eigenfunction corresponding to 0 is the scalar multiple of a strictly positive function. If $Ah = i\alpha h$, $h \neq 0$, $\alpha \neq 0$ we consider $\tilde{h}(x) := h(x)/|h(x)|$. Assuming that $\tilde{h}(X)$ is not not dense in Γ, there exists a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that

$$p_n(z) \to 1/z \text{ uniformly in } z \in \tilde{h}(X). \tag{3.8}$$

It follows that $h(x)\cdot p_n(\tilde{h}(x)) \to |h|(x)$ uniformly in $x \in X$. Obviously, $h\cdot p_n(h)$ is a linear combination of $h^{[1]}$, $h^{[2]}$, $h^{[3]}$, ..., that is, it is an element of $\text{span}\{\cup_{k=1}\ker(ik\alpha - A)\}$ (cf. Thm.2.4). By (3.7) the linear form Ψ vanishes on $\ker(\lambda - A)$ whenever $\lambda \neq 0$. Therefore $\langle h\cdot p_n(h),\Psi\rangle = 0$ and we have
$0 < \langle |h|,\Psi\rangle = \lim_{n\to\infty}\langle h\cdot p_n(h),\Psi\rangle = 0$ which is a contradiction.

□

The group $P\sigma(A)\cap i\mathbb{R}$ need not be discrete. For example, the semigroup described in Ex.2.6(d) satisfies the assumptions of Thm.3.6 if α/β is irrational. In this case $P\sigma(A) = i\alpha\mathbb{Z} + i\beta\mathbb{Z}$ is a dense subgroup of $i\mathbb{R}$. Actually one can show that for every subgroup H of $i\mathbb{R}$ there is an irreducible semigroup on $C(G)$, $G := (H_d)^\wedge$, such that $P\sigma(A) = H$. Here $(H_d)^\wedge$ denotes the dual of the abelian group H equipped with the discrete topology. For details see [Greiner (1982), p.62].

An immediate consequence of assertion (d) of Theorem 3.6 is the following corollary.

<u>Corollary</u> 3.7. Suppose T satisfies the hypotheses of Thm.3.6 and let A be its generator. If k is a bounded continuous real-valued function, M_k the corresponding multiplication operator, then for $B := A + M_k$ we have $\sigma(B) + P\sigma(A)\cap i\mathbb{R} = \sigma(B)$.
In particular, $s(B) + P\sigma(A)\cap i\mathbb{R} \subset \sigma(B)$.

The next two corollaries are essentially consequences of assertion (c) of Theorem 3.6. The statement of the first one can be summarized as follows: In case there are non-real eigenvalues in the boundary spectrum then the semigroup 'contains' the semigroup of rotations on Γ.

<u>Corollary</u> 3.8. Suppose that the hypotheses of Thm.3.6 are satisfied and that there is an eigenvalue $i\alpha$ of A with $\alpha > 0$.
Let $\tau := 2\pi/\alpha$.
Then there exists a continuous injective lattice homomorphism $j : C(\Gamma) \to C_o(X)$ such that the diagram