(2.7a) $P_t(x,.)$ is a probability measure for $x \in X$, $t > 0$;

(2.7b) $P_t(.,C)$ is a measurable function for $C \in \Sigma$, $t > 0$;

(2.7c) $P_{t+s}(x,C) = \int_K P_s(y,C)P_t(x,dy)$ for all $s,t>0$, $x \in K$, $C \in \Sigma$.

We assume that (P_t) possesses an invariant probability measure μ , i.e. we assume

(2.7d) $\mu(C) = \int P_t(x,C)\, d\mu(x)$ for every $C \in \Sigma$.

Finally we assume that the following continuity condition holds true.

(2.7e) For every $C \in \Sigma$ one has $\lim_{t\to 0} P_t(x,C) = 1_C(x)$ μ-a.e. .

Given $h \in L^1(\mu)$ we define a measure P_th on Σ by $P_th(C) := \int P_t(x,C)h(x)\, d\mu(x)$. In case $\mu(C) = 0$ then by (2.7d) $P_t(x,C) = 0$ μ-a.e. on X hence $P_th(C) = 0$. That is, P_th is absolutely continuous with respect to μ . By the Radon-Nikodym theorem P_th has an integrable density with respect to μ . We define $T(t)h$ to be this density (which is unique as an element of $L^1(\mu)$) . Thus for $h \in L^1(\mu)$, $C \in \Sigma$ we have

(2.8) $\int_C (T(t)h)(x)\, d\mu(x) = \int P_t(x,C)h(x)\, d\mu(x)$ for all $C \in \Sigma$.

It is not difficult to see that $T(t)$ is a positive linear contraction on $L^1(\mu)$. We have $T(t)'1_X = 1_X$ and $T(t)1_X = 1_X$ for all $t \geq 0$ and $T(t)T(s) = T(t+s)$ for t , $s \geq 0$. This follows from (2.7a) , (2.7d) and (2.7c) respectively. Moreover (2.7e) implies strong continuity of the semigroup $(T(t))_{t\geq 0}$. In fact by Prop.1.23 of Davies (1980) we only have to show weak continuity at $t = 0$. Since the characteristic functions are total in $L^\infty(\mu)$ this is true provided that $\lim_{t\to 0} \langle T(t)h,1_C\rangle = \langle h,1_C\rangle$ for $h \in L^1(\mu)$, $C \in \Sigma$. Given $h \in L^1(\mu)$, then by (2.7e) $\lim_{t\to 0} P_t(x,C)h(x) = 1_C(x)h(x)$ μ-a.e.. By Lebesgue's Theorem $\langle T(t)h,1_C\rangle = \int P_t(x,C)h(x)\, d\mu(x)$ tends to $\int 1_C(x)h(x)\, d\mu(x) = \langle h,1_C\rangle$ as $t \to 0$ and we have weak hence strong continuity.

Therefore a Markov transition function satisfying all the assumptions of (2.7) induces a strongly continuous semigroup on $L^1(\mu)$, and by interpolation on $L^p(\mu)$, which satisfies the hypotheses of Thm.2.6.

In the following corollaries of Thm.2.6 we give criteria which ensure convergence on the whole space. In view of Cor.2.7 it is enough to show $X_2 = \emptyset$.

<u>Corollary</u> 2.9. Let $(T(t))_{t\geq 0}$ be a positive semigroup of contractions on the Banach lattice $L^1(\mu)$ and assume that there exists a strictly positive eigenfunction $e \in \ker A$.