(0) $|T(t) - T(t+\tau)|e_o \downarrow 0$ for $e_o = e\cdot 1_{X_o}$ as $t \to \infty$,

(2) $|T(t) - T(t+\tau)|e_2 = 2e_2$ for $e_2 = e\cdot 1_{X_2}$ and all $t \geq 0$.

(b) In case the semigroup is irreducible then for every $\tau > 0$ one has the alternative

(0) $|T(t) - T(t+\tau)|e \downarrow 0$ as $t \to \infty$ or

(2) $|T(t) - T(t+\tau)|e = 2e$ for all $t \geq 0$.

This '0-2 law' can be used in order to obtain results on convergence of positive semigroups.

Corollary 2.7. Assume that (in addition to the assumptions made in Thm.2.6) $P\sigma(A)\cap i\mathbb{R} = \{0\}$. If we decompose $X = X_o \cup X_2$ for some $\tau > 0$ according to assertion (a), then $\lim_{t\to\infty} T(t)f$ exists for every $f \in L^p(\mu)$ vanishing μ-a.e. on X_2 .

Proof. From $T(t)e_j \leq T(t)e = e$ we obtain $T(t)e_j \leq e_j$ since X_0 and X_2 are $(T(t))$-invariant. Then $T(t)e_o + T(t)e_2 = T(t)e = e$ implies $T(t)e_j = e_j$ $(j=0,2)$. Thus we can assume $X = X_o$, $e = e_o$.
Given $g \in L^p(\mu)$ such that $|g| \leq e$ we have
$|T(t)(Id - T(\tau))g| \leq |T(t) - T(t+\tau)|e \downarrow 0$ for $t \to \infty$.
Since $\{g \in L^p(\mu) : |g| \leq e\}$ is a total subset of E (e is strictly positive) and $(T(t))_{t\geq 0}$ is bounded we conclude

(2.6) $\lim_{t\to\infty} T(t)f = 0$ for every $f \in \overline{\mathrm{im}(Id - T(\tau))}$.

The assumption $P\sigma(A)\cap i\mathbb{R} = \{0\}$ implies $\ker(Id - T(\tau)) = \ker A$ (cf. A-III,Cor.6.4), hence we have convergence on $\ker(Id - T(\tau))$.
Since $T(\tau)$ is a contraction on a reflexive Banach space we have $L^p(\mu) = \ker(Id - T(\tau)) \oplus \overline{\mathrm{im}(Id - T(\tau))}$ (see Krengel(1985),p.74) which finally proves the convergence on the whole space.

□

Typical examples for which Thm.2.6 and Cor.2.7 can be applied occur in the theory of stochastic processes (see also B-IV,Ex.2.6). We briefly describe this situation and remind that in this context the sets X_o and X_2 have a probabilistic meaning (see Greiner-Nagel (1982) or the Supplement in Krengel (1985)).

Example 2.8. Let X be a set and Σ be a σ-algebra of subsets of X . We consider a Markov transition function $(P_t)_{t\geq 0}$ on (X,Σ) , i.e. each P_t is a real-valued function on $X \times \Sigma$ such that