4. SEMIGROUPS OF LATTICE HOMOMORPHISMS

As we have seen in Section 2 the boundary spectrum of a many positive semigroups is a cyclic set. However, there are hardly any restrictions on the set $\{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda < s(A)\}$, except that it is symmetric with respect to the real axis. For semigroups of lattice homomorphisms the situation is quite different. We will show that the whole spectrum is an imaginary additively cyclic subset of $\mathbb{C}$ (see Def.2.5). A complete proof of this results requires some facts of the theory of Banach lattices, therefore, we postpone it to Part C (see C-III,Thm.4.2).

<u>Theorem</u> 4.1. If A is the generator of a semigroup of lattice homomorphisms, then $\sigma(A)$, $A\sigma(A)$ and $P\sigma(A)$ are cyclic subsets of $\mathbb{C}$.

<u>1^{st} part of the proof</u>. We prove the assertion concerning $A\sigma(A)$ and $P\sigma(A)$. Assume that $Ah = (\alpha + i\beta)h$, $\alpha,\beta \in \mathbb{R}$, $h \neq 0$, then $T(t)h = e^{\alpha t}e^{i\beta t}h$ for all $t \geq 0$ (A-III,Cor.6.4). Since $T(t)$ is a lattice homomorphism we have $T(t)|h| = |T(t)h| = e^{\alpha t}|h|$ $(t \geq 0)$ or $A|h| = \alpha|h|$, hence $Ah^{[n]} = (\alpha + in\beta)h^{[n]}$ for all $n \in \mathbb{Z}$ by Thm.2.4(b). We have shown that $P\sigma(A)$ is cyclic.
To prove that $A\sigma(A)$ is cyclic as well, one considers a semigroup F-product E_F^T of E (see A-III,4.5). It is easy to see that E_F^T is a Banach lattice and $(T_F(t))$ is a semigroup of lattice homomorphisms. The proposition in A-III,3.5 implies $A\sigma(A) = P\sigma(A_F)$. Thus the assertion follows from the cyclicity of point spectrum.

□

Performing a similar construction as in Ex.2.6(f) one can show that every closed cyclic subset of $\mathbb{C}$ which is contained in a left half-plane is the spectrum of a suitable semigroup of lattice homomorphisms. For details see Derndinger-Nagel (1979).
In the following we restrict ourselves to the case of compact spaces. Then a semigroup of lattice homomorphisms can be described explicitly by a semi-flow ϕ and real-valued functions h and p (see B-II, Thms.3.5 & 3.6). The function p has no influence on spectral properties (cf. B-II,(3.7)). Therefore we will assume that $(T(t))$ has the following form (cf. B-II,Thm.3.5):

(4.1) $T(t)f = h_t \cdot f\circ\phi_t$ $(t \geq 0,\ f \in C(K))$ where
$\phi = (\phi_t) : \mathbb{R}_+ \times K \to K$ is a continuous semiflow and
$h_t(x) = \exp \int_0^t h(\phi(s,x))\,ds$ $(t \geq 0,\ x \in K)$ for some
continuous function $h : K \to \mathbb{R}$.