by M say. Then $\{z \in \mathbb{C} : \operatorname{Re} z = s(A)\} \subseteq \rho(A)$ and $\|R(z,A)\| \leq M$ for z with $\operatorname{Re} z = s(A)$. It follows that $\{z \in \mathbb{C} : |\operatorname{Re} z - s(A)| < M^{-1}\} \subseteq \rho(A)$, which is absurd by the definition of $s(A)$.

□

Corollary 1.5. Suppose that $s(A)$ is a pole of order m of the resolvent $R(\lambda,A)$. Then m is a majorant for the order of any other pole on the line $s(A) + i\mathbb{R}$.

Proof. Without loss of generality we may assume that $s(A) = 0$. By (1.5) we have $\|R(\varepsilon+i\beta,A)\| \leq \|R(\varepsilon,A)\|$ for every $\beta \in \mathbb{R}$, $\varepsilon > 0$. Therefore $\lim_{\varepsilon\to 0}\|\varepsilon^k R(\varepsilon+i\beta,A)\| \leq \lim_{\varepsilon\to 0}\|\varepsilon^k R(\varepsilon,A)\| = 0$ for $k > m$.

□

The spectrum of a positive semigroup may be empty (see B-III, Ex.1.2(a)) and the spectrum of a general group may be empty as well (see [Hille-Phillips (1957), Sec.23.16]). However, for positive groups this cannot occur. More precisely, we have the following result:

Corollary 1.6. If A is the generator of a positive group then $\sigma(A) \cap \mathbb{R} \neq \emptyset$.

Proof. Both A and $-A$ are generators of positive semigroups, hence if $\sigma(A) = \emptyset$, then $s(A) = s(-A) = -\infty$ and (1.5) implies that $\{R(\lambda,A) : \operatorname{Re}\lambda \geq 0\} \cup \{R(\lambda,-A) : \operatorname{Re}\lambda \geq 0\}$ is bounded in $L(E)$, i.e., $\{R(\lambda,A) : \lambda \in \mathbb{C}\}$ is bounded. By Liouville's Theorem the function $\lambda \to R(\lambda,A)$ is constant, hence identically zero because $\lim_{\lambda\to\infty} R(\lambda,A) = 0$. Thus we arrive at a contradiction.

□

We conclude this section by indicating possible extensions and further consequences of the results stated above.

Remarks 1.7.(a) Many of the results of this section remain true for positive semigroups on ordered Banach spaces more general than Banach lattices. The interested reader is referred to Greiner-Voigt-Wolff (1981) .

(b) From Thm.1.2 one can easily deduce that for positive semigroups on L^1-spaces, spectral bound and growth bound coincide. To prove the analoguous result for L^2-spaces one makes use of Cor.1.3. For details we refer to C-IV,Thm.1.1.