(a') $(T(t))_{t\geq 0}$ is stable and bounded.

(b') $(T(t))_{t\geq 0}$ is uniformly stable.

(c') $(T(t))_{t\geq 0}$ is bounded and there is a dense subspace D such that $\int_0^\infty T(t)f\,dt$ exists for every $f \in D$.

Proof. If $(T(t))_{t\geq 0}$ is stable, then, by Thm.1.12, $\ker A = \{0\}$ and $\int_0^t T(s)Af\,ds = T(t)f - f \to -f$ as $t \to \infty$. Therefore (a) implies (b). On the other hand, if $\int_0^t T(s)Af\,ds$ converges as $t \to \infty$, then, by the equation above, $g := \lim_{t\to\infty} T(t)f$ exists. But $\ker A = \{0\}$ and therefore $g = 0$. This proves "(b) $\to$ (a)".

The implication "(a') $\to$ (b')" is obvious. If $T(t)f \to 0$ for every $f \in E$, then $\|T(t)\| \leq M$ and $0 \notin R\sigma(A)$ (Thm.1.12). Therefore $D := \operatorname{im} A$ is dense and $\int_0^\infty T(t)f\,dt$ exists for every $f \in D$. This proves "(b') $\to$ (c')". We have to show that (c') implies (a'). Define $G := \{h \in E : h = \int_0^\infty T(t)g\,dt$ for some $g \in D\}$. We will show that G is dense in E. First we notice that $g - T(s)g \in D$ whenever $g \in D$ and $s \in \mathbb{R}_+$.

Define $h_s = \frac{1}{s}\cdot\int_0^\infty T(t)(g - T(s)g)\,dt = \frac{1}{s}\cdot\int_0^s T(t)g\,dt$. Then $h_s \in G$ and $h_s \to g$ as $s \to 0$. Therefore $D \subset \bar{G}$ or $\bar{G} = E$. Now let $h \in G$. Then $T(t)h = T(t)\int_0^\infty T(s)g\,ds = \int_t^\infty T(s)g\,ds \to 0$ as $t \to \infty$. But $\|T(t)\| \leq M$ and therefore $T(t)f \to 0$ for every $f \in E$.

□

Remarks 1.17. (a) If A is the generator of a stable semigroup $(T(t))_{t\geq 0}$ on a Banach space E, then, by the previous theorem,

$$\operatorname{im} A \subset \{f \in E : \int_0^\infty T(t)f\,dt \text{ exists}\} =: H .$$

If $g \in H$, then $\int_0^\infty T(t)g\,dt \in D(A)$ and $A\int_0^\infty T(t)g\,dt = -g$. Therefore $g \in \operatorname{im} A$ and we obtain that the dense subspace $\operatorname{im} A$ is given by

$$\operatorname{im} A = \{f \in E : \int_0^\infty T(t)f\,dt \text{ exists}\} \tag{1.14}$$

in case that A is the generator of a stable semigroup $(T(t))_{t\geq 0}$.

(b) If $\omega(f) < 0$ for every $f \in D(A)$, then $(T(t))$ is stable (but might not be exponentially stable if $0 = \omega_1(A) = \sup\{\omega(f) : f \in D(A)\}$. In this case it can be seen by a proof similar to the one of Thm.1.4, that $\sigma(A)$ has to be contained in the open left half-plane; i.e. $\operatorname{Re}\lambda < 0$ for $\lambda \in \sigma(A)$.

(c) If one defines a semigroup $(T(t))_{t\geq 0}$ to be <u>weakly stable</u> if $\langle T(t)f,\Phi\rangle \to 0$ as $t \to \infty$ for every $f \in D(A)$ and $\Phi \in E'$ or as <u>weakly uniformly stable</u> if $\langle T(t)f,\Phi\rangle \to 0$ as $t \to \infty$ for every $f \in E$ and $\Phi \in E'$, then Theorem 1.13 and 1.16 can be reformulated in a