Examples 3.10. (a) We assume that the semigroup $(T(t))$ satisfies the hypotheses of Thm.3.6 and that it is given by

$$(T(t)f)(x) = \int_X k(t,x,y)f(y)\,d\mu(y) ,$$

where μ is a positive measure and k is a positive continuous function (see Ex.3.4(b)). We will show that $P\sigma(A)\cap(s(A) + i\mathbb{R}) = \{s(A)\}$. Assuming the contrary, by Thm.3.6(d) there exist $\alpha \neq 0$, $h \in C_o(X)$ such that

$$(3.10)\quad S_{\bar{h}}\circ T(t)\circ S_h = e^{i\alpha t}\cdot T(t) \quad \text{for all} \quad t \geq 0 .$$

This implies that k satisfies

$$(3.11)\quad \frac{\overline{h(x)}}{|h(x)|}\cdot\frac{h(y)}{|h(y)|}\cdot k(t,x,y) = e^{i\alpha t}\, k(t,x,y) \quad (t > 0 ,\ x,y \in X) .$$

It follows that for $0 < |s-t| < 2\pi/\alpha$ $k(t,.,.)$ and $k(s,.,.)$ have disjoint support. This is impossible if k is continuous.

(b) Let Ω be a domain in $\mathbb{R}^n$ and define L_o as follows:

$L_o f := \sum_{i,j=1}^n a_{ij}\, f''_{ij} + \sum_{i=1}^n b_i\, f'_i + cf$,
with domain $D(L_o) := \{f \in C_o(\Omega) : f \text{ is } C^\infty ,\ L_o f \in C_o(\Omega)\}$.
(Here f'_i stands for $\partial f/\partial x_i$, thus $f''_{ij} = \partial^2 f/\partial x_i \partial x_j$.)

Suppose that L_o is elliptic, a_{ij}, b_i, c are real-valued C^∞-functions with $\lambda_o := \sup c < \infty$, assume further that the closure L of L_o is the generator of a positive semigroup on $C_o(\Omega)$ which has compact resolvent. For example this is true if $\partial\Omega$ is C^∞ and $a_{ij} \in C^\infty(\bar{\Omega})$ (cf Thm.4.8.3 of Fattorini (1983)). We will show that $P\sigma(A)\cap(s(A) + i\mathbb{R}) = \{s(A)\}$. In order to apply Thm.3.6 we have to show that the corresponding semigroup $(T(t))$ is irreducible:
Given $0 < f \in E$ then there is $g \in D(L_o)$ such that $0 < g \leq f$. $h := R(\lambda,L)g$ is C^∞ (Weyl's Lemma) and satisfies $L_o h - \lambda h = -g < 0$. Assuming that $\lambda > \lambda_o$ then h is positive, even strictly positive by the maximum principle [Protter-Weinberger (1967), Chap.2, Thm.6]. It follows from $R(\lambda,L)f \geq R(\lambda,L)g = h >> 0$ that $(T(t))$ is irreducible. Next we apply Thm.3.6(d) in order to show that the spectral bound is a dominant eigenvalue. We can assume that $s(L) = 0$. If $s(L)$ is not dominant, then by Thm.3.6(d) we have

$$(3.12)\quad L_o h = i\alpha h ,\ L_o|h| = 0 ,\ L_o\bar{h} = -i\alpha\bar{h} \quad \text{for some} \quad h \neq 0 ,\ \alpha > 0$$