(g) $\rightarrow$ (a): Since the vector states are dense in the predual of M ([Takesaki (1979), Theorem II.2.6]) and since the preadjoint semigroup of $\mathcal{T}$ is strongly continuous, it is easy to see that the integral

$$\int_0^\infty \phi(T(s)x)\,ds$$

exists for all $x \in M$ and $\phi \in M_*$. Therefore, the resolvent $R(0,A)$ exists and is positive, hence $s(A) < 0$.

□

3. CONVERGENCE OF POSITIVE SEMIGROUPS

In this section the asymptotic behavior of positive semigroups $(T(t))_{t\geq 0}$ will be described in more detail. Essentially we distinguish three cases:

1. The Cesàro means $\frac{1}{s}\int_0^s T(t)\,dt$ converge strongly to a projection P onto the fixed space of $(T(t))_{t\geq 0}$ (see Proposition 3.3 and 3.4)

2. The maps $T(t)$ converge strongly to P (see Proposition 3.7, 3.8 and 3.9).

3. The maps $T(t)$ behave asymptotically as a periodic group (Theorem 3.11).

Much of the following is based on the theory of weakly compact operator semigroups. Therefore the following compactness criterium is quite useful.

Proposition 3.1. Let M be a W*-algebra, $\mathcal{T}$ a bounded semigroup of positive maps on M_* and suppose that there exists a faithful family Φ of $\mathcal{T}$-subinvariant states in M_*. Then $\mathcal{T}$ is relatively compact in the weak operator topology of $\mathcal{L}(M_*)$. In particular, $\mathcal{T}$ is strongly ergodic, i.e. $\lim_{s\to\infty} \frac{1}{s}\int_0^s T(t)x\,dt$ exists for every x in M and yields a projection onto $\mathrm{Fix}(\mathcal{T})$.

Proof. Since the positive cone of M_* is generating, it is enough to show that for every $0\leq\psi\in M_*$ the orbit $\{T(t)\psi : t\in\mathbb{R}_+\}$ is relatively weak compact. For this we use [Takesaki(1979),Theorem III.5.4.(iii)].