Math. Ann. 298, 349–371 (1994)

Mathematische Annalen

Weak compactness in the dual of a C*-algebra is determined commutatively

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Received April 20, 1992; in revised form May 24, 1993

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1 Introduction

The main result of this article, Theorem 1, deals with a topic of functional analysis that can be classified between Banach space theory and C*-algebra theory.

A particular question which motivated our work was the question whether L(H), the Banach space of bounded linear operators on a Hilbert space H, is a Grothendieck space or not.

Before we state basic definitions and theorems in Sect. 2 we make some introductory comments on the proof of our main result, which should make up for the inconveniences caused by a lot of technical details that can possibly hide the ideas in it.

In 1953 Grothendieck [16, Theorem 2.3] proved a criterion for w-compactness in the dual of C(K)-spaces: If (and only if) a bounded set $\mathcal{K} \subset \mathrm{C}(K)'$ is not relatively w-compact there is a sequence (O_n) of pairwise disjoint open subsets of K such that $\sup_{\mu\in\mathcal{K}} |\mu(O_n)| \not\to 0$ (see e.g. also [10, p. 98]). By Urysohn's lemma it is easily seen that in the last formula the O_n's can be replaced by positive continuous functions $f_n \in \mathrm{C}(K)$ with supports in the O_n's. Since these continuous functions span an "orthogonal" c_0-copy Grothendieck's criterion says that w-compactness in the dual of C(K) can be tested on (orthogonal) c_0-copies in C(K). Independently, some years later Pełczyński [27] gave a measure theoretic proof of Grothendieck's criterion and defined that, roughly speaking, a Banach space allowing such a criterion has property (V). Our aim is to generalize Grothendieck's result to non-commutative C^*-algebras and in particular to prove property (V) for C^*-algebras.

Let's first have a brief look at Pełczyński's proof, in particular at Lemma 1 in [27]. In essence, Pełczyński uses Urysohn's lemma and the regularity of Borel measures (in the sense that the measure of any Borel set can be approximated by the measures of closed sets contained in it) for the following situation: Two orthogonal measures μ and ν on a compact set K are supported by two disjoint Borel sets $E, F \subset K$. Putting up with an arbitrary small error one can replace E and F by two closed sets contained in E and F, respectively, and Urysohn's lemma serves to finally get two disjoint open sets $\tilde{E}$ and $\tilde{F}$ which almost support μ and ν, i.e. $\mu(\tilde{E}) \approx \mu(E) = \|\mu\|$ and $\mu(\tilde{F}) \approx \mu(F) = \|\nu\|$. Moreover there are normed positive functions $f, g \in C(K)$ whose supports are in $\tilde{E}$ and $\tilde{F}$, respectively – in particular f and g are orthogonal – such that $\mu(f) \approx \|\mu\|$ and $\nu(g) \approx \|\nu\|$.

So if we want to generalize Pełczyński's proof to the non-commutative setting we should first of all dispose of an appropriate counterpart of an open set. To do so we identify a subset E of the compact set K with its characteristic function χ_E. Then χ_E is a projection in $C(K)''$; in the special case when E is open χ_E is, by Urysohn's lemma, even a lower semicontinuous function on K and can be approximated from below by positive elements of $C(K)$. It is this aspect of semicontinuity of open sets which was generalized by Akemann [1] when he defined the notion of open projections as the non-commutative counterpart of open sets. (For the exact definition see 2.2.10 below.)

Unfortunately, this generalization of open sets does not allow a non-commutative version of regularity of Borel measures. I thank Ch. Akemann for having pointed out to me that this kind of regularity definitely fails in arbitrary C^*-algebras.

Nevertheless we try to imitate Pełczyński's lemma in [27] to construct c_0-copies. Our approach was also influenced by the construction of c_0-copies in two papers of Bourgain [5, Lemma 3; 6, Sect. 2]. In these papers as well as in [27] the authors localize the construction of c_0 reducing it to the construction of $l^\infty(n)$'s by using the natural M-structure of the space in question. The $l^\infty(n)$'s in their turn are "fetched down from the bidual" either by the principle of local reflexivity [5, 6] or by using regularity and Urysohn's lemma (as in [27]; note, for example, that in the above sketch concerning Lemma 1 of [27] we have $\mathrm{lin}\{f, g\} = l^\infty(2)$). In the case of C^*-algebras we will construct the $l^\infty(n)$'s with the aid of the natural local M-structure, hidden in the classical Hahn-Jordan decomposition and we will dodge regularity by considering it as a kind of measure theoretic counterpart of Goldstine's theorem (because given $\mu \in C(K)'$ and $\chi_E \in C(K)''$ regularity yields an element f of $C(K)$ such that $\chi_E(\mu) \approx \mu(f)$). So it fits our purpose well to look at $\mathscr{K}$ as a set containing an l^1-basis (as a consequence of the w-sequential completeness of A' and Rosenthal's l^1-theorem), to look at two orthogonal states as a basis of $l^1(2)$ and at their orthogonal supports as a basis of $l^\infty(2)$ in A'' which we want to get down to A itself. The proof of Lemma 4, though fairly short and special, contains all these ingredients. Therefore Lemma 4, and Lemma 5 as its rather natural generalization, contain the core of our proof, namely, in some sense, the local version of property (V).

Some tedious obstacles stem from perturbation arguments due to the fact that $\mathscr{K}$, if it is not relatively w-compact, gives rise to an only $(1+\varepsilon)$-isomorphic copy of l^1. But the point is that the interplay between the Banach spaces $l^\infty(n)$, $l^1(n)$ on the one side and the algebraic concept of orthogonality on the other side still works if we have only almost isometric copies and not isometric copies of $l^\infty(n)$'s and $l^1(n)$'s. Again, this appears already in Lemma 4.

2 Notation and preliminaries

Since our subject lies between Banach space theory and C*-algebra theory we follow the referees' suggestion to take into account that the specialist in one of these theories might not be so familiar with the background material of the other field. We try to make up for this in the following a bit lengthy explanations.

2.1 Banach spaces

Basic properties and definitions concerning Banach space theory which are not explained here can be found in [22–24] or in [12].

2.1.1 The dual of a Banach space X is denoted by X', the corresponding elements by $x, y, \ldots \in X$ and $x', y', \ldots \in X'$, respectively; $x'|_Y$ stands for the restriction of x' to the subspace $Y \subset X$. The w-topology of X is the $\sigma(X, X')$-topology, the w^*-topology of X' is the $\sigma(X', X)$-topology. By virtue of Goldstine's theorem the unit ball of (the canonical image of) X is w^*-dense in X''.

To conform to the usual terminology we use the term operator for a linear bounded map and by a subspace of a Banach space we mean a closed subspace.

2.1.2 We will use the fact that a bounded non-relatively w-compact subset of a w-sequentially complete Banach space contains a basic sequence equivalent to the canonical l^1-basis. This follows from two well known theorems. The theorem of Eberlein-Šmulian [10, p. 18] says that a subset of a Banach space is relatively w-compact if and only if it is relatively sequentially w-compact. Rosenthal's l^1-theorem [10, Chap. XI] states that in a Banach space a bounded sequence without w-Cauchy subsequence contains a subsequence equivalent to the canonical l^1-basis.

2.1.3 Property (V), *Grothendieck spaces.* A series $\sum x_i$ in a Banach space X is called w-unconditionally Cauchy (wuC for short) if it satisfies $\sum |x'(x_i)| < \infty$ for all $x' \in X'$. If $\sum x_i$ is a non-trivial wuC-series, i.e. one which does not converge weakly, then, as has been proved by Bessaga and Pełczyński, it is essentially a c_0-basis in the sense that (x_i) contains a subsequence equivalent to the canonical c_0-basis [10, Chap. V].

Pełczyński defined property (V) in [27]:

Definition. *A Banach space X has property (V) if for any (bounded) non relatively w-compact set $\mathcal{K} \subset X'$ there is a wuC-series $\sum x_i$ in X such that $\sup_{x' \in \mathcal{K}} |x'(x_i)|$ does not converge to zero.*

Pełczyński showed that X having (V) is equivalent to the fact that any $T \in \mathrm{L}(X, Y)$ (Y any Banach space) is either weakly compact or fixes a copy of c_0. (We say T fixes a copy of c_0 if X contains a subspace isomorphic to c_0 such that the restriction of T to this subspace is an isomorphism.) See also [28, Satz. 3.4]. Furthermore it is easy to see that X' is w-sequentially complete if X has (V) [27]. Property (V) passes to quotients [27] and is clearly stable under isomorphisms.

Among the classical spaces, spaces of the form $\mathrm{C}(K)$, K compact, (thus in particular $\mathrm{L}^\infty(\mu)$-spaces) and $\mathrm{H}^\infty$are known to have property (V). (See [27; 5].)

Definition. *A Banach space is called Grothendieck space if in its dual each w^*-convergent sequence converges weakly.*

See [28, Sects. 1, 3] for some elementary properties of Grothendieck spaces: Their duals are w-sequentially complete, and quotients of Grothendieck spaces are Grothendieck spaces, too. Since the w^*-convergent (w-convergent) sequences in the dual of a Banach space X are in one-to-one correspondence with the bounded (w-compact) operators from X into c_0, X is a Grothendieck space if and only if each $T \in \mathrm{L}(X, c_0)$ is w-compact; in particular c_0 (or any other non-reflexive separable Banach space as can be easily shown) cannot be a quotient of a Grothendieck space [16, p. 169; 10, p. 114]. From this it is easy to infer that dual spaces with property (V) are Grothendieck spaces [9, p. 40; 28, Sect. 3]. Consequently l^∞, $\mathrm{L}^\infty(\mu)$, H^∞ are Grothendieck spaces.

Special constructions and examples of Grothendieck spaces can be found in [18, 31] (Grothendieck spaces of the form $\mathrm{C}(K)$ without a copy of l^∞), [3, 8, 20, 21, 28, 29].

2.1.4 For the sake of completeness we cite James' distortion theorem [19]. Inequality (1) follows from the inequalities preceding it.

Theorem (James). *Let $\delta > 0$ and let (x_n) be a normalized l^1-basis in a (real or complex) Banach space with a constant $r > 0$ such that*

$$r \sum |\alpha_n| \leqq \left\| \sum \alpha_n x_n \right\| \leqq \sum |\alpha_n|$$

where the α_n are arbitrary scalars. Then there is a sequence (λ_i) in the scalar field and a sequence (F_n) of pairwise disjoint finite subsets of $\mathbb{N}$ such that

$$\begin{gathered}(1-\delta) \sum_{n \geqq 1} |\alpha_n| \leqq \left\| \sum_{n \geqq 1} \alpha_n y_n \right\| \leqq \sum_{n \geqq 1} |\alpha_n|, \\ \sum_{i \in F_n} |\lambda_i| \leqq \frac{1}{r} \quad \forall n \in \mathbb{N},\end{gathered} \tag{1}$$

where $y_n = \sum\limits_{i \in F_n} \lambda_i x_i$.

2.2 C^*-*algebras*

If not stated otherwise the basic knowledge we will use in the sequel is found in [25] and [30].

2.2.1 A_1 denotes the unit ball of a C^*-algebra A; A_{sa}, A_+ denote the selfadjoint and positive elements, respectively, of A, similarly A_1^{sa}, A_1^+ and A'_1, $A'_{\mathrm{sa}} = (A')_{\mathrm{sa}}$ and so on are to be understood. Note that $(A')_{\mathrm{sa}} = (A_{\mathrm{sa}})'$ [25, 3.2.1].

2.2.2 A C^*-algebra A decomposes as $A = A_{\mathrm{sa}} + \iota A_{\mathrm{sa}}$, where A_{sa} is a *real* Banach space and ι denotes the imaginary unit. A W^*-algebra W (or, equivalently, a von Neumann algebra W) is a C^*-algebra which is a dual Banach space; it can be shown that the predual W_* of W is unique and w-sequentially complete [30, III.3.9, III.5.2]; w-sequential completeness follows also from the more general viewpoint that W_* is an L-summand in its bidual [30, III.2.14; 13].

2.2.3 Each C*-algebra can be viewed as a concrete C*-algebra via its universal representation [30, I.9.18, III.2.4]. As usual we identify the bidual of a C*-algebra with its universal enveloping von Neumann algebra [25, 3.7; 30, III.2, III.3]. In this way the bidual of a C*-algebra becomes a W*-algebra. (See also [4, Sect. 12].) Thus the dual of a C*-algebra is w-sequentially complete being the predual of a W*-algebra. In slight contrast with the usual notation in C*-algebra theory (but conform to our notation in 2.1.1) we write $a'', b'', \ldots$ for the elements of A'' of a C*-algebra A in order to underline the rôle of A'' as a bidual; only projections of A'' are denoted, for convenience, by $p, q, \ldots$.

Considering a C*-algebra A as a C*-subalgebra of A'' we write $\tilde{A}$ for the C*-subalgebra generated by (the canonical image of) A and the unit $\mathbf{1} \in A''$. (Note that A'', being a dual C*-algebra, has always a unit e.g. because its unit ball contains extreme points [30, I.10.2]; we may also consider this unit as the unit in the bicommutant of the universal representation of A.)

2.2.4 By Gelfand's theorem each commutative C*-algebra is of the form $C_0(K)$ where K is locally compact in general and compact if the algebra is unital. By χ_E the characteristic function of a set E is meant. If $E \subset K$ is a Borel set then $\chi_E \in C(K)''$ [26, Chap. 6].

According to functional calculus we use the notation $a = a^+ - a^-$, $|a| = a^+ + a^-$ for $a \in A_{sa}$. (As a reference for functional calculus we suggest [26, 4.4–4.5].) It is completely elementary but important to note that in this case a^+ and a^- are orthogonal and therefore span an isometric copy of $l^\infty(2)$ if $a^+ \neq 0 \neq a^-$. (Compare with the dual remark in 2.2.8 below.)

2.2.5 Occasionally we distinguish general functionals $a', b', \ldots \in A'$ on a C*-algebra in our notation from selfadjoint and positive ones, which we write as $f, g, \ldots \in A'_{sa}$ and $\phi, \psi, \omega \in A'_+$, respectively; sometimes, when the Hahn-Jordan decomposition (see 2.2.8 below) is used, the positive functionals are denoted by $f^+, f^-, g^+, \ldots$, in this case we write for example $|f| = f^+ + f^-$.

Positive functionals of norm one are called states, the state space is the set of all states of a C*-algebra.

The expression $x''a'y'' \in A'$ means the functional defined by $x''a'y''(z'') = a'(y''z''x'')$, where $x'', y'', z'' \in A''$, $a' \in A'$ [4, Sect. 12; 30, p. 123]; here we use the w^*-continuity of the multiplication by an element of A'' [25, 2.1.2] to see that $x''a'y''$ is indeed w^*-continuous on A''.

2.2.6 The following facts on positive functionals will be used throughout the article. Positive functionals on unital C*-algebras attain their norms on the unit [30, I.9.9.]. From this one deduces easily that the norm is additive on positive functionals meaning that $\|\phi + \psi\| = \|\phi\| + \|\psi\|$ for positive functionals ϕ, ψ on any C*-algebra [11, 4.22.16].

For any $a, b, x \in A$, A a C*-algebra, $x \geqq 0$ implies $a^*xa \geqq 0$ and $0 \leqq x \leqq a$ implies both $0 \leqq b^*xb \leqq b^*ab$ and $\|x\| \leqq \|a\|$ [25, 1.3.5]. Thus if $\omega \in A'$ is positive and if $x'' = x''^*$ then the functional $x''\omega x'' \in A'$ is positive, too. Each functional $a' \in A'$ is the linear combination of two selfadjoint functionals, namely $a' = \frac{1}{2}(a' + a'^*) + \iota \frac{1}{2\iota}(a' - a'^*)$, and thus, by the Hahn-Jordan decomposition (see 2.2.8 below), each $a' \in A'$ is the linear combination of four positive functionals each of whose norms is majorized by that of a'.

2.2.7 Cauchy-Schwarz inequality. There is a variant of the Cauchy-Schwarz inequality for W*-algebras W saying that $|(y^*x)(\omega)|^2 \leqq (y^*y)(\omega) \cdot (x^*x)(\omega)$ for $x, y \in W$, $\omega \in (W_*)_+$.

2.2.8 The Hahn-Jordan decomposition is a generalization of the commutative case proved by Grothendieck and runs as follows [30, III.4.2; 25, 3.3]:

Theorem (Hahn-Jordan decomposition). *Each selfadjoint functional $f \in W_*$ in the predual W_* of a W^*-algebra W can be uniquely written as the difference of two orthogonal positive functionals $\phi, \psi \in (W_*)_+$, more precisely $f = \phi - \psi$, $\|f\| = \|\phi\| + \|\psi\|$ $(= \|\phi + \psi\| = \||f|\|)$. Moreover ϕ and ψ have (analogously to the commutative case) "disjoint supports" which means that there are two orthogonal projections $p, q \in W$ such that $p(\phi) = \|\phi\|$ and $q(\psi) = \|\psi\|$ and $p(\psi) = 0 = q(\phi)$, in particular $\|f\| = (p - q)(f)$.*

Remark. It is elementary but important to note that ϕ and ψ in the above theorem span an ($\mathbb{R}$)-isometric copy of $l^1(2)$ and that, dually, $\mathrm{lin}_{\mathbb{R}}\{p, q\} \cong l^\infty(2)$.

2.2.9 Projections in A'' are written as $p, q, \ldots \in A''$; $p^c = \mathbf{1} - p \in A''$ is called the complement of the projection p. (As to the fact that $\mathbf{1} \in A''$ whether A is unital or not, see 2.2.3.)

If $M \subset A_{\mathrm{sa}}$, A a C*-algebra, is a set of selfadjoint elements, then M^m (respectively M^σ) and M_m (respectively M_δ) stands for the set of those elements of A'' which are limits of monotone increasing and, respectively, decreasing nets (respectively sequences) of M; the limit is to be understood in the w^*-topology of A'', but since we identify A'' with the universal enveloping von Neumann algebra of A this amounts to the convergence in the strong operator topology. Recall that *bounded monotone nets of selfadjoint elements have limits in the strong operator topology* [25, 2.2.3], and that on bounded sets the latter one is finer than the w^*-topology of A'' [30, II.2.5]. For example, for a bounded sequence $(x_n) \subset A$ of pairwise orthogonal selfadjoint nonzero elements in a C*-algebra A the sum $\sum x_n$ exists in A'' both in the w^*-topology and in the strong operator topology of A''; in particular $\sum x_n$ is a wuC-series and the closed linear span of (x_n) is an isometric c_0-copy. This can be seen easily for $C_0(K)$-spaces; with the aid of Gelfand's theorem the non-commutative case reduces to the commutative one since the operators in question commute pairwise and are therefore contained in a commutative subalgebra.

2.2.10 Open and closed projections. Due to the idea according to which projections in non-commutative C*-algebra correspond to the subsets of K in the commutative C*-algebra C(K), Akemann [1] (see also [25, 3.11] and [30, Chap. III.6]) defined the notion of open projections:

Definition (Open and closed projections). *A projection $p \in A''$ (A a C*-algebra) is called open if $p \in (A_+)^m$, i.e. if there is an increasing net $a_\alpha \in A_+$ with $p = \lim a_\alpha$.*

A projection $p \in A''$ is called closed if its complement is open.

In particular we have $\omega(a_\alpha) \nearrow p(\omega)$ for all positive functionals $\omega \in A'_+$. By Urysohn's lemma the open projections coincide indeed with the open subsets of K if the latter ones are identified with their characteristic functions contained in the bidual of C(K).

The closure $\bar{p}$ (or $\bar{p}^A$ to make clear to which algebra the closure refers) of a projection $p \in A''$ is the infimum of all closed projections in A'' majorizing p.

See [25, 3.11.9] for the proof of the following theorem. Roughly speaking, it says that A and $\tilde{A}$ have the same open projections.

Theorem (Open projections). *For a projection $p \in A''$ (A a unital or non-unital C^*-algebra) the following assertions are equivalent:*

(i) *p is open,*

(ii) *$p \in (\tilde{A}_{sa})^m$,*

(iii) *p lies in the closure in the strong operator topology of the hereditary subalgebra $B = pA''p \cap A$ in A'', in particular $B'' = pA''p$. (B is called hereditary if for all $a \in A_+$ and $b \in B_+$ such that $0 \leqq a \leqq b$ we have $a \in B$.)*

Since each C^*-algebra A contains an approximate unit, which, being a bounded increasing net, converges in $\sigma(A'', A')$ to the unit, the bidual of each C^*-algebra A contains the unit as an open projection of A.

Remark. Although the supremum of two closed projections need not be closed [1, Theorem II.6], Akemann [1, Theorem II.7] proved that the finite sum $\sum_{n=1}^{n} p_i$ of pairwise orthogonal closed projections $p_i \in A''$ is closed. (For the sake of completeness we refer to the theorem on open projections in this paragraph in order to reduce the non-unital case (which is not treated in [1]) to the unital one: Let A be non-unital, $p_1, p_2 \in A''$ be orthogonal and closed i.e. $p_1^c, p_2^c \in (\tilde{A}_+)^m$, then the unital case gives that $(p_1 + p_2)^c \in (\tilde{A}_+)^m$, whence $(p_1 + p_2)^c \in (A_+)^m$ is open and $p_1 + p_2$ is closed.)

2.2.11 Range projections are a special kind of open projections: the range projection of an operator $x \in A_1^+$ is defined as $\mathrm{rp}(x) = \chi_{(0,1]}(x)$ [25, 2.2.7]. (If $T \in L(H)$, then $\mathrm{rp}(T)$ is the projection on the closure of the range of T.) Since $\chi_{(0,1]}$ is approximated pointwise by the increasing sequence of continuous functions $\chi_n \in C[0,1]$ with $\chi_n(t) = \left(\frac{1}{n} + t\right)^{-1} t$, the projection $\mathrm{rp}(x)$ is open and commutes with x. (Here by functional calculus for Borel functions the pointwise convergence of the χ_n in C[0, 1] translates into the convergence of $\chi_n(x)$ in the w^*-topology and in the strong operator topology of A''.) We note also that the χ_n are operator monotone which means that $0 \leqq \chi_n(x) \leqq \chi_n(y)$ if $0 \leqq x \leqq y \leqq \mathbf{1}$ [30, I.7.2]. Thus $0 \leqq \mathrm{rp}(x) \leqq \mathrm{rp}(y) \leqq \mathbf{1}$ because obviously operator monotonicity passes to monotone limites of operator monotone functions [25, 1.3, in particular 1.3.12]. Moreover functional calculus gives $\mathrm{rp}(x)\,\mathrm{rp}(y) = 0$ for $xy = 0$, $x, y \in A_1^+$. Note also that $x \in A_1^+$ implies $\mathrm{rp}(x) \geqq x$.

3 The main result

Theorem 1. *In the dual of a C^*-algebra w-compactness is determined commutatively.*

More precisely: Let A be a C^*-algebra, $\mathcal{K} \subset A'$ be bounded. Then $\mathcal{K}$ is not relatively w-compact if and only if there is a sequence $(x_n) \subset A_1$ of pairwise orthogonal selfadjoint elements such that $\sup_{a' \in \mathcal{K}} |a'(x_n)| \not\to 0$; in particular in this case there is a maximal commutative C^*-subalgebra $B \subset A$ such that $\mathcal{K}|_B$ $(= \{a'|_B \mid a' \in \mathcal{K}\})$ is not relatively w-compact.

For the proof (which starts after Lemma 5 and the remark following it) we prepare some lemmas.

Proposition II.3 of [1] provides a simple way to get open or closed projections including a certain "control on the measure".

Lemma 2. *Let $x \in A_1^+$ be a positive element in the unit ball of a C^*-algebra A, let $\phi \in A'_+$ be a positive functional and $1 > \varepsilon > 0$. Then $p = \chi_{(\varepsilon,1]}(x)$ is an open projection in A'' such that*

$$p(\phi) \geqq \phi(x) - \varepsilon\|\phi\| . \tag{2}$$

and $q = \chi_{[\delta,1]}(x)$ is a closed projection for all $\delta \in (0, 1]$.

Proof. The characteristic functions $\chi_{(\varepsilon,1]}$ and $\chi_{[\delta,1]}$ are pointwise limits of the monotone increasing respectively decreasing sequences of continuous functions χ_n, $\tau_n \in C[0,1]$ with

$$\chi_n(t) = \begin{cases} 0 & 0 \leqq t \leqq \varepsilon \\ \text{linear} & \varepsilon \leqq t \leqq \varepsilon + \frac{1}{n} \\ 1 & \varepsilon + \frac{1}{n} \leqq t \leqq 1 , \end{cases}$$

$$\tau_n(t) = \begin{cases} 0 & 0 \leqq t \leqq \delta - \frac{1}{n} \\ \text{linear} & \delta - \frac{1}{n} \leqq t \leqq \delta \\ 1 & \delta \leqq t \leqq 1 . \end{cases}$$

Functional calculus says that $p = \chi_{(\varepsilon,1]}(x)$ is open, and again by functional calculus we get (2) from $\| \mathrm{id}_{[0,\varepsilon]}(x)\| \leqq \varepsilon$ and $\chi_{(\varepsilon,1]}(t) \geqq (\mathrm{id}_{[0,1]} - \mathrm{id}_{[0,\varepsilon]})(t)$, $t \in [0,1]$.

If A has a unit then $q = \chi_{[\delta,1]}(x)$ is closed because the complement of q is open being the limit of the increasing sequence $\mathbf{1} - \tau_n(x) \in A$. If A is not unital then the latter argument applies to the unital algebra $\tilde{A}$ and gives $q^c \in (\tilde{A}_+)^m$ hence $q^c \in (A_+)^m$ by the theorem of 2.2.10 and $q \in A''$is closed. □

If a positive functional ω on a C^*-algebra A attains its norm on a positive element $a \in A_1^+$ then it seems to be well known (although I do not know a reference in the literature for it) that $\omega = a\omega a$. The following lemma proves a straightforward quantified version of this.

Lemma 3. *Let $\omega \in W_*$ be a positive and $f \in W_*$ a selfadjoint normal functional on a W^*-algebra W (for example $\omega, f \in A'$, A a C^*-algebra with $W = A''$). Let further $\delta \geqq 0$, $a \in W_1^+$, $b \in W_1^{\mathrm{sa}}$, $x \in W_1$, $a(\omega) \geqq \|\omega\| - \delta$ and $b(f) \geqq \|f\| - \delta$. Then we have*

$$|(axa)(\omega) - x(\omega)| \leqq 2\delta^{1/2}\|\omega\|^{1/2} , \tag{3}$$
$$|(bxb)(f) - x(f)| \leqq 8\delta^{1/2}\|f\|^{1/2} . \tag{4}$$

Proof. Inequality (3) follows with the aid of Cauchy-Schwarz's inequality from

$$\begin{aligned} |(ax(1-a))(\omega)|^2 &\leqq (ax(ax)^*)(\omega) \cdot ((1-a)^2)(\omega) \\ &\leqq \|\omega\| \cdot (1-a)(\omega) \leqq \delta\|\omega\| , \\ |((1-a)x)(\omega)|^2 &\leqq ((1-a)^2)(\omega) \cdot (x^*x)(\omega) \\ &\leqq \|\omega\| (1-a)(\omega) \leqq \delta\|\omega\| \end{aligned}$$

and from $x - axa = ax(1-a) + (1-a)x$.

For the second part of the assertion we use the Hahn-Jordan decomposition $f = \phi - \psi$ (see 2.2.8) and the decomposition $b = b^+ - b^-$ with $b^+, b^- \in A_1^+$. From

$$\begin{aligned} \|\phi\| + \|\psi\| = \|f\| &\geqq b(f) = b^+(\phi) + b^-(\psi) - (b^-(\phi) + b^+(\psi)) \\ &\geqq \|f\| - \delta = \|\phi\| + \|\psi\| - \delta \end{aligned}$$

we deduce

$$\begin{aligned} &b^+(\phi) \geqq \|\phi\| - \delta + \|\psi\| - b^-(\psi) \geqq \|\phi\| - \delta\,, \\ &b^-(\psi) \geqq \|\psi\| - \delta\,, \end{aligned}$$

and $b^-(\phi) + b^+(\psi) \leqq b^+(\phi) - \|\phi\| + b^-(\psi) - \|\psi\| + \delta$, hence

$$b^-(\phi) \leqq \delta\,, \qquad b^+(\psi) \leqq \delta\,. \tag{5}$$

Similarly as in the proof of the first part we see (by (5), by the Cauchy-Schwarz inequality and recalling that $(b^-)^2 \leqq b^-$, $(b^+)^2 \leqq b^+$) that for any $c \in A_1$ all of the four expressions $|(cb^-)(\phi)|$, $|(b^-c)(\phi)|$, $|(cb^+)(\psi)|$, $|(b^+c)(\phi)|$ are smaller than $\delta^{1/2}\|\phi\|^{1/2}$ and $\delta^{1/2}\|\psi\|^{1/2}$, respectively. Together with the first part of the assertion this yields

$$\begin{aligned} |(bxb)(\phi) - x(\phi)| &= |(b^+xb^+ - b^+xb^- - b^-xb)(\phi) - x(\phi)| \\ &\leqq |(b^+xb^+)(\phi) - x(\phi)| + |(b^+xb^-)(\phi)| + |(b^-xb)(\phi)| \\ &\leqq 2\delta^{1/2}\|\phi\|^{1/2} + 2\delta^{1/2}\|\phi\|^{1/2} \end{aligned}$$

and analogously

$$|(bxb)(\psi) - x(\psi)| \leqq 4\delta^{1/2}\|\psi\|^{1/2}\,,$$

hence (4) holds because

$$\begin{aligned} |(bxb)(f) - x(f)| &\leqq |(bxb)(\phi) - x(\phi)| + |(bxb)(\psi) - x(\psi)| \\ &\leqq 4\delta^{1/2}(\|\phi\|^{1/2} + \|\psi\|^{1/2}) \leqq 4\delta^{1/2}) \cdot 2\|f\|^{1/2}\,. \end{aligned}$$ □

Despite its shortness, the following Lemma 4 contains the essential idea of how commutativity comes into play. This lemma (and, alike, Lemma 5) gives a quantified version of the Hahn-Jordan decomposition, and, what is more important, links two orthogonal positive functionals in A' (spanning $l^1(2)$) to two orthogonal elements (spanning $l^\infty(2)$) which are not only in A'' but even in A. (Compare the proof of Lemma 4 with the one of [25, 3.2.3].)

Lemma 4. *Let two positive functionals $\phi, \psi \in A'$ on a* C**-algebra A and a number $\delta > 0$ satisfy the condition*

$$\|\phi - \psi\| > \|\phi + \psi\| - \delta\,.$$

Then there are two orthogonal positive elements $a, b \in A_1^+$ (i.e. $ab = 0$) and two orthogonal open projections $p, q \in A''$, such that

$$\begin{aligned} &\phi(a) > \|\phi\| - 2\delta\,, \qquad p(\phi) > \|\phi\| - 2\delta\,, \\ &\psi(b) > \|\psi\| - 2\delta\,, \qquad q(\psi) > \|\psi\| - 2\delta\,. \end{aligned}$$

Proof. There is a selfadjoint element $x \in A_1^{sa}$ such that

$$(\phi - \psi)(x) > \|\phi + \psi\| - \delta\,.$$

From

$$(\phi-\psi)(x)=(\phi+\psi)(x^{+}+x^{-})-2(\phi(x^{-})+\psi(x^{+}))>\|\phi+\psi\|-\delta \tag{6}$$

and $|x|\leqq \mathbf{1}$ we infer

$$\|\phi+\psi\|\geqq(\phi+\psi)(|x|)\overset{(6)}{>}\|\phi+\psi\|+2(\phi(x^{-})+\psi(x^{+}))-\delta$$

and hence $(\phi+\psi)(|x|)>\|\phi+\psi\|-\delta$ and $\phi(x^{-})+\psi(x^{+})<\delta$, so

$$\begin{aligned}\phi(|x|)&>\|\phi\|-\delta\,, & \psi(|x|)&>\|\psi\|-\delta\,,\\ \phi(x^{+})&>\|\phi\|-2\delta\,, & \phi(x^{-})&>\|\phi\|-2\delta\,.\end{aligned}$$

Put $a=x^{+}$, $b=x^{-}$. Choose $\eta>0$ such that $\phi(a)>\|\phi\|-2\delta+2\eta\|\phi\|$ and $\psi(b)>\|\psi\|-2\delta+2\eta\|\psi\|$. The projections $p=\chi_{(\eta,1]}(a)$ and $q=\chi_{(\eta,1]}(b)$ are open, satisfy the desired inequalities by Lemma 2, and are orthogonal because a and b and thus also rp(a) and rp(b) are orthogonal and because $p\leqq$ rp(a) and $q\leqq$ rp(b). □

Lemma 5 relies on the preceding one, seems natural but a bit cumbersome to prove.

Lemma 5. *Let n be a natural number. For each $\varepsilon>0$ there is a number $\delta=\delta(n,\varepsilon)>0$ with the following property:*

If selfadjoint functionals $f_1,\ldots,f_n\in A'_{\mathrm{sa}}$ on a C^*-algebra A span an l^1-copy depending on δ, more precisely if

$$(1-\delta)\sum_{k=1}^{n}|\alpha_k|\leqq\left\|\sum_{k=1}^{n}\alpha_k f_k\right\|\leqq\sum_{k=1}^{n}|\alpha_k|\,, \tag{7}$$

then for $k=1,\ldots,n$ there are pairwise orthogonal elements $a_k,b_k\in A_{\mathbf{1}}^{+}$, i.e.

$$a_k b_i=0\,,\qquad a_k a_j=0=b_k b_j\quad\forall i,k,j\leqq n,k\neq j\,, \tag{8}$$

such that

$$f_k(a_k-b_k)>\|f_k\|-\varepsilon\quad\forall k\leqq n\,. \tag{9}$$

Proof. We prove this by induction and start with the case $n=1$:

The Hahn-Jordan decomposition (see 2.2.8) of $f_1=\phi-\psi$ with $\|\phi-\psi\|=\|\phi+\psi\|$ satisfies the assumption of Lemma 4 for each $\delta>0$: There are two elements a_1,b_1 as wanted, such that $\phi(a_1)>\|\phi\|-2\delta$, $\psi(b_1)>\|\psi\|-2\delta$, $\phi(b_1)\leqq\phi(\mathbf{1}-a_1)<2\delta$, $\psi(a_1)<2\delta$, hence $f_1(a_1-b_1)>\|\phi-\psi\|-8\delta$. Put $\delta=\delta(1,\varepsilon)=\varepsilon/8$.

Since we need the case $n=2$ for the induction step, we treat this case separately.

Consider the Hahn-Jordan decompositions $f_i=\phi_i-\psi_i (i=1,2)$ and the four positive functionals $\phi_1+\phi_2$, $\psi_1+\psi_2$, $\phi_1+\psi_2$, $\phi_2+\psi_1$. Choose $\delta=\delta(2,\varepsilon)<1$ such that $0<260\delta^{1/2}\leqq\varepsilon$. By assumption we have $\|f_1\pm f_2\|\geqq 2(1-\delta)$, $\|f_i\|\leqq 1 (i=1,2)$ and therefore

$$\begin{aligned}\|(\phi_1+\phi_2)-(\psi_1+\psi_2)\|&=\|f_1+f_2\|\geqq 2-2\delta\geqq\|f_1\|+\|f_2\|-2\delta\\&=\|\phi_1\|+\|\psi_1\|+\|\phi_2\|+\|\psi_2\|-2\delta\\&=\|(\phi_1+\phi_2)+(\psi_1+\psi_2)\|-2\delta\\&>\|(\phi_1+\phi_2)+(\psi_1+\psi_2)|-\tfrac{5}{2}\,\delta\end{aligned} \tag{10}$$

(recall that the norm is additive on $(A')_+$, cf. 2.2.6) and analogously

$$\|(\phi_1+\psi_2)-(\phi_2+\psi_1)>\|(\phi_1+\psi_2)+(\phi_2+\psi_1)\|-\tfrac{5}{2}\,\delta\,. \tag{11}$$

[To get an idea of what is going on it might be useful to have a look at the easy case where $\delta = 0$. Then the calculations of (10) and (11) mean that $(\phi_1 + \phi_2)$ and $(\psi_1 + \psi_2)$ are orthogonal and that $(\phi_1 + \psi_2)$ and $(\phi_2 + \psi_1)$ are orthogonal, too. By the Hahn-Jordan theorem these two pairs of positive orthogonal functionals are supported by projections p_1, p_2 and p_3, p_4 where $p_1 p_2 = 0$ and $p_3 p_4 = 0$. In other words, $\psi_2 = p_2 \psi_2 p_2$, whence $p_1 \psi_2 p_1 = 0$ since $p_1 p_2 = 0$. Similarly one sees that $\phi_1 = p_1 \phi_1 p_1$, $p_2 \phi_1 p_2 = 0$. Hence ϕ_1 and ψ_2 are orthogonal, and in fact $\phi_1, \phi_2, \psi_1, \psi_2$ are all pairwise orthogonal, as is seen in the same way. Now we repeat the same reasoning with a side glance at Lemma 4 and conclude that these four functionals are actually suported, up to a small error, by four orthogonal elements of A_1^+. Of course, for $\delta > 0$ the argument becomes more technical.]

By Lemma 4 there are two orthogonal open projections s_1, s_2 and two orthogonal open projections s_3, s_4 in A'' such that

$$\begin{aligned} s_1(\phi_1 + \phi_2) &> \|\phi_1 + \phi_2\| - 5\delta, \\ s_2(\psi_1 + \psi_2) &> \|\psi_1 + \psi_2\| - 5\delta, \\ s_3(\phi_1 + \psi_2) &> \|\phi_1 + \psi_2\| - 5\delta, \\ s_4(\phi_2 + \psi_1) &> \|\phi_2 + \psi_1\| - 5\delta. \end{aligned}$$

hence

$$\begin{aligned} s_1(\phi_1) &> \|\phi_1 + \phi_2\| - 5\delta - s_1(\phi_2) \\ &= \|\phi_1\| - 5\delta + \|\phi_2\| - s_1(\phi_2) \geqq \|\phi_1\| - 5\delta, \\ s_2(\phi_1) &\leqq (\mathbf{1} - s_1)(\phi_1) \leqq \|\phi_1\| - s_1(\phi_1) < 5\delta \end{aligned}$$

(as to the unit in the last formula we recall that for non-unital A there is a unit in A'', see 2.2.3) and similarly

$$\begin{aligned} &s_1(\phi_2) > \|\phi_2\| - 5\delta, && s_2(\phi_2) < 5\delta, \\ &s_2(\psi_i) > \|\psi_i\| - 5\delta, && s_1(\psi_i) < 5\delta, \qquad i = 1, 2 \\ &s_3(\phi_1) > \|\phi_1\| - 5\delta, && s_4(\phi_2) > \|\phi_2\| - 5\delta, \\ &s_4(\psi_1) > \|\psi_1\| - 5\delta, && s_3(\psi_2) > \|\psi_2\| - 5\delta. \end{aligned}$$

From these inequalities we derive on the one hand

$$\begin{aligned} \|\phi_1 - \psi_2\| &\geqq (s_1 - s_2)(\phi_1 - \psi_2) > \|\phi_1\| - 5\delta - 5\delta + \|\psi_2\| - 5\delta - 5\delta \\ &= \|\phi_1 + \psi_2\| - 20\delta \qquad (12) \\ \|\phi_2 - \psi_1\| &> \|\phi_2 + \psi_1\| - 20\delta, \qquad (13) \end{aligned}$$

and on the other hand by Lemma 3 we have the estimates

$$\|\phi_1 - s_3\phi_1 s_3\| \leqq 2(5\delta)^{1/2}\|\phi_1\|^{1/2} \leqq 5\delta^{1/2}\|f_1\|^{1/2} \leqq 5\delta^{1/2}. \qquad (14)$$

$$\|\psi_2 - s_3\psi_2 s_3\| \leqq 5\delta^{1/2}. \qquad (15)$$

Consider the subalgebra $B_3 = s_3 A'' s_3 \cap A$. Since s_3 is open B_3 is hereditary and $B_3'' = s_3 A'' s_3$ (see the theorem in 2.2.10). Consequently for any selfadjoint functional $g \in A'$ we have that

$$\|g|_{B_3}\|_{B_3} = \|s_3 g s_3\|. \qquad (16)$$

Now we get

$$\begin{aligned}
\|(\phi_1 - \psi_2)|_{B_3}\|_{B_3} &\overset{(16)}{=} \|s_3\phi_1 s_3 - s_3\psi_2 s_3\| \\
&\geqq \|\phi_1 - \psi_2\| - (\|\phi_1 - s_3\phi_1 s_3\| + \|\psi_2 - s_3\psi_2 s_3\|) \\
&\overset{(14)(15)}{\geqq} \|\phi_1 - \psi_2\| - 10\delta^{1/2} \\
&\overset{(12)}{=} \|\phi_1 + \psi_2\| - 20\delta - 10\delta^{1/2} \\
&> \|(\phi_1 + \psi_2)|_{B_3}\|_{B_3} - 30\delta^{1/2}\,. \qquad (17)
\end{aligned}$$

In a similar way the estimate

$$\|(\phi_2 - \psi_1)|_{B_4}\|_{B_4} > \|(\phi_2 + \psi_1)|_{B_4}\|_{B_4} - 30\delta^{1/2} \qquad (18)$$

follows for the subalgebra $B_4 = s_4 A'' s_4 \cap A$.

Now we use Lemma 4 and (17) in order to choose two orthogonal elements $a_1, b_2 \in (B_3)_{\mathbf{1}}^+ \subset A_{\mathbf{1}}^+$ for the two positive functionals $\phi_1|_{B_3}$ and $\psi_2|_{B_3}$ on the C*-algebra B_3 such that

$$\begin{aligned}
\phi_1(a_1) &> \|\phi_1|_{B_3}\|_{B_3} - 60\delta^{1/2} \\
&\overset{(16)}{=} \|s_3\phi_1 s_3\| - 60\delta^{1/2} \\
&\overset{(14)}{\geqq} \|\phi_1\| - 65\delta^{1/2}\,, \qquad (19) \\
\psi_2(b_2) &> \|\psi_2|_{B_3}\|_{B_3} - 60\delta^{1/2} \\
&\overset{(15)(16)}{\geqq} \|\psi_2\| - 65\delta^{1/2}\,. \qquad (20)
\end{aligned}$$

Analogously we obtain two orthogonal elements $a_2, b_1 \in (B_4)_{\mathbf{1}}^+ \subset A_{\mathbf{1}}^+$ such that

$$\begin{aligned}
\phi_2(a_2) &> \|\phi_2\| - 65\delta^{1/2}\,, \qquad (21) \\
\psi_1(b_1) &> \|\psi_1\| - 65\delta^{1/2}\,. \qquad (22)
\end{aligned}$$

By construction one has $a_1 b_2 = 0 = a_2 b_1$ and

$$\begin{aligned}
a_1 &= s_3 a_1 s_3\,, \quad a_2 = s_4 a_2 s_4\,, \\
b_2 &= s_3 b_2 s_3\,, \quad b_1 = s_4 b_1 s_4\,,
\end{aligned}$$

which gives $(8, n = 2)$. By $\phi_1(b_1) \leqq \phi_1(\mathbf{1} - a_1) < 65\delta^{1/2}$ and $\psi_1(a_1) \leqq \psi_1(\mathbf{1} - b_1) < 65\delta^{1/2}$ the inequalities (19–22) imply

$$\begin{aligned}
f_1(a_1 - b_1) &= \phi_1(a_1 - b_1) - \psi_1(a_1 - b_1) \\
&> \|\phi_1\| + \|\psi_1\| - 4 \cdot 65\delta^{1/2} \\
&= \|f_1\| - 260\delta^{1/2} \qquad (23)
\end{aligned}$$

and analogously

$$f_2(a_2 - b_2) > \|f_2\| - 260\delta^{1/2}\,. \qquad (24)$$

Induction step $n \to n + 1$:

Let $\varepsilon > 0$ and let $f_1, \dots, f_{n+1} \in A'_{\mathrm{sa}}$ be selfadjoint functionals on a C*-algebra A such that

$$(1-\delta_{n+1}) \sum_{k=1}^{n+1} |\alpha_k| \leqq \left\| \sum_{k=1}^{n+1} \alpha_k f_k \right\| \leqq \sum_{k=1}^{n+1} |\alpha_k|, \tag{25}$$

where $\delta_{n+1} > 0$ is arbitrary for the moment and will be determined later. By induction hypothesis we choose $\delta_n = \delta_n(n, \varepsilon/2) > 0$.

Set $f = \frac{1}{n} \sum_{k=1}^{n} f_k$, hence

$$(1-\delta_{n+1})(|\alpha_1| + |\alpha_2|) \leqq \|\alpha_1 f + \alpha_2 f_{n+1}\| \leqq |\alpha_1| + |\alpha_2| \tag{26}$$

for all scalars α_1, α_2. The case $n = 2$ just proved (in particular see (19–22) yields four pairwise orthogonal elements $a, b, a_{n+1}, b_{n+1} \in A_{\mathbf{1}}^{+}$ such that

$$f^+(a) > \|f^+\| - 65\delta_{n+1}^{1/2} + \eta, \quad f_{n+1}^+(a_{n+1}) > \|f_{n+1}^+\| - 65\delta_{n+1}^{1/2} + \eta,$$
$$f^-(b) > \|f^-\| - 65\delta_{n+1}^{1/2} + \eta, \quad f_{n+1}^-(b_{n+1}) > \|f_{n+1}^-\| - 65\delta_{n+1}^{1/2} + \eta,$$

where $\eta > 0$ can be chosen appropriately. By functional calculus the projections

$$s_1 = \chi_{(\eta,1]}(a), \quad p_{n+1} = \chi_{(\eta,1]}(a_{n+1}),$$
$$s_2 = \chi_{(\eta,1]}(b), \quad q_{n+1} = \chi_{(\eta,1]}(b_{n+1})$$

are pairwise orthogonal and by Lemma 2 they are open. From Lemma 2 we infer also that $s_1(f^+) \geqq f^+(a) - \eta\|f^+\| > \|f^+\| - 65\delta_{n+1}^{1/2}$ and similarly $s_2(f^-) > \|f^-\| - 65\delta_{n+1}^{1/2}$, therefore $s_2(f^+) \leqq (\mathbf{1} - s_1)(f^+) < 65\delta_{n+1}^{1/2}$ and $s_1(f^-) < 65\delta_{n+1}^{1/2}$; altogether we have

$$(s_1 - s_2)(f) > \|f\| - 260\delta_{n+1}^{1/2}. \tag{27}$$

In the same way it follows that

$$f_{n+1}(a_{n+1} - b_{n+1}) > \|f_{n+1}\| - 260\delta_{n+1}^{1/2}. \tag{28}$$

Define $\tilde{f} = (s_1 - s_2) f (s_1 - s_2)$, $\tilde{f}_k = (s_1 - s_2) f_k (s_1 - s_2)$ for $k \leqq n$. Note that $\|\tilde{f}_k\| \leqq \|f_k\| \leqq 1$. Put $\tau = \delta_{n+1} + 260\delta_{n+1}^{1/2}$. From

$$1 - \tau \overset{(26)}{\leqq} \|f\| - 260\delta^{1/2} \overset{(27)}{<} (s_1 - s_2)(f) = (s_1 - s_2)(\tilde{f})$$
$$= \sum_{k=1}^{n} \frac{1}{n} (s_1 - s_2)(\tilde{f}_k)$$

and from $(s_1 - s_2)(\tilde{f}_k) \leqq 1$ we infer

$$(s_1 - s_2)(\tilde{f}_k) > 1 - n\tau \geqq \|f_k\| - n\tau \quad \forall k \leqq n. \tag{29}$$

By Lemma 3 we have

$$\|\tilde{f}_k - f_k\| \leqq 8(n\tau)^{1/2} \|f_k\|^{1/2} \leqq 8(n\tau)^{1/2} \tag{30}$$

for $k \leqq n$.

If one considers $\tilde{g} = (s_1 - s_2) g (s_1 - s_2)$ for any $g \in A'_{\mathrm{sa}}$, then $\|\tilde{g}\| \leqq \|g|_B\|$, where B means the subalgebra $B = (s_1 + s_2) A''(s_1 + s_2) \cap A$, because $\tilde{g}(x) =$

$g((s_1 - s_2)x(s_1 - s_2)) = g((s_1 + s_2)(s_1 - s_2)x(s_1 - s_2)(s_1 + s_2)) \leqq \|g|_{B''}\|_{B''}$ for all $x \in A_1^{sa}$ and because $\|g|_{B''}\|_{B''} = \|g|_B\|_B$ since $s_1 + s_2$ is open and thus $B'' = (s_1+s_2)A''(s_1+s_2)$ (see 2.2.10). This observation explains the second inequality of the following calculations:

$$0 \leqq \left\| \sum_{k=1}^{n} \alpha_k f_k \right\| - \left\| \sum_{k=1}^{n} \alpha_k f_k|_B \right\|_B \leqq \left\| \sum_{k=1}^{n} \alpha_k f_k \right\| - \left\| \sum_{k=1}^{n} \alpha_k \tilde{f}_k \right\|$$
$$\leqq \left\| \sum_{k=1}^{n} \alpha(f_k - \tilde{f}_k) \right\| \leqq \max_{k \leqq n} \|f_k - \tilde{f}_k\| \sum_{k=1}^{n} |\alpha_k| \overset{(30)}{\leqq} 8(n\tau)^{1/2} \sum_{k=1}^{n} |\alpha_k|$$

and therefore (25) implies

$$(1 - \delta_{n+1} - 8(n\tau)^{1/2}) \sum_{k=1}^{n} |\alpha_k| \leqq \left\| \sum_{k=1}^{n} \alpha_k f_k|_B \right\|_B \leqq \sum_{k=1}^{n} |\alpha_k| . \tag{31}$$

Since δ_{n+1} was arbitrary we may choose it to be such that both $\delta_{n+1} + 8(n\tau)^{1/2} < \delta_n$ and $n\tau < \varepsilon/2$ with δ_n as already chosen above. Apply the induction hypothesis to n, $\frac{\varepsilon}{2}$, B and $f_k|_B (k \leqq n)$. There are pairwise orthogonal elements a_k, $b_k \in B_1^+ (k \leqq n)$ such that by $(s_1 - s_2)(f_k) = (s_1 - s_2)(\tilde{f}_k)$

$$\begin{aligned} f_k(a_k - b_k) &= (f_k|_B)(a_k - b_k) > \|f_k|_B\|_B - \frac{\varepsilon}{2} \\ &\geqq (s_1 - s_2)(\tilde{f}_k) - \frac{\varepsilon}{2} \\ &\overset{(29)}{\geqq} \|f_k\| - \varepsilon \ \forall k \leqq n . \end{aligned}$$

Together with (28) we have $(9, n+1)$.

The pairwise orthogonality of the elements in question, i.e. $(8, n+1)$, follows from the induction hypothesis (i.e. from the orthognality of $a_k, b_k, k \leqq n$), from the fact that $s_1, s_2, a_{n+1}, b_{n+1}$ are pairwise orthogonal and from the fact that by construction

$$a_k = (s_1 + s_2) a_k (s_1 + s_2) \ \forall k \leqq n ,$$
$$b_k = (s_1 + s_2) b_k (s_1 + s_2) \ \forall k \leqq n . \qquad \square$$

We will use Lemma 5 in the following form:

Remark. For each $n \in \mathbb{N}$ and $\varepsilon > 0$ there is $\delta = \delta(n, \varepsilon) > 0$ with the following property:

If B is a hereditary C*-subalgebra of a C*-algebra A and if selfadjoint functionals $f_1, \ldots, f_n \in (A')_1^{sa}$ span an $l^1(n)$ on B, such that

$$(1 - \delta) \sum_{k=1}^{n} |\alpha_k| \leqq \left\| \sum_{k=1}^{n} \alpha_k f_k|_B \right\|_B \leqq \sum_{k=1}^{n} |\alpha_k| , \tag{32}$$

then for $k = 1, \ldots, n$ there are open projections $p_k, q_k \in B''$ with pairwise orthogonal closures in A, that is

$$\left. \begin{aligned} \overline{p_k}^A, \overline{q_j}^A \in B'' \\ \overline{p_k}^A \overline{q_i}^A = \overline{p_k}^A \overline{p_j}^A = \overline{q_k}^A \overline{q_j}^A = 0 \end{aligned} \right\} \quad \forall i, k, j \leqq n, k \neq j , \tag{33}$$

and such that

$$(p_k - q_k)(f_k) > \|f_k\| - \varepsilon \qquad \forall k \leqq n\,. \tag{34}$$

Proof of the remark. Set $\tilde{f}_k = f_k|_B \in B'$. From Lemma 5, applied to B, we infer the existence of $\delta = \delta(n, \varepsilon/8) > 0$ and of orthogonal elements $a_k, b_k \in B_1^+$ satisfying (8) and such that $\delta < \varepsilon/2$ and

$$\tilde{f}_k(a_k - b_k) > \|\tilde{f}_k\|_B - \frac{\varepsilon}{8}\,.$$

Then it follows that

$$\begin{aligned}(\tilde{f}_k)^+(a_k) + (\tilde{f}_k)^-(b_k) &= \tilde{f}_k(a_k - b_k) + ((\tilde{f}_k)^+(b_k) + (\tilde{f}_k)^-(a_k))\\ &> \|\tilde{f}_k\|_B + ((\tilde{f}_k)^+(b_k) + (\tilde{f}_k)^-(a_k)) - \frac{\varepsilon}{8}\\ &\geqq \|(\tilde{f}_k)^+\|_B + \|(\tilde{f}_k)^-\|_B - \frac{\varepsilon}{8}\,,\end{aligned}$$

and therefore there is $\eta > 0$ such that for $k \leqq n$

$$\begin{aligned}(\tilde{f}_k)^+(a_k) &> \|(\tilde{f}_k)^+\|_B - \frac{\varepsilon}{8} + \eta\|(\tilde{f}_k)^+\|_B\,,\\ (\tilde{f}_k)^-(b_k) &> \|(\tilde{f}_k)^-\|_B - \frac{\varepsilon}{8} + \eta\|(\tilde{f}_k)^-\|_B\,.\end{aligned}$$

Similarly as in the proof of Lemma 4 we define pairwise orthogonal open projections $p_k = \chi_{(\eta,1]}(a_k)$, $q_k = \chi_{(\eta,1]}(b_k)$. Note that $p_k, q_k \in B'' \subset A''$. The fact that $\overline{p_k}^A \in B'' \subset A''$ (and analogously $\overline{q_k}^A \in B'' \subset A''$) for each $k \leqq n$ follows from the facts that $\chi_{[\eta,1]}(a_k)$ is closed in A'' (Lemma 2), that $\overline{p_k}^A \leqq \chi_{[\eta,1]}(a_k) \leqq \mathrm{rp}(a_k) \in B''$ and that B is hereditary. By Lemma 2 we have

$$\begin{aligned}p_k((\tilde{f}_k)^+) &> \|(\tilde{f}_k)^+\|_B - \frac{\varepsilon}{8}\,, \quad p_k((\tilde{f}_k)^-) \leqq (\mathbf{1} - q_k)((\tilde{f}_k)^-) < \frac{\varepsilon}{8}\,,\\ q_k((\tilde{f}_k)^-) &> \|(\tilde{f}_k)^-\|_B - \frac{\varepsilon}{8}\,, \quad q_k((\tilde{f}_k)^+) \leqq (\mathbf{1} - p_k)((\tilde{f}_k)^+) < \frac{\varepsilon}{8}\,.\end{aligned}$$

Thus (34) follows from

$$\begin{aligned}(p_k - q_k)(f_k) &= (p_k - q_k)(\tilde{f}_k)\\ &> \|(\tilde{f}_k)^+\|_B + \|(\tilde{f}_k)^-\|_B - \frac{\varepsilon}{2}\\ &= \|(\tilde{f}_k)\|_B - \frac{\varepsilon}{2}\\ &\overset{(32)}{\geqq} \|f_k\| - \delta - \frac{\varepsilon}{2} \geqq \|f_k\| - \varepsilon\,.\end{aligned}$$

□

Proof of Theorem 1. We prove only the if-implication, because the other implication follows easily, for example, from Theorem III.5.4 in [30]. A more direct argument yields that the quotient map $q: A' \to B'$ maps a relatively w-compact set $\mathscr{K} \subset A'$ onto the relatively w-compact set $\mathscr{K}|_B$.

For the other implication let A be a C*-algebra, $\mathscr{K} \subset A'$ a bounded non-relatively w-compact set.

Since A' is weakly sequentially complete (see 2.2.3) the theorems of Eberlein-Šmuljan and of Rosenthal (see 2.1.2) imply that $\mathcal{K}$ contains an l^1-basis (a'_k). If we show the existence of a number $\theta > 0$, of a subsequence (a'_{m_n}) and a c_0-basis (x_n) such that

$$x_n \in A_{\mathbf{1}}^{\mathrm{sa}}, \qquad x_\imath x_n = 0 = x_n x_\imath \quad |a'_{m_n}(x_n)| > \theta \;\; \forall i, n \in \mathbb{N}, i \neq n\,. \tag{35}$$

we will be done because then by Zorn's lemma there exists a maximal abelian selfadjoint subalgebra B that contains all the x_n.

Without loss of generality the a'_k are selfadjoint, because if $a'_k = f_k + \imath g_k$ is the canonical decomposition of a'_k in $A = A_{\mathrm{sa}} + \imath A_{\mathrm{sa}}$ then (after passing to appropriate subsequences) at least one of the sequences (f_k) and (g_k), say (f_k), has no w-converging subsequence and contains therefore, again by Rosenthal's theorem, an l^1-basis (f_{k_n}), and if $\sum x_n$ is a wuC-series consisting of selfadjoint elements such that $|f_{k_n}(x_n)| > \theta\, \forall n \in \mathbb{N}$, then $|a'_{k_n}(x_n)| \geqq |\mathrm{Re}(f_{k_n}(x_n) + \imath g_{k_n}(x_n))| = |f_{k_n}(x_n)| > \theta\, \forall n \in \mathbb{N}$.

Let $r > 0$ be a constant such that the l^1-basis $\dfrac{a'_k}{\|a'_k\|}$ fulfills

$$r \sum_{k \geqq 1} |\alpha_k| \leqq \left\| \sum_{k \geqq 1} \alpha_k \frac{a'_k}{\|a'_k\|} \right\| \leqq \sum_{k \geqq 1} |\alpha_k| \;\; \forall (\alpha_k) \subset \mathbb{R}\,, \tag{36}$$

in particular we have that $r \leqq 1$.

Choose a number ε with $0 < \varepsilon < 1$ and a sequence (ε_n) of positive numbers such that $\sum \varepsilon_n = \varepsilon$ and $\varepsilon_n \leqq \frac{3}{4}$ for all $n \in \mathbb{N}$. Set $\theta = (1 - \varepsilon) r \inf_{k \in \mathbb{N}} \|a'_k\|$. Without loss of generality we may assume

$$\|a'_k\| = 1\,.$$

By induction over $n = 1, 2, \ldots$ we construct sequences (p_n), $(q_n) \subset A''$ of open projections, a sequence of indices (m_n), a decreasing sequence $(N_\imath)$ of infinite subsets of $\mathbb{N}$, i.e. $\ldots \subset N_{i+1} \subset N_i \subset \ldots \subset N_1 \subset N_0 = \mathbb{N}$, such that we have for all $n \in \mathbb{N}$:

$$\overline{p_i}\,\overline{p_n} = 0 = \overline{q_i}\,\overline{q_n}\,, \qquad \overline{p_j}\,\overline{q_n} = 0 = \overline{q_j}\,\overline{p_n} \;\; \forall i < n, j \leqq n\,, \tag{37}$$

$$(\overline{p_n} + \overline{q_n})(|a'_m|) < \tfrac{1}{144} r^2 \varepsilon_n^4 \qquad \forall m \in N_n \tag{38}$$

$$(p_n - q_n)(a'_{m_n}) > r\left(1 - \sum_1^n \varepsilon_\imath\right) \tag{39}$$

$$m_n \in N_{n-1}\,. \tag{40}$$

We start the induction with $n = 1$.

Choose $j_1 \in \mathbb{N}$ with $1/j_1 < r^2 \varepsilon_1^4/144$. For j_1 and $\varepsilon_1/2$ the remark after the proof of Lemma 5 yields a number $\delta_1 = \delta_1(j_1, \varepsilon_1/2) > 0$ and without loss of generality we assume $\delta_1 \leqq \varepsilon_1/2$. By James' theorem there are pairwise disjoint finite sets $F_k^{(1)} \subset N_0 = \mathbb{N}$, a sequence $(\lambda_i^{(1)}) \subset \mathbb{R}$ and functionals $f_k^{(1)} = \sum_{F_k^{(1)}} \lambda_i^{(1)} a'_\imath$ for $k \in \mathbb{N}$, such that

$$\sum_{F_k^{(1)}} |\lambda_i^{(1)}| \leqq \frac{1}{r}\,, \tag{41}$$

$$(1 - \delta_1) \sum_{k \geqq 1} |\alpha_k| \leqq \left\| \sum_{k \geqq 1} \alpha_k f_k^{(1)} \right\| \leqq \sum_{k \geqq 1} |\alpha_k|\,. \tag{42}$$

Again by the remark following Lemma 5 there are pairwise orthogonal open projections $p_k^{(1)}, q_k^{(1)} \in A''$, $k \leqq j_1$, such that

$$(p_k^{(1)} - q_k^{(1)})(f_k^{(1)}) > \|f_k^{(1)}\| - \frac{\varepsilon_1}{2} \ \forall k \leqq j_1 . \tag{43}$$

and since the projections can be chosen to have orthogonal closures we have $0 \leqq \sum_1^{j_1} \overline{p_k^{(1)}} + \overline{q_k^{(1)}} \leqq \mathbf{1}$ and

$$\left(\sum_1^{j_1} \frac{1}{j_1} (\overline{p_k^{(1)}} + \overline{q_k^{(1)}})\right)(|a'_m|) \leqq |a'_m|\left(\frac{1}{j_1} \cdot \mathbf{1}\right) < \|a'_m\| \frac{r^2 \varepsilon_1^4}{144} \ \forall m \in N_0 .$$

Therefore there exist a $k_1 \leqq j_1$ and an inifnite set $N_1 \subset N_0$ such that

$$(\overline{p_{k_1}^{(1)}} + \overline{q_{k_1}^{(1)}})(|a'_m|) < \frac{r^2 \varepsilon_1^4}{144} \ \forall m \in N_1 . \tag{44}$$

Set $p_1 = p_{k_1}^{(1)}$, $q_1 = q_{k_1}^{(1)}$, $f_1 = f_{k_1}^{(1)}$, $F_1 = F_{k_1}^{(1)}$. Now we infer that

$$(p_1 - q_1)(f_1) \overset{(43)}{>} \|f_1\| - \frac{\varepsilon_1}{2} \overset{(42)}{>} \left(1 - \delta_1 - \frac{\varepsilon_1}{2}\right) \geqq 1 - \varepsilon_1 ,$$

which in turn yields the existence of an index $m_1 \in F_1 \subset N_0$ as desired in (39) and (40) for $n = 1$, because otherwise we would have

$$\begin{aligned} |(p_1 - q_1)(f_1)| &= \left| \sum_{F_1} \lambda_i^{(1)} (p_1 - q_1)(a'_i) \right| \\ &\leqq \max_{i \in F_1} |(p_1 - q_1)(a'_i)| \sum_{F_1} |\lambda_i^{(1)}| \overset{(41)}{\leqq} r(1 - \varepsilon_1)\frac{1}{r} . \end{aligned}$$

In order to see $(37, n = 1)$ we verify only that $\overline{p_1}\,\overline{q_1} = 0$ which follows from the construction. Inequality $(38, n = 1)$ corresponds to (44). The first induction step is done.

Induction step $n \longrightarrow n + 1$:

Suppose p_k, q_k, N_k, m_k to be constructed for $k \leqq n$ according to (37–40). Consider the projection $s_n = \left(\sum_1^n \overline{p_k} + \overline{q_k}\right)^c$. Since the $\overline{p_k}$ and the $\overline{q_k}$ are orthogonal, $\sum_1^n \overline{p_k} + \overline{q_k}$ is closed (see the remark in 2.2.10) and therefore s_n is open. Set $B_{n+1} = s_n A'' s_n \cap A$ and $\tilde{a}'_m = s_n a'_m s_n$ for $m \in N_n$. Then B_{n+1} is a hereditary C*-subalgebra of A whose strong closure contains s_n (see the theorem in 2.2.10). Note that the a'_m and the $\tilde{a}'_m$ are equal on B_{n+1}, more precisely $\tilde{a}'_m|_{B_{n+1}} = a'_m|_{B_{n+1}}$ for $m \in N_n$, and note that

$$\|a'_m\| = \|\tilde{a}'_m|_{B_{n+1}}\|_{B_{n+1}} . \tag{45}$$

One of the main observations in this induction step is formula (46) which, compared with (36), says that the $\dfrac{\tilde{a}'_m}{\|\tilde{a}'_m\|}$ behave on B_{n+1} almost like the a'_m on A. This is a consequence of the fact that the a'_m, $m \in N_n$, are small on the projections constructed up to now – see the induction hypothesis (38) – and must therefore be supported almost entirely by "the complement" B_{n+1}. Having (46) in hand we will then work in B_{n+1} and construct two orthogonal projections p_{n+1} and q_{n+1} in the bidual of B_{n+1} similarly as in the first induction step; since these two projections will lie in B''_{n+1} they are automatically orthogonal to the projections constructed so far, as required in $(37, n+1)$.

Claim. The normalized functionals $\left(\frac{\tilde{a}'_m}{\|\tilde{a}'_m\|}\right)_{m\in N_n}$ form an l^1-basis with

$$r\left(1-\sum_1^n \varepsilon_i^2\right)\sum_{m\in N_n}|\alpha_m| \leqq \left\|\sum_{m\in N_n}\alpha_m\frac{\tilde{a}'_m}{\|\tilde{a}'_m\|}\right\| \leqq \sum_{m\in N_n}|\alpha_m| \ \ \forall(\alpha_m)_{m\in N_n}\subset\mathbb{R}. \tag{46}$$

Consider the following calculations:

$$s_n^c(|a'_m|) = \left(\sum_1^n \overline{p_k}+\overline{q_k}\right)(|a'_m|) \overset{(38)}{<} \frac{1}{144} r^2 \sum_1^n \varepsilon_i^4 \ \ \forall m\in N_n\,.$$

thus $(s_n(|a'_m|) > \|(a'_m)^+\| + \|(a'_m)^-\| - \frac{1}{144} r^2 \sum_1^n \varepsilon_i^4$ and

$$s_n((a'_m)^+) > \|(a'_m)^+\| - \frac{1}{144} r^2 \sum_1^n \varepsilon_i^4 \ \ \forall m\in N_n\,, \tag{47}$$

$$s_n((a'_m)^-) > \|(a'_m)^-\| - \frac{1}{144} r^2 \sum_1^n \varepsilon_i^4 \ \ \forall m\in N_n\,, \tag{48}$$

$$\begin{aligned}\|s_n a'_m s_n - a'_m\| &\leqq \|s_n(a'_m)^+ s_n - (a'_m)^+\| + \|s_n(a'_m)^- s_n - (a'_m)^-\| \\ &\overset{3(47)(48)}{\leqq} 2\left(\frac{r^2\sum_1^n\varepsilon_i^4}{144}\right)^{1/2}(\|(a'_m)^+\|^{1/2}+\|(a'_m)^-\|^{1/2}) \\ &\leqq \frac{r}{6}\left(\sum_1^n\varepsilon_i^2\right)\cdot 2 = \frac{r}{3}\sum_1^n\varepsilon_i^2 \ \ \forall m\in N_n\,;\end{aligned} \tag{49}$$

further we note that for all $m\in N_n$

$$\|\tilde{a}'_m\| \leqq \|a'_m\| = 1\,, \tag{50}$$

$$0 \leqq 1-\|\tilde{a}'_m\| = \|a'_m\|-\|\tilde{a}'_m\| \leqq \|a'_m-\tilde{a}'_m\| \overset{(49)}{\leqq} \frac{r}{3}\sum_1^n\varepsilon_i^2\,, \tag{51}$$

$$\|\tilde{a}'_m\| \geqq \|a'_m\|-\|a'_m-\tilde{a}'_m\| \overset{(49)}{\geqq} 1-\frac{r}{3}\sum_1^n\varepsilon_i^2 > \frac{2}{3}\,, \tag{52}$$

hence

$$\begin{aligned}\left\|\frac{\tilde{a}'_m}{\|\tilde{a}'_m\|}-\tilde{a}'_m\right\| &\overset{(50)}{\leqq} \frac{1}{\|\tilde{a}'_m\|}-1 = (1-\|\tilde{a}'_m\|)\frac{1}{\|\tilde{a}'_m\|} \\ &\overset{(51)(52)}{\leqq} \frac{3}{2}\cdot\frac{r}{3}\sum_1^n\varepsilon_i^2\end{aligned} \tag{53}$$

and

$$\left\| \frac{\tilde{a}'_m}{\|\tilde{a}'_m\|} - a'_m \right\| \leqq \left\| \frac{\tilde{a}'_m}{\|\tilde{a}'_m\|} - \tilde{a}'_m \right\| + \|\tilde{a}'_m - a'_m\|$$
$$\overset{(53)(49)}{\leqq} \left(\frac{r}{2} + \frac{r}{3}\right) \sum_1^n \varepsilon_i^2 < r \sum_1^n \varepsilon_i^2 \,. \tag{54}$$

Then by

$$\left\| \sum_{N_n} \alpha_m \frac{\tilde{a}'_m}{\|\tilde{a}'_m\|} - \sum_{N_n} \alpha_m a'_m \right\| \leqq \sup_{N_n} \left\| \frac{\tilde{a}'_m}{\|\tilde{a}'_m\|} - a'_m \right\| \sum_{N_n} |\alpha_m|$$
$$\overset{(54)}{\leqq} r \left(\sum_1^n \varepsilon_i^2 \right) \left(\sum_{N_n} |\alpha_m| \right) \quad \forall (\alpha_m)_{m \in N_n} \subset \mathbb{R}$$

formula (46) follows from (36) and the Claim is established.

Choose a number $j_{n+1} \in \mathbb{N}$ with $1/j_{n+1} < r^2 \varepsilon_{n+1}^4/144$. For j_{n+1} and $\varepsilon_{n+1}^2/2$ we may further choose by the remark that follows Lemma 5 a number $\delta_{n+1} = \delta_{n+1}(j_{n+1}, \varepsilon_{n+1}^2/2) > 0$ (and without loss of generality such that $\delta_{n+1} \leqq \varepsilon_{n+1}^2/2$). Now we apply James' theorem. By (46) there are pairwise disjoint finite sets $F_k^{(n+1)} \subset N_n$, a sequence $(\lambda_i^{(n+1)}) \subset \mathbb{R}$ and functionals $f_k^{(n+1)} = \sum_{F_k^{(n+1)}} \lambda_i^{(n+1)} \frac{\tilde{a}'_i}{\|\tilde{a}'_i\|}$ for each $k \in \mathbb{N}$ such that $\|f_k^{(n+1)}\| = \|f_k^{(n+1)}|_{B_{n+1}}\|_{B_{n+1}}$ by (45) and

$$\sum_{F_k^{(n+1)}} |\lambda_i^{(n+1)}| \leqq \frac{1}{r\left(1 - \sum_1^n \varepsilon_i^2\right)} \quad \forall k \in \mathbb{N} \,. \tag{55}$$

$$(1 - \delta_{n+1}) \sum_{k \geqq 1} |\alpha_k| \leqq \left\| \sum_{k \geqq 1} \alpha_k f_k^{(n+1)} \right\|$$
$$\overset{(45)}{=} \left\| \sum_{k \geqq 1} \alpha_k f_k^{(n+1)}|_{B_{n+1}} \right\|_{B_{n+1}} \leqq \sum_{k \geqq 1} |\alpha_k| \,. \tag{56}$$

Again by the remark following Lemma 5, applied to the hereditary algebra $B_{n+1} \subset A$, to the functionals $f_k^{(n+1)} \in A'$ and to (56), there exist open projections $p_k^{(n+1)}$, $q_k^{(n+1)} \in B''_{n+1}$, $k \leqq j_{n+1}$, with pairwise orthogonal closures (in A'') such that

$$(p_k^{(n+1)} - q_k^{(n+1)})(f_k^{(n+1)}) > \|f_k^{(n+1)}|_{B_{n+1}}\|_{B_{n+1}} - \frac{\varepsilon_{n+1}^2}{2} \quad \forall k \leqq j_{n+1} \,. \tag{57}$$

Since the projections have orthogonal closures we have $0 \leqq \sum_1^{j_{n+1}} (\overline{p_k^{(n+1)}} + \overline{q_k^{(n+1)}}) \leqq \mathbf{1}$ and

$$\left(\sum_1^{j_{n+1}} \frac{1}{j_{n+1}} (\overline{p_k^{(n+1)}} + \overline{q_k^{(n+1)}}) \right) (|a'_m|) \leqq \|a'_m\| \frac{r^2 \varepsilon_{n+1}^4}{144} \quad \forall m \in N_n \,.$$

Therefore there exist an index $k_{n+1} \leqq j_{n+1}$ and an infinite subset $N_{n+1} \subset N_n$ such that

$$|a'_m|\,(\overline{p^{(n+1)}_{k_{n+1}}} + \overline{q^{(n+1)}_{k_{n+1}}}) \leqq \frac{r^2 \varepsilon^4_{n+1}}{144} \ \ \forall m \in N_{n+1}\,.$$

Set $p_{n+1} = p^{(n+1)}_{k_{n+1}}$, $q_{n+1} = q^{(n+1)}_{k_{n+1}}$, $f_{n+1} = f^{(n+1)}_{k_{n+1}}$, $F_{n+1} = F^{(n+1)}_{k_{n+1}}$. Now we infer that

$$\begin{aligned}(p_{n+1} - q_{n+1})(f_{n+1}) &\overset{(57)}{>} \|f_{n+1}|_{B_{n+1}}\|_{B_{n+1}} - \frac{\varepsilon^2_{n+1}}{2} \\ &\overset{(56)}{\geqq} 1 - \delta_{n+1} - \frac{\varepsilon^2_{n+1}}{2} \geqq 1 - \varepsilon^2_{n+1}\,, \end{aligned} \tag{58}$$

hence there is an index $m_{n+1} \in F_{n+1} \subset N_n$ such that

$$(p_{n+1} - q_{n+1})\left(\frac{\tilde{a}'_{m_{n+1}}}{\|\tilde{a}'_{m_{n+1}}\|}\right) > r(1 - \varepsilon^2_{n+1})\left(1 - \sum_1^n \varepsilon^2_i\right). \tag{59}$$

because otherwise the following estimates would contradict (58):

$$\begin{aligned}|(p_{n+1} - q_{n+1})(f_{n+1})| &= \left|\sum_{i \in F_{n+1}} \lambda^{(n+1)}_i (p_{n+1} - q_{n+1})\left(\frac{\tilde{a}'_i}{\|\tilde{a}'_i\|}\right)\right| \\ &\leqq \max_{i \in F_{n+1}} \left|(p_{n+1} - q_{n+1})\left(\frac{\tilde{a}'_i}{\|\tilde{a}'_i\|}\right)\right| \sum_{F_{n+1}} |\lambda^{(n+1)}_i| \\ &\overset{(55)}{\leqq} r(1 - \varepsilon^2_{n+1})\left(1 - \sum_1^n \varepsilon^2_i\right) \frac{1}{r\left(1 - \sum_1^n \varepsilon^2_i\right)} \\ &= 1 - \varepsilon^2_{n+1}\,.\end{aligned}$$

In order to show $(39, n+1)$ we consider

$$\begin{aligned}(p_{n+1} - q_{n+1})(a'_{m_{n+1}}) &= (p_{n+1} - q_{n+1})(\tilde{a}'_{m_{n+1}}) \\ &\overset{(59)}{>} \|\tilde{a}'_{m_{n+1}}\|\, r(1 - \varepsilon^2_{n+1})\left(1 - \sum_1^n \varepsilon^2_i\right) \\ &\overset{(51)}{\geqq} \left(1 - \frac{r}{3}\sum_1^n \varepsilon^2_i\right) r(1 - \varepsilon^2_{n+1})\left(1 - \sum_1^n \varepsilon^2_i\right) \\ &> r\left(1 - \sum_1^{n+1} \varepsilon_i\right)\end{aligned}$$

where the last inequality follows from the following completely elementary estimates:

$$\left(1-\frac{r}{3}\sum_1^n \varepsilon_i^2\right)(1-\varepsilon_{n+1}^2)\left(1-\sum_1^n \varepsilon_i^2\right)$$
$$=\left(1-\sum_1^{n+1}\varepsilon_i^2\right)-\frac{r}{3}\left[1-(1-\varepsilon_{n+1}^2)\sum_1^n\varepsilon_i^2-\left(1+\frac{3}{r}\right)\varepsilon_{n+1}^2\right]\sum_1^n\varepsilon_i^2$$
$$>\left(1-\sum_1^{n+1}\varepsilon_i^2\right)-\frac{r}{3}\sum_1^n\varepsilon_i^2$$
$$>\left(1-\sum_1^{n+1}\varepsilon_i^2\right)-\frac{1}{3}\sum_1^{n+1}\varepsilon_i^2$$
$$=\left(1-\sum_1^{n+1}\varepsilon_i\right)+\sum_1^{n+1}\varepsilon_i\left(1-\varepsilon_i-\frac{\varepsilon_i}{3}\right)$$
$$\geqq 1-\sum_1^{n+1}\varepsilon_i$$

since we assumed $\varepsilon' \leqq \frac{3}{4}$ for all $i \in \mathbb{N}$. Thus $(39, n+1)$ is proved.

Since $\overline{p_{n+1}}, \overline{q_{n+1}}$ are in B''_{n+1} they are orthogonal to s_n^c, and hence orthogonal to $\overline{p_i}, \overline{q_i}, i \leqq n$. Together with $(37, n)$ we infer $(37, n+1)$ for $i, j < n+1$; $\overline{p_{n+1}}\,\overline{q_{n+1}} = 0$ holds true by construction. This ends the induction.

It remains to define the x_n of (35) via the projections p_n and q_n. Set $x_n = a_n - b_n \in A$ where by (39) and by the fact that the p_n and q_n are open, a_n and b_n may be chosen such that $0 \leqq a_n \leqq p_n$, $0 \leqq b_n \leqq q_n$ and

$$a'_{m_n}(x_n) > r\left(1+\sum_1^n \varepsilon_i\right) > (1-\varepsilon)r = \theta\,.$$

The orthogonality of the projections implies $(a_n - b_n)(a_i - b_i) = 0$ for $i \neq n$, which finally gives all of (35). □

4 Some consequences of Theorem 1

In 2.1.3 we recalled the definitions of property (V) and of Grothendieck spaces.

It is known that K(H), the space of compact operators on a Hilbert space, has property (V). This follows e.g. from the fact that K(H) is an M-ideal in its bidual L(H), the space of bounded operators on a Hilbert space [17, ex.I.1.4d], and from the fact that M-embedded Banach spaces have property (V) [14, 15].

Corollary 6. *C*-algebras have property* (V).

Proof. Let $\mathcal{K} \subset A'$ and (x_n) be as in Theorem 1. Then $\sum x_n$ is a wuC-series (2.2.9) and meets the requirements of property (V). □

A partial result of the following corollary is contained in [2]; it states that a w^*-convergent sequence of positive functionals in the dual of a von Neumann algebra converges weakly.

Corollary 7. *W*-algebras – and thus in particular* $L(H)$ *– are Grothendieck spaces.*

Proof. W*-algebras are dual spaces. Dual spaces with property (V) are Grothendieck spaces (2.1.3). □

Corollary 8. *JBW*-triples (see* [32] *for the definition) have property* (V) *and are Grothendieck spaces.*

Proof. Every JBW*-triple is, as a Banach space, isomorphic to a complemented subspace of a von Neumann algebra [7]. On the one hand property (V) and the Grothendieck property pass to complemented subspaces, on the other hand von Neumann algebras have both properties, as seen in the Corollaries 6 and 7. □

Question 8. *Do JB*-triples have property* (V)*?*

In light of the last corollary and given so many issues that have been generalized from C*-algebras to JB*-triples one tends to conjecture that the answer to this question is affirmative.

Corollary 10. *If A is a W^*-algebra and $\mathcal{K} \subset A'$ is bounded and not relatively w-compact then there is a commutative W^*-subalgebra B such that $\mathcal{K}|_B$ is not relatively w-compact and $B \cong l^\infty$ is the w^*-closure of the linear span of a sequence of pairwise orthogonal projections.*

Proof. Being a von Neumann algebra A contains the range projections of the a_n and b_n which appear in the end of the proof of Theorem 1. They form a sequence of pairwise orthogonal projections because the a_n and b_n are pairwise orthogonal. Take B as the w^*-closure of the linear span of these projections. □

Acknowledgements. I thank the Arbeitsgruppe Behrends of the Freie Universität Berlin and the équipe d'analyse of Paris VI, in particular it is a pleasure to thank D. Werner and G. Godefroy for many helpful discussions. I am also very grateful to the referees for spotting many mistakes in earlier versions of this paper and to Ch. Akemann for useful comments.

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