The eigenvalues of M_q can be characterized as follows: $\lambda \in P\sigma(M_q)$ if and only if the set $\{x \in X : q(x) = \lambda\}$ has non empty interior (analogously for $P\sigma(T(t))$). For example, it follows that $P\sigma(M_q) = \emptyset$ for $E = C_o(\mathbb{R}_+)$ and $q(x) = -x$, $x \in \mathbb{R}_+$.

On $E = L^p(X,\Sigma,\mu)$ analogous results are valid, but their exact formulation - using the notion 'essential range', see Goldstein (1985a) - is left to the reader.

2.4 The Spectrum of Translation Semigroups.
First we consider the translation semigroup

$$T(t)f(x) := f(x+t)$$

on $E = C_o(\mathbb{R}_+)$ (or $L^p(\mathbb{R}_+)$, see A-I,2.4). Its generator A is the first derivative and for every $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda < 0$, the function $\varepsilon_\lambda : x \to e^{\lambda x}$ belongs to $D(A)$ and satisfies

$$A\varepsilon_\lambda = \lambda\varepsilon_\lambda ,$$

hence $\lambda \in P\sigma(A)$. Since $\mathcal{T} = (T(t))_{t\geq 0}$ is a contraction semigroup it follows that $\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\}$ and $i\mathbb{R} \subset A\sigma(A)$ (use Prop. 2.2.(i) or show directly that $f_n(x) = e^{i\alpha x}\cdot e^{-x/n}$ defines an approximate eigenvector for $i\alpha$, $\alpha \in \mathbb{R}$). Using the same functions one obtains

$$P\sigma(T(t)) = \{ e^{\lambda t} : \mathrm{Re}\lambda < 0 \} = \{ z \in \mathbb{C} : |z| < 1 \}$$

and $\sigma(T(t)) = \{ z \in \mathbb{C} : |z| \leq 1 \}$ for every $t > 0$.
In the case of the translation group on $E = C_o(\mathbb{R})$ one has that $\sigma(A) \subset i\mathbb{R}$. As above one obtains approximate eigenvectors for every $\alpha \in \mathbb{R}$ from $f_n(x) = e^{i\alpha x}.e^{-|x|/n}$, hence

$$\sigma(A) = A\sigma(A) = i\mathbb{R}.$$

The generator A of the nilpotent translation semigroup A-I,2.6 has empty spectrum by A-I,Prop.1.11 . The resolvent is given by $R(\lambda,A)f(x) = e^{\lambda x}\int_x^\tau e^{-\lambda s}f(s)\,ds$ $(f \in L^p([0,\tau]$, $\lambda \in \mathbb{C})$.
Finally the generator of the periodic translation group from A-I,2.5 on $E = \{ f \in C[0,1] : f(0) = f(1) \}$ has point spectrum

$$P\sigma(A) = 2\pi i\mathbb{Z}$$

with eigenfunctions $\varepsilon_n(x) := \exp(2\pi inx)$. In Section 5 we show that

$$\sigma(A) = 2\pi i\mathbb{Z} .$$

We now return to the general theory and recall from Corollary 1.2 that it is very useful (e.g., for stability theory) to be able to convert