Define $m : (a,b) \to \mathbb{R}$ by

$$m(x) = \begin{cases} 0 & \text{if } x \in K \\ 1/r_n'(x) & \text{if } x \in (a_n,b_n) \ , \ n \in J \end{cases}$$

Then m is continuous and admissible. The given flow coincides with the one constructed from m in Theorem 3.17. Thus $\delta = \delta_m$.

□

Remark. Let $m : (a,b) \to \mathbb{R}$ be continuous. Then m is admissible if and only if the initial value problem

$$\dot{y}(t) = m(y(t)) \quad (t \in \mathbb{R}) \quad ; \quad y(0) = x$$

has a unique solution $y \in C^1(\mathbb{R},(a,b))$ which depends continuously on the initial value x (i.e., if $x_n \to x$ in (a,b) then the solution $y_n \in C^1(\mathbb{R},(a,b))$ with initial value $y_n(0) = x_n$ satisfies $y_n(t) \to y(t)$ $(n \to \infty)$ for all $t \in \mathbb{R}$) . This is not difficult to see.

As we have seen above the operators δ_m , where m is an admissible function, do not exhaust all generators of automorphism groups. But one can obtain every such generator by a similarity transformation (see A-I,3.0) from some δ_m .

Theorem 3.24. Let $-\infty \le a < b \le \infty$. An operator δ on $C_o(a,b)$ is the generator of an automorphism group on $C_o(a,b)$ if and only if there exists an algebra isomorphism V from $C_o(a,b)$ onto $C_o(a,b)$ and an admissible function $m : (a,b) \to \mathbb{R}$ such that $\delta = V^{-1}\delta_m V$.

Proof. In order to prove the non-trivial implication let $(T(t))_{t\in\mathbb{R}}$ be an automorphism group on $C_o(a,b)$ with generator δ . Let ϕ be the continuous flow on (a,b) such that $T(t)f = f\circ\phi_t$ $(f \in C_o(a,b)$, $t \in \mathbb{R})$. Then ϕ is of the form given in Prop. 3.21. For every $n \in J$ choose a C^1-diffeomorphism q_n from (a_n,b_n) onto $(-\infty,\infty)$ satisfying $q_n'(x) > 0$ for all $x \in (a_n,b_n)$ in the case when r_n is increasing and $q_n'(x) < 0$ for all $x \in (a_n,b_n)$ in the case when r_n is decreasing. Then $\beta_n := r_n^{-1}\circ q_n$ is a homeomorphism from (a_n,b_n) onto itself satisfying $\lim_{x\downarrow a_n}\beta_n(x) = a_n$ and $\lim_{x\uparrow b_n}\beta_n(x) = b_n$.

Let $\beta : (a,b) \to (a,b)$ be defined by

$$\beta(x) = \begin{cases} x & \text{if } x \in K \\ \beta_n(x) & \text{if } x \in (a_n,b_n) \ , \ n \in J. \end{cases}$$

Then β is a homeomorphism from (a,b) onto (a,b) and $\psi_t := \beta^{-1}\circ\phi_t\circ\beta$ $(t \in \mathbb{R})$ defines a continuous flow on (a,b) .