(i) eventually compact semigroups,
(ii) eventually differentiable semigroups,
(iii) holomorphic semigroups,
(iv) uniformly continuous semigroups.

Another application of the above spectral mapping theorem can be made to tensor product semigroups (see A-I,3.7). Let $\mathcal{T}_1 = (T_1(t))_{t\geq 0}$, $\mathcal{T}_2 = (T_2(t))_{t\geq 0}$ be strongly continuous semigroups on Banach spaces E_1, E_2 with generator A_1, A_2. The tensor product semigroup $\mathcal{T} = T_1 \otimes \mathcal{T}_2$ on some (appropriate) tensor product $E := E_1 \tilde{\otimes} E_2$ has the generator $A = A_1 \otimes Id + Id \otimes A_2$, but in general the spectrum of A is not determined by the spectra of A_1, A_2. But with an additional hypothesis the following can be proved.

Corollary 6.8. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are eventually norm continuous then

$$\sigma(A) = \sigma(A_1) + \sigma(A_2) ,$$

where A is the generator of the tensor product semigroup

$$\mathcal{T}_1 \otimes \mathcal{T}_2 = (T_1(t) \otimes T_2(t))_{t\geq 0} .$$

Proof. Clearly, the tensor product semigroup is eventually norm continuous and hence the spectral mapping theorem 6.6 is valid for all three semigroups $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}$. Moreover the spectrum of the tensor product of bounded operators is the product of the spectra [Reed-Simon (1978), XIII.9]. Therefore

$$\sigma(T_1(t)\otimes T_2(t)) = \sigma(T_1(t))\cdot\sigma(T_2(t)) , \quad t \geq 0 .$$

Consequently we have the following identity for every $t \geq 0$:

$$\begin{aligned} e^{t\cdot\sigma(A)} &= \sigma(T_1(t)\otimes T_2(t)) \setminus \{0\} \\ &= \sigma(T_1(t))\cdot\sigma(T_2(t)) \setminus \{0\} \\ &= e^{t\cdot\sigma(A_1)}\cdot e^{t\cdot\sigma(A_2)} \\ &= e^{t(\sigma(A_1)+\sigma(A_2))}. \end{aligned}$$

From this identity we want to deduce $\sigma(A) = \sigma(A_1) + \sigma(A_2)$.
" $\subset$ ": Take $\xi \in \sigma(A)$. Then for every $t > 0$ there exist $\mu_t \in \sigma(A_1)$, $\lambda_t \in \sigma(A_2)$ and $n_t \in \mathbb{Z}$ such that $\xi = \mu_t + \lambda_t + 2\pi i n_t/t$.
Since the real parts of μ_t, λ_t are bounded above, they lie in some interval $[a,b]$. But

$$\sigma(A_i) \cap ([a,b] + i\mathbb{R})$$

is compact for $i = 1, 2$, since A_i is the generator of an eventually norm continuous semigroup (see A-II, Thm.1.20). By taking t