<u>Proposition</u> 1.11. Let A be the generator of a positive, stable semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E. Let $F(\cdot)$ be a locally integrable function from $\mathbb{R}_+$ into E. If there are $G(\cdot) \in C_o(\mathbb{R}_+,\mathbb{R}_+)$, $f_o \in \operatorname{im} A$ and $g_o \in \operatorname{im} A_+$ such that $|F(s) - f_o| \leq G(s)g_o$ for every $s \geq 0$, then every mild solution $u(\cdot)$ of (1.6) converges as $t \to \infty$ and $\lim_{t\to\infty} u(t) = -h$ where $h \in D(A)$ with $Ah = -f_o$.

<u>Proof</u>. Recall that every solution of (1.6) satisfies

$$(1.7) \qquad u(t) = T(t)f + \int_0^t T(t-s)f_o\, ds + \int_0^t T(t-s)(F(s) - f_o)\, ds .$$

By the stability of the semigroup and $f \in D(A)$, the first term converges to zero as $t \to \infty$. Since $f_o \in \operatorname{im} A$, the second term converges to $h := \int_0^\infty T(s)f_o\, ds \in \operatorname{im} A$ (A-IV,Thm.1.16) and $Ah = -f_o$. Define $H(s) := F(s) - f_o = H_+(s) - H_-(s)$. We have to show that $\int_0^t T(t-s)H_\pm(s)\, ds \to 0$ as $t \to \infty$. Again, the assumption $g_o \in \operatorname{im} A$ is equivalent to the existence of $\int_0^\infty T(t)g_o\, dt$. Choose

(i) a constant M such that
$$0 \leq H_\pm(s) \leq H_+(s) + H_-(s) = |H(s)| \leq G(s)g_o \leq Mg_o$$

(ii) a constant t' such that $\|\int_{t'}^\infty T(s)g_o\, ds\| \leq \varepsilon/(2M)$ and
$$G(s) \leq \varepsilon/2\, \|\int_0^\infty T(s)g_o\, ds\| \text{ for every } s \geq t' .$$

Then, for $t > 2t'$,

$$\begin{aligned} 0 &\leq \int_0^t T(t)H_\pm(s)\, ds \leq \int_0^t T(t)G(s)g_o\, ds \\ &= \int_0^{t'} T(t)G(s)g_o\, ds + \int_{t'}^t T(t)G(s)g_o\, ds \\ &\leq M \int_{t-t'}^t T(t)g_o\, ds + \varepsilon/2\, \|\int_0^\infty T(t)g_o\, ds\|^{-1} \int_0^{t-t'} T(t)g_o\, ds \\ &\leq M \int_{t'}^\infty T(t)g_o\, ds + \varepsilon/2\, \|\int_0^\infty T(t)g_o\, ds\|^{-1} \int_0^\infty T(t)g_o\, ds . \end{aligned}$$

Hence $\|\int_0^t T(t)H_\pm(s)\, ds\| \leq \varepsilon$ for every $t > 2t'$.

□

We conclude with a result similar to the previous proposition. Instead of uniform stability we now require $s(A) < 0$ while the assumption on the forcing term is weaker than in Prop.1.11.

<u>Proposition</u> 1.12. Let $(T(t))_{t\geq 0}$ be a positive semigroup with $s(A) < 0$. Assume that the forcing term F has values in $D(A)$, that it is continuous with respect to the graph norm and that
$f_o := \|\cdot\|_A\text{-}\lim_{t\to\infty} F(t)$ exists.
Then for every solution $u(\cdot)$ of (1.6) we have $\lim_{t\to\infty} u(t) = -A^{-1}f_o$.

(Note, that the assumptions imply that (1.6) has a unique strong solution, see [Pazy (1983),Thm.4.2.4].)