Chapter 1

Basic Results on Semigroups and Operator Algebras

This is not a systematic introduction to the theory of strongly continuous semigroups on C*- and W*-algebras. We only prepare for the subsequent chapters on spectral theory and asymptotic by fixing the notations and introducing some standard constructions.

1 Notations

1. Let M denote a C*-algebra with unit $\mathbb{1}$, where $M^{sa} := \{x \in M : x^* = x\}$ is the self-adjoint part of M and $M_+ := \{x^*x : x \in M\}$ is the positive cone in M. If M' is the dual of M, then $M'_+ := \{\psi \in M' : \psi(x) \geqslant 0, x \in M_+\}$ is a weak*-closed generating cone in M' and $S(M) := \{\varphi \in M'_+ : \varphi(\mathbb{1}) = 1\}$ is called the state space of M. For the theory of C*-algebras and related notions see Pedersen (1979).

2. We say that M is a W*-algebra if there exists a Banach space M_* such that its dual $(M_*)'$ is (isomorphic to) M. We call M_* the *predual* of M, and $\psi \in M_*$ a *normal linear functional*. It is known that M_* is unique (Sakai, 1971, 1.13.3). For other properties of M_*, see (Takesaki, 1979, Chapter III).

3. A map $T \in \mathcal{L}(M)$ is called *positive* (in symbols $T \geqslant 0$) if $T(M_+) \subseteq M_+$. $T \in \mathcal{L}(M)$ is called n*-positive* ($n \in \mathbb{N}$) if $T \otimes \mathrm{Id}_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where Id_n is the identity map on the C*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive.

We call a contraction $T \in \mathcal{L}(M)$ a Schwarz map if T satisfies the so called *Schwarz-inequality*

$$T(x)T(x)^* \leqslant T(xx^*)$$

for all $x \in M$. It is well known that every n-positive contraction, $n \geqslant 2$ and that every positive contraction on a commutative C*-algebra is a Schwarz map ((Takesaki, 1979, Corollary IV. 3.8.)). As we shall see, the Schwarz inequality is crucial for our investigations.

4. If M is a C^*-algebra, we assume that $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup (abbreviated as semigroup), while for W^*-algebras we consider weak*-semigroups, i.e. the mapping $(t \mapsto T(t)x)$ is continuous from $\mathbb{R}_+$ into $(M, \sigma(M, M_*))$, where M_* is the predual of M, and every $T(t) \in \mathcal{T}$ is $\sigma(M, M_*)$-continuous. Note that the preadjoint semigroup

$$\mathcal{T}_* = \{T(t)_* \colon T(t) \in \mathcal{T}\}$$

is weakly, hence strongly continuous on M_* (see e.g., (Davies, 1980, Prop. 1.23)). We call $\mathcal{T}$ identity preserving if $T(t)\mathbb{1} = \mathbb{1}$ and of *Schwarz type* if every $T(t)$ is a Schwarz map.

For the notations concerning one-parameter semigroups we refer to Part A. In addition, we recommend to compare the results of this section of the book with the corresponding results for commutative C^*-algebras, i.e. for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ (see Part B).

2 A Fundamental Inequality for the Resolvent

If $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup of Schwarz maps on a C^*-algebra M (resp. a weak*-semigroup of Schwarz type on a W^*-algebra M) with generator A, then the spectral bound $s(A) \leqslant 0$. Then $\Re(\lambda) > 0$ for $\lambda \in \mathbb{C}$ and there exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \,\mathrm{d}t \quad (x \in M)$$

where the integral exists in the norm topology. Here we relate the domination of two semigroups to an inequality for the corresponding resolvent operator. This inequality will be needed later.

Theorem 2.1. *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a semigroup of Schwarz type with generator A and $\mathcal{S} = (S(t))_{t\geqslant 0}$ a semigroup with generator B on a C^*-algebra M. If*

$$(S(t)x)(S(t)x)^* \leqslant T(t)(xx^*) \tag{*}$$

for all $x \in M$ and $t \in \mathbb{R}_+$. Then

$$(\mu R(\mu, B)x)\,(\mu R(\mu, B)x)^* \leqslant \mu R(\mu, A)xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$. The same result holds if $\mathcal{T}$ is a weak-semigroup of Schwarz type and $\mathcal{S}$ is a weak*-semigroup on a W^*-algebra M such that $(*)$ is fulfilled.*

Proof. From the assumption $(*)$ it follows that

$$\begin{aligned} 0 \leqslant (S(r)x - S(t)x)\,(S(r)x - S(t)x)^* \\ &= (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* \\ &\quad - (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \\ &\leqslant T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* \end{aligned}$$

for every $r, t \in \mathbb{R}_+$ and therefore

$$(S(r)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \leqslant T(r)xx^* + T(t)xx^*.$$

Obviously, $\|S(t)\| \leqslant 1$ for all $t \in \mathbb{R}_+$. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$

$$\begin{aligned}
&(R(\mu, B)x)\,(R(\mu, B)x)^* = \left(\int_0^\infty e^{-\mu r} S(r)x\,\mathrm{d}r\right)\left(\int_0^\infty e^{-\mu t} S(t)x\,\mathrm{d}t\right)^* \\
&= \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}((S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^*\,\mathrm{d}r\,\mathrm{d}t\right) \\
&\leqslant \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*)\,\mathrm{d}r\,\mathrm{d}t\right) \\
&= \left(\int_0^\infty e^{-\mu s} ds\right)\left(\int_0^\infty e^{-\mu t} T(t)xx^*\,\mathrm{d}t\right) = \mu^{-1}R(\mu, A)xx^*
\end{aligned}$$

where the handling of the integral is justified by Bourbaki (1955, Chap. V, §8, n° 4, Proposition 9) or Bourbaki (2004)). The claim is obtained by multiplying both sides by μ^2. ☐

Corollary 2.2. *Let $\mathcal{T}$ be a semigroup of Schwarz maps (resp., weak*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$:*

$$(R(\lambda, A)x)\,(R(\lambda, A)x)^* \leqslant \Re(\lambda)^{-1} R(\Re(\lambda), A)xx^*\,,\ x \in M.$$

In particular for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in M$

$$(\mu R(\mu + \mathrm{i}\alpha, A)x)(\mu R(\mu + \mathrm{i}\alpha, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Then the semigroup

$$S \coloneqq \left(e^{-\mathrm{i}\Im(\lambda)t} T(t)\right)_{t \geqslant 0}$$

fulfills the assumption of Thm. 2.1 and $B \coloneqq A - \mathrm{i}\lambda$ is the generator of S. Consequently $R(\lambda, A) = R(\Re\lambda, B)$ and the corollary follows from Thm. 2.1. ☐

Remark 2.3. Since

$$T(t)x = \lim_n \left(\frac{n}{t} R\left(\frac{n}{t}, A\right)\right)^n x, \quad x \in M,$$

it follows from above, that $\mathcal{T}$ is a semigroup of Schwarz-type, if an only if $\mu R(\mu, A)$ is a Schwarz-operator for every $\mu \in \mathbb{R}_+$.

As in Section C-III the following notion will be an important tool for the spectral theory of semigroups ond C*- and W*-algebras.

Definition 2.4. Let E be a Banach space and let D be a non-empty open subset of $\mathbb{C}$. A family $\mathcal{R} \colon D \mapsto L(E)$ is called a *pseudo-resolvent* on D with values in E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu) \qquad \text{(Resolvent Equation)}$$

for all λ, μ in D and $R \in \mathcal{R}$.

If $\mathcal{R}$ is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \Re(\lambda) > 0\}$ with values in a C*- or W*-algebra, then $\mathcal{R}$ is called of Schwarz type if

$$(R(\lambda)x)(R(\lambda)x)^* \leqslant (\Re\lambda)^{-1} R(\Re\lambda) x x^*$$

and *identity preserving* if $\lambda R(\lambda)\mathbb{1} = \mathbb{1}$ for all $\lambda \in D$ and $R \in \mathcal{R}$. For examples and properties of a pseudo-resolvent, see C-III, 2.5.

We state what will be used without further reference.

(i) If $\alpha \in \mathbb{C}$ and $x \in E$ such that $(\alpha - \lambda)R(\lambda)x = x$ for some $\lambda \in D$, then $(\alpha - \mu)R(\mu)x = x$ for all $\mu \in D$ (use the *resolvent equation*).

(ii) If F is a closed subspace of E such that $R(\lambda)F \subseteq F$ for some $\lambda \in D$, then $R(\mu)F \subseteq F$ for all μ in a neighborhood of λ. This follows from the fact that for all $\mu \in D$ near λ the pseudo-resolvent in μ is given by

$$R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}.$$

Definition 2.5. We call a semigroup $\mathcal{T}$ on the *predual* M_* of a W*-algebra M *identity preserving and of Schwarz type* if its adjoint weak*-semigroup has these properties. Similarly, a pseudo-resolvent R on $D = \{\lambda \in \mathbb{C} \colon \Re(\lambda) > 0\}$ with values in M_* is said to be identity preserving and of Schwarz type if R' has these properties.

Since for a semigroup of contractions on a Banach space

$$\mathrm{Fix}(T) = \bigcap_{t \geqslant 0} \ker(\mathrm{Id} - T(t)) =$$

$$= \ker(\mathrm{Id} - \lambda R(\lambda, A)) = \mathrm{Fix}(\lambda R(\lambda, A))$$

for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, it follows that a semigroup of contractions on M is identity preserving if and only if the (pseudo)-resolvent on $D = \{\lambda \in \mathbb{C} \colon \Re(\lambda) > 0\}$ given by

$$R(\lambda) := R(\lambda, A)_{|D}$$

is identity preserving. By Corollary 2.2 an analogous statement holds for *Schwarz type*.

3 Induction and Reduction

1. If E is a Banach space and $\mathcal{S} \subseteq \mathcal{L}(E)$ is a semigroup of bounded operators, then a closed subspace F is called $\mathcal{S}$-invariant, if $SF \subseteq F$ for all $S \in \mathcal{S}$. We call the semigroup $\mathcal{S}_{|F} := \{S_{|F} \colon S \in \mathcal{S}\}$ the reduced semigroup. Note that for a one-parameter semigroup $\mathcal{T}$ (resp., pseudo-resolvent $\mathcal{R}$) the reduced semigroup is again strongly continuous (resp. $\mathcal{R}_{|F}$ is again a pseudo-resolvent) (compare the construction in A-I,3.2).

2. Let M be a W*-algebra, $p \in M$ a projection and $S \in \mathcal{L}(M)$ such that $S(p^\perp M) \subseteq p^\perp M$ and $S(Mp^\perp) \subseteq Mp^\perp$, where $p^\perp := \mathbb{1} - p$. Since for all $x \in M$:

$$p[S(x) - S(pxp)] = p[S(p^\perp xp) + S(xp^\perp)]p = 0,$$

we obtain $p(Sx)p = p(S(pxp))p$. Therefore, the map

$$S_p := (x \mapsto p(Sx)p) \colon pMp \to pMp$$

is well defined and we call S_p the induced map. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$.

3. If $\mathcal{T} = (T(t))_{t \geqslant 0}$ is a weak*-semigroup on M which is of Schwarz type and if $T(t)(p^\perp) \leqslant p^\perp$ for all $t \in \mathbb{R}_+$, then T leaves $p^\perp M$ and $Mp^\perp$ invariant. One can verify that the induced semigroup $T_p = (T(t)p)_{t \geqslant 0}$ is again a weak*-semigroup.

If R is an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} \colon \Re(\lambda) > 0\}$ with values in M such that $R(\mu)p^\perp \leqslant p^\perp$ for some $\mu \in \mathbb{R}_+$, then $p^\perp M$ and $Mp^\perp$ are R-invariant. It follows directly that the induced pseudo-resolvent R_p has both the Schwarz type property and is identity preservation.

4. Let φ be a positive normal linear functional on a W*-algebra M such that $T_*\varphi = \varphi$ for some identity preserving Schwarz map T on M with preadjoint $T_* \in L(M_*)$. Then $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$ where $s(\varphi)$ is the support projection of φ.

Let $L_\varphi := x \in M : \varphi(xx^)) = 0$ and $M_\varphi := L_\varphi \cap L_\varphi$ be defined. Since φ is T_*-invariant, and T is a Schwarz map, the subspaces L_φ and M_φ are T-invariant. From $M_\varphi = s(\varphi)^\perp M s(\varphi)^\perp$ and $T(s(\varphi)^\perp) \leqslant 1$ it follows that $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$.

Let $T_{s(\varphi)}$ be the induced map on $M_{s(\varphi)}$. If

$$s(\varphi)M_*s(\varphi) := \{\psi \in M_* \colon \psi = s(\varphi)\psi s(\varphi)\}$$

where $\langle s(\varphi)\psi s(\varphi), x\rangle := \langle \psi, s(\varphi)xs(\varphi)\rangle$ ($x \in M$). For any $\psi \in s(\varphi)M_s(\varphi)$ and all $x \in M$, the following equalities holds:

$$\begin{aligned}(T_*\psi)(x) &= \psi(Tx) = \langle \psi, s(\varphi)(Tx)s(\varphi)\rangle \\ &= \langle \psi, s(\varphi)(T(s(\varphi)xs(\varphi)))s(\varphi)\rangle = \langle T_*\psi, s(\varphi)xs(\varphi)\rangle,\end{aligned}$$

hence $T_*\psi \in s(\varphi)M_*s(\varphi)$. Since the dual of $s(\varphi)M_*s(\varphi)$ is $M_{s(\varphi)}$, it follows that the adjoint of the reduced map $T_{*|}$ is identity preserving and of Schwarz type.

For example, if T is an identity preserving semigroup of Schwarz type on M_* such that $\varphi \in \mathrm{Fix}(T)$, then the semigroup $T_{|(s(\varphi)M_*s(\varphi))}$ is again identity preserving and of Schwarz type. Furthermore, if R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \Re(\lambda) > 0\}$ with values in M_* which is identity preserving and of Schwarz type such that $R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$, then $R_{|s(\varphi)M_*s(\varphi)}$ has the same properties.

Notes

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