Moreover, the semigroup $(T(t))_{t\geq 0}$ is strongly continuous. This can be seen as follows: In view of Prop.1.23 in Davies (1980) we only have to show that $\lim_{t\downarrow 0}\langle T(t)f-f,\nu\rangle = 0$ for every $f \in C(K)$, $\nu \in M(K)$. Due to Lebesgue's Dominated Convergence Theorem this is true whenever $\lim_{t\downarrow 0}(T(t)f)(x) = f(x)$ for every $f \in C(K)$, $x \in K$. Given f, x and $\varepsilon > 0$ there exists an open neighborhood U of x such that $|f(x) - f(y)| < \varepsilon$ for every $y \in U$. Then we have

$(T(t)f)(x) - f(x) = \int_K f(y)P_t(x,dy) - \int_K f(x)P_t(x,dy) =$
$\int_U (f(y)-f(x))P_t(x,dy) + \int_{K\setminus U} (f(y)-f(x))P_t(x,dy) \leq$
$\varepsilon\cdot P_t(x,U) + 2\|f\|_\infty\cdot P_t(x,K\setminus U)$.

Since $P_t(x,U) \leq 1$ and $\lim_{t\downarrow 0} P_t(x,U) = 1 = P_t(x,K)$ this estimate implies $\limsup_{t\downarrow 0}((T(t)f)(x) - f(x)) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary we have pointwise convergence hence strong continuity of the semigroup.
Finally we observe that every operator $T(t)$ defined by (2.4) has the strong Feller property since $T(t)''\chi_C = P_t(\cdot,C)$ for every Borel set $C \subset K$ (see Prop.2.4(a)).
Thus Thm.2.5 can be applied in this situation.

We now turn our interest from eventually compact semigroups to quasi-compact semigroups. While "eventually compact" means that the operators $T(t)$ with $t \geq t_o$ have to be compact, "quasi-compactness" only means that $T(t)$ approaches the compact operators as $t \to \infty$. To make this precise we introduce the following notations.
For a Banach space G the ideal of all compact linear operators on G is denoted by $\mathcal{K}(G)$. For $T \in \mathcal{L}(G)$ we define

$$\operatorname{dist}(T,\mathcal{K}(G)) := \inf\{\|T - K\| : K \in \mathcal{K}(G)\} .$$

<u>Definition</u> 2.7. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space G is called <u>quasi-compact</u> if $\lim_{t\to\infty}\operatorname{dist}(T(t),\mathcal{K}(G)) = 0$.

Quasi-compactness can be characterized in different ways. Two of them are stated in the following proposition. The first one uses the notion of the essential growth bound $\omega_{ess}(\mathcal{T})$ of a semigroup $\mathcal{T}$ which was introduced in A-III,3.7.

<u>Proposition</u> 2.8. For a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on a Banach space G the following conditions are equivalent:

(i) $\mathcal{T}$ is quasi-compact;

(ii) $\omega_{ess}(\mathcal{T}) < 0$;

(iii) There exist $t_o > 0$, $K \in \mathcal{K}(G)$ such that $\|T(t_o) - K\| < 1$.