- Since $\lim_{n\to\infty}\|T(t)f_n - rf_n\| = 0$ and $\|f_n\| = 1$ we have $\lim_{n\to\infty}\|T(t)f_n\| = r > 0$.

□

For semigroups satisfying the assumption of Prop.4.5 $\sigma(A)$ is additively cyclic (by Thm.4.2) and $\sigma(T(t))$ is multiplicatively cyclic (by Schaefer (1974), V.Thm.4.4). Then the relation (4.7) implies that decompositons of the spectrum by vertical lines allow a spectral decomposition of the semigroup (cf. A-III,Def.3.1). (One simply performs a spectral decomposition of a single operator $T(t)$). In the following we will show that for positive groups (on arbitrary Banach lattices) spectral decompositions of this type always exist. Moreover, it will turn out that the decomposition is compatible with the lattice structure. The proof of this result uses Kato's equality (see Sec.5 of C-II). As a consequence of C-II,Cor.5.8 we have the following:

Let E be a Banach lattice with order continuous norm and $(T(t))_{t\in\mathbb{R}}$ be a group of positive operators on E with generator A .
Then the domain $D(A)$ is a sublattice of E and

(4.8) $A|f| = \mathrm{Re}[(\mathrm{sign}\ \bar{f})Af]$ for every $f \in D(A)$.

For real μ one has $\mu|f| = \mathrm{Re}[(\mathrm{sign}\ \bar{f})\mu f]$, hence

$(\mu - A)|f| = \mathrm{Re}[(\mathrm{sign}\ \bar{f})(\mu - A)f]$ for $\mu \in \mathbb{R}$, $f \in D(A)$.

The relations $f^+ = \frac{1}{2}(|f| + f)$, $f^- = \frac{1}{2}(|f| - f)$ yield

$(\mu - A)f^+ = \frac{1}{2}\big((\mathrm{sign}\ f)(\mu - A)f + (\mu - A)f\big)$ and

$(\mu - A)f^- = \frac{1}{2}\big((\mathrm{sign}\ f)(\mu - A)f - (\mu - A)f\big)$,

in case f is contained in the underlying real Banach lattice $E_{\mathbb{R}}$.
For $\mu \in \rho(A)\cap\mathbb{R}$, we can apply $R(\mu,A)$ on both sides and the substitution $f = R(\mu,A)g$ finally leads to

$$(4.9)\quad \begin{array}{l} (R(\mu,A)g)^+ = \frac{1}{2}R(\mu,A)\big((\mathrm{sign}\ R(\mu,A)g)g + g\big) \\ (R(\mu,A)g)^- = \frac{1}{2}R(\mu,A)\big((\mathrm{sign}\ R(\mu,A)g)g - g\big) \end{array} \quad \text{for all } g \in E_{\mathbb{R}} .$$

If we set $g_1 := \frac{1}{2}\cdot\big(g + (\mathrm{sign}\ R(\mu,A)g)g\big)$ and

$g_2 := \frac{1}{2}\cdot\big(g - (\mathrm{sign}\ R(\mu,A)g)g\big)$,

then obviously $g = g_1 + g_2$. Moreover, g is positive if and only if both, g_1 and g_2 are positive. We summarize these considerations in the following lemma.