(d) (cf. Ex.2.14(b)) Consider $Af = f'$ on $E = C[-1,0]$ with $D(A_\Psi) = \{f \in C^1 : f'(0) = \Psi(f)\}$ where the linear functional Ψ satisfies $\Psi + \alpha\delta_o \geq 0$ for some $\alpha \in \mathbb{R}$ (see B-II,Ex.1.22) . The corresponding semigroup is irreducible if and only if $-1 \in \operatorname{supp} \Psi$.

(e) The second derivative $Af = f''$ generates an irreducible semigroup on $C_o(\mathbb{R})$ and on $C_o(0,1)$ (cf. A-I,2.7). With Neumann boundary conditions (or more generally: $f'(0) = \alpha_o f(0)$, $f'(1) = \alpha_1 f(1)$ where α_o , $\alpha_1 \in \mathbb{R}$) the second derivative generates an irreducible semigroup on $C[0,1]$ (cf. A-I,2.7).
The operator $Af = f'' - Vf$ on $C_o(\mathbb{R})$, where V is continuous, real-valued with $\inf V(x) > -\infty$ (see Example 2.14(a)) also generates an irreducible semigroup. This can be derived from the maximum principle as follows: For $\lambda > -\inf V(x)$, $f \in C_o(\mathbb{R})$, $g := R(\lambda,A)f$ we have $g \in C^2$ and $g'' - (\lambda + V)g = -f$. If $f > 0$, then $g > 0$, hence [Protter-Weinberger (1967), Chap.I, Thm.3] implies that g is strictly positive.

(f) The Laplacian Δ generates an irreducible semigroup on $C_o(\mathbb{R}^n)$ as can be seen easily from A-I,2.8. More general elliptic operators will be discussed below (see Ex.3.10(b)).

We now return to the general situation and show that irreducible semigroups possess several interesting properties.

Proposition 3.5. Suppose A is the generator of a strongly continuous semigroup on $C_o(X)$ which is irreducible. Then the following assertions are true:

(a) $\sigma(A) \neq \emptyset$;
(b) every positive eigenfunction of A is strictly positive;
(c) every positive eigenvector of A' is strictly positive;
(d) if $\ker(s(A) - A')$ contains a positive element (e.g., if X is compact (cf. Thm.1.6)), then $\dim(\ker(s(A) - A)) \leq 1$;
(e) if $s(A)$ is a pole of the resolvent, then it is algebraically simple. The residue has the form $P = \phi \otimes u$ where $\phi \in E'$ and $u \in E$ are strictly positive eigenvectors of A' and A , respectively, satisfying $\langle u,\phi\rangle = 1$.

Proof. (a) Take any $f_o \in C_o(X)$ which is positive and has compact support. If $\lambda > s(A)$, then $R(\lambda,A)f_o$ is strictly positive (by Def.3.1(v)), hence there exists $\varepsilon > 0$ such that $R(\lambda,A)f_o \geq \varepsilon f_o$. It follows that $R(\lambda,A)^n f_o \geq \varepsilon^n f_o \geq 0$ for all $n \in \mathbb{N}$ and therefore