<u>Proposition</u> 3.18. Suppose that A is the generator of a positive semigroup and that $K \in L(E)$ is a positive linear operator.
If K is A-compact (i.e., if $KR(\lambda_o,A)$ is compact for some $\lambda_o \in \rho(A)$) and if $s(A+K) > s(A)$ then $B := A + K$ satisfies the assumptions of Thm.3.14.
If, in addition, K is irreducible then $s(B)$ is a dominant eigenvalue and the semigroup generated by B is irreducible.

<u>Proof</u>. The resolvent equation $R(\lambda,A) = R(\lambda_o,A)\big(1 - (\lambda-\lambda_o)R(\lambda,A)\big)$ implies that $KR(\lambda,A)$ is a compact operator for every $\lambda \in \rho(A)$. For $\lambda > s(A)$ we have $\lambda - B = \big(1 - KR(\lambda,A)\big)(\lambda-A)$ and $(1 - KR(\lambda,A))^{-1}$ exists for $\lambda > s(B)$. Therefore Thm.XIII.13 of Reed-Simon (1979) implies that $R(\lambda,B) = R(\lambda,A)\big(1 - KR(\lambda,A)\big)^{-1}$ has only poles of finite algebraic multiplicity in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > s(A)\}$. This proves the first claim. In order to prove the second, we denote the semigroup corresponding to A and B by $(T(t))$ and $(S(t))$ respectively.
It follows from Prop.3.3 that $(S(t))$ is irreducible and we have $S(t) = T(t) + \int_0^t T(t-s)KS(s)\,ds$ (see A-II,(1.9)) . Iterating this identity we obtain for every $m \in \mathbb{N}$, $t \geq 0$:

$$(3.21)\quad S(t) = \textstyle\sum_{n=0}^{m-1} T_n(t) + R_m(t) \quad \text{where}$$

$$T_0(t) := T(t) \quad , \quad T_n(t) := \int_0^t T(t-s)KT_{n-1}(s)\,ds \quad (n \in \mathbb{N}) ,$$

$$R_m(t) := \int_0^t\int_0^{t_1} .. \int_0^{t_{m-1}} T(t-t_1)KT(t_1-t_2)K \,..\, KS(t_{m-1}-t_m)\,dt_m..dt_1$$

We fix $0 < f \in E$, $0 < \phi \in E'$, $t > 0$. By Thm.3.2(a), $S(t)f > 0$.
Since K is irreducible there exists $m \in \mathbb{N}$ such that $\langle K^m S(t)f,\phi\rangle > 0$. Thus the integrand appearing in the the representation (3.21) of $\langle R_m(t)f,\phi\rangle$ is non-zero at $t_1=t_2= .. =t_{m-1}=t$, $t_m= t$.
Since the integrand is positive and continuous we conclude

$$(3.22)\quad \langle S(t)f,\phi\rangle \geq \langle R_m(t)f,\phi\rangle > 0 \quad \text{for } 0 < f ,\ 0 < \phi ,\ t > 0$$

It follows that $(e^{-ts(B)}S(t))_{t\geq 0}$ cannot contain the rotation semigroup on Γ . On the other hand, assuming that $s(B)$ is not dominant, then $\dim\big(\ker(\exp(\tau\cdot s(B)) - S(\tau))\big) > 1$ for some $\tau > 0$. Hence the restriction $(e^{-ts(B)}S(t)_{|F})_{t\geq 0}$ where $F := \ker(\exp(\tau\cdot s(B)) - S(\tau))$, contains the rotation semigroup by Cor.3.9.

□

We conclude this section considering once again Example 3.4(d).
The generator considered there is $B = (A - M) + K$, where K is