It follows from (2.25) and (2.26) that for $\lambda > \|\Psi\|$ $R(\lambda,A_\Psi)$ exists and satisfies $\|R(\lambda,A_\Psi)\| \leq \lambda/(\lambda-\|\Psi\|)\cdot 1/\lambda = 1/(\lambda-\|\Psi\|)$. Then the Hille-Yosida Theorem (A-II,Thm.1.7) implies that A_Ψ generates a semigroup $(T(t))$ satisfying $\|T(t)\| \leq \exp(\|\Psi\|t)$. Moreover, this semigroup is eventually norm continuous (see B-IV,Cor.3.3).
By B-II,Ex.1.22 we have the following equivalence:

(2.29) A_Ψ generates a positive semigroup if and only if
$\Psi + r\delta_o \geq 0$ for some $r \in \mathbb{R}$.

Thus Cor.2.12 is applicable if $\Psi + r\delta_o \geq 0$ for some $r \in \mathbb{R}$. Since every eigenvalue of A_Ψ is an eigenvalue of A_m and since $\ker(\lambda-A_m) = \{\alpha e_\lambda : \alpha \in \mathbb{C}\}$, the spectral bound $s(A_\Psi)$ is determined by the (unique) real $\lambda \in \mathbb{R}$ such that $e_\lambda \in D(A_\Psi)$ or equivalently, λ is a solution of the so-called <u>characteristic equation</u>

(2.30) $\lambda = \Psi(e_\lambda)$, $\lambda \in \mathbb{R}$.

(The assumption $\Psi + r\delta_o \geq 0$ implies that the function $\lambda \to \Psi(e_\lambda)$ is strictly decreasing and $\lim_{\lambda\to\infty}\langle e_\lambda,\Psi\rangle > -\infty$, $\lim_{\lambda\to-\infty}\langle e_\lambda,\Psi\rangle = \infty$ unless $\Psi = r_o\delta_o$ for some $r_o \in \mathbb{R}$.)

We conclude this section with some additional remarks related to Thm.2.9 and its corollaries.

<u>Remarks</u> 2.15.(a) If $s(A)$ is a pole of the resolvent, then for generators of positive semigroups one has the following equivalences:

(i) $s(A)$ is a first order pole.

(ii) For every $0 < f \in \ker(s(A) - A)$ there exists
$0 \leq \Psi \in \ker(s(A) - A')$ such that $\langle f,\Psi\rangle > 0$.

(iii) For every $0 < \Psi \in \ker(s(A) - A')$ there exists
$0 \leq f \in \ker(s(A) - A)$ such that $\langle f,\Psi\rangle > 0$.

In particular, if $\ker(s(A) - A)$ contains a strictly positive function or if $\ker(s(A) - A')$ contains a strictly positive measure, then $s(A)$ is a first order pole.