Chapter 1

Asymptotics of Positive Semigroups on C^*- and W^*-Algebras

1.1 Stability of Positive Semigroups

As explained in A-III, Section 1, it is possible to deduce uniform exponential stability of strongly continuous semigroups from the location of the spectrum of its generator if the spectral bound s(A) and the growth bound ω coincide. In this section we prove "s(A) = ω" for positive semigroups on C*-algebras and preduals of W*-algebras. A more general discussion of the "s(A) = ω" problem can be found in [Greiner-Voigt-Wolff (1981)]. For the results of this section the existence of a unit is essential.

Theorem 1.1. *Let M be a* C^**-algebra with unit and $T = (T(t))_{t\geq 0}$ a positive semigroup on M. Then*

$$-\infty < s(A) = \omega \in \sigma(A)$$

Proof. For every $t \geq 0$ there exists ϕ_t in the state space $S(M)$ of M such that

$$T(t)'\phi_t = r(T(t))\phi_t = \exp(\omega t)\phi_t$$

(see, e.g., [Groh (1981), 2.1]). Let $n \in \mathbb{N}$ and

$$E_n := \{\phi \in S(M) : T(2^{-n})\phi = \exp(\omega 2^{-n})\phi\}$$

Then $\emptyset \neq E_{n+1} \subseteq E_n$ ($n \in \mathbb{N}$). Since $S(M)$ is $\sigma(M, M')$-compact there exists $\phi \in \bigcap_{n\in\mathbb{N}} E_n$ and $T(t)'\phi = \exp(\omega t)\phi$ follows because the adjoint semigroup $(T(t)')_{t\geq 0}$ is a weak*-semigroup on M'. Suppose $-\infty = \omega$. Then for $t > 0$, $r(T(t)) = 0$ (A-III,Prop.1.1) or $T(t)'\phi = 0$, in particular $\phi(T(t)1) = 0$. From this we obtain the contradiction $\phi(1) = 0$. Hence $-\infty < \omega$ and $\exp(\omega t) \in \rho(T(t)')$ for every $t \in \mathbb{R}_+$. Thus $\omega \in \sigma(A)$ or $\omega = s(A)$. $\square$

Remark 1.2. (a) If we consider the nilpotent translation semigroup on the C^*-algebra $C_0([0,1))$ then $\sigma(A) = \emptyset$ and $\omega = -\infty$. This shows that the existence of a unit is essential.

(b) "s(A) = ω" still holds for positive semigroups on commutative C*-algebras without unit (see B-IV, Rem.1.2.b).

Theorem 1.3. *Let M be a W^*-algebra with predual M_* and let $(T(t))_{t\geq 0}$ be a positive semigroup on M_*. Then $s(A) = \omega_0$.*

Proof. For all $\lambda > s(A)$ and $\phi \in M_*$

$$R(\lambda, A)\phi = \int_0^\infty e^{-\lambda t} T(s)\phi \, ds$$

which follows as in C-III, Section 1 or [Greiner-Voigt-Wolff (1981), Theorem 3]. Since $\|\phi\| = \phi(1)$ for every $\phi \in M_*^+$ and since the norm is additive on the positive cone of M_* the integral

$$\int_0^\infty e^{\lambda t} \|T(s)\phi\| \, ds$$

exists for all $\phi \in M_*$ and all $\lambda > s(A)$. From this the assumption follows by A-IV,Thm.1.11. □

Corollary 1.4. *Let M be a C^*-algebra and $(T(t))_{t\geq 0}$ a positive semigroup on M'. Then $s(A) = \omega_0$ holds.*

This follows from the fact that the bidual of a C^*-algebra is a W^*-algebra (see [Takesaki (1979), Theorem III.2.4.]).

Remark 1.5. A simple modification of A-III, Example 1.4 (take c_0 instead of ℓ^2) shows that Theorem 1.3 is no longer true for non-positive semigroups (for details see [Groh-Neubrander (1981), Beispiel 2.5]).

While the growth bound ω characterizes uniform exponential stability of the semigroup there are other (and weaker) stability concepts (cf. A-IV, Section 1).

Hier ist die LaTeX-Konvertierung mit allen mathematischen Symbolen:

Definition 1.6. Let E be a Banach space and $(T(t))_{t\geq 0}$ a semigroup on E. We call the semigroup

(1) uniformly exponentially stable, if $\|T(t)\| \leq Me^{-\omega t}$ for some ω, $M > 0$ and all $t \geq 0$.

(2) uniformly stable, if $\lim_{t\to\infty} T(t) = 0$ in the strong operator topology.

(3) weakly stable, if $\lim_{t\to\infty} T(t) = 0$ in the weak operator topology.

Surprisingly all these properties coincide for positive semigroups on C*-algebras with unit.

Theorem 1.7. *Let M be a C*-algebra with unit and $(T(t))_{t\geq 0}$ a positive semigroup on M. Then the following assertions are equivalent:*

(1) $s(A) < 0$.

(2) *The semigroup* $(T(t))_{t\geq 0}$ *is uniformly exponentially stable.*

(3) *The semigroup* $(T(t))_{t\geq 0}$ *is uniformly stable.*

(4) *The semigroup* $(T(t))_{t\geq 0}$ *is weakly stable.*

Proof. Since "$s(A) = \omega_0$" by Theorem 1.3, it suffices to show that 4. implies 1. For $t > 0$ there exists $\phi \in S(M)$ such that

$$T(t)'\phi = r(T(t))\phi$$

Then for $x \in M$

$$\phi(T(t)^n x) = (r(T(t)))^n \phi(x) \to 0$$

as $n \to \infty$. Therefore $r(T(t)) < 1$ or $\omega < 0$. Since $s(A) \leq \omega$ the assertion follows. $\square$