Then $(T(t))_{t\geq 0}$ extends continuously to a strongly continuous semigroup $(T_{-1}(t))_{t\geq 0}$ on E_{-1} and the above diagram can be extended to the negative integers.

3.6. The $\mathcal{F}$-Product Semigroup

It is a very successful mathematical method to consider a sequence of points in a certain space as a point in a new and larger space. In particular such a method can serve to convert an approximate eigenvector of a linear operator into an eigenvector. Occasionally we will need such a construction and refer to Section V.1 of Schaefer (1974) for the details.

If we try to adapt this construction to strongly continuous semigroups we encounter the difficulty that the semigroup extended to the larger space will not remain strongly continuous. An idea already used in 3.4 will help to overcome this difficulty.

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space E. Denote by $m(E)$ the Banach space of all bounded E-valued sequences endowed with the norm

$$\|(f_n)_{n\in\mathbb{N}}\| := \sup\{\|f_n\| : n \in \mathbb{N}\} .$$

It is clear that every $T(t)$ extends canonically to a bounded linear operator

$$\hat{T}(t)(f_n) := (T(t)f_n)$$

on $m(E)$, but the semigroup $(\hat{T}(t))_{t\geq 0}$ is strongly continuous if and only if $\mathcal{T}$ has a bounded generator. Therefore we restrict our attention to the closed, $(\hat{T}(t))$-invariant subspace

$$m^{\mathcal{T}}(E) := \{(f_n) \in m(E) : \lim_{t\to 0} \|T(t)f_n - f_n\| = 0 \text{ uniformly for } n \in \mathbb{N}\}.$$

Then the restricted semigroup

$$\tilde{T}(t)(f_n) = (T(t)f_n) , \quad (f_n) \in m^{\mathcal{T}}(E),$$

is strongly continuous and we denote its generator by $(\tilde{A}, D(\tilde{A}))$. The following lemma shows that $\tilde{A}$ is obtained canonically from A.

Lemma. For the generator $\tilde{A}$ of $(\tilde{T}(t))_{t\geq 0}$ on $m^{\mathcal{T}}(E)$ one has:

(i) $D(\tilde{A}) = \{(f_n) \in m^{\mathcal{T}}(E) : f_n \in D(A) \text{ and } (Af_n) \in m^{\mathcal{T}}(E)\}$,

(ii) $\tilde{A}(f_n) = (Af_n)$ for $(f_n) \in D(\tilde{A})$.

For the proof we refer to Lemma 1.4. of Derndinger (1980).

Now let $\mathcal{F}$ be any filter on $\mathbb{N}$ finer than the Frechet filter (i.e. the filter of sets with finite complement). (In most cases $\mathcal{F}$ will be either the Frechet filter or some free ultra filter.) Then the sub-