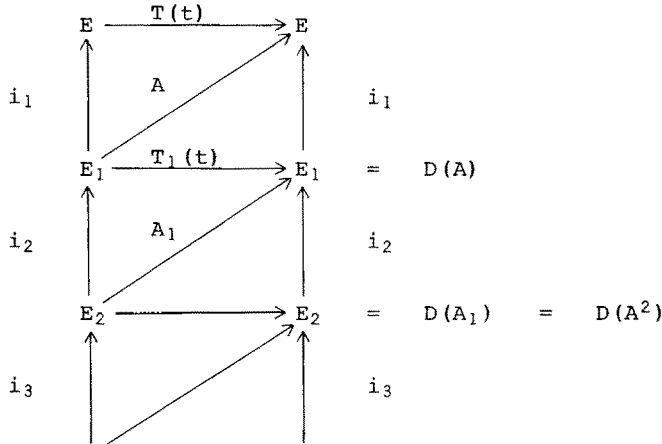


$$D(A_1) = \{f \in E_1 : Af \in E_1\} = D(A^2) \quad \text{and}$$

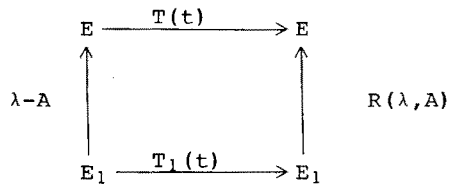
$$A_1 f = Af \quad \text{for every } f \in D(A_1) .$$

It is now possible to repeat this construction in order to obtain Banach spaces E_n and semigroups $(T_n(t))_{t \geq 0}$ with generators $(A_n, D(A_n))$ which are related as visualized in the following diagram:



For the translation semigroup on $L^p(\mathbb{R})$ (see 2.3) the above construction leads to the usual 'Sobolev spaces'. Therefore we might call E_n the n -th Sobolev space and $(T_n(t))_{t \geq 0}$ the n -th Sobolev semigroup associated to E and $(T(t))_{t \geq 0}$.

Remarks: 1. For $\lambda \in \rho(A)$ the operator $(\lambda - A)$ and the resolvent $R(\lambda, A)$ are isomorphisms from E_1 onto E , resp. from E onto E_1 (show that $\|\cdot\|_1$ and $\|\cdot\|_\lambda$ with $\|\cdot\|_\lambda := \|(\lambda - A) \cdot\|$ are equivalent). In addition, the diagram



commutes. Therefore all Sobolev semigroups $(E_n, (T_n(t))_{t \geq 0})$, $n \in \mathbb{N}$, are isomorphic.

2. For $\lambda \in \rho(A)$ consider the norm

$$\|f\|_{-1} := \|R(\lambda, A)f\|$$

for every $f \in E$ and define E_{-1} as the completion of E for $\|\cdot\|_{-1}$.