Finally, we come back to Corollary 5.9. If in condition (ii) we demand that $D(A)$ is not only a sublattice but an ideal of E we obtain a characterization of multiplication semigroups.

Here we call a semigroup $(T(t))_{t\geq 0}$ <u>multiplication semigroup</u> if $T(t)$ is a multiplication operator (i.e., an element of the center) for every $t > 0$.

<u>Theorem</u> 5.13. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a σ-order complete real or complex Banach lattice E. The following assertions are equivalent.

(i) $(T(t))_{t\geq 0}$ is a multiplication semigroup.

(ii) There exists $\lambda \in \rho(A)$ such that $R(\lambda,A)$ is a multiplication operator.

(iii) $R(\lambda,A)$ is a multiplication operator for all $\lambda \in \rho(A)$.

(iv) A is local and $D(A)$ is an ideal in E.

(v) If $f \in D(A)$ then $Pf \in D(A)$ for every band projection P on E and $APf = PAf$.

<u>Proof</u>. Assume that (i) holds and let $\lambda > \omega(A)$. Since $R(\lambda,A)$ is the Laplace transform of the semigroup, it follows that $R(\lambda,A)$ is local since $T(t)$ is local for all $t \geq 0$. This implies $R(\lambda,A) \in Z(E)$ (see C-I,Sec.9).
We show that (ii) implies (v). Assume that $\lambda \in \rho(A)$ such that $R(\lambda,A)$ is a multiplication operator. Let P be a band projection. Then $PR(\lambda,A) = R(\lambda,A)P$. Let $f \in D(A)$, $g := (\lambda-A)f$. Then $Pf = PR(\lambda,A)g = R(\lambda,A)Pg$. Hence $Pf \in D(A)$ and $(\lambda-A)Pf = Pg$. Thus $APf = \lambda Pf - Pg = P(\lambda f - g) = PAf$.
We show that (v) implies (iii). Let $\lambda \in \rho(A)$ and P be a band projection. We have to show that $PR(\lambda,A) = R(\lambda,A)P$. Let $g \in E$, $f := R(\lambda,A)g$. Then $Pf \in D(A)$ and $APf = PAf$. Hence $PR(\lambda,A)g = Pf = R(\lambda,A)(\lambda-A)Pf = R(\lambda,A)P(\lambda-A)f = R(\lambda,A)Pg$. It follows from C-I,Sec.9 that $R(\lambda,A) \in Z(E)$.
(iii) implies (i) since $T(t) = \lim_{n\to\infty} [n/tR(n/t,A)]^n$ strongly for all $t > 0$.
It remains to show the equivalence of (iv) and (v). Assume that (iv) holds, let $f \in D(A)$ and P be a band projection. Then $Pf \in D(A)$ and $(Id-P)f \in D(A)$ by the assumption. Since A is local we have $APf = PAPf + (Id-P)APf = PAPf = PAPf + PA(Id-P)f = PAf$. Conversely, assume (v). Let $f \in D(A)$ and $|g| \leq |f|$. Then there exists a band projection P such that $Pf = g$. Hence $g \in D(A)$. We have shown