tion seems to be justified by the fact that for each of the following subsets of $\sigma(A)$ there exist canonical constructions converting the corresponding spectral values into eigenvalues (see Prop. 2.2.ii and Prop. 4.5 below).

Definition 2.1. For a closed, densely defined, linear operator A with domain $D(A)$ in the Banach space E denote by the

(i) point spectrum $P\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not injective.

(ii) approximate point spectrum $A\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not injective or $(\lambda - A)D(A)$ is not closed in E.

(iii) residual spectrum $R\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $(\lambda - A)D(A)$ is not dense in E.

From these definitions it follows that $\lambda \in \mathbb{C}$ is an eigenvalue of A, i.e. $\lambda \in P\sigma(A)$, if and only if there exists an eigenvector $0 \neq f \in D(A)$ such that $Af = \lambda f$. It follows from the Open Mapping Theorem that $\lambda \in A\sigma(A)$ if and only if λ is an approximate eigenvalue, i.e. there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset D(A)$, called an approximate eigenvector, such that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|Af_n - \lambda f_n\| = 0$.
Clearly we have $P\sigma(A) \subset A\sigma(A)$ and $\sigma(A) = A\sigma(A) \cup R\sigma(A)$ where the union need not be disjoint.

The following proposition is a first indication that the subdivision we made implies nice properties.

Proposition 2.2. For a closed, densely defined, linear operator $(A,D(A))$ in a Banach space E the following holds:

(i) The topological boundary $\partial\sigma(A)$ of $\sigma(A)$ is contained in $A\sigma(A)$.

(ii) $R\sigma(A) = P\sigma(A')$ for the adjoint operator A' on E'.

Proof. (i) Take $\lambda_o \in \partial\sigma(A)$ and $\lambda_n \in \rho(A)$ such that $\lambda_n \to \lambda_o$. Since $\|R(\lambda_n,A)\| \geq (\mathrm{dist}(\lambda_n,\sigma(A)))^{-1}$ (see Prop. 2.5.(ii)), by the uniform boundedness principle we find $f \in E$ such that

$$\lim_{n\to\infty} \|R(\lambda_n,A)f\| = \infty .$$