the identity on $L^p(\mu)$, $1 \leq p < \infty$, is a kernel operator if and only if the measure space (X,Σ,μ) is purely atomic, i.e. $L^p(\mu) \cong \ell^p_I$ for some index set I. Moreover, from Axmann (1980) we quote the following result (see Satz 3.5 l.c.):

(2.9) If T is an irreducible kernel operator then $\inf\{T^n,T^m\} > 0$ for some $n, m \in \mathbb{N}$, $n \neq m$.

<u>Corollary</u> 2.11. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a semigroup on $L^p(\mu)$ satisfying the assumptions of Thm.2.6 and assume that one operator $T(t_o)$ is a kernel operator. Then $\lim_{t\to\infty} T(t)f$ exists for every $f \in L^p(\mu)$.

<u>Proof</u>. First we note that $\ker A = \mathrm{Fix}(\mathcal{T})$ is a sublattice of $L^p(\mu)$, hence is itself an L^p-space. Since $T(t_o)_{|\ker A} = \mathrm{Id}$ we conclude that $\ker A \cong \ell^p_I$. Thus $L^p(\mu)$ contains an orthogonal system $\{e_j \in \ker A : j \in I\}$ of atoms such that $\sup_{j\in I} e_j = e$. The closed principal ideal E_j generated by e_j in $L^p(\mu)$ is $(T(t))$-invariant and the restriction of $(T(t))_{t\geq 0}$ to this ideal yields an irreducible semigroup $(T_j(t))_{t\geq 0}$ having generator A_j. From C-III,Cor.3.9 we conclude that $P\sigma(A_j)\cap i\mathbb{R} = \{0\}$. It follows that $T_j(t_o)$ is an irreducible kernel operator hence by (2.9) all the assumptions of Cor.2.10 are satisfied. Thus we have convergence on the the principal ideal E_j. Since the semigroup is bounded and the union of these ideals is total in $L^p(\mu)$ we have convergence on the whole space.

□

In all the results obtained so far we had to show or to assume that $P\sigma(A)\cap i\mathbb{R} = \{0\}$. This is not surprising since for an eigenvector $g \in E$ corresponding to $i\alpha \neq 0$, $\alpha \in \mathbb{R}$, we have $T(t)g = e^{i\alpha t}g$ and $\lim_{t\to\infty} T(t)g$ does not exist. Nevertheless in some cases with $P\sigma(A)\cap i\mathbb{R} \neq \{0\}$ one can describe the asymptotic behavior of $(T(t))_{t\geq 0}$ for large t. Instead of convergence to an equilibrium point one obtains that $T(t)f$ 'converges to a periodic function'.

To that purpose we consider a bounded, irreducible semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ of positive operators on some Banach lattice E having order continuous norm. Under the assumption that the spectral bound $s(A) = 0$ is a pole of the resolvent we can apply Theorem 3.12 of Chapter C-III. In particular, if 0 is not the only point in the boundary spectrum $\sigma(A)\cap i\mathbb{R}$ we obtain that

$$\sigma(A)\cap i\mathbb{R} = P\sigma(A)\cap i\mathbb{R} = i\alpha\mathbb{Z} \text{ for some } 0 < \alpha \in \mathbb{R}.$$