$T_1(t)'|\phi| = |T_1(t)'\phi| = |e^{i\alpha t}\phi|$ or equivalently $A_1^*|\phi| = 0$. Now we can apply Thm.2.2 and obtain $i\alpha\mathbb{Z} \subset P\sigma(A_1^*) = R\sigma(A_1)$.

□

<u>Theorem</u> 4.2. Let A be the generator of a semigroup $(T(t))_{t\geq 0}$ of lattice homomorphisms on a Banach lattice E . Then $\sigma(A)$, $A\sigma(A)$ and $P\sigma(A)$ are imaginary additively cyclic subsets of $\mathbb{C}$.

<u>Proof</u>. We first consider the point spectrum. If $\lambda \in P\sigma(A)$, $\lambda = \alpha + i\beta$ $(\alpha,\beta \in \mathbb{R})$, then there exists $f \in E$, $f \neq 0$ such that $Af = \lambda f$. It follows that $T(t)f = e^{\lambda t}f$ $(t \geq 0)$ hence $T(t)|f| = |T(t)f| = e^{\alpha t}|f|$ $(t \geq 0)$, or equivalently, $A|f| = \alpha|f|$. Now Thm.2.2 is applicable and we obtain $A(f^{[n]}) = (\alpha + in\beta)f^{[n]}$ for all $n \in \mathbb{Z}$.

To prove the assertion for $A\sigma(A)$ we consider an $\mathcal{F}$-product semigroup in order to reduce the problem to the point spectrum. We use the notation of A-I,3.6 . Obviously the space $m(E)$ is a Banach lattice and every operator $\hat{T}(t)$ is a lattice homomorphism. We have $\big||T(t)|f| - |f|\big| = \big||T(t)f| - |f|\big| \leq |T(t)f - f|$ $(f \in E)$, hence $(|f_n|) \in m^{\mathcal{T}}(E)$ whenever $(f_n) \in m^{\mathcal{T}}(E)$. This proves that $m^{\mathcal{T}}(E)$ is a sublattice, hence a Banach lattice as well. Obviously, $c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E)$ is an order ideal. Thus $E_{\mathcal{F}}^{\mathcal{T}}$ is a Banach lattice and $(T_{\mathcal{F}}(t))$ is a semigroup of lattice homomorphisms. It follows that $P\sigma(A_{\mathcal{F}})$ is cyclic hence $A\sigma(A)$ is cyclic by A-III,4.5 .

Cyclicity of the entire spectrum now follows from the cyclicity of $A\sigma(A)$ and Lemma 4.1 .

□

One can use Thm.4.2 in order to prove cyclicity for the eigenvalues in the boundary spectrum of positive semigroups. We list some typical cases in the following corollary.

<u>Corollary</u> 4.3. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a positive semigroup on a Banach lattice E which is bounded. Each of the following conditions implies that $P\sigma(A) \cap i\mathbb{R}$ is imaginary additively cyclic.

(a) E is weakly sequentially complete (e.g. $E = L^p(\mu)$, $1 \leq p < \infty$);

(b) Every operator $T(t)$ is mean ergodic (i.e. the Césaro means $\frac{1}{n}\sum_{k=0}^{n-1} T(t)^k$ converge strongly as $n \to \infty$);

(c) There is a strictly positive linear form which is $\mathcal{T}$-invariant.