Chapter 2

Characterization of Positive Semigroups on W^*-Algebras

Since the positive cone of a C*-algebra has non-empty interior many results of Chapter B-II can be applied verbatim to the characterization of the generator of positive semigroups on C*-algebras. On the other hand a concrete and detailed representation of such generators has been found only in the uniformly continuous case (see **?**lindblad:1976). A third area of active research has been the following: Which maps on C^*-algebras (in particular, which derivations) commuting with certain automorphism groups are automatically generators of strongly continuous positive semigroups. For more information we refer to the survey article of Evans (1984).

1 Semigroups on Properly Infinite W*-Algebras

The aim of this section is to show that strongly continuous semigroups of Schwarz maps on properly infinite W*-algebras are already uniformly continuous. In particular, our theorem is applicable to such semigroups on $B(H)$.

It is worthwhile to remark, that the result of Lotz (1985) on the uniform continuity of every strongly continuous semigroup on L^∞ (see A-II, Sec.3) does not extend to arbitrary W*-algebras.

Example 1.1. Take $M = B(H)$, H infinite dimensional, and choose a projection $p \in M$ such that Mp is topologically isomorphic to H. Therefore $M = H \oplus M_0$, where $M_0 = \ker(x \mapsto xp)$. Next take a strongly, but not uniformly continuous, semigroup S on H and consider the strongly continuous semigroup $S \oplus \mathrm{Id}$ on M.

For results from the classification theory of W^*-algebras needed in our approach we refer to Sakai (1971, 2.2) and Takesaki (1979, V.1).

Theorem 1.2. *Every strongly continuous one-parameter semigroup of Schwarz type on a properly infinite W*-algebra M is uniformly continuous.*

Proof. Let $T = (T(t)_{t \geqslant 0})$ be strongly continuous on M and suppose T not to be uniformly continuous. Then there exists a sequence $(T_n) \subset T$ and $\epsilon > 0$ such that $\|T_n - \mathrm{Id}\,\| \geqslant \epsilon$ but $T_n \to \mathrm{Id}$ in the strong operator topology. We claim that for every sequence (P_k) of mutually orthogonal projections and all bounded sequences (x_k) in M

$$\lim_n \|(T_n - \mathrm{Id})(P_k x_k P_k)\| = 0$$

uniformly in $k \in \mathbb{N}$. This follows from an application of the *Lemma of Phillips* and the fact that the sequence $(P_k x_k P_k)$ is summable in the $s^*(M, M_*)$-topology (compare Elliot (1972)).

Let (P_k) be a sequence of mutually orthogonal projections in M such that every P_k is equivalent to 1 via some $u_k \in M$ (Sakai, 1971, 2.2). Without loss of generality we may assume $\|(T_n - \mathrm{Id})(u_n)\| \leqslant n^{-1}$ since the semigroup T is strongly continuous. Thus we obtained the following:

(i) $\lim_n \|(T_n - \mathrm{Id})(P_k x_k P_k)\| = 0$ uniformly in $k \in \mathbb{N}$ for every bounded sequence (x_k) in M.

(ii) Every projection P_k is equivalent to 1 via some $u_k \in M$.

(iii) $\|(T_n - \mathrm{Id})u_n\| \leqslant n^{-1}$ for all $n \in \mathbb{N}$.

For the following construction see A-I,3.6 and D-II,Sec.2. Let

(i) $\widehat{M}$ be an ultrapower of M,

(ii) let $p := \widehat{(P_k)} \in \widehat{M}$,

(iii) $T := \widehat{(T_n)} \in L(\widehat{M})$

(iv) and $u := \widehat{(u_k)} \in \widehat{M}$.

Then T is identity preserving and of Schwarz type on $\widehat{M}$. Since $u^*u = p$ and $uu^* = 1$ it follows $pu^* = u^*$ and $(uu^*)x(uu^*) = x$ for all $x \in \widehat{M}$. Finally, $T(pxp) = pxp$ for all $x \in \widehat{M}$, which follows from (i), and $T(u^*) = T(pu^*) = pu^* = u^*$ and $T(u) = u$, which follows from (iii). Using the Schwarz inequality we obtain

$$T(uu^*) = T(1) \leqslant 1 = uu^* = T(u)T(u)^*.$$

Using D-III, Lemma 1.1. we conclude $T(ux) = uT(x)$ and $T(xu^*) = T(x)u^*$ for all $x \in \widehat{M}$. Hence

$$\begin{aligned} T(x) &= T(uu^*xuu^*) = uT(u^*xu)u^* = uT(pu^*xup)u^* \\ &= upu^*xupu^* = uu^*xuu^* = x \end{aligned}$$

for all $x \in \widehat{M}$. From this we obtain that for every bounded sequence (x_k) in M

$$\lim_k \|T_k x_k - x_k\| = 0$$

for some subsequence of the T_n's and of the x_k's. This conflicts with our assumption at the beginning, hence the theorem is proved. □

Bibliography

D. Evans. Quantum dynamical semigroups, symmetry groups, and locality. *Acta Applicandae Mathematicae*, 2:333–352, 1984.

H. P. Lotz. Uniform convergence of operators on L^∞ and similar spaces. *Math. Z.*, 190:207–220, 1985.

S. Sakai. *C^*-Algebras and W^*-Algebras.* Springer, Berlin-Heidelberg-New York, 1971.

M. Takesaki. *Theory of Operator Algebras I.* Springer, New York-Heidelberg-Berlin, 1979.