In addition, we saw in Prop.1.1 that the validity of such a spectral mapping theorem implies

$$s(A) = \omega(A)$$

for the spectral- and growth bounds and therefore guarantees that the location of the spectrum of A determines the asymptotic behavior of $\mathcal{T}$. As we have seen in Examples 1.3 and 1.4 the last statement does not hold in general. We therefore present a detailed analysis, where and why it fails and what additional assumptions are needed for its validity. Before doing so we have another look at the examples.

6.1 <u>The counterexamples revisited</u>.

(i) Take the nilpotent translation semigroup from A-I,2.6. Then $\sigma(A) = \emptyset$ and $\sigma(T(t)) = 0$ for every $t > 0$. By this trivial example and since $e^z \neq 0$ for every $z \in \mathbb{C}$, it is natural to read the 'spectral mapping theorem' modulo the addition of $\{0\}$, i.e.

$$\sigma(T(t)) \cup \{0\} = \exp(t\sigma(A)) \cup \{0\} \quad \text{for} \quad t \geq 0 .$$

(ii) The spectrum of the generator A of the τ-periodic rotation group $(R_\tau(t))_{t\geq 0}$ on $C(\Gamma)$ is $\sigma(A) = 2\pi i/\tau \cdot \mathbb{Z}$ and $\exp(2\pi int/\tau)$, $n \in \mathbb{Z}$, is an eigenvalue of $R_\tau(t)$ for every $t \geq 0$ (see Example 5.6). If t/τ is irrational these eigenvalues form a dense subset of Γ. Since the spectrum is closed we obtain $\sigma(T(t)) = \Gamma$ for these t. Therefore in this example the spectral mapping theorem is valid only in the following 'weak' form:

$$\sigma(T(t)) = \overline{\exp(t\sigma(A))} , \quad t \geq 0 .$$

(iii) By Example 1.3 there exists a semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ with generator A such that $s(A) = -1$ and $\omega(\mathcal{T}) = 0$. This implies that for preassigned real numbers $\alpha < \beta$ there exists a semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ with generator B such that $s(B) = \alpha$ and $\omega(\mathcal{S}) = \beta$: Take $S(t) := e^{\beta t}T((\beta - \alpha)t)$ and observe that $B = (\beta-\alpha)A + \beta\mathrm{Id}$. In that case $\exp(t\sigma(B))$ is contained in the circle about 0 with radius $e^{\alpha t}$ by Lemma 1.1; hence there must be points in $\sigma(S(t))$ which are not in the closure of $\exp(t\sigma(B))$.

(iv) The Example 1.3 can be strengthend in order to yield a semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ with generator A such that $\sigma(A) = \emptyset$ but $\|T(t)\| = r(T(t)) = 1$ for $t \geq 0$, i.e. $s(A) = -\infty$, $\omega = 0$ and $s(A) < \omega$: