2. A FUNDAMENTAL INEQUALITY FOR THE RESOLVENT

If $\mathcal{T} = (T(t))_{t\geqq 0}$ is a C_o-semigroup of Schwarz maps on a C*-algebra M (resp. a C_o*-semigroup on a W*-algebra M) with generator A, then $s(A) \leqq 0$. Thus the resolvent $R(\lambda,A)$ has for all $\lambda\varepsilon\mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ the representation

$$R(\lambda,A)x = \int_0^\infty e^{-\lambda t}\, T(t)xdt \qquad (x\varepsilon M)$$

where the integral exists in the norm topology. This observation is quite useful for our first result which will be fundamental for our approach.

<u>THEOREM</u> 2.1. Let $\mathcal{T} = (T(t))_{t\geqq 0}$ be a C_o-semigroup of Schwarz type and $\mathcal{S} = (S(t))_{t\geqq 0}$ a C_o-semigroup on a C*-algebra M with generators A and B, respectively. If

$$(S(t)x)(S(t)x)^* \leqq T(t)(xx^*) \qquad (*)$$

for all $x\varepsilon M$ and $t\varepsilon\mathbb{R}_+$, then

$$(\mu R(\mu,B)x)(\mu R(\mu,B)x)^* \leqq \mu R(\mu,A)xx^*$$

for all $x\varepsilon M$ and $\mu\varepsilon\mathbb{R}_+$. The same result holds if $\mathcal{T}$ is a C_o*-semigroup of Schwarz type and $\mathcal{S}$ is a C_o*-semigroup on a W*-algebra M such that (*) is fulfilled.

Proof. From the assumption (*) it follows

$$0 \leqq (S(r)x - S(t)x)(S(r)x - S(t)x)^* =$$

$$= (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* -$$

$$- (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \leqq$$

$$\leqq T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* -$$

$$- (S(t)x)(S(r)x)^*$$

for every $r,t \in \mathbb{R}_+$. Hence

$$(S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^* \leqq T(r)xx^* + T(t)xx^*$$

Obviously, $||S(t)|| \leqq 1$ for all $t \in \mathbb{R}_+$. Then for all $0 \leqq \leqq \mu \in \mathbb{R}_+$ and $x \in M$::

$$(R(\mu,B)x)(R(\mu,B)x)^* = (\int_0^\infty e^{-\mu r} S(r)x dr)(\int_0^\infty e^{-\mu t} S(t)x dt)^* =$$

$$= \frac{1}{2}(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}((S(r)x)(S(t)x)^* +$$

$$+ (S(t)x)(S(r)x)^*) drdt \leqq$$

$$\leqq \frac{1}{2}(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*) drdt =$$

$$= (\int_0^\infty e^{-\mu s} ds)(\int_0^\infty e^{-\mu t} T(t)xx^* dt) = \mu^{-1} R(\mu,A)(xx^*) .$$

where the handling of the integral is justified by [5,§8, n° 4, Proposition 9.]. □

<u>COROLLARY</u> 2.2. Let $\mathcal{T}$ be a C_o-semigroup of Schwarz maps (res. C_o*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$:

$$(R(\lambda,A)x)(R(\lambda,A)x)^* \leqq (\mathrm{Re}\lambda)^{-1} R(\mathrm{Re}\lambda,A)xx^* .$$

In particular for all $(\mu,\alpha) \in \mathbb{R}_+ \times \mathbb{R}$:

$$(\mu R(\mu+i\alpha,A)x)(\mu R(\mu+i\alpha,A)x)^* \leqq \mu R(\mu,A)(xx^*)$$

<u>Proof</u>. Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Then the semigroup $\mathcal{S} :=$ $:= (e^{-i\mathrm{Im}(\lambda)t}T(t))_{t \geqq 0}$ fulfils the assumption of Theorem 2.2. and $B := A - i\mathrm{Im}(\lambda)I$ is the generator of $\mathcal{S}$. Consequently $R(\lambda,A) = R(\mathrm{Re}\lambda,B)$ and the corollary is proved. □

We now recall some spectral theoretic notions. If $\mathcal{T}$ is a C_o-semigroup on a Banach space E with generator A then $e^{t\sigma(A)} \setminus \{0\} \subseteq \sigma(T(t))$ for all $t \in \mathbb{R}_+$ ([8, Theorem 2.16]) and the inclusion can be proper ([8, Theorem 2.17, Example 2.18]). On the other hand, if $\lambda \in \rho(A)$, then

$$\sigma(R(\lambda,A)) \cup \{0\} = \{(\lambda - \mu)^{-1} : \mu \in \sigma(A)\} \cup \{0\}.$$

This "spectral mapping theorem" remains valid also for the <u>point spectrum</u> $P\sigma(A)$, <u>approximative point spectrum</u> $A\sigma(A)$

and the <u>residual spectrum</u> $R\sigma(A)$ of A and $R(\lambda,A)$ respectively ([12, Proposition 1.1]). In addition, other properties of $\mathfrak{I}$ such as ergodicity can be expressed with the help of the resolvent (see, e.g. [22, XVIII; 42, VIII. 4.]). These observations, the first resolvent inequality for $R(.,T)$ and Corollary 2.2. motivate the introduction of a more general concept (see also [12, 34]):

<u>Definition</u> 2.3. Let E be a Banach space and $\emptyset \neq D$ an open subset of $\mathbb{C}$. A family $R: D \mapsto L(E)$ is called a <u>pseudoresolvent on</u> D <u>with values in</u> E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu)$$

for all λ and μ in D . If R is a pseudoresolvent on $D = \{\lambda\varepsilon C : Re(\lambda) > 0\}$ with values in a C*- or W*-algebra, then R is called of <u>Schwarz type</u> , if

$$(R(\lambda,A)x)(R(\lambda,A)x)^* \leq (Re\lambda)^{-1} R(Re\lambda)xx^*$$

for all $\lambda\varepsilon D$ and $x\varepsilon M$. R is called <u>identity preserving</u> , if $\lambda R(\lambda)1 = 1$ for all $\lambda\varepsilon D$.

For properties of a pseudoresolvent R with values in a Banach space we refer to [12, Proposition 1.2; 34; 42, VIII.4.]. We collect some facts which we will use without further reference in the rest of this paper:

(a) If $\alpha \varepsilon \mathbb{C}$ and $x \varepsilon E$ such that $(\alpha - \lambda) R(\lambda) x = x$ for some $\lambda \varepsilon D$, then $(\alpha - \mu) R(\mu) x = x$ for all $\mu \varepsilon D$ (use the "resolvent equation").

(b) If F is a closed subspace of E such that $R(\lambda) F \subseteq F$ for some $\lambda \varepsilon D$, then $R(\mu) F \subseteq F$ for all μ in a neighbourhood of λ. This follows from the fact that for all $\mu \varepsilon D$ such that $|\lambda - \mu| \leqq ||R(\lambda)||^{-1}$, the pseudoresolvent in μ is given by $R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}$.

For conditions which guarantee that a pseudoresolvent is the resolvent of a densely defined operator see [34, 42].

<u>DEFINITION</u> 2.4. We call a C_o-semigroup $\mathcal{T}$ on the predual M_* of a W*-algebra M <u>identity preserving and of Schwarz type</u> , if its adjoint C_o*-semigroup has these properties. Likewise, a pseudoresolvent R on $D = \{\lambda \varepsilon \mathbb{C} : Re(\lambda) > 0\}$ with values in M_* is called <u>identity preserving and of Schwarz type</u>, if R' has these properties.

Since for a C_o-semigroup of contractions on a Banach space

$$\mathrm{Fix}(\mathcal{T}) = \bigcup_{t \geqq o} \ker(I - T(t)) =$$

$$= \ker(I - \lambda R(\lambda, A)) = \mathrm{Fix}(\lambda R(\lambda, A))$$

for all $\lambda \varepsilon \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, a C_o-semigroup of contractions on M is identity preserving and of Schwarz type iff the pseudoresolvent on $D = \{\lambda \varepsilon \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ given by $R(\lambda) := R(\lambda, A)_{|D}$ is identity preserving and of Schwarz type.

<u>INDUCTION AND REDUCTION</u> 2.5. (a) If E is a Banach space E and $\mathcal{S} \subseteq L(E)$ a semigroup of bounded operators, then a closed subspace F is called $\mathcal{S}$-invariant, if $SF \subseteq F$ for all $S \varepsilon \mathcal{S}$. We call the semigroup $\mathcal{S}_| := \{S_{|F} : S \varepsilon \mathcal{S}\}$ the <u>reduced</u> semigroup. Note that for a C_o-semigroup $\mathcal{T}$ (resp. pseudoresolvent R) the reduced semigroup $\mathcal{T}_|$ is again strongly continuous (resp. $R_|$ is again a pseudoresolvent).

(b) Let M be a W*-algebra, $p \varepsilon M$ a projection and $S \varepsilon L(M)$ such that $S(p^\perp M) \subseteq p^\perp M$ and $S(Mp^\perp) \subseteq Mp^\perp$, where $p^\perp := 1-p$. Since for all $x \varepsilon M$:

$$p[S(x) - S(pxp)] = p[S(p^\perp xp) + S(xp^\perp)]p = 0 ,$$

we obtain $p(Sx)p = p(S(pxp)p$. Therefore the map

$$S_p := (x \mapsto p(Sx)p) : pMp \mapsto pMp$$

is well defined. We call S_p the <u>induced</u> map. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$. If

$\mathcal{T}$ is a C_o*-semigroup on M which is of Schwarz type and if $T(t)(p^\perp) \leqq p^\perp$ for all $t\varepsilon\mathbb{R}_+$, then $\mathcal{T}$ leaves $p^\perp M$ and $Mp^\perp$ invariant. It is easy to see that the induced semigroup $\mathcal{T}_p$ is again a C_o*-semigroup. If an identity preserving pseudoresolvent of Schwarz type on $D = \{\lambda\varepsilon\mathbb{C} :$ $: Re(\lambda) > 0\}$ with values in M such that $\mu R(\mu)p^\perp \leqq p^\perp$ for some $\mu\varepsilon\mathbb{R}_+$ then $p^\perp M$ and $Mp^\perp$ are R-invariant. Again, the induced pseudoresolvent R_p is of Schwarz type and identity preserving.

(c) Let ϕ be a normal linear functional on a W*-algebra M such that $T_*\phi = \phi$ for some identity preserving Schwarz map T on M with preadjoint $T_*\varepsilon L(M_*)$ .Then $T(s(\phi)^\perp) \leqq$ $\leqq s(\phi)^\perp$ where $s(\phi)$ is the support projection of ϕ . To see this let $L_\phi := \{x\varepsilon M : \phi(xx^*) = 0\}$ and $M_\phi := L_\phi \cap L_\phi{}^*$. Since ϕ is T_*-invariant, and T is a Schwarz map, the subspaces L_ϕ and M_ϕ are T-invariant. Because $M_\phi =$ $= s(\phi)^\perp M s(\phi)^\perp$ and $T(s(\phi)^\perp) \leqq 1$ it follows $T(s(\phi)^\perp) \leqq$ $\leqq s(\phi)^\perp$. Let $T_{s(\phi)}$ be the induced map on $M_{s(\phi)}$. If

$$s(\phi)M_*s(\phi) := \{\psi\varepsilon M_* : \psi = s(\phi)\psi s(\phi)\}$$

where $\langle s(\phi)\psi s(\phi),x\rangle := \langle\psi,s(\phi)xs(\phi)\rangle$ $(x\varepsilon M)$, and if $\psi\varepsilon s(\phi)M_*s(\phi)$, then for all $x\varepsilon M$:

$$(T_*\psi)(x) = \psi(Tx) = \langle\psi,s(\phi)(Tx)s(\phi)\rangle =$$

$$= \langle\psi,s(\phi)(T(s(\phi)xs(\phi)))s(\phi)\rangle = \langle T_*\psi,s(\phi)xs(\phi)\rangle ,$$

hence $T_*\psi\varepsilon s(\phi)M_*s(\phi)$. Since the dual of $s(\phi)M_*s(\phi)$ is $M_{s(\phi)}$, it follows that the adjoint of the reduced map $T_*|$ is identity preserving and of Schwarz type. For example, if $\mathcal{T}$ is an identity preserving C_o-semigroup of Schwarz type on M_* such that $\phi\varepsilon\mathrm{Fix}(\mathcal{T})$, then $\mathcal{T}|(s(\phi)M_*s(\phi))$ is again a C_o-semigroup which is identity preserving and of Schwarz type. Furthermore, if R is a pseudoresolvent on $D = \{\lambda\varepsilon\mathbb{C}: : \mathrm{Re}(\lambda) > 0\}$ with values in M_* which is identity preserving and of Schwarz type such that $\mu R(\mu)\phi = \phi$ for some $\mu\varepsilon\mathbb{R}_+$, then $R|s(\phi)M_*s(\phi)$ has the same properties.