A detailed analysis of $\rho_F(T)$ can be found in Section IV.5.6 of Kato (1966). In particular we recall that the poles of $R(\cdot,T)$ with finite algebraic multiplicity belong to $\rho_F(T)$. Conversely, an element of the unbounded component of $\rho_F(T)$ either belongs to $\rho(T)$ or is a pole of finite algebraic multiplicity. Thus $r_{ess}(T)$ can be characterized as follows

(3.8) $r_{ess}(T)$ is the smallest $r \in \mathbb{R}_+$ such that every $\lambda \in \sigma(T)$, $|\lambda| > r$ is a pole of finite algebraic multiplicity.

Now, if $\mathcal{T} = (T(t))_{t\geq 0}$ is a strongly continuous semigroup then VIII.1, Lemma 4 of Dunford-Schwartz (1958) applied to the function $t \to \log \|T(t)\|_{ess}$ ensures that

(3.9) $\omega_{ess}(\mathcal{T}) := \lim_{t\to\infty} \frac{1}{t} \log\|T(t)\|_{ess} = \inf \{\frac{1}{t} \log\|T(t)\|_{ess} : t > 0\}$

is well defined (possibly $-\infty$). By the definition of $\omega_{ess}(\mathcal{T})$ and by (3.6) we have

(3.10) $r_{ess}(T(t)) = \exp(t\cdot\omega_{ess}(\mathcal{T}))$, $t \geq 0$.

Obviously, $\omega_{ess} \leq \omega$ and equality occurs if and only if $r_{ess}(T(t)) = r(T(t))$ for $t \geq 0$.
If $\omega_{ess} < \omega$ there exists an eigenvalue λ of $T(t)$ satisfying $|\lambda| = r(T(t))$, hence by Theorem 6.3 below there exists $\lambda_1 \in P\sigma(A)$ such that $\operatorname{Re}\lambda_1 = \omega$. Thus $\omega_{ess} < \omega$ implies $s(A) = \omega(\mathcal{T})$, i.e., we have

(3.11) $$\omega(\mathcal{T}) = \max\{\omega_{ess}(\mathcal{T}), s(A)\} .$$

As a final observation we point out that

(3.12) $$\omega_{ess}(\mathcal{T}) = \omega_{ess}(\mathcal{S})$$

whenever $\mathcal{T}$ is generated by A and $\mathcal{S}$ is generated by $A + K$ for some compact operator K (see Prop.2.8 and Prop.2.9 of B-IV).

4. THE SPECTRUM OF INDUCED SEMIGROUPS

In the previous section we tried to decompose a semigroup into the direct sum of two, hopefully simpler objects. Here we present other methods to reduce the complexity of a semigroup and its generator. Forming subspace or quotient semigroups as in A-I,3.2, A-I,3.3 are such methods. But also the constructions of new semigroups on canonically associated spaces such as the dual space, see A-I,3.4, or the F-product, see A-I,3.6, might be helpful. We review these construc-