A more general result on cyclicity of the eigenvalues in the boundary spectrum will be proved in Sect.4 (see Cor.4.3) . In the remaining part of this section we focus our interest on the entire boundary spectrum. We will prove that it is cyclic provided that the resolvent $R(\lambda,A)$ does not grow too fast as $\lambda \downarrow s(A)$. We start with some preparations. An important tool in the proof are pseudo-resolvents.

Definition 2.4. Let D be an open (non-empty) subset of $\mathbb{C}$ and let G be a Banach space. A mapping $R : D \to L(G)$ which satisfies

(2.8) $$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu) \qquad (\lambda,\mu \in D)$$

is called a pseudo-resolvent on G .

An equivalent (often quite useful) version of (2.8) is the following:

(2.9) $$\big(1 - (\lambda - \mu)R(\lambda)\big)\big(1 - (\mu - \lambda)R(\mu)\big) = 1 \qquad (\lambda,\mu \in D)$$

Obviously, the resolvent of a closed linear operator A on G is a pseudo-resolvent on $D = \rho(A)$. In general a pseudo-resolvent need not be the resolvent of an operator. Further information can be found in Hille-Phillips (1957), Pazy (1983) or Yosida (1974). For our purposes the following examples are of particular interest:

Example 2.5.(a) Suppose A is a densely defined linear operator on G with $\rho(A) \neq \emptyset$ and let G_F be an F-product of G (cf. A-I,3.6). Then the canonical extensions $R(\lambda,A)_F$ of $R(\lambda,A)$ form a pseudo-resolvent R_F on G_F with $\rho(A)$ as domain of definition. If A is unbounded, then $0 \in A\sigma(R(\lambda,A))$ hence $0 \in P\sigma(R_F(\lambda,A))$ (cf. A-III, 4.5). It follows that R_F is not the resolvent of an operator on G .

(b) If $\{R(\lambda)\}_{\lambda\in D}$ is a pseudo-resolvent on G , then $\{R(\lambda)'\}_{\lambda\in D}$ is a pseudo-resolvent on G' . Moreover, if H is a closed linear subspace of G which is $\{R(\lambda)\}_{\lambda\in D}$-invariant ($R(\lambda)H \subset H$ for all $\lambda \in D$), then the operators on H and G/H induced by $R(\lambda)$ in the canonical way form a pseudo-resolvent on H and G/H respectively.

In the following we list several simple properties. We assume that $R : D \to L(G)$ is a pseudo-resolvent on a Banach space G .

(2.10) Given $\lambda_o \in D$, $\mu \in \mathbb{C}$ there exists at most one operator $S \in L(G)$ such that

$$R(\lambda_o) - S = -(\lambda_o-\mu)R(\lambda_o)S = -(\lambda_o-\mu)SR(\lambda_o) \ .$$

In case such an operator exists we have

$$R(\lambda) - S = -(\lambda-\mu)R(\lambda)S = -(\lambda-\mu)SR(\lambda) \quad \text{for all } \lambda \in D$$