Chapter 1

Asymptotics of Semigroups on Banach Spaces

In this chapter we study the asymptotic behavior of the solutions of the initial value problem

$$\dot{u}(t) = Au(t) + F(t), \quad u(0) = f$$

with respect to time $t \geq 0$. Here A will be a generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E and $F(\cdot)$ is a function from $\mathbb{R}_+$ with values in E.

In Section 1 we investigate whether and how fast a solution $T(\cdot)f$ of the homogeneous problem tends to the zero solution as $t \to \infty$. In Section 2 we consider the long term behavior of the solutions of $(*)$ for different classes of forcing terms F.

1.1 Stability: Homogeneous Case

Let $(T(t))_{t\geq 0}$ be a semigroup on E with generator A. An initial value $f \in D(A)$ is called *stable* if the solution $t \mapsto T(t)f$ of

$$\dot{u}(t) = Au(t), \quad u(0) = f$$

converges to zero as t tends to infinity. The semigroup is called *stable* if every solution converges to zero; i.e., if every initial value $f \in D(A)$ is stable.

If the space E is finite dimensional, then the stability of the semigroup implies that the decay is exponential. More precisely, the statements

(a) $\|T(t)f\| \to 0$ for every $f \in \mathbb{C}^n$,

(b) $\|T(t)\| \leq Me^{-\omega t}$ for some $\omega > 0$