(iii) Define $D(A^n) := \{f \in D(A^{n-1}) : Af \in D(A^{n-1})\}$, $D(A^1) = D(A)$. Then $D(A^\infty) := \bigcap_{n\in\mathbb{N}} D(A^n)$ is dense in E and a core for A .

Example 1.10. Property (iii) above does not hold for general densely defined closed operators. Take $E = C[0,1]$, $D(B) = C^1[0,1]$ and $Bf = q\cdot f'$ for some nowhere differentiable function $q \in C[0,1]$. Then B is closed, but $D(B^2) = (0)$.

Proposition 1.11. For the generator A of a strongly continous semigroup $(T(t))_{t\geq 0}$ on a Banach space E the following holds.
If $\int_0^\infty e^{-\lambda t}T(t)f\,dt$ exists for every $f \in E$ and some $\lambda \in \mathbb{C}$, then $\lambda \in \rho(A)$ and $R(\lambda,A)f = \int_0^\infty e^{-\lambda t}T(t)f\,dt$. In particular,

$$(1.7)\quad R(\lambda,A)^{n+1}f = \frac{(-1)^n}{n!}\left(\frac{d}{d\lambda}\right)^n R(\lambda,A)f = \int_0^\infty e^{-\lambda t}\, t^n/n!\; T(t)f\,dt$$

for every $f \in E$, $n \geq 0$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$.

Remarks 1.12. (1) For continuous Banach space valued functions such as $t \to T(t)f$ we consider the Riemann integral and define $\int_0^\infty T(t)f\,dt$ as $\lim_{t\to\infty} \int_0^t T(s)f\,ds$. Sometimes such integrals for strongly continuous semigroups $(T(t))_{t\geq 0}$ are written as $\int_a^b T(t)\,dt$ and understood in the strong sense.
(2) Since the generator $(A,D(A))$ determines the semigroup $(T(t))_{t\geq 0}$ uniquely, we will speak occasionally of the growth bound of the generator instead of the semigroup, i.e. we write $\omega = \omega(A) = \omega((T(t))_{t\geq 0})$.
(3) For one-parameter groups it might seem to be more natural to define the generator as the 'derivative' rather than just the 'right derivative' at $t = 0$. This yields the same operator as the following result shows:
The strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator A can be extended to a strongly continuous one-parameter group $(U(t))_{t\in\mathbb{R}}$ if and only if $-A$ generates a semigroup $(S(t))_{t\geq 0}$.

In that case $(U(t))_{t\in\mathbb{R}}$ is obtained as

$$U(t) := \begin{cases} T(t) & \text{for } t \geq 0 \\ S(-t) & \text{for } t \leq 0 . \end{cases}$$

We refer to [Davies (1980), Prop.1.14] for the details.