Particularily important are semigroups such that for every $t \geqq 0$ we have $\|T(t)\| \leqq M$ (<u>bounded semigroups</u>) or $\|T(t)\| \leqq 1$ (<u>contraction semigroups</u>). In both cases we have $\omega \leqq 0$.

It follows from the subsequent examples and from 3.1 that ω may be any number $-\infty \leqq \omega < +\infty$. Moreover the reader should observe that the infimum in (1.1) need not be attained and that M may be larger than 1 even for bounded semigroups.

<u>Examples</u> 1.4. (i) Take $E = \mathbb{C}^2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then for the 1-norm on E we obtain $\|T(t)\| = 1 + t$, hence $(T(t))_{t\geqq 0}$ is an unbounded semigroup having growth bound $\omega = 0$.
(ii) Take $E = L^1(\mathbb{R})$ and for $f \in E$, $t \geqq 0$ define

$$T(t)f(x) := \begin{cases} 2\cdot f(x+t) & \text{if } x \in [-t,0] \\ f(x+t) & \text{otherwise .} \end{cases}$$

Each $T(t)$, $t > 0$, satisfies $\|T(t)\| = 2$ as can be seen by taking $f := 1_{[0,t]}$. Therefore $(T(t))_{t\geqq 0}$ is a strongly continuous semigroup which is bounded, hence has $\omega = 0$, but the constant M in (1.1) cannot be choosen to be 1 .

The most important object associated to a strongly continuous semigroup $(T(t))_{t\geqq 0}$ is its 'generator' which is obtained as the (right)derivative of the map $t \to T(t)$ at $t = 0$. Since for strongly continuous semigroups the functions $t \to T(t)f$, $f \in E$, are continuous but not always differentiable we have to restrict our attention to those $f \in E$ for which the desired derivative exists. We then obtain the 'generator' as a not necessarily everywhere defined operator.

<u>Definition</u> 1.5. To every semigroup $(T(t))_{t\geqq 0}$ there belongs an operator $(A,D(A))$, called the <u>generator</u> and defined on the domain

$$D(A) := \{f \in E : \lim_{h\to 0} \frac{T(h)f - f}{h} \text{ exists in } E\}$$

by $$Af := \lim_{h\to 0} \frac{T(h)f - f}{h} \quad \text{for } f \in D(A).$$

Clearly, $D(A)$ is a linear subspace of E and A is linear from $D(A)$ into E . Only in certain special cases (see 2.1) the generator