Theorem 3.14. An operator A on $C_o(X)$ is the generator of a positive group $(T(t))_{t\in\mathbb{R}}$ if and only if there exist a derivation δ on $C_o(X)$ which is the generator of a group, a function $h \in C^b(X)$ and $p \in C^b(X)$ satisfying $\inf_{x\in X} p(x) > 0$ such that

$$(3.18) \quad A = V\delta V^{-1} + h$$

where $V : C_o(X) \to C_o(X)$ is given by $Vf = p\cdot f$. In that case one has

$$(3.19) \quad (T(t)f)(x) = [p(x)/p(\phi_t(x))]\cdot(\exp \int_0^t h(\phi(s,x))ds)\cdot f(\phi_t(x))$$

for all $f \in C_o(X)$, $t \in \mathbb{R}$, $x \in X$.

Note: (3.18) means that $D(A) = \{ f : V^{-1}f \in D(\delta) \}$ and
$Af = V\delta V^{-1}f + hf$.

Proof. Assume that A is given by (3.18). Since V is a lattice isomorphism, it is clear that $V^{-1}\delta V$ generates a positive group; and consequently, A does so as well (cf. the proof of Theorem 3.5). Conversely, let $(T(t))_{t\in\mathbb{R}}$ be a positive group with generator A . By Prop. 3.9 and Lemma 3.12 we know that there exist a continuous flow ϕ , $0 \ll p \in C^b(X)$ and $h \in C^b(X)$ such that (3.19) holds. Let δ be the generator of the automorphism group defined by ϕ . We have to show that (3.18) holds. As in Theorem 3.5 one sees that $\delta + h$ generates the group $(S(t))_{t\in\mathbb{R}}$ given by
$(S(t)f)(x) = \exp(\int_0^t h(\phi(s,x))ds)\cdot f(\phi_t(x))$. Hence
$V\delta V^{-1} + h = V(\delta + h)V^{-1}$ generates $(VS(t)V^{-1})_{t\in\mathbb{R}} = (T(t))_{t\in\mathbb{R}}$. This is (3.18).

□

Since every generator of a positive group is the perturbation of a derivation, we now look for examples of derivations which generate a group.

Example 3.15. Let $X = \mathbb{R}^n$. Consider a function $F \in C^1(\mathbb{R}^n,\mathbb{R}^n)$ such that $\sup_{x\in\mathbb{R}^n} \|DF(x)\| < \infty$ where $DF(x) \in L(\mathbb{R}^n)$ denotes the derivative of F in x . Then there exists a continuous flow ϕ on $\mathbb{R}^n$ such that

$$(3.20) \quad \frac{\partial}{\partial t}\phi(t,x) = F(\phi(t,x)) \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^n .$$

Consider the automorphism group $(T_o(t))_{t\in\mathbb{R}}$ given by $T_o(t)f = f\circ\phi_t$