Define $g_n \in D(A)$ by

$$g_n := \|R(\lambda_n,A)f\|^{-1} R(\lambda_n,A)f$$

and use the identity

$$(\lambda_o - A)g_n = (\lambda_o - \lambda_n)g_n + (\lambda_n - A)g_n$$

to show that $(g_n)_{n\in\mathbb{N}}$ is an approximate eigenvector corresponding to λ_o.

(ii) This is a simple consequence of the Hahn-Banach theorem.

□

In order to illuminate the above definitons we now return to the Standard Examples introduced in Section 2 of A-I and discuss the fine structure of the spectrum of these strongly continuous semigroups, i.e. of their generators and their semigroup operators.

2.3 The Spectrum of Multiplication Semigroups.

Take $E = C_o(X)$ for some locally compact space X and take a continuous function $q : X \to \mathbb{C}$ whose real part is bounded above. As observed in A-I,2.3 the multiplication operator

$$M_q : f \to q\cdot f$$

with maximal domain $D(M_q)$ generates the multiplication semigroup

$$T(t)f := e^{tq}\cdot f \quad , \ f \in E .$$

Since M_q is bounded if and only if q is bounded we conclude that M_q is invertible (with bounded inverse $M_{1/q}$) if and only if

$$0 \notin \overline{\{q(x) : x \in X\}} .$$

Therefore we obtain

$$\sigma(M_q) = \overline{q(X)} = \overline{\{q(x) : x \in X\}}$$

and

$$\sigma(T(t)) = \overline{\{\exp(tq(x)) : x \in X\}} .$$

In particular the following 'weak spectral mapping theorem' is valid:

$$\sigma(T(t)) = \overline{\exp(t\sigma(M_q))} .$$

In addition we observe that to each spectral value of A (resp. of $T(t)$) there exists an approximate eigenvector and hence

$$\sigma(A) = A\sigma(A) \qquad \text{and} \qquad \sigma(T(t)) = A\sigma(T(t)).$$

Since each Dirac functional is an eigenvector for the adjoint multiplication operator we obtain

$$q(X) \subset R\sigma(M_q) \quad \text{and} \quad e^{tq(X)} \subset R\sigma(T(t)) .$$