PART B

POSITIVE SEMIGROUPS ON $C_o(X)$

CHAPTER B-I

B A S I C   R E S U L T S   O N   S P A C E S $C_o(X)$

by

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This part.of the book is devoted to a study of one-parameter semi-
groups of operators on spaces of continuous functions of type $C_o(X)$,
spaces which are Banach lattices of a very special kind. We treat this
case separately since we feel that an intermingling with the abstract
Banach lattice situation considered in Part C would have made it dif-
ficult to appreciate the easy accessibility and the pilot function of
methods and results available here. In this chapter we want to fix the
notation we are going to use and to collect some basic facts about the
spaces we are considering.

If X is a locally compact topological space, then $C_o(X)$ denotes
the space of all continuous complex-valued functions on X which
vanish at infinity, endowed with the supremum-norm. If X is compact,
then any continuous function on X "vanishes at infinity" and $C_o(X)$
is the space of all continuous functions on X . We often write $C(X)$
instead of $C_o(X)$ in this situation. We sometimes study real-valued
functions and write the corresponding real spaces as $C_o(X,\mathbb{R})$ and
$C(X,\mathbb{R})$, and the notations $C_o(X,\mathbb{C})$ and $C(X,\mathbb{C})$ are used if there is
the possibility of confusion between both cases.

1. ALGEBRAIC AND ORDER-STRUCTURE; IDEALS AND QUOTIENTS

Any space $C_o(X)$ is a commutative C*-algebra under the sup-norm and
the pointwise multiplication, and by the Gelfand Representation
Theorem any commutative C*-algebra can, on the other hand, be canoni-
cally represented as an algebra $C_o(X)$ on a suitable locally compact
space X . The algebraic notions of ideal, quotient, homomorphism are

well known and need not be explained further. Another natural and
important structure of $C_o(X)$ is the pointwise ordering, under which
$C_o(X,\mathbb{R})$ is a (real) Banach lattice and $C_o(X,\mathbb{C})$ a complex Banach
lattice in the sense explained in Chapter C-I. Concerning the order
structure of $C_o(X)$ we use the following notations: For a function f
in $C_o(X,\mathbb{R})$

> $f \geqq 0$ means $f(t) \geqq 0$ for all $t \in X$ (f is then called
>   positive),
> $f > 0$ means that   f is positive but does not vanish iden-
>   tically,
> $f \gg 0$ means that $f(t) > 0$ for all t in X (f is then
>   called strictly positive).

The notion of an order ideal explained in Chapter C-I applies of
course to the Banach lattices $C_o(X)$ and might give rise to confusion
with the corresponding algebraic notion. However, since we are mainly
considering closed ideals and since a closed linear subspace I of
$C_o(X)$ is a lattice ideal if and only if I is an algebraic ideal, we
may and will simply speak of closed ideals without specifying whether
we think of the algebraic or the order theoretic meaning of this word.
An important and frequently used characterization of these objects is
the following: A subspace I of $C_o(X)$ is a closed ideal if and only
if there exists a closed subset A of X such that a function f
belongs to I if and only if f vanishes on A . A is of course
uniquely determined by I and is called the support of I . If
$I = I_A$ is a closed ideal with support A then I_A is naturally
isomorphic to $C_o(X\backslash A)$ and the quotient $C_o(X)/_I$ is (under the
natural quotient structure) again a Banach algebra and a Banach
lattice that can be identified canonically (via the map $f + I \to f_{|A}$)
with $C_o(A)$.

2. LINEAR FORMS AND DUALITY

The Riesz Representation Theorem asserts that the dual of $C_o(X)$ can
be identified in a natural way with the space of bounded regular Borel
measures on X . While there is no natural algebra structure on this
dual, the dual ordering (see C-I) makes $C_o(X)'$ into a Banach lat-
tice. We will occasionally make use of the order structure of $C_o(X)'$
but since at least its measure theoretic interpretation is supposed to
be well-known, it may suffice here to refer to Chapter C-I, Sections 3

and 7, for a more detailed discussion and to recall some basic nota-
tions here: If μ is a linear form on $C_o(X,\mathbb{R})$ then

   $\mu \geqq 0$ means $\mu(f) \geqq 0$ for all $f \geqq 0$; μ is then called
       positive (positivity automatically implies continuity),

   $\mu > 0$ means that $\mu \geqq 0$, but μ does not vanish identically,

   $\mu \gg 0$ means that $\mu(f) > 0$ for any $f > 0$; μ is then called
       strictly positive.

If μ is a linear form on $C_o(X,\mathbb{C})$, then μ can be written uniquely
as $\mu = \mu_1 + i\mu_2$ where μ_1 and μ_2 map $C_o(X,\mathbb{R})$ into $\mathbb{R}$ (decompo-
sition into real and imaginary parts). We call μ positive (strictly
positive) and use the above notations if $\mu_2 = 0$ and μ_1 is positive
(strictly positive). We point out that strictly positive linear forms
need not exist on $C_o(X)$, but if X is separable then a strictly
positive linear form is obtained by summing a suitable sequence of
point measures.

The coincidence of the notions of a closed algebraic and a closed
lattice ideal in $C_o(X)$ has of course its effect on the algebraic and
the lattice theoretic notions of a homomorphism. The case of a homo-
morphism into another space $C_o(Y)$ will be discussed below. As for
homomorphisms into the scalar field, we have essentially coincidence
between the algebraic and the order theoretic meaning of this word,
more exactly: A linear form $\mu \neq 0$ on $C_o(X)$ is a lattice homomor-
phism if and only if μ is, up to normalization, an algebra homomor-
phism (algebra homomorphisms $\neq 0$ must necessarily have norm 1).
Since the algebra homomorphisms $\neq 0$ on $C_o(X)$ are known to be the
point measures (denoted by δ_t) on X and since on the other hand μ
is a lattice homomorphism if and only if $|\mu(f)|$ equals $\mu(|f|)$ for
all f , it follows that this latter condition on μ is equivalent to
$\mu = \alpha\delta_t$ for a suitable t in X and a positive real number α .
This can be summarized by saying that X can be canonically ident-
ified, via the map $t \rightarrow \delta_t$, with the subset of the dual $C_o(X)'$ con-
sisting of the non-zero algebra homomorphisms, which is also the set
of all normalized lattice homomorphisms. This identification is a
topological isomorphism with respect to the weak*-topology of
$C_o(X)'$.

3. <u>LINEAR OPERATORS</u>

A linear mapping T from $C_o(X,\mathbb{R})$ into $C_o(Y,\mathbb{R})$ is called

> <u>positive</u> (notation: $T \geq 0$), if Tf is a positive function
> whenever f is positive,
> a <u>lattice homomorphism</u> if $|Tf| = T|f|$ for all f ,
> a <u>Markov-operator</u> if X and Y are compact and T is a
> positive operator mapping 1_X to 1_Y .

In the case of complex scalars T can be decomposed into real and
imaginary parts. We call T positive in this situation if the imag-
inary part of T is = 0 and the real part is positive. The terms
"Markov operator" and "lattice homomorphism" are defined formally in
the same way as above. Note that a complex lattice homomorphism is
necessarily positive, and that the complexification of a real lattice
homomorphism is a complex lattice homomorphism. Positive Operators are
always continuous.

Since the adjoint of a Markov operator T maps positive normalized
measures into positive normalized measures while the adjoint of an
algebra homomorphism (lattice homomorphism) maps point measures into
(multiples of) point measures, the adjoint of a Markov lattice homo-
morphism as well as the adjoint of an algebra homomorphism induces a
continuous map ϕ from Y (viewed as a subset of the weak dual
$C(Y)'$) into X (viewed as a subset of $C(X)'$). This mapping ϕ
determines T in a natural and unique way, so that the following are
equivalent assertions on a linear mapping T from a space C(X) into
a space C(Y) :

> (a)   T is a Markov lattice homomorphism
> (b)   T is a Markov algebra homomorphism
> (c)   There exists a continuous map ϕ from Y into X such
> $Tf = f \circ \phi$ for all $f \in C(X)$.

If T is in addition bijective, then the mapping ϕ in (c) is a
homeomorhism from Y onto X . This characterization of homomorphisms
carries over mutatis mutandis to situations where the conditions on
X , Y or T are less restrictive. For later reference we explicitly
state:
(i) Let K be compact. Then $T \in L(C(K))$ is a lattice homomorphism
if and only if there is a mapping ϕ from K into K and a function

$h \in C(K)$ such that $Tf(s) = h(s)f(\phi(s))$ holds for all $s \in K$. ϕ is
continuous in every point t with $h(t) \neq 0$.

(ii)   Let X be locally compact, $T \in L(C_o(X))$. T is a lattice
isomorphism if and only if there is a homeomorhism ϕ from X onto
X and a   bounded   continuous   function h on X such that
$h(s) \geq \delta > 0$ for all s and $Tf(s) = h(s)f(\phi(s))$ ($s \in X$). T is
an algebraic *-isomorphism if and only if T is a lattice isomorphism
and the function h above is $\equiv 1$.