Proof. Since every closed ideal is of the form $\{f \in E : f_{|M} = 0\}$ where $M \subset X$ is a closed subset (cf. Sec.1 of B-I) it is clear that all closed ideals are invariant under the multiplication operator M_h and M_{-h} respectively. Thus the assertion follows from the expansions which are true for λ sufficiently large.

$$R(\lambda,B) = (1 - R(\lambda,A)M_h)^{-1}R(\lambda,A) = \sum_{n=0}^{\infty} (R(\lambda,A)M_h)^n R(\lambda,A)$$
$$R(\lambda,A) = (1 - R(\lambda,B)M_{-h})^{-1}R(\lambda,B) = \sum_{n=0}^{\infty} (R(\lambda,B)M_{-h})^n R(\lambda,B)$$

□

Before discussing further properties of irreducible semigroups we consider several examples.

Examples 3.4.(a) (cf. B-II,Sec.3). Suppose $(T(t))_{t\geq 0}$ is governed by a continuous semiflow $\phi : \mathbb{R}_+ \times X \to X$, i.e., $T(t)f = f\circ\phi_t$ $(f\in C_o(X))$. Then the following assertions are equivalent:

(i) $(T(t))_{t\geq 0}$ is irreducible.

(ii) There is no closed subset of X which is ϕ-invariant except $\emptyset$ and X.

(iii) Every orbit $\{\phi(t,x) : t \in \mathbb{R}_+\}$ is dense in X.

More generally, these equivalences still hold if the semigroup $(T(t))$ is given by $T(t)f = h_t\cdot(f\circ\phi_t)$ where h_t are suitable continuous, strictly positive, bounded functions on X.

(b) Suppose that the semigroup $(T(t))_{t\geq 0}$ has the following form: There exist a positive measure μ on X and a positive continuous function $k : (0,\infty) \times X \times X \to \mathbb{R}$ such that

(3.1) $$(T(t)f)(x) = \int_X k(t,x,y)f(y)\, d\mu(y) \quad (t > 0,\ f \in C_o(X),\ x \in X).$$

Then $(T(t))_{t\geq 0}$ is irreducible if and only if $\cup_{t>0}\mathrm{supp}\{k(t,x,.)\}$ is dense in X for every $x \in X$.

(c) We consider the first derivative $Af = f'$ (cf. A-I,2.4). If $E = C_o(\mathbb{R})$, then the corresponding semigroup $(T(t))_{t\geq 0}$ is not irreducible. Note however, that there is no closed invariant ideal I with $\{0\} \subsetneqq I \subsetneqq E$ which is invariant under the group $(T(t))_{t\in\mathbb{R}}$ generated by A.

For $E = C_o[0,\infty)$ and $E = C_o(-\infty,0)$ the corresponding semigroups are reducible (i.e. not irreducible) as well. If $E = C_{2\pi}(\mathbb{R})$ (i.e. the 2π-periodic functions), then $Af = f'$ generates an irreducible semigroup on E. It is (isomorphic to) the semigroup of rotations on the unit circle.