Proof. (i)$\to$(iii) is obvious by the definition of quasi-compactness.
(iii)$\to$(ii): Recalling the definition of the essential spectral radius from A-III,(3.6), assertion (iii) implies $r_{ess}(T(t_o)) \leq \|T(t_o)\|_{ess} < 1$. Then $\omega_{ess}(\mathcal{T}) < 0$ by A-III,(3.10).
(ii)$\to$(i): By A-III,(3.10) we have $r_{ess}(T(1)) < 1$. Then A-III,(3.6) implies $\lim_{n\to\infty}\|T(n)\|_{ess}^{1/n} < 1$, where $\|T\|_{ess} = \mathrm{dist}(T,\mathcal{K}(G))$. Thus for suitable $n_o \in \mathbb{N}$, $a < 1$ we have $\|T(n)\|_{ess} < a^n$ for $n \geq n_o$. Choosing a sequence $K_n \in \mathcal{K}(G)$ such that $\|T(n) - K_n\| < a^n$ for $n \geq n_o$ and defining $M := \sup_{0\leq s\leq 1}\|T(s)\|$ we obtain for $t \in [n,n+1]$ $(n \geq n_o)$ $\|T(t) - T(t-n)K_n\| \leq \|T(t-n)\|\|T(n) - K_n\| \leq M\cdot a^n$. This implies that $\lim_{t\to\infty}\mathrm{dist}(T(t),\mathcal{K}(G)) = 0$.

□

A typical situation where quasi-compact semigroups occur is the following. If $\mathcal{T} = (T(t))_{t\geq 0}$ is a strongly continuous semigroup with $\omega_{ess}(\mathcal{T}) < \omega(\mathcal{T})$ then the rescaled semigroup $(\exp(-\omega(\mathcal{T}))T(t))_{t\geq 0}$ is quasi-compact. Obviously every semigroup with growth bound less than zero is quasi-compact. A more interesting situation is the following: If $(T_o(t))_{t\geq 0}$ is a semigroup with growth bound less than zero and A_o is its generator, then for every compact operator K the perturbed operator $A := A_o + K$ generates a quasi-compact semigroup. More generally we have the following result:

<u>Proposition</u> 2.9. If $(T(t))_{t\geq 0}$ is a quasi-compact semigroup on a Banach space G with generator A and K is a compact operator then $A + K$ generates a quasi-compact semigroup.

<u>Proof</u>. If $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are the semigroups generated by A and $A + K$ respectively we have $S(t) = T(t) + \int_0^t T(t-s)KS(s)\,ds$. In view of Prop.2.8(iii) it is enough to show that $\int_0^t T(t-s)KS(s)\,ds$ is a compact operator.
Since the mapping $(t,x) \to T(t)x$ is jointly continuous on $\mathbb{R}_+ \times G$ and since K is compact the set $M := \{T(s)Kx : 0 \leq s \leq t,\ \|x\| \leq 1\}$ is relatively compact in G. Having in mind that $\int_0^t T(t-s)KS(s)x\,ds$ $(x \in G)$ is the norm limit of Riemann sums, one observes that $(ct)^{-1}\int_0^t T(t-s)KS(s)x\,ds$ is an element of the closed convex hull $\overline{\mathrm{co}}\,M$ of M, provided that $c := \sup\{\|S(s)\| : 0 \leq s \leq t\}$ and $\|x\| \leq 1$. Since $\overline{\mathrm{co}}\,M$ is compact (see II.4.3 in Schaefer (1966)) the assertion follows.

□