Contents

Chapter 1

Updated Notes Part A

Updated Notes A-I

Among recent books on C_0-semigroups on Banach spaces we mention:

- Arendt et al. [14] approaches semigroups via the Laplace transform and the resolvent of the generator.

- The first part of Engel and Nagel [53] contains the theory of C_0-semigroups via generation results, perturbation and approximation, spectral theory, and asymptotic behavior; hence, it is very analogous to the present lecture notes. The second half, under the headline "Semigroups Everywhere" and partly written by other authors, shows how different evolution equations can be treated using the theory of semigroups (see Engel and Nagel [53, Chap. VI]).

- Operator semigroups on Hilbert spaces can also be studied via the theory of forms. We refer to the monographs of Ouhabaz [103] and W. Arendt, H. Vogt, and J. Voigt: *Form Methods for Evolution Equations* (Birkhäuser, to appear).

- The role of one-parameter semigroups in the theory of dynamical systems is studied in great detail in the monograph Chicone and Latushkin [36].

- As textbooks suited for graduate courses, we recommend, e.g., Engel and Nagel [54], Applebaum [3], and Sinha and Srivastava [113].

- While all semigroups in these texts are assumed to be strongly continuous, in many situations semigroups appear—under various names—that are continuous only for some weaker topology. The concept of "bicontinuous semigroups", covering these different notions, is proposed in Kühnemund [83].

- The $\mathcal{F}$-product in Section 3.7 on page **??** and the corresponding extension of a C_0-semigroup is a special case of the so-called *ultraproduct construction* of Banach spaces (see, e.g., Heinrich [69] or Sims [112]). This technique is useful for spectral theory, converting the approximate point spectrum into point spectrum. Its application to the spectral theory of C_0-semigroups, as in A-III.6.6, was started with the aforementioned work of Derndinger [43] and extended in Krupa [81].

Updated Notes A-II

• General properties of dissipative operators and the Hille-Yosida Theorem are discussed in Chapter II.2 of Engel and Nagel [53]. Here we mention the following version of the Lumer-Phillips Theorem (A-II, Theorem 2.11). An operator A is invertible and generates a contractive semigroup on a Banach space if and only if it is dissipative and surjective. For the proof we refer to Arendt et al. [18], where also convergence results for semigroups are proved.

• For Lotz's Theorem (A-II, Theorem 3.5) to hold, the operator A need not be the generator of a semigroup. Indeed, if A is densely defined such that the resolvent $R(\lambda, A)$ exists and $\lambda R(\lambda, A)$ is uniformly bounded for $\lambda \geqslant \lambda_0$, and the underlying space is L^∞, then A is bounded. See Theorem 4.3.18 in Arendt et al. [14].

• An overview on Grothendieck spaces can be found in González and Kania [65]. For the Dunford-Pettis property, Diestel [45] remains a valuable resource, although many of the open problems posed therein have since been resolved. See also Castillo and González [34].

• The asymptotic behaviour of a semigroup as $t \to 0$ is related to various regularity properties of the semigroup (such as holomorphy, or having a bounded generator), see Chalendar et al. [35] for a survey.

Updated Notes A-III

• The validity or failure of the spectral mapping theorem from A-III, Sect. 6 and 7,

$$\sigma(T(t)) \setminus \{0\} = \mathrm{e}^{t \cdot \sigma(A)} \quad \text{for every } t \geqslant 0,$$

and the identity

$$s(A) = \omega_0$$

remain important and interesting topics. We refer to van Neerven [122, Section 2] or Engel and Nagel [53, Chapter IV] for a systematic and more recent study. In contrast to the usual continuity or growth assumptions, Latushkin and Montgomery-Smith [88] and Räbiger and Schnaubelt [108] proved that the spectral mapping theorem always holds for so-called evolution semigroups. See also Engel and Nagel [53, Chapter VI, Theorem 9.18].

• The monograph Haase [68] treats the spectral theory of semigroups in the view of functional calculus.

Updated Notes A-IV

Our leitmotif in this chapter has been "The spectrum of the generator A determines the asymptotic behavior of the semigroup $(T(t))$."

• This is pursued in the monographs Engel and Nagel [53, Chapter V], van Neerven [122, Sections 3 and 4] and Eisner [47, Chapter III]. A different approach with emphasis on the resolvent of the generator is taken in Arendt et al. [14, Chapter 5] while Emel'yanov [51] provides a "non-spectral" asymptotic analysis.

• That a "countable imaginary spectrum" of the generator can imply strong stability of the semigroup has been discovered by Arendt and Batty [4] and by Lyubich and Phóng [92], see also Arendt et al. [14, Theorem 5.5.5] or Engel and Nagel [53, Theorem 2.21].

Here we mention a basic result on stability as a special case of the ABLV-Theorem. Let A be the generator of a bounded semigroup $(T(t))_{t\geqslant 0}$ on a reflexive Banach space E such that the boundary spectrum is countable. Then the semigroup $(T(t))_{t\geqslant 0}$ is stable (i.e., converges strongly to 0 as $t \to \infty$) if and only if there is no point spectrum on the imaginary line. An analogous result is valid for power bounded operators. We refer to Arendt et al. [14, Theorem 5.5.5] or Engel and Nagel [54, Theorem 2.21]. While countability of the boundary spectrum is not necessary for stability, Nagel and Räbiger [99] and Huang and Räbiger [73] characterize countability in terms of "super stability" (i.e., stability of all the semigroups induced on an ultra power of the underlying Banach space).

• There are more general versions of the ABLV-Theorem whith implications on the asymptotic behavior of the semigroup, we refer to Engel and Nagel [53, Theorem 2.21] and Arendt et al. [14, Theorem 5.5.5]. They have applications to positive semigroups where cyclicity of the boundary spectrum can be used (see the updated notes to Section C-IV).

• The asymptotic behaviour of a semigroup as $t \to \infty$ with respect to the weak topology is studied in Eisner et al. [48] and in the monograph Eisner [47].

Chapter 2

Updated Notes Part B

Updated Notes B-I

For the abstract characterization of spaces of continuous functions as commutative C*-algebras, i.e., the Gelfand-Naimark theorem, see Takesaki [118, Chapter I-3]. For concepts such as ideals, their connections with closed sets, and the representation of lattice or algebra homomorphisms, we refer to Semadeni [111]. The various types of positive operators on these algebras are discussed in Eisner et al. [49, Chapter 4]. Semigroups on spaces of continuous functions generated by elliptic operators in non-divergence form are treated in the monograph Lunardi [90].

Updated Notes B-II

• Today many examples of positive semigroups generated by differential operators on spaces of continuous functions are known. For elliptic operators in divergence form with Dirichlet boundary conditions we refer to Arendt and Bénilan [7], for Robin boundary conditions to Arendt et al. [13] and Nittka [100]. In contrast to the situation in L^p, irreducibility on spaces of continuous functions is not so easy to prove, we refer to Arendt et al. [17] for a Banach lattice argument which works for elliptic operators in divergence form. Elliptic operators in non-divergence form generate an irreducible, positive, holomorphic semigroup on $C_0(\Omega)$ if Ω is connected and satisfies the uniform exterior cone condition, see Arendt and Schätzle [11].

• The Dirichlet-to-Neumann operator is an example of a non-local operator generating a positive semigroup on $C(\partial\Omega)$, whenever Ω is a bounded, open set with Lipschitz boundary $\partial\Omega$, see Arendt and ter Elst [12], where even general elliptic operators are considered. The semigroup is irreducible whenever Ω is connected. This is surprising since the boundary may not be connected (think of a ring). Thus, the notion of irreducibility reflects the non-local character of the Dirichlet-to-Neumann operator. So far it is unknown whether Lipschitz continuity of the boundary implies holomorphy of such a semigroup. However, for a slightly better boundary it does, see ter Elst and Ouhabaz [119].

It was discovered by Daners [40] that the Dirichlet-to-Neumann operator on $C(\partial\Omega)$ with respect to the Laplace operator perturbed by a potential has unexpected properties concerning positivity. In fact, there are cases where the semigroup is merely eventually positive but not positive. This lead to a systematic investigation of semigroups which are merely positive after some time (called *eventually positive semigroups*) and similar concepts, see Daners et al. [42], Glück [61], Daners and Glück [41], Glück [58] and Herzog and Kunstmann [72] for some recent results in this direction.

- There are also non-local versions of Dirichlet boundary conditions (see Kunze [87]) and of Robin and Wentzell boundary conditions (see Kunze et al. [85]) leading to positive semigroups.

- Positive contractive semigroups acting on spaces of continuous functions, called *Feller semigroups*, are of great importance for stochastic processes. As an example for the rich litterature we mention the monographs Taira [117], Van Casteren [121], Jacob [75], Jacob [76] and Jacob [77] .

Perturbation results for Feller semigroups are obtained in Kunze [86] and Kühn and Kunze [82], approximation of Feller semigroups is studied in Budde et al. [32].

- In B-II, Example 3.15 the solution flow of a nonlinear differential equation on $\mathbb{R}^n$ leads to a C_0-(semi-)group of positive operators on a Banach lattice of continuous functions. Its generator is a linear differential operator given by Formula (3.12). Such *Markov lattice semigroups*, see B-II, Definition 3.3, are now frequently called *Koopman semigroups*. This kind of linearization of nonlinear partial differential operators became popular, e.g, by the work of I. Mezic [97] in the context of numerical problems using the *dynamical mode decomposition*. A solid mathematical setting for such Koopman semigroups on $C_0(X)$, X not locally compact, needed for the solution flow of a partial differential equation, is proposed by Farkas and Kreidler [55] . An introduction to Koopman semigroups is in Chapter 16 of Bátkai et al. [28]. Further semigroups induced by semiflows are studied e.g. in Miana and Poblete [98].

- An interesting method of decomposing resolvents and Feller semigroups is presented in Gregosiewicz [66], where also applications to Brownian motion are given.

- Finally we mention a recent perturbation theory for generators of positive semigroups on AM- and AL-spaces presented in Barbieri and Engel [25].

Updated Notes B-III

- The question whether the boundary spectrum of a positive semigroup is additively cyclic is still open, even on the space $C(K)$. See also the updated notes to C-III for more details. Concerning the set of all eigenvalues in the boundary spectrum, i.e., the set $P\sigma_b(A)$, the situation is different. In B-III Proposition 2.7 a condition is given implying its cyclicity and B-II Example 2.13 shows that an additional condition is needed in general. There exists even a semigroup of Markov operators on $C(K)$ such that $P\sigma_b(A)$ is not cyclic, see Glück [57].

- The right notion for eventually positive semigroups corresponding to irreducibility is *persistent irreducibility*, as introduced in Arora and Glück [20]. The authors extend various results to this more general situation. For example, as in C-III, Proposition 3.5, persistent irreducibilty implies that the generator has non-empty spectrum if the underlying space is $C_0(X)$.

Updated Notes B-IV

Concerning the asymptotic behaviour of positive semigroups generated by elliptic operators we refer to the updated notes of B-II. In view of probabilistic interpretation, convergence of Feller semigroups is of interest. This is shown for example in Budde et al. [32]. The asymptotic behavior of Feller semigroups with non-local Dirichlet boundary conditions is studied in Arendt et al. [15], whereas non-local Robin boundary conditions are the subject of Arendt et al. [16]. In Kunze et al. [85] it is shown that elliptic operaotrs with non-local Robin-Wentzell boundary conditions generate a positive semigroup on spaces of continuous functions, whose asymptotic behaviour as $t \to 0$ is investigated.

On the space $C(K)$, or more generally on an ordered Banach space, whose positive cone has non empty interior, exponential stability can be characterized in the spirit of the Collatz-Krein formula for matrices, see Glück and Mironchenko [64].

A general reference to delay equations using semigroups as in Chapter B-IV, Section 3 is the monograph Diekmann et al. [44]. A large part of the book by Bátkai and Piazzera [26] is devoted to the asymptotic behavior of the solutions of delay equations, again by semigroup methods.

Chapter 3

Updated Notes Part C

Updated Notes C-I

- Our main source for the theory of Banach lattices and positive operators is Schaefer [110]. Other useful references are Aliprantis and Burkinshaw [1], Meyer-Nieberg [96], and Zaanen [127].

- A gentle introduction to semigroups of positive operators is Bátkai et al. [28], starting from finite dimensions and leading to many concrete applications.

- Motivated by concrete PDEs (see, e.g., Daners et al. [42]), "eventually positive" semigroups form another very active research area. We refer to the survey article by Glück [61].

Updated Notes C-II

- It is interesting that some properties of semigroups are preserved by domination. We mention the following result by Glück [62].

> *Let $(T(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$ be positive semigroups on a Banach lattice E such that $S(t) \leqslant T(t)$ for all $t \geqslant 0$. If the semigroup $(T(t))_{t\geqslant 0}$ is holomorphic, then so is the semigroup $(S(t))_{t\geqslant 0}$.*

The proof uses a result by Räbiger and Wolff [109] about the preservation of spectral and asymptotic behaviour of semigroups under domination.

Also mean ergodicity is preserved under domination if the underlying Banach lattice E has order continuous norm, see Arendt and Batty [5]. Specifically, this is valid for complex Banach lattices and even if the semigroup S is not necessarily positive (which is needed for the preceding result, though). Thus the weaker domination property $|S(t)f| \leqslant T(t)|f|$ for all $t \geqslant 0$, $f \in E$ suffices. However, on a space of type $C(K)$

mean ergodicity is not necessarily inherited from a dominating semigroup, see Section 3 in Arendt and Batty [5].

Another interesting result is proved by Räbiger [106]:

> *Let $(T(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$ be positive semigroups on a Banach lattice E with order continuous norm such that $S(t) \leqslant T(t)$ for all $t \geqslant 0$. If $T(t)f$ converges to Pf as $t \to 0$ for all $f \in E$ and P has finite rank, then also $S(t)f$ converges as $t \to 0$ for all $f \in E$.*

For further properties inherited by domination we refer to Räbiger [107] and the literature mentioned there.

- Kato's classical inequality is frequently used to prove uniqueness results. A generalisation of Kato's inequality has been proved by Brezis and Ponce [31]. The abstract Kato inequality (K) in C-II, Theorem 3.8 for generators of positive semigroups has interesting applications to semi-linear evolution equations, see Arendt and Daners [8].

- Form methods are important for generation of holomorphic semigroups on a Hilbert space. The Beurling-Deny criterion is a most efficient tool to characterise positivity of a semigroup on L^2 which is associated with a form. Ouhabaz [101] extended this criterion to describe invariance of arbitrary closed convex sets in the underlying Hilbert space. This allows him to characterize irreducibility of the associated semigroups in a very simple way. We refer to Ouhabaz' monograph Ouhabaz [103] for this and a comprehensive theory of forms. In particular, semigroups generated by elliptic operators under diverse boundary conditions on L^2 can be described efficiently by form methods.

- Domination can be proved most conveniently for semigroups associated with a form, see e.g., Manavi et al. [93], Ouhabaz [103]. More general criteria for domination, valid in ordered Banach spaces, are given by Herzog and Kunstmann [71] . The modulus semigroups has been determined in a series of concrete cases, see Vogt and Voigt [124], Stein and Voigt [114], Stein et al. [115]

- Kernel estimates for positive semigroups, and in particular Gaussian estimates, play an important role. They imply that a semigroup defined and holomorphic on L^2 extends to all L^p-spaces and is holomorphic on each of these spaces (and in particular on L^1), see Ouhabaz [102]. Even the spectrum of the generator is independent on p in this case, see Kunstmann [84]. In Daners [38] Gaussian estimates are proved for semigroups generated by elliptic opertors with measurable coefficients under several boundary conditions. A comprehensive account is given in Ouhabaz [103].

- It is most remarkable that a positive contractive semigroup on L^p for $1 < p < \infty$ enjoys *maximal regularity*, an important property much studied in the past two decades. This result is due to Weis [126]. We refer to Chapter 17 in the monograph Hytönen et al. [74] for a comprehensive treatment of maximal regularity.

- The Dirichlet-to-Neumann operator generates a holomorphic positive irreducible semigroup on $L^2(\partial\Omega)$ whenever Ω is a bounded, connected Lipschitz domain (see

Arendt and Mazzeo [10] and the Updated Notes of B-II). This can be proved by form methods. Kernel estimates are obtained in ter Elst and Ouhabaz [120]

• Important research has been done on so-called Ornstein-Uhlenbeck semigroups which are explicitly given by a Gaussian kernel. Such a semigroup acts on all L^p-spaces with respect to the Lebesgue measure and also with respect to the invariant measure μ when the drift matrix A is real with eigenvalues in the open left halfplane. The domain of its generator can be described explicitely, see Prüss et al. [105]. For regularity properties and the spectrum of Ornstein-Uhlenbeck operators we refer the reader to the survey article Lunardi et al. [91] and the monograph Lorenzi [89]. For quantitative and qualitative properties of more general Kolmogorov operators we refer to Lorenzi [89], Metafune et al. [95] and Metafune et al. [94].

• An elliptic operator with Robin boundary conditions (also called boundary conditions of the third kind) generates a positive semigroup for very general functions defining the Robin boundary, see Daners [39] .

Also non-local boundary conditions lead to positive semigroups, see e.g. Kunze et al. [85].

• The survey article Banasiak [23] shows which role positivity plays in models and also gives some new perturbation results (in Section 6). Further results can be found in Arora et al. [21].

• Semigroups of lattice homomorphisms from the Koopman point of view on L^p-spaces are the subject of Edeko et al. [46]; see also the extended notes of Chapter B-II concerning Koopman semigroups.

• In C-II, Proposition 5.16 it is shown that any strongly continuous group in the center of a real Banach lattice has a bounded generator. In this context it is interesting to mention the *Markov conjecture*: *Any generator of a strongly continuous positive semigroup on ℓ^1 which is norm-preserving on the positive cone and which extends to a group has a bounded generator.* This conjecture is still open, but a special case has been proved by Glück [59].

• Perturbation of positive semigroups is systematically studied in the monograph Banasiak and Arlotti [24].

Updated Notes C-III

• The question whether the generator of a positive semigroup on any Banach lattice has always additively cyclic boundary spectrum is still open. As in C-III (and B-III) additional assumptions, essentially on the growth of the resolvent, are needed. In the analogous case of a bounded positive operator they are relaxed in Glück [57].

• Concerning the additive cyclicity of the boundary point spectrum the situation is clearer. C-III, Corollary 4.3 establishes additive cyclicity under additional assumptions, while C-III, Example 4.4 shows that the boundary point spectrum may not be additively cyclic, in general. If a positive semigroup is irreducible and bounded and if $s(A) = 0$, then the boundary point spectrum $P\sigma_b(A)$ of its generator A is a subgroup of $\mathrm{i}\mathbb{R}$. This is a consequence of C-III, Theorem 3.8, see also Proposition 3.1 in Glück [60]. However, there exists a bounded, irreducible, positive semigroup on an L^1-space, which preserves the norm on the positive cone, such that the boundary spectrum $\sigma_b(A)$ of its generator A is not a subgroup of $\mathrm{i}\mathbb{R}$, see Theorem 3.2 in Glück [60]. This solves the problem formulated before B-III, Theorem 3.11.

• There is also the notion of the *ergodic spectrum* $E\sigma(A)$ of a bounded semigroup $(T(t))_{t\geqslant 0}$ consisting of all points $s \in \mathbb{R}$ such that the semigroup $(\mathrm{e}^{-\mathrm{i}st}T(t))_{t\geqslant 0}$ is not mean ergodic. If the semigroup is positive and the underlying Banach lattice has order continuous norm, then $E\sigma(A)$ is additively cyclic, see Arendt and Batty [6]. This is no longer true on $C(K)$.

• Part of the results of Chapter C-III have been extended to bounded, uniformly eventually positive semigroups with $s(A) = 0$. By Theorem 4.7 in Arora [19] their generator has cyclic boundary spectrum. Moreover, assume that such a semigroup $(T(t))_{t\geqslant 0}$, defined on a Banach lattice E, is *persistently irreducible* (i.e., if J is a closed ideal such that $T(t)J \subset J$ for all $t \geqslant t_0$ for some $t_0 > 0$, then $J = 0$ or $J = E$). Then the following holds. If $s(A) = 0$ is a pole of the resolvent, then $P\sigma(A) = i\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$, see Theorem 4.3 in Arora [19] . More information on persistently irreducible semigroups is given in Arora and Glück [20].

Updated Notes C-IV

• The problem formulated after C-IV, Theorem 1.1 has been solved by Weis [125]: The growth bound and spectral bound coincide for positive semigroups on all L^p-spaces, for $1 \leqslant p < \infty$. This proof is reproduced with more details in the monograph van Neerven [122], and a different proof is given in Arendt et al. [14, Theorem 5.3.6]. Recently a short and elegant proof of Weis' Theorem has been found by Vogt [123], which is even valid for eventually positive semigroups.

• A survey on the asymptotic behaviour of positive semigroups can be found in Arendt and Glück [9], where also the condition of a countable boundary spectrum is discussed.

• In C-IV, Corollary 2.12 non-spectral conditions imply strong convergence of a semigroup as $t \to \infty$. The essential property is that one operator $T(t_0)$ is a kernel operator. This surprising phenomenon has been systematically studied in Gerlach and Glück [56], where various generalizations and different arguments are given. The main hypothesis is that one of the semigroup operators $T(t_0)$ is AM-compact (which includes kernel operators and compact operators). These ideas are developed further in Glück and Haase [63].

• In C-IV, Theorem 2.14, conditions are given implying that a positive, irreducible, bounded semigroup converges strongly to a periodic group. Further results of alalogous asymptotic behaviour are given in Keicher and Nagel [78]. Also, certain flows on a network converge to a periodic flow as shown in Kramar and Sikolya [80]. Many of such results are based on the Jacobs-DeLeeuw-Glicksberg Theorem, see Engel and Nagel [53, Theorem V.2.8] for an introduction taylored for one-parameter semigroups, and see Eisner et al. [49] for a more complete presentation of this theorem. The strongest results of this sort are obtained if the semigroup has strongly compact orbits, see Engel and Nagel [53, Theorem V.2.14]. A different approach to such a decomposition in a group part and a part which converges to 0 is given in Arendt et al. [14, Chapter V], see also the notes to Section 5.4 in that book. Glück and Haase [63] introduce the notion *semigoup at infinity* which allows them not only to generalize the results by Gerlach and Glück [56] mentioned above, but also to give structure theorems for positive groups.

Chapter 4

Updated Notes Part D

Updated Notes D-I

An overview on positive operators on operator algebras can be found in Størmer [116], but there seems to be no systematic reference for such positive C_0-semigroups on operator algebras. However, many papers deal with Markov semigroups (see, e.g., Bratteli and Robinson [30]) or with so-called E-semigroups (see Arveson [22]).

Updated Notes D-II

As we have seen in Chapter A-II, Section 3, strongly continuous semigroups on commutative W^*-algebras, that is, on L^∞, are already norm-continuous. The proof depends heavily on the Grothendieck property and the Dunford-Pettis property of these Banach spaces. Therefore, it is natural to ask what happens in the noncommutative case.

The positive result is that every W^*-algebra has the Grothendieck property. This was shown by Pfitzner [104] and an alternative approach can be found in Chu et al. [37]. Surprisingly, if every strongly continuous C_0-semigroup on a C^*-algebra has a bounded generator, then it is a Grothendieck space. To prove this, one uses the fact that a C^*-algebra is a Grothendieck space if and only if c_0 is not a complemented subspace (see González and Kania [65, Prop. 3.1.13 and Prop. 4.2.1]). But on $\mathcal{B}(H)$, H an infinite-dimensional Hilbert space, there always exist strongly but not uniformly continuous C_0-semigroups (see the example in D-II-1.1).

In contrast to this, one can easily see that e.g., $\mathcal{B}(H)$ with H infinite-dimensional does not have the Dunford-Pettis property, using example D-II-1.1, since the Hilbert space H is a direct factor of $\mathcal{L}(H)$.

To understand what happens one needs the representation of finite type I W^*-algebras. By Takesaki [118, Thm. V.1.27] such a W^*-algebra has a representation $\oplus_j(\mathcal{L}(H_j)\overline{\otimes}M_j)$, where the M_j are commutative W^*-algebras.

In Chu et al. [37] it is shown that a W^*-algebra has the Dunford-Pettis property if and only if it is of finite type I and has a representation with $\sup_j \dim H_j < \infty$. On the other hand, in Bunce [33] it is shown that the predual of every finite type I W^*-algebra has the Dunford-Pettis property without further restrictions. Thus, as a consequence of A-II Theorem 3.5, in this case strongly continuous semigroups have a bounded generator and, again, that the type I W^*-algebra $\mathcal{B}(H)$ does not have the Dunford-Pettis property.

In contrast to all of the above, a strongly continuous C_0-semigroup of completely positive operators on a W^*-algebra is always norm-continuous and thus has a bounded generator. This follows from Elliott [50]: for sequences of completely positive maps, strong and norm convergence to the identity operator is equivalent on W^*-algebras (even on the larger class of AW^*-algebras).

Updated Notes D-III

- Using Batty and Robinson [29] or Greiner et al. [67], one can derive properties such as $s(A) \in \sigma(A)$ or $s(A) = \omega_0$ from the theory of semigroups of positive operators on ordered Banach spaces since the positive cone of a C^*-algebra has the required properties. Remaining in the category of C^*-algebras, these results are summarized in [xy].

- While W^*-algebras are not stable under the ultraproduct construction, see Heinrich [69, p. 79] or Henson and Moore [70], their preduals are and can be used for spectral theoretic purposes. More on ultraproducts of W^*-algebras can be found in Ando and Haagerup [2].

- Another approach to the spectral theory on W^*-algebras is in Bátkai et al. [27], where a Jacobs-de Leeuw-Glicksberg decomposition is constructed. This leads to a noncommutative version of the Perron-Frobenius theorem for W^*-algebras and is applied to the asymptotics of W^*-dynamical systems. A similar approach is in Kielanowicz and Luczak [79].

Updated Notes D-IV

As in D-III, results from Glück and Mironchenko [64] on semigroups on ordered Banach spaces with a normal cone and order unit can be applied. More precise asymptotic results on W^*-algebras and their preduals are in Emel'yanov and Wolff [52].

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