Theorem 1.8. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E. Assume that
a) there exists $w \in \mathbb{R}$ such that $\|T(t)\| \leq e^{wt}$ for all $t \geq 0$;
b) there exists a core D_o of A such that $f \in D_o$ implies $|f| \in D_o$.
If the restriction of A to D_o satisfies the positive minimum principle, then the semigroup is positive.

Remark. Elementary examples show that neither a) nor b) hold for generators of positive semigroups, in general.

The proof of Theorem 1.8 is based on the following proposition.

Proposition 1.9. Let A be a densely defined dissipative operator which possesses a core D_o such that $f \in D_o$ implies $|f| \in D_o$.
If the restriction of A to D_o satisfies the positive minimum principle (P), then A is dispersive.

Proof. By A-II,Prop.2.9, it is enough to show that $A_o := A|D_o$ is dispersive.
Let $f \in D_o$ and $\phi \in dN^+(f)$. Then $\phi \in E'_+$, $\|\phi\| \leq 1$ and $\langle f,\phi\rangle = \|f^+\|$. Hence, $\langle f^-,\phi\rangle = \langle f^-,\phi\rangle + \langle f,\phi\rangle - \|f^+\| = \langle f^+,\phi\rangle - \|f^+\| \leq 0$. Thus $\langle f^-,\phi\rangle = 0$. Consequently, $\langle f^+,\phi\rangle = \langle f,\phi\rangle = \|f^+\|$; and so $\phi \in dN(f^+)$. Since A is dissipative it follows that $\langle Af^+,\phi\rangle \leq 0$. Moreover, since A satisfies (P) we have $\langle Af^-,\phi\rangle \geq 0$. So we finally obtain, $\langle Af,\phi\rangle = \langle Af^+\phi\rangle - \langle Af^-,\phi\rangle \leq 0$.

□

Proof of Theorem 1.8. The operator $A - w$ satisfies (P) as well. So it follows from Proposition 1.9 that $A - w$ is dispersive. Consequently, the semigroup $(e^{-wt}T(t))_{t\geq 0}$, which is generated by $A - w$, is positive. Thus $(T(t))_{t\geq 0}$ is positive as well.

□

Next we give a reformulation of the positive minimum principle. For $0 < u \in E_+$ we denote by E_u the principal ideal generated by u. If $g \in E_+$, then $g \in \overline{E_u}$ if and only if $\lim_{n\to\infty} \|u - nu\wedge g\| = 0$.

Lemma 1.10. An operator A on E satisfies (P) if and only if

$$(1.8) \qquad (Au)^- \in \overline{E_u} \quad \text{for all } u \in D(A)_+ := D(A) \cap E_+ .$$