Example 4.16. Let A_o be the generator of a positive semigroup on an order complete Banach lattice E and $M \in Z(E)$. The semigroup generated by $A_o + M$ possesses a modulus semigroup. Its generator is $A_o + \mathrm{Re}M$. (This can be proved as the assertion in Example 4.15.)

If a semigroup has a bounded regular generator, then it possesses a modulus semigroup. Its generator is bounded too (see C-I, Sec.6 for the notion of regular operators).

Theorem 4.17. Let B be a regular, bounded operator on an order complete complex Banach lattice E. The semigroup $(e^{tB})_{t\geq 0}$ possesses a modulus semigroup. Its generator is $A = |B_o| + \mathrm{Re}M$, where $B = B_o + M$ is the unique decomposition of B in $L^r(E)$ satisfying $M \in Z(E)$, $B_o \in Z(E)^d$.

For the proof we need the following result which is of independent interest.

Lemma 4.18. Let A be the generator of a positive semigroup on a Banach lattice E. If $Af \geq 0$ for all $f \in D(A)_+$, then A is bounded.

Proof. There exists $M \geq 1$ such that $\|R(\lambda,A)\| \leq M/\lambda$ for all $\lambda \geq \omega(A)+1$. Fix $\mu \geq \omega(A)+1$. Then $AR(\mu,A)Af = \mu R(\mu,A)Af - Af = \mu^2 R(\mu,A)f - \mu f - Af$; hence $0 \leq Af \leq \mu^2 R(\mu,A)f$ whenever $f \in D(A)_+$. Thus $\|Af\| \leq c\|f\|$ for all $f \in D(A)_+$ (where $c := \mu^2\|R(\mu,A)\|$). Consequently,
$\|(\lambda R(\lambda,A) - Id)f\| = \|AR(\lambda,A)f\| \leq c\|R(\lambda,A)f\| \leq (Mc/\lambda)\|f\|$ for all $f \in E_+$ and all $\lambda \geq \omega(A)+1$. Hence
$\|(\lambda R(\lambda,A) - Id)g\| \leq Mc/\lambda(\|g^+\| + \|g^-\|) \leq (2Mc/\lambda)\|g\|$ for all $g \in E$.
Thus $\lambda R(\lambda,A)$ is invertible if λ is large enough and $D(A) = \mathrm{im}(\lambda R(\lambda,A)) = E$.

□

Proof of Thm.4.17. Let $A = |B_o| + \mathrm{Re}M$. It has been shown in Example 4.4b that $(e^{tA})_{t\geq 0}$ dominates $(e^{tB})_{t\geq 0}$. Let $(U(t))_{t\geq 0}$ be a positive semigroup dominating $(e^{tB})_{t\geq 0}$ and C its generator. We first show that C is bounded.
Let $f \in D(C)_+$. Then $\mathrm{Re}(Bf) = \lim_{t\downarrow 0} 1/t(\mathrm{Re}(e^{tB}f) - f) \leq \lim_{t\downarrow 0} 1/t(U(t)f - f) = Cf$. Hence $(C + |B|)f \geq (C - \mathrm{Re}B)f \geq 0$ for all $f \in D(C)_+$. By Lemma 4.18 this implies that $C + |B|$ is bounded. Hence C is bounded as well.