Then integration by parts yields for $f \in D(A) = D(C)$

$$\langle Cf,1\rangle = \int_{1/4}^{1/2} (-xf'(x) + k(x)f(2x))w(x)\,dx - \int_{1/2}^{1} xf'(x)w(x)\,dx = 0 .$$

Thus $1 \in D(C')$ and $C'1 = 0$, equivalently $S(t)'1 = 1$ for all t. This shows that $(S(t))$ is a semigroup of contractions on E.
It remains to show that there is $\alpha > 0$ such that $i\alpha \in \sigma(C)$.
In fact, considering $\alpha := 2\pi(\log 2)^{-1}$ then $i\alpha$ is an eigenvalue of C. A corresponding eigenfunction is given by $h_1(x) := x^{-i\alpha}h_o(x)$, where h_o is the eigenfunction corresponding to 0 defined as

$$(2.15)\qquad h_o(x) := \begin{cases} \displaystyle\int_{1/4}^{x} \frac{k(y)}{y}\,dy & \text{for } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 1 & \text{for } \frac{1}{2} \leq x \leq 1 . \end{cases}$$

The verification of these statements is left as an excercise.

In several of the above results we had to assume that the positive semigroup $(T(t))_{t\geq 0}$ is bounded and has spectral bound zero. In general, these conditions are difficult to verify, in particular, when only the generator is known. In the final example we described a method how to cope with this problem: If $s(A)$ is an eigenvalue of the adjoint A' with a strictly positive eigenvector ϕ, then $(T(t))_{t\geq 0}$ induces in a canonical way a positive semigroup $(T_\phi(t))_{t\geq 0}$ on the AL-space (E,ϕ). This semigroup satisfies $\|T_\phi(t)\| \leq \exp(t\cdot s(A))$ and has spectral bound $s(A)$. Hence one may apply the results of this section to the rescaled semigroup $(\exp(-t\cdot s(A))T_\phi(t))_{t\geq 0}$ thus obtaining convergence of $(T(t))_{t\geq 0}$ for the weaker topology on E which is induced by (E,ϕ).