If $(T(t))_{t\geq 0}$ is eventually norm-continuous then $\lim_{t\to\infty} T(t)f$ exists for every $f \in L^1(\mu)$.

Proof. Since the semigroup is positive and eventually norm-continuous its boundary spectrum is cyclic and bounded, i.e. we have $P\sigma(A)\cap i\mathbb{R} = \{0\}$. Moreover there exist $t_o > 0$ and $\tau > 0$ such that $\|T(t_o) - T(t_o+\tau)\| < 1$.
For bounded linear operators $S \in \mathcal{L}(L^1)$ one has $\|S\| = \||S|\|$ (see IV,Thm.1.5 of Schaefer (1974)) hence $\||T(t_o) - T(t_o+\tau)|f\| < \|f\|$ for every $f \in L^1(\mu)$, $f \neq 0$. This shows that condition (2) of Thm.2.6(a) can be true only when $e_2 = 0$, i.e., $X_2 = \emptyset$.

□

Corollary 2.10. Let $(T(t))_{t\geq 0}$ be an irreducible semigroup on $L^p(\mu)$ satisfying the assumptions of Thm.2.6.
If $P\sigma(A)\cap i\mathbb{R} = \{0\}$ and if there exist $0 \leq r < s$, such that $\inf\{T(r),T(s)\} > 0$ then there exists a strictly positive function $h \in L^q(\mu)$ $(p^{-1}+q^{-1} = 1)$ such that $\lim_{t\to\infty} T(t)f = \langle f,h\rangle e$ for every $f \in L^p(\mu)$.

Proof. Since $\inf\{T(r),T(s)\} > 0$ we have $(\inf\{T(r),T(s)\})e > 0$ for the strictly positive fixed vector e . Since the regular operators on $L^p(\mu)$ form a vector lattice it follows by [Schaefer (1974), II.1.4, Formula (5) & (5')] that $|T(r) - T(s)|e = T(r)e + T(s)e - 2(\inf\{T(r),T(s)\})e < 2e$. Consequently the first alternative of Thm.2.6(b) holds true with $\tau := s-r$. Equivalently, we have $X_2 = \emptyset$ and by Cor.2.7 $Pf := \lim_{t\to\infty} T(t)f$ exists for every $f \in L^p(\mu)$.
The limit P is a positive projection, satisfying $PT(t) = T(t)P = P$ for all $t \geq 0$. It follows that $\operatorname{im} P \subset \ker A$ and $\operatorname{im} P' \subset \ker A'$. Since $P \neq 0$ $(Pe = e)$ we conclude that $\ker A'$ contains positive elements. Now C-III,Prop.3.5(a)-(c) implies that P has the form $P = h\otimes e$ for a strictly positive function $h \in L^q(\mu) = (L^p(\mu))'$.

□

In a last corollary we consider the case where one operator $T(t_o)$ is a kernel operator, i.e., $T(t_o)$ is induced by a $\mu\otimes\mu$-measurable kernel on $X\times X$. The corollary is of particular interest for semigroups on spaces ℓ^p , $1 \leq p < \infty$, where every positive operator is a kernel operator. For a precise definition and fundamental properties of kernel operators we refer to Sec.IV.9 of Schaefer (1974) or Chap.13 of Zaanen (1983). In particular we recall that the restriction of a kernel operator to a sublattice is again a kernel operator and that