Proposition 3.4. Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a W*-algebra M . Then the following assertions are equivalent.

(a) $\mathcal{T}$ is irreducible and $P\sigma(A) \cap i\mathbb{R} \neq \emptyset$.

(b) $\mathcal{T}$ is relatively compact in the weak operator topology and the fixed space of $\mathcal{T}$ is generated by a faithful state.

(c) $\mathcal{T}$ is strongly ergodic and the fixed space of $\mathcal{T}$ is generated by a faithul state.

(d) The fixed space of $\mathcal{T}$ is generated by a faithful state.

Proof. Suppose (a) is satisfied. Since $\mathrm{Fix}(\mathcal{T}) \neq \{0\}$ there exists a faithful normal state ϕ on M such that $\mathrm{Fix}(\mathcal{T}) = \phi\mathbb{C}$ (D-III, Thm.1.10.). Therefore $\mathcal{T}$ is relatively compact in the weak operator topology by Proposition 3.1., whence (b) holds.

The implications from (b) to (c) and (c) to (d) are trivial.

Suppose that (d) holds. Let ϕ be a faithful normal state on M such that $\mathrm{Fix}(\mathcal{T}) = \phi\mathbb{C}$. By Proposition 3.1 the semigroup $\mathcal{T}$ is strongly ergodic. Therefore the fixed space of $\mathcal{T}$ separates the points of $\mathrm{Fix}(\mathcal{T}')$. Consequently $\mathrm{Fix}(\mathcal{T}') = \mathbb{C}1$. Thus the ergodic projection associated with $\mathcal{T}$ is given by $P = 1 \otimes \phi$, i.e. $P\psi = \psi(1)\phi$ for all $\psi \in M_*$. Let F be a closed $\mathcal{T}$-invariant face of M_*^+ . If $0 \neq \psi \in F$ then

$$\lim_{s\to\infty} C(s)\psi = \psi(1)\phi \in F .$$

Hence $\phi \in F$ and therefore $F = M_*^+$ by the faithfulness of ϕ which proves (a).

□

The next theorem is an extension of D-III,Thm.1.10 and shows the usefulness of the theory of semitopological semigroups. Assume $\mathcal{T} \subseteq L(M_*)$ to be relatively compact in the weak operator topology. Since $\mathcal{T}$ is commutative its closure $\mathcal{S} = (\mathcal{T})^- \subseteq L_w(M_*)$ contains a unique minimal ideal K , called the kernel of $\mathcal{S}$, which is a compact Abelian group ([DeLeeuw-Glicksberg (1961); Junghenn (1971); Krengel (1985), § 2.4]. The identity Q of K is a projection onto