$u \in D(A_{\mathbb{R}})$. Moreover, $\lim_{t\to o} T(t)u = u$ uniformly. Thus there exists $t_o > 0$ such that $T(t)u$ is strictly positive for all $t \in [0,t_o]$. Let $f \in D(A_{\mathbb{R}})$. For $\varepsilon > 0$ let $f_\varepsilon = f + i\varepsilon u$. Then $T(t)f_\varepsilon \in D(A)$ and $|T(t)f_\varepsilon|$ is strictly positive for all $t \in [0,t_o]$. By hypothesis, $|T(t)f_\varepsilon| \in D(A)$ and $\mathrm{Re}((\mathrm{sign}\ (T(t)\bar{f}_\varepsilon))AT(t)f_\varepsilon) = A|T(t)f_\varepsilon|$ for all $t \in [0,t_o]$. One sees as in the proof of Thm. 2.5 that this implies that $|T(t)f_\varepsilon| = T(t)|f_\varepsilon|$ for all $t \in [0,t_o]$. Letting $\varepsilon \to 0$ one obtains that $|T(t)f| = T(t)|f|$ $(t \in [o,t_o])$. Since $D(A)$ is dense in $C(K,\mathbb{R})$ we conclude that $|T(t)f| = T(t)|f|$ for all $f \in C(K,\mathbb{R})$ and all $t \in [o,t_o]$. Let $s > t_o$. Then $s/n \leq t_o$ for some $n \in \mathbb{N}$. Hence $|T(s)f| = |T(s/n)^n f| = T(s/n)^n|f| = T(s)|f|$ for all $f \in C(K,\mathbb{R})$. We have shown that $T_{\mathbb{R}}(t)$ is a lattice homomorphism for all $t \geq 0$; hence $T(t)$ is so as well (cf. Rem. 2.1).

□

Corollary 2.11. Let A be the generator of a lattice semigroup on $C(K,\mathbb{C})$ (K compact). Assume that $m \in C(K)$ is strictly positive. Then $m\cdot A$ with domain $D(m\cdot A) = D(A)$ generates a lattice semigroup.

Proof. By Theorem 1.20 $m\cdot A$ is the generator of a strongly continuous semigroup. It remains to show that it is a lattice semigroup. We use Theorem 2.10. Let $f \in D(m\cdot A) = D(A)$ such that $f(x) \neq 0$ for all $x \in K$. Then $\mathrm{Re}[(\mathrm{sign}\ \bar{f})m\cdot Af] = m\cdot \mathrm{Re}[(\mathrm{sign}\ \bar{f})Af] = m\cdot A|f|$.

□

Example 2.12. The operator A_{max} on the (real or complex space) $C[-1,0]$ given by $A_{max}f = f'$ with domain $D(A_{max}) = C^1[-1,0]$ satisfies Kato's equality; i.e.,

$$\langle \mathrm{Re}[(\widehat{\mathrm{sign}}\ \bar{f})(A_{max}f)],\phi\rangle = \langle |f|,A'_{max}\phi\rangle \quad (f \in D(A_{max}),\ \phi \in D(A'_{max})) . \tag{2.16}$$

Moreover, $(\lambda - A_{max})$ is surjective for $\lambda \geq 0$ (cf. Ex. 1.22). Thus, since $\ker(\lambda - A_{max}) = \mathbb{C}e_\lambda$ $(e_\lambda(x) = e^{\lambda x})$, Kato's equality does not have as strong consequences as the positive minimum principle (which by Thm. 1.13 would imply that A_{max} is a generator).

Proof. It is not difficult to prove that the adjoint A'_{max} of A_{max} is given by

$$A'_{max}\ \phi = \phi(0)\delta_o - \phi(-1)\delta_{-1} - d\phi \tag{2.17}$$