(b) $s(A_2) < 0$ and $\|R(\lambda,A_2)\|$ is uniformly bounded in each semiplane $\{\lambda \in \mathbb{C}\ ,\ \mathrm{Re}\lambda > s(A_2)+\varepsilon\}$ with $\varepsilon > 0$.

(c) E_1 is a closed sublattice of E and $\mathcal{T}_1$ is a periodic, irreducible, positive semigroup on E_1 . In particular, $(E_1,\mathcal{T}_1)$ is isomorphic to $(L,R_\tau(t))$ where L is a function lattice between $C(\Gamma)$ and $L^1(\Gamma)$ and $R_\tau(t)$ is the rotation group with period $\tau = 2\pi/\alpha$.

Proof. (a) has been derived above while (b) follows immediately from (2.10). The properties of $\mathcal{T}_1$ mentioned in (c) have been stated above. Hence the representation of $\mathcal{T}_1$ as a rotation group follows from C-III,Cor.3.9.

□

For Hilbert spaces $L^2(\mu)$ property (b) of the above lemma and A-III, Cor.7.11 imply that the growth bound $\omega(A_2)$ is less than zero. Therefore we obtain the following result on the asymptotic behavior of $\mathcal{T}$.

Proposition 2.13. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a bounded, irreducible, positive semigroup on a Hilbert lattice $E = L^2(\mu)$. Assume that $s(A) = 0$ is a pole of the resolvent of the generator A and that $i\alpha \in \sigma(A)$ for some $0 \neq \alpha \in \mathbb{R}$. Then $\mathcal{T}$ behaves asymptotically as the rotation group $(R_\tau(t))_{t\geq 0}$ with period $\tau = 2\pi n/\alpha$ for some $n \in \mathbb{N}$ on $L^2(\Gamma)$.
More precisely, we can identify $L^2(\Gamma)$ with a sublattice of E , which is the range of a strictly positive projection Q and we find constants $\varepsilon > 0$ and $M \geq 1$ such that for every $f \in E$ we have

(2.11) $\|T(t)f - R_\tau(t)g\| \leq Me^{-\varepsilon t}\|f\|$ for every $t \geq 0$ where $g := Qf$.

For L^p-spaces the analogous statement can be shown only for a weaker type of convergence. The proof of this result uses interpolation for operators, mainly the Riesz Convexity Theorem (see the remarks preceding Cor.1.2).

Theorem 2.14. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a bounded, irreducible positive semigroup on a Banach lattice $E = L^p(\mu)$, $1 \leq p < \infty$. Assume that $s(A) = 0$ is a pole of the resolvent of the generator A and that $i\alpha \in \sigma(A)$ for some $0 \neq \alpha \in \mathbb{R}$. Then $\mathcal{T}$ behaves asymptotically as the rotation group $(R_\tau(t))_{t\geq 0}$ with period $\tau > 0$ on $L^p(\Gamma)$, i.e., we can identify $L^p(\Gamma)$ with a sublattice of E such that for every $f \in E$ there exists $g \in L^p(\Gamma)$ satisfying

(2.12) $$\lim_{t\to\infty} \|T(t)f - R_\tau(t)g\| = 0 \quad .$$