Proof. One proves as in Theorem 4.2 that (i) implies (ii). It is trivial that (ii) implies (iii). Assume that (iii) holds. Let $\lambda > \lambda_o = \max\{s(A), s(B), 0\}$. In a similar way as (4.6) one shows that for all $f \in D_o$

$$\lambda f - Bf = g \quad \text{implies} \quad f^+ \leq R(\lambda,A)g^+ . \tag{4.10}$$

Since D_o is a core of B it follows that (4.10) also holds for all $f \in D(B)$. This implies that $(R(\lambda,B)g)^+ \leq R(\lambda,A)g^+$ for all $g \in E$, $\lambda > \lambda_o$. Consequently, $0 \leq R(\lambda,B) \leq R(\lambda,A)$ for all $\lambda > \lambda_o$. Hence (i) holds.

□

In the following example we apply Theorem 4.3 to Schrödinger operators. Here the range condition is proved by an elegant argument due to Kato (1986) with the help of Kato's classical inequality.

Example 4.7 (Schrödinger operators on L^p).
Let $E = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and $V \in L^p_{loc}(\mathbb{R}^n)$ such that $\mathrm{Re}V \geq 0$. Define B on E by $Bf = \Delta f - Vf$ with domain $D(B) = C_c^\infty(\mathbb{R}^n)$. Then B is closable and $\bar{B}$ is the generator of a semigroup $(S(t))_{t\geq 0}$ which is dominated by the diffusion semigroup (Example 1.5d and A-I, 2.8). If $V \geq 0$, then $(S(t))_{t\geq 0}$ is positive.

Proof. Denote by A the generator of the diffusion semigroup. Then $C_c^\infty := C_c^\infty(\mathbb{R}^n)$ is a core of A and $Af = \Delta f$ for $f \in C_c^\infty$ (see Example 1.5d). Let $0 \leq \phi \in D(A')$. Then
$\mathrm{Re}\langle(\mathrm{sign}\bar{f})Bf,\phi\rangle = \mathrm{Re}\langle(\mathrm{sign}\bar{f})Af,\phi\rangle - \langle(\mathrm{Re}V)|f|,\phi\rangle \leq \mathrm{Re}\langle(\mathrm{sign}\bar{f})Af,\phi\rangle \leq \langle|f|,A'\phi\rangle$ for all $f \in C_c^\infty$ by Theorem 2.4. Thus (4.4) holds.
We show that $(\lambda - B)$ has dense range for $\lambda > 0$. If not, then there exists $0 \neq \phi \in E' = L^q(\mathbb{R}^n)$ such that $\langle(\lambda - \Delta + V)f,\phi\rangle = 0$ for all $f \in C_c^\infty$; i.e., $(\lambda - \Delta + V)\phi = 0$ in the sense of distributions. By Kato's classical inequality (see Example 2.5) this implies that
$(\lambda - \Delta + \mathrm{Re}V)|\phi| \leq \lambda|\phi| - \mathrm{Re}[(\mathrm{sign}\bar{\phi})(\lambda\phi - \Delta\phi + V\phi)] = 0$ (here we use that $\Delta\phi = \lambda\phi + V\phi \in L^1_{loc}$). Hence $(\lambda - \Delta)|\phi| \leq -(\mathrm{Re}V)|\phi| \leq 0$. Since $(\lambda - \Delta)^{-1}$ is a positive linear mapping from $\mathcal{S}(\mathbb{R}^n)'$ onto $\mathcal{S}(\mathbb{R}^n)'$, this implies that $\phi = 0$. It follows from Thm. 4.3 that $\bar{B}$ is the generator of a semigroup $(S(t))_{t\geq 0}$ which is dominated by the semigroup generated by A.
If $V = \mathrm{Re}V \geq 0$, we may consider the real space $L^p(\mathbb{R}^n)$. Then for every $f \in C_c^\infty$, $0 \leq \phi \in D(A')$ we have