We sketch the proof of (i) <=> (ii) assuming that $s(A) = 0$. If 0 is a first order pole, then the residue P is a positive projection satisfying $PE = \ker A$, $P'E' = \ker A'$ (see A-III,3.6) . Thus given $0 < f \in \ker A$ and any $0 \leq \phi \in E'$ such that $\langle f,\phi\rangle > 0$, we have for $\Psi := P'\phi$: $\langle f,\Psi\rangle = \langle f,P'\phi\rangle = \langle Pf,\phi\rangle = \langle f,\phi\rangle > 0$. To prove the reverse direction we first observe that the highest coefficient Q_k of the Laurent expansion is a positive operator. Thus if 0 is a pole of order $k \geq 2$ we choose $0 < h \in E$ such that $f := Q_k h > 0$. Then $Af = AQ_k h = 0$ and for every $\Psi \in \ker A'$ we have
$\langle f,\Psi\rangle = \langle Q_k h,\Psi\rangle = \langle h,Q_k'\Psi\rangle = \langle h,Q_{k-1}'A'\Psi\rangle = 0$.

(b) If a linear operator S on $C_o(X)$ is weakly compact, then S^2 is compact (see B-IV,Prop.2.4(b)). Therefore every non-zero spectral value of a weakly compact operator is a pole of the resolvent. This shows that Thm.2.9 is applicable if either $T(t_o)$ is weakly compact for some t_o or $R(\lambda,A)$ is weakly compact for some $\lambda \in \rho(A)$. We quote two criteria for weak compactness:

(2.31) If $T \in L(C(K))$, K compact, is positive, then it is weakly compact if and only if its biadjoint T'' maps the bounded Borel functions into $C(K)$ (see B-IV,Prop.2.4).

(2.32) A positive operator T on $C_o(X)$ which is dominated by a finite rank operator, is weakly compact.
(Actually, its adjoint T' is dominated by a finite rank operator as well, hence it maps the unit ball in an order interval. It follows that T' is weakly compact hence so is T .)

(c) Stronger results than Thm.2.9 will be proved in Chapter C-III . Actually, assuming only that $s(A)$ is a pole of finite algebraic multiplicity one can show that $\sigma_b(A)$ contains only poles of finite multiplicity (C-III,Thm.3.13). In C-III,Cor.2.12 we will show that $\sigma_b(A)$ is cyclic whenever $s(A)$ is a pole of the resolvent.

(d) Example 2.14 (b) can be extended to systems of functional differential equations even the infinite dimensional case. For details we refer to Sec.3 of Chapter B-IV .