For the important relation of semigroups to abstract Cauchy problems we refer to A-II,Section 1. Here we only point out that the above theorem implies that a semigroup is uniquely determined by its generator.

While the generator is bounded only for uniformly continuous semigroups (see 2.1 below), it always enjoys a weaker but useful property.

Definition 1.8. An operator B with domain $D(B)$ on a Banach space E is called closed if $D(B)$ endowed with the graph norm

$$\|f\|_B := \|f\| + \|Bf\|$$

becomes a Banach space. Equivalently, $(B,D(B))$ is closed if and only if its graph $\{(f,Bf) : f \in D(B)\}$ is closed in $E \times E$, i.e.

(1.5) $f_n \in D(B)$, $f_n \to f$ and $Bf_n \to g$ implies $f \in D(B)$ and $Bf = g$.

It is clear from this definition that the 'closedness' of an operator B depends very much on the size of the domain $D(B)$. For example, a bounded and densely defined operator $(B,D(B))$ is closed if and only if $D(B) = E$.

On the other hand it may happen that $(B,D(B))$ is not closed but has a closed extension $(C,D(C))$, i.e. $D(B) \subset D(C)$ and $Bf = Cf$ for every $f \in D(B)$. In that case, B is called closable, a property which is equivalent to the following:

(1.6) $f_n \in D(B)$, $f_n \to 0$ and $Bf_n \to g$ implies $g = 0$.

The smallest closed extension of $(B,D(B))$ will be called the closure $\bar{B}$ with domain $D(\bar{B})$. In other words, the graph of $\bar{B}$ is the closure of $\{(f,Bf) : f \in D(B)\}$ in $E \times E$.

Finally we call a subset D_o of $D(B)$ a core for B if D_o is $\|.\|_B$-dense in $D(B)$. This means that a closed operator is determined (via closure) by its restriction to a core in its domain.

We now collect the fundamental topological properties of semigroup generators, their domains (see also A-II,Cor.1.34) and their resolvents.

Proposition 1.9. For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ the following holds:

(i) The generator A is a closed operator.

(ii) If a subspace D_o of the domain $D(A)$ is dense in E and $(T(t))$-invariant, then it is a core for A.