Proof. Since p is a strict half-norm we can assume that $\|f\| = \|f\|_p$ for all $f \in E$. (i) implies (ii) by Theorem 2.6.
Now suppose that (ii) holds. Then it follows from Lemma 2.10 that $\mu \in \rho(A)$ and $\|\mu R(\mu,A)\| \leq 1$ whenever $\mu > 0$ such that $\mathrm{im}(\mu - A) = E$. So by hypothesis $\lambda \in \rho(A)$ and $\mathrm{dist}(\lambda,\sigma(A)) \geq \|R(\lambda,A)\|^{-1} \geq \lambda$. Hence $(0,2\lambda) \subset \rho(A)$. Iterating this argument we see that $(0,\infty) \subset \rho(A)$. It follows from the Hille-Yosida theorem that A generates a contraction semigroup $(T(t))_{t\geq 0}$. Finally, from Thm. 2.6 it follows that $(T(t))_{t\geq 0}$ is p-contractive.

□

Of course, the norm function N given by $N(f) = \|f\|$ is a strict half-norm. In the case when $p = N$, Theorem 2.11 is due to Lumer and Phillips (1961). It turns out to be extremely useful in showing that a concrete operator is a generator. Because of its importance we state this special case explicitly below (including the complex case). Before that let us formulate Theorem 2.11 for the case when the operator is merely given on a core.

Corollary 2.12. Let p be a strict half-norm and A be a densely defined operator. If A is p-dissipative and $(\lambda - A)$ has dense range for some $\lambda > 0$, then A is closable and the closure $\bar{A}$ of A generates a p-contraction semigroup.

Proof. It follows from Prop. 2.9 that A is closable and the closure $\bar{A}$ is p-dissipative. Lemma 2.10 implies that $(\lambda - \bar{A})D(\bar{A}) = E$. So Thm. 2.11 yields the desired conclusion.

□

We conclude this section indicating the results for the complex case.

Let E be a complex Banach space and $p : E \to \mathbb{R}_+$ be a seminorm on E (i.e., $p(f+g) \leq p(f) + p(g)$ and $p(\lambda f) = |\lambda| p(f)$ holds for all $f, g \in E$, $\lambda \in \mathbb{C}$). The subdifferential $dp(f)$ of p in $f \in E$ is defined by

$$(2.15)\quad dp(f) = \{\phi \in E' : \mathrm{Re}\langle g,\phi\rangle \leq p(g) \text{ for all } g \in E \text{ and } \langle f,\phi\rangle = p(f)\}.$$

We assume in addition that p is continuous. Then it follows from the Hahn-Banach theorem that $dp(f) \neq \emptyset$ for any $f \in E$.