Then $\tilde{v}$ is a solution of (ACP). It follows that $\tilde{v}(s) = T_1(s)f$ for all $s \geq 0$. Hence $v = \eta(f)$. We have shown that $\eta$ has a closed graph and so $\eta$ is continuous. This implies the continuity of $T_1(t)$. It remains to show that $A_1$ is the generator of $(T_1(t))_{t\geq 0}$.

We first show that for $f \in D(A^2)$ one has

(1.1) $\quad AT_1(t)f = T_1(t)Af.$

In fact, let $v(t) = f + \int_0^t u(s,Af)\,ds$. Then
$\dot{v}(t) = u(t,Af) = Af + \int_0^t Au(s,Af)\,ds = A(f + \int_0^t u(s,Af)\,ds) = Av(t)$.
Since $v(0) = f$, it follows that $v(t) = u(t,f)$.
Hence $Au(t,f) = Av(t) = \dot{v}(t) = u(t,Af)$. This is (1.1).
Now denote by $B$ the generator of $(T_1(t))_{t\geq 0}$.
For $f \in D(A^2)$ we have

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af$$

and by (1.1),

$$\lim_{t\to 0} A\, \frac{T_1(t)f - f}{t} = \lim_{t\to 0} \frac{T_1(t)Af - Af}{t} = A^2 f \quad \text{in the norm of } E.$$

Hence $\lim_{t\to 0} \dfrac{T_1(t)f - f}{t} = Af$ in the norm of $E_1$.

This shows that $A_1 \subset B$. In order to show the converse, let $f \in D(B)$.

Then $\lim_{t\to 0} A\, \dfrac{T_1(t)f - f}{t}$ exists in the norm of $E$.

Since $\lim_{t\to 0} \dfrac{T_1(t)f - f}{t} = Af$ in the norm of $E$, it follows

that $Af \in D(A)$, since $A$ is closed. Thus $f \in D(A^2) = D(A_1)$.
We have shown that $B = A_1$.

(ii) implies (i).
Assume that $A_1$ is the generator of a strongly continuous semigroup $(T_1(t))_{t\geq 0}$ on $E_1$. Let $f \in D(A)$ and set $u(t) = T_1(t)f$.
Then $u \in C([0,\infty),E)$ and $Au(\cdot) \in C([0,\infty),E)$.
Moreover, $\int_0^t u(s)ds = \int_0^t T_1(s)f ds \in D(A_1) = D(A^2)$ and $A\int_0^t u(s)ds = u(t) - u(0) = u(t) - f$ (by A-I,(1.3)).
Consequently, $u(t) = f + A\int_0^t u(s)ds = f + \int_0^t Au(s)ds$.
Hence $u \in C^1([0,\infty),E)$ and $\dot{u}(t) = Au(t)$. Thus $u$ is a solution of (ACP). We have shown existence.