bounded semigroup $(S(t))_{t\geq 0}$ given by $S(t) = T(z_o t)$ $(t\geq 0)$ (where again we denote by T the holomorphic extension of $(T(t))_{t\geq 0}$ on $S(\alpha)$).

As an application of Theorem 1.12. we prove the following.

<u>Corollary</u> 1.13. Let A be the generator of a bounded group. Then A^2 generates a bounded holomorphic semigroup of angle $\pi/2$.

<u>Proof</u>. Let $0 < \alpha_1 < \pi/2$; $\lambda \in S(\alpha_1+\pi/2)$. There exist $r \geq 0$ and $\beta \in (-\beta_1,\beta_1)$, where $\beta_1 := (\alpha_1+\pi/2)/2$, such that $\lambda = r^2 e^{i2\beta}$. Then $(\lambda-A^2) = (re^{i\beta}-A)(re^{i\beta}+A)$; so it follows that $\lambda \in \rho(A^2)$ and

(1.7) $R(\lambda,A^2) = R(re^{i\beta},A)R(re^{i\beta},-A)$.

Since A generates a bounded group, there exists a constant $N \geq 0$ such that $\|R(\mu,A)\| \leq N/\mathrm{Re}\mu$, $\|R(\mu,-A)\| \leq N/\mathrm{Re}\mu$ for all $\mu \in S(\pi/2)$. Consequently, $\|R(\lambda,A^2)\| \leq N^2/r^2(\cos\beta)^2 \leq 1/r^2\cdot[N/\cos\beta_1]^2 = M/|\lambda|$.
□

The corollary will be extended below. We first consider an example.

<u>Example</u> (The Laplacian on $E = C_o(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)$ $(1\leq p<\infty)$).

a) Let $n = 1$. Then $(U(t)f)(x) = f(x+t)$ $(t\in\mathbb{R},\ x\in\mathbb{R})$ defines an isometric group on E. Its generator A is given by $Af = f'$ with $D(A) = \{f \in C^1(\mathbb{R}) \cap C_o(\mathbb{R}) : f' \in C_o(\mathbb{R})\}$ in the case $E = C_o(\mathbb{R})$ and $D(A) = \{f \in E \cap AC(\mathbb{R}) : f' \in E\}$ in the case $E = L^p$ (see A-I,2.4). Thus A^2 generates a bounded holomorphic semigroup by Cor.1.13.

b) Let $E = C_o(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)$ $(1\leq p<\infty)$.
For $i\in \{1,\dots,n\}$ denote by $(U_i(t))_{t\in\mathbb{R}}$ the group given by $(U_i(t)f)(x) = f(x_1,\dots,x_{i-1},x_i+t,\dots,x_n)$ $(x\in\mathbb{R}^n,\ t\in\mathbb{R})$ and by A_i its generator. Since $U_i(t)U_j(s) = U_j(s)U_i(t)$ $(s,t\in\mathbb{R},\ i,j\in\{1,\dots,n\})$ it follows that the resolvents of A_i commute. So the same is true for the resolvents of A_i^2 (cf.(1.7) and A-I,3.8).
Denote by $(T_i(t))_{t\geq 0}$ the semigroup generated by A_i^2 $(i=1,\dots,n)$. Then for $z,z' \in S(\pi/2)$ one has $T_i(z)T_j(z') = T_j(z')T_i(z)$ $(i,j=1,\dots,n)$. Consequently, $T(t) := T_1(t)\circ\dots\circ T_n(t)$ $(t\geq 0)$ defines a holomorphic semigroup of angle $\pi/2$. According to A-I,3.8 the domain of its generator A contains $D(A_1^2)\cap\dots\cap D(A_n^2)$; in particular $D_o = \{f \in E \cap C^2(\mathbb{R}^n) : D^\alpha f \in E$ for every multiindex α with $|\alpha| \leq 2\} \subset D(A)$ and the generator is given by

$$Af = (A_1^2+\dots+A_n^2)f = \sum_{i=1}^n \frac{\partial^2}{(\partial x_i)^2} f = \Delta f \quad \text{for all } f \in D_0.$$