The translation semigroup

$$T(t)f(x) := f(x+t)$$

is strongly continuous on E and one shows as in A-I,2.4 that its generator is given by

$$Af = f' \ , \quad D(A) = \{ f \in E : f \in C^1(\mathbb{R}_+), \ f' \in E \} .$$

First we observe that $\|T(t)\| = 1$ for every $t \geq 0$, hence $\omega(\mathcal{T}) = 0$. Moreover it is clear that λ is an eigenvalue of A as soon as $\mathrm{Re}\lambda < -1$ (in fact : the function

$$x \to \varepsilon_\lambda(x) := e^{\lambda x}$$

belongs to $D(A)$ and is an eigenvector of A), hence $s(A) \geq -1$. For $f \in E$, $\mathrm{Re}\lambda > -1$,

$$\|\cdot\|_1\text{-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)f \, ds$$

exists since $\|T(s)f\|_1 \leq e^{-s}\|f\|_1$, $s \geq 0$, and

$$\|\cdot\|_\infty\text{-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)f \, ds$$

exists since $\int_0^\infty e^x |f(x)| \, dx < \infty$. Therefore $\int_0^\infty e^{-\lambda s} T(s)f \, ds$ exists in E for every $f \in E$, $\mathrm{Re}\lambda > -1$. As we observed in A-I,Prop.1.11 this implies $\lambda \in \rho(A)$. Therefore $\mathcal{T} = (T(t))_{t\geq 0}$ is a semigroup having $s(A) = -1$ but $\omega(\mathcal{T}) = 0$.

Example 1.4. (Hilbert space, Zabczyk (1975)) For every $n \in \mathbb{N}$ consider the n-dimensional Hilbert space $E_n := \mathbb{C}^n$ and operators $A_n \in \mathcal{L}(E_n)$ defined by the matrices

$$A_n = \begin{pmatrix} 0 & 1 & \cdot & \cdot & 0 \\ \cdot & 0 & 1 & & \cdot \\ & \cdot & \cdot & \cdot & \\ \cdot & & & \cdot & 1 \\ 0 & \cdot & & \cdot & 0 \end{pmatrix}_{n\times n} .$$

These matrices are nilpotent and therefore $\sigma(A_n) = \{0\}$. The elements $x_n := n^{-1/2}(1, \ldots , 1) \in E_n$ satisfy the following properties :

(i) $\|x_n\| = 1$ for every $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \|A_n x_n - x_n\| = 0$,

(iii) $\lim_{n\to\infty} \|\exp(tA_n)x_n - e^t x_n\| = 0$.

Consider now the Hilbert space $E := \oplus_{n\in\mathbb{N}} E_n$ and the operator $A := (A_n + 2\pi in)_{n\in\mathbb{N}}$ with maximal domain in E . Analogously we define a semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ by

$$T(t) := (e^{2\pi int}\exp(tA_n))_{n\in\mathbb{N}} .$$