$$\begin{aligned}\langle P_{(f^+)}Bf,\phi\rangle &= \langle P_{(f^+)}Af,\phi\rangle - \langle Vf^+,\phi\rangle\\ &\le \langle P_{(f^+)}Af,\phi\rangle\\ &\le \langle f^+,A'\phi\rangle\end{aligned}$$

by (3.6). It follows from Prop. 4.5 that $(S(t))_{t\geq 0}$ is positive. □

Finally, if it is known that the semigroup $(S(t))_{t\geq 0}$ is positive, domination can be characterized as follows.

Proposition 4.8. Let E be a real Banach lattice, $(T(t))_{t\geq 0}$ a positive semigroup with generator A and $(S(t))_{t\geq 0}$ a positive semigroup with generator B. Consider the following conditions.

(i) $S(t) \leq T(t)$ $(t \geq 0)$.

(ii) $\langle Bf,\phi\rangle \leq \langle f,A'\phi\rangle$ for all $f \in D(B)_+$, $\phi \in D(A')_+$.

(iii) $Bf \leq Af$ for $0 \leq f \in D(A) \cap D(B)$.

Then (i) and (ii) are equivalent and imply (iii).
Moreover, if $D(A) \subset D(B)$ or $D(B) \subset D(A)$, then (iii) implies (i).

Proof. Assume that (i) holds. Then for $f \in D(B)_+$, $\phi \in D(A')_+$,

$$\langle Bf,\phi\rangle = \lim_{t\to 0} 1/t \langle S(t)f - f, \phi\rangle \leq \lim_{t\to 0} 1/t \langle T(t)f - f, \phi\rangle = \langle f, A'\phi\rangle .$$

So (ii) holds. (iii) is proved similarly.
Now assume (ii). Let $\lambda > \max\{s(A), s(B)\}$. Let $g \in E_+$, $\psi \in E'_+$.
Then $\langle R(\lambda,B)g - R(\lambda,A)g, \psi\rangle$

$$= \langle R(\lambda,A)g, \lambda R(\lambda,B)'\psi - \psi\rangle - \langle \lambda R(\lambda,A)g - g, R(\lambda,B)'\psi\rangle$$
$$= \langle f, B'\phi\rangle - \langle Af, \phi\rangle \leq 0 ,$$

where $f = R(\lambda,A)g \in D(A)_+$ and $\phi = R(\lambda,B)'\psi \in D(B')_+$. Hence $R(\lambda,B) \leq R(\lambda,A)$ and (i) follows.
Finally, we prove that (iii) implies (i) if $D(B) \subset D(A)$, say.
Let $\lambda > \max\{s(A),s(B)\}$. Then $(A - B)R(\lambda,B)$ is a positive operator.
Hence $R(\lambda,A) - R(\lambda,B) = R(\lambda,A)(A - B)R(\lambda,B) \geq 0$. This implies (i). □

The preceding results can be applied to the perturbation by multiplication operators. Let (X,μ) be a σ-finite measure space and $E = L^p(X,\mu)$ $(1 \leq p < \infty)$. Consider a positive semigroup $(T(t))_{t\geq 0}$ with generator A . Let $m : X \to \mathbb{R}$ be a measurable function such that $m(x) \leq 0$ for all $x \in X$. Let $D(m) = \{f \in E : f\cdot m \in E\}$. Define the operator B with domain $D(B) = D(A) \cap D(m)$ by $Bf = Af + m\cdot f$ $(f \in D(B))$.