It suffices to show the translation property (T) for $f \in D(A)$ only. To that purpose we treat two cases separately.

1. Let $t \geq 0$, $s \in [-1,0]$ and $t+s > 0$. It suffices to prove $T(-s)g(s) = g(0)$ for $g := T(t+s)f$. For arbitrary $g \in D(A)$ we define the map

$$h: [-t,0] \to F \quad \text{by} \quad h(r) = \delta_r T(-r)g \text{ ,}$$

where δ_r denotes the point evaluation $f \to f(r)$ on E.

For $\theta \neq 0$ we have

$$
\begin{aligned}
1/\theta\cdot(h(r+\theta) - h(r)) &= 1/\theta\cdot(T(-r-\theta)g(r+\theta) - T(-r)g(r)) \\
(1) \qquad &= 1/\theta\cdot(T(-r-\theta)g(r) - T(-r)g(r)) \\
(2) \qquad &\quad + 1/\theta\cdot(\delta_{r+\theta} - \delta_r)(T(-r-\theta)g - T(-r)g) \\
(3) \qquad &\quad + 1/\theta\cdot(T(-r)g(r+\theta) - T(-r)g(r)) \text{ .}
\end{aligned}
$$

As $\theta \to 0$, (1) converges to $-A[T(-r)g](r)$, (2) converges to zero and (3) converges to $A[T(-r)g](r)$. Thus h is continuously differentiable with derivative zero, whence $h(r) = h(0)$ for all $r \in [-t,0]$. Taking $r = s$ yields $T(-s)g(s) = g(0)$.

2. Let $t \geq 0$, $s \in [-1,0]$ and $t+s \leq 0$. As in the first case we show that the map $k: [0,t] \to F : r \to [T(r)f](t+s-r)$ is continuously differentiable with derivative zero.

Thus $f(t+s) = k(0) = k(t) = T(t)f(s)$.

□

The translation property (T) enables us to specify the correspondence between the semigroup $(T(t))_{t\geq 0}$ generated by the operator in (3.1) and the solution of the retarded Cauchy problem (RCP).

Corollary 3.2. For $g \in D(A)$ define $u : [-1,\infty) \to F$ by

$$u(t) := \begin{cases} g(t) & \text{if } -1 \leq t \leq 0 \\ T(t)g(0) & \text{if } 0 < t \text{ .} \end{cases}$$

Then u is the unique solution of (RCP).

Proof. Evidently $u \in C([-1,\infty),F)$ for $g \in D(A)$.

From A-I,Prop.1.6.(iii) and the definition of $D(A)$ we obtain

$$
\begin{aligned}
T(t)g(0) - g(0) &= [A(\textstyle\int_0^t T(s)g\,ds)](0) \\
&= B[(\textstyle\int_0^t T(s)g\,ds)(0)] + \Phi(\textstyle\int_0^t T(s)g\,ds) \\
&= B(\textstyle\int_0^t T(s)g(0)\,ds) + \textstyle\int_0^t \Phi T(s)g\,ds \\
&= B(\textstyle\int_0^t u(s)\,ds) + \textstyle\int_0^t \Phi T(s)g\,ds,
\end{aligned}
$$

since $\int_0^t T(s)g\,ds \in D(A)$.