

Chapter 1

Characterization of Positive Semigroups on W^* -Algebras

Since the positive cone of a C^* -algebra has non-empty interior, many results of Chapter B-II can be applied verbatim to the characterization of the generator of positive semigroups on C^* -algebras. On the other hand, a concrete and detailed representation of such generators has been found only in the uniformly continuous case (see [Lindblad \(1976\)](#)). A third area of active research has been the following: Which maps on C^* -algebras (in particular, which derivations) commuting with certain automorphism groups are automatically generators of strongly continuous positive semigroups. For more information we refer to the survey article of [Evans \(1984\)](#).

1 Semigroups on Properly Infinite W^* -Algebras

The aim of this section is to show that strongly continuous semigroups of Schwarz maps on properly infinite W^* -algebras are already uniformly continuous. In particular, our theorem is applicable to such semigroups on $B(H)$.

It is worthwhile to remark that the result of [Lotz \(1985\)](#) on the uniform continuity of every strongly continuous semigroup on L^∞ (see A-II, Sec.3) does not extend to arbitrary W^* -algebras.

Example 1.1. Take $M = \mathcal{B}(H)$, H infinite dimensional, and choose a projection $p \in M$ such that Mp is topologically isomorphic to H . Therefore $M = H \oplus M_0$, where $M_0 = \text{Ker}(x \mapsto xp)$. Next, take a strongly, but not uniformly continuous semigroup \mathcal{T} on H and consider the strongly continuous semigroup $\mathcal{T} \oplus \text{Id}$ on M .

For results on the classification theory of W^* -algebras needed in our approach we refer to [Sakai \(1971, 2.2\)](#) and [Takesaki \(1979, V.1\)](#).

Theorem 1.2. *Every strongly continuous one-parameter semigroup of Schwarz type on a properly infinite W^* -algebra M is uniformly continuous.*

Proof. Let $\mathcal{T} = (T(t)_{t \geq 0})$ be strongly continuous on M and suppose \mathcal{T} not to be uniformly continuous. Then there exists a sequence (T_n) in \mathcal{T} and $\varepsilon > 0$ such that $\|T_n - \text{Id}\| \geq \varepsilon$, but $T_n \rightarrow \text{Id}$ in the strong operator topology. We claim that for every sequence (p_k) of mutually orthogonal projections and all bounded sequences (x_k) in M

$$\lim_n \|(T_n - \text{Id})(p_k x_k p_k)\| = 0$$

uniformly in $k \in \mathbb{N}$. This follows from the *Lemma of Phillips* (Schaefer (1974)) and the fact that the sequence $(p_k x_k p_k)$ is summable in the $s^*(M, M_*)$ -topology (compare Elliot (1972), Lemma 2.).

Let (p_k) be a sequence of mutually orthogonal projections in M such that every p_k is equivalent to $\mathbb{1}$ via some $u_k \in M$ (Sakai, 1971, 2.2). Without loss of generality we may assume $\|(T_n - \text{Id})(u_n)\| \leq n^{-1}$ since the semigroup T is strongly continuous. Thus we obtained the following.

- (i) $\lim_n \|(T_n - \text{Id})(p_k x_k p_k)\| = 0$ uniformly in $k \in \mathbb{N}$ for every bounded sequence (x_k) in M .
- (ii) Every projection p_k is equivalent to 1 via some $u_k \in M$.
- (iii) $\|(T_n - \text{Id})u_n\| \leq n^{-1}$ for all $n \in \mathbb{N}$.

For the following construction see A-I,3.6 and D-II,Sec.2. Take

- (i) \widehat{M} be an ultrapower of M ,
- (ii) let $p := \widehat{(p_k)} \in \widehat{M}$,
- (iii) let $T := \widehat{(T_n)} \in L(\widehat{M})$
- (iv) and let $u := \widehat{(u_k)} \in \widehat{M}$.

Then T is identity preserving and of Schwarz type on \widehat{M} .

Since $u^*u = p$ and $uu^* = \mathbb{1}$, it follows $pu^* = u^*$ and $(uu^*)x(uu^*) = x$ for all $x \in \widehat{M}$. Finally, $T(ppp) = ppp$ for all $x \in \widehat{M}$ which follows from (i), and $T(u^*) = T(pu^*) = pu^* = u^*$ and $T(u) = u$, which follows from (iii). Using the Schwarz inequality we obtain

$$T(uu^*) = T(\mathbb{1}) \leq \mathbb{1} = uu^* = T(u)T(u)^*.$$

From D-III, Lemma 1.1., we conclude $T(ux) = uT(x)$ and $T(xu^*) = T(x)u^*$ for all $x \in \widehat{M}$. Hence

$$\begin{aligned} T(x) &= T(uu^*xuu^*) = uT(u^*xu)u^* = uT(pu^*xup)u^* \\ &= upu^*xupu^* = uu^*xuu^* = x \end{aligned}$$

for all $x \in \widehat{M}$. From this we obtain that for every bounded sequence (x_k) in M

$$\lim_k \|T_k x_k - x_k\| = 0$$

for some subsequence of the T_k 's and of the x_k 's. This conflicts with our assumption at the beginning, hence the theorem is proved. \square

Notes

Let M be a W^* -algebra and H be an infinite-dimensional Hilbert-space. Then the W^* -tensor product $N := M \overline{\otimes} \mathcal{B}(H)$ is a properly infinite W^* -algebra (Sakai (1971, Thm. 2.6.6)). Let \mathcal{S} be the semigroup

$$S(t) = T(t) \otimes \text{Id}_H \quad (t \geq 0).$$

Then $S(t)$ is a Schwarz-map on N and contractive (Takesaki (1979, Prop. IV.5.13.)), hence the smigroup \mathcal{S} is equicontinuous in $\mathcal{L}(N)$.

Let $x \in M$ and $\xi \in H$. Since the norm on N is a cross-norm, we obtain

$$\lim_{t \rightarrow 0} \|(S(t) - \text{Id})x \otimes \xi\| = \lim_{t \rightarrow 0} \|(S(t) - \text{Id})x\| \|\xi\| = 0.$$

From Schaefer (1966, III.4.5) it follows that \mathcal{S} is strongly-continuous, hence norm-continuous on N from which we conclude, that \mathcal{T} is norm-continuous on M .

Remark 1.3. If M is a finite W^* -algebra of Type I, then M is a Grothendieck space and has the Dunford-Pettis property. Hence we can apply the results of Lotz (1985). However, has W^* -algebra have the Dunford-Pettis property iff it is finite and of Type I (Chu and Iochum (1990)). But is known that every W^* -algebra is a Grothendieck space (Pfitzner (1994)).

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