Theorem 1.17. A strongly continuous semigroup $(T(t))_{t\geq 0}$ is eventually differentiable if and only if its generator A satisfies the following: there exist constants $c>0$, $b>0$, $M>0$ such that

$$\Sigma := \{\lambda \in \mathbb{C} : ce^{-b\cdot \mathrm{Re}\,\lambda} \leqq |\mathrm{Im}\lambda|\} \subset \rho(A)$$

and $\|R(\lambda,A)\| \leqq M\cdot|\mathrm{Im}\lambda|$ for all $\lambda \in \Sigma$ satisfying $\mathrm{Re}\,\lambda \leqq \omega(A)$.

Theorem 1.18. A strongly continuous semigroup $(T(t))_{t\geq 0}$ is differentiable if and only if its generator A satisfies the following: for all $b>0$ there exist $c>0$, $M>0$ such that

$$\Sigma := \{\lambda \in \mathbb{C} : ce^{-b\cdot \mathrm{Re}\,\lambda} \leqq |\mathrm{Im}\lambda|\} \subset \rho(A)$$

and $\|R(\lambda,A)\| \leqq M\cdot|\mathrm{Im}\lambda|$ for all $\lambda \in \Sigma$ satisfying $\mathrm{Re}\,\lambda \leqq \omega(A)$.

For the proofs of these two theorems we refer to [Pazy (1983), Chap.3, Theorem 4.7 and 4.8].

Norm continuous semigroups

Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup and $t' > 0$. If $\lim_{t\downarrow t'} \|T(t) - T(t')\| = 0$, then it follows from the semigroup property, that the function $t \to T(t)$ is norm continuous on the whole half line (t',∞).

Definition 1.19. A semigroup $(T(t))_{t\geq 0}$ is called *eventually norm continuous* if there exists $t' \geq 0$ such that the function $t \to T(t)$ from (t',∞) into $L(E)$ is norm continuous. The semigroup is called *norm continuous* if t' can be chosen equal to 0.

The spectrum of generators of eventually norm continuous semigroups still is compact in every right half-plane.

Theorem 1.20. Let A be the generator of an eventually norm continuous semigroup. Then for every $b \in \mathbb{R}$ the set

$$\{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda \geqq b\}$$

is bounded.