subeigenvectors of A'. In fact, let $\psi \in N'$, $\phi = R(\lambda,A)'\psi$. Then $A'\phi = \lambda\phi - \psi \leq \lambda\phi$.

□

The fact that $\phi \in D(A')_+$ is a subeigenvector can be expressed by the semigroup (if it is positive).

Proposition 3.6. Assume that A is the generator of a positive semigroup $(T(t))_{t\geq 0}$ on a real Banach lattice E. Let $\phi \in D(A')_+$ and $\lambda \in \mathbb{R}$. Then

$$A'\phi \leq \lambda\,\phi \quad \text{if and only if} \quad T(t)'\phi \leq e^{\lambda t}\phi \quad (t \geq 0).$$

Proof. If $T(t)'\phi \leq e^{\lambda t}\,\phi$ for all $t \geq 0$, then

$A'\phi = \sigma(E',E)\text{-}\lim_{t\to 0} 1/t\,(T(t)'\phi - \phi) \leq \lim_{t\to 0} 1/t\,(e^{\lambda t}\phi - \phi) = \lambda\phi$.

For the converse let $f \in E_+$. Then

$$\begin{aligned}\langle f,T(t)'\phi\rangle &= \langle f,\phi\rangle + \int_0^t \langle f,T(s)'A'\phi\rangle\,ds\\ &\leq \langle f,\phi\rangle + \lambda \int_0^t \langle f,T(s)'\phi\rangle\,ds.\end{aligned}$$

It follows from Gronwall's lemma that $\langle f,T(t)'\phi\rangle \leq e^{\lambda t}\langle f,\phi\rangle$.

□

Remark 3.7. a) Using Prop. 3.6 it is immediately clear that $(T(t))_{t\geq 0}$ is irreducible if and only if every positive subeigenvector of A' is strictly positive (cf. C-III,Def.3.1).

b) In the proof of the "only if" - part of Prop. 3.6 we needed the positivity of the semigroup in order to be able to apply Gronwall's lemma. However, if instead of assuming that the semigroup is positive we suppose that A satisfies Kato's inequality and $A'\phi \leq \lambda\phi$ for some strictly positive $\phi \in D(A')$ then we will show that $T(t)'\phi \leq e^{\lambda t}\,\phi$ and that the semigroup is positive (see Cor.3.9).

The following is our characterization.

Theorem 3.8. Let $(T(t))_{t\geq 0}$ be a semigroup on a σ-order complete real Banach lattice E. The semigroup is positive if and only if its generator A satisfies the following condition.
There exists a core D_o of A and a strictly positive set M' of subeigenvectors of A' such that

(K) $\quad \langle (\operatorname{sign} f)\,Af,\phi\rangle \leq \langle |f|,A'\phi\rangle \quad$ for all $f \in D_o$, $\phi \in M'$.