We will now show that for quasi-compact semigroups one can give a description of the asymptotic behavior similar to the one stated for eventually compact semigroups in Thm.2.1 . One obtains a representation as in (2.1) with a remainder of exponential decay but the rate of the decay cannot be chosen arbitrarily large.

Theorem 2.10. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a quasi-compact semigroup on a Banach space G with generator A . Then $\{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda \geq 0\}$ is a finite set (possibly empty) and contains only poles of finite algebraic multiplicity. Denoting the eigenvalues with nonnegative real part $\lambda_1, \lambda_2, \ldots, \lambda_m$, the corresponding residues $P_1, P_2, \ldots, P_m$ and the orders of the poles $k(1), k(2), \ldots, k(m)$ we have

$$T(t) = T_1(t) + T_2(t) + \ldots + T_m(t) + R(t) \quad \text{where}$$

$$(2.5)\qquad T_n(t) = \exp(\lambda_n t)\cdot\sum_{j=0}^{k(n)-1} \frac{1}{j!}\cdot t^j (A - \lambda_n)^j \circ P_n \quad (t\geq 0) \quad \text{and}$$

$$\|R(t)\| \leq C\cdot e^{-\varepsilon t} \quad \text{for suitable constants } \varepsilon>0\ ,\ C\geq 1\ .$$

Proof. We have $\omega_{ess}(\mathcal{T}) < 0$ hence $r_{ess}(T(1)) < 1$ (see A-III,(3.10)). Therefore $\{z \in \sigma(T(1)) : |z| \geq 1\}$ is a finite set and contains only poles of finite algebraic multiplicity (cf. A-III,(3.8)). Let P denote the spectral projection of $T(1)$ corresponding to $\{z \in \sigma(T(1)) : |z| \geq 1\}$. Then A-III,Cor.6.5 implies that $\{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda \geq 0\}$ is a finite set, it contains only poles of $R(.,A)$ of finite algebraic multiplicity and $P = P_1+P_2+ \ldots +P_m$. One can now prove the representation of $T(t)$ stated in (2.5) in the same way as statement (2.1) .

□

In case we consider positive quasi-compact semigroups on $C_o(X)$ one can combine Thm.2.10 with the results of B-III . For example, B-III, Cor.2.11 assures that, in case there is at least one eigenvalue with nonnegative real part, the generator has a strictly dominant eigenvalue $r \in \mathbb{R}$. Thus in (2.5) the operators $T_j(t)$ belonging to $\lambda_j = r$ will determine the long term behavior of $(T(t))$. More precisely one has the following.

Corollary 2.11. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a positive semigroup on $C_o(X)$ which is quasi-compact and let A be its generator.

(a) Let r be an eigenvalue of A admitting a stricly positive eigenfunction and satisfying $\mathrm{Re}\, r \geq 0$. Then $r = \omega(\mathcal{T}) = s(A)$ and there is a positive projection P of finite rank such that for