(2.5) $\lim_{t\to\infty} T(t)f = 0$ for every $f \in \mathrm{im}(Id - T(1))$.

That is, $\mathrm{im}(Id - T(1)) \subset F$. Since $\ker A = \bigcap_{t\geq 0} \ker(Id - T(t)) = \ker(Id - T(1))$ (cf. A-III,Cor.6.4) we have
$\mathrm{im}(Id - T(1)) + \ker(Id - T(1)) \subset F$.
Since power bounded operators on a reflexive Banach space are mean ergodic (e.g., see Krengel (1985), Chap.2,Thm.1.2) we obtain that $\mathrm{im}(Id - T(1)) + \ker(Id - T(1))$ is dense in E , hence $F = E$.

□

Strong convergence of the semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ implies strong convergence of the Césaro means $C(t)f := \frac{1}{t}\cdot\int_0^t T(s)f\,ds$, $f \in E$ which (by definition) is mean ergodicity of the semigroup $\mathcal{T}$ (see Davies (1980), Chap.5.1). On the other hand an inspection of the proof of Thm.2.5 shows that reflexivity of the underlying space can be replaced by the assumption that $\mathcal{T}$ is a mean ergodic semigroup.
This remark also shows where to look for examples of semigroups not converging as $t \to \infty$: Consider the positive contraction R defined by $(Rf)(x) := f(x+1)$ on $E = L^1(\mathbb{R}))$. Then $T(t) := e^{t(R-Id)}$ defines a positive norm-continuous semigroup on E . Since $\ker(R - Id) = \mathrm{Fix}\, R = \{0\}$ but $\|T(t)f\| = e^{-t}\sum_{n=0}^{\infty} \|R^n f\| t^n/n! = \|f\| > 0$ for every $0 < f \in E$ we see that $\lim_{t\to\infty} T(t)$ does not exist for the strong operator topology.
Finally we remark that in Thm.2.5 'eventual norm-continuity' is crucial as well. This can be seen by considering the translation (semi-) groups on $L^p(\mathbb{R})$.

In the next few results we study semigroups which are not necessarily eventually norm-continuous, but restrict our attention to positive semigroups on L^p-spaces ($1 \leq p < \infty$). The essential tool will be the following '0-2 law' which we quote from Greiner (1982), Thm.3.7.

If (X,Σ,μ) is a measure space and $(T(t))_{t\geq 0}$ is a positive semigroup on $L^p(\mu)$ then we call a subset $C \in \Sigma$ <u>$(T(t))$- invariant</u> if the principal ideal generated by the characteristic function 1_C is $(T(t))$-invariant in the usual sense.

<u>Theorem</u> 2.6. Let $(T(t))_{t\geq 0}$ be a positive contraction semigroup on $L^p(\mu)$, $1 \leq p < \infty$, and assume that there exists a strictly positive fixed function $e \in \ker A$. Then the following holds:

(a) For every $\tau > 0$ there exists a disjoint decomposition $X = X_o \cup X_2$ into $(T(t))$-invariant measurable subsets such that