Then $x_q \in D(A)$ and

$$\|T(t)x_q\|^2 = \sum_{n=1}^{\infty} n^2 q^{2n} \|\exp(tA_n)e_n\|^2 \geq$$
$$\geq \sum_{n=1}^{\infty} n^2 q^{2n}\left(\tfrac{1}{n^2}\cdot\sum_{i=o}^{n-1}\tfrac{1}{i!}(2t)^i\right)$$
$$= \sum_{i=0}^{\infty}\sum_{n=i+1}^{\infty}\left(q^{2n}\cdot\tfrac{1}{i!}(2t)^i\right)$$
$$= \sum_{i=0}^{\infty} q^{2i+2}\cdot(1-q^2)^{-1}\cdot\tfrac{1}{i!}(2t)^i =$$
$$= \frac{q^2}{1-q^2}\cdot\sum_{i=0}^{\infty}\frac{1}{i!}(2tq^2)^i = \frac{q^2}{1-q^2}\cdot e^{2tq^2}\ .$$

It follows that $\omega(x_q) \geq q^2$ . Thus

$$1 = \sup\{\omega(x_q) : 0 < q < 1\} \leq \omega_1(A) \leq \omega(A) = 1\ .$$

Rescaling the semigroup (i.e. looking at $e^{-3/2\cdot t}\,T(t)$) we obtain a semigroup generator $A$ on the Hilbert space $E$ with $-3/2 = s(A)$ and $\omega_1(A) = \omega(A) = -1/2$ . On the other hand, Example 1.2.(2) yields a semigroup on a Banach space $F$ with generator $B$ such that $-1 = s(B) = \omega_1(B)$ while $\omega(B) = 0$ . Now the operator $C := A \oplus B$ on the Banach space $E \oplus F$ is a semigroup generator for which $s(C) = \max\{s(A),s(B)\} = -1$ , $\omega_1(C) = \max\{\omega_1(A),\omega_1(B)\} = -1/2$ and $\omega(C) = \max\{\omega(A),\omega(B)\} = 0$ .

(1.7) Important remark: For eventually norm continuous semigroups, in particular for compact, differentiable, holomorphic or nilpotent semigroups the spectral mapping theorem $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$ holds, and therefore

$$s(A) = \omega_1(A) = \omega(A) \tag{1.8}$$

is valid (Cor.1.5 and A-III,Cor.6.7).
Hence, if $A$ is the generator of an eventually norm continuous semigroup, then the exponential growth bounds of the strong and the mild solutions of $\dot{u}(t) = Au(t)$ are determined by the spectral bound $s(A) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ .

In general, the growth bound $\omega(A)$ can be obtained through the Hille-Yosida theorem (see A-II,Thm.1.7) as

$$\omega(A) = \inf\{w : \|R(\lambda,A)^n\| \leq M\cdot(\mathrm{Re}\,\lambda - w)^{-n} \text{ for some } M \text{ and every } n \in \mathbb{N} \text{ and } \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\,\lambda > w\}\ . \tag{1.9}$$

In view of the difficulties in estimating all powers of the resolvent this equation is of little practical interest. If $A$ is the generator