Title:

The Elliptical Integral and the Large Angle Pendulum

Page Count:

20

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1 Introduction

A pendulum is a mass suspended from a fixed point so that it can swing back and forth under the influence of gravity. It's behaviour, specifically the period T (the time it takes to complete one oscillation in seconds) is instrumental to understanding harmonic motion. We learn that in the small angle approximation sinθ ≈ θ and the period T is independent of the initial opening angle of the pendulum, I will show this phenomenon below:

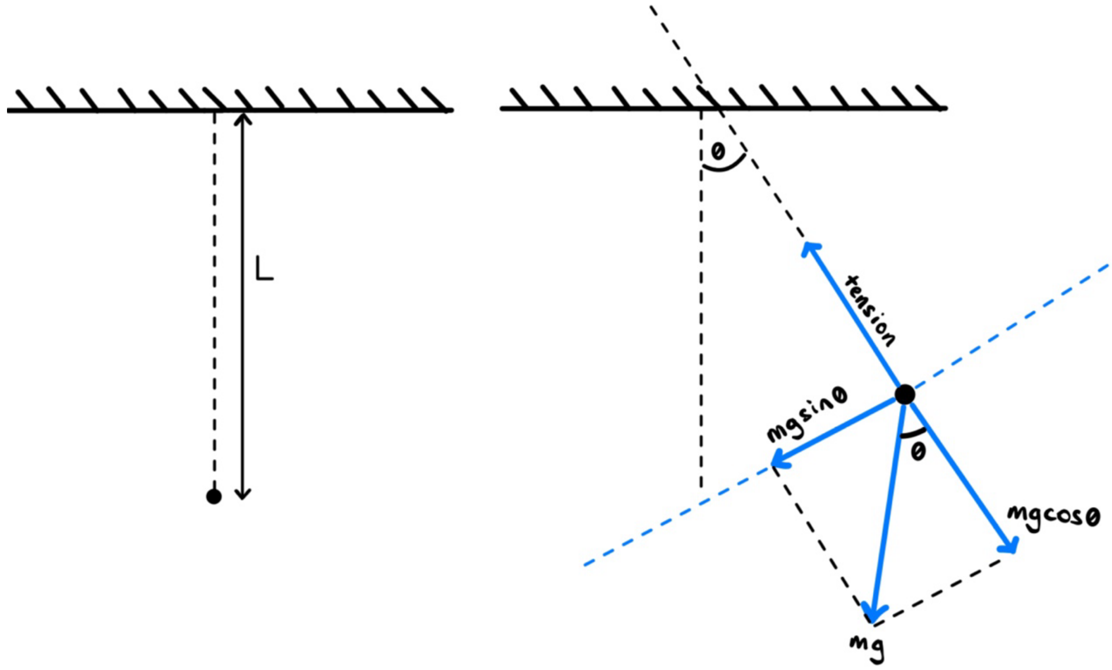


Figure 1: Force diagram of a simple gravity pendulum

When a pendulum is displaced by angle θ from the vertical, the force that acts to return it to the equilibrium position is the component of gravitational force acting along the arc of the pendulum's swing. This force is:

$$F = mgsin\theta$$

(Tsokos, 2014)

The acceleration *a* of the pendulum bob is found through the resultant force:

$$ma = -mgsin\theta$$

where *m* is the mass of the bob, *g* is the acceleration due to gravity, and the negative sign indicates that the force is restoring, acting opposite to the displacement. Simplifying to:

$$a = -gsin\theta$$

The displacement the pendulum bob follows is an arc. The length of the arc *x* is related to the angle θ

$$x = L\theta$$

where *L* is the length of the pendulum
(Tsokos, 2014)

Substituting the following into acceleration, I get:

$$a = -gsin\left(\frac{x}{L}\right)$$

This brings me to the concept of **small angle approximation**. For small angles (less than about 15 degrees of 0.26 radians, the sine of the angle is approximately equal to the angle itself when the angle is measured in radians. Therefore:

$$\sin\left(\frac{x}{L}\right) \approx \frac{x}{L}$$

I now test the following theory with some values to test out this approximation:

Sine values in radians	
sin(**0.5**) = 0.4794 ≈ **0.50**	sin(**1.25**) = 0.9490 ≉ **1.25**

Hence, for small angles, the acceleration can be approximated as:

$$a \approx -\frac{g}{L}x$$

A system undergoing Simple Harmonic Motion (SHM) experiences an acceleration that is proportional to its displacement from the equilibrium but in the opposite direction. This is expressed as:

$$a \approx -\omega^2 x$$

where ω is the angular frequency
(Tsokos, 2014)

By comparing this with the acceleration equation from the **small angle approximation** and **simple harmonic motion** definition.

$$-\frac{g}{L} = -\omega^2$$

This implies that the angular frequency is:

$$\omega = \sqrt{\frac{g}{L}}$$

The period T of an oscillator in simple harmonic motion, the time for one complete cycle/oscillation is given by:

$$T = \frac{2\pi}{\omega}$$

Substituting our expression for ω gives us the period of the pendulum for small oscillations.

$$T_0 = 2\pi\sqrt{\frac{L}{g}}$$

Eq.1

where L is the length of the pendulum, g is the acceleration due to gravity.

This shows that under small angle approximation, Eq.1 (the period T) is **independent of the initial angle** meaning it does not matter how far from the equilibrium position the pendulum starts. It will take the same amount of time to complete one oscillation, if the angle is small (Tsokos, 2014).

In this IA, I aim to investigate the behaviour of a large angle pendulum and how it deviates from the theoretical predictions of the small angle approximation. Traditional understanding, as outlined, shows that period T is independent of the initial angle. My aim is to determine the extent to which the period of a pendulum varies when released from larger angles surpassing the 15-degree threshold by showing how the elliptical integral arises in the problem of the large angle pendulum.

I show and prove in detail how the elliptical integral can be written as an infinite series. I also conduct numerical comparisons of infinite series with alternative methods to calculate the elliptic integral. I wrote a computer program to facilitate the calculations of K(k). I also use the general purpose of the program Desmos to calculate the integrals numerically. I specifically focus on the elliptical integral below and its sensitivity to different orders of k^2. My early prediction is that for larger angles, the period will deviate significantly from the first and second-order approximations.

2 Background Information

In exploring the dynamics of a pendulum's swing at large angles, I found that the period T depends on elliptic integrals, a concept diverging significantly from the SHM applicable to small angles. My journey began with a required practical in my physics class where I experimented with pendulums to determine the gravitational constant *g*. This involved timing the period of oscillations using strings of various lengths and a metallic ball. I was instructed to not go above the 15° which made me wonder what would happen when the angles are close to 180°. I came to the observation that for the first few oscillations, it was clearly shown that it takes a lot longer to complete one period. For larger angles, the dynamics of the pendulum's swing are much different. I came across that the period T then depends on elliptic integrals. I will show below that this period T is for large angles:

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} \qquad \text{Eq.2}$$

where, $k^2 = \sin\left(\frac{\theta_0}{2}\right)$ where θ_0 is the opening angle of the pendulum.

In Eq.2, T is the period of a pendulum swing at large angles. The integral is an elliptic integral of the first kind, which cannot be expressed in terms of elementary functions. It calculates the period T for large angle oscillations, highlighting that the period T is dependent of the θ_0 contrary to the case for small angles where T is independent of θ_0. Elliptical functions are integral to the pendulum's theoretical framework for large angle oscillations, where the path of the bob is non-linear, and the trigonometric approximation is no longer applicable mentioned above is no longer applicable. These pendulums are found in practical applications all around the world: in amusement parks, playground swings and even clocks.

Jacob Bernoulli found in the 17th century that the arc length of a spiral is related to the elliptic integral. Euler and Fagnano studied the arc length of ellipses and lemniscates where elliptical integrals naturally occur. However, the connections of pendulums to elliptical functions emerged in the 18th century with the work of mathematicians like Gauss who investigated the integral of the arc sine function which relates to the period T of a pendulum. This integral couldn't be expressed in terms of elementary functions.

3 Derivation of expression for period T

I derive an equation of motion using energy conservation, it's important to consider that air friction, although small, can affect the pendulum's dynamics. In a theoretical model, I will be ignoring air friction to simplify calculations. Therefore, the gravitational potential energy (GPE) caused by the height of the pendulum bob is converted into kinetic energy (KE) as the bob moves down and inversely when the bob moves up again.

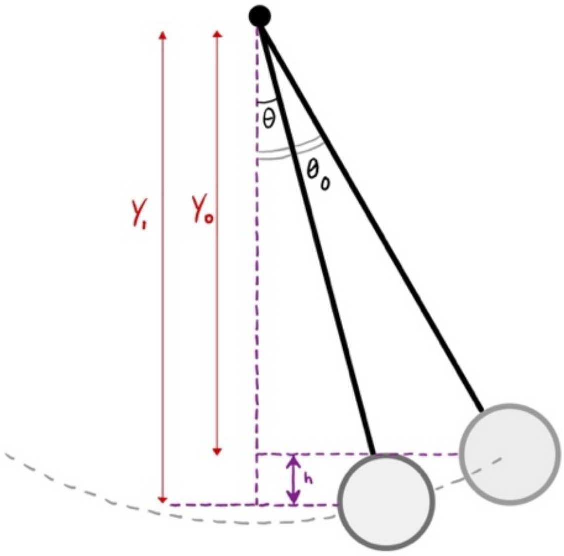


Figure 2: Simple pendulum displaying the change in height h caused by the movement.

When an object falls a vertical height *h*, the kinetic energy ($\frac{1}{2}mv^2$) it gains, becomes equivalent to the gravitational potential energy lost (mgh). Therefore:

$$\frac{1}{2}mv^2 = \mathrm{mgh}$$

I then have an expression for the bob's velocity:

$$v = \sqrt{2gh}$$

In the introduction, I showed that the displacement travelled of a pendulum is the following:

$$s = L\theta$$

I then differentiate with respect to time.

$$\frac{ds}{dt} = L \cdot \frac{d\theta}{dt}$$

Hence,

$$v = L \cdot \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{\sqrt{2gh}}{L}$$

From Figure 1 above using trigonometry:

$$h = \mathrm{y}_1 - \mathrm{y}_0 = \mathrm{L}\,(\cos\theta - \cos\theta_0)$$

Then solving for dt:

$$\frac{d\theta}{dt} = \frac{\sqrt{2g\mathrm{L}\,(\cos\theta - \cos\theta_0)}}{L}$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{L}} \cdot \sqrt{(\cos\theta - \cos\theta_0)}$$

$$\mathrm{dt} = \frac{d\theta}{\sqrt{\frac{2g}{L}} \cdot \sqrt{(\cos\theta - \cos\theta_0)}}$$

$$dt = \sqrt{\frac{L}{2g}} \cdot \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}}$$

I then integrate the above to get an expression for period T. Notably, the pendulum swings from its initial angle θ_0 to the vertical in a quarter of its period. Thus,

$$T = 4\sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{(cos\theta - cos\theta_0)}} \qquad \text{Eq.3}$$

I want to show that this expression is equivalent to an elliptical integral Eq.2. I then use the following trigonometric identities and substitution in order to demonstrate that this is also in fact an elliptical integral.

$$sinu = \frac{sin\frac{\theta}{2}}{sin\frac{\theta_0}{2}} \quad and\ k = sin\frac{\theta_0}{2}$$

By applying the double angle formula for cosine:

$$cos2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

Substitute $\frac{\theta}{2}$ for θ,

$$cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta}{2}$$

Subsequently allows us to express $\cos\theta_0$ as:

$$cos\theta_0 = 1 - 2\sin^2\frac{\theta_0}{2}$$

Examining the integrant and incorporating the substitutions and identities, I find:

$$cos\theta - cos\theta_0 = 2\sin^2\frac{\theta}{2} - 2\sin^2\frac{\theta_0}{2}$$

By substituting $\sin^2\frac{\theta}{2} = k^2\sin^2 u$, I get:

$$cos\theta - cos\theta_0 = 2(\mathrm{k}^2\sin^2 u) - 2k^2 = 2(k^2\sin^2 u - k^2)$$

Taking the total differential of $sinu = \frac{\sin\left(\frac{\theta}{2}\right)}{k}$

$$\frac{d}{du}(sinu) = \frac{1}{k} \cdot \frac{d}{d\theta}(\sin\frac{\theta}{2})$$

$$cosu\ du = \frac{1}{2k} \cdot cos\frac{\theta}{2} \cdot d\theta$$

Upon rearrangement, I obtain:

$$d\theta = \frac{2\mathrm{k} \cdot cosu \cdot du}{cos\frac{\theta}{2}}$$

I recognize that:

$$cosu = \sqrt{1 - \sin^2 u}$$

$$cos\frac{\theta}{2} = \sqrt{1 - \sin^2\frac{\theta}{2}}$$

And since:

$$\sin^2\frac{\theta}{2} = k^2 \sin^2 u$$

It follows that:

$$cos\frac{\theta}{2} = \sqrt{1 - k^2 \sin^2 u}$$

Thus, I express dθ as:

$$d\theta = \frac{2\mathrm{k} \cdot \sqrt{1 - \sin^2 u} \cdot du}{\sqrt{1 - k^2 \sin^2 u}}$$

Refocusing on Eq. 3, I can now replace dθ and $\cos\theta - \cos\theta_0$ with terms involving u.

$$T = 4\sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{(cos\theta - cos\theta_0)}}$$

Considering the integral:

$$\int_0^{\theta_0} \frac{\frac{2\mathrm{k} \cdot \sqrt{1 - \sin^2 u} \cdot du}{\sqrt{1 - k^2 \sin^2 u}}}{\sqrt{(2(k^2 \sin^2 u - k^2))}}$$

I adjust the limits of integration based on angles:

$$\theta = 0, \mathrm{u} = 0$$
$$\theta_0 = 0, \mathrm{sinu} = 1, \mathrm{u} = \frac{\pi}{2}$$

This leads us to:

$$\int_0^{\frac{\pi}{2}} \frac{2k \cdot \sqrt{1 - \sin^2 u}}{\sqrt{1 - k^2 \sin^2 u} \cdot \sqrt{(2(k^2 \sin^2 u - k^2))}} du$$

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{(2k)^2} \cdot \sqrt{1 - \sin^2 u}}{\sqrt{1 - k^2 \sin^2 u} \cdot \sqrt{2(k^2 \sin^2 u - k^2)}} du$$

Upon simplifying, I arrive at:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{2} \cdot \sqrt{2k^2 \sin^2 u - 2k^2}}{\sqrt{1 - k^2 \sin^2 u} \cdot \sqrt{2k^2 \sin^2 u - 2k^2}} du$$

Ultimately, I reformulate:

$$T = 4\sqrt{\frac{L}{2g}} \sqrt{2} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 u}} du$$

Which simplifies to Eq.3

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 u}} du$$

3.1 Complete elliptic integral of the first kind

The eccentricity of an ellipse is a measure of its deviation from being a circle. It is defined as the ratio of the distance between the two foci of the ellipse and the length of the major axis (BYJUS, n.d.).

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

where a is the length of the semi-major axis and b is the length of the semi-minor axis of the ellipse

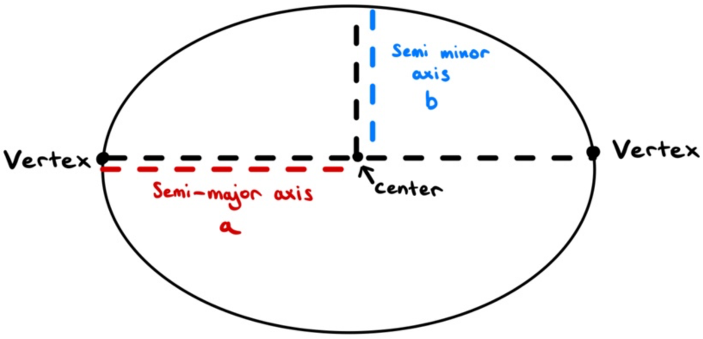


Figure 3: Eccentricity of an ellipse

The complete elliptic integral of the first kind usually denoted as ‘K’ or ‘K(k)’ is a special mathematical function used in areas of math and physics. The parameter 'k' is known as the elliptic modulus which lies in the range $0 \leq k \leq 1$ and is directly related to the eccentricity *e* of the ellipse.

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1 - k^2 \sin^2 \theta)}} \qquad \text{Eq.4}$$

The relationship between them is that $k = \sqrt{e}$. The parameter here also determines the shape of the elliptic integral. Eq. 4 indicates that K(k) spans the integral limits from 0 to $\frac{\pi}{2}$ which is why it is termed 'complete'.

Alternatively, considering that the eccentricity is k^2, the elliptic modulus k is the square root of the eccentricity. Thus, as k approaches 0, the ellipse becomes more circular (since e approaches 0) and as k approaches 1, the ellipse becomes more elongated (since e approaches 1). This relationship highlights the role of k in determining the shape of the elliptic arc described by K(k).

Turning to physics, specifically pendulum motion, an expression for the period T of a pendulum using the complete elliptic integral is as follows:

$$T = 4\sqrt{\frac{L}{g}} \cdot K(k) \qquad \text{Eq. 5}$$

Using ‘Tables of Integrals and other mathematical data’ by Herbert Dwight, I found that the integral K(k) above can be written in terms of an infinite series (Dwight, 1961).

In the following section, I will rigorously prove that K(k) is:

$$K(k) = \frac{\pi}{2}\left(1 + \frac{1}{2^2}k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}k^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}k^6 + \cdots\right) \qquad \text{Eq.6}$$

The power series in Equation 6 offers a precise expression for K(k) that converges to the true value of the integral ensuring accuracy in calculating the pendulum's period at large angles up to 180°,

The proof is the focus of this IA.

3.2 Proof: Step 1 - Write K(k) as a series in $k^{2n} \int_0^{\frac{\pi}{2}} \boldsymbol{sin^{2n}x\, dx}$

I start with the integrant of Equation 2:

$$(1 - k^2 \sin^2 x)^{-\frac{1}{2}} \qquad \text{Eq.7}$$

To simplify the expression, I do the following substitution and make a Taylor expansion.

$$y = k^2 \sin^2 x$$

$$(1-y)^{-\frac{1}{2}} = 1 + \binom{-\frac{1}{2}}{1}(-y) + \binom{-\frac{1}{2}}{2}(-y)^2 + \binom{-\frac{1}{2}}{3}(-y)^3 + \binom{-\frac{1}{2}}{4}(-y)^4 + \cdots$$

Obviously, this is an infinite series in y.
I then evaluate the binomial coefficients and substitute $k^2 \sin^2 x$ back. Then Eq.7 becomes:

$$(1 - k^2 \sin^2 x)^{-\frac{1}{2}} =$$

$$= 1 + \left(-\tfrac{1}{2}\right)(-k^2 \sin^2 x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-k^2 \sin^2 x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-k^2 \sin^2 x)^3 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}(-k^2 \sin^2 x)^4 \qquad \text{Eq.8}$$

Further evaluating these binomial coefficients, I start to see a pattern:

$$(1 + \frac{1}{2}k^2 \sin^2 x + \frac{1}{2^2} \cdot \frac{1 \cdot 3}{2!} k^4 \sin^4 x + \frac{1}{2^3} \cdot \frac{1 \cdot 3 \cdot 5}{3!} k^6 \sin^6 x + \frac{1}{2^4} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{4!} k^8 \sin^8 x + \cdots)$$

$$= (1 + \frac{1}{2}k^2 \sin^2 x + \frac{1 \cdot 3}{(2 \cdot 2)(2 \cdot 1)} k^4 \sin^4 x + \frac{1 \cdot 3 \cdot 5}{(2 \cdot 3)(2 \cdot 2)(2 \cdot 1)} k^6 \sin^6 x + \frac{1 \cdot 3 \cdot 5 \cdot 7}{(2 \cdot 4)(2 \cdot 3)(2 \cdot 2)(2 \cdot 1)} k^8 \sin^8 x + \cdots)$$

I introduce the double factorial concept below and define it as the following:

$$N!! = N \cdot (N-2)!!$$

It would also be good to note that:

$$N!! = 1 \; if \; n = 0 \; or \; n = 1$$

I then use the double factorial to express the Taylor series as the following:

$$= (1 + \frac{1!!}{2!!} k^2 \sin^2 x + \frac{3!!}{4!!} k^4 \sin^4 x + \frac{5!!}{6!!} k^6 \sin^6 x + \frac{7!!}{9!!} k^8 \sin^8 x + \cdots)$$

To find a general expression for the n^th^ term in expansion, I start with the general combinatorial formula such that:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

(Awada, 2019)

As I can see from these coefficients of the Taylor series, there is a consistent pattern meaning I can determine a general expression for the n[th] term in expansion:

$$\binom{-\frac{1}{2}}{n}(-1)^n = \frac{\left(-\frac{1}{2}\right)!}{n!\left(-\frac{1}{2}-n\right)!}(-1)^n$$

$$\binom{-\frac{1}{2}}{n}(-1)^n = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)\left(-\frac{1}{2}-n\right)!}{n!\left(-\frac{1}{2}-n\right)!}(-1)^n$$

$$\binom{-\frac{1}{2}}{n}(-1)^n = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}(-1)^n$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{(2n-1)}{2}\right)}{n!}(-1)^n$$

$$= \frac{1}{2^n}\cdot(-1)^n\cdot\frac{(2n-1)!!}{n!}\cdot(-1)^n$$

$$= \frac{1}{2^n}\cdot\frac{(2n-1)!!}{n!} = \frac{(2n-1)!!}{(2n)\cdot 2(n-1)\cdots(2\cdot 1)}$$

$$= \frac{(2n-1)!!}{(2n)!!}$$

Hence, Eq. 8 becomes:

$$(1-k^2\sin^2 x)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot k^{2n}\cdot\sin^{2n} x \qquad Eq.9$$

I obtain K(k) by integrating both sides of Equation 9

$$K(k) = \int_0^{\frac{\pi}{2}}(1-k^2\sin^2 x)^{-\frac{1}{2}}\,dx = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot k^{2n}\cdot\int_0^{\frac{\pi}{2}}\sin^{2n} x\,dx$$

3.3 Proof: Step 2 – Show that integral $\int_0^{\frac{\pi}{2}} sin^{2n} dx = \frac{\pi}{2}\cdot\frac{(2n-1)!!}{(2n)!!}$

In this section I focus on the integral $\int_0^{\frac{\pi}{2}}\sin^{2n} x\; dx$ as shown in Equation 6 above. I integrate this integral for n is 1, 2 and 3 to see if I can find a pattern. For n = 1, I have the following integration:

$$\int_0^{\frac{\pi}{2}} \sin^2 x \ dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - cos2x) \ dx$$

I then used the trigonometric double angle identity:

$$\sin^2 x = \frac{1}{2}(1 - cos2x) \qquad \text{Eq.10}$$

Then apply the boundaries.

$$\int_0^{\frac{\pi}{2}} \sin^2 x \ dx = = \frac{1}{2}x - \frac{1}{2}sin2x \ \Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Same process is repeated but now for n = 2 meaning I now have the following to integration:

$$\int_0^{\frac{\pi}{2}} \sin^4 x \ dx = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \sin^2 x \ dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\right)^2 (1 - cos2x)^2 \ dx$$

$$= \left(\frac{1}{2}\right)^2 \int_0^{\frac{\pi}{2}} (1 - 2cos2x + \cos^2 2x) \ dx \qquad \text{Eq.11}$$

I use the following trigonometric identity to deal with the $\cos^2 2x$ term.

$$cosA \cdot cosB = \frac{1}{2}[\cos(A + B) + \cos(A - B)] \qquad \text{Eq.12}$$

Therefore, I get the following:

$$\cos^2 2x = \frac{1}{2}[\cos(4x) + \cos(0)] = \frac{1}{2}cos4x + \frac{1}{2}$$

I now have the following expression for $\sin^4$ x:

$$\sin^4 x = \left(\frac{1}{2}\right)^2 \left(1 - 2cos2x + \frac{1}{2}cos4x + \frac{1}{2}\right)$$

$$= \left(\frac{1}{2}\right)^2 [\frac{3}{2} + \frac{1}{2}cos4x - 2cos2x] \qquad \text{Eq.13}$$

I continue with $\sin^6$ x. I do this by using the integration result from $\sin^2$x and $\sin^4$x.

$$\sin^6 x = sin^2 \cdot \sin^4 x$$

$$= \frac{1}{2}(1 - cos2x)\left(\frac{1}{2}\right)^2 \cdot [\frac{3}{2} + \frac{1}{2}cos4x - 2cos2x]$$

$$= \left(\frac{1}{2}\right)^3 (-\frac{3}{2}cos2x - \frac{1}{2}cos2x \ cos4x + 2\cos^2 2x + \frac{3}{2} + \frac{1}{2}cos4x - 2cos2x)$$

Now I use Eq.12 to deal with $\cos^2 2x$ and the mixed term cos2x · cos4x:

$$=\left(\frac{1}{2}\right)^3\left(-\frac{7}{2}cos2x+\frac{3}{2}+\frac{1}{2}cos4x+2\left(\frac{1+cos4x}{2}\right)-\frac{1}{2}\left(\frac{1}{2}(cos6x+cos2x)\right)\right.$$

$$=\left(\frac{1}{2}\right)^3\left(\frac{1}{4}\right)[10+6cos4x-15cos2x-cos6x] \qquad Eq.14$$

Looking at the expressions $\sin^2 x, \sin^4 x, \sin^6 x$ I see that the only relevant term is the constant term. *This is because upon integration of these expressions over the interval 0 to $\frac{\pi}{2}$ the cos2x, cos4x, cos6x terms vanish*. Therefore, I only need to pay attention to the constant term. Hence,

$$\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x\, dx = \left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cdot\frac{1\cdot 3}{2\cdot 4}$$

$$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx = \left(\frac{1}{2}\right)^3\left(\frac{1}{4}\right)(10)\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}$$

I again notice the pattern which is the same as that of the series in section 3.1 (which seems to be a remarkable coincidence as they are not related).

From the expressions for $\sin^2 x, \sin^4 x, \sin^6 x$, I suspect that $\sin^{2n} x$ can be written as:

$$\sin^{2n} x = C + A_2 cos2x + A_4 cos4x + \cdots + A_{2n}\cos(2nx) \qquad Eq.15$$

Where C, A_2, A_4, … A_{2n} are constants.

This is our induction hypothesis P(n).

I have proved previously P(1), P(2) and P(3). Our next step is to assume that P(n) is true,

And now I'll prove P(n+1). So, I need to prove that $\sin^{2(n+1)} x$ can be written as a constant term plus terms with cos2x, cos4x and so on until $\cos(2(n+1))x$. Using P(n):

$$\sin^{2(n+1)} x = \sin^2 x \cdot (C + A_2 cos2x + A_4 cos4x + A_{2n}\cos(2nx))$$

$$=\frac{1}{2}(1-cos2x)\cdot(C + A_2 cos2x + A_4 cos4x + A_{2n}\cos(2nx))$$

I now multiply out the two factors in the brackets. The 1 in the first factor gives a constant terms plus terms that include cos2x, cos4x, …, cos2nx. Multiplying out the cos2x term in the first bracket gives a cos2x term, a $\cos^2(2x)$

term and mixed terms of the type cos2x · cos4x, … cos2x · cos 2nx. The $\cos^2 x$ term and all the mixed cosine terms can be converted to cos2x, cos 4x, … $cos2(n+1)x$ terms using Equation 12. Taking everything together $\sin^{2(n+1)} x$ can be written as a constant term plus a linear combination of cos2x, cos 4x, … $cos2(n+1)x$. Herewith P(n+1) is proven.

Finally, using Equation 15,

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \frac{\pi}{2} \cdot C \qquad Eq.16$$

Now I need to find an expression for the constant term C in $\sin^{2n}x$. I could work out an expression for $\sin^{2n}x$ continuing equations 13 and 14, but this is very tedious. There is a faster way to determine the constant C. For this I express $\sin x$ using Euler's formula:

$$e^{ix} = cosx + isinx$$

I then take the complex conjugate of both sides. (Since, cosine is an even function and sine is an odd function, the cosine remains unchanged while the sign in front of the sine changes)

$$e^{-ix} = cosx - isinx$$

Now I have two equations and to isolate sinx, I subtract the complex conjugate from Euler's formula. So,

$$e^{ix} - e^{-ix} = (cosx + isinx) - (cosx - isinx)$$
$$e^{ix} - e^{-ix} = 2i \cdot sinx$$

Lastly,

$$sinx = \frac{e^{ix} - e^{-ix}}{2i}$$

Consider the binomial expansion of:

$$\sin^{2n} x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^{2n}$$

Only the n^{th} term is independent of x because only then e^{inx} and e^{-inx} cancel out. The n^{th} term is:

$$\binom{2n}{n}\left(\frac{1}{2i}\right)^{2n}(e^{inx})(-1)^n(e^{-inx})$$

$$= \binom{2n}{n}\left(\frac{1}{2^{2n}}\right)\left(\frac{1}{-1}\right)^n(-1)^n$$

$$= \binom{2n}{n}\left(\frac{1}{2^{2n}}\right)$$

$$= \frac{(2n)!}{n!\,n!} \cdot \frac{1}{2^{2n}} = \frac{(2n)!}{2n \cdot 2(n-1) \cdots 2 \cdot 2n \cdot 2(n-1) \cdots 2} = \frac{(2n)!}{(2n)!!\,(2n)!!}$$

$$= \frac{(2n)(2n-1)(2n-2)\cdots(2)}{(2n)!!\,(2n)!!}$$

$$= \frac{(2n-1)!!}{(2n)!!}$$

A factor (2n)!! can be divided out, and only (2n - 1)!! remains in the numerator. Now, Eq.16 becomes:

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!}$$

And finally, equation 9 becomes:

$$K(k) = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 x)^{-\frac{1}{2}} dx = \frac{\pi}{2} + \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \cdot k^{2n}$$

$$= \frac{\pi}{2} \left(1 + \frac{1^2}{2^2} k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} k^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} k^6 + \cdots \right) \qquad Eq.17$$

Which is the result that I set out to prove.

3.4 Convergence properties of K(k)

The series of Eq.17 has an infinite number of terms, and I would like to understand for what k this series converges. By exploring convergence, I can pinpoint the threshold where the model remains valid, specifically for angles up to 180 degrees, ensuring that the period calculations are reliable.

In this section I study for which k^2 converges (recall that $k = \sin\left(\frac{\theta_0}{2}\right)$)

Note that:

$$\frac{(2n-1)!!}{(2n)!!} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \cdots$$

Each of the factors of the RHS are less than 1. So,

$$\frac{(2n-1)!!}{(2n)!!} < 1 \; and \; also \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 < 1$$

Then, multiply both sides by k^{2n}

$$\left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \cdot k^{2n} < k^{2n}$$

After taking the sum of n to each side,

$$\sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \cdot k^{2n} < \sum_{n=1}^{\infty} k^{2n}$$

And adding 1

$$1 + \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \cdot k^{2n} < 1 + \sum_{n=1}^{\infty} k^{2n}$$

I then see that the LHS of the inequality expression is equal to K(k) up to a factor $\frac{\pi}{2}$ and, that the sum on the RHS equals to $\frac{1}{1-k^2}$. Hence,

$$K(k) < \frac{\pi}{2} \cdot \frac{1}{1-k^2}$$

Hence, K(k) is convergent for $k^2 < 1$

3.5 Divergent properties of K(k)

It is good to wonder at what happens when $k^2 = 1$. This is when the initial angle (θ_0) is 180°. In this case K(k) does not converge. This can be seen most easily by calculation of the elliptical integral directly (this is not possible for $k^2 \neq 1$)

I consider $k^2 = 1$ and use the following integral to prove this.

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{(1-\sin^2 u)}} = 4\sqrt{\frac{L}{g}} \cdot K(1)$$

Focusing on the integral itself and make the following substitutions to see what we get when $k^2 = 1$.

$$K(1) = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{(1-\sin^2 u)}}$$

And with the following substitutions:

$$v = sinu$$

$$dv = \cos u \, du = \sqrt{(1-v^2)}\, du$$

I get,

$$K(1) = \int_0^1 \frac{dv}{\sqrt{(1-v^2)}} \cdot \frac{1}{\sqrt{(1-v^2)}}$$

I then separate this into partial fractions to integrate.

$$\int_0^1 \frac{dv}{(1-v^2)} = \int_0^1 \frac{A}{(1-v)} + \frac{B}{(1+v)}$$

$$= \int_0^1 \left(\frac{1}{2(1-v)} + \frac{1}{2(1+v)}\right) dv$$

$$= \frac{1}{2}\int_0^1 \left(\frac{1}{(1-v)} + \frac{1}{(1+v)}\right) dv$$

I now integrate the following and apply the boundaries:

$$K(1) = \frac{1}{2}\Big[\ln(1-v) + \ln(1+v)\Big|_0^1 = \infty$$

So, for $k^2 = 1$, the elliptic integral diverges. Consequently, since $T \propto K(k)$, this is what I expect for the pendulum (with a solid rod): if the pendulum is perfectly balanced at $\theta_0 = 180°$, it'll never come down and $T = \infty$.

4 Numerical studies

In the realm of mathematics, physics, and engineering. One often tries to find a formula for such a phenomenon. If the full expression is very complicated and unwieldy, one would try to find an approximation. For instance, the period T of a pendulum $T = 2\pi\sqrt{\frac{L}{g}}$ in the first order approximation. In the second order approximation it is $T =$

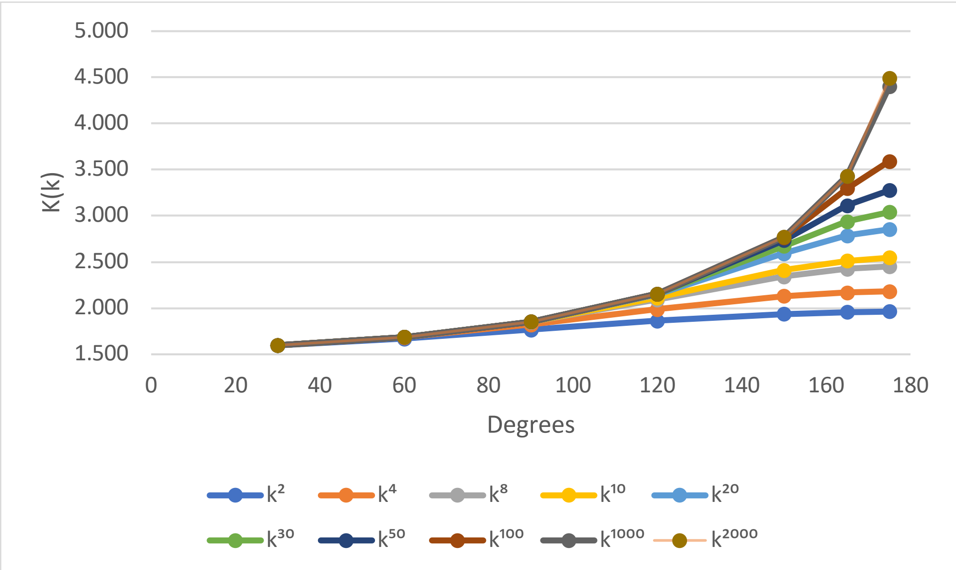
$2\pi\sqrt{\frac{L}{g}}\cdot\left(1+\frac{\theta_0{}^2}{16}\right)$. But how good are such approximations? The critical question I tackle in this part of my study is the accuracy of such approximations.

In this section, I conduct numerical studies to validate the results I have obtained and to ensure that our mathematical analysis aligns. The goal of these studies is to show that our expression for the elliptical integral, presented in Equation 2, accurately describes the behaviour of the large angle pendulum.

4.1 Numerical calculations of K(k) up to very high orders of k^2

I write a computer program in Python to calculate the complete elliptic integral of the first kind (K(k)) for a given opening angle θ_0. It employs a series expansion method with a specified number of terms (Nterms) to approximate the integral. I initialize numerator and denominator coefficients, also calculate the elliptic modulus k and its square to iteratively compute the coefficients for the series expansion. It evaluates the elliptic integral using a loop by adding up the terms to obtain an approximation. It can add up 1000s of orders in k^2. This code offers a practical means of numerically estimating K(k) for different initial angles to support the analysis of large angle pendulums.

In graph 1 below I use the python program to calculate K(k) as a function of the opening angle. **Note that the python computer program is stated in Appendix A.** Recall that $k = sin\left(\frac{\theta_0}{2}\right)$.

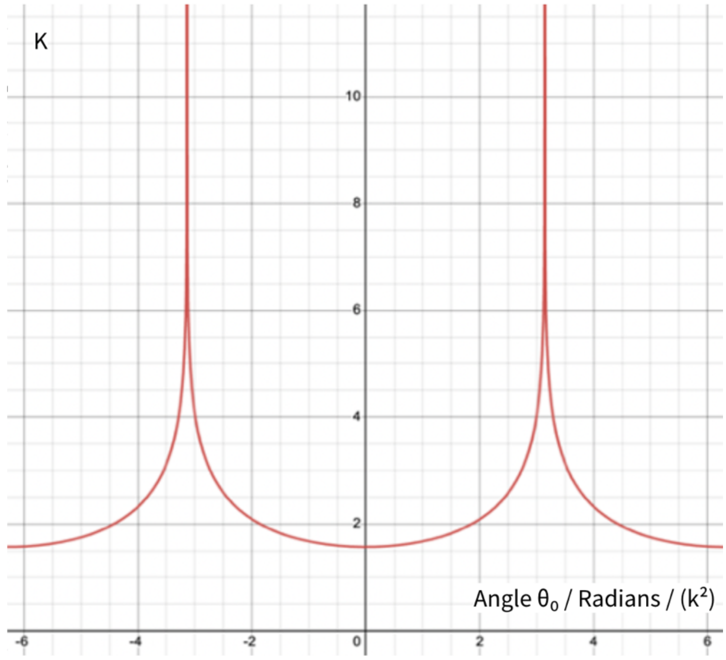


Graph 1: The graph here displays the function K(k) vs Initial angle (θ_0 in degrees). The multiple lines represent different powers of: k^2, k^4, k^8, k^{10}, k^{20}, k^{30}, k^{50}, k^{100}, k^{1000}, k^{2000}.

Graph 1 clearly shows that for small angles, say up to 60° the K(k) series converges quickly, indicating that fewer terms are needed in Eq.17 for a precise approximation of the pendulum's period T. A closer look at the graph reveals a marked slowdown in convergence as the angle increases past 60° degrees. This deceleration in convergence is noticeable as k approaches its asymptote (180°). For example, at $\sin\left(\frac{150°}{2}\right) = 0.966$. Since the value is less than 1, I know that at such high values of k, the K(k) series converges much more slowly, necessitating more terms for an accurate approximation of T. The observation of slow convergence at large angles like 150 degrees offers a critical reflection point, it reinforces the limitations of using series approximations for large-angle pendulum motions. As well as suggest the need for alternative methods or more terms in the series to ensure accuracy.

4.2 Comparison of K(k) with Desmos computer program to evaluate integrals numerically.

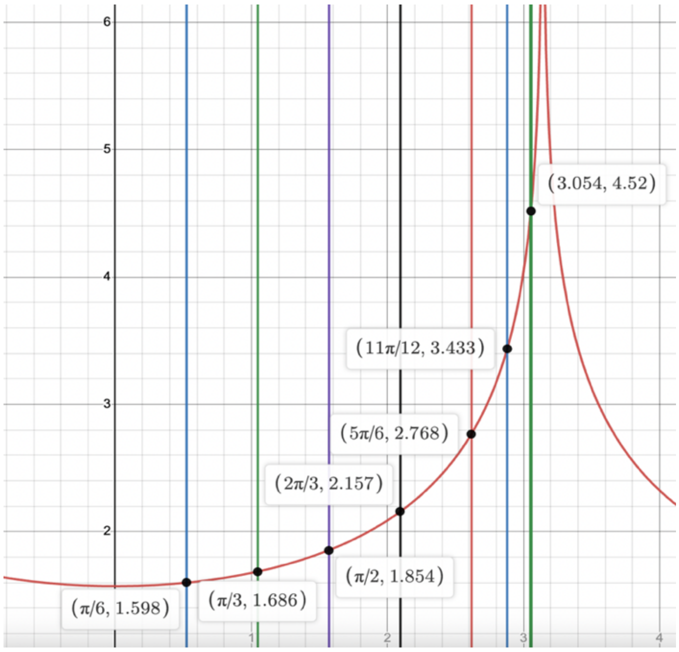
Small angle approximations for T will not work here, and to obtain accurate results full numerical calculations are required. One notable feature of the graph is the asymptote at $k^2 = 1$. As this k^2 approaches 1, as proven in 3.4 Convergence properties of K(k), the elliptical integral is approaching infinity highlighting that the initial angle of the pendulum is exactly 180° as the integral K(k) diverges. This corresponds to the pendulum being balanced in a vertical position and at a state of unstable equilibrium, resulting in an infinite period T. In graph 2 I plot K(k), but only focus on the domain from $0 < \theta_0 < \pi$.



Graph 2: K plotted against k^2 in Desmos. *The graph's y-axis represents K (Eq.4), while the x-axis represents k^2 ($sin\frac{\theta_0}{2}$). The focus of this graph is the domain where $0 < \theta_0 < \pi$.*

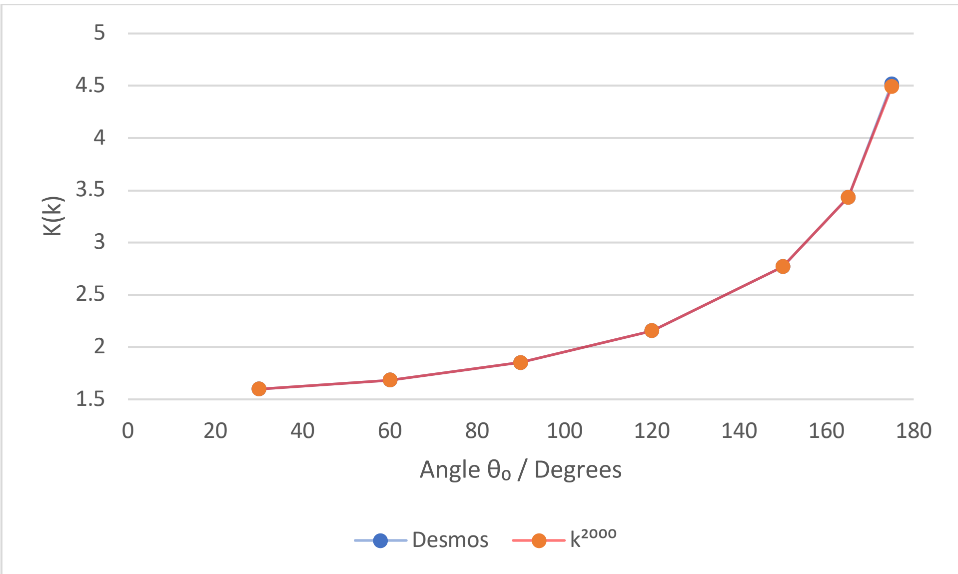
It's noteworthy that when $k^2 = 1$ the integral diverges (shown in Section 3.5), indicating an asymptotic behaviour at π. This divergence is significant as mentioned before, it describes the scenario where the pendulum is perfectly vertical, a state that lacks a definable period. This insight reinforces an understanding of the pendulum's behaviour at extreme positions.

Next, I determine the value of $K\left(sin\left(\frac{\theta_0}{2}\right)\right)$ for several values of θ_0.The results are shown below.



Graph 3: K(k) (Eq.4) represented in Graph 2 along with the several lines corresponding to different initial angles of the pendulum ranging from $\frac{\pi}{6}$ to $\frac{35\pi}{36}$.

The intersection points represent the specific values of k^2. At these points, the complete elliptic integral of the first kind is equal to a particular constant. These constants are crucial as they are used to calculate the period of a pendulum.



Graph 4: The graph is displaying the value of K(k) (Eq.6) up to the term with k^{2000} vs the integrals calculated numerically by Desmos.

I can now compare K(k) for up to order k^{2000} with the result from Desmos. See graph 4. The graph demonstrates a near-perfect congruence between the two datasets, as evidenced by the overlapping lines. The blue line (Desmos) isn't perfectly visible as the orange line (k^{2000}) overlaps it. The only difference is the 175° where both points aren't overlapping but still in agreement.

5 Conclusions

Studying the empirical behaviour of the large angle pendulums, I was intrigued by the role of elliptical integrals in their description. My investigation centered on the complete elliptic integral of the first kind. I proved that such an integral (Eq.2) can be written as an infinite series (Eq.6). We further found that the convergence of the function $K\left(sin\left(\frac{\theta_0}{2}\right)\right)$ particularly for opening angles θ_0 greater than $150°$, where thousands of terms must be calculated to get a precise result.

Reflecting on my early prediction, it is evident that the period of a pendulum released from large angles diverges noticeably from simpler first and second-order approximations. This realization reinforces the importance of numerical methods in modern physics and engineering. The use of Desmos to calculate the elliptical integral directly yielded results that were in excellent agreement with the series calculations.

However, it's crucial to acknowledge that tools like Desmos, Excel and Programming applications were not at the disposal of Euler or Bernoulli, nor available physicists and mathematicians of the past. Their absence necessitated a greater ingenuity in deriving precise results, often through simplifications or approximations. This reflection not only highlights the advancements in computational technology that helped my current analyses, but inspired an appreciation for the historical mathematical developments that laid the groundwork for our current understanding.

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7 Appendix A

```
import math

Nterms = 10 #the number of terms we are going to calculate

Num = []
Den = []
Num = [0 for i in range(Nterms+1)]
Den = [0 for i in range(Nterms+1)]

Theta0 =  math.pi / 2 #Input opening angle
k = math.sin(Theta0/2)
ksquared = k ** 2

Num[1] = 1
Den[0] = 1

for i in range(1, Nterms):
    Num[i+1] = Num[i] * (2*i+1) #Calculating coefficients (Numerator)
    Den[i] = Den[i-1] * 2*i #Calculating coefficients (Denomenator)

Den[Nterms] = Den[Nterms - 1] * 2 * Nterms

BigK = 1
for i in range(1, Nterms + 1):
    BigK = BigK + (Num[i]/Den[i]) ** 2 * ksquared ** i

print(BigK * math.pi / 2)
```

Figure 3: Python computer program to calculate K(k) using Eq.6