus we apply X

   $$(\mathbb{I} \otimes \mathbb{I}) |01\rangle \otimes X(\beta |0\rangle + \alpha |1\rangle) = |00\rangle \otimes (\alpha |0\rangle + \beta |1\rangle) = |01\rangle \otimes |\psi\rangle$$

   (c) $|10\rangle$: there is a negative phase multiplying β, to fix it we apply Z

   $$(\mathbb{I} \otimes \mathbb{I}) |10\rangle \otimes Z(\alpha |0\rangle - \beta |1\rangle) = |10\rangle \otimes (\alpha |0\rangle + \beta |1\rangle) = |10\rangle \otimes |\psi\rangle$$

   (d) $|11\rangle$: α and β got flipped, and there is a negative phase in front of β, therefore we apply X and Z

   $$(\mathbb{I} \otimes \mathbb{I}) |11\rangle \otimes ZX(\beta |0\rangle - \alpha |1\rangle) = |11\rangle \otimes (\alpha |0\rangle + \beta |1\rangle) = |11\rangle \otimes |\psi\rangle$$

   In the end, the Quantum Teleportation protocol is: