and denote by δ its generator.
Then $D_o = \{ f \in C_o(\mathbb{R}^n) \cap C^1(\mathbb{R}^n) : \lim_{\|x\|\to\infty} \|(\text{grad } f)(x)\| = 0 \}$ is a core of δ and

$$(3.21) \quad (\delta f)(x) = ((\text{grad } f)(x) \,|\, F(x)) \quad \text{for all } f \in D_o , \ x \in \mathbb{R}^n ,$$

where $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^n$

Proof. Let $f \in D_o$. Then $g = f - (\text{grad } f \mid F) \in C_o(\mathbb{R}^n)$ and $(R(1,\delta)g)(x) = \int_0^\infty e^{-t} f(\phi(x,t))dt - \int_0^\infty e^{-t}((\text{grad}f)(\phi(t,x))\,|\,F(\phi(t,x)))dt$ $= f(x)$ by integrating by parts. Hence $f \in D(\delta)$ and $f - \delta f = g$; i.e. $\delta f = (\text{grad } f|F)$. This proves (3.21).
Next we show $T_o(t)D_o \subset D_o$ for all $t \geq 0$, which implies that D_o is a core of δ by A-I,Thm.1.9 (or A-II,Cor.1.34).
Since $F \in C^1(\mathbb{R}^n,\mathbb{R}^n)$, it follows that $\phi \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ (see e.g., [Hirsch-Smale (1974), 15.2]). Moreover for each $x \in \mathbb{R}^n$, $\frac{d}{dt}(D\phi_t(x))$ $= DF(\phi_t(x))\cdot D\phi_t(x)$ and $D\phi_0(x) = \text{Id}$, (see [Hirsch-Smale (1974), p. 300] ; here $\text{Id} \in L(\mathbb{R}^n)$ denotes the identity operator. Hence $D\phi_t(x) = \text{Id} + \int_0^t DF(\phi_s(x))\cdot D\phi_s(x)ds$. Consequently $\|D\phi_t(x)\| \leq 1 + \int_0^t M\cdot\|D\phi_s(x)\|ds$ for all $t \geq 0$ and $x \in \mathbb{R}^n$; where $M := \sup_{x\in\mathbb{R}^n} \|DF(x)\| < \infty$ by hypothesis. Hence by Gronwall's inequality, $\|D\phi_t(x)\| \leq e^{Mt}$ $(t \geq 0)$ for all $x \in \mathbb{R}^n$. Now let $f \in D_o$, $t \geq 0$. Then $[\text{grad } (f\circ\phi_t)](x) = [(\text{grad } f)(\phi_t(x))]\cdot D\phi_t(x)$. Hence $\|[\text{grad } (f\circ\phi_t)](x)\| \leq e^{Mt}\|(\text{grad } f)(\phi_t(x))\|$, and so $\lim_{\|x\|\to\infty} \|[\text{grad } (f\circ\phi_t)](x)\| \leq e^{Mt} \lim_{\|x\|\to\infty} \|(\text{grad } f)(\phi_t(x))\| = 0$. Thus $f\circ\phi_t \in D_o$ for all $t \geq 0$.

□

As a second class of examples we consider derivations on $C_o(a,b)$. Eventually we will determine all derivations on $C_o(a,b)$, which are generators of a group. We start by looking at differential operators of first order. Let $-\infty \leq a < b \leq \infty$ and let $m : (a,b) \to \mathbb{R}$ be a continuous function. We consider the operator δ_m on $C_o(a,b)$ given by

$$(\delta_m f)(x) = \begin{cases} m(x)f'(x) & \text{if } m(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

with domain $D(\delta_m) = \{ f \in C_o(a,b) : f$ is differentiable in x if $m(x) \neq 0$ and $\delta_m f \in C_o(a,b) \}$.
Note that δ_m is a derivation on $C_o(a,b)$.