Proof. $D(A)$ is dense and one can show in an analoguous manner as in b) that A is dispersive. We know from d) that $C_c^\infty(\Omega) \subset (Id - A)D(A)$. Thus $(Id - A)D(A)$ is dense in E and the claim follows from Cor.1.3.

□

We now turn to the problem to characterize generators of arbitrary (not necessarily contractive) positive semigroups. Of course, as in B-II, Sec.1 one sees that a semigroup $(T(t))_{t\geq 0}$ is positive if and only if $R(\lambda,A) \geq 0$ for all $\lambda > \omega(A)$ where A denotes the generator of $(T(t))_{t\geq 0}$. We are looking for an intrinsic condition on A.

The positive minimum principle which is characteristic for generators of strongly continuous semigroups on $C(K)$ (see B-II,Thm.1.6) can be reformulated on an arbitrary Banach lattice E.

Definition 1.6. An operator A on E satisfies the positive minimum principle if for all $f \in D(A)_+$, $\phi \in E'_+$,

(P) $\langle f,\phi\rangle = 0$ implies $\langle Af,\phi\rangle \geq 0$.

Remark. It is easy to see that this definition coincides with that given in B-II,Sec.1 in the case when $E = C(K)$ (K compact). [In fact, suppose that for all $f \in D(A)_+$ and $x \in K$, $f(x) = 0$ implies $(Af)(x) \geq 0$. Let $g \in D(A)_+$, $\mu \in M(K)_+$ such that $\langle g,\mu\rangle = 0$. Then $g(x) = 0$ for all $x \in \operatorname{supp}\mu$. Thus by hypothesis, $(Ag)(x) \geq 0$ for all $x \in \operatorname{supp}\mu$. Consequently $\langle Ag,\mu\rangle \geq 0$. This proves one direction. The other is obvious by considering point measures.]

Proposition 1.7. The generator of a strongly continuous positive semigroup satisfies the positive minimum principle (P).

Proof. Let $(T(t))_{t\geq 0}$ be a strongly continuous positive semigroup with generator A and $0 \leq f \in D(A)$, $\phi \in E'_+$ such that $\langle f,\phi\rangle = 0$. Then $\langle Af,\phi\rangle = \lim_{t\to 0} 1/t \langle T(t)f - f, \phi\rangle = \lim_{t\to 0} 1/t \langle T(t)f, \phi\rangle \geq 0$.

□

We will see that the positive minimum principle is not sufficient for the positivity of the semigroup, in general (Remark 3.16). However, the following special case is of interest.