In fact, A-I,(3.1) implies $(e^{rt} - T(r))R(r,A) \geq 0$ hence
$T(t)'\psi = rR(r,A)'T(t)'|\phi| \leq r\cdot e^{rt}R(r,A)'|\phi| = e^{rt}\psi$.
Moreover, the inequality $T(t)'|\phi| \geq |\phi|$ $(t \geq 0)$ implies
$T(t)'\psi = rR(r,A)'T(t)'|\phi| \geq rR(r,A)'|\phi| = \psi$ and
$\psi = rR(r,A)'|\phi| = r\int_0^\infty e^{-rt}T(t)'|\phi|\,dt \geq r\int_0^\infty e^{-rt}|\phi|\,dt = |\phi|$.

Considering the AL-space (E,ψ) (see C-I,Sec.4) the first inequality of (4.1) implies that $(T(t))_{t\geq 0}$ induces a strongly continuous semigroup $(T_1(t))_{t\geq 0}$ on (E,ψ) .
That is we have

$$\begin{tikzcd} E \arrow[r, "T(t)"] \arrow[d, "q_\psi"'] & E \arrow[d, "q_\psi"] \\ (E,\psi) \arrow[r, "T_1(t)"'] & (E,\psi) \end{tikzcd}$$

(4.2) $\quad T_1(t)\circ q_\psi = q_\psi\circ T(t) \quad (t \geq 0)$

Denoting by A_1 the generator of $(T_1(t))$ we have $R\sigma(A_1) \subset R\sigma(A)$.
Indeed, $A_1{}^*\chi = \lambda\chi$ implies $T_1(t)'\chi = e^{\lambda t}\chi$ hence by (4.2) $T(t)'q_\psi'(\chi) = e^{\lambda t}q_\psi'(\chi)$ or equivalently $A^*(q_\psi'(\chi)) = q_\psi'(\chi)$. Thus it remains to show that either $i\alpha\mathbb{Z}$ or $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}$ is contained in $R\sigma(A_1)$. Obviously, $(T_1(t))$ is a semigroup of lattice homomorphisms as well. The second inequality of (4.1) implies

(4.3) $\quad \|T_1(t)f\|_\psi = \langle|T_1(t)f|,\psi\rangle = \langle|f|,T_1(t)'\psi\rangle \geq \langle|f|,\psi\rangle = \|f\|_\psi$.

Then for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda < 0$ we have
$\|(e^{\lambda t}-T_1(t))f\|_\psi \geq \|T_1(t)f\|_\psi - \|e^{\lambda t}f\|_\psi \geq (1-|e^{\lambda t}|)\|f\|_\psi \quad (f \in (E,\psi))$
and we obtain for the corresponding generator

(4.4) $$\|(\lambda-A_1)f\|_\psi = \lim_{t\to 0}\|\tfrac{1}{t}(e^{-\lambda t}T_1(t)f-f)\|_\psi \geq \lim_{t\to 0}\tfrac{1}{t}(e^{-t\mathrm{Re}\lambda}-1)\|f\|_\psi$$
$$= -\mathrm{Re}\lambda\cdot\|f\|_\psi \quad \text{for } \mathrm{Re}\,\lambda < 0 \text{ and } f \in (E,\psi) .$$

It follows from (4.3) and (4.4) that $A\sigma(T_1(t))\cap\{z \in \mathbb{C} : |z| < 1\} = \emptyset$ and $A\sigma(A_1)\cap\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} = \emptyset$. Since the toplogical boundary of the spectrum is always contained in the approximate point spectrum (see A-III,Prop.2.2) and $R\sigma(T(t))\setminus\{0\} = \exp(tR\sigma(A))$ (see A-III, Thm.6.3), precisely one of the following two cases occurs :

(A) $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} \subset \rho(A_1)$ and $\{z \in \mathbb{C} : |z| < 1\} \subset \rho(T_1(t))$;
(B) $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} \subset R\sigma(A_1)$ and $\{z \in \mathbb{C} : |z| < 1\} \subset R\sigma(T_1(t))$.

We mentioned above that $R\sigma(A_1) \subset R\sigma(A)$. Thus we only have to analyze case (A). In this case each operator $T_1(t)$ is an invertible lattice homomorphism hence a lattice isomorphism. It follows that $T_1(t)'$ is a lattice isomorphism as well. The third inequality in (4.1) implies that ϕ can be considered as an element of $(E,\psi)'$ and $T(t)'\phi = e^{i\alpha t}\phi$ $(t\geq 0)$ implies $T_1(t)'\phi = e^{i\alpha t}\phi$. Furthermore, we have