Extended Notes for B-II.

Section 4 Positive semigroups generated by elliptic operators on spaces of continuous functions

Important examples of semigroups on $C_0(\Omega)$ or $C(\bar{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open, bounded, are generated by elliptic differential operators. In the following we put together a series of results starting with the Laplacian subject to Dirichlet and to Robin boundary conditions and ending with the Dirichlet-to-Neumann operator on $C(\partial\Omega)$. Each time we obtain a positive irreducible semigroup. We consider $\mathbb{K} = \mathbb{R}$ throughout this section.

The Laplacian

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. We say that Ω is Dirichlet-regular if for every $g \in C(\partial\Omega)$ there exists a (unique) function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\Delta u = 0 \quad \text{and} \quad u_{|\partial\Omega} = g.$$

This means that the Dirichlet problem is well-posed.

This property is very well understood and precise characterizations in terms of barriers or of capacity are known.

If Ω has Lipschitz boundary, then Ω is Dirichlet regular.

In dimension $d=2$ it suffices that Ω is simply connected.

We refer to Arendt – Urban [AU24, Section 6.9] or Gilbarg – Trudinger [GT83, Section 2.8] for further information on the Dirichlet Problem.

The Dirichlet Laplacian Δ_0 on $C_0(\Omega)$ is defined by

$$\Delta_0 u := \Delta u$$

$$D(\Delta_0) := \{ u \in C_0(\Omega) : \Delta u \in C_0(\Omega) \}.$$

Here Δu is to be understood in the sense of distributions.

<u>Theorem</u> 4.1 TFAE

(a) Ω is Dirichlet regular;

(b) Δ_0 generates a positive semigroup T on $C_0(\Omega)$.

In that case the semigroup T is holomorphic of angle $\pi/2$. Moreover $T(t)$ is compact for all $t > 0$ and $\omega_0(\Delta_0) < 0$. If Ω is connected,

then the semigroup is irreducible.
Moreover, $\|T(t)\| \leq M e^{-\varepsilon t}$ $(t \geq 0$
for some $\varepsilon > 0$, $M \geq 1$.

This result is due to
Arendt - Bénilan [ArBe99] besides
irreducibility on which we comment later.
In Example C-II 1.5 e) the generation result was obtained
if Ω has C^2-boundary.

The implication (a) $\Rightarrow$ (b) of Theorem 4.1 is proved below in order to show how the Dirichlet problem comes into play and leads to a result with minimal regularity assumptions on the boundary of Ω.

We use the following abstract generation result which is of independent interest.
By C-II, Theorem 1.2 a densely defined operator A generates a contractive positive semigroup iff A is dispersive and $(\lambda - A)$ is surjective for some $\lambda > 0$.
We now describe the case $\lambda = 0$.

Theorem 4.2. Let A be a densely defined operator on a real or complex Banach lattice E. T.F.A.E.

(a) A generates a positive, contractive semigroup and $s(A) < 0$.

(b) A is dispersive and surjective.

In particular, (b) implies that A is closed.

Dispersive operators are defined before

C-II, Theorem 1.2. A densely defined operator A on $C_0(\Omega)$ is dispersive iff for $u \in D(A)$, $x_0 \in \Omega$

$$u(x_0) = \sup_{x \in \Omega} u(x) > 0 \text{ implies } (Au)(x_0) \leq 0.$$

Proof of Theorem 4.2. (b) $\Rightarrow$ (a)

Consider the equivalent norm $|||u||| := \|u^+\| + \|u^-\|$ on E. Since A is dispersive it is dissipative with respect to this new norm as is easy to see. Now Theorem 4.5 of Arendt, Chalendar and Moletsane [ACM 24] implies that A generates a contraction semigroup T and A is invertible. Since A is dispersive, it follows from C-II, Theorem 1.2 that T is positive and contractive (with respect to the original norm). Since $R(\lambda, A) \geq 0$ for $\lambda > 0$, it follows that $-A^{-1} \geq 0$. Now C-III, Theorem 1.1 (ii) implies that $s(A) < 0$.

(a) $\Rightarrow$ (b) is obvious from C-II, Theorem 1.2. $\square$

Proof of Theorem 4.1. (a) $\Rightarrow$ (b)

The operator Δ_0 is dispersive by the maximum principle. If Ω is Dirichlet regular, then Δ_0 is surjective.

In fact, let $f \in C_0(\Omega)$. Extend f by 0 to $\mathbb{R}^n$ and let $w = \Gamma * f$, where Γ is the fundamental solution of Laplace's equation, see Gilbarg and Trudinger [GT83, (2.12)]. Then $w \in C(\mathbb{R}^n)$ and $\Delta w = f$ in the sense of distributions. Let $g = w_{|\partial\Omega}$ and let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be the solution of the