standard of these "function spaces", we mention the space $C_o(X)$ of all continuous complex valued functions vanishing at infinity on a locally compact space X, or the spaces $L^p(X,\Sigma,\mu)$, $1 \leqq p \leqq \infty$, of all (equivalence classes of) p-integrable functions on a σ-finite measure space (X,Σ,μ).

On these function spaces $E = C_o(X)$, resp. $E = L^p(X,\Sigma,\mu)$, there is a simple way to define "multiplication operators" : Take a continuous, resp. measurable function $q : X \to \mathbb{C}$ and define

$$M_q f := q \cdot f \ , \quad \text{i.e.} \quad M_q f(x) := q(x) \cdot f(x) \quad \text{for } x \in X \ ,$$

for every f in the "maximal" domain $D(M_q) := \{g \in E : q \cdot g \in E\}$. This natural domain is a dense subspace of $C_o(X)$, resp. $L^p(X,\Sigma,\mu)$, for $1 \leqq p < \infty$. Moreover, $(M_q, D(M_q))$ is a closed operator. This is easy in case $E = C_o(X)$. For $E = L^p(\mu)$, $1 \leqq p < \infty$, we consider a sequence $(f_n) \subset E$ such that $\lim_{n\to\infty} f_n = f \in E$ and $\lim_{n\to\infty} qf_n =: g \in E$. Choose a subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f_{n(k)}(x) = f(x)$ and $\lim_{k\to\infty} q(x)f_{n(k)}(x) = g(x)$ for μ-almost every $x \in X$. Then $g = q \cdot f$ and $f \in D(M_q)$, i.e. M_q is closed.
For such multiplication operators many properties can be checked quite directly. For example, the following statements are equivalent:

(a) M_q is bounded. (b) q is (μ-essentially) bounded.

One has $\|M_q\| = \|q\|_\infty$ in this situation.

Observe that on spaces $C(K)$, K compact, there are no densely defined, unbounded multiplication operators.

By defining the <u>multiplication semigroups</u>

$$T(t)f(x) := \exp(t \cdot q(x))f(x) \quad , \quad x \in X \ , \ f \in E \ ,$$

one obtains the following characterizations.

<u>Proposition</u>. Let M_q be a multiplication operator on $E = C_o(X)$ or $E = L^p(X,\Sigma,\mu)$, $1 \leqq p < \infty$. Then the properties (a) and (b), resp. (a') and (b'), are equivalent:

(a) M_q generates a strongly continuous semigroup.

(b) $\sup\{\operatorname{Re} q(x) : x \in X\} < \infty$.

(a') M_q generates a uniformly continuous semigroup.

(b') $\sup\{|q(x)| : x \in X\} < \infty$.