$(T(t))_{t\geq 0}$ is a semigroup which is not positive. Nevertheless its generator A satisfies Kato's inequality. Even the equality is valid; i.e.,

(3.10) $\langle(\mathrm{sign}\, f)\, Af,\phi\rangle = \langle|f|,A'\phi\rangle$ for all $f \in D(A)$, $0 \leq \phi \in D(A')$.

Proof. It is not difficult to see that

$$D(A') = \{\phi \in AC[0,1] : \phi' \in L^q[0,1],\ \phi(0) = d\phi(1)\}$$
(3.11)
$$A'\phi = -\phi' \quad \text{for all } \phi \in D(A').$$

where $1/p + 1/q = 1$. Let $\phi \in D(A')_+$. Since $d < 0$, it follows that $\phi(0) = \phi(1) = 0$. Hence for $f \in D(A)$,

$$\begin{aligned}\langle(\mathrm{sign}\, f)Af,\phi\rangle &= \langle(\mathrm{sign}\, f)f',\phi\rangle = \langle|f|',\phi\rangle \\ &= \int_0^1 |f|'(x)\,\phi(x)\,dx \\ &= |f(1)|\phi(1) - |f(0)|\phi(0) - \int_0^1 |f(x)|\,\phi'(x)\,dx \\ &= |f(1)|\phi(1) - |f(0)|\phi(0) + \langle|f|,A'\phi\rangle \\ &= \langle|f|,A'\phi\rangle\end{aligned}$$

□

Remark 3.15. The equality (3.10) does not hold for all $\phi \in D(A')$. In fact, this would imply that $|f| \in D(A)$ and $(\mathrm{sign}\, f)Af = A|f|$ for all $f \in D(A)$. Thus by Cor. 5.8 below the semigroup would be positive. The reason why in this example the equality holds will be explained from a more general point of view in Section 5 (see Rem.5.12).

Relation (3.10) shows that A also satisfies Kato's inequality formally in the strong sense. In order to formulate this more precisely, observe that it follows from (3.8) that $D(A_{max}) = D(A) + \mathbb{R}\cdot e_\lambda$ (for any fixed $0 < \lambda \in \rho(A)$). Thus the extension A_{max} of A satisfies the following.

(3.12) A_{max} is closed.

(3.13) $D(A_{max})$ is a sublattice of E.

(3.14) $D(A)$ has codimension one in $D(A_{max})$.

(3.15) $(\mathrm{sign}\, f)Af = A_{max}|f|$ for all $f \in D(A)$.

It is also remarkable that there exists a dense sublattice $D_o := \{f \in D(A) : f(0) = f(1) = 0\}$ of E which is included in $D(A)$. But D_o is not a core of A (this would imply the positivity of the semigroup by Thm.1.8 if $|d| \leq 1$).