If $\lambda = \mu + i\nu$ with μ , ν real and $\mu > s(A)$ we have for arbitrary $f \in E$, $\phi \in E'$:

$|<\int_r^t e^{-\lambda s}T(s)f\, ds,\phi>| \le \int_r^t e^{-\mu s}<T(s)|f|,|\phi|>ds$ hence

$\|<\int_r^t e^{-\lambda s}T(s)f\, ds\| \le \|\int_r^t e^{-\mu s}T(s)|f|\, ds\|$ which shows that

$\lim_{t\to\infty} \int_0^t e^{-\lambda s}T(s)f\, ds$ exists .

Thus $R(\lambda,A)f = \int_0^\infty e^{-\lambda s}\, T(s)f\, ds$ by A-I,Prop.1.11.

It remains to prove that the net $(\int_0^r \exp(-\mu s)T(s)\, ds)_{r\ge 0}$ converges with respect to the operator norm. We fix μ such that $s(A) < \mu < \mathrm{Re}\,\lambda$. As we have seen above the map $s \to e^{-\mu s}<T(s)f,\phi>$ is Lebesgue integrable for every $(f,\phi) \in E \times E'$, thus defining a bilinear map $b : E \times E' \to L^1(\mathbb{R}_+)$. Using the closed graph theorem it is easy to see that b is separately continuous, hence jointly continuous by [Schaefer (1966), III.Thm.5.1] . Thus there is a constant M such that

$$(1.4) \qquad \int_0^\infty e^{-\mu s}|<T(s)f,\phi>|\, ds = \|b(f,\phi)\| \le M\|f\|\|\phi\| \qquad (f \in E\ ,\ \phi \in E')$$

Given $0 \le t < r$ and setting $\varepsilon := \mathrm{Re}\,\lambda - \mu$ we have:

$|\int_t^r e^{-\lambda s}<T(s)f,\phi>\, ds| \le \int_t^r \exp(-(\mathrm{Re}\lambda-\mu)s)e^{-\lambda s}|<T(s)f,\phi>|\, ds$

$\le e^{-\varepsilon t}\int_t^r e^{-\lambda s}|<T(s)f,\phi>|\, ds \le e^{-\varepsilon t}\, M\|f\|\|\phi\|$.

It follows that $\|\int_t^r e^{-\lambda s}T(s)\, ds\| \le Me^{-\varepsilon t}$, hence $(\int_0^t e^{-\lambda s}T(s)\, ds)_{t\ge 0}$ is a Cauchy net with respect to the operator norm.

□

Theorem 1.2 has many consequences. In particular, we can conclude that $s(A) \in \sigma(A)$ whenever $s(A) > -\infty$ (without using the analogous result for bounded operators, cf. Cor.1.4 below). In each of the following corollaries we assume that A is the generator of a positive semigroup $(T(t))_{t\ge 0}$ on a Banach lattice E .

<u>Corollary</u> 1.3. If $\mathrm{Re}\,\lambda > s(A)$ then we have

$$(1.5) \qquad |R(\lambda,A)f| \le R(\mathrm{Re}\lambda,A)|f| \qquad (f \in E) .$$

The proof is an immediate consequence of Thm.1.2.

<u>Corollary</u> 1.4. We have $s(A) \in \sigma(A)$ unless $s(A) = -\infty$.

<u>Proof</u>. Assume that $s(A) > -\infty$ and $s(A) \notin \sigma(A)$, then it follows from (1.5) that $\{R(\lambda,A) : \mathrm{Re}\lambda > s(A)\}$ is uniformly bounded in $L(E)$,