Take the translation semigroup on the Banach space

$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^{x^2}dx)$

with $\|f\| := \sup\{ |f(x)| : x \in \mathbb{R}_+ \} + \int_0^\infty |f(x)|e^{x^2}\,dx$

(see Greiner-Voigt-Wolff (1981)).

(v) Another modification of Example 1.3 yields a group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ satisfying $s(A) < \omega$. Therefore the spectral mapping theorem does not hold (see Wolff (1981)).

The next few theorems form the core of this chapter. We show that only one part of the spectrum and one inclusion is responsible for the failure of the spectral mapping theorem. The usefulness of this detailed analysis will become clear in the subsequent chapter on stability and asymptotics.

6.2. Spectral Inclusion Theorem. Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on some Banach space E. Then

$$\exp(t\sigma(A)) \subset \sigma(T(t)) \quad \text{for} \quad t \geq 0 .$$

More precisely we have the following inclusions:

(6.1) $\exp(t\cdot P\sigma(A)) \subset P\sigma(T(t))$,

(6.2) $\exp(t\cdot A\sigma(A)) \subset A\sigma(T(t))$,

(6.3) $\exp(t\cdot R\sigma(A)) \subset R\sigma(T(t))$.

Proof. Since $e^{\lambda t} - T(t) = (\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)\,ds$ (see A-I,(3.1)) it follows that $(e^{\lambda t} - T(t))$ is not bijective if $(\lambda - A)$ fails to be bijective, which proves the main inclusion.

The inclusion (6.1) becomes evident from the following proof of (6.2): Take $\lambda \in A\sigma(A)$ and an associated approximate eigenvector $(f_n) \subset D(A)$. Then

$$g_n := e^{\lambda t}f_n - T(t)f_n = \int_0^t e^{\lambda(t-s)}T(s)(\lambda-A)f_n\,ds$$

converges to zero as $n \to \infty$. Consequently, $e^{\lambda t} \in A\sigma(T(t))$ and in fact, the same approximate eigenvector (f_n) does the job for all $t \geq 0$.

For the proof of (6.3) we take $\lambda \in R\sigma(A)$ and observe that

$(e^{\lambda t} - T(t))f = (\lambda - A)(\int_0^t e^{\lambda(t-s)}T(s)f\,ds) \in (\lambda - A)D(A)$

for every $f \in E$.

□