2. THE BOUNDARY SPECTRUM

In this section we restrict our attention to the boundary spectrum $\sigma_b(A)$ of a generator A, which, by definition, is the intersection of $\sigma(A)$ with the line $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = s(A)\}$. Thus $\sigma_b(A)$ contains all spectral values of A which have maximal real part. Note that in general the boundary spectrum is a proper subset of the topological boundary of $\sigma(A)$. Our aim is to prove results ensuring that $\sigma_b(A)$ is a cyclic set (see Def.2.5).

While most of the results of the preceding section were obtained by transforming the problem to a resolvent operator $R(\lambda,A)$ ($\lambda \in \mathbb{R}$ large enough), this procedure fails here. The reason is that there is no one-to-one correspondence between the boundary spectrum of A und the peripheral spectrum of $R(\lambda,A)$. Actually, from Thm.1.1 and A-III, Prop.2.5 it follows that the peripheral spectrum of $R(\lambda,A)$ (i.e., the set of spectral values having maximal absolute value) is trivial, since it only contains the spectral radius $r(R(\lambda,A)) = (\lambda - s(A))^{-1}$.

We begin our discussion with two lemmas.

Lemma 2.1. Suppose K, L are compact and $T : C(K) \to C(L)$ is a linear operator satisfying $T1_K = 1_L$.
Then we have $T \geq 0$ if and only if $\|T\| \leq 1$.

Proof. If T is positive, then

$$(2.1) \quad |Tf| \leq T|f| \leq T(\|f\|\cdot 1_K) = \|f\|\cdot T(1_K), \quad f \in C(K),$$

hence $\|T\| = \|T1_K\|$, whenever T is positive. This shows that $T \geq 0$ implies $\|T\| \leq 1$ whenever $T1_K = 1_L$.

To prove the reverse direction, we first observe that for complex numbers and hence for complex-valued functions the following equivalence holds:

$$(2.2) \quad \begin{array}{l} -1 \leq f \leq 1 \text{ if and only if} \\ \|f - i\cdot r\cdot 1\| \leq \rho_r := (1+r^2)^{1/2} \text{ for every } r \in \mathbb{R}. \end{array}$$

Now suppose $f \in C(K)$, $0 \leq f \leq 2\cdot 1_K$. Then we have $-1_K \leq f - 1_K \leq 1_K$ hence by (2.2) $\|f - 1_K - i\cdot r\cdot 1_K\| \leq \rho_r$ for every $r \in \mathbb{R}$. From $T1_K = 1_L$ and $\|T\| \leq 1$ it follows that $\|Tf - 1_L - i\cdot r\cdot 1_L\| \leq \rho_r$ for every $r \in \mathbb{R}$. Using (2.2) once again, we obtain $-1_L \leq Tf - 1_L \leq 1_L$ or $0 \leq Tf \leq 2\cdot 1_L$.

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