$$(1.16)\quad (r - T')\Psi_n = \|R(\lambda_n,T)'\Psi\|^{-1}\cdot\big((r-\lambda_n) + (\lambda_n - T')\big)R(\lambda_n,T)'\Psi =$$
$$= (r-\lambda_n)\Psi_n + \|R(\lambda_n,T)'\Psi\|^{-1}\Psi \to 0 .$$

Since $r - T'$ is $\sigma(M(K),C(K))$-continuous, (1.16) implies that every $\sigma(M(K),C(K))$ cluster point of (Ψ_n) is a positive eigenvector, provided that it is non-zero. Because K is compact we have $\{\phi \in M(K) : \phi \geq 0 , \|\phi\| = 1\} = \{\phi \in M(K) : \phi \geq 0 , \langle\phi,1\rangle = 1\}$ which shows that the set of probability measures is $\sigma(M(K),C(K))$-compact. Therefore the sequence (Ψ_n) has non-zero cluster points.

□

This theorem implies that for positive semigroups on $C(K)$ the growth and spectral bounds coincide (cf. A-III,4.4). Actually, this is true for locally compact spaces as well and can be proved directly (see B-IV,Thm.1.4). Using this result one can prove Thm.1.6 by applying the classical Krein-Rutman theorem to any resolvent operator $R(\lambda,A)$ for $\lambda \in \mathbb{R}$ sufficiently large.
The theorem ensures that A' always has eigenvalues. The generator itself may have no eigenvalue at all. Multiplication operators have no eigenvalues unless the multiplier is constant on an open subset. Thm.1.6 fails to be true for locally compact spaces as the following example shows:

<u>Examples</u> 1.7. Consider $E = C_o(\mathbb{R}^n)$ and the semigroup $(T(t))_{t\geq 0}$ generated by the Laplacian (cf. A-I,2.8). From the explicit representation of $T(t)$,

$$(1.17)\quad (T(t)f)(x) = (4\pi t)^{-n/2}\int_{\mathbb{R}^n} \exp(-(x-y)^2/4t)\cdot f(y)\,dy ,$$

it follows that $\lim_{t\to\infty} T(t)f = 0$ for every $f \in C_o(\mathbb{R}^n)$ (Note that $\|T(t)f\| \leq (4\pi t)^{-n/2}\int_{\mathbb{R}^n} |f(y)|dy \to 0$ provided that f has compact support and that $\|T(t)\| = 1$ for all $t \geq 0$).
If ϕ is an eigenvector of A' corresponding to $s(A) = \omega(A) = 0$, we have $T(t)'\phi = \phi$ for all $t \geq 0$, hence
$\langle\phi,f\rangle = \lim_{t\to\infty}\langle T(t)f,\phi\rangle = 0$ for every f, i.e., $\phi = 0$.