dix of Schaefer (1966)) to the expansion given in (2.11) one can conclude that $R$ has an extension to the halfplane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. This shows that without loss of generality one can assume that the domain of a positive pseudo-resolvent contains the halfplane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Proposition 2.7. Suppose $R : \Delta \to L(E)$ is a positive pseudo-resolvent on a Banach lattice $E$ and $\Delta := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.
If for some $\beta \in \mathbb{R}$, $h \in E$ we have

$\lambda R(\lambda+i\beta)h = h$ and $\lambda R(\lambda)|h| = |h|$ $(\lambda \in \Delta)$, then

$\lambda R(\lambda+in\beta)h^{[n]} = h^{[n]}$ for all $n \in \mathbb{Z}$, $\lambda \in \Delta$.

Proof. At first we prove the following domination property which is the extension of (1.5) to pseudo-resolvents.

$$(2.13)\quad |R(\lambda)f| \leq R(\operatorname{Re}\lambda)|f| \quad \text{for every } \lambda \in \Delta,\ f \in E.$$

To do this we fix $\lambda \in \Delta$. Then there exists $r_o > 0$ such that $|r-\lambda| < r$ whenever $r > r_o$. Therefore $R(\lambda) = \sum_{n=0}^{\infty}(r-\lambda)^n R(r)^{n+1}$ for $r > r_o$, which implies for $f \in E$
$|R(\lambda)f| \leq \sum_{n=0}^{\infty}|r-\lambda|^n R(r)^{n+1}|f| = \sum_{n=0}^{\infty}(r - (r-|r-\lambda|))^n R(r)^{n+1}|f|$
$= R(r - |\lambda-r|)|f|$. Since $\lim_{r\to\infty}(r - |\lambda-r|) = \operatorname{Re}\lambda$, we obtain (2.13). As a consequence of (2.13) and the assumption $rR(r)|h| = |h|$ we have that the principal ideal $E_{|h|}$ is $\{R(\lambda)\}_{\lambda\in\Delta}$-invariant. Identifying, according to the Kakutani-Krein Theorem $E_{|h|}$ with a space $C(K)$, $K$ compact, and by restricting the operators $R(\lambda)$ to $E_{|h|} \cong C(K)$ we obtain a positive pseudo-resolvent $\tilde{R} : \Delta \to L(C(K))$. Then we have for every $\alpha > 0$ and $f \in E$:
$\alpha\tilde{R}(\alpha+i\beta)h = h$, $\alpha\tilde{R}(\alpha)|h| = |h| = 1_K$, $\alpha|\tilde{R}(\alpha+i\beta)f| \leq \alpha\tilde{R}(\alpha)|f|$.
Applying B-III, Lemma 2.3 we obtain $\tilde{R}(\alpha) = S_{\tilde{h}}^{-1}\tilde{R}(\alpha + i\beta)S_{\tilde{h}}$ for every $\alpha > 0$ and using the uniqueness theorem for holomorphic functions we get $\tilde{R}(z) = S_{\tilde{h}}^{-1}\tilde{R}(z + i\beta)S_{\tilde{h}}$ for every $z \in \Delta$. Iterating this identity we obtain:

$$(2.14)\quad \tilde{R}(z) = S_{\tilde{h}}^{-n}\tilde{R}(z + in\beta)S_{\tilde{h}}^{n} \quad \text{for all } z \in \Delta,\ n \in \mathbb{Z}$$

In particular, $S_{\tilde{h}}^{n}|h| = S_{\tilde{h}}^{-n}z\tilde{R}(z)|h| = z\tilde{R}(z+in\beta)S_{\tilde{h}}^{n}|h|$.
In terms of the initial space this means precisely
$h^{[n]} = zR(z+in\beta)h^{[n]}$, and the propositon is proved.

□

We will prove cyclicity of the boundary spectrum under a growth condition which is stated in the following definition.