Theorem 7.4. Let $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ be a strongly continuous group on some Banach space E such that $\|T(t)\| \leq p(t)$ for some polynomial p and all $t \in \mathbb{R}$. Then the Weak Spectral Mapping Theorem holds, i.e.,

$$\sigma(T(t)) = \overline{\exp(t\cdot\sigma(A))} \quad \text{for all } t \in \mathbb{R} . \tag{7.2}$$

From the proof we isolate a series of lemmas for which we always assume the hypothesis made in Thm.7.4. Moreover we recall from Fourier analysis that the Fourier transformation $\phi \to \vec{\phi}$, $\vec{\phi}(\alpha) := \int_{-\infty}^{\infty} \phi(x)e^{-i\alpha x}\,dx$, and its inverse $\Psi \to \overleftarrow{\Psi}$, $\overleftarrow{\Psi}(x) := 1/2\pi\cdot\int_{-\infty}^{\infty} \Psi(\alpha)\cdot e^{i\alpha x}\,d\alpha$ are topological isomorphisms of the Schwartz space $\mathcal{S}$ ($= \mathcal{S}(\mathbb{R})$) . Since the subspace $\mathcal{D}$ of all functions having compact support is dense in $\mathcal{S}$ it follows that $\{\phi \in \mathcal{S} : \overleftarrow{\phi} \in \mathcal{D}\}$ is dense in $\mathcal{S}$.

Lemma 7.5. For every function $\phi \in \mathcal{S}$ we obtain an operator $T(\phi) \in \mathcal{L}(E)$ by

$$T(\phi)f := \int_{-\infty}^{\infty} \phi(s)T(s)f\,ds \quad , \quad f \in E .$$

This operator can be represented as

(7.3)

$$T(\phi)f = \lim_{\varepsilon\to 0} 1/2\pi\cdot\int_{-\infty}^{\infty} \overleftarrow{\phi}(\alpha)[R(\varepsilon-i\alpha,A)f - R(-\varepsilon-i\alpha,A)f]\,d\alpha \quad , \quad f \in E .$$

Proof. That $T(\phi)$ is well-defined follows from the polynomial boundedness of $(T(t))_{t\in\mathbb{R}}$. In fact, $\phi \to T(\phi)$ is continuous from $\mathcal{S}$ into $(\mathcal{L}(E), \|\cdot\|)$.

We obtain

$$\begin{aligned} T(\phi)f &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \phi(s)e^{-\varepsilon|s|}T(s)f\,ds \\ &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} 1/2\pi \int_{-\infty}^{\infty} \overleftarrow{\phi}(\alpha)e^{i\alpha s}e^{-\varepsilon|s|}T(s)f\,d\alpha\,ds \\ &= \lim_{\varepsilon\to 0} 1/2\pi \int_{-\infty}^{\infty} \overleftarrow{\phi}(\alpha) \int_{-\infty}^{\infty} e^{i\alpha s}e^{-\varepsilon|s|}\,T(s)f\,ds\,d\alpha \\ &= \lim_{\varepsilon\to 0} 1/2\pi \int_{-\infty}^{\infty} \overleftarrow{\phi}(\alpha)[R(\varepsilon-i\alpha,A)f - R(-\varepsilon-i\alpha,A)f]\,d\alpha . \end{aligned}$$

Here we used that polynomially bounded semigroups have growth bound 0 , hence $\omega(A) = \omega(-A) = 0$. Hence the integral representation of $R(\varepsilon-i\alpha,A)$ (cf. A-I,Prop.1.11) exists for $\varepsilon \neq 0$.

□

Lemma 7.6. If $E \neq \{0\}$, then $\sigma(A) \neq \emptyset$.

Proof. If $\sigma(A) = \emptyset$ then (7.3) implies $T(\phi) = 0$ whenever $\overleftarrow{\phi}$ has compact support. Since these functions form a dense subspace of $\mathcal{S}$ we conclude that $T(\phi) = 0$ for all $\phi \in \mathcal{S}$.