Proposition 1.10. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. Then

(1.3) $T(t)f = \lim_{n\to\infty} (Id - t/nA)^{-n}f = \lim_{n\to\infty} (n/t\cdot R(n/t,A))^n f$

for all $f \in E$ and $t \geq 0$.

Holomorphic semigroups

We now describe a hierarchy of smoothness conditions on the semigroup, starting with the most restrictive class; namely, holomorphic semigroups. The generators of these semigroups can be characterized by a particularly simple condition.

For $\alpha \in (0,\pi]$ we define the sector $S(\alpha)$ in the complex plane by $S(\alpha) = \{re^{i\theta} : r\geq 0,\ \theta\in(-\alpha,\alpha)\}$.

Definition 1.11. Let $\alpha \in (0,\pi/2]$. A strongly continuous semigroup $(T(t))_{t\geq 0}$ is called a bounded holomorphic semigroup of angle α if $T(\cdot)$ is the restriction of a holomorphic function

$$T : S(\alpha) \to L(E)$$

satisfying

(1.4) $T(z)T(z') = T(z+z') \quad (z,z' \in S(\alpha))$

(1.5) For each $\alpha_1 \in (0,\alpha)$ the set $\{T(z) : z \in S(\alpha_1)\}$ is uniformly bounded and $\lim_{n\to\infty} T(z_n)f = f$ for every null-sequence (z_n) in $S(\alpha_1)$ and every $f \in E$.

Remark. A function $T : S(\alpha) \to L(E)$ is holomorphic with respect to the operator norm if and only if it is strongly holomorphic if and only if it is weakly holomorphic [Yosida (1965);V.3].

Theorem 1.12. Let A be a densely defined operator on a Banach space E and $\alpha \in (0,\pi/2]$. Then A is the generator of a bounded holomorphic semigroup of angle α if and only if

$$S(\alpha + \pi/2) \subset \rho(A)$$

and for every $\alpha_1 \in (0,\alpha)$ there exists a constant M such that

(1.6) $\|R(\lambda,A)\| \leq M/|\lambda| \quad (\lambda\in S(\alpha_1+\pi/2))$.

For the proof we refer to [Davies (1980)], for example.

Remark. Let A be the generator of a bounded holomorphic semigroup $(T(t))_{t\geq 0}$ of angle α, and let $z_0 \in S(\alpha)$. Then z_0A generates a