

Wolfgang Arendt, Annette Grabosch, Günther Greiner, Ulrich Groh, Heinrich P. Lotz, Ulrich Moustakas, Rainer Nagel, Frank Neubrander, Ulf Schlotterbeck

# One-parameter Semigroups of Positive Operators

Edited by R. Nagel

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*This revised edition of  
“One-Parameter Semigroups of  
Positive Operators” is dedicated to the  
memory of our co-authors, Heinrich  
P. Lotz (1934–2010) and Ulf  
Schlotterbeck (1941–2021). Their  
contributions to the first edition  
remain an inspiration to us all. We  
miss their presence and remain  
grateful for the legacy they have left in  
this work.*



# Preface

## Preface to the Second Edition

When the first edition of these lecture notes appeared in 1986, the theory of one-parameter semigroups of positive operators was undergoing rapid development, stimulated by applications in ergodic theory, evolution equations, stochastics, and mathematical physics. Our goal at the time was to provide a systematic and accessible account of the foundations and structure theory of positive semigroups, with particular emphasis on Banach lattices and  $C^*$ -algebras. We were gratified by the positive reception the volume received and the extent to which it found use in research and graduate instruction.

Over the past four decades, the mathematical community has continued to draw on the results and techniques developed in these notes. Despite the appearance of newer texts and the evolution of the field, this volume has remained widely cited and used—likely due to its thorough and methodical treatment of a core area in functional analysis. The sustained interest from both researchers and students has encouraged us to prepare this second edition.

We have preserved the structure and exposition of the original edition but have made a number of editorial improvements, including transferring the entire book into  $\text{\LaTeX}$ . Obvious misprints have been corrected, references have been updated where appropriate, and the notes at the end of each chapter have been significantly expanded in the chapter *Updated Notes* to highlight subsequent developments. However, we have refrained from substantially altering the original content in order to retain the historical character and coherence of the text.

Two of our original co-authors, Heinrich P. Lotz (1934–2010) and Ulf Schlotterbeck (1941–2021), sadly passed away in the years following the publication of the first edition. In addition, our former co-authors Annette Grabosch and Ulrich Moustakas are no longer engaged in active mathematical research at this time. To preserve the integrity and historical authorship of the original text, we have attributed all chapters

to their original authors. Members of the present author team carefully reviewed these contributions for this second edition:

- The chapters originally written by Ulf Schlotterbeck were reviewed and updated by Rainer Nagel.
- The chapters originally written by Heinrich P. Lotz were reviewed and updated by Wolfgang Arendt.
- The chapters originally written by Annette Grabosch and Ulrich Moustakas were reviewed and updated by Günther Greiner.

The original authors of all other chapters participated directly in this new edition. We remain grateful for the foundational work of all nine original authors and hope that the clarity regarding authorship and review responsibilities will support the scholarly use of both editions.

We gratefully acknowledge the efforts of our co-author, Ulrich Groh, who guided the transfer of the manuscript into L<sup>A</sup>T<sub>E</sub>X, with the assistance of Klaus-Georg Kuhn and the support of Claude, an artificial intelligence model developed by Anthropic.

It is our hope that this new edition will continue to serve as a valuable resource for those working in operator theory, functional analysis, and their many applications. We remain deeply grateful to our colleagues and readers who have provided feedback and encouragement over the years.

## Preface to the First Edition

As early as 1948 in the first edition of his fundamental treatise on *Semigroups and Functional Analysis*, E. Hille expressed the need for

*... developing an adequate theory of transformation semigroups operating in partially ordered spaces* (l.c., Foreword).

In the meantime the theory of one-parameter semigroups of positive linear operators has grown continuously. Motivated by problems in probability theory and partial differential equations W. Feller (1952) and R. S. Phillips (1962) laid the first cornerstones by characterizing the generators of special positive semigroups. In the 60's and 70's the theory of positive operators on ordered Banach spaces was built systematically and is well documented in the monographs of H. H. Schaefer (1974) and A. C. Zaanen (1983). But in this process the original ties with the applications and, in particular, with initial value problems were at times obscured. Only in recent years an adequate and up-to-date theory emerged, largely based on the techniques developed for positive operators and thus recombining the functional analytic theory with the investigation of Cauchy problems having positive solutions to each positive initial value. Even though this development — in particular with respect to applications to concrete evolution equations in transport theory, mathematical biology, and physics —

is far from being complete, the present volume is a first attempt to shape the multitude of available results into a coherent theory of one-parameter semigroups of positive linear operators on ordered Banach spaces.

The book is organized as follows. We concentrate our attention on three subjects of semigroup theory: *characterization*, *spectral theory* and *asymptotic behavior*. By *characterization*, we understand the problem of describing special properties of a semigroup, such as positivity, through the generator. By *spectral theory* we mean the investigation of the spectrum of a generator. *Asymptotic behavior* refers to the orbits of the initial values under a given semigroup and phenomena such as stability.

This program (characterization, spectral theory, asymptotic behavior) is worked out on four different types of underlying spaces.

- (A) On Banach spaces—Here we present the background for the subsequent discussions related to order.
- (B) On spaces  $C_0(X)$  ( $X$  locally compact), which constitute an important class of ordered Banach spaces and where our results can be presented in a form which makes them accessible also for the non-expert in order-theory.
- (C) On Banach lattices, which admit a rich theory and are still sufficiently general as to include many concrete spaces appearing in analysis; e.g.,  $C_0(X)$ ,  $\mathcal{L}^p(k)$  or  $l^p$ .
- (D) On non-commutative operator algebras such as  $C^*$ - or  $W^*$ -algebras, which are not lattice ordered but still possess an interesting order structure of great importance in mathematical physics.

In each of these cases we start with a short collection of basic results and notations, so that the contents of the book may be visualized in the form of a  $4 \times 4$  matrix in a way which will allow “row readers” (interested in semigroups on certain types of spaces) and “column readers” (interested in certain aspects) to find a path through the book corresponding to their interest.

We display this matrix, together with the names of the authors contributing to the subjects defined through this scheme. This “matrix of contents” has been an indispensable guide line in our discussions on the scope and the spirit of the various contributions. However, we would not have succeeded in completing this manuscript, as a collection of independent contributions (personally accounted for by the authors), under less favorable conditions than we have actually met. For one thing, Rainer Nagel was an unfaltering and undisputed spiritus rector from the very beginning of the project. On the other hand we gratefully acknowledge the influence of Helmut H. Schaefer and his pioneering work on order structures in analysis. It was the team spirit produced by this common mathematical background which, with a little help from our friends, made it possible to overcome most difficulties.

We have prepared the manuscript with the aid of a word processor, but we confess that without the assistance of Klaus-Georg Kuhn the pitfalls of such a system would have been greater than its benefits.

	I Basic Results	II Characterization	III Spectral Theory	IV Asymptotics
A. Banach Spaces	R. Nagel U. Schlotterbeck	W. Arendt H. P. Lotz	G. Greiner R. Nagel	F. Neubrander
B. $C_0(X)$	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas F. Neubrander
C. Banach Lattices	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas R. Nagel F. Neubrander
D. Operator Algebras	U. Groh	U. Groh	U. Groh	U. Groh

*The authors*



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# List of Symbols

$E_{\mathbb{R}}, E_{\mathbb{C}}$	real, complex Banach lattice
$E_+$	positive cone of an ordered vector space
$E'$	dual Banach space
$E^*$	semigroup dual
$m(E), \ell^\infty(E)$	bounded $E$ -valued sequences
$E_{\mathcal{T}}$	$\mathcal{F}$ -product of $E$ with respect to the semigroup $\mathcal{T}$
$E_{\mathcal{F}}$	$\mathcal{F}$ -product of $E$
$E_f$	see C-I, 4
$(E, \varphi)$	see C-I, 4
$E \otimes F$	tensor product
$\mathcal{L}(E)$	bounded linear operators on $E$
$\mathcal{Z}(E)$	center of $E$
$E_n$	$n$ -th Sobolev space
$\mathcal{B}(H)$	$W^*$ -algebra of all bounded linear operators on $H$
$S(M)$	state space of a $C^*$ -algebra $M$
$M_+$	positive cone of the $C^*$ -algebra $M$
$M_*$	predual of a $W^*$ -algebra $M$
$M^{sa}$	self-adjoint part of a $C^*$ -algebra
$M_n$	$C^*$ -algebra of all $n \times n$ -matrices
$AC$	absolutely continuous functions
$BV$	functions of bounded variation
$K$	compact topological space
$X$	locally compact topological space
$C(K), C(K, E)$	continuous functions (with values in $E$ )
$C_c(X)$	continuous functions with compact support
$C_0(X)$	continuous functions vanishing at infinity
$C^b(X)$	bounded continuous functions
$C_{bu}(X)$	bounded uniformly continuous functions
$C^n, C^{(n)}$	continuous differentiable functions ( $n$ -times)

$C_c^\infty(\mathbb{R}^n)$	infinitely differentiable functions with compact support
$L^p(\mu)$	p-integrable functions
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space
$M(X)$	regular Borel measures
$M_b(X)$	bounded regular Borel measures
$\mathcal{T} = (T(t))_{t \geq 0}$	(one-parameter) semigroup
$T $	subspace (reduced) semigroup
$T/$	quotient semigroup
$\text{Fix}(\mathcal{T})$	fixed space of $\mathcal{T}$
$A$	generator of a $C_0$ -semigroup
$A'$	adjoint operator of $A$
$A^*$	adjoint generator
$\sigma(A)$	spectrum of $A$
$\varrho(A)$	resolvent set of $A$
$\sigma_{ess}(A)$	essential spectrum of $A$
$\sigma_b(A)$	boundary spectrum of $A$
$P\sigma(A)$	point spectrum of $A$
$P\sigma_b(A)$	boundary point spectrum
$A\sigma(A)$	approximate point spectrum of $A$
$R\sigma(A)$	residual spectrum
$\omega_0; \omega_0(A); \omega_0(\mathcal{T}); \omega_0(T(t))$	growth bound
$s(A)$	spectral bound
$\omega_I(A)$	growth bound of the solution of the (ACP)
$\omega(f)$	growth bound of $T(\cdot)f$
$r(T)$	spectral radius of $T$
$\omega_{ess}(\mathcal{T})$	essential growth bound of $\mathcal{T}$
$r_{ess}(T)$	essential spectral radius of $T$
$R(\lambda, A)$	resolvent operator of $A$
$I^d, \{I^d\}^{dd}$	orthogonal band of $I$ (of $I^d$ )
$\wedge$	infimum
$\vee$	supremum
$ T $	modulus of a regular operator
$\hat{f}, \check{f}$	Fourier (inverse Fourier) transformation
$dp(f)$	subdifferential of $p$ in $f$
$dN(f)$	subdifferential of the norm in $f$
$dN^+(f)$	subdifferential of the canonical half-norm in $f$
$\text{im}(T)$	range of $T$
$\ker(T)$	null-space of $T$
$\text{Im } z$	imaginary part of $z$
$\text{Re } z$	real part of $z$
$\text{Re}(f), \text{Im}(f)$	see C-I, 7
$\text{Re } T, \text{Im } T$	see C-I, 7
$\bar{f}$	complex conjugate of $f$
$S_f$	signum operator with respect to $f$
$\text{sign}(f)$	signum of $f$

$f^{[n]}$	see B-III,2.2 ; C-III,2.1
$ f $	absolute value of $f$
$f^+$	positive part of $f$
$f^-$	negative part of $f$
$\text{Id}$	identity operator
$M_P$	multiplication operator
$\mathbb{1}$	function identically 1
$\mathbb{1}_B$	characteristic function of the set $B$
$\delta_x$	Dirac measure in $x$
$\text{tr}$	trace
$\text{span } M$	linear subspace generated by $M$
$S(\alpha)$	sector in the complex plane
$(ACP)$	abstract Cauchy problem
$(P)$	positive minimum principle
$(P')$	see B-II,1.21
$(K)$	Kato's (equality) inequality
$(RCP)$	retarded Cauchy problem
$(RE)$	retarded equation
$(T)$	translation property



**Part A**

**One-parameter Semigroups on Banach  
Spaces**



## Chapter A-I

# Basic Results on Semigroups on Banach Spaces

by  
Rainer Nagel and Ulf Schlotterbeck

Since the basic theory of one-parameter semigroups can be found in several excellent books (e.g., Davies [1], Goldstein [3], Pazy [8] or Hille and Phillips [5]), we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, to collect briefly some important definitions and results (Section 1), to present a list of *standard examples* in Section 2 and to discuss standard constructions of new semigroups from a given one in Section 3 on p. 18.

In the entire chapter we denote by  $E$  a (real or) complex Banach space and consider one-parameter semigroups of bounded linear operators  $T(t)$  on  $E$ . By this we understand a subset  $\{T(t) : t \in \mathbb{R}_+\}$  of  $\mathcal{L}(E)$ , usually written as  $(T(t))_{t \geq 0}$ , such that

$$\begin{aligned} T(0) &= \text{Id}, \\ T(s+t) &= T(s) \cdot T(t) \text{ for all } s, t \in \mathbb{R}_+. \end{aligned}$$

In more abstract terms this means that the map  $t \mapsto T(t)$  is a homomorphism from the additive semigroup  $(\mathbb{R}_+, +)$  into the multiplicative semigroup  $(\mathcal{L}(E), \cdot)$ . Similarly, a one-parameter group  $(T(t))_{t \in \mathbb{R}}$  is a homomorphism from the group  $(\mathbb{R}, +)$  to  $(\mathcal{L}(E), \cdot)$ .

## 1 Standard Definitions and Results

We consider a one-parameter semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  and observe that the domain  $\mathbb{R}_+$  and the range  $\mathcal{L}(E)$  of the (semigroup) homomorphism  $\tau : t \mapsto T(t)$  are topological semigroups for the natural topology on  $\mathbb{R}_+$  and any one of the

standard operator topologies on  $\mathcal{L}(E)$ . We single out the strong operator topology on  $\mathcal{L}(E)$  and require  $\tau$  to be continuous.

**Definition 1.1** A one-parameter semigroup  $(T(t))_{t \geq 0}$  is called *strongly continuous* if the map  $t \mapsto T(t)$  is continuous for the strong operator topology on  $\mathcal{L}(E)$ , e.g.,

$$\lim_{t \rightarrow t_0} \|T(t)f - T(t_0)f\| = 0$$

for every  $f \in E$  and  $t_0 \geq 0$ .

Clearly one defines in a similar way *weakly continuous*, resp. *uniformly continuous* (compare A-II, Definition 1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology.

If not stated otherwise, a *semigroup* is a strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

**Remarks 1.2** (i) A one-parameter semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  is strongly continuous if and only if for any  $f \in E$  it is true that  $T(t)f \rightarrow f$  as  $t \rightarrow 0$ .

(ii) For every strongly continuous semigroup there exist constants  $M \geq 1$ ,  $\omega \in \mathbb{R}$ , such that  $\|T(t)\| \leq M \cdot e^{\omega t}$  for every  $t \geq 0$ .

(iii) If  $(T(t))_{t \geq 0}$  is a one-parameter semigroup such that  $\|T(t)\|$  is bounded for  $0 < t \leq \delta$  then it is strongly continuous if and only if  $\lim_{t \rightarrow 0} T(t)f = f$  for every  $f$  in a total subset of  $E$ .

The exponential estimate from Remark 1.2(ii) for the growth of  $\|T(t)\|$  can be used to define an important characteristic of the semigroup.

**Definition 1.3** By the *growth bound* (or type) of the semigroup  $T(t)_{t \geq 0}$  we understand the number  $\omega_0$ ,

$$\begin{aligned} \omega_0 &:= \inf\{\omega \in \mathbb{R} : \text{there exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leq M \cdot e^{\omega t} \text{ for } t \geq 0\} \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T(t)\| = \inf_{t > 0} \frac{1}{t} \log \|T(t)\|. \end{aligned} \tag{1.1}$$

Particularly important are semigroups such that for every  $t \geq 0$  we have  $\|T(t)\| \leq M$  (*bounded semigroups*) or  $\|T(t)\| \leq 1$  (*contraction semigroups*). In both cases we have  $\omega_0 \leq 0$ .

It follows from the subsequent examples and from Definition 1.3 that  $\omega_0$  may be any number  $-\infty \leq \omega < +\infty$ . Moreover the reader should observe that the infimum in

Definition 1.3 need not be attained and that  $M$  may be larger than 1 even for bounded semigroups.

**Examples 1.4** (i) Take

$$E = \mathbb{C}^2, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then for the  $\ell^1$ -norm on  $E$  we obtain  $\|T(t)\| = 1 + t$ , hence  $(T(t))_{t \geq 0}$  is an unbounded semigroup having growth bound  $\omega_0 = 0$ .

(ii) Take  $E = L^1(\mathbb{R})$  and for  $f \in E$ ,  $t \geq 0$  define

$$T(t)f(x) := \begin{cases} 2 \cdot f(x+t) & \text{if } x \in [-t, 0] \\ f(x+t) & \text{otherwise.} \end{cases}$$

Each  $T(t)$ ,  $t > 0$ , satisfies  $\|T(t)\| = 2$  as can be seen by taking  $f := \mathbb{1}_{[0,t]}$ . Therefore  $(T(t))_{t \geq 0}$  is a strongly continuous semigroup which is bounded, hence has  $\omega_0 = 0$ , but the constant  $M$  in Eq. (1.1) on p. 4 cannot be chosen to be 1.

The most important object associated to a strongly continuous semigroup  $(T(t))_{t \geq 0}$  is its *generator* which is obtained as the (right) derivative of the map  $t \mapsto T(t)$  at  $t = 0$ . Since for strongly continuous semigroups the functions  $t \mapsto T(t)f$ ,  $f \in E$ , are continuous but not always differentiable, we have to restrict our attention to those  $f \in E$  for which the desired derivative exists. We then obtain the *generator* as a not necessarily everywhere defined operator.

**Definition 1.5** To every semigroup  $(T(t))_{t \geq 0}$  there belongs an operator  $(A, D(A))$ , called the *generator* and defined on the *domain*

$$D(A) := \left\{ f \in E : \lim_{h \rightarrow 0} \frac{T(h)f - f}{h} \text{ exists in } E \right\} \text{ by}$$

$$Af := \lim_{h \rightarrow 0} \frac{T(h)f - f}{h} \quad (f \in D(A)).$$

Clearly,  $D(A)$  is a linear subspace of  $E$  and  $A$  is linear from  $D(A)$  into  $E$ . Only in certain special cases (see Section 2.1) the generator is everywhere defined and therefore bounded (use Proposition 1.9 (ii) on p. 7). In general, the precise extent of the domain  $D(A)$  is essential for the characterization of the generator. But since the domain is canonically associated to the generator of a semigroup, we shall write in most cases  $A$  instead of  $(A, D(A))$ . As a first result we collect some information on the domain of the generator.

**Proposition 1.6** For the generator  $A$  of a semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  the following assertions hold.

- (i) If  $f \in D(A)$ , then  $T(t)f \in D(A)$  for every  $t \geq 0$ .
- (ii) The map  $t \mapsto T(t)f$  is differentiable on  $\mathbb{R}_+$  if and only if  $f \in D(A)$ . In that case one has

$$\frac{d}{dt}T(t)f = AT(t)f = T(t)Af. \quad (1.2)$$

- (iii) For every  $f \in E$  and  $t > 0$  the element  $\int_0^t T(s)f \, ds$  belongs to  $D(A)$  and one has

$$A \int_0^t T(s)f \, ds = T(t)f - f. \quad (1.3)$$

- (iv) If  $f \in D(A)$ , then

$$\int_0^t T(s)Af \, ds = T(t)f - f. \quad (1.4)$$

- (v) The domain  $D(A)$  is dense in  $E$ .

The identity (1.2) is of great importance and shows how semigroups are related to certain Cauchy problems. We state this explicitly in the following theorem.

**Theorem 1.7** *Let  $(A, D(A))$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on the Banach space  $E$ . Then the abstract Cauchy problem*

$$\frac{d}{dt}\xi(t) = A\xi(t), \quad \xi(0) = f_0 \quad (1.5)$$

*has for every  $f_0 \in D(A)$  a unique solution  $\xi: \mathbb{R}_+ \rightarrow D(A)$  in  $C^1(\mathbb{R}_+, E)$ . In fact, this solution is given by  $\xi(t) := T(t)f_0$ .*

For more on the relation of semigroups to abstract Cauchy problems we refer to A-II, Section 1. Here we only point out that the above theorem implies that a semigroup is uniquely determined by its generator.

While the generator is bounded only for uniformly continuous semigroups (see Section 2 below), it always enjoys a weaker but useful property.

**Definition 1.8** An operator  $B$  with domain  $D(B)$  on a Banach space  $E$  is called *closed* if  $D(B)$  endowed with the *graph norm*

$$\|f\|_B := \|f\| + \|Bf\|$$

becomes a Banach space. Equivalently,  $(B, D(B))$  is closed if and only if its *graph*  $\{(f, Bf): f \in D(B)\}$  is closed in  $E \times E$ , i.e.,

$$(f_n) \subset D(B), f_n \rightarrow f \text{ and } Bf_n \rightarrow g \text{ implies } f \in D(B) \text{ and } Bf = g.$$

It is clear from this definition that the *closedness* of an operator  $B$  depends very much on the size of the domain  $D(B)$ . For example, a bounded and densely defined operator  $(B, D(B))$  is closed if and only if  $D(B) = E$ .

On the other hand it may happen that  $(B, D(B))$  is not closed but has a closed *extension*  $(C, D(C))$ , i.e.,  $D(B) \subseteq D(C)$  and  $Bf = Cf$  for every  $f \in D(B)$ . In that case,  $B$  is called *closable*, a property which is equivalent to

$$(f_n) \subset D(B), f_n \rightarrow 0 \text{ and } Bf_n \rightarrow g \text{ implies } g = 0.$$

The smallest closed extension of  $(B, D(B))$  will be called the *closure*  $\bar{B}$  with domain  $D(\bar{B})$ . In other words, the graph of  $\bar{B}$  is the closure of  $\{(f, Bf) : f \in D(B)\}$  in  $E \times E$ .

Finally, we call a subset  $D_0$  of  $D(B)$  a *core* for  $B$  if  $D_0$  is  $\|\cdot\|_B$ -dense in  $D(B)$ . This means that a closed operator is determined (via closure) by its restriction to a core in its domain.

We now collect the fundamental topological properties of semigroup generators, their domains (see also A-II, Corollary 1.34) and their resolvents.

**Proposition 1.9** *For the generator  $A$  of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  the following hold.*

- (i) *The generator  $A$  is a closed operator.*
- (ii) *If a subspace  $D_0$  of the domain  $D(A)$  is dense in  $E$  and  $(T(t))$ -invariant, then it is a core for  $A$ .*
- (iii) *Define*

$$\begin{aligned} D(A^1) &= D(A) \\ D(A^n) &= \{f \in D(A^{n-1}) : Af \in D(A^{n-1})\} \text{ for } n \geq 2, \end{aligned}$$

*Then*

$$D(A^\infty) := \bigcap_{n \in \mathbb{N}} D(A^n)$$

*is dense in  $E$  and a core for  $A$ .*

**Example 1.10** Property (iii) above does not hold for general densely defined closed operators. Take  $E = C[0, 1]$ ,  $D(B) = C^1[0, 1]$  and  $Bf = q \cdot f'$  for some nowhere differentiable function  $q \in C[0, 1]$ . Then  $B$  is closed, but  $D(B^2) = \{0\}$ .

**Proposition 1.11** *For the generator  $A$  of a strongly continuous semigroup on a Banach space  $E$  the following hold. If*

$$\int_0^\infty e^{-\lambda t} T(t) f \, dt$$

exists for every  $f \in E$  and some  $\lambda \in \mathbb{C}$ , then  $\lambda \in \varrho(A)$  and

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t)f \, dt.$$

In particular,

$$R(\lambda, A)^{n+1}f = \frac{(-1)^n}{n!} \left( \frac{d}{d\lambda} \right)^n R(\lambda, A)f = \int_0^\infty e^{-\lambda t} \frac{t^n}{n!} T(t)f \, dt \quad (1.6)$$

for every  $f \in E$ ,  $n \geq 0$  and  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > \omega_0$ .

**Remarks 1.12** (i) For continuous Banach space valued functions such as  $t \mapsto T(t)f$  we consider the Riemann integral and define

$$\int_0^\infty T(t)f \, dt \quad \text{as} \quad \lim_{t \rightarrow \infty} \int_0^t T(s)f \, ds.$$

Sometimes such integrals for strongly continuous semigroups are written as  $\int_a^b T(t) \, dt$  but understood in the strong sense.

(ii) Since the generator  $(A, D(A))$  determines the semigroup  $(T(t))_{t \geq 0}$  uniquely, we will speak occasionally of the *growth bound of the generator* instead of the semigroup, i.e., we write  $\omega_0 = \omega_0(A) = \omega_0(\mathcal{T})$  where  $\mathcal{T} = (T(t))_{t \geq 0}$  denotes the semigroup.

(iii) For one-parameter groups it might seem to be more natural to define the generator as the *derivative* rather than just the *right derivative* at  $t = 0$ . This yields the same operator as the following result shows.

The strongly continuous semigroup  $(T(t))_{t \geq 0}$  with generator  $A$  can be extended to a strongly continuous one-parameter group  $(U(t))_{t \in \mathbb{R}}$  if and only if  $-A$  generates a semigroup  $(S(t))_{t \geq 0}$ . In that case  $(U(t))_{t \in \mathbb{R}}$  is obtained as

$$U(t) = \begin{cases} T(t) & \text{for } t \geq 0, \\ S(-t) & \text{for } t \leq 0. \end{cases}$$

We refer to Davies [1, Proposition 1.14] for the details.

## 2 Standard Examples

In this section we list and discuss briefly the most basic examples of semigroups together with their generators. These semigroups will reappear throughout this book and will be used to illustrate the theory. We start with the class of semigroups mentioned after Definition 1.1 on p. 4.

## 2.1 Uniformly Continuous Semigroups

It follows from elementary operator theory that for every bounded operator  $A$  in  $\mathcal{L}(E)$  the sum

$$\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} =: e^{tA}$$

exists and determines a unique uniformly continuous (semi)group  $(e^{tA})_{t \in \mathbb{R}}$  having  $A$  as its generator. Conversely, any uniformly continuous semigroup is of this form.

If the semigroup  $(T(t))_{t \geq 0}$  is uniformly continuous, then

$$\frac{1}{t} \int_0^t T(s) \, ds$$

uniformly converges to  $T(0) = \text{Id}$  as  $t \rightarrow 0$ . Therefore for some  $t' > 0$  the operator

$$\frac{1}{t'} \int_0^{t'} T(s) \, ds$$

is invertible and every  $f \in E$  is of the form

$$f = \frac{1}{t'} \int_0^{t'} T(s) g \, ds$$

for some  $g \in E$ . But these elements belong to  $D(A)$  by (1.3), hence  $D(A) = E$ . Since the generator  $A$  is closed and everywhere defined, it must be bounded. Remark that bounded operators are always generators of groups, not just semigroups. Moreover, the growth bound  $\omega_0$  satisfies  $|\omega_0| \leq \|A\|$  in this situation.

The above characterization of the generators of uniformly continuous semigroups as the bounded operators shows that these semigroups are—at least in many aspects—rather simple objects.

## 2.2 Matrix Semigroups

The above considerations especially apply in the situation  $E = M_n(\mathbb{C})$ . If  $n = 2$  and  $A \in E$ , one can derive an explicit formula for  $e^{tA}$ .

Let

$$s = \text{trace}(A), \quad d = \det(A) \quad \text{and} \quad D^2 := \left( \frac{s^2}{4} - d \right).$$

If

$$p_A(\lambda) := \det(\lambda - A) = \lambda^2 - s \cdot \lambda + d$$

is the characteristic polynomial of  $A$ , then  $p_A(A) = 0$  by *Cayley-Hamilton*, hence

$$(A - s/2 \cdot \text{Id})^2 = D^2 \cdot \text{Id},$$

where  $\text{Id}$  denotes the unit matrix.

Now we have to consider two cases.

$D^2 = 0$ : Then  $(A - s/2 \cdot \text{Id})^k = D^k \cdot \text{Id} = 0$  for all  $(k \geq 2)$  and this implies

$$e^{t \cdot (A - s/2 \cdot \text{Id})} = \text{Id} + t(A - s/2 \cdot \text{Id})$$

or

$$e^{tA} = e^{t \cdot s/2} \left[ \left(1 - \frac{s}{2} t\right) \cdot \text{Id} + t \cdot A \right].$$

$D^2 \neq 0$ : In this case we obtain for every  $k \in \mathbb{N}$

$$\begin{aligned} (A - s/2 \cdot \text{Id})^{2k} &= D^{2k} \cdot \text{Id} \\ (A - s/2 \cdot \text{Id})^{2k+1} &= \frac{1}{D} D^{2k+1} (A - s/2 \cdot \text{Id}). \end{aligned}$$

Thus

$$\begin{aligned} e^{t(A - s/2 \cdot \text{Id})} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} (A - s/2 \cdot \text{Id})^k = \sum_{k \text{ even}} (\dots) + \sum_{k \text{ odd}} (\dots) \\ &= \cosh(tD) \cdot \text{Id} + \frac{1}{D} \sinh(tD) (A - s/2 \cdot \text{Id}) \\ &= \frac{1}{D} \sinh(tD) \cdot A + \left( \cosh(tD) - \frac{s}{2D} \sinh(tD) \right) \cdot \text{Id} \end{aligned}$$

by using the power series representation of the hyperbolic functions  $\sinh$  and  $\cosh$ .

As summary we obtain

$$e^{tA} = \begin{cases} e^{t \cdot s/2} \left[ \frac{1}{D} \sinh(tD) \cdot A + \left( \cosh(tD) - \frac{s}{2D} \sinh(tD) \right) \cdot \text{Id} \right], & \text{if } D \neq 0, \\ e^{t \cdot s/2} \left[ \left(1 - \frac{s}{2} t\right) \cdot \text{Id} + t \cdot A \right], & \text{if } D = 0. \end{cases}$$

In case  $A$  is a real  $2 \times 2$ -matrix and if  $D \neq 0$ , then  $D^2 = s^2/4 - d$  can be positive or negative.

$D^2 > 0$ : Since  $\sinh$  is an odd function, we can choose  $D > 0$  in the formula for  $e^{tA}$ . Furthermore,  $\sinh$  and  $\cosh$  are (by definition) linear combinations of  $e^{tD}$  and  $e^{-tD}$  simplifying our formula (see below).

$D^2 < 0$ : In this case  $D = \pm i|D|$ . For  $z \in \mathbb{C}$  we have the following identities

$$\sinh(iz) = \frac{1}{2}(e^{iz} - e^{-iz}) = i \sin(z) \quad \text{and} \quad \cosh(iz) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos(z).$$

Hence  $\sinh$  is an odd function,  $\sinh(i|D|t) = i \sin(t|D|)$  and  $\cosh(i|D|t) = \cos(|D|t)$ .

Here are some simple examples.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \implies e^{tA} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \implies e^{tA} = \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies e^{tA} = \begin{pmatrix} \cos(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}.$$

### 2.3 Multiplication Semigroups

Many Banach spaces appearing in applications are Banach spaces of (real or) complex valued functions over a set  $X$ . As the most standard examples of these “function spaces”, we mention the space  $C_0(X)$  of all continuous complex valued functions vanishing at infinity on a locally compact space  $X$ , or the spaces  $L^p(X, \Sigma, \mu)$ ,  $1 \leq p \leq \infty$ , of all (equivalence classes of)  $p$ -integrable functions on a  $\sigma$ -finite measure space  $(X, \Sigma, \mu)$ .

On these function spaces  $E = C_0(X)$ , resp.  $E = L^p(X, \Sigma, \mu)$ , there is a simple way to define *multiplication operators*.

Take a continuous, resp. measurable function  $q: X \rightarrow \mathbb{C}$  and define

$$M_q f := q \cdot f, \quad \text{i.e.,} \quad M_q f(x) := q(x) \cdot f(x) \quad \text{for } x \in X$$

and for every  $f$  in the *maximal* domain  $D(M_q) := \{g \in E : q \cdot g \in E\}$ .

This natural domain is a dense subspace of  $C_0(X)$ , resp.  $L^p(X, \Sigma, \mu)$ , for  $1 \leq p < \infty$ . Moreover,  $(M_q, D(M_q))$  is a closed operator. This is easy in case  $E = C_0(X)$ .

For  $E = L^p(X, \Sigma, \mu)$ ,  $1 \leq p < \infty$ , we consider a sequence  $(f_n) \subset E$  such that  $\lim_{n \rightarrow \infty} f_n = f \in E$  and  $\lim_{n \rightarrow \infty} q f_n =: g \in E$ . Choose a subsequence  $(f_{n(k)})_{k \in \mathbb{N}}$  such that  $\lim_{k \rightarrow \infty} f_{n(k)}(x) = f(x)$  and  $\lim_{k \rightarrow \infty} q(x) f_{n(k)}(x) = g(x)$  for  $\mu$ -almost every  $x \in X$ . Then  $g = q \cdot f$  and  $f \in D(M_q)$ , i.e.,  $M_q$  is closed.

For such multiplication operators many properties can be checked quite directly. For example, the following statements are equivalent.

- (a)  $M_q$  is bounded.
- (b)  $q$  is  $(\mu$ -essentially) bounded.

One has  $\|M_q\| = \|q\|_\infty$  in this situation. Observe that on spaces  $C(K)$ ,  $K$  compact, there are no densely defined, unbounded multiplication operators.

By defining the multiplication semigroups

$$T(t)f(x) := \exp(t \cdot q(x))f(x), \quad x \in X, f \in E,$$

one obtains the following characterizations.

**Proposition** *Let  $M_q$  be a multiplication operator on  $E = C_0(X)$  or  $E = L^p(X, \Sigma, \mu)$ ,  $1 \leq p < \infty$ . Then the properties (a) and (b), resp. (a') and (b'), are equivalent.*

- (a)  $M_q$  generates a strongly continuous semigroup.
- (b)  $\sup\{\operatorname{Re}(q(x)) : x \in X\} < \infty$ .
- (a')  $M_q$  generates a uniformly continuous semigroup.
- (b')  $\sup\{|q(x)| : x \in X\} < \infty$ .

As a consequence one computes the growth bound of a multiplication semigroup as

$$\omega_0 = \sup\{\operatorname{Re}(q(x)) : x \in X\}$$

in the case  $E = C_0(X)$  and

$$\omega_0 = \mu\text{-ess-}\sup\{\operatorname{Re}(q(x)) : x \in X\}$$

in the case  $E = L^p(\mu)$ . It is a nice exercise to characterize those multiplication operators which generate strongly continuous groups.

We point out that the above results cover the cases of sequence spaces such as  $c_0$  or  $\ell^p$ ,  $1 \leq p < \infty$ . An abstract characterization of generators of multiplication semigroups will be given in C-II, Theorem 5.13.

## 2.4 Translation (Semi)Groups

Let  $E$  to be one of the following function spaces  $C_0(\mathbb{R}_+)$ ,  $C_0(\mathbb{R})$ ,  $L^p(\mathbb{R}_+)$  or  $L^p(\mathbb{R})$  for  $1 \leq p < \infty$ . Define  $T(t)$  to be the (left) translation operator

$$T(t)f(x) := f(x + t)$$

for  $x, t \in \mathbb{R}_+$ , resp.  $x, t \in \mathbb{R}$  and  $f \in E$ . Then  $(T(t))_{t \geq 0}$  is a strongly continuous semigroup, resp. group of contractions on  $E$  and its generator is the first derivative  $\frac{d}{dx}$  with *maximal* domain. In order to be more precise we have to distinguish the cases  $E = C_0$  and  $E = L^p$ .

The generator of the translation (semi)group on  $E = C_0(\mathbb{R}_+)$  is

$$Af := \frac{d}{dx}f = f'$$

$$D(A) := \{f \in E : f \text{ differentiable and } f' \in E\}.$$

**Proof** For  $f \in D(A)$  it follows that for every  $x \in \mathbb{R}_{(+)}$

$$\lim_{h \rightarrow 0} \frac{T(h)f(x) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

(uniformly in  $x$ ) and coincides with  $Af(x)$ . Therefore  $f$  is differentiable and  $f' \in E$ .

On the other hand, take  $f \in E$  differentiable such that  $f' \in E$ . Then

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leq \frac{1}{h} \int_x^{x+h} |f'(y) - f'(x)| dy,$$

where the last expression tends to zero uniformly in  $x$  as  $h \rightarrow 0$ . Thus  $f \in D(A)$  and  $f' = Af$ .  $\square$

The generator of the translation (semi)group on  $E = L^p(\mathbb{R}_+)$ ,  $1 \leq p < \infty$ , is

$$Af := \frac{d}{dx}f = f',$$

$$D(A) := \{f \in E : f \text{ absolutely continuous, } f' \in E\}.$$

**Proof** Take  $f \in D(A)$  such that  $\lim_{h \rightarrow 0} \frac{1}{h}(T(h)f - f) = g \in E$ . Since integration is continuous, we obtain for every  $a, b \in \mathbb{R}_{(+)}$  that

$$(*) \quad \frac{1}{h} \int_b^{b+h} f(x) dx - \frac{1}{h} \int_a^{a+h} f(x) dx = \int_a^b \frac{f(x+h) - f(x)}{h} dx$$

converges to  $\int_a^b g(x) dx$  as  $h \rightarrow 0+$ . But for almost all  $a, b$  the left hand side of  $(*)$  converges to  $f(b) - f(a)$ . By redefining  $f$  on a nullset we obtain

$$f(y) = \int_a^y g(x) dx + f(a), \quad y \in \mathbb{R}_{(+)},$$

which is an absolutely continuous function whose derivative is (almost everywhere) equal to  $g$ .

On the other hand, let  $f$  be absolutely continuous such that  $f' \in L^p$ . Then

$$\begin{aligned}
\lim_{h \rightarrow 0} \int \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right|^p dx &= \lim_{h \rightarrow 0} \int \left| \frac{1}{h} \int_0^h (f'(x+s) - f'(x)) ds \right|^p dx \\
&= \lim_{h \rightarrow 0} \int \left| \int_0^1 (f'(x+uh) - f'(x)) du \right|^p dx \\
&\leq \lim_{h \rightarrow 0} \int \int_0^1 |f'(x+uh) - f'(x)|^p du dx \\
&= \int_0^1 \lim_{h \rightarrow 0} \int |f'(x+uh) - f'(x)|^p dx du = 0,
\end{aligned}$$

hence  $f \in D(A)$ .  $\square$

## 2.5 Rotation Groups

On  $E = C(\Gamma)$ , resp.  $E = L^p(\Gamma, m)$ ,  $1 \leq p < \infty$ ,  $m$  Lebesgue measure we have canonical groups defined by rotations of the unit circle  $\Gamma$  with a certain period, i.e., for  $0 < \tau \in \mathbb{R}$  the operators

$$R_\tau(t)f(z) := f(e^{2\pi i t/\tau} \cdot z), \quad z \in \Gamma$$

yield a group  $(R_\tau(t))_{t \in \mathbb{R}}$  having period  $\tau$ , i.e.,  $R_\tau(\tau) = \text{Id}$ . As in Example 2.4 one shows that its generator has the form

$$\begin{aligned}
D(A) &= \{f \in E : f \text{ absolutely continuous, } f' \in E\}, \\
Af(z) &= (2\pi i/\tau) \cdot z \cdot f'(z).
\end{aligned}$$

An isomorphic copy of the group  $(R_\tau(t))_{t \in \mathbb{R}}$  is obtained if we consider

$$E = \{f \in C[0, 1] : f(0) = f(1)\}$$

resp.

$$E = L^p([0, 1])$$

and the group of *periodic translations*

$$T(t)f(x) := f(y) \quad \text{for } y \in [0, 1], y = x + t \bmod 1$$

with generator

$$D(A) := \{f \in E : f \text{ absolutely continuous, } f' \in E, Af := f'.\}$$

## 2.6 Nilpotent Translation Semigroups

Take  $E = L^p([0, \tau], m)$  for  $1 \leq p < \infty$  and define

$$T(t)f(x) := \begin{cases} f(x+t) & \text{if } x+t \leq \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then  $(T(t))_{t \geq 0}$  is a semigroup satisfying  $T(t) = 0$  for  $t \geq \tau$ . Its generator is still the first derivative  $A = \frac{d}{dx}$ , but with domain

$$D(A) = \{f \in E : f \text{ absolutely continuous, } f' \in E, f(\tau) = 0\}.$$

In fact, if  $f \in D(A)$ , then  $f$  is absolutely continuous with  $f' \in E$ . By Proposition 1.6(i), it follows that  $T(t)f$  is absolutely continuous and hence  $f(\tau) = 0$ .

## 2.7 One-dimensional Diffusion Semigroup

For the second derivative

$$Bf(x) := \frac{d^2}{dx^2}f(x) = f''(x)$$

we take the domain

$$D(B) := \{f \in C^2[0, 1] : f'(0) = f'(1) = 0\}$$

in the Banach space  $E = C[0, 1]$ . Then  $D(B)$  is dense in  $C[0, 1]$ , but closed for the graph norm. Obviously, each function

$$e_n(x) := \cos \pi n x, \quad n \in \mathbb{Z},$$

is contained in  $D(B)$  and is an eigenfunction of  $B$  pertaining to the eigenvalue  $\lambda_n := -\pi^2 n^2$ . The linear hull  $\text{span}\{e_n : n \in \mathbb{Z}\} =: E_0$  forms a subalgebra of  $D(B)$  which by the Stone-Weierstrass theorem is dense in  $E$ .

We now use  $e_n$  to define bounded linear operators

$$e_n \otimes e_n : f \mapsto \left( \int_0^1 f(x) e_n(x) dx \right) e_n = (f|e_n) e_n$$

satisfying  $\|e_n \otimes e_n\| \leq 1$  and  $(e_n \otimes e_n)(e_m \otimes e_m) = \delta_{n,m}(e_n \otimes e_n)$  for  $n \in \mathbb{Z}$ .

For  $t > 0$  we define

$$\begin{aligned}
T(t) &:= \sum_{n \in \mathbb{Z}} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n \\
&= e_0 \otimes e_0 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n,
\end{aligned}$$

or

$$T(t)f(x) = \int_0^1 k_t(x, y)f(y)dy$$

$$\text{where } k_t(x, y) = 1 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cos \pi n x \cos \pi n y.$$

The Jacobi identity

$$\begin{aligned}
w_t(x) &:= \left( \frac{1}{4\pi t} \right)^{\frac{1}{2}} \sum_{m \in \mathbb{Z}} \exp(-(x + 2m)^2 / 4t) \\
&= \frac{1}{2} + \sum_{n \in \mathbb{N}} \exp(-\pi^2 n^2 t) \cos \pi n x
\end{aligned}$$

and trigonometric relations show that

$$k_t(x, y) = w_t(x + y) + w_t(x - y)$$

which is a positive function on  $[0, 1]^2$ . Therefore  $T(t)$  is a bounded operator on  $C[0, 1]$  with

$$\|T(t)\| = \|T(t)\mathbb{1}\| = \sup_{x \in [0, 1]} \int_0^1 k_t(x, y)dy = 1.$$

From the behavior of  $T(t)$  on the dense subspace  $E_0$  it follows that  $(T(t))_{t \geq 0}$  with  $T(0) = \text{Id}$  is a strongly continuous semigroup on  $E$  and its generator  $A$  coincides with  $B$  on  $E_0$ . Finally, we observe that  $E_0$  is a core for  $(A, D(A))$  by Proposition 1.9(ii).

Consequently,  $(T(t))_{t \geq 0}$  is the semigroup generated by  $B$ .

## 2.8 n-dimensional Diffusion Semigroup

On  $E = L^p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ , the operators

$$\begin{aligned}
T(t)f(x) &:= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|x - y|^2 / 4t) f(y)dy \\
&= \mu_t * f(x)
\end{aligned}$$

for  $x \in \mathbb{R}^n$ ,  $t > 0$  and  $\mu_t(x) := (4\pi t)^{-n/2} \exp(-|x|^2/4t)$  form a strongly continuous semigroup.

In fact, the integral exists for every  $f \in L^p(\mathbb{R}^n)$  since  $\mu_t$  is an element of the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  of all rapidly decreasing smooth functions on  $\mathbb{R}^n$ .

Moreover,

$$\|T(t)f\|_p \leq \|\mu_t\|_1 \|f\|_p = \|f\|_p$$

by Young's inequality (Reed and Simon [10, p.28]). Hence  $\|T(t)\| \leq 1$  for every  $t > 0$ .

Next we observe that  $\mathcal{S}(\mathbb{R}^n)$  is dense in  $E$  and invariant under each  $T(t)$ . Therefore we can apply the Fourier transformation  $\mathbf{F}$  which leaves  $\mathcal{S}(\mathbb{R}^n)$  invariant and yields

$$\mathbf{F}(\mu_t * f) = (2\pi)^{n/2} \mathbf{F}(\mu_t) \cdot \mathbf{F}(f) = (2\pi)^{n/2} \hat{\mu}_t \cdot \hat{f}$$

where  $f \in \mathcal{S}(\mathbb{R}^n)$ ,  $\hat{f} = \mathbf{F}f \in \mathcal{S}(\mathbb{R}^n)$ .

In other words,  $\mathbf{F}$  transforms  $(T(t)|_{\mathcal{S}(\mathbb{R}^n)})_{t \geq 0}$  into a multiplication semigroup on  $\mathcal{S}(\mathbb{R}^n)$  which is pointwise continuous for the usual topology of  $\mathcal{S}(\mathbb{R}^n)$ . The generator, i.e., the right derivative at 0, of this semigroup is the multiplication operator

$$B\hat{f}(x) := -|x|^2 \hat{f}(x) \quad (x \in \mathbb{R}^n)$$

for every  $f \in \mathcal{S}(\mathbb{R}^n)$ .

Applying the inverse Fourier transformation and observing that the topology of  $\mathcal{S}(\mathbb{R}^n)$  is finer than the topology induced from  $L^p(\mathbb{R}^n)$ , we obtain that  $(T(t))_{t \geq 0}$  is a semigroup which is strongly continuous (use Remark 1.2 (iii) on p. 4).

Its generator  $A$  coincides with

$$\Delta f(x) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f(x_1, \dots, x_n)$$

for every  $f \in \mathcal{S}(\mathbb{R}^n)$ . Since  $\mathcal{S}(\mathbb{R}^n)$  is  $(T(t))$ -invariant, we have determined the generator on a core of its domain (see Proposition 1.9 (ii)). In particular, the above semigroup solves the initial value problem for the *heat equation*

$$\frac{\partial}{\partial t} f(x, t) = \Delta f(x, t), \quad f(x, 0) = f_0(x), \quad x \in \mathbb{R}^n.$$

For the analogous discussion of the unitary group on  $L^2(\mathbb{R}^n)$  generated by

$$C := i\Delta$$

we refer to Section IX.7 in Reed and Simon [10].

Examples analogous to 2.7 are valid in  $L^p[0, 1]$ , resp. to 2.8 in  $C_0(\mathbb{R}^n)$ .

### 3 Standard Constructions

Starting with a semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  it is possible to construct new semigroups on spaces naturally associated with  $E$ . Such constructions will be important technical devices in many of the subsequent proofs. Although most of these constructions are rather routine, we present in the sequel a systematic account of them for the convenience of the reader.

We always start with a semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ , and denote its generator by  $A$  on the domain  $D(A)$ .

#### 3.1 Similar Semigroups

There is an easy way how to obtain different (but isomorphic) semigroups out of a given semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ .

Let  $V$  be an isomorphism from  $E$  onto  $E$ . Then  $S(t) := VT(t)V^{-1}$ ,  $t \geq 0$ , defines a strongly continuous semigroup. If  $A$  is the generator of  $(T(t))_{t \geq 0}$ , then

$$B := VAV^{-1} \text{ with domain } D(B) := \{f \in E : V^{-1}f \in D(A)\}$$

is the generator of  $(S(t))_{t \geq 0}$ .

#### 3.2 The Rescaled Semigroup

For fixed  $\lambda \in \mathbb{C}$  and  $\alpha > 0$  the operators

$$S(t) := \exp(\lambda t)T(\alpha t)$$

yield a new semigroup having generator

$$B := \alpha A + \lambda \text{Id} \text{ with } D(B) = D(A).$$

This *rescaled semigroup* enjoys most of the properties of the original semigroup and the same is true for the corresponding generators. However, by using this procedure certain constants associated with  $(T(t))_{t \geq 0}$  and  $A$  can be normalized. For example, by this rescaling we may suppose, in many cases without loss of generality, that the growth bound  $\omega_0$  is zero.

Another application is the following. For  $\lambda \in \mathbb{C}$  and  $S(t) := \exp(-\lambda t)T(t)$  the formulas (1.3) and (1.4) yield:

$$e^{-\lambda t}T(t)f - f = (A - \lambda) \int_0^t e^{-\lambda s}T(s)f \, ds \quad \text{or}$$

$$(e^{\lambda t} - T(t))f = (\lambda - A) \int_0^t e^{\lambda(t-s)}T(s)f \, ds \quad \text{for } f \in E,$$

and

$$e^{-\lambda t}T(t)f - f = \int_0^t e^{-\lambda s}T(s)(A - \lambda)f \, ds \quad \text{or}$$

$$(e^{\lambda t} - T(t))f = \int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)f \, ds \quad \text{for } f \in D(A).$$

### 3.3 The Subspace Semigroup

Assume  $F$  to be a closed  $(T(t))$ -invariant or, equivalently,  $R(\lambda, A)$ -invariant for  $\lambda \in \mathbb{C}$ ,  $\operatorname{Re}(\lambda) > \omega_0$ , subspace of  $E$ . Then the semigroup  $(T(t)|_F)_{t \geq 0}$  of all restrictions  $T(t)|_F := T(t)|_F$  is strongly continuous on  $F$ . If  $(A, D(A))$  denotes the generator of  $(T(t))_{t \geq 0}$  it follows from the  $(T(t))$ -invariance and closedness of  $F$  that  $A$  maps  $D(A) \cap F$  into  $F$ . Therefore

$$A|_F := A|_{(D(A) \cap F)} \text{ with domain } D(A|_F) := D(A) \cap F$$

is the generator of  $(T(t)|_F)$ . Conversely, if  $F$  is a closed *linear subspace* of  $E$  with  $A(D(A) \cap F) \subset F$  such that  $A|_F$  is a generator on  $F$ , then  $F$  is  $(T(t))$ -invariant.

An  $A$ -invariant subspace need not necessarily be  $(T(t))$ -invariant: Take for example the translation group with  $(T(t)f)(x) = f(x + t)$  on  $E = C_0(\mathbb{R})$  and the subspace  $F := \{f \in E : f(x) = 0 \text{ for } x \leq 0\}$ .

### 3.4 The Quotient Semigroup

Let  $F$  be a closed  $(T(t))$ -invariant subspace of  $E$  and consider the quotient space  $E/_F := E/F$  with quotient map  $q: E \rightarrow E/_F$ . The quotient operators

$$T(t)_/q(f) := q(T(t)f), \quad f \in E,$$

are well defined and form a strongly continuous semigroup  $(T(t)_/)_{t \geq 0}$  on  $E/_F$ . For the generator  $(A/_/, D(A/_/))$  of  $(T(t)_/)_{t \geq 0}$  the following holds:

$$D(A/_/) = q(D(A)) \quad \text{and} \quad A/_/q(f) = q(Af)$$

for every  $f \in D(A)$ . Here we use the fact that every  $\hat{f} := q(f) \in D(A_)$  can be written as

$$\hat{f} = \int_0^\infty e^{-\lambda s} \hat{T}(s) \hat{g} \, ds = \int_0^\infty e^{-\lambda s} q(T(s)g) \, ds = q\left(\int_0^\infty e^{-\lambda s} T(s)g \, ds\right) = q(h)$$

where  $h \in D(A)$  and  $\lambda > \omega$  (see Proposition 1.6). In particular we point out that for every  $\hat{f} \in D(A_)$  there exist representatives  $f \in \hat{f}$  belonging to  $D(A)$ .

**Example** We start with the Banach space  $E = L^1(\mathbb{R})$  and the translation semigroup  $(T(t))_{t \geq 0}$  where  $T(t)f(x) := f(x+t)$  (see Example 2.4). Then  $L^1((-\infty, 1])$  can be identified with the closed,  $(T(t))$ -invariant subspace

$$J := \{f \in E : f(x) = 0 \text{ for } 1 < x < \infty\}.$$

There we obtain the subspace semigroup

$$(T(t)|_J)f(x) = \mathbb{1}_{(-\infty, 1]}(x) \cdot f(x+t),$$

where  $f \in L^1((-\infty, 1])$ ,  $-\infty < x \leq 1$  and  $t \geq 0$ .

By 2.4 and 3.2 its generator is

$$A|_J f := f'$$

for  $f \in D(A|_J) := \{f \in E : f \in \text{AC with } f' \in E \text{ and } f(x) = 0 \text{ for } x \geq 1\}$ .

Next we identify  $L^1([0, 1])$  with the quotient space  $L^1((-\infty, 1])I$  where

$$I := \{f \in L^1((-\infty, 1]) : f(x) = 0 \text{ for } 0 \leq x \leq 1\}.$$

Again  $I$  is invariant for the restricted semigroup  $(T(t)|_I)$  and the quotient semigroup  $(T(t)|_J)$  on  $L^1([0, 1])$  is the nilpotent translation semigroup as in Example 2.6. In particular it follows that the domain of its generator is

$$D(A|_J) = \{f \in L^1([0, 1]) : f \in \text{AC with } f' \in L^1([0, 1]) \text{ and } f(1) = 0\}.$$

### 3.5 The Adjoint Semigroup

The adjoint operators  $(T(t)')_{t \geq 0}$  of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  form a semigroup on  $E'$  which need, however, not be strongly continuous.

**Example** Take the translation operators  $T(t)f(x) = f(x+t)$  on  $E = L^1(\mathbb{R})$  (see Example 2.4) and their adjoints

$$T(t)'f(x) = f(x-t)$$

on  $E' = L^\infty(\mathbb{R})$ . Then  $(T(t)')_{t \in \mathbb{R}}$  is a one-parameter group which is not strongly continuous on  $L^\infty(\mathbb{R})$  (take any non-trivial characteristic function).

Since the semigroup  $(T(t)')_{t \geq 0}$  is obviously *weak\*-continuous* in the sense that

$$\lim_{t \rightarrow s} \langle f, (T(t)') - T(s)'\rangle \varphi = 0$$

for every  $f \in E$ ,  $\varphi \in E'$  and  $s, t \geq 0$ , it is natural to associate  $(T(t)')_{t \geq 0}$  its *weak\*-generator*

$$\begin{aligned} A'\varphi &:= \sigma(E', E)\text{-}\lim \frac{1}{h} (T(h)'\varphi - \varphi) \text{ for every } \varphi \text{ in the domain} \\ D(A') &:= \{\varphi \in E' : \sigma(E', E)\text{-}\lim \frac{1}{h} (T(h)'\varphi - \varphi) \text{ exists}\}. \end{aligned}$$

This operator coincides with the *adjoint* of the generator  $(A, D(A))$ , i.e.,

$$D(A') = \{\varphi \in E' : \text{there exists } \psi \in E' \text{ such that } \langle f, \psi \rangle = \langle Af, \varphi \rangle \text{ for all } f \in D(A)\}$$

and  $A'\varphi = \psi$ . In particular,  $A'$  is a closed and  $\sigma(E', E)$ -densely defined operator in  $E'$ .

It follows that the resolvent  $R(\lambda, A')$  of  $A'$  is  $R(\lambda, A)'$  (Kato [6, Theorem III.5.30]). In particular, the spectra  $\sigma(A)$  and  $\sigma(A')$  coincide.

However, it is still necessary in some situations to have strong continuity for the adjoint semigroup. In order to achieve this we restrict  $T(t)'$  to an appropriate subspace of  $E'$ .

**Definition** (Phillips [9]) The *semigroup dual* of the Banach space  $E$  with respect to the strongly continuous semigroup  $(T(t))_{t \geq 0}$  is

$$E^* := \{\varphi \in E' : \|\cdot\| \text{-}\lim_{t \rightarrow 0} T(t)'\varphi = \varphi\}.$$

The adjoint semigroup on  $E^*$  is given by the operators

$$T(t)^* := T(t)'|_{E^*}, \quad t \geq 0.$$

Since  $(T(t)^*)_{t \geq 0}$  is strongly continuous on  $E^*$  we call its generator  $(A^*, D(A^*))$  the *adjoint generator*.

The above definition makes sense since  $E^*$  is norm-closed in  $E'$  and  $(T(t)')$ -invariant. The main point is that  $E^*$  is still reasonably large. In fact, since  $\int_0^t T(s)'\varphi \, ds$ , understood in the weak sense, is contained in  $E^*$  for every  $\varphi \in E'$  and  $t \geq 0$ , it follows that

$$\sup\{\langle f, \varphi \rangle : \varphi \in E^*, \|\varphi\| \leq 1\} \leq \|f\| \leq M \cdot \sup\{\langle f, \varphi \rangle : \varphi \in E^*, \|\varphi\| \leq 1\}$$

where  $M := \limsup_{t \rightarrow 0} \|T(t)\|$ . In particular,  $E^*$  separates  $E$ , i.e.,  $E^*$  is  $\sigma(E', E)$ -dense in  $E'$ . In addition, the norm estimate given above yields

$$\|T(t)^*\| \leq \|T(t)\| \leq M\|T(t)^*\| \quad \text{for all } t \geq 0.$$

In the following proposition we describe the relation between  $A^*$  and  $A'$ .

**Proposition** *For the adjoint generator  $A^*$  of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on  $E$  the following assertions hold.*

- (i)  $E^*$  is the  $\|\cdot\|$ -closure of  $D(A')$ .
- (ii)  $D(A^*) = \{\varphi \in D(A') : A'\varphi \in E^*\}$ .
- (iii)  $A^*$  and  $A'$  coincide on  $D(A^*)$ .

**Proof** (i) Take  $\varphi \in D(A')$  fixed. For every  $f \in D(A)$  with  $\|f\| \leq 1$  we define a continuously differentiable function

$$t \mapsto \xi_f(t) := \langle T(t)f, \varphi \rangle$$

on  $[0, 1]$  with derivative  $\xi'_f(t) = \langle T(t)Af, \varphi \rangle = \langle T(t)f, A'\varphi \rangle$ .

Since  $\{\xi'_f(t) : t \in [0, 1], f \in D(A), \|f\| \leq 1\}$  is bounded, it follows that the set

$$\{\xi_f : f \in D(A), \|f\| \leq 1\}$$

is equicontinuous at 0, i.e., for every  $\varepsilon > 0$  there exists  $0 < t_0 < 1$  such that

$$|\xi_f(s) - \xi_f(0)| = |\langle f, T(s)' \varphi - \varphi \rangle| < \varepsilon$$

for every  $0 \leq s \leq t_0$  and  $f \in D(A)$ ,  $\|f\| \leq 1$ . But this implies  $\|T(s)' \varphi - \varphi\| < \varepsilon$  and hence  $\varphi \in E^*$ .

Conversely, take  $\psi \in E^*$ . Then  $\frac{1}{t} \int_0^t T(s)' \psi \, ds$ ,  $t > 0$ , belongs to  $D(A')$  and norm converges to  $\psi$  as  $t \rightarrow 0$ , i.e.,  $\psi$  belongs to the norm closure of  $D(A')$ .

(ii) and (iii): Since the weak\* topology on  $E'$  is weaker than the norm topology, it follows that  $A'$  is an extension of  $A^*$ . Now take  $\varphi \in D(A')$  such that  $A'\varphi \in E^*$ . As above define the functions  $\xi_f$ . The assumption on  $\varphi$  implies the set of all derivatives

$$\{\xi'_f : f \in D(A), \|f\| \leq 1\}$$

to be equicontinuous at  $t = 0$ . This means that for every  $\varepsilon > 0$  there exists  $0 < t_0 < 1$  such that  $|f'_f(0) - f'_f(s)| < \varepsilon$  for every  $f \in D(A)$ ,  $\|f\| \leq 1$  and  $0 < s < t_0$ . In particular,

$$\varepsilon > \left| f'_f(0) - \frac{1}{s} (\xi_f(s) - \xi_f(0)) \right| = \left| \left\langle f, A'\varphi - \frac{1}{s} (T(s)' \varphi - \varphi) \right\rangle \right|,$$

hence

$$\varepsilon > \left\| A' \varphi - \frac{1}{s} (T(s)' \varphi - \varphi) \right\|$$

for all  $0 \leq s \leq t_0$ . From this it follows that  $\varphi \in D(A^*)$ .  $\square$

On reflexive Banach spaces we have  $A^* = A'$  by the above proposition. In other cases this construction is more interesting.

**Example (continued)** The adjoints of the (left) translation  $T(t)$  on  $E = L^1(\mathbb{R})$  are the (right) translations  $T(t)'$  on  $E' = L^\infty(\mathbb{R})$ . The largest subspace of  $L^\infty(\mathbb{R})$  on which these translations form a strongly-continuous semigroup with respect to the sup-norm, is the space of all bounded uniformly continuous functions on  $\mathbb{R}$ , i.e.,  $E^* = C_{bu}(\mathbb{R})$ .

Calculating  $D(A')$  and  $D(A^*)$  respectively, one obtains

$$\begin{aligned} D(A') &= \{f \in L^\infty(\mathbb{R}) : f \in AC, f' \in L^\infty(\mathbb{R})\}, \\ D(A^*) &= \{f \in L^\infty(\mathbb{R}) : f \in C^1(\mathbb{R}), f' \in C_{bu}(\mathbb{R})\}. \end{aligned}$$

Obviously, the function  $x \mapsto |\sin x|$  belongs to  $D(A')$ , but not to  $D(A^*)$ .

### 3.6 The Associated Sobolev Semigroups

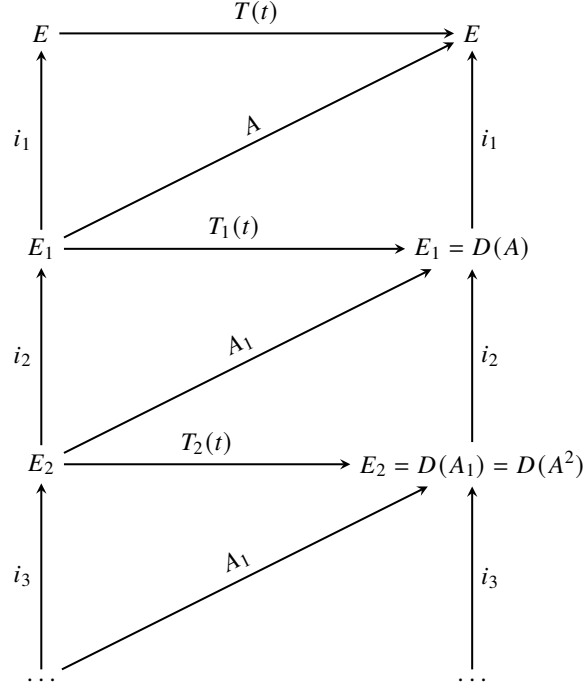
Since the generator  $A$  of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  is closed, its domain  $D(A)$  becomes a Banach space for the graph norm

$$\|f\|_1 := \|f\| + \|Af\|.$$

We denote this Banach space by  $E_1$  and the continuous injection from  $E_1$  into  $E$  by  $i_1$ . Since  $E_1$  is invariant under  $(T(t))_{t \geq 0}$ , apply Proposition 1.6 (i), it makes sense to consider the semigroup  $(T_1(t))_{t \geq 0}$  of all restrictions  $T_1(t) := T(t)|_{E_1}$ . The results of Proposition 1.6 imply that  $T_1(t) \in \mathcal{L}(E_1)$  and  $\|T_1(t)f - f\|_1 \rightarrow 0$  as  $t \rightarrow 0$  for every  $f \in E_1$ . Thus  $(T_1(t))_{t \geq 0}$  is a strongly continuous semigroup on  $E_1$  and has a generator denoted by  $(A_1, D(A_1))$ .

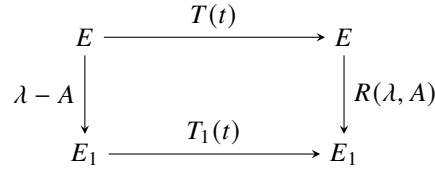
Using 1.6 again we see that  $A_1$  is the restriction of  $A$  to  $E_1$  with maximal domain, i.e.,  $D(A_1) = \{f \in E_1 : Af \in E_1\} = D(A^2)$  and  $A_1 f = Af$  for every  $f \in D(A_1)$ .

It is now possible to repeat this construction in order to obtain Banach spaces  $E_n$  and semigroups  $(T_n(t))_{t \geq 0}$  with generators  $(A_n, D(A_n))$  which are related as visualized in the following diagram.



For the translation semigroup on  $L^p(\mathbb{R})$  (see 2.3) the above construction leads to the usual *Sobolev spaces*. Therefore we might call  $E_n$  the *n-th Sobolev space* and  $(T_n(t))_{t \geq 0}$  the *n-th Sobolev semigroup* associated to  $E$  and  $(T(t))_{t \geq 0}$ .

**Remark** (i) For  $\lambda \in \varrho(A)$  the operator  $(\lambda - A)$  and the resolvent  $R(\lambda, A)$  are isomorphisms from  $E_1$  onto  $E$ , resp. from  $E$  onto  $E_1$  (show that  $\|\cdot\|_1$  and  $\|\cdot\|_\lambda$  with  $\|\cdot\|_\lambda := \|(\lambda - A) \cdot\|$  are equivalent). In addition, the following diagram commutes.



Therefore all Sobolev semigroups  $(E_n, T_n(t))_{t \geq 0}$ ,  $n \in \mathbb{N}$ , are isomorphic.

(ii) For  $\lambda \in \varrho(A)$  consider the norm

$$\|f\|_{-1} := \|R(\lambda, A)f\|$$

for every  $f \in E$  and define  $E_{-1}$  as the completion of  $E$  for  $\|\cdot\|_{-1}$ . Then  $(T(t))_{t \geq 0}$  extends continuously to a strongly continuous semigroup  $(T_{-1}(t))_{t \geq 0}$  on  $E_{-1}$  and the above diagram can be extended to the negative integers.

### 3.7 The $\mathcal{F}$ -Product Semigroup

It is standard in functional analysis to consider a sequence of points in a certain space as a point in a new and larger space. In particular such a method can serve to convert an approximate eigenvector of a linear operator into an eigenvector. Occasionally we will need such a construction and refer to Section V.1 of Schaefer [11].

If we try to adapt this construction to strongly continuous semigroups we encounter the difficulty that the semigroup extended to the larger space will not remain strongly continuous. An idea already used in 3.4 will help to overcome this difficulty.

Let  $\mathcal{T} = (T(t))_{t \geq 0}$  be a strongly continuous semigroup on the Banach space  $E$ . Denote by  $m(E)$  the Banach space of all bounded  $E$ -valued sequences endowed with the norm

$$\|(f_n)_{n \in \mathbb{N}}\| := \sup\{\|f_n\| : n \in \mathbb{N}\}.$$

It is clear that every  $T(t)$  extends canonically to a bounded linear operator

$$\hat{T}(t)(f_n) := (T(t)f_n)$$

on  $m(E)$ , but the semigroup  $(\hat{T}(t))_{t \geq 0}$  is strongly continuous if and only if  $T$  has a bounded generator. Therefore we restrict our attention to the closed,  $(\hat{T}(t))$ -invariant subspace

$$m^{\mathcal{T}}(E) := \{(f_n) \in m(E) : \lim_{t \rightarrow 0} \|T(t)f_n - f_n\| = 0 \text{ uniformly for } n \in \mathbb{N}\}.$$

Then the restricted semigroup

$$\hat{T}(t)(f_n) = (T(t)f_n), \quad (f_n) \in m^{\mathcal{T}}(E)$$

is strongly continuous and we denote its generator by  $(\hat{A}, D(\hat{A}))$ .

The following lemma shows that  $\hat{A}$  is obtained canonically from  $A$ .

**Lemma** *For the generator  $\hat{A}$  of  $(\hat{T}(t))_{t \geq 0}$  on  $m^{\mathcal{T}}(E)$  one has the following properties.*

- (i)  $D(\hat{A}) = \{(f_n) \in m^{\mathcal{T}}(E) : f_n \in D(A) \text{ and } (Af_n) \in m^{\mathcal{T}}(E)\},$
- (ii)  $\hat{A}(f_n) = (Af_n) \text{ for } (f_n) \in D(\hat{A}).$

For the proof we refer to Lemma 1.4. of Derndinger [2].

Now let  $\mathcal{F}$  be any filter on  $\mathbb{N}$  finer than the Fréchet filter (i.e., the filter of sets with finite complement. In most cases  $\mathcal{F}$  will be either the Fréchet filter or some free ultra filter). The space of all  $\mathcal{F}$ -null sequences in  $m(E)$ , i.e.,

$$c_{\mathcal{F}}(E) := \{(f_n) \in m(E) : \mathcal{F}\text{-}\lim \|f_n\| = 0\}$$

is closed in  $m(E)$  and invariant under  $(\hat{T}(t))_{t \geq 0}$ . We call the quotient spaces

$$E_{\mathcal{F}} := m(E)/c_{\mathcal{F}}(E) \quad \text{and} \quad E_{\mathcal{F}}^T := m^T(E)/(c_{\mathcal{F}}(E) \cap m^T(E))$$

the  $\mathcal{F}$ -product of  $E$  and the  $\mathcal{F}$ -product of  $E$  with respect to the semigroup  $T$ , respectively.

Thus  $E_{\mathcal{F}}^T$  can be considered as a closed linear subspace of  $E_{\mathcal{F}}$ . We have  $E_{\mathcal{F}}^T = E_{\mathcal{F}}$  if (and only if)  $T$  has a bounded generator.

The canonical quotient norm on  $E_{\mathcal{F}}$  is given by

$$\|(f_n) + c_{\mathcal{F}}(E)\| = \mathcal{F}\text{-}\limsup \|f_n\|.$$

We can apply Subsection 3.4 in order to define the  $\mathcal{F}$ -product semigroup  $(T_{\mathcal{F}}(t))_{t \geq 0}$  on  $E_{\mathcal{F}}^T$  by

$$T_{\mathcal{F}}(t)((f_n) + c_{\mathcal{F}}(E)) := (T(t)f_n) + (c_{\mathcal{F}}(E) \cap m^T(E)).$$

Thus  $T_{\mathcal{F}}(t)$  is the restriction of  $T(t)_F$  where  $T(t)_F$  denotes the canonical extension of  $T(t)$  to the  $\mathcal{F}$ -product  $E_{\mathcal{F}}$ . But note that  $(T(t)_F)_{t \geq 0}$  is not strongly continuous unless  $T$  has a bounded generator.

With the canonical injection  $j: f \mapsto (f, f, f, \dots) + c_{\mathcal{F}}(E)$  from  $E$  into  $E_{\mathcal{F}}^T$  the operators  $T_{\mathcal{F}}(t)$  are extensions of  $T(t)$  satisfying  $\|T_{\mathcal{F}}(t)\| = \|T(t)\|$ . The basic facts about the generator  $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$  of  $(T_{\mathcal{F}}(t))_{t \geq 0}$  follow from 3.3 and are collected in the following proposition.

**Proposition** *For the generator  $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$  of the  $\mathcal{F}$ -product semigroup the following holds.*

- (i)  $D(A_{\mathcal{F}}) = \{(f_n) + c_{\mathcal{F}}(E) : f_n \in D(A); (f_n), (Af_n) \in m^T(E)\},$
- (ii)  $A_{\mathcal{F}}((f_n) + c_{\mathcal{F}}(E)) = (Af_n) + c_{\mathcal{F}}(E).$

In case  $A$  is a bounded operator then  $D(A_{\mathcal{F}}) = E_{\mathcal{F}}^T = E_{\mathcal{F}}$  and  $A_{\mathcal{F}}$  is the canonical extension of  $A$  to  $E_{\mathcal{F}}$ .

We will show in A-III, Section 4.2 that the above construction preserves and even improves many spectral properties of the semigroup and its generator.

### 3.8 The Tensor Product Semigroup

Real- or complex-valued functions of two variables  $x, y$  are often limits of functions of the form  $\sum_{i=1}^n f_i(x)g_i(y)$  which, to some extent, allows one to consider the variables  $x$  and  $y$  separately. Since algebraic manipulation with these latter functions is governed by the formal rules of a tensor product, it is customary to identify (for example) the function

$$(x, y) \mapsto f(x)g(y)$$

with the tensor product  $f \otimes g$  and to consider limits of linear combinations of such functions as elements of a completed tensor product.

To be more precise, we briefly present the most important examples for this situation.

**Examples** (i) Let  $(X, \Sigma, \mu)$  and  $(Y, \Omega, \nu)$  be measure spaces. If we identify for  $f_i \in L^p(\mu)$ ,  $g_i \in L^p(\nu)$  the elements  $\sum_{i=1}^n f_i \otimes g_i$  of the tensor product

$$L^p(\mu) \otimes L^p(\nu)$$

with the (class of  $\mu \times \nu$ -a.e.-defined) functions

$$(x, y) \mapsto \sum_{i=1}^n f_i(x)g_i(y),$$

then  $L^p(\mu) \otimes L^p(\nu)$  becomes a dense subspace of  $L^p(X \times Y, \Sigma \times \Omega, \mu \times \nu)$  for  $1 \leq p < \infty$ .

(ii) Similarly, let  $X, Y$  be compact spaces. Then  $C(X) \otimes C(Y)$  becomes a dense subspace of  $C(X \times Y)$  by identifying, for  $f \in C(X)$  and  $g \in C(Y)$ ,  $f \otimes g$  with the function

$$(x, y) \mapsto f(x)g(y).$$

We do not intend to go deeper into the quite sophisticated problems related to normed tensor products of general Banach spaces, but will rather confine ourselves to the discussion of certain special cases. These will always be related to one of the following standard methods to define a norm on the tensor product of two Banach spaces  $E, F$ .

Let  $u := \sum_{i=1}^n f_i \otimes g_i$  be an element of  $E \otimes F$ . Then

(i)  $\|u\|_\pi := \inf\{\sum_{j=1}^m \|h_j\| \|k_j\| : u = \sum_{j=1}^m h_j \otimes k_j, h_j \in E, k_j \in F\}$  defines the *greatest cross norm*  $\pi$  on  $E \otimes F$ .

(ii)  $\|u\|_\varepsilon := \sup\{\langle u, \varphi \otimes \psi \rangle : \varphi \in E', \psi \in F', \|\varphi\|, \|\psi\| \leq 1\}$  defines the *least cross norm*  $\varepsilon$  on  $E \otimes F$ . Here,  $\langle u, \varphi \otimes \psi \rangle$  denotes the canonical bilinear form on  $(E \otimes F) \times (E' \otimes F')$ , i.e.,  $\langle \sum_{i=1}^n f_i \otimes g_i, \varphi \otimes \psi \rangle = \sum_{i=1}^n \langle f_i, \varphi \rangle \langle g_i, \psi \rangle$ .

(iii) If  $E$  and  $F$  are Hilbert spaces,  $\|u\|_h = (u|u)_h^{1/2}$ , where the scalar product  $(\cdot|\cdot)_h$  is defined as in (ii), defines the *Hilbert norm*  $h$  on  $E \otimes F$ .

In the following we write  $E \otimes_\alpha F$  for the tensor product of  $E$  and  $F$  endowed—with if applicable—with one of the norms  $\pi, \varepsilon, h$  just defined. In each case one has  $\|f \otimes g\| = \|f\| \|g\|$  for  $f \in E, g \in F$ .

By  $E \widetilde{\otimes}_\alpha F$  we mean the completion of  $E \otimes_\alpha F$ . Moreover we recall how examples (i) and (ii) above fit into this pattern

$$L^1(\mu \otimes \nu) = L^1(\mu) \widetilde{\otimes}_\pi L^1(\nu), \quad L^2(\mu \otimes \nu) = L^2(\mu) \widetilde{\otimes}_h L^2(\nu),$$

$$C(X \times Y) = C(X) \widetilde{\otimes}_\varepsilon C(Y).$$

Finally, we point out that for any  $S \in \mathcal{L}(E)$ ,  $T \in \mathcal{L}(F)$ , the mapping

$$\sum_{i=1}^n f_i \otimes g_i \mapsto \sum_{i=1}^n S f_i \otimes T g_i$$

defined on  $E \otimes F$  is linear and continuous on  $E \otimes_\alpha F$ , hence has a continuous extension to  $E \widetilde{\otimes}_\alpha F$ . This operator, as well as its continuous extension, will be denoted by  $S \otimes T$  and satisfies  $\|S \otimes T\| = \|S\| \|T\|$ . The notation  $A \otimes B$  will also be used in the obvious way if  $A$  and  $B$  are not necessarily bounded operators on  $E$  and  $F$ . We are now ready to consider semigroups induced on the tensor product.

**Proposition** *Let  $(S(t))_{t \geq 0}$  and  $(T(t))_{t \geq 0}$  be strongly continuous semigroups on Banach spaces  $E$ ,  $F$ , and let  $A$ ,  $B$  be their generators. Then the family  $(S(t) \otimes T(t))_{t \geq 0}$  is a strongly continuous semigroup on  $E \widetilde{\otimes}_\alpha F$ . The closure of  $A \otimes \text{Id} + \text{Id} \otimes B$ , defined on the core  $D(A) \otimes D(B)$ , is its generator.*

**Proof** It is immediately verified that  $(S(t) \otimes T(t))_{t \geq 0}$  is in fact a semigroup of operators on  $E \widetilde{\otimes}_\alpha F$ . The strong continuity need only be verified at  $t = 0$  and on elements of the form  $u = f \otimes g \in E \otimes F$ .

This verification being straightforward, there remains to show that the generator of  $(S(t) \otimes T(t))_{t \geq 0}$  is obtained as the closure of

$$(A \otimes \text{Id} + \text{Id} \otimes B, D(A) \otimes D(B)).$$

To this end, let  $f \in D(A)$  and  $g \in D(B)$ . Then

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{1}{h} (T(h) \otimes S(h)(f \otimes g) - f \otimes g) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} (T(h)f \otimes (S(h)g - g) + (T(h)f - f) \otimes g) \\ &= (f \otimes Bg) + (Af \otimes g). \end{aligned}$$

Since the elements of the form  $f \otimes g$ ,  $f \in D(A)$ ,  $g \in D(B)$ , generate the linear subspace  $D(A) \otimes D(B)$  of  $E \otimes_\alpha F$ , this subspace belongs to the domain of the generator. Moreover,  $D(A) \otimes D(B)$  is dense in  $E \widetilde{\otimes}_\alpha F$  and invariant under  $(S(t) \otimes T(t))_{t \geq 0}$ , hence it is a core of  $A \otimes \text{Id} + \text{Id} \otimes B$  by Proposition 1.9(ii).  $\square$

### 3.9 The Product of Commuting Semigroups

Let  $(S(t))_{t \geq 0}$  and  $(T(t))_{t \geq 0}$  be semigroups with generators  $A$  and  $B$ , respectively on some Banach space  $E$ . It is not difficult to see that the following assertions are equivalent.

- (a)  $S(t)T(t) = S(t)T(t)$  for all  $t \geq 0$ .
- (b)  $R(\mu, A)R(\mu, B) = R(\mu, B)R(\mu, A)$  for some  $\mu \in \varrho(A) \cap \varrho(B)$ .
- (c)  $R(\mu, A)R(\mu, B) = R(\mu, B)R(\mu, A)$  for all  $\mu \in \varrho(A) \cap \varrho(B)$ .

In that case  $U(t) = S(t)T(t)$  ( $t \geq 0$ ) defines a semigroup  $(U(t))_{t \geq 0}$ . Using Proposition 1.9(ii) on p. 7 one easily shows that  $D_0 := D(A) \cap D(B)$  is a core for its generator  $C$  and  $Cf = Af + Bf$  for all  $f \in D_0$ .

## Notes

For more complete information on semigroup theory we refer the reader to Hille and Phillips [5], to the monographs by Davies [1], Goldstein [3] and Pazy [8], to the survey article by Krein and Khazan [7], to the bibliography by Goldstein [4].

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## Chapter A-II

# Characterization of Semigroups on Banach Spaces

In this chapter two different problems are treated:

- (i) to characterize generators of strongly continuous semigroups;
- (ii) to characterize various properties of strongly continuous semigroups in terms of their generators.

In Section 1 the first problem is solved by finding conditions on the Cauchy problem associated with  $A$  and also by finding conditions on the resolvent of  $A$ . The second problem is treated by considering a hierarchy of smoothness properties of the semigroup.

Contraction semigroups are considered in Section 2. Here, the first problem has a simple and extremely useful solution: A densely defined operator  $A$  is the generator of a contraction semigroup if and only if  $A$  is dissipative and satisfies a range condition.

Our approach is quite general. We not only consider contractions with respect to the norm but also with respect to *half-norms*. This will allow us to obtain results on positive contraction semigroups simultaneously by choosing a suitable half-norm (cf. C-II, Section 1).

The last section contains a surprising result: on certain Banach spaces (e.g.,  $L^\infty$ ) only bounded operators are generators of strongly continuous semigroups.

## 1 The Abstract Cauchy Problem, Special Semigroups and Perturbation

by Wolfgang Arendt

Linear differential equations in Banach spaces are intimately connected with the

theory of one-parameter semigroups. In fact, given a closed linear operator  $A$  with dense domain  $D(A)$ , the following statement is true (with some reservation regarding a technical detail).

The abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) \quad (t \geq 0) \\ u(0) &= f \end{aligned}$$

has a unique solution for every  $f \in D(A)$  if and only if  $A$  is the generator of a strongly continuous semigroup.

This is one characterization of generators which illustrates their important role for applications. The fundamental Hille-Yosida theorem gives a different characterization in terms of the resolvent and yields a powerful tool for actually proving that a given operator is the generator of a semigroup.

Another problem we will treat here is how diverse properties of a semigroup can be described in terms of its generator. This is a reasonable question from the theoretical point of view (since the generator uniquely determines the semigroup). It is of interest from the practical point of view as well: the generator is the given object, defined by the differential equation. It is useful to dispose of conditions of the generator itself giving information on the solutions (which might not be known explicitly). We discuss smoothness properties such as analyticity, differentiability, norm continuity and compactness of the semigroup.

A frequent method to obtain new generators out of a given one is by perturbation. We will have a brief look at this circle of problems at the end of this section.

The results are explained and illustrated by examples. Proofs are only given when new aspects are presented which are not yet contained in the literature, otherwise we refer to the recent monographs Davies [11], Goldstein [15], Pazy [31].

### 1.1 The abstract Cauchy problem

Let  $A$  be a closed operator on a Banach space  $E$  and consider the *Abstract Cauchy Problem (ACP)*

$$(ACP) \quad \begin{cases} \dot{u}(t) &= Au(t) \quad (t \geq 0) \\ u(0) &= f. \end{cases}$$

By a solution of (ACP) for the initial value  $f \in D(A)$  we understand a continuously differentiable function  $u: [0, \infty) \rightarrow E$  satisfying  $u(0) = f$  and  $u(t) \in D(A)$  for all  $t \geq 0$  such that  $u'(t) = Au(t)$  for  $t \geq 0$ .

By A-I, Theorem.1.7 there exists a unique solution of (ACP) for all initial values in the domain  $D(A)$  whenever  $A$  is the generator of a strongly continuous semigroup. The converse does not hold (see Example 1.4. below). However, for the operator  $A_1$  on the Banach space  $E_1 = D(A)$  (see A-I, 3.6) with domain  $D(A_1) = D(A^2)$  given by  $A_1 f = Af$  ( $f \in D(A_1)$ ) the following holds.

**Theorem 1.1** *The following assertions are equivalent.*

- (a) *For every  $f \in D(A)$  there exists a unique solution of (ACP).*
- (b)  *$A_1$  is the generator of a strongly continuous semigroup.*

**Proof** (a)  $\implies$  (b): Assume that (a) holds, i.e., for every  $f \in D(A)$  there exists a unique solution  $u(\cdot, f) \in C^1([0, \infty), E)$  of (ACP). For  $f \in E_1$  define

$$T_1(t)f := u(t, f).$$

By the uniqueness of the solutions it follows that  $T_1(t)$  is a linear operator on  $E_1$  and  $T_1(s+t) = T_1(s)T_1(t)$ . Moreover, since  $u(\cdot, f) \in C^1$ , it follows that  $t \mapsto T_1(t)f$  is continuous from  $[0, \infty)$  into  $E_1$ . We show that  $T_1(t)$  is a continuous operator for all  $t > 0$ .

Let  $t > 0$ . Consider the mapping  $\eta: E_1 \rightarrow C([0, t], E_1)$  given by

$$\eta(f) = T_1(\cdot)f = u(\cdot, f).$$

We show that  $\eta$  has a closed graph.

In fact, let  $f_n \rightarrow f$  in  $E_1$  and  $\eta(f_n) = u(\cdot, f_n) \rightarrow v$  in  $C([0, t], E_1)$ . Then

$$u(s, f_n) = f_n + \int_0^s Au(r, f_n) dr.$$

Letting  $n \rightarrow \infty$  we obtain  $v(s) = f + \int_0^s Av(r) dr$  for  $0 \leq s \leq t$ .

Let

$$\tilde{v}(s) = \begin{cases} T_1(s-t)v(t) & \text{for } s > t, \\ \tilde{v}(s) = v(s) & 0 \leq s \leq t. \end{cases}$$

Then  $\tilde{v}$  is a solution of (ACP). It follows that  $\tilde{v}(s) = T_1(s)f$  for all  $s \geq 0$ . Hence  $v = \eta(f)$ .

We have shown that  $\eta$  has a closed graph and so  $\eta$  is continuous. This implies the continuity of  $T_1(t)$ . It remains to show that  $A_1$  is the generator of  $(T_1(t))_{t \geq 0}$ .

We first show that for  $f \in D(A^2)$  one has

$$AT_1(t)f = T_1(t)Af. \tag{1.1}$$

In fact, let  $v(t) = f + \int_0^t u(s, Af) ds$ . Then

$$\dot{v}(t) = u(t, Af) = Af + \int_0^t Au(s, Af) ds = A(f + \int_0^t u(s, Af) ds) = Av(t).$$

Since  $v(0) = f$ , it follows that  $v(t) = u(t, f)$ . Hence  $Au(t, f) = Av(t) = \dot{v}(t) = u(t, Af)$ . This is (1.1). Now denote by  $B$  the generator of  $(T_1(t))_{t \geq 0}$ . For  $f \in D(A^2)$ , we have

$$\lim_{t \rightarrow 0} \frac{T_1(t)f - f}{t} = Af$$

and by (1.1),

$$\lim_{t \rightarrow 0} A \frac{T_1(t)f - f}{t} = \lim_{t \rightarrow 0} \frac{T_1(t)Af - Af}{t} = A^2f$$

in the norm of  $E$ . Hence

$$\lim_{t \rightarrow 0} \frac{T_1(t)f - f}{t} = Af$$

in the norm of  $E_1$ . This shows that  $A_1 \subset B$ .

In order to show the converse, let  $f \in D(B)$ . Then

$$\lim_{t \rightarrow 0} \frac{T_1(t)f - f}{t}$$

exists in the norm of  $E$ . Since

$$\lim_{t \rightarrow 0} \frac{T_1(t)f - f}{t} = Af \text{ in the norm of } E,$$

it follows that  $Af \in D(A)$ , since  $A$  is closed. Thus  $f \in D(A^2) = D(A_1)$ . We have shown that  $B = A_1$ .

(b)  $\implies$  (a): Assume that  $A_1$  is the generator of a strongly continuous semigroup  $(T_1(t))_{t \geq 0}$  on  $E_1$ . Let  $f \in D(A)$  and set  $u(t) = T_1(t)f$ . Then  $u \in C([0, \infty), E)$  and  $Au(\cdot) \in C([0, \infty), E)$ .

Moreover,

$$\int_0^t u(s) ds = \int_0^t T_1(s)f ds \in D(A_1) = D(A^2)$$

and

$$A \int_0^t u(s) ds = u(t) - u(0) = u(t) - f$$

(by A-I, (1.3)). Consequently,  $u(t) = f + \int_0^t Au(s) ds$ . Hence  $u \in C^1([0, \infty), E)$  and  $\dot{u}(t) = Au(t)$ . Thus  $u$  is a solution of (ACP). We have shown existence.

In order to show uniqueness, assume that  $u$  is a solution of (ACP) with initial value 0. We have to show that  $u \equiv 0$ .

Let  $v(t) = \int_0^t u(s) ds$ . Then  $v(t) \in D(A)$  and  $Av(t) = \int_0^t Au(s) ds = \int_0^t \dot{u}(s) ds = u(t) \in D(A)$ . Consequently,  $v(t) \in D(A^2)$  for all  $t \geq 0$ . Moreover,  $\dot{v}(t) = u(t) =$

$Av(t)$  and  $\frac{d}{dt}Av(t) = Au(t) = A_1\dot{v}(t) = A^2v(t)$ . Thus  $v \in C^1([0, \infty), E_1)$  and  $\dot{v}(t) = A_1v(t)$ . Since  $v(0) = 0$ , it follows that  $v \equiv 0$ . Thus  $u \equiv v \equiv 0$ .  $\square$

If (ACP) has a unique solution for every initial value in  $D(A)$ , then  $A$  is the generator of a strongly continuous semigroup only if some additional assumptions on the solutions (continuous dependence from the initial value) or on  $A$  ( $\varrho(A) \neq \emptyset$ ) are made.

**Corollary 1.2** *Let  $A$  be a closed operator. Consider the following existence and uniqueness condition.*

$$(EU) \quad \begin{cases} \text{For every } f \in D(A) \text{ there exists a unique solution} \\ u(\cdot, f) \in C^1([0, \infty), E) \text{ of the Cauchy problem associated with } A \\ \text{having the initial value } u(0, f) = f. \end{cases}$$

*The following assertions are equivalent.*

- (a)  $A$  is the generator of a strongly continuous semigroup.
- (b)  $A$  satisfies (EU) and  $\varrho(A) \neq \emptyset$ .
- (c)  $A$  satisfies (EU) and for every  $\mu \in \mathbb{R}$  there exists  $\lambda > \mu$  such that  $(\lambda - A)D(A) = E$ .
- (d)  $A$  satisfies (EU), has dense domain and for every sequence  $(f_n)$  in  $D(A)$  satisfying  $\lim_{n \rightarrow \infty} f_n = 0$  one has  $\lim_{n \rightarrow \infty} u(t, f_n) = 0$  uniformly in  $t \in [0, 1]$ .

**Proof** It is clear that (a) implies the remaining assertions. So assume that  $A$  satisfy (EU). Then by Theorem 1.1,  $A_1$  is a generator. If there exists  $\lambda \in \varrho(A)$ , then  $(\lambda - A)$  is an isomorphism from  $E_1$  onto  $E$  and  $A$  is similar to  $A_1$  via this isomorphism since  $D(A_1) = \{(\lambda - A)^{-1}f : f \in D(A)\}$  and  $Af = (\lambda - A)A_1(\lambda - A)^{-1}f$  for all  $f \in D(A)$ , see A-I, 3.1. Thus  $A$  is a generator on  $E$  and we have shown that (b) implies (a).

If (c) holds, then there exists  $\lambda > s(A_1)$  such that  $(\lambda - A)D(A) = E$ . We show that  $(\lambda - A)$  is injective. Then  $\lambda \in \varrho(A)$  since  $A$  is closed. Assume that  $\lambda f = Af$  for some  $f \in D(A)$ . Then  $f \in D(A^2) = D(A_1)$ , and so  $f = 0$  since  $\lambda \in \varrho(A_1)$ . This proves that (c) implies (b).

It remains to show that (d) implies (a). Assertion (d) implies that for all  $t \geq 0$  there exist bounded operators  $T(t) \in \mathcal{L}(E)$  such that  $u(t, f) = T(t)f$  if  $f \in D(A)$ . Moreover,  $\sup_{0 \leq t \leq 1} \|T(t)\| < \infty$ . It follows that  $T(\cdot)f$  is strongly continuous for all  $f \in E$  (since it is so for  $f \in D(A)$  and  $D(A)$  is dense). Let  $t > 1$ . There exist unique  $n \in \mathbb{N}$  and  $s \in [0, 1)$  such that  $t = n + s$ . Let  $T(t) := T(1)^n T(s)$ . From the uniqueness of the solutions it follows that  $T(t)f = u(t, f)$  for all  $t \geq 0$  as well as  $T(t + s)f = T(s)T(t)f$  for all  $f \in D(A)$  and  $s, t \geq 0$ . Thus  $T$  is a semigroup.

Denote by  $B$  its generator. It follows from the definition that  $A \subset B$ . Moreover,  $D(A)$  is invariant under the semigroup. So by A-I, Proposition 1.9 (ii)  $D(A)$  is a core of  $B$ . Since  $A$  is closed this implies that  $A = B$ .  $\square$

**Remark 1.3** It is surprising that from condition (b) and (c) in the corollary it follows automatically that  $D(A)$  is dense. On the other hand, this condition cannot be omitted in (d). In fact, consider any generator  $\tilde{A}$  and its restriction  $A$  with domain  $D(A) = \{0\}$ . Then  $\tilde{A}$  satisfies the remaining conditions in (d) but is not a generator (if  $\dim E > 0$ ).

**Example 1.4** We give a densely defined closed operator  $A$  such that there exists a unique solution of (ACP) for all initial values in  $D(A)$ , but  $A$  is not a generator.

Let  $B$  be a densely defined unbounded closed operator on a Banach space  $F$ . Consider  $E = F \oplus F$  and  $A$  on  $E$  given by

$$A := \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix}$$

with domain  $F \times D(B)$ .

Then  $D(A^2) = \{(f, g) \in F \times D(B) : Bg \in F\} = D(A)$  and so  $A_1 \in \mathcal{L}(E_1)$ . In particular,  $A_1$  is a generator while  $A$  is not. For instance condition (b) in Corollary 1.2 does not hold since, for each  $\lambda \in \mathbb{C}$

$$(\lambda - A)D(A) = \{(\lambda f - Bg, \lambda g) : f \in F, g \in D(B)\} \subset F \times D(B) \neq F \times F = E,$$

so  $\varrho(A) = \emptyset$ .

As a further illustration, we note that the solution of the corresponding abstract Cauchy problem for the initial value  $(f, g) \in F \times D(B)$  is given by

$$u(t) = (f + tBg, g).$$

Since  $B$  is unbounded, condition (d) of Corollary 1.2 is clearly violated.

**Remark** It may happen that a generator  $A$  can be extended to a closed operator  $B$ . Then one can consider the abstract Cauchy problem  $\text{ACP}(B)$  associated with  $B$ . It also has a solution for every initial value in  $D(B)$ , but none of the solutions is unique unless  $A = B$ .

In fact, denote by  $(T(t))_{t \geq 0}$  the semigroup generated by  $A$ . Let  $f \in D(B)$ . Let  $\lambda > \omega(A)$ . Then there exists  $g \in D(A)$  such that  $(\lambda - B)f = (\lambda - A)g$ . Let  $h = f - g$ . Then  $h \in \ker(\lambda - B)$ . Define  $u$  by  $u(t) = e^{\lambda t}h + T(t)g$ . Then  $u$  is a solution  $\text{ACP}(B)$  with initial value  $f$ . It follows from Corollary 1.2 that there exists a non-trivial solution for the initial value 0.

## 1.2 One-parameter groups

Generators of one-parameter groups can be characterized similarly by existence and uniqueness of the solutions of the associated Cauchy problem. However, here the

assumption of continuous dependence on the initial values can be relaxed (in fact, one has no longer to assume that the continuous dependence is uniform in  $t$ ).

**Theorem 1.6** *Let  $A$  be a closed densely defined operator. The following assertions are equivalent.*

- (a)  $A$  is generator of a strongly continuous one-parameter group.
- (b) For every  $f \in D(A)$  there exists a unique function  $u(\cdot, f) \in C^1(\mathbb{R})$  satisfying  $u(t, f) \in D(A)$  for all  $t \in \mathbb{R}$  and  $u(0, f) = f$  such that  $\frac{d}{dt}u = Au(t, f)$ , and if  $f_n \in D(A)$  such that  $\lim_{n \rightarrow \infty} f_n = 0$ , then  $\lim_{n \rightarrow \infty} u(t, f_n) = 0$  for all  $t \in \mathbb{R}$ .

**Proof** It is clear that (a) implies (b). If (b) holds then there exist operators  $T(t) \in \mathcal{L}(E)$  such that  $u(t, f) = T(t)f$  ( $t \in \mathbb{R}, f \in D(A)$ ). It follows from the uniqueness of the solutions that  $T(t+s) = T(t)T(s)$  ( $s, t \in \mathbb{R}$ ). Let  $f \in E$ . There exist  $(f_n) \in D(A)$  such that  $\lim_{n \rightarrow \infty} f_n = f$ .

Then  $\lim_{n \rightarrow \infty} T(t)f_n = T(t)f$  for all  $t \in \mathbb{R}$ . Since  $T(\cdot)f$  is continuous, it follows that  $T(\cdot)f$  is measurable. Hence by Hille and Phillips [21, 10.2.1]  $\sup_{t \in J} \|T(t)\| < \infty$  for every compact interval  $J \subset (0, \infty)$ . By the group property this implies that  $T(\cdot)$  is norm bounded on bounded subsets of  $\mathbb{R}$  and  $T(\cdot)f$  is continuous if  $f \in D(A)$ . Since  $D(A)$  is dense, this implies the strong continuity of  $(T(t))_{t \in \mathbb{R}}$ .  $\square$

### 1.3 The Hille-Yosida Theorem

Given an operator  $A$ , frequently it is easier to obtain information about its resolvent than to solve the Cauchy problem. Therefore the following theorem is central in the theory of one-parameter semigroups.

**Theorem 1.7 (Hille-Yosida)** *Let  $A$  be an operator on a Banach space  $E$ . The following conditions are equivalent.*

- (a)  $A$  is the generator of a strongly continuous semigroup.
- (b) The domain  $D(A)$  of  $A$  is dense in  $E$  and there exist  $w, M \in \mathbb{R}$  such that  $(w, \infty) \subset \rho(A)$  and

$$\|(\lambda - w)^n (\lambda - A)^{-n}\| \leq M$$

for all  $\lambda > w$  and  $n \in \mathbb{N}$ .

In general it is not easy to give an estimate for the powers of the resolvent in order to apply Theorem 1.7. However, there is an important case when it suffices to consider merely the resolvent.

**Corollary 1.8** *For an operator  $A$  on a Banach space  $E$  the following assertions are equivalent.*

- (a)  $A$  is the generator of a strongly continuous contraction semigroup.
- (b)  $A$  is densely defined,  $(0, \infty) \subset \varrho(A)$  and  $\|\lambda R(\lambda, A)\| \leq 1$  for all  $\lambda > 0$ .

The difficult part in the proof of Theorem 1.7 is to show that (b) implies (a). One has to construct the semigroup out of the resolvent. We mention two formulas which can be used for the proof.

**Proposition 1.9** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$ . For  $\lambda > 0$  let*

$$A(\lambda) = \lambda^2 R(\lambda, A) - \lambda \text{Id} = \lambda A R(\lambda, A).$$

*Then*

$$T(t)f = \lim_{\lambda \rightarrow \infty} e^{tA(\lambda)} f \quad (1.2)$$

*for all  $f \in E$  and  $t \geq 0$ .*

Yosida's proof consists in showing that the limit in (1.2) exists under the hypothesis (b) of Theorem 1.7 (see Davies [11], Goldstein [16] or Pazy [31]).

The proof of Hille (see Kato [22]) is inspired by the following formula.

**Proposition 1.10** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$ . Then*

$$T(t)f = \lim_{n \rightarrow \infty} (\text{Id} - t/nA)^{-n} f = \lim_{n \rightarrow \infty} (n/t \cdot R(n/t, A))^n f \quad (1.3)$$

*for all  $f \in E$  and  $t \geq 0$ .*

## 1.4 Holomorphic semigroups

We now describe a hierarchy of smoothness conditions on the semigroup, starting with the most restrictive class; namely, holomorphic semigroups. The generators of these semigroups can be characterized by a particularly simple condition.

For  $\alpha \in (0, \pi]$  we define the sector  $S(\alpha)$  in the complex plane by

$$S(\alpha) = \{re^{i\vartheta} : r \geq 0, \vartheta \in (-\alpha, \alpha)\}.$$

**Definition 1.11** Let  $\alpha \in (0, \pi/2]$ . A strongly continuous semigroup  $(T(t))_{t \geq 0}$  is called a bounded holomorphic semigroup of angle  $\alpha$  if  $T(\cdot)$  is the restriction of a holomorphic function

$$T: S(\alpha) \rightarrow \mathcal{L}(E)$$

satisfying

$$T(z)T(z') = T(z + z') \quad (z, z' \in S(\alpha)), \quad (1.4)$$

For each  $\alpha_1 \in (0, \alpha)$  the set  $\{T(z) : z \in S(\alpha_1)\}$  is uniformly bounded, and

$$\lim_{n \rightarrow \infty} T(z_n)f = f \text{ for every null-sequence } (z_n) \text{ in } S(\alpha_1) \text{ and every } f \in E. \quad (1.5)$$

**Remark** A function  $T : S(\alpha) \rightarrow \mathcal{L}(E)$  is holomorphic with respect to the operator norm if and only if it is strongly holomorphic if and only if it is weakly holomorphic Yosida [40, V.3].

**Theorem 1.12** *Let  $A$  be a densely defined operator on a Banach space  $E$  and  $\alpha \in (0, \pi/2]$ . Then  $A$  is the generator of a bounded holomorphic semigroup of angle  $\alpha$  if and only if*

$$S(\alpha + \pi/2) \subset \varrho(A)$$

*and for every  $\alpha_1 \in (0, \alpha)$  there exists a constant  $M$  such that*

$$\|R(\lambda, A)\| \leq M/|\lambda| \quad (\lambda \in S(\alpha_1 + \pi/2)). \quad (1.6)$$

For the proof we refer to Davies [11], for example.

**Remark** Let  $A$  be the generator of a bounded holomorphic semigroup  $(T(t))_{t \geq 0}$  of angle  $\alpha$ , and let  $z_0 \in S(\alpha)$ . Then  $z_0 A$  generates a bounded semigroup  $(S(t))_{t \geq 0}$  given by  $S(t) = T(z_0 t)$  ( $t \geq 0$ ) (where again we denote by  $T(z)$  the holomorphic extension of  $(T(t))_{t \geq 0}$  on  $S(\alpha)$ ).

As an application of Theorem 1.12 we prove the following.

**Corollary 1.13** *Let  $A$  be the generator of a bounded group. Then  $A^2$  generates a bounded holomorphic semigroup of angle  $\pi/2$ .*

**Proof** Let  $0 < \alpha_1 < \pi/2$  and  $\lambda \in S(\alpha_1 + \pi/2)$ . There exists  $r > 0$  and  $\beta \in (-\beta_1, \beta_1)$ , where  $\beta_1 := (\alpha_1 + \pi/2)/2$  such that  $\lambda = r^2 e^{i2\beta}$ . Then

$$(\lambda - A^2) = (r e^{i\beta} - A)(r e^{i\beta} + A).$$

It follows that  $\lambda \in \varrho(A)$  and

$$R(\lambda, A^2) = R(r e^{i\beta}, A) R(r e^{-i\beta}, -A). \quad (1.7)$$

Since  $A$  generates a bounded group, there exists a constant  $N \geq 0$  such that  $\|R(\mu, A)\| \leq N/\operatorname{Re}(\mu)$ ,  $\|R(\mu, -A)\| \leq N/\operatorname{Re}(\mu)$  for all  $\mu \in S(\pi/2)$ . Consequently,  $\|R(\lambda, A^2)\| \leq N^2/r^2(\cos \beta)^2 \leq 1/r^2[N/\cos \beta]^2 = M/|\lambda|$ .  $\square$

The corollary will be extended below. We first consider an example.

**Example** (The Laplacian on  $E = C_0(\mathbb{R}^n)$  or  $L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ))

(i) Let  $n = 1$ . Then  $(U(t)f)(x) = f(x + t)$  ( $x \in \mathbb{R}$ ) defines an isometric group on  $E$ . Its generator  $A$  is given by

$$Af = f'$$

with

$$D(A) = \{f \in C^1(\mathbb{R}) \cap C_0(\mathbb{R}) : f' \in C_0(\mathbb{R})\} \quad \text{in the case } E = C_0(\mathbb{R})$$

and

$$D(A) = \{f \in E \cap AC(\mathbb{R}) : f' \in E\} \quad \text{in the case } E = L^p$$

(see A-I, 2.4). Thus  $A^2$  generates a bounded holomorphic semigroup by Corollary 1.13.

(ii) Let  $E = C_0(\mathbb{R}^n)$  or  $L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ). For  $i \in \{1, \dots, n\}$  denote by  $(U_i)_{t \geq 0}$  the group given by

$$(U_i(t)f)(x) = f(x_1, \dots, x_{i-1}, x_i + t, \dots, x_n) \quad (x \in \mathbb{R}^n, t \in \mathbb{R})$$

and by  $A_i$  its generator. Since

$$U_i(t)U_j(s) = U_j(s)U_i(t) \quad (s, t \in \mathbb{R}, i, j \in \{1, \dots, n\}),$$

it follows that the resolvents of  $A_i$  commute. So the same is true for the resolvents of  $A_i^2$  (cf. (1.7) and A-I, 3.9).

Denote by  $(T_i(t))_{t \geq 0}$  the semigroup generated by  $A_i^2$  ( $i = 1, \dots, n$ ). Then for  $z, z' \in S(\pi/2)$  one has  $T_i(z)T_j(z') = T_j(z')T_i(z)$  ( $i, j = 1, \dots, n$ ). Consequently,  $T(t) := T_1(t) \circ \dots \circ T_n(t)$  ( $t \geq 0$ ) defines a holomorphic semigroup of angle  $\pi/2$ . According to A-I, 3.9 the domain of its generator  $A$  contains  $D(A_1^2) \cap \dots \cap D(A_n^2)$  and, in particular

$$D_0 = \{f \in E \cap C^2(\mathbb{R}^n) : D^\alpha f \in E \text{ for every multiindex } \alpha \text{ with } |\alpha| \leq 2\} \subset D(A).$$

On  $D_0$  the generator is given by

$$Af = (A_1^2 + \dots + A_n^2)f = \sum_{i=1}^n \frac{\partial^2}{(\partial x_i)^2} f = \Delta f \quad \text{for all } f \in D_0.$$

Let  $\alpha \in (0, \pi/2]$ . A semigroup  $(T(t))_{t \geq 0}$  is called *holomorphic of angle*  $\alpha$  if it possesses an extension  $T : S(\alpha) \rightarrow \mathcal{L}(E)$  for some  $\alpha \in (0, \pi/2]$  which satisfies all the requirements of Definition 1.11 except that it is not required to be bounded on any sector  $S(\alpha_1)$ .

**Theorem 1.14** *A densely defined operator  $A$  is the generator of a holomorphic semigroup if and only if there exist  $M > 0$  and  $r \geq 0$  such that  $\lambda \in \varrho(A)$  and  $\|R(\lambda, A)\| \leq M/|\lambda|$  whenever  $\operatorname{Re}(\lambda) > 0$ ,  $|\lambda| \geq r$ .*

**Proof** It is not difficult to show that  $A$  generates a holomorphic semigroup of angle  $\alpha$  if and only if for every  $\alpha_1 \in (0, \alpha)$  there exists  $w \in \mathbb{R}$  such that  $A - w$  generates a bounded holomorphic semigroup of angle  $\alpha_1$  (cf. Reed and Simon [35, p.252]). As a

consequence one obtains the following. A densely defined operator  $A$  generates a holomorphic semigroup of angle  $\alpha \in (0, \pi/2]$  if and only if for every  $\alpha_1 \in [0, \alpha[$  there exist a constant  $M \geq 0$  and  $r \geq 0$  such that

$$S(\alpha_1 + \pi/2) \setminus B(r) \subset \varrho(A) \quad (\text{where } B(r) = \{z \in \mathbb{C} : |z| \leq r\})$$

and

$$\|R(\lambda, A)\| \leq M/|\lambda| \quad \text{for all } \lambda \in S(\alpha_1 + \pi/2) \setminus B(r).$$

This shows that the condition of the theorem is necessary. Conversely, assume that the condition holds. Since  $\|R(\lambda, A)\| \rightarrow \infty$  when  $\lambda$  approaches  $\sigma(A)$  (cf. Lemma 1.21 below), it follows that  $\lambda \in \varrho(A)$  and  $\|R(\lambda, A)\| \leq M/|\lambda|$  if  $\operatorname{Re}(\lambda) = 0$  and  $|\lambda| > r$  as well.

Let  $c = 1/(2M)$ . If  $\xi, \eta \in \mathbb{R}$  satisfy  $|\xi| \leq c|\eta|$ ,  $|\eta| \geq r$ , then

$$\|\xi R(i\eta, A)\| \leq |\xi| \cdot M/|\eta| \leq c \cdot M = 1/2.$$

Hence  $R(\xi + i\eta, A) = \sum_{n=0}^{\infty} (-\xi)^n R(i\eta, A)^{n+1}$  exists and

$$\begin{aligned} \|R(\xi + i\eta, A)\| &\leq (|\xi + i\eta|)^{-1} \cdot |\xi + i\eta| \cdot \sum_{n=0}^{\infty} |\xi|^n M^{n+1} / |\eta|^{n+1} \\ &\leq (|\xi + i\eta|)^{-1} \cdot 2M(|\xi|^2 + |\eta|^2)^{1/2} / |\eta| \cdot \sum_{n=0}^{\infty} M^n c^n \\ &\leq (4M \cdot (c^2 + 1)^{1/2}) / |\xi + i\eta| \\ &\leq N / |\xi + i\eta|. \end{aligned}$$

This together with the assumption implies that there exist  $N' \geq 0$  and  $\alpha \in ]0, \pi/2]$  such that  $\lambda \in \varrho(A)$  and  $\|R(\lambda, A)\| \leq N'/|\lambda|$  for all  $\lambda \in S(\alpha + \pi/2)$ .  $\square$

Compared with the Hille-Yosida theorem, Theorem 1.14 gives a very simple criterion for an operator to be the generator of a (holomorphic) semigroup. Merely the resolvent and not its powers have to be estimated. However, the resolvent has to be known in a right half-plane instead of a right half-line.

On the other hand, given a strongly continuous semigroup, merely an estimate on a vertical line implies that the semigroup is holomorphic. More precisely, the following holds.

**Corollary** *A strongly continuous semigroup with generator  $A$  is holomorphic if and only if there exist  $w > \omega(A)$  and  $M \geq 0$  such that one has*

$$\|R(w + i\eta, A)\| \leq \frac{M}{|\eta|} \quad \text{for all } \eta \in \mathbb{R}.$$

**Proof** In fact, assume that the condition holds. Since  $A - w$  is the generator of a bounded semigroup, one has  $\|R(\lambda, A - w)\| \leq N/\operatorname{Re}(\lambda)$  for some  $N > 0$  and all

$\lambda \in \mathbb{C}$  satisfying  $\operatorname{Re}(\lambda) > 0$ . Consequently, for every  $\alpha \in (0, \pi/2)$ ,

$$\|R(\lambda, A - w)\| \leq (|\lambda|/\operatorname{Re}(\lambda))N/|\lambda| \leq N(\cos \alpha)^{-1}/|\lambda| \text{ for all } \lambda \in S(\alpha).$$

The remaining estimate for a sector symmetric to the imaginary axis is given by the proof of Theorem 1.14 and shows that  $A - w$  generates a holomorphic semigroup. The reverse implication is clear.  $\square$

We now prove the following extension of Corollary 1.13

**Theorem 1.15** *Let  $A$  be the generator of a strongly continuous group. Then  $A^2$  generates a holomorphic semigroup of angle  $\pi/2$ .*

**Proof** There exists  $w \geq 0$  such that  $(\pm A - w)$  generates a bounded semigroup. Consequently, there exists  $M \geq 0$  such that  $\|R(\mu, \pm A - w)\| \leq M/\operatorname{Re}(\mu)$  whenever  $\operatorname{Re}(\mu) > 0$ .

Let  $\alpha \in (0, \pi/2)$ . There exist  $r_0 \geq 0$  and  $\beta \in (0, \pi/2)$  such that

$$S(\alpha + \pi/2) \setminus B(r_0) \subset \{z^2 : z \in S(\beta) + w\}.$$

In fact, the line  $\{w + r(\cos \beta + i \sin \beta) : r \geq 0\}$  can be parameterized by

$$z(t) = w + t + i \cdot t/\varepsilon \quad (t \geq 0),$$

where  $\varepsilon > 0$  depends on  $\beta$ . Then

$$z(t)^2 = (w + t)^2 - t^2/\varepsilon^2 + i2t(w + t)/\varepsilon.$$

Thus  $\lim_{t \rightarrow \infty} \operatorname{Im} z(t)^2 / \operatorname{Re} z(t)^2 = 2\varepsilon/(\varepsilon^2 - 1)$ . Choose  $\beta \in (\pi/4, \pi/2)$  such that  $\tan(\alpha + \pi/2) > 2\varepsilon/(\varepsilon^2 - 1)$ .

Now let  $\lambda \in S(\alpha + \pi/2) \setminus B(r_0)$ . Then there exist  $\vartheta \in (-\beta, \beta)$  and  $r \geq 0$  such that  $\lambda = (re^{i\vartheta} + w)^2$ . Thus  $(\lambda - A^2) = (re^{i\vartheta} + w - A)(re^{i\vartheta} + w + A)$ . Hence  $\lambda \in \varrho(A^2)$  and  $R(\lambda, A^2) = R(re^{i\vartheta}, A - w)R(re^{i\vartheta}, -A - w)$ . We conclude that

$$|\lambda| \cdot \|R(\lambda, A^2)\| \leq |\lambda| \cdot M^2/(\cos \vartheta)^2 r^2 \leq (|\lambda|/r^2) \cdot M^2/(\cos \beta)^2.$$

Thus  $|\lambda| \cdot \|R(\lambda, A^2)\|$  is uniformly bounded for  $\lambda \in S(\alpha + \pi/2) \setminus B(r_0)$ .  $\square$

**Remark** In Theorem 1.15 the assumption that  $\pm A$  are generators can be relaxed. In fact, the proof shows the following. If  $A$  is a densely defined operator such that  $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\} \subset \varrho(\pm A - w)$  and  $\|R(\lambda, \pm A - w)\| \leq M/\operatorname{Re}(\lambda)$  for some  $M \geq 0, w \geq 0$ , then  $A^2$  generates a holomorphic semigroup of angle  $\pi/2$ .

### 1.5 Differentiable semigroups

Let  $(T(t))_{t \geq 0}$  be a strongly continuous semigroup with generator  $A$ . Let  $t_0 \geq 0$  and  $f \in E$ . Then the function  $t \mapsto T(t)f$  is right sided differentiable at  $t_0$  if and only if  $T(t_0)f \in D(A)$ ; and in that case it is differentiable at every  $s > t_0$  and the derivative is  $AT(s)f$  (this follows from A-I, Proposition 1.6 (ii)).

**Definition 1.16** A strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  is called *eventually differentiable* if there exists  $t_0 \geq 0$  such that the function  $t \mapsto T(t)f$  from  $(t_0, \infty)$  into  $E$  is differentiable for every  $f \in E$ . The semigroup is called *differentiable* if  $t_0$  can be chosen 0.

It is not difficult to see that if  $(T(t))_{t \geq 0}$  is differentiable for  $t > t_0$ , then for  $n \in \mathbb{N}$  it is  $n$ -times differentiable at all  $s > nt_0$  and  $T(s)E \subset D(A^n)$ . If  $(T(t))_{t \geq 0}$  is differentiable, then the function  $t \mapsto T(t)f$  from  $(0, \infty)$  into  $E$  is infinitely often differentiable for every  $f \in E$ .

Generators of (eventually) differentiable semigroups can be characterized by the spectral behavior of the resolvent, in a similar way as it has been done for holomorphic semigroups in the last section. Whereas the spectrum of the generator of a holomorphic semigroup is included in a sector, the spectrum of the generator of an eventually differentiable semigroup is limited by a function of exponential growth (instead of a line).

**Theorem 1.17** *A strongly continuous semigroup  $(T(t))_{t \geq 0}$  is eventually differentiable if and only if its generator  $A$  satisfies the following.  
There exist constants  $c > 0$ ,  $b > 0$ ,  $M > 0$  such that*

$$\Sigma := \{\lambda \in \mathbb{C}: ce^{-b \cdot \operatorname{Re}(\lambda)} \leq |\operatorname{Im}(\lambda)|\} \subset \varrho(A)$$

and

$$\|R(\lambda, A)\| \leq M \cdot |\operatorname{Im}(\lambda)| \text{ for all } \lambda \in \Sigma \text{ satisfying } \operatorname{Re}(\lambda) \leq \omega_0(A).$$

**Theorem 1.18** *A strongly continuous semigroup  $(T(t))_{t \geq 0}$  is differentiable if and only if its generator  $A$  satisfies the following.  
For all  $b > 0$  there exist  $c > 0$ ,  $M > 0$  such that*

$$\Sigma := \{\lambda \in \mathbb{C}: ce^{-b \cdot \operatorname{Re}(\lambda)} \leq |\operatorname{Im}(\lambda)|\} \subset \varrho(A)$$

and

$$\|R(\lambda, A)\| \leq M \cdot |\operatorname{Im}(\lambda)| \text{ for all } \lambda \in \Sigma \text{ satisfying } \operatorname{Re}(\lambda) \leq \omega_0(A).$$

For the proofs of these two theorems we refer to Pazy [31, Chapter 3, Theorem 4.7 and 4.8].

### 1.6 Norm continuous semigroups

Let  $(T(t))_{t \geq 0}$  be a strongly continuous semigroup and  $t' > 0$ . If  $\lim_{t \rightarrow t'} \|T(t) - T(t')\| = 0$ , then it follows from the semigroup property, that the function  $t \mapsto T(t)$  is norm continuous on the whole half line  $(t', \infty)$ .

**Definition 1.19** A semigroup  $(T(t))_{t \geq 0}$  is called *eventually norm continuous* if there exists  $t' \geq 0$  such that the function  $t \mapsto T(t)$  from  $(t', \infty)$  into  $\mathcal{L}(E)$  is norm continuous. The semigroup is called *norm continuous* if  $t'$  can be chosen equal to 0.

The spectrum of generators of eventually norm continuous semigroups still is compact in every right half-plane.

**Theorem 1.20** *Let  $A$  be the generator of an eventually norm continuous semigroup. Then for every  $b \in \mathbb{R}$  the set*

$$\{\lambda \in \sigma(A) : \operatorname{Re}(\lambda) \geq b\}$$

*is bounded.*

For the proof of Theorem 1.20 we use the following lemma.

**Lemma 1.21** *Let  $A$  be an operator and  $\lambda \in \rho(A)$ . Then*

$$\operatorname{dist}(\lambda, \sigma(A)) = r(R(\lambda, A))^{-1}.$$

**Proof** One has  $\{0\} \cup \sigma(R(\lambda, A)) = \{0\} \cup \{(\lambda - \mu)^{-1} : \mu \in \sigma(A)\}$  (Davies [11, Lemma 2.11]). Hence

$$\begin{aligned} r(R(\lambda, A)) &= \sup \{|\lambda - \mu|^{-1} : \mu \in \sigma(A)\} \\ &= (\inf \{|\lambda - \mu| : \mu \in \sigma(A)\})^{-1} = \operatorname{dist}(\lambda, \sigma(A))^{-1}. \end{aligned}$$

**Proof (Proof of Theorem 1.20)** It is enough to show the following. Let  $a > \omega(A)$ . Then for every  $\varepsilon > 0$  there exist  $n \in \mathbb{N}$  and  $r_0 \geq 0$  such that

$$\|R(a + ir, A)^n\|^{1/n} < \varepsilon \quad \text{for all } r \in \mathbb{R} \text{ satisfying } |r| \geq r_0.$$

In fact, then we have by the lemma,

$$\operatorname{dist}(a + ir, \sigma(A)) = r(R(a + ir, A))^{-1} \geq \varepsilon^{-1} \quad \text{whenever } |r| \geq r_0.$$

So let  $\varepsilon > 0$ . If  $\operatorname{Re}(\lambda) > \omega(A)$ , then by A-I, Proposition 1.11,

$$R(\lambda, A)^{n+1} = \frac{1}{n!} \int_0^\infty e^{-\lambda t} t^n T(t) dt \quad (n \in \mathbb{N}).$$

Let  $t' > 0$  such that  $t \mapsto T(t)$  is norm continuous on  $(t', \infty)$ . Let  $w \in (\omega(A), a)$ . There exists  $M \geq 1$  such that  $\|T(t)\| \leq M e^{wt}$  for all  $t \geq 0$ . Let

$$N := M \cdot \int_0^{t'} e^{-at} e^{wt} dt.$$

Since  $\lim_{n \rightarrow \infty} c^n/n! = 0$  for all  $c > 0$ , there exists  $n \in \mathbb{N}$  such that

$$N \cdot \frac{(t')^n}{n!} < \frac{1}{3} \varepsilon^{n+1}.$$

Choose  $T \geq t'$  such that  $\frac{1}{n!} \int_T^\infty t^n e^{-at} \|T(t)\| dt < \varepsilon^{n+1}/3$ .

Since  $(T(t))_{t \geq 0}$  is norm continuous for  $t \geq t'$ , it follows from the Riemann-Lebesgue lemma that there exists  $r_0 \geq 0$  such that

$$\left\| \frac{1}{n!} \int_{t'}^T t^n e^{-irt} e^{-at} T(t) dt \right\| < \varepsilon^{n+1}/3$$

whenever  $|r| \geq r_0$ .

All together we obtain for  $|r| \geq r_0$ ,

$$\begin{aligned} \|R(a + ir, A)^{n+1}\| &= \frac{1}{n!} \cdot \left\| \int_0^\infty e^{-(a+ir)t} t^n T(t) dt \right\| \\ &\leq \frac{1}{n!} \cdot \int_0^{t'} e^{-at} t^n \|T(t)\| dt \\ &\quad + \frac{1}{n!} \cdot \left\| \int_{t'}^T t^n e^{-irt} e^{-at} T(t) dt \right\| \\ &\quad + \frac{1}{n!} \cdot \int_T^\infty e^{-at} t^n \|T(t)\| dt \\ &\leq \frac{1}{n!} \cdot (t')^n \int_0^{t'} e^{-at} M e^{wt} dt + \frac{2}{3} \cdot \varepsilon^{n+1} \\ &\leq N \cdot \frac{(t')^n}{n!} + \frac{2}{3} \cdot \varepsilon^{n+1} \\ &\leq \varepsilon^{n+1}. \end{aligned}$$

A complete characterization of eventually norm continuous semigroups in terms of their generator seems not to be known. Eventually norm continuous semigroups are of particular interest in spectral theory (cf. A-III, Theorem 6.6). Moreover their asymptotic behavior is easy to describe (see A-IV, (1.8)).

Next we describe special norm continuous semigroups.

### 1.7 Compact semigroups

Let  $(T(t))_{t \geq 0}$  be a strongly continuous semigroup and  $t_0 > 0$ . If  $T(t_0)$  is compact, then it follows from the semigroup property that  $T(t)$  is compact for all  $t \geq t_0$ . Moreover,  $t \mapsto T(t)$  is norm continuous at every  $t > t_0$ .

In fact, since  $T(h) \rightarrow \text{Id}$  strongly with  $h \downarrow 0$ , it follows that  $\lim_{h \downarrow 0} T(h)f = f$  uniformly on every compact subset  $K$  of  $E$ . Now let  $t \geq t_0$ . Then  $K = T(t)(U)$  is compact (where  $U$  denotes the unit ball of  $E$ ). Hence  $\lim_{h \downarrow 0} T(h+t)f = \lim_{h \downarrow 0} T(h)T(t)f$  uniformly for  $f \in U$ . So the semigroup is right-sided norm continuous on  $[t_0, \infty)$  and so norm continuous on  $(t_0, \infty)$ .

**Definition 1.22** A strongly continuous semigroup  $(T(t))_{t \geq 0}$  is called *compact* if  $T(t)$  is compact for all  $t > 0$ . It is called *eventually compact* if there exists  $t_0 > 0$  such that  $T(t_0)$  is compact (and hence  $T(t)$  is compact for all  $t \geq t_0$ ).

We want to find a relation between the compactness of the semigroup and the compactness of the resolvent of its generator.

**Definition 1.23** Let  $A$  be an operator and  $\varrho(A) \neq \emptyset$ . We say,  $A$  has a compact resolvent if  $R(\lambda, A)$  is compact for one (and hence all)  $\lambda \in \varrho(A)$ .

**Proposition 1.24** Let  $(T(t))_{t \geq 0}$  be a strongly continuous semigroup and assume that its generator has a compact resolvent. If  $t \mapsto T(t)$  is norm continuous at  $t_0$ , then  $T(t)$  is compact for all  $t \geq t_0$ .

**Proof** Considering  $(e^{-wt}T(t))_{t \geq 0}$  for some  $w > 0$  if necessary, we can assume that  $\sigma(A) < 0$ . Let  $S(t) \in \mathcal{L}(E)$  be given by  $S(t)f = \int_0^t T(s)f \, ds$  ( $t \geq 0$ ). Then  $AS(t)f = T(t)f - f$  for all  $f \in E$ , and so  $S(t) = R(0, A)(\text{Id} - T(t))$  is compact for all  $t \geq 0$ .

Since  $t \mapsto T(t)$  is norm continuous for  $t \geq t_0$ , one has  $\lim_{h \downarrow 0} \frac{1}{h}(S(t_0 + h) - S(t_0)) = T(t_0)$  in the operator norm. Thus  $T(t_0)$  is compact as limit of compact operators.  $\square$

**Theorem 1.25** A strongly continuous semigroup  $(T(t))_{t \geq 0}$  is compact if and only if it is norm continuous and its generator  $A$  has compact resolvent.

**Proof** Assume that  $(T(t))_{t \geq 0}$  is compact. Then  $T(\cdot)$  is norm continuous on  $(0, \infty)$ , and so

$$\int_0^t e^{-ws}T(s) \, ds$$

is compact as the norm limit of linear combinations of compact operators, whenever  $w > \omega_0(A)$ . Since

$$R(w, A) = \lim_{t \rightarrow \infty} \int_0^t e^{-ws}T(s) \, ds$$

in the operator norm, it follows that  $R(w, A)$  is compact. This proves one implication. The other follows from Proposition 1.24.  $\square$

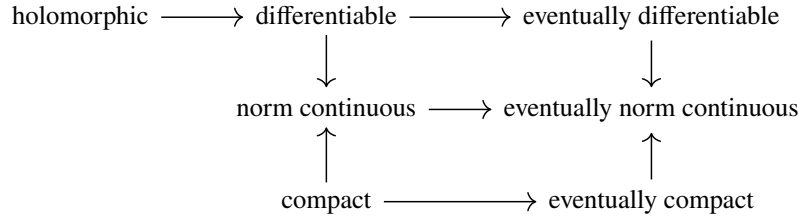
**Remark 1.26** (i) Generators of eventually compact semigroups do not necessarily have compact resolvent. Consider the nilpotent translation semigroup  $(T(t))_{t \geq 0}$  on  $F := L^1([0, 1])$  (see A-I, Example 2.6). Let  $E = F \tilde{\otimes}_\pi F = L^1([0, 1] \times [0, 1])$  and  $S(t) = T(t) \otimes \text{Id}$  ( $t \geq 0$ ). Then  $(S(t))_{t \geq 0}$  is a strongly continuous semigroup (see A-I, 3.8), with  $B$  as its generator. Clearly  $(S(t))_{t \geq 0}$  is a nilpotent semigroup (so it is eventually compact), but  $R(\lambda, B) = R(\lambda, A) \otimes \text{Id}$  is not compact.

(ii) It is obvious that a group  $(T(t))_{t \in \mathbb{R}}$  is eventually norm continuous if and only if it is norm continuous at 0, i.e., its generator is bounded.

On the other hand, the generator of the rotation group (A-I, Example 2.5) has a compact resolvent. Hence this condition does not imply any smoothness property of the semigroup.

Positive eventually compact semigroups have remarkable properties in the setting of the Perron-Frobenius theory (see e.g., B-III, Corollary 2.12).

The following scheme indicates the relation between the different classes of semigroups defined so far.



All these classes are different. This is shown by the following examples.

**Example 1.27** The nilpotent shift semigroup (A-I, 2.6) is obviously eventually differentiable, eventually compact and eventually norm continuous. But it is not norm continuous and consequently not differentiable or compact.

**Example 1.28** We consider multiplication semigroups (see A-I, 2.3). Let  $E = C_0(X)$ , where  $X$  is a locally compact space, or  $E = L^p(X, \Sigma, \mu)$ , where  $1 \leq p < \infty$  and  $(X, \Sigma, \mu)$  is a  $\sigma$ -finite measure space. Let  $m: X \rightarrow \mathbb{R}$  be continuous [resp., measurable] such that  $[\text{ess}]\text{-}\sup_{x \in X} \text{Re}(m(x)) < \infty$ .

Then  $Af = m \cdot f$  with domain  $D(A) = \{f \in E : m \cdot f \in E\}$  is the generator of the semigroup  $(T(t))_{t \geq 0}$  given by

$$(T(t)f)(x) = e^{tm(x)} f(x), \quad (t \geq 0, x \in X, f \in E).$$

Observe that  $\sigma(A) = \overline{m(X)}$  in case  $E = C_0(X)$  and

$$\sigma(A) = [\text{ess}]\text{-image}(m) := \{\lambda \in \mathbb{C} : \mu(\{x \in X : |m(x) - \lambda| < \varepsilon\}) \neq 0 \text{ for all } \varepsilon > 0\}$$

If  $E = L^p$  (see A-II, 2.3). Consequently,  $s(A) = \omega(A) = [\text{ess}]\text{-}\sup_{x \in X} \text{Re}(m(x))$ .

(i) The semigroup is norm continuous for  $t > 0$  if and only if it is eventually norm continuous if and only if  $\{\lambda \in \sigma(A) : \operatorname{Re}(\lambda) \geq b\}$  is bounded for every  $b \in \mathbb{R}$ . Thus the property proved in Theorem 1.20 characterizes generators of eventually norm continuous semigroups in the case that the semigroup consists of multiplication operators.

**Proof** Assume that  $\{\lambda \in \sigma(A) : \operatorname{Re}(\lambda) \geq b\}$  is bounded for every  $b \in \mathbb{R}$ . Let  $t' > 0$ . We show that the semigroup is norm continuous at  $t'$ . Take  $\varepsilon > 0$  and  $b \in \mathbb{R}$  such that  $2e^{(t'+1)b} < \varepsilon$ .

If  $\operatorname{Re}(m(x)) \leq b$ , then

$$\left| e^{tm(x)} - e^{t'm(x)} \right| \leq e^{t \operatorname{Re}(m(x))} + e^{t' \operatorname{Re}(m(x))} \leq 2e^{(t'+1)b} < \varepsilon$$

whenever  $|t - t'| \leq 1$ .

By hypothesis, the set  $H := \{m(x) : x \in X, \operatorname{Re}(m(x)) \geq b\}$  in the case  $E = C_0(X)$  and  $H := \{m(x) : \operatorname{Re}(\lambda) \geq b \text{ and for all } \eta > 0, \mu(\{x \in X : |m(x) - \lambda| < \eta\}) \neq \emptyset\}$  in the case  $E = L^P$  is a bounded subset of  $\mathbb{C}$ . Thus  $\lim_{t \rightarrow t'} |e^{tz} - e^{t'z}| = 0$  uniformly for  $z \in H$ . Hence there exists  $\delta \in ]0, 1]$  such that

$$[\text{ess}]\text{-ess sup} \left\{ \left| e^{tm(x)} - e^{t'm(x)} \right| : x \in X, \operatorname{Re}(m(x)) > b \right\} < \varepsilon$$

whenever  $|t - t'| < \delta$ . Together with the inequality above, we obtain that

$$\|T(t) - T(t')\| = [\text{ess}]\text{-sup} \left\{ \left| e^{tm(x)} - e^{t'm(x)} \right| : x \in X \right\} < \varepsilon$$

whenever  $|t - t'| < \delta$ . We have shown that the semigroup is norm continuous for  $t > 0$  whenever  $\{\lambda \in \sigma(A) : \operatorname{Re}(\lambda) \geq b\}$  is bounded for all  $b \in \mathbb{R}$ .  $\square$

(ii) The semigroup is right-sided differentiable at a point  $t > 0$  if and only if there exists  $c > 0$  such that  $\{\lambda \in \mathbb{C} : |\operatorname{Im}(\lambda)| > c \cdot e^{-t \operatorname{Re}(\lambda)}\} \subset \varrho(A)$ .

**Proof** The semigroup is right-sided differentiable at  $t$  if and only if  $T(t)E \subset D(A)$  if and only if  $e^{tm} \cdot f \cdot m \in E$  for all  $f \in E$  if and only if  $e^{tm} \cdot m$  is [essentially] bounded if and only if  $e^{t \operatorname{Re}(m)} \cdot \operatorname{Im}(m)$  is [essentially] bounded if and only if there exists  $c > 0$  such that  $[\text{ess}]\text{-} \operatorname{im}(m) \subset \{\lambda \in \mathbb{C} : e^{t \operatorname{Re}(\lambda)} |\operatorname{Im}(\lambda)| \leq c\}$  if and only if there exists  $c > 0$  such that  $\{\lambda \in \mathbb{C} : |\operatorname{Im}(\lambda)| > c \cdot e^{-t \operatorname{Re}(\lambda)}\} \subset \varrho(A)$ .  $\square$

(iii)  $(T(t))_{t \geq 0}$  is a bounded holomorphic semigroup of angle  $\vartheta$  if and only if  $S(\vartheta + \pi/2) \subset \varrho(A)$ .

**Proof** The condition is necessary by Theorem 1.12. Conversely, if  $S(\vartheta + \pi/2) \subset \varrho(A)$ , then one verifies directly that

$$(T(z)f)(x) = e^{z \cdot m(x)} f(x) \quad (f \in E, x \in X)$$

defines a family  $(T(z))_{z \in S(\vartheta)}$  of bounded operators satisfying conditions (1.4) and (1.5).  $\square$

(iv) Choosing  $X = \mathbb{N}$  and the counting measure we have  $E = c_0$  or  $\ell^p$ . Then  $A$  has a compact resolvent if  $\lim_{n \rightarrow \infty} |m(n)| = \infty$ .

In fact, let  $\lambda > s(A)$ . Then  $(R(\lambda, A)f)(n) = (\lambda - m(n))^{-1}f(n)$ . Hence  $R(\lambda, A)$  is compact if and only if  $((\lambda - m(n))^{-1})_{n \in \mathbb{N}} \in c_0$ .

The semigroup is compact if and only if it is eventually compact if and only if

$$\lim_{n \rightarrow \infty} \operatorname{Re}(m(n)) = -\infty.$$

(v) Now it is easy to give concrete examples. Again let  $X = \mathbb{N}$ , so that  $E = c_0$  or  $\ell^p$ . Let  $m(n) = -n + i \cdot \exp(n^2)$ . Then the semigroup is compact and (consequently) norm continuous for  $t > 0$ , but it is not eventually differentiable. Let  $m(n) = -n + i e^{t'n}$ . Then the semigroup is differentiable for  $t > t'$  but not differentiable at  $t \in [0, t')$ . If  $m(n) = -n + i \cdot n^2$ , then the semigroup is differentiable but not holomorphic.

## 1.8 Perturbation of Generators

A useful way to construct new semigroups out of a given one is by additive perturbation.

**Theorem 1.29** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  and let  $B \in \mathcal{L}(E)$ . Then  $A + B$  with domain  $D(A + B) = D(A)$  is the generator of a strongly continuous semigroup  $(S(t))_{t \geq 0}$ .*

It is possible to express the new semigroup  $(S(t))_{t \geq 0}$  by known objects. The product formula

$$S(t)f = \lim_{n \rightarrow \infty} \left( T(t/n) e^{t/n \cdot B} \right)^n f \quad (1.8)$$

holds for all  $t \geq 0$  and  $f \in E$ .

Moreover,  $S(t)$  is the solution of the following integral equation

$$S(t)f = T(t)f + \int_0^t T(t-s)BS(s)f \, ds \quad (t \geq 0, f \in E). \quad (1.9)$$

Let  $S_0(t) = T(t)$  and

$$S_n(t)f = \int_0^t T(t-s)BS_{n-1}(s)f \, ds \quad (f \in E) \quad (1.10)$$

for  $n \in \mathbb{N}$ . Then

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \quad (1.11)$$

where the series converges in the operator norm uniformly on bounded intervals. We refer to Davies [11, III.1], Goldstein [15, I.6] or Pazy [31, Chapter 3] for these results.

Several special properties discussed above are preserved by bounded perturbations.

**Theorem 1.30** *Let  $(T(t))_{t \geq 0}$  be a strongly continuous semigroup with generator  $A$ , and let  $B \in \mathcal{L}(E)$ .*

- (i) *If  $(T(t))_{t \geq 0}$  is holomorphic or norm continuous or compact, then so is the semigroup  $(S(t))_{t \geq 0}$  generated by  $A + B$ .*
- (ii) *If  $A$  has a compact resolvent then so has  $A + B$ .*
- (iii) *Let  $t_0 \geq 0$ . If  $(T(t))_{t \geq 0}$  is norm continuous for  $t > t_0$  and if  $B$  is compact, then  $(S(t))_{t \geq 0}$  is also norm continuous for  $t > t_0$ .*

**Proof** If  $(T(t))_{t \geq 0}$  is norm continuous for  $t > 0$ , then  $S_n(t)$  in (1.10) is norm continuous at  $t > 0$  for every  $n$ . Thus  $(S(t))_{t \geq 0}$  is norm continuous at  $t > 0$  by (1.11). There exists  $\lambda_0 \in \mathbb{R}$  such that  $\|R(\lambda, A)\| \leq (2\|B\|)^{-1}$  for  $\operatorname{Re}(\lambda) \geq \lambda_0$ . Hence  $(\operatorname{Id} - BR(\lambda, A))^{-1}$  exists for  $\operatorname{Re}(\lambda) \geq \lambda_0$ . Since

$$(\lambda - (A + B))f = (\operatorname{Id} - BR(\lambda, A))(\lambda - A)f \quad \text{for all } f \in D(A),$$

it follows that  $(\lambda - (A + B))^{-1}$  exists and is given by

$$R(\lambda, A + B) = R(\lambda, A)(\operatorname{Id} - BR(\lambda, A))^{-1} \quad (1.12)$$

whenever  $\operatorname{Re}(\lambda) \geq \lambda_0$ . Now if  $A$  generates a holomorphic semigroup, there exists  $M \geq 0$  such that  $\|R(\lambda_0 + i\eta, A)\| \leq M/|\eta|$  for all  $\eta \in \mathbb{R}$ . Consequently,

$$\|R(\lambda_0 + i\eta, A + B)\| \leq \|(\operatorname{Id} - BR(\lambda_0 + i\eta, A))^{-1}\| \cdot 2M/|\eta| \leq 2M/|\eta|$$

for all  $\eta \in \mathbb{R}$ . Thus  $A + B$  generates a holomorphic semigroup by the corollary of Theorem 1.14. Moreover, it follows from (1.12) that  $R(\lambda, A + B)$  is compact whenever  $R(\lambda, A)$  is compact. Consequently, by Theorem 1.25 and the assertion proved above,  $(S(t))_{t \geq 0}$  is compact whenever  $(T(t))_{t \geq 0}$  is compact.

Finally assume that  $B$  is compact and  $t_0 \geq 0$  such that  $(T(t))_{t \geq 0}$  is norm continuous for  $t > t_0$ . Fix  $t > t_0$ . Denote by  $U$  the unit ball of  $E$  and fix  $s \in (0, t]$ . Then

$$\lim_{h \rightarrow 0} (T(t + s - h) - T(t - s))f = 0$$

for all  $f \in \overline{BS(s)U} =: K$ .

Since  $K$  is compact it follows that the limit exists uniformly with respect to  $f \in K$ , i.e.,  $\lim_{h \rightarrow 0} \|(T(t + s - h) - T(t - s))BS(s)\| = 0$ . It follows from the dominated convergence theorem that

$$\lim_{h \rightarrow 0} \int_0^t \|(T(t+s-h) - T(t-s))BS(s)\| \, ds = 0. \quad (1.13)$$

Using (1.9) we obtain

$$\begin{aligned} \|S(t+h) - S(t)\| &\leq \|T(t+h) - T(t)\| \\ &\quad + \left\| \int_0^{t+h} T(t+h-s)BS(s) \, ds - \int_0^t (T(t-s)BS(s) \, ds \right\| \\ &\leq \|T(t+h) - T(t)\| + \int_t^{t+h} \|T(t+h-s)BS(s)\| \, ds \\ &\quad + \int_0^t \|(T(t+h-s) - T(t-s))BS(s)\| \, ds \rightarrow 0 \end{aligned}$$

when  $h \rightarrow 0$ . □

In C-IV, Example 2.15 a generator  $A$  of an eventually differentiable and eventually compact semigroup and a bounded operator  $B$  will be given such that the semigroup generated by  $A + B$  is not eventually norm continuous.

Using Theorem 1.29 we now prove a perturbation result due to Desch and Schappacher [12]. Instead of assuming that  $B \in \mathcal{L}(E)$  we assume that  $B \in \mathcal{L}(D(A))$ . The short proof given below is due to G. Greiner.

**Theorem 1.31** *Let  $(T(t))_{t \geq 0}$  be a strongly continuous semigroup with generator  $A$ . Assume that  $B: D(A) \rightarrow D(A)$  is linear and continuous for the graph norm on  $D(A)$ . Then  $A + B$  with domain  $D(A + B) = D(A)$  is the generator of a strongly continuous semigroup. Moreover, there exists a bounded operator  $C$  on  $E$  such that  $A + B$  is similar to  $A + C$ .*

**Proof** We first show that  $(\text{Id} - BR(\lambda, A))$  is invertible for some  $\lambda \in \mathbb{C}$ . Choose  $\lambda_0 \in \rho(A)$ . Then  $S := (\lambda_0 - A)BR(\lambda_0, A) \in \mathcal{L}(E)$ . Let  $\lambda > s(A)$  be so large such that  $\|SR(\lambda, A)\| < 1$ . Then

$$1 - (\lambda_0 - A)BR(\lambda_0, A)R(\lambda, A) = (1 - SR(\lambda, A))$$

is invertible. Consequently, also  $(1 - BR(\lambda, A))^{-1}$  exists (since

$$\sigma(TR(\lambda_0, A)) \setminus \{0\} = \sigma(R(\lambda_0, A)T) \setminus \{0\}$$

where  $T = (\lambda_0 - A)BR(\lambda, A)$ ).

Let  $C = (A - \lambda)B(A - \lambda)^{-1} \in \mathcal{L}(E)$ . Then  $A + C$  is the generator of a strongly continuous semigroup by Theorem 1.29. We show that  $A + B$  is similar to  $A + C$ . In fact, let  $U = (1 - BR(\lambda, A))$ . Then  $U$  is an isomorphism on  $E$  such that  $U(D(A)) = D(A)$ . Moreover,

$$\begin{aligned}
U(A + C)U^{-1} &= U(A - \lambda + C)U^{-1} + \lambda \\
&= U[(A - \lambda) - (A - \lambda)BR(\lambda, A)]U^{-1} + \lambda \\
&= U(A - \lambda)[1 - BR(\lambda, A)]U^{-1} + \lambda \\
&= U(A - \lambda) + \lambda \\
&= A - \lambda + B + \lambda \\
&= A + B.
\end{aligned}$$

**Corollary 1.32** *Keeping the hypotheses and notations of Theorem 1.31 denote by  $(S(t))_{t \geq 0}$  the semigroup generated by  $A + B$ . If  $(T(t))_{t \geq 0}$  is norm continuous or compact or holomorphic, then  $(S(t))_{t \geq 0}$  has the corresponding properties. If  $B$  is compact as an operator on  $D(A)$  endowed with the graph norm and if  $(T(t))_{t \geq 0}$  is eventually norm continuous, then so is  $(S(t))_{t \geq 0}$ .*

**Proof** This follows from Theorem 1.30 since  $(US(t)U^{-1})_{t \geq 0}$  has  $A + C$  as generator.  $\square$

## 1.9 Domains of Uniqueness

Given a semigroup  $(T(t))_{t \geq 0}$ , it is frequently difficult to determine the precise domain of its generator  $A$ . So it is important to know which (possibly strict) subspaces of  $D(A)$  determine the semigroup uniquely. This can be formulated more precisely in the following way.

Let  $D_0$  be a subspace of  $D(A)$  and consider the restriction  $A_0$  of  $A$  to  $D_0$ . Under which condition on  $D_0$  is  $A$  the only extension of  $A_0$  which is a generator? One obvious condition is that  $D_0$  is a core. [In fact, in that case,  $A$  is the closure of  $A_0$ . Since every generator  $B$  extending  $A_0$  is closed, it follows that  $A \subset B$  and hence  $A = B$  since  $\varrho(A) \cap \varrho(B) \neq \emptyset$ ].

We now show that cores are the only domains of uniqueness.

**Theorem 1.33** *Let  $A$  be the generator of a semigroup and  $D_0$  a subspace of  $D(A)$ . Consider the restriction  $A_0$  of  $A$  to  $D_0$ . If  $D_0$  is not a core of  $A$ , then there exists an infinite number of extensions of  $A_0$  which are generators.*

**Proof** If  $D_0$  is not dense in  $D(A)$  with respect to the graph norm, then there exists a non-zero linear form  $\varphi$  on  $D(A)$  which is continuous for the graph norm such that  $\varphi(f) = 0$  for all  $f \in D_0$ . Let  $u \in D(A)$  and  $B: D(A) \rightarrow D(A)$  be given by  $Bf = \varphi(f)u$  for all  $f \in D(A)$ . Then  $B$  is continuous for the graph norm. So by Theorem 1.31 the operator  $A + B$  with domain  $D(A)$  is a generator. Clearly,  $A + B \neq A$  if  $u \neq 0$  but  $Af + Bf = Af$  for all  $f \in D_0$ . It is obvious that an infinite number of generators can be constructed in that way.  $\square$

**Corollary 1.34** *Let  $(T(t))_{t \geq 0}$  be a strongly continuous semigroup with generator  $A$ . Let  $D_0$  be a dense subspace of  $E$ . Assume that  $D_0 \subset D(A)$  and  $T(t)D_0 \subset D_0$  for all  $t \geq 0$ . Then  $D_0$  is a core.*

**Proof** Let  $(S(t))_{t \geq 0}$  be a semigroup with generator  $B$  such that  $B|_{D_0} = A|_{D_0}$ , and take  $f \in D_0$ . Then  $u(t) := T(t)f$  satisfies

$$u(0) = f \text{ and } \dot{u}(t) = AT(t)f = BT(t)f = Bu(t) \quad (t \geq 0).$$

Since  $v(t) = S(t)f$  ( $t \geq 0$ ) also is a solution of the Cauchy problem defined by  $B$  with initial value  $f$  it follows that  $S(t)f = T(t)f$  ( $t \geq 0$ ). Since  $D_0$  is dense in  $E$ , it follows that  $S(t) = T(t)$  ( $t \geq 0$ ).  $\square$

## 2 Contraction Semigroups and Dissipative Operators

by Wolfgang Arendt

The Hille-Yosida theorem gives a characterization of generators in terms of the resolvent of the operator. However, given an operator  $A$ , frequently it is difficult to compute the resolvent (and its powers). So it is desirable to find conditions more immanent on  $A$ . This is possible for generators of contraction semigroups.

For later purposes (see B-II and C-II) it will be useful not only to consider semigroups which are contractive with respect to the norm but to consider more general sublinear functionals than the norm as well.

So our setting is the following. By  $E$  we denote a real Banach space throughout and  $p: E \rightarrow \mathbb{R}$  is a continuous sublinear function; i.e.,  $p$  satisfies

$$p(f + g) \leq p(f) + p(g) \quad (f, g \in E), \quad (2.1)$$

$$p(\lambda f) = \lambda p(f) \quad (f \in E, \lambda \geq 0). \quad (2.2)$$

The continuity of  $p$  implies that there exists a constant  $c > 0$  such that

$$|p(f)| \leq c\|f\| \quad (f \in E). \quad (2.3)$$

Moreover, it follows from (2.1) and (2.2) that

$$p(f) + p(-f) \geq p(0) = 0 \quad (f \in E). \quad (2.4)$$

A bounded operator  $T$  on  $E$  is called  $p$ -contractive if  $p(Tf) \leq p(f)$  for all  $f \in E$ . Similarly, a semigroup  $(T(t))_{t \geq 0}$  is called  $p$ -contractive if  $T(t)$  is  $p$ -contractive for all  $t \geq 0$ . Of course, the most important case in this section is when  $p$  is the norm function  $N$  given by  $N(f) = \|f\|$  ( $f \in E$ ). An  $N$ -contractive operator is just a contraction in the usual sense.

**Remark** However in Chapter B-II and C-II it will be important to dispose of a variety of sublinear functionals other than  $N$ . For example, we will consider  $N^+$  on  $C[0, 1]$  given by  $N^+(f) = \sup_{x \in [0, 1]} f(x)$ . Then a bounded operator  $T$  is  $N^+$ -contractive if and only if  $T$  is positive and  $\|T\| \leq 1$ .

We first want to solve the following problem. Given the generator  $A$  of a semigroup  $(T(t))_{t \geq 0}$  find a condition on  $A$  which is equivalent to  $T(t)$  being  $p$ -contractive for all  $t \geq 0$ .

The *subdifferential*  $dp$  of  $p$  at  $f$  is defined by

$$dp(f) = \{\varphi \in E' : \langle g, \varphi \rangle \leq p(g) \text{ for all } g \in E, \langle f, \varphi \rangle = p(f)\}. \quad (2.5)$$

It follows from the Hahn-Banach theorem that  $dp(f) \neq \emptyset$  for all  $f \in E$ .

**Definition 2.1** An operator  $A$  on  $E$  is called *p-dissipative* if for all  $f \in D(A)$  there exists  $\varphi \in dp(f)$  such that  $\langle Af, \varphi \rangle \leq 0$  and  $A$  is called *strictly p-dissipative* if for all  $f \in D(A)$  the inequality  $\langle Af, \varphi \rangle \leq 0$  holds for all  $\varphi \in dp(f)$ .

For convenience we want to have a distinctive name for the norm function. So we denote by  $N : E \rightarrow \mathbb{R}$  the function given by  $N(f) = \|f\|$  throughout. Then (2.5) can be written in the form

$$dN(f) = \{\varphi \in E' : \|\varphi\| \leq 1, \langle f, \varphi \rangle = \|f\|\}. \quad (2.6)$$

A (strictly)  $N$ -dissipative operator is simply called (strictly) *dissipative*, which is in accordance with the usual nomenclature.

### Examples 2.2

(i) Let  $E = C[0, 1]$ ,  $f \in E$ . Then there exists  $x \in [0, 1]$  such that  $|f(x)| = \|f\|_\infty$ . Define  $\varphi \in E'$  by  $\langle g, \varphi \rangle = (\text{sign } f(x))g(x)$ . Then  $\varphi \in dN(f)$ . Note that  $dN(f)$  may be an infinite set.

(ii) Let  $H$  be a Hilbert space,  $f \in H$ ,  $f \neq 0$ . Then  $dN(f) = \{\varphi_f\}$  where  $\langle g, \varphi_f \rangle = 1/\|f\| \langle g, f \rangle$ .

(iii)  $A - \|A\| \text{Id}$  is strictly dissipative for every bounded operator  $A$ .

**Proposition 2.3** Let  $A$  be an operator on  $E$ . Then  $A$  is  $p$ -dissipative if and only if

$$p(f) \leq p(f - tAf) \text{ for all } f \in D(A), t > 0. \quad (2.7)$$

If in particular  $(w, \infty) \subset \varrho(A)$  for some  $w \in \mathbb{R}$ , then  $A$  is  $p$ -dissipative if and only if

$$p(\lambda R(\lambda, A)f) \leq p(f) \text{ for all } f \in E, \lambda > w. \quad (2.8)$$

**Proof** Assume that  $A$  is  $p$ -dissipative. Let  $f \in D(A)$ ,  $t > 0$ . There exists  $\varphi \in dp(f)$  such that  $\langle Af, \varphi \rangle \leq 0$ . Hence,

$$p(f) = \langle f, \varphi \rangle = \langle f - tAf + tAf, \varphi \rangle \leq \langle f - tAf, \varphi \rangle \leq p(f - tAf).$$

So (2.7) holds.

Conversely, let  $f \in D(A)$ . For every  $t > 0$  choose  $\varphi_t \in dp(f - tAf)$ . Then  $\pm \langle g, \varphi_t \rangle \leq p(\pm g) \leq c\|g\|$  for all  $g \in E$ ,  $t > 0$ . Thus the net  $(\varphi_t)_{t>0}$  is bounded. Consequently it possesses a  $\sigma(E', E)$ -limit point  $\varphi$  as  $t \rightarrow 0$ . We show that  $\varphi \in dp(f)$  and  $\langle Af, \varphi \rangle \leq 0$ .

Since  $\langle g, \varphi_t \rangle \leq p(g)$  for all  $t > 0$ , it follows that  $\langle g, \varphi \rangle \leq p(g)$  ( $g \in E$ ). Moreover,  $\langle f, \varphi_t \rangle - t\langle Af, \varphi_t \rangle = p(f - tAf)$  ( $t > 0$ ). Letting  $t \rightarrow 0$  yields  $\langle f, \varphi \rangle = p(f)$ .

We have proved that  $\varphi \in dp(f)$ . By hypothesis we have for all  $t > 0$ ,

$$p(f) \leq p(f - tAf) = \langle f - tAf, \varphi_t \rangle = \langle f, \varphi_t \rangle - t\langle Af, \varphi_t \rangle \leq p(f) - t\langle Af, \varphi_t \rangle.$$

Consequently  $\langle Af, \varphi_t \rangle \leq 0$  for all  $t > 0$ . Thus  $\langle Af, \varphi \rangle \leq 0$ .  $\square$

**Remark 2.4** The function  $p$  is convex. So the one-sided Gateaux-derivatives

$$D_g^+ p(f) = \lim_{t \downarrow 0} \frac{1}{t} (p(f + tg) - p(f)) \quad \text{and} \\ D_g^- p(f) = \lim_{t \uparrow 0} \frac{1}{t} (p(f + tg) - p(f))$$

exist and satisfy  $D_g^- p(f) \leq D_g^+ p(f)$  for all  $f, g \in E$  (cf. Moreau [30]). Moreover,

$$D_g^+ p(f) = \sup\{\langle g, \varphi \rangle : \varphi \in dp(f)\}, \quad (2.9)$$

$$D_g^- p(f) = \inf\{\langle g, \varphi \rangle : \varphi \in dp(f)\}. \quad (2.10)$$

Thus  $A$  is  $p$ -dissipative if and only if  $D_{Af}^- p(f) \leq 0$ , and  $A$  is strictly  $p$ -dissipative if and only if  $D_{Af}^+ p(f) \leq 0$  for all  $f \in D(A)$ .

**Corollary 2.5** *Let  $A$  be a closable operator. If  $A$  is  $p$ -dissipative, then so is its closure.*

**Theorem 2.6** *Let  $p$  be a continuous sublinear functional on a real Banach space  $E$ . Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$ . The following assertions are equivalent.*

- (a)  $p(T(t)f) \leq p(f)$  for all  $t \geq 0$ ,  $f \in E$ .
- (b)  $A$  is strictly  $p$ -dissipative.
- (c) There exists a core  $D$  of  $A$  such that  $A|_D$  is  $p$ -dissipative.

**Proof** Assume that (a) holds. Let  $f \in D(A)$ ,  $\varphi \in dp(f)$ . Then

$$\begin{aligned} \langle Af, \varphi \rangle &= \lim_{t \rightarrow 0} \frac{1}{t} (\langle T(t)f, \varphi \rangle - \langle f, \varphi \rangle) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (\langle T(t)f, \varphi \rangle - p(f)) \\ &\leq \limsup_{t \rightarrow 0} \frac{1}{t} (p(T(t)f) - p(f)) \leq 0. \end{aligned}$$

This proves (b).

It is trivial that (b) implies (c). So let us assume (c). Then it follows from Corollary 2.5 that  $A$  is  $p$ -dissipative. Hence, by (2.8),  $p(\lambda R(\lambda, A)g) \leq p(g)$  for all  $g \in E$ ,  $\lambda > \omega_0(A)$ . Hence  $\lambda R(\lambda, A)$  is  $p$ -contractive for  $\lambda > \omega_0(A)$ . This implies that  $T(t)$  is  $p$ -contractive by the formula (1.3),

$$T(t) = \lim_{t \rightarrow 0} \left( \frac{n}{t} R\left(\frac{n}{t}, A\right) \right)^n \text{ (strongly) for } t \geq 0.$$

We have shown that for generators,  $p$ -dissipativity is equivalent to  $p$ -contractivity of the semigroup. Now we will consider a  $p$ -dissipative operator  $A$  (which is not a generator a priori) and investigate under which additional hypotheses  $A$  is the generator of a (necessarily  $p$ -contractive) semigroup. At first we present some consequences of  $p$ -dissipativity.

**Theorem 2.7** *Let  $A$  be a  $p$ -dissipative operator. If  $D(A)$  is dense, then  $A$  is strictly  $p$ -dissipative.*

**Proof** Let  $f \in D(A)$ ,  $\varphi \in dp(f)$ . Then for every  $t > 0$  and  $g \in D(A)$  we have

$$\begin{aligned} \langle Af, \varphi \rangle &= \frac{1}{t} (\langle f + tAf, \varphi \rangle - \langle f, \varphi \rangle) \leq \frac{1}{t} (p(f + tAf) - p(f)) \\ &\leq \frac{1}{t} (p(f + tg) + tp(Af - g) - p(f)) \\ &\leq \frac{1}{t} (p((\text{Id} - tA)(f + tg)) + tp(Af - g) - p(f)) \quad (\text{by (2.7)}) \\ &\leq \frac{1}{t} (p(f) + tp(g - Af) + t^2 p(-Ag) + tp(Af - g) - p(f)) \\ &\leq \frac{1}{t} (2tc\|g - Af\| + t^2 c\|Ag\|) \quad (\text{by (2.3)}) \\ &= 2c\|g - Af\| + tc\|Ag\|. \end{aligned}$$

Letting  $t \rightarrow 0$  we obtain  $\langle Af, \varphi \rangle \leq 2c\|g - Af\|$  for all  $g \in D(A)$ . Since  $D(A)$  is dense in  $E$ , this implies that  $\langle Af, \varphi \rangle \leq 0$ .  $\square$

We now impose stronger conditions on  $p$ . A continuous sublinear function  $p: E \rightarrow \mathbb{R}$  is called *half-norm* if

$$p(f) + p(-f) > 0 \text{ whenever } f \neq 0; \quad (2.11)$$

and  $p$  is called a *strict half-norm* if in addition there exists some constant  $d > 0$  such that

$$p(f) + p(-f) \geq d\|f\| \text{ for all } f \in E. \quad (2.12)$$

If  $p$  is a half-norm, then

$$\|f\|_p = p(f) + p(-f) \quad (f \in E) \quad (2.13)$$

defines a norm on  $E$  which is equivalent to the given norm if and only if  $p$  is strict.

**Remark 2.8** Every half-norm  $p$  induces a closed proper cone

$$E_p := \{f \in E : p(-f) \geq 0\}$$

on  $E$ . Any  $p$ -contractive operator  $T$  on  $E$  leaves the cone  $E_p$  invariant (i.e.,  $T$  is positive for the corresponding ordering).

Conversely, given a closed proper cone  $E_+$  on  $E$ , then

$$p(f) := \text{dist}(-f, E_+) = \inf\{\|f + g\| : g \in E_+\}$$

defines a half-norm on  $E$  such that  $E_+ = E_p$ . This half-norm is called the *canonical half-norm* on the ordered Banach space  $(E, E_+)$ . The canonical half-norm is strict if and only if the cone  $E_+$  is *normal* (this is equivalent to the fact that for every  $\varphi \in E'$  there exist positive linear forms  $\varphi_1$  and  $\varphi_2$  on  $E$  such that  $\varphi = \varphi_1 - \varphi_2$  (see Batty and Robinson [3] or Schaefer [37, Chapter V])).

**Proposition 2.9** *Let  $A$  be a  $p$ -dissipative operator where  $p$  is a half-norm. If  $D(A)$  is dense, then  $A$  is closable (and the closure of  $A$  is  $p$ -dissipative as well (by Corollary 2.5)).*

**Proof** Let  $f_n \in D(A)$ ,  $\lim_{n \rightarrow \infty} f_n = 0$ ,  $\lim_{n \rightarrow \infty} Af_n = g$ . We have to show that  $g = 0$ . To this end let  $h \in D(A)$ . Then (2.7) gives

$$p(f_n + th) \leq p(f_n + th - tA(f_n + th)) \quad (t > 0).$$

Letting  $n \rightarrow \infty$  we obtain  $p(th) \leq p(th - tg - t^2Ah) \quad (t > 0)$ . Hence

$$p(h) \leq p((h - g) - tAh) \quad (t > 0)$$

by positive homogeneity. Letting  $t \downarrow 0$  finally we obtain  $p(h) \leq p(h - g)$  for all  $h \in D(A)$ . Since  $D(A)$  is dense by hypothesis, we can approximate  $g$  by  $h \in D(A)$  and conclude that  $p(g) \leq p(0) = 0$ . Since  $\lim_{n \rightarrow \infty} A(-f_n) = -g$ , we have  $p(-g) \leq 0$  by symmetry. Hence  $p(g) + p(-g) \leq 0$  which implies  $g = 0$  by (2.11).  $\square$

**Lemma 2.10** *Let  $p$  be a half-norm and  $A$  a  $p$ -dissipative operator. Then*

$$\lambda \|f\|_p \leq \|(\lambda - A)f\|_p \text{ for all } f \in D(A), \lambda > 0. \quad (2.14)$$

In particular,  $(\lambda - A)$  is injective for all  $\lambda > 0$ . If  $p$  is strict and  $A$  is closed, then  $\text{im}(\lambda - A)$  is closed for all  $\lambda > 0$ .

**Proof** Let  $\lambda > 0$ ,  $f \in D(A)$ . Then by (2.7),  $\lambda p(\pm f) \leq p((\lambda - A)(\pm f))$ . Hence

$$\lambda \|f\|_p = \lambda p(f) + \lambda p(-f) \leq p((\lambda - A)f) + p(-(\lambda - A)f) = \|(\lambda - A)f\|_p.$$

Thus (2.14) is proved. Now suppose that  $p$  is strict. Then  $\|\cdot\|_p$  is equivalent to the given norm. Let  $\lambda > 0$  and  $g \in \overline{\text{im}(\lambda - A)}$ . Then  $g = \lim_{n \rightarrow \infty} (\lambda - A)f_n$  for some sequence  $(f_n)_{n \in \mathbb{N}} \subset D(A)$ . It follows from (2.14) that  $(f_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. Let  $f = \lim_{n \rightarrow \infty} f_n$ . Then

$$\lim_{n \rightarrow \infty} Af_n = \lambda \lim_{n \rightarrow \infty} f_n - \lim_{n \rightarrow \infty} (\lambda - A)f_n = \lambda f - g$$

exists. If  $A$  is closed, this implies that  $f \in D(A)$  and  $Af = \lambda f - g$ . Hence  $g = (\lambda - A)f \in \text{im}(\lambda - A)$ . We have shown that  $\text{im}(\lambda - A)$  is closed.  $\square$

The following is the main theorem of this section.

**Theorem 2.11** *Let  $p$  be a strict half-norm and  $A$  an operator on  $E$ . The following assertions are equivalent.*

- (a)  $A$  is the generator of a  $p$ -contraction semigroup.
- (b)  $D(A)$  is dense,  $A$  is  $p$ -dissipative and  $\text{im}(\lambda - A) = E$  for some  $\lambda > 0$ .

**Proof** Since  $p$  is a strict half-norm, we can assume that  $\|f\| = \|f\|_p$  for all  $f \in E$ . By Theorem 2.6, condition (a) implies (b).

Now suppose that (b) holds. Then it follows from Lemma 2.10 that  $\mu \in \varrho(A)$  and  $\|\mu R(\mu, A)\| \leq 1$  whenever  $\mu > 0$  such that  $\text{im}(\mu - A) = E$ . So by hypothesis  $\lambda \in \varrho(A)$  and  $\text{dist}(\lambda, \sigma(A)) \geq \|R(\lambda, A)\|^{-1} \geq \lambda$ . Hence  $(0, 2\lambda) \subset \varrho(A)$ . Iterating this argument we see that  $(0, \infty) \subset \varrho(A)$ . It follows from the Hille-Yosida theorem that  $A$  generates a contraction semigroup  $(T(t))_{t \geq 0}$ . Finally, from 2.6 it follows that  $(T(t))_{t \geq 0}$  is  $p$ -contractive.  $\square$

Of course, the norm function  $N$  given by  $N(f) = \|f\|$  is a strict half-norm. In the case when  $p = N$ , Theorem 2.11 is due to Lumer and Phillips [28]. It turns out to be extremely useful in showing that a concrete operator is a generator. Because of its importance we state this special case explicitly below (including the complex case). Before that let us formulate Theorem 2.11 for the case when the operator is merely given on a core.

**Corollary 2.12** *Let  $p$  be a strict half-norm and  $A$  be a densely defined operator. If  $A$  is  $p$ -dissipative and  $(\lambda - A)$  has dense range for some  $\lambda > 0$ , then  $A$  is closable and the closure  $\bar{A}$  of  $A$  generates a  $p$ -contraction semigroup.*

**Proof** It follows from Proposition 2.9 that  $A$  is closable and the closure  $\overline{A}$  is  $p$ -dissipative. Lemma 2.10 implies that  $(\lambda - \overline{A})D(\overline{A}) = E$ . So Theorem 2.11 yields the desired conclusion.  $\square$

We conclude this section indicating the results for the complex case.

Let  $E$  be a complex Banach space and  $p: E \rightarrow \mathbb{R}_+$  be a seminorm on  $E$  (i.e.,  $p(f+g) \leq p(f) + p(g)$  and  $p(\lambda f) = |\lambda|p(f)$  holds for all  $f, g \in E, \lambda \in \mathbb{C}$ ). The subdifferential  $dp(f)$  of  $p$  in  $f \in E$  is defined by

$$dp(f) = \{\varphi \in E': \operatorname{Re}\langle g, \varphi \rangle \leq p(g) \text{ for all } g \in E \text{ and } \langle f, \varphi \rangle = p(f)\}. \quad (2.15)$$

We assume in addition that  $p$  is continuous. Then it follows from the Hahn-Banach theorem that  $dp(f) \neq \emptyset$  for any  $f \in E$ . A linear operator  $A$  on  $E$  is called  $p$ -dissipative if for all  $f \in D(A)$  there exists  $\varphi \in dp(f)$  such that  $\operatorname{Re}\langle Af, \varphi \rangle \leq 0$ .

The arguments given above show that also in the situation considered here  $A$  is  $p$ -dissipative if and only if

$$p((1-tA)f) \geq p(f)$$

for all  $f \in D(A), t \geq 0$ .

The results of this section carry over if they are appropriately modified. We explicitly state the most important result for the case when  $p$  is the norm. A linear operator  $A$  is simply called *dissipative* if it is  $N$ -dissipative where  $N(f) = \|f\|$  ( $f \in E$ ).

**Theorem 2.13 (Lumer-Phillips)** *Let  $A$  be a densely defined operator on a complex Banach space  $E$ . The following assertions are equivalent.*

- (a)  *$A$  is closable and the closure of  $A$  is the generator of a contraction semigroup.*
- (b)  *$A$  is dissipative and  $(\lambda - A)$  has dense range for some  $\lambda > 0$ .*

### 3 Semigroups on $L^\infty$ and $H^\infty$

by Heinrich P. Lotz

In this section we shall prove that on  $L^\infty$ , on  $H^\infty$ , and on some other classical Banach spaces every strongly continuous semigroup of operators is uniformly continuous.

**Lemma 3.1** *Let  $\mathcal{T} = (T(t))_{t \geq 0}$  be a one-parameter semigroup of operators on a Banach space  $E$ . Suppose that  $s = \limsup_{t \rightarrow 0} \|T(t) - \operatorname{Id}\|$  is finite. Then  $\mathcal{T}$  is uniformly continuous, if  $\lim_{t \rightarrow 0} \|(T(t) - \operatorname{Id})^2\| = 0$ .*

**Proof** The identity  $2(T(t) - \text{Id}) = T(2t) - \text{Id} - (T(t) - \text{Id})^2$  shows that

$$2\|T(t) - \text{Id}\| - \|(T(t) - \text{Id})^2\| \leq \|T(2t) - \text{Id}\|.$$

Hence  $2s \leq \limsup_{t \downarrow 0} \|T(2t) - \text{Id}\|$ . Obviously,  $\limsup_{t \downarrow 0} \|T(2t) - \text{Id}\| = s$  and so,  $2s \leq s$ . Consequently,  $s = 0$ .  $\square$

**Remark**

(i) If, in Lemma 3.1,  $\mathcal{T} = (T(t))_{t \geq 0}$  is strongly continuous, in which case  $s < \infty$ , one can replace  $\lim_{t \rightarrow 0} \|(T(t) - \text{Id})^2\| = 0$  by the weaker condition

$$\limsup_{t \rightarrow 0} r(T(t) - \text{Id}) < 1,$$

see Lotz [27, Lemma 2], where  $r$  denotes the spectral radius.

(ii) The condition  $s < \infty$  in Lemma 3.1 is essential as the following example shows: Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be non-continuous with  $f(s+t) = f(s) + f(t)$  for all  $s, t \in \mathbb{R}$  (see Hamel [18]). Then  $\{(t, f(t)): t \in \mathbb{R}\}$  is dense in  $\mathbb{R}^2$ . Hence for the semigroups  $\mathcal{T} = (T(t))_{t \geq 0}$  on  $\mathbb{R}^2$  with

$$T(t) = \begin{pmatrix} 1 & f(t) \\ 0 & 1 \end{pmatrix} \quad \text{for } t \geq 0$$

we have  $s = \infty$ . Therefore  $\mathcal{T}$  is not uniformly continuous. However,  $(T(t) - \text{Id})^2 = 0$  for all  $t \geq 0$ .

**Lemma 3.2** Let  $\mathcal{T} = (T(t))_{t \geq 0}$  be a one-parameter semigroup of operators on a Banach space  $E$ . Then the following assertions are equivalent

- (a)  $\mathcal{T}' = (T(t)')_{t \geq 0}$  is a strongly continuous semigroup on the dual  $E'$ .
- (b)  $((T(t_n) - \text{Id})x_n)$  converges weakly to zero for every bounded sequence  $(x_n)$  in  $E$  and every sequence  $(t_n)$  in  $[0, \infty)$  with  $\lim t_n = 0$ .

Moreover, (a) implies

- (c)  $\mathcal{T}$  is strongly continuous.

**Proof** Let  $x' \in E'$  and  $t_n \geq 0$  be given. Then  $\lim \|(T(t_n) - \text{Id})'x'\| = 0$  if and only if  $\lim \langle x_n, (T(t_n) - \text{Id})'x' \rangle = 0$  for every bounded sequence  $(x_n)$  in  $E$ . This implies the equivalence of (a) and (b). In particular, (a) implies that  $((T(t_n) - \text{Id})x)$  converges weakly to zero for every sequence  $(t_n)$  in  $[0, \infty)$  with  $\lim t_n = 0$  and every  $x \in E$ . Hence  $\mathcal{T}$  is strongly continuous by Proposition 1.23 in Davies [11].  $\square$

We recall that a Banach space  $E$  is called a *Grothendieck space* if every weak\* convergent sequence in  $E'$  converges weakly.

**Theorem 3.3** Let  $E$  be a Grothendieck space. If  $\mathcal{T} = (T(t))_{t \geq 0}$  is a strongly continuous semigroup in  $E$ , then  $\mathcal{T}'' = (T(t)'')_{t \geq 0}$  is strongly continuous in  $E''$ .

**Proof** Suppose that  $(x'_n)$  is a bounded sequence in  $E'$  and that  $t_n \geq 0$  with  $\lim t_n = 0$ . Put  $V_n := T(t_n) - \text{Id}$ . Then  $\lim \|V_n x\| = 0$  and therefore  $\lim \langle x, V'_n x'_n \rangle = 0$  for every  $x \in E$ . Hence  $(V'_n x'_n)$   $w^*$ -converges to zero. Since  $E$  is a Grothendieck space,  $(V'_n x'_n)$  converges weakly to zero. Now Lemma 3.2 implies that  $(T(t))''$  is strongly continuous.  $\square$

Recall now that a Banach space  $E$  is said to have the *Dunford-Pettis property* if  $\lim \langle x_n, x'_n \rangle = 0$  whenever  $(x_n)$  in  $E$  and  $(x'_n)$  in  $E'$  converge weakly to zero.

**Theorem 3.4** *Let  $E$  be a Banach space with the Dunford-Pettis property and let  $\mathcal{T} = (T(t))_{t \geq 0}$  be a one-parameter semigroup of operators on  $E$ . If  $\mathcal{T}'' = (T(t))''_{t \geq 0}$  is strongly continuous in  $E''$ , then  $\mathcal{T}$  is uniformly continuous.*

**Proof** Suppose that  $\mathcal{T}''$  is a strongly continuous semigroup. Then Lemma 3.2 implies that  $\mathcal{T}'$  and  $\mathcal{T}$  are strongly continuous. Hence by the uniform boundedness principle,  $\limsup_{t \rightarrow 0} \|(T(t) - \text{Id})\|$  is finite. By Lemma 3.1 it suffices to show that  $\lim_{t \rightarrow 0} \|(T(t) - \text{Id})^2\| = 0$ . Let  $t_n \geq 0$  with  $\lim t_n = 0$  be given. Then there exists a bounded sequence  $(x_n)$  in  $E$  and a bounded sequence  $(x'_n)$  in  $E'$  such that

$$\|(T(t_n) - \text{Id})^2\| - \frac{1}{n} \leq \langle (T(t_n) - \text{Id})x_n, (T(t_n) - \text{Id})'x'_n \rangle.$$

Since  $\mathcal{T}'$  and  $\mathcal{T}''$  are strongly continuous, Lemma 3.2 implies that  $((T(t_n) - \text{Id})x_n)$  and  $((T(t_n) - \text{Id})'x'_n)$  converge weakly to zero. Since  $E$  has the Dunford-Pettis property,  $\lim \|(T(t_n) - \text{Id})^2\| = 0$ . Consequently,  $\lim_{t \rightarrow 0} \|(T(t) - \text{Id})^2\| = 0$ .  $\square$

An immediate consequence of Theorem 3.3 and Theorem 3.4 is the following.

**Theorem 3.5** *Let  $E$  be a Grothendieck space with the Dunford-Pettis property. Then every strongly continuous semigroup of operators on  $E$  is uniformly continuous.*

A compact Hausdorff space is called an *F-space* if the closures of two disjoint open  $F_\sigma$ -sets are disjoint and is called a *Stonean* (res.,  $\sigma$ -Stonean) space if the closure of every open set (res., open  $F_\sigma$ -set) is open. Every  $\sigma$ -Stonean space is an F-space.

**Theorem 3.6** *Every strongly continuous semigroups of operators on one of the following Banach spaces is uniformly continuous.*

- (i)  $C(K)$ , where  $K$  is a compact F-space.
- (ii)  $L^\infty(S, \Sigma, \mu)$  for any measure space  $(S, \Sigma, \mu)$ .
- (iii) The Banach space  $B(S, \Sigma)$  of all bounded  $\Sigma$ -measurable functions on  $S$  if  $\Sigma$  is a  $\sigma$ -algebra of subsets of  $S$ .
- (iv) The Banach space  $\mathcal{H}(O)$  of all bounded continuous solutions of

$$\sum_{1 \leq i \leq n} (\partial^2 f / \partial x_i^2) = 0$$

on an open subset  $O$  of  $\mathbb{R}^n$ .

(v) The Banach space  $\mathcal{W}(O)$  of all bounded continuous solutions of

$$\sum_{1 \leq i \leq n} (\partial^2 f / \partial x_i^2) = (\partial f / \partial x_{n+1})$$

on an open subset  $O$  of  $\mathbb{R}^{n+1}$ .

(vi) The Banach space  $H^\infty(O)$  of bounded analytic functions on a finitely connected domain  $O$  of the complex plane.

**Proof** By Theorem 3.5 it suffices to show that the spaces listed above are Grothendieck spaces with the Dunford-Pettis property.

(i) If  $K$  is compact, then  $C(K)$  has the Dunford-Pettis property (cf. Grothendieck [17, Théorème 4]). If  $K$  is a compact F-space, then  $C(K)$  is a Grothendieck space cf. Seever [38, Theorem 2.5]. The special cases for Stonean and  $\sigma$ -Stonean spaces are due to Grothendieck [17, Théorème 9] and Ando [1], respectively.

(ii) and (iii) It is well known that every  $\sigma$ -order complete AM-space with unit is isometric to a space  $C(K)$  where  $K$  is a compact  $\sigma$ -Stonean space. Obviously, the spaces under (ii) and (iii) are  $\sigma$ -order complete AM-spaces with unit and therefore, by proof of (i), are Grothendieck spaces with the Dunford-Pettis property.

(iv) and (v) These spaces are order complete vector lattices. This follows from Bauer [4, pp.18-22, Standardbeispiele 1 and 2 p.55]. Since these spaces contain the constant functions on  $O$ , they are complete for the supremum-norm. Indeed, if  $(f_n)$  is a Cauchy-sequence for this norm, it is easily seen that  $(f_n)$  converges in norm to  $\inf_n \sup(f_k : n < k)$ . Therefore these spaces are  $\sigma$ -order complete AM-spaces with unit and so as before Grothendieck spaces with the Dunford-Pettis property.

(vi) In Bourgain [7] it is shown, that  $H^\infty(D)$  is a Grothendieck space, where  $D$  is the open unit disc  $\{z : |z| < 1\}$ , and in Bourgain [8], that this Banach space has the Dunford-Pettis property (see also the summary of Bourgain in Blei and Sidney [6]).

If  $O$  is a finitely connected domain and  $H^\infty$  does not only contain the constant functions, then  $H^\infty(O)$  is isomorphic to a finite direct sum of copies of  $H^\infty(D)$ . (Note that  $H^\infty(D)$  is isomorphic to  $\{f \in H^\infty(D) : f(0) = 0\}$  via the map  $f \mapsto zf$ . Then use Grothendieck [17, p.77 and Proposition 4.4.1]). Hence  $H^\infty(O)$  is a Grothendieck space with the Dunford-Pettis property.

**Remark (Final)** It follows from Theorem 3.6 that on  $L^\infty$  the infinitesimal generator of a strongly continuous semigroup is necessarily bounded. It is not obvious that on  $L^\infty([0, 1])$  there exist closed densely defined unbounded operators.

To see this let  $A$  be a closed densely defined unbounded operator from  $\ell^2$  into  $L^\infty([0, 1])$  with domain  $D$  (such operators can easily be constructed). By the Khintchine inequality, the map  $R : (a_n) \mapsto \sum a_n r_n$ , where  $r_n$  denotes the  $n^{\text{th}}$

Rademacher function, from  $\ell^2$  into  $L^1([0, 1])$  is a topological isomorphism. Hence  $T = R'$  maps  $L^\infty([0, 1])$  onto  $\ell^2$ .

Banach's homomorphism theorem implies that  $T^{-1}(D)$  is dense in  $L^\infty([0, 1])$  and that  $AT$  is a closed densely defined unbounded operator on  $L^\infty([0, 1])$  with domain  $T^{-1}(D)$ . This solves a problem raised by R. Kaufman.

H. Porta and the author have shown that if a Banach space  $E$  has an infinite dimensional separable quotient space and  $F$  is an infinite dimensional Banach space, then there always exists a closed densely defined unbounded operator from  $E$  into  $F$ .

## Notes

*Section 1:* The abstract Cauchy problem is treated systematically in the monographs of Krein [24] and Fattorini [13]. We refer to these books for more details and historical notes. One implication of Theorem 1.1 is proved in Krein [24, Theorem 2.11].

The Hille-Yosida Theorem has been proved independently by Hille [20] and Yosida [39] for contraction semigroups. The extension to arbitrary strongly continuous semigroups is independently due to Feller [14], Miyadera [29] and Phillips [32]. Thus our terminology is slightly incorrect, and some authors refer to the general version as the Hille-Yosida-Phillips theorem which is more correct.

Holomorphic semigroups belong to the standard material of the theory of one-parameter semigroups. Our Theorem 1.14 deviates from the usual presentation since the condition on the resolvent is merely required on a half-plane.

Differentiable semigroups are treated in detail in the book of Pazy [31] who discovered Theorem 1.17 and 1.18. The spectral property of eventually norm continuous semigroups given in Theorem 1.20 is contained in Hille and Phillips [21, Theorem 16.4.2] with a proof depending on Gelfand theory. For norm continuous semigroups it is contained in Pazy [31] with a simpler proof. The elementary proof we give here is due to G. Greiner.

Theorem 1.29 on the perturbation by bounded operators is due to Phillips [32] who also investigated permanence of smoothness properties by this kind of perturbation. We also refer to Pazy [31, Section 3.1].

The observation that eventually norm continuity is preserved by perturbation by a compact operator (see Theorem 1.30) seems to be new.

The perturbation by operators being continuous on the graph of the generator is due to Desch and Schappacher [12]. The short proof we give here is due to G. Greiner and has the advantage to yield the same permanence for smoothness properties as in the classical case, see Corollary 1.32.

The characterization of a core as “domain of uniqueness” given in Theorem 1.33 seems to be new. In this section we have presented part of the standard theory

of one-parameter semigroups including some new aspects. A very elegant brief introduction to one-parameter semigroups is given in the treatise of Kato [22] where one can also find all the results on perturbation theory going beyond the elementary facts we discuss here. A complete information on the general theory can be obtained by consulting the books of Davies [11], Goldstein [15] and Pazy [31]. The monograph of Goldstein [15] contains a variety of examples and applications.

*Section 2:* Dissipative operators were introduced by Lumer and Phillips [28]. The analogous notion of dispersiveness is due to Phillips [33]. Our approach follows closely Arendt et al. [2] where half-norms were introduced. Related previous results were obtained by Calvert [9], Hasegawa [19], Sato [36], B  nilan and Picard [5] and Picard [34], where the two last consider non-linear semigroups. A further investigation of half-norms can be found in Batty and Robinson [3] who consider ordered Banach spaces other than Banach lattices in great detail. We also refer to the historical notes given there.

*Section 3:* It had been proved by Kishimoto and Robinson [23] that every generator of a positive semigroup on  $L^\infty$  is bounded. That every strongly continuous semigroup on  $L^\infty$  is uniformly continuous was first shown by Lotz [25], Lotz [26] and Lotz [27]. The proof of Lemma 3.1 was communicated to the author of this section by T. Coulhon, who independently obtained a particular case of Theorem 3.5 (Coulhon [10]).

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## Chapter A-III

# Spectral Theory

by  
Günther Greiner and Rainer Nagel

### 1 Introduction

In this chapter, we begin a systematic analysis of the spectrum of a strongly continuous semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  on a complex Banach space  $E$ . By the spectrum of the semigroup we mean the spectrum  $\sigma(A)$  of its generator  $A$ . In particular, we are interested in the precise relations between  $\sigma(A)$  and  $\sigma(T(t))$ . The heuristic formula

$$T(t) = e^{tA}$$

serves as a leitmotif and suggests relations of the form

$$\sigma(T(t)) = e^{t\sigma(A)} = \{e^{t\lambda} : \lambda \in \sigma(A)\},$$

called *spectral mapping theorem*. These — or similar — relations will be of great use in Chapter IV enabling us to determine the asymptotic behavior of the semigroup  $\mathcal{T}$  by examining the spectrum of its generator.

As motivation and also as a preliminary step, we focus here on the *spectral radius*

$$r(T(t)) := \sup\{|\lambda| : \lambda \in \sigma(T(t))\}, \quad t \geq 0, \quad (1.1)$$

and demonstrate its relationship with the *spectral bound*

$$s(A) := \sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\} \quad (1.2)$$

of the generator  $A$  and to the *growth bound*

$$\omega_0 := \inf\{\omega \in \mathbb{R}: \|T(t)\| \leq M_\omega \cdot e^{\omega t} \text{ for all } t \geq 0 \text{ and suitable } M_\omega\} \quad (1.3)$$

of the semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$ . (Note that we sometimes write  $\omega_0(\mathcal{T})$  or  $\omega_0(A)$  instead of  $\omega_0$ ). The Examples 1.3 and 1.4 below illustrate the main difficulties to be encountered.

**Proposition 1.1** *Let  $\omega_0$  be the growth bound of the strongly continuous semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$ . Then*

$$r(T(t)) = e^{\omega_0 t} \quad (1.4)$$

for every  $t \geq 0$ .

**Proof** From A-I, (1.1) we know that

$$\omega_0(\mathcal{T}) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T(t)\|.$$

Since the spectral radius of  $T(t)$  is given as

$$r(T(t)) = \lim_{n \rightarrow \infty} \|T(nt)\|^{1/n},$$

we obtain for  $t > 0$

$$r(T(t)) = \lim_{n \rightarrow \infty} \exp\left(\frac{t}{nt} \log \|T(nt)\|\right) = e^{\omega_0 t}.$$

It was shown in A-I, Proposition 1.11 that the spectral bound  $s(A)$  is always dominated by the growth bound  $\omega_0$  and therefore  $e^{s(A)t} \leq r(T(t))$ . If the above mentioned spectral mapping theorem holds—as is the case for bounded generators (e.g., see Theorem VII.3.11 of Dunford and Schwartz [5])—we obtain the equality

$$e^{s(A)t} = r(T(t)) = e^{\omega_0(\mathcal{T})t},$$

hence  $s(A) = \omega_0(\mathcal{T})$ . Therefore, the following corollary is a consequence of the definitions of  $s(A)$  and  $\omega_0(\mathcal{T})$ .

**Corollary 1.2** *Consider the semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  generated by some bounded linear operator  $A \in \mathcal{L}(E)$ . If  $\operatorname{Re}(\lambda) < 0$  for each  $\lambda \in \sigma(A)$ , then  $\lim_{t \rightarrow \infty} \|T(t)\| = 0$ .*

Through this corollary we have re-established Liapunov's famous result, which assures that the solutions of the linear Cauchy problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in \mathbb{C}^n \quad \text{and} \quad A = (a_{ij})_{n \times n}$$

are *stable*, i.e., they converge to zero as  $t \rightarrow \infty$  if (and only if) the real parts of all eigenvalues of the matrix  $A$  are less than zero.

The situation is much more difficult for unbounded generators, and  $s(A)$  may differ drastically from  $\omega_0(\mathcal{T})$ .

**Example 1.3** (Banach function space, Greiner et al. [11]) Consider the Banach space  $E$  of all complex valued continuous functions on  $\mathbb{R}_+$  which vanish at infinity and are integrable for  $e^x dx$ , i.e.,

$$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x dx)$$

endowed with the norm

$$\|f\| := \|f\|_\infty + \|f\|_1 = \sup\{|f(x)| : x \in \mathbb{R}_+\} + \int_0^\infty |f(x)|e^x dx.$$

The translation semigroup

$$T(t)f(x) := f(x+t)$$

is strongly continuous on  $E$  and one shows as in A-I, 2.4 that its generator is given by

$$Af = f', \quad D(A) = \{f \in E : f \in C^1(\mathbb{R}_+), f' \in E\}.$$

First we observe that  $\|T(t)\| = 1$  for every  $t \geq 0$ , hence  $\omega_0(\mathcal{T}) = 0$ . Moreover it is clear that  $\lambda$  is an eigenvalue of  $A$  as soon as  $\operatorname{Re}(\lambda) < -1$  (in fact: the function

$$x \mapsto e_\lambda(x) := e^{\lambda x}$$

belongs to  $D(A)$  and is an eigenvector of  $A$ ), hence  $s(A) \geq -1$ . For  $f \in E$  and  $\operatorname{Re}(\lambda) > -1$

$$\|\cdot\|_1\text{-}\lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} T(s)f \, ds$$

exists since  $\|T(s)f\|_1 \leq e^{-s}\|f\|_1$  for  $s \geq 0$ , and

$$\|\cdot\|_\infty\text{-}\lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} T(s)f \, ds$$

exists since  $\int_0^\infty e^x |f(x)| \, dx < \infty$ . Therefore  $\int_0^\infty e^{-\lambda s} T(s)f \, ds$  exists in  $E$  for every  $f \in E$ ,  $\operatorname{Re}(\lambda) > -1$ .

As we observed in A-I, Proposition 1.11, this implies  $\lambda \in \varrho(A)$ . Therefore  $\mathcal{T} = (T(t))_{t \geq 0}$  is a semigroup having  $s(A) = -1$ , but  $\omega_0(\mathcal{T}) = 0$ .

**Example 1.4** (Hilbert space, Zabczyk [27]) For every  $n \in \mathbb{N}$  consider the  $n$ -dimensional Hilbert space  $H_n := \mathbb{C}^n$  and operators  $A_n \in \mathcal{L}(H_n)$  defined by the matrices

$$A_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdot & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & 0 \end{pmatrix}_{n \times n}.$$

These matrices are nilpotent and therefore  $\sigma(A_n) = \{0\}$ . The elements

$$x_n := n^{-1/2}(1, \dots, 1) \in H_n$$

satisfy the following properties.

- (i)  $\|x_n\| = 1$  for every  $n \in \mathbb{N}$ ,
- (ii)  $\lim_{n \rightarrow \infty} \|A_n x_n - x_n\| = 0$ ,
- (iii)  $\lim_{n \rightarrow \infty} \|\exp(tA_n)x_n - e^t x_n\| = 0$ .

Consider now the Hilbert space

$$H := \bigoplus_{n \in \mathbb{N}} H_n \text{ and the operator } A := (A_n + 2\pi i n)_{n \in \mathbb{N}}$$

with maximal domain in  $H$ .

Analogously we define a semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  by

$$T(t) := (e^{2\pi i n t} \exp(tA_n))_{n \in \mathbb{N}}.$$

Since  $\|\exp(tA_n)\| \leq e^t$  for every  $n \in \mathbb{N}$ ,  $t \geq 0$ , and since  $t \mapsto T(t)x$  is continuous on each component  $E_n$ , it follows that  $\mathcal{T}$  is strongly continuous. Its generator is the operator  $A$  as defined above.

For  $\lambda \in \mathbb{C}$ ,  $\operatorname{Re}(\lambda) > 0$ , we have  $\lim_{n \rightarrow \infty} \|R(\lambda - 2\pi i n, A_n)\| = 0$ , hence

$$(R(\lambda, A_n + 2\pi i n))_{n \in \mathbb{N}} = (R(\lambda - 2\pi i n, A_n))_{n \in \mathbb{N}}$$

is a bounded operator on  $H$  representing the resolvent  $R(\lambda, A)$ . Therefore we obtain  $s(A) \leq 0$ . On the other hand, each  $2\pi i n$  is an eigenvalue of  $A$ , hence  $s(A) = 0$ .

Take now  $x_n \in H_n$  as above and consider the sequence  $(x_n)_{n \in \mathbb{N}}$ . From (iii) it follows that for  $t > 0$  the number  $e^t$  is an approximate eigenvalue of  $T(t)$  with approximate eigenvector  $(x_n)_{n \in \mathbb{N}}$  (see Definition 2.1 below). Therefore

$$e^t \leq r(T(t)) \leq \|T(t)\|$$

and hence  $\omega_0(\mathcal{T}) \geq 1$ . On the other hand, it is easy to see that  $\|T(t)\| = e^t$ , hence  $\omega_0(\mathcal{T}) = 1$ .

Finally, if we take  $S(t) := e^{-t/2}T(t)$ , we obtain a semigroup  $\mathcal{S}$  having spectral bound  $-\frac{1}{2}$  but satisfying  $\lim_{t \rightarrow \infty} \|S(t)\| = \infty$  in contrast with Corollary 1.2.

These examples show that neither the conclusion of Corollary 1.2, i.e., “ $s(A) < 0$  implies stability”, nor the “spectral mapping theorem”

$$\sigma(T(t)) = \exp(t \cdot \sigma(A))$$

is valid for arbitrary strongly continuous semigroups. A careful analysis of the general situation will be given in Section 6 below, but we will first develop systematically the necessary spectral theoretic tools for unbounded operators.

## 2 The Fine Structure of the Spectrum

As usual, to a closed linear operator  $A$  with dense domain  $D(A)$  in a Banach space  $E$ , we associate its spectrum  $\sigma(A)$ , its resolvent set  $\varrho(A)$  and its resolvent

$$\lambda \mapsto R(\lambda, A) := (\lambda - A)^{-1}$$

which is a holomorphic map from  $\varrho(A)$  into  $\mathcal{L}(E)$ . In contrast to the finite dimensional situation, where a linear operator fails to be surjective if and only if it fails to be injective, we now have to distinguish different cases of *non-invertibility* of  $\lambda - A$ . This distinction gives rise to a subdivision of  $\sigma(A)$  into different subsets. We point out that these subsets need not be disjoint. Our definitions are justified by the fact that for each of the following subsets of  $\sigma(A)$  there exist canonical constructions converting the corresponding spectral values into eigenvalues (see Proposition 2.2 (ii) and Proposition 4.4 below).

**Definition 2.1** For a closed, densely defined, linear operator  $A$  with domain  $D(A)$  in the Banach space  $E$  denote by the

- (i) *point spectrum*  $P\sigma(A)$  the set of all  $\lambda \in \mathbb{C}$  such that  $A - \lambda$  is not injective.
- (ii) *approximate point spectrum*  $A\sigma(A)$  the set of all  $\lambda \in \mathbb{C}$  such that  $A - \lambda$  is not injective or  $(A - \lambda)D(A)$  is not closed in  $E$ .
- (iii) *residual spectrum*  $R\sigma(A)$  the set of all  $\lambda \in \mathbb{C}$  such that  $(A - \lambda)D(A)$  is not dense in  $E$ .

From these definitions it follows that  $\lambda \in P\sigma(A)$  if and only if there exists a non-zero *eigenvector*  $f \in D(A)$  such that  $Af = \lambda f$ , i.e.,  $\lambda$  is an *eigenvalue*. It follows from the *Open Mapping Theorem* that  $\lambda \in A\sigma(A)$  if and only if  $\lambda$  is an *approximate eigenvalue*, i.e., there exists a sequence  $(f_n)_{n \in \mathbb{N}} \subset D(A)$ , called an *approximate eigenvector*, such that  $\|f_n\| = 1$  and  $\lim_{n \rightarrow \infty} \|Af_n - \lambda f_n\| = 0$ .

Clearly we have  $P\sigma(A) \subset A\sigma(A)$  and  $\sigma(A) = A\sigma(A) \cup R\sigma(A)$  where the union need not be disjoint.

The following proposition is a first indication that the subdivision we made yields nice properties.

**Proposition 2.2** For a closed, densely defined, linear operator  $(A, D(A))$  in a Banach space  $E$  the following holds.

- (i) The topological boundary  $\partial\sigma(A)$  of  $\sigma(A)$  is contained in  $A\sigma(A)$ .

(ii)  $R\sigma(A) = P\sigma(A')$  for the adjoint operator  $A'$  on  $E'$ .

**Proof** (i) Take  $\lambda_0 \in \partial\sigma(A)$  and  $\lambda_n \in \varrho(A)$  such that  $\lambda_n \rightarrow \lambda_0$ . Since

$$\|R(\lambda_n, A)\| \geq r(R(\lambda_n, A)) = (\text{dist}(\lambda_n, \sigma(A)))^{-1}$$

(see Proposition 2.5 (ii)), by the uniform boundedness principle we find  $f \in E$  such that

$$\lim_{n \rightarrow \infty} \|R(\lambda_n, A)f\| = \infty.$$

Define  $g_n \in D(A)$  by

$$g_n := \|R(\lambda_n, A)f\|^{-1} R(\lambda_n, A)f$$

and use the identity

$$(\lambda_0 - A)g_n = (\lambda_0 - \lambda_n)g_n + (\lambda_n - A)g_n$$

to show that  $(g_n)_{n \in \mathbb{N}}$  is an approximate eigenvector corresponding to  $\lambda_0$ .

(ii) This is a simple consequence of the Hahn-Banach theorem.  $\square$

In order to illuminate the above definitions we now return to the Standard Examples introduced in Section 2 of A-I and discuss the fine structure of the spectrum of these strongly continuous semigroups, i.e., of their generators and their semigroup operators.

**Example 2.3** (The Spectrum of Multiplication Semigroups) Take  $E = C_0(X)$  for some locally compact space  $X$  and take a continuous function  $q: X \rightarrow \mathbb{C}$  whose real part is bounded above. As observed in A-I, 2.3 the multiplication operator

$$M_q: f \mapsto q \cdot f$$

with maximal domain  $D(M_q)$  generates the multiplication semigroup

$$T(t)f := e^{tq} \cdot f, \quad f \in E.$$

Since  $M_q$  is bounded if and only if  $q$  is bounded, we conclude that  $M_q$  is invertible (with bounded inverse  $M_{1/q}$ ) if and only if

$$0 \notin \overline{\{q(x) : x \in X\}}.$$

Therefore we obtain

$$\sigma(M_q) = \overline{q(X)} = \overline{\{q(x) : x \in X\}},$$

and

$$\sigma(T(t)) = \overline{\{\exp(tq(x)) : x \in X\}}.$$

In particular the following *weak spectral mapping theorem* is valid

$$\sigma(T(t)) = \overline{\exp(t\sigma(M_q))}.$$

In addition, we observe that to each spectral value of  $A$  (resp. of  $T(t)$ ) there exists an approximate eigenvector and hence

$$\sigma(A) = A\sigma(A) \text{ and } \sigma(T(t)) = A\sigma(T(t)).$$

Since each Dirac functional is an eigenvector for the adjoint multiplication operator, we obtain

$$q(X) \subset R\sigma(M_q) \text{ and } e^{tq(X)} \subset R\sigma(T(t)).$$

The eigenvalues of  $M_q$  can be characterized as follows.

$\lambda \in P\sigma(M_q)$  if and only if the set  $\{x \in X : q(x) = \lambda\}$  has non empty interior (analogously for  $P\sigma(T(t))$ ).

For example, it follows that  $P\sigma(M_q) = \emptyset$  for  $E = C_0(\mathbb{R}_+)$  and  $q(x) = -x$ ,  $x \in \mathbb{R}_+$ .

On  $E = L^p(X, \Sigma, \mu)$  analogous results are valid, but their exact formulation—using the notion *essential range*, see Goldstein [8]—is left to the reader.

#### Example 2.4 (The Spectrum of Translation Semigroups)

We consider the translation semigroup

$$T(t)f(x) := f(x + t)$$

on  $E = C_0(\mathbb{R}_+)$  (or  $L^p(\mathbb{R}_+)$ , see A-I, 2.4). Its generator  $A$  is the first derivative and for every  $\lambda \in \mathbb{C}$ ,  $\operatorname{Re}(\lambda) < 0$ , the function  $\varepsilon_\lambda : x \mapsto e^{\lambda x}$  belongs to  $D(A)$  and satisfies

$$A\varepsilon_\lambda = \lambda\varepsilon_\lambda,$$

hence  $\lambda \in P\sigma(A)$ .

Since  $\mathcal{T} = (T(t))_{t \geq 0}$  is a contraction semigroup, it follows that

$$\sigma(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\} \text{ and } i\mathbb{R} \subset A\sigma(A)$$

(use Proposition 2.2 (i) or show directly that  $f_n(x) = e^{i\alpha x}e^{-x/n}$  defines an approximate eigenvector for  $i\alpha$ ,  $\alpha \in \mathbb{R}$ ). Using the same functions one obtains

$$\begin{aligned} P\sigma(T(t)) &= \{e^{\lambda t} : \operatorname{Re}(\lambda) < 0\} = \{z \in \mathbb{C} : |z| < 1\}, \\ \sigma(T(t)) &= \{z \in \mathbb{C} : |z| \leq 1\} \text{ for every } t > 0. \end{aligned}$$

In the case of the translation group on  $E = C_0(\mathbb{R})$  one has  $\sigma(A) \subset i\mathbb{R}$ . As above one obtains approximate eigenvectors for every  $\alpha \in \mathbb{R}$  from  $f_n(x) = e^{i\alpha x}e^{-|x|/n}$ , hence

$$\sigma(A) = A\sigma(A) = i\mathbb{R}.$$

The generator  $A$  of the nilpotent translation semigroup A-I, 2.6 has empty spectrum by A-I, Proposition 1.11. The resolvent is given by

$$R(\lambda, A)f(x) = e^{\lambda x} \int_x^\infty e^{-\lambda s} f(s) ds \quad (f \in L^p([0, \tau]), \lambda \in \mathbb{C}).$$

Finally, the generator of the periodic translation group from A-I, 2.5 on

$$E = \{f \in C[0, 1] : f(0) = f(1)\}$$

has point spectrum

$$P\sigma(A) = 2\pi i\mathbb{Z}$$

with eigenfunctions  $\varepsilon_n(x) := \exp(2\pi i n x)$ . In Section 5 we show that  $\sigma(A) = 2\pi i\mathbb{Z}$ .

We now return to the general theory and recall from Corollary 1.2 that it is very useful (e.g., for stability theory) to be able to convert spectral values of the generator  $A$  into spectral values of the semigroup operator  $T(t)$  and vice versa. As shown in Examples 1.3 and 1.4 this is not possible in general. Therefore we tackle first a much easier *spectral mapping theorem*: the relation between  $\sigma(A)$  and  $\sigma(R(\lambda_0))$ , where  $R(\lambda_0) := R(\lambda_0, A)$  for some  $\lambda_0 \in \varrho(A)$ .

**Proposition 2.5** *Let  $(A, D(A))$  be a densely defined closed linear operator with non-empty resolvent set  $\varrho(A)$ . For each  $\lambda_0 \in \varrho(A)$  the following assertions hold.*

- (i)  $\sigma(R(\lambda_0)) \setminus \{0\} = (\lambda_0 - \sigma(A))^{-1}$ , in particular,  $r(R(\lambda_0)) = (\text{dist}(\lambda_0, \sigma(A)))^{-1}$ .
- (ii) Analogous statements hold for the point-, approximate point-, residual spectra of  $A$  and  $R(\lambda_0, A)$ .
- (iii) The point  $\alpha$  is isolated in  $\sigma(A)$  if and only if  $(\lambda_0 - \alpha)^{-1}$  is isolated in  $\sigma(R(\lambda_0))$ . In that case the residues (resp. the pole orders) in  $\alpha$  and in  $(\lambda_0 - \alpha)^{-1}$  coincide.

**Proof** (i) is well known. It can be found, for example, in Dunford and Schwartz [5, VII.9.2].

(ii) We show that  $\alpha \in A\sigma(A)$  if  $(\lambda_0 - \alpha)^{-1} \in A\sigma(R(\lambda_0))$  and leave the proof of the remaining statements to the reader.

Take  $(f_n)_{n \in \mathbb{N}} \subset E$  such that  $\|f_n\| = 1$ ,  $\|(\lambda_0 - \alpha)^{-1} f_n - R(\lambda_0, A)f_n\| \rightarrow 0$  and  $\|R(\lambda_0, A)f_n\| \geq \frac{1}{2}|\lambda_0 - \alpha|^{-1}$ . Define

$$g_n := \|R(\lambda_0, A)f_n\|^{-1} R(\lambda_0, A)f_n \in D(A)$$

and deduce from

$$\begin{aligned} (\alpha - A)g_n &= \|R(\lambda_0, A)f_n\|^{-1} \cdot [(\lambda_0 - A) - (\lambda_0 - \alpha)]R(\lambda_0, A)f_n \\ &= \|R(\lambda_0, A)f_n\|^{-1} \cdot (\lambda_0 - \alpha)[(\lambda_0 - \alpha)^{-1} - R(\lambda_0, A)]f_n \end{aligned}$$

that  $(g_n)$  is an approximate eigenvector of  $A$  to the eigenvalue  $\alpha$ .

(iii) First we recall the well known *resolvent equation*.

For any  $z, \lambda_0 \in \varrho(A)$  we have

$$R(\lambda_0, A) - R(z, A) = -(\lambda_0 - z)R(\lambda_0, A)R(z, A).$$

From this it follows that

$$(\lambda_0 - z)^2 \cdot R(z, A) = \left( (\lambda_0 - z)^{-1} - R(\lambda_0, A) \right)^{-1} - (\lambda_0 - z).$$

If we now take a circle  $\Gamma$  with center  $\alpha$  and sufficiently small radius, then the residue  $P$  of  $R(\cdot, A)$  at  $\alpha$  is

$$\begin{aligned} P &= \frac{1}{2\pi i} \int_{\Gamma} R(z, A) dz = \\ &= \frac{1}{2\pi i} \left[ \int_{\Gamma} (\lambda_0 - z)^{-2} R \left( (\lambda_0 - z)^{-1}, R(\lambda_0, A) \right) dz - \int_{\Gamma} (\lambda_0 - z)^{-1} dz \right]. \end{aligned}$$

If  $\lambda_0$  lies in the exterior of  $\Gamma$ , the second integral is zero. The substitution

$$\tilde{z} := (\lambda_0 - z)^{-1}$$

yields a path  $\tilde{\Gamma}$  around  $(\lambda_0 - \alpha)^{-1}$  and we obtain

$$P = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} R(\tilde{z}, R(\lambda_0, A)) d\tilde{z}$$

which is the residue of  $R(\cdot, R(\lambda_0, A))$  at  $(\lambda_0 - \alpha)^{-1}$ . The final assertion on the pole order follows from the identities

$$V_{-n} = ((\lambda_0 - \alpha)^{-1} R(\lambda_0, A))^{n-1} U_{-n} \quad (n \in \mathbb{N}),$$

where  $U_n$ , resp.  $V_n$  stand for the  $n$ -th coefficient in the Laurent series of  $R(\cdot, A)$ , resp.  $R(\cdot, R(\lambda_0, A))$  at  $\alpha$ , resp.  $(\lambda_0 - \alpha)^{-1}$ . This has already been proved for  $n = 1$  and follows for  $n > 1$  by induction using the relations

$$U_{-n-1} = (A - \alpha)U_{-n} \quad \text{and} \quad V_{-n-1} = \left( R(\lambda_0, A) - (\lambda_0 - \alpha)^{-1} \right) V_{-n}.$$

### 3 Spectral Decomposition

In the next two sections we develop some important techniques for our further investigation of semigroups and their generators. Even though these methods are well known (compare, e.g., Section VII.3 of Dunford and Schwartz [5]) or rather technical, it is useful to present them in a coherent way.

Our interest in this section is the following: Let  $E$  be a Banach space and  $\mathcal{T} = (T(t))_{t \geq 0}$  a strongly continuous semigroup with generator  $A$ . Suppose that the spectrum  $\sigma(A)$  splits into the disjoint union of two closed subsets  $\sigma_1$  and  $\sigma_2$ . Does there exist a corresponding decomposition of the space  $E$  and the semigroup  $\mathcal{T}$ ?

In the following definition, we explain what we understand by “corresponding decomposition”.

**Definition 3.1** Assume that  $\sigma(A)$  is the disjoint union

$$\sigma(A) = \sigma_1 \cup \sigma_2$$

of two non-empty closed subsets  $\sigma_1, \sigma_2$ . A decomposition

$$E = E_1 \oplus E_2$$

of  $E$  into the direct sum of two non-trivial closed  $\mathcal{T}$ -invariant subspaces is called a *spectral decomposition* corresponding to  $\sigma_1 \cup \sigma_2$  if the spectrum  $\sigma(A_i)$  of the generator  $A_i$  of  $\mathcal{T}_i := (T(t)|_{E_i})_{t \geq 0}$  coincides with  $\sigma_i$  for  $i = 1, 2$ .

For a better understanding of the above definition we recall that to every direct sum decomposition  $E = E_1 \oplus E_2$  there corresponds a continuous projection  $P \in \mathcal{L}(E)$  such that  $PE = E_1$  and  $P^{-1}(0) = E_2$ . Moreover, the subspaces  $E_1, E_2$  are  $\mathcal{T}$ -invariant if and only if  $P$  commutes with the semigroup  $\mathcal{T}$ , i.e.,  $T(t)P = PT(t)$  for every  $t \geq 0$ . In this case it follows that the domain  $D(A)$  of the generator  $A$  splits analogously and  $D(A) \cap E_i$  is the domain  $D(A_i)$  of the generator  $A_i$  of the restricted semigroup  $\mathcal{T}_i$ ,  $i = 1, 2$ . We write

$$A = A_1 \oplus A_2.$$

and say that  $A$  *commutes with*  $P$  and call  $P$  a *spectral projection*. In terms of the generator  $A$  this means that for  $f \in D(A)$  we have  $Pf \in D(A)$  and  $APf = PAf$ .

The existence of such projections reduces the semigroup  $\mathcal{T}$  into two (possibly simpler) semigroups  $\mathcal{T}_1, \mathcal{T}_2$  such that

$$\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \quad \text{and} \quad \sigma(T(t)) = \sigma(T_1(t)) \cup \sigma(T_2(t)).$$

For example, in some cases (see Theorem 3.3 below) it can be shown that one of the reduced semigroups has additional properties.

In order to achieve such decompositions we will assume that  $\sigma(A)$  decomposes into sets  $\sigma_1$  and  $\sigma_2$  and will then try to find a corresponding spectral projection. Unfortunately such spectral decompositions do not exist in general.

**Example 3.2** Take the rotation semigroup from A-I, 2.5 on the Banach space  $L^p(\Gamma)$ ,  $1 \leq p < \infty$ ,  $\tau = 2\pi$ . It was stated in Example 2.4 and will be proved in Section 5 that

its generator  $A$  has spectrum

$$\sigma(A) = P\sigma(A) = i\mathbb{Z}$$

where  $\varepsilon_k(z) := z^k$  spans the eigenspace corresponding to  $ik$ ,  $k \in \mathbb{Z}$ .

Now,  $\sigma(A)$  is the disjoint union of  $\sigma_1 := \{0, i, 2i, \dots\}$  and  $\sigma_2 := \{-i, -2i, \dots\}$ . By a result of M. Riesz there is no projection  $P \in \mathcal{L}(L^1(\Gamma))$  satisfying  $P\varepsilon_k = \varepsilon_k$  for  $k \geq 0$ ,  $P\varepsilon_k = 0$  for  $k < 0$ , hence there is no spectral decomposition of  $L^1(\Gamma)$  corresponding to  $\sigma_1, \sigma_2$  (Lindenstrauss and Tzafriri [19, p.165]).

On the other hand, for  $L^p(\Gamma)$ ,  $1 < p < \infty$ , such a spectral projection exists (l.c., 2.c.15). As long as  $p \neq 2$  we can always decompose  $\sigma(A)$  into suitable subsets admitting no spectral decomposition (l.c., remark before 2.c.15). Clearly, for  $p = 2$  such spectral decompositions always exist.

In the above example both subsets  $\sigma_1, \sigma_2$  of  $\sigma(A)$  are unbounded. But as soon as one of these sets is bounded a corresponding spectral decomposition can always be found.

**Theorem 3.3** *Let  $\mathcal{T}$  be a strongly continuous semigroup on a Banach space  $E$  and assume that the spectrum  $\sigma(A)$  of the generator  $A$  can be decomposed into the disjoint union of two non-empty closed subsets  $\sigma_1$  and  $\sigma_2$ .*

*If  $\sigma_1$  is compact, then there exists a unique corresponding spectral decomposition  $E = E_1 \oplus E_2$  such that the restricted semigroup  $\mathcal{T}_1$  has a bounded generator.*

**Proof** We assume the reader to be familiar with the spectral decomposition theorem for bounded operators (see, e.g., Dunford and Schwartz [5, p.572]) and apply the spectral mapping theorem for the resolvent (Proposition 2.5 (i)) in order to decompose  $R(\lambda, A)$  instead of  $A$ .

For  $\lambda_0 > \omega_0(\mathcal{T})$  it follows from Proposition 2.5 that  $\sigma(R(\lambda_0, A)) \setminus \{0\} = (\lambda_0 - \sigma(A))^{-1}$ . From  $\sigma(A) = \sigma_1 \cup \sigma_2$  we obtain a decomposition of  $\sigma(R(\lambda_0, A)) \setminus \{0\}$  into

$$\tau_1 := (\lambda_0 - \sigma_1)^{-1}, \quad \tau_2 := (\lambda_0 - \sigma_2)^{-1}.$$

Since  $\sigma_1$  is compact, the set  $\tau_1$  is compact and does not contain 0. Only in the case that  $\sigma_2$  is unbounded, the point 0 will be an accumulation point of  $\tau_2$ . Therefore  $\sigma(R(\lambda_0, A)) \cup \{0\}$  is the disjoint union of the closed sets  $\tau_1$  and  $\tau_2 \cup \{0\}$ .

Take now  $P$  to be the spectral projection of  $R(\lambda_0, A)$  corresponding to this decomposition. Then  $P$  commutes with  $R(\lambda_0, A)$  (by definition), with  $R(\lambda, A)$  for every  $\lambda > \omega_0(\mathcal{T})$  (use the series representation of the resolvent), with  $T(t)$  for each  $t \geq 0$  (use A-II, Proposition 1.10) and therefore with the generator  $A$  (in the sense explained above). In particular, we obtain

$$R(\lambda_0, A)P = R(\lambda_0, A_1), \quad R(\lambda_0, A)(\text{Id} - P) = R(\lambda_0, A_2)$$

for the generator  $A_1$  of  $T_1 = (T(t)P)_{t \geq 0}$  and  $A_2$  of  $T_2 = (T(t)(\text{Id} - P))_{t \geq 0}$ . Applying the spectral mapping theorem for the resolvent (Theorem 2.5) we conclude

$$\sigma(A_1) = \sigma_1 \text{ and } \sigma(A_2) = \sigma_2,$$

i.e.,  $P$  is a spectral projection corresponding to  $\sigma_1, \sigma_2$ . Finally, the above spectral decomposition of  $R(\lambda_0, A)$  is unique and satisfies  $0 \notin \sigma(R(\lambda_0, A_1))$ . Therefore  $R(\lambda_0, A_1)^{-1} = (\lambda_0 - A_1)$  is bounded.  $\square$

**Example** If we do not require  $\mathcal{T}_1$  to be uniformly continuous, the above spectral decomposition need not be unique, as can be seen from the following example.

Consider a decomposition  $E = E_1 \oplus E_2$  and add a direct summand  $E_3$  with a strongly continuous semigroup  $T_3$  whose generator  $A_3$  has empty spectrum (e.g., A-I, 2.6). Then still  $\sigma(A) = \sigma_1 \cup \sigma_2$ , but  $E_1 \oplus (E_2 \oplus E_3)$  and  $(E_1 \oplus E_3) \oplus E_2$  are two different spectral decompositions corresponding to  $\sigma_1, \sigma_2$ .

The importance of the above theorem stems from the fact that  $\mathcal{T}_1$  has a bounded generator and therefore is easy to deal with. In particular, the asymptotic behavior of  $\mathcal{T}_1$  can be deduced from the location of  $\sigma_1$ .

**Corollary 3.4** *Assume that  $\sigma(A)$  splits into non-empty closed sets  $\sigma_1, \sigma_2$  where  $\sigma_1$  is compact and consider the corresponding spectral decomposition  $E = E_1 \oplus E_2$  for which  $\mathcal{T}_1$  is uniformly continuous.*

*For all constants  $\nu, \omega \in \mathbb{R}$  satisfying*

$$\nu < \inf\{\text{Re}(\lambda) : \lambda \in \sigma_1\} \leq \sup\{\text{Re}(\lambda) : \lambda \in \sigma_1\} < \omega$$

*there exist  $m \leq 1, M \geq 1$  such that*

$$m \cdot e^{\nu t} \|f\| \leq \|T_1(t)f\| \leq M \cdot e^{\omega t} \|f\|$$

*for every  $f \in E_1, t \geq 0$ .*

**Proof** Since the generator  $A_1$  of  $\mathcal{T}_1$  is bounded, we have  $T_1(t) = \exp(tA_1)$  and  $\sigma(T_1(t)) = \exp(t\sigma(A_1))$ . Therefore by the remark following Proposition 1.1, the spectral bound  $s(A_1)$  coincides with the growth bound  $\omega_0(T_1)$  and we have the upper estimate. The lower estimate is obtained by applying the same reasoning to  $-A_1$  which generates the semigroup  $(T_1(t)^{-1})_{t \geq 0}$  on  $E_1$ .  $\square$

It is clear from Examples 1.3 and 1.4 on page 69 that no norm estimates for  $(T_2(t))_{t \geq 0}$  can be obtained from the location of  $\sigma_2$ . Only by adding appropriate hypotheses we will achieve spectral decompositions admitting norm estimates on both components (see Theorem 6.6 below).

Another way of obtaining such norm estimates is by constructing spectral decompositions starting from a semigroup operator  $T(t_0)$  (instead of  $A$ , and  $R(\lambda, A)$ , resp., as in Theorem 3.3).

**Corollary 3.5** *If  $\sigma(T(t_0)) = \tau_1 \cup \tau_2$  for two non-empty, closed, disjoint sets  $\tau_1, \tau_2$  and if  $P$  is the spectral projection corresponding to  $T(t_0)$  and  $\tau_1, \tau_2$ , then  $\sigma(A)$  splits into closed subsets  $\sigma_1, \sigma_2$  and  $P$  is the corresponding spectral projection for  $\mathcal{T}$  and  $\sigma_1, \sigma_2$ .*

**Proof** The spectral projection  $P$  of  $T(t_0)$  is obtained by integrating  $R(\lambda, T(t_0))$  (see, e.g., Dunford and Schwartz [5, Section VII.3]). Since every  $T(t)$ ,  $t \geq 0$ , commutes with  $T(t_0)$ , it must commute with  $R(\lambda, T(t_0))$ , hence with  $P$ . The statement on the decomposition  $\sigma(A) = \sigma_1 \cup \sigma_2$  follows from the Spectral Inclusion Theorem (see 6.2) below.

This decomposition can be applied to the study of the asymptotic behavior of  $\mathcal{T}$ . In the situation of Corollary 3.5 assume

$$\sup\{|\lambda| : \lambda \in \tau_2\} < \alpha < \inf\{|\lambda| : \lambda \in \tau_1\}$$

for some  $\alpha > 0$ . If we set  $\beta := (\log \alpha)/t_0$  and use Pazy [21, Chap.I, Theorem 6.5], we obtain  $\omega_0(\mathcal{T}_2) < \beta$  and  $\omega_0(\mathcal{T}_1^{-1}) < \beta$  by Proposition 1.1. Therefore we have constants  $m, M$  with  $m \leq 1 \leq M$  such that

$$\begin{aligned} \|T(t)f\| &\leq M \cdot e^{\beta t} \|f\| \quad \text{for } f \in E_2, \\ \|T(t)f\| &\geq m \cdot e^{-\beta t} \|f\| \quad \text{for } f \in E_1. \end{aligned}$$

As nice as they might look, results of this type are unsatisfactory. We need information on the semigroup in order to estimate its asymptotic behavior. In Chapter IV we will try to obtain such results by exploiting information about the generator only.

### Example 3.6 (Isolated singularities and poles)

In case that  $\lambda_0$  is an isolated point of  $\sigma(A)$  the holomorphic function  $\lambda \mapsto R(\lambda, A)$  can be expanded as a Laurent series

$$R(\lambda, A) = \sum_{n=-\infty}^{+\infty} U_n(\lambda - \lambda_0)^n \quad \text{for } 0 < |\lambda - \lambda_0| < \delta \text{ and some } \delta > 0.$$

The coefficients  $U_n$  are bounded linear operators given by

$$U_n = \frac{1}{2\pi i} \int_{\Gamma} (z - \lambda_0)^{-(n+1)} R(z, A) dz, \quad n \in \mathbb{Z}, \quad (3.1)$$

where  $\Gamma = \{z \in \mathbb{C} : |z - \lambda_0| = \delta/2\}$ . The coefficient  $U_{-1}$  is the spectral projection corresponding to the spectral set  $\{\lambda_0\}$  (see Definition 3.1). It is called the *residue* of  $R(\cdot, A)$  at  $\lambda_0$  and will be denoted by  $P$ . From (3.1) one deduces

$$U_{-(n+1)} = (A - \lambda_0)^n \circ P \quad \text{and} \quad U_{-(n+1)} \circ U_{-(m+1)} = U_{-(n+m+1)} \quad \text{for } n, m \geq 0.$$

If there exists  $k > 0$  such that  $U_{-k} \neq 0$  while  $U_{-n} = 0$  for all  $n > k$ , the point  $\lambda_0$  is called a *pole of  $R(\cdot, A)$  of order  $k$* , otherwise it is called an *essential singularity*. In

view of (3.6) this is true if  $U_{-k} \neq 0$  and  $U_{-(k+1)} = 0$ . In this case one can retrieve  $U_{-k}$  as

$$U_{-k} = \lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^k R(\lambda, A). \quad (3.2)$$

The dimension of  $PE$  (i.e., the dimension of the spectral subspace corresponding to  $\{\lambda_0\}$ ) is called the *algebraic multiplicity*  $m_a$  of  $\lambda_0$ , while the *geometric multiplicity* is  $m_g := \dim \ker(\lambda_0 - A)$ . In case  $m_a = 1$ , we call  $\lambda_0$  an *algebraically simple pole*. If  $k$  is the pole order ( $k = \infty$  in case of an essential singularity), we have

$$\max\{m_g, k\} \leq m_a \leq k \cdot m_g \quad (3.3)$$

where  $\infty \cdot 0 = \infty$ .

These inequalities yield the following implications.

- (i)  $m_a < \infty$  if and only if  $\lambda_0$  is a pole with  $m_g < \infty$ ,
- (ii) If  $\lambda_0$  is a pole with order  $k$ , then  $\lambda_0 \in P\sigma(A)$  and  $PE = \ker(\lambda_0 - A)^k$ .

If  $A$  has compact resolvent, then every point of  $\sigma(A)$  is a pole of finite algebraic multiplicity. This is a consequence of Proposition 2.5 (iii) and the well-known Riesz-Schauder Theory for compact operators (see Dunford and Schwartz [5, VII.4.5]).

**Example 3.7 (The essential spectrum)** For an operator  $T \in \mathcal{L}(E)$  the *Fredholm domain*  $\varrho_F(T)$  is

$$\begin{aligned} \varrho_F(T) &:= \{\lambda \in \mathbb{C} : \lambda - T \text{ is a Fredholm operator}\} \\ &= \{\lambda \in \mathbb{C} : \ker(\lambda - T) \text{ and } E/\text{im}(\lambda - T) \text{ are finite dimensional}\}. \end{aligned} \quad (3.4)$$

An equivalent characterization of  $\varrho_F(T)$  is obtained through the *Calkin algebra*  $\mathcal{L}(E)/\mathcal{K}(E)$ , where  $\mathcal{K}(E)$  stands for the closed ideal of all compact operators. In fact,  $\varrho_F(T)$  coincides with the resolvent set of the canonical image of  $T$  in the Calkin algebra. The complement of  $\varrho_F(T)$  is called *essential spectrum* of  $T$  and denoted by  $\sigma_{\text{ess}}(T)$ . The corresponding spectral radius, called *essential spectral radius*, satisfies

$$r_{\text{ess}}(T) := \sup\{|\lambda| : \lambda \in \sigma_{\text{ess}}(T)\} = \lim_{n \rightarrow \infty} \|T^n\|_{\text{ess}}^{1/n}, \quad (3.5)$$

where

$$\|T\|_{\text{ess}} = \text{dist}(T, \mathcal{K}(E)) := \inf\{\|T - K\| : K \in \mathcal{K}(E)\}$$

is the norm of  $T$  in  $\mathcal{L}(E)/\mathcal{K}(E)$ .

For every compact operator  $K$  we have  $\|T - K\|_{\text{ess}} = \|T\|_{\text{ess}}$ , hence

$$r_{\text{ess}}(T - K) = r_{\text{ess}}(T). \quad (3.6)$$

A detailed analysis of  $\varrho_F(T)$  can be found in Section IV.5.6 of Kato [16]. In particular we recall that the poles of  $R(\cdot, T)$  with finite algebraic multiplicity belong to  $\varrho_F(T)$ .

Conversely, an element of the unbounded component of  $\varrho_F(T)$  either belongs to  $\varrho(T)$  or is a pole of finite algebraic multiplicity.

Thus  $r_{\text{ess}}(T)$  can be characterized as

$$r_{\text{ess}}(T) \text{ is the smallest } r \in \mathbb{R}_+ \text{ such that every } \lambda \in \sigma(T), |\lambda| > r \text{ is a pole of finite algebraic multiplicity.} \quad (3.7)$$

Now, if  $\mathcal{T} = (T(t))_{t \geq 0}$  is a strongly continuous semigroup, then VIII.1, Lemma 4 of Dunford and Schwartz [5] applied to the function  $t \mapsto \log \|T(t)\|_{\text{ess}}$  ensures that

$$\omega_{\text{ess}}(\mathcal{T}) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T(t)\|_{\text{ess}} = \inf \left\{ \frac{1}{t} \log \|T(t)\|_{\text{ess}} : t > 0 \right\} \quad (3.8)$$

is well defined (possibly  $-\infty$ ). By the definition of  $\omega_{\text{ess}}(\mathcal{T})$  and (3.5) we have

$$r_{\text{ess}}(T(t)) = \exp(t\omega_{\text{ess}}(\mathcal{T})), \quad t \geq 0. \quad (3.9)$$

Obviously,  $\omega_{\text{ess}} \leq \omega_0$  and equality occurs if and only if  $r_{\text{ess}}(T(t)) = r(T(t))$  for  $t \geq 0$ .

If  $\omega_{\text{ess}} < \omega_0$ , there exists an eigenvalue  $\lambda$  of  $T(t)$  satisfying  $|\lambda| = r(T(t))$ , hence by Theorem 6.3 below there exists  $\lambda_1 \in P\sigma(A)$  such that  $\text{Re}(\lambda_1) = \omega_0$ . Thus  $\omega_{\text{ess}} < \omega_0$  implies  $s(A) = \omega_0(\mathcal{T})$ , i.e., we have

$$\omega_0(\mathcal{T}) = \max\{\omega_{\text{ess}}(\mathcal{T}), s(A)\}. \quad (3.10)$$

As a final observation we point out that

$$\omega_{\text{ess}}(\mathcal{T}) = \omega_{\text{ess}}(\mathcal{S}), \quad (3.11)$$

whenever  $\mathcal{T}$  is generated by  $A$  and  $\mathcal{S}$  is generated by  $A + K$  for some compact operator  $K$  (see Proposition 2.8 and Proposition 2.9 of B-IV).

## 4 The Spectrum of Induced Semigroups

In the previous section we tried to decompose a semigroup into the direct sum of two, hopefully simpler objects. Here we present other methods to reduce the complexity of a semigroup and its generator. Forming subspace or quotient semigroups as in A-I, 3.3, A-I, 3.4 are such methods. But also the constructions of new semigroups on canonically associated spaces such as the dual space, see A-I, 3.5, or the  $\mathcal{F}$ -product, see A-I, 3.7, might be helpful. We review these constructions under the spectral theoretical point of view and collect a number of technical properties for later use.

We start by studying the spectrum of subspace and quotient semigroups. To that purpose assume that the strongly continuous semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  leaves invariant some closed subspace  $N$  of the Banach space  $E$ . There are canonically induced semigroups  $\mathcal{T}|_N$  on  $N$ , resp.  $\mathcal{T}_I$  on  $E/N$  and their generators  $A|_N$ , resp.  $A_I$  are canonically obtained from the generator  $A$  of  $\mathcal{T}$  (see A-I, Section 3). The following example shows that the spectra of  $A$ ,  $A|_N$  and  $A_I$  may differ quite drastically.

**Example 4.1** As in the example in A-I, 2.4 we consider the translation semigroup on  $E = L^1(\mathbb{R})$  and the invariant subspace

$$N := \{f \in E : f(x) = 0 \text{ for } x \geq 1\}.$$

Then  $\sigma(A) = i\mathbb{R}$ , but  $\sigma(A|_N) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\}$ . Next we take the translation invariant subspace

$$M := \{f \in N : f(x) = 0 \text{ for } 0 \leq x \leq 1\}$$

and obtain  $\sigma(A|_M) = \emptyset$  for the generator  $A|_M$  of the quotient semigroup  $\mathcal{T}_I$  (use the fact that  $\mathcal{T}_I$  is nilpotent).

In the next proposition we collect the information on  $\sigma(A)$  which in general can be obtained from the “subspace spectrum”  $\sigma(A|_N)$  and the “quotient spectrum”  $\sigma(A_I)$ .

**Proposition 4.2** *Using the standard notations the following inclusions hold.*

- (i)  $\varrho(A) \subset [\varrho(A|_N) \cap \varrho(A_I)] \cup [\sigma(A|_N) \cap \sigma(A_I)]$ ,
- (ii)  $[\varrho(A|_N) \cap \varrho(A_I)] \subset \varrho(A)$ ,
- (iii)  $\varrho_+(A) \subset [\varrho(A|_N) \cap \varrho(A_I)]$ ,

where  $\varrho_+(A)$  denotes the connected component of  $\varrho(A)$  which is unbounded to the right.

**Proof** (i) Assume  $\lambda \in \varrho(A)$ , i.e.,  $(\lambda - A)$  is a bijection from  $D(A)$  onto  $E$ . Since  $N$  is  $T$ -invariant, we have  $D(A|_N) = D(A) \cap N$  and  $(\lambda - A)D(A|_N) \subset N$ . If  $(\lambda - A)D(A|_N) = N$ , then  $R(\lambda, A)N = D(A|_N)$  and the induced operators  $R(\lambda, A)|_N$ , resp.  $R(\lambda, A)_I$  are the inverses of  $(\lambda - A|_N)$ , resp.  $(\lambda - A_I)$ . If  $(\lambda - A)D(A|_N) \neq N$ , then  $\lambda \in \sigma(A|_N)$ .

In addition there exists  $f \in D(A) \setminus N$  such that  $g := (\lambda - A)f \in N$ . Hence for  $\hat{f} := f + N$ ,  $\hat{g} := g + N \in E_I$  it follows that  $(\lambda - A_I)\hat{f} = \hat{g} = 0$ , i.e.,  $\lambda \in \sigma(A_I)$ .

(ii) Take  $\lambda \in \varrho(A|_N) \cap \varrho(A_I)$ . Then  $(\lambda - A)$  is injective since  $(\lambda - A)f = 0$  implies  $(\lambda - A_I)\hat{f} = 0$ , hence  $\hat{f} = 0$ , i.e.,  $f \in N$  and therefore  $f = 0$ .

In addition,  $(\lambda - A)$  is surjective: For  $g \in E$  there exists  $\hat{f} \in E_I$  such that  $(\lambda - A_I)\hat{f} = \hat{g}$ , i.e., there exists  $h \in N$  such that  $(\lambda - A)f - g = h = (\lambda - A)k$  for some  $k \in D(A|_N)$ . Therefore we obtain  $(\lambda - A)(f - k) = g$ .

(iii) The integral representation of the resolvent for  $\lambda > \omega_0(\mathcal{T})$  (see A-I, Proposition 1.11) shows that  $R(\lambda, A)N \subset N$ . By the power series expansion for holomorphic functions this extends to all  $\lambda \in \varrho_+(A)$ . Therefore the restriction  $R(\lambda, A)|_N$  coincides with the resolvent  $R(\lambda, A|_N)$ . On the other hand  $R(\lambda, A)_I$  is well defined on  $E_I$  and satisfies

$$R(\lambda, A)_I(f + N) = R(\lambda, A)f + N$$

(use again the integral representation). This proves that  $R(\lambda, A)_I = R(\lambda, A|_N)$ .  $\square$

**Corollary 4.3** *Under the above assumptions take a point  $\mu$  in the closure of  $\varrho_+(A)$ . Then*

- (i)  $\mu \in \sigma(A)$  if and only if  $\mu \in \sigma(A|_N)$  or  $\mu \in \sigma(A_I)$ .
- (ii)  $\mu$  is a pole of  $R(\cdot, A)$  if and only if  $\mu$  is a pole of  $R(\cdot, A|_N)$  and of  $R(\cdot, A_I)$ .

*In that case,*

$$\max\{k|, k_I\} \leq k \leq k| + k_I$$

*for the respective pole orders. Note that hereby pole orders 0 are allowed.*

**Proof** (i) This follows from Proposition 4.2 (ii) and (iii).

(ii) By the previous assertion we may assume that for some  $\delta > 0$  the pointed disc

$$\{\lambda \in \mathbb{C} : 0 < |\lambda - \mu| < \delta\}$$

is contained in  $\varrho(A) \cap \varrho(A|_N) \cap \varrho(A_I)$ .

Call  $U_n$  the coefficients of the Laurent expansion of  $R(\cdot, A)$ . Since  $N$  is  $R(\lambda, A)$ -invariant for  $\lambda \in \varrho_+(A)$ , the same holds for each  $U_n$ . With the obvious notations we have  $R(\lambda, A) = \sum_n U_n(\lambda - \mu)^n$ ,  $R(\lambda, A)|_N = \sum U_n|_N(\lambda - \mu)^n$  and  $R(\lambda, A)_I = \sum U_n|_I(\lambda - \mu)^n$  which shows  $\max\{k|, k_I\} \leq k$ .

If  $R(\cdot, A)|_N$  has a pole in  $\mu$  of order  $\ell$ , then  $U_{-(\ell+1)|_N} = 0$ , i.e.,  $U_{-(\ell+1)}N = \{0\}$ . Similarly, it follows that  $U_{-(m+1)}E \subset N$  if  $R(\cdot, A)_I$  has a pole in  $\mu$  of order  $m$ . Therefore  $U_{-(\ell+1)} \circ U_{-(m+1)} = 0$ .

The relations (3.6) imply  $U_{-(m+\ell+1)} = 0$ , hence the pole order of  $R(\cdot, A)$  is dominated by  $\ell + m$ .  $\square$

## 4.1 Spectrum of the adjoint semigroup

We recall from A-I, 3.5 that to every strongly continuous semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  there corresponds a strongly continuous adjoint semigroup  $\mathcal{T}^* = (T(t)^*)_{t \geq 0}$  on the semigroup dual

$$E^* = \{\varphi \in E' : \lim_{t \rightarrow \infty} \|T(t)'\varphi - \varphi\| = 0\}.$$

Its generator  $A^*$  is the maximal restriction of the adjoint  $A'$  to  $E^*$ . For these operators the spectra coincide, or more precisely.

- (i)  $\sigma(T(t)) = \sigma(T(t)') = \sigma(T(t)^*)$ ,  
 $R\sigma(T(t)) = P\sigma(T(t)') = P\sigma(T(t)^*)$ ,
- (ii)  $\sigma(A) = \sigma(A') = \sigma(A^*)$ ,  $R\sigma(A) = P\sigma(A') = P\sigma(A^*)$ ,
- (iii)  $s(A) = s(A^*)$ ,  $\omega_0(A) = \omega_0(A^*)$ .

**Proof** The left part of these equalities is either well known or has been stated in 2.2 (ii). The first statement of (iii) follows from (ii), while the second is an immediate consequence of the estimate  $\|T(t)^*\| \leq \|T(t)\| \leq M \cdot \|T(t)^*\|$  given in A-I, 3.5.

As a sample for the remaining assertions we show that  $0 \notin \sigma(A)$  if and only if  $0 \notin \sigma(A^*)$ : If  $A$  and therefore  $A'$  is invertible, it follows from A-I, 3.5 that  $A^*$  is a bijection from  $D(A^*)$  onto  $E^*$ .

Conversely assume that  $A^*$  is invertible. Then  $A'$  must be injective by the Proposition in A-I, 3.5. Moreover  $A'(D(A'))$  contains  $A^*(D(A^*)) = E^*$  and is  $\sigma(E', E)$ -dense in  $E'$ . By standard duality arguments it follows that  $A$  is injective with dense image. Next we show that  $A(D(A))$  is closed: For  $f \in D(A)$  choose  $\varphi \in D(A')$  such that  $\|\varphi\| \leq 1$  and  $|\langle f, \varphi \rangle| \geq \frac{1}{2}\|f\|$ . Then

$$\begin{aligned} \|(A^*)^{-1}\| \|Af\| &\geq \|(A^*)^{-1}\| |\langle Af, \varphi \rangle| \geq |\langle Af, (A^*)^{-1}\varphi \rangle| \\ &= |\langle f, \varphi \rangle| \geq \frac{1}{2}\|f\|, \end{aligned}$$

hence

$$\|Af\| \geq \frac{1}{2}\|(A^*)^{-1}\|^{-1}\|f\|,$$

and  $A(D(A))$  is closed since  $A$  is closed. □

## 4.2 Spectrum of the $\mathcal{F}$ -product semigroup

As stated in A-I, 3.7 the  $\mathcal{F}$ -product semigroup  $\mathcal{T}_{\mathcal{F}} = (T_{\mathcal{F}}(t))_{t \geq 0}$  on  $E_{\mathcal{F}}^{\mathcal{T}}$  of a strongly continuous semigroup  $\mathcal{T}$  on  $E$  serves to convert sequences in  $E$  into points in  $E_{\mathcal{F}}^{\mathcal{T}}$ . In particular it can be used to convert approximate eigenvectors of the generator  $A$  into eigenvectors of  $A_{\mathcal{F}}$ .

**Proposition 4.4** *Let  $A$  be the generator of a strongly continuous semigroup. Then the generator  $A_{\mathcal{F}}$  of the  $\mathcal{F}$ -product semigroup satisfies.*

- (i)  $A\sigma(A) = A\sigma(A_{\mathcal{F}}) = P\sigma(A_{\mathcal{F}})$ ,
- (ii)  $\sigma(A) = \sigma(A_{\mathcal{F}})$ .

**Remark 4.5** In case  $A$  is bounded, then the canonical extension  $A_{\mathcal{F}}$  is a generator and  $E_{\mathcal{F}}^{\mathcal{T}} = E_{\mathcal{F}}$  (cf. A-I, 3.7). Thus the proposition applies to bounded linear operators and their canonical extensions to the  $\mathcal{F}$ -product  $E_{\mathcal{F}}$ .

**Proof (Proof of the proposition)** (i) The inclusion  $P\sigma(A_{\mathcal{F}}) \subset A\sigma(A_{\mathcal{F}})$  holds trivially.

We show that  $A\sigma(A_{\mathcal{F}}) \subset A\sigma(A)$ : Take  $\lambda \in A\sigma(A_{\mathcal{F}})$  and an associated approximate eigenvector  $(\hat{f}^m)_{m \in \mathbb{N}}$ , i.e.,  $\hat{f}^m \in D(A_{\mathcal{F}})$ ,  $\|\hat{f}^m\| = 1$  and  $(\lambda - A_{\mathcal{F}})\hat{f}^m \rightarrow 0$  as  $m \rightarrow \infty$ .

By the considerations in A-I, 3.7 we can represent each  $\hat{f}^m$  as a normalized sequence  $(f_n^m)_{n \in \mathbb{N}}$  in  $D(A)$  such that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \|(\lambda - A)f_n^m\| = 0.$$

Therefore we can find a sequence  $g_k = f_k^{m(k)}$  satisfying

$$\lim_{k \rightarrow \infty} \|(\lambda - A)g_k\| = 0,$$

i.e.,  $\lambda \in A\sigma(A)$ .

Finally we show  $A\sigma(A) \subset P\sigma(A_{\mathcal{F}})$ : For  $\lambda \in A\sigma(A)$  take a corresponding approximate eigenvector  $(f_n)$ . By A-I, (3.2) we have

$$\begin{aligned} \|T(t)f_n - f_n\| &\leq \|T(t)f_n - e^{\lambda t}f_n\| + |e^{\lambda t} - 1| \\ &= \left\| \int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)f_n \, ds \right\| + |e^{\lambda t} - 1| \end{aligned}$$

which converges to zero uniformly in  $n$  as  $t \rightarrow 0$ , i.e.,  $(f_n) \in m^{\mathcal{T}}(E)$ . From the characterization of  $D(A_{\mathcal{F}})$  given in A-I, 3.7 it follows that

$$\hat{f} := (f_n) + c_F(E) \in D(A_{\mathcal{F}}) \quad \text{and} \quad A_{\mathcal{F}}\hat{f} = \lambda\hat{f}$$

i.e.,  $\lambda \in P\sigma(A_{\mathcal{F}})$ .

(ii) The inclusion  $A\sigma(A) \subset \sigma(A_{\mathcal{F}})$  follows from (i).

Now we show  $R\sigma(A) \subset R\sigma(A_{\mathcal{F}})$ . For  $\lambda \in R\sigma(A)$  choose  $f \in E$  such that  $\|(\lambda - A)g - f\| \geq 1$  for every  $g \in D(A)$ . Then  $\|(\lambda - A_{\mathcal{F}})\hat{g} - \hat{f}\| \geq 1$  for every  $\hat{g} \in D(A_{\mathcal{F}})$  and  $\hat{f} = (f, f, \dots) + c_F(E)$ . Therefore  $\lambda \in R\sigma(A_{\mathcal{F}})$ .

We now show  $\varrho(A) \subset \varrho(A_{\mathcal{F}})$ : Assume  $\lambda \in \varrho(A)$ . By (i)  $(\lambda - A_{\mathcal{F}})$  has to be injective. Choose  $\hat{f} = (f_1, f_2, \dots) + c_{\mathcal{F}}(E)$  such that  $(f_n) \in m^{\mathcal{T}}(E)$ . Then  $(R(\lambda, A)f_n) \in m^{\mathcal{T}}(E)$  and  $(\lambda - A_{\mathcal{F}})((R(\lambda, A)f_n) + c_{\mathcal{F}}(E)) = (f_n) + c_{\mathcal{F}}(E)$ , i.e.,  $(\lambda - A_{\mathcal{F}})$  is surjective and  $\lambda \in \varrho(A_{\mathcal{F}})$ .  $\square$

Applying this proposition to a single operator  $T(t)$ , we obtain

$$A\sigma(T(t)) = P\sigma(T(t)_{\mathcal{F}}).$$

Note that in general  $A\sigma(T(t)) \neq P\sigma(T_{\mathcal{F}}(t))$  (see the Examples 1.3 and 1.4 in combination with Theorem 6.3).

## 5 The Spectrum of Periodic Semigroups

In this section we determine the spectrum of a particularly simple class of strongly continuous semigroups and thereby achieve a rather complete description of the semigroup itself. Besides being nice and simple these semigroups gain their importance as building blocks for the general theory.

**Definition 5.1** A strongly continuous semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  on a Banach space  $E$  is called *periodic* if  $T(t_0) = \text{Id}$  for some  $t_0 > 0$ .

The *period*  $\tau$  of  $\mathcal{T}$  is obtained as

$$\tau := \inf\{t_0 > 0 : T(t_0) = \text{Id}\}.$$

We immediately observe that periodic semigroups are groups with inverses  $T(t)^{-1} = T(n\tau - t)$  for  $0 \leq t \leq n\tau$ ,  $\tau$  the period of  $\mathcal{T}$ . Moreover, they are bounded, hence the growth bound is zero and  $\sigma(A) \subset i\mathbb{R}$ .

**Lemma 5.2** *Let  $T$  be a strongly continuous semigroup with period  $\tau > 0$  and generator  $A$ . Then*

$$\sigma(A) \subset 2\pi i/\tau \cdot \mathbb{Z}$$

and

$$R(\mu, A) = (1 - e^{-\mu\tau})^{-1} \int_0^\tau e^{-\mu s} T(s) \, ds \quad (5.1)$$

for  $\mu \notin 2\pi i/\tau \cdot \mathbb{Z}$ .

**Proof** From the basic identities A-I, (3.1) and A-I, (3.2) for  $t = \tau$ , it follows that  $(\mu - A)$  has a left and right inverse if  $\mu \neq 2\pi in/\tau$ ,  $n \in \mathbb{Z}$ , and that the inverse is given by the above expression.  $\square$

The representation of  $R(\mu, A)$  given in A-I, Proposition 1.11 shows that the resolvent of the generator of a periodic semigroup is a meromorphic function having only poles of order one and the residues

$$P_n := \lim_{\mu \rightarrow \mu_n} (\mu - \mu_n) R(\mu, A) \quad \text{in} \quad \mu_n := 2\pi in/\tau, \quad n \in \mathbb{Z},$$

are

$$P_n = \frac{1}{\tau} \int_0^\tau \exp(-\mu_n s) T(s) \, ds. \quad (5.2)$$

Moreover, it follows that the spectrum of  $A$  consists of eigenvalues only and each  $P_n$  is the spectral projection belonging to  $\mu_n$  (see 3.6). Another way of looking

at  $P_n$  is given by saying that  $P_n$  is the  $n$ -th Fourier coefficient of the  $\tau$ -periodic function  $s \mapsto T(s)$ . From this it follows that no non-zero  $\varphi \in E'$  vanishes on all  $P_n E$  simultaneously. By the Hahn-Banach theorem we conclude that  $\text{span } \cup_{n \in \mathbb{Z}} P_n E$  is dense in  $E$ .

Since  $P_n E \subset D(A)$ , we obtain from A-I, (3.1) that

$$AP_n f = \mu_n P_n f \quad (5.3)$$

for every  $f \in E$ ,  $n \in \mathbb{Z}$ . This and A-I, (3.2) imply

$$T(t)P_n f = \exp(\mu_n t) \cdot P_n f \quad (5.4)$$

for every  $t \geq 0$ . Therefore  $\mu_n$  is an eigenvalue of  $A$  and  $\exp(\mu_n t)$  is an eigenvalue of  $T(t)$  if and only if  $P_n \neq 0$ . In that case,  $P_n E$  is the corresponding eigenspace and we have the following lemma.

**Lemma 5.3** *For a  $\tau$ -periodic semigroup  $\mathcal{T}$  we take  $\mu_n := 2\pi i n / \tau$ ,  $n \in \mathbb{Z}$ , and consider*

$$P_n := \frac{1}{\tau} \int_0^\tau \exp(-\mu_n s) T(s) ds.$$

*Then the following assertions are equivalent.*

- (a)  $P_n \neq 0$ ,
- (b)  $\mu_n \in P\sigma(A)$ ,
- (c)  $\exp(\mu_n t) \in P\sigma(T(t))$  for every  $t > 0$ .

The action of  $A$ , resp.  $T(t)$  in the subspaces  $P_n E$ ,  $n \in \mathbb{Z}$ , is determined by (5.3) and (5.4). Moreover,

$$P_m P_n f = \frac{1}{\tau} \int_0^\tau \exp(-\mu_m s) T(s) P_n f ds = \frac{1}{\tau} \int_0^\tau \exp((\mu_n - \mu_m)s) P_n f ds = 0$$

for  $n \neq m$ , hence the subspaces  $P_n E$  are “orthogonal”. Since their union is total in  $E$ , one expects to be able to extend the representations (5.3) and (5.4) of  $A$  and  $T(t)$ . This is possible if

$$\sum_{n=-\infty}^{+\infty} P_n = \text{Id},$$

where the series should be summable for the strong operator topology.

Unfortunately, this is false in general since the family of projections

$$Q_H := \sum_{n \in H} P_n,$$

where  $H$  runs through all finite subsets of  $\mathbb{Z}$ , may be unbounded (see the example below). Nevertheless the following is true.

**Theorem 5.4** Let  $\mathcal{T} = (T(t))_{t \geq 0}$  be a  $\tau$ -periodic semigroup on a Banach space  $E$  with generator  $A$  and associated spectral projections

$$P_n := \frac{1}{\tau} \int_0^\tau \exp(-\mu_n s) T(s) \, ds, \quad \mu_n := 2\pi i n / \tau, \quad n \in \mathbb{Z}.$$

For every  $f \in D(A)$  one has  $f = \sum_{-\infty}^{+\infty} P_n f$  and therefore

- (i)  $T(t)f = \sum_{-\infty}^{+\infty} \exp(\mu_n t) P_n f$  if  $f \in D(A)$ ,
- (ii)  $Af = \sum_{-\infty}^{+\infty} \mu_n P_n f$  if  $f \in D(A^2)$ .

**Proof** It suffices to prove the first statement. Then (i) and (ii) follow by (5.3) and (5.4).

We assume  $\tau = 2\pi$  and show first that  $\sum_{-\infty}^{+\infty} P_n f$  is summable for  $f \in D(A)$ : For  $g := Af$  we obtain  $P_n g = P_n A f = A P_n f = i n P_n f$ . Take  $H$  to be a finite subset of  $\mathbb{Z} \setminus \{0\}$  and  $\varphi \in E'$ . Then

$$\left| \sum_{n \in H} \langle P_n f, \varphi \rangle \right| = \left| \sum_{n \in H} \frac{1}{i n} \langle P_n g, \varphi \rangle \right| \leq \left( \sum_{n \in H} \frac{1}{n^2} \right)^{1/2} \left( \sum_{n \in H} |\langle P_n g, \varphi \rangle|^2 \right)^{1/2}.$$

From Bessel's inequality we obtain for the second factor

$$\sum_{n \in H} |\langle P_n g, \varphi \rangle|^2 \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} |T(s)g, \varphi|^2 \, ds \leq \|\varphi\|^2 \cdot \frac{1}{2\pi} \cdot \int_0^{2\pi} \|T(s)g\|^2 \, ds.$$

With the constant  $c := \left( \frac{1}{2\pi} \cdot \int_0^{2\pi} \|T(s)g\|^2 \, ds \right)^{1/2}$  we obtain

$$\left\| \sum_{n \in H} P_n f \right\| \leq c \left( \sum_{n \in H} n^{-2} \right)^{1/2}$$

for every finite subset  $H$  of  $\mathbb{Z}$ , i.e.,  $\sum_{-\infty}^{+\infty} P_n f$  is summable.

Next we set  $h := \sum_{-\infty}^{+\infty} P_n f$  and observe that for every  $\varphi' \in E'$  the Fourier coefficients of the continuous,  $\tau$ -periodic functions  $s \mapsto \langle T(s)h, \varphi' \rangle$  and  $s \mapsto \langle T(s)f, \varphi' \rangle$  coincide. Therefore these functions are identical for  $s \geq 0$  and in particular for  $s = 0$ , i.e.,  $\langle h, \varphi' \rangle = \langle f, \varphi' \rangle$ . By the Hahn-Banach Theorem we obtain  $f = h$ .  $\square$

The above theorem contains rather precise information on periodic semigroups. In particular, it characterizes periodic semigroups by the fact that  $\sigma(A)$  is contained in  $i\alpha\mathbb{Z}$  for some  $\alpha \in \mathbb{R}$  and the eigenfunctions of  $A$  form a total subset of  $E$ .

For a periodic semigroup with bounded generator only a finite number of spectral projections  $P_n$  are distinct from 0 and we have the following characterization.

**Corollary 5.5** Let  $\mathcal{T} = (T(t))_{t \geq 0}$  be a semigroup with bounded generator on some Banach space  $E$ .

This semigroup has period  $\tau/k$  for some  $k \in \mathbb{N}$  if and only if there exist finitely many pairwise orthogonal projections  $P_n$ ,  $-m \leq n \leq m$ ,  $P_{-m} \neq 0$  or  $P_m \neq 0$ , such that

- (i)  $\sum_{-m}^{+m} P_n = \text{Id}$ ,
- (ii)  $T(t) = \sum_{-m}^{+m} \exp(2\pi i n t / \tau) P_n$ ,
- (iii)  $A = \sum_{-m}^{+m} (2\pi i n / \tau) P_n$ .

**Example 5.6** From A-I, 2.5 we recall briefly the rotation group

$$R_\tau(t)f(z) := f(\exp(2\pi i t / \tau) \cdot z)$$

on  $E = C(\Gamma)$ , resp.  $E = L^p(\Gamma, m)$  for  $1 \leq p < \infty$ . The spectrum of the generator  $Af(z) = (2\pi i / \tau)z \cdot f'(z)$  is  $\sigma(A) = (2\pi i / \tau) \cdot \mathbb{Z}$ . The eigenfunctions  $\varepsilon_n(z) := z^n$  yield the projections

$$P_n = (1/2\pi i) \cdot \varepsilon_{-(n+1)} \otimes \varepsilon_n, \text{ i.e.,}$$

$$P_n f(z) = (1/2\pi i) \cdot \left( \int_{\Gamma} f(w) w^{-(n+1)} dw \right) \cdot z^n.$$

It is left as an exercise to compute the norms of  $Q_m := \sum_{-m}^{+m} P_n$  in  $L^p(\Gamma, m)$  for various  $p$  and then check the assertions of Theorem 5.4.

Clearly, this proves some classical convergence theorems for Fourier series (compare Davies [3, Chap.8.1]).

## 6 Spectral Mapping Theorems

We now return to the question posed in the introduction to this chapter: In which form and under which conditions is it true that the spectrum  $\sigma(T(t))$  of the semigroup operators is obtained—via the exponential map—from the spectrum  $\sigma(A)$  of the generator, or briefly

$$\text{Do we have } \sigma(T(t)) = \exp(t\sigma(A)) \text{ or at least } \sigma(T(t)) = \overline{\exp(t\sigma(A))} \text{ ?}$$

This and similar statements will be called *spectral mapping theorems* for the semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  and its generator  $A$ . In addition, we saw in Proposition 1.1 that the validity of such a spectral mapping theorem implies

$$s(A) = \omega_0(A)$$

for the spectral- and growth bounds and therefore guarantees that the location of the spectrum of  $A$  determines the asymptotic behavior of  $\mathcal{T}$ . As we have seen in Examples 1.3 and 1.4 the last statement does not hold in general. We therefore present a detailed analysis, where and why it fails and what additional assumptions are needed for its validity. Before doing so, we have another look at the examples.

**Example 6.1 (The counterexamples revisited)**

(i) Take the nilpotent translation semigroup from A-I, 2.6. Then  $\sigma(A) = \emptyset$  and  $\sigma(T(t)) = 0$  for every  $t > 0$ . By this trivial example and since  $e^z \neq 0$  for every  $z \in \mathbb{C}$ , it is natural to read the spectral mapping theorem modulo the addition of  $\{0\}$ , i.e.,

$$\sigma(T(t)) \setminus \{0\} = \exp(t\sigma(A)) \text{ for } t \geq 0.$$

(ii) The spectrum of the generator  $A$  of the  $\tau$ -periodic rotation group  $(R_\tau(t))_{t \geq 0}$  on  $C(\Gamma)$  is  $\sigma(A) = 2\pi i/\tau \cdot \mathbb{Z}$  and  $\exp(2\pi i n t/\tau)$ ,  $n \in \mathbb{Z}$ , is an eigenvalue of  $R_\tau(t)$  for every  $t \geq 0$  (see Example 5.6). If  $t/\tau$  is irrational, these eigenvalues form a dense subset of  $\Gamma$ . Since the spectrum is closed, we obtain  $\sigma(T(t)) = \Gamma$  for these  $t$ . Therefore in this example the spectral mapping theorem is valid only in the following “weak” form

$$\sigma(T(t)) = \overline{\exp(t\sigma(A))}, \quad t \geq 0.$$

(iii) By Example 1.3 there exists a semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  with generator  $A$  such that  $s(A) = -1$  and  $\omega_0(\mathcal{T}) = 0$ . This implies that for preassigned real numbers  $\alpha < \beta$  there exists a semigroup  $\mathcal{S} = (S(t))_{t \geq 0}$  with generator  $B$  such that  $s(B) = \alpha$  and  $\omega_0(\mathcal{S}) = \beta$ . Indeed, take  $S(t) = e^{\beta t} T((\beta - \alpha)t)$  and observe that  $B = (\beta - \alpha)A + \beta \text{Id}$ . In that case  $\exp(t\sigma(B))$  is contained in the circle about 0 with radius  $e^{\alpha t}$  while  $\sigma(S(t))$  has spectral values satisfying  $|\lambda| = r(S(t)) = e^{\beta t} > e^{\alpha t}$ .

(iv) The Example 1.3 can be strengthened in order to yield a semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  with generator  $A$  such that  $\sigma(A) = \emptyset$ , but  $\|T(t)\| = r(T(t)) = 1$  for  $t \geq 0$ , i.e.,  $s(A) = -\infty$ ,  $\omega_0 = 0$  and  $s(A) < \omega_0$ . Take the translation semigroup on the Banach space

$$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^{x^2} dx)$$

with  $\|f\| := \sup\{|f(x)| : x \in \mathbb{R}_+\} + \int_0^\infty |f(x)|e^{x^2} dx$  (see Greiner et al. [11]).

(v) Another modification of Example 1.3 yields a group  $\mathcal{T} = (T(t))_{t \in \mathbb{R}}$  satisfying  $s(A) < \omega_0$ . Therefore the spectral mapping theorem does not hold in the setting of groups (see Wolff (1981)).

The next few theorems form the core of this chapter. We show that only one part of the spectrum and one inclusion is responsible for the failure of the spectral mapping theorem. The usefulness of this detailed analysis will become clear in the subsequent chapters on stability and asymptotics.

**Proposition 6.2 (Spectral Inclusion Theorem)**

Let  $A$  be the generator of a strongly continuous semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  on some Banach space  $E$ . Then

$$\exp(t\sigma(A)) \subset \sigma(T(t)) \text{ for } t \geq 0.$$

More precisely we have the following inclusions.

$$\exp(t \cdot P\sigma(A)) \subset P\sigma(T(t)), \quad (6.1)$$

$$\exp(t \cdot A\sigma(A)) \subset A\sigma(T(t)), \quad (6.2)$$

$$\exp(t \cdot R\sigma(A)) \subset R\sigma(T(t)). \quad (6.3)$$

**Proof** Since  $e^{\lambda t} - T(t) = (\lambda - A) \int_0^t e^{\lambda(t-s)} T(s) \, ds$  (see A-I, (3.1)), it follows that  $(e^{\lambda t} - T(t))$  is not bijective if  $(\lambda - A)$  fails to be bijective proving the main inclusion.

The inclusion (6.1) becomes evident from the following proof of (6.2). Take  $\lambda \in A\sigma(A)$  and an associated approximate eigenvector  $(f_n) \subset D(A)$ . Then

$$g_n := e^{\lambda t} f_n - T(t) f_n = \int_0^t e^{\lambda(t-s)} T(s) (\lambda - A) f_n \, ds$$

converges to zero as  $n \rightarrow \infty$ . Consequently,  $e^{\lambda t} \in A\sigma(T(t))$  and, in fact, the same approximate eigenvector  $(f_n)$  does the job for all  $t \geq 0$ .

For the proof of (6.3) we take  $\lambda \in R\sigma(A)$  and observe that  $(e^{\lambda t} - T(t))f = (\lambda - A) \left( \int_0^t e^{\lambda(t-s)} T(s) f \, ds \right) \in (\lambda - A)D(A)$  for every  $f \in E$ .  $\square$

As we know from the Examples 6.1, the converse inclusions do not hold in general, i.e., not every spectral value of a semigroup operator  $T(t)$  comes—via the exponential map—from a spectral value of the generator. But at least this is true for some important parts of the spectrum.

### Theorem 6.3 (Spectral Mapping Theorem for Point and Residual Spectrum)

Let  $A$  be the generator of a strongly continuous semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$ . Then

$$\exp(t \cdot P\sigma(A)) = P\sigma(T(t)) \setminus \{0\}, \quad (6.4)$$

$$\exp(t \cdot R\sigma(A)) = R\sigma(T(t)) \setminus \{0\} \text{ for } t \geq 0. \quad (6.5)$$

**Proof** For the proof of (6.4), take  $t_0 > 0$  and  $0 \neq \lambda \in P\sigma(T(t_0))$ .

After rescaling the semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  to the semigroup

$$(\exp(-t \cdot \log \lambda / t_0) T(t))_{t \geq 0},$$

we may assume  $\lambda = 1$ . Then the closed,  $\mathcal{T}$ -invariant subspace

$$G := \{g \in E : T(t_0)g = g\}$$

is non trivial. The restricted semigroup  $T|_G$  is periodic with period  $\tau \leq t_0$  and the spectrum of its generator  $A|_G$  contains at least one eigenvalue  $\mu = 2\pi i n / t_0$  for some  $n \in \mathbb{Z}$  (see Lemma 5.3). Since every eigenvalue of  $A|_G$  is an eigenvalue of  $A$ , we obtain that  $1 \in \exp(t_0 \cdot P\sigma(A))$ . The converse inclusion has been proved in (6.1).

In fact, even more can be said: Let  $g \in G$  be an eigenvector of  $T(t_0)$  corresponding to the eigenvalue  $\lambda = 1$ . For each  $n \in \mathbb{Z}$  define

$$g_n := P_n g = 1/t_0 \cdot \int_0^{t_0} \exp(-2\pi i n s/t_0) T(s) g \, ds \in G$$

as in Section 5. If  $g_n \neq 0$ , then  $g_n$  is an eigenvector of  $A|_G$ , hence of  $A$  with eigenvalue  $2\pi i n/t_0$  as soon as  $g_n$  is distinct from zero. Since  $D(A|_G)$  is dense in  $G$  it follows from Theorem 5.4 that this holds for at least one  $n \in \mathbb{Z}$ . And from the proof of (6.1) we know that this  $g_n$  is in fact an eigenvector for each  $T(t)$ ,  $t \geq 0$ .

Since  $R\sigma(A) = P\sigma(A^*)$  and  $R\sigma(T(t)) = P\sigma(T(t)^*)$  (see Proposition 4.4) the assertion (6.5) follows from (6.4).  $\square$

Note that the proof is essentially an application of the structure theorem for periodic semigroups as given in Theorem 5.4. The information gained there can be reformulated into statements on the eigenspaces of  $A$  and  $T(t)$ .

**Corollary 6.4** *For the eigenspaces of the generator  $A$ , resp. of the semigroup operators  $T(t)$ ,  $t > 0$ , the following holds for  $\mu \in \mathbb{C}$ .*

- (i)  $\ker(\mu - A) = \bigcap_{s \geq 0} \ker(e^{\mu s} - T(s))$ ,
- (ii)  $\ker(e^{\mu t} - T(t)) = \overline{\text{span}_{n \in \mathbb{Z}} \{\ker(\mu + 2\pi i n/t - A)\}}$ .

We note that an analogous statement is valid for  $\ker(\mu - A')$  and  $\ker(e^{\mu t} - T(t)')$  if we take in (ii) the  $\sigma(E', E)$ -closure.

Without proof (see Greiner [9, Proposition 1.10]) we add another corollary showing that poles of the resolvent of  $T(t)$  correspond necessarily to poles of the resolvent of the generator. Again the converse is not true as shown by Example 5.6.

**Corollary 6.5** *Assume that  $e^{\mu t}$  is a pole of order  $k$  of  $R(\cdot, T(t))$  with residue  $P$  and  $Q$  as the  $k$ -th coefficient of the Laurent series. Then*

- (i)  $\mu + 2\pi i n/t$  is a pole of  $R(\cdot, A)$  of order  $\leq k$  for every  $n \in \mathbb{Z}$ ,
- (ii) the residues  $P_n$  in  $\mu + 2\pi i n/t$  yield  $PE = \overline{\text{span}_{n \in \mathbb{Z}} \{P_n E\}}$ ,
- (iii) the  $k$ -th coefficient of the Laurent series of  $R(\cdot, A)$  at  $\mu + 2\pi i n/t$  is

$$Q_n = (t \cdot e^{\mu t})^{1-k} \cdot Q \circ (1/t) \int_0^t e^{-(\mu + 2\pi i n/t)s} T(s) \, ds.$$

From Proposition 6.2 and Theorem 6.3 it follows that the approximate point spectrum is the trouble maker in the sense that not every approximate eigenvalue of  $T(t)$  corresponds to an approximate eigenvalue of the generator  $A$ . Since nothing more can be said in general, we now look for additional hypotheses on the semigroup implying the spectral mapping theorem.

As a simple example we assume  $T(t_0)$  to be compact for some  $t_0 > 0$ . Then  $\sigma(T(t)) \setminus \{0\} = P\sigma(T(t)) \setminus \{0\}$  for  $t \geq t_0$  and the spectral mapping theorem is valid

by (6.4). A different class of semigroups verifying the spectral mapping theorem is given by the uniformly continuous semigroups (compare Corollary 1.2).

Both cases, and many more, are included in the following result.

**Theorem 6.6 (Spectral Mapping Theorem for Eventually Norm Continuous Semigroups)** *Let  $\mathcal{T} = (T(t))_{t \geq 0}$  be an eventually norm continuous semigroup with generator  $A$ . Then the spectral mapping theorem is valid, i.e.,*

$$\sigma(T(t)) \setminus \{0\} = e^{t \cdot \sigma(A)} \text{ for every } t \geq 0. \quad (6.6)$$

**Proof** By the previous considerations it suffices to show that  $A\sigma(T(t)) \setminus \{0\} \subset e^{t \cdot \sigma(A)}$  for  $t > 0$ . This will be done by converting approximate eigenvectors into eigenvectors in the semigroup  $\mathcal{F}$ -product (see Subsection 4.2). The assertion then follows from (6.4) and Proposition 4.4 (ii).

Assume  $t \mapsto T(t)$  to be norm continuous for  $t \geq t_0$ . Moreover it suffices to consider  $1 \in A\sigma(T(t_1))$  for some  $t_1 > 0$ , i.e., we have a normalized sequence  $(f_n)_{n \in \mathbb{N}} \subset E$  such that

$$\lim_{n \rightarrow \infty} \|T(t_1)f_n - f_n\| = 0.$$

Choose  $k \in \mathbb{N}$  such that  $kt_1 > t_0$  and define  $g_n := T(kt_1)f_n$ . Then

$$\lim_{n \rightarrow \infty} \|g_n\| = \lim_{n \rightarrow \infty} \|T(t_1)^k f_n\| = \lim_{n \rightarrow \infty} \|f_n\| = 1$$

and

$$\lim_{n \rightarrow \infty} \|T(t_1)g_n - g_n\| = 0,$$

i.e.,  $(g_n)_{n \in \mathbb{N}}$  yields an approximate eigenvector of  $T(t_1)$  with approximate eigenvalue 1. But the semigroup  $\mathcal{T}$  is uniformly continuous on sets of the form  $T(t_0)V$ ,  $V$  bounded in  $E$ . In particular, it is uniformly continuous on the sequence  $(g_n)_{n \in \mathbb{N}}$ , which therefore defines an element  $g$  in the semigroup  $\mathcal{F}$ -product  $E_{\mathcal{F}}$ .

Obviously,  $g$  is an eigenvector of  $T_{\mathcal{F}}(t_1)$  with eigenvalue 1 and by (6.4) we obtain an eigenvalue  $2\pi i n/t_1$  of  $A_{\mathcal{F}}$  for some  $n \in \mathbb{Z}$ . The coincidence of  $\sigma(A)$  and  $\sigma(A_{\mathcal{F}})$  proves the assertion.  $\square$

We point out that the above spectral mapping theorem implies the coincidence of spectral bound and growth bound for eventually norm continuous semigroups, hence we have generalized the Liapunov Stability Theorem (see 1.2) to a much larger class of semigroups. As mentioned before, this will be of great use in many applications. Therefore we state explicitly the spectral mapping theorem for several important classes of semigroups all of which are eventually norm continuous (cf. the diagram preceding A-II, Example 1.27).

**Corollary 6.7** *The Spectral Mapping Theorem 6.6 holds for each of the following classes of strongly continuous semigroups.*

- (i) *eventually compact semigroups*,
- (ii) *eventually differentiable semigroups*,
- (iii) *holomorphic semigroups*,
- (iv) *uniformly continuous semigroups*.

Another application of the above spectral mapping theorem can be made to tensor product semigroups (see A-I, 3.8). Let  $\mathcal{T}_1 = (T_1(t))_{t \geq 0}$ ,  $\mathcal{T}_2 = (T_2(t))_{t \geq 0}$  be strongly continuous semigroups on Banach spaces  $E_1, E_2$  with generator  $A_1, A_2$ . The tensor product semigroup  $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$  on some (appropriate) tensor product  $E := E_1 \otimes E_2$  has the generator  $A = A_1 \otimes \text{Id} + \text{Id} \otimes A_2$ , but in general the spectrum of  $A$  is not determined by the spectra of  $A_1, A_2$ . But with an additional hypothesis the following can be proved.

**Corollary 6.8** *If  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are eventually norm continuous, then*

$$\sigma(A) = \sigma(A_1) + \sigma(A_2),$$

where  $A$  is the generator of the tensor product semigroup

$$\mathcal{T}_1 \otimes \mathcal{T}_2 = (T_1(t) \otimes T_2(t))_{t \geq 0}.$$

**Proof** Clearly, the tensor product semigroup is eventually norm continuous and hence the Spectral Mapping Theorem 6.6 is valid for all three semigroups  $\mathcal{T}_1, \mathcal{T}_2$  and  $\mathcal{T}$ . Moreover the spectrum of the tensor product of bounded operators is the product of the spectra (Reed and Simon [23, XIII.9]). Therefore

$$\sigma(T_1(t) \otimes T_2(t)) = \sigma(T_1(t)) \cdot \sigma(T_2(t)), \quad t \geq 0.$$

Consequently we have the following identity for every  $t \geq 0$

$$\begin{aligned} e^{t \cdot \sigma(A)} &= \sigma(T_1(t) \otimes T_2(t)) \setminus \{0\} \\ &= \sigma(T_1(t)) \cdot \sigma(T_2(t)) \setminus \{0\} \\ &= e^{t \cdot \sigma(A_1)} \cdot e^{t \cdot \sigma(A_2)} \\ &= e^{t(\sigma(A_1) + \sigma(A_2))}. \end{aligned}$$

From this identity we want to deduce  $\sigma(A) = \sigma(A_1) + \sigma(A_2)$ .

“ $\subseteq$ ” Take  $\xi \in \sigma(A)$ . Then for every  $t > 0$  there exist  $\mu_t \in \sigma(A_1)$ ,  $\lambda_t \in \sigma(A_2)$  and  $n_t \in \mathbb{Z}$  such that  $\xi = \mu_t + \lambda_t + 2\pi i n_t / t$ .

Since the real parts of  $\mu_t, \lambda_t$  are bounded above, they lie in some interval  $[a, b]$ . But  $\sigma(A_k) \cap ([a, b] + i\mathbb{R})$  is compact for  $k = 1, 2$  since  $A_k$  is the generator of an eventually norm continuous semigroup (see A-II, Theorem 1.20). By taking  $t$  sufficiently small, we conclude that  $n_{t'} = 0$  for some  $t' > 0$ , i.e.,  $\xi = \mu_{t'} + \lambda_{t'}$ .

“ $\supseteq$ ” Choose  $\mu \in \sigma(A_1)$ ,  $\lambda \in \sigma(A_2)$ . For every  $t > 0$  there exist  $\eta_t \in \sigma(A)$ ,  $m_t \in \mathbb{Z}$  such that  $\mu + \lambda = \eta_t + 2\pi i m_t / t$ . Since  $\operatorname{Re}(\mu) + \operatorname{Re}(\lambda) = \operatorname{Re}(\eta_t)$  and  $\{\operatorname{Im}(\eta_t) : t > 0\}$  is bounded,  $\mathcal{T} = (T_1(t) \otimes T_2(t))_{t \geq 0}$  being eventually norm continuous, it follows that  $m_{t'} = 0$  for some  $t' > 0$ .  $\square$

## 7 Weak Spectral Mapping Theorems

In the previous section we showed under which hypotheses a spectral mapping theorem of the form

$$\sigma(T(t)) \setminus \{0\} = e^{t \cdot \sigma(A)}, \quad t \geq 0, \quad (7.1)$$

is valid for the generator  $A$  of a strongly continuous semigroup  $(T(t))_{t \geq 0}$ .

Among the various examples showing that (7.1) does not hold in general we recall the following. Take the Banach space  $E = c_0$ , the multiplication operator  $A(x_n)_{n \in \mathbb{N}} = (inx_n)_{n \in \mathbb{N}}$  with maximal domain and the corresponding semigroup  $T(t)(x_n)_{n \in \mathbb{N}} = (e^{int}x_n)_{n \in \mathbb{N}}$ . Then  $\sigma(A) = \{in : n \in \mathbb{N}\}$  and the spectral mapping theorem is valid only in the following weak form

$$\sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))}, \quad t \geq 0. \quad (7.2)$$

In this section we prove similar weak spectral mapping theorems. We start with a generalization of the above example.

Consider the Banach space  $E = C_0(X, \mathbb{C}^n)$  of all continuous  $\mathbb{C}^n$ -valued functions vanishing at infinity on some locally compact space  $X$ . In analogy to A-I, 2.3 we associate to every continuous function  $q : X \rightarrow M(n)$ , where  $M(n)$  denotes the space of all complex  $n \times n$ -matrices, a “multiplication operator”  $M_q : f \rightarrow q \cdot f$  such that  $(q \cdot f)(x) = q(x) \cdot f(x)$ ,  $x \in X$ , on the maximal domain  $D(M_q) = \{f \in E : q \cdot f \in E\}$ . If  $\|e^{tq(x)}\|$  is uniformly bounded for  $0 \leq t \leq 1$  and  $x \in X$ , it follows that  $M_q$  generates the multiplication semigroup

$$(T(t)f)(x) = e^{tq(x)} \cdot f(x), \quad f \in E, \quad x \in X, \quad t \geq 0.$$

Since  $M_q$  has a bounded inverse if and only if  $q(x)^{-1}$  exists and is uniformly bounded for  $x \in X$ , it follows that the eigenvalues of each matrix  $q(x)$  are always contained in  $\sigma(M_q)$ . In fact, much more can be said in case the function is bounded.

**Lemma 7.1** *The spectrum of the matrix valued multiplication operator  $M_q$ , where  $q : X \rightarrow M(n)$  is bounded, is given by  $\sigma(M_q) = \overline{\bigcup_{x \in X} \sigma(q(x))}$ .*

**Proof** It remains to show that  $0 \notin \overline{\bigcup_{x \in X} \sigma(q(x))}$  implies  $0 \notin \sigma(M_q)$ . Since  $\det q(x)$  is the product of  $n$  eigenvalues (according to their multiplicity) of  $q(x)$ , the hypothesis implies that  $d := \inf\{|\det q(x)| : x \in X\} > 0$ . By Formula 4.12 in Chapter I of Kato

[16] we obtain

$$\|q(x)^{-1}\| \leq \gamma \cdot \|q(x)\|^{n-1} \cdot |\det q(x)|^{-1} \leq \gamma/d \cdot \|M_q\|^{n-1}$$

for every  $x \in X$  and a constant  $\gamma$  depending only on the norm chosen on  $\mathbb{C}^n$ . Therefore,  $x \mapsto q(x)^{-1}$  defines a bounded continuous function on  $X$  which obviously yields the inverse of  $M_q$ , i.e.,  $0 \notin \sigma(M_q)$ .  $\square$

**Theorem 7.2** *Let  $A = M_q$  be a matrix multiplication operator on  $C_0(X, \mathbb{C}^n)$  generating a strongly continuous semigroup  $(T(t))_{t \geq 0}$ ,*

$$T(t) := M_{e^{tq}} \quad \text{for } t \geq 0.$$

*Then the Weak Spectral Mapping Theorem 7.2 holds true, i.e.,*

$$\sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))}.$$

**Proof** By the Spectral Inclusion Proposition 6.2 we always have  $\exp(t\sigma(A)) \subset \sigma(T(t))$ . Since  $T(t)$  is a matrix multiplication operator with a bounded function, we obtain from Lemma 7.1

$$\sigma(T(t)) = \overline{\bigcup_{x \in X} \sigma(\exp(tq(x)))} = \overline{\bigcup_{x \in X} \exp(t\sigma(q(x)))} \subset \overline{\exp(t\sigma(A))}$$

which proves the assertion.  $\square$

**Corollary 7.3** *The growth bound  $\omega_0(A)$  and the spectral bound  $s(A)$  coincide for matrix multiplication semigroups.*

The above results remain valid for other Banach spaces of  $\mathbb{C}^n$ -valued functions such as  $L^p(X, \mathbb{C}^n)$ ,  $1 \leq p < \infty$ .

The example given at the beginning of this section can be generalized in a different way. In fact,  $A(x_n) := (inx_n)$  on  $E = c_0$  generates a bounded group, and we will show that this property too ensures that the Weak Spectral Mapping Theorem 7.2 holds. Without any boundedness assumption on  $(T(t))_{t \in \mathbb{R}}$  this result cannot be true (see Hille and Phillips [14, Sec.23.16] or Wolff [25]).

**Theorem 7.4** *Let  $\mathcal{T} = (T(t))_{t \in \mathbb{R}}$  be a strongly continuous group on some Banach space  $E$  such that  $\|T(t)\| \leq p(t)$  for some polynomial  $p$  and all  $t \in \mathbb{R}$ . Then the Weak Spectral Mapping Theorem 7.2 holds, i.e.,*

$$\sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))} \text{ for all } t \in \mathbb{R}.$$

From the proof we isolate a series of lemmas assuming the hypothesis made in Theorem 7.4. Moreover we recall from Fourier analysis that the Fourier transformation  $\varphi \mapsto \hat{\varphi}$ ,

$$\hat{\varphi}(\alpha) := \int_{-\infty}^{\infty} \varphi(x) e^{-i\alpha x} dx$$

and its inverse  $\Psi \mapsto \check{\Psi}$ ,

$$\check{\Psi}(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\alpha) e^{i\alpha x} d\alpha$$

are topological isomorphisms of the Schwartz space  $\mathcal{S}(=\mathcal{S}(\mathbb{R}))$ . Since the subspace  $\mathcal{D}$  of all functions having compact support is dense in  $\mathcal{S}$ , it follows that  $\{\varphi \in \mathcal{S} : \check{\varphi} \in \mathcal{D}\}$  is also dense in  $\mathcal{S}$ .

**Lemma 7.5** *For every function  $\varphi \in \mathcal{S}$  we obtain an operator  $T(\varphi) \in \mathcal{L}(E)$  by*

$$T(\varphi)f := \int_{-\infty}^{\infty} \varphi(s)T(s)f ds, \quad f \in E.$$

*This operator can be represented as*

$$T(\varphi)f = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{\varphi}(\alpha) [R(\varepsilon - i\alpha, A)f - R(-\varepsilon - i\alpha, A)f] d\alpha, \quad f \in E. \quad (7.3)$$

**Proof** That  $T(\varphi)$  is well-defined follows from the polynomial boundedness of  $(T(t))_{t \in \mathbb{R}}$ . In fact,  $\varphi \rightarrow T(\varphi)$  is continuous from  $\mathcal{S}$  into  $(\mathcal{L}(E), \|\cdot\|)$ . We obtain

$$\begin{aligned} T(\varphi)f &= \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \varphi(s) e^{-\varepsilon|s|} T(s)f ds \\ &= \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{\varphi}(\alpha) e^{i\alpha s} e^{-\varepsilon|s|} T(s)f d\alpha ds \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{\varphi}(\alpha) \int_{-\infty}^{\infty} e^{i\alpha s} e^{-\varepsilon|s|} T(s)f ds d\alpha \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{\varphi}(\alpha) [R(\varepsilon - i\alpha, A)f - R(-\varepsilon - i\alpha, A)f] d\alpha. \end{aligned}$$

Here we used that polynomially bounded groups have growth bound 0, hence  $\omega_0(A) = \omega_0(-A) = 0$ . Therefore the integral representation of  $R(\varepsilon - i\alpha, A)$  (cf. A-I, Proposition 1.11) exists for  $\varepsilon \neq 0$ .  $\square$

**Lemma 7.6** *If  $E \neq \{0\}$ , then  $\sigma(A) \neq \emptyset$ .*

**Proof** If  $\sigma(A) = \emptyset$ , then (7.3) implies  $T(\varphi) = 0$  whenever  $\check{\varphi}$  has compact support. Since these functions form a dense subspace of  $\mathcal{S}$ , we conclude that  $T(\varphi) = 0$  for all  $\varphi \in \mathcal{S}$ . Choosing an approximate identity  $(\psi_n)_{n \in \mathbb{N}} \subset \mathcal{D}$ , we obtain

$$f = T(0)f = \lim_{n \rightarrow \infty} T(\psi_n)f = 0$$

for every  $f \in E$ .  $\square$

**Proof (Proof of Theorem 7.4 (1<sup>st</sup> part))** By the Spectral Inclusion (see Proposition 6.2), we have to show that every spectral value of  $T(t)$  can be approximated by exponentials of spectral values of  $A$ . In view of the rescaling procedure it suffices to prove this when  $-1 \in \varrho(T(\pi))$ , provided that the following condition is satisfied.

$$\text{There exists } \varepsilon > 0 \text{ such that } \bigcup_{k \in \mathbb{Z}} i [2k + 1 - 2\varepsilon, 2k + 1 + 2\varepsilon] \subset \varrho(A). \quad (7.4)$$

Assume now that (7.4) holds. Then each of the sets

$$\sigma_k := \{i\alpha \in \sigma(A) : \alpha \in [2k - 1, 2k + 1]\}$$

is a spectral set of  $A$  with corresponding spectral projection  $P_k$ . If we choose  $\varphi_0 \in \mathcal{D}$  such that

$$\begin{aligned} \text{supp } (\varphi_0) &\subseteq [-1 + \varepsilon, 1 - \varepsilon], \\ \varphi_0(x) &= 1 \quad \text{for } x \in [-1 + 2\varepsilon, 1 - 2\varepsilon], \end{aligned}$$

it follows from (7.3) and the integral representation of  $P_k$  (cf. (3.1)) that  $P_0 = T(\check{\varphi}_0)$ .

More generally, since

$$\left( e^{i2k} \check{\varphi}_0 \right)^\wedge(\alpha) = \varphi_0(\alpha - 2k),$$

the assertions (7.3) and (7.4) imply

$$P_k = \int_{-\infty}^{\infty} e^{i2ks} \check{\varphi}_0(s) T(s) ds \quad \text{for } k \in \mathbb{Z}. \quad (7.5)$$

At this point we isolate another lemma.

**Lemma 7.7**  $\text{span } \bigcup_{k \in \mathbb{Z}} P_k E$  is dense in  $E$ .

**Proof** The closure of  $\text{span } \bigcup_{k \in \mathbb{Z}} P_k E$  is a  $\mathcal{T}$ -invariant subspace  $G$  of  $E$ . Consider the quotient group  $(T(t)_/)_t \in \mathbb{R}$  induced on  $E/G$ . The spectrum of its generator  $A_/_$  is contained in  $\sigma(A)$  by Proposition 4.2 (ii). Moreover, the spectral projection corresponding to  $\sigma(A_/) \cap \sigma_k$  is the quotient operator  $P_{k/_}$ . Obviously  $P_{k/_} = 0$ , hence  $\sigma(A_/) \cap \sigma_k = \emptyset$  for every  $k \in \mathbb{Z}$  and  $\sigma(A_/) = \emptyset$ . By Lemma 7.6 this implies  $E/G = \{0\}$ , i.e.,  $G = E$ .  $\square$

**Proof (Proof of Theorem 7.4 (2<sup>nd</sup> part))** We return to the situation of the first part of the proof. Using (7.5), the spectral projection  $P_k$  can be transformed into

$$\begin{aligned}
P_k &= \int_{-\infty}^{\infty} e^{i2ks} \check{\varphi}_0(s) T(s) \, ds \\
&= \sum_{m \in \mathbb{Z}} \int_{(m-1/2)\pi}^{(m+1/2)\pi} e^{i2ks} \check{\varphi}_0(s) T(s) \, ds \\
&= \int_{-\pi/2}^{\pi/2} e^{i2ks} \sum_{m \in \mathbb{Z}} \check{\varphi}_0(s + m\pi) T(s + m\pi) \, ds,
\end{aligned}$$

i.e.,  $P_k f$  is the  $k$ -th Fourier coefficient of the  $\pi$ -periodic, continuous function  $\xi_f: s \mapsto \sum_{m \in \mathbb{Z}} \check{\varphi}_0(s + m\pi) T(s + m\pi) f$ ,  $f \in E$ . Since the projections  $P_k$  are mutually orthogonal, i.e.,  $P_k P_m = 0$  for  $k \neq m$ , it follows that  $g = \sum_{n \in \mathbb{Z}} P_n g$  for every  $g \in \text{span } \bigcup_{k \in \mathbb{Z}} P_k E$ . In particular, the Fourier coefficients of the function  $\xi_g$  are absolutely summable, hence the Fourier series of  $\xi_g$  converges to  $\xi$ .

For  $s = 0$  we obtain

$$g = \sum_{n \in \mathbb{Z}} P_n g \cdot e^{-in0} = \sum_{m \in \mathbb{Z}} \check{\varphi}_0(0 + m\pi) T(0 + m\pi) g \quad \left( g \in \text{span } \bigcup_{k \in \mathbb{Z}} P_k E \right).$$

Since  $\text{span } \bigcup_{k \in \mathbb{Z}} P_k E$  is dense (Lemma 7.7), we conclude that

$$\sum_{m \in \mathbb{Z}} \check{\varphi}_0(m\pi) T(m\pi) = \text{Id}. \quad (7.6)$$

As the final step we construct the inverse operator of  $\text{Id} + T(\pi)$  showing that  $-1 \in \varrho(T(\pi))$ . We define  $\psi_0(\alpha) := \varphi_0(\alpha) \cdot (1 + e^{i\pi\alpha})^{-1}$ ,  $\alpha \in \mathbb{R}$ . Then we have  $\psi_0 \in \mathcal{S}$  and  $\psi_0 \cdot (1 + e^{i\pi\cdot}) = \varphi_0$ , hence  $\check{\psi}_0(x) + \check{\psi}_0(x + \pi) = \check{\varphi}_0(x)$  for all  $x \in \mathbb{R}$ . Then (7.6) implies

$$\begin{aligned}
\text{Id} &= \sum_{m \in \mathbb{Z}} \check{\varphi}_0(m\pi) T(m\pi) \\
&= \sum_{m \in \mathbb{Z}} (\check{\psi}_0(m\pi) + \check{\psi}_0((m+1)\pi)) T(m\pi) \\
&= \left[ \sum_{m \in \mathbb{Z}} \check{\psi}_0(m\pi) T(m\pi) \right] (\text{Id} + T(\pi)).
\end{aligned}$$

□

In the rest of this section we discuss the behavior of the single spectral values  $\lambda$  of  $T(t)$ ,  $t > 0$ . The goal is a characterization of  $\sigma(T(t))$  involving only properties of the generator. By the rescaling procedure A-I, 3.2 we may assume  $\lambda = 1$  and  $t = 2\pi$ .

From the Spectral Inclusion Theorem (Proposition 6.2 on page 90) we know that  $1 \in \varrho(T(2\pi))$  implies  $i\mathbb{Z} \subset \varrho(A)$ . As seen in many examples the converse does not hold and we are now looking for additional conditions. Henceforth we assume

$i\mathbb{Z} \subset \varrho(A)$  and define for  $k \in \mathbb{Z}$

$$Q_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-iks} T(s) ds = \frac{1}{2\pi} (1 - T(2\pi)) R(ik, A) \quad (7.7)$$

(cf. Formula A-I, (3.1)).

Our approach is based on Fejér's Theorem (for Banach space valued functions). Let us recall this result. Suppose  $\xi: [0, 2\pi] \rightarrow E$  is a continuous function and let  $\xi_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-iks} \xi(s) ds$  be its  $k$ -th Fourier coefficient. Then the Fourier series is Césaro summable to  $\xi$  in every point  $t \in (0, 2\pi)$ . Moreover one has

$$\frac{1}{2}(\xi(0) + \xi(2\pi)) = C_1 \cdot \sum_{k \in \mathbb{Z}} \xi_k := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left( \sum_{k=-n}^n \xi_k \right). \quad (7.8)$$

This result enables us to prove the following proposition.

**Proposition 7.8** *Let  $(T(t))_{t \geq 0}$  be a semigroup on a Banach space  $E$  and denote its generator by  $A$ . Then the following conditions are equivalent.*

- (a)  $1 \in \varrho(T(2\pi))$ ,
- (b)  $i\mathbb{Z} \subset \varrho(A)$  and the series  $\sum_{k \in \mathbb{Z}} R(ik, A)f$  is Césaro-summable for every  $f \in E$ ,
- (c)  $i\mathbb{Z} \subset \varrho(A)$  and the series  $\sum_{k \in \mathbb{Z}} R(ik, A)Q_k f$  is Césaro-summable for every  $f \in E$ .

**Proof** (a)  $\Rightarrow$  (b) The Spectral Inclusion Theorem 6.2 implies  $i\mathbb{Z} \subset \varrho(A)$ . By (7.7) we have  $R(ik, A) = 2\pi \cdot (1 - T(2\pi))^{-1} Q_k$ . Since  $\sum_{k \in \mathbb{Z}} Q_k f$  is Césaro-summable (towards  $\frac{1}{2}(f + T(2\pi)f)$ ) (see (7.8), it follows that  $\sum_{k \in \mathbb{Z}} R(ik, A)f$  is Césaro-summable as well.

(b)  $\Leftrightarrow$  (c) If we use A-I, (3.1) and integrate by parts, we obtain

$$\begin{aligned} R(ik, A)Q_k f &= \frac{1}{2\pi} \int_0^{2\pi} e^{-iks} T(s) R(ik, A) f ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[ R(ik, A) f - \int_0^s e^{-ikt} T(t) f dt \right] ds \\ &= R(ik, A) f - \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} (2\pi - t) T(t) f dt. \end{aligned}$$

Fejér's theorem ensures that

$$\sum_{k \in \mathbb{Z}} (1/2\pi) \int_0^{2\pi} e^{-ikt} (2\pi - t) T(t) f dt$$

is Césaro summable. Hence  $\sum_{k \in \mathbb{Z}} R(ik, A)Q_k f$  is Césaro-summable if and only if  $\sum_{k \in \mathbb{Z}} R(ik, A)f$  is.

(b)  $\Rightarrow$  (a) We have  $Q_k = \frac{1}{2\pi}(1 - T(2\pi))R(ik, A)$ . Furthermore we know by (7.7) and (7.8) that  $\sum_{k \in \mathbb{Z}} Q_k f$  is Césaro-summable towards  $\frac{1}{2}(f + T(2\pi)f)$ . If we define  $S: E \rightarrow E$  by  $Sf := \frac{1}{2}f + \frac{1}{2\pi} \cdot C_1 \cdot \sum_k R(ik, A)f$ , then we have

$$\begin{aligned} (1 - T(2\pi))Sf &= \frac{1}{2}(f - T(2\pi)f) + \frac{1}{2\pi} \cdot C_1 \cdot \sum_k (1 - T(2\pi))R(ik, A)f \\ &= \frac{1}{2}(f - T(2\pi)f) + \frac{1}{2}(f + T(2\pi)f) = f. \end{aligned}$$

Since  $S$  commutes with  $T(2\pi)$ , it follows that  $S$  is the inverse of  $(1 - T(2\pi))$  thus  $1 \in \varrho(T(2\pi))$ .  $\square$

Based on the equivalence of (a) and (b), one can state a characterization of the spectrum of  $T(t)$  in terms of the generator and its resolvent only. However, in general it is difficult to verify the summability condition stated in (b).

In Hilbert spaces the boundedness of the resolvents will suffice (see Theorem 7.10 below).

**Lemma 7.9** *Let  $(T(t))_{t \geq 0}$  be a semigroup on some Hilbert space  $H$  and assume  $i\mathbb{Z} \subset \varrho(A)$  for the generator  $A$ . Then we have*

- (i)  $(Q_k f)_{k \in \mathbb{Z}} \subset \ell^2(H)$  for every  $f \in H$ , and
- (ii) if  $\sup_{k \in \mathbb{Z}} \|R(ik, A)\| < \infty$ , then  $\sum_{k \in \mathbb{Z}} R(ik, A)f_k$  is summable whenever  $(f_k)_{k \in \mathbb{Z}} \in \ell^2(H)$ .

**Proof** (i) We consider the Hilbert space  $L^2([0, 2\pi], H)$  and obtain

$$\begin{aligned} 0 &\leq \left\| T(\cdot)f - \sum_{k=-n}^n Q_k f \cdot e^{ik\cdot} \right\|^2 \\ &= \int_0^{2\pi} \|T(s)f\|^2 ds - \int_0^{2\pi} \sum_{k=-n}^n (T(s)f | e^{iks} Q_k f) ds - \\ &\quad \int_0^{2\pi} \sum_{k=-n}^n (e^{iks} Q_k f | T(s)f) ds + \int_0^{2\pi} \left( \sum_{k=-n}^n e^{iks} Q_k f \middle| \sum_{\ell=-n}^n e^{i\ell s} Q_\ell f \right) ds \\ &= \int_0^{2\pi} \|T(s)f\|^2 ds - 2\pi \sum_{k=-n}^n \|Q_k f\|^2 \text{ (use (7.5))}. \end{aligned}$$

It follows that  $\sum_{k \in \mathbb{Z}} \|Q_k f\|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} \|T(s)f\|^2 ds < \infty$ .

(ii) Fix  $\lambda > 0$  sufficiently large and set

$$g_k := (1 + \lambda R(ik, A))f_k, \quad k \in \mathbb{Z}.$$

Using the resolvent equation and then (A-I, (3.1)), we obtain

$$R(ik, A)f_k = R(\lambda + ik, A)g_k = [1 - e^{-2\pi\lambda}T(2\pi)]^{-1} \int_0^{2\pi} e^{-\lambda s} e^{-iks} T(s)g_k ds.$$

This yields for every finite subset  $N$  of  $\mathbb{Z}$  that

$$\begin{aligned} \left\| \sum_{k \in N} R(ik, A)f_k \right\| &\leq \|(1 - e^{-2\pi\lambda}T(2\pi))^{-1}\| \cdot \int_0^{2\pi} \|T(s)\| \left\| \sum_{k \in N} e^{-iks} g_k \right\| ds \\ &\leq \|(1 - e^{-2\pi\lambda}T(2\pi))^{-1}\| \cdot \left( \int_0^{2\pi} \|T(s)\|^2 ds \right)^{1/2} \cdot \left( \int_0^{2\pi} \left\| \sum_{k \in N} e^{-iks} g_k \right\|^2 dx \right)^{1/2} \\ &= c \left( \sum_{k \in N} \|g_k\|^2 \right)^{1/2} \leq c(1 + \lambda M) \left( \sum_{k \in N} \|f_k\|^2 \right)^{1/2}. \end{aligned}$$

Here  $c := \|(1 - e^{-2\pi\lambda}T(2\pi))^{-1}\| \cdot \left( \int_0^{2\pi} \|T(s)\|^2 ds \right)^{1/2}$  and  $M := \sup_{k \in \mathbb{Z}} \|R(ik, A)\|$  do the job.  $\square$

**Theorem 7.10** *Let  $A$  be the generator of a semigroup  $(T(t))_{t \geq 0}$  on some Hilbert space  $H$ . Then the following form of the spectral mapping theorem is valid.*

$e^{\lambda t} \in \sigma(T(t)) \setminus \{0\}$  if and only if either  $\mu_k := \lambda + 2\pi ik/t \in \sigma(A)$  for some  $k \in \mathbb{Z}$  or if  $(\|R(\mu_k, A)\|)_{k \in \mathbb{Z}}$  is unbounded.

**Proof** If  $e^{\lambda t} \notin \sigma(T(t))$ , it follows from the Spectral Inclusion Theorem 6.2 that  $\mu_k \notin \sigma(A)$  for every  $k \in \mathbb{Z}$  and from Formula (3.1) in A-I, that  $\|R(\mu_k, A)\|$  is bounded. For the converse inclusion it suffices to assume  $t = 2\pi$  and  $\lambda = 0$  (use the rescaling procedure A-I, 3.2). Assuming that  $i\mathbb{Z} \subset \varrho(A)$  and  $\|R(ik, A)\|$  is bounded, then  $\sum_{k \in \mathbb{Z}} R(ik, A)Q_k f$  is summable by Lemma 7.9. Since every summable series is Cesàro-summable, condition (c) of Proposition 7.8 is satisfied and we conclude  $1 \in \varrho(T(2\pi))$ .  $\square$

As an immediate consequence we obtain an interesting characterization of the growth bound  $\omega_0$  of semigroups on Hilbert spaces.

**Corollary 7.11** *The growth bound of a semigroup  $(T(t))_{t \geq 0}$  on a Hilbert space  $H$  satisfies*

$$\omega_0 = \inf \{ \lambda \in \mathbb{R} : \lambda + i\mathbb{R} \subset \varrho(A) \text{ and } \|R(\lambda + i\mu, A)\| \text{ is bounded for } \mu \in \mathbb{R} \}. \quad (7.9)$$

The Example 1.3 above in combination with C-III, Corollary 1.3 shows that (7.9) is not valid in arbitrary Banach spaces.

## Notes

*Section 1:* It was already known to Hille and Phillips [14] that for strongly continuous semigroups  $(T(t))_{t \geq 0}$  with generator  $A$  the Spectral Mapping Theorem “ $\sigma(T(t)) = \exp(t\sigma(A))$ ” may be violated in various ways [l.c., Sec. 23.16]. The simple Examples 1.3 and 1.4 are due to Wolff (see Greiner et al. [11]) and Zabczyk [27]. A more sophisticated example of a positive group with “ $s(A) < \omega_0(A)$ ” is given in Wolff [25].

*Section 2:* In Definition 2.1 we define the residual spectrum of  $A$  in such a way that it coincides with the point spectrum of the adjoint  $A'$  (see Proposition 2.2 (ii)). It therefore differs slightly from the one used, e.g., by Schaefer [24]. The spectral mapping theorem for the resolvent (Theorem 2.5) is well known and can, e.g., be deduced from Lemma 9.2 and Theorem 3.11 of Chap. VII in Dunford and Schwartz [5].

*Section 3:* The general theory of spectral decompositions can be found in Kato [16], Chap. III, § 6.4]. Applications to isolated singularities like Example 3.6 are discussed extensively in [l.c., Chap. III, § 6.5] and Yosida [26, Chap. VIII, Sec. 8]. There are many ways to introduce an “essential spectrum” (see the footnote on page 243 of Kato [16]). However each one yields the same “essential spectral radius”.

*Section 4:* These results are taken from Derndinger [4] and Greiner [9].

*Section 5:* Periodic semigroups are studied explicitly in Bart [1], but most of the results of this section seem to be well known.

*Section 6:* The *Spectral Inclusion Theorem* 6.2 and the *Spectral Mapping Theorem* 6.6 for eventually norm continuous semigroups date back to Hille and Phillips [14]. Davies [3] gives an elegant proof using Banach algebra techniques. Tensor products of operators and their spectral theory have been studied by Ichinose and others (see Chap. XIII.9 of Reed and Simon [23]). The spectral mapping theorem in Corollary 6.8 generalizes Theorem XIII.35 of Reed and Simon [23] (see also Herbst [12]).

*Section 7:* Matrix valued multiplication semigroups appear as solution, via Fourier transformation, of systems of partial differential equations. Kreiss initiated a systematic investigation (see, e.g., Kreiss [17], Kreiss [18], Miller and Strang [20]) and the Weak Spectral Mapping Theorem 7.2 must have been known to him. The direct proof of the Weak Spectral Mapping Theorem 7.4 for polynomially bounded groups seems to be new. The result can also be deduced from the theory of spectral subspaces of group representations (see, e.g., Combes and Delaroche [2]), since the Arveson spectrum of a strongly continuous one-parameter group can be identified with the spectrum of the generator (see Evans [6]). The final part of this section is taken from Greiner [10] and yields a new approach to Gearhart’s characterization of the spectrum of semigroups on Hilbert spaces (Gearhart [7]). Different proofs can be found in Herbst [13], Howland [15] and Prüss [22].

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## Chapter A-IV

# Asymptotics of Semigroups on Banach Spaces

by  
Frank Neubrander

In this chapter, we study the asymptotic behavior of the solutions of the inhomogeneous initial value problem

$$\dot{u}(t) = Au(t) + F(t) \text{ for } t \geq 0 \text{ and } u(0) = f, \quad (*)$$

where  $A$  is the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ , and  $F(\cdot)$  is a function from  $\mathbb{R}_+$  into  $E$ .

In Section 1, we investigate whether – and at what rate – a solution  $T(\cdot)f$  of the homogeneous problem converges to the zero solution as  $t \rightarrow \infty$ . In Section 2, we consider the long-term behavior of the solutions of  $(*)$  for different classes of forcing terms  $F$ .

### 1 Stability: Homogeneous Case

Let  $(T(t))_{t \geq 0}$  be a semigroup on  $E$  with generator  $A$ . An initial value  $f \in D(A)$  is said to be *stable* if the solution  $t \mapsto T(t)f$  of

$$\dot{u}(t) = Au(t), \quad u(0) = f \quad (\text{ACP})$$

converges to zero as  $t \rightarrow \infty$ . The semigroup is called *stable* if every solution converges to zero, i.e., if every initial value  $f \in D(A)$  is stable.

If the space  $E$  is finite-dimensional, then the stability of the semigroup implies exponential decay. More precisely, the following statements are equivalent.

- (a)  $\|T(t)f\| \rightarrow 0$  for every  $f \in \mathbb{C}^n$ ,
- (b)  $\|T(t)\| \leq Me^{-\omega t}$  for some  $M \geq 1$  and  $\omega > 0$ .

Moreover, these statements hold if and only if

$$(c) \quad s(A) = \sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\} < 0,$$

see A-III, Corollary 1.2.

As discussed in Chapter A-III, the situation becomes significantly more intricate in the infinite-dimensional setting. For unbounded generators, we must distinguish between *strong* and *generalized (mild)* solutions of  $\dot{u}(t) = Au(t)$ , as well as between various notions of stability. If  $A$  is the generator of a strongly continuous semigroup on a Banach space  $E$  and  $f \in D(A)$ , then  $T(\cdot)f$  is the unique solution or, equivalently, the strong solution of (ACP) with initial value  $f$ ; see A-II, Corollary 1.2. For an arbitrary  $f \in E$ , the function  $T(\cdot)f$  is referred to as a generalized or mild solution of (ACP). Next, we introduce several constants that characterize the growth of solutions of (ACP).

**Definition 1.1** (1<sup>st</sup> part) Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . Then we define the following constants.

- (i)  $\omega(f) := \inf\{\omega : \|T(t)f\| \leq Me^{\omega t} \text{ for some } M \text{ and every } t \geq 0\}$  is called the *exponential growth bound* of  $T(\cdot)f$ ,
- (ii)  $\omega_1(A) := \sup\{\omega(f) : f \in D(A)\}$  is called the *exponential growth bound for the solutions of the Cauchy problem*  $\dot{u}(t) = Au(t)$ ,
- (iii)  $\omega_0(A) := \sup\{\omega(f) : f \in E\}$  is called the (exponential) *growth bound for the mild solutions of the Cauchy problem*  $\dot{u}(t) = Au(t)$ .

Note that, by the Principle of Uniform Boundedness,

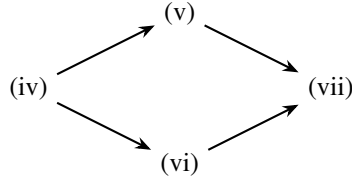
$$\sup\{\omega(f) : f \in E\} = \inf\{\omega : \|T(t)\| \leq Me^{\omega t} \text{ for some } M \text{ and every } t \geq 0\}.$$

Hence,  $\omega_0(A)$  coincides with the growth bound of the semigroup  $(T(t))_{t \geq 0}$  as defined in A-I, 1.3. Using the constants defined above, we obtain the following stability concepts.

**Definition 1.1** (2<sup>nd</sup> part) The semigroup is called

- (iv) *uniformly exponentially stable* if  $\omega_0(A) < 0$ ,
- (v) *exponentially stable* if  $\omega_1(A) < 0$ ,
- (vi) *uniformly stable* if  $\|T(t)f\| \rightarrow 0$  as  $t \rightarrow \infty$  for every  $f \in E$ ,
- (vii) *stable* if  $\|T(t)f\| \rightarrow 0$  as  $t \rightarrow \infty$  for every  $f \in D(A)$ .

The interrelation between these stability concepts is given by



If  $A$  is a bounded operator, that is, if  $D(A) = E$ , then

$$(iv) \Leftrightarrow (v) \text{ and } (vi) \Leftrightarrow (vii).$$

However, if  $A$  is unbounded, all stability notions may differ, as illustrated in the following examples.

**Example 1.2** (i) Let  $E = c_0$ . Then

$$A: (x_n)_{n \in \mathbb{N}} \mapsto (-1/n \cdot x_n)_{n \in \mathbb{N}}$$

generates the semigroup

$$T(t)(x_n)_{n \in \mathbb{N}} = (e^{-t/n} x_n)_{n \in \mathbb{N}}.$$

It is easy to see that  $\|T(t)\| = 1$  and that  $\|T(t)f\| \rightarrow 0$  for every  $f \in c_0$ .

Moreover, since  $A$  is a bounded operator,  $D(A) = E$ . This provides an example of a (uniformly) stable but not exponentially stable semigroup.

The translation semigroups generated by the first derivative on  $C_0(\mathbb{R}_+)$  or  $L^p(\mathbb{R}_+)$  for  $1 < p < \infty$  offer further examples of (uniformly) stable but not exponentially stable semigroups.

Moreover, as shown in A-II, Example 1.14, the Laplacian  $\Delta$  on  $C_0(\mathbb{R}^n)$  generates a bounded holomorphic semigroup given by

$$T(t)f(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y) dy.$$

This semigroup is not exponentially stable because  $0 \in \sigma(\Delta)$  ( $\text{Im}(\Delta) \neq C_0(\mathbb{R}^n)$ ); see Corollary 1.5 below. To see that the semigroup is (uniformly) stable, observe that for every fixed  $x \in \mathbb{R}^n$ , the kernel  $k_t(x, y) := (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$  defines a probability density with  $\int_{\mathbb{R}^n} k_t(x, y) dy = 1$ . Hence,  $\|T(t)\| \leq 1$ ; in fact,  $\|T(t)\| = 1$  since  $k_t$  also forms an approximate identity. Let  $f \in C_0(\mathbb{R}^n)$  and  $\varepsilon > 0$ . Then there exists a compactly supported  $g \in C_0(\mathbb{R}^n)$  such that  $\|f - g\| \leq \varepsilon$ . Therefore,

$$\|T(t)f\| \leq \|T(t)\| \|f - g\| + \|T(t)g\| \leq \varepsilon + (4\pi t)^{-n/2} \int_{\mathbb{R}^n} |g(y)| dy,$$

which implies  $\|T(t)f\| \rightarrow 0$  as  $t \rightarrow \infty$  for all  $f \in C_0(\mathbb{R}^n)$ ; see also B-III, Example 1.7. This shows that the Laplacian on  $C_0(\mathbb{R}^n)$  (and also on  $L^p(\mathbb{R}^n)$  for  $1 < p < \infty$ , see

Example 1.15 below) generates a uniformly stable but not exponentially stable semigroup.

(ii) Note that the condition

$$0 \leq \omega_0(A) = \inf\{\omega : \|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0\}$$

does not exclude the possibility that the semigroup is exponentially stable.

To see this, consider  $E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x dx)$ . Then, as shown in A-III, Example 1.3, the translation semigroup satisfies  $\|T(t)\| = 1$ , and hence  $\omega_0(A) = 0$ . For every  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > -1$  and every  $f \in E$ , the resolvent of the generator is given as  $R(\lambda, A)f = \int_0^\infty e^{\lambda t} T(t)f dt$ . From the Equation A-I, (3.2), it follows that

$$T(t)f = e^{\lambda t} \left( f - \int_0^t e^{-\lambda s} T(s)(\lambda - A)f ds \right),$$

and from the existence of the limit

$$\lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} T(s)(\lambda - A)f ds,$$

it follows that  $\|T(t)f\| \leq Me^{\lambda t}$  for every  $f \in D(A)$  and some constant  $M$  depending on  $f$ . This yields  $\omega_1(A) \leq -1 < 0 = \omega_0(A)$ . Thus, we have a semigroup that is exponentially stable, but not uniformly exponentially stable.

(iii) Rescaling this semigroup (see A-I, 3.2), we obtain a semigroup with  $\omega_1(A) = -1/2$  and  $\omega_0(A) = 1/2$ . Therefore, there exist exponentially stable (and hence, stable) semigroups that are not bounded, and hence, not uniformly stable. This example illustrates that there may be a significant difference between the long-term behavior of the semigroup  $(T(t))_{t \geq 0}$  (i.e., the set of all mild solutions) and the long-term behavior of the strong solutions  $\{T(\cdot)f : f \in D(A)\}$  of (ACP).

In what follows, we characterize the exponential growth bounds  $\omega(f)$ ,  $\omega_1(A)$ , and  $\omega_0(A)$  in terms of certain abscissas of simple or absolute convergence of the Laplace transform of  $T(\cdot)f$ . These characterizations will serve as one of the tools for establishing that, for certain semigroups beyond the class of eventually norm-continuous semigroups (see A-III, Theorem 6.6), the growth bounds  $\omega_0(A)$  and/or  $\omega_1(A)$  coincide with the spectral bound  $s(A) = \sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\}$ . For results of this type, see, for example, B-IV, C-IV, and D-IV.

We begin by observing that  $s(A)$  can be interpreted as the abscissa of holomorphy of the Laplace transform  $\lambda \mapsto \int_0^\infty e^{-\lambda t} T(t) dt$  of the semigroup  $(T(t))_{t \geq 0}$ . Furthermore, we recall that the Laplace transform exists for every  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > \operatorname{Re}(\mu)$ , provided it exists for some  $\mu \in \mathbb{C}$ . This follows from

$$\begin{aligned} \int_0^t e^{-\lambda s} f(s) \, ds &= e^{-(\lambda-\mu)t} \int_0^t e^{\mu s} f(s) \, ds \\ &+ (\lambda - \mu) \int_0^t e^{-(\lambda-\mu)s} \int_0^s e^{\mu r} f(r) \, dr \, ds. \end{aligned} \quad (1.1)$$

Note that even boundedness of

$$\left\{ \int_0^t e^{-\mu s} f(s) \, ds : t > 0 \right\}$$

implies the existence of the Laplace transform for  $\operatorname{Re}(\lambda) > \operatorname{Re}(\mu)$ . Therefore, the subset of  $\mathbb{C}$  for which the Laplace transform exists is always a half-plane  $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > \gamma\} \cup H$ , where  $H$  is a subset of the line  $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) = \gamma\}$ .

In the following theorem, we show that the bound of the half-plane for which the Laplace transform of  $T(\cdot)f$  ( $f \in E$ ) exists absolutely, and the bound of the half-plane for which the Laplace transform of  $T(\cdot)Af$  ( $f \in D(A)$ ) exists strongly, coincide with the growth bound  $\omega(f) = \inf\{\omega : \|T(t)f\| \leq Me^{\omega t} \text{ for all } t \geq 0\}$ .

**Theorem 1.3** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . Then, for every  $f \in E$ ,*

$$\omega(f) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|T(t)f\|, \quad (1.2)$$

and

$$(i) \quad \omega(f) = \inf\{\operatorname{Re}(\lambda) : \int_0^\infty \|e^{-\lambda t} T(t)f\| \, dt \text{ exists}\}.$$

If  $\ker(A) = \{0\}$ , then for every  $f \in D(A)$  we have

$$(ii) \quad \omega(f) = \inf\{\operatorname{Re}(\lambda) : \int_0^\infty e^{-\lambda t} T(t)Af \, dt \text{ exists}\}.$$

**Proof** The proof of (1.2) is omitted (see Hille and Phillips [11, p.306]. To prove (i) and (ii), we need the following lemma.

**Lemma** *Let  $F \in C(\mathbb{R}_+, \mathbb{R}_+)$  be such that  $\int_0^\infty F(r) \, dr$  exists. If there exist positive constants  $m$  and  $n$  such that  $F$  satisfies the local growth condition  $F(t+s) \leq m \cdot F(s)$  for all  $s \geq 0$  and  $t \in [0, n]$ , then  $\lim_{s \rightarrow \infty} F(s) = 0$ .  $\square$*

**Proof (Proof of Lemma)** Let  $\varepsilon > 0$ . Since  $\int_0^\infty F(r) \, dr < \infty$ , there exists  $a > 0$  such that

$$A(a) := \int_a^\infty F(r) \, dr < \frac{n}{m} \varepsilon.$$

Now fix any  $s > a+n$ . We claim that there exists  $r \in [s-n, s]$  such that  $F(r) \leq \frac{1}{n} A(a)$ . Indeed, suppose that  $F(r) > \frac{1}{n} A(a)$  for all  $r \in [s-n, s]$ . Then

$$A(a) = \int_{s-n}^s \frac{1}{n} A(a) \, dr < \int_{s-n}^s F(r) \, dr \leq \int_a^\infty F(r) \, dr = A(a),$$

which is a contradiction. Hence, such an  $r \in [s-n, s]$  must exist. Finally, if  $s > a+n$  and  $r \in [s-n, s]$ , then  $0 \leq s-r \leq n$  and, therefore,  $F(s) = F(s-r+r) \leq m \cdot F(r) \leq m \cdot \frac{A(a)}{n} < \varepsilon$ . This shows that  $\lim_{s \rightarrow \infty} F(s) = 0$ .  $\square$

To prove part (i) of Theorem 1.3, define

$$b := \inf \left\{ \operatorname{Re}(\lambda) : \int_0^\infty \|\mathrm{e}^{-\lambda t} T(t) f\| dt \text{ exists} \right\}.$$

A straightforward application of the lemma shows that  $\omega(f) \leq b$ . The definition of  $\omega(f)$  yields the reverse inequality.

It remains to prove part (ii) of Theorem 1.3. Assume that  $\ker(A) = \{0\}$  and let  $f \in D(A)$ ,  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > \omega(f)$ . From the equation

$$\int_0^t \mathrm{e}^{-\lambda s} T(s) A f ds = \mathrm{e}^{-\lambda t} T(t) f - f + \lambda \int_0^t \mathrm{e}^{-\lambda s} T(s) f ds$$

it follows that  $\int_0^\infty \mathrm{e}^{-\lambda t} T(t) A f dt$  exists. Therefore,

$$c := \inf \{ \operatorname{Re}(\lambda) : \int_0^\infty \mathrm{e}^{-\lambda t} T(t) A f dt \text{ exists} \} \leq \omega(f).$$

Next, we show that  $c < 0$  implies  $c = \omega(f)$ . For  $c < 0$ , it follows from (\*) that  $\int_0^\infty T(s) A f ds$  exists. Since  $\int_0^r T(s) A f ds = T(r) f - f$ , it follows that

$$g := \lim_{r \rightarrow \infty} T(r) f$$

exists. But for every  $t \geq 0$ ,  $T(t)g = g$  which implies that  $g \in \ker(A) = \{0\}$  or  $g = 0$ . Hence,  $\int_0^\infty T(s) A f ds = -f$ . Now, choosing  $r < 0$ ,  $b < r < 0$ , and integrating by parts, we obtain

$$\begin{aligned} -T(t)f &= \lim_{u \rightarrow \infty} \int_t^u \mathrm{e}^{rs} \mathrm{e}^{-rs} T(s) A f ds \\ &= \lim_{u \rightarrow \infty} \left( \mathrm{e}^{ru} \int_0^u \mathrm{e}^{-rs} T(s) A f ds - \mathrm{e}^{rt} \int_0^t \mathrm{e}^{-rs} T(s) A f ds \right. \\ &\quad \left. - r \int_t^u \mathrm{e}^{rs} \int_0^s \mathrm{e}^{-rv} T(v) A f dv ds \right) \\ &= -\mathrm{e}^{rt} \int_0^t \mathrm{e}^{-rs} T(s) A f ds - r \int_t^\infty \mathrm{e}^{rs} \int_0^s \mathrm{e}^{-rv} T(v) A f dv ds. \end{aligned}$$

From  $\left\| \int_0^t \mathrm{e}^{-rs} T(s) A f ds \right\| \leq M$  for some  $M$  and every  $t \geq 0$  we conclude that  $\|T(t)f\| \leq \tilde{M} \mathrm{e}^{rt}$  for all  $t \geq 0$  and some constant  $\tilde{M}$ . Hence,  $\omega(f) \leq r$  for every  $c < r < 0$ , i.e.,  $\omega(f) \leq c$ .

If  $c \geq 0$  and  $w > c$ , then  $\left\| \int_0^t \mathrm{e}^{-ws} T(s) A f ds \right\| \leq M$  for all  $t \geq 0$ . By

$$\begin{aligned}
T(t)f - f &= \int_0^t e^{ws} e^{-ws} T(s) A f \, ds \\
&= e^{wt} \int_0^t e^{-ws} T(s) A f \, ds - w \int_0^t e^{ws} \int_0^s e^{-wr} T(r) A f \, dr \, ds,
\end{aligned}$$

we obtain  $\|T(t)f - f\| \leq M e^{wt} + M(e^{wt} - 1) \leq 2M e^{wt}$ . Hence,  $\omega(f) \leq w$  for every  $w > c$ , that is,  $\omega(f) \leq c$ .  $\square$

Finally, from (1.2) and the Uniform Boundedness Principle, it follows that the growth bound

$$\omega_1(A) = \sup\{\omega(f) : f \in D(A)\}$$

satisfies

$$\begin{aligned}
\omega_1(A) &= \inf\{\omega : \text{for every } f \in D(A) \text{ there exists a constant } M \text{ such that} \\
&\quad \|T(t)f\| \leq M e^{\omega t} \text{ for every } t \geq 0\} \\
&= \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|T(t)R(\lambda, A)\| \quad (\lambda \in \varrho(A)).
\end{aligned} \tag{1.3}$$

The following theorem plays a central role in the stability theory of positive semigroups. We show that the constant  $\omega_1(A)$  coincides both with the abscissa of simple convergence of the Laplace transform of the semigroup and with the abscissa of absolute convergence of the Laplace transform of the strong solutions of (ACP).

**Theorem 1.4** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . Then*

$$\begin{aligned}
\omega_1(A) &= \inf\{\operatorname{Re}(\lambda) : \int_0^\infty e^{-\lambda t} T(t) f \, dt \text{ exists as an } \dots \\
&\quad \dots \text{ improper Riemann integral for every } f \in E\} \\
&= \inf\{\operatorname{Re}(\lambda) : \int_0^\infty \|e^{-\lambda t} T(t) f\| \, dt \text{ exists for every } f \in D(A)\}.
\end{aligned} \tag{1.4}$$

**Remark** (i) One can show that the abscissas of uniform, strong, and weak convergence of the Laplace transform coincide (see C-III, Theorem I.2, last part of the proof). Therefore, by Theorem 1.4,

$$\begin{aligned}
\omega_1(A) &= \inf\left\{\operatorname{Re}(\lambda) : \text{weak-} \lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} T(s) \, ds \text{ exists}\right\} \\
&= \inf\left\{\operatorname{Re}(\lambda) : \text{uniform-} \lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} T(s) \, ds \text{ exists}\right\}.
\end{aligned} \tag{1.5}$$

(ii) In Equations (1.4) and (1.5), the term “ $\operatorname{Re}(\lambda)$ ” may be replaced by “ $\lambda \in \mathbb{R}$ ” (use (\*)).

**Proof** (Proof of Theorem 1.4) The equality

$$\omega_1(A) = \inf \left\{ \operatorname{Re}(\lambda) : \int_0^\infty \|e^{-\lambda t} T(t) f\| dt \text{ exists for all } f \in D(A) \right\}$$

follows from the definition of  $\omega_1(A)$  and the lemma in the proof of Theorem 1.3. We aim to prove that

$$\omega_1(A) = \inf \left\{ \operatorname{Re}(\lambda) : \int_0^\infty e^{-\lambda s} T(s) f ds \text{ exists for every } f \in E \right\} =: b.$$

The identity

$$T(t)f = e^{\lambda t} \left\{ f - \int_0^t e^{-\lambda s} T(s)(\lambda - A)f ds \right\}$$

yields

$$\omega_1(A) \leq \inf \left\{ \operatorname{Re}(\lambda) : \int_0^\infty e^{-\lambda t} T(t) f dt \text{ exists for every } f \in \operatorname{im}(\lambda - A) \right\}.$$

Therefore,  $\omega_1(A) \leq b$ . Take  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > \omega_1(A)$ . Then  $\int_0^\infty e^{-\lambda t} T(t) f dt$  exists for every  $f \in D(A)$ . Define  $g := \int_0^\infty e^{-\lambda t} T(t) f dt$ . Then  $g \in D(A)$  and  $\int_0^n e^{-\lambda t} T(t) f dt = \sum_{k=0}^{n-1} e^{-\lambda k} T(k)g$ . Since  $\operatorname{Re}(\lambda) > \omega_1(A)$ , the sum converges for every  $g \in D(A)$ . Therefore, the integral converges as  $n \rightarrow \infty$  for every  $f \in E$ . For every  $t \in \mathbb{R}_+$ , define a bounded operator  $T_t$  by  $f \mapsto \int_0^t e^{-\lambda s} T(s) f ds$ . As seen above,  $T_n f$  converges as  $n \rightarrow \infty$  for every  $f \in E$ . It follows from the Uniform Boundedness Principle that  $(T_n)_{n \in \mathbb{N}}$  is uniformly bounded.

For every  $t \in \mathbb{R}_+$ , there exist  $n \in \mathbb{N}$  and  $t' \in [0, 1)$  such that  $T_t = T_{t'} + e^{-\lambda t'} T(t') T_n$ . Since the operator families on the right side are uniformly bounded, the same is true for  $(T_t f)_{t \geq 0}$ . Since  $(T_t f)_{t \geq 0}$  converges for every  $f \in D(A)$ , it follows that  $(T_t f)_{t \geq 0}$  converges for every  $f \in E$ . Thus,  $b \leq \omega_1(A)$ .  $\square$

The inequality

$$\omega_0(A) \geq \inf \left\{ \operatorname{Re}(\lambda) : \int_0^\infty \|e^{-\lambda t} T(t) f\| dt \text{ exists for every } f \in E \right\}$$

combined with the lemma of Theorem 1.3 implies that the growth bound  $\omega_0(A)$  coincides with the abscissa of absolute convergence of the Laplace transform of  $(T(t))_{t \geq 0}$ , i.e.,

$$\omega_0(A) = \inf \left\{ \operatorname{Re}(\lambda) : \int_0^\infty \|e^{-\lambda t} T(t) f\| dt \text{ exists for every } f \in E \right\}. \quad (1.6)$$

As seen in A-I, Proposition 1.11, if  $\int_0^\infty e^{-\lambda t} T(t) f dt$  exists for every  $f \in E$ , then  $\lambda \in \varrho(A)$  and  $R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f dt$ . This and Theorem 1.4 yield the following corollary.

**Corollary 1.5** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . Then*

$$s(A) \leq \omega_1(A) \leq \omega_0(A).$$

Example 1.2(c) shows that uniform exponential stability is not equivalent to  $\sigma(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq q < 0\}$ . The following example shows that even strong solutions can be unstable when  $s(A) < 0$ . We construct a semigroup where  $s(A) < \omega_1(A) < \omega_0(A)$ .

**Example 1.6** As in A-III, Example 1.4, consider the semigroup  $(T(t))_{t \geq 0}$  on the Hilbert space  $E = \{(x^1, x^2, \dots) : x^n \in \mathbb{C}^n : \sum_{j=1}^\infty \|x^j\|^2 < \infty\}$ , defined by

$$T(t) := (e^{2\pi i n t} \cdot \exp(t A_n))_{n \in \mathbb{N}},$$

where

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \vdots \\ & & & & 0 & 1 \\ 0 & & & & & 0 \end{bmatrix}_{n \times n}.$$

This semigroup satisfies  $\|T(t)\| = e^t$  for all  $t \geq 0$ . Hence,  $\omega_0(A) = 1$ , while the generator  $A = (2\pi i n + A_n)_{n \in \mathbb{N}}$  has spectral bound  $s(A) = 0$ . We first show that  $\omega_1(A) = \omega_0(A)$ ; this will later be used to construct a semigroup for which  $s(A) < \omega_1(A) < \omega_0(A)$ . Let  $e_n = n^{-1/2}(1, \dots, 1) \in \mathbb{C}^n$ .

Then, for fixed  $n$  and  $t \geq 0$ ,

$$\begin{aligned} \|\exp(t A_n) \cdot e_n\|^2 &= \\ &= \frac{1}{n} \left\| \left( 1 + t + \dots + \frac{t^{n-1}}{(n-1)!}, 1 + t + \dots + \frac{t^{n-2}}{(n-2)!}, \dots, 1 + t, 1 \right) \right\|^2 \\ &= \frac{1}{n} \sum_{r=0}^{n-1} \left( \sum_{j=0}^r \frac{1}{j!} t^j \right)^2 = \frac{1}{n} \sum_{r=0}^{n-1} \sum_{j,s=0}^r \frac{1}{j!s!} t^{j+s} = \frac{1}{n} \sum_{r=0}^{n-1} \sum_{i=0}^{2r} t^i \sum_{\substack{j+s=i \\ 0 \leq j,s \leq r}} \frac{1}{j!s!} \\ &\geq \frac{1}{n} \sum_{r=0}^{n-1} \sum_{i=0}^r \frac{(2t)^i}{i!} = \frac{1}{n} \sum_{i=0}^{n-1} (n-i) \frac{(2t)^i}{i!} \geq \frac{1}{n^2} \sum_{i=0}^{n-1} \frac{(2t)^i}{i!}. \end{aligned}$$

For  $0 < q < 1$ , define  $x_q \in E$  as  $x_q := (q e_1, 2q^2 e_2, \dots, nq^n e_n, \dots)$ . Then  $x_q \in D(A)$  and

$$\begin{aligned}
\|T(t)x_q\|^2 &= \sum_{n=1}^{\infty} n^2 q^{2n} \|\exp(tA)e_n\|^2 \geq \sum_{n=1}^{\infty} n^2 q^{2n} \left( \frac{1}{n^2} \sum_{i=0}^{n-1} \frac{1}{i!} (2t)^i \right) \\
&= \sum_{i=0}^{\infty} \sum_{n=i+1}^{\infty} \left( q^{2n} \frac{1}{i!} (2t)^i \right) = \sum_{i=0}^{\infty} q^{2i+2} (1-q^2)^{-1} \frac{1}{i!} (2t)^i \\
&= \frac{q^2}{1-q^2} \sum_{i=0}^{\infty} \frac{1}{i!} (2tq^2)^i = \frac{q^2}{1-q^2} e^{2tq^2}.
\end{aligned}$$

It follows that  $\omega(x_q) \geq q^2$ . Thus,

$$1 = \sup\{\omega(x_q) : 0 < q < 1\} \leq \omega_1(A) \leq \omega_0(A) = 1.$$

Rescaling the semigroup (i.e., looking at  $e^{-3/2 \cdot t} T(t)$ ), we obtain a semigroup generator  $A$  on the Hilbert space  $E$  with  $-3/2 = s(A)$  and  $\omega_1(A) = \omega_0(A) = -1/2$ . On the other hand, Example 1.2(c) produces a semigroup in a Banach space  $F$  with generator  $B$  such that  $-1 = s(B) = \omega_1(B)$  while  $\omega_0(B) = 0$ . Now the operator  $C := A \oplus B$  on the Banach space  $E \oplus F$  is a semigroup generator for which

$$\begin{aligned}
s(C) &= \max\{s(A), s(B)\} = -1, \quad \omega_1(C) = \max\{\omega_1(A), \omega_1(B)\} = -1/2 \\
&\text{and} \quad \omega_0(C) = \max\{\omega_0(A), \omega_0(B)\} = 0.
\end{aligned}$$

**Remark 1.7** For eventually norm continuous semigroups—particularly compact, differentiable, holomorphic, or nilpotent ones—the spectral mapping theorem

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \quad (1.7)$$

holds. Consequently,

$$s(A) = \omega_1(A) = \omega_0(A) \quad (1.8)$$

is valid (Corollary 1.5 and A-III, Corollary 6.7). Hence, if  $A$  is the generator of an eventually norm-continuous semigroup, the exponential growth bounds of the strong and mild solutions of the abstract Cauchy problem  $\dot{u}(t) = Au(t)$ ,  $u(0) = x$  are determined by the spectral bound  $s(A) = \sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\}$ .

**Remark 1.8** In general, the growth bound  $\omega_0(A)$  can be obtained using the Hille-Yosida theorem (see A-II, Theorem 1.7) as

$$\begin{aligned}
\omega_0(A) &= \inf\{w : \|R(\lambda, A)^n\| \leq M(\operatorname{Re}(\lambda) - w)^{-n} \text{ for some } M \text{ and} \\
&\quad \text{every } n \in \mathbb{N} \text{ and } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}(\lambda) > w\}.
\end{aligned} \quad (1.9)$$

Due to the difficulty of estimating all powers of the resolvent, this characterization is of limited practical use. However, if  $A$  is the generator of a semigroup on a Hilbert space  $H$ , then it is shown in A-III, Corollary 7.11 that

$$\omega_0(A) = \inf\{w : \|R(\lambda, A)\| \leq M_w \text{ for } \operatorname{Re}(\lambda) > w\}. \quad (1.10)$$

Unfortunately, the identity (1.10) does not hold on arbitrary Banach spaces. However, as we will see in Section 1 of C-IV, the identity

$$s(A) = \omega_1(A) = \inf\{w: \|R(\lambda, A)\| \leq M_w \text{ for } \operatorname{Re}(\lambda) > w\} \quad (1.11)$$

holds for every positive semigroup on a Banach lattice. Consequently, for positive semigroups with  $s(A) = \omega_1(A) < \omega_0(A)$  (see Example 1.2 (ii)), the identity (1.10) is not applicable. Nevertheless, we can establish the following theorem.

**Theorem 1.9** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . Suppose there exist constants  $a \geq 0$  and  $q \geq s(A)$ , and that there exist  $C > 0$  and  $n \in \mathbb{N}$  such that*

$$\|R(\lambda, A)\| \leq C|\lambda|^{n-2}$$

*for all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > q$  and  $|\operatorname{Im}(\lambda)| > a$ . Then*

$$\sup\{\omega(f), f \in D(A^n)\} \leq q.$$

**Proof** The hypothesis  $\|R(\lambda, A)\| \leq C|\lambda|^{n-2}$  is invariant under rescaling. That is, the resolvent  $R(\lambda, -b + A)$  of the generator  $-b + A$  of the rescaled semigroup  $e^{-bt}T(t)$  satisfies  $\|R(\lambda, -b + A)\| \leq \tilde{C}|\lambda|^{n-2}$  for every  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > q - b$  and  $|\operatorname{Im}(\lambda)| > a + 2b$ , for a suitable constant  $\tilde{C}$ . Therefore, we may assume, without loss of generality, that  $b := \max(\omega_0(A), q) < 0$ . Let  $\omega_0(A) < p < 0$ , and set  $p' := \max\{p, q\} < 0$ . Then, for every  $f \in D(A)$ , the inversion formula for the Laplace transform yields

$$T(t)f = \frac{1}{2\pi i} \int_{p'-i\infty}^{p'+i\infty} e^{\lambda t} R(\lambda, A) f d\lambda. \quad (1.12)$$

(For a proof of the vector-valued inversion formula, see Widder [24, p.66]; also refer to the notes in this section.) Using the resolvent equation, we obtain

$$R(\lambda, A)^n R(0, A) = \sum_{k=1}^n (-1)^{k+1} \lambda^{-k} R(0, A)^{n+1-k} + (-1)^n \lambda^{-n} R(\lambda, A).$$

Since  $\frac{1}{2\pi i} \int_{p'-i\infty}^{p'+i\infty} e^{\lambda t} \lambda^{-k} d\lambda = 0$  for  $k \geq 1$ ,  $p' < 0$  and  $t > 0$ , it follows that

$$T(t)R(0, A)^n f = (-1)^n \frac{1}{2\pi i} \int_{p'-i\infty}^{p'+i\infty} e^{\lambda t} \lambda^{-n} R(\lambda, A) f d\lambda \quad (1.13)$$

for every  $f \in E$  and  $t > 0$ .

If  $q < p'$ , then by Cauchy's Integral Theorem and the growth bound  $\|R(\lambda, A)\| \leq C|\lambda|^{n-2}$ , we can shift the path of integration to  $\operatorname{Re}(\lambda) = q$ , yielding

$$T(t)R(0, A)^n f = (-1)^n \frac{1}{2\pi i} \int_{q-i\infty}^{q+i\infty} e^{st} \lambda^{-n} R(\lambda, A) f d\lambda.$$

Thus, we estimate

$$\|T(t)R(0, A)^n f\| \leq c' e^{qt} \|f\| \int_{-\infty}^{\infty} (q^2 + s^2)^{-1} ds = M e^{qt} \|f\|.$$

Equivalently,

$$\|T(t)f\| \leq M e^{qt} \|A^n f\| \text{ for } f \in D(A^n).$$

In view of the characterizations given in Section 1 of A-II, the semigroups considered in the theorem above are holomorphic if  $n = 1$ . In this case, one may apply (1.7) to obtain the stronger statement (1.8).

Rather than imposing conditions on the resolvent of  $A$ , we now adopt a different perspective and characterize the property “ $\omega_0(A) < 0$ ” directly in terms of the semigroup  $(T(t))_{t \geq 0}$ .

**Proposition 1.10** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . Then the following statements are equivalent.*

- (a)  $\omega_0(A) < 0$ .
- (b)  $\lim_{t \rightarrow \infty} \|T(t)\| = 0$ .
- (c)  $\|T(t')\| < 1$  for some  $t' > 0$ .

**Proof** The implications (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c) are immediate; To prove (c)  $\Rightarrow$  (a), note that  $\omega_0(A) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|T(t)\|$  (see A-I, (1.1)). Assume  $\|T(t')\| < 1$  for some  $t' > 0$ . For  $t = nt' + s$  with  $s \in [0, t']$ , we have  $\|T(t)\| \leq \|T(t')\|^n \|T(s)\|$ , so that

$$\frac{\log \|T(t)\|}{t} \leq \frac{n \log \|T(t')\|}{nt' + s} + \frac{\log \|T(s)\|}{nt' + s}.$$

Since  $\|T(s)\|$  is bounded on compact intervals, the second term tends to zero as  $n \rightarrow \infty$ . The first term tends to  $\frac{\log \|T(t')\|}{t'} < 0$ . Thus,  $\omega_0(A) < 0$ , which proves (c)  $\Rightarrow$  (a).  $\square$

Other less obvious characterizations of the property “ $\omega_0(A) < 0$ ” are provided in the following theorem. The equivalence of (a) and (c) is known as *Datko's Theorem*.

**Theorem 1.11** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . Then the following statements are equivalent.*

- (a)  $\omega_0(A) < 0$ .
- (b)  $s(A) < 0$  and there is  $t_0 > 0$  such that  $|\lambda| < 1$  for every  $\lambda \in A\sigma(T(t_0))$ .

(c) For some (equivalently, every)  $p \geq 1$  the integral  $\int_0^\infty \|T(t)f\|^p dt$  exists for every  $f \in E$ .

**Proof** The implication “(a)  $\Rightarrow$  (b)” follows from the fact that  $r(T(t)) = e^{\omega_0(A)t} < 1$  and  $s(A) \leq \omega_0(A) < 0$ . For the point and residual spectrum, the spectral mapping theorem is valid (see A-III, Theorem 6.3). Since the approximate point spectrum is closed, the additional assumption in (b) implies that  $|\lambda| \leq r < 1$  for all  $\lambda \in A\sigma(T(t_0))$ . Consequently,

$$\exp(\omega_0(A) \cdot t_0) = r(T(t_0)) \leq \max\{\exp(t_0 \cdot s(A)), r\} < 1,$$

which implies  $\omega_0(A) < 0$ . This proves “(b)  $\Rightarrow$  (a)”. For a proof of the equivalence of (a) and (c), we refer to Datko [4] or Pazy [17, Theorem 4.4.1].  $\square$

By rescaling a given semigroup  $(T(t))_{t \geq 0}$ , one obtains the following corollary from (1.1) and statement (c) of the preceding theorem.

**Corollary 1.12** *Let  $(T(t))_{t \geq 0}$  be a strongly continuous semigroup on a Banach space  $E$ . Then the set of  $\lambda \in \mathbb{C}$  for which*

$$\int_0^\infty \|e^{-\lambda t} T(t)f\| dt$$

*exists for every  $f \in E$  is an open right half-plane.*

In the next theorem, we present two necessary conditions for the stability of the semigroup  $(T(t))_{t \geq 0}$  in terms of its generator  $A$ .

We will see in Chapter C-IV that for positive semigroups a condition similar to statement (ii) below is even sufficient for stability.

It is important to note that stable semigroups need not be uniformly bounded (see Example 1.2(c)) and that the equality  $s(A) = \omega_0(A) = 0$  does not imply boundedness or even stability of the semigroup (see also A-I, Example 1.4.(i)).

**Theorem 1.13** *Let  $A$  be the generator of a stable semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . Then the following assertions hold.*

- (i)  $s(A) \leq 0$  and  $\operatorname{Re}(\lambda) < 0$  for every  $\lambda \in P\sigma(A) \cup R\sigma(A)$ .
- (ii)  $\lim_{\lambda \downarrow 0} \lambda R(\lambda, A)f$  exists for every  $f \in D(A)$ .

**Proof** (i) If  $(T(t))_{t \geq 0}$  is stable, then  $\|T(t)f\| \leq M_f$  for every  $f \in D(A)$ . Therefore,  $s(A) \leq \omega_1(A) \leq 0$ .

Assume, by contradiction, that there exists  $\lambda \in P\sigma(A)$  with  $\operatorname{Re}(\lambda) = 0$ . Then, by A-III, Corollary 6.4, there exists  $g \neq 0$  such that  $T(t)g = e^{\lambda t}g$  for all  $t \geq 0$ . Since  $|e^{\lambda t}| = 1$ , this contradicts the stability of the semigroup.

Now assume there exists  $\lambda \in R\sigma(A) = P\sigma(A')$  with  $\operatorname{Re}(\lambda) = 0$ . Then there exists  $0 \neq \varphi \in E'$  with  $T(t)^*\varphi = \exp(\lambda t)\varphi$  for all  $t \geq 0$ . Choose  $f \in D(A)$  such that

$\langle f, \varphi \rangle \neq 0$ . Then  $|\langle T(t)f, \varphi \rangle| = |\langle f, \varphi \rangle| > 0$  for every  $t \geq 0$ , which again contradicts the stability of the semigroup.

(ii) From the stability of the semigroup and the identity  $\int_0^t T(s)Af \, ds = T(t)f - f$ , it follows that  $\int_0^\infty T(s)Af \, ds$  exists for every  $f \in D(A)$ . Since  $\omega_1(A) \leq 0$ , we may apply Theorem 1.4 to write  $R(\lambda, A)Af = \int_0^\infty e^{-\lambda s} T(s)Af \, ds$  for every  $\lambda > 0$ . By a classical theorem of Laplace transform theory, (for a proof of the vector-valued version one may follow Widder [25, p.196]), we conclude that  $\lim_{\lambda \rightarrow 0^+} R(\lambda, A)Af$  exists and is equal to  $\int_0^\infty T(s)Af \, ds$ . The identity  $R(\lambda, A)Af = \lambda R(\lambda, A)f - f$  yields the existence of  $\lim_{\lambda \rightarrow 0^+} \lambda R(\lambda, A)f$  for every  $f \in D(A)$ .  $\square$

Bounded holomorphic semigroups (see A-II, Definition 1.11) satisfy the estimate

$$\|AT(t)\| \leq \frac{m}{t}$$

(see Goldstein [9, p.33]), and hence  $T(t)f \rightarrow 0$  as  $t \rightarrow \infty$  for every  $f \in \text{im}(A)$ . If  $\text{im}(A)$  is dense (i.e.,  $0 \notin R\sigma(A)$ ), then we obtain uniform stability and the following corollary.

**Corollary 1.14** *Let  $A$  be the generator of a bounded holomorphic semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . Then the following statements are equivalent.*

- (a)  $0 \notin P\sigma(A) \cup R\sigma(A)$ .
- (b)  $(T(t))_{t \geq 0}$  is uniformly stable.

**Example 1.15** The Laplacian  $\Delta$  generates a bounded holomorphic semigroups on  $L^p(\mathbb{R}^n)$  for  $1 \leq p < \infty$  (see the example proceeding Corollary 1.13 in Chapter A-II). All solutions of the equation  $\Delta f = 0$  are either constant or unbounded, so  $0 \notin P\sigma(\Delta)$ . If  $1 < p < \infty$ , then the adjoint of the Laplacian on  $L^p(\mathbb{R}^n)$  is the Laplacian on  $L^q(\mathbb{R}^n)$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ . Therefore,  $0 \notin P\sigma(\Delta) \cup R\sigma(\Delta)$ , and by Corollary 1.14, the Laplacian generates a uniformly stable semigroup on  $L^p(\mathbb{R}^n)$  for  $1 < p < \infty$ . However, since  $\text{im}(\Delta) \neq L^p(\mathbb{R}^n)$ , Corollary 1.5 implies that the semigroup is not exponentially stable.

As seen in Theorem 1.4, exponential stability can be characterized by the condition that the abscissa of convergence of the Laplace transform of  $(T(t))_{t \geq 0}$  is less than zero. This should be compared to the following result, which characterizes (uniform) stability in terms of integrability and the kernel of the generator.

**Theorem 1.16** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . The following assertions are equivalent.*

- (a)  $(T(t))_{t \geq 0}$  is stable.
- (b)  $\ker(A) = \{0\}$  and  $\int_0^\infty T(t)f \, dt$  exists for all  $f \in \text{im}(A)$ .

Furthermore the following statements are equivalent,

(a')  $(T(t))_{t \geq 0}$  is stable and bounded.

(b')  $(T(t))_{t \geq 0}$  is uniformly stable.

(c')  $(T(t))_{t \geq 0}$  is bounded and there is a dense subspace  $D$  such that  $\int_0^\infty T(t)f \, dt$  exists for every  $f \in D$ .

**Proof** If  $(T(t))_{t \geq 0}$  is stable, then by Theorem 1.13,  $\ker(A) = \{0\}$ , and

$$\int_0^t T(s)Af \, ds = T(t)f - f \rightarrow -f \text{ as } t \rightarrow \infty.$$

Hence, (a)  $\Rightarrow$  (b).

Conversely, suppose  $\int_0^t T(s)Af \, ds$  converges as  $t \rightarrow \infty$ . Then, by the identity above, the limit  $g := \lim_{t \rightarrow \infty} T(t)f$  exists. Since  $\ker(A) = \{0\}$ , it follows that  $g = 0$ , so  $T(t)f \rightarrow 0$ . Thus, (b)  $\Rightarrow$  (a).

The implication (a')  $\Rightarrow$  (b') is immediate.

If  $T(t)f \rightarrow 0$  for every  $f \in E$ , then  $\|T(t)\| \leq M$ , and  $0 \notin R\sigma(A)$  by Theorem 1.13. Therefore,  $D := \text{im}(A)$  is dense in  $E$ , and for every  $f \in D$ , the integral  $\int_0^\infty T(t)f \, dt$  exists. This proves (b')  $\Rightarrow$  (c').

It remains to prove (c')  $\Rightarrow$  (a'). Define

$$G := \{h \in E : h = \int_0^\infty T(t)g \, dt \text{ for some } g \in D\}.$$

We claim that  $G$  is dense in  $E$ . Indeed, for any  $g \in D$  and  $s > 0$ ,  $g - T(s)g \in D$ . Define  $h_s = \frac{1}{s} \int_0^\infty T(t)(g - T(s)g) \, dt = \frac{1}{s} \int_0^s T(t)g \, dt$ . Then  $h_s \in G$ , and  $h_s \rightarrow g$  as  $s \rightarrow 0$ . Therefore,  $D \subset \overline{G}$  or  $\overline{G} = E$  since  $D$  is dense in  $E$ . Now, let  $h \in G$ , so  $h = \int_0^\infty T(t)g \, dt$  for some  $g \in D$ . Then  $T(t)h = T(t) \int_0^\infty T(s)g \, ds = \int_t^\infty T(s)g \, ds \rightarrow 0$  as  $t \rightarrow \infty$ . Since  $\|T(t)\| \leq M$ , it follows that  $T(t)f \rightarrow 0$  for every  $f \in E$ .  $\square$

**Remark 1.17** (i) If  $A$  is the generator of a stable semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ , then by the previous theorem,

$$\text{im}(A) \subset \{f \in E : \int_0^\infty T(t)f \, dt \text{ exists}\} =: H.$$

If  $g \in H$ , then  $\int_0^\infty T(t)g \, dt \in D(A)$  and  $A \int_0^\infty T(t)g \, dt = -g$ . Thus,  $g \in \text{im}(A)$ , and the dense subspace  $\text{im}(A)$  is given by

$$\text{im}(A) = \{f \in E : \int_0^\infty T(t)f \, dt \text{ exists}\} \quad (1.14)$$

whenever  $A$  generates a stable semigroup  $(T(t))_{t \geq 0}$ .

(ii) If  $\omega(f) < 0$  for every  $f \in D(A)$ , then  $(T(t))$  is stable, but might not be exponentially stable if  $0 = \omega_1(A) = \sup\{\omega(f) : f \in D(A)\}$ . In this case, one can

show—via a proof similar to that of Theorem 1.4—that the spectrum  $\sigma(A)$  must be contained in the open left half-plane, i.e.,  $\operatorname{Re}(\lambda) < 0$  for all  $\lambda \in \sigma(A)$ .

(iii) If one defines a semigroup  $(T(t))_{t \geq 0}$  to be weakly stable if  $\langle T(t)f, \varphi \rangle \rightarrow 0$  as  $t \rightarrow \infty$  for all  $f \in D(A)$  and  $\varphi \in E'$  or as weakly uniformly stable if the above holds for all  $f \in E$  and  $\varphi \in E'$ , then Theorems 1.13 and 1.16 can be reformulated in a weak form (i.e., replacing stable by weakly stable and *lim* by *weak-lim*). The proofs require only obvious modifications. If  $A$  has a compact resolvent, or if  $A$  generates a bounded holomorphic semigroup, then weak stability implies stability. In general, this implication fails; e.g., the translation semigroup on  $L^2(\mathbb{R})$  is weakly uniformly stable but not stable (see also B-IV, Example 1.2).

## 2 Stability: Inhomogeneous Case

Using the results of the previous section, we now investigate the long-term behavior of solutions to the inhomogeneous initial value problem

$$\dot{u}(t) = Au(t) + F(t) \quad , \quad u(0) = f, \quad (2.1)$$

where  $A$  generates a strongly continuous semigroup on a Banach space  $E$  and  $F(\cdot)$  is a locally integrable function from  $\mathbb{R}_+$  into  $E$ , referred to as the *forcing term*. A function  $u(\cdot)$  is called a (*strong*) *solution* of (2.1) if  $u(\cdot): \mathbb{R}_+ \rightarrow D(A)$ ,  $u(\cdot) \in C^1(\mathbb{R}_+, E)$ , and (2.1) holds for all  $t \geq 0$ . The assumption that  $A$  generates a semigroup  $(T(t))_{t \geq 0}$  guarantees the uniqueness of the solution to (2.1). If  $u(\cdot)$  is a solution, then for fixed  $t > 0$ , the function  $v(s) := T(t-s)u(s)$ ,  $0 \leq s \leq t$ , is differentiable and  $v'(s) = T(t-s)F(s)$ . Since  $F(\cdot)$  is locally integrable, it follows that

$$\int_0^t T(t-s)F(s) \, ds = v(t) - v(0) = u(t) - T(t)f.$$

Thus, the solution  $u(t)$  of (2.1) is given by

$$u(t) = T(t)f + \int_0^t T(t-s)F(s) \, ds. \quad (2.2)$$

**Example** Let  $(T(t))_{t \geq 0}$  be a semigroup that is not eventually differentiable. Then there exists  $g \in E$  such that  $t \mapsto T(t)g$  is not differentiable on  $(0, \infty)$ . Consider the initial value problem  $\dot{u}(t) = Au(t) + T(t)g$ ,  $u(0) = 0$ . This problem has no (strong) solution  $u(\cdot)$ , because otherwise we would have

$$u(t) = \int_0^t T(t-s)T(s)g \, ds = tT(t)g,$$

which would imply that  $t \mapsto T(t)g$  is differentiable on  $\mathbb{R}_+$ , a contradiction.

Whenever the expression (2.2) is well-defined, we refer to it as a *generalized* (or *mild*) solution of (2.1). If  $F(\cdot)$  is continuous and  $f \in D(A)$ , then the generalized solution of (2.1) is a strong solution if and only if the function  $v(t) := \int_0^t T(t-s)F(s) ds$  is differentiable (see Pazy [17, Chapter 4,2.4]. There are several sufficient conditions on the generator  $A$ , the forcing term  $F(\cdot)$ , or the space  $E$  under which every mild solution is a strong solution of (2.1) (see Travis [21] or Pazy [17, Section 4.2]).

Our aim in this section is to study the asymptotic behavior of the solutions of (2.1) as  $t \rightarrow \infty$ . To that end, we consider absolutely integrable or periodic forcing terms  $F(\cdot)$ , and assume that the semigroup is uniformly stable.

Similar results for integrable and convergent forcing terms  $F(\cdot)$  can be obtained under the assumption of uniform stability (see Pazy [17, p.119] or Neubrander [16]). However, if the semigroup is positive, these results remain valid even under the weaker assumption of stability (see Section C-IV). Recall from Theorem 1.13 (i) that for stable semigroups, the range  $\text{im}(A)$  is dense in  $E$ .

**Theorem 2.1** *Let  $A$  be the generator of a uniformly stable semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . If there exists  $g \in \text{im}(A)$  such that  $\int_0^\infty \|F(s) - g\| ds < \infty$ , then every generalized solution  $u(\cdot)$  of (2.1) converges as  $t \rightarrow \infty$ , and  $\lim_{t \rightarrow \infty} u(t) = -h$ , where  $h \in D(A)$  satisfies  $Ah = g$ .*

**Proof** If  $u(\cdot)$  is a generalized solution of (2.1), then by (2.2), we have

$$u(t) = T(t)f + \int_0^t T(s)g ds + \int_0^t T(t-s)(F(s) - g) ds.$$

By uniform stability and the identity  $\int_0^t T(s)Ah ds = T(t)h - h$  (see A-I, Proposition 1.6), the first term converges to zero and the second term converges to  $-h$ . It remains to show that the third term also converges to zero. Let  $\varepsilon > 0$  and define  $G(s) := F(s) - g$ . Then for any  $r > 0$ ,

$$\begin{aligned} \left\| \int_0^t T(t-s)G(s) ds \right\| &\leq \left\| \int_0^r T(t-r+r-s)G(s) ds \right\| + \left\| \int_r^t T(t-s)G(s) ds \right\| \\ &\leq \left\| T(t-r) \int_0^r T(r-s)G(s) ds \right\| + M \int_r^\infty \|G(s)\| ds, \end{aligned}$$

where  $\|T(t)\| \leq M$  for all  $t \geq 0$ . Since the semigroup is uniformly stable, we obtain  $T(t-r) \int_0^r T(r-s)G(s) ds \rightarrow 0$  as  $t \rightarrow \infty$  for every  $r \geq 0$ . Therefore,  $\left\| \int_0^t T(t-s)G(s) ds \right\| \leq \varepsilon$  for all sufficiently large  $t$ .  $\square$

In the following theorem, we show that if  $A$  generates a uniformly stable semigroup, the forcing term  $F(\cdot)$  is  $p$ -periodic and  $\int_0^p T(p-s)F(s) ds \in \text{im}(\text{Id} - T(p))$ , then (2.1) admits a unique  $p$ -periodic mild solution that is *asymptotically stable*; i.e., for every generalized solution  $v(\cdot)$  of (2.1),

$$\lim_{t \rightarrow \infty} \|v(t) - u(t)\| = 0.$$

(Notice that, by Theorem 1.13 and A-III, Lemma 5.3,  $\overline{\text{im}((\text{Id} - T(p)))} = E$ .)

**Lemma 2.2** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ , and let  $F(\cdot)$  be a  $p$ -periodic, locally integrable function with  $p > 0$ . Then the following statements are equivalent.*

- (a)  $\dot{u}(t) = Au(t) + F(t)$  admits a (unique) generalized  $p$ -periodic solution.
- (b) There exists a (unique)  $f \in E$  such that  $(\text{Id} - T(p))f = \int_0^p T(p-s)F(s) ds$ .

**Proof** (a)  $\Rightarrow$  (b): Let  $f := u(0)$  be the initial value for which (2.1) admits a  $p$ -periodic mild solution. Then for every  $t \geq 0$ ,

$$\begin{aligned} u(t) &= u(t+p) \\ &= T(t)T(p)f + \int_0^p T(t+p-s)F(s) ds + \int_p^{t+p} T(t+p-s)F(s) ds \\ &= T(t) \left[ T(p)f + \int_0^p T(p-s)F(s) ds \right] + \int_0^t T(t-s)F(s) ds. \end{aligned}$$

Since  $u(t) = T(t)f + \int_0^t T(t-s)F(s) ds$ , it follows that

$$T(t)f = T(t) \left[ T(p)f + \int_0^p T(p-s)F(s) ds \right].$$

This implies  $f = T(p)f + \int_0^p T(p-s)F(s) ds$ . If  $u(\cdot)$  is a unique periodic solution with  $u(0) = f$ , then  $f$  is the unique element in  $E$  for which this identity holds.

(b)  $\Rightarrow$  (a): Define  $u(\cdot)$  as in (2.2). Then

$$u(t+p) = T(t) \left[ T(p)f + \int_0^p T(p-s)F(s) ds \right] + \int_0^t T(t-s)F(s) ds = u(t).$$

If  $f$  is unique, then, by the above considerations, the solution is also unique.  $\square$

**Remark 2.3** Let  $A$  be the generator of a strongly continuous semigroup for which the spectral mapping theorem holds (see A-III, Section 6), and let  $F$  be a  $p$ -periodic forcing term. If  $\frac{2\pi in}{p} \in \varrho(A)$  for every  $n \in \mathbb{Z}$ , then, by Lemma 2.2, (2.1) admits a unique  $p$ -periodic solution with initial value  $(\text{Id} - T(p))^{-1} \left( \int_0^p T(p-s)F(s) ds \right)$ .

As a consequence of Theorem 1.13 and A-III, Corollary 6.4, a uniformly stable semigroup admits at most one  $f \in E$  satisfying

$$(\text{Id} - T(p))f = \int_0^p T(p-s)F(s) ds.$$

This fact, together with Lemma 2.2, is used to prove the following theorem.

**Theorem 2.4** Let  $A$  be the generator of a uniformly stable semigroup  $(T(t))_{t \geq 0}$  and let  $F(\cdot)$  be a  $p$ -periodic, locally integrable function such that

$$(\text{Id} - T(p))f = \int_0^p T(p-s)F(s) \, ds \text{ for some } f \in E.$$

Then the unique  $p$ -periodic generalized solution

$$u(t) = T(t)f + \int_0^t T(t-s)F(s) \, ds$$

is asymptotically stable.

**Example 2.5** Let  $E$  be the Banach space  $C_0(\mathbb{R}_+)$  of continuous functions vanishing at infinity. Define  $A = \frac{d}{dx}$  with domain  $D(A) = \{f \in E : f' \in C^1 \text{ and } f' \in E\}$  is the generator of the uniformly stable translation semigroup  $T(t)f(x) := f(t+x)$ . Applying (1.14), we obtain  $\text{im}(A) = \{f : \int_0^\infty f(x) \, dx \text{ exists}\}$  is dense in  $C_0(\mathbb{R}_+)$ . Let  $r \in \text{im}(A)$  and let  $F(\cdot)$  be a  $p$ -periodic, real-valued function. We apply Theorem 2.4 to the initial value problem

$$\frac{d}{dt}u(t, x) = \frac{d}{dx}u(t, x) + r(x)F(x+t), \quad u(0, \cdot) \in D(A). \quad (*)$$

We rewrite  $(*)$  as

$$\dot{v}(t) = Av(t) + G(t), \quad (**)$$

where  $v(t) = u(t, \cdot)$  and  $G : \mathbb{R}_+ \rightarrow E$  is defined by  $G(t)(x) = r(x)F(x+t)$ . Then  $G$  is  $p$ -periodic with values in  $E$  and  $h_0 := \int_0^p T(p-t)G(t) \, dt$  is the function  $x \mapsto \left[ \int_0^p T(p-t)G(t) \, dt \right](x) = F(x) \int_x^{x+p} r(s) \, ds$ . Now define  $f := \sum_{k=0}^\infty T(kp)h_0$ , which corresponds to the function  $x \mapsto F(x) \int_x^\infty r(s) \, ds$ . It is then clear that  $(\text{Id} - T(p))f = h_0$ . Therefore,  $(**)$  admits a unique  $p$ -periodic generalized solution by Theorem 2.4, even though  $i\mathbb{R} \subset \sigma(A)$  (cf. Remark 2.3).

The unique  $p$ -periodic generalized solution  $u(t, \cdot)$  is given explicitly by

$$u(t, x) = F(x+t) \int_{x+t}^\infty r(s) \, ds + F(x+t) \int_x^{x+t} r(s) \, ds = F(x+t) \int_x^\infty r(s) \, ds.$$

Finally, for every solution  $v(t, \cdot)$  of  $(*)$ , Theorem 2.4 implies that

$$\sup \left\{ \left| v(t, x) - F(x+t) \int_x^\infty r(s) \, ds \right| : x \in \mathbb{R}_+ \right\} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

## Notes

*Section 1:* The exponential growth bounds  $\omega(f)$  and  $\omega_0(A)$  as well as the characterizations (1.2), (1.6) and Theorem 1.3 (i) can be found in Hille and Phillips [11]. Growth bounds similar to  $\omega_1(A)$  were first considered in D’Jacenko [6] and in Zabczyk [28, Proposition 2]. Example 1.2(ii) is taken from Wolff [26]; other *counterexamples* can be found in Hille and Phillips [11], Foias [8], Triggiani [22], Zabczyk [27] and Greiner et al. [10]. Statements (1.2), (1.6) and Theorem 1.3 (i) are semigroup versions of results in classical Laplace transform theory, see Hille and Phillips [11] and Widder [24]. Theorem 1.3 (ii) is a semigroup version of Theorem 1.2.7 and 1.2.8 in Doetsch [7]. The lemma in the proof of Theorem 1.3 is taken from Mil’stein [13]. Theorem 1.4 and Corollary 1.5 can be found in Neubrandner [15]. Example 1.6 follows Remark 2 in Zabczyk [27]. Statement (1.8) is sometimes called the *spectrum determined growth assumption*, see, for example, Triggiani [23]. Theorem 1.9 is due to Slemrod [20]. The proof presented here is based on the following sharper version of the inversion formula for the Laplace transform, which improves on the one given in Hille and Phillips [11, p.349]. Using Widder [24, p.66] or Doetsch [7, p.212] one can establish the following theorem (see Neubrandner [14]).

**Theorem 2.6** *Let  $A$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$ . For every  $f \in D(A)$  and  $p > \omega_1(A)$  we have*

$$T(t)f = \frac{1}{2\pi i} \int_{p-i\infty}^{p+i\infty} e^{\mu t} R(\mu, A) f d\mu.$$

The equivalence of the statements (1.12), (1.13) and  $\omega_0(A) < 0$  were observed by many authors, see for example, Balakrishnan [1, p.178], or Benchimol [2]. Theorem 1.11 is due to Datko [3]; for a proof see Pazy [17, p.116]. Theorems 1.13 and 1.16 can be found in Neubrandner [16] and Corollary 1.14 is due to Komatsu [12]. An example of an unstable semigroup generator  $A$  with  $\operatorname{Re}(\mu) < 0$  for all  $\mu \in \sigma(A)$  is given in Datko [5].

*Section 2:* For a discussion of well-posedness of inhomogeneous Cauchy problems, we refer to Goldstein [9, p.83], and Pazy [17, p.105]. Further results on the asymptotic behavior of the solutions of the inhomogeneous problem can be found in Rao and Hengartner [19], Zaidman [29], Pazy [17], and Neubrandner [16]. Results similar to Lemma 2.2 and Theorem 2.4 are due to Prüss [18]. For a discussion of the asymptotic behavior of the solutions of  $\dot{u}(t) = A(t)u(t) + F(t)$  see Datko [4] and Pazy [17, p.172].

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## **Part B**

### **Positive Semigroups on Spaces $C_0(X)$**



## Chapter B-I

### Basic Results of Spaces $C_0(X)$

by

Rainer Nagel and Ulf Schlotterbeck

This part of the book is devoted to one-parameter semigroups of operators on spaces of continuous functions of type  $C_0(X)$ . These spaces are Banach lattices of a particularly special kind. We treat this case separately because mixing it with the abstract Banach lattice theory considered in Part C would obscure the accessibility and exemplary power of the methods and results presented here. In this chapter we fix the notation we are going to use and collect some basic facts about the spaces we are considering.

If  $X$  is a locally compact space, then  $C_0(X)$  denotes the space of all continuous complex-valued functions on  $X$  that vanish at infinity, endowed with the supremum norm. If  $X$  is compact, then any continuous function on  $X$  vanishes at infinity, and  $C_0(X)$  is the space of all continuous functions on  $X$ . We often write  $C(X)$  instead of  $C_0(X)$  in this case.

We sometimes study real-valued functions and denote the corresponding spaces as  $C_0(X, \mathbb{R})$  and  $C(X, \mathbb{R})$ ; we use the notations  $C_0(X, \mathbb{C})$  and  $C(X, \mathbb{C})$  when we need to distinguish between the two cases.

#### 1 Algebraic and Order-Structure: Ideals and Quotients

Any space  $C_0(X)$  is a commutative  $C^*$ -algebra under the sup-norm and the pointwise multiplication, and by the *Gelfand Representation Theorem* any commutative  $C^*$ -algebra can, on the other hand, be canonically represented as an algebra  $C_0(X)$  on a suitable locally compact space  $X$ . The algebraic notions of ideal, quotient, homomorphism are well known and need not be explained further.

Another natural and important structure of  $C_0(X)$  is the *pointwise* ordering under which  $C_0(X, \mathbb{R})$  is a (real) Banach lattice and  $C_0(X, \mathbb{C})$  a complex Banach lattice in the sense explained in Chapter C-I. Concerning the order structure of  $C_0(X)$  we use the following notations. For a function  $f$  in  $C_0(X, \mathbb{R})$

- (i) A function  $f$  is called *positive*,  $f \geq 0$ , if  $f(t) \geq 0$  for all  $t \in X$ ,
- (ii) We write  $f > 0$  if  $f$  is positive but does not vanish identically,
- (iii) We call  $f$  *strictly positive*,  $f \gg 0$ , if  $f(t) > 0$  for all  $t \in X$ .

The notion of an order ideal explained in Chapter C-I applies of course to the Banach lattices  $C_0(X)$  and might give rise to confusion with the corresponding algebraic notion.

However, since we are mainly considering closed ideals and since a closed linear subspace  $I$  of  $C_0(X)$  is a lattice ideal if and only if  $I$  is an algebraic ideal, we may and will simply speak of closed ideals, without specifying whether we think of the algebraic or the order theoretic meaning of this word.

An important and frequently used characterization of these objects is the following: A subspace  $I$  of  $C_0(X)$  is a closed ideal if and only if there exists a closed subset  $A$  of  $X$  such that a function  $f$  belongs to  $I$  if and only if  $f$  vanishes on  $A$ . The set  $A$  is of course uniquely determined by  $I$  and is called the *support* of  $I$ . If  $I = I_A$  is a closed ideal with support  $A$ , then  $I_A$  is naturally isomorphic to  $C_0(X \setminus A)$  and the quotient  $C_0(X)/I$  (under the natural quotient structure) is again a Banach algebra and a Banach lattice that can be identified canonically (via the map  $f + I \rightarrow f|_A$ ) with  $C_0(A)$ .

## 2 Linear Forms and Duality

The *Riesz Representation Theorem* asserts that the dual of  $C_0(X)$  can be identified in a natural way with the space of bounded regular Borel measures on  $X$ . While there is no natural algebra structure on this dual, the dual ordering (see Chapter C-I) makes  $C_0(X)'$  into a Banach lattice. We will occasionally make use of the order structure of  $C_0(X)'$ , but since at least its measure theoretic interpretation is supposed to be well-known, it may suffice here to refer to Chapter C-I, Sections 3 and 7, for a more detailed discussion. We recall only some basic notations here.

If  $\mu$  is a linear form on  $C_0(X, \mathbb{R})$ , then

- (i)  $\mu \geq 0$  means  $\mu(f) \geq 0$  for all  $f \geq 0$ ;  $\mu$  is then called *positive*,
- (ii)  $\mu > 0$  means that  $\mu$  is positive but does not vanish identically,
- (iii)  $\mu \gg 0$  means that  $\mu(f) > 0$  for any  $f > 0$ ;  $\mu$  is then called *strictly positive*.

If  $\mu$  is a linear form on  $C_0(X, \mathbb{C})$ , then  $\mu$  can be written uniquely as

$$\mu = \mu_1 + i\mu_2$$

where  $\mu_1$  and  $\mu_2$  map  $C_0(X, \mathbb{R})$  into  $\mathbb{R}$  (decomposition into *real* and *imaginary parts*).

We call  $\mu$  *positive* (*strictly positive*) and use the above notations if  $\mu_2 = 0$  and  $\mu_1$  is positive (strictly positive). We point out that strictly positive linear forms need not exist on  $C_0(X)$ , but if  $X$  is separable, then a strictly positive linear form is obtained by summing a suitable sequence of point measures.

The coincidence of the notions of a closed algebraic and a closed lattice ideal in  $C_0(X)$  has of course its effect on the algebraic and the lattice theoretic notions of a homomorphism. The case of a homomorphism into another space  $C_0(Y)$  will be discussed below. As for homomorphisms into the scalar field, we have essentially coincidence between the algebraic and the order theoretic meaning of this word, more exactly:

*A linear form  $\mu \neq 0$  on  $C_0(X)$  is a lattice homomorphism if and only if  $\mu$  is, up to normalization, an algebra homomorphism (algebra homomorphisms  $\neq 0$  must necessarily have norm 1).*

Since the algebra homomorphisms  $\neq 0$  on  $C_0(X)$  are known to be the point measures (denoted by  $\delta_t$ ) on  $X$  and, since on the other hand,  $\mu$  is a lattice homomorphism if and only if

$$|\mu(f)| = \mu(|f|) \quad \text{for all } f,$$

it follows that this latter condition on  $\mu$  is equivalent to  $\mu = \alpha \delta_t$  for a suitable  $t$  in  $X$  and a positive real number  $\alpha$ .

This can be summarized by saying that  $X$  can be canonically identified, via the map  $t \rightarrow \delta_t$ , with the subset of the dual  $C_0(X)'$  consisting of the non-zero algebra homomorphisms, which is also the set of all normalized lattice homomorphisms. This identification is a topological isomorphism with respect to the weak\*-topology of  $C_0(X)'$ .

### 3 Linear Operators

A linear mapping  $T$  from  $C_0(X, \mathbb{R})$  into  $C_0(Y, \mathbb{R})$  is called

- (i) *positive* (notation:  $T \geq 0$ ) if  $Tf$  is positive whenever  $f$  is positive,
- (ii) *lattice homomorphism* if  $|Tf| = T|f|$  for all  $f$ ,
- (iii) *Markov-operator* if  $X$  and  $Y$  are compact and  $T$  is a positive operator mapping  $\mathbb{1}_X$  to  $\mathbb{1}_Y$ .

In the case of complex scalars,  $T$  can be decomposed into real and imaginary parts. We call  $T$  positive in this situation if the imaginary part of  $T$  is  $= 0$  and the real part

is positive. The terms *Markov operator* and *lattice homomorphism* are defined as above. Note that a complex lattice homomorphism is necessarily positive, and that the *complexification* of a real lattice homomorphism is a complex lattice homomorphism. Positive operators are always continuous.

Note that the adjoint of a Markov operator  $T$  maps positive normalized measures into positive normalized measures while the adjoint of an algebra homomorphism (lattice homomorphism) maps point measures into (multiples of) point measures. Therefore the adjoint of a Markov lattice homomorphism as well as the adjoint of an algebra homomorphism induces a continuous map  $\varphi$  from  $Y$  (viewed as a subset of the weak dual  $C(Y)'$ ) into  $X$  (viewed as a subset of  $C(X)'$ ).

This mapping  $\varphi$  determines  $T$  in a natural and unique way, so that the following are equivalent assertions on a linear mapping  $T$  from a space  $C(X)$  into a space  $C(Y)$ .

- (a)  $T$  is a Markov lattice homomorphism.
- (b)  $T$  is a Markov algebra homomorphism.
- (c) There exists a continuous map  $\varphi$  from  $Y$  into  $X$  such that  $Tf = f \circ \varphi$  for all  $f \in C(X)$ .

If  $T$  is, in addition, bijective, then the mapping  $\varphi$  in (c) is a homeomorphism from  $Y$  onto  $X$ . This characterization of homomorphisms carries over *mutatis mutandis* to situations where the conditions on  $X$ ,  $Y$  or  $T$  are less restrictive. For later reference we explicitly state the following.

- (i) Let  $K$  be compact. Then  $T \in \mathcal{L}(C(K))$  is a lattice homomorphism if and only if there is a mapping  $\varphi$  from  $K$  into  $K$  and a function  $h \in C(K)$  such that  $Tf(s) = h(s)f(\varphi(s))$  holds for all  $s \in K$ . The mapping  $\varphi$  is continuous in every point  $t$  with  $h(t) \neq 0$ .
- (ii) Let  $X$  be locally compact and  $T \in \mathcal{L}(C_0(X))$ . Then  $T$  is a lattice isomorphism if and only if there is a homeomorphism  $\varphi$  from  $X$  onto  $X$  and a bounded continuous function  $h$  on  $X$  such that  $h(s) \geq \delta > 0$  for all  $s$  and  $Tf(s) = h(s)f(\varphi(s))$  ( $s \in X$ ). Moreover,  $T$  is an algebraic  $*$ -isomorphism if and only if  $T$  is a lattice isomorphism and the function  $h$  above is  $\equiv \mathbb{1}$ .

## References

## Chapter B-II

# Characterization of Positive Semigroups on $C_0(X)$

by  
Wolfgang Arendt

It lies in the very nature of the theory of one-parameter semigroups that frequently an operator  $A$  is known to be a generator but the semigroup is not known explicitly. Thus, since the semigroup is uniquely determined by the generator, it is a central task in the theory to express properties of the semigroup in terms of its generator. In this chapter we do this for two properties. We characterize generators of positive semigroups and generators of lattice semigroups.

In Section 1 we consider a semigroup  $(T(t))_{t \geq 0}$  on the real space  $C(K)$  ( $K$  compact). It is shown that the semigroup consists of positive operators if and only if its generator satisfies a positive minimum principle (P). Even without assuming a priori that  $A$  is a generator the positive minimum principle has strong consequences. Together with a range condition it implies that  $A$  is a generator (of a positive semigroup). Moreover, we show that for a densely defined operator  $A$  to be the generator of a positive semigroup it is already sufficient that the resolvent  $R(\lambda, A)$  of  $A$  exists and is positive for all sufficiently large real  $\lambda$ . For all these results it is essential to assume that  $K$  is compact. Concerning the characterization of positive semigroups on  $C_0(X)$  ( $X$  locally compact, non-compact) we follow a completely different line and will treat this case in the context of general Banach lattices in Chapter C-II.

A special class of positive semigroups are lattice semigroups; i.e., semigroups of lattice homomorphisms. We show in Section 2 that  $(T(t))_{t \geq 0}$  is a lattice semigroup if and only if its generator  $A$  satisfies an identity (K), the so-called Kato's Equality (Theorem ??).

We refer to Chapter C-II for a discussion of this identity and classical analogs for the Laplacian due to ? ].

After the abstract characterization in Section 2 we show that every continuous semiflow on  $X$  together with a cocycle defines a lattice semigroup in a canonical way, and on  $C(K)$ , every lattice semigroup can be so represented. This furnishes a wide class of examples. Furthermore, positive one-parameter groups on  $C_0(X)$  (which form a particular type of lattice semigroups) are discussed. Their generators are similar to a derivation perturbed by a multiplication operator (Section 3).

## 1 Generators of Positive Semigroups on $C_0(X)$

Let  $X$  be a locally compact space. Throughout this section we denote by  $C_0(X)$  the space of all real-valued continuous functions on  $C_0(X)$  which vanish at infinity. Recall that a semigroup  $(T(t))_{t \geq 0}$  on  $C_0(X)$  is called *positive* if  $T(t) \geq 0$  for all  $t \geq 0$ . It is easy to describe the positivity of  $(T(t))_{t \geq 0}$  in terms of the resolvent  $R(\lambda, A)$  of its generator  $A$  because of the close relation between these two objects. In fact, the resolvent is expressed by the semigroup by

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt \quad (\lambda > \omega_0(A)); \quad (1.1)$$

and conversely, the semigroup by the resolvent via the formula

$$T(t) = \lim_{n \rightarrow \infty} (n/t R(n/t, A))^n \quad \text{strongly} \quad (1.2)$$

(cf. A-II, Proposition 1.10). So we obtain the following.

**Proposition 1.1** *Let  $(T(t))_{t \geq 0}$  be a semigroup with generator  $A$ . The semigroup is positive if and only if  $R(\lambda, A) \geq 0$  for all sufficiently large real  $\lambda$ .*

It is more difficult and more interesting to characterize the positivity of the semigroup by intrinsic conditions on the generator. This is the purpose of this section. As a first orientation we consider bounded generators. We need the following lemma.

**Lemma 1.2** *Let  $X$  be a locally compact space,  $x \in X$  and  $\mu$  a regular bounded Borel measure on  $X$  such that  $\mu(\{x\}) = 0$ . Then  $\mu \geq 0$  if and only if  $\langle f, \mu \rangle \geq 0$  for all  $f \in C_0(X)_+$  satisfying  $f(x) = 0$ .*

We omit the easy proof.

**Theorem 1.3** *Let  $X$  be locally compact and  $A$  be a bounded operator on  $C_0(X)$ . The following assertions are equivalent.*

- (a)  $e^{tA} \geq 0$  ( $t \geq 0$ ).
- (b) For  $0 \leq f \in C_0(X)$  and  $x \in X$ ,  $f(x) = 0$  implies  $(Af)(x) \geq 0$ .

(c)  $A + \|A\| \text{Id} \geq 0$ .

**Proof** ?? implies ?? . Let  $f \in C_0(X)_+$  and  $x \in X$  such that  $f(x) = 0$ . Then

$$\begin{aligned} (Af)(x) &= \lim_{t \rightarrow 0} 1/t ((e^{tA}f(x) - f(x))) \\ &= \lim_{t \rightarrow 0} 1/t ((e^{tA}f(x))) \geq 0 \end{aligned}$$

?? implies ?? . Let  $x \in X$ . We have to show that  $(Af)(x) + \|A\|f(x) \geq 0$  for all  $f \in C_0(X)$ . Let  $A'\delta_x = \mu + c\delta_x$  where  $\mu \in M(X)$  such that  $\mu(\{x\}) = 0$  and  $c \in \mathbb{R}$ . We claim that  $\mu \geq 0$ . Let  $0 \leq f \in C_0(X)$  such that  $f(x) = 0$ . Then  $\langle f, \mu \rangle = \langle f, A'\delta_x \rangle = (Af)(x) \geq 0$  by ?? . Thus  $\mu \geq 0$  by Lemma ?? . Moreover,  $|c| = \|c\delta_x\| \leq \|c\delta_x + \mu\| = \|A'\delta_x\| \leq \|A\|$ . Hence, for  $f \in C_0(X)_+$ ,

$$(Af)(x) + \|A\|f(x) = \langle f, A'\delta_x + \|A\|\delta_x \rangle = \langle f, \mu + (c + \|A\|)\delta_x \rangle \geq 0.$$

This shows ?? to hold.

?? implies ?? . We have  $e^{tA} = e^{-t\|A\|}e^{t(A+\|A\|\text{Id})} \geq e^{-t\|A\|} \text{Id}$  for all  $t \geq 0$ . □

**Example 1.4** (i) Let  $B$  be a positive operator on  $C_0(X)$  and  $m: X \rightarrow \mathbb{R}$  be a continuous and bounded mapping. Let  $Af = Bf - m \cdot f$  ( $f \in C_0(X)$ ). Then  $e^{tA} \geq 0$  for all  $t \geq 0$ .

(ii) Let  $A$  be an  $n \times n$  - matrix. Then  $e^{tA} \geq 0$  for all  $t \geq 0$  if and only if  $a_{ij} \geq 0$  for  $i \neq j$ . This is the linear version of Kamke's theorem (see ? ).

Now we come to the actual subject of this section, the characterization of strongly continuous positive semigroups on  $C(K)$ . Here  $K$  denotes a compact space and  $C(K)$  the space of all real-valued continuous functions on  $K$ . It will be essential that  $K$  is compact for all what follows since it will be needed that the positive cone of  $C(K)$  has interior points.

We reformulate condition ?? of Theorem ?? for unbounded operators.

**Definition 1.5** An (unbounded) operator  $A$  on  $C(K)$  is said to satisfy the *positive minimum principle* if

(P) for every  $0 \leq f \in D(A)$  and  $x \in K$ ,  
 $f(x) = 0$  implies  $(Af)(x) \geq 0$

Our next theorem shows that the positive minimum principle characterizes the positivity of the semigroup; and in fact, the proof is very elementary. Using more involved arguments we will later prove a much stronger result (Theorem ?? ).

**Theorem 1.6** Let  $A$  be the generator of a strongly continuous semigroup on  $C(K)$ . Then the semigroup is positive if and only if the generator  $A$  satisfies the positive minimum principle (P).

**Proof** The necessity of the condition is proved as "?? implies ?? " in Theorem ?? . Assume that (P) holds. We claim that  $R(\lambda, A) \geq 0$  for sufficiently large real  $\lambda$ . (This implies the positivity of the semigroup by Proposition ??).

Let  $s := \inf\{\lambda \in \mathbb{R} : [\lambda, \infty) \subset \varrho(A)\}$ . Then  $s \leq \omega_0(A) < \infty$ .

Let  $0 \ll u \in C(K)$ . Then

$$\lambda_0 := \inf\{\lambda > s : R(\mu, A)u \gg 0 \text{ for all } \mu \in (\lambda, \infty)\} < \infty$$

since  $\lim_{\mu \rightarrow \infty} \mu R(\mu, A)u = u$ .

We claim that  $\lambda_0 = s$ .

In fact, if this is not true, then  $[\lambda_0, \infty) \subset \varrho(A)$  and  $R(\lambda_0, A)u \geq 0$  but  $R(\lambda_0, A)u$  is not strictly positive. Consequently there exists  $x \in K$  such that  $(R(\lambda_0, A)u)(x) = 0$ . Then (P) implies that  $A(R(\lambda_0, A)u)(x) \geq 0$ . Hence,

$$0 < u(x) = \lambda_0(R(\lambda_0, A)u)(x) - A(R(\lambda_0, A)u)(x) \leq 0,$$

a contradiction. We have shown that  $R(\lambda, A)u \gg 0$  for all  $u \gg 0$  and  $\lambda > s$ . Since  $\{u \in C(K) : u \gg 0\}$  is dense in  $C(K)_+$ , it follows that  $R(\lambda, A) \geq 0$  for all  $\lambda > s$ .  $\square$

**Remark 1.7** The proof of Theorem ?? shows that for the generator  $A$  of a positive semigroup on  $C(K)$ ,  $R(\lambda, A)u \gg 0$  whenever  $0 \ll u \in C(K)$  and  $[\lambda, \infty) \subset \varrho(A)$ . In particular,  $R(\lambda, A) \geq 0$  whenever  $[\lambda, \infty) \subset \varrho(A)$ .

If  $A$  is a generator, then the positivity of the resolvent  $R(\lambda, A)$  for large real  $\lambda$  implies the positivity of the semigroup (by Proposition. ??). On  $C(K)$  much more is true. Even if  $A$  is not supposed to be a generator, the existence and positivity of  $R(\lambda, A)$  for large real  $\lambda$  implies that  $A$  is a generator (of a positive semigroup). This is surprising, because it means that in the case when the resolvent is positive, the norm condition on the resolvent

$$\sup_{\lambda > \omega} \{ \|(\lambda - w)^n R(\lambda, A)^n\| : n \in \mathbb{N} \} < \infty$$

which appears in the Hille-Yosida Theorem (A-II, Theorem 1.7) is automatically fulfilled.

**Theorem 1.8** Let  $K$  be compact and  $A$  be a densely defined operator on  $C(K)$ . Suppose that there exists  $w \in \mathbb{R}$  such that  $[w, \infty) \subset \varrho(A)$  and  $R(\lambda, A) \geq 0$  for all  $\lambda \geq w$ . Then  $A$  is the generator of a strongly continuous positive semigroup. Moreover,

$$\omega_0(A) \leq w. \quad (1.3)$$

**Proof** (i) Assume that  $w < 0$ . Denote by 1 the constant-1-function. Let  $u = R(0, A)1$ . We claim that  $u \gg 0$ . If not, then there exists  $x \in K$  such that  $u(x) = 0$ . Let  $f \in C(K)$ . Then  $|f| \leq \|f\|1$ . Consequently,  $|R(0, A)f| \leq R(0, A)|f| \leq \|f\|R(0, A)1 = \|f\|u$ . Hence  $(R(0, A)f)(x) = 0$  for all  $f \in C(K)$ . Since  $D(A) = R(0, A)C(K)$ , it follows that  $D(A)$  is not dense, a contradiction. Define

$$\|f\|_0 = \inf\{\lambda > 0: |f| \leq \lambda u\} = \|f/u\|_\infty.$$

Then  $\|\cdot\|_0$  is an equivalent norm on  $C(K)$ . Moreover,  $\|f\|_0 \leq 1$  if and only if  $f \in [-u, u]$ . By the resolvent equation we have

$$\lambda R(\lambda, A)u = \lambda R(\lambda, A)R(0, A)1 = R(0, A)1 - R(\lambda, A)1 \leq R(0, A)1 = u$$

for all  $\lambda \geq 0$ . This implies that  $\lambda R(\lambda, A)$  is contractive for the norm  $\|\cdot\|_0$ . Thus by the Hille-Yosida Theorem  $A$  is the generator of a semigroup which is contractive with respect to the norm  $\|\cdot\|_0$  and so is bounded with respect to the supremum norm on  $C(K)$ .

(ii) If  $w$  is arbitrary, let  $\lambda > w$  and consider  $A - \lambda$ . Then  $[w - \lambda, \infty) \subset \varrho(A - \lambda)$  and  $R(\mu, A - \lambda) = R(\mu + \lambda, A) \geq 0$  for all  $\mu \in [w - \lambda, \infty)$ . Thus by a),  $A - \lambda$  is the generator of a bounded positive semigroup. Consequently,  $A$  is a generator as well and  $\omega_0(A) \leq \lambda$ .

In Theorem ?? it is enough to assume that  $R(\lambda_n, A) \geq 0$  for some sequence  $(\lambda_n) \subset \varrho(A) \cap \mathbb{R}$  satisfying  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ . This follows from the following lemma.

**Lemma 1.9** *Let  $B$  be an operator on  $C(K)$  (more generally, on a Banach lattice). If  $\mu_1, \mu_2 \in \varrho(B) \cap \mathbb{R}$  such that  $0 \leq R(\mu_1, B)$ ,  $0 \leq R(\mu_2, B)$  and  $\mu_1 < \mu_2$ , then  $[\mu_1, \mu_2] \subset \varrho(B)$  and*

$$0 \leq R(\mu_2, B) \leq R(\mu, B) \leq R(\mu_1, B) \quad \text{for all } \mu \in [\mu_1, \mu_2]$$

**Proof** Let

$$M := \{\mu \in \varrho(B) \cap [\mu_1, \mu_2]: [\mu, \mu_2] \subset \varrho(B) \text{ and } R(\lambda, B) \geq 0 \text{ for all } \lambda \in [\mu, \mu_2]\}$$

(i) The set  $M$  is open. In fact, let  $\mu \in M$ . Then for small  $h > 0$  one has  $R(\mu - h, B) = \sum_{n=0}^{\infty} h^n R(\mu, B)^{n+1} \geq 0$ .

(ii)  $M$  is closed. In fact, by the resolvent equation one has for  $\mu \in M$ ,

$$R(\mu_1, B) - R(\mu, B) = (\mu - \mu_1)R(\mu_1, B)R(\mu, B) \geq 0,$$

hence  $R(\mu, B) \leq R(\mu_1, B)$ . Consequently,

$$\text{dist}(\mu, \sigma(B)) \geq 1/\|R(\mu, B)\| \geq 1/\|R(\mu_1, B)\|$$

for all  $\mu \in M$ . This implies that  $M$  is closed.

The assertions (i) and (ii) imply that  $M = [\mu_1, \mu_2]$ . □

**Remark** (i) The lemma shows in particular that the resolvent of the generator  $A$  of a positive semigroup is decreasing on  $(s(A), \infty)$ .

(ii) There exists a linear operator  $B$  on  $\mathbb{R}^n$  such that  $R(\mu, B) \geq 0$  on some interval  $[\mu_1, \mu_2] \subset \varrho(B) \cap \mathbb{R}$  but  $(e^{tB})_{t \geq 0}$  is not positive (see ? ).

**Remark** Theorem ?? does not hold in  $C_0(X)$ , in general. In fact, for  $\alpha \in (0, 1)$ , the operator  $A$  on  $C_0(0, 1]$  given by

$$Af(x) = f'(x) + \frac{\alpha}{x}f(x) \quad (x \in (0, 1])$$

with domain

$$D(A) = \{f \in C^1[0, 1] : f'(0) = f(0) = 0\}$$

satisfies  $\varrho(A) = \mathbb{C}$  and  $R(\lambda, A) \geq 0$  for all  $\lambda \in \mathbb{R}$ . But  $A$  is not the generator of a semigroup (even if more general classes than  $C_0$ -semigroups are admitted). See ? ] for this example and a general theory of resolvent positive operators. Another example is given by ? ].

Next we investigate consequences of the positive minimum principle for a densely defined operator which is not a priori assumed to be a generator. For that we will make use of the theory of half-norms developed in A-II, Section 2.

For  $0 \ll u \in C(K)$  let

$$p_u(f) = \inf\{\lambda \in \mathbb{R}_+ : f \leq \lambda u\} = \sup_{x \in K} \frac{f^+(x)}{u(x)}. \quad (1.4)$$

Then  $p_u$  is a strict half-norm on  $C(K)$  (see A-II, Section 2). Note that

$$p_u(f)u - f \geq 0 \quad (f \in C(K)). \quad (1.5)$$

For  $x \in K$ , define  $\varphi_x \in C(K)'$  by  $\langle f, \varphi_x \rangle = f(x)/u(x)$ . Let  $f \in C(K)$  such that  $-f$  is not strictly positive. Then there exists  $x \in K$  such that  $f(x)/u(x) = p_u(f)$ . For such an  $x$  we have

$$\varphi_x \in \mathbf{d}p_u(f) \quad (1.6)$$

(see A-II, Section 2 for the definition of the subdifferential  $\mathbf{d}p_u$ ).

Note that for  $f \in C(K)$  one has  $f \geq 0$  if and only if  $p_u(-f) \leq 0$  (i.e., the half-norm  $p_u$  induces the given ordering on  $C(K)$  (cf. A-II, Remark 2.8)). As a consequence, every  $p_u$ -contractive bounded operator  $T$  on  $C(K)$  is positive.

**Proposition 1.10** *Let  $A$  be a densely defined operator on  $C(K)$ . Then there exists a strictly positive  $u \in D(A)$ . For any such  $u$  the following assertions are equivalent.*

- (a)  $A$  is  $p_u$ -dissipative.
- (b)  $Au \leq 0$  and  $A$  satisfies (P).

**Proof** Since  $\{u \in C(K) : u \gg 0\}$  is open and non-empty and  $D(A)$  is dense, there exists  $0 \ll u \in D(A)$ .

?? implies ??. One has  $p_u(u) = 1$ . Let  $x \in K$ . It follows from (??) that  $\varphi_x \in dp_u(u)$ . Since  $D(A)$  is dense, it follows from A-II, Theorem 2.7 that  $A$  is strictly  $p_u$ -dissipative. Hence  $\langle Au, \varphi_x \rangle \leq 0$ . Thus  $(Au)(x) \leq 0$ . We now show (P). Let  $0 \leq f \in D(A)$  and  $x \in K$  such that  $f(x) = 0$ . We have to show that  $(Af)(x) \geq 0$ . Since  $f(x) = 0$  and  $p_u(-f) = 0$  we have by (??)  $\varphi_x \in dp_u(-f)$ . Since  $A$  is strictly  $p_u$ -dissipative we conclude that  $-u(x)(Af)(x) = \langle A(-f), \varphi_x \rangle \leq 0$ . Hence  $(Af)(x) \geq 0$ .

?? implies ??. Let  $f \in D(A)$ . If  $p_u(f) = 0$  then  $\varphi := 0 \in dp_u(f)$  and  $\langle Af, \varphi \rangle \leq 0$ . If  $p_u(f) > 0$ , then there exists  $x \in K$  such that  $\varphi_x \in dp_u(f)$ . Hence,  $0 \leq p_u(f)u - f$  and  $(p_u(f)u - f)(x) = 0$ . It follows from (P) that  $p_u(f)(Au)(x) - (Af)(x) \geq 0$ . Hence  $(Af)(x) \leq p_u(f)(Au)(x) \leq 0$  (by ??); i.e.,  $\langle Af, \varphi_x \rangle \leq 0$ .  $\square$

**Corollary 1.11** *Let  $A$  be a densely defined operator on  $C(K)$ . If  $A$  satisfies (P) then  $A$  is closable and the closure of  $A$  satisfies (P) as well.*

**Proof (Proof of Corollary ??)** Let  $u \in D(A)$  be strictly positive. Then there exists  $\lambda \in \mathbb{R}$  such that  $Au \leq \lambda u$ . The operator  $B = A - \lambda$  satisfies (P) as well and  $Bu \leq 0$ . Then by Proposition ??,  $B$  is  $p_u$ -dissipative. Hence  $B$  is closable and the closure  $\overline{B}$  of  $B$  is  $p_u$ -dissipative as well (by A-II Proposition 2.9). Then by Proposition ??  $\overline{B}$  satisfies (P). Thus  $A$  is closable and its closure  $\overline{A} = \overline{B} + \lambda$  satisfies (P) as well.  $\square$

**Corollary 1.12** *Let  $A : C(K) \rightarrow C(K)$  be linear. If  $A$  satisfies (P) then  $A$  is bounded and  $A + \|A\| \text{Id} \geq 0$ .*

**Proof** It follows from Corollary ?? that  $A$  is closed. Hence  $A$  is bounded. Since  $A$  satisfies (P), it follows from Theorem ?? that  $A + \|A\| \text{Id} \geq 0$ .  $\square$

The next result is a strengthened form of Theorem 1.6. It is somewhat similar to the Lumer-Phillips Theorem (A-II, Theorem 2.13). Note that, however, in contrast to the condition of dissipativity,  $A + w$  satisfies (P) for any  $w \in \mathbb{R}$  whenever (P) holds for  $A$ . Thus (P) is not a "metric" condition; that is, it does not imply any norm estimate for the semigroup. We also point out that, if  $(T(t))_{t \geq 0}$  is a positive semigroup on  $C(K)$ , then in general none of the semigroups  $(e^{-wt}T(t))_{t \geq 0}$  ( $w \in \mathbb{R}$ ) is contractive (see ? ] or ? ]).

**Theorem 1.13** *Let  $A$  be a densely defined operator on  $C(K)$  which satisfies (P). Then*

$$\lambda_0 := \inf\{\lambda \in \mathbb{R} : Au \leq \lambda u \text{ for some } 0 \ll u \in D(A)\} < \infty.$$

- (i) *If  $(\lambda - A)D(A)$  is dense for some  $\lambda > \lambda_0$ , then  $A$  is closable and the closure  $\overline{A}$  of  $A$  is the generator of a positive semigroup.*
- (ii) *If  $\lambda - A$  is surjective for some  $\lambda > \lambda_0$ , then  $A$  is the generator of a positive semigroup.*

**Proof** It follows from Proposition ?? that  $\lambda_0 < \infty$ .

Assume that  $(\lambda - A)D(A)$  is dense, where  $\lambda > \lambda_0$ . Let  $\lambda_0 < \mu < \lambda$  and  $B = A - \mu$ . Then  $B$  satisfies (P) and  $Bu \leq 0$  for some strictly positive  $u \in D(B) = D(A)$ . Thus  $B$  is  $p_u$ -dissipative by Proposition ??. Moreover,  $((\lambda - \mu) - B)D(B)$  is dense. Thus by A-II, Corollary 2.12 the closure  $\overline{B}$  of  $B$  generates a  $p_u$ -contraction semigroup. Hence the closure  $\overline{A} = \overline{B} + \mu$  of  $A$  generates a positive semigroup of type  $\omega_0(\overline{A}) \leq \lambda$ . If  $(\lambda - A)$  is surjective, then  $A = \overline{A}$ .  $\square$

The proof of Theorem ?? yields estimates for the growth bound of a positive semigroup (see A-III, (1.3)) which we state explicitly in the next corollary.

**Corollary 1.14** *Let  $A$  be the generator of a strongly continuous positive semigroup on  $C(K)$ . Then*

$$-\infty < s(A) = \omega_0(A) \in \sigma(A). \quad (1.7)$$

Moreover,

$$s(A) = \inf\{\lambda \in \mathbb{R}: Au \leq \lambda u \text{ for some } 0 \ll u \in D(A)\} \quad (1.8)$$

and

$$s(A) \geq \sup\{\mu \in \mathbb{R}: Af \geq \mu f \text{ for some } 0 < f \in D(A)\}. \quad (1.9)$$

**Proof** Let  $s := \inf\{\lambda \in \mathbb{R}: [\lambda, \infty) \subset \varrho(A)\}$ . Clearly,  $s \leq s(A)$ . Moreover, by Remark ??,  $R(\lambda, A) \geq 0$  for all  $\lambda > s$ . Hence it follows from (??) that  $\omega_0(A) \leq s$ . Consequently,  $s = s(A) = \omega_0(A)$ .

Next we prove (??). Let  $0 < f \in D(A)$  such that  $Af \geq \mu f$ . Assume that  $\mu > s(A)$ . Then  $R(\mu, A) \geq 0$ . Hence,  $f = R(\mu, A)(\mu - A)f \leq 0$ , a contradiction.

Since  $D(A)$  is dense, there exists a strictly positive  $u \in D(A)$ . Then  $Au \geq \mu u$  for some  $\mu \in \mathbb{R}$ . Hence,  $-\infty < s(A)$  by (??). Since  $s(A) = s$  it is clear that  $s(A) \in \sigma(A)$ .

It remains to show (??). Let  $\lambda > s(A)$  and  $u = R(\lambda, A)1$ . Then  $u$  is strictly positive (by Remark ??) and  $Au = \lambda u - 1 \leq \lambda u$ . This proves one inequality in (??). Assume now that  $u \in D(A)$  is strictly positive such that  $Au \leq \lambda u$ . Then by the proof of Theorem ?? we have  $\omega_0(A) \leq \lambda$ . This proves the other inequality in (??).  $\square$

**Remark 1.15** If  $A$  has compact resolvent, then by the Krein-Rutman Theorem there exists a positive eigenvector  $u$  of  $A$  corresponding to a real eigenvalue. So the equality is valid in (??) and the supremum is a maximum. If in addition the semigroup is irreducible (see B-III, Section 3 below), then  $u$  is strictly positive and in (??) the infimum is attained as well.

Conversely, if in (??) the infimum is attained, then  $s(A)$  is an eigenvalue.

**Example 1.16** Let  $A = (a_{ij})$  be an  $n \times n$ -matrix such that  $a_{ij} \geq 0$  whenever  $i \neq j$  (see Example ????). Then by Corollary ??

$$\begin{aligned}
s(A) &= \inf\{\lambda \in \mathbb{R}: Au \leq \lambda u \text{ for some strictly positive } u\} \\
&= \inf_{u \gg 0} \inf\{\lambda \in \mathbb{R}: Au \leq \lambda u\} \\
&= \inf \left\{ \sup_i \sum_{j=1}^n a_{ij} u_j / u_i : u \gg 0 \right\}.
\end{aligned}$$

This formula is due to [?] (see also [?, Chapter I, Exercise 20] and [?]).

**Corollary 1.17** *Let  $(T(t))_{t \geq 0}$  be a strongly continuous positive semigroup on  $C(K)$ . Then  $T(t)u \gg 0$  for all  $u \gg 0$ ,  $t \geq 0$ .*

**Proof** Denote by  $A$  the generator of  $(T(t))_{t \geq 0}$ . Then by the proof of Theorem ?? there exist  $u \gg 0$  and  $\lambda \in \mathbb{R}$  such that  $A - \lambda$  is  $p_u$ -dissipative. This implies that  $p_u(T(t)f) \leq e^{\lambda t} p_u(f)$ . Observing that  $f \gg 0$  if and only if  $p_u(-f) < 0$  the claim follows.  $\square$

**Remark 1.18** Corollaries ?? and ?? do not hold on  $C_0(X)$ . For example, let  $X = [0, 1]$  and

$$(T(t)f)(x) = \begin{cases} f(x+t) & \text{if } x+t \leq 1 \\ 0 & \text{if } x+t > 1 \end{cases}.$$

Then  $(T(t))_{t \geq 0}$  is a positive semigroup on  $C_0(X)$  and  $T(t) = 0$  for all  $t \geq 1$ . The generator  $A$  of  $(T(t))_{t \geq 0}$  has empty spectrum, so that (??) is violated. However, it is still true that  $s(A) = \omega_0(A)$  for generators of positive semigroups on  $C_0(X)$  (see B-IV, Theorem 1.4).

**Remark 1.19** So far, the results of this section do not depend on the lattice structure of  $C(K)$ . They also hold on an ordered Banach space  $E$  with normal cone  $E_+$  which has non-empty interior. We refer to [?] and to [?] for this more general setting.

Next we apply Theorem ?? to prove a result on the multiplicative perturbation of a generator  $A$  which is due to [?] in the case when  $A$  is dissipative.

**Theorem 1.20** *Let  $A$  be the generator of a positive semigroup on  $C(K)$  and  $m \in C(K)$  be strictly positive. Then the operator  $m \cdot A$  given by  $(m \cdot A)f = m \cdot (Af)$  on the domain  $D(m \cdot A) = D(A)$  is the generator of a positive semigroup. Moreover,*

$$\|m^{-1}\|_{\infty}^{-1} \omega_0(A) \leq \omega_0(m \cdot A) \leq \|m\|_{\infty} \omega_0(A). \quad (1.10)$$

**Proof** We can assume that  $\|m\|_{\infty} \leq 1$  (in fact, if  $B := (m/\|m\|_{\infty}) \cdot A$  is the generator of a positive semigroup, then by A-I, 3.2, the operator  $m \cdot A = \|m\|_{\infty} B$  also generates a positive semigroup). The assertion of the theorem holds for  $A$  if and only if it is valid for  $A - w$  ( $w \in \mathbb{R}$ ). So by the proof of Theorem ?? we can assume that there exists  $0 \ll u \in C(K)$  such that  $A$  is  $p_u$ -dissipative. We first show the following. Let  $0 \ll q \in C(K)$ .

If  $B$  is a  $p_u$ -dissipative operator, then  $q \cdot B$  is  $p_u$ -dissipative. (1.11)

Let  $f \in D(q \cdot B) = D(B)$ . There exists  $x \in K$  such that  $\varphi_x \in \text{dp}_u(f)$  (by (??)). Hence  $\langle Bf, \varphi_x \rangle \leq 0$ . Consequently,  $\langle q \cdot Bf, \varphi_x \rangle = q(x)\langle Bf, \varphi_x \rangle \leq 0$ .

Next we show,

if  $B$  is the generator of a  $p_u$ -contraction semigroup and  
 $1 \geq q \in C(K)_+$  is such that  $\|1 - q\|_\infty < 1/2$ , (1.12)  
 then  $q \cdot B$  generates a  $p_u$ -contraction semigroup.

Because of (??) we only have to show that  $(I - q \cdot B)$  is surjective. Note that  $1 \in \rho(B)$ . We have

$$(\text{Id} - q \cdot B) = (\text{Id} - B - (q - 1)B) = (\text{Id} - (q - 1)BR(1, B))(\text{Id} - B)$$

Thus it suffices to show that  $\text{Id} - (q - 1)BR(1, B)$  is invertible. The norm

$$\|f\|_u = \max\{p_u(f), p_u(-f)\} = \sup_{x \in K} |f(x)|/u(x)$$

is equivalent to the supremum norm. Denote by  $\|T\|_u$  the operator norm corresponding to  $\|\cdot\|_u$  ( $T \in \mathcal{L}(C(K))$ ). Then it is enough to show that  $\|(q - 1)BR(1, B)\|_u = \|(q - 1)(R(1, B) - I)\|_u < 1$ .

For  $r \in C(K)_+$  denote by  $M_r$  the multiplication operator given by  $M_r f = r \cdot f$ . Then

$$\begin{aligned} \|M_r\|_u &= \sup\{\|r \cdot f\|_u : \|f\|_u \leq 1\} \\ &= \sup\{\sup_{x \in K} r(x)|f(x)|/u(x) : \|f\|_u \leq 1\} \\ &\leq \|r\|_\infty. \end{aligned}$$

Since  $B$  is  $p_u$ -dissipative we have  $\|R(1, B)\|_u \leq 1$  (by A-II, Lemma 2.10). This gives  $\|(q - 1)(R(1, B) - I)\|_u \leq \|M_{(1-q)}\|_u(\|R(1, B)\|_u + 1) \leq 2\|1 - q\|_\infty < 1$ .

The proof of (??) is complete.

There exists  $k \in \mathbb{N}$  such that  $\|1 - m^{1/k}\|_\infty < 1/2$ . Applying now (??) successively to  $B = m^{1/k} \cdot A$  and  $q = m^{1/k}$  ( $l = 1, \dots, k - 1$ ) we obtain that  $m \cdot A$  generates a  $p_u$ -contraction semigroup (which in particular is positive).

Finally we show (??) to hold. Let  $0 \ll u \in D(A) = D(m \cdot A)$  and  $Au \leq \lambda u$ . Then  $m \cdot Au \leq \|m\|_\infty \lambda u$ . So (??) implies that  $\omega_0(m \cdot A) \leq \|m\|_\infty \omega_0(A)$ . This is one part of (??). The other part follows from this since  $\omega_0(A) = \omega_0(m^{-1} \cdot m \cdot A) \leq \|m^{-1}\|_\infty \omega_0(m \cdot A)$ .  $\square$

In the following lemma a condition (P') is introduced which is dual to the positive minimum principle.

**Lemma 1.21** *Let  $A$  be the generator of a strongly continuous positive semigroup on  $C(K)$ . Then*

(P')  $f \in C(K)_+, 0 \leq \mu \in D(A'), \langle f, \mu \rangle = 0$  implies  $\langle f, A'\mu \rangle \geq 0$ .

**Proof**  $\langle f, A'\mu \rangle = \lim_{t \rightarrow 0} \frac{1}{t} \langle T(t)f - f, \mu \rangle = \lim_{t \rightarrow 0} \frac{1}{t} \langle T(t)f, \mu \rangle \geq 0$ .  $\square$

**Example 1.22** Let  $K = [-1, 0]$ . Let  $\alpha \in \mathbb{R}$  and  $\mu$  be a measure on  $[-1, 0]$  such that  $\mu(\{0\}) = 0$ . Define the operator  $A$  on  $C[-1, 0]$  by  $Af = f'$  with domain  $D(A) = \{f \in C^1[-1, 0] : f'(0) = \alpha f(0) + \langle f, \mu \rangle\}$ .

Claim:  $A$  is the generator of a positive semigroup if and only if  $\mu \geq 0$ .

**Proof (Proof of the claim)** Assume that  $A$  generates a positive semigroup. By the definition of  $A$  one has  $\delta_0 \in D(A')$  and  $A'\delta_0 = \alpha\delta_0 + \mu$ . So it follows from (P') that  $\langle f, \mu \rangle = \langle f, A'\delta_0 \rangle \geq 0$  for all  $f \in C[-1, 0]_+$  such that  $f(0) = 0$ . By Lemma ?? this implies that  $\mu \geq 0$ .

In order to show the converse assume that  $\mu \geq 0$ .

a) We show that  $A$  is densely defined. Consider the normed space  $F = C^1[-1, 0]$  with the supremum norm. Then  $\psi : F \rightarrow \mathbb{R}$  given by  $\psi(f) = f'(0) - \alpha f(0) - \langle f, \mu \rangle$  is a discontinuous linear form on  $F$ . Consequently  $D(A) = \ker(\psi)$  is dense in  $F$ . Since  $F$  is dense in  $C[-1, 0]$ ,  $D(A)$  is dense in  $C[-1, 0]$  as well.

b)  $A$  satisfies (P) (see Definition ??). In fact, let  $f \in D(A)_+$  and  $x \in [-1, 0]$  such that  $f(x) = 0$ . It is clear that  $Af(x) = f'(x) \geq 0$  if  $x < 0$ . But if  $f(0) = 0$ , then  $Af(0) = f'(0) = \langle f, \mu \rangle \geq 0$  since  $f \in D(A)$ .

c) We show that  $(\lambda - A)$  is bijective for  $\lambda > \alpha + \|\mu\|$ . Let  $g \in C[-1, 0]$ . The solutions of the equation  $\lambda f - f' = g$  ( $f \in C[-1, 0]$ ) are given by  $f(x) = e^{\lambda x} [\int_x^0 e^{-\lambda y} g(y) dy + c]$  where  $c \in \mathbb{R}$ . Moreover,  $f \in D(A)$  if and only if

$$c(\lambda - \alpha - \int_{-1}^0 e^{\lambda x} d\mu(x)) = g(0) + \int_{-1}^0 e^{\lambda x} \int_x^0 e^{-\lambda y} g(y) dy d\mu(x). \quad (1.13)$$

If  $\lambda > \alpha + \|\mu\|$ , then  $\lambda - \alpha - \int_{-1}^0 e^{\lambda x} d\mu(x) \neq 0$  and there exists exactly one  $c \in \mathbb{R}$  satisfying (1.13). We have shown that  $(\lambda - A)$  is bijective for  $\lambda > \alpha + \|\mu\|$ .

By Theorem ??, it follows from a), b) and c) that  $A$  generates a positive semigroup.  $\square$

Let us mention in addition that it follows from a) in the proof that  $(\alpha + \|\mu\|, \infty) \subset \varrho(A)$ , since  $A$  is closed. By Remark ?? we thus have

$$s(A) \leq \alpha + \|\mu\|. \quad (1.14)$$

**Example 1.23** Let  $E = C([-1, 0], \mathbb{R}^n)$ . Then  $u \in E$  is given by  $u = (u_1, \dots, u_n)$  where  $u_i \in C[-1, 0]$  ( $i = 1, \dots, n$ ). Let  $A$  be defined by  $Au = u' = (u'_1, \dots, u'_n)$  with domain  $D(A) = \{u \in C^1([-1, 0], \mathbb{R}^n) : u'(0) = Lu\}$ .

Here  $L$  is defined by

$$Lu = \begin{pmatrix} L_{11}u_1 + \dots + L_{1n}u_n \\ \vdots \\ L_{n1}u_1 + \dots + L_{nn}u_n \end{pmatrix}$$

where  $L_{ij} \in C[-1, 0]'$  ( $1 \leq i, j \leq n$ ). Let  $L_{ii} = \alpha_i \delta_0 + \mu_i$  with  $\mu_i(\{0\}) = 0$  ( $i = 1, \dots, n$ ). Then  $A$  generates a positive semigroup if and only if

$$L_{ij} \geq 0 \text{ for } i \neq j \text{ and } \mu_i \geq 0 \quad (i, j = 1, \dots, n).$$

This can be proved in a similar way as the claim in Example ?? (see ? ).

**Example 1.24** Let  $A$  on  $C[0, 1]$  be given by  $Af = f''$  with domain

$$D(A) = \{f \in C^2[0, 1] : f'(0) + \alpha f(0) = 0, f'(1) + \beta f(1) = 0\},$$

where  $\alpha, \beta \in \mathbb{R}$ . Then  $A$  is the generator of positive semigroup.

**Proof** The operator  $A$  satisfies (P). In fact, let  $0 \leq f \in D(A)$  and  $f(a) = 0$  where  $a \in [0, 1]$ . If  $a \in (0, 1)$  then  $f''(a) \geq 0$  since  $f$  has a minimum in  $a$ . If  $a = 0$  then

$$f'(0) = f'(0) + \alpha f(0) = 0$$

since  $f \in D(A)$ . Consequently,  $f(x) = \int_0^x (x-y)f''(y) dy \geq 0$  for all  $x \geq 0$ . This implies  $f''(0) \geq 0$ . The argument for  $a = 1$  is analogous. It remains to show that  $\mu - A$  is surjective for large real  $\mu$ . Let  $g \in C[0, 1]$ . Let  $\lambda > 0$  and

$$k = \frac{1}{2}\lambda \left[ e^{\lambda x} \int_x^1 e^{-\lambda y} g(y) dy - e^{-\lambda x} \int_x^1 e^{\lambda y} g(y) dy \right].$$

Then  $k \in C^2[0, 1]$  and  $\lambda^2 k - k'' = g$ . Let  $h = ae^{\lambda x} + be^{-\lambda x}$ , where  $a, b \in \mathbb{R}$ . Then  $h \in C^2[0, 1]$  and  $\lambda^2 h - h'' = 0$ . Let  $f = k + h$ . Then  $\lambda^2 f - f'' = g$ . The condition that  $f \in D(A)$  leads to two linear equations in  $a$  and  $b$ , and it is easy to see that they have a solution  $(a, b) \in \mathbb{R}^2$  if  $(\lambda + \alpha)(\beta - \lambda) + (\lambda - \alpha)(\lambda + \beta) \exp(\lambda^2) \neq 0$ . Thus there exists a solution if  $\lambda$  is large enough, and  $(\lambda^2 - A)$  is surjective.  $\square$

## 2 Lattice Semigroups on $C_0(X)$

Throughout this section  $X$  denotes a locally compact space and  $C_0(X, \mathbb{R})$  (resp.,  $C_0(X, \mathbb{C})$ ) the space of all real-valued (resp., complex-valued) continuous functions on  $X$  which vanish at infinity. If we do not want to specify the field we simply write  $C_0(X)$ .

Recall from B-I, Section 3 that a linear bounded operator  $T$  on  $C_0(X)$  is positive if and only if

$$|Tf| \leq T|f| \quad \text{for all } f \in C_0(X). \quad (2.1)$$

The operator  $T$  is a lattice homomorphism if and only if in (2.1) equality holds; i.e.,

$$|Tf| = T|f| \quad \text{for all } f \in C_0(X). \quad (2.2)$$

**Remark 2.1** if  $T$  is a lattice homomorphism on  $C_0(X, \mathbb{C})$  then  $T$  leaves  $C_0(X, \mathbb{R})$  invariant and the restriction  $T_{\mathbb{R}}$  of  $T$  to  $C_0(X, \mathbb{R})$  is a lattice homomorphism. Conversely, the linear extension  $T$  of a lattice homomorphism  $T_{\mathbb{R}}$  on  $C_0(X, \mathbb{R})$  to  $C_0(X, \mathbb{C})$  is a lattice homomorphism (see B-I, Section 3).

A semigroup  $(T(t))_{t \geq 0}$  is called *lattice semigroup* if  $T(t)$  is a lattice homomorphism for all  $t \geq 0$ . In Section 3 we will give a concrete representation of lattice-semigroups which shows that there is a large variety of examples. This section is devoted to the characterization of lattice semigroups in terms of their generators. The heuristic idea is the following. Let  $(T(t))_{t \geq 0}$  be a lattice semigroup with generator  $A$ .

Let  $f \in D(A)$  and assume that the modulus function  $\Theta$  given by  $\Theta(g) = |g|$  is differentiable at  $f$  (in some sense which has to be made precise). Then one expects that a chain rule holds so that  $\Theta(T(t)f) = |T(t)f|$  is differentiable at  $t = 0$ . Since  $|T(t)f| = T(t)|f|$ , this implies  $|f| \in D(A)$  and

$$A|f| = d/dt|_{t=0} \Theta(T(t)f) = D_{Af} \Theta(f) d/dt|_{t=0} T(t)f = (D_{Af} \Theta(f))Af$$

(where the precise meaning of  $(D_{Af} \Theta(f))Af$  depends on the chain rule which we will have to establish). So we obtain an identity for the generator  $A$  which corresponds exactly to the lattice property  $|T(t)f| = T(t)|f|$  of the semigroup. We will see in C-II, Section 5 that in a Banach lattice with order continuous norm the above argument is rigorous (for all  $f \in D(A)$ ). On  $C_0(X)$  we have to use a weak form of the argument and  $|f| \in D(A)$  only holds for special  $f \in D(A)$  (see Corollary ??).

We start by investigating differentiability of the modulus and by establishing a chain rule. For later use we formulate the following definition and proposition for a general Banach space  $G$  even though only  $G = \mathbb{C}$  will be considered in this section.

**Definition 2.2** Let  $G$  be a Banach space and  $\Theta: G \rightarrow G$  a mapping. Let  $f \in G$ ,  $u \in G$ . Then  $\Theta$  is called *right-sided Gateaux differentiable at  $f$  in direction  $u$*  if

$$D_u \Theta(f) := \lim_{t \rightarrow 0} 1/t (\Theta(f + tu) - \Theta(f)) \text{ exists.} \quad (2.3)$$

The mapping  $\Theta$  is *right-sided Gateaux differentiable at  $f$*  if  $D_u \Theta(f)$  exists for all directions  $u \in G$ ; and if  $\Theta$  is right-sided Gateaux-differentiable at every point  $f \in G$  then we call  $\Theta$  *right-sided Gateaux differentiable*.

**Proposition 2.3** (chain rule). Let  $G$  be a Banach space and  $k: \mathbb{R} \rightarrow G$  be right-sided differentiable at  $a \in \mathbb{R}$  (with right derivative  $k'(a)$ ). Suppose that  $\Theta: G \rightarrow G$  is a

*Lipschitz continuous mapping. If  $\Theta$  is right-sided Gateaux-differentiable at  $k(a)$  in the direction of  $k'(a)$ , then  $\Theta \circ k: \mathbb{R} \rightarrow G$  is right-sided differentiable at  $a$  and has a right derivative*

$$(\Theta \circ k)'(a) = D_{k'(a)}\Theta(k(a)). \quad (2.4)$$

**Proof** There exists  $L \geq 0$  such that  $\|\Theta(x) - \Theta(y)\| \leq L\|x - y\|$  for all  $x, y \in G$ . Then

$$\begin{aligned} & \lim_{t \downarrow 0} \|1/t(\Theta(k(a+t)) - \Theta(k(a))) - D_{k'(a)}\Theta(k(a))\| \\ & \leq \limsup_{t \downarrow 0} \|1/t(\Theta(k(a+t)) - \Theta(k(a) + tk'(a)))\| \\ & \quad + \limsup_{t \downarrow 0} \|1/t[\Theta(k(a) + tk'(a)) - \Theta(k(a)) - D_{k'(a)}\Theta(k(a))]\| \\ & \leq \limsup_{t \downarrow 0} L \cdot \|1/t(k(a+t) - k(a) - tk'(a))\| + 0 \\ & = 0. \end{aligned}$$

□

For  $z \in \mathbb{C}$  we let

$$\text{sign } z = \begin{cases} z/|z| & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \quad (2.5)$$

**Lemma 2.4** *The function  $\Theta: \mathbb{C} \rightarrow \mathbb{C}$  given by  $\Theta(z) = |z|$  is right-sided Gateaux differentiable and*

$$D_u\Theta(z) = \begin{cases} \text{Re}[(\text{sign } \bar{z}) \cdot u] & \text{if } z \neq 0 \\ |u| & \text{if } z = 0 \end{cases} \quad (2.6)$$

**Proof** If  $z = 0$ , relation (??) is obvious from the definition. Let  $z = (x_0 + iy_0) \neq 0$ . We identify  $\mathbb{C}$  and  $\mathbb{R}^2$ . Then  $\Theta(x, y) = (x^2 + y^2)^{1/2}$  is differentiable in  $z$  and

$$\begin{aligned} D_u\Theta(z) &= (\Theta(x_0, y_0)|u) = 1/|z| ((x_0, y_0)|(u_1, u_2)) \\ &= 1/|z| (x_0u_1 + y_0u_2) \\ &= 1/|z| \text{Re}((x_0 - iy_0) \cdot (u_1 + iu_2)) \\ &= \text{Re}[(\text{sign } \bar{z}) \cdot u], \end{aligned}$$

where  $u = u_1 + iu_2 = (u_1, u_2) \in \mathbb{C} = \mathbb{R}^2$  and  $(v|u)$  denotes the canonical scalar product of  $v, u \in \mathbb{R}^2$ . □

Let  $f, g \in C_0(X)$ . We denote by  $(\widehat{\text{sign } f})(g)$  the bounded Borel function given by

$$[(\widehat{\text{sign } f})(g)](x) := \begin{cases} (\text{sign } f(x)) \cdot g(x) & \text{if } f(x) \neq 0 \\ |g(x)| & \text{if } f(x) = 0 \end{cases} \quad (2.7)$$

Similarly,  $(\text{sign } f)(g)$  is defined by

$$[(\text{sign } f)(g)](x) = (\text{sign } f(x)) \cdot g(x). \quad (2.8)$$

We identify the dual space of  $C_0(X)$  with  $M(X)$ , the space of all bounded regular Borel measures on  $X$ . We extend the duality by setting

$$\langle h, \varphi \rangle = \int h(x) d\varphi(x)$$

for every bounded Borel function  $h$  on  $X$  and every  $\varphi \in M(X)$ .

After these preparations we now can show that the lattice property  $|T(t)f| = T(t)|f|$  of the semigroup corresponds to the identity (??) below for the generator, which we call Kato's equality (cf. Remark ??).

**Theorem 2.5** *A strongly continuous semigroup  $(T(t))_{t \geq 0}$  on  $C_0(X)$  is a lattice semigroup if and only if its generator  $A$  satisfies*

$$\langle \operatorname{Re}[(\widehat{\operatorname{sign} \bar{f}})(Af)], \varphi \rangle = \langle |f|, A'\varphi \rangle \quad (\text{Kato's equality}). \quad (2.9)$$

for all  $f \in D(A)$ ,  $\varphi \in D(A')$

From the proof of the theorem we isolate the following lemma.

**Lemma 2.6** *Let  $(T(t))_{t \geq 0}$  be a semigroup on  $C_0(X)$  with generator  $A$ . Then for every  $f \in D(A)$ ,  $\varphi \in M(X)$ ,*

$$\frac{d}{dt} \Big|_{t=0} \langle |T(t)f|, \varphi \rangle = \langle \operatorname{Re}[(\widehat{\operatorname{sign} \bar{f}})(Af)], \varphi \rangle. \quad (2.10)$$

**Proof** Let  $f \in D(A)$  and  $x \in X$ . Define the function  $k(t) = (T(t)f)(x)$  for all  $t \geq 0$ . Then  $k$  is right-sided differentiable at 0 with derivative  $k'(0) = (Af)(x)$ . It follows from the chain rule Proposition ?? that

$$\frac{d}{dt} \Big|_{t=0} |(T(t)f)(x)| = \operatorname{Re}[(\widehat{\operatorname{sign} \bar{f}})(Af)](x) \quad (2.11)$$

Moreover,  $\frac{1}{t} ||T(t)f| - |f|| \leq \frac{1}{t} |T(t)f - f|$ . Thus  $\sup_{0 < t \leq 1} \frac{1}{t} ||T(t)f| - |f|| < \infty$ ; i.e., the functions  $k_t \in C_0(X)$  given by

$$k_t(x) = 1/t (|(T(t)f)(x)| - |f(x)|) \quad (x \in X) \quad (2.12)$$

are uniformly on  $(0, 1]$  dominated by a constant. The dominated convergence theorem and (??) imply that

$$\frac{d}{dt} \Big|_{t=0} \langle |T(t)f|, \varphi \rangle = \lim_{t \downarrow 0} \langle k_t, \varphi \rangle = \langle \operatorname{Re}[(\widehat{\operatorname{sign} \bar{f}})(Af)], \varphi \rangle$$

**Proof (Proof of Theorem ??)** Assume that  $(T(t))_{t \geq 0}$  is a lattice semigroup. Let  $f \in D(A)$ ,  $\varphi \in D(A')$ . It follows from the preceding lemma that

$$\langle \operatorname{Re}[(\widehat{\operatorname{sign} f})(Af)], \varphi \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle |T(t)f|, \varphi \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle T(t)|f|, \varphi \rangle = \langle |f|, A'\varphi \rangle$$

Conversely, assume that (??) holds. Let  $t > 0$ ,  $f \in C_0(X)$ . We have to show that  $|T(t)f| = T(t)|f|$ . Since  $D(A)$  is dense in  $C_0(X)$ , we can assume that  $f \in D(A)$ . Moreover, since  $D(A')$  is  $\sigma(M(X), C_0(X))$ -dense in  $M(X)$ , it suffices to show that

$$\langle |T(t)f|, \varphi \rangle = \langle T(t)|f|, \varphi \rangle \quad (2.13)$$

for all  $\varphi \in D(A')$ .

Let  $\varphi \in D(A')$  and define the function  $k(s) = \langle T(t-s)|T(s)f|, \varphi \rangle$  ( $s \in [0, t]$ ). We claim that  $k$  is right-sided differentiable with derivative  $k'(s) = 0$  for all  $s \in [0, t]$ . This implies that  $k(0) = k(t)$  which is (??).

Since  $\varphi \in D(A')$  we have

$$\lim_{h \downarrow 0} 1/h \langle g, (T(t-(s+h)) - T(t-s))'\varphi \rangle = -\langle g, A'T(t-s)'\varphi \rangle \quad (2.14)$$

for all  $g \in C_0(X)$ . Consequently,

$$\overline{\lim}_{h \downarrow 0} 1/h \|(T(t-(s+h)) - T(t-s))'\varphi\| < \infty$$

by the uniform boundedness principle. Hence, since  $\lim_{h \downarrow 0} |T(s+h)f| = |T(s)f|$ , (??) implies that

$$\begin{aligned} \lim_{h \downarrow 0} 1/h \langle |T(s+h)f|, (T(t-(s+h)) - T(t-s))'\varphi \rangle \\ = -\langle |T(s)f|, A'T(t-s)'\varphi \rangle \end{aligned} \quad (2.15)$$

Using this we obtain

$$\begin{aligned} \lim_{h \downarrow 0} 1/h (k(s+h) - k(s)) \\ = \lim_{h \downarrow 0} 1/h (\langle T(t-(s+h))|T(s+h)f|, \varphi \rangle - \langle T(t-s)|T(s+h)f|, \varphi \rangle) \\ + \lim_{h \downarrow 0} 1/h \langle T(t-s)|T(s+h)f| - T(t-s)|T(s)f|, \varphi \rangle \\ = -\langle |T(s)f|, A'T(t-s)'\varphi \rangle + \lim_{t \downarrow 0} 1/h \langle (|T(s+h)f| - |T(s)f|), T(t-s)'\varphi \rangle. \end{aligned}$$

By Lemma ?? the last term is

$$-\langle |T(s)f|, A'T(t-s)'\varphi \rangle + \langle \operatorname{Re}[(\widehat{\operatorname{sign} T(s)f})(AT(s)f)], T(t-s)'\varphi \rangle,$$

and this is 0 by hypothesis.  $\square$

**Remark 2.7** We will see in Chapter C-II that the inequality  $|T(t)f| \leq T(t)|f|$ , which holds precisely for positive semigroups, implies the inequality corresponding to (??)

if  $\varphi \geq 0$ . For  $A = \Delta$  (the Laplacian) this is a version of the classical Kato's inequality.

**Corollary 2.8** *Let  $(T(t))_{t \geq 0}$  be a lattice semigroup on  $C_0(X)$  with generator  $A$ . If  $f \in D(A)$  and  $f(x) \neq 0$  for all  $x \in X$ , then*

$$|f| \in D(A) \text{ and } \operatorname{Re}[(\operatorname{sign} \bar{f})(Af)] = A|f|.$$

**Proof** If  $f \in D(A)$  and  $f(x) \neq 0$  for all  $x \in X$ , then  $(\widehat{\operatorname{sign} \bar{f}})(Af) = (\operatorname{sign} \bar{f})(Af) \in C_0(X)$ . Hence by (??),  $\langle \operatorname{Re}[(\operatorname{sign} \bar{f})(Af)], \varphi \rangle = \langle |f|, A'\varphi \rangle$  for all  $\varphi \in D(A')$ . So the assertion follows from the following lemma.  $\square$

**Lemma 2.9** *Let  $A$  be a densely defined closed operator on a (real or complex) Banach space  $G$ . Let  $f, g \in G$  such that  $\langle f, \varphi \rangle = \langle g, A'\varphi \rangle$  for all  $\varphi \in D(A')$ . Then  $g \in D(A)$  and  $Ag = f$ .*

**Proof** Denote by  $G(A) := \{(h, Ah) : h \in D(A)\} \subset G \times G$  the graph of  $A$ . Assume that  $(g, f) \notin G(A)$ . Since  $G(A)$  is closed, it follows from the Hahn-Banach theorem that there exists  $(\psi_1, \psi_2) \in G' \times G'$  such that  $\langle h, \psi_1 \rangle + \langle Ah, \psi_2 \rangle = 0$  for all  $h \in D(A)$  and  $\langle g, \psi_1 \rangle + \langle f, \psi_2 \rangle \neq 0$ . By the definition of  $A'$  this implies that  $\psi_2 \in D(A')$  and  $A'\psi_2 = -\psi_1$ . Hence  $\langle f, \psi_2 \rangle \neq -\langle g, \psi_1 \rangle = \langle g, A'\psi_2 \rangle$ . So the condition in the lemma is violated.  $\square$

Next we prove a converse of Corollary ??.

**Theorem 2.10** *Let  $A$  be the generator of a real semigroup  $(T(t))_{t \geq 0}$  on  $C(K, \mathbb{C})$ , where  $K$  is compact. Then  $(T(t))_{t \geq 0}$  is a lattice semigroup if and only if  $f \in D(A)$ ,  $f(x) \neq 0$  for all  $x \in K$  implies  $|f| \in D(A)$  and  $A|f| = \operatorname{Re}((\operatorname{sign} \bar{f})Af)$ .*

**Remark** Although we assume that  $(T(t))_{t \geq 0}$  is a real semigroup (i.e.,  $T(t)C(K, \mathbb{R}) \subset C(K, \mathbb{R})$  for all  $t \geq 0$ ), it is important for the proof that we consider the space of all complex-valued continuous functions on  $K$ . In fact, if  $K$  is connected, the condition in the theorem is always trivially satisfied for all  $f \in C(K, \mathbb{R})$ .

**Proof (Proof of Theorem 2.10)** It follows from Corollary ?? that the condition is necessary. So assume that the condition is satisfied. Since  $(T(t))_{t \geq 0}$  is real, the restriction  $T_{\mathbb{R}}(t)$  of  $T(t)$  to  $C(K, \mathbb{R})$  ( $t \geq 0$ ) defines a strongly continuous semigroup. Its generator  $A_{\mathbb{R}}$  is a restriction of  $A$ . Since  $D(A_{\mathbb{R}})$  is dense in  $C(K, \mathbb{R})$ , there exists a strictly positive  $u \in D(A_{\mathbb{R}})$ . Moreover,  $\lim_{t \rightarrow 0} T(t)u = u$  uniformly. Thus there exists  $t_0 > 0$  such that  $T(t)u$  is strictly positive for all  $t \in [0, t_0]$  and  $\varepsilon > 0$  small enough..

Let  $f \in D(A_{\mathbb{R}})$ . For  $\varepsilon > 0$  let  $f_{\varepsilon} := f + i\varepsilon u$ . Then  $T(t)f_{\varepsilon} \in D(A)$  and  $|T(t)f_{\varepsilon}|$  is strictly positive for all  $t \in [0, t_0]$ . By hypothesis,  $|T(t)f_{\varepsilon}| \in D(A)$  and

$$\operatorname{Re}[(\operatorname{sign} T(t)\overline{f_{\varepsilon}})AT(t)f_{\varepsilon}] = A|T(t)f_{\varepsilon}| \quad \text{for all } t \in [0, t_0].$$

One sees as in the proof of Theorem ?? that this implies that  $|T(t)f_\varepsilon| = T(t)|f_\varepsilon|$  for all  $t \in [0, t_0]$ . Letting  $\varepsilon \rightarrow 0$  one obtains that  $|T(t)f| = T(t)|f|$  ( $t \in [0, t_0]$ ). Since  $D(A)$  is dense in  $C(K, \mathbb{R})$  we conclude that  $|T(t)f| = T(t)|f|$  for all  $f \in C(K, \mathbb{R})$  and all  $t \in [0, t_0]$ .

Let  $s > t_0$ . Then  $s/n \leq t_0$  for some  $n \in \mathbb{N}$ . Hence  $|T(s)f| = |T(s/n)^n f| = T(s/n)^n |f| = T(s)|f|$  for all  $f \in C(K, \mathbb{R})$ . We have shown that  $T_{\mathbb{R}}(t)$  is a lattice homomorphism for all  $t \geq 0$ ; hence  $T(t)$  is so as well (cf. Remark 2.1).  $\square$

**Corollary 2.11** *Let  $A$  be the generator of a lattice semigroup on  $C(K, \mathbb{C})$  ( $K$  compact). Assume that  $m \in C(K)$  is strictly positive. Then  $m \cdot A$  with domain  $D(m \cdot A) = D(A)$  generates a lattice semigroup.*

**Proof** By Theorem 1.20, the operator  $m \cdot A$  is the generator of a strongly continuous semigroup. It remains to show that it is a lattice semigroup. We use Theorem ??. Let  $f \in D(m \cdot A) = D(A)$  such that  $f(x) \neq 0$  for all  $x \in K$ . Then

$$\operatorname{Re}[(\operatorname{sign} \bar{f})m \cdot Af] = m \cdot \operatorname{Re}[(\operatorname{sign} \bar{f})Af] = m \cdot A|f|.$$

**Example 2.12** The operator  $A_{\max}$  on the (real or complex space)  $C[-1, 0]$  given by  $A_{\max}f = f'$  with domain  $D(A_{\max}) = C^1[-1, 0]$  satisfies Kato's equality; i.e.,

$$\langle \operatorname{Re}[(\widehat{\operatorname{sign} \bar{f}})(A_{\max}f)], \varphi \rangle = \langle |f|, A'_{\max} \varphi \rangle \quad (2.16)$$

( $f \in D(A_{\max})$ ,  $\varphi \in D(A'_{\max})$ ).

Moreover,  $(\lambda - A_{\max})$  is surjective for  $\lambda \geq 0$  (cf. Example ??). Thus, since  $\ker(\lambda - A_{\max}) = \mathbb{C}e_\lambda$  ( $e_\lambda(x) = e^{\lambda x}$ ), Kato's equality does not have as strong consequences as the positive minimum principle (which by Theorem 1.13 would imply that  $A_{\max}$  is a generator).

**Proof** It is not difficult to prove that the adjoint  $A'_{\max}$  of  $A_{\max}$  is given by

$$A'_{\max} \varphi = \varphi(0)\delta_0 - \varphi(-1)\delta_{-1} - d\varphi \quad (2.17)$$

with domain  $D(A'_{\max}) = BV[-1, 0]$  (the space of all functions of bounded variation on  $[-1, 0]$ ). Here we identify  $BV[-1, 0] \subset L^1[-1, 0]$  with a subspace of  $C[-1, 0]'$  by setting  $\langle f, \varphi \rangle = \int_{-1}^0 f(x)\varphi(x) \, dx$  ( $f \in C[-1, 0]$ ,  $\varphi \in BV[-1, 0]$ ). For  $\varphi \in BV[-1, 0]$ ,  $d\varphi$  denotes the linear form on  $C[-1, 0]$  given by  $f \mapsto \int_{-1}^0 f(x) \, d\varphi(x)$ .

We now show (?). Let  $f \in D(A_{\max}) = C^1[-1, 0]$ ,  $\varphi \in D(A'_{\max}) = BV[-1, 0]$ . By Lemma ?? and the chain rule (Proposition ??) we have

$$|f(x)|' := d^+/dt|_{t=x}|f(t)| = \operatorname{Re}[(\widehat{\operatorname{sign} \bar{f}})f'](x)$$

(where  $f'(x) = (\operatorname{Re} f)'(x) + i(\operatorname{Im} f)'(x)$  in the complex case). Thus

$$\begin{aligned}
\langle \operatorname{Re}[(\widehat{\operatorname{sign} f})Af], \varphi \rangle &= \int_{-1}^0 |f(x)|' \varphi(x) \, dx = \int_0^1 \varphi(x) \, d|f(x)| = \\
&= \varphi(0)|f(0)| - \varphi(-1)|f(-1)| - \int_{-1}^0 |f(x)| \, d\varphi(x) \\
&= \langle |f|, A'_{\max} \varphi \rangle.
\end{aligned}$$

**Example 2.13** Let  $A$  on (the real or complex) space  $C[-1, 0]$  be given by  $Af = f'$  with domain  $D(A) = \{f \in C^1[-1, 0] : f'(0) = Lf\}$  where  $L \in M[-1, 0] = C[-1, 0]'$ . Then  $A$  is the generator of a lattice semigroup if and only if  $L = \alpha\delta_0$  for some  $\alpha \geq 0$ .

**Proof** Assume that  $A$  is the generator of a lattice semigroup  $(T(t))_{t \geq 0}$ . There exists  $\mu \in M[-1, 0]$  satisfying  $\mu(\{0\}) = 0$  and  $\alpha \in \mathbb{R}$  such that  $L = \alpha\delta_0 + \mu$ . We claim that

$$|\langle f, \mu \rangle| = \langle |f|, \mu \rangle \quad \text{for all } f \in D(A) \text{ satisfying } f(0) = 0. \quad (2.18)$$

In fact, by the definition of  $A$  we have

$$\delta_0 \in D(A') \text{ and } A'\delta_0 = L. \quad (2.19)$$

Moreover, by Theorem ??,  $A$  satisfies Kato's Equality (?). Since  $f(0) = 0$  this implies

$$\begin{aligned}
|\langle f, \mu \rangle| &= |f'(0)| = \operatorname{Re}[(\widehat{\operatorname{sign} f})(f')](0) \\
&= \langle \operatorname{Re}[(\widehat{\operatorname{sign} f})(Af)], \delta_0 \rangle = \langle |f|, A'\delta_0 \rangle \quad (\text{by } (?)) \\
&= \langle |f|, \mu \rangle.
\end{aligned}$$

Since  $\varphi(f) = f'(0) - \langle f, \mu \rangle$  defines a linear form on the space

$$F = \{f \in C^1[-1, 0] : f(0) = 0\}$$

which is discontinuous for the supremum norm, the space  $D(A) = \ker(\varphi)$  is dense in  $F$  and consequently dense in  $C_0[-1, 0)$ . It follows that (??) holds for all  $f \in C_0[-1, 0)$ . So by B-I, Section 2, there exist  $\beta \geq 0$  and  $x \in [-1, 0)$  such that  $\mu = \beta\delta_x$ . Assume that  $\beta \neq 0$ . It is easy to see that there exists a real function  $f \in C^1[-1, 0]$  satisfying  $f'(0) = \alpha f(0) + \beta f(x)$  and  $f(0)f(x) < 0$ . Hence  $f \in D(A)$  but

$$\begin{aligned}
\langle \operatorname{Re}[(\widehat{\operatorname{sign} f})(Af)], \delta_0 \rangle &= (\operatorname{sign} f(0))f'(0) = (\operatorname{sign} f(0))(\alpha f(0) + \beta f(x)) \\
&= \alpha|f(0)| + \beta(\operatorname{sign} f(0))f(x) \\
&\neq \alpha|f(0)| + \beta|f(x)| = \langle |f|, \alpha\delta_0 + \beta\delta_x \rangle \\
&= \langle |f|, A'\delta_0 \rangle.
\end{aligned}$$

This contradicts (?). We have shown that  $\beta = 0$ ; i.e.,  $L = \alpha\delta_0$ .

The converse can be shown by using Theorem ?? again. However, if  $L = \alpha\delta_0$ , then it is easy to see that  $A$  generates the semigroup  $(T(t))_{t \geq 0}$  given by

$$(T(t)f)(x) = \begin{cases} f(x+t) & \text{if } x+t < 0 \\ e^{t\alpha} f(0) & \text{if } x+t \geq 0 \end{cases}$$

So  $(T(t))_{t \geq 0}$  is clearly a lattice semigroup.  $\square$

### 3 Semiflows, Flows and Positive Groups

In this section we establish a relation between generators of lattice homomorphisms and derivations. On the space  $C_0(\mathbb{R})$ , for example, this will enable us, to give a detailed description of all generators of positive groups.

At first we consider a compact space  $K$  and denote by  $C(K) = C(K, \mathbb{R})$  the space of all real valued continuous functions on  $K$ . The extension of the subsequent results to the complex space is obvious.

A lattice homomorphism  $T$  on  $C(K)$  is an algebra homomorphism if and only if  $T\mathbb{1} = \mathbb{1}$  (see B-I, Section 3). We start by describing semigroups of algebra homomorphisms on  $C(K)$ .

**Definition 3.1** A mapping  $\varphi: [0, \infty) \times K \rightarrow K$  is called *semiflow* if for each  $t \geq 0$  the mapping  $\varphi_t$  given by  $\varphi_t(x) = \varphi(t, x)$  is continuous and

$$\varphi_s \circ \varphi_t = \varphi_{s+t} \text{ for all } s, t \geq 0 \quad (3.1)$$

$$\varphi_0(x) = x \quad (x \in K) \quad (3.2)$$

A semiflow  $\varphi$  on  $K$  induces a family  $(T(t))_{t \geq 0}$  of algebra homomorphisms on  $C(K)$  by

$$T(t)f = f \circ \varphi_t. \quad (3.3)$$

Then clearly  $T(t)T(s) = T(t+s)$  ( $t, s \geq 0$ ); i.e.,  $(T(t))_{t \geq 0}$  has the semigroup property. Conditions for strong continuity are given in the following lemma.

**Lemma 3.2** *The following assertions are equivalent:*

- (a) *The mapping  $\varphi: \mathbb{R}_+ \times K \rightarrow K$  is continuous (where  $\mathbb{R}_+ \times K$  carries the product topology).*
- (b) *The mapping  $\varphi$  is separately continuous.*
- (c)  *$(T(t))_{t \geq 0}$  is a strongly continuous semigroup on  $C(K)$ .*

**Proof** ?? trivially implies ?. If ? holds, then  $t \mapsto T(t)f$  is weakly continuous for every  $f \in C(K)$  (by the theorem of dominated convergence). This implies

strong continuity (see for example [?], Proposition 1.23]). It remains to show that [?] implies [?]. Because of [?] it suffices to show that the restriction  $\tilde{\varphi}$  of  $\varphi$  to  $[0, 1] \times K$  is continuous. By hypothesis, the mapping  $W: f \mapsto (t \mapsto T(t)f)$  from  $C(K)$  into  $C([0, 1], C(K))$  is continuous. Identifying  $C([0, 1], C(K))$  canonically with  $C([0, 1] \times K)$  the mapping  $W$  obtains the form  $f \mapsto f \circ \tilde{\varphi}$ . Since  $W$  is continuous,  $\tilde{\varphi}$  is continuous as well.  $\square$

A semiflow is called *continuous* if it satisfies the equivalent conditions of Lemma [?].

**Definition 3.3** An operator  $\delta$  on  $C(K)$  is called *derivation* if  $D(\delta)$  is a subalgebra of  $C(K)$  such that

$$\delta(f \cdot g) = (\delta f)g + f(\delta g) \quad \text{for all } f, g \in D(\delta). \quad (3.4)$$

$$\mathbb{1} \in D(\delta) \quad (3.5)$$

Note that [?] implies  $\delta \mathbb{1} = 0$ .

A lattice semigroup  $(T(t))_{t \geq 0}$  on  $C(K)$  is called *Markovian* if  $T(t)\mathbb{1} = \mathbb{1}$  for all  $t \geq 0$ .

**Theorem 3.4** Let  $(T(t))_{t \geq 0}$  be a semigroup on  $C(K)$  with generator  $A$ . The following assertions are equivalent.

- (a)  $(T(t))_{t \geq 0}$  is a Markovian lattice semigroup.
- (b)  $T(t)$  is an algebra homomorphism for every  $t \geq 0$ .
- (c) There exists a continuous semiflow  $\varphi$  on  $K$  such that  $T(t)f = f \circ \varphi_t$  ( $t \geq 0$ ).
- (d)  $A$  is a derivation.

**Proof** [?] and [?] are equivalent by the remark at the beginning of this section. Assume that [?] holds. Then there exists a continuous mapping  $\varphi_t: K \rightarrow K$  such that  $T(t)f = f \circ \varphi_t$  for all  $f \in C(K)$  (see B-I, Section 3). The semigroup property implies that  $(\varphi_t)_{t \geq 0}$  is a continuous semiflow. This shows [?] to hold.

If [?] holds, then  $T(t)\mathbb{1} = \mathbb{1}$  for all  $t \geq 0$ . Hence  $\mathbb{1} \in D(A)$  and  $A\mathbb{1} = 0$ . Let  $f, g \in D(A)$ . Then  $\frac{d}{dt}\big|_{t=0} T(t)(f \cdot g) = \frac{d}{dt}\big|_{t=0} (T(t)f) \cdot (T(t)g) = (Af) \cdot g + f \cdot (Ag)$ . Thus  $f \cdot g \in D(A)$  and [?] holds. Hence  $A$  is a derivation.

Finally assume that [?] holds. We prove [?], i.e., we have to show that  $T(t)(f \cdot g) = T(t)f \cdot T(t)g$  for  $t > 0$ . Since  $D(A)$  is a dense subalgebra, we can assume that  $f, g \in D(A)$ . Define  $\eta: [0, t] \rightarrow C(K)$  by  $\eta(s) := T(t-s)[T(s)f \cdot T(s)g]$ . Then  $\eta(0) = T(t)(f \cdot g)$  and  $\eta(t) = T(t)f \cdot T(t)g$ . Since  $A$  is a derivation,  $\eta'(s) = 0$  for  $s \in [0, t]$ . Hence  $\eta(0) = \eta(t)$ . This shows [?] to hold.  $\square$

If  $\delta$  is the generator of a semigroup  $(T(t))_{t \geq 0}$  given by  $T(t)f = f \circ \varphi_t$ , then we call  $\varphi$  given by  $\varphi(t, x) = \varphi_t(x)$  the *semiflow associated with  $(T(t))_{t \geq 0}$*  (or *associated*

with  $\delta$ ). We now can describe the generator of any lattice semigroup as a perturbation of a derivation. If  $\mathbb{1}$  is in the domain of the generator, an additive perturbation (by a multiplication operator) suffices; in general a similarity transformation has to be applied in addition. This is the assertion of the following two theorems.

**Theorem 3.5** *Let  $A$  be a generator of a semigroup  $(T(t))_{t \geq 0}$  on  $C(K)$ . Suppose that  $\mathbb{1} \in D(A)$ . Then the following assertions are equivalent.*

- (a)  $(T(t))_{t \geq 0}$  is a lattice semigroup.
- (b) *There exist a derivation  $\delta$  generating a semigroup (of algebra homomorphisms) and a multiplier  $h \in C(K)$  such that  $A = \delta + h$  (i.e.,  $D(A) = D(\delta)$  and  $Af = \delta f + h \cdot f$  for  $f \in D(A)$ ).*

Moreover, if ?? holds, then  $(T(t))_{t \geq 0}$  is given by

$$(T(t)f)(x) = \exp \left( \int_0^t h(\varphi(s, x)) \, ds \right) \cdot f(\varphi(t, x)) \quad (3.6)$$

where  $\varphi$  is the semiflow associated with  $\delta$ .

**Proof** Let  $h = A\mathbb{1}$  and  $\delta = A - h$ . Then the semigroup  $(T_0(t))_{t \geq 0}$  generated by  $\delta$  is a lattice semigroup if and only if  $(T(t))_{t \geq 0}$  is a lattice semigroup. In fact, by A-II, (1.8)

$$T_0(t)f = \lim_{n \rightarrow \infty} \left( e^{-t/n \cdot h} \cdot T(t/n) \right)^n f \quad \text{and} \quad T(t)f = \lim_{n \rightarrow \infty} \left( e^{t/n \cdot h} \cdot T_0(t/n) \right)^n f$$

for all  $t \geq 0$ ,  $f \in C(K)$

Since  $\delta\mathbb{1} = 0$  one has  $T_0(t)\mathbb{1} = \mathbb{1}$  for all  $t \geq 0$  and the equivalence of ?? and ?? follows with the help of Theorem ??.

Now assume that ?? and ?? hold. Let

$$(S(t)f)(x) = \exp \left( \int_0^t h(\varphi(s, x)) \, ds \right) \cdot f(\varphi(t, x))$$

$x \in K$ ,  $f \in C(K)$ ,  $t \geq 0$ . Then one easily shows that  $(S(t))_{t \geq 0}$  is a strongly continuous semigroup. Denote by  $B$  its generator. For  $f \in D(\delta)$ ,  $\frac{d}{dt}|_{t=0} S(t)f = h \cdot f + \delta f$ . Hence  $\delta + h \subset B$ . Since  $\delta + h$  also is a generator, it follows that  $\delta + h = B$ .  $\square$

**Theorem 3.6** *An operator  $A$  is generator of a lattice semigroup on  $C(K)$  if and only if there exists a derivation  $\delta$  which is a generator, a function  $h \in C(K)$  and a strictly positive function  $p \in C(K)$  such that*

$$A = M\delta M^{-1} + h \quad (3.7)$$

where  $M \in \mathcal{L}((C(K)))$  is given by  $Mf = p \cdot f$ .

**Proof** In order to show the non-trivial implication assume that  $A$  generates a lattice semigroup. Since  $D(A)$  is dense in  $C(K)$  there exists  $0 \ll p \in D(A)$ . Let

$h(x) = (Ap)(x)/p(x)$ . The operator given by  $Mf = f \cdot p$  is a lattice isomorphism. Thus  $\delta := M^{-1}(A - h)M$  generates a lattice semigroup. Since  $M\mathbb{1} = p \in D(A)$  one has  $\mathbb{1} \in D(\delta)$  and  $\delta\mathbb{1} = M^{-1}(A - h)p = 0$ . Thus  $\delta$  is the generator of a semigroup of algebra homomorphisms, hence a derivation by Theorem ??.

At the end of this section we will show that any derivation on  $C[0, 1]$  which generates a group is similar to a differential operator of first order. This in connection with Theorem ?? will enable us to describe all generators of positive groups as perturbations of a differential operator.

In Section 1 we had obtained a very simple condition describing generators of positive semigroups on  $C(K)$  by the positive minimum principle and a range condition. This result yields a characterization of generators of automorphism groups by "locality" and a range condition. By an *automorphism* we understand an algebra isomorphism of  $C(K)$  onto itself.

**Theorem 3.7** *Let  $A$  be a densely defined operator on  $C(K)$ . The following assertions are equivalent.*

- (a)  *$A$  is the generator of an automorphism group.*
- (b)  *$\mathbb{1} \in D(A)$  and  $A\mathbb{1} = 0$ ;  $(\pm\mathbb{1} - A)D(A) = C(K)$  and  $A$  is local, in the sense that for  $0 \leq f \in D(A)$ ,  $f(x) = 0$  implies  $(Af)(x) = 0$  ( $x \in K$ ).*

**Proof** An invertible operator  $T$  such that  $T \geq 0$  and  $T^{-1} \geq 0$  is an automorphism if and only if  $T\mathbb{1} = \mathbb{1}$ . Hence  $A$  is the generator of an automorphism group if and only if  $A$  and  $-A$  generate a positive group,  $\mathbb{1} \in D(A)$  and  $A\mathbb{1} = 0$ . Thus Theorem ?? follows from Theorem ??.

**Remark** It is remarkable that from locality, the range condition and  $\mathbb{1} \in D(A)$ ,  $A\mathbb{1} = 0$  it follows that  $D(A)$  actually is a subalgebra of  $C(K)$  and  $A$  is a derivation. The "order-theoretical" property of locality is in some aspects stronger than the algebraic property of being a derivation. For example a local, densely defined operator is closable (by Corollary ??); but there exist derivations on  $C[0, 1]$  which are not closable (see ? ).

**Remark (an excursion to C\*-algebras)** Theorem ?? also holds for non-commutative C\*-algebras. More precisely: Let  $\mathcal{A}$  be a C\*-algebra with unit  $\mathbb{1}$  and let  $\mathcal{A}_h$  be the real Banach space of all hermitian elements in  $\mathcal{A}$ . Then  $\mathcal{A}_h$  is a real ordered Banach space and  $\mathbb{1}$  is an interior point of  $(\mathcal{A}_h)_+$ . Let  $A$  be a densely defined operator on  $\mathcal{A}_h$ . Then  $A$  is the generator of an automorphism group if and only if  $\mathbb{1} \in D(A)$  and  $A\mathbb{1} = 0$ ;  $(\pm\mathbb{1} - A)(D(A)) = \mathcal{A}_h$  and  $A$  is local in the sense that for  $0 \leq x \in D(A)$ ,  $0 \leq \varphi \in (\mathcal{A}_h)'$ ,  $\varphi(x) = 0$  implies  $\varphi(Ax) = 0$ .

The proof of Theorem ?? can be carried over to this case if one notices the following. A strongly continuous group  $T(t)_{t \in \mathbb{R}}$  on  $\mathcal{A}_h$  is an automorphism group if and only if it is positive and  $T(t)\mathbb{1} = \mathbb{1}$  for all  $t \in \mathbb{R}$  (see ?, Corollary 3.2.21).

Now we let  $X$  be a locally compact space and consider positive groups on  $C_0(X) = C_0(X, \mathbb{R})$ , the space of all continuous real-valued functions on  $X$  which vanish at infinity. Our aim is to describe their generators as perturbations of generators of automorphism groups; i.e., we will extend Theorem 3.6 by allowing  $X$  to be noncompact but restrict ourselves to positive groups (or equivalently semigroups of lattice isomorphisms). And in fact, it is not difficult to show by an example that the corresponding result is wrong for lattice semigroups in general.

A mapping  $\varphi: \mathbb{R} \times X \rightarrow X$  is called a *flow* on  $X$  if the partial maps  $\varphi_t: X \rightarrow X$  given by  $\varphi_t(x) = \varphi(t, x)$  are continuous and satisfy

$$\varphi_0(x) = x \quad (x \in X) \quad (3.8)$$

$$\varphi_s \circ \varphi_t = \varphi_{s+t} \quad (s, t \in \mathbb{R}) \quad (3.9)$$

It follows from the definition that each  $\varphi_t$  is a homeomorphism on  $X$  and  $\varphi_{-t} = (\varphi_t)^{-1}$ .

A flow  $\varphi$  is called *continuous* if it is continuous with respect to the product topology on  $\mathbb{R} \times X$ .

Given a flow  $\varphi$  a family  $(h_t)_{t \in \mathbb{R}} \subset C^b(X)$  is called a *cocycle* of  $\varphi$  if

$$h_0 = \mathbb{1} \quad (3.10)$$

$$h_{t+s} = h_t \cdot (h_s \circ \varphi_t) \quad (s, t \in \mathbb{R}) \quad (3.11)$$

It follows from (??) and (??) that  $h_t(x) \neq 0$  for all  $x \in X$  and  $1/h_t(x) = h_{-t}(\varphi_t(x))$  ( $t \in \mathbb{R}$ ). The cocycle is called *continuous* if the mapping  $(t, x) \mapsto h_t(x)$  from  $\mathbb{R} \times X$  into  $\mathbb{R}$  is continuous with respect to the product topology on  $\mathbb{R} \times X$ .

Let  $\varphi$  be a flow and  $(h_t)_{t \in \mathbb{R}}$  a cocycle of  $\varphi$ . Then

$$T(t)f = h_t \cdot f \circ \varphi_t \quad (3.12)$$

defines a bounded operator  $T(t)$  on  $C_0(X)$  ( $t \in \mathbb{R}$ ). Clearly  $T(s+t) = T(s)T(t)$  for all  $s, t \in \mathbb{R}$ .

**Proposition 3.8** *Let  $\varphi: \mathbb{R} \times X \rightarrow X$  be a flow and  $(h_t)_{t \in \mathbb{R}}$  a cocycle of  $\varphi$ . If for every  $x \in X$  the mappings  $t \mapsto \varphi_t(x)$  and  $t \mapsto h_t(x)$  are continuous, then (??) defines a strongly continuous group.*

**Proof** We first note that  $\|T(t)\|$  is bounded on compact intervals of  $\mathbb{R}$ . This follows from [?, 7.4.1] since  $q(t) = \log \|T(t)\|$  defines a subadditive, measurable function from  $\mathbb{R}$  into  $\mathbb{R}$  [In fact,  $\|T(t)\| = \sup_{x \in X} |h_t(x)|$  for  $t \in \mathbb{R}$ . So it follows from the assumption that  $t \mapsto \|T(t)\|$  is lower semicontinuous and hence measurable]. If  $f \in C_0(X)$ , then by hypothesis the function  $t \mapsto h_t(x)f(\varphi(t, x)) = (T(t)f)(x)$  is continuous on  $\mathbb{R}$ . It follows from the dominated convergence theorem that  $T(\cdot)f$  is

weakly continuous. Hence  $(T(t))_{t \in \mathbb{R}}$  is strongly continuous (see e.g., [?], Proposition 1.23)).  $\square$

The group defined by (??) is positive whenever  $(h_t)_{t \in \mathbb{R}} \subset C^b(X)_+$ . We now show that every positive group on  $C_0(X)$  is of the form (??).

**Proposition 3.9** *Let  $(T(t))_{t \in \mathbb{R}}$  be a strongly continuous group of positive operators on  $C_0(X)$ . Then there exist a continuous flow  $\Phi$  on  $X$  and a continuous cocycle  $(h_t)_{t \in \mathbb{R}}$  of  $\Phi$  such that (??) holds.*

**Proof** Since  $T(t)$  and  $T(t)^{-1} = T(-t)$  are positive operators,  $T(t)$  actually is a lattice isomorphism. Then there exist a homeomorphism  $\Phi_t$  on  $X$  and  $h_t \in C^b(X)_+$  such that  $T(t)f = h_t \cdot f \circ \Phi_t$  for all  $f \in C_0(X)$  ( $t \in \mathbb{R}$ ). The group property of  $(T(t))_{t \in \mathbb{R}}$  then implies that  $\Phi(t, x) := \Phi_t(x)$  defines a flow on  $X$  and that  $(h_t)_{t \in \mathbb{R}}$  is a cocycle of  $\Phi$ . It remains to show that  $\Phi$  and  $(h_t)_{t \in \mathbb{R}}$  are continuous.

At first we consider the flow. Since we have  $\Phi_{t+s} = \Phi_t \circ \Phi_s$  and every  $\Phi_t$  is a homeomorphism on  $X$ , it is enough to establish continuity of  $\Phi$  at points  $(0, x_0) \in \mathbb{R} \times X$ . Given a compact neighbourhood  $V$  of  $x_0 = \Phi(0, x_0)$ , there exists a continuous function  $f: X \rightarrow [0, 1]$  satisfying  $f(x_0) = 1$  and  $\text{supp } f \subset V$ . There exists  $t_0 > 0$  such that  $\|T(t)f - f\| < \frac{1}{2}$  for  $|t| \leq t_0$ . Let  $W := \{x \in X: |f(x)| > \frac{1}{2}\}$ ; then for  $|t| \leq t_0$  and  $x \in W$  we have  $|h_t(x) \cdot f(\Phi(t, x)) - f(x)| < \frac{1}{2}$  and  $|f(x)| > \frac{1}{2}$ ; hence  $f(\Phi(t, x)) > 0$ . This implies that  $\Phi(t, x) \in V$  whenever  $|t| \leq t_0$  and  $x \in W$ .

To prove continuity of the cocycle we first remark that by strong continuity of  $(T(t))_{t \in \mathbb{R}}$  the mapping  $(t, x) \mapsto (T(t)f)(x)$  is continuous on  $\mathbb{R} \times X$  for every fixed  $f \in C_0(X)$ . Given compact subsets  $A \subset \mathbb{R}, B \subset X$ , the set  $C := \Phi(A \times B)$  is compact; hence there exists  $f \in C_0(X)$  such that  $f|_C = 1$ . For  $(t, x) \in A \times B$  we have  $h_t(x) = (T(t)f)(x)$ . Thus  $(t, x) \mapsto h_t(x)$  is continuous on  $A \times B$ .  $\square$

**Corollary 3.10** *Let  $\varphi$  be a separately continuous flow. Then  $\varphi$  is continuous. If  $(h_t)_{t \in \mathbb{R}}$  is a cocycle of  $\varphi$  such that  $t \mapsto h_t(x)$  is continuous for every  $x \in X$ , then  $(h_t)_{t \in \mathbb{R}}$  is continuous.*

This follows from Proposition ?? and Proposition ?? .

**Example 3.11** Let  $\varphi$  be a continuous flow on  $X$ .

(i) Let  $p$  be a continuous function on  $X$  such that  $\inf_{x \in X} p(x) > 0$  and  $\sup_{x \in X} p(x) < \infty$ . Then

$$p_t := p / (p \circ \varphi_t) \quad (t \in \mathbb{R}) \quad (3.13)$$

defines a continuous cocycle of  $\varphi$ .

(ii) For  $h \in C^b(X)$  define

$$h_t(x) := \exp \left( \int_0^t h(\varphi(s, x)) \, ds \right). \quad (3.14)$$

Then  $(h_t)_{t \in \mathbb{R}}$  is a continuous cocycle of  $\varphi$  (compare (??)).

Cocycles as defined by (??) are always globally bounded. In general this is false for cocycles of the second type. On the other hand, a cocycle described by (??) is differentiable with respect to  $t$ . This is not satisfied by cocycles of the first type in general. Thus neither (??) nor (??) gives a description of arbitrary cocycles. However every positive cocycle is a product of cocycles of the form (??) and (??). More precisely, we have the following lemma.

**Lemma 3.12** *Let  $\varphi$  be a continuous flow on  $X$  and  $(k_t)_{t \in \mathbb{R}} \subset C^b(X)_+$  a continuous cocycle of  $\varphi$ . Then there exist  $p \in C^b(X)$  satisfying  $\inf_{x \in X} p(x) > 0$  and  $h \in C^b(X)$  such that*

$$k_t(x) = (p(x)/p(\varphi(t, x))) \cdot \exp \left( \int_0^t h(\varphi(s, x)) \, ds \right) \quad (3.15)$$

for all  $t \in \mathbb{R}$ ,  $x \in X$ .

**Proof** We first note that there exist constants  $M, \omega \geq 1$  such that

$$(Me^{(\omega-1)|t|})^{-1} \leq k_t(x) \leq Me^{(\omega-1)|t|} \quad \text{for all } t \in \mathbb{R}, x \in X. \quad (3.16)$$

In fact, let  $(T(t))_{t \in \mathbb{R}}$  be the group given by  $T(t)f = k_t \cdot f \circ \varphi_t$  ( $t \in \mathbb{R}$ ,  $f \in C_0(X)$ ). Then there exist constants  $M, \omega \geq 1$  such that

$$\|T(t)\| \leq Me^{(\omega-1)|t|} \quad (3.17)$$

for all  $t \in \mathbb{R}$ . Since  $\|T(t)\| = \sup_{x \in X} k_t(x)$  the right inequality of (??) follows. Moreover,  $k_{-t} = 1/(k_t \circ \varphi_{-t})$ . Hence  $\|T(-t)\| = \sup_{x \in X} 1/k_t(x) = [\inf_{x \in X} k_t(x)]^{-1}$ . So (??) also implies the first inequality in (??).

Now we define  $p$  and  $h$  by

$$p(x) := \int_0^\infty e^{-ws} k_s(x) \, ds, \quad h(x) := w - 1/p(x) \quad (x \in X).$$

Then  $p$  is a continuous function and we have

$$\begin{aligned} (M(2w-1))^{-1} &= \int_0^\infty e^{-ws} \left( Me^{(w-1)s} \right)^{-1} ds \\ &\leq p(x) \\ &\leq \int_0^\infty e^{-ws} Me^{(w-1)s} ds \\ &= M \text{ for all } x \in X. \end{aligned}$$

In particular, it follows that  $h \in C^b(X)$ .

For all  $x \in X$ ,  $t \in \mathbb{R}$  we have

$$k_t(x) \cdot p(\varphi(t, x)) = \int_0^\infty e^{-ws} k_{t+s}(x) ds = e^{wt} \int_t^\infty e^{-ws} k_s(x) ds.$$

Now fix  $x \in X$  and define  $f: \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(t) := k_t(x) p(\varphi(t, x)) / p(x) = [e^{wt} / p(x)] \cdot \int_t^\infty e^{-ws} k_s(x) ds.$$

The function  $f$  is differentiable and satisfies the following differential equation

$$f'(t) = wf(t) - k_t(t) / p(x) = h(\varphi(t, x)) f(t).$$

Moreover  $f(0) = 1$ . Hence  $f(t) = \exp\left(\int_0^t h(\varphi(s, x)) ds\right)$  for every  $t \in \mathbb{R}$ . This is (??).  $\square$

As before we call a group  $(T_0(t))_{t \in \mathbb{R}}$  on  $C_0(X)$  an *automorphism group* if each  $T_0(t)$  is an algebra isomorphism on  $C_0(X)$ . Analogously an operator  $\delta$  on  $C_0(X)$  is called a *derivation* if  $D(\delta)$  is a subalgebra of  $C_0(X)$  and  $\delta(f \cdot g) = (\delta f) \cdot g + f \cdot (\delta g)$  for all  $f, g \in D(\delta)$ .

**Proposition 3.13** *Let  $(T_0(t))_{t \in \mathbb{R}}$  be a group on  $C_0(X)$ . The following assertions are equivalent:*

- (a)  $(T_0(t))_{t \in \mathbb{R}}$  is an automorphism group.
- (b) There exists a continuous flow  $\varphi$  on  $X$  such that  $T_0(t)f = f \circ \varphi_t$  for all  $f \in C_0(X)$ , and all  $t \in \mathbb{R}$ .
- (c) The generator of  $(T_0(t))_{t \in \mathbb{R}}$  is a derivation.

**Proof** Every automorphism group is positive. So by Proposition ?? it is defined via (??) by some continuous flow and cocycle. It is easy to see that the cocycle is identically 1. Thus ?? implies ?. One shows as in Theorem ?? that ? implies ? and ? implies ?.

If  $(T_0(t))_{t \in \mathbb{R}}$  is an automorphism group with generator  $\delta$ , the  $\varphi$  in ?? of Proposition ?? is called *the flow associated with  $(T_0(t))_{t \in \mathbb{R}}$*  (or *associated with  $\delta$* ).

Now we can show that every generator of a positive group is a perturbation of a derivation.

**Theorem 3.14** *An operator  $A$  on  $C_0(X)$  is the generator of a positive group  $(T(t))_{t \in \mathbb{R}}$  if and only if there exist a derivation  $\delta$  on  $C_0(X)$  which is the generator of a group, a function  $h \in C^b(X)$  and  $p \in C^b(X)$  satisfying  $\inf_{x \in X} p(x) > 0$  such that*

$$A = V\delta V^{-1} + h \tag{3.18}$$

where  $V: C_0(X) \rightarrow C_0(X)$  is given by  $Vf = p \cdot f$ . In that case one has

$$(T(t)f)(x) = \frac{p(x)}{p(\varphi_t(x))} \cdot \left( \exp \int_0^t h(\varphi(s, x)) \, ds \right) \cdot f(\varphi_t(x)) \quad (3.19)$$

for all  $f \in C_0(X)$ ,  $t \in \mathbb{R}$ ,  $x \in X$ .

Note: (??) means that  $D(A) = \{f : V^{-1}f \in D(\delta)\}$  and  $Af = V\delta V^{-1}f + hf$ .

**Proof** Assume that  $A$  is given by (??). Since  $V$  is a lattice isomorphism, it is clear that  $V^{-1}\delta V$  generates a positive group; and consequently,  $A$  does so as well (cf. the proof of Theorem ??). Conversely, let  $(T(t))_{t \in \mathbb{R}}$  be a positive group with generator  $A$ . By Proposition ?? and Lemma ?? we know that there exist a continuous flow  $\varphi$ ,  $0 \ll p \in C^b(X)$  and  $h \in C^b(X)$  such that (??) holds. Let  $\delta$  be the generator of the automorphism group defined by  $\varphi$ . We have to show that (??) holds. As in Theorem ?? one sees that  $\delta + h$  generates the group  $(S(t))_{t \in \mathbb{R}}$  given by

$$(S(t)f)(x) = \exp \left( \int_0^t h(\varphi(s, x)) \, ds \right) \cdot f(\varphi_t(x)).$$

Hence  $V\delta V^{-1} + h = V(\delta + h)V^{-1}$  generates  $(VS(t)V^{-1})_{t \in \mathbb{R}} = (T(t))_{t \in \mathbb{R}}$ . This is (??).  $\square$

Since every generator of a positive group is the perturbation of a derivation, we now look for examples of derivations which generate a group.

**Example 3.15** Let  $X = \mathbb{R}^n$ . Consider a function  $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$  such that  $\sup_{x \in \mathbb{R}^n} \|DF(x)\| < \infty$  where  $DF(x) \in \mathcal{L}(\mathbb{R}^n)$  denotes the derivative of  $F$  at  $x$ . Then there exists a continuous flow  $\varphi$  on  $\mathbb{R}^n$  such that

$$\frac{\partial}{\partial t} \varphi(t, x) = F(\varphi(t, x)) \text{ for all } t \in \mathbb{R}, x \in \mathbb{R}^n \quad (3.20)$$

Consider the automorphism group  $(T_o(t))_{t \in \mathbb{R}}$  given by  $T_o(t)f = f \circ \varphi_t$  and denote by  $\delta$  its generator. Then

$$D_0 = \{f \in C_0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n) : \lim_{\|x\| \rightarrow \infty} \|(\text{grad } f)(x)\| = 0\}$$

is a core of  $\delta$  and

$$(\delta f)(x) = ((\text{grad } f)(x) | F(x)) \text{ for all } f \in D_0, x \in \mathbb{R}^n, \quad (3.21)$$

where  $(\cdot | \cdot)$  denotes the scalar product in  $\mathbb{R}^n$

**Proof** Let  $f \in D_0$ . Then  $g = f - (\text{grad } f | F) \in C_0(\mathbb{R}^n)$  and

$$\begin{aligned} (R(1, \delta)g)(x) &= \int_0^\infty e^{-t} f(\varphi(t, x)) \, dt - \int_0^\infty e^{-t} ((\text{grad } f)(\varphi(t, x)) | F(\varphi(t, x))) \, dt \\ &= f(x) \end{aligned}$$

by integrating by parts. Hence  $f \in D(\delta)$  and  $f - \delta f = g$ ; i.e.,  $\delta f = (\text{grad } f|F)$ . This proves (??). Next we show  $T_o(t)D_0 \subset D_0$  for all  $t \geq 0$ , which implies that  $D_0$  is a core of  $\delta$  by A-I, Theorem 1.9 (or A-II, Corollary 1.34). Since  $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ , it follows that  $\varphi \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$  (see e.g., ? , 15.2). Moreover for each  $x \in \mathbb{R}^n$ ,  $(D\varphi_t(x))' = DF(\varphi_t(x)) \cdot D\varphi_t(x)$  and  $D\varphi_0(x) = \text{Id}$ , (see ? , p. 300); here  $\text{Id} \in \mathcal{L}(\mathbb{R}^n)$  denotes the identity operator. Hence  $D\varphi_t(x) = \text{Id} + \int_0^t DF(\varphi_s(x)) \cdot D\varphi_s(x) ds$ . Consequently  $\|D\varphi_t(x)\| \leq 1 + \int_0^t M \cdot \|D\varphi_s(x)\| ds$  for all  $t \geq 0$  and  $x \in \mathbb{R}^n$ ; where  $M := \sup_{x \in \mathbb{R}^n} \|DF(x)\| < \infty$  by hypothesis. Hence by Gronwall's inequality,  $\|D\varphi_t(x)\| \leq e^{Mt}$  ( $t \geq 0$ ) for all  $x \in \mathbb{R}^n$ . Now let  $f \in D_0$ ,  $t \geq 0$ . Then  $[\text{grad } (f \circ \varphi_t)](x) = [(\text{grad } f)(\varphi_t(x))] \cdot D\varphi_t(x)$ . Hence  $\|[\text{grad } (f \circ \varphi_t)](x)\| \leq e^{Mt} \|(\text{grad } f)(\varphi_t(x))\|$ , and so

$$\lim_{\|x\| \rightarrow \infty} \|[\text{grad } (f \circ \varphi_t)](x)\| \leq e^{Mt} \lim_{\|x\| \rightarrow \infty} \|(\text{grad } f)(\varphi_t(x))\| = 0.$$

Thus  $f \circ \varphi_t \in D_0$  for all  $t \geq 0$ . □

As a second class of examples we consider derivations on  $C_0(a, b)$ . Eventually we will determine all derivations on  $C_0(a, b)$ , which are generators of a group. We start by looking at differential operators of first order. Let  $-\infty \leq a < b \leq \infty$  and let  $m: (a, b) \rightarrow \mathbb{R}$  be a continuous function. We consider the operator  $\delta_m$  on  $C_0(a, b)$  given by

$$(\delta_m f)(x) = \begin{cases} m(x)f'(x) & \text{if } m(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

with domain

$$D(\delta_m) = \{f \in C_0(a, b) : f \text{ is differentiable at } x \text{ if } m(x) \neq 0 \text{ and } \delta_m f \in C_0(a, b)\}$$

Note that  $\delta_m$  is a derivation on  $C_0(a, b)$ .

**Definition 3.16** A function  $m: (a, b) \rightarrow \mathbb{R}$  is *admissible* if it is continuous and the following holds. Whenever  $a \leq c < d \leq b$  such that  $m(x) \neq 0$  for  $x \in (c, d)$  and

$$c > -\infty, m(c) = 0 \quad \text{or} \quad a = c = -\infty$$

and

$$d < \infty, m(d) = 0 \quad \text{or} \quad d = b = \infty,$$

then

$$\int_c^z 1/|m(x)| dx = \int_z^d 1/|m(x)| dx = \infty \text{ for } z \in (c, d).$$

Note: Every Lipschitz continuous function is admissible.

**Theorem 3.17** Let  $m: (a, b) \rightarrow \mathbb{R}$  be a continuous function. The operator  $\delta_m$  is generator of an automorphism group on  $C_0(a, b)$  if and only if  $m$  is admissible. In that case  $D_0(\delta_m) := \{f \in D(\delta_m) : f \text{ is differentiable on } (a, b)\}$  is a core of  $\delta_m$ .

Additional properties: If  $m$  is admissible, then the flow  $\varphi$  defining the group generated by  $\delta_m$  can be described explicitly: The set  $\{x \in (a, b) : m(x) \neq 0\}$  is the union of a finite or countable number of disjoint intervals  $(a_n, b_n)$  ( $n \in J$ ). Let  $c_n \in (a_n, b_n)$  and  $g_n(x) := \int_{c_n}^x 1/m(y) dy$  ( $x \in (a_n, b_n), n \in J$ ). Since  $m$  is admissible,  $g_n$  is a homeomorphism from  $(a_n, b_n)$  onto  $\mathbb{R}$ . Now the flow  $\varphi$  is defined by

$$\varphi(t, x) = \begin{cases} x & \text{if } m(x) = 0 \\ g_n^{-1}(g_n(x) + t) & \text{if } x \in (a_n, b_n) \end{cases} \quad (3.22)$$

for all  $t \in \mathbb{R}$ .

We first prove a special case of Theorem ??.

**Proposition 3.18** Suppose that  $m(x) \neq 0$  for all  $x \in (a, b)$ . Then  $\delta_m$  is the generator of a group on  $C_0(a, b)$  if and only if  $m$  is admissible. In that case the group generated by  $\delta_m$  is similar to the translation group on  $C_0(\mathbb{R})$ .

**Proof** Let  $q \in C^1(a, b)$  such that  $q'(x) = 1/m(x)$  for all  $x \in (a, b)$ . Then  $q$  is a  $C^1$ -diffeomorphism from  $(a, b)$  onto an interval  $(a', b')$ . By  $Vf = f \circ q$  one defines an isomorphism from  $C_0(a', b')$  onto  $C_0(a, b)$ . Let  $B$  on  $C_0(a', b')$  be given by  $B = V^{-1}\delta_m V$ . Then

$$\begin{aligned} D(B) &= \{g \in C_0(a', b') : g \circ q \in D(\delta_m)\} \\ &= \{g \in C_0(a', b') \cap C^1(a', b') : g' \circ q = m \cdot (g \circ q)' \in C_0(a, b)\} \\ &= \{g \in C_0(a', b') \cap C^1(a', b') : g' \in C_0(a', b')\} \end{aligned}$$

and  $Bg = V^{-1}\delta_m V = V^{-1}(m(g \circ q)') = V^{-1}(g' \circ q) = g'$ . Now observe that  $m$  is admissible if and only if  $a' = -\infty$  and  $b' = \infty$ . If  $a' = -\infty$  and  $b' = \infty$ , then  $B$  is the generator of the translation group on  $C_0(\mathbb{R})$ . Hence also  $\delta_m$  is the generator of a group  $(T(t))_{t \in \mathbb{R}}$  on  $C_0(a, b)$ . Conversely, assume that  $B$  generates a group  $(T(t))_{t \in \mathbb{R}}$ . Assume that  $a' > -\infty$ . Then  $C_0(a', b')$  is a closed subspace of  $C_0[a', b']$ . Let

$$(T_1(t)f)(x) = \begin{cases} f(x+t) & \text{for } x+t < b' \\ 0 & \text{for } x+t \geq b' \end{cases}$$

for all  $f \in C_0[a', b']$ ,  $x \in [a', b']$ ,  $t \geq 0$ . Then  $(T_1(t))_{t \geq 0}$  is a semigroup on  $C_0[a', b']$  with generator  $B_1$  given by  $B_1 f = f'$  with domain

$$D(B_1) = \{f \in C_0[a', b'] \cap C^1(a', b) : \lim_{x \rightarrow b'} f'(x) = 0\}.$$

If we consider  $B$  as an operator on  $C_0[a', b']$ , then  $B \subset B_1$ . Let  $f \in D(B)$ . Then  $u(t) := T(t)f \in D(B) \subset D(B_1)$  for all  $t \geq 0$ ; and  $\dot{u}(t) = Bu(t) = B_1 u(t)$ ;  $u(0) = f$ . It follows from A-I, Theorem 1.7. (or A-II, Corollary 1.2.) that  $T_1(t)f = u(t)$ . Hence  $T_1(t)f \in C_0(a', b')$  for all  $t \geq 0$  and  $f \in D(B)$ . This is impossible since  $a' > -\infty$ . Similary one shows that  $b' = \infty$ .  $\square$

**Proof (Proof of Theorem ??)** Suppose that  $m$  is admissible. It is easy to see that (??) then defines a continuous flow on  $(a, b)$ . Moreover, for every  $x \in (a, b)$  the function  $\varphi(\cdot, x)$  is differentiable and satisfies

$$\frac{\partial}{\partial t} \varphi(t, x) = m(\varphi(t, x)) \quad (x \in (a, b), t \in \mathbb{R}). \quad (3.23)$$

Denote by  $(T(t))_{t \in \mathbb{R}}$  the group on  $C_0(a, b)$  given by  $T(t)f = f \circ \varphi_t$  ( $t \in \mathbb{R}, f \in C_0(a, b)$ ) and let  $A$  be its generator. Take  $g \in C_0(a, b)$  and  $f = R(1, A)g$ . Then  $f(x) = \int_0^\infty e^{-t} g(\varphi(t, x)) dt$ ,  $x \in (a, b)$ . If  $m(x) = 0$  then  $f(x) = \int_0^\infty e^{-t} g(x) dt = g(x)$ . If  $x \in (a_n, b_n)$  ( $n \in J$ ), then

$$f(x) = \int_0^\infty e^{-t} g(q_n^{-1}(q_n(x) + t)) dt = e^{q_n(x)} \int_{q_n(x)}^\infty e^{-s} g(q_n^{-1}(s)) ds.$$

Thus  $f$  is differentiable at  $x$  and  $f'(x) = (1/m(x))(f(x) - g(x))$ . Consequently  $f \in D(\delta_m)$  and  $\delta_m f = f - g$ . This shows that  $A \subset \delta_m$ . In order to show the converse inclusion, let  $f \in D(\delta_m)$  and  $g = f - \delta_m(f) \in C_0(a, b)$ . Then  $R(1, A)g(x) = f(x)$  if  $m(x) = 0$  and

$$\begin{aligned} R(1, A)g(x) &= \int_0^\infty e^{-t} f(\varphi(t, x)) dt - \int_0^\infty e^{-t} m(\varphi(t, x)) f'(\varphi(t, x)) dt \\ &= \int_0^\infty e^{-t} f(\varphi(t, x)) dt - \int_0^\infty e^{-t} \frac{\partial}{\partial t} f(\varphi(t, x)) dt \quad (\text{by (??)}) \\ &= f(x) \end{aligned}$$

by integrating by parts. This shows that  $f = R(1, A)g \in D(A)$  and that  $\delta_m$  is the generator of the group  $(T(t))_{t \in \mathbb{R}}$ . Finally we show that  $T(t)D_0(\delta_m) \subset D_0(\delta_m)$ , which implies that  $D_0(\delta_m)$  is a core (by A-II, Corollary 1.34.). Let  $t \in \mathbb{R}$ . The function  $\varphi_t(\cdot)$  is differentiable on  $(a, b)$  and  $m(x) \frac{\partial}{\partial x} \varphi(t, x) = m(\varphi(t, x))$  for all  $x \in (a, b)$ . Let  $f \in D_0(\delta_m) = D(\delta_m) \cap C^1$ ,  $t \in \mathbb{R}$ . Then  $T(t)f = f \circ \varphi_t$  is differentiable and so in  $D_0(\delta_m)$ .

Conversely, assume that  $\delta_m$  is generator of a group  $(T(t))_{t \in \mathbb{R}}$  on  $C_0(a, b)$ . Since  $\delta_m$  is a derivation, there exists a continuous flow  $(\varphi_t)_{t \in \mathbb{R}}$  on  $(a, b)$  such that  $T(t)f = f \circ \varphi_t$  for all  $f \in C_0(a, b)$ ,  $t \in \mathbb{R}$ . In order to show that  $m$  is admissible let  $a \leq c < d \leq b$  such that  $m(x) \neq 0$  for all  $x \in (c, d)$  and  $m(c) = 0$  or  $a = c = -\infty$  and  $m(d) = 0$  or  $d = b = \infty$ . If  $a < c$  then  $m(c) = 0$ ; consequently  $(\delta_m f)(c) = 0$  for all  $f \in D(\delta_m)$ . Thus  $(T(t)f)(c) = f(c)$  for all  $f \in D(\delta_m)$  and  $t \in \mathbb{R}$ . This shows that  $\varphi(t, c) = c$  for all  $t \in \mathbb{R}$ . Consequently  $\varphi_t(a, c) \subset (a, c)$  for all  $t \in \mathbb{R}$ . Similarly  $\varphi_t(d, b) \subset (d, b)$  for all  $t \in \mathbb{R}$ . Thus the space  $E_o := \{f \in C_0(a, b) : f \text{ vanishes off } (c, d)\}$  is invariant under the group  $(T(t))_{t \in \mathbb{R}}$ . We denote the group restricted to  $E_o$  by  $(T_o(t))_{t \in \mathbb{R}}$  and by  $A_o$  its generator. Then  $D(A_o) = \{f \in E_o \cap D(\delta_m) : \delta_m f \in E_o\}$ . Identifying  $E_o$  with  $C_0(c, d)$  we obtain  $A_o = \delta_{m'}$ , where  $m'$  denotes the restriction of  $m$  to  $(c, d)$ . So it follows from Proposition ?? that  $m'$  is admissible.  $\square$

**Remark 3.19** If  $\varphi$  is a flow on  $(a, b)$ , a point  $x \in (a, b)$  is called *stationary* if  $\varphi(t, x) = x$  for all  $t \in \mathbb{R}$ . Let  $\delta$  be the generator of the group  $(T(t))_{t \in \mathbb{R}}$  associated with  $\varphi$ . Then  $x \in (a, b)$  is a stationary point if and only if  $(\delta f)(x) = 0$  for all  $f \in D(\delta)$ . If  $m$  is an admissible function on  $(a, b)$  then we have seen that  $x \in (a, b)$  is a stationary point of the flow associated with  $\delta_m$  if and only if  $m(x) = 0$ . This does no longer hold for functions which are not admissible as the following example shows.

**Example 3.20** Consider the flow  $\varphi(t, x) = (x^{1/3} + t)^3$  on  $\mathbb{R}$  and the group  $(T(t))_{t \in \mathbb{R}}$  induced by this flow on  $C_0(\mathbb{R})$ . One can easily see that the generator  $\delta$  of  $(T(t))_{t \in \mathbb{R}}$  is the following operator. Let  $m(x) = 3x^{2/3}$ . Then  $(\delta f)(x) = m(x)f'(x)$  for  $x \neq 0$  and

$$D(\delta) = \left\{ f \in C_0(\mathbb{R}) : \begin{array}{l} f \text{ is differentiable at } x \neq 0 \text{ and} \\ m(x)f'(x) \text{ has a continuous extension in } C_0(\mathbb{R}) \end{array} \right\}.$$

However the function  $m$  is not admissible. And in fact  $m(0) = 0$  but 0 is not a stationary point of  $\varphi$ . In particular, there exists a function  $f \in D(\delta)$  such that  $(\delta f)(0) \neq 0$ .

Next we describe an arbitrary continuous flow on an open interval.

**Proposition 3.21** Let  $-\infty \leq a < b \leq \infty$ . A mapping  $\varphi: \mathbb{R} \times (a, b) \rightarrow (a, b)$  defines a continuous flow if and only if there exists a finite or countable set of disjoint intervals  $(a_n, b_n) \subset (a, b)$  ( $n \in J$ ) and for every  $n \in J$  there exists a homeomorphism  $r_n$  from  $(a_n, b_n)$  onto  $(-\infty, \infty)$  such that

$$\varphi(t, x) = \begin{cases} x & \text{if } x \notin \bigcup_{n \in J} (a_n, b_n) \\ r_n^{-1}(r_n(x) + t) & \text{if } x \in (a_n, b_n), n \in J \end{cases}$$

for all  $t \in \mathbb{R}$

Note:  $J = \emptyset$  if and only if  $\varphi(t, x) = x$  for all  $x \in (a, b)$  and  $t \in \mathbb{R}$ .

**Proof** It is not difficult to see that the construction in the proposition defines a continuous flow on  $(a, b)$ . Now let  $\varphi$  be a continuous flow. The set

$$K = \{x \in (a, b) : \varphi(t, x) = x \text{ for all } t \in \mathbb{R}\}$$

is closed in  $(a, b)$ . Thus  $(a, b) \setminus K$  is the union of a finite or countable set of disjoint intervals  $(a_n, b_n)$  ( $n \in J$ ). Pick  $x_n \in (a_n, b_n)$  ( $n \in J$ ). Then  $\alpha_n(t) := \varphi(t, x_n)$  defines an injective mapping from  $\mathbb{R}$  into  $(a_n, b_n)$ . Thus  $\alpha_n$  is strictly monotonous. It is easy to see that  $\lim_{t \rightarrow \infty} \varphi(t, x_n)$  is an element of  $K$  whenever the limit exists in  $(a, b)$ ; similarly for the limit as  $t \rightarrow -\infty$ . Consequently,  $\alpha_n(\mathbb{R}) = (a_n, b_n)$  i.e.,  $\alpha_n$  is a homeomorphism from  $\mathbb{R}$  onto  $(a_n, b_n)$ . Define  $r_n$  to be the inverse of  $\alpha_n$ . Let  $x \in (a_n, b_n)$ . Then

$$\begin{aligned}
\varphi(t, x) &= \varphi(t, \alpha_n(r_n(x))) = \varphi(t, \varphi(r_n(x), x_n)) \\
&= \varphi(t + r_n(x), x_n) = \alpha_n(t + r_n(x)) \\
&= r_n^{-1}(r_n(x) + t)
\end{aligned}$$

for all  $t \in \mathbb{R}$ . This proves that  $\varphi$  has the desired form.  $\square$

If  $m$  is an admissible function on  $(a, b)$ , then  $D(\delta)$  contains many differentiable functions. This can be expressed by two facts:

- (i)  $C_c^1(a, b) := \{f \in C^1(a, b) : f \text{ vanishes in a neighbourhood of } a \text{ and } b\}$  is contained in  $D(\delta_m)$  (this follows from the definition of  $\delta_m$ ); and
- (ii) the set  $D_0(\delta_m)$  of all differentiable functions in  $D_0(\delta_m)$  is a core of  $\delta_m$  (this follows from Theorem ??).

We will show below that these two properties are characteristic for the operators  $\delta_m$ . For other generators of automorphism groups they can be violated dramatically as the following example shows.

**Example 3.22** There exists a generator  $\delta$  of an automorphism group on  $C_0(\mathbb{R})$  such that  $D(\delta) \cap C^1(\mathbb{R}) = \{0\}$ . In fact, consider a strictly increasing continuous map  $q$  from  $\mathbb{R}$  onto  $\mathbb{R}$  such that  $q'(x) = 0$  a.e. Then  $V : C_0(\mathbb{R}) \rightarrow C_0(\mathbb{R})$  given by  $Vf = f \circ q$  is an algebra isomorphism. Let  $A$  be the generator of the translation group on  $C_0(\mathbb{R})$  and  $\delta = V^{-1}AV$ . Then

$$D(\delta) = \{f \in C_0(\mathbb{R}) : Vf \in D(A)\} = \{f \in C_0(\mathbb{R}) : f \circ q \in C^1(\mathbb{R}), (f \circ q)' \in C_0(\mathbb{R})\}.$$

Let  $f \in C^1(\mathbb{R}) \cap D(\delta)$ . If  $f \neq 0$ , then  $f$  is not constant. Hence there exists  $x_o \in \mathbb{R}$  such that  $f'(x_o) \neq 0$ . Then  $f$  has a continuously differentiable inverse in some open neighbourhood of  $x_o$ . Since  $f \circ q \in C^1(\mathbb{R})$ , it follows that  $q$  is continuously differentiable in some neighborhood of  $q^{-1}(x_o)$ . This is a contradiction since  $q'(y) = 0$  a.e.

**Theorem 3.23** Let  $\delta$  be the generator of an automorphism group on  $C_0((a, b))$ , where  $-\infty \leq a < b \leq \infty$ . The following assertions are equivalent.

- (a) There exists a continuous admissible function  $m : (a, b) \rightarrow \mathbb{R}$  such that  $\delta = \delta_m$ .
- (b)  $C_c^1(a, b) \subset D(\delta)$  and  $D_0(\delta) = \{f \in D(\delta) : f \text{ is differentiable}\}$  is a core of  $\delta$ .

**Proof** We have already pointed out that as a consequence of Theorem ??, ?? implies ??. So assume that ?? holds. Let  $(T(t))_{t \in \mathbb{R}}$  be the group generated by  $\delta$  and  $\varphi$  the continuous flow associated with the group. We can assume that  $\varphi$  is of the form given in Proposition ??. Let  $n \in J$ . We show that  $r_n^{-1} : \mathbb{R} \rightarrow (a_n, b_n)$  is continuously differentiable. Let  $x_o \in (a_n, b_n)$ . There exists  $f \in C_c^1(a, b)$  such that  $f(x) = x$  in a neighborhood of  $x_o$ . Then  $r_n^{-1}(r_n(x_o) + t) = f(\varphi(t, x_o)) = (T(t)f)(x_o)$  for all  $t$  in some neighborhood of 0. Since  $f \in D(\delta)$  it follows that the function  $t \mapsto r_n^{-1}(r_n(x_o) + t)$  is continuously differentiable in some neighborhood of 0 and so

$r_n^{-1}$  is continuously differentiable at  $r_n(x_o)$ . Since  $r_n: (a_n, b_n) \rightarrow \mathbb{R}$  is surjective this proves the claim. Next we show  $(r_n^{-1})'(t) \neq 0$  for all  $t \in \mathbb{R}$ . In fact, let  $x_o \in (a_n, b_n)$  and assume that  $(r_n^{-1})'(r_n(x_o)) = 0$ . Then for all  $f \in D_0(\delta)$  one has

$$(\delta f)(x_o) = \frac{\partial}{\partial t} \Big|_{t=0} f(r_n^{-1}(r_n(x_o) + t)) = f'(x_o)(r_n^{-1})'(r_n(x_o)) = 0.$$

Since  $D_0(\delta)$  is a core of  $\delta$  this implies that  $\varphi(t, x_o) = x_o$  for all  $t \in \mathbb{R}$ . Hence  $x_o \in K$ , a contradiction. It follows that  $r_n: (a_n, b_n) \rightarrow \mathbb{R}$  is a  $C^1$ -diffeomorphism for all  $n \in J$ .

Define  $m: (a, b) \rightarrow \mathbb{R}$  by

$$m(x) = \begin{cases} 0 & \text{if } x \in K \\ 1/r'_n(x) & \text{if } x \in (a_n, b_n), n \in J \end{cases}$$

Then  $m$  is continuous and admissible. The given flow coincides with the one constructed from  $m$  in Theorem ???. Thus  $\delta = \delta_m$ .  $\square$

**Remark** Let  $m: (a, b) \rightarrow \mathbb{R}$  be continuous. Then  $m$  is admissible if and only if the initial value problem

$$\begin{aligned} \dot{y}(t) &= m(y(t)) \quad (t \in \mathbb{R}) \\ y(0) &= x \end{aligned}$$

has a unique solution  $y \in C^1(\mathbb{R}, (a, b))$  which depends continuously on the initial value  $x$  (i.e., if  $x_n \rightarrow x$  in  $(a, b)$  then the solution  $y_n \in C^1(\mathbb{R}, (a, b))$  with initial value  $y_n(0) = x_n$  satisfies  $y_n(t) \rightarrow y(t)$  ( $n \rightarrow \infty$ ) for all  $t \in \mathbb{R}$ ). This is not difficult to see.

As we have seen above the operators  $\delta_m$ , where  $m$  is an admissible function, do not exhaust all generators of automorphism groups. But one can obtain every such generator by a similarity transformation (see A-I, 3.1) from some  $\delta_m$ .

**Theorem 3.24** Let  $-\infty \leq a < b \leq \infty$ . An operator  $\delta$  on  $C_0(a, b)$  is the generator of an automorphism group on  $C_0(a, b)$  if and only if there exists an algebra isomorphism  $V$  from  $C_0(a, b)$  onto  $C_0(a, b)$  and an admissible function  $m: (a, b) \rightarrow \mathbb{R}$  such that  $\delta = V^{-1}\delta_m V$ .

**Proof** In order to prove the non-trivial implication let  $(T(t))_{t \in \mathbb{R}}$  be an automorphism group on  $C_0(a, b)$  with generator  $\delta$ . Let  $\varphi$  be the continuous flow on  $(a, b)$  such that  $T(t)f = f \circ \varphi_t$  ( $f \in C_0(a, b), t \in \mathbb{R}$ ). Then  $\varphi$  is of the form given in Proposition ???. For every  $n \in J$  choose a  $C^1$ -diffeomorphism  $q_n$  from  $(a_n, b_n)$  onto  $(-\infty, \infty)$  satisfying  $q'_n(x) > 0$  for all  $x \in (a_n, b_n)$  in the case when  $r_n$  is increasing and  $q'_n(x) < 0$  for all  $x \in (a_n, b_n)$  in the case when  $r_n$  is decreasing. Then  $\beta_n := r_n^{-1} \circ q_n$  is a homeomorphism from  $(a_n, b_n)$  onto itself satisfying  $\lim_{x \downarrow a_n} \beta_n(x) = a_n$  and  $\lim_{x \uparrow b_n} \beta_n(x) = b_n$ . Let  $\beta: (a, b) \rightarrow (a, b)$  be defined by

$$\beta(x) = \begin{cases} x & \text{if } x \in K \\ \beta_n(x) & \text{if } x \in (a_n, b_n), n \in J \end{cases}$$

Then  $\beta$  is a homeomorphism from  $(a, b)$  onto  $(a, b)$  and  $\psi_t := \beta^{-1} \circ \varphi_t \circ \beta$  ( $t \in \mathbb{R}$ ) defines a continuous flow on  $(a, b)$ . Define  $m: (a, b) \rightarrow \mathbb{R}$  by

$$m(x) = \begin{cases} 0 & \text{if } x \in K \\ 1/q'_n(x) & \text{if } x \in (a_n, b_n) \end{cases}$$

Then  $m$  is continuous and admissible and the flow  $\psi$  coincides with the flow constructed from  $m$  in Theorem ?? . Hence  $\delta_m$  is the generator of the group  $(S(t))_{t \in \mathbb{R}}$  given by  $S(t)f = f \circ \psi_t = f \circ \beta^{-1} \circ \varphi_t \circ \beta = VT(t)V^{-1}f$ , where  $V$  is the isomorphism on  $C_0(a, b)$  given by  $Vf = f \circ \beta$ . Consequently,  $\delta = V^{-1}\delta_m V$ .  $\square$

Now we are able to describe arbitrary generators of positive groups on  $C_0(a, b)$ .

**Theorem 3.25** *Let  $-\infty \leq a < b \leq \infty$ . An operator  $A$  generates a positive group on  $C_0(a, b)$  if and only if there exist*

- (i) *a lattice isomorphism  $V$  on  $C_0(a, b)$ ,*
- (ii) *an admissible function  $m$  on  $(a, b)$ ,*
- (iii) *a bounded continuous function  $h: (a, b) \rightarrow \mathbb{R}$  such that*

$$A = V^{-1}\delta_m V + h. \quad (3.24)$$

**Proof** Let  $A$  be the generator of a positive group on  $C_0(a, b)$ . By Theorem ?? there exist a continuous bounded function  $p: (a, b) \rightarrow \mathbb{R}$  such that  $\inf_{x \in (a, b)} p(x) > 0$  and  $h \in C^b(a, b)$  and the generator  $\delta$  of an automorphism group such that  $A = M\delta M^{-1} + h$  where  $M \in \mathcal{L}(C_0(a, b))$  is given by  $Mf = p \cdot f$ . By Theorem ?? there exist an admissible continuous function  $m: (a, b) \rightarrow \mathbb{R}$  and a lattice isomorphism  $U \in \mathcal{L}(C_0(a, b))$  such that  $\delta = U\delta_m U^{-1}$ . Setting  $V = MU$  we obtain  $A = V\delta_m V^{-1} + h$ .  $\square$

Finally we consider compact intervals. Let  $-\infty < a < b < \infty$  and  $\varphi$  be a continuous flow on  $[a, b]$ . Then it is easy to see that  $\varphi(a, t) = a$  and  $\varphi(b, t) = b$  for all  $t \in \mathbb{R}$ . So the restriction  $\varphi_0$  of  $\varphi$  to  $(a, b)$  is a continuous flow on  $(a, b)$ . Conversely, if  $\varphi_0$  is a continuous flow on  $(a, b)$  the extension  $\varphi_0$  to  $\varphi: \mathbb{R} \times [a, b] \rightarrow [a, b]$  by setting  $\varphi(t, a) = a$ ;  $\varphi(t, b) = b$  for all  $t \in \mathbb{R}$  defines a continuous flow on  $[a, b]$ . This consideration allows us to extend easily the preceding results to the space  $C[a, b]$ . Let  $m: (a, b) \rightarrow \mathbb{R}$  be a continuous function. We define the operator  $\tilde{\delta}_m$  on  $C[a, b]$  by  $\tilde{\delta}_m f = g$  such that

$$g(x) = \begin{cases} m(x)f'(x) & \text{if } m(x) \neq 0 \\ 0 & \text{if } m(x) = 0 \end{cases} \quad (3.25)$$

for all  $x \in (a, b)$  and

$$D(\tilde{\delta}_m) = \left\{ f \in C[a, b] : \begin{array}{l} f \text{ is differentiable at } x \in (a, b) \text{ whenever } m(x) \neq 0 \\ \text{and there exists a (necessarily unique) } g \in C[a, b] \\ \text{such that (??) holds} \end{array} \right\}.$$

**Theorem 3.26** *Let  $m$  be a continuous function on  $(a, b)$ . The operator  $\tilde{\delta}_m$  is generator of an automorphism group on  $C[a, b]$  if and only if  $m$  is admissible.*

**Proof** If  $\tilde{\delta}_m$  generates an automorphism group  $(T(t))_{t \in \mathbb{R}}$  then by the remark above  $T(t)C_0(a, b) = C_0(a, b)$  ( $t \in \mathbb{R}$ ). The generator of the restricted group has the domain  $\{f \in C_0(a, b) \cap D(\tilde{\delta}_m) : \tilde{\delta}_m f \in C_0(a, b)\} = D(\delta_m)$ . Hence  $\delta_m$  is a generator and so  $m$  is admissible by Theorem ???. Conversely, if  $m$  is admissible, then  $\delta_m$  generates a group on  $C_0(a, b)$  given by a flow  $\varphi_0$  on  $(a, b)$ . Extending  $\varphi_0$  to  $[a, b]$  as above one obtains a continuous flow  $\varphi$  on  $[a, b]$  which defines a group  $(T(t))_{t \in \mathbb{R}}$ . It is easy to verify, that the generator of this group is  $\tilde{\delta}_m$ .  $\square$

**Theorem 3.27** *Let  $\delta$  be the generator of an automorphism group on  $C[a, b]$ . Then there exists an admissible function  $m : (a, b) \rightarrow \mathbb{R}$  and an algebra isomorphism  $V$  from  $C[a, b]$  onto  $C[a, b]$  such that  $\delta = V^{-1}\tilde{\delta}_m V$ .*

**Proof** The restriction  $\delta_o$  of  $\delta$  to  $C_0(a, b)$  is the generator of an automorphism group. Thus by Theorem ??? there exists a continuous admissible function  $m : (a, b) \rightarrow \mathbb{R}$  and an algebra isomorphism  $V_o$  from  $C_0(a, b)$  onto  $C_0(a, b)$  such that  $\delta_o = V_o^{-1}\delta_m V_o$ . Let  $V$  be the unique algebra isomorphism on  $C[a, b]$  which extends  $V_o$ . Then it is easy to see that  $\delta = V^{-1}\tilde{\delta}_m V$ .  $\square$

**Theorem 3.28** *An operator  $A$  on  $C[a, b]$  is the generator of a positive group on  $C[a, b]$  if and only if there exist*

- (i) *a lattice isomorphism  $V$  on  $C[a, b]$*
- (ii) *an admissible function  $m : (a, b) \rightarrow \mathbb{R}$*
- (iii) *and a function  $h \in C[a, b]$  such that  $A = V^{-1}\delta_m V + h$ .*

The proof follows from Theorem ??? via Theorem ??? in the same way as Theorem ??? (via Theorem ???).

## Notes

*Section 1:* Concerning bounded generators of positive semigroups and the positive minimum principle we refer to the corresponding notes in Chapter C-II, Theorem ?? and ?? are due to ? ], but we give a more direct proof here. Theorem ?? and its corollary are from the same source. In the case when  $A$  is dissipative Theorem ?? is

due to [?]. We use precisely Dorroh's arguments to verify the range condition. Other extensions of Dorroh's result have been given by [?] and [?].

*Section 2:* A characterization of generators of lattice semigroups by Kato's equality is due to [?] if the underlying space has order continuous norm (see C-II, Section 5), for general Banach spaces and  $C_0(X)$  in particular the problem has been considered in [?]. Theorem ?? is due to [?].

*Section 3:* The characterization of generators of lattice semigroups as perturbation of a derivation (Theorem ?? and ??) is due to [?]. The corresponding result for positive groups on  $C_0(X)$  (Theorem ??) was obtained by [?]. [?] consider multiplicative perturbations of a generator of an automorphism group on  $C(K)$  ( $K$  compact) by a function  $m$  which has a finite number of zeros. The function  $m$  is assumed to satisfy the "generalized Osgood condition" which is similar to being *admissible* (in our sense) but in addition the given flow is involved in the definition.

[?] determined all densely defined derivations  $\delta$  on  $C[0, 1]$  which are well-behaved (i.e.,  $\pm\delta$  is dispersive) by a representation similar to Theorem ??. In contrast to Batty, here we assume that  $\delta$  is the generator of a group. This simplifies the matter considerably since all continuous flows on an interval are easy to determine (Proposition ??). Our approach is inspired by [?] to whom Theorem ?? is due.

For simplicity we confined ourselves to groups. [?] determined all semi-flows on an interval.

In the sequel of Batty's work (loc.cit.) a characterization of all densely defined closed derivations on  $C[0, 1]$  has been obtained by Kurose in a series of papers [? ? ?].

## References