Lemma 7.1. The spectrum of the matrix valued multiplication operator M_p where $p : X \to M(n)$ is bounded is given by $\sigma(M_p) = \overline{\bigcup_{x\in X} \sigma(p(x))}$.

Proof. It remains to show that $0 \notin \overline{\bigcup_{x\in X} \sigma(p(x))}$ implies $0 \notin \sigma(M_p)$. Since $\det p(x)$ is the product of n eigenvalues (according to their multiplicity) of $p(x)$ the hypothesis implies that $d := \inf\{|\det p(x)| : x \in X\} > 0$. By Formula 4.12 in Chapter I of Kato (1966) we obtain

$$\|p(x)^{-1}\| \leq \gamma\cdot\|p(x)\|^{n-1}\cdot|\det p(x)|^{-1} \leq \gamma/\alpha\cdot\|M_p\|^{n-1}$$

for every $x \in X$ and a constant γ depending only on the norm chosen on $\mathbb{C}^n$. Therefore $x \to p(x)^{-1}$ defines a bounded continuous function on X which obviously yields the inverse of M_p , i.e., $0 \notin \sigma(M_p)$.

□

Theorem 7.2. Let $A = M_q$ be a matrix multiplication operator on $C_o(X,\mathbb{C}^n)$ generating a strongly continuous semigroup $(T(t))_{t\geq 0} = (e^{tq(\cdot)})_{t\geq 0}$. Then the Weak Spectral Mapping Theorem of the form

$$\sigma(T(t)) = \overline{\exp(t\cdot\sigma(A))} \tag{7.2}$$

is valid.

Proof. By the Spectral Inclusion Theorem 6.2 we always have $\exp(t\sigma(A)) \subset \sigma(T(t))$. Since $T(t)$ is a matrix multiplication operator with a bounded function we obtain from Lemma 7.1

$$\sigma(T(t)) = \overline{\bigcup_{x\in X} \sigma(\exp(tq(x)))} = \overline{\bigcup_{x\in X} \exp(t\sigma(q(x)))} \subset \overline{\exp(t\sigma(A))} ,$$

which proves the assertion.

□

Corollary 7.3. The growth bound $\omega(A)$ and the spectral bound $s(A)$ coincide for matrix multiplication semigroups.

Remark. The above results remain valid for other Banach spaces of $\mathbb{C}^n$-valued functions such as $L^p(X,\mathbb{C}^n)$, $1 \leq p < \infty$.

The example given at the beginning of this section can be generalized in a different way. In fact, $A(x_n) := (inx_n)$ on $E = c_o$ generates a bounded group, and we will show that this property too ensures that the Weak Spectral Mapping Theorem (7.2) holds. Without any boundedness assumption on $(T(t))_{t\in\mathbb{R}}$ this result cannot be true (see [Hille-Phillips (1957),Sec.23.16] or [Wolff(1981)]).