Corollary 6.4. For the eigenspaces of the generator A , resp. of the semigroup operators $T(t)$, $t > 0$, the following holds:

(i) $\ker(\mu - A) = \bigcap_{s\geq 0} \ker(e^{\mu s} - T(s))$,

(ii) $\ker(e^{\mu t} - T(t)) = \overline{\mathrm{span}}_{n\in\mathbb{Z}} \ker(\mu + 2\pi in/t - A)$, $\mu \in \mathbb{C}$.

Remark that analogous statements are valid for $\ker(\mu - A')$ and $\ker(e^{\mu t} - T(t)')$ if we take in (ii) the $\sigma(E',E)$-closure.

Without proof (see Greiner (1981), Prop.1.10) we add another corollary showing that poles of the resolvent of $T(t)$ correspond necessarily to poles of the resolvent of the generator. Again the converse is not true as shown by Example 5.6 .

Corollary 6.5. Assume that $e^{\mu t}$ is a pole of order k of $R(\cdot,T(t))$ with residue P and Q as the k-th coefficient of the Laurent series. Then

(i) $\mu + 2\pi in/t$ is a pole of $R(\cdot,A)$ of order $\leq k$ for every $n \in \mathbb{Z}$,

(ii) the residues P_n in $\mu + 2\pi in/t$ yield $PE = \overline{\mathrm{span}}_{n\in\mathbb{Z}} P_nE$,

(iii) the k-th coefficient of the Laurent series of $R(\cdot,A)$ at $\mu + 2\pi in/t$ is

$$Q_n = (t\cdot e^{\mu t})^{1-k}\cdot Q\circ(1/t)\int_0^t e^{-(\mu+2\pi in/t)s}T(s)\,ds \ .$$

From Theorem 6.2 and 6.3 it follows that the approximate point spectrum is the trouble maker in the sense that not every approximate eigenvalue of $T(t)$ corresponds to an approximate eigenvalue of the generator A . Since nothing more can be said in general we now look for additional hypotheses on the semigroup implying the spectral mapping theorem.

As a simple example we assume $T(t_o)$ to be compact for some $t_o > 0$. Then $\sigma(T(t)) \setminus \{0\} = P\sigma(T(t)) \setminus \{0\}$ for $t \geq t_o$ and the spectral mapping theorem is valid by (6.4). A different class of semigroups verifying the spectral mapping theorem is given by the uniformly continuous semigroups (compare Cor.1.2).

Both cases, and many more , are included in the following result.