2. COMPACT AND QUASI-COMPACT SEMIGROUPS

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Using the Riesz-Schauder Theory for compact operators (see e.g. Chapter VII.4 of Dunford-Schwartz (1958) or Section 26 of Pietsch (1978)) and the results of Chapter A-III, one can easily describe the asymptotic behavior of eventually compact semigroups. Since no positivity is involved we state the fundamental result for arbitrary Banach spaces.

<u>Theorem</u> 2.1. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space G which is eventually compact (i.e., there is $t_o > 0$ such that $T(t_o)$ is a compact operator). Then the spectrum of the generator A is a countable set (possibly finite or empty) and contains only poles of finite algebraic multiplicity. Furthermore, the set $\{\mu \in \sigma(A) : \mathrm{Re}\,\mu \geq r\}$ is finite for every $r \in \mathbb{R}$. Thus $\sigma(A) = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ with $\mathrm{Re}\,\lambda_{n+1} \leq \mathrm{Re}\,\lambda_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathrm{Re}\,\lambda_n = -\infty$ if $\sigma(A)$ is infinite.
Denoting the pole order at λ_n by $k(n)$ and the corresponding residue by P_n, we have for every $m \in \mathbb{N}$

$$T(t) = T_1(t) + T_2(t) + \ldots + T_m(t) + R_m(t) \quad , \text{ where}$$

$$T_n(t) = \exp(\lambda_n t)\cdot\sum_{j=0}^{k(n)-1} \frac{1}{j!}\cdot t^j (A-\lambda_n)^j \circ P_n \quad (t\geq 0) \quad , \tag{2.1}$$

$\|R_m(t)\| \leq C\cdot\exp((\varepsilon+\mathrm{Re}\,\lambda_m)t)$ for $t\geq 0$, $\varepsilon>0$ and a suitable constant $C=C(\varepsilon,m)$.

<u>Proof</u>. Fix $r \in \mathbb{R}$. By the Riesz-Schauder Theory we know that $\{z \in \sigma(T(t_o)) : |z| \geq \exp(rt_o)\}$ is a finite set and contains only poles of finite algebraic multiplicity. Thus the first assertion follows from A-III,Cor.6.5 .
To prove the remaining assertion we fix $m \in \mathbb{N}$ and apply the spectral decomposition as described in Section 3 of Chapter A-III. For simplicity we assume $\mathrm{Re}\,\lambda_{m+1} < \mathrm{Re}\,\lambda_m$. Let P be the spectral projection of $T(t_o)$ corresponding to the spectral set $\{z \in \sigma(T(t_o)) : |z| \geq \exp(\mathrm{Re}\lambda_m\cdot t_o)\}$. Then P reduces the semigroup and we have $\sigma(T(t_o)_{|\ker P}) \subset \{z \in \mathbb{C} : |z| < \exp(\mathrm{Re}\,\lambda_m\cdot t_o)\}$. Hence the type of $(T(t_o)_{|\ker P})$ is less than $\mathrm{Re}\,\lambda_m$. Then there exists a constant C_o such that