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One-parameter Semigroups of Positive Operators

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This Latex version of the book "One-Parameter Semigroups of Positive Operators" is dedicated to the memory of our co-authors, Heinrich P. Lotz, Ulrich Moustakas, and Ulf Schlotterbeck. Their contributions to the first edition remain an inspiration to us all. We miss their presence and remain grateful for the legacy they have left in this work.

Preface

As early as 1948 in the first edition of his fundamental treatise on Semigroups and Functional Analysis, E. Hille expressed the need for "developing an adequate theory of transformation semigroups operating in partially ordered spaces" (l.c., Foreword). In the meantime the theory of one-parameter semigroups of positive linear operators has grown continuously. Motivated by problems in probability theory and partial differential equations W. Feller (1952) and R. S. Phillips (1962) laid the first cornerstones by characterizing the generators of special positive semigroups. In the 60's and 70's the theory of positive operators on ordered Banach spaces was built systematically and is well documented in the monographs of H. H. Schaefer (1974) and A. C. Zaanen (1983). But in this process the original ties with the applications and, in particular, with initial value problems were at times obscured. Only in recent years an adequate and up-to-date theory emerged, largely based on the techniques developed for positive operators and thus recombining the functional analytic theory with the investigation of Cauchy problems having positive solutions to each positive initial value. Even though this development — in particular with respect to applications to concrete evolution equations in transport theory, mathematical biology, and physics — is far from being complete, the present volume is a first attempt to shape the multitude of available results into a coherent theory of one-parameter semigroups of positive linear operators on ordered Banach spaces.

The book is organized as follows. We concentrate our attention on three subjects of semigroup theory: *characterization*, *spectral theory* and *asymptotic behavior*. By *characterization*, we understand the problem of describing special properties of a semigroup, such as positivity, through the generator. By *spectral theory* we mean the investigation of the spectrum of a generator. *Asymptotic behavior* refers to the orbits of the initial values under a given semigroup and phenomena such as stability.

This program (characterization, spectral theory, asymptotic behavior) is worked out on four different types of underlying spaces:

(A) On Banach spaces — Here we present the background for the subsequent discussions related to order.

(B) On spaces $C_0(X)$ (X locally compact), which constitute an important class of ordered Banach spaces and where our results can be presented in a form which makes them accessible also for the non-expert in order-theory.
(C) On Banach lattices, which admit a rich theory and are still sufficiently general as to include many concrete spaces appearing in analysis; e.g., $C_0(X)$, $\mathcal{L}^p(k)$ or l^p.
(D) On non-commutative operator algebras such as C^*- or W^*-algebras, which are not lattice ordered but still possess an interesting order structure of great importance in mathematical physics.

In each of these cases we start with a short collection of basic results and notations, so that the contents of the book may be visualized in the form of a 4×4 matrix in a way which will allow "row readers" (interested in semigroups on certain types of spaces) and "column readers" (interested in certain aspects) to find a path through the book corresponding to their interest.

We display this matrix, together with the names of the authors contributing to the subjects defined through this scheme:

	I Basic Results	II Characterization	III Spectral Theory	IV Asymptotics
A. Banach Spaces	R. Nagel U. Schlotterbeck	W. Arendt H. P. Lotz	G. Greiner R. Nagel	F. Neubrander
B. $C_0(X)$	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas F. Neubrander
C. Banach Lattices	R. Nagel U. Schlotterbeck	G. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas R. Nagel F. Neubrander
D. Operator Algebras	U. Groh	U. Groh	U. Groh	U. Groh

This "matrix of contents" has been an indispensable guide line in our discussions on the scope and the spirit of the various contributions. However, we would not have succeeded in completing this manuscript, as a collection of independent contributions (personally accounted for by the authors), under less favorable conditions than we have actually met. For one thing, Rainer Nagel was an unfaltering and undisputed spiritus rector from the very beginning of the project. On the other hand we gratefully acknowledge the influence of Helmut H. Schaefer and his pioneering work on order structures in analysis. It was the team spirit produced by this common mathematical background which, with a little help from our friends, made it possible to overcome most difficulties.

We have prepared the manuscript with the aid of a word processor, but we confess that without the assistance of Klaus Kuhn the pitfalls of such a system would have been greater than its benefits.

The authors

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Part A
One-parameter Semigroups on Banach Spaces

Chapter 1
Basic Results on Semigroups on Banach Spaces

Since the basic theory of one-parameter semigroups can be found in several excellent books (e.g. Davies (1980), Goldstein (1985a), Pazy (1983) or Hille-Phillips (1957)) we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, to collect briefly some important definitions and results (Section 1), to present a list of standard examples (Section 2) and to discuss standard constructions of new semigroups from a given one (Section 3). In the entire chapter we denote by E a (real or) complex Banach space and consider one-parameter semigroups of bounded linear operators $T(t)$ on E. By this we understand a subset $\{T(t) : t \in \mathbb{R}_+\}$ of $L(E)$, usually written as $(T(t))_{t\geq 0}$, such that

$$T(0) = Id$$
$$T(s+t) = T(s) \cdot T(t) \text{ for all } s, t \in \mathbb{R}_+.$$

In more abstract terms this means that the map $t \mapsto T(t)$ is a homomorphism from the additive semigroup $\mathbb{R}_+$ into the multiplicative semigroup $(L(E), \cdot)$. Similarly, a one-parameter group $(T(t))_{t\in\mathbb{R}}$ will be a homomorphic image of the group $(\mathbb{R}, +)$ in $(L(E), \cdot)$.

1.1 Standard Definitions and Results

We consider a one-parameter semigroup $(T(t))_{t\geq 0}$ on a Banach space E and observe that the domain $\mathbb{R}_+$ and the range $L(E)$ of the (semigroup) homomorphism $\tau : t \mapsto T(t)$ are topological semigroups for the natural topology on $\mathbb{R}_+$ and any one of the standard operator topologies on $L(E)$. We single out the strong operator topology on $L(E)$ and require τ to be continuous.

Definition 1.1 A one-parameter semigroup $(T(t))_{t\geq 0}$ is called strongly continuous if the map $t \mapsto T(t)$ is continuous for the strong operator topology on $L(E)$, i.e., $\lim_{t\to t_0} \|T(t)f - T(t_0)f\| = 0$ for every $f \in E$ and $t, t_0 \geq 0$.

Clearly one defines in a similar way weakly continuous, resp. uniformly continuous (compare A-II, Def. 1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology: From now on 'semigroup' always means strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

Remark 1.2 (i) A one-parameter semigroup $(T(t))_{t\geq 0}$ on a Banach space E is strongly continuous if and only if for any $f \in E$ it is true that $T(t)f \to f$ as $t \to 0$.

(ii) For every strongly continuous semigroup $(T(t))_{t\geq 0}$ there exist constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq M \cdot e^{\omega t}$ for every $t \geq 0$.

(iii) If $(T(t))_{t\geq 0}$ is a one-parameter semigroup such that $\|T(t)\|$ is bounded for $0 \leq t \leq \delta$ then it is strongly continuous if and only if $\lim_{t\to 0} T(t)f = f$ for every f in a total subset of E.

Definition 1.3 By the growth bound (or type) of the semigroup $(T(t))_{t\geq 0}$ we understand the number

$$\omega := \inf\{w \in \mathbb{R} : \text{There exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leq Me^{wt} \text{ for } t \geq 0\} \quad (1.1)$$

$$= \lim_{t\to\infty} \tfrac{1}{t} \cdot \log \|T(t)\| = \inf_{t>0} \tfrac{1}{t} \cdot \log \|T(t)\|$$

Particularly important are semigroups such that for every $t \geq 0$ we have $\|T(t)\| \leq M$ (bounded semigroups) or $\|T(t)\| \leq 1$ (contraction semigroups). In both cases we have $\omega \leq 0$. It follows from the subsequent examples and from 3.1 that ω may be any number $-\infty \leq \omega < +\infty$. Moreover the reader should observe that the infimum in (1.1) need not be attained and that M may be larger than 1 even for bounded semigroups.

Example 1.4 (i) Take $E = \mathbb{C}^2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then for the 1-norm on E we obtain $\|T(t)\| = 1 + t$, hence $(T(t))_{t\geq 0}$ is an unbounded semigroup having growth bound $\omega = 0$.

(ii) Take $E = L^1(\mathbb{R})$ and for $f \in E$, $t \geq 0$ define

$$T(t)f(x) := \begin{cases} 2 \cdot f(x+t) & \text{if } x \in [-t, 0] \\ f(x+t) & \text{otherwise.} \end{cases}$$

Each $T(t)$, $t > 0$, satisfies $\|T(t)\| = 2$ as can be seen by taking $f := 1_{[0,t]}$. Therefore $(T(t))_{t\geq 0}$ is a strongly continuous semigroup which is bounded, hence has $\omega = 0$, but the constant M in (1.1) cannot be chosen to be 1.

Definition 1.5 To every semigroup $(T(t))_{t\geq 0}$ there belongs an operator $(A, D(A))$, called the generator and defined on the domain

$$D(A) := \left\{ f \in E : \lim_{h \to 0} \frac{T(h)f - f}{h} \text{ exists in } E \right\}$$

by $Af := \lim_{h \to 0} \frac{T(h)f-f}{h}$ for $f \in D(A)$.

Clearly, $D(A)$ is a linear subspace of E and A is linear from $D(A)$ into E. Only in certain special cases (see 2.1) the generator is everywhere defined and therefore bounded (use Prop. 1.9(i)). In general the precise extent of the domain $D(A)$ is essential for the characterization of the generator. But since the domain is canonically associated to the generator of a semigroup we shall write in most cases A instead of $(A, D(A))$.

Proposition 1.6 *For the generator A of a semigroup $(T(t))_{t \geq 0}$ on a Banach space E the following assertions hold:*

(i) If $f \in D(A)$ then $T(t)f \in D(A)$ for every $t \geq 0$.
(ii) The map $t \mapsto T(t)f$ is differentiable on $\mathbb{R}_+$ if and only if $f \in D(A)$. In that case one has

$$\frac{d}{dt}T(t)f = AT(t)f = T(t)Af. \tag{1.2}$$

(iii) Every $f \in E$ one has $\int_0^t T(s)f\,ds \in D(A)$ and

$$A \int_0^t T(s)f\,ds = T(t)f - f. \tag{1.3}$$

(iv) If $f \in D(A)$ then

$$T(t)f = f + \int_0^t AT(s)f\,ds = f + \int_0^t T(s)Af\,ds.$$

(v) The domain $D(A)$ is dense in E.

Theorem 1.7 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on the Banach space E. Then the 'abstract Cauchy problem' (ACP)*

$$\frac{d}{dt}\xi(t) = A\xi(t), \quad \xi(0) = f_0,$$

has a unique solution $\xi : \mathbb{R}_+ \to D(A)$ in $C^1(\mathbb{R}_+, E)$ for every $f_0 \in D(A)$. In fact, this solution is given by $\xi(t) := T(t)f_0$.

For the important relation of semigroups to abstract Cauchy problems we refer to A-II, Section 1. Here we only point out that the above theorem implies that a semigroup is uniquely determined by its generator. While the generator is bounded only for uniformly continuous semigroups (see 2.1 below), it always enjoys a weaker but useful property.

Definition 1.8 An operator B with domain $D(B)$ on a Banach space E is called closed if $D(B)$ endowed with the graph norm

$$\|f\|_B := \|f\| + \|Bf\|$$

becomes a Banach space. Equivalently, $(B, D(B))$ is closed if and only if its graph $\{(f, Bf) : f \in D(B)\}$ is closed in $E \times E$, i.e.

$$f_n \in D(B), f_n \to f \text{ and } Bf_n \to g \text{ implies } f \in D(B) \text{ and } Bf = g. \tag{1.4}$$

It is clear from this definition that the 'closedness' of an operator B depends very much on the size of the domain $D(B)$. For example, a bounded and densely defined operator $(B, D(B))$ is closed if and only if $D(B) = E$. On the other hand it may happen that $(B, D(B))$ is not closed but has a closed extension $(C, D(C))$, i.e., $D(B) \subset D(C)$ and $Bf = Cf$ for every $f \in D(B)$. In that case, B is called closable, a property which is equivalent to the following:

$$f_n \in D(B), f_n \to 0 \text{ and } Bf_n \to g \text{ implies } g = 0. \tag{1.5}$$

The smallest closed extension of $(B, D(B))$ will be called the closure $\bar{B}$ with domain $D(\bar{B})$. In other words, the graph of $\bar{B}$ is the closure of $\{(f, Bf) : f \in D(B)\}$ in $E \times E$. Finally we call a subset D_0 of $D(B)$ a core for B if D_0 is $\|\cdot\|_B$-dense in $D(B)$. This means that a closed operator is determined (via closure) by its restriction to a core in its domain.

Proposition 1.9 *For the generator A of a strongly continuous semigroup $(T(t))_{t \geq 0}$ the following holds:*

(i) The generator A is a closed operator.
(ii) If a subspace D_0 of the domain $D(A)$ is dense in E and $(T(t))$-invariant, then it is a core for A.
(iii) Define $D(A^n) := \{f \in D(A^{n-1}) : Af \in D(A^{n-1})\}$, $D(A^1) = D(A)$. Then $D(A^\infty) := \bigcap_{n \in \mathbb{N}} D(A^n)$ is dense in E and a core for A.

Example 1.10 Property (iii) above does not hold for general densely defined closed operators. Take $E = C[0,1]$, $D(B) = C^1[0,1]$ and $Bf = q \cdot f$ for some nowhere differentiable function $q \in C[0,1]$. Then B is closed, but $D(B^2) = (0)$.

Proposition 1.11 *For the generator A of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space E the following holds. If $\int_0^\infty e^{-\lambda t} T(t) f\, dt$ exists for every $f \in E$ and some $\lambda \in \mathbb{C}$, then $\lambda \in \rho(A)$ and $R(\lambda, A) f = \int_0^\infty e^{-\lambda t} T(t) f\, dt$. In particular,*

$$R(\lambda, A)^{n+1} f = \frac{(-1)^n}{n!} \left(\frac{d}{d\lambda}\right)^n R(\lambda, A) f = \int_0^\infty e^{-\lambda t} \frac{t^n}{n!} T(t) f\, dt \tag{1.6}$$

for $n \in \mathbb{N}$, $f \in E$ and all λ with $\operatorname{Re} \lambda > \omega$.

Remark 1.12 (i) For continuous Banach space valued functions such as $t \mapsto T(t)f$ we consider the Riemann integral and define $\int_0^\infty T(t) f\, dt$ as $\lim_{t \to \infty} \int_0^t T(s) f\, ds$. Sometimes such integrals for strongly continuous semigroups $(T(t))_{t \geq 0}$ are written as $\int_a^b T(t)\, dt$ and understood in the strong sense.

(ii) Since the generator $(A, D(A))$ determines the semigroup $(T(t))_{t>0}$ uniquely, we will speak occasionally of the growth bound of the generator instead of the semigroup, i.e., we write $\omega = \omega(A) = \omega((T(t))_{t\geq 0})$.

(iii) For one-parameter groups it might seem to be more natural to define the generator as the 'derivative' rather than just the 'right derivative' at $t = 0$. This yields the same operator as the following result shows: The strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator A can be extended to a strongly continuous one-parameter group $(U(t))_{t\in\mathbb{R}}$ if and only if $-A$ generates a semigroup $(S(t))_{t\geq 0}$. In that case $(U(t))_{t\in\mathbb{R}}$ is obtained as

$$U(t) := \begin{cases} T(t) & \text{for } t \geq 0 \\ S(-t) & \text{for } t \leq 0 \end{cases}$$

We refer to [Davies (1980), Prop. 1.14] for the details.

1.2 Standard Examples

In this section we list and discuss briefly the most basic examples of semigroups together with their generators. These semigroups will reappear throughout this book and will be used to illustrate the theory. We start with the class of semigroups mentioned after Definition 1.1.

1.2.1 Uniformly Continuous Semigroups

It follows from elementary operator theory that for every bounded operator $A \in L(E)$ the sum exists and determines a unique uniformly continuous (semi)group $(e^{tA})_{t\in\mathbb{R}}$ having A as its generator.

$$\sum_{n=0}^{m} \frac{t^n A^n}{n!} =: e^{tA}$$

Conversely, any uniformly continuous semigroup is of this form: If the semigroup $(T(t))_{t\geq 0}$ is uniformly continuous, then $\frac{1}{t}\int_0^t T(s)ds$ uniformly converges to $T(0) = Id$ as $t \longrightarrow 0$. Therefore for some $t' \neq 0$ the operator $\frac{1}{t}\int_0^t T(s)ds$ is invertible and every $f \in E$ is of the form $f = \frac{1}{t}\int_0^{t'} T(s)g ds$ for some $g \in E$. But these elements belong to $D(A)$ by (1.3), hence $D(A) = E$. Since the generator A is closed and everywhere defined it must be bounded. Remark that bounded operators are always generators of groups, not just semigroups. Moreover the growth bound ω satisfies $|\omega| \leq \|A\|$ in this situation.

The above characterization of the generators of uniformly continuous semigroups as the bounded operators shows that these semigroups are - at least in many aspects - rather simple objects.

1.2.2 Matrix Semigroups

The above considerations especially apply in the situation $E = \mathbb{C}^n$. If $n = 2$ and $A = (a_{ij})_{2\times 2}$ the following explicit formulas for e^{tA} might be of interest: Set $s :=$ trace A, $d :=$ det A and $D := (s^2 - 4d)^{1/2}$. Then

$$e^{tA} = \begin{cases} e^{ts/2} \cdot [D^{-1} 2\sinh(tD/2) \cdot A + (\cosh(tD/2) - sD^{-1}\sinh(tD/2)) \cdot Id] & \text{if } D \neq 0, \\ e^{ts/2} \cdot [tA + (1 - ts/2) \cdot Id] & \text{if } D = \emptyset \text{ resp.} \end{cases}$$

1.2.3 Multiplication Semigroups

Many Banach spaces appearing in applications are Banach spaces of (real or) complex valued functions over a set X. As the most standard of these "function spaces", we mention the space $C_0(X)$ of all continuous complex valued functions vanishing at infinity on a locally compact space X, or the space $L^p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, of all (equivalence classes of) p-integrable functions on a σ-finite measure space (X, Σ, μ).

On these function spaces $E = C_0(X)$, resp. $E = L^p(X, \Sigma, \mu)$, there is a simple way to define "multiplication operators": Take a continuous, resp. measurable function $q : X \to \mathbb{C}$ and define

$$M_q f := q \cdot f, \text{ i.e. } M_q f(x) := q(x) \cdot f(x) \text{ for } x \in X,$$

for every f in the "maximal" domain $D(M_q) := \{g \in E : q \cdot g \in E\}$.

For such multiplication operators many properties can be checked quite directly. For example, the following statements are equivalent:

(a) M_q is bounded.
(b) q is (p-essentially) bounded.

One has $\|M_q\| = \|q\|_\infty$ in this situation.

Observe that on spaces $C(K)$, K compact, there are no densely defined, unbounded multiplication operators.

By defining the multiplication semigroups

$$T(t)f(x) := \exp(t \cdot q(x))f(x), x \in X, f \in E,$$

one obtains the following characterizations.

Proposition 1.13 *Let M_q be a multiplication operator on $E = C_0(X)$ or $E = L^p(X, \Sigma, \mu)$, $1 \leq p < \infty$. Then the properties (a) and (b), resp. (a') and (b'), are equivalent:*

(a) *M_q generates a strongly continuous semigroup.*
(b) $\sup\{Re\ q(x) : x \in X\} < \infty$.

(a′) *M_q generates a uniformly continuous semigroup.*
(b′) $\sup\{|q(x)| : x \in X\} < \infty$.

As a consequence one computes the growth bound of a multiplication semigroup as follows:

$$\omega = \sup\{\text{Re } q(x) : x \in X\} \text{ in the case } E = C_0(X),$$

$$\omega = \text{ess-sup}\{\text{Re } q(x) : x \in X\} \text{ in the case } E = L^p(\mu).$$

It is a nice exercise to characterize those multiplication operators which generate strongly continuous groups.

1.2.4 Translation (Semi)Groups

Let E be one of the following function spaces $C_0(\mathbb{R}_+)$, $C_0(\mathbb{R})$ or $L^p(\mathbb{R}_+)$, $L^p(\mathbb{R})$ for $1 \leq p < \infty$. Define $T(t)$ to be the (left) translation operator

$$T(t)f(x) := f(x+t)$$

for $x, t \in \mathbb{R}_+$, resp. $x, t \in \mathbb{R}$ and $f \in E$. Then $(T(t))_{t \geq 0}$ is a strongly continuous semigroup, resp. group of contractions on E and its generator is the first derivative $\frac{d}{dx}$ with 'maximal' domain. In order to be more precise we have to distinguish the cases $E = C_0$ and $E = L^p$:

(i) The generator of the translation (semi)group on $E = C_0(\mathbb{R}_+)$ is

$$Af := \frac{d}{dx} f = f',$$

with domain

$$D(A) := \{f \in E : f \text{ differentiable and } f' \in E\}$$

Proof. For $f \in D(A)$ it follows that for every $x \in \mathbb{R}_{(+)}$

$$\lim_{h \to 0} \frac{T(h)f(x) - f(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

(uniformly in x) and coincides with $Af(x)$. Therefore f is differentiable and $f' \in E$. On the other hand, take $f \in E$ differentiable such that $f' \in E$. Then

$$\left|\frac{f(x+h)-f(x)}{h}-f'(x)\right| \leq \frac{1}{h}\int_x^{x+h} |f'(y)-f'(x)|dy$$

where the last expression tends to zero uniformly in x as $h \to 0$. Thus $f \in D(A)$ and $f' = Af$. □

(ii) The generator of the translation (semi)group on $E = L^p(\mathbb{R}_{(+)})$, $1 \leq p < \infty$, is

$$Af := \frac{d}{dx}f = f',$$

with domain

$$D(A) := \{f \in E : f \text{ absolutely continuous}, f' \in E\}$$

1.2.5 Rotation Groups

On $E = C(\mathbb{T})$, resp. $E = L^p(\mathbb{T}, m)$, $1 \leq p < \infty$, m Lebesgue measure we have canonical groups defined by rotations of the unit circle $\mathbb{T}$ with a certain period. For $0 < \tau \in \mathbb{R}$ the operators

$$R_\tau(t)f(z) := f(e^{2\pi it/\tau} \cdot z)$$

yield a group $(R_\tau(t))_{t\in\mathbb{R}}$ having period τ, i.e. $R_\tau(\tau) = Id$. As in Example 2.4 one shows that its generator has the form

$$D(A) = \{f \in E : f \text{ absolutely continuous}, f' \in E\},$$

$$Af(z) = (2\pi i/\tau) \cdot z \cdot f'(z).$$

An isomorphic copy of the group $(R_\tau(t))_{t\in\mathbb{R}}$ is obtained if we consider $E = \{f \in C[0,1] : f(0) = f(1)\}$, resp. $E = L^p([0,1])$ and the group of 'periodic translations'

$$T(t)f(x) := f(y) \text{ for } y \in [0,1], y = x + t \text{ mod } 1$$

with generator

$$D(A) := \{f \in E : f \text{ absolutely continuous}, f' \in E\},$$

$$Af := f'.$$

1.2.6 Nilpotent Translation Semigroups

Take $E = L^p([0,\tau], m)$ for $1 \leq p < \infty$ and define

$$T(t)f(x) := \begin{cases} f(x+t) & \text{if } x+t \le \tau \\ 0 & \text{otherwise} \end{cases}$$

Then $(T(t))_{t\ge 0}$ is a semigroup satisfying $T(t) = 0$ for $t \ge \tau$. Its generator is still the first derivative $A = \frac{d}{dx}$, but its domain is

$$D(A) = \{f \in L^p([0,\tau]) : f \text{ absolutely continuous}, f' \in L^p([0,\tau]), f(\tau) = 0\}.$$

In fact, if $f \in D(A)$ then f is absolutely continuous with $f' \in E$. By Prop.1.6.i it follows that $T(t)f$ is absolutely continuous and hence $f(\tau) = 0$.

1.2.7 One-dimensional Diffusion Semigroup

For the second derivative

$$Bf(x) := \frac{d^2}{dx^2} f(x) = f''(x)$$

we take the domain

$$D(B) := \{f \in C^2[0,1] : f'(0) = f'(1) = 0\}$$

in the Banach space $E = C[0,1]$. Then $D(B)$ is dense in $C[0,1]$, but closed for the graph norm. Obviously, each function

$$e_n(x) := \cos \pi n x, \quad n \in \mathbb{Z},$$

is contained in $D(B)$ and an eigenfunction of B pertaining to the eigenvalue $\lambda_n := -\pi^2 n^2$. The linear hull

$$\operatorname{span}\{e_n : n \in \mathbb{Z}\} =: E_0$$

forms a subalgebra of $D(B)$ which by the Stone-Weierstrass theorem is dense in E.

We now use e_n to define bounded linear operators

$$e_n \otimes e_n : f \to \left(\int_0^1 f(x) e_n(x)\, dx \right) e_n = \langle f, e_n \rangle e_n$$

satisfying $\|e_n \otimes e_n\| \le 1$ and $(e_n \otimes e_n)(e_m \otimes e_m) = \delta_{n,m}(e_n \otimes e_n)$ for $n \in \mathbb{Z}$.

For $t > 0$ we define

$$\begin{aligned} T(t) &:= \sum_{n \in \mathbb{Z}} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n \\ &= e_0 \otimes e_0 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n \end{aligned}$$

or

$$T(t)f(x) = \int_0^1 k_t(x,y)f(y)dy$$
$$\text{where } k_t(x,y) = 1 + 2\sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t)\cos \pi nx \cos \pi ny.$$

Die Jacobi-Identität

$$\begin{aligned} w_t(x) &:= 1/(4\pi t)^{\frac{1}{2}} \sum_{m\in\mathbb{Z}} \exp(-(x+2m)^2/4t) \\ &= \frac{1}{2} + \sum_{n\in\mathbb{N}} \exp(-\pi^2 n^2 t)\cos \pi nx \end{aligned}$$

und trigonometrische Beziehungen zeigen, dass

$$k_t(x,y) = w_t(x+y) + w_t(x-y)$$

welches eine positive Funktion auf $[0,1]^2$ ist. Daher ist $T(t)$ ein beschränkter Operator auf $C[0,1]$ mit

$$\|T(t)\| = \|T(t)1\| = \sup_{x\in[0,1]} \int_0^1 k_t(x,y)dy = 1.$$

1.2.8 n-dimensional Diffusion Semigroup

On $E = L^p(\mathbb{R}^n)$, $1 \le p < \infty$, the operators

$$\begin{aligned} T(t)f(x) &:= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|x-y|^2/4t)f(y)dy \\ &:= \psi_t * f(x) \end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$ and $\psi_t(x) := (4\pi t)^{-n/2}\exp(-|x|^2/4t)$ form a strongly continuous semigroup.

In fact the integral exists for every $f \in L^p(\mathbb{R}^n)$, since ψ_t is an element of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of all rapidly decreasing smooth functions on $\mathbb{R}^n$.

Moreover,

$$\|T(t)f\|_p \le \|\psi_t\|_1 \|f\|_p = \|f\|_p$$

by Young's inequality [Reed-Simon (1975), p.28], hence $\|T(t)\| \le 1$ for every $t > 0$.

Next we observe that $\mathcal{S}(\mathbb{R}^n)$ is dense in E and invariant under each $T(t)$. Therefore we can apply the Fourier transformation $\mathcal{F}$ which leaves $\mathcal{S}(\mathbb{R}^n)$ invariant and yields

$$\mathcal{F}(\psi_t * f) = (2\pi)^{n/2}\mathcal{F}(\psi_t)\cdot\mathcal{F}(f) = (2\pi)^{n/2}\hat{\psi}_t \cdot \hat{f}$$

where $f \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f} = \mathcal{F} f \in \mathcal{S}(\mathbb{R}^n)$.

In other words, $\mathcal{F}$ transforms $(T(t)|_{\mathcal{S}(\mathbb{R}^n)})_{t\geq 0}$ into a multiplication semigroup on $\mathcal{S}(\mathbb{R}^n)$ which is pointwise continuous for the usual topology of $\mathcal{S}(\mathbb{R}^n)$. The generator, i.e. the right derivative at 0, of this semigroup is the multiplication operator

$$B\hat{f}(x) := -|x|^2 \hat{f}(x)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Applying the inverse Fourier transformation and observing that the topology of $\mathcal{S}(\mathbb{R}^n)$ is finer than the topology induced from $L^p(\mathbb{R}^n)$, we obtain that $(T(t))_{t\geq 0}$ is a semigroup which is strongly continuous (use Remark 1.2, (3)) and its generator A coincides with

$$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x_1, \ldots, x_n)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Since $\mathcal{S}(\mathbb{R}^n)$ is $(T(t))$-invariant we have determined the generator on a core of its domain (see Prop.1.9.ii).

In particular the above semigroup 'solves' the initial value problem for the 'heat equation'

$$\frac{\partial}{\partial t} f(x,t) = \Delta f(x,t), \quad f(x,0) = f_0(x), \quad x \in \mathbb{R}^n.$$

For the analogous discussion of the unitary group on $L^2(\mathbb{R}^n)$ generated by

$$C := i\Delta$$

we refer to Section IX.7 in Reed-Simon (1975).

1.3 Standard Constructions

Starting with a semigroup $(T(t))_{t\geq 0}$ on a Banach space E it is possible to construct new semigroups on spaces naturally associated with E. Such constructions will be important technical devices in many of the subsequent proofs. Although most of these constructions are rather routine, we present in the sequel a systematic account of them for the convenience of the reader.

We always start with a semigroup $(T(t))_{t\geq 0}$ on a Banach space E, and denote its generator by A on the domain $D(A)$.

1.3.1 Similar Semigroups

There is an easy way how to obtain different (but isomorphic) semigroups: Take any isomorphism $S \in L(E, F)$ between two Banach spaces E and F. Then

$$U(t) := ST(t)S^{-1}, \quad t \geq 0,$$

defines a semigroup on F which is strongly continuous if and only if $(T(t))_{t\geq 0}$ has this property. In this case the generator B of $(U(t))_{t\geq 0}$ is given by

$$B = SAS^{-1} \text{ with } D(B) = SD(A).$$

Part B
Positive Semigroups on Spaces $C_0(X)$

Part C
Positive Semigroups on Banach Lattices

Part D
Positive Semigroups on C^*- and W^*-Algebras

Chapter 1
Basic Results on Semigroups and Operator Algebras

This is not a systematic introduction into the theory of strongly continuous semigroups on C^* and W^*-algebras. For that we refer to Bratteli-Robinson (1979), Davies (1976) and the survey article of Oseledets (1984). We only prepare for the subsequent chapters on spectral theory and asymptotics by fixing the notations and introducing some standard constructions.

1.1 Notations

1. By M we shall denote a C^*-algebra with unit 1. $M^{\mathrm{sa}} := \{x \in M : x^* = x\}$ is the selfadjoint part of M and $M_+ := \{x^*x : x \in M\}$ the positive cone in M. If M' is the dual of M, then $M'_+ = \{\psi \in M' : \psi(x) \geq 0, x \in M_+\}$ is a weak*-closed generating cone in M'. $S(M) := \{\psi \in M'_+ : \psi(1) = 1\}$ is called the state space of M. For the theory of C^*-algebras and related notions we refer to [Pedersen (1979)]. M is called a W^*-algebra, if there exists a Banach space M_*, such that its dual $(M_*)'$ is (isomorphic to) M. We call M_* the predual of M and $\psi \in M_*$ a normal linear functional. It is known that M_* is unique [Sakai (1971), 1.13.3.]. For further properties of M_* we refer to [Takesaki (1979), Chapter III].

2. A map $T \in L(M)$ is called positive (in symbols $T \geq 0$) if $T(M_+) \leq M_+$. $T \in L(M)$ is called n-positive ($n \in \mathbb{N}$) if $T \otimes id_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where id_n is the identity map on the C^*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive. We call $T \in L(M)$ a Schwarz map if T satisfies the inequality

$$T(x)T(x)^* \leq T(xx^*), x \in M$$

Note that such T is necessarily a contraction. It is well known that every n-positive contraction, $n \geq 2$ and that every positive contraction on a commutative C^*-algebra is a Schwarz map [Takesaki (1979), Corollary IV.3.8.]. As we shall see, the Schwarz inequality is crucial for our investigations.

3. If M is a C^*-algebra we assume $T = (T(t))_{t\geq 0}$ to be a strongly continuous semigroup (abbreviated semigroup) while on W^*-algebras we consider weak*-semigroups, i.e. the mapping $(t \to T(t)x)$ is continuous from $\mathbb{R}_+$ into $(M, \sigma(M, M_*))$, M_* the predual of M, and every $T(t) \in T$ is $\sigma(M, M_*)$-continuous. Note that the preadjoint semigroup

$$T_* = \{T(t)_* : T(t) \in T\}$$

is weakly, hence strongly continuous on M_* (see e.g., Davies (1980), Prop.1.23). We call T identity preserving if $T(t)1 = 1$ and of Schwarz type if every $T(t) \in T$ is a Schwarz map. For the notations concerning one-parameter semigroups we refer to part A. In addition we recommend to compare the results of this section of the book with the corresponding results for commutative C^*-algebras, i.e. for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ (see Part B).

1.2 A Fundamental Inequality for the Resolvent

If $T = (T(t))_{t\geq 0}$ is a strongly continuous semigroup of Schwarz maps on a C^*-algebra M (resp. a weak*-semigroup of Schwarz type on a W^*-algebra M) with generator A, then the spectral bound $s(A) \leq 0$. Then for $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$, there exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x dt, \quad x \in M$$

where the integral exists in the norm topology.

In [Bratteli-Robinson (1979)] it is shown that T is a semigroup of Schwarz type if and only if $R(\mu, A)$ is a Schwarz map for every $\mu \in \mathbb{R}_+$. Here we relate the domination of two semigroups to an inequality for the corresponding resolvent operator. This inequality will be needed later.

Theorem 1.1 *Let $T = (T(t))_{t\geq 0}$ be a semigroup of Schwarz type and $S = (S(t))_{t\geq 0}$ a semigroup on a C^*-algebra M with generators A and B, respectively. If*

$$(S(t)x)(S(t)x)^* \leq T(t)xx^*$$

for all $x \in M$ and $t \in \mathbb{R}_+$, then

$$(\mu R(\mu, B)x)(\mu R(\mu, B)x)^* \leq \mu R(\mu, A)xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$. The same result holds if T is a weak-semigroup of Schwarz type and S is a weak*-semigroup on a W^*-algebra M such that (*) is fulfilled.*

Proof. From the assumption (*) it follows that

$$\begin{aligned}0 &\le (S(r)x - S(t)x)(S(r)x - S(t)x)^* \\ &= (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \\ &\le T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^*\end{aligned}$$

for every $r, t \in \mathbb{R}_+$. Hence

$$(S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^* \le T(r)xx^* + T(t)xx^*$$

Obviously, $\|S(t)\| \le 1$ for all $t \in \mathbb{R}_+$. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$:

$$\begin{aligned}&(R(\mu, B)x)(R(\mu, B)x)^* \\ &= \left(\int_0^\infty e^{-\mu r} S(r)x dr\right)\left(\int_0^\infty e^{-\mu t} S(t)x dt\right)^* \\ &= \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}((S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^*) dr dt\right) \\ &\le \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*) dr dt\right) \\ &= \left(\int_0^\infty e^{-\mu s} ds\right)\left(\int_0^\infty e^{-\mu t} T(t)xx^* dt\right) = \mu^{-1} R(\mu, A)xx^*\end{aligned}$$

where the handling of the integral is justified by [Bourbaki (1955), §8, $n°$ 4, Proposition 9]. □

Corollary 1.2 *Let T be a semigroup of Schwarz maps (resp., weak*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with* $\mathrm{Re}(\lambda) > 0$*:*

$$(R(\lambda, A)x)(R(\lambda, A)x)^* \le (\mathrm{Re}\,\lambda)^{-1} R(\mathrm{Re}\,\lambda, A)xx^*, \quad x \in M$$

In particular for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}, x \in M$:

$$(\mu R(\mu + i\alpha, A)x)(\mu R(\mu + i\alpha, A)x)^* \le \mu R(\mu, A)(xx^*)$$

Proof. Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Then the semigroup

$$S := (e^{-i\,\mathrm{Im}(\lambda)t} T(t))_{t \ge 0}$$

fulfils the assumption of Thm 2.1. and $B := A - i\lambda$ is the generator of S. Consequently $R(\lambda, A) = R(\mathrm{Re}\,\lambda, B)$ and the corollary follows from Theorem 2.1. □

As in section C-III the following notion will be an important tool for the spectral theory of semigroups.

Definition 1.3 Let E be a Banach space and $\emptyset \ne D$ an open subset of $\mathbb{C}$. A family $R : D \to L(E)$ is called a pseudo-resolvent on D with values in E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu)$$

for all λ and μ in D. If R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in a C^*- or W^*-algebra, then R is called of Schwarz type if

$$(R(\lambda)x)(R(\lambda)x)^* \leq (\mathrm{Re}\,\lambda)^{-1} R(\mathrm{Re}\,\lambda)xx^*$$

for all $\lambda \in D$ and $x \in M$. R is called identity preserving if $\lambda R(\lambda)1 = 1$ for all $\lambda \in D$.

For examples and properties of a pseudo-resolvent see C-III, 2.5. We state what will be used without further reference.

1. If $a \in \mathbb{C}$ and $x \in E$ such that $(a - \lambda)R(\lambda)x = x$ for some $\lambda \in D$, then $(a - \mu)R(\mu)x = x$ for all $\mu \in D$ (use the "resolvent equation").
2. If F is a closed subspace of E such that $R(\lambda)F \subseteq F$ for some $\lambda \in D$, then $R(\mu)F \subseteq F$ for all μ in a neighbourhood of λ. This follows from the fact that for all $\mu \in D$ near λ the pseudo-resolvent in μ is given by

$$R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}$$

Definition 1.4 We call a semigroup T on the predual M_* of a W^*-algebra M identity preserving and of Schwarz type, if its adjoint weak*-semigroup has these properties. Likewise, a pseudo-resolvent R on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in M_* is called identity preserving and of Schwarz type, if R' has these properties.