A little bit more calculation is necessary to check that in case of (d) the functional Ψ_k defined by

$$\Psi_k(f) := \int_T^{T+\tau} \exp(-i\cdot\tfrac{2\pi k}{\tau}\cdot t)\cdot h_t(x_o)\cdot f(\phi(t,x_o))\, dt \quad (k \in \mathbb{Z}\ ,\ f \in C(K))$$

is an eigenvector of A' corresponding to $h^\wedge(x_o) + i\cdot\frac{2\pi k}{\tau}$.

(e) Given $\beta \in \mathbb{R}$ we will show that $h^\wedge + i\beta \in A\sigma(A') \subseteq \sigma(A)$. For $n,m \in \mathbb{N}$ we define a linear functional Ψ_{nm} as follows:

$$\Psi_{nm}(f) := \tfrac{1}{n}\cdot\int_0^n \exp(-(h^\wedge+i\beta)t)\cdot h_t(\phi(m,x_o))\cdot f(\phi(m+t,x_o))\, dt\ ,\ f\in C(K)\ .$$

For $f \in D(A)$ we have

$$<(h^\wedge+i\beta-A)f,\Psi_{nm}> =$$
$$= \tfrac{1}{n}\cdot\big(f(\phi(m,x_o)) - \exp(-i\beta n)\exp(\int_m^{m+n} (h(\phi(s,x_o))-h^\wedge)ds)f(\phi(m+n,x_o))\big).$$

It follows that $\phi_{nm} \in D(A')$ and, since $\lim_{t\to\infty} h(\phi,x_o)) = h^\wedge$,

(4.13) $\limsup_{m\to\infty} \|(h^\wedge+i\beta-A')\Psi_{nm}\| \leq 1/n$ for every $n \in \mathbb{N}$.

Because the orbit is infinite we have

$$\|\Psi_{nm}\| = \tfrac{1}{n}\cdot\int_0^n |e^{-(h^\wedge+i\beta)t}\ h_t(\phi(m,x_o))|dt =$$
$$= \tfrac{1}{n}\cdot\int_0^n \exp(\int_m^{m+t} (h(\phi(s,x_o))-h^\wedge)ds$$

which shows that

(4.14) $\lim_{m\to\infty} \|\Psi_{nm}\| = 1$ for every $n \in \mathbb{N}$.

In view of (4.13) and (4.14) it is not difficult to find a subsequence $k(n)$ of $\mathbb{N}$ such that $(\Psi_{n,k(n)})$ is an approximate eigenvector of A' corresponding to $h^\wedge + i\beta$.

□

We are now going to apply the results obtained so far to the special case where $h = 0$, i.e., we consider semigroups of lattice homomorphisms which are Markov operators.

Theorem 4.9. Suppose $\mathcal{T}$ is a semigroup of Markov lattice homomorphisms on $C(K)$ governed by the semiflow ϕ .

(a) If $\phi|K_\infty$ is not injective or if $K_t \neq K_\infty$ for every $t < \infty$, then $\sigma(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$.

(b) If $K_\infty = K_s$ for some s and $\phi|K_\infty$ is injective, then $\sigma(A)$ is a cyclic closed subset of $i\mathbb{R}$. Moreover, we have $\sigma(A) \neq i\mathbb{R}$ if and only if there is a $T < \infty$ such that every orbit of ϕ has length less than T (i.e., $\phi(\mathbb{R}_+,x) = \phi([0,T],x)$ for every $x \in K$).