Example 1.27. The nilpotent shift semigroup (A-I,2.6) is obviously eventually differentiable, eventually compact and eventually norm continuous. But it is not norm continuous and consequently not differentiable or compact.

Example 1.28. We consider multiplication semigroups (see A-I,2.3). Let $E = C_o(X)$, where X is a locally compact space, or $E = L^p(X,\Sigma,\mu)$, where $1 \leq p < \infty$ and (X,Σ,μ) is a σ-finite measure space. Let $m : X \to \mathbb{R}$ be continuous [resp.,measurable] such that $[\text{ess}]\text{-}\sup_{x\in X} \operatorname{Re}(m(x)) < \infty$.

Then $Af = m\cdot f$ with domain $D(A) = \{f \in E : m\cdot f \in E\}$ is the generator of the semigroup $(T(t))_{t\geq 0}$ given by $(T(t)f)(x) = e^{tm(x)}f(x)$ $(t\geq 0,\ x\in X,\ f\in E)$. Observe that $\sigma(A) = \overline{m(X)}$ in case $E = C_o(X)$ and $\sigma(A) = [\text{ess}]\text{-image}(m) := \{\lambda \in \mathbb{C} : \mu((\{x \in X : |m(x)-\lambda| < \varepsilon\}) \neq 0$ for all $\varepsilon > 0\}$ if $E = L^p$ (see A-II,2.3). Consequently, $s(A) = \omega(A) = [\text{ess}]\text{-}\sup_{x\in X} \operatorname{Re}(m(x))$.

a) The semigroup is norm continuous for $t > 0$ if and only if it is eventually norm continuous if and only if $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geq b\}$ is bounded for every $b \in \mathbb{R}$. Thus the property proved in Theorem 1.20 characterizes generators of eventually norm continuous semigroups in the case that the semigroup consists of multiplication operators.

Proof. Assume that $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geq b\}$ is bounded for every $b \in \mathbb{R}$. Let $t' > 0$. We show that the semigroup is norm continuous in t'. Let $\varepsilon > 0$. Let $b \in \mathbb{R}$ such that $2e^{(t'+1)b} < \varepsilon$.

If $\operatorname{Re}(m(x)) \leq b$, then

$$|e^{tm(x)} - e^{t'm(x)}| \leq e^{t\cdot\operatorname{Re}(m(x))} + e^{t'\cdot\operatorname{Re}(m(x))} \leq 2e^{(t'+1)b} < \varepsilon$$

whenever $|t-t'| \leq 1$.

By hypothesis, the set $H := \{m(x) : x \in X,\ \operatorname{Re}(m(x)) \geq b\}$ in the case $E = C_o(X)$ and $H := \{m(x) : \operatorname{Re}\lambda \geq b$ and for all $\eta>0$ $\mu(\{x \in X : |m(x)-\lambda| < \eta\}) \neq 0\}$ in the case $E = L^p(X,\Sigma,\mu)$ is a bounded subset of $\mathbb{C}$.

Thus $\lim_{t\to t'} |e^{tz} - e^{t'z}| = 0$ uniformly for $z \in H$. Hence there exists $\delta \in (0,1]$ such that $[\text{ess}]\text{-}\sup\,\{|e^{tm(x)} - e^{t'm(x)}| : x \in X,\ \operatorname{Re}(m(x)) > b\} < \varepsilon$ whenever $|t-t'| < \delta$. Together with the inequality above, we obtain that $\|T(t) - T(t')\| = [\text{ess}]\text{-}\sup\,\{|e^{tm(x)} - e^{t'm(x)}| : x \in X\} < \varepsilon$ whenever $|t-t'| < \delta$. We have shown that the semigroup is norm continuous for $t > 0$ whenever $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geq b\}$ is bounded for all $b \in \mathbb{R}$.

□