4 Positive semigroups generated by elliptic operators on spaces of continuous functions

Important examples of semigroups on $C_0(\Omega)$ or $C(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open and bounded, are generated by elliptic differential operators. In the following we put together a series of results starting with the Laplacian subject to Dirichlet and to Robin boundary conditions and ending with the Dirichlet-to-Neumann operator on $C(\partial\Omega)$. Each time we obtain a positive irreducible semigroup. We consider $\mathbb{K} = \mathbb{R}$ throughout this section.

4.1 The Laplacian

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. We say that Ω is *Dirichlet-regular* if for every $g \in C(\partial\Omega)$ there exists a (unique) function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{aligned} \Delta u &= 0 \quad \text{and} \\ u|_{\partial\Omega} &= g. \end{aligned}$$

This means that the Dirichlet problem is well-posed. This property is very well understood and precise characterizations in terms of barriers or of capacity are known. If Ω has Lipschitz boundary, then Ω is Dirichlet regular. In dimension $d = 2$ it suffices that Ω is simply connected.

We refer to Arendt-Urban [9], Section 6.9 or Gilbarg-Trudinger [18], Section 2.8 for further information on the Dirichlet Problem.

The Dirichlet Laplacian Δ_0 on $C_0(\Omega)$ is defined by

$$\begin{aligned} \Delta_0 u &\coloneqq \Delta u \\ D(\Delta_0) &\coloneqq \{u \in C_0(\Omega) \colon \Delta u \in C_0(\Omega)\}. \end{aligned}$$

Here Δu is to be understood in the sense of distributions.

In thes section we consider always real spaces. Then a semigroup is called *holomorphic* if its extension to the coerresponding complexification (here $C_0(\Omega, \mathbb{C})$) is holomorphic.

Theorem 4.1. *The following are equivalent.*

(a) *Ω is Dirichlet regular;*

(b) *Δ_0 generates a positive semigroup $\mathcal{T}$ on $C_0(\Omega)$.*

In that case the semigroup $\mathcal{T}$ is holomorphic of angle $\pi/2$. Moreover $T(t)$ is compact for all $t > 0$. If Ω is connected, then the semigroup is irreducible. Moreover,

$$\|T(t)\| \leqslant Me^{-\epsilon t} \quad (t \geqslant 0) \tag{4.1}$$

for some $\epsilon > 0$, $M \geqslant 1$.

This result is due to Arendt-Bénilan [2] besides irreducibility on which we comment later. In Example C-II.1.5 (e), the generation result was obtained if Ω has C^2-boundary.

The implication (a) $\Rightarrow$ (b) of Theorem 4.1 is proved below in order to show how the Dirichlet problem comes into play and leads to a result with minimal regularity assumptions on the boundary of Ω.

We use the following abstract generation result which is of independent interest. By C-II, Theorem 1.2 a densely defined operator A generates a contractive positive semigroup if and only if A is dispersive and $(\lambda - A)$ is surjective for some $\lambda > 0$. We now describe the case $\lambda = 0$.

Theorem 4.2. *Let A be a densely defined operator on a real or complex Banach lattice E. The following are equivalent.*

(a) *A generates a positive, contractive semigroup and $s(A) \leqslant 0$.*

(b) *A is dispersive and surjective.*

In particular, (b) implies that A is closed.

Dispersive operators are defined before C-II, Theorem 1.2. A densely defined operator A on $C_0(\Omega)$ is dispersive iff for $u \in D(A)$, $x_0 \in \overline{\Omega}$:

$$u(x_0) = \sup_{x \in \overline{\Omega}} u(x) > 0 \text{ implies } (Au)(x_0) \leqslant 0.$$

Proof. (Theorem 4.2.) (b) $\Rightarrow$ (a)

Consider the equivalent norm

$$\|u\|_1 := \|u^+\| + \|u^-\|$$

on E. Since A is dispersive it is dissipative with respect to this new norm as is easy to see. Now Theorem 4.5 of Arendt, Chalendar and Moletsare [11] implies that A generates a contraction semigroup $\mathcal{T}$ and A is invertible. Since A is dispersive, it follows from C-II, Theorem 1.2 that $\mathcal{T}$ is positive and contractive (with respect to the original norm). Since $R(\lambda, A) \geqslant 0$ for $\lambda > 0$, it follows that $-A^{-1} \geqslant 0$. Now C-I, Theorem 1.1 (vi) implies that $s(A) \leqslant 0$.

(a) $\Rightarrow$ (b) is obvious from C-II, Theorem 1.2.

□

Proof. (Theorem 4.1.) (a) $\Rightarrow$ (b)

The operator Δ_0 is dispersive by the maximum principle. If Ω is Dirichlet regular, then Δ_0 is surjective. In fact, let $f \in C_0(\Omega)$. Extend f by 0 to $\mathbb{R}^n$ and let $w = \Gamma * f$, where Γ is the fundamental solution of Laplace's equation (see Gilbarg and Trudinger [18, 2.12]). Then $w \in C(\mathbb{R}^n)$ and $\Delta w = f$ in the sense of distributions. Let $g = w|_{\partial\Omega}$ and let $v \in C^2(\overline{\Omega}) \cap C(\overline{\Omega})$ be the solution of the Dirichlet problem, i.e.,

$$v|_{\partial\Omega} = g \quad \text{and} \quad \Delta v = 0 \text{in } \Omega.$$

Then $u := w - v \in D(\Delta_0)$ and $\Delta u = f$.

We have shown that Δ_0 satisfies condition (b) of Theorem 4.2. Thus Δ_0 generates a positive, contractive C_0-semigroup $(T(t))_{t\geqslant 0}$ on $C_0(\Omega)$ and $s(\Delta_0) \leqslant 0$. Since by C-IV Theorem 1.1 (iv) $s(\Delta_0) = \omega_0(\Delta_0)$, it is exponentially stable.

We refer to Arendt and Bénilan [2] for the proof of (b) $\Rightarrow$ (a). $\square$

We want to add two further comments on the Dirichlet Laplacian Δ_0 on $C_0(\Omega)$. The first concerns its domain

$$D(\Delta_0) = \{u \in C_0(\Omega) \colon \Delta u \in C_0(\Omega)\}.$$

This distributional domain is not contained in $C^2(\Omega)$ for any open set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, see Arendt-Urban [9, Theorem 6.60].

Our second comment concerns the proof of holomorphy. It can be given via Gaussian estimates (see the Extended Notes for C-II). In our context, a short proof based on Kato's inequality of C-II, Section 2 is more appealing (see Arendt-Batty [1]).

Finally, we comment on irreducibility. On $C_0(\Omega)$ it is a strong property. By C-III, Theorem 3.2 (ii) it means that for $0 \leqslant f \in C_0(\Omega)$, $f \neq 0$,

$$(T(t)f)(x) > 0 \text{ for all } x \in \Omega, t > 0.$$

On $L^2(\Omega)$ irreducibility is much weaker (meaning that $(T(t)f)(x) > 0$ x-a.e.), but easy to prove (see the Extended Notes to C-I). In the paper Arendt, ter Elst, Glück [10] an argument based on Banach lattice technique shows how irreducibility on $L^2(\Omega)$ can be carried over to $C_0(\Omega)$ or even to $C(\overline{\Omega})$ in the case of Robin boundary conditions which we consider now.

By

$$H^1(\Omega) := \{u \in L^2(\Omega) \colon \partial_j u \in L^2(\Omega) \text{ for } j = 1, \dots, n\}$$

we denote the first Sobolev space. We assume that Ω has Lipschitz boundary. Then there exists a unique bounded operator

$$\mathrm{tr}\colon H^1(\Omega) \to L^2(\partial\Omega)$$

such that tr $(u) = u|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega})$. It is called the *trace operator.*

Here the space $L^2(\partial\Omega)$ is defined with respect to the surface measure (i.e. the $(d-1)$-dimensional Hausdorff measure) on $\partial\Omega$.

The normal derivative $\partial_\nu u$ of u is defined as follows. Let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Let $h \in L^2(\partial\Omega)$. We say that h is the *(outer) normal derivative* of u and write $\partial_\nu u = h$ if

$$\int_\Omega \Delta u v + \int_\Omega \nabla u \nabla v = \int_{\partial\Omega} hv$$

for all $v \in C^1(\overline{\Omega})$.

If $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$ we say $\partial_\nu u \in L^2(\partial\Omega)$ if there exists $h \in L^2(\partial\Omega)$ such that $\partial_\nu u = h$.

Remark. Since Ω has Lipschitz boundary the outer normal $\nu(z)$ exists for almost all $z \in \partial\Omega$ and $\nu \in L^\infty(\partial\Omega)$. But we do not use this outer normal and rather define $\partial_\nu u$ weakly by the validity of Green's formula.

Let $\beta \in L^\infty(\partial\Omega)$. We define the Laplacian Δ^β with Robin boundary conditions as follows:

$$D(\Delta^\beta) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \partial_\nu u + \beta \mathrm{tr}(u) = 0\} \tag{4.2}$$

$$\Delta^\beta u := \Delta u. \tag{4.3}$$

We call Δ^β briefly the Robin-Laplacian. Note that for $\beta = 0$, we obtain Neumann boundary conditions, and $\Delta^0 =: \Delta^N$ is the Neumann Laplacian.

The following result is valid.

Theorem 4.3 (4.3)**.** *Assume that $\Omega \subset \mathbb{R}^d$ is bounded, open, connected with Lipschitz boundary, and let $\beta \in L^\infty(\partial\Omega)$. Then Δ^β generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ on $C(\overline{\Omega})$. Moreover, $T(t)$ is compact for all $t > 0$.*

The generation property on $C(\overline{\Omega})$ is due to Nittka [21]. A major point is to show that the resolvent of the corresponding operator on $L^2(\Omega)$ leaves $C(\overline{\Omega})$ invariant. Given $f \in C(\overline{\Omega})$, $u \in H^1(\Omega)$ such that $u - \Delta u = f$, $\partial_\nu u + \beta u|_{\partial\Omega} = 0$.

Theorem 4.4 (4.4)**.** *Assume (4.2) and (4.4). Then $\Delta - V$ generates a positive, irreducible semigroup on $C(\partial\Omega)$. If $V \geqslant 0$, then the semigroup is contractive.*

If Ω is of class C^∞ similar results have been obtained by Escher [17] and Engel [16]. Under the very general conditions here, Theorem 4.8 is due to Arendt and ter Elst [8]. There it is shown that N_V is resolvent-

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One has to show that $u \in C(\overline{\Omega})$. Nittka extends u to an open set $\widetilde{\Omega}$ containing $\overline{\Omega}$ by reflecting u along the graph. Then u becomes the solution of an elliptic problem on $\widetilde{\Omega}$. Continuity on $\widetilde{\Omega}$, and hence on $\overline{\Omega}$, follows from the De Giorgi-Nash Theorem.

Irreducibility is due to Arendt, ter Elst and Glück [10], Theorem 4.5. These results have first been proved for $\beta \geqslant 0$. Daves [**?**] has shown how one can treat general $\beta \in L^\infty(\delta\Omega)$.

Since the semigroup is holomorphic, by C-II, Theorem 3.2 (ii), it implies that

$$\inf_{x\in\overline{\Omega}} (T(t)f)(x) > 0 \tag{4.1}$$

for all $t > 0$ and $0 \leqslant f \in C(\overline{\Omega})$, $f \neq 0$.

Denote by $s(\Delta^\beta)$ the spectral bound of Δ^β. By C-III, Theorem 3.8 (iv), $s(\Delta^\beta)$ is the unique eigenvalue with a positive eigenfunction $u_0 \geqslant 0$, $u_0 \neq 0$. It follows from (4.1) that u_0 is strictly positive; i.e.,

$$\inf_{x \in \overline{\Omega}} u_0(x) > 0,$$

a remarkable property, which has important applications to semi-linear problems, see Arendt-Daners [3].

The spectral bound $s(\Delta^\beta)$ determines the asymptotic behavior of the semigroup $\mathcal{T}$. In fact, the following corollary follows from B-III Proposition 3.5.

Corollary 4.5. *There exist a strictly positive Borel measure μ on $\overline{\Omega}$, $M \geqslant 0$ and $\epsilon > 0$ such that $\langle \mu, u_0 \rangle = 1$ and*

$$\|T(t) - e^{s(\Delta^\beta)t} P\| \leqslant M e^{-\epsilon t}$$

for all $t \geqslant 0$, where $P \in \mathcal{L}(C(\overline{\Omega}))$ is given by

$$Pf = \langle \mu, f \rangle u_0.$$

The theorem says that the rescaled semigroup $(e^{-s(\Delta^\beta)t} T(t))_{t \geqslant 0}$ converges in the operator norm to the rank-1-projection P exponentially fast.

4.2 Elliptic operators in divergence form

The preceding results extend to elliptic operators in divergence form with bounded measurable coefficients. Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

Let $a_{k,\ell}, b_k, c_k, c_0 \in L^\infty(\Omega)$, $k, \ell = 1, \dots, n$ such that for some $\alpha > 0$

$$\sum_{k,\ell=1}^{n} a_{k,\ell}(x)\xi_k\xi_\ell \geqslant \alpha|\xi|^2$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$, where $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$. Let

$$H^1_{\mathrm{loc}}(\Omega) := \{u \in L^2_{\mathrm{loc}}(\Omega) \colon \partial_k u \in L^2_{\mathrm{loc}}(\Omega), k = 1, \dots, n\}.$$

Define $\mathcal{A} \colon H^1_{\mathrm{loc}}(\Omega) \to C_c^\infty(\Omega)'$ by

$$\langle \mathcal{A}u, v\rangle = \int_\Omega \left(\sum_{k,\ell=1}^{d} \partial_k(a_{k\ell}\partial_\ell u) + \sum_{k=1}^{d} \partial_k(b_k u) + \sum_{k=1}^{d} c_k \partial_k u + c_0 u \right) \mathrm{d}x.$$

We define A_0 as the part of $\mathcal{A}$ in $C_0(\Omega)$; i.e.,

$$\begin{aligned} D(A_0) &:= \{u \in C_0(\Omega) \cap H^1_{\mathrm{loc}}(\Omega) \colon \mathcal{A}u \in C_0(\Omega)\} \\ A_0 u &:= \mathcal{A}u. \end{aligned}$$

Then Theorem 4.1 holds with Δ_0 replaced by A_0. It is remarkable that Dirichlet regularity of Ω is the right regularity condition at the boundary, a discovery due to Stampacchia. We refer to Arendt and Bénilan [2], Section 4 for a proof of the following result.

Theorem 4.6. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded, open, connected, Dirichlet regular set. Then A_0 generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ on $C_0(\Omega)$. Moreover, $T(t)$ is compact for all $t > 0$.*

Remark. The proof of holomorphy depends on Gaussian estimates, which in [2] were merely known if the $b_k \in W^{1,\infty}(\Omega)$. Later, it was shown by Davies [12] that they always hold.

Also the results for Robin boundary conditions Theorem 4.3 and 4.4 can be extended to elliptic operators in divergence form on $C(\overline{\Omega})$; see Theorem 4.5 in Arendt, ter Elst, Glück [10]. It uses results of Nittka [21].

4.3 Elliptic operators in non-divergence forms

The techniques for elliptic operators in non-divergence form are quite different than those used in the divergence-case form. But the results are similar.

Let $\Omega \subset \mathbb{R}^n$ be open and connected. We assume that Ω satisfies the uniform exterior cone condition. This means the following. There exists a finite, right circular cone V such that for each $x \in \partial\Omega$ there exists a cone V_x which is congruent to V such that $V_x \cap \overline{\Omega} = \{x\}$.

Let $a_{k\ell} = a_{\ell k} \in C(\overline{\Omega})$, $b_k \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$, $c \leqslant 0$ such that

$$\sum_{k,\ell=1}^{n} a_{k\ell}(x)\xi_k\xi_\ell \geqslant \mu|\xi|^2$$

for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$ and some $\mu > 0$.

For $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ we define

$$\mathcal{A}u = \sum_{k,\ell=1}^{n} \partial_k a_{k\ell}\partial_\ell u + \sum_{k=1}^{n} b_k\partial_k u + cu.$$

Thus $\mathcal{A}\colon W^{2,n}_{\mathrm{loc}}(\Omega) \to L^n_{\mathrm{loc}}(\Omega)$ is linear. Here

$W^{2,n}_{\mathrm{loc}}(\Omega) := \{u \in L^n_{\mathrm{loc}}(\Omega)\colon \partial_k u \in L^n_{\mathrm{loc}}(\Omega), \partial_k\partial_\ell u \in L^n_{\mathrm{loc}}(\Omega)$ for all $k, \ell = 1, \dots, n\}$.

We consider the operator A on $C_0(\Omega)$ defined by

$$D(A) := \{u \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)\colon \mathcal{A}u \in C_0(\Omega)\}$$

$$Au := \mathcal{A}u.$$

Then the following holds.

Theorem 4.7. *The operator A generates a positive, irreducible, contractive holomorphic semigroup $(T(t))_{t\geqslant 0}$ on $C_0(\Omega)$. Moreover*

$$\|T(t)\| \leqslant Me^{-\epsilon t} \quad (t \geqslant 0)$$

for some $\epsilon > 0$, $M \geqslant 1$. The resolvent of A is compact.

This result is proved by Arendt and Schätzle [5], Proposition 4.7.

Positivity and irreducibility are a consequence of the Alexandrov maximum principle. For C^2-boundary also results on $L^p(\Omega)$ spaces are obtained by Denk, Hieber and Prüss [15] whose main interest lies in proving maximal regularity and establishing a bounded H^∞-calculus.

However, in the situation of Theorem 4.6, without assuming merely the uniform exterior cone condition on Ω, it seems not to be known whether the semigroup extends to a strongly continuous semigroup on $L^p(\Omega)$ for some $p \in [1, \infty)$.

Theorem 4.6 is extended by Arendt and Schätzle [6] to unbounded open sets which satisfy the locally uniform exterior cone condition. However, in the case of unbounded Ω the semigroup converges merely strongly to 0 (and not exponentially fast).

The monograph of Lunardi [20] is devoted to the study of holomorphic semigroups generated by elliptic operators in non-divergence form.

4.4 The Dirichlet-to-Neumann operator on $C(\partial\Omega)$

Let Ω be a bounded, open, connected subset of $\mathbb{R}^n$ with Lipschitz boundary and let $V \in L^\infty(\Omega)$. We consider the Dirichlet-to-Neumann operator with respect to $\Delta - V$ on the space $C(\partial\Omega)$. For that we first establish well-posedness of the Dirichlet Problem.

We assume throughout this subsection that

$$u \in C_0(\Omega), \Delta u - Vu = 0 \text{ implies } u = 0. \tag{4.2}$$

This is exactly the condition that the solutions of the Dirichlet problem with respect to $\Delta - V$ formulated in Proposition 4.7 are unique. An equivalent condition is

$$u \in H_0^1(\Omega), \Delta u - Vu = 0 \text{ implies } u = 0; \tag{4.3}$$

(which means that $0 \notin \sigma(\Delta_0 - V)$ where Δ_0 is the Dirichlet Laplacian on $L^2(\Omega)$).

Proposition 4.8. *Assume (4.2). Let $g \in C(\partial\Omega)$. Then there exists a unique $u_g \in C(\overline{\Omega})$ such that*

$$(\Delta - V)u_g = 0 \quad \textit{and} \quad u_g|_{\partial\Omega} = g.$$

Thus, u_g is a harmonic function with respect to $\Delta - V$ which has to be understood in the sense of distributions; i.e.,

$$\int_\Omega u_g \Delta\phi - \int_\Omega V u_g \phi = 0$$

for all $\phi \in C_c^\infty(\Omega)$.

For a simple proof of Proposition 4.7 we refer to [7].

Next we define the Dirichlet-to-Neumann operator N_V with respect to $\Delta - V$ on $C(\partial\Omega)$ as follows.

$$D(N_V) := \{g \in C(\partial\Omega) : u_g \in H^1(\Omega), \text{ and } \partial_\nu u_g \in C(\partial\Omega)\} \tag{4.4}$$

$$N_V g := -\partial_\nu u_g. \tag{4.5}$$

Recall that $\partial_\nu u_g \in C(\partial\Omega)$ means that there exists $h \in C(\partial\Omega)$ such that

$$\int_\Omega \Delta u_g \phi + \int_\Omega \nabla u_g \nabla \phi = \int_{\partial\Omega} h\phi$$

for all $\phi \in C^1(\overline{\Omega})$. Then we put $\partial_\nu u_g := h$.

We will need the hypothesis that $-\Delta_0 + V$ is form-positive i.e.

$$\int_\Omega (|\nabla u|^2 + V|u|^2) \geqslant 0 \tag{4.4}$$

for all $u \in H_0^1(\Omega)$.

Theorem 4.9. *Assume (4.2) and (4.4). Then $\Delta - N_V$ generates a positive, irreducible semigroup on $C(\partial\Omega)$. If $V \geqslant 0$, then the semigroup is contractive.*

If Ω is of class C^∞ similar results have been obtained by Escher [17] and Engel [16]. Under the very general conditions here, Theorem 4.8 is due to Arendt and ter Elst [8]. There it is shown that N_V is resolvent-positive and that the domain is dense (which is the main difficulty). Then by B-II, Theorem 1.8 N_V generates a positive semigroup. Irreducibility is surprising. In fact, even though Ω is supposed to be connected, $\partial\Omega$ might not be connected (consider a ring for example). The fact that the semigroup is irreducible shows that the operator N_V is non-local in quite a dramatic way.

A first result on irreducibility (on $L^2(\partial\Omega)$) was obtained by Arendt and Mazzeo [4].

It is not known so far whether the semigroup generated by N_V is holomorphic if $\partial\Omega$ has Lipschitz boundary. If the boundary is of class $C^{n+\alpha}$ with $\alpha > 0$, then it is holomorphic of angle $\pi/2$. This is due to ter Elst and Ouhabaz [22].

The operator N_V is also called *voltage-to-current map* and has physical meaning. One version of the famous Calderón-Problem is the question whether for $V_1, V_2 \in$

$L^\infty(\Omega)$, such that $0 \notin \sigma(\Delta_{V_1}) \cup \sigma(\Delta_{V_2})$,

$$N_{V_1} = N_{V_2} \text{ implies } V_1 = V_2.$$

This is true under the only assumption that Ω has Lipschitz boundary; see Theorem 1.1 by Krupchyk and Uhlmann [19].

Finally we mention that N_V may generate a positive semigroup even if (4.4) is violated. This and other surprising phenomena were discovered by Daners [13], and led to the new theory of eventually positive semigroups; see e.g. [14].

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