6.6 <u>Spectral Mapping Theorem for Eventually Norm Continuous Semigroups</u>.

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be an eventually norm continuous semigroup with generator A . Then the spectral mapping theorem is valid, i.e.

(6.6) $\qquad \sigma(T(t)) \setminus \{0\} = e^{t\cdot\sigma(A)} \qquad$ for every $t \geq 0$.

<u>Proof</u>. By the previous considerations it suffices to show that $A\sigma(T(t)) \setminus \{0\} \subset e^{t\cdot\sigma(A)}$ for $t > 0$. This will be done by converting approximate eigenvectors into eigenvectors in the semigroup $\mathcal{F}$-product (see 4.5). The assertion then follows from (6.4) and assertion (ii) of the Proposition in 4.5.

Assume $t \to T(t)$ to be norm continuous for $t \geq t_o$. Moreover it suffices to consider $1 \in A\sigma(T(t_1))$ for some $t_1 > 0$, i.e. we have a normalized sequence $(f_n)_{n\in\mathbb{N}} \subset E$ such that

$$\lim_{n\to\infty} \|T(t_1)f_n - f_n\| = 0 .$$

Choose $k \in \mathbb{N}$ such that $kt_1 > t_o$ and define $g_n := T(kt_1)f_n$. Then

$$\lim_{n\to\infty}\|g_n\| = \lim_{n\to\infty}\|T(t_1)^k f_n\| = \lim_{n\to\infty}\|f_n\| = 1$$

and $$\lim_{n\to\infty} \|T(t_1)g_n - g_n\| = 0 ,$$

i.e. $(g_n)_{n\in\mathbb{N}}$ yields an approximate eigenvector of $T(t_1)$ with approximate eigenvalue 1 . But the semigroup $\mathcal{T}$ is uniformly continuous on sets of the form $T(t_o)V$, V bounded in E . In particular, it is uniformly continuous on the sequence $(g_n)_{n\in\mathbb{N}_\mathcal{T}}$, which therefore defines an element $\hat{g}$ in the semigroup $\mathcal{F}$-product $E_\mathcal{F}^\mathcal{T}$.

Obviously, $\hat{g}$ is an eigenvector of $T_\mathcal{F}(t_1)$ with eigenvalue 1 and by (6.4) we obtain an eigenvalue $2\pi in/t_1$ of $A_\mathcal{F}$ for some $n \in \mathbb{Z}$. The coincidence of $\sigma(A)$ and $\sigma(A_\mathcal{F})$ proves the assertion.

□

We point out that the above spectral mapping theorem implies the coincidence of spectral bound and growth bound for eventually norm continuous semigroups, hence we have generalized the Liapunov Stability Theorem 1.2 to a much larger class of semigroups. As mentioned before this will be of great use in many applications. Therefore we state explicitly the spectral mapping theorem for several important classes of semigroups all of which are eventually norm continuous (cf. the diagram preceding A-II,Ex.1.27).

<u>Corollary</u> 6.7. The spectral mapping theorem

(6.6) $\qquad \sigma(T(t)) \setminus \{0\} = e^{t\cdot\sigma(A)} , \quad t \geq 0 ,$

holds for each of the following classes of strongly continuous semigroups: