that we need that the modulus function is differentiable. If $E = L^p(X,\Sigma,\mu)$ $(1 \leq p < \infty)$ this had been proved in Section 2 (Ex.2.3). We extend this result to Banach lattices with order continuous norm.

Proposition 5.6. Let E be a real or complex Banach lattice with order continuous norm. Then the modulus function $\Theta : E \to E$ (given by $\Theta(h) = |h|$) is right-sided Gateaux differerentiable and

$$D_g\Theta(f) = \mathrm{Re}((\mathrm{si\hat{g}n}\ \bar{f})g) \qquad (f,\ g \in E) \ . \tag{5.6}$$

Proof. Let $f,\ g \in E$. Define $k : \mathbb{R} \to E$ by $k(t) = |f+tg| - |f|$. Then $k(0) = 0$ and k is convex (i.e., $k(\lambda s + (1-\lambda)t) \leq \lambda k(s) + (1-\lambda)k(t)$ for all $s,\ t \in \mathbb{R},\ \lambda \in [0,1]$).
We show that

$$k(s)/s \leq k(t)/t \tag{5.7}$$

whenever $s < t$, $s,t \neq 0$.

First case: $s < t < 0$.
Choose $\lambda = t/s \in (0,1)$. Then $t = (1-\lambda)0 + \lambda s$. Consequently,
$k(t) \leq (1-\lambda)k(0) + \lambda k(s) = t/s\ k(s)$.
Second case: $s < 0 < t$.
Let $0 < \lambda := t/(t-s) < 1$. Then $0 = \lambda s + (1-\lambda)t$. Hence $0 = k(0) \leq \lambda k(s) + (1-\lambda)k(t) = t/(t-s)\ k(s) - s/(t-s)\ k(t)$, which implies (5.7).
Third case: $0 < s < t$.
Let $\lambda = s/t \in (0,1)$. Then $s = (1-\lambda)0 + \lambda t$. Consequently, $k(s) \leq (1-\lambda)k(0) + \lambda k(t) = s/t\ k(t)$, which implies (5.7).

It follows from (5.7) that the net $(k(t)/t)_{t>0}$ is decreasing and bounded below (by $-k(-1)$, for instance). Since E has order continuous norm, it follows that $D_g\Theta(f) = \lim_{t\to 0+} k(t)/t$ exists.

It remains to show that $D_g\Theta(f) = \mathrm{Re}(\mathrm{si\hat{g}n}\ \bar{f})g$.
First of all denote by P the band projection onto $\{f\}^{dd}$. Then it follows from the definition of $D_g\Theta(f)$ that $D_g\Theta(f) = PD_g\Theta(f) + (Id-P)D_g\Theta(f) = D_{Pg}\Theta(f) + |(Id-P)g|$. Thus it remains to show that

$$D_h\Theta(f) = \mathrm{Re}((\mathrm{sign}\ \bar{f})h) \qquad \text{whenever} \quad h \in \{f\}^{dd} \ . \tag{5.8}$$

According to the Kakutani-Krein theorem there exists a compact space K such that $E_{|f|}$ can be identified with $C(K)$. Then by B-II, Lemma 2.4

$$\lim_{t\to 0+} 1/t(|f+th| - |f|)(x) = \mathrm{Re}(\mathrm{sign}(\overline{f(x)})h(x)) \qquad (x \in K). \tag{5.9}$$

Let $\phi \in E'_+$. Then ϕ restricted to $E_{|f|}$ can be identified with a regular Borel measure μ on $C(K)$.