<u>Theorem</u> 5.4. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a $\tau$-periodic semigroup on a Banach space $E$ with generator $A$ and associated spectral projections

$P_n := \tau^{-1}\cdot\int_0^\tau \exp(-\mu_n s)T(s)\,ds$ , $\mu_n := 2\pi i n/\tau$ , $n \in \mathbb{Z}$ .

For every $f \in D(A)$ one has $f = \sum_{-\infty}^{+\infty} P_n f$ and therefore

(i) $\quad T(t)f = \sum_{-\infty}^{+\infty} \exp(\mu_n t)P_n f \qquad$ if $f \in D(A)$ ,

(ii) $\quad Af = \sum_{-\infty}^{+\infty} \mu_n P_n f \qquad$ if $f \in D(A^2)$ .

<u>Proof</u>. It suffices to prove the first statement. Then (i) and (ii) follow by (5.3) and (5.4).

We assume $\tau = 2\pi$ and show first that $\sum_{-\infty}^{+\infty} P_n f$ is summable for $f \in D(A)$ : For $g := Af$ we obtain $P_n g = P_n Af = AP_n f = inP_n f$ . Take $H$ to be a finite subset of $\mathbb{Z} \setminus \{0\}$ and $\phi \in E'$ . Then

$$\begin{aligned}|\textstyle\sum_{n\in H} \langle P_n f,\phi\rangle| &= |\textstyle\sum_{n\in H} (in)^{-1} \langle P_n g,\phi\rangle| \\ &\leq (\textstyle\sum_{n\in H} n^{-2})^{1/2}(\textstyle\sum_{n\in H} |\langle P_n g,\phi\rangle|^2)^{1/2} .\end{aligned}$$

From Bessel's inequality we obtain for the second factor

$$\begin{aligned}\textstyle\sum_{n\in H} |\langle P_n g,\phi\rangle|^2 &\leq 1/2\pi\cdot\int_0^{2\pi} |\langle T(s)g,\phi\rangle|^2\,ds \\ &\leq \|\phi\|^2\cdot 1/2\pi\cdot \int_0^{2\pi} \|T(s)g\|^2\,ds .\end{aligned}$$

With the constant $c := (1/2\pi\cdot \int_0^{2\pi} \|T(s)g\|^2\,ds)^{1/2}$ we obtain

$$\|\textstyle\sum_{n\in H} P_n f\| \leq c(\textstyle\sum_{n\in H} n^{-2})^{1/2}$$

for every finite subset $H$ of $\mathbb{Z}$ , i.e. $\sum_{-\infty}^{+\infty} P_n f$ is summable.

Next we set $h := \sum_{-\infty}^{+\infty} P_n f$ and observe that for every $\phi' \in E'$ the Fourier coefficients of the continuous, $\tau$-periodic functions

$$s \to \langle T(s)h,\phi\rangle \quad \text{and} \quad s \to \langle T(s)f,\phi\rangle$$

coincide. Therefore these functions are identical for $s \geq 0$ and in particular for $s = 0$ , i.e. $\langle h,\phi\rangle = \langle f,\phi\rangle$ . By the Hahn-Banach Theorem we obtain $f = h$ .

□

The above theorem contains rather precise information on periodic semigroups. In particular, it characterizes periodic semigroups by the fact that $\sigma(A)$ is contained in $i\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$ and the eigenfunctions of $A$ form a total subset of $E$ .

If we suppose in addition that a periodic semigroup has a bounded generator it follows that the spectrum of its generator is bounded.