(i) $(Q_k f)_{k\in\mathbb{Z}} \subset \ell^2(H)$ for every $f \in H$, and

(ii) if $\sup_{k\in\mathbb{Z}}\|R(ik,A)\| < \infty$, then $\sum_{k\in\mathbb{Z}} R(ik,A)f_k$ is summable whenever $(f_k)_{k\in\mathbb{Z}} \in \ell^2(H)$.

Proof. (i) We consider the Hilbert space $L^2([0,2\pi],H)$ and obtain

$$\begin{aligned}0 &\leq \|T(\cdot)f - \textstyle\sum_{k=-n}^{n} Q_k f\cdot e^{ik\cdot}\|^2\\ &= \int_0^{2\pi} \|T(s)f\|^2\, ds - \int_0^{2\pi} \textstyle\sum_{k=-n}^{n} (T(s)f|e^{iks}Q_k f)\, ds -\\ &\quad \int_0^{2\pi} \textstyle\sum_{k=-n}^{n} (e^{iks}Q_k f|T(s)f)ds + \int_0^{2\pi} (\sum_{k=-n}^{n} e^{iks}Q_k f|\sum_{\ell-n}^{n} e^{i\ell s}Q_\ell f)\, ds\\ &= \int_0^{2\pi} \|T(s)f\|^2\, ds - 2\pi\textstyle\sum_{k=-n}^{n} \|Q_k f\|^2, \quad \text{(use (7.5))}.\end{aligned}$$

It follows that $\sum_{k\in\mathbb{Z}} \|Q_k f\|^2 \leq \frac{1}{2\pi}\cdot\int_0^{2\pi} \|T(s)f\|^2\, ds < \infty$.

(ii) Fix $\lambda > 0$ sufficiently large and set

$$g_k := (1 + \lambda R(ik,A))f_k, \quad k \in \mathbb{Z}.$$

Using the resolvent equation and then (A-I,(3.1)) we obtain

$$R(ik,A)f_k = R(\lambda + ik,A)g_k = [1 - e^{-2\pi\lambda}T(2\pi)]^{-1}\int_0^{2\pi} e^{-\lambda s}e^{-iks}T(s)g_k\, ds.$$

This yields for every finite subset N of $\mathbb{Z}$

$$\begin{aligned}\|\textstyle\sum_{k\in N} R(ik,A)f_k\| &\leq \|(1 - e^{-2\pi\lambda}T(2\pi))^{-1}\|\cdot\int_0^{2\pi}\|T(s)\|\,\|\sum_{k\in N} e^{-iks}g_k\|\, ds \leq\\ &\leq \|(1 - e^{-2\pi\lambda}T(2\pi))^{-1}\|\cdot(\textstyle\int_0^{2\pi}\|T(s)\|^2\, ds)^{1/2}\cdot(\int_0^{2\pi}\|\sum_{k\in N} e^{-iks}g_k\|^2\, dx)^{1/2}\\ &= c(\textstyle\sum_{k\in N} \|g_k\|^2)^{1/2} \leq c(1 + \lambda M)(\sum_{k\in N} \|f_k\|^2)^{1/2}.\end{aligned}$$

Here $c := \|(1 - e^{-2\pi\lambda}T(2\pi))^{-1}\|\cdot(\int_0^{2\pi}\|T(s)\|^2\, ds)^{1/2}$ and $M := \sup_{k\in\mathbb{Z}}\|R(ik,A)\|$.

□

Theorem 7.10. Let A be the generator of a semigroup $(T(t))_{t\geq 0}$ on some Hilbert space H. Then the following form of the spectral mapping theorem is valid

$$\sigma(T(t))\setminus\{0\} = \{e^{\lambda t} : \text{either } \mu_k := \lambda + 2\pi ik/t \in \sigma(A) \text{ for some } k \in \mathbb{Z} \text{ or } (\|R(\mu_k,A)\|)_{k\in\mathbb{Z}} \text{ is unbounded}\}.$$

Proof. If $e^{\lambda t} \notin \sigma(T(t))$ it follows from the spectral inclusion theorem that $\mu_k \notin \sigma(A)$ for every $k \in \mathbb{Z}$ and from A-I,3.1, Formula (3.1), that $\|R(\mu_k,A)\|$ is bounded. For the converse inclusion it suffices to assume $t = 2\pi$ and $\lambda = 0$ (use the rescaling procedure A-I,3.1). Assuming that $i\mathbb{Z} \subset \rho(A)$ and $\|R(ik,A)\|$ is bounded then $\sum_{k\in\mathbb{Z}} R(ik,A)Q_k f$ is summable by Lemma 7.9. Since every summable series is Césaro-summable condition (c) of Prop. 7.8 is satisfied and we conclude $1 \in \rho(T(2\pi))$.

□