Therefore $U_{-(\ell+1)} \circ U_{-(m+1)} = 0$ . The relations (3.2) imply $U_{-(m+l+1)} = 0$ , hence the pole order of $R(.,A)$ is dominated by $\ell + m$ .

□

4.4. Spectrum of the adjoint semigroup.

We recall from A-I,3.4 that to every strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ there corresponds a strongly continuous adjoint semigroup $\mathcal{T}^* = (T(t)^*)_{t \geq 0}$ on the semigroup dual

$$E^* = \{\phi \in E' : \lim_{t \to \infty} \|T(t)'\phi - \phi\| = 0\} .$$

Its generator $A^*$ is the maximal restriction of the adjoint $A'$ to $E^*$ . For these operators the spectra coincide, or more precisely

(i) $\sigma(T(t)) = \sigma(T(t)') = \sigma(T(t)^*)$ ,

$R\sigma(T(t)) = P\sigma(T(t)') = P\sigma(T(t)^*)$ .

(ii) $\sigma(A) = \sigma(A') = \sigma(A^*)$ , $R\sigma(A) = P\sigma(A') = P\sigma(A^*)$ .

(iii) $s(A) = s(A^*)$ , $\omega(A) = \omega(A^*)$ .

The left part of these equalities is either well known or has been stated in Prop.2.2(ii). The first statment of (iii) follows from (ii), while the second is an immediate consequence of the estimate $\|T(t)^*\| \leq \|T(t)\| \leq M \cdot \|T(t)^*\|$ given in A-I,3.4. As a sample for the remaining assertions we show that $0 \notin \sigma(A)$ if and only if $0 \notin \sigma(A^*)$: If $A$ and therefore $A'$ is invertible it follows from A-I,3.4 that $A^*$ is a bijection from $D(A^*)$ onto $E^*$ . Conversely assume that $A^*$ is invertible. Then $A'$ must be injective by the Proposition in A-I,3.4. Moreover $A'(D(A'))$ contains $A^*(D(A^*)) = E^*$ and is $\sigma(E',E)$-dense in $E'$ . By standard duality arguments follows that $A$ is injective with dense image. We show that $A(D(A))$ is closed: For $f \in D(A)$ choose $\phi \in D(A')$ such that $\|\phi\| \leq 1$ and $|\langle f,\phi\rangle| \geq \frac{1}{2}\|f\|$ . Then

$$\|(A^*)^{-1}\| \, \|Af\| \geq \|(A^*)^{-1}\| \, |\langle Af,\phi\rangle| \geq |\langle Af,(A^*)^{-1}\phi\rangle|$$
$$= |\langle f,\phi\rangle| \geq \frac{1}{2}\|f\| ,$$

hence

$$\|Af\| \geq \frac{1}{2}\|(A^*)^{-1}\|^{-1}\|f\| ,$$

and $A(D(A))$ is closed since $A$ is closed.

□

4.5 Spectrum of the $\mathcal{F}$-product semigroup.

As stated in A-I,3.6 the $\mathcal{F}$-product semigroup $\mathcal{T}_{\mathcal{F}} = (T_{\mathcal{F}}(t))_{t \geq 0}$ on $E_{\mathcal{F}}^{\mathcal{T}}$ of a strongly continuous semigroup $\mathcal{T}$ on $E$ serves to convert sequences in $E$ into points in $E_{\mathcal{F}}^{\mathcal{T}}$ . In particular it can be used to