$r(R(\lambda,A)) = \lim\|R(\lambda,A)^n\|^{1/n} \geq \varepsilon > 0$.
The assertion now follows from A-III,Prop.2.5 .

(b) Suppose $Ah = rh$ where $h \neq 0$ is positive. Then r has to be real and we have $T(t)h = e^{rt}h$ (A-III,Cor.6.4). For $|f| \leq n\cdot h$ $(n \in \mathbb{N})$ we have

$$(3.2) \quad |T(t)f| \leq T(t)|f| \leq n\cdot T(t)h = n\cdot e^{rt}h .$$

This shows that the ideal generated by h is invariant, hence dense by irreducibility. This is true if and only if h is strictly positive.

(c) Suppose $A'\phi = r\phi$ for some $0 < \phi \in E'$. Again r has to be real and $T(t)'\phi = e^{rt}\phi$ $(t \geq 0)$. From

$$(3.3) \quad \langle|T(t)f|,\phi\rangle \leq \langle T(t)|f|,\phi\rangle = \langle|f|,e^{rt}\phi\rangle , \quad f \in E$$

it follows that $I := \{f \in E : \phi(|f|) = 0\}$ is an invariant ideal. We have $I \neq E$ (because $\phi \neq 0$), hence the irreducibility implies $I = \{0\}$, i.e., ϕ is strictly positive.

(d) By (a) we know that $s(A) > -\infty$ hence we can assume without loss of generality that $s(A) = 0$. By (c) there exists a strictly positive $\Psi \in E'$ such that $A'\Psi = 0$. It follows from (2.14) and (2.15) that

$$(3.4) \quad h \in \ker A \text{ implies } |h| \in \ker A .$$

Assuming $\dim(\ker A) \geq 2$, then there is an eigenfunction $h \in \ker A$, $h \neq 0$ which has at least one zero in X $(h := h_1(x_o)\cdot h_2 - h_2(x_o)\cdot h_1$, where h_1 , h_2 are linearly independent, $x_o \in X$). By (3.4) $|h|$ is a positive eigenfunction but not strictly positive. This is a contradiction with (b).

(e) If $s(A)$ is a pole, then there exists a corresponding positive eigenfunction (see the proof of Cor.1.4). By (b) it is even strictly positive, thus $s(A)$ is a first order pole by Rem.2.15(a) . The residue P is a positive operator satisfying $PE = \ker(s(A) - A)$ and $P'E' = \ker(s(A) - A')$, therefore the remaining assertion follows from (e), (b) and (d).

□

In the remainder of this section we focus our interest on the boundary spectrum of irreducible semigroups, more precisely, on the eigenvalues and the corresponding eigenfunctions of the boundary spectrum. In view of assertion (a) of Prop.3.5 the assumption "$s(A) = 0$" is not crucial in the following theorem. However, it allows a simpler formulation.