group on I (see A-I,3.2). Now let I be a closed ideal of E.
(ii)$\to$(i). If I is $\mathcal{T}$-invariant then $A_{|I}$ generates a semigroup on I. The restrictions $K_{|I}$ and $M_{|I}$ of K and M respectively are bounded linear operators on I. (Note that each closed ideal is invariant for M, cf. C-I,Sec.8.) Thus $B_{|I} = A_{|I} + M_{|I} + K_{|I}$ with domain $D(A_{|I}) = D(A) \cap I = D(B) \cap I$ is the generator of a semigroup on I. It follows that I is invariant for the semigroup generated by B.
(i)$\to$(ii). Without loss of generality we assume $M \geq 0$. Then we have $0 \leq T(t) \leq S(t)$ for all $t \geq 0$. It follows that I is $\mathcal{T}$-invariant. Thus for $x \in D(A) \cap I = D(B) \cap I$ we have $Kx = Bx - Ax - Mx$. This shows that $K(D(B) \cap I) \subset I$. Since $B_{|I}$ is a generator $D(B) \cap I$ is dense in I. Then by continuity we have $KI \subset I$; i.e., I is K-invariant.

□

Next we consider some concrete examples.

Examples 3.4. (a) Suppose that on $E = L^p(\mu)$ $(1 \leq p < \infty)$ the semigroup $(T(t))$ is given by

$$(3.1) \quad (T(t)f)(x) = \int_X k(t,x,y)f(y)\, d\mu(y) \quad (x \in X,\ t > 0)$$

for some measurable function $k : \mathbb{R}_+ \times X \times X \to \mathbb{R}_+$.
Then $(T(t))$ is irreducible if and only if for any two measurable sets M and N with $0 < \mu(M) < \infty$, $0 < \mu(N) < \infty$, $\mu(M \cap N) = 0$ there exist $t_o > 0$ such that $\int_M \int_N k(t_o,x,y)\, d\mu(x)\,d\mu(y) > 0$

(b) Consider the first derivative on $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{R}_{2\pi} \cong \Gamma$ as operator on the corresponding L^p-space (with respect to the Lebesgue measure.) Then the statements made in B-III,Ex.3.4(c) are true. The same is true for B-III,Ex.3.5(e) and (f) (second order differential operator) when the corresponding L^p-spaces are considered.

(c) Let $E = L^1[-1,0]$ and for $g \in L^\infty$ consider the operator A_g given by

$$(3.2) \quad A_g f := f',\ D(A_g) := \{f \in W^1[-1,0] : f(0) = \int_{-1}^0 f(x)g(x)\, dx\}$$

If $g \geq 0$ then A_g generates a positive semigroup. In case there exist $\varepsilon > 0$ such that g vanishes a.e. on $[-1,-1+\varepsilon]$, then $I := \{f \in L^1 : f_{|[-1+\varepsilon,0]} = 0\}$ is a non-trivial closed ideal which is invariant under the semigroup. It is not difficult to see that the condition on g stated above is also necessary for $(T(t))$ to be reducible (i.e., not irreducible.)