Corollary 1.4. Let $(T(t))_{t\geq 0}$ be a positive semigroup on a (real or complex) Banach lattice with generator A. Each of the following conditions implies that the solutions of the abstract Cauchy problem are exponentially stable, i.e., there is $\delta > 0$ such that $\lim_{t\to\infty} e^{\delta t}T(t)f = 0$ for every $f \in D(A)$.

(a) $\lambda - A$ is invertible for every $\lambda \geq 0$;

(b) A is invertible and $A^{-1} \leq 0$.

Proof. In case of a real Banach lattice we consider the complexification (see Sec.7 of C-I). Note that both, the hypotheses and the satement remain preserved.
Since $s(A) \in \sigma(A)$ assertion (a) implies $s(A) < 0$. If (b) is satisfied then $R(0,A) \geq 0$, hence $s(A) < 0$ by C-III,Thm.1.1(b). It follows from Thm.1.3 that $\sup\{\omega(f) : f \in D(A)\} = \omega_1(A) < 0$.

□

In the following we give a spectral characterization of stability for eventually norm-continuous positive semigroups. An important tool in the proof is the following result on power bounded operators due to Katznelson-Tzafriri (1984):

(1.4) Let R be a linear operator on a Banach space such that $\sup_{n\in\mathbb{N}} \|R^n\| < \infty$. Then one has $\sigma(R)\cap\Gamma \subseteq \{1\}$ if and only if $\lim_{n\to\infty} \|R^n - R^{n+1}\| = 0$.

Theorem 1.5. Let $(T(t))_{t\geq 0}$ be a positive semigroup on a Banach lattice E which is bounded and eventually norm-continuous.
The following two assertions are equivalent:

(i) $(T(t))_{t\geq 0}$ is uniformly stable;
(ii) $0 \notin R\sigma(A)$ (i.e., $\ker A' = \{0\}$).

In case E is reflexive (i) and (ii) are equivalent to

(iii) $0 \notin P\sigma(A)$ (i.e., $\ker A = \{0\}$).

Proof. (i)→(ii) was proven in A-IV,Thm.1.12 in a more general setting.
(ii)→(i) In case $\omega(A) < 0$ one trivially has (i). Therefore we can assume $\omega(A) = 0$. By Cor.2.13 and Prop.2.9 of C-III we have $\sigma(A)\cap i\mathbb{R} = \{0\}$. Since the spectral mapping theorem holds (cf. Thm.6.6