Proposition 2.9. Let A be a p-dissipative operator where p is a half-norm. If $D(A)$ is dense, then A is closable (and the closure of A is p-dissipative as well (by Cor. 2.5)).

Proof. Let $f_n \in D(A)$, $\lim_{n\to\infty} f_n = 0$, $\lim_{n\to\infty} Af_n = g$. We have to show that $g = 0$. To this end let $h \in D(A)$. Then (2.7) gives $p(f_n + th) \leq p(f_n + th - tA(f_n + th))$ $(t > 0)$. Letting $n \to \infty$ we obtain $p(th) \leq p(th - tg - t^2 Ah)$ $(t > 0)$.
Hence $p(h) \leq p((h-g) - tAh)$ $(t > 0)$ by positive homogeneity. Letting $t \downarrow 0$ finally we obtain $p(h) \leq p(h - g)$ for all $h \in D(A)$. Since $D(A)$ is dense by hypothesis, we can approximate g by $h \in D(A)$ and conclude that $p(g) \leq p(0) = 0$. Since $\lim_{n\to\infty} A(-f_n) = -g$, we have $p(-g) \leq 0$ by symmetry. Hence $p(g) + p(-g) \leq 0$ which implies $g = 0$ by (2.11).

□

Lemma 2.10. Let p be a half-norm and A a p-dissipative operator. Then

(2.14) $$\lambda\|f\|_p \leq \|(\lambda - A)f\|_p \quad \text{for all } f \in D(A),\ \lambda > 0 .$$

In particular, $(\lambda-A)$ is injective for all $\lambda > 0$.
If p is strict and A is closed, then $\operatorname{im}(\lambda - A)$ is closed for all $\lambda > 0$.

Proof. Let $\lambda > 0$, $f \in D(A)$. Then by (2.7), $\lambda p(\pm f) \leq p((\lambda - A)(\pm f))$. Hence $\lambda\|f\|_p = \lambda p(f) + \lambda p(-f) \leq p((\lambda - A)f) + p(-(\lambda-A)f) = \|(\lambda-A)f\|_p$. Thus (2.14) is proved. Now suppose that p is strict. Then $\|\ \|_p$ is equivalent to the given norm. Let $\lambda > 0$ and $g \in (\operatorname{im}(\lambda - A))^-$. Then $g = \lim_{n\to\infty} (\lambda - A)f_n$ for some sequence $(f_n)_{n\in\mathbb{N}} \subset D(A)$. It follows from (2.14) that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $f = \lim_{n\to\infty} f_n$. Then $\lim_{n\to\infty} Af_n = \lambda\lim_{n\to\infty} f_n - \lim_{n\to\infty} (\lambda - A)f_n = \lambda f - g$ exists. If A is closed, this implies that $f \in D(A)$ and $Af = \lambda f - g$. Hence $g = (\lambda - A)f \in \operatorname{im}(\lambda - A)$. We have shown that $\operatorname{im}(\lambda - A)$ is closed.

□

The following is the main theorem of this section.

Theorem 2.11. Let p be a strict half-norm and A an operator on E. The following assertions are equivalent.

(i) A is the generator of a p-contraction semigroup.
(ii) $D(A)$ is dense, A is p-dissipative and $\operatorname{im}(\lambda - A) = E$ for some $\lambda > 0$.