<u>Proposition</u> 2.3. Let T be a semigroup of contractions on a Banach space E with generator A. Then the following assertions are equivalent:

(a) Each $i\alpha$, $\alpha \in \mathbb{R}$, is a pole of the resolvent $R(.,A)$ such that the corresponding residue has finite rank.

(b) $\dim \mathrm{Fix}((\lambda - i\alpha)\hat{R}(\lambda,A)) < \infty$ for some (hence all) $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$ and the canonical extensions $\hat{R}(\lambda,A)$ of $R(\lambda,A)$ to some ultrapower.

<u>Proof</u>. Let P_α be the residue of the resolvent $R(.,A)$ in $i\alpha$. Then $P_\alpha = \lim_{\lambda \to i\alpha}(\lambda - i\alpha)R(\lambda,A)$ in the operator norm of $L(E)$. Since the canonical map $(T \to \hat{T})$ is isometric and since $\hat{E}$ is an ultrapower, we obtain

$$\hat{P}_\alpha = \lim_{\lambda \to i\alpha}(\lambda - i\alpha)\hat{R}(\lambda,A)$$

in $L(\hat{E})$ and $\mathrm{rank}(P_\alpha) = \mathrm{rank}(\hat{P}_\alpha)$. Because of

$$\hat{P}_\alpha(\hat{E}) = \mathrm{Fix}((\lambda - i\alpha)\hat{R}(\lambda))$$

one part of the corollary is proved. The other follows from Lemma 2.2. □

<u>Remarks</u> 2.4. (a) By the results in [Lin (1974)] a semigroup of contractions on a Banach space is uniformly ergodic if and only if 0 is a pole of the generator with order ≤ 1. The residue of the resolvent in 0 and the associated ergodic projection are identical.

(b) Let M be a W*-algebra with predual M_*, $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$ and $\hat{M}$ (resp. $(M_*)^\wedge$) the ultrapower of M (resp. M_*) with respect to $\mathcal{U}$. Then it is easy to see that $c_{\mathcal{U}}(M)$ is a two sided ideal in $\ell^\infty(M)$ hence $\hat{M}$ is a C*-algebra, but in general not a W*-algebra. Note that the unit of $\hat{M}$ is the canonical image of 1. For $\hat{x} \in \hat{M}$ and $\hat{\phi} \in (M_*)^\wedge$ let $J\colon (M_*)^\wedge \to \hat{M}'$ be defined by

$$\langle \hat{x}, J(\hat{\phi}) \rangle := \lim_{\mathcal{U}} \phi_n(x_n)\ ,\quad (x_n) \in \hat{x}\ ,\quad (\phi_n) \in \hat{\phi}\ .$$

J is well defined and is an isometric embedding. It turns out that $J((M_*)^\wedge)$ is a translation invariant subspace of $(M')^\wedge$. Hence there exists a central projection $z \in \hat{M}''$ such that $J((M_*)^\wedge) = \hat{M}''z$ [Groh (1984), Proposition 2.2].