Proof. Let $u \in D(A)_+$, $g = Au$. Assume that $(Au)^- \in \overline{E_u^-}$. Then, if $0 \leq \phi \in E'_+$ such that $\langle u,\phi\rangle = 0$ one has $\langle f,\phi\rangle = 0$ for all $f \in \overline{E_u^-}$, hence $\langle (Au)^-,\phi\rangle = 0$ and consequently $\langle Au,\phi\rangle \geq 0$. This proves one direction. To prove the other assume that $g^- \notin \overline{E_u^-}$. Then there exists $\phi \in (E_u)^o$ such that $\langle g^-,\phi\rangle \neq 0$. Since $(E_u)^o$ has a generating cone (by [Schaefer (1974),II,4.7]), we can assume that $\phi > 0$.

Define $\psi_o(f) = \sup \phi([0,f] \cap E_{(g^-)})$ for $f \in E_+$. Then ψ_o is positive homogeneous on E_+. Thus the linear extension of ψ_o defines a positive linear form ψ on E . We have $\langle g^-,\psi\rangle = \langle g^-,\phi\rangle > 0$ and $\langle g^+,\psi\rangle = 0$. Thus $\langle Au,\psi\rangle = -\langle g^-,\psi\rangle < 0$. But $\langle u,\psi\rangle \leq \langle u,\phi\rangle = 0$. Thus (P) does not hold.

□

Bounded generators of positive semigroups can now be characterized as follows.

Theorem 1.11. Let A be a bounded operator on a Banach lattice E . The following assertions are equivalent:

(i) $e^{tA} \geq 0$ $(t \geq 0)$.

(ii) $f \in E_+$, $\phi \in E'_+$, $\langle f,\phi\rangle = 0$ implies $\langle Af,\phi\rangle \geq 0$.

(iii) $(Af)^- \in \overline{E_f}$ for all $f \in D(A)_+$.

(iv) $A + \|A\|\cdot Id \geq 0$.

Proof. It follows by Proposition 1.7 that (i) implies (ii). Since $\|e^{tA}\| \leq e^{t\|A\|}$ $(t\geq 0)$, (ii) implies (i) by Theorem 1.8. The equivalence of (ii) and (iii) is established by Lemma 1.10. If (iv) holds, then $e^{t(A+\|A\|)} \geq 0$ $(t\geq 0)$. Thus $e^{tA} = e^{-t\|A\|}e^{t(A+\|A\|)} \geq 0$ $(t\geq 0)$. We have shown that (i), (ii) and (iii) are equivalent and (iv) implies (i).

It remains to show that (i) implies (iv). Since assertions (i) and (iv) are satisfied for A if and only if they are satisfied for A' , we can assume that E is order complete (considering A' instead of A if necessary). Assume that (i) holds. Then by what we have proved above (iii) holds as well. In particular

$$(1.9) \qquad (Au)^- \in \{u\}^{dd} \quad \text{for all } u \in E_+ .$$

Let $\lambda \geq 0$ and $f \in E_+$ such that $g = (A + \lambda)f \ngeq 0$. We have to show that $\lambda \leq \|A\|$. Denote by P the band projection onto the band generated by g^-. Then $PAf + \lambda Pf = Pg = g^- < 0$. Since by (1.9), $[A(Id-P)f]^- \in (Id-P)E$, it follows $0 > \lambda Pf + PAf = \lambda Pf + PAPf + PA(Id-P)f = \lambda Pf + PAPf + P(A(Id-P)f)^+ \geq \lambda Pf + PAPf$.