tions ξ_f . The assumption on ϕ implies the set of all derivatives

$$\{\xi_f' : f \in D(A) \ , \ \|f\| \leq 1\}$$

to be equicontinuous at $t = 0$. This means that for every $\varepsilon > 0$ there exists $0 < t_o < 1$ such that $|\xi_f'(0) - \xi_f'(s)| < \varepsilon$ for every $f \in D(A)$, $\|f\| \leq 1$ and $0 < s < t_o$.

In particular,

$$\varepsilon > |\xi_f'(0) - \tfrac{1}{s}(\xi_f(s)-\xi_f(0))| = |\langle f, A'\phi - \tfrac{1}{s}(T(s)'\phi-\phi)\rangle| \ ,$$

hence

$$\varepsilon > \|A'\phi - \tfrac{1}{s}(T(s)'\phi-\phi)\|$$

for all $0 \leq s \leq t_o$. From this it follows that $\phi \in D(A^*)$.

□

On reflexive Banach spaces we have $A^* = A'$ by the above proposition. In other cases this construction is more interesting.

Example (continued). The adjoints of the (left) translation $T(t)$ on $E = L^1(\mathbb{R})$ are the (right) translations $T(t)'$ on $E' = L^\infty(\mathbb{R})$. The largest subspace of $L^\infty(\mathbb{R})$ on which these translations form a semigroup which is strongly continuous with respect to the sup-norm, is the space of all bounded uniformly continuous functions on $\mathbb{R}$, i.e. $E^* = C_{bu}(\mathbb{R})$.

Calculating $D(A')$ and $D(A^*)$ respectively, one obtains

$$D(A') = \{f \in L^\infty(\mathbb{R}) : f \in AC \ , \ f' \in L^\infty(\mathbb{R})\} \ ,$$

$$D(A^*) = \{f \in L^\infty(\mathbb{R}) : f \in C^1(\mathbb{R}) \ , \ f' \in C_{bu}(\mathbb{R})\} \ .$$

Obviously, the function $x \to |\sin x|$ belongs to $D(A')$ but not to $D(A^*)$.

3.5. The Associated Sobolev Semigroups

Since the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ is closed, its domain $D(A)$ becomes a Banach space for the graph norm

$$\|f\|_1 := \|f\| + \|Af\|.$$

We denote this Banach space by E_1 and the continuous injection from E_1 into E by i_1 . Since E_1 is invariant under $(T(t))_{t\geq 0}$ - apply Prop.1.6.i - it makes sense to consider the semigroup $(T_1(t))_{t\geq 0}$ of all restrictions $T_1(t) := T(t)_{|E_1}$. The results of Prop.1.6 imply that $T_1(t) \in L(E_1)$ and $\|T_1(t)f-f\|_1 \to 0$ as $t \to 0$ for every $f \in E_1$. Thus $(T_1(t))_{t\geq 0}$ is a strongly continuous semigroup on E_1 and has a generator denoted by $(A_1, D(A_1))$. Using Prop.1.6 again we see that A_1 is the restriction of A to E_1 with maximal domain, i.e.