Remark. Frequently a generator A can be extended to a closed operator B. Then one can consider the abstract Cauchy problem ACP(B) associated with B. It also has a solution for every initial value in $D(B)$, but none of the solutions is unique unless $A = B$.
[In fact, denote by $(T(t))_{t\geq 0}$ the semigroup generated by A. Let $f \in D(B)$. Let $\lambda > \omega(A)$. Then there exists $g \in D(A)$ such that $(\lambda-B)f = (\lambda-A)g$. Let $h = f - g$. Then $h \in \ker(\lambda-B)$. Define u by $u(t) = e^{\lambda t}h + T(t)g$. Then u is a solution ACP(B) with initial value f. It follows from Cor.1.2 that there exists a non-trivial solution for the initial value 0 .]

One-parameter groups

Generators of one-parameter groups can be characterized similarly by existence and uniqueness of the solutions of the associated Cauchy problem. However, here the assumption of continuous dependence on the initial values can be relaxed (in fact, one has no longer to assume that the continuous dependence is uniform in t).

Theorem 1.6. Let A be a closed densely defined operator. The following assertions are equivalent.

(i) A is generator of a strongly continuous one-parameter group.

(ii) For every $f \in D(A)$ there exists a unique function $u(\cdot,f) \in C^1(\mathbb{R})$ satisfying $u(t,f) \in D(A)$ for all $t \in \mathbb{R}$ and $u(0,f) = f$ such that $\frac{d}{dt}u(t,f) = Au(t,f)$; and if $f_n \in D(A)$ such that $\lim_{n\to\infty} f_n = 0$, then $\lim_{n\to\infty} u(t,f_n) = 0$ for all $t \in \mathbb{R}$.

Proof. It is clear that (i) implies (ii). If (ii) holds then there exist operators $T(t) \in \mathcal{L}(E)$ such that $u(t,f) = T(t)f$ $(t \in \mathbb{R}$, $f \in D(A))$. It follows from the uniqueness of the solutions that $T(t+s) = T(t)T(s)$ $(s,t \in \mathbb{R})$. Let $f \in E$. There exist $f_n \in D(A)$ such that $\lim_{n\to\infty} f_n = f$.
Then $\lim_{n\to\infty} T(t)f_n = T(t)f$ for all $t \in \mathbb{R}$. Since $T(\cdot)f$ is continuous, it follows that $T(\cdot)f$ is measureable. Hence by [Hille-Phillips (1975),10.2.1] $\sup_{t\in J}\|T(t)\| < \infty$ for every compact interval $J \subset (0,\infty)$. Because of the group property this implies that $T(\cdot)$ is norm bounded on bounded subsets of $\mathbb{R}$. $T(\cdot)f$ is continuous if $f \in D(A)$. Since $D(A)$ is dense this implies the strong continuity of $(T(t))_{t\in\mathbb{R}}$.

□