Proposition 7.8. Let $(T(t))_{t\geq 0}$ be a semigroup on a Banach space E and denote its generator by A. Then the following conditions are equivalent:

(a) $1 \in \rho(T(2\pi))$,

(b) $i\mathbb{Z} \subset \rho(A)$ and the series $\sum_{k\in\mathbb{Z}} R(ik,A)f$ is Césaro-summable for every $f \in E$,

(c) $i\mathbb{Z} \subset \rho(A)$ and the series $\sum_{k\in\mathbb{Z}} R(ik,A)Q_k f$ is Césaro-summable for every $f \in E$.

Proof. (a) $\to$ (b): The Spectral Inclusion Theorem implies $i\mathbb{Z} \subset \rho(A)$. By (7.7) we have $R(ik,A) = 2\pi\cdot(1-T(2\pi))^{-1}Q_k$. Since $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable (towards $1/2(f + T(2\pi)f)$) (see (7.8)) it follows that $\sum_{k\in\mathbb{Z}} R(ik,A)f$ is Césaro-summable as well.

(b) <=> (c): If we use A-I,(3.1) and integrate by parts, we obtain

$$\begin{aligned} R(ik,A)Q_k f &= 1/2\pi\int_0^{2\pi} e^{-iks}\ T(s)R(ik,A)f\ ds \\ &= 1/2\pi\int_0^{2\pi} [R(ik,A)f - \int_0^s e^{-ikt}\ T(t)f\ dt]\ ds \\ &= R(ik,A)f - 1/2\pi\int_0^{2\pi} e^{-ikt}\ (2\pi-t)\ T(t)f\ dt\ . \end{aligned}$$

Fejer's theorem ensures that $\sum_{k\in\mathbb{Z}} (1/2\pi)\int_0^{2\pi} e^{-ikt}\ (2\pi-t)\ T(t)f\ dt$ is Césaro summable. Hence $\sum_{k\in\mathbb{Z}} R(ik,A)Q_k f$ is Césaro-summable if and only if $\sum_{k\in\mathbb{Z}} R(ik,A)f$ is.

(b) $\to$ (a): We have $Q_k = \frac{1}{2\pi}(1 - T(2\pi))R(ik,A)$. Furthermore we know by (7.7) and (7.8) that $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable towards $\frac{1}{2}(f + T(2\pi)f)$.

If we define $S : E \to E$ by $Sf := \frac{f}{2} + \frac{1}{2\pi}\cdot C_1\text{-}\sum R(ik,A)f$ then we have

$$\begin{aligned} (1 - T(2\pi))Sf &= \tfrac{1}{2}(f - T(2\pi)f) + \tfrac{1}{2\pi}\cdot C_1\text{-}\textstyle\sum (1 - T(2\pi))R(ik,A)f = \\ &= \tfrac{1}{2}(f - T(2\pi)f) + \tfrac{1}{2}(f + T(2\pi)f) = f\ . \end{aligned}$$

Since S commutes with $T(2\pi)$ it follows that S is the inverse of $(1 - T(2\pi))$ thus $1 \in \rho(T(2\pi))$.

□

Based on the equivalence of (a) and (b), one can state a characterization of the spectrum of $T(t)$ in terms of the generator and its resolvent only. However, in general it is difficult to verify the summability condition stated in (b).

In Hilbert spaces the boundedness of the resolvents will suffice (see Thm.7.10 below).

Lemma 7.9. Let $(T(t))_{t\geq 0}$ be a semigroup on some Hilbert space H and assume $i\mathbb{Z} \subset \rho(A)$ for the generator A . Then we have