$\operatorname{supp} f_o \cap \operatorname{supp} T(t)'\delta_y = \emptyset$ for every $t \geq 0$. Hence $(T(t)f_o)(y) = \langle f_o, T(t)'\delta_y\rangle > 0$, that is, $y \notin \bigcup_{t\geq 0}\{x \in X : (T(t)f_o)(x) > 0\}$.

(vi) → (v): Given $0 < f \in E$, $\lambda > \omega(A)$, $y \in X$, there exists $t_o \geq 0$ such that $\{x : f(x) > 0\} \cap \operatorname{supp} T(t_o)'\delta_y \neq \emptyset$. Hence, $(T(t_o)(f)(y) = \langle f, T(t_o)'\delta_y\rangle > 0$ and therefore $(R(\lambda,A)f)(y) = \int_0^\infty e^{-\lambda t}(T(t)f)(y)\,dt > 0$. Since $\lambda \to R(\lambda,A)f$ is decreasing in the interval $(s(A),\infty)$ (use the resolvent equation and the fact that $R(\lambda,A)$ is positive) we have $R(\lambda,A)f \gg 0$ for all $\lambda > s(A)$.

(v) → (vi): If I is a $R(\lambda,A)$-invariant ideal and $0 < f \in I$, then $g := R(\lambda,A)f \in I$. By (v) g is strictly positive thus I has to be dense (it contains all functions of compact support).

(iv) → (i): At first we recall that a closed linear subspace which is invariant for $R(\lambda_o,A)$ $(\lambda_o \in \rho(A))$, is invariant for $R(\lambda,A)$ whenever λ and λ_o belong to the same component of $\rho(A)$. Hence by A-I,3.2 every $R(\lambda_o,A)$-invariant subspace where $\lambda_o \in \rho_+(A)$ is $\mathcal{T}$-invariant and vice versa.

Remark 3.2. Obviously, irreducibility of a semigroup $(T(t))_{t\geq 0}$ is implied by the following condition:

(vii) $T(t)f \gg 0$ whenever $f > 0$ and $t > 0$.

The rotation semigroup (see A-I,2.5) is irreducible but it does not satisfy condition (vii). However, assuming that the semigroup $(T(t))$ is holomorphic, then (vii) of Def.3.1 is equivalent to irreducibility. We will give a proof of this result in the more general situation of Banach lattices (see C-III,Thm.3.2(b)).

A semigroup $(T(t))_{t\geq 0}$ is irreducible if and only if $(e^{-\alpha t}T(t))_{t\geq 0}$, $\alpha \in \mathbb{R}$ is. More generally, irreducibility is invariant under perturbations by multiplication operators. In fact, we have the following result:

Proposition 3.3. Suppose A generates a positive semigroup $\mathcal{T}$ on $C_o(X)$ and let h be a continuous, bounded real-valued function on X . Then the semigroup $\mathcal{S}$ generated by $B := A + M_h$ is irreducible if and only if $\mathcal{T}$ has this property.