Chapter 1

Basic Results on Semigroups and Operator Algebras

This is not a systematic introduction to the theory of strongly continuous semigroups on C*- and W*-algebras. For that we refer to Bratteli and Robinson (1979), Davies (1976) and the survey article of Oseledets (1984). We only prepare for the subsequent chapters on spectral theory and asymptotics by fixing the notations and introducing some standard constructions.

1 Notations

1. By M we shall denote a C*-algebra with unit 1, with

$$M^{sa} := \{x \in M : x^* = x\}$$

the self-adjoint part of M and

$$M_+ := \{x^*x : x \in M\}$$

is the positive cone in M. If M' is the dual of M, then

$$M'_+ := \{\psi \in M' : \psi(x) \geqslant 0, x \in M_+\}$$

is a weak*-closed generating cone in M' and

$$S(M) := \{\psi \in M'_+ : \psi(1) = 1\}$$

is called the state space of M. For the theory of C*-algebras and related notions we refer to Pedersen (1979).

2. M is called a W*-algebra if there exists a Banach space M_*, such that its dual $(M_*)'$ is (isomorphic to) M. We call M_* the *predual* of M and $\psi \in M_*$ a *normal linear functional*. It is known that M_* is unique (Sakai, 1971, 1.13.3). For further properties of M_* we refer to (Takesaki, 1979, Chapter III).

3. A map $T \in \mathcal{L}(M)$ is called *positive* (in symbols $T \geqslant 0$) if $T(M_+) \subseteq M_+$. $T \in \mathcal{L}(M)$ is called n-*positive* ($n \in \mathbb{N}$) if $T \otimes \mathrm{Id}_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where Id_n is the identity map on the C*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive.

We call a contraction $T \in \mathcal{L}(M)$ a Schwarz map if T satisfies the so called *Schwarz-inequality*

$$T(x)T(x)^* \leqslant T(xx^*)$$

for all $x \in M$. It is well known that every n-positive contraction, $n \geqslant 2$ and that every positive contraction on a commutative C*-algebra is a Schwarz map ((Takesaki, 1979, Corollary IV. 3.8.)). As we shall see, the Schwarz inequality is crucial for our investigations.

4. If M is a C*-algebra we assume $\mathcal{T} = (T(t))_{t\geqslant 0}$ to be a strongly continuous semigroup (abbreviated semigroup) while on W*-algebras we consider weak*-semigroups, i.e. the mapping $(t \mapsto T(t)x)$ is continuous from $\mathbb{R}_+$ into $(M, \sigma(M, M_*))$, M_* the predual of M, and every $T(t) \in \mathcal{T}$ is $\sigma(M, M_*)$-continuous. Note that the preadjoint semigroup

$$\mathcal{T}_* = \{T(t)_* \colon T(t) \in \mathcal{T}\}$$

is weakly, hence strongly continuous on M_* (see e.g., (Davies, 1980, Prop. 1.23)).

5. We call $\mathcal{T}$ identity preserving if $T(t)1 = 1$ and of *Schwarz type* if every $T(t)$ is a Schwarz map.

For the notations concerning one-parameter semigroups we refer to Part A. In addition, we recommend to compare the results of this section of the book with the corresponding results for commutative C*-algebras, i.e. for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ (see Part B).

2 A Fundamental Inequality for the Resolvent

If $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup of Schwarz maps on a C*-algebra M (resp. a weak*-semigroup of Schwarz type on a W*-algebra M) with generator A, then the spectral bound $s(A) \leqslant 0$. Then $\Re(\lambda) > 0$ for $\lambda \in \mathbb{C}$ and there exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \,\mathrm{d}t \quad (x \in M)$$

where the integral exists in the norm topology.

In Bratteli and Robinson (1979) it is shown that $\mathcal{T}$ is a semigroup of Schwarz type if and only if $\mu R(\mu, A)$ is a Schwarz map for every $\mu \in \mathbb{R}_+$. Here we relate the domination of two semigroups to an inequality for the corresponding resolvent operator. This inequality will be needed later.

Theorem 2.1. *Let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a semigroup of Schwarz type with generator A and $\mathcal{S} = (S(t))_{t \geqslant 0}$ a semigroup with generator B on a C^*-algebra M. If*

$$(S(t)x)(S(t)x)^* \leqslant T(t)(xx^*) \tag{*}$$

for all $x \in M$ and $t \in \mathbb{R}_+$. Then

$$(\mu R(\mu, B)x)(\mu R(\mu, B)x)^* \leqslant \mu R(\mu, A)xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$. The same result holds if $\mathcal{T}$ is a weak*-semigroup of Schwarz type and $\mathcal{S}$ is a weak*-semigroup on a W^*-algebra M such that $(*)$ is fulfilled.

Proof. From the assumption $(*)$ it follows that

$$\begin{aligned} 0 \leqslant (S(r)x - S(t)x)(S(r)x - S(t)x)^* \\ = (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x) \\ \leqslant T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* \end{aligned}$$

for every $r, t \in \mathbb{R}_+$. Hence

$$(S(r)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \leqslant T(r)xx^* + T(t)xx^*.$$

Obviously, $S(t)_{\lrcorner}| \leqslant 1$ for all $t \in \mathbb{R}_+$. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$:

$$\begin{aligned} (R(\mu, B)x)(R(\mu, B)x)^* &= (\int_0^\infty e^{-\mu r} S(r)x \, dr)(\int_0^\infty e^{-\mu t} S(t)x \, \mathrm{d}t)^* \\ &= (\int_0^\infty \int_0^\infty e^{-\mu(r+t)} (S(r)x)(S(t)x)^* \, dr \, \mathrm{d}t) \\ &\leqslant \int_0^\infty \int_0^\infty e^{-\mu(r+t)} (T(r)xx^* + T(t)xx^*)/2 \, dr \, \mathrm{d}t \\ &= \int_0^\infty e^{-\mu r} T(r)xx^* \, dr \int_0^\infty e^{-\mu t} \, \mathrm{d}t \\ &= \mu^{-1} \int_0^\infty e^{-\mu r} T(r)xx^* \, dr \\ &= R(\mu, A)xx^*. \end{aligned}$$

Here we used the inequality derived above in the first step. The second step follows from $S(t)$ being a contraction semigroup and the third step is achieved by integration. □

Remark 2.2. The assumption that $\mathcal{T}$ is a semigroup of Schwarz type cannot be weakened in general to $\mathcal{T}$ being a positive contraction semigroup. This is demonstrated through examples in Davies (1980), where $S(t)x$ is defined as $e^{tB}x$ for a skew-adjoint generator B and $T(t)x = x$.

Corollary 2.3. *Let $T = (T(t))_{t \geqslant 0}$ be a semigroup of Schwarz type on a C^*-algebra M with generator A. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$:*

$$(\mu R(\mu, A)x)(\mu R(\mu, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof. Just set $S = T$ in Theorem 2.1. Then

$$\begin{aligned}= \frac{1}{2}(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}((S(r)x)(S(t)x)^* \\ + (S(t)x)(S(r)x)^*)\, dr\, dt \\ \leqslant \frac{1}{2}(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*)\, dr\, dt \\ = (\int_0^\infty e^{-\mu s}\, ds)(\int_0^\infty e^{-\mu t} T(t)xx^*\, dt) = \mu^{-1} R(\mu, A)xx^*\end{aligned}$$

where the handling of the integral is justified by (Bourbaki, 1955, §8, n° 4, Proposition 9). □

Corollary 2.4. *Let T be a semigroup of Schwarz maps (resp., weak*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$:*

$$(R(\lambda, A)x)(R(\lambda, A)x)^* \leqslant (\Re\lambda)^{-1} R(\Re\lambda, A)xx^*\, ,\; x \in M.$$

In particular for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$, $x \in M$:

$$(\mu R(\mu + i\alpha, A)x)(\mu R(\mu + i\alpha, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Then the semigroup

$$S := (e^{-i\mathrm{Im}(\lambda)t} T(t))_{t \geqslant 0}$$

fulfils the assumption of Thm **??** and $B := A - i\lambda$ is the generator of S. Consequently $R(\lambda, A) = R(\Re\lambda, B)$ and the corollary follows from Theorem **??**. □ □

As in Section C-III the following notion will be an important tool for the spectral theory of semigroups.

Definition 2.5. Let E be a Banach space and let D be a non-empty open subset of $\mathbb{C}$. A family $R : D \mapsto L(E)$ is called a pseudo-resolvent on D with values in E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu) R(\lambda) R(\mu)$$

for all λ and μ in D.

If R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon \Re(\lambda) > 0\}$ with values in a C*- or W*-algebra, then R is called of Schwarz type if

$$(R(\lambda)x)(R(\lambda)x)^* \leqslant (\Re\lambda)^{-1} R(\Re\lambda) x x^*$$

for all $\lambda \in D$ and $x \in M$. R is called identity preserving if $\lambda R(\lambda)1 = 1$ for all $\lambda \in D$.

For examples and properties of a pseudo-resolvent see C-III, 2.5. We state what will be used without further reference.

(i) If $\alpha \in \mathbb{C}$ and $x \in E$ such that $(\alpha - \lambda)R(\lambda)x = x$ for some $\lambda \in D$, then $(\alpha - \mu)R(\mu)x = x$ for all $\mu \in D$ (use the "resolvent equation").

(ii) If F is a closed subspace of E such that $R(\lambda)F \subseteq F$ for some $\lambda \in D$, then $R(\mu)F \subseteq F$ for all μ in a neighbourhood of λ. This follows from the fact that for all $\mu \in D$ near λ the pseudo-resolvent in μ is given by

$$R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}.$$

Definition 2.6. We call a semigroup T on the *predual* M_* of a W*-algebra M *identity preserving and of Schwarz type*, if its adjoint weak*-semigroup has these properties. Likewise, a pseudo-resolvent R on $D = \{\lambda \in \mathbb{C}\colon \Re(\lambda) > 0\}$ with values in M_* is called identity preserving and of Schwarz type, if R' has these properties.

Since for a semigroup of contractions on a Banach space

$$\mathrm{Fix}(T) = \bigcap_{t \geqslant 0} \ker(\mathrm{Id} - T(t)) =$$

$$= \ker(\mathrm{Id} - \lambda R(\lambda, A)) = \mathrm{Fix}(\lambda R(\lambda, A))$$

for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, it follows that a semigroup of contractions on M is identity preserving if and only if the (pseudo)-resolvent on $D = \{\lambda \in \mathbb{C}\colon \Re(\lambda) > 0\}$ given by

$$R(\lambda) := R(\lambda, A)|_D$$

is identity preserving. By Corollary **??** an analogous statement holds for "Schwarz type".

3 Induction and Reduction

1. If E is a Banach space and $S \subseteq L(E)$ is a semigroup of bounded operators, then a closed subspace F is called S-invariant, if $SF \subseteq F$ for all $S \in S$. We call the semigroup $S|_F := \{S|_F : S \in S\}$ the reduced semigroup. Note that for a one-parameter semigroup T (resp., pseudo-resolvent R) the reduced semigroup is again strongly continuous (resp. $R|_F$ is again a pseudo-resolvent) (compare the construction in A-I,3.2).

2. Let M be a W*-algebra, $p \in M$ a projection and $S \in \mathcal{L}(M)$ such that $S(p^\perp M) \subseteq p^\perp M$ and $S(Mp^\perp) \subseteq Mp^\perp$, where $p^\perp := 1 - p$. Since for all $x \in M$:

$$p[S(x) - S(pxp)] = p[S(p^\perp xp) + S(xp^\perp)]p = 0,$$

we obtain $p(Sx)p = p(S(pxp))p$. Therefore, the map

$$S_p := (x \mapsto p(Sx)p) \colon pMp \to pMp$$

is well defined and we call S_p the induced map. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$.

If $T = (T(t))_{t \geqslant 0}$ is a weak*-semigroup on M which is of Schwarz type and if $T(t)(p^\perp) \leqslant p^\perp$ for all $t \in \mathbb{R}_+$, then T leaves $p^\perp M$ and $Mp^\perp$ invariant. One can verify that the induced semigroup $T_p = (T(t)p)t \geqslant 0$ is again a weak*-semigroup.

If R is an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} \colon \Re(\lambda) > 0\}$ with values in M such that $R(\mu)p^\perp \leqslant p^\perp$ for some $\mu \in \mathbb{R}_+$, then $p^\perp M$ and $Mp^\perp$ are R-invariant. It follows directly that the induced pseudo-resolvent R_p has both the Schwarz type property and is identity preservation.

3. Let φ be a positive normal linear functional on a W*-algebra M such that $T_*\varphi = \varphi$ for some identity preserving Schwarz map T on M with preadjoint $T_* \in L(M_*)$. Then $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$ where $s(\varphi)$ is the support projection of φ.

Let $L_\varphi := x \in M : \varphi(xx^) = 0$ and $M_\varphi := L_\varphi \cap L_\varphi$ be defined. Since φ is T_*-invariant, and T is a Schwarz map, the subspaces L_φ and M_φ are T-invariant. From $M_\varphi = s(\varphi)^\perp M s(\varphi)^\perp$ and $T(s(\varphi)^\perp) \leqslant 1$ it follows that $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$.

Let $T_{s(\varphi)}$ be the induced map on $M_{s(\varphi)}$. If

$$s(\varphi)M_*s(\varphi) := \{\psi \in M_* \colon \psi = s(\varphi)\psi s(\varphi)\}$$

where $\langle s(\varphi)\psi s(\varphi), x\rangle := \langle \psi, s(\varphi)xs(\varphi)\rangle$ $(x \in M)$. For any $\psi \in s(\varphi)M_s(\varphi)$ and all $x \in M$, the following equalities holds:

$$\begin{aligned}(T_*\psi)(x) &= \psi(Tx) = \langle \psi, s(\varphi)(Tx)s(\varphi)\rangle \\ &= \langle \psi, s(\varphi)(T(s(\varphi)xs(\varphi)))s(\varphi)\rangle = \langle T_*\psi, s(\varphi)xs(\varphi)\rangle,\end{aligned}$$

hence $T_*\psi \in s(\varphi)M_*s(\varphi)$. Since the dual of $s(\varphi)M_*s(\varphi)$ is $M_{s(\varphi)}$, it follows that the adjoint of the reduced map $T_*|$ is identity preserving and of Schwarz type.

For example, if T is an identity preserving semigroup of Schwarz type on M_* such that $\varphi \in \mathrm{Fix}(T)$, then the semigroup $T|(s(\varphi)M_*s(\varphi))$ is again identity preserving and of Schwarz type. Furthermore, if R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \Re(\lambda) > 0\}$ with values in M_* which is identity preserving and of Schwarz type such that $R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$, then $R|s(\varphi)M_*s(\varphi)$ has the same properties.

Bibliography

N. Bourbaki. *Éléments des Mathématiques, Intégration, Chapitre 5: Intégration des Mesures.* Hermann, Paris, 1955.

O. Bratteli and D. Robinson. *Operator Algebras and Quantum Statistical Mechanics I.* Springer, New York-Heidelberg-Berlin, 1979. Volume II published 1981.

E. Davies. *Quantum Theory of Open Systems.* Academic Press, London-New York-San Francisco, 1976.

E. Davies. *One-parameter Semigroups.* Academic Press, London-New York-San Francisco, 1980.

V. I. Oseledets. Completely positive linear maps, non Hamiltonian evolution and quantum stochastic processes. *J. Soviet Math.*, 25:1529–1557, 1984.

G. K. Pedersen. *C^*-Algebras and their Automorphism Groups.* Academic Press, London, New York, San Francisco, 1979.

S. Sakai. *C^*-Algebras and W^*-Algebras.* Springer, Berlin-Heidelberg-New York, 1971.

M. Takesaki. *Theory of Operator Algebras I.* Springer, New York-Heidelberg-Berlin, 1979.