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One-parameter Semigroups of Positive Operators

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Lecture Notes in Mathematics

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This Latex version of the book "One-Parameter Semigroups of Positive Operators" is dedicated to the memory of our co-authors, Heinrich P. Lotz and Ulf Schlotterbeck. Their contributions to the first edition remain an inspiration to us all. We miss their presence and remain grateful for the legacy they have left in this work.

Preface

As early as 1948 in the first edition of his fundamental treatise on *Semigroups and Functional Analysis*, E. Hille expressed the need for

> . . . *developing an adequate theory of transformation semigroups operating in partially ordered spaces* (l.c., Foreword).

In the meantime the theory of one-parameter semigroups of positive linear operators has grown continuously. Motivated by problems in probability theory and partial differential equations W. Feller (1952) and R. S. Phillips (1962) laid the first cornerstones by characterizing the generators of special positive semigroups. In the 60's and 70's the theory of positive operators on ordered Banach spaces was built systematically and is well documented in the monographs of H. H. Schaefer (1974) and A. C. Zaanen (1983). But in this process the original ties with the applications and, in particular, with initial value problems were at times obscured. Only in recent years an adequate and up-to-date theory emerged, largely based on the techniques developed for positive operators and thus recombining the functional analytic theory with the investigation of Cauchy problems having positive solutions to each positive initial value. Even though this development — in particular with respect to applications to concrete evolution equations in transport theory, mathematical biology, and physics — is far from being complete, the present volume is a first attempt to shape the multitude of available results into a coherent theory of one-parameter semigroups of positive linear operators on ordered Banach spaces.

The book is organized as follows. We concentrate our attention on three subjects of semigroup theory: *characterization*, *spectral theory* and *asymptotic behavior*. By *characterization*, we understand the problem of describing special properties of a semigroup, such as positivity, through the generator. By *spectral theory* we mean the investigation of the spectrum of a generator. *Asymptotic behavior* refers to the orbits of the initial values under a given semigroup and phenomena such as stability.

This program (characterization, spectral theory, asymptotic behavior) is worked out on four different types of underlying spaces.

(A) On Banach spaces—Here we present the background for the subsequent discussions related to order.
(B) On spaces $C_0(X)$ (X locally compact), which constitute an important class of ordered Banach spaces and where our results can be presented in a form which makes them accessible also for the non-expert in order-theory.
(C) On Banach lattices, which admit a rich theory and are still sufficiently general as to include many concrete spaces appearing in analysis; e.g., $C_0(X)$, $\mathcal{L}^p(k)$ or l^p.
(D) On non-commutative operator algebras such as C*- or W*-algebras, which are not lattice ordered but still possess an interesting order structure of great importance in mathematical physics.

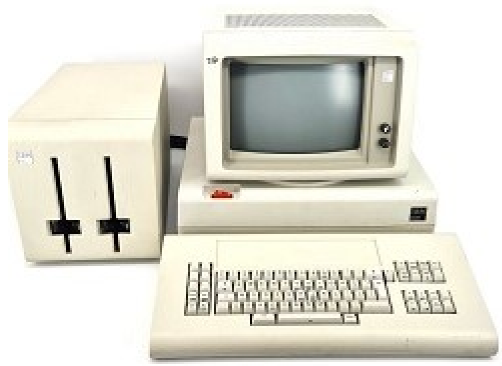
In each of these cases we start with a short collection of basic results and notations, so that the contents of the book may be visualized in the form of a 4×4 matrix in a way which will allow "row readers" (interested in semigroups on certain types of spaces) and "column readers" (interested in certain aspects) to find a path through the book corresponding to their interest.

We display this matrix, together with the names of the authors contributing to the subjects defined through this scheme.

	I Basic Results	II Characterization	III Spectral Theory	IV Asymptotics
A. Banach Spaces	R. Nagel U. Schlotterbeck	W. Arendt H. P. Lotz	G. Greiner R. Nagel	F. Neubrander
B. $C_0(X)$	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas F. Neubrander
C. Banach Lattices	R. Nagel U. Schlotterbeck	G. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas R. Nagel F. Neubrander
D. Operator Algebras	U. Groh	U. Groh	U. Groh	U. Groh

This "matrix of contents" has been an indispensable guide line in our discussions on the scope and the spirit of the various contributions. However, we would not have succeeded in completing this manuscript, as a collection of independent contributions (personally accounted for by the authors), under less favorable conditions than we have actually met. For one thing, Rainer Nagel was an unfaltering and undisputed spiritus rector from the very beginning of the project. On the other hand we gratefully acknowledge the influence of Helmut H. Schaefer and his pioneering work on order structures in analysis. It was the team spirit produced by this common mathematical background which, with a little help from our friends, made it possible to overcome most difficulties.

We have prepared the manuscript with the aid of a word processor, but we confess that without the assistance of Klaus Kuhn the pitfalls of such a system would have been greater than its benefits.



The authors

Contents

Acronyms

$E_{\mathbb{R}}$, $E_{\mathbb{C}}$ real, complex Banach lattice
E_+ positive cone
E' dual Banach space
E^* semigroup dual
E_F^T $\mathcal{F}$-product of E with respect to semigroup $\mathcal{T}$
E_F $\mathcal{F}$-product of E
E_f see C-I,4
(E, φ) see C-I,4
$E \otimes F$ tensor product
$\mathcal{L}(E)$ Banach space of bounded linear operators on E
$\mathcal{Z}(E)$ center of E
E_n n-th Sobolev space
$B(H)$ W*-algebra of bounded linear operators on a Hilbert space H

Part A
One-parameter Semigroups on Banach Spaces

Chapter A-I
Basic Results on Semigroups on Banach Spaces

Since the basic theory of one-parameter semigroups can be found in several excellent books (e.g. **?**, **?**, **?** or **?**), we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, to collect briefly some important definitions and results (Section 1), to present a list of *standard examples* in Section 2 and to discuss standard constructions of new semigroups from a given one in Section 3 on p. 15.

In the entire chapter we denote by E a (real or) complex Banach space and consider one-parameter semigroups of bounded linear operators $T(t)$ on E. By this we understand a subset $\{T(t)\colon t \in \mathbb{R}_+\}$ of $\mathcal{L}(E)$, usually written as $(T(t))_{t\geqslant 0}$, such that

$$\begin{aligned} T(0) &= \mathrm{Id}, \\ T(s+t) &= T(s)\cdot T(t) \text{ for all } s,t \in \mathbb{R}_+. \end{aligned}$$

In more abstract terms this means that the map $t \mapsto T(t)$ is a homomorphism from the additive semigroup $(\mathbb{R}_+, +)$ into the multiplicative semigroup $(\mathcal{L}(E), \cdot)$. Similarly, a one-parameter group $(T(t))_{t\in\mathbb{R}}$ will be a homomorphic image of the group $(\mathbb{R}, +)$ in $(\mathcal{L}(E), \cdot)$.

1 Standard Definitions and Results

We consider a one-parameter semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E and observe that the domain $\mathbb{R}_+$ and the range $\mathcal{L}(E)$ of the (semigroup) homomorphism $\tau\colon t \mapsto T(t)$ are topological semigroups for the natural topology on $\mathbb{R}_+$ and any one of the standard operator topologies on $\mathcal{L}(E)$. We single out the strong operator topology on $\mathcal{L}(E)$ and require τ to be continuous.

Definition 1.1 A one-parameter semigroup $(T(t))_{t\geqslant 0}$ is called *strongly continuous* if the map $t \mapsto T(t)$ is continuous for the strong operator topology on $\mathcal{L}(E)$, e.g.,

$$\lim_{t \to t_0} \|T(t)f - T(t_0)f\| = 0$$

for every $f \in E$ and $t, t_0 \geqslant 0$.

Clearly one defines in a similar way *weakly continuous*, resp. *uniformly continuous* (compare A-II, Def. 1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology.

If not stated otherwise, a *semigroup* is a strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

Remark 1.2 (i) A one-parameter semigroup $(T(t))_{t \geqslant 0}$ on a Banach space E is strongly continuous if and only if for any $f \in E$ it is true that $T(t)f \to f$ if $t \to 0$.
(ii) For every strongly continuous semigroup there exist constants $M \geqslant 1$, $w \in \mathbb{R}$ such that $\|T(t)\| \leqslant M \cdot e^{wt}$ for every $t \geqslant 0$.
(iii) If $(T(t))_{t \geqslant 0}$ is a one-parameter semigroup such that $\|T(t)\|$ is bounded for $0 < t \leqslant \delta$ then it is strongly continuous if and only if $\lim_{t \to 0} T(t)f = f$ for every f in a total subset of E.

The exponential estimate from Remark 1.2 (ii) for the growth of $\|T(t)\|$ can be used to define an important characteristic of the semigroup.

Definition 1.3 By the growth bound (or type) of the semigroup $(T(t))_{t \geqslant 0}$ we understand the number

$$\begin{aligned} \omega_0 := {} & \inf\{w \in \mathbb{R} \colon \text{There exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leqslant M e^{wt} \text{ for } t \geqslant 0\} \\ = {} & \lim_{t \to \infty} \frac{1}{t} \log \|T(t)\| = \inf_{t > 0} \frac{1}{t} \log \|T(t)\|. \end{aligned}$$

Particularly important are semigroups such that for every $t \geqslant 0$ we have $\|T(t)\| \leqslant M$ (*bounded semigroups*) or $\|T(t)\| \leqslant 1$ (*contraction semigroups*). In both cases we have $\omega_0 \leqslant 0$.

It follows from the subsequent examples and from Def. 1.3 that ω_0 may be any number $-\infty \leqslant \omega < +\infty$. Moreover the reader should observe that the infimum in Def. 1.3 need not be attained and that M may be larger than 1 even for bounded semigroups.

Examples 1.4 (i) Take $E = \mathbb{C}^2$,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then for the ℓ^1-norm on E we obtain $\|T(t)\| = 1+t$, hence $(T(t))_{t \geqslant 0}$ is an unbounded semigroup having growth bound $\omega_0 = 0$.

(ii) Take $E = L^1(\mathbb{R})$ and for $f \in E$, $t \geqslant 0$ define

$$T(t)f(x) := \begin{cases} 2 \cdot f(x+t) & \text{if } x \in [-t, 0] \\ f(x+t) & \text{otherwise.} \end{cases}$$

Each $T(t)$, $t > 0$, satisfies $\|T(t)\| = 2$ as can be seen by taking $f := \chi_{[0,t]}$. Therefore $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup which is bounded, hence has $\omega_0 = 0$, but the constant M in (1.3) cannot be chosen to be 1.

The most important object associated to a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is its *generator* which is obtained as the (right)derivative of the map $t \mapsto T(t)$ at $t = 0$. Since for strongly continuous semigroups the functions $t \mapsto T(t)f$, $f \in E$, are continuous but not always differentiable, we have to restrict our attention to those $f \in E$ for which the desired derivative exists. We then obtain the *generator* as a not necessarily everywhere defined operator.

Definition 1.5 To every semigroup $(T(t))_{t\geqslant 0}$ there belongs an operator $(A, D(A))$, called the *generator* and defined on the *domain*

$$D(A) := \{f \in E \colon \lim_{h\to 0} \frac{T(h)f - f}{h} \text{ exists in } E\} \text{ by}$$
$$Af := \lim_{h\to 0} \frac{T(h)f - f}{h} \quad (f \in D(A)).$$

Clearly, $D(A)$ is a linear subspace of E and A is linear from $D(A)$ into E. Only in certain special cases (see 2.1) the generator is everywhere defined and therefore bounded (use Prop. 1.9 (ii)) on p. 6). In general, the precise extent of the domain $D(A)$ is essential for the characterization of the generator. But since the domain is canonically associated to the generator of a semigroup, we shall write in most cases A instead of $(A, D(A))$.

As a first result we collect some information on the domain of the generator.

Proposition 1.6 *For the generator A of a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E the following assertions hold.*

(i) If $f \in D(A)$, then $T(t)f \in D(A)$ for every $t \geqslant 0$.
(ii) The map $t \mapsto T(t)f$ is differentiable on $\mathbb{R}_+$ if and only if $f \in D(A)$. In that case one has

$$\frac{d}{dt}T(t)f = AT(t)f = T(t)Af. \tag{1.1}$$

(iii) For every $f \in E$ and $t > 0$ the element $\int_0^t T(s)f\,\mathrm{d}s$ belongs to $D(A)$ and one has

$$A\int_0^t T(s)f\,\mathrm{d}s = T(t)f - f. \tag{1.2}$$

(iv) If $f \in D(A)$, then

$$\int_0^t T(s)Af\,\mathrm{d}s = T(t)f - f. \tag{1.3}$$

(v) The domain $D(A)$ is dense in E.

The identity (1.1) is of great importance and shows how semigroups are related to certain Cauchy problems. We state this explicitly in the following theorem.

Theorem 1.7 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on the Banach space E. Then the* abstract Cauchy problem

$$\frac{d}{dt}\xi(t) = A\xi(t), \quad \xi(0) = f_0 \tag{1.4}$$

has a unique solution $\xi\colon \mathbb{R}_+ \to D(A)$ in $C^1(\mathbb{R}_+, E)$ for every $f_0 \in D(A)$. In fact, this solution is given by $\xi(t) := T(t)f_0$.

For more on the relation of semigroups to abstract Cauchy problems we refer to A-II,Section 1. Here we only point out that the above theorem implies that a semigroup is uniquely determined by its generator.

While the generator is bounded only for uniformly continuous semigroups (see Sec. 2 below), it always enjoys a weaker but useful property.

Definition 1.8 An operator B with domain $D(B)$ on a Banach space E is called *closed* if $D(B)$ endowed with the *graph norm*

$$\|f\|_B := \|f\| + \|Bf\|$$

becomes a Banach space. Equivalently, $(B, D(B))$ is closed if and only if its *graph* $\{(f, Bf)\colon f \in D(B)\}$ is closed in $E \times E$, i.e.,

$$f_n \in D(B), f_n \to f \text{ and } Bf_n \to g \text{ implies } f \in D(B) \text{ and } Bf = g.$$

It is clear from this definition that the *closedness* of an operator B depends very much on the size of the domain $D(B)$. For example, a bounded and densely defined operator $(B, D(B))$ is closed if and only if $D(B) = E$.

On the other hand it may happen that $(B, D(B))$ is not closed but has a closed *extension* $(C, D(C))$, i.e., $D(B) \subseteq D(C)$ and $Bf = Cf$ for every $f \in D(B)$. In that case, B is called *closable*, a property which is equivalent to

$$f_n \in D(B), f_n \to 0 \text{ and } Bf_n \to g \text{ implies } g = 0.$$

The smallest closed extension of $(B, D(B))$ will be called the *closure* $\overline{B}$ with domain $D(\overline{B})$. In other words, the graph of $\overline{B}$ is the closure of $\{(f, Bf)\colon f \in D(B)\}$ in $E \times E$.

Finally we call a subset D_0 of $D(B)$ a *core* for B if D_0 is $\|\cdot\|_B$-dense in $D(B)$. This means that a closed operator is determined (via closure) by its restriction to a core in its domain.

We now collect the fundamental topological properties of semigroup generators, their domains (see also A-II,Cor.1.34) and their resolvents.

Proposition 1.9 *For the generator A of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ the following hold.*

(i) The generator A is a closed operator.
(ii) If a subspace D_0 of the domain $D(A)$ is dense in E and $(T(t))$-invariant, then it is a core for A.
(iii) Define $D(A^n) := \{f \in D(A^{n-1})\colon Af \in D(A^{n-1})\}$, $D(A^1) = D(A)$. Then $D(A^\infty) := \bigcap_{n\in\mathbb{N}} D(A^n)$ is dense in E and a core for A.

Example 1.10 Property (iii) above does not hold for general densely defined closed operators. Take $E = C[0,1]$, $D(B) = C^1[0,1]$ and $Bf = q \cdot f'$ for some nowhere differentiable function $q \in C[0,1]$. Then B is closed, but $D(B^2) = \{0\}$.

Proposition 1.11 *For the generator A of a strongly continous semigroup on a Banach space E the following hold.*

If $\int_0^\infty e^{-\lambda t} T(t) f\, dt$ exists for every $f \in E$ and some $\lambda \in \mathbb{C}$, then $\lambda \in \varrho(A)$ and $R(\lambda, A) f = \int_0^\infty e^{-\lambda t} T(t) f\, dt$. In particular,

$$R(\lambda, A)^{n+1} f = \frac{(-1)^n}{n!} \left(\frac{d}{d\lambda}\right)^n R(\lambda, A) f = \int_0^\infty e^{-\lambda t} \frac{t^n}{n!} T(t) f\, dt \tag{1.5}$$

for every $f \in E$, $n \geqslant 0$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega$.

Remark 1.12 (i) For continuous Banach space valued functions such as $t \mapsto T(t) f$ we consider the Riemann integral and define

$$\int_0^\infty T(t) f \ \mathrm{d}t \quad \text{as} \quad \lim_{t\to\infty} \int_0^t T(s) f \ \mathrm{d}s.$$

Sometimes such integrals for strongly continuous semigroups are written as $\int_a^b T(t)\, dt$ but understood in the strong sense.
(ii) Since the generator $(A, D(A))$ determines the semigroup uniquely, we will speak occasionally of the growth bound of the generator instead of the semigroup, i.e., we write $\omega_0 = \omega_0(A) = \omega_0((T(t))_{t\geqslant 0}$.
(iii) For one-parameter groups it might seem to be more natural to define the generator as the *derivative* rather than just the *right derivative* at $t = 0$. This yields the same operator as the following result shows.

The strongly continuous semigroup $(T(t))_{t\geqslant 0}$ with generator A can be extended to a strongly continuous one-parameter group $(U(t))_{t\in\mathbb{R}}$ if and only if $-A$ generates a semigroup $(S(t))_{t\geqslant 0}$.

In that case $(U(t))_{t\in\mathbb{R}}$ is obtained as

$$U(t) = \begin{cases} T(t) & \text{for } t \geqslant 0, \\ S(-t) & \text{for } t \leqslant 0. \end{cases}$$

We refer to **?**, Prop.1.14 for the details.

2 Standard Examples

In this section we list and discuss briefly the most basic examples of semigroups together with their generators. These semigroups will reappear throughout this book and will be used to illustrate the theory. We start with the class of semigroups mentioned after Definition 1.1 on p. 3.

2.1 Uniformly Continuous Semigroups

It follows from elementary operator theory that for every bounded operator A in $\mathcal{L}(E)$ the sum

$$\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} =: e^{tA}$$

exists and determines a unique uniformly continuous (semi)group $(e^{tA})_{t\in\mathbb{R}}$ having A as its generator. Conversely, any uniformly continuous semigroup is of this form.

If the semigroup $(T(t))_{t\geqslant 0}$ is uniformly continuous, then $\frac{1}{t}\int_0^t T(s)\,\mathrm{d}s$ uniformly converges to $T(0) = \mathrm{Id}$ as $t \to 0$. Therefore for some $t' > 0$ the operator $\frac{1}{t'}\int_0^{t'} T(s)\,\mathrm{d}s$ is invertible and every $f \in E$ is of the form $f = \frac{1}{t'}\int_0^{t'} T(s)g\,\mathrm{d}s$ for some $g \in E$. But these elements belong to $D(A)$ by (1.3), hence $D(A) = E$. Since the generator A is closed and everywhere defined, it must be bounded.

Remark that bounded operators are always generators of groups, not just semigroups. Moreover, the growth bound ω satisfies $|\omega| \leqslant \|A\|$ in this situation.

The above characterization of the generators of uniformly continuous semigroups as the bounded operators shows that these semigroups are—at least in many aspects—rather simple objects.

2.2 Matrix Semigroups

The above considerations expecially apply in the situation $E = \mathbb{C}^n$. If $n = 2$ and $A = (a_{ij})_{2\times 2}$ the following explicit formulas for e^{tA} might be of interest.

Set (i) $s := \operatorname{trace} A$, (ii) $d := \det A$ (iii) and $D := (s^2 - 4d)^{1/2}$. Then if $D \neq 0$

$$e^{tA} = e^{ts/2} \cdot [D^{-1} 2\sinh(tD/2) \cdot A + (\cosh(tD/2) - sD^{-1}\sinh(tD/2)) \cdot \mathrm{Id}]$$

and if $D = 0$

$$e^{ts/2} \cdot [tA + (1 - ts/2) \cdot \mathrm{Id}].$$

2.3 Multiplication Semigroups

Many Banach spaces appearing in applications are Banach spaces of (real or) complex valued functions over a set X. As the most standard examples of these "function spaces", we mention the space $C_0(X)$ of all continuous complex valued functions vanishing at infinity on a locally compact space X, or the spaces $L^p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, of all (equivalence classes of) p-integrable functions on a σ-finite measure space (X, Σ, μ).

On these function spaces $E = C_0(X)$, resp. $E = L^p(X, \Sigma, \mu)$, there is a simple way to define *multiplication operators*.

Take a continuous, resp. measurable function $q \colon X \to \mathbb{C}$ and define

$$M_q f := q \cdot f, \quad \text{i.e.,} \quad M_q f(x) := q(x) \cdot f(x) \quad \text{for } x \in X$$

and for every f in the *maximal* domain $D(M_q) := \{g \in E \colon q \cdot g \in E\}$.

This natural domain is a dense subspace of $C_0(X)$, resp. $L^p(X, \Sigma, \mu)$, for $1 \leqslant p < \infty$. Moreover, $(M_q, D(M_q))$ is a closed operator. This is easy in case $E = C_0(X)$.

For $E = L^p(X, \Sigma, \mu)$, $1 \leqslant p < \infty$, we consider a sequence $(f_n) \subset E$ such that $\lim_{n\to\infty} f_n = f \in E$ and $\lim_{n\to\infty} q f_n =: g \in E$. Choose a subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f_{n(k)}(x) = f(x)$ and $\lim_{k\to\infty} q(x) f_{n(k)}(x) = g(x)$ for μ-almost every $x \in X$. Then $g = q \cdot f$ and $f \in D(M_q)$, i.e., M_q is closed.

For such multiplication operators many properties can be checked quite directly. For example, the following statements are equivalent.

(a) M_q is bounded.
(b) q is (μ-essentially) bounded.

One has $\|M_q\| = \|q\|_\infty$ in this situation. Observe that on spaces $C(K)$, K compact, there are no densely defined, unbounded multiplication operators.

By defining the multiplication semigroups

$$T(t)f(x) := \exp(t \cdot q(x)) f(x), \quad x \in X, f \in E,$$

one obtains the following characterizations.

Proposition 2.1 *Let M_q be a multiplication operator on $E = C_0(X)$ or $E = L^p(X, \Sigma, \mu)$, $1 \leqslant p < \infty$. Then the properties (a) and (b), resp. (a') and (b'), are equivalent.*

(a) M_q generates a strongly continuous semigroup.
(b) $\sup\{\operatorname{Re}(q(x)) \colon x \in X\} < \infty$.
(a') M_q generates a uniformly continuous semigroup.
(b') $\sup\{|q(x)| \colon x \in X\} < \infty$.

As a consequence one computes the growth bound of a multiplication semigroup as

$$\omega_0 = \sup\{\operatorname{Re}(q(x)) \colon x \in X\}$$

in the case $E = C_0(X)$ and

$$\omega_0 = \mu\text{-ess-}\sup\{\mathrm{Re}(q(x)) : x \in X\}$$

in the case $E = L^p(\mu)$. It is a nice exercise to characterize those multiplication operators which generate strongly continuous groups.

We point out that the above results cover the cases of sequence spaces such as c_0 or ℓ^p, $1 \leqslant p < \infty$. An abstract characterization of generators of multiplication semigroups will be given in C-II,Thm.5.13.

2.4 Translation (Semi)Groups

Let E to be one of the following function spaces $C_0(\mathbb{R}_+)$, $C_0(\mathbb{R})$ or $L^p(\mathbb{R}_+)$, $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$. Define $T(t)$ to be the (left) translation operator

$$T(t)f(x) := f(x+t)$$

for x ,, $t \in \mathbb{R}_+$, resp. $x, t \in \mathbb{R}$ and $f \in E$. Then $(T(t))_{t\geqslant 0}$ is a strongly continous semigroup, resp. group of contractions on E and its generator is the first derivative $\frac{d}{dx}$ with *maximal* domain. In order to be more precise we have to distinguish the cases $E = C_0$ and $E = L^p$.

The generator of the translation (semi)group on $E = C_0(\mathbb{R}_+)$ is

$$\begin{aligned} Af &:= \frac{d}{\mathrm{d}x} f = f' \\ D(A) &:= \{f \in E : f \text{ differentiable and } f' \in E\}. \end{aligned}$$

Proof For $f \in D(A)$ it follows that for every $x \in \mathbb{R}_{(+)}$

$$\lim_{h\to 0} \frac{T(h)f(x) - f(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

(uniformly in x) and coincides with $Af(x)$. Therefore f is differentiable and $f' \in E$.

On the other hand, take $f \in E$ differentiable such that $f' \in E$. Then

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leqslant \frac{1}{h} \int_x^{x+h} |f'(y) - f'(x)| \,\mathrm{d}y,$$

where the last expression tends to zero uniformly in x as $h \to 0$. Thus $f \in D(A)$ and $f' = Af$. □

The generator of the translation (semi)group on $E = L^p(\mathbb{R}_+)$, $1 \leqslant p < \infty$, is

$$\begin{aligned} Af &:= \frac{d}{\mathrm{d}x} f = f', \\ D(A) &:= \{f \in E : f \text{ absolutely continuous}, f' \in E\}. \end{aligned}$$

Proof Take $f \in D(A)$ such that $\lim_{h\to 0} \frac{1}{h}(T(h)f - f) = g \in E$. Since integration is continuous, we obtain for every $a, b \in \mathbb{R}_{(+)}$ that

$$(*) \quad \frac{1}{h}\int_{b+h}^{b} f(x)\,\mathrm{d}x - \frac{1}{h}\int_{a+h}^{a} f(x)\,\mathrm{d}x = \int_a^b \frac{f(x+h) - f(x)}{h}\,\mathrm{d}x$$

converges to $\int_a^b g(x)\,\mathrm{d}x$ as $h \to 0+$. But for almost all a, b the left hand side of $(*)$ converges to $f(b) - f(a)$. By redefining f on a nullset we obtain

$$f(y) = \int_a^y g(x)\,\mathrm{d}x + f(a), \quad y \in \mathbb{R}_{(+)},$$

which is an absolutely continuous function whose derivative is (almost everywhere) equal to g.

On the other hand, let f be absolutely continuous such that $f' \in L^p$. Then

$$\begin{aligned}
\lim_{h\to 0}\int \left|\frac{f(x+h)-f(x)}{h} - f'(x)\right|^p \mathrm{d}x &= \lim_{h\to 0}\int \left|\frac{1}{h}\int_0^h (f'(x+s) - f'(x))\,\mathrm{d}s\right|^p \mathrm{d}x \\
&= \lim_{h\to 0}\int \left|\int_0^1 (f'(x+uh) - f'(x))\,\mathrm{d}u\right|^p \mathrm{d}x \\
&\leqslant \lim_{h\to 0}\int \int_0^1 |f'(x+uh) - f'(x)|^p\,\mathrm{d}u\,\mathrm{d}x \\
&= \int_0^1 \lim_{h\to 0}\int |f'(x+uh) - f'(x)|^p\,\mathrm{d}x\,\mathrm{d}u = 0,
\end{aligned}$$

hence $f \in D(A)$. □

2.5 Rotation Groups

On $E = C(\Gamma)$, resp. $E = L^p(\Gamma, m)$, $1 \leqslant p < \infty$, m Lebesgue measure we have canonical groups defined by rotations of the unit circle Γ with a certain period, i.e., for $0 < \tau \in \mathbb{R}$ the operators

$$R_\tau(t)f(z) := f(e^{2\pi i t/\tau} \cdot z), \quad z \in \Gamma$$

yield a group $(R_\tau(t))_{t\in\mathbb{R}}$ having period τ, i.e., $R_\tau(\tau) = \mathrm{Id}$. As in Example 2.4 one shows that its generator has the form

$$\begin{aligned}
D(A) &= \{f \in E : f \text{ absolutely continuous}, f' \in E\}, \\
Af(z) &= (2\pi i/\tau) \cdot z \cdot f'(z).
\end{aligned}$$

An isomorphic copy of the group $(R_\tau(t))_{t\in\mathbb{R}}$ is obtained if we consider $E = \{f \in C[0,1] : f(0) = f(1)\}$, resp. $E = L^p([0,1])$ and the group of *periodic translations*

$$T(t)f(x) := f(y) \quad \text{for } y \in [0,1]\,,\, y = x + t \bmod 1$$

with generator

$$D(A) := \{f \in E\colon f \text{ absolutely continuous}, f' \in E\}, \quad Af := f'.$$

2.6 Nilpotent Translation Semigroups

Take $E = L^p([0,\tau], m)$ for $1 \leqslant p < \infty$ and define

$$T(t)f(x) := \begin{cases} f(x+t) & \text{if } x + t \leqslant \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then $(T(t))_{t \geqslant 0}$ is a semigroup satisfying $T(t) = 0$ for $t \geqslant \tau$. Its generator is still the first derivative $A = \frac{d}{\mathrm{d}x}$, but with domain is

$$D(A) = \{f \in E\colon f \text{ absolutely continuous}, f' \in E, f(\tau) = 0\}.$$

In fact, if $f \in D(A)$, then f is absolutely continuous with $f' \in E$. By Prop. 1.6 (i) on p. 5 it follows that $T(t)f$ is absolutely continuous and hence $f(\tau) = 0$.

2.7 One-dimensional Diffusion Semigroup

For the second derivative

$$Bf(x) := \frac{d^2}{dx^2} f(x) = f''(x)$$

we take the domain

$$D(B) := \{f \in C^2\,[0,1] : f'(0) = f'(1) = 0\}$$

in the Banach space $E = C\,[0,1]$. Then $D(B)$ is dense in $C\,[0,1]$, but closed for the graph norm. Obviously, each function

$$e_n(x) := \cos \pi n x, \quad n \in \mathbb{Z},$$

is contained in $D(B)$ and is an eigenfunction of B pertaining to the eigenvalue $\lambda_n := -\pi^2 n^2$. The linear hull span $\{e_n : n \in \mathbb{Z}\} =: E_0$ forms a subalgebra of $D(B)$ which by the Stone-Weierstrass theorem is dense in E.

We now use e_n to define bounded linear operators

$$e_n \otimes e_n \colon f \mapsto \left(\int_0^1 f(x) e_n(x) \, \mathrm{d}x \right) e_n = (f|e_n) e_n$$

satisfying $\|e_n \otimes e_n\| \leqslant 1$ and $(e_n \otimes e_n)(e_m \otimes e_m) = \delta_{n,m}(e_n \otimes e_n)$ for $n \in \mathbb{Z}$.

For $t > 0$ we define

$$\begin{aligned} T(t) &:= \sum_{n \in \mathbb{Z}} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n \\ &= e_0 \otimes e_0 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n, \end{aligned}$$

or

$$T(t) f(x) = \int_0^1 k_t(x,y) f(x) dy$$

$$\text{where } k_t(x,y) = 1 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cos \pi n x \cos \pi n y.$$

The Jacobi identity

$$\begin{aligned} w_t(x) &:= 1/(4\pi t)^{\frac{1}{2}} \sum_{m \in \mathbb{Z}} \exp(-(x+2m)^2/4t) \\ &= \frac{1}{2} + \sum_{n \in \mathbb{N}} \exp(-\pi^2 n^2 t) \cos \pi n x \end{aligned}$$

and trigonometric relations show that

$$k_t(x,y) = w_t(x+y) + w_t(x-y)$$

which is a positive function on $[0,1]^2$. Therefore $T(t)$ is a bounded operator on $C[0,1]$ with

$$\|T(t)\| = \|T(t)\mathbb{1}\| = \sup_{x \in [0,1]} \int_0^1 k_t(x,y) dy = 1.$$

From the behavior of $T(t)$ on the dense subspace E_0 it follows that $(T(t))_{t \geqslant 0}$ with $T(0) = \mathrm{Id}$ is a strongly continuous semigroup on E and its generator A coincides with B on E_0. Finally, we observe that E_0 is a core for $(A, D(A))$ by Prop.1.9(ii).

Consequently, $(T(t))_{t \geqslant 0}$ is the semigroup generated by the closure of the second derivative with domain $D(B)$.

2.8 n-dimensional Diffusion Semigroup

On $E = L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, the operators

$$\begin{aligned} T(t)f(x) &:= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|x-y|^2/4t) f(y) dy \\ &= \mu_t * f(x) \end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$ and $\mu_t(x) := (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ form a strongly continuous semigroup:

In fact the integral exists for every $f \in L^p(\mathbb{R}^n)$ since μ_t is an element of the Schwartz space $S(\mathbb{R}^n)$ of all rapidly decreasing smooth functions on $\mathbb{R}^n$.

Moreover,

$$\|T(t)f\|_p \leqslant \|\mu_t\|_1 \|f\|_p = \|f\|_p$$

by Young's inequality, **?**, p.28, hence $\|T(t)\| \leqslant 1$ for every $t > 0$. Next we observe that $S(\mathbb{R}^n)$ is dense in E and invariant under each $T(t)$. Therefore we can apply the Fourier transformation F which leaves $S(\mathbb{R}^n)$ invariant and yields

$$F(\mu_t * f) = (2\pi)^{n/2} F(\mu_t) \cdot F(f) = (2\pi)^{n/2} \hat{\mu}_t \cdot \hat{f}$$

where $f \in S(\mathbb{R}^n)$, $\hat{f} = Ff \in S(\mathbb{R}^n)$.

In other words, F transforms $(T(t)|_{S(\mathbb{R}^n)})_{t \geqslant 0}$ into a multiplication semigroup on $S(\mathbb{R}^n)$ which is pointwise continuous for the usual topology of $S(\mathbb{R}^n)$. The generator, i.e., the right derivative at 0, of this semigroup is the multiplication operator

$$B\hat{f}(x) := -|x|^2 \hat{f}(x) \quad (x \in \mathbb{R}^n)$$

for every $f \in S(\mathbb{R}^n)$.

Applying the inverse Fourier transformation and observing that the topology of $S(\mathbb{R}^n)$ is finer than the topology induced from $L^p(\mathbb{R}^n)$, we obtain that $(T(t))_{t \geqslant 0}$ is a semigroup which is strongly continuous (use Rem. 1.2 (iii) on p. 4).

Its generator A coincides with

$$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x_1, \ldots, x_n)$$

for every $f \in S(\mathbb{R}^n)$. Since $S(\mathbb{R}^n)$ is $(T(t))$-invariant, we have determined the generator on a core of its domain (see Prop.1.9.ii). In particular, the above semigroup *solves* the initial value problem for the *heat equation*

$$\frac{\partial}{\partial t} f(x,t) = \Delta f(x,t), \quad f(x,0) = f_0(x), \quad x \in \mathbb{R}^n.$$

For the analogous discussion of the unitary group on $L^2(\mathbb{R}^n)$ generated by

$$C := \mathrm{i}\Delta$$

we refer to Section IX.7 in **?**.

Analogous examples to 2.7 are valid in $L^p\ [0,1]$, resp. to 2.8 in $C_0(\mathbb{R}^n)$.

3 Standard Constructions

Starting with a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E it is possible to construct new semigroups on spaces naturally associated with E. Such constructions will be important technical devices in many of the subsequent proofs. Although most of these constructions are rather routine, we present in the sequel a systematic account of them for the convenience of the reader.

We always start with a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E, and denote its generator by A on the domain $D(A)$.

3.1 Similar Semigroups

There is an easy way how to obtain different (but isomorphic) semigroups out of a given semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E.

Let V be an isomorphism from E onto E. Then $S(t) := VT(t)V^{-1}$, $t \geqslant 0$, defines a strongly continuous semigroup. If A is the generator of $(T(t))_{t\geqslant 0}$ then

$$B := VAV^{-1} \text{ with domain } D(B) := \{f \in E : V^{-1}f \in D(A)\}$$

is the generator of $(S(t))_{t\geqslant 0}$.

3.2 The Rescaled Semigroup

For fixed $\lambda \in \mathbb{C}$ and $\alpha > 0$ the operators

$$S(t) := \exp(\lambda t)T(\alpha t)$$

yield a new semigroup having generator

$$B := \alpha A + \lambda \,\mathrm{Id} \;\text{ with } D(B) = D(A).$$

This *rescaled semigroup* enjoys most of the properties of the original semigroup and the same is true for the corresponding generators. However, by using this procedure certain constants associated with $(T(t))_{t\geqslant 0}$ and A can be normalized. For example, by this rescaling we may in many cases suppose without loss of generality that the growth bound ω_0 is zero.

Another application is the following. For $\lambda \in \mathbb{C}$ and $S(t) := \exp(-\lambda t)T(t)$ the formulas *(1.3)* and *(1.4)* yield:

$$e^{-\lambda t}T(t)f - f = (\lambda - A)\int_0^t e^{-\lambda s}T(s)f\,\mathrm{d}s \text{ or}$$
$$(e^{\lambda t} - T(t))f = (\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)f\,\mathrm{d}s \quad \text{for } f \in E,$$

and

$$e^{-\lambda t}T(t)f - f = \int_0^t e^{-\lambda s}T(s)(\lambda - A)f\,\mathrm{d}s\text{or}$$

$$(e^{\lambda t} - T(t))f = \int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)f\,\mathrm{d}s \quad \text{for } f \in D(A).$$

3.3 The Subspace Semigroup

Assume F to be a closed $(T(t))$-invariant or, equivalently, $R(\lambda, A)$-invariant for $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > \omega_0$, subspace of E. Then the semigroup $(T(t)_|)_{t\geqslant 0}$ of all restrictions $T(t)_| := T(t)_{|F}$ is strongly continuous on F. If $(A, D(A))$ denotes the generator of $(T(t))_{t\geqslant 0}$ it follows from the $(T(t))$-invariance and closedness of F that A maps $D(A) \cap F$ into F. Therefore

$$A_| := A_{|(D(A)\cap F)} \text{ with domain } D(A_|) := D(A) \cap F$$

is the generator of $(T(t)_|)$. Conversely, if F is a closed *linear subspace* of E with $A(D(A) \cap F) \subset F$ such that $A_|$ is a generator on F, then F is $(T(t))$-invariant.

An A-invariant subspace need not necessarily be $(T(t))$-invariant: Take for example the translation group with $T(t)f(x) = f(x + t)$ on $E = C_0(\mathbb{R})$ and $F := \{f \in E : f(x) = 0 \text{ for } x \leqslant 0\}$.

3.4 The Quotient Semigroup

Let F be a closed $(T(t))$-invariant subspace of E and consider the quotient space $E_/ := {}^{E}\!/_{F}$ with quotient map $q : E \to E_/$. The quotient operators

$$T(t)_/ q(f) := q(T(t)f), \quad f \in E,$$

are well defined and form a strongly continuous semigroup $(T(t)_/)_{t\geqslant 0}$ on $E_/$. For the generator $(A_/, D(A_/))$ of $(T(t)_/)_{t\geqslant 0}$ the following holds:

$$D(A_/) = q(D(A)) \quad \text{and} \quad A_/ q(f) = q(Af)$$

for every $f \in D(A)$. Here we use the fact that every $\hat{f} := q(f) \in D(A_/)$ can be written as

$$\hat{f} = \int_0^\infty e^{-\lambda s}\hat{T}(s)_/\hat{g}\,\mathrm{d}s = \int_0^\infty e^{-\lambda s} q(T(s)g)\,\mathrm{d}s = q\big(\int_0^\infty e^{-\lambda s} T(s)g\,\mathrm{d}s\big) = q(h)$$

where $h \in D(A)$ and $\lambda > \omega$ (see Proposition–1.6). In particular we point out that for every $\hat{f} \in D(A/)$ there exist representatives $f \in \hat{f}$ belonging to $D(A)$.

Example 3.1 We start with the Banach space $E = L^1(\mathbb{R})$ and the translation semigroup $(T(t))_{t\geqslant 0}$ where $T(t)f(x) := f(x+t)$ (see Example 2.4). Then $L^1((-\infty,1])$ can be identified with the closed, $(T(t))$-invariant subspace

$$J := \{f \in E : f(x) = 0 \text{ for } 1 < x < \infty\}.$$

There we obtain the subspace semigroup

$$T(t)|_{(-\infty,1]}(x) \cdot f(x+t),$$

where $f \in L^1((-\infty,1])$, $-\infty < x \leqslant 1$ and $t \geqslant 0$.

By 2.4 and 3.2 its generator is

$$A| f := f'$$

for $f \in D(A|) := \{f \in E : f \in \mathrm{AC} \text{ with } f' \in E \text{ and } f(x) = 0 \text{ for } x \geqslant 1\}$.

Next we identify $L^1([0,1])$ with the quotient space $L^1((-\infty,1])/I$ where

$$I := \{f \in L^1((-\infty,1]) : f(x) = 0 \text{ for } 0 \leqslant x \leqslant 1\}.$$

Again I is invariant for the restricted semigroup $(T(t)_|)$ and the quotient semigroup $(T(t)|/)$ on $L^1([0,1])$ is the nilpotent translation semigroup as in Example 2.6. In particular it follows that the domain of its generator is

$$D(A_{|/}) = \{f \in L^1([0,1]) : f \in \mathrm{AC} \text{ with } f' \in L^1([0,1]) \text{ and } f(1) = 0\}.$$

3.5 The Adjoint Semigroup

The adjoint operators $(T(t)')_{t\geqslant 0}$ of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E form a semigroup on E' which need, however, not be strongly continuous.

Example 3.2 Take the translation operators $T(t)f(x) = f(x+t)$ on $E = L^1(\mathbb{R})$ (see Example 2.4) and their adjoints

$$T(t)'f(x) = f(x-t)$$

on $E' = L^\infty(\mathbb{R})$. Then $(T(t)')_{t\in\mathbb{R}}$ is a one-parameter group which is not strongly continuous on $L^\infty(\mathbb{R})$ (take any non-trivial characteristic function).

Since the semigroup $(T(t)')_{t\geqslant 0}$ is obviously *weak*-continuous* in the sense that $\lim_{t\to s}\langle f, (T(t)' - T(s)')\varphi\rangle = 0$ for every $f \in E$, $\varphi \in E'$ and $s, t \geqslant 0$, it is natural to associate $(T(t)')_{t\geqslant 0}$ its a *weak*-generator*

$$A'\varphi := \sigma(E', E)\text{-}\lim \frac{1}{h}(T(h)'\varphi - \varphi) \text{ for every } \varphi \text{ in the domain}$$

$$D(A') := \{\varphi \in E' : \sigma(E', E)\text{-}\lim \frac{1}{h}(T(h)'\varphi - \varphi) \text{ exists}\}.$$

This operator coincides with the *adjoint* of the generator $(A, D(A))$, i.e.,

$$D(A') = \{\varphi \in E' : \text{there exists } \psi \in E' \text{ such that } \langle f, \psi\rangle = \langle Af, \varphi\rangle \text{ for all } f \in D(A)\}$$

and $A'\varphi = \psi$. In particular, A' is a closed and $\sigma(E', E)$-densely defined operator in E'.

It follows from **?**, Thm.III.5.30 that the resolvent $R(\lambda, A')$ of A' is $R(\lambda, A)'$. In particular, the spectra $\sigma(A)$ and $\sigma(A')$ coincide.

However, it is still necessary in some situations to have strong continuity for the adjoint semigroup. In order to achieve this we restrict $T(t)'$ to an appropriate subspace of E'.

Definition 3.3 (**?**) The *semigroup dual* of the Banach space E with respect to the strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is

$$E^* := \{\varphi \in E' : \|\cdot\|\text{-}\lim_{t\to 0} T(t)'\varphi = \varphi\}.$$

The adjoint semigroup on E^* is given by the operators

$$T(t)^* := T(t)'|_{E^*}, \quad t \geqslant 0.$$

Since $(T(t)^*)_{t\geqslant 0}$ is strongly continuous on E^* we call its generator $(A^*, D(A^*))$ the *adjoint generator*.

The above definition makes sense since E^* is norm-closed in E' and $(T(t)')$-invariant. The main point is that E^* is still reasonably large. In fact, since $\int_0^t T(s)'\varphi \, \mathrm{d}s$, understood in the weak sense, is contained in E^* for every $\varphi \in E'$ and $t \geqslant 0$, it follows that

$$\sup\{\langle f, \varphi\rangle : \varphi \in E^*, \|\varphi\| \leqslant 1\} \leqslant \|f\| \leqslant M \cdot \sup\{\langle f, \varphi\rangle : \varphi \in E^*, \|\varphi\| \leqslant 1\}$$

where $M := \limsup_{t\to 0}\|T(t)\|$. In particular, E^* separates E, i.e., E^* is $\sigma(E', E)$-dense in E'. In addition the estimate of $\|\cdot\|$ given above yields

$$\|T(t)^*\| \leqslant \|T(t)\| \leqslant M\|T(t)^*\| \quad \text{for all } t \geqslant 0.$$

In the following proposition we describe the relation between A^* and A'.

Proposition 3.4 *For the adjoint generator A^* of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on E the following assertions hold.*

(i) E^* is the $\|\cdot\|$-closure of $D(A')$.
(ii) $D(A^*) = \{\varphi \in D(A') \colon A'\varphi \in E^*\}$.
(iii) A^* and A' coincide on $D(A^*)$.

Proof (i) Take $\varphi \in D(A')$ fixed. For every $f \in D(A)$ with $\|f\| \leqslant 1$ we define a continuously differentiable function

$$t \mapsto \xi_f(t) := \langle T(t)f, \varphi\rangle$$

on $[0, 1]$ with derivative $\xi'_f(t) = \langle T(t)Af, \varphi\rangle = \langle T(t)f, A'\varphi\rangle$.

Since $\{\xi'_f(t) \colon t \in [0, 1], f \in D(A), \|f\| \leqslant 1\}$ is bounded, it follows that the set

$$\{\xi_f \colon f \in D(A), \|f\| \leqslant 1\}$$

is equicontinuous at 0, i.e., for every $\varepsilon > 0$ there exists $0 < t_0 < 1$ such that

$$|\xi_f(s) - \xi_f(0)| = |\langle f, T(s)'\varphi - \varphi\rangle| < \varepsilon$$

for every $0 \leqslant s \leqslant t_0$ and $f \in D(A)$, $\|f\| \leqslant 1$. But this implies $\|T(s)'\varphi - \varphi\| < \varepsilon$ and hence $\varphi \in E^*$.

Conversely, take $\psi \in E^*$. Then $\frac{1}{t}\int_0^t T(s)'\psi \,\mathrm{d}s$, $t > 0$, belongs to $D(A')$ and norm converges to ψ as $t \to 0$, i.e., ψ belongs to the norm closure of $D(A')$.

(ii) and (iii): Since the weak* topology on E' is weaker than the norm topology, it follows that A' is an extension of A^*. Now take $\varphi \in D(A')$ such that $A'\varphi \in E^*$. As above define the functions ξ_f. The assumption on φ implies the set of all derivatives

$$\{\xi'_f \colon f \in D(A), \|f\| \leqslant 1\}$$

to be equicontinuous at $t = 0$. This means that for every $\varepsilon > 0$ there exists $0 < t_o < 1$ such that $|f'_f(0) - f'_f(s)| < \varepsilon$ for every $f \in D(A)$, $\|f\| \leqslant 1$ and $0 < s < t_o$. In particular,

$$\varepsilon > |f'_f(0) - \frac{1}{s}(\xi_f(s) - \xi_f(0))| = | < f, A'\varphi - \frac{1}{s}(T(s)'\varphi - \varphi) > |,$$

hence

$$\varepsilon > \|A'\varphi - \frac{1}{s}(T(s)'\varphi - \varphi)\|$$

for all $0 \leqslant s \leqslant t_o$. From this it follows that $\varphi \in D(A^*)$. □

On reflexive Banach spaces we have $A^* = A'$ by the above proposition. In other cases this construction is more interesting.

Example 3.5 (continued) The adjoints of the (left) translation $T(t)$ on $E = L^1(\mathbb{R})$ are the (right) translations $T(t)'$ on $E' = L^\infty(\mathbb{R})$. The largest subspace of $L^\infty(\mathbb{R})$ on which these translations form a strongly-continuos semigroup with respect to the sup-norm, is the space of all bounded uniformly continuous functions on $\mathbb{R}$, i.e., $E^* = C_{bu}(\mathbb{R})$.

Calculating $D(A')$ and $D(A^*)$ respectively, one obtains

$$D(A') = \{f \in L^\infty(\mathbb{R}) \colon f \in AC, f' \in L^\infty(\mathbb{R})\},$$
$$D(A^*) = \{f \in L^\infty(\mathbb{R}) \colon f \in C^1(\mathbb{R}), f' \in C_{bu}(\mathbb{R})\}.$$

Obviously, the function $x \mapsto |\sin x|$ belongs to $D(A')$, but not to $D(A^*)$.

3.6 The Associated Sobolev Semigroups

Since the generator A of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is closed, its domain $D(A)$ becomes a Banach space for the graph norm

$$\|f\|_1 := \|f\| + \|Af\|.$$

We denote this Banach space by E_1 and the continuous injection from E_1 into E by i_1. Since E_1 is invariant under $(T(t))_{t\geqslant 0}$, apply Prop. 1.6 (i), it makes sense to consider the semigroup $(T_1(t))_{t\geqslant 0}$ of all restrictions $T_1(t) := T(t)|_{E_1}$. The results of Prop. 1.6 imply that $T_1(t) \in \mathrm{L}E_1$ and $\|T_1(t)f - f\|_1 \to 0$ as $t \to 0$ for every $f \in E_1$. Thus $(T_1(t))_{t\geqslant 0}$ is a strongly continuous semigroup on E_1 and has a generator denoted by $(A_1, D(A_1))$.

Using 1.6 again we see that A_1 is the restriction of A to E_1 with maximal domain, i.e., $D(A_1) = \{f \in E_1 \colon Af \in E_1\} = D(A^2)$ and $A_1 f = Af$ for every $f \in D(A_1)$.

It is now possible to repeat this construction in order to obtain Banach spaces E_n and semigroups $(T_n(t))_{t\geqslant 0}$ with generators $(A_n, D(A_n))$ which are related as visualized in the following diagram.

$$
\begin{tikzcd}
E \arrow[r, "T(t)"] \arrow[d, "i_1"'] & E \arrow[d, "i_1"] \\
E_1 \arrow[r, "T_1(t)"] \arrow[ur, "A"] \arrow[d, "i_2"'] & E_1 = D(A) \arrow[d, "i_2"] \\
E_2 \arrow[r, "T_2(t)"] \arrow[ur, "A_1"] & E_2 = D(A_1) = D(A^2)
\end{tikzcd}
$$

For the translation semigroup on $L^p(\mathbb{R})$ (see 2.3) the above construction leads to the usual *Sobolev spaces*. Therefore we might call E_n the *n-th Sobolev space* and $(T_n(t))_{t\geqslant 0}$ the *n-th Sobolev semigroup* associated to E and $(T(t))_{t\geqslant 0}$.

Remark 3.6 For $\lambda \in \varrho(A)$ the operator $(\lambda - A)$ and the resolvent $R(\lambda, A)$ are isomorphisms from E_1 onto E, resp. from E onto E_1 (show that $\|\cdot\|_1$ and $\|\cdot\|_\lambda$ with $\|\cdot\|_\lambda := \|(\lambda - A)\cdot\|$ are equivalent). In addition, the following diagram commutes.

$$
\begin{tikzcd}
E \arrow[r, "T(t)"] \arrow[d, "\lambda - A"'] & E \arrow[d, "R(\lambda, A)"] \\
E_1 \arrow[r, "T_1(t)"] & E_1
\end{tikzcd}
$$

Therefore all Sobolev semigroups $(E_n, T_n(t))_{t\geqslant 0}$, $n \in \mathbb{N}$, are isomorphic.

Remark 3.7 For $\lambda \in \varrho(A)$ consider the norm

$$\|f\|_{-1} := \|R(\lambda, A)f\|$$

for every $f \in E$ and define E_{-1} as the completion of E for $\|\cdot\|_{-1}$.

Then $(T(t))_{t\geqslant 0}$ extends continuously to a strongly continuous semigroup $(T_{-1}(t))_{t\geqslant 0}$ on E_{-1} and the above diagram can be extended to the negative integers.

4 The $\mathcal{F}$-Product Semigroup

It is standard in functional analysis to consider a sequence of points in a certain space as a point in a new and larger space. In particular such a method can serve to convert an approximate eigenvector of a linear operator into an eigenvector. Occasionally we will need such a construction and refer to Section V.1 of **?**.

If we try to adapt this construction to strongly continuous semigroups we encounter the difficulty that the semigroup extended to the larger space will not remain strongly continuous. An idea already used in 3.4 will help to overcome this difficulty.

Let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a strongly continuous semigroup on the Banach space E. Denote by $m(E)$ the Banach space of all bounded E-valued sequences endowed with the norm

$$\|(f_n)_{n\in\mathbb{N}}\| := \sup\{\|f_n\| : n \in \mathbb{N}\}.$$

It is clear that every $T(t)$ extends canonically to a bounded linear operator

$$\hat{T}(t)(f_n) := (T(t)f_n)$$

on $m(E)$, but the semigroup $(\hat{T}(t))_{t\geqslant 0}$ is strongly continuous if and only if T has a bounded generator. Therefore we restrict our attention to the closed, $(\hat{T}(t))$-invariant subspace

$$m^{\mathcal{T}}(E) := \{(f_n) \in m(E) : \lim_{t\to 0} \|T(t)f_n - f_n\| = 0 \text{ uniformly for } n \in \mathbb{N}\}.$$

Then the restricted semigroup

$$\hat{T}(t)(f_n) = (T(t)f_n), \quad (f_n) \in m^{\mathcal{T}}(E)$$

is strongly continuous and we denote its generator by $(\hat{A}, D(\hat{A}))$.

The following lemma shows that $\hat{A}$ is obtained canonically from A.

Lemma 4.1 *For the generator $\hat{A}$ of $(\hat{T}(t))_{t\geqslant 0}$ on $m^{\mathcal{T}}(E)$ one has the following properties.*

(i) $D(\hat{A}) = \{(f_n) \in m^{\mathcal{T}}(E) : f_n \in D(A)$ *and* $(Af_n) \in m^{\mathcal{T}}(E)\}$,
(ii) $\hat{A}(f_n) = (Af_n)$ *for* $(f_n) \in D(\hat{A})$.

For the proof we refer to Lemma 1.4. of **?**.

Now let $\mathcal{F}$ be any filter on $\mathbb{N}$ finer than the Frechét filter (i.e., the filter of sets with finite complement. In most cases F will be either the Frechét filter or some free ultra filter.) The space of all $\mathcal{F}$-null sequences in $m(E)$, i.e.,

$$c_{\mathcal{F}}(E) := \{(f_n) \in m(E) : \mathcal{F}\text{-}\lim \|f_n\| = 0\}$$

is closed in $m(E)$ and invariant under $(\hat{T}(t))_{t\geqslant 0}$. We call the quotient spaces

$$E_{\mathcal{F}} := m(E)/c_{\mathcal{F}}(E) \quad \text{and} \quad E^{T}_{\mathcal{F}} := m^{\mathcal{T}}(E)/c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E)$$

the *$\mathcal{F}$-product of E* and the *$\mathcal{F}$-product of E with respect to the semigroup T*, respectively.

Thus $E_{\mathcal{F}}^T$ can be considered as a closed linear subspace of $E_{\mathcal{F}}$. We have $E_{\mathcal{F}}^T = E_{\mathcal{F}}$ if (and only if) T has a bounded generator.

The canonical quotient norm on $E_{\mathcal{F}}$ is given by

$$\|(f_n) + c_{\mathcal{F}}(E)\| = \mathcal{F}\text{-}\limsup \|f_n\|.$$

We can apply Subsec. 3.4 in order to define the *$\mathcal{F}$-product semigroup* $(T_{\mathcal{F}}(t))_{t\geqslant 0}$ on $E_{\mathcal{F}}^T$ by

$$T_{\mathcal{F}}(t)((f_n) + c_{\mathcal{F}}(E)) := (T(t)f_n) + c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E)$$

Thus $T_{\mathcal{F}}(t)$ is the restriction of $T(t)_F$ where $T(t)_F$ denotes the canonical extension of $T(t)$ to the $\mathcal{F}$-product $E_{\mathcal{F}}$. But note that $(T(t)_F)_{t\geqslant 0}$ is not strongly continuous unless T has a bounded generator.

With the canonical injection $j\colon f \mapsto (f, f, f, \ldots) + c_{\mathcal{F}}(E)$ from E into $E_{\mathcal{F}}^T$ the operators $T_{\mathcal{F}}(t)$ are extensions of $T(t)$ satisfying $\|T_{\mathcal{F}}(t)\| = \|T(t)\|$. The basic facts about the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of $(T_{\mathcal{F}}(t))_{t\geqslant 0}$ follow from 3.3 and are collected in the following proposition.

Proposition 4.2 *For the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of the $\mathcal{F}$-product semigroup the following holds.*

(i) $D(A_{\mathcal{F}}) = \{(f_n) + c_{\mathcal{F}}(E)\colon f_n \in D(A); (f_n), (Af_n) \in m^{\mathcal{T}}(E)\}$,
(ii) $A_{\mathcal{F}}((f_n) + c_{\mathcal{F}}(E)) = (Af_n) + c_{\mathcal{F}}(E)$.

In case A is a bounded operator then $D(A_{\mathcal{F}}) = E_{\mathcal{F}}^T = E_{\mathcal{F}}$ and $A_{\mathcal{F}}$ is the canonical extension of A to $E_{\mathcal{F}}$.

We will show in A-III,4.5 that the above construction preserves and even improves many spectral properties of the semigroup and its generator.

5 The Tensor Product Semigroup

Real- or complex-valued functions of two variables x, y are often limits of functions of the form $\sum_{i=1}^n f_i(x)g_i(y)$ which, to some extent, allows one to consider the variables x and y separately. Since algebraic manipulation with these latter functions is governed by the formal rules of a tensor product, it is customary to identify (for example) the function

$$(x, y) \mapsto f(x)g(y)$$

with the tensor product $f \otimes g$ and to consider limits of linear combinations of such functions as elements of a completed tensor product.

To be more precise, we briefly present the most important examples for this situation.

Examples 5.1 (i) Let (X, Σ, μ) and (Y, Ω, ν) be measure spaces. If we identify for $f_i \in L^p(\mu)$, $g_i \in L^p(\nu)$ the elements $\sum_{i=1}^n f_i \otimes g_i$ of the tensor product

$$L^p(\mu) \otimes L^p(\nu)$$

with the (class of $\mu \times \nu$-a.e.-defined) functions

$$(x, y) \mapsto \sum_{i=1}^{n} f_i(x) g_i(y),$$

then $L^p(\mu) \otimes L^p(\nu)$ becomes a dense subspace of $L^p(X \times Y, \Sigma \times \Omega, \mu \times \nu)$ for $1 \leqslant p < \infty$.
(ii) Similarly, let X,Y be compact spaces. Then $C(X) \otimes C(Y)$ becomes a dense subspace of $C(X \times Y)$ by identifying, for $f \in C(X)$ and $g \in C(Y)$, $f \otimes g$ with the function

$$(x, y) \mapsto f(x) g(y).$$

We do not intend to go deeper into the quite sophisticated problems related to normed tensor products of general Banach spaces, but will rather confine ourselves to the discussion of certain special cases. These will always be related to one of the following standard methods to define a norm on the tensor product of two Banach spaces E, F.

Let $u := \sum_{i=1}^{n} f_i \otimes g_i$ be an element of $E \otimes F$. Then

(i) $\|u\|_\pi := \inf\{\sum_{j=1}^{m} \|h_j\| \|k_j\| : u = \sum_{j=1}^{m} h_j \otimes k_j, h_j \in E, k_j \in F\}$ defines the *greatest cross norm* π on $E \otimes F$.
(ii) $\|u\|_\varepsilon := \sup\{\langle u, \varphi \otimes \psi\rangle : \varphi \in E', \psi \in F', \|\varphi\|, \|\psi\| \leqslant 1\}$ defines the *least cross norm* ε on $E \times F$. Here, $\langle u, \varphi \otimes \psi\rangle$ denotes the canonical bilinear form on $(E \otimes F) \times (E' \otimes F')$, i.e., $\langle \sum_{i=1}^{n} f_i \otimes g_i, \varphi \otimes \psi\rangle = \sum_{i=1}^{n} \langle f_i, \varphi\rangle \langle g_i, \psi\rangle$.
(iii) if E and F are Hilbert spaces, $\|u\|_h = (u|u)_h^{1/2}$, where the scalar product $(\cdot|\cdot)_h$ is defined as in (ii), defines the *Hilbert norm* h on $E \otimes F$.

In the following we write $E \otimes_\alpha F$ for the tensor product of E and F endowed—if applicable—with one of the norms π, ε, h just defined. In each case one has $\|f \otimes g\| = \|f\| \|g\|$ for $f \in E$, $g \in F$.

By $E \widetilde{\otimes}_\alpha F$ we mean the completion of $E \otimes_\alpha F$. Moreover we recall how examples (i) and (ii) above fit into this pattern

$$L^1(\mu \otimes \nu) = L^1(\mu) \widetilde{\otimes}_\pi L^1(\nu), \quad L^2(\mu \otimes \nu) = L^2(\mu) \widetilde{\otimes}_h L^2(\nu),$$

$$C(X \otimes Y) = C(X) \widetilde{\otimes}_\varepsilon C(Y).$$

Finally, we point out that for any $S \in \mathcal{L}(E)$, $T \in \mathrm{L}F$, the mapping

$$\sum_{i=1}^{n} f_i \otimes g_i \mapsto \sum_{i=1}^{n} S f_i \otimes T g_i$$

defined on $E \otimes F$ is linear and continuous on $E \otimes_\alpha F$, hence has a continuous extension to $E \widetilde{\otimes}_\alpha F$. This operator, as well as its continuous extension, will be denoted by $S \otimes T$ and satisfies $\|S \otimes T\| = \|S\| \|T\|$. The notation $A \otimes B$ will also be used in the obvious way if A and B are not necessarily bounded operators on E and F. We are now ready to consider semigroups induced on the tensor product.

Proposition 5.2 *Let $(S(t))_{t\geqslant 0}$ and $(T(t))_{t\geqslant 0}$ be strongly continuous semigroups on Banach spaces E, F, and let A, B be their generators. Then the family $(S(t)\otimes T(t))_{t\geqslant 0}$ is a strongly continuous semigroup on $E\widetilde{\otimes}_\alpha F$. The closure of $A\otimes \mathrm{Id}+\mathrm{Id}\otimes B$, defined on the core $D(A)\otimes D(B)$, is its generator.*

Proof It is immediately verified that $(S(t)\otimes T(t))_{t\geqslant 0}$ is in fact a semigroup of operators on $E\widetilde{\otimes}_\alpha F$. The strong continuity need only be verified at $t=0$ and on elements of the form $u=f\otimes g\in E\otimes F$.

This verification being straightforward, there remains to show that the generator of $(S(t)\otimes T(t))_{t\geqslant 0}$ is obtained as the closure of

$$(A\otimes \mathrm{Id}+\mathrm{Id}\otimes B, D(A)\otimes D(B)).$$

To this end, let $f\in D(A)$ and $g\in D(B)$. Then

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}(T(h)\otimes S(h)(f\otimes g)-f\otimes g)\\
&=\lim_{h\to 0}\frac{1}{h}(T(h)f\otimes (S(h)g-g)+(T(h)f-f)\otimes g)\\
&=(f\otimes Bg)+(Af\otimes g).
\end{aligned}$$

Since the elements of the form $f\otimes g$, $f\in D(A)$, $g\in D(B)$, generate the linear subspace $D(A)\otimes D(B)$ of $E\otimes_\alpha F$, this subspace belongs to the domain of the generator. Moreover, $D(A)\otimes D(B)$ is dense in $E\widetilde{\otimes}_\alpha F$ and invariant under $(S(t)\otimes T(t))_{t\geqslant 0}$, hence it is a core of $A\otimes \mathrm{Id}+\mathrm{Id}\otimes B$ by Prop. 1.9 (ii). □

6 The Product of Commuting Semigroups

Let $(S(t))_{t\geqslant 0}$ and $(T(t))_{t\geqslant 0}$ be semigroups with generators A and B, respectively on some Banach space E. It is not difficult to see that the following assertions are equivalent.

(a) $S(t)T(t)=S(t)T(t)$ for all $t\geqslant 0$.
(b) $R(\mu,A)R(\mu,B)=R(\mu,B)R(\mu,A)$ for some $\mu\in\varrho(A)\cap\varrho(B)$.
(c) $R(\mu,A)R(\mu,B)=R(\mu,B)R(\mu,A)$ for all $\mu\in\varrho(A)\cap\varrho(B)$.

In that case $U(t)=S(t)T(t)$ $(t\geqslant 0)$ defines a semigroup $(U(t))_{t\geqslant 0}$. Using Prop. 1.9 (ii) on p. 6 one easily shows that $D_0:=D(A)\cap D(B)$ is a core for its generator C and $Cf=Af+Bf$ for all $f\in D_0$.

Notes

For more complete information on semigroup theory we refer the reader to **?**, to the monographs by **?**, **?** and **?**, to the survey article by **?**, to the bibliography by **?** and to **?**.

Chapter A-II
Characterization of Semigroups on Banach Spaces

Chapter A-III
Spectral Theory

1 Introduction

In this chapter we start a systematic analysis of the spectrum of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ on a complex Banach space E. By the spectrum of the semigroup we understand the spectrum $\sigma(A)$ of the generator A of $\mathcal{T}$. In particular we are interested in precise relations between $\sigma(A)$ and $\sigma(T(t))$. The heuristic formula

$$T(t) = e^{tA}$$

serves as a leitmotiv and suggests relations of the form

$$\sigma(T(t)) = e^{t\sigma(A)} = \{e^{t\lambda} \colon \lambda \in \sigma(A)\},$$

called *spectral mapping theorem*. These—or similar—relations will be of great use in Chapter IV and enable us to determine the asymptotic behavior of the semigroup $\mathcal{T}$ by the spectrum of the generator.

As a motivation as well as a preliminary step we concentrate here on the spectral radius

$$r(T(t)) := \sup\{|\lambda| : \lambda \in \sigma(T(t))\}, \quad t \geqslant 0, \tag{1.1}$$

and show how it is related to the spectral bound

$$s(A) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\} \tag{1.2}$$

of the generator A and to the growth bound

$$\omega := \inf\{\omega \in \mathbb{R} : \|T(t)\| \leqslant M_\omega \cdot e^{\omega t} \text{ for all } t \geqslant 0 \text{ and suitable } M_\omega\} \tag{1.3}$$

of the semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$. (Recall that we sometimes write $\omega(\mathcal{T})$ or $\omega(A)$ instead of ω). The Examples 1.3 and 1.4 below illustrate the main difficulties to be encountered.

Proposition 1.1 *Let ω_0 be the growth bound of the strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$. Then*
$$r(T(t)) = e^{\omega_0 t} \tag{1.4}$$
for every $t \geqslant 0$.

Proof From A-I, (1.1) we know that

$$\omega(\mathcal{T}) = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\| \,,$$

Since the spectral radius of $T(t)$ is given as

$$r(T(t)) = \lim_{n\to\infty} \|T(nt)\|^{1/n} \,,$$

we obtain for $t > 0$

$$r(T(t)) = \lim_{n\to\infty} \exp(t(nt)^{-1} \log \|T(nt)\|) = e^{\omega_0 t} \,.$$

It was shown in A-I, Prop.1.11 that the spectral bound $s(A)$ is always dominated by the growth bound ω_0 and therefore $e^{s(A)t} \leqslant r(T(t))$. If the above mentioned spectral mapping theorem holds—as is the case for bounded generators (e.g., see Thm. VII.3.11 of Dunford and Schwartz (1958))—we obtain the equality

$$e^{s(A)t} = r(T(t)) = e^{\omega(\mathcal{T})t} \,,$$

hence $s(A) = \omega(\mathcal{T})$. Therefore the following corollary is a consequence of the definitions of $s(A)$ and $\omega(\mathcal{T})$.

Corollary 1.2 *Consider the semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ generated by some bounded linear operator $A \in LE$. If $\mathrm{Re}(\lambda) < 0$ for each $\lambda \in \sigma(A)$, then $\lim_{t\to\infty} \|T(t)\| = 0$.*

Through this corollary we have re-established a famous result of Liapunov which assures that the solutions of the linear Cauchy problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in \mathbb{C}^n \quad \text{and} \quad A = (a_{ij})_{n\times n}$$

are *stable*, i.e., they converge to zero as $t \to \infty$, if the real parts of all eigenvalues of the matrix A are smaller than zero.

For unbounded generators the situation is much more difficult and $s(A)$ may differ drastically from $\omega(\mathcal{T})$.

Example 1.3 (Banach function space, Greiner et al. (1981)) Consider the Banach space E of all complex valued continuous functions on $\mathbb{R}_+$ which vanish at infinity and are integrable for $e^x\,\mathrm{d}x$, i.e.,

$$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x\,\mathrm{d}x)$$

endowed with the norm

$$\|f\| := \|f\|_\infty + \|f\|_1 = \sup\{|f(x)| : x \in \mathbb{R}_+\} + \int_0^\infty |f(x)|e^x \, dx \, .$$

The translation semigroup

$$T(t)f(x) := f(x+t)$$

is strongly continuous on E and one shows as in A-I,2.4 that its generator is given by

$$Af = f', \quad D(A) = \{f \in E : f \in C^1(\mathbb{R}_+), f' \in E\} .$$

First we observe that $\|T(t)\| = 1$ for every $t \geqslant 0$, hence $\omega(\mathcal{T}) = 0$. Moreover it is clear that λ is an eigenvalue of A as soon as $\mathrm{Re}(\lambda) < -1$ (in fact: the function

$$x \mapsto e_\lambda(x) := e^{\lambda x}$$

belongs to $D(A)$ and is an eigenvector of A), hence $s(A) \geqslant -1$. For $f \in E$, $\mathrm{Re}(\lambda) > -1$,

$$\| \cdot \|_1\text{-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s) f \; ds$$

exists since $\|T(s)f\|_1 \leqslant e^{-s}\|f\|_1$, $s \geqslant 0$, and

$$\| \cdot \|_\infty\text{-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s) f \; ds$$

exists since $\int_0^\infty e^x |f(x)| \, dx < \infty$. Therefore $\int_0^\infty e^{-\lambda s} T(s) f \; ds$ exists in E for every $f \in E$, $\mathrm{Re}(\lambda) > -1$. As we observed in A-I,Prop.1.11, this implies $\lambda \in \varrho(A)$. Therefore $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a semigroup having $s(A) = -1$ but $\omega(\mathcal{T}) = 0$.

Example 1.4 (Hilbert space, Zabczyk (1975)) For every $n \in \mathbb{N}$ consider the n-dimensional Hilbert space $E_n := \mathbb{C}^n$ and operators $A_n \in L(E_n)$ defined by the matrices

$$A_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdot & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & 0 \end{pmatrix}_{n\times n} .$$

These matrices are nilpotent and therefore $\sigma(A_n) = \{0\}$. The elements

$$x_n := n^{-1/2}(1, \ldots, 1) \in E_n$$

satisfy the following properties:

(i) $\|x_n\| = 1$ for every $n \in \mathbb{N}$,
(ii) $\lim_{n\to\infty} \|A_n x_n - x_n\| = 0$,
(iii) $\lim_{n\to\infty} \| \exp(tA_n)x_n - e^t x_n\| = 0$.

Consider now the Hilbert space $E := \bigoplus_{n\in\mathbb{N}} E_n$ and the operator $A := (A_n + 2\pi i n)_{n\in\mathbb{N}}$ with maximal domain in E. Analogously we define a semigroup

$\mathcal{T} = (T(t))_{t \geqslant 0}$ by

$$T(t) := (e^{2\pi int} \exp(tA_n))_{n\in\mathbb{N}} \,.$$

Since $\| \exp(tA_n)\| \leqslant e^t$ for every $n \in \mathbb{N}$, $t \geqslant 0$, and since $t \mapsto T(t)x$ is continuous on each component E_n, it follows that $\mathcal{T}$ is strongly continuous. Its generator is the operator A as defined above.

For $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$, we have $\lim_{n\to\infty} \|R(\lambda - 2\pi in, A_n)\| = 0$, hence

$$(R(\lambda, A_n + 2\pi in))_{n\in\mathbb{N}} = (R(\lambda - 2\pi in, A_n))_{n\in\mathbb{N}}$$

is a bounded operator on E representing the resolvent $R(\lambda, A)$. Therefore we obtain $s(A) \leqslant 0$. On the other hand, each $2\pi in$ is an eigenvalue of A, hence $s(A) = 0$.

Take now $x_n \in E_n$ as above and consider the sequence $(x_n)_{n\in\mathbb{N}}$. From (iii) it follows that for $t > 0$ the number e^t is an approximate eigenvalue of $T(t)$ with approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ (see Def.2.1 below). Therefore $e^t \leqslant r(T(t)) \leqslant \|T(t)\|$ and hence $\omega(\mathcal{T}) \geqslant 1$. On the other hand, it is easy to see that $\|T(t)\| = e^t$, hence $\omega(\mathcal{T}) = 1$.

Finally if we take $S(t) := e^{-t/2}T(t)$ we obtain a semigroup $\mathcal{S}$ having spectral bound $-\frac{1}{2}$ but satisfying $\lim_{t\to\infty} \|S(t)\| = \infty$ in contrast with Cor. 1.2.

These examples show that neither the conclusion of Cor.1.2, i.e., "$s(A) < 0$ implies stability", nor the "spectral mapping theorem"

$$\sigma(T(t)) = \exp(t \cdot \sigma(A))$$

is valid for arbitrary strongly continuous semigroups. A careful analysis of the general situation will be given in Section 6 below, but we will first develop systematically the necessary spectral theoretic tools for unbounded operators.

2 The Fine Structure of the Spectrum

As usual, with a closed linear operator A with dense domain $D(A)$ in a Banach space E, we associate its spectrum $\sigma(A)$, its resolvent set $\varrho(A)$ and its resolvent

$$\lambda \mapsto R(\lambda, A) := (\lambda - A)^{-1}$$

which is a holomorphic map from $\varrho(A)$ into $L(E)$. In contrast to the finite dimensional situation, where a linear operator fails to be surjective if and only if it fails to be injective, we now have to distinguish different cases of *non-invertibility* of $\lambda - A$. This distinction gives rise to a subdivision of $\sigma(A)$ into different subsets. We point out that these subsets need not be disjoint. Our definitions are justified by the fact that for each of the following subsets of $\sigma(A)$ there exist canonical constructions converting the corresponding spectral values into eigenvalues (see Prop. 2.2.(ii) and Prop. 4.6 below).

Definition 2.1 For a closed, densely defined, linear operator A with domain $D(A)$ in the Banach space E denote by the

(i) *point spectrum* $P\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not injective.
(ii) *approximate point spectrum* $A\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not injective or $(A - \lambda)D(A)$ is not closed in E.
(iii) *residual spectrum* $R\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $(A - \lambda)D(A)$ is not dense in E.

From these definitions it follows that $\lambda \in P\sigma(A)$ if and only if there exists a non-zero *eigenvector* $f \in D(A)$ such that $Af = \lambda f$ i.e., λ is an *eigenvalue*. It follows from the Open Mapping Theorem that $\lambda \in A\sigma(A)$ if and only if λ is an *approximate eigenvalue*, i.e., there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset D(A)$, called an *approximate eigenvector*, such that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|Af_n - \lambda f_n\| = 0$.

Clearly we have $P\sigma(A) \subset A\sigma(A)$ and $\sigma(A) = A\sigma(A) \cup R\sigma(A)$ where the union need not be disjoint.

The following proposition is a first indication that the subdivision we made yields nice properties.

Proposition 2.2 *For a closed, densely defined, linear operator $(A, D(A))$ in a Banach space E the following holds.*

(i) The topological boundary $\partial\sigma(A)$ of $\sigma(A)$ is contained in $A\sigma(A)$.
(ii) $R\sigma(A) = P\sigma(A')$ for the adjoint operator A' on E'.

Proof (i) Take $\lambda_0 \in \partial\sigma(A)$ and $\lambda_n \in \varrho(A)$ such that $\lambda_n \to \lambda_0$.

Since $\|R(\lambda_n, A)\| \geqslant (\mathrm{dist}(x, \sigma(A)))^{-1}$ (see Prop. 2.5.(ii)), by the uniform boundedness principle we find $f \in E$ such that

$$\lim_{n\to\infty} \|R(\lambda_n, A)f\| = \infty\,.$$

Define $g_n \in D(A)$ by

$$g_n := \|R(\lambda_n, A)f\|^{-1} R(\lambda_n, A)f$$

and use the identity

$$(\lambda_0 - A)g_n = (\lambda_0 - \lambda_n)g_n + (\lambda_n - A)g_n$$

to show that $(g_n)_{n\in\mathbb{N}}$ is an approximate eigenvector corresponding to λ_0.

(ii) This is a simple consequence of the Hahn-Banach theorem. $\square$

In order to illuminate the above definitons we now return to the Standard Examples introduced in Section 2 of A-I and discuss the fine structure of the spectrum of these strongly continuous semigroups, i.e., of their generators and their semigroup operators.

Example 2.3 (The Spectrum of Multiplication Semigroups)

Take $E = C_0(X)$ for some locally compact space X and take a continuous function $q\colon X \mapsto \mathbb{C}$ whose real part is bounded above. As observed in A-I,2.3 the multiplication operator

$$M_q\colon f \mapsto q \cdot f$$

with maximal domain $D(M_q)$ generates the multiplication semigroup

$$T(t)f := e^{tq} \cdot f\,,\ f \in E$$

Since M_q is bounded if and only if q is bounded, we conclude that M_q is invertible (with bounded inverse $M_{1/q}$) if and only if

$$0 \notin \overline{\{q(x)\colon x \in X\}}\,.$$

Therefore we obtain

$$\sigma(M_q) = \overline{q(X)} = \overline{\{q(x)\colon x \in X\}}\,,$$

and

$$\sigma(T(t)) = \overline{\{\exp(tq(x))\colon x \in X\}}\,.$$

In particular the following *weak spectral mapping theorem* is valid

$$\sigma(T(t)) = \overline{\exp(t\sigma(M_q))}\,.$$

In addition, we observe that to each spectral value of A (resp. of $T(t)$) there exists an approximate eigenvector and hence

$$\sigma(A) = A\sigma(A) \text{ and } \sigma(T(t)) = A\sigma(T(t))\,.$$

Since each Dirac functional is an eigenvector for the adjoint multiplication operator, we obtain

$$q(X) \subset R\sigma(M_q) \text{ and } e^{tq(X)} \subset R\sigma(T(t))\,.$$

The eigenvalues of M_q can be characterized as follows.

$\lambda \in P\sigma(M_q)$ if and only if the set $\{x \in X\colon q(x) = \lambda\}$ has non empty interior (analogously for $P\sigma(T(t))$).

For example, it follows that $P\sigma(M_q) = \emptyset$ for $E = C_0(\mathbb{R}_+)$ and $q(x) = -x$, $x \in \mathbb{R}_+$.

On $E = L^p(X, \Sigma, \mu)$ analogous results are valid, but their exact formulation—using the notion *essential range*, see Goldstein (1985)—is left to the reader.

Example 2.4 (The Spectrum of Translation Semigroups) We consider the translation semigroup

$$T(t)f(x) := f(x+t)$$

on $E = C_0(\mathbb{R}_+)$ (or $L^p(\mathbb{R}_+)$, see A-I,2.4). Its generator A is the first derivative and for every $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) < 0$, the function $\varepsilon_\lambda\colon x \mapsto e^{\lambda x}$ belongs to $D(A)$ and satisfies

$$A\varepsilon_\lambda = \lambda\varepsilon_\lambda\,,$$

hence $\lambda \in P\sigma(A)$. Since $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a contraction semigroup it follows that $\sigma(A) = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant 0\}$ and $\mathrm{i}\mathbb{R} \subset A\sigma(A)$ (use Prop. 2.2.(i)) or show directly that $f_n(x) = e^{i\alpha x}e^{-x/n}$ defines an approximate eigenvector for $i\alpha$, $\alpha \in \mathbb{R}$). Using the same functions one obtains

$$P\sigma(T(t)) = \{e^{\lambda t}\colon \operatorname{Re}\lambda < 0\} = \{z \in \mathbb{C}\colon |z| < 1\},$$
$$\sigma(T(t)) = \{z \in \mathbb{C}\colon |z| \leqslant 1\} \text{ for every } t > 0\,.$$

In the case of the translation group on $E = C_0(\mathbb{R})$ one has that $\sigma(A) \subset \mathrm{i}\mathbb{R}$. As above one obtains approximate eigenvectors for every $\alpha \in \mathbb{R}$ from $f_n(x) = e^{i\alpha x}e^{-|x|/n}$, hence

$$\sigma(A) = A\sigma(A) = \mathrm{i}\mathbb{R}\,.$$

The generator A of the nilpotent translation semigroup A-I,2.6 has empty spectrum by A-I,Prop.1.11. The resolvent is given by

$$R(\lambda, A)f(x) = e^{\lambda x}\int_x^\infty e^{-\lambda s}f(s)\,\mathrm{d}s \quad (f \in L^p([0,\tau]), \lambda \in \mathbb{C})\,.$$

Finally, the generator of the periodic translation group from A-I,2.5 on $E = \{f \in C[0,1]\colon f(0) = f(1)\}$ has point spectrum

$$P\sigma(A) = 2\pi\mathrm{i}\mathbb{Z}$$

with eigenfunctions $\varepsilon_n(x) := \exp(2\pi\mathrm{i}nx)$. In Section 5 we show that $\sigma(A) = 2\pi\mathrm{i}\mathbb{Z}$.

We now return to the general theory and recall from Corollary 1.2 that it is very useful (e.g., for stability theory) to be able to convert spectral values of the generator A into spectral values of the semigroup operator $T(t)$ and vice versa. As shown in Examples 1.3 and 1.4 this is not possible in general. Therefore we tackle first a much easier *spectral mapping theorem*: the relation between $\sigma(A)$ and $\sigma(R(\lambda_0))$, where $R(\lambda_0) := R(\lambda_0, A)$ for some $\lambda_0 \in \varrho(A)$.

Proposition 2.5 *Let $(A, D(A))$ be a densely defined closed linear operator with non-empty resolvent set $\varrho(A)$. For each $\lambda_0 \in \varrho(A)$ the following assertions hold*

(i) $\sigma(R(\lambda_0)) \setminus \{0\} = (\lambda_0 - \sigma(A))^{-1}$, in particular, $r(R(\lambda_0)) = (dist(\lambda_0, \sigma(A)))^{-1}$.
(ii) Analogous statements hold for the point-, approximate point-, residual spectra of A and $R(\lambda_0, A)$.
(iii) The point α is isolated in $\sigma(A)$ if and only if $(\lambda_0 - \alpha)^{-1}$ is isolated in $\sigma(R(\lambda_0))$. In that case the residues (resp. the pole orders) in α and in $(\lambda_0 - \alpha)^{-1}$ coincide.

Proof (i) is well known. It can be found for example in Dunford and Schwartz (1958, VII.9.2).

(ii) We show that $\alpha \in A\sigma(A)$ if $(\lambda_0 - \alpha)^{-1} \in A\sigma(R(\lambda_0))$ and leave the proof of the remaining statements to the reader.

Take $(f_n)_{n\in\mathbb{N}} \subset E$ such that $\|f_n\| = 1$, $\|(\lambda_0 - \alpha)^{-1}f_n - R(\lambda_0, A)f_n\| \to 0$ and $\|R(\lambda_0, A)f_n\| \geqslant \frac{1}{2}|\lambda_0 - \alpha|^{-1}$. Define

$$g_n := \|R(\lambda_0, A) f_n\|^{-1} R(\lambda_0, A) f_n \in D(A)$$

and deduce from

$$\begin{aligned}(\alpha - A) g_n &= \|R(\lambda_0, A) f_n\|^{-1} \cdot [(\lambda_0 - A) - (\lambda_0 - \alpha)] R(\lambda_0, A) f_n \\ &= \|R(\lambda_0, A) f_n\|^{-1} \cdot (\lambda_0 - \alpha)[(\lambda_0 - \alpha)^{-1} - R(\lambda_0, A)] f_n\end{aligned}$$

that (g_n) is an approximate eigenvector of A to the eigenvalue α.

(iii) Take a circle Γ with center α and sufficiently small radius. Then the residue P of $R(\cdot, A)$ at α is

$$\begin{aligned}P &= \frac{1}{2\pi i} \int_\Gamma R(z, A)\, \mathrm{dz} \\ &= \frac{1}{2\pi i} \int_\Gamma (\lambda_0 - z)^{-2} R((\lambda_0 - z)^{-1}, R(\lambda_0, A))\, \mathrm{dz} \\ &- \frac{1}{2\pi i} \int_\Gamma (\lambda_0 - z)^{-1} \mathrm{dz}\, (\text{use } \$\$)\,.\end{aligned}$$

If λ_0 lies in the exterior of Γ, the second integral is zero. The substitution $\tilde{z} := (\lambda_0 - z)^{-1}$ yields a path $\tilde{\Gamma}$ around $(\lambda_0 - \alpha)^{-1}$ and we obtain

$$P = \frac{1}{2\pi i} \int_\Gamma R(\tilde{z}, R(\lambda_0, A))\, \mathrm{d}\tilde{z}$$

which is the residue of $R(\cdot, R(\lambda_0, A))$ at $(\lambda_0 - \alpha)^{-1}$. The final assertion on the pole order follows from the identities

$$V_{-n} = ((\lambda_0 - \alpha)^{-1} R(\lambda_0, A))^{n-1} U_{-n}, \quad n \in \mathbb{N}\,.$$

where U_n, resp. V_n stand for the n-th coefficient in the Laurent series of $R(\cdot, A)$, resp. $R(\cdot, R(\lambda_0, A))$ at α, resp. $(\lambda_0 - \alpha)^{-1}$. This has already been proved for $n = 1$ and follows for $n > 1$ by induction, using the relations

$$U_{-n-1} = (A - \alpha) U_{-n} \quad \text{and} \quad V_{-n-1} = (R(\lambda_0, A) - (\lambda_0 - \alpha)^{-1}) V_{-n}\,.$$

3 Spectral Decomposition

In the next two sections we develop some important techniques for our further investigation of semigroups and their generators. Even though these methods are well known (compare, e.g. Section VII.3 of Dunford and Schwartz (1958)) or rather technical, it is useful to present them in a coherent way.

Our interest in this section is the following: Let E be a Banach space and $\mathcal{T} = (T(t))_{t \geqslant 0}$ a strongly continuous semigroup with generator A. Suppose that the spectrum $\sigma(A)$ splits into the disjoint union of two closed subsets σ_1 and σ_2. Does there exist a corresponding decomposition of the space E and the semigroup $\mathcal{T}$?

In the following definition we explain what we understand by "corresponding decomposition".

Definition 3.1 Assume that $\sigma(A)$ is the disjoint union

$$\sigma(A) = \sigma_1 \cup \sigma_2$$

of two non-empty closed subsets σ_1, σ_2. A decomposition

$$E = E_1 \oplus E_2$$

of E into the direct sum of two non-trivial closed $\mathcal{T}$-invariant subspaces is called a *spectral decomposition* corresponding to $\sigma_1 \cup \sigma_2$ if the spectrum $\sigma(A_i)$ of the generator A_i of $\mathcal{T}_i := (T(t)|_{E_i})_{t\geqslant 0}$ coincides with σ_i for $i = 1, 2$.

For a better understanding of the above definition we recall that to every direct sum decomposition $E = E_1 \oplus E_2$ there corresponds a continuous projection $P \in \mathcal{L}(E)$ such that $PE = E_1$ and $P^{-1}(0) = E_2$. Moreover, the subspaces E_1, E_2 are T-invariant if and only if P commutes with the semigroup $\mathcal{T}$, i.e., $T(t)P = PT(t)$ for every $t \geqslant 0$. In this case it follows that the domain $D(A)$ of the generator A splits analogously and $D(A) \cap E_i$ is the domain $D(A_i)$ of the generator A_i of the restricted semigroup T_i, $i = 1, 2$. We write

$$A = A_1 \oplus A_2 \,.$$

and say that "A commutes with P" and call P a *spectral projection*. In terms of the generator A this means that for $f \in D(A)$ we have $Pf \in D(A)$ and $APf = PAf$.

The existence of such projections reduces the semigroup $\mathcal{T}$ into two (possibly simpler) semigroups $\mathcal{T}_1, \mathcal{T}_2$ such that

$$\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \quad \text{and} \quad \sigma(T(t)) = \sigma(T_1(t)) \cup \sigma(T_2(t)) \,.$$

For example, in some cases (see Theorem 3.3 below) it can be shown that one of the reduced semigroups has additional properties.

In order to achieve such decompositions we will assume that $\sigma(A)$ decomposes into sets σ_1 and σ_2 and will then try to find a corresponding spectral projection. Unfortunately such spectral decompositions do not exist in general.

Example 3.2 Take the rotation semigroup from A-I,2.4 on the Banach space $L^p(\Gamma)$, $1 \leqslant p < \infty$, $\tau = 2\pi$. It was stated in 2.4 and will be proved in Section 5 that its generator A has spectrum

$$\sigma(A) = P\sigma(A) = \mathrm{i}\mathbb{Z}$$

where $\varepsilon_k(z) := z^k$ spans the eigenspace corresponding to $\mathrm{i}k$, $k \in \mathbb{Z}$.

Now, $\sigma(A)$ is the disjoint union of $\sigma_1 := \{0, \mathrm{i}, 2\mathrm{i}, \ ldots\}$ and $\sigma_2 := \{-\mathrm{i}, -2\mathrm{i}, \ldots\}$. By a result of M. Riesz there is no projection $P \in \mathrm{L}L^1(\Gamma)$ satisfying $P\varepsilon_k = \varepsilon_k$ for

$k \geqslant 0$, $P\varepsilon_k = 0$ for $k < 0$, hence there is no spectral decomposition of $L^1(\Gamma)$ corresponding to σ_1, σ_2 (Lindenstrauss and Tzafriri (1979, p.165)).

On the other hand, for $L^p(\Gamma)$, $1 < p < \infty$, such a spectral projection exists (l.c., 2.c.15). As long as $p \neq 2$ we can always decompose $\sigma(A)$ into suitable subsets admitting no spectral decomposition (l.c., remark before 2.c.15). Clearly, for $p = 2$ such spectral decompositions always exist.

In the above example both subsets σ_1, σ_2 of $\sigma(A)$ are unbounded. But as soon as one of these sets is bounded a corresponding spectral decomposition can always be found.

Theorem 3.3 *Let $\mathcal{T}$ be a strongly continuous semigroup on a Banach space E and assume that the spectrum $\sigma(A)$ of the generator A can be decomposed into the disjoint union of two non-empty closed subsets σ_1, σ_2.*

If σ_1 is compact, then there exists a unique corresponding spectral decomposition $E = E_1 \oplus E_2$ such that the restricted semigroup $\mathcal{T}_1$ has a bounded generator.

Proof We assume the reader to be familiar with the spectral decomposition theorem for bounded operators (see e.g. Dunford and Schwartz (1958, p.572)) and apply the "spectral mapping theorem" for the resolvent (A-III,Thm.2.5) in order to decompose $R(\lambda, A)$ instead of A.

For $\lambda_0 > \omega(\mathcal{T})$ it follows from A-III,Thm.2.5 that $\sigma(R(\lambda_0, A)) \setminus \{0\} = (\lambda_0 - \sigma(A))^{-1}$. From $\sigma(A) = \sigma_1 \cup \sigma_2$ we obtain a decomposition of $\sigma(R(\lambda_0, A)) \setminus \{0\}$ into

$$\tau_1 := (\lambda_0 - \sigma_1)^{-1}, \quad \tau_2 := (\lambda_0 - \sigma_2)^{-1} .$$

Since σ_1 is compact the set τ_1 is compact and does not contain 0. Only in the case that σ_2 is unbounded the point 0 will be an accumulation point of τ_2. Therefore $\sigma(R(\lambda_0, A)) \cup \{0\}$ is the disjoint union of the closed sets τ_1 and $\tau_2 \cup \{0\}$.

Take now P to be the spectral projection of $R(\lambda_0, A)$ corresponding to this decomposition. Then P commutes with $R(\lambda_0, A)$ (by definition), with $R(\lambda, A)$ for every $\lambda > \omega(\mathcal{T})$ (use the series representation of the resolvent), with $T(t)$ for each $t \geqslant 0$ (use A-II, Prop.1.10) and therefore with the generator A (in the sense explained above). In particular, we obtain

$$R(\lambda_0, A)P = R(\lambda_0, A_1), \quad R(\lambda_0, A)(Id - P) = R(\lambda_0, A_2)$$

for the generator A_1 of $T_1 = (T(t)P)_{t \geqslant 0}$ and A_2 of $T_2 = (T(t)(Id - P))_{t \geqslant 0}$. Applying the Spectral Mapping Theorem 2.5 we conclude

$$\sigma(A_1) = \sigma_1 \text{ and } \sigma(A_2) = \sigma_2$$

i.e., P is a spectral projection corresponding to σ_1, σ_2. Finally, the above spectral decomposition of $R(\lambda_0, A)$ is unique and satisfies $0 \notin \sigma(R(\lambda_0, A_1))$. Therefore $R(\lambda_0, A_1)^{-1} = (\lambda_0 - A_1)$ is bounded. □

If we do not require $\mathcal{T}_1$ to be uniformly continuous, the above spectral decomposition need not be unique, which can be seen from the following example.

Consider a decomposition $E = E_1 \oplus E_2$ and add a direct summand E_3 with a strongly continuous semigroup T_3 whose generator A_3 has empty spectrum (e.g., A-I,Example 2.6). Then still $\sigma(A) = \sigma_1 \cup \sigma_2$ but $E_1 \oplus (E_2 \oplus E_3)$ and $(E_1 \oplus E_3) \oplus E_2$ are two different spectral decompositions corresponding to σ_1, σ_2.

The importance of the above theorem stems from the fact that $\mathcal{T}_1$ has a bounded generator and therefore is easy to deal with. In particular the asymptotic behavior of $\mathcal{T}_1$ can be deduced from the location of σ_1.

Corollary 3.4 *Assume that $\sigma(A)$ splits into non-empty closed sets σ_1, σ_2 where σ_1 is compact and consider the corresponding spectral decomposition $E = E_1 \oplus E_2$ for which $\mathcal{T}_1$ is uniformly continuous.*

For all constants ν, $\omega \in \mathbb{R}$ satisfying

$$\nu < \inf\{\mathrm{Re}\lambda \colon \lambda \in \sigma_1\} \leqslant \sup\{\mathrm{Re}\lambda \colon \lambda \in \sigma_1\} < \omega$$

there exist m, $M \geqslant 1$ such that

$$m \cdot e^{\nu t}\|f\| \leqslant \|T_1(t)f\| \leqslant M \cdot e^{\omega t}\|f\|$$

for every $f \in E_1$, $t \geqslant 0$.

Proof Since the generator A_1 of $\mathcal{T}_1$ is bounded, we have $T_1(t) = \exp(tA_1)$ and $\sigma(T_1(t)) = \exp(t\sigma(A_1))$. Therefore by the remark following Prop.1.1 the spectral bound $s(A_1)$ coincides with the growth bound $\omega(T_1)$ and we have the upper estimate. The lower estimate is obtained by applying the same reasoning to $-A_1$ which generates the semigroup $(T_1(t)^{-1})_{t\geqslant 0}$ on E_1. □

It is clear from Examples 1.3 and 1.4 on page 31, that no norm estimates for $(T_2(t))_{t\geqslant 0}$ can be obtained from the location of σ_2. Only by adding appropriate hypotheses we will achieve spectral decompositions admitting norm estimates on both components (see A-III,6.6).

Another way of obtaining such norm estimates is by constructing spectral decompositions starting from a semigroup operator $T(t_0)$ (instead of A, and $R(\lambda, A)$ resp., as in Thm.3.3).

Corollary 3.5 *If $\sigma(T(t_0)) = \tau_1 \cup \tau_2$ for two non-empty, closed, disjoint sets τ_1, τ_2 and if P is the spectral projection corresponding to $T(t_0)$ and τ_1, τ_2, then $\sigma(A)$ splits into closed subsets σ_1, σ_2 and P is the corresponding spectral projection for $\mathcal{T}$ and σ_1, σ_2.*

Proof The spectral projection P of $T(t_0)$ is obtained by integrating $R(\lambda, T(t_0))$ (see e.g. Dunford and Schwartz (1958, Section VII.3)). Since every $T(t)$, $t \geqslant 0$, commutes with $T(t_0)$ it must commute with $R(\lambda, T(t_0))$, hence with P. The statement on the decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$ follows from the Spectral Inclusion Theorem 6.2 below. □

This decomposition can be applied to the study of the asymptotic behavior of $\mathcal{T}$: In the situation of Cor.3.5 assume

$$\sup\{|\lambda| : \lambda \in \tau_2\} < \alpha < \inf\{|\lambda| : \lambda \in \tau_1\}\,.$$

for some $\alpha > 0$. If we set $\beta := (\log \alpha)/t_0$ and use Pazy (1983, Chap.I,Thm.6.5) we obtain $\omega(\mathcal{T}_2) < \beta$ and $\omega(\mathcal{T}_1^{-1}) < \beta$ by Prop.1.1. Therefore we have constants m, M with $m \le 1 \le M$ such that

$$\begin{aligned}\|T(t)f\| &\leqslant M \cdot e^{\beta t}\|f\| \quad \text{for } f \in E_2,\\ \|T(t)f\| &\geqslant m \cdot e^{-\beta t}\|f\| \quad \text{for } f \in E_1\end{aligned}$$

As nice as they might look, results of this type are unsatisfactory. We need information on the semigroup in order to estimate its asymptotic behavior. In Chapter IV we will try to obtain such results by exploiting information about the generator only.

Example 3.6 **Isolated singularities and poles** In case that λ_0 is an isolated point of $\sigma(A)$ the holomorphic function $\lambda \mapsto R(\lambda, A)$ can be expanded as a Laurent series

$$R(\lambda, A) = \sum_{n=-\infty}^{+\infty} U_n(\lambda - \lambda_0)^n \text{ for } 0 < |\lambda - \lambda_0| < \delta \text{ and some } \delta > 0\,.$$

The coefficients U_n are bounded linear operators given by

$$U_n = \frac{1}{2\pi i}\int_\Gamma (z-\lambda_0)^{-(n+1)} R(z, A)\, \underset{\cdot}{z}\,,\ n \in \mathbb{Z}\,. \tag{3.1}$$

where $\Gamma = \{z \in \mathbb{C} : |z - \lambda_0| = \delta/2\}$. The coefficient U_{-1} is the spectral projection corresponding to the spectral set $\{\lambda_0\}$ (see Def.3.1) is called the *residue* of $R(\cdot, A)$ at λ_0, and will be denoted by P. From (3.1) one deduces

$$U_{-(n+1)} = (A - \lambda_0)^n \circ P \quad \text{and} \quad U_{-(n+1)} \circ U_{-(m+1)} = U_{-(n+m+1)} \text{ for } n, m \geqslant 0\,. \tag{3.2}$$

If there exists $k > 0$ such that $U_{-k} \neq 0$ while $U_{-n} = 0$ for all $n > k$ the point λ_0 is called a *pole of* $R(\cdot, A)$ *of order* k. In view of (3.2) this is true if $U_{-k} \neq 0$ and $U_{-(k+1)} = 0$. In this case one can retrieve U_{-k} as

$$U_{-k} = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^k R(\lambda, A)\,. \tag{3.3}$$

The dimension of PE (i.e., the dimension of the spectral subspace corresponding to $\{\lambda_0\}$) is called the *algebraic multiplicity* m_a of λ_0, while the *geometric multiplicity* is $m_g := \dim\ker(\lambda_0 - A)$. In case $m_a = 1$, we call λ_0 an *algebraically simple pole*.

If k is the pole order ($k = \infty$ in case of an essential singularity), we have

$$\max\{m_g, k\} \leqslant m_a \leqslant k \cdot m_g \tag{3.4}$$

where $\infty \cdot 0 = \infty$.

These inequalities yield the following implications:

(i) $m_a < \infty$ if and only if λ_0 is a pole with $m_g < \infty$;

(ii) if λ_0 is a pole with order k, then $\lambda_0 \in P\sigma(A)$ and $PE = \ker(\lambda_0 - A)^k$.

If A has compact resolvent then every point of $\sigma(A)$ is a pole of finite algebraic multiplicity. This is a consequence of Prop.2.5(iii) and the well-known Riesz-Schauder Theory for compact operators (see Dunford and Schwartz (1958, VII.4.5)).

Example 3.7 (The essential spectrum)

For an operator $T \in \mathcal{L}(E)$ the *Fredholm domain* $\varrho_F(T)$ is

$$\begin{aligned} \varrho_F(T) &:= \{\lambda \in \mathbb{C} \colon \lambda - T \text{ is a Fredholm operator}\} \\ &= \{\lambda \in \mathbb{C} \colon \ker(\lambda - T) \text{ and } E/\mathrm{im}(\lambda - T) \text{ are finite dimensional}\}. \end{aligned} \tag{3.5}$$

An equivalent characterization of $\varrho_F(T)$ is obtained through the Calkin algebra $\mathcal{L}(E)/\mathcal{K}(E)$, where $\mathcal{K}(E)$ stands for the closed ideal of all compact operators. In fact, $\varrho_F(T)$ coincides with the resolvent set of the canonical image of T in the Calkin algebra. The complement of $\varrho_F(T)$ is called *essential spectrum* of T and denoted by $\sigma_{\mathrm{ess}}(T)$. The corresponding spectral radius, called *essential spectral radius*, satisfies

$$r_{\mathrm{ess}}(T) := \sup\{|\lambda| \colon \lambda \in \sigma_{\mathrm{ess}}(T)\} = \lim_{n\to\infty} \|T^n\|_{\mathrm{ess}}^{1/n}, \tag{3.6}$$

where $\|T\|_{\mathrm{ess}} = \mathrm{dist}(T, K(E)) := \inf\{\|T - K\| \colon K \in K(E)\}$ is the norm of T in $L(E)/K(E)$.

For every compact operator K we have $\|T - K\|_{\mathrm{ess}} = \|T\|_{\mathrm{ess}}$, hence

$$r_{\mathrm{ess}}(T - K) = r_{\mathrm{ess}}(T)\,. \tag{3.7}$$

A detailed analysis of $\varrho_F(T)$ can be found in Section IV.5.6 of Kato (1966). In particular we recall that the poles of $R(\cdot, T)$ with finite algebraic multiplicity belong to $\varrho_F(T)$. Conversely, an element of the unbounded component of $\varrho_F(T)$ either belongs to $\varrho(T)$ or is a pole of finite algebraic multiplicity. Thus $r_{\mathrm{ess}}(T)$ can be characterized as

$$\begin{aligned} &r_{\mathrm{ess}}(T) \text{ is the smallest } r \in \mathbb{R}_+ \text{ such that every } \lambda \in \sigma(T), |\lambda| > r \\ &\qquad\qquad\qquad\qquad \text{is a pole of finite algebraic multiplicity.} \end{aligned} \tag{3.8}$$

Now, if $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup, then VIII.1, Lemma 4 of Dunford and Schwartz (1958) applied to the function $t \mapsto \log \|T(t)\|_{\mathrm{ess}}$ ensures that

$$\omega_{\mathrm{ess}}(\mathcal{T}) := \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|_{\mathrm{ess}} = \inf\{\frac{1}{t} \log \|T(t)\|_{\mathrm{ess}} \colon t > 0\} \tag{3.9}$$

is well defined (possibly $-\infty$). By the definition of $\omega_{\mathrm{ess}}(\mathcal{T})$ and by (3.6) we have

$$r_{\mathrm{ess}}(T(t)) = \exp(t\omega_{\mathrm{ess}}(\mathcal{T})), \quad t \geqslant 0\,. \tag{3.10}$$

Obviously, $\omega_{\mathrm{ess}} \leqslant \omega$ and equality occurs if and only if $r_{\mathrm{ess}}(T(t)) = r(T(t))$ for $t \geqslant 0$.

If $\omega_{\mathrm{ess}} < \omega$ there exists an eigenvalue λ of $T(t)$ satisfying $|\lambda| = r(T(t))$, hence by Theorem 6.3 below there exists $\lambda_1 \in P\sigma(A)$ such that $\mathrm{Re}\lambda_1 = \omega$. Thus $\omega_{\mathrm{ess}} < \omega$

implies $s(A) = \omega(\mathcal{T})$, i.e., we have

$$\omega(\mathcal{T}) = \max\{\omega_{\text{ess}}(T), s(A)\}\,. \tag{3.11}$$

As a final observation we point out that

$$\omega_{\text{ess}}(\mathcal{T}) = \omega_{\text{ess}}(\mathcal{S})\,, \tag{3.12}$$

whenever $\mathcal{T}$ is generated by A and $\mathcal{S}$ is generated by $A + K$ for some compact operator K (see Prop.2.8 and Prop.2.9 of B-IV).

4 The Spectrum of Induced Semigroups

In the previous section we tried to decompose a semigroup into the direct sum of two, hopefully simpler objects. Here we present other methods to reduce the complexity of a semigroup and its generator. Forming subspace or quotient semigroups as in A-I,3.2, A-I,3.3 are such methods. But also the constructions of new semigroups on canonically associated spaces such as the dual space, see A-I,3.4, or the $\mathcal{F}$-product, see A-I,3.6, might be helpful. We review these constructions under the spectral theoretical point of view and collect a number of technical properties for later use.

We start by studying the spectrum of subspace and quotient semigroups. To that purpose assume that the strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ leaves invariant some closed subspace N of the Banach space E. There are canonically induced semigroups $\mathcal{T}_|$ on N, resp. $\mathcal{T}_/$ on E/N and their generators $A_|$, resp. $A_/$ are canonically obtained from the generator A of $\mathcal{T}$ (see A-I, Section 3). The following example shows that the spectra of A, $A_|$ and $A_/$ may differ quite drastically.

Example 4.1 As in the example in A-I,3.3 we consider the translation semigroup on $E = L^1(\mathbb{R})$ and the invariant subspace $N := \{f \in E\colon f(x) = 0 \text{ for } x \geqslant 1\}$. Then $\sigma(A) = \mathrm{i}\mathbb{R}$ but $\sigma(A_|) = \{\lambda \in \mathbb{C}\colon \mathrm{Re}(\lambda) \leqslant 0\}$. Next we take the translation invariant subspace $M := \{f \in N\colon f(x) = 0 \text{ for } 0 \leqslant x \leqslant 1\}$ and obtain $\sigma(A_{|/}) = \emptyset$ for the generator $A_{|/}$ of the quotient semigroup $\mathcal{T}_{|/}$ (use the fact that $\mathcal{T}_{|/}$ is nilpotent).

In the next proposition we collect the information on $\sigma(A)$ which in general can be obtained from the "subspace spectrum" $\sigma(A_|)$ and the "quotient spectrum" $\sigma(A_/)$.

Proposition 4.2 *Using the standard notations the following inclusions hold*

(i) $\varrho(A) \subset [\varrho(A_|) \cap \varrho(A_/)] \cup [\sigma(A_|) \cap \sigma(A_/)]\,,$
(ii) $[\varrho(A_|) \cap \varrho(A_/)] \subset \varrho(A)\,,$
(iii) $\varrho_+(A) \subset [\varrho(A_|) \cap \varrho(A_/)]\,,$

where $\varrho_+(A)$ *denotes the connected component of* $\varrho(A)$ *which is unbounded to the right.*

Proof (i) Assume $\lambda \in \varrho(A)$, i.e., $(\lambda - A)$ is a bijection from $D(A)$ onto E. Since N is T-invariant, we have $D(A_|) = D(A) \cap N$ and $(\lambda - A)D(A_|) \subset N$. If $(\lambda - A)D(A_|) = N$, then $R(\lambda, A)N = D(A_|)$ and the induced operators $R(\lambda, A)_|$, resp. $R(\lambda, A)_/$ are the inverses of $(\lambda - A_|)$, resp. $(\lambda - A_/)$. If $(\lambda - A)D(A_|) \neq N$, then $\lambda \in \sigma(A_|)$.

In addition there exists $f \in D(A)\backslash N$ such that $g := (\lambda - A)f \in N$. Hence for $\hat{f} := f + N$, $\hat{g} := g + N \in E_/$ it follows that $(\lambda - A_/)\hat{f} = \hat{g} = 0$, i.e., $\lambda \in \sigma(A_/)$

(ii) Take $\lambda \in \varrho(A_|) \cap \varrho(A_/)$. Then $(\lambda - A)$ is injective, since $(\lambda - A)f = 0$ implies $(\lambda - A_/)\hat{f} = 0$, hence $\hat{f} = 0$, i.e., $f \in N$ and therefore $f = 0$. In addition, $(\lambda - A)$ is surjective: For $g \in E$ there exists $\hat{f} \in E_/$ such that $(\lambda - A_/)\hat{f} = \hat{g}$, i.e., there exists $h \in N$ such that $(\lambda - A)f - g = h = (\lambda - A)k$ for some $k \in D(A_|)$. Therefore we obtain $(\lambda - A)(f - k) = g$.

(iii) The integral representation of the resolvent for $\lambda > \omega(\mathcal{T})$ (see A-I, Prop.1.11) shows that $R(\lambda, A)N \subset N$. By the power series expansion for holomorphic functions this extends to all $\lambda \in \varrho_+(A)$. Therefore the restriction $R(\lambda, A)_|$ coincides with the resolvent $R(\lambda, A_|)$. On the other hand $R(\lambda, A)_/$ is well defined on $E_/$ and satisfies

$$R(\lambda, A)_/(f + N) = R(\lambda, A)f + N$$

(use again the integral representation). This proves that $R(\lambda, A)_/ = R(\lambda, A_/)$. □

Corollary 4.3 *Under the above assumptions take a point μ in the closure of $\varrho_+(A)$. Then*

(i) $\mu \in \sigma(A)$ if and only if $\mu \in \sigma(A_|)$ or $\mu \in \sigma(A_/)$.
(ii) μ is a pole of $R(\cdot, A)$ if and only if μ is a pole of $R(\cdot, A_|)$ and of $R(\cdot, A_/)$.
In that case,

$$\max\{k_|, k_/\} \leqslant k \leqslant k_| + k_/$$

for the respective pole orders. Note that hereby pole orders 0 are allowed.

Proof (i) follows from Prop.4.2, inclusions (ii) and (iii)

(ii) By the previous assertion we may assume that for some $\delta > 0$ the pointed disc

$$\{\lambda \in \mathbb{C} : 0 < |\lambda - \mu| < \delta\}$$

is contained in $\varrho(A) \cap \varrho(A_|) \cap \varrho(A_/)$.

Call U_n the coefficients of the Laurent expansion of $R(\cdot, A)$. Since N is $R(\lambda, A)$-invariant for $\lambda \in \varrho_+(A)$, the same holds for each U_n. With the obvious notations we have $R(\lambda, A) = \sum_n U_n(\lambda - \mu)^n$, $R(\lambda, A)_| = \sum U_{n|}(\lambda - \mu)^n$ and $R(\lambda, A)_/ = \sum U_{n/}(\lambda - \mu)^n$ which shows $\max\{k_|, k_/\} \leqslant k$.

If $R(\cdot, A)_|$ has a pole in μ of order ℓ, then $U_{-(\ell+1)|} = 0$, i.e., $U_{-(\ell+1)}N = \{0\}$.

Similarly it follows that $U_{-(m+1)}E \subset N$ if $R(\cdot, A)_/$ has a pole in μ of order m.

Therefore $U_{-(\ell+1)} \circ U_{-(m+1)} = 0$.

The relations (3.2) imply $U_{-(m+\ell+1)} = 0$, hence the pole order of $R(\cdot, A)$ is dominated by $\ell + m$. □

Example 4.4 (Spectrum of the adjoint semigroups) We recall from A-I,3.4 that to every strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ there corresponds a strongly continuous adjoint semigroup $\mathcal{T}^* = (T(t)^*)_{t\geqslant 0}$ on the semigroup dual

$$E^* = \{\varphi \in E' : \lim_{t\to\infty} \|T(t)'\varphi - \varphi\| = 0\} .$$

Its generator A^* is the maximal restriction of the adjoint A' to E^*. For these operators the spectra coincide, or more precisely

(i) $\sigma(T(t)) = \sigma(T(t)') = \sigma(T(t)^*)$,
$R\sigma(T(t)) = P\sigma(T(t)') = P\sigma(T(t)^*)$,
(ii) $\sigma(A) = \sigma(A') = \sigma(A^*)$, $R\sigma(A) = P\sigma(A') = P\sigma(A^*)$,
(iii) $s(A) = s(A^*)$, $\omega(A) = \omega(A^*)$.

Proof The left part of these equalities is either well known or has been stated in Prop.2.2(ii). The first statement of (iii) follows from (ii), while the second is an immediate consequence of the estimate $\|T(t)^*\| \leqslant \|T(t)\| \leqslant M \cdot \|T(t)^*\|$ given in A-I,3.4.

As a sample for the remaining assertions we show that $0 \notin \sigma(A)$ if and only if $0 \notin \sigma(A^*)$: If A and therefore A' is invertible, it follows from A-I,3.4 that A^* is a bijection from $D(A^*)$ onto E^*.

Conversely assume that A^* is invertible. Then A' must be injective by the Proposition in A-I,3.4. Moreover $A'(D(A'))$ contains $A^*(D(A^*)) = E^*$ and is $\sigma(E', E)$-dense in E'. By standard duality arguments it follows that A is injective with dense image. Next we show that $A(D(A))$ is closed: For $f \in D(A)$ choose $\varphi \in D(A')$ such that $\|\varphi\| \leqslant 1$ and $|\langle f, \varphi\rangle| \geqslant \frac{1}{2}\|f\|$. Then

$$\begin{aligned}\|(A^*)^{-1}\|\|Af\| &\geqslant \|(A^*)^{-1}\||\langle Af, \varphi\rangle| \geqslant |\langle Af, (A^*)^{-1}\varphi\rangle| \\ &= |\langle f, \varphi\rangle| \geqslant \frac{1}{2}\|f\| ,\end{aligned}$$

hence

$$\|Af\| \geqslant \frac{1}{2}\|(A^*)^{-1}\|^{-1}\|f\| ,$$

and $A(D(A))$ is closed since A is closed. □

Example 4.5 (Spectrum of the $\mathcal{F}$-product semigroup) As stated in A-I,3.6 the $\mathcal{F}$-product semigroup $\mathcal{T}_F = (T_F(t))_{t\geqslant 0}$ on $E^{\mathcal{T}}_{\mathcal{F}}$ of a strongly continuous semigroup $\mathcal{T}$ on E serves to convert sequences in E into points in $E^{\mathcal{T}}_{\mathcal{F}}$. In particular it can be used to convert approximate eigenvectors of the generator A into eigenvectors of $A_{\mathcal{F}}$.

Proposition 4.6 *Let A be the generator of a strongly continuous semigroup. Then the generator $A_{\mathcal{F}}$ of the $\mathcal{F}$-product semigroup satisfies:*

(i) $A\sigma(A) = A\sigma(A_{\mathcal{F}}) = P\sigma(A_{\mathcal{F}})$,
(ii) $\sigma(A) = \sigma(A_{\mathcal{F}})$.

Remark 4.7 In case A is bounded, then the canonical extension A_F is a generator and $E_{\mathcal{F}}^{\mathcal{T}} = E_{\mathcal{F}}$ (cf. A-I,3.6). Thus the proposition applies to bounded linear operators and their canonical extensions to the $\mathcal{F}$-product $E_{\mathcal{F}}$.

Proof (Proof of the proposition) (i) The inclusion $P\sigma(A_{\mathcal{F}}) \subset A\sigma(A_{\mathcal{F}})$ holds trivially.

We show that $A\sigma(A_{\mathcal{F}}) \subset A\sigma(A)$: Take $\lambda \in A\sigma(A_{\mathcal{F}})$ and an associated approximate eigenvector $(\hat{f}^m)_{n\in\mathbb{N}}$, i.e., $\hat{f}^m \in D(A_{\mathcal{F}})$, $\|\hat{f}^m\| = 1$ and $(\lambda - A_{\mathcal{F}})\hat{f}^m \to 0$ as $m \to \infty$.

By the considerations in A-I,3.6 we can represent each $\hat{f}^m$ as a normalized sequence $(f_n^m)_{n\in\mathbb{N}}$ in $D(A)$ such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|(\lambda - A)f_n^m\| = 0\,.$$

Therefore we can find a sequence $g_k = f_k^{m(k)}$ satisfying

$$\lim_{k\to\infty} \|(\lambda - A)g_k\| = 0\,,$$

i.e., $\lambda \in A\sigma(A)$.

Finally we show $A\sigma(A) \subset P\sigma(A_{\mathcal{F}})$: For $\lambda \in A\sigma(A)$ take a corresponding approximate eigenvector (f_n). By A-I,(3.2) we have

$$\begin{aligned}\|T(t)f_n - f_n\| &\leqslant \|T(t)f_n - e^{\lambda t}f_n\| + |e^{\lambda t} - 1| \\ &= \|\int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)f_n \, \mathrm{d}s\| + |e^{\lambda t} - 1|\end{aligned}$$

which converges to zero uniformly in n as $t \to 0$, i.e., $(f_n) \in m^{\mathcal{T}}(E)$. From the characterization of $D(A_{\mathcal{F}})$ given in A-I,3.6 it follows that

$$\hat{f} := (f_n) + c_F(E) \in D(A_{\mathcal{F}})$$

and $A_{\mathcal{F}}\hat{f} = \lambda\hat{f}$, i.e., $\lambda \in P\sigma(A_{\mathcal{F}})$.

(ii) The inclusion $A\sigma(A) \subset \sigma(A_{\mathcal{F}})$ follows from (i). Now we show $R\sigma(A) \subset R\sigma(A_{\mathcal{F}})$: For $\lambda \in R\sigma(A)$ choose $f \in E$ such that $\|(\lambda - A)g - f\| \geqslant 1$ for every $g \in D(A)$. Then $\|(\lambda - A_{\mathcal{F}})g - \hat{f}\| \geqslant 1$ for every $\hat{g} \in D(A_{\mathcal{F}})$ and $\hat{f} = (f, f, \ldots) + c_F(E)$. Therefore $\lambda \in R\sigma(A_{\mathcal{F}})$.

We now show $\varrho(A) \subset \varrho(A_{\mathcal{F}})$: Assume $\lambda \in \varrho(A)$. By (i) $(\lambda - A_{\mathcal{F}})$ has to be injective. Choose $\hat{f} = (f_1, f_2, \ldots) + c_F(E)$ such that $(f_n) \in m^{\mathcal{T}}(E)$. Then $(R(\lambda, A)f_n) \in m^{\mathcal{T}}(E)$ and $(\lambda - A_{\mathcal{F}})((R(\lambda, A)f_n) + c_F(E)) = (f_n) + c_F(E)$, i.e., $(\lambda - A_{\mathcal{F}})$ is surjective and $\lambda \in \varrho(A_{\mathcal{F}})$. □

Applying the proposition to a single operator $T(t)$ we obtain

$$A\sigma(T(t)) = P\sigma(T_F(t)).$$

Note that in general $A\sigma(T(t)) \neq P\sigma(T_F(t))$ (see the Examples 1.3 and 1.4 in combination with Theorem 6.3).

5 The Spectrum of Periodic Semigroups

In this section we determine the spectrum of a particularly simple class of strongly continuous semigroups and thereby achieve a rather complete description of the semigroup itself. Besides being nice and simple these semigroups gain their importance as building blocks for the general theory.

Definition 5.1 A strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ on a Banach space E is called *periodic* if $T(t_0) = \mathrm{Id}$ for some $t_0 > 0$.

The *period* τ of $\mathcal{T}$ is obtained as

$$\tau := \inf\{t_0 > 0 \colon T(t_0) = \mathrm{Id}\}\,.$$

We immediately observe that periodic semigroups are groups with inverses $T(t)^{-1} = T(n\tau - t)$ for $0 \leqslant t \leqslant n\tau$, τ the period of $\mathcal{T}$. Moreover, they are bounded, hence the growth bound is zero and $\sigma(A) \subset \mathrm{i}\mathbb{R}$.

Lemma 5.2 *Let T be a strongly continuous semigroup with period $\tau > 0$ and generator A. Then*

$$\sigma(A) \subset 2\pi i/\tau \cdot \mathbb{Z}$$

and

$$R(\mu, A) = (1 - e^{-\mu\tau})^{-1} \int_0^\tau e^{-\mu s} T(s)\, \mathrm{d}s \tag{5.1}$$

for $\mu \notin 2\pi i/\tau \cdot \mathbb{Z}$.

Proof From the basic identities A-I,(3.1) and A-I,(3.2) for $t = \tau$, it follows that $(\mu - A)$ has a left and right inverse if $\mu \neq 2\pi i n/\tau$, $n \in \mathbb{Z}$, and that the inverse is given by the above expression. □

The representation of $R(\mu, A)$ given in A-I,Prop.1.11 shows that the resolvent of the generator of a periodic semigroup is a meromorphic function having only poles of order one and the residues

$$P_n := \lim_{\mu \to \mu_n} (\mu - \mu_n) R(\mu, A) \quad \text{in} \quad \mu_n := 2\pi i n/\tau, \quad n \in \mathbb{Z}$$

, are

$$P_n = \frac{1}{\tau} \int_0^\tau \exp(-\mu_n s) T(s)\, \mathrm{d}s\,. \tag{5.2}$$

Moreover, it follows that the spectrum of A consists of eigenvalues only and each P_n is the spectral projection belonging to μ_n (see 3.6). Another way of looking at P_n is given by saying that P_n is the n-th Fourier coefficient of the τ-periodic function $s \mapsto T(s)$. From this it follows that no non-zero $\varphi \in E'$ vanishes on all $P_n E$

simultaneously. By the Hahn-Banach theorem we conclude that span $\cup_{n\in\mathbb{Z}} P_n E$ is dense in E.

Since $P_n E \subset D(A)$ we obtain from A-I,(3.1) that

$$AP_n f = \mu_n P_n f \tag{5.3}$$

for every $f \in E$, $n \in \mathbb{Z}$. This and A-I,(3.2) imply

$$T(t)P_n f = \exp(\mu_n t) \cdot P_n f \tag{5.4}$$

for every $t \geqslant 0$. Therefore μ_n is an eigenvalue of A and $\exp(\mu_n t)$ is an eigenvalue of $T(t)$ if and only if $P_n \neq 0$. In that case, $P_n E$ is the corresponding eigenspace and we have the following lemma.

Lemma 5.3 *For a τ-periodic semigroup $\mathcal{T}$ we take $\mu_n := 2\pi i n/\tau$, $n \in \mathbb{Z}$, and consider*

$$P_n := \frac{1}{\tau}\int_0^\tau \exp(-\mu_n s)T(s)\,\mathrm{d}s$$

For $n \in \mathbb{Z}$ the following assertions are equivalent.

(a) $P_n \neq 0$,
(b) $\mu_n \in P\sigma(A)$,
(c) $\exp(\mu_n t) \in P\sigma(T(t))$ *for every* $t > 0$.

The action of A, resp. $T(t)$ on the subspaces $P_n E$, $n \in \mathbb{Z}$, is determined by (5.3), resp. (5.4). Moreover,

$$\begin{aligned} P_m P_n f &= \frac{1}{\tau}\int_0^\tau \exp(-\mu_m s)T(s)P_n f\,\mathrm{d}s \\ &= \frac{1}{\tau}\int_0^\tau \exp((\mu_n - \mu_m)s)P_n f\,\mathrm{d}s = 0 \end{aligned}$$

for $n \neq m$, i.e., the subspaces $P_n E$ are "orthogonal". Since their union is total in E one expects to be able to extend the representations (5.3) and (5.4) of A and $T(t)$. This is possible if

$$\sum_{-\infty}^{+\infty} P_n = \mathrm{Id}\,,$$

where the series should be summable for the strong operator topology.

Unfortunately this is false in general since the family of projections

$$Q_H := \sum_{n\in H} P_n\,,$$

where H runs through all finite subsets of $\mathbb{Z}$, may be unbounded (see the example below). Nevertheless the following is true.

Theorem 5.4 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a τ-periodic semigroup on a Banach space E with generator A and associated spectral projections*

$$P_n := \frac{1}{\tau}\int_0^\tau \exp(-\mu_n s)T(s)\,\mathrm{d}s, \quad \mu_n := 2\pi in/\tau, \quad n \in \mathbb{Z}\,.$$

For every $f \in D(A)$ one has $f = \sum_{-\infty}^{+\infty} P_n f$ and therefore

(i) $T(t)f = \sum_{-\infty}^{+\infty} \exp(\mu_n t)P_n f$ if $f \in D(A)$,,
(ii) $Af = \sum_{-\infty}^{+\infty} \mu_n P_n f$ if $f \in D(A^2)$.

Proof It suffices to prove the first statement. Then (i) and (ii) follow by (5.3) and (5.4).

We assume $\tau = 2\pi$ and show first that $\sum_{-\infty}^{+\infty} P_n f$ is summable for $f \in D(A)$: For $g := Af$ we obtain $P_n g = P_n Af = AP_n f = inP_n f$. Take H to be a finite subset of $\mathbb{Z} \setminus \{0\}$ and $\varphi \in E'$. Then

$$\begin{aligned}\left|\sum_{n\in H}\langle P_n f, \varphi\rangle\right| &= \left|\sum_{n\in H}(in)^{-1}\langle P_n g, \varphi\rangle\right| \\ &\leqslant (\sum_{n\in H} n^{-2})^{1/2}\left(\sum_{n\in H}|\langle P_n g, \varphi\rangle|^2\right)^{1/2}.\end{aligned}$$

From Bessel's inequality we obtain for the second factor

$$\begin{aligned}\sum_{n\in H}|\langle P_n g, \varphi\rangle|^2 &\leqslant 1/2\pi \cdot \int_0^{2\pi} |\langle T(s)g, \varphi\rangle|^2\,\mathrm{d}s \\ &\leqslant \|\varphi\|^2 \cdot 1/2\pi \cdot \int_0^{2\pi} \|T(s)g\|^2\,\mathrm{d}s\,.\end{aligned}$$

With the constant $c := (1/2\pi \cdot \int_0^{2\pi} \|T(s)g\|^2\,\mathrm{d}s)^{1/2}$ we obtain

$$\|\sum_{n\in H} P_n f\| \leqslant c(\sum_{n\in H} n^{-2})^{1/2}$$

for every finite subset H of $\mathbb{Z}$, i.e., $\sum_{-\infty}^{+\infty} P_n f$ is summable.

Next we set $h := \sum_{-\infty}^{+\infty} P_n f$ and observe that for every $\varphi' \in E'$ the Fourier coefficients of the continuous, τ-periodic functions $s \mapsto \langle T(s)h, \varphi\rangle$ and $s \mapsto \langle T(s)f, \varphi\rangle$ coincide. Therefore these functions are identical for $s \geqslant 0$ and in particular for $s = 0$, i.e., $\langle h, \varphi\rangle = \langle f, \varphi\rangle$. By the Hahn-Banach Theorem we obtain $f = h$. □

The above theorem contains rather precise information on periodic semigroups. In particular, it characterizes periodic semigroups by the fact that $\sigma(A)$ is contained in $i\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$ and the eigenfunctions of A form a total subset of E.

For a periodic semigroup with bounded generator only a finite number of spectral projections P_n are distinct from 0 and we have the following characterization.

Corollary 5.5 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a semigroup with bounded generator on some Banach space E.*

This semigroup has period τ/k for some $k \in \mathbb{N}$ if and only if there exist finitely many pairwise orthogonal projections P_n, $-m \leqslant n \leqslant m$, $P_{-m} \neq 0$ or $P_m \neq 0$, such that

(i) $\sum_{-m}^{+m} P_n = \mathrm{Id}$,
(ii) $T(t) = \sum_{-m}^{+m} \exp(2\pi int/\tau)P_n$,
(iii) $A = \sum_{-m}^{+m}(2\pi in/\tau)P_n$.

Example 5.6 From A-I,2.5 we recall briefly the rotation group

$$R_\tau(t)f(z) := f(\exp(2\pi int/\tau) \cdot z)$$

on $E = C(\Gamma)$, resp. $E = L^p(\Gamma, m)$ for $1 \leqslant p < \infty$. The spectrum of the generator $Af(z) = (2\pi i/\tau)z \cdot f'(z)$ is $\sigma(A) = (2\pi i/\tau) \cdot \mathbb{Z}$. The eigenfunctions $\varepsilon_n(z) := z^n$ yield the projections

$$P_n = (1/2\pi i) \cdot \varepsilon_{-(n+1)} \otimes \varepsilon_n, \text{ i.e.,}$$

$$P_n f(z) = (1/2\pi i) \cdot (\int_\Gamma f(w)w^{-(n+1)}\, dw) \cdot z^n .$$

It is left as an exercise to compute the norms of $Q_m := \sum_{-m}^{+m} P_n$ in $L^p(\Gamma)$ for various p and then check the assertions of Theorem 5.4.

Clearly, this proves some classical convergence theorems for Fourier series (compare Davies (1980, Chap.8.1)).

6 Spectral Mapping Theorems

We now return to the question posed in the introduction to this chapter: In which form and under which conditions is it true, that the spectrum $\sigma(T(t))$ of the semigroup operators is obtained—via the exponential map—from the spectrum $\sigma(A)$ of the generator, or briefly

$$\text{Do we have } \sigma(T(t)) = \exp(t\sigma(A)) \text{ or at least } \sigma(T(t)) = \overline{\exp(t\sigma(A))}?$$

This and similar statements will be called *spectral mapping theorems* for the semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ and its generator A. In addition, we saw in Proposition 1.1 that the validity of such a spectral mapping theorem implies

$$s(A) = \omega(A)$$

for the spectral- and growth bounds and therefore guarantees that the location of the spectrum of A determines the asymptotic behavior of $\mathcal{T}$. As we have seen in Examples 1.3 and 1.4 the last statement does not hold in general. We therefore present a detailed analysis, where and why it fails and what additional assumptions are needed for its validity. Before doing so, we have another look at the examples.

Example 6.1 (The counterexamples revisited)

(i) Take the nilpotent translation semigroup from A-I,2.6. Then $\sigma(A) = \emptyset$ and $\sigma(T(t)) = 0$ for every $t > 0$. By this trivial example and since $e^z \neq 0$ for every $z \in \mathbb{C}$, it is natural to read the "spectral mapping theorem" modulo the addition of $\{0\}$, i.e.,

$$\sigma(T(t)) \cup \{0\} = \exp(t\sigma(A)) \cup \{0\} \text{ for } t \geqslant 0\,.$$

(ii) The spectrum of the generator A of the τ-periodic rotation group $(R_\tau(t))_{t\geqslant 0}$ on $C(\Gamma)$ is $\sigma(A) = 2\pi i/\tau \cdot \mathbb{Z}$ and $\exp(2\pi int/\tau)$, $n \in \mathbb{Z}$, is an eigenvalue of $R_\tau(t)$ for every $t \geqslant 0$ (see Example 5.6). If t/τ is irrational, these eigenvalues form a dense subset of Γ. Since the spectrum is closed, we obtain $\sigma(T(t)) = \Gamma$ for these t. Therefore in this example the spectral mapping theorem is valid only in the following "weak" form

$$\sigma(T(t)) = \overline{\exp(t\sigma(A))}, \quad t \geqslant 0\,.$$

(iii) By Example 1.3 there exists a semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ with generator A such that $s(A) = -1$ and $\omega(\mathcal{T}) = 0$. This implies that for preassigned real numbers $\alpha < \beta$ there exists a semigroup $\mathcal{S} = (S(t))_{t\geqslant 0}$ with generator B such that $s(B) = \alpha$ and $\omega(S) = \beta$. Indeed, take $S(t) = e^{\beta t}T((\beta - \alpha)t)$ and observe that $B = (\beta - \alpha)A + \beta\,\mathrm{Id}$ In that case $\exp(t\sigma(B))$ is contained in the circle about 0 with radius $e^{\alpha t}$ by Lemma 1.1. Hence there must be points in $\sigma(S(t))$ which are not in the closure of $\exp(t\sigma(B))$.

(iv) The Example 1.3 can be strengthend in order to yield a semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ with generator A such that $\sigma(A) = \emptyset$, but $\|T(t)\| = r(T(t)) = 1$ for $t \geqslant 0$, i.e., $s(A) = -\infty$, $\omega = 0$ and $s(A) < \omega$ **???RAINER???** take the translation semigroup on the Banach space

$$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^{x^2}\,\mathrm{d}x)$$

with $\|f\| := \sup\{|f(x)| : x \in \mathbb{R}_+\} + \int_0^\infty |f(x)|e^{x^2}\,\mathrm{d}x$ (see Greiner et al. (1981)).

(v) Another modification of Example 1.3 yields a group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ satisfying $s(A) < \omega$. Therefore the spectral mapping theorem does not hold (see Wolff (1981)).

The next few theorems form the core of this chapter. We show that only one part of the spectrum and one inclusion is responsible for the failure of the spectral mapping theorem. The usefulness of this detailed analysis will become clear in the subsequent chapter on stability and asymptotics.

Proposition 6.2 (Spectral Inclusion Theorem)

Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ on some Banach space E. Then

$$\exp(t\sigma(A)) \subset \sigma(T(t)) \textit{ for } t \geqslant 0\,.$$

More precisely we have the following inclusions

$$\exp(t \cdot P\sigma(A)) \subset P\sigma(T(t)), \tag{6.1}$$

$$\exp(t \cdot A\sigma(A)) \subset A\sigma(T(t)), \tag{6.2}$$

$$\exp(t \cdot R\sigma(A)) \subset R\sigma(T(t)). \tag{6.3}$$

Proof Since $e^{\lambda t} - T(t) = (\lambda - A) \int_0^t e^{\lambda(t-s)} T(s)\, \mathrm{d}s$ (see A-I,(3.1)) it follows that $(e^{\lambda t} - T(t))$ is not bijective if $(\lambda - A)$ fails to be bijective, which proves the main inclusion.

The inclusion (6.1) becomes evident from the following proof of (6.2): Take $\lambda \in A\sigma(A)$ and an associated approximate eigenvector $(f_n) \subset D(A)$. Then

$$g_n := e^{\lambda t} f_n - T(t) f_n = \int_0^t e^{\lambda(t-s)} T(s)(\lambda - A) f_n \, \mathrm{d}s$$

converges to zero as $n \to \infty$. Consequently, $e^{\lambda t} \in A\sigma(T(t))$ and in fact, the same approximate eigenvector (f_n) does the job for all $t \geqslant 0$.

For the proof of (6.3) we take $\lambda \in R\sigma(A)$ and observe that $(e^{\lambda t} - T(t)) f = (\lambda - A)(\int_0^t e^{\lambda(t-s)} T(s) f \, \mathrm{d}s) \in (\lambda - A) D(A)$ for every $f \in E$. □

As we know from the Examples 6.1, the converse inclusions do not hold in general, i.e., not every spectral value of a semigroup operator $T(t)$ comes - via the exponential map - from a spectral value of the generator. But at least this is true for some important parts of the spectrum.

Theorem 6.3 (Spectral Mapping Theorem for Point and Residual Spectrum) *Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$. Then*

$$\exp(t \cdot P\sigma(A)) = P\sigma(T(t)) \setminus \{0\}, \tag{6.4}$$

$$\exp(t \cdot R\sigma(A)) = R\sigma(T(t)) \setminus \{0\} \text{ for } t \geqslant 0. \tag{6.5}$$

Proof For the proof of (6.4) take $t_0 > 0$ and $0 \neq \lambda \in P\sigma(T(t_0))$.

After rescaling the semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ to the semigroup

$$(\exp(-t \cdot \log \lambda / t_0) T(t))_{t \geqslant 0},$$

we may assume $\lambda = 1$. Then the closed, $\mathcal{T}$-invariant subspace

$$G := \{g \in E : T(t_0) g = g\}$$

is non trivial. The restricted semigroup $T_|$ is periodic with period $\tau \leqslant t_0$ and the spectrum of its generator $A_|$ contains at least one eigenvalue $\mu = 2\pi i n / t_0$ for some $n \in \mathbb{Z}$ (see Lemma 5.3). Since every eigenvalue of $A_|$ is an eigenvalue of A, we obtain that $1 \in \exp(t_0 \cdot P\sigma(A))$. The converse inclusion has been proved in (6.1).

In fact, even more can be said: Let $g \in G$ be an eigenvector of $T(t_0)$ corresponding to the eigenvalue $\lambda = 1$. For each $n \in \mathbb{Z}$ define

$$g_n := P_n g = 1/t_0 \cdot \int_0^{t_0} \exp(-2\pi i n s / t_0) T(s) g \, \mathrm{d}s \in G$$

as in Section 5. If $g_n \neq 0$, then G_n is an eigenvector of $A_|$, hence of A with eigenvalue $2\pi i n/t_0$ as soon as g_n is distinct from zero. Since $D(A_|)$ is dense in G it follows from Theorem 5.4 that this holds for at least one $n \in \mathbb{Z}$. And from the proof of (6.1) we know that this g_n is in fact an eigenvector for each $T(t)$, $t \geqslant 0$.

Since $R\sigma(A) = P\sigma(A^*)$ and $R\sigma(T(t)) = P\sigma(T(t)^*)$ (see (4.4)) the assertion (6.5) follows from (6.4). □

Note that the proof is essentially an application of the structure theorem for periodic semigroups as given in Thm.5.4. The information gained there can be reformulated into statements on the eigenspaces of A and $T(t)$.

Corollary 6.4 *For the eigenspaces of the generator A, resp. of the semigroup operators $T(t)$, $t > 0$, the following holds for $\mu \in \mathbb{C}$*

(i) $\ker(\mu - A) = \bigcap_{s \geqslant 0} \ker(e^{\mu s} - T(s))$,
(ii) $\ker(e^{\mu t} - T(t)) = \overline{\operatorname{span}_{n\in\mathbb{Z}}\{\ker(\mu + 2\pi \mathrm{i} n/t - A)\}}$.

Remark that analogous statements are valid for $\ker(\mu - A')$ and $\ker(e^{\mu t} - T(t)')$ if we take in (ii) the $\sigma(E', E)$-closure.

Without proof (see Greiner (1981, Prop.I.10)) we add another corollary showing that poles of the resolvent of $T(t)$ correspond necessarily to poles of the resolvent of the generator. Again the converse is not true as shown by Example 5.6.

Corollary 6.5 *Assume that $e^{\mu t}$ is a pole of order k of $R(\cdot, T(t))$ with residue P and Q as the k-th coefficient of the Laurent series. Then*

(i) $\mu + 2\pi \mathrm{i} n/t$ is a pole of $R(\cdot, A)$ of order $\leqslant k$ for every $n \in \mathbb{Z}$,
(ii) the residues P_n in $\mu + 2\pi \mathrm{i} n/t$ yield $PE = \overline{\mathit{span}}_{n\in\mathbb{Z}}\{P_n\}E$,
(iii) the k-th coefficient of the Laurent series of $R(\cdot, A)$ at $\mu + 2\pi \mathrm{i} n/t$ is

$$Q_n = (t \cdot e^{\mu t})^{1-k} \cdot Q \circ (1/t) \int_0^t e^{-(\mu + 2\pi \mathrm{i} n/t)s} T(s)\, \mathrm{d}s \, .$$

From Theorem 6.2 and 6.3 it follows that the approximate point spectrum is the trouble maker in the sense that not every approximate eigenvalue of $T(t)$ corresponds to an approximate eigenvalue of the generator A. Since nothing more can be said in general we now look for additional hypotheses on the semigroup implying the spectral mapping theorem.

As a simple example we assume $T(t_0)$ to be compact for some $t_0 > 0$. Then $\sigma(T(t)) \setminus \{0\} = P\sigma(T(t)) \setminus \{0\}$ for $t \geqslant t_0$ and the spectral mapping theorem is valid by (6.4). A different class of semigroups verifying the spectral mapping theorem is given by the uniformly continuous semigroups (compare Cor.1.2).

Both cases, and many more, are included in the following result.

Theorem 6.6 (Spectral Mapping Theorem for Eventually Continuous Semigroups) *Let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be an eventually norm continuous semigroup with generator A. Then the spectral mapping theorem is valid, i.e.,*

$$\sigma(T(t)) \setminus \{0\} = e^{t \cdot \sigma(A)} \text{ for every } t \geqslant 0 \, . \tag{6.6}$$

Proof By the previous considerations it suffices to show that $A\sigma(T(t)) \setminus \{0\} \subset e^{t\cdot\sigma(A)}$ for $t > 0$. This will be done by converting approximate eigenvectors into eigenvectors in the semigroup $\mathcal{F}$-product (see 4.5). The assertion then follows from (6.4) and assertion (ii) of the Proposition in 4.5.

Assume $t \mapsto T(t)$ to be norm continuous for $t \geqslant t_0$. Moreover it suffices to consider $1 \in A\sigma(T(t_1))$ for some $t_1 > 0$, i.e., we have a normalized sequence $(f_n)_{n\in\mathbb{N}} \subset E$ such that

$$\lim_{n\to\infty} \|T(t_1)f_n - f_n\| = 0\,.$$

Choose $k \in \mathbb{N}$ such that $kt_1 > t_0$ and define $g_n := T(kt_1)f_n$. Then

$$\lim_{n\to\infty} \|g_n\| = \lim_{n\to\infty} \|T(t_1)^k f_n\| = \lim_{n\to\infty} \|f_n\| = 1$$

and

$$\lim_{n\to\infty} \|T(t_1)g_n - g_n\| = 0\,,$$

i.e., $(g_n)_{n\in\mathbb{N}}$ yields an approximate eigenvector of $T(t_1)$ with approximate eigenvalue 1. But the semigroup $\mathcal{T}$ is uniformly continuous on sets of the form $T(t_0)V$, V bounded in E. In particular, it is uniformly continuous on the sequence $(g_n)_{n\in\mathbb{N}}$, which therefore defines an element g in the semigroup $\mathcal{F}$-product $E_{\mathcal{F}}$.

Obviously, g is an eigenvector of $T_F(t_1)$ with eigenvalue 1 and by (6.4) we obtain an eigenvalue $2\pi i n/t_1$ of $A_{\mathcal{F}}$ for some $n \in \mathbb{Z}$. The coincidence of $\sigma(A)$ and $\sigma(A_{\mathcal{F}})$ proves the assertion. ◻

We point out that the above spectral mapping theorem implies the coincidence of spectral bound and growth bound for eventually norm continuous semigroups, hence we have generalized the Liapunov Stability Theorem 1.2 to a much larger class of semigroups. As mentioned before this will be of great use in many applications. Therefore we state explicitly the spectral mapping theorem for several important classes of semigroups all of which are eventually norm continuous (cf. the diagram preceding A-II,Ex.1.27).

Corollary 6.7 *The spectral mapping theorem*

$$\sigma(T(t)) \setminus \{0\} = e^{t\cdot\sigma(A)}, \quad t \geqslant 0\,,$$

holds for each of the following classes of strongly continuous semigroups

(i) eventually compact semigroups,
(ii) eventually differentiable semigroups,
(iii) holomorphic semigroups,
(iv) uniformly continuous semigroups.

Another application of the above spectral mapping theorem can be made to tensor product semigroups (see A-I,3.7). Let $\mathcal{T}_1 = (T_1(t))_{t\geqslant 0}$, $\mathcal{T}_2 = (T_2(t))_{t\geqslant 0}$ be strongly continuous semigroups on Banach spaces E_1, E_2 with generator A_1, A_2. The tensor product semigroup $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$ on some (appropriate) tensor product $E := E_1 \otimes E_2$ has the generator $A = A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$, but in general the spectrum of A is not determined by the spectra of A_1, A_2. But with an additional hypothesis the following can be proved.

Corollary 6.8 *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are eventually norm continuous, then*

$$\sigma(A) = \sigma(A_1) + \sigma(A_2)$$

where A is the generator of the tensor product semigroup

$$\mathcal{T}_1 \otimes \mathcal{T}_2 = (T_1(t) \otimes T_2(t))_{t \geqslant 0}\,.$$

Proof Clearly, the tensor product semigroup is eventually norm continuous and hence the Spectral Mapping Theorem 6.6 is valid for all three semigroups $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}$. Moreover the spectrum of the tensor product of bounded operators is the product of the spectra Reed and Simon (1978, XIII.9). Therefore

$$\sigma(T_1(t) \otimes T_2(t)) = \sigma(T_1(t)) \cdot \sigma(T_2(t)), \quad t \geqslant 0\,.$$

Consequently we have the following identity for every $t \geqslant 0$

$$\begin{aligned} e^{t\cdot\sigma(A)} &= \sigma(T_1(t) \otimes T_2(t)) \setminus \{0\} \\ &= \sigma(T_1(t)) \cdot \sigma(T_2(t)) \setminus \{0\} \\ &= e^{t\cdot\sigma(A_1)} \cdot e^{t\cdot\sigma(A_2)} \\ &= e^{t(\sigma(A_1)+\sigma(A_2))}\,. \end{aligned}$$

From this identity we want to deduce $\sigma(A) = \sigma(A_1) + \sigma(A_2)$.

"$\subset$" Take $\xi \in \sigma(A)$. Then for every $t > 0$ there exist $\mu_t \in \sigma(A_1)$, $\lambda_t \in \sigma(A_2)$ and $n_t \in \mathbb{Z}$ such that $\xi = \mu_t + \lambda_t + 2\pi \mathrm{i} n_t/t$.

Since the real parts of μ_t, λ_t are bounded above, they lie in some interval $[a, b]$. But $\sigma(A_i) \cap ([a, b] + \mathrm{i}\mathbb{R})$ is compact for $i = 1, 2$ since A_i is the generator of an eventually norm continuous semigroup (see A-II, Thm.1.20). By taking t sufficiently small, we conclude that $n_{t'} = 0$ for some $t' > 0$, i.e., $\xi = \mu_{t'} + \lambda_{t'}$.

"$\supset$" Choose $\mu \in \sigma(A_1)$, $\lambda \in \sigma(A_2)$. For every $t > 0$ there exist $\eta_t \in \sigma(A)$, $m_t \in \mathbb{Z}$ such that $\mu + \lambda = \eta_t + 2\pi \mathrm{i}_t/t$. Since $\mathrm{Re}\mu + \mathrm{Re}\lambda = \mathrm{Re}\eta_t$ and $\{\mathrm{Im}\eta_t : t > 0\}$ is bounded - $\mathcal{T} = (T_1(t) \otimes T_2(t))_{t \geqslant 0}$ is eventually norm continuous - it follows that $m_{t'} = 0$ for some $t' > 0$. □

7 Weak Spectral Mapping Theorems

In the previous section we showed under which hypotheses a spectral mapping theorem of the form

$$\sigma(T(t)) \setminus \{0\} = e^{t\cdot\sigma(A)}, \quad t \geqslant 0\,, \tag{7.1}$$

is valid for the generator A of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$.

Among the various examples showing that (7.1) does not hold in general we recall the following. Take the Banach space $E = c_0$, the multiplication operator $A(x_n)_{n\in\mathbb{N}} = (\mathrm{i}x_n)_{n\in\mathbb{N}}$ with maximal domain and the corresponding semigroup $T(t)(x_n)_{n\in\mathbb{N}} = (e^{\mathrm{i}t}x_n)_{n\in\mathbb{N}}$. Then $\sigma(A) = \{in\colon n \in \mathbb{N}\}$ and the spectral mapping theorem is valid only in the following weak form

$$\sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))}, \quad t \geqslant 0\,. \tag{7.2}$$

In this section we prove similar weak spectral mapping theorems. We start with a generalization of the above example.

Consider the Banach space $E = C_0(X, \mathbb{C}^n)$ of all continuous $\mathbb{C}^n$-valued functions vanishing at infinity on some locally compact space X. In analogy to A-I,2.3 we associate to every continuous function $q\colon X \to M(n)$, where $M(n)$ denotes the space of all complex $n\times n$-matrices, a "multiplication operator" $M_q\colon f \to q\cdot f$ such that $(q\cdot f)(x) = q(x)\cdot f(x), x \in X$, on the maximal domain $D(M_q) = \{f \in E\colon q\cdot f \in E\}$. If $\|e^{tq(x)}\|$ is uniformly bounded for $0 \leqslant t \leqslant 1$ and $x \in X$, it follows that M_q generates the multiplication semigroup

$$(T(t)f)(x) = e^{tq(x)}(f(x)), \quad f \in E, \quad x \in X, \quad t \geqslant 0\,.$$

Since M_q has a bounded inverse if and only if $q(x)^{-1}$ exists and is uniformly bounded for $x \in X$, it follows that the eigenvalues of each matrix $q(x)$ are always contained in $\sigma(M_q)$. In fact, much more can be said in case the function is bounded.

Lemma 7.1 *The spectrum of the matrix valued multiplication operator M_p, where $p\colon X \to M(n)$, is bounded is given by $\sigma(M_p) = \overline{\bigcup_{x\in X}\sigma(p(x))}$.*

Proof It remains to show that $0 \notin \overline{\bigcup_{x\in X}\sigma(p(x))}$ implies $0 \notin \sigma(M_p)$. Since $\det p(x)$ is the product of n eigenvalues (according to their multiplicity) of $p(x)$, the hypothesis implies that $d := \inf\{|\det p(x)|\colon x \in X\} > 0$. By Formula 4.12 in Chapter I of Kato (1966) we obtain

$$\|p(x)^{-1}\| \leqslant \gamma \cdot \|p(x)\|^{n-1} \cdot |\det p(x)|^{-1} \leqslant \gamma/d \cdot \|M_p\|^{n-1}$$

for every $x \in X$ and a constant γ depending only on the norm chosen on $\mathbb{C}^n$. Therefore $x \mapsto p(x)^{-1}$ defines a bounded continuous function on X which obviously yields the inverse of M_p, i.e., $0 \notin \sigma(M_p)$. □

Theorem 7.2 *Let $A = M_q$ be a matrix multiplication operator on $C_0(X, \mathbb{C}^n)$ generating a strongly continuous semigroup $(T(t))_{t\geqslant 0} = (e^{tq(\cdot)})_{t\geqslant 0}$. Then the Weak Spectral Mapping Theorem (7.2) holds true, i.e.,*

$$\sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))}\,.$$

Proof By the Spectral Inclusion Theorem 6.2 we always have $\exp(t\sigma(A)) \subset \sigma(T(t))$. Since $T(t)$ is a matrix multiplication operator with a bounded function, we obtain from Lemma 7.1

$$\sigma(T(t)) = \overline{\bigcup_{x\in X} \sigma(\exp(tq(x)))} = \overline{\bigcup_{x\in X} \exp(t\sigma(q(x)))} \subset \overline{\exp(t\sigma(A))}$$

which proves the assertion. □

Corollary 7.3 *The growth bound $\omega(A)$ and the spectral bound $s(A)$ coincide for matrix multiplication semigroups.*

The above results remain valid for other Banach spaces of $\mathbb{C}^n$-valued functions such as $L^p(X, \mathbb{C}^n)$, $1 \leqslant p < \infty$.

The example given at the beginning of this section can be generalized in a different way. In fact, $A(x_n) := (inx_n)$ on $E = c_0$ generates a bounded group, and we will show that this property too ensures that the Weak Spectral Mapping Theorem (7.2) holds. Without any boundedness assumption on $(T(t))_{t\in\mathbb{R}}$ this result cannot be true (see Hille and Phillips (1957, Sec.23.16) or [Wolff(1981)]).

Theorem 7.4 *Let $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ be a strongly continuous group on some Banach space E such that $\|T(t)\| \leqslant p(t)$ for some polynomial p and all $t \in \mathbb{R}$. Then the Weak Spectral Mapping Theorem (7.2) holds, i.e.,*

$$\sigma(T(t)) = \overline{\exp(t\cdot\sigma(A))} \text{ for all } t \in \mathbb{R}\,.$$

From the proof we isolate a series of lemmas for which we always assume the hypothesis made in Thm.7.4. Moreover we recall from Fourier analysis that the Fourier transformation $\varphi \to \hat{\varphi}$,

$$\hat{\varphi}(\alpha) := \int_{-\infty}^{\infty} \varphi(x)e^{-i\alpha x}\,\mathrm{d}x$$

and its inverse $\Psi \to \check{\Psi}$,

$$\check{\Psi}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\alpha)e^{i\alpha x}d\alpha$$

are topological isomorphisms of the Schwartz space $\mathcal{S}(= \mathcal{S}(\mathbb{R}))$. Since the subspace $\mathcal{D}$ of all functions having compact support is dense in $\mathcal{S}$, it follows that $\{\varphi \in \mathcal{S} : \hat{\varphi} \in \mathcal{D}\}$ is also dense in $\mathcal{S}$.

Lemma 7.5 *For every function $\varphi \in \mathcal{S}$ we obtain an operator $T(\varphi) \in \mathcal{L}(E)$ by*

$$T(\varphi)f := \int_{-\infty}^{\infty} \varphi(s)T(s)f\,\mathrm{d}s, \quad f \in E\,.$$

This operator can be represented as

$$T(\varphi)f = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\alpha)[R(\epsilon - i\alpha, A)f - R(-\epsilon - i\alpha, A)f]d\alpha, \quad f \in E\,. \quad (7.3)$$

Proof That $T(\varphi)$ is well-defined follows from the polynomial boundedness of $(T(t))_{t\in\mathbb{R}}$. In fact, $\varphi \to T(\varphi)$ is continuous from $\mathcal{S}$ into $(L(E), \|\cdot\|)$. We obtain

$$
\begin{aligned}
T(\varphi)f &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\varphi(s)e^{-\epsilon|s|}T(s)f\,\mathrm{d}s\\
&= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\varphi}(\alpha)e^{i\alpha s}e^{-\epsilon|s|}T(s)f\,d\alpha\,\mathrm{d}s\\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\varphi}(\alpha)\int_{-\infty}^{\infty}e^{i\alpha s}e^{-\epsilon|s|}T(s)f\,\mathrm{d}s d\alpha\\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\varphi}(\alpha)[R(\epsilon-i\alpha,A)f-R(-\epsilon-i\alpha,A)f]\,d\alpha\,.
\end{aligned}
$$

Here we used that polynomially bounded semigroups have growth bound 0, hence $\omega(A)=\omega(-A)=0$. Therefore the integral representation of $R(\epsilon-i\alpha,A)$ (cf. A-I,Prop.1.11) exists for $\epsilon\neq 0$. □

Lemma 7.6 *If $E\neq\{0\}$, then $\sigma(A)\neq\emptyset$.*

Proof If $\sigma(A)=\emptyset$, then (7.3) implies $T(\varphi)=0$ whenever $\hat{\varphi}$ has compact support. Since these functions form a dense subspace of $\mathcal{S}$, we conclude that $T(\varphi)=0$ for all $\varphi\in\mathcal{S}$. Choosing an approximate identity $(\psi_n)_{n\in\mathbb{N}}\subset\mathcal{D}$, we obtain

$$f=T(0)f=\lim_{n\to\infty}T(\psi_n)f=0$$

for every $f\in E$. □

Proof (Proof of Theorem 7.4 (1st part)) By the Spectral Inclusion Theorem 6.2, we have to show that every spectral value of $T(t)$ can be approximated by exponentials of spectral values of A. In view of the rescaling procedure it suffices to prove this when $-1\in\varrho(T(\pi))$, provided that the following condition is satisfied.

$$\text{There exists } \epsilon>0 \text{ such that } \bigcup_{k\in\mathbb{Z}} i[2k+1-2\epsilon,2k+1+2\epsilon]\subset\varrho(A)\,. \tag{7.4}$$

Assume now that (7.4) holds. Then each of the sets

$$\sigma_k := \{i\alpha\in\sigma(A)\colon \alpha\in[2k-1,2k+1]\}$$

is a spectral set of A with corresponding spectral projection P_k. If we choose $\varphi_0\in\mathcal{D}$ such that

$$\operatorname{supp}(\varphi_0)\subset[-1+\epsilon,1-\epsilon] \text{ and } \varphi_0(x)=1 \text{ for } x\in[-1+2\epsilon,1-2\epsilon]$$

it follows from (7.3) and the integral representation of P_k (cf. (3.1)) that $P_0=T(\varphi_0)$. More generally, since $(e^{i2k\cdot}\hat{\varphi}_0)\check{}(\alpha)=\varphi_0(\alpha-2k)$, the assertions (7.3) and (7.4) imply

$$P_k=\int_{-\infty}^{\infty}e^{i2ks}\hat{\varphi}_0(s)T(s)\,\mathrm{d}s \text{ for } k\in\mathbb{Z}\,. \tag{7.5}$$

At this point we isolate another lemma.

Lemma 7.7 span $\bigcup_{k\in\mathbb{Z}}P_kE$ *is dense in E.*

Proof The closure of span $\bigcup_{k\in\mathbb{Z}} P_k E$ is a $\mathcal{T}$-invariant subspace G of E. Consider the quotient group $(T(t)_/)_{t\in\mathbb{R}}$ induced on E/G. The spectrum of its generator $A_/$ is contained in $\sigma(A)$ by Prop.4.2.iii. Moreover the spectral projection corresponding to $\sigma(A_/)\cap\sigma_k$ is the quotient operator $P_{k/}$. Obviously $P_{k/}=0$, hence $\sigma(A_/)\cap\sigma_k=\emptyset$ for every $k\in\mathbb{Z}$ and $\sigma(A_/)=\emptyset$. By Lemma 7.6 this implies $E/G=\{0\}$, i.e., $G=E$. □

Proof (Proof of Theorem 7.4 (2nd part)) We return to the situation of the first part of the proof. Using (7.5) the spectral projection P_k can be transformed into

$$\begin{aligned} P_k &= \int_{-\infty}^{\infty} e^{i2ks}\hat{\varphi}_0(s)T(s)\,\mathrm{d}s \\ &= \sum_{m\in\mathbb{Z}} \int_{(m-1/2)\pi}^{(m+1/2)\pi} e^{i2ks}\hat{\varphi}_0(s)T(s)\,\mathrm{d}s \\ &= \int_{-\pi/2}^{\pi/2} e^{i2ks} \sum_{m\in\mathbb{Z}} \hat{\varphi}_0(s+m\pi)T(s+m\pi)\,\mathrm{d}s\,. \end{aligned}$$

i.e., $P_k f$ is the k-th Fourier coefficient of the π-periodic, continuous function $\xi_f\colon s\mapsto \sum_{m\in\mathbb{Z}}\hat{\varphi}_0(s+m\pi)T(s+m\pi)f$, $f\in E$. Since the projections P_k are mutually orthogonal, i.e., $P_kP_m=0$ for $k\neq m$, it follows that $g=\sum_{n\in\mathbb{Z}}P_n g$ for every $g\in \operatorname{span}\bigcup_{k\in\mathbb{Z}}P_kE$. In particular, the Fourier coefficients of the function ξ_g are absolutely summable, hence the Fourier series of ξ_g converges to ξ.

For $s=0$ we obtain

$$g=\sum_{n\in\mathbb{Z}}P_n g\cdot e^{-in0}=\sum_{m\in\mathbb{Z}}\hat{\varphi}_0(0+m\pi)T(0+m\pi)g \qquad \left(g\in \operatorname{span}\bigcup_{k\in\mathbb{Z}}P_kE\right).$$

Since span $\bigcup_{k\in\mathbb{Z}} P_k E$ is dense (Lemma 7.7), we conclude that

$$\sum_{m\in\mathbb{Z}}\varphi_0(m\pi)T(m\pi)=\mathrm{Id}\,. \tag{7.6}$$

As the final step we construct the inverse operator of $\mathrm{Id}+T(\pi)$ showing that $-1\in\varrho(T(\pi))$. We define $\psi_0(\alpha):=\varphi_0(\alpha)\cdot(1+e^{i\pi\alpha})^{-1}$, $\alpha\in\mathbb{R}$. Then we have $\psi_0\in S$ and $\psi_0(1+e^{i\pi})=\varphi_0$, hence $\hat{\psi}_0(x)+\hat{\psi}_0(x+\pi)=\hat{\varphi}_0(x)$ for all $x\in\mathbb{R}$. Then (7.6) implies

$$\begin{aligned} \mathrm{Id} &= \sum_{m\in\mathbb{Z}}\hat{\varphi}_0(m\pi)T(m\pi) \\ &= \sum_{m\in\mathbb{Z}}(\hat{\psi}_0(m\pi)+\hat{\psi}_0((m+1)\pi))T(m\pi) \\ &= [\sum_{m\in\mathbb{Z}}\hat{\psi}_0(m\pi)T(m\pi)](\mathrm{Id}+T(\pi))\,. \end{aligned}$$

In the rest of this section we discuss the behavior of the single spectral values λ of $T(t)$, $t > 0$. The goal is a characterization of $\sigma(T(t))$ involving only properties of the generator. By the rescaling procedure A-I,3.1 we may assume $\lambda = 1$ and $t = 2\pi$.

From the Spectral Inclusion Theorem 6.2 we know that $1 \in \varrho(T(2\pi))$ implies $\mathrm{i}\mathbb{Z} \subset \varrho(A)$. As seen in many examples the converse does not hold and we are now looking for additional conditions. Henceforth we assume $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ and define for $k \in \mathbb{Z}$

$$Q_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}ks}T(s)\,\mathrm{d}s = \frac{1}{2\pi}(1 - T(2\pi))R(\mathrm{i}, A) \tag{7.7}$$

(cf. Formula A-I, (3.1)).

Our approach is based on Fejér's Theorem (for Banach space valued functions). Let us recall this result. Suppose $\xi\colon [0, 2\pi] \to E$ is a continuous function and let $\xi_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}s}\xi(s)\,\mathrm{d}s$ be its k-th Fourier coefficient. Then the Fourier series is Césaro summable to ξ in every point $t \in (0, 2\pi)$. Moreover one has

$$\frac{1}{2}(\xi(0) + \xi(2\pi)) = C_1\text{-}\sum_{k\in\mathbb{Z}} \xi_n := \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}\Big(\sum_{k=-n}^{n} \xi_k\Big)\,. \tag{7.8}$$

This result enables us to prove the following proposition.

Proposition 7.8 *Let $(T(t))_{t\geqslant 0}$ be a semigroup on a Banach space E and denote its generator by A. Then the following conditions are equivalent.*

(a) $1 \in \varrho(T(2\pi))$,
(b) $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ *and the series* $\sum_{k\in\mathbb{Z}} R(\mathrm{i}, A)f$ *is Césaro-summable for every* $f \in E$,
(c) $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ *and the series* $\sum_{k\in\mathbb{Z}} R(\mathrm{i}, A)Q_k f$ *is Césaro-summable for every* $f \in E$.

Proof (a) $\Rightarrow$ (b) The Spectral Inclusion Theorem implies $\mathrm{i}\mathbb{Z} \subset \varrho(A)$. By (7.7) we have $R(ik, A) = 2\pi \cdot (1 - T(2\pi))^{-1}Q_k$. Since $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable (towards $\frac{1}{2}(f + T(2\pi)f)$) (see (7.8)), it follows that $\sum_{k\in\mathbb{Z}} R(\mathrm{i}, A)f$ is Césaro-summable as well.

(b) $\Leftrightarrow$ (c) If we use A-I,(3.1) and integrate by parts, we obtain

$$\begin{aligned} R(\mathrm{i}, A)Q_k f &= \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}s}T(s)R(\mathrm{i}, A)f\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Big[R(\mathrm{i}, A)f - \int_0^s e^{-\mathrm{i}t}T(t)f\,\mathrm{d}t\Big]\,\mathrm{d}s \\ &= R(\mathrm{i}, A)f - \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}t}(2\pi - t)T(t)f\,\mathrm{d}t\,. \end{aligned}$$

Fejér's theorem ensures $\sum_{k\in\mathbb{Z}}(1/2\pi)\int_0^{2\pi} e^{-\mathrm{i}t}(2\pi - t)T(t)f\,\mathrm{d}t$ is Césaro summable. Hence $\sum_{k\in\mathbb{Z}} R(\mathrm{i}, A)Q_k f$ is Césaro-summable if and only if $\sum_{k\in\mathbb{Z}} R(\mathrm{i}, A)f$ is.

(b) $\Rightarrow$ (a) We have $Q_k = \frac{1}{2\pi}(1 - T(2\pi))R(\mathrm{i}, A)$. Furthermore we know by (7.7) and (7.8) that $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable towards $\frac{1}{2}(f + T(2\pi)f)$. If we define

$S: E \to E$ by $Sf := \frac{f}{2} + \frac{1}{2\pi} \cdot C_1\text{-}\sum_k R(\mathrm{i}, A)f$, then we have

$$\begin{aligned}(1 - T(2\pi))Sf &= \frac{1}{2}(f - T(2\pi)f) + \frac{1}{2\pi} \cdot C_1\text{-}\sum_k (1 - T(2\pi))R(\mathrm{i}, A)f \\ &= \frac{1}{2}(f - T(2\pi)f) + \frac{1}{2}(f + T(2\pi)f) = f\,.\end{aligned}$$

Since S commutes with $T(2\pi)$, it follows that S is the inverse of $(1 - T(2\pi))$ thus $1 \in \varrho(T(2\pi))$. □

Based on the equivalence of (a) and (b), one can state a characterization of the spectrum of $T(t)$ in terms of the generator and its resolvent only. However, in general it is difficult to verify the summability condition stated in (b).

In Hilbert spaces the boundedness of the resolvents will suffice (see Thm.7.10 below).

Lemma 7.9 *Let $(T(t))_{t\geqslant 0}$ be a semigroup on some Hilbert space H and assume* $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ *for the generator A. Then we have*

(i) $(Q_k f)_{k\in\mathbb{Z}} \subset \ell^2(H)$ *for every $f \in H$, and*
(ii) if $\sup_{k\in\mathbb{Z}} \|R(\mathrm{i}, A)\| < \infty$ *, then* $\sum_{k\in\mathbb{Z}} R(\mathrm{i}, A)f_k$ *is summable whenever* $(f_k)_{k\in\mathbb{Z}} \in \ell^2(H)$ *.*

Proof (i) We consider the Hilbert space $L^2([0, 2\pi], H)$ and obtain

$$\begin{aligned}0 &\leqslant \left\| T(\cdot)f - \sum_{k=-n}^{n} Q_k f \cdot e^{\mathrm{i}\cdot} \right\|^2 \\ &= \int_0^{2\pi} \|T(s)f\|^2\,\mathrm{d}s - \int_0^{2\pi} \sum_{k=-n}^{n} (T(s)f | e^{\mathrm{i}s} Q_k f)\,\mathrm{d}s - \\ &\quad \int_0^{2\pi} \sum_{k=-n}^{n} (e^{\mathrm{i}s} Q_k f | T(s)f)\,\mathrm{d}s + \int_0^{2\pi} \left(\sum_{k=-n}^{n} e^{\mathrm{i}s} Q_k f \Big| \sum_{\ell=-n}^{n} e^{i\ell s} Q_\ell f \right) \mathrm{d}s \\ &= \int_0^{2\pi} \|T(s)f\|^2\,\mathrm{d}s - 2\pi \sum_{k=-n}^{n} \|Q_k f\|^2 \text{ (use (7.5))}\,.\end{aligned}$$

It follows that $\sum_{k\in\mathbb{Z}} \|Q_k f\|^2 \leqslant \frac{1}{2\pi} \int_0^{2\pi} \|T(s)f\|^2\,\mathrm{d}s < \infty$.

(ii) Fix $\lambda > 0$ sufficiently large and set

$$g_k := (1 + \lambda R(\mathrm{i}, A))f_k,\ k \in \mathbb{Z}\,.$$

Using the resolvent equation and then (A-I,(3.1)), we obtain

$$R(\mathrm{i}, A)f_k = R(\lambda + \mathrm{i}, A)g_k = [1 - e^{-2\pi\lambda} T(2\pi)]^{-1} \int_0^{2\pi} e^{-\lambda s} e^{-\mathrm{i}s} T(s) g_k\,\mathrm{d}s\,.$$

This yields for every finite subset N of $\mathbb{Z}$ that

$$\left\|\sum_{k\in N} R(\mathrm{i}k, A) f_k\right\| \leqslant \|(1-e^{-2\pi\lambda}T(2\pi))^{-1}\| \cdot \int_0^{2\pi} \|T(s)\| \left\|\sum_{k\in N} e^{-\mathrm{i}ks} g_k\right\| ds$$

$$\leqslant \|(1-e^{-2\pi\lambda}T(2\pi))^{-1}\| \cdot \left(\int_0^{2\pi} \|T(s)\|^2 ds\right)^{1/2} \cdot \left(\int_0^{2\pi} \left\|\sum_{k\in N} e^{-iks} g_k\right\|^2 ds\right)^{1/2}$$

$$= c\left(\sum_{k\in N} \|g_k\|^2\right)^{1/2} \leqslant c(1+\lambda M)\left(\sum_{k\in N} \|f_k\|^2\right)^{1/2} .$$

Here

$$c := \|(1-e^{-2\pi\lambda}T(2\pi))^{-1}\| \cdot (\int_0^{2\pi} \|T(s)\|^2 \,\mathrm{d}s)^{1/2}$$

and $M := \sup_{k\in\mathbb{Z}} \|R(\mathrm{i}, A)\|$. □

Theorem 7.10 *Let A be the generator of a semigroup $(T(t))_{t\geqslant 0}$ on some Hilbert space H. Then the following form of the spectral mapping theorem is valid*

$$\sigma(T(t)) \setminus \{0\} = \{e^{\lambda t} : \text{ either } \mu_k := \lambda + 2\pi\mathrm{i}k/t \in \sigma(A) \text{ for some } k \in \mathbb{Z} \text{ or } (\|R(\mu_k, A)\|)_{k\in\mathbb{Z}} \text{ is unbounded}\} .$$

Proof If $e^{\lambda t} \notin \sigma(T(t))$, it follows from the spectral inclusion theorem that $\mu_k \notin \sigma(A)$ for every $k \in \mathbb{Z}$ and from Formula (3.1) in A-I, that $\|R(\mu_k, A)\|$ is bounded. For the converse inclusion it suffices to assume $t = 2\pi$ and $\lambda = 0$ (use the rescaling procedure A-I,3.1). Assuming that $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ and $\|R(\mathrm{i}, A)\|$ is bounded, then $\sum_{k\in\mathbb{Z}} R(\mathrm{i}, A)Q_k f$ is summable by Lemma 7.9. Since every summable series is Cesàro-summable, condition (c) of Prop. 7.8 is satisfied and we conclude $1 \in \varrho(T(2\pi))$. □

As an immediate consequence we obtain an interesting characterization of the growth bound ω_0 of semigroups on Hilbert spaces.

Corollary 7.11 *The growth bound of a semigroup $(T(t))_{t\geqslant 0}$ on a Hilbert space H satisfies*

$$\omega_0 = \inf\{\lambda \in \mathbb{R} : \lambda + \mathrm{i}\mathbb{R} \subset \varrho(A) \text{ and } \|R(\lambda + \imath\mu, A)\| \text{ is bounded for } \mu \in \mathbb{R}\} . \quad (7.9)$$

The Example 1.3 above in combination with C-III,Cor.1.3 shows that (7.9) is not valid in arbitrary Banach spaces.

NOTES

Section 1: It was already known to Hille and Phillips (1957) that for strongly continuous semigroups $(T(t))_{t\geqslant 0}$ with generator A the spectral mapping theorem

"$\sigma(T(t)) = \exp(t\sigma(A))$" may be violated in various ways [l.c.,Sec.23.16]. The simple Examples 1.3 and 1.4 are due to Wolff (see Greiner et al. (1981)) and Zabczyk (1975). A more sophisticated example of a positive group with "$s(A) < \omega(A)$" is given in Wolff (1981).

Section 2: In Definition 2.1 we define the residual spectrum of A in such a way that it coincides with the point spectrum of the adjoint A' (see Prop. 2.2.(ii)). It therefore differs slightly from the one used, e.g., by Schaefer (1974). The spectral mapping theorem for the resolvent (Thm.2.5) is well known and can, e.g., be deduced from Lemma 9.2 and Thm.3.11 of Chap.VII in Dunford and Schwartz (1958).

Section 3: The general theory of spectral decompositions can be found in Kato (1966), Chap.III,§ 6.4]. Applications to isolated singularities like 3.6 are discussed extensively in [l.c., Chap. III, §6.5] and Yosida (1965, Chap.VIII, Sec.8). There are many ways to introduce an "essential spectrum" (see the footnote on page 243 of Kato (1966)). However each one yields the same "essential spectral radius".

Section 4: These results are taken from Derndinger (1980) and Greiner (1981).

Section 5: Periodic semigroups are studied explicitly in Bart (1977), but most of the results of this section seem to be well known.

Section 6: The Spectral Inclusion Theorem 6.2 and the Spectral Mapping Theorem 6.6 for eventually norm continuous semigroups date back to Hille and Phillips (1957). Davies (1980) gives an elegant proof using Banach algebra techniques. Tensor products of operators and their spectral theory have been studied by Ichinose and others (see Chap. XIII.9 of Reed and Simon (1978)). The spectral mapping theorem in Corollary 6.8 generalizes Thm.XIII.35 of Reed and Simon (1978) (see also Herbst (1982)).

Section 7: Matrix valued multiplication semigroups appear as solution, via Fourier transformation, of systems of partial differential equations. Kreiss initiated a systematic investigation (see, e.g., Kreiss (1958), Kreiss (1959), Miller and Strang (1966)) and the Weak Spectral Mapping Theorem 7.2 must have been known to him. The direct proof of the Weak Spectral Mapping Theorem 7.4 for polynomially bounded groups seems to be new. The result can also be deduced from the theory of spectral subspaces of group representations (see, e.g., Combes and Delaroche (1978)), since the Arveson spectrum of a strongly continuous one-parameter group can be identified with the spectrum of the generator (see Evans (1976)). The final part of this section is taken from Greiner (1985) and yields a new approach to Gearhart's characterization of the spectrum of semigroups on Hilbert spaces Gearhart (1978). Different proofs can be found in Herbst (1983), Howland (1984) and Prüss (1984).

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Chapter A-IV
Asymptotics of Semigroups on Banach Spaces

Part B
Positive Semigroups on Spaces $C_0(X)$

Chapter B-I
Basic Results on $C_0(X)$

This part of the book is devoted to one-parameter semigroups of operators on spaces of continuous functions of type $C_0(X)$. Such spaces are Banach lattices of a very special kind. We treat this case separately since we feel that an intermingling with the abstract Banach lattice situation considered in Part C would have made it difficult to appreciate the easy accessibility and the pilot function of methods and results available here. In this chapter we want to fix the notation we are going to use and to collect some basic facts about the spaces we are considering.
If X is a locally compact topological space, then $C_0(X)$ denotes the space of all continuous complex-valued functions on X which vanish at infinity, endowed with the supremum-norm. If X is compact, then any continuous function on X "vanishes at infinity" and $C_0(X)$ is the space of all continuous functions on X. We often write $C(X)$ instead of $C_0(X)$ in this situation.
We sometimes study real-valued functions and write the corresponding real spaces as $C_0(X,\mathbb{R})$ and $C(X,\mathbb{R})$, and the notations $C_0(X,\mathbb{C})$ and $C(X,\mathbb{C})$ are used if there might be confusion between both cases.

1 Algebraic and Order-Structure: Ideals and Quotients

Any space $C_0(X)$ is a commutative C*-algebra under the sup-norm and the pointwise multiplication, and by the *Gelfand Representation Theorem* any commutative C*-algebra can, on the other hand, be canonically represented as an algebra $C_0(X)$ on a suitable locally compact space X. The algebraic notions of ideal, quotient, homomorphism are well known and need not be explained further.
Another natural and important structure of $C_0(X)$ is the *pointwise* ordering, under which $C_0(X,\mathbb{R})$ is a (real) Banach lattice and $C_0(X,\mathbb{C})$ a complex Banach lattice in the sense explained in Chapter C-I. Concerning the order structure of $C_0(X)$ we use the following notations. For a function f in $C_0(X,\mathbb{R})$

1. A function f is called *positive*, $f \geqslant 0$, if $f(t) \geqslant 0$ for all $t \in X$,
2. We write $f > 0$ if f is positive but does not vanish identically,

3. We call f *strictly positive* if $f(t) > 0$ for all $t \in X$.

The notion of an order ideal explained in Chapter C-I applies of course to the Banach lattices $C_0(X)$ and might give rise to confusion with the corresponding algebraic notion.
However, since we are mainly considering closed ideals and since a closed linear subspace I of $C_0(X)$ is a lattice ideal if and only if I is an algebraic ideal, we may and will simply speak of closed ideals without specifying whether we think of the algebraic or the order theoretic meaning of this word.
An important and frequently used characterization of these objects is the following: A subspace I of $C_0(X)$ is a closed ideal if and only if there exists a closed subset A of X such that a function f belongs to I if and only if f vanishes on A. The set A is of course uniquely determined by I and is called the *support* of I. If $I = I_A$ is a closed ideal with support A, then I_A is naturally isomorphic to $C_0(X \setminus A)$ and the quotient $C_0(X)/I$ (under the natural quotient structure) is again a Banach algebra and a Banach lattice that can be identified canonically (via the map $f + I \to f_{|A}$) with $C_0(A)$.

2 Linear Forms and Duality

The *Riesz Representation Theorem* asserts that the dual of $C_0(X)$ can be identified in a natural way with the space of bounded regular Borel measures on X. While there is no natural algebra structure on this dual, the dual ordering (see Chapter C-I) makes $C_0(X)'$ into a Banach lattice. We will occasionally make use of the order structure of $C_0(X)'$, but since at least its measure theoretic interpretation is supposed to be well-known, it may suffice here to refer to Chapter C-I, Sections 3 and 7, for a more detailed discussion and to recall only some basic notations here.
If μ is a linear form on $C_0(X, \mathbb{R})$, then

$\mu \geqslant 0$ means $\mu(f) \geqslant 0$ for all $f \geqslant 0$; μ is then called *positive*,
$\mu > 0$ means that μ is positive but does not vanish identically,
$\mu \gg 0$ means that $\mu(f) > 0$ for any $f > 0$; μ is then called *strictly positive*.

If μ is a linear form on $C_0(X, \mathbb{C})$, then μ can be written uniquely as $\mu = \mu_1 + i\mu_2$ where μ_1 and μ_2 map $C_0(X, \mathbb{R})$ into $\mathbb{R}$ (decomposition into *real* and *imaginary parts*). We call μ positive (strictly positive) and use the above notations if $\mu_2 = 0$ and μ_1 is positive (strictly positive). We point out that strictly positive linear forms need not exist on $C_0(X)$, but if X is separable, then a strictly positive linear form is obtained by summing a suitable sequence of point measures.
The coincidence of the notions of a closed algebraic and a closed lattice ideal in $C_0(X)$ has of course its effect on the algebraic and the lattice theoretic notions of a homomorphism. The case of a homomorphism into another space $C_0(Y)$ will be discussed below.
As for homomorphisms into the scalar field, we have essentially coincidence between the algebraic and the order theoretic meaning of this word, more exactly:

A linear form $\mu \neq 0$ on $C_0(X)$ is a lattice homomorphism if and only if μ is, up to normalization, an algebra homomorphism (algebra homomorphisms $\neq 0$ must necessarily have norm 1). Since the algebra homomorphisms $\neq 0$ on $C_0(X)$ are known to be the point measures (denoted by δ_t) on X and since on the other hand μ is a lattice homomorphism if and only if $|\mu(f)|$ equals $\mu(|f|)$ for all f, it follows that this latter condition on μ is equivalent to $\mu = \alpha\delta_t$ for a suitable t in X and a positive real number α.
This can be summarized by saying that X can be canonically identified, via the map $t \to \delta_t$, with the subset of the dual $C_0(X)'$ consisting of the non-zero algebra homomorphisms, which is also the set of all normalized lattice homomorphisms. This identification is a topological isomorphisms with respect to the weak*-topology of $C_0(X)'$.

3 Linear Operators

A linear mapping T from $C_0(X, \mathbb{R})$ into $C_0(Y, \mathbb{R})$ is called

positive (notation: $T \geqslant 0$) if Tf is a positive whenever f is positive,
lattice homomorphism if $|Tf| = T|f|$ all f,
Markov-operator if X and Y are compact and T is a positive operator mapping 1_X to 1_Y.

In the case of complex scalars, T can be decomposed into real and imaginary parts. We call T positive in this situation if the imaginary part of T is $= 0$ and the real part is positive. The terms *Markov operator* and *lattice homomorphism* are defined as above. Note that a complex lattice homomorphism is necessarily positive, and that the *complexification* of a real lattice homomorphism is a complex lattice homomorphism. Positive Operators are always continuous.
Note that the adjoint of a Markov operator T maps positive normalized measures into positive normalized measures while the adjoint of an algebra homomorphism (lattice homomorphism) maps point measures into (multiples of) point measures. Therefore the adjoint of a Markov lattice homomorphism as well as the adjoint of an algebra homomorphism induces a continuous map φ from Y (viewed as a subset of the weak dual $C(Y)'$) into X (viewed as a subset of $C(X)'$).
This mapping φ determines T in a natural and unique way, so that the following are equivalent assertions on a linear mapping T from a space $C(X)$ into a space $C(Y)$.

(a) T is a Markov lattice homomorphism.
(b) T is a Markov algebra homomorphism.
(c) There exists a continuous map φ from Y into X such that $Tf = f \circ \varphi$ for all $f \in C(X)$.

If T is, in addition, bijective, then the mapping φ in (c) is a homeomorphism from Y onto X. This characterization of homomorphisms carries over mutatis mutandis to situations where the conditions on X, Y or T are less restrictive. For later reference we explicitly state the following.

(i) Let K be compact. Then $T \in \mathrm{L}C(K)$ is a lattice homomorphism if and only if there is a mapping φ from K into K and a function $h \in C(K)$ such that $Tf(s) = h(s)f(\varphi(s))$ holds for all $s \in K$. The mapping φ is continuous in every point t with $h(t) \neq 0$.

(ii) Let X be locally compact and $T \in \mathrm{L}C_0(X)$. Then T is a lattice isomorphism if and only if there is a homeomorphism φ from X onto X and a bounded continuous function h on X such that $h(s) \geqslant \delta > 0$ for all s and $Tf(s) = h(s)f(\varphi(s))$ $(s \in X)$. Moreover, T is an algebraic $*$-isomorphism if and only if T is a lattice isomorphism and the function h above is $\equiv 1$.

Notes

For the representation theory of commutative C*-algebras we refer to Takesaki (1979). This and the other mentioned properties like algebraic ideals, their connections with closed sets, the representation of latice or algebraic homomorpism etc. we refer to the excellent book Semadeni (1971).

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Chapter B-II
Characterization of Positive Semigroups on $C_0(X)$

Chapter B-III
Spectral Theory of Positive Semigroups on $C_0(X)$

Chapter B-IV
Asymptotics of Positive Semigroups on $C_0(X)$

Part C
Positive Semigroups on Banach Lattices

Chapter C-I
Basic Results on Banach Lattices and Positive Operators

This introductory chapter is intended to give a brief exposition of those results on Banach lattices and ordered Banach spaces which are indispensable for an understanding of the subsequent chapters. We do not give proofs of the results, since these can easily be found in the literature (e.g., in Schaefer (1974)). We rather want to give the reader, who is unfamiliar with the results or the terminology used in this book, the necessary information for an intelligent reading of the main discussions. Since relatively few general results on ordered Banach spaces are needed, we will primarily talk about Banach lattices. The scalar field will be $\mathbb{R}$ except for the last three sections, where complex Banach lattices will be discussed.

The notion of a Banach lattice was devised to obtain a common abstract setting within which one could talk about phenomena related to positivity. This has previously been studied in various types of spaces of real-valued functions, such as the spaces $C(K)$ of continuous functions on a compact topological space K, the Lebesgue spaces $L^1(\mu)$ or more generally the spaces $L^p(\mu)$ constructed over a measure space (X, Σ, μ) for $1 \leqslant p \leqslant \infty$. Thus it is a good idea to think of such spaces first in order to get a feeling for the concrete meaning of the abstract notions we introduce. Later we will see that the connections between *abstract* Banach lattices and the *concrete* function lattices $C(K)$ and $L^1(\mu)$ are closer than one might think at first. We will use without further explanation the terms *order relation* (ordering), *ordered set*, *majorant*, *minorant*, *supremum*, *infimum*.

By definition, a Banach lattice is a Banach space $(E, \|\cdot\|)$ which is endowed with an order relation, usually written $\leqslant$, such that $(E, \leqslant)$ is a lattice and the ordering is compatible with the Banach space structure of E. We elaborate this in more detail now. The axioms of compatibility between the linear structure of E and the order are

$$f \leqslant g \text{ implies } f + h \leqslant g + h \text{ for all } f, g, h \text{ in } E, \tag{LO1}$$

$$f \geqslant 0 \text{ implies } \lambda f \geqslant 0 \text{ for all } f \text{ in } E \text{ and } \lambda \geqslant 0. \tag{LO2}$$

Any (real) vector space with an ordering satisfying (LO_1) and (LO_1) is called an *ordered vector space*. The property expressed in (LO_1) is sometimes called *trans-*

lation invariance and implies that the ordering of an ordered vector space E is completely determined by the positive part $E_+ = \{f \in E : f \geqslant 0\}$ of E. In fact, one has $f \leqslant g$ if and only if $g - f \in E_+$. (LO_1) together with (LO_2) furthermore imply that the positive part of E is a convex set and a cone with vertex 0 (often called the *positive cone* of E). It is easily verified that conversely any proper convex cone C with vertex 0 in E is the positive part of E with respect to a uniquely determined compatible ordering.

An ordered vector space E is called a *vector lattice* if any two elements f, g in E have a supremum, which is denoted by $\sup(f, g)$ or by $f \vee g$, and an infimum, denoted by $\inf(f, g)$ or by $f \wedge g$. It is obvious that the existence of, e.g., the supremum of any two elements in an ordered vector space implies the existence of the supremum of any finite number of elements in E and, since $f \leqslant g$ is equivalent to $-g \leqslant -f$ this automatically implies the existence of finite infima. However, suprema (infima) of infinite majorized subsets need not exist in a vector lattice. If they do, then the vector lattice is called *order complete* (*countably order complete* or *σ-order complete* if suprema of countable majorized subsets exist). At any rate, the binary relations *sup* and *inf* in a vector lattice automatically satisfy the (infinite) distributive laws

$$\inf(\sup_\alpha f_\alpha, h) = \sup_\alpha(\inf(f_\alpha, h)),$$
$$\sup(\inf_\alpha f_\alpha, h) = \inf_\alpha(\sup(f_\alpha, h)),$$

whenever one side exists. This gives rise to the following definitions.

$\sup(f, -f) = |f|$ is called the *absolute value* of f,
$\sup(f, 0) = f^+$ is called the *positive part* of f,
$\sup(-f, 0) = f^-$ is called the *negative part* of f.

Note that the negative part of f is positive. We call two elements f, g of a vector lattice *orthogonal* or *lattice disjoint* and write $f \perp g$ if $\inf(|f|, |g|) = 0$.

Apart from this, the above definitions allow us to formulate the axiom of compatibility between norm and order requested in a Banach lattice in the following short way.

$$|f| \leqslant |g| \text{ implies } \|f\| \leqslant \|g\|. \tag{LN}$$

A norm on a vector lattice is called a *lattice norm* if it satisfies (LN). With these notations we can now give the definition of a Banach lattice as follows.

A Banach lattice *is a Banach space E endowed with an ordering $\leqslant$ such that $(E, \leqslant)$ is a vector lattice and the norm on E is a lattice norm. By a* normed vector lattice *we understand a vector lattice endowed with a lattice norm.*

There is a number of elementary, but very important formulas valid in any vector lattice, such as

$$f = f^+ - f^- \qquad\qquad |f + g| \leqslant |f| + |g|$$
$$|f| = f^+ + f^- \qquad\qquad f + g = \sup(f, g) + \inf(f, g)$$

(see, e.g., Schaefer (1974)). Let us note in passing the following consequences.

(i) The lattice operations $(f, g) \mapsto \sup(f, g)$ and $(f, g) \mapsto \inf(f, g)$ and the mappings $f \mapsto f^+$, $f \mapsto f^-$, $f \mapsto |f|$ are uniformly continuous.
(ii) The positive cone is closed.
(iii) *Order intervals*, i.e., sets of the form

$$[f, g] = \{h \in E : f \leqslant h \leqslant g\}$$

are closed and bounded.

Instead of dwelling upon a detailed discussion of the above equalities and inequalities let us rather formulate the following principle, which allows us to verify any of them and to invent, prove or disprove new ones whenever necessary.

> *Any general formula relating a finite number of* variables *to each other by means of lattice operations and/or linear operations is valid in any Banach lattice as soon as it is valid in the real number system.*

In fact, we see below that any Banach lattice E is, as a vector lattice, *locally* of type $C(X)$, more exactly: Given any finite number $x_1, \ldots, x_n$ of elements in E, there is a compact topological space X and a vector sublattice J of E which is isomorphic to $C(X)$ and contains $x_1, \ldots, x_n$ (see Section. 4). The above principle is an easy consequence of the following: In a space $C(X)$ it is clear that a formula of the type considered need only be verified pointwise, i.e., in $\mathbb{R}$.
The reader may now be prepared to follow a concise presentation of the most basic facts on Banach lattices.

1 Sublattices, Ideals, Bands

The notion of a *vector sublattice* of a vector lattice E is self-explanatory, but it should be pointed out that a vector subspace F of E which is a vector lattice for the ordering induced by E need not be a vector sublattice of E (formation of suprema may differ in E and in F), and that a vector sublattice need not contain (or may lead to different) infinite suprema and infima. The following are necessary and sufficient conditions on a vector subspace G of E to be a vector sublattice.

(a) $|h| \in G$ for all $h \in G$,
(b) $h^+ \in G$ for all $h \in G$,
(c) $h^- \in G$ for all $h \in G$.

A subset B of a vector lattice is called *solid* if $f \in B$, $|g| \leqslant |f|$ implies $g \in B$. Thus a norm on a vector lattice is a lattice norm if and only if its unit ball is solid. A solid linear subspace is called an *ideal*. Ideals are automatically vector sublattices since $|\sup(f,g)| \leqslant |f| + |g|$. On the other hand, a vector sublattice F is an ideal in E if $g \in F$ and $0 \leqslant f \leqslant g$ imply $f \in F$. A *band* in a vector lattice E is an ideal which contains arbitrary suprema, or more exactly:

> *B is a band in E if B is an ideal in E and* $\sup M$ *is contained in B whenever M is contained in B and has a supremum in E.*

Since the notions of sublattice, ideal, band are invariant under the formation of arbitrary intersections there exists, for any subset B of E, a uniquely determined smallest sublattice (ideal, band) of E containing B, i.e., the *sublattice* (*ideal*, *band*) *generated by B*.
If we denote by B^d the set $\{h \in E\colon \inf(|h|,|f|) = 0 \text{ for all } f \in B\}$, then B^d is a band for any subset B of E, and $(B^d)^d = B^{dd}$ is a band containing B. If E is a normed vector lattice (more generally, if E is archimedean ordered, see e.g., Schaefer (1974)), then B^{dd} is the band generated by B.
If two ideals I, J of a vector lattice E have trivial intersection $\{0\}$, then I and J are *lattice disjoint*, i.e., $I \subset J^d$. Thus if E is the direct sum of two ideals I, J, then one has automatically $I = J^d$ and $J = I^d$, hence $I = I^{dd}$ and $J = J^{dd}$ must be bands in this situation. In general, an ideal I need not have a complementary ideal J even if $I = I^{dd}$ is a band. This amounts to the same as saying that even if $I = I^{dd}$ (which is always true if I is a band in a normed vector lattice) one need not necessarily have $E = I + I^d$.
An ideal I is called a *projection band* if it does have a complementary ideal, and in this case the projection of E onto I with kernel I^d is called the *band projection* belonging to I. An example of a band which is not a projection band is furnished by the subspace of $C([0,1])$ consisting of the functions vanishing on $[0,1/2]$.
The *Riesz Decomposition Theorem* asserts that in an order complete vector lattice every band is a projection band. As a consequence, if E is order complete and B is an arbitrary subset of E, then E is the direct sum of the complementary bands B^d and B^{dd}.
This theorem, which is quite easy to prove, is widely used in analysis and gives an abstract background to, e.g., the decomposition of a measure into atomic and diffuse parts (the atomic measures being those contained in the band generated by the point measures, the diffuse measures those disjoint to the latter). Or, more specifically, to the well-known decomposition of a measure on $[a,b]$ into an atomic part and a diffuse part, which latter can in turn be decomposed into the sum of a measure which is *absolutely continuous* (which means, contained in the band generated by Lebesgue measure) and a *singular measure* (which means, a diffuse measure disjoint to Lebesgue measure).
A band in a normed vector lattice is necessarily closed. By contrast, an ideal need not be closed, but the closure of an ideal is again an ideal. The situation, where every closed ideal is a band, will be briefly discussed in Section 5.

2 Order Units, Weak Order Units, Quasi-Interior Points

An element u in the positive cone of a vector lattice E is called an *order unit* if the ideal generated by u is all of E. If the band generated by u is all of E (which is equivalent to $\{u\}^d = 0$ whenever E is archimedean, hence in particular if E is a normed vector lattice), then u is called a *weak order unit* of E. If E is a Banach lattice, then any order unit in E is an interior point of the positive cone E_+, and conversely any interior point of E_+ must be an order unit of E. Every space $C(K)$ has order units (namely, the strictly positive functions), and conversely by the Kakutani-Krein Representation Theorem (see Section 4), every Banach lattice with an order unit is isomorphic to a space $C(K)$.
If an element u in the positive cone of a Banach lattice E has the property that the closed ideal generated by u is all of E, then u is called a *quasi-interior point* of E_+. Quasi-interior points of the positive cone exist, e.g., in any separable Banach lattice. If $E = C(K)$, then the quasi-interior points and the interior points of E_+ coincide, while the weak order units of E are the (positive) functions vanishing on a nowhere dense subset of K. If E is a space $L^p(\mu)$ with σ-finite μ and $1 \leqslant p < \infty$, then the weak order units and the quasi-interior points of E_+ coincide with the functions strictly positive μ-a.e., while E_+ does not contain any interior point.

3 Linear Forms and Duality

A linear functional φ on a vector lattice E is called

order-bounded if φ is bounded on order intervals of E,
positive if $\varphi(f) \geqslant 0$ for all $f \geqslant 0$,
strictly positive if $\varphi(f) > 0$ for all $f > 0$.

Any positive linear functional is order bounded, and the positive functionals form a proper convex cone with vertex 0 in the linear space $E^\#$ of all order bounded functionals, thus defining a natural ordering (given by $\varphi \leqslant \psi$ if and only if $\varphi(f) \leqslant \psi(f)$ for all $f \in E_+$) under which $E^\#$ is an order complete vector lattice. In particular, positive part, negative part and absolute value exist for any order bounded functional on E, the absolute value of $\varphi \in E^\#$ being given by

$$|\varphi|(f) = \sup\{\varphi(h) \colon |h| \leqslant f \text{ for } f \in E_+ \ .$$

As a consequence, one has $|\varphi(f)| \leqslant |\varphi|(|f|)$ for all f in E whenever φ is order bounded, and $|\varphi(f)| \leqslant \varphi(|f|)$ if and only if φ is positive. An order bounded linear functional φ is called *order-continuous* (*σ-order-continuous*) if both positive and negative part of φ have the property that they transform any decreasing net (any decreasing sequence) with infimum 0 into a net (sequence) converging to 0 in $\mathbb{R}$. The order-continuous (σ-order-continuous) functionals form a band in $E^\#$.

In general, a vector lattice E need not admit any non-zero order-bounded linear functional. However, if E is a normed lattice, then any continuous functional is order-bounded, and if E is a Banach lattice, then one has coincidence between $E^{\#}$ and E'. Still, order-continuous functionals $\neq 0$ need not exist on a Banach lattice. Situations where every order-bounded functional is order-continuous will be briefly discussed in Section 5.

If E is a Banach lattice, then the dual norm on $E' = E^{\#}$ is a lattice norm, hence E' is an order-complete Banach lattice under the natural norm and order. The evaluation map from E into the second dual E'' is a lattice homomorphism (for the definition see Section 6) into the band of order-continuous functionals on E'. In particular, every dual Banach lattice E admits sufficiently many order-continuous functionals to separate the points of E.

4 AM- and AL-Spaces

If the norm on a Banach lattice E satisfies

$$\|\sup(f,g)\| = \sup(\|f\|,\|g\|) \text{ for } f,g \in E_+, \tag{M}$$

then E is called an abstract M-space or an *AM-space*. If, in addition, the unit ball of E contains a largest element u, then u must be an order unit of E and E is then called an *(AM)-space with unit*. Condition (M) in E implies that in the dual of E one has

$$\|f+g\| = \|f\| + \|g\| \text{ for } f,g \in E'_+. \tag{L}$$

Any Banach lattice satisfying (L) is called an abstract L-space or an *AL-space*. Thus the dual of an AM-space is an AL-space.

It is quite easy to verify that, on the other hand, the dual of an AL-space is an AM-space with unit, the unit being the uniquely determined linear functional that coincides with the norm on the positive cone. Putting this together, one gets that the second dual of an AM-space E is an AM-space with unit. If E already has a unit u, then u is also the unit of E'', so that the ideal of E'' generated by E is all of E''. By contrast, if E is an AL-space, then E is an ideal (even a band) in E''. Infinite-dimensional AL- or AM-spaces are never reflexive.

The importance of AL- and AM-spaces in the general theory of Banach lattices is due to the fact that these spaces have very special concrete representations as function lattices and that, on the other hand, any general Banach lattice E is in a very intimate way connected to certain families of AL- and AM-spaces canonically associated with E. Let us first discuss the natural representations of AM- and AL-spaces.

If E is an AM-space with unit u, then the set K of lattice homomorphisms from E into $\mathbb{R}$ taking the value 1 on u is a non-empty, $\sigma(E',E)$-compact subset of E' and the natural evaluation map from E into $\mathbb{R}^K$ maps E isometrically onto the continuous real-valued functions on K (cf. Section 6). This is the *Kakutani-Krein Repre-*

sentation Theorem, which is an order-theoretic counterpart to the Gelfand Representation Theorem in the theory of commutative C*-algebras. If E is an AM-space without unit, then the second dual of E has a unit and thus gives a representation of E as a closed sublattice of a space $C(K)$.
If E is an AL-space, then the representation of the dual of E as a space $C(K)$ leads to an interpretation of the elements of the bidual of E as Radon measures on K. If E_+ has a quasi-interior point h, then in this interpretation E consists exactly of the measures absolutely continuous with respect to (the measure corresponding to) h, thus by the *Radon-Nikodym-Theorem*, $E = L^1(K, h)$. In general, a similar argument leads to a representation of E as a space $L^1(X, \mu)$ constructed over a locally compact space X.
If E is an arbitrary Banach lattice and $f \in E_+$, then the ideal I generated by f in E (which is the union of the positive multiples of the interval $[-f, f]$) can be made into an AM-space with unit f by endowing it with the gauge function p_f of $[-f, f]$. We denote (I, p_f) by E_f. On the other hand, if f' is a positive linear functional on E, then the mapping $q_{f'}: f \mapsto \langle |f|, f' \rangle$ is a semi-norm on E. The kernel J of $q_{f'}$ is an ideal in E, and the completion of E/J with respect to the norm canonically derived from $q_{f'}$ becomes an AL-space which we denote by (E, x'). A good illustration for these constructions is the following.
If $E = C(K)$ and if μ is a positive linear form (Radon measure) on E, then (E, μ) is just $L^1(K, \mu)$; if $E = L^p(\mu)$ $(1 \leqslant p < \infty$, μ finite), then $E_{1_X} = L^\infty(\mu)$.

5 Special Connections Between Norm and Order

If an increasing net $(x_\alpha)_{\alpha \in A}$ in a normed vector lattice is convergent, then its limit must be the supremum as a consequence of the closedness of the positive cone. On the other hand, if $\{x_\alpha : \alpha \in A\}$ has a supremum, the net $(x_\alpha)_{\alpha \in A}$ need not converge. A Banach lattice is said to have *order-continuous norm* (σ*-order-continuous norm*) if any increasing net (sequence) which has a supremum is automatically convergent. This is of course equivalent to saying that any decreasing net (sequence) with an infimum is convergent. Since infimum and limit must coincide, the order continuity (σ-order continuity) of the norm in a Banach lattice is also equivalent to the propperty that any decreasing net (sequence) with infimum 0 converges to 0.
A Banach lattice with order-continuous norm must be order complete, but σ-order-continuity of the norm need not imply order completeness. At any rate, one has the following characterization.
A Banach lattice E has order-continuous norm if and only if any one of the following equivalent assertions holds.

(a) E is σ-order complete and has σ-order-continuous norm.
(b) Every order interval in E is weakly compact.
(c) E is (under evaluation) an ideal in E''.
(d) Every continuous linear form on E is order continuous.

(e) Every closed ideal in E is a projection band.

An even more stringent condition than order-continuity of the norm is that any increasing norm-bounded net be convergent. This condition is satisfied if and only if any one of the following equivalent assertions holds.

(a) E is (under evaluation) a band in E''.
(b) E is weakly sequentially complete.
(c) Every order-continuous linear form on E' belongs to E.
(d) No closed sublattice of E is isomorphic to c_0.

The most important examples of non-reflexive Banach lattices with this property are the AL-spaces.

6 Positive Operators, Lattice Homomorphisms

A linear mapping T from an ordered Banach space E into an ordered Banach space F is called *positive* (notation: $T \geqslant 0$) if $Tf \in F_+$ for all $f \in E_+$; T is called *strictly positive* if $T \geqslant 0$ and $\{f \in E: T|f| = 0\} = \{0\}$. The set of all positive linear mappings is a convex cone in the space LE, F of all linear mappings from E into F defining the *natural ordering* of LE, F. The linear subspace of LE, F generated by the positive maps (i.e. the space of linear maps that can be written as differences of positive maps) is denoted by $\mathcal{L}^r(E; F)$ and its elements are called *regular* mappings. If E and F are Banach lattices, then any regular mapping from E into F is continuous, but $\mathcal{L}^r(E; F)$ is in general a proper subspace of the space LE, F of all continuous linear mappings. One has coincidence of $\mathcal{L}^r(E; F)$ and LE, F, e.g., when $E = F$ is an order complete AM-space with unit or an AL-space. At any rate, if F is order complete, then $\mathcal{L}^r(E; F)$ under the natural ordering is an order-complete vector lattice, and a Banach lattice under the norm

$$T \mapsto \|T\|_r = \||T|\|,$$

the right hand side denoting the operator norm of the absolute value of T. The absolute value of $T \in \mathcal{L}^r(E; F)$, if it exists, is given by

$$|T|(f) := \sup\{Th: |h| \leqslant f,\ f \in E_+\}.$$

Thus T is positive if and only if $|Tf| \leqslant T|f|$ holds for any f in E.
An operator $T \in$ LE, F is called a *lattice homomorphism* if $|Tf| = T|f|$ holds for all $f \in E$. Lattice homomorphisms are alternatively characterized by any one of the following conditions holding for all f, and $g \in E$.

(i) $(Tf)^+ = T(f^+)$,
(ii) $(Tf)^- = T(f^-)$,
(iii) $T(f \vee g) = Tf \vee Tg$,
(iv) $T(f \wedge g) = Tf \wedge Tg$,

(v) $T(f^+) \wedge T(f^-) = 0$.

The kernel of a lattice homomorphism is an ideal. If T is bijective, then T is a lattice homomorphism if and only if T and T^{-1} are positive.

7 Complex Banach Lattices

Although the notion of a Banach lattice is intrinsically related to the real number system, it is possible and often desirable to extend discussions to complexifications of Banach lattices in such a way that the order-related terms introduced in the real situation essentially retain their meaning. Thus we define a *complex Banach lattice* E to be the complexification of a real Banach lattice $E_{\mathbb{R}}$ in the sense that

$$E = E_{\mathbb{R}} \oplus \mathrm{i}E_{\mathbb{R}}$$

which means more exactly $E = E_{\mathbb{R}} \times E_{\mathbb{R}}$ with scalar multiplication

$$(\alpha + \mathrm{i}\beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y).$$

The space $E_{\mathbb{R}}$ will sometimes be called the *underlying real Banach lattice* or the *real part* of E. The classical complex Banach spaces $C(X)$, $L^p(\mu)$ are complex Banach lattices in this sense, the underlying real Banach lattices being the corresponding (real) subspaces of real-valued functions. We want to extend the formation of absolute values, which is a priori defined only in the real part of E, in such a way that in the classical situation $E = C(X)$ or $E = L^p(\mu)$ the usual absolute value of a function is obtained. This is in fact possible by putting, for $h = f + ig$ in E,

$$|h| = \sup\{\mathrm{Re}\left(e^{\mathrm{i}\vartheta} h\right) : 0 \leqslant \vartheta \leqslant 2\pi\}.$$

The only problem with this definition being the existence of the right hand side without the assumption of order-completeness on $E_{\mathbb{R}}$. But for this we just have to observe that the set $M = \{\mathrm{Re}(e^{\mathrm{i}\vartheta} h) \colon 0 \leqslant \vartheta \leqslant 2\pi\}$ is contained and order bounded in the ideal generated in $E_{\mathbb{R}}$ by $|f| + |g|$, which in turn is by the Kakutani-Krein Representation Theorem isomorphic to a space $C_{\mathbb{R}}(X)$ under the pointwise ordering. Now the pointwise supremum of M in $\mathbb{R}^X$ is readily seen to be a continuous function (namely, the modulus of the complex valued continuous function corresponding to $f + \mathrm{i}g$), so that M has a supremum in $C_{\mathbb{R}}(X) = (E_{\mathbb{R}})_{|f|+|g|}$.
Since the mapping $f \mapsto |f|$ now has a meaning in E, the definition of an ideal can be extended formally unchanged to the complex situation. We observe that $|f + \mathrm{i}g| = |f - \mathrm{i}g| \leqslant |f| + |g|$ implies that any ideal J in a complex Banach lattice is conjugation invariant and itself the complexification of the ideal $J \cap E_{\mathbb{R}}$ of the real part of E.
Suffice it now to say that the meaning of most of the terms introduced for real Banach lattices can be extended to the complex situation under retention (mutatis

mutandis) of the corresponding results valid in the real case by either using the complex modulus or else, if the formation of suprema or infima is involved, by relating them to real parts. For example $f \in E$ is called *positive* if $f = |f|$ which means that f is a positive element of $E_{\mathbb{R}}$, E is called order complete if $E_{\mathbb{R}}$ is order complete, and an ideal J is called a band if the real part of J is a band. We refer to Chapter II, Section 11 of Schaefer (1974) for a detailed discussion of this and restrict ourselves to a short discussion of linear mappings.

Let E and F be complex Banach lattices with real parts $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$. Then a linear mapping T from E into F is determined by its restriction T_0 to $E_{\mathbb{R}}$, and T_0 can be written in the form $T_0 = T_1 + iT_2$ with real-linear mappings T_j from $E_{\mathbb{R}}$ into $F_{\mathbb{R}}$. Thus $L(E, F)$ is the complexification of the real linear space $L(E_{\mathbb{R}}, F_{\mathbb{R}})$. With the above notation, T is called *real* if T_2 is $= 0$, *positive* if T is real and T_1 is positive, and a *lattice homomorphism* if T is real and T_1 is a lattice homomorphism. Lattice homomorphisms are characterized by the equality $|Th| = T|h|$ as in the real case.

8 The Signum Operator

We discuss in some detail how a mapping of the form

$$g \mapsto (\operatorname{sign} f)g$$

which has an obvious meaning, depending on f, in spaces $C(K)$, can be defined in an abstract complex Banach lattice. We prove the following

Let E be a complex Banach lattice and let $f \in E$. If either E is order-complete or $|f|$ is a quasi-interior point in E_+, then there exists a unique linear mapping S_f, called the *signum operator* with respect to f, with the following properties.

(i) $S_f \bar{f} = |f|$, where $\bar{f} = \operatorname{Re}(f) - \mathrm{i} \cdot \operatorname{Im}(f)$,
(ii) $|S_f g| \leqslant |g|$ for every g in E,
(iii) $S_f g = 0$ for every g in E orthogonal to f.

In fact, if $E = C(K)$ and if $|f|$ is a quasi-interior point in E, then $|f|$ is a strictly positive function and multiplication with the function $\operatorname{sign} f = f \cdot |f|^{-1}$ has the desired properties. Uniqueness follows from Zaanen (1983, Chap. 20). We reduce the general situation to the case just considered in the following way.

— If $|f|$ is quasi-interior to E_+, then $E_{|f|}$ is a dense subspace of E isomorphic to a space $C(K)$, and with the above arguments one gets a uniquely determined operator S_0 on $E_{|f|}$ with the desired properties. Since (ii) implies the continuity of S_0 for the norm induced by E, S_0 can be extended to E.
— If f is arbitrary, then, as above, one gets an operator S_0 on $E_{|f|}$ with (i) and (ii). If E is order complete, an extension S_f of S_0 to E satisfying (i)–(iii) is possible as soon as S_0 can be extended to the band $\{x\}^{dd}$ of E.

  - On the complementary band $\{x\}^d$ one has necessarily the values $= 0$ for S_f.

- The extension to $\{x\}^{dd}$ is obtained as follows: If S_0 is positive (which means $f \geqslant 0$), then

$$S_f h = \sup\{S_f g : g \in [0, h] \cap E_{|f|} \text{ for } h \geqslant 0\}$$

will do.

In general, the problem can be reduced to this situation by decomposing S_0 into a sum of the form $S_0 = (S_1 - S_2) + i(S_3 - S_4)$ with positive operators S_j. Such a decomposition of S_0 exists since the order completeness of E implies the order completeness of $E_{|f|} = C(K)$ and since every continuous linear operator on a space $C(K)$ is necessarily order-bounded.

9 The Center of $\mathcal{L}(E)$

We give a short description of a special, but important class of operators.
Let E be a (complex) Banach lattice. For $T \in \mathcal{L}(E)$ the following conditions are equivalent.

(a) $f \perp g$ implies $Tf \perp g \quad (f, g \in E)$,
(b) $\pm T \leqslant \|T\| \mathrm{Id}$,
(c) $TJ \subseteq J$ for every ideal J in E.

If E is countably order complete, then this is equivalent to:

(d) $TJ \subseteq J$ for every projection band J in E.

The last assertion also means that T commutes with every band projection.
The set of all $T \in \mathcal{L}(E)$ satisfying the above conditions is called the *center* of $\mathcal{L}(E)$ and denoted $\mathcal{Z}(E)$. Under its natural ordering, the operator norm and the composition product is $\mathcal{Z}(E)$ isomorphic to a Banach lattice algebra $C(K)$ with K compact. The following examples may illustrate the situation and explain why the term *multiplication operator* is often used for operators in $\mathcal{Z}(E)$.

(i) If $E = L^p(X, \Sigma, \mu)$ $(1 \leqslant p \leqslant \infty)$ with σ-finite μ, then $Z(E)$ is isomorphic to $L^\infty(\mu)$ via the natural identification of a function $f \in L^\infty(\mu)$ with the multiplication operator $g \mapsto f \cdot g$ on E.
(ii) If X is locally compact, $E = C_0(X)$, then similarly $\mathcal{Z}(E) \cong C^b(X)$ via the identification of $f \in C^b(X)$ with the mapping $g \mapsto f \cdot g$ $(g \in C_0(X))$.

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Chapter C-II

Characterization of Positive Semigroups on Banach Lattices and Positive Operators

Chapter C-III

Spectral Theory of Positive Semigroups on Banach Lattices

Chapter C-IV

Asymptotics of Positive Semigroups on Banach Lattices

Part D
Positive Semigroups on C*- and W*-Algebras

Chapter D-I
Basic Results on Semigroups and Operator Algebras

This is not a systematic introduction to the theory of strongly continuous semigroups on C*- and W*-algebras. We only prepare for the following chapters on spectral and asymptotic theory by fixing the notations and introducing some standard constructions. For results on strongly continuous semigroups on Banach spaces, we refer to Chapter A-I.

1 Notations

1. Let M denote a C*-algebra with unit $\mathbb{1}$, where $M^{sa} := \{x \in M : x^* = x\}$ is the self-adjoint part of M and $M_+ := \{x^*x : x \in M\}$ is the positive cone in M. If M' is the dual of M, then $M'_+ := \{\varphi \in M' : \varphi(x) \geqslant 0, x \in M_+\}$ is a weak*-closed generating cone in M' and $S(M) := \{\varphi \in M'_+ : \varphi(\mathbb{1}) = 1\}$ is called the state space of M. For the theory of C*-algebras and related notions see Pedersen (1979).
2. We say that M is a W*-algebra if there exists a Banach space M_* such that its dual $(M_*)'$ is (isomorphic to) M. We call M_* the *predual* of M and $\varphi \in M_*$ a *normal linear functional*. It is known that M_* is unique. For this and other properties of M_*, see Takesaki (1979, Chapter III).
3. A map $T \in \mathrm{L}M$ is called *positive* (in symbols $T \geqslant 0$) if $T(M_+) \subseteq M_+$. It is called *n-positive* ($n \in \mathbb{N}$) if $T \otimes \mathrm{Id}_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where Id_n is the identity map on the C*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive.
We call a contraction $T \in \mathrm{L}M$ a *Schwarz map* if T satisfies the so called *Schwarz-inequality*
$$T(x)T(x)^* \leqslant T(xx^*)$$
for all $x \in M$. It is well known that every n-positive contraction, for $n \geqslant 2$ and every positive contraction on a commutative C*-algebra is a Schwarz map. (Takesaki (1979, Chapter IV)) As we shall see, the Schwarz inequality is crucial for our investigations.

4. If M is a C*-algebra, we assume that $\mathcal{T} = (T(t))_{t \geqslant 0}$ is a strongly continuous semigroup (abbreviated as semigroup), while for W*-algebras we consider weak*-semigroups, i.e. the mapping $(t \mapsto T(t)x)$ is continuous from $\mathbb{R}_+$ into $(M, \sigma(M, M_*))$, where M_* is the predual of M, and every $T(t) \in \mathcal{T}$ is $\sigma(M, M_*)$-continuous. Note that the preadjoint semigroup

$$\mathcal{T}_* = \{T(t)_* : T(t) \in \mathcal{T}\}$$

is weakly, hence strongly continuous on M_*. (Chapter A-I, **??**)
5. We call the semigroup $\mathcal{T}$ *identity preserving* if $T(t)\mathbb{1} = \mathbb{1}$ and of *Schwarz type* if every $T(t)$ is a Schwarz map.

For the notations concerning one-parameter semigroups we refer to Part A. In addition, we recommend to compare the results of this section with the corresponding results for commutative C*-algebras, i.e., for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ in Part B.

2 A Fundamental Inequality for the Resolvent

If $\mathcal{T} = (T(t))_{t \geqslant 0}$ is a strongly continuous semigroup of Schwarz maps on a C*-algebra M (resp. a weak*-semigroup of Schwarz type on a W*-algebra M) with generator A, then the spectral bound satisfies $s(A) \leqslant 0$. The resolvent $R(\lambda, A)$ exists for $\mathrm{Re}(\lambda) > 0$ and is positive for $\lambda \in \mathbb{R}_+$. There exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \,\mathrm{d}t \quad (x \in M)$$

where the integral exists in the norm topology.
The next theorem relates the domination of two semigroups to an inequality for the corresponding resolvent operators. This inequality will be needed later and can be used to characterize semigroups of Schwarz type on C*-algebras.

Theorem 2.1 *Let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a semigroup of Schwarz type with generator A and $\mathcal{S} = (S(t))_{t \geqslant 0}$ a semigroup with generator B on a C*-algebra M. If*

$$(S(t)x)(S(t)x)^* \leqslant T(t)(xx^*) \qquad (*)$$

for all $x \in M$ and $t \in \mathbb{R}_+$. Then

$$(\mu R(\mu, B)x)\,(\mu R(\mu, B)x)^* \leqslant \mu R(\mu, A)xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$. The same result holds if $\mathcal{T}$ is a weak-semigroup of Schwarz type and $\mathcal{S}$ is a weak*-semigroup on a W*-algebra M such that $(*)$ is fulfilled.*

Proof From the assumption $(*)$ it follows that

$$\begin{aligned}0 \leqslant (S(r)x - S(t)x)\,(S(r)x - S(t)x)^* \\ &= (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* \\ &\quad - (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \\ &\leqslant T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^*\end{aligned}$$

for every $r, t \in \mathbb{R}_+$ and therefore

$$(S(r)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \leqslant T(r)xx^* + T(t)xx^*.$$

Obviously, $\|S(t)\| \leqslant 1$ for all $t \in \mathbb{R}_+$. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$

$$\begin{aligned}(R(\mu, B)x)\,(R(\mu, B)x)^* &= \left(\int_0^\infty e^{-\mu r} S(r)x\,\mathrm{d}r\right)\left(\int_0^\infty e^{-\mu t} S(t)x\,\mathrm{d}t\right)^* \\ &= \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}((S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^*\,\mathrm{d}r\,\mathrm{d}t\right) \\ &\leqslant \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*)\,\mathrm{d}r\,\mathrm{d}t\right) \\ &= \left(\int_0^\infty e^{-\mu s} ds\right)\left(\int_0^\infty e^{-\mu t} T(t)xx^*\,\mathrm{d}t\right) = \mu^{-1} R(\mu, A)xx^*\end{aligned}$$

where the handling of the integral is justified by Bourbaki (1955, Chap. V, §8, n° 4, Proposition 9). The claim is obtained by multiplying both sides by μ^2. □

Corollary 2.2 *Let $\mathcal{T}$ be a semigroup of Schwarz maps (resp. weak*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with* Re $(\lambda) > 0$ *we have*

$$(R(\lambda, A)x)\,(R(\lambda, A)x)^* \leqslant \mathrm{Re}(\lambda)^{-1} R(\mathrm{Re}(\lambda), A)xx^*\,,\ x \in M.$$

In particular for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in M$

$$(\mu R(\mu + \mathrm{i}\alpha, A)x)(\mu R(\mu + \mathrm{i}\alpha, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Then the semigroup

$$S := \left(e^{-\mathrm{i}(\lambda)t} T(t)\right)_{t \geqslant 0}$$

fulfills the assumption of Thm. 2.1 and $B := A - \mathrm{i}\lambda$ is the generator of S. Consequently $R(\lambda, A) = R(\mathrm{Re}\lambda, B)$ and the corollary follows from Thm. 2.1. □

Remark 2.3 Since

$$T(t)x = \lim_n \left(\frac{n}{t} R\left(\frac{n}{t}, A\right)\right)^n x, \quad x \in M,$$

it follows from above, that $\mathcal{T}$ is a semigroup of Schwarz-type, if and only if $\mu R(\mu, A)$ is a Schwarz-operator for every $\mu \in \mathbb{R}_+$.

As in Section C-III the following notion will be an important tool for the spectral theory of semigroups on C*- and W*-algebras.

Definition 2.4 Let E be a Banach space and let D be a non-empty open subset of $\mathbb{C}$. A family $\mathcal{R}\colon D \mapsto L(E)$ is called a *pseudo-resolvent* on D with values in E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu) \qquad \text{(Resolvent Equation)}$$

for all λ, μ in D and $R \in \mathcal{R}$.

If $\mathcal{R}$ is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$ with values in a C*- or W*-algebra, then $\mathcal{R}$ is called of Schwarz type if

$$(R(\lambda)x)(R(\lambda)x)^* \leqslant (\operatorname{Re}\lambda)^{-1} R(\operatorname{Re}\lambda)xx^*$$

and *identity preserving* if $\lambda R(\lambda)\mathbb{1} = \mathbb{1}$ for all $\lambda \in D$ and $R \in \mathcal{R}$. For examples and properties of a pseudo-resolvent, see C-III, 2.5.
We state what will be used without further reference.

(i) If $\alpha \in \mathbb{C}$ and $x \in E$ such that $(\alpha - \lambda)R(\lambda)x = x$ for some $\lambda \in D$, then $(\alpha - \mu)R(\mu)x = x$ for all $\mu \in D$ (use the *resolvent equation*).
(ii) If F is a closed subspace of E such that $R(\lambda)F \subseteq F$ for some $\lambda \in D$, then $R(\mu)F \subseteq F$ for all μ in a neighborhood of λ. This follows from the fact that for all $\mu \in D$ near λ the pseudo-resolvent in μ is given by

$$R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}.$$

Definition 2.5 We call a semigroup $\mathcal{T}$ on the *predual* M_* of a W*-algebra M *identity preserving and of Schwarz type* if its adjoint weak*-semigroup has these properties. Similarly, a pseudo-resolvent $\mathcal{R}$ on $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$ with values in M_* is said to be identity preserving and of Schwarz type if $\mathcal{R}'$ has these properties.

For a semigroup of contractions on a Banach space we have

$$\begin{aligned}\operatorname{Fix}(T) &= \bigcap_{t \geqslant 0} \ker(\operatorname{Id} - T(t)) \\ &= \ker(\operatorname{Id} - \lambda R(\lambda, A)) = \operatorname{Fix}((\lambda R(\lambda, A))\end{aligned}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$. Therefore a semigroup of contractions on M is identity preserving, if and only if the pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$ given by

$$R(\lambda) := R(\lambda, A)_{|D}$$

is identity preserving. By Corollary 2.2 an analogous statement holds for *Schwarz type*.

3 Induction and Reduction

1. If E is a Banach space and $\mathcal{S} \subseteq \mathcal{L}(E)$ is a semigroup of bounded operators, then a closed subspace F is called $\mathcal{S}$-invariant, if $SF \subseteq F$ for all $S \in \mathcal{S}$. We call the semigroup $\mathcal{S}_{|F} := \{S_{|F} : S \in \mathcal{S}\}$ the reduced semigroup. Note that for a one-parameter semigroup $\mathcal{T}$ (resp., pseudo-resolvent $\mathcal{R}$) the reduced semigroup is again strongly continuous (resp. $\mathcal{R}_{|F}$ is again a pseudo-resolvent). (Compare A-I, 3.2).
2. Let M be a W*-algebra, $p \in M$ a projection and $S \in \mathrm{L}M$ such that

$$S(p^{\perp}M) \subseteq p^{\perp}M \quad \text{and} \quad S(Mp^{\perp}) \subseteq Mp^{\perp},$$

where $p^{\perp} := \mathbb{1} - p$. Since for all $x \in M$

$$p[S(x) - S(pxp)] = p[S(p^{\perp}xp) + S(xp^{\perp})]p = 0,$$

we obtain $p(Sx)p = p(S(pxp))p$. Therefore, the map

$$S_p := (x \mapsto p(Sx)p) : pMp \to pMp$$

is well defined and we call S_p the *induced map*. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$.
3. If $\mathcal{T} = (T(t))_{t \geqslant 0}$ is a weak*-semigroup on M which is of Schwarz type and if $T(t)(p^{\perp}) \leqslant p^{\perp}$ for all $t \in \mathbb{R}_+$, then T leaves $p^{\perp}M$ and $Mp^{\perp}$ invariant. One can verify that the induced semigroup $T_p = (T(t)p)_{t \geqslant 0}$ is again a weak*-semigroup. If $\mathcal{R}$ is an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in M such that $R(\mu)p^{\perp} \leqslant p^{\perp}$ for some $\mu \in \mathbb{R}_+$, then $p^{\perp}M$ and $Mp^{\perp}$ are $\mathcal{R}$-invariant. It follows directly that the induced pseudo-resolvent $\mathcal{R}_p$ has both the Schwarz type property and is identity preservation.
4. Let φ be a positive normal linear functional on a W*-algebra M such that $T_*\varphi = \varphi$ for some identity preserving Schwarz map T on M with preadjoint $T_* \in L(M_*)$. Then $T(s(\varphi)^{\perp}) \leqslant s(\varphi)^{\perp}$ where $s(\varphi)$ is the support projection of φ.
Let

$$L_{\varphi} := \{x \in M : \varphi(xx^*) = 0\} \quad \text{and} \quad M_{\varphi} := L_{\varphi} \cap L_{\varphi}^*.$$

Since φ is T_*-invariant and T is a Schwarz map, the subspaces L_{φ} and M_{φ} are T-invariant. From $M_{\varphi} = s(\varphi)^{\perp}Ms(\varphi)^{\perp}$ and $T(s(\varphi)^{\perp}) \leqslant 1$ it follows that $T(s(\varphi)^{\perp}) \leqslant s(\varphi)^{\perp}$.

Let $T_{s(\varphi)}$ be the induced map on $M_{s(\varphi)}$ and define

$$s(\varphi)M_*s(\varphi) := \{\psi \in M_* : \psi = s(\varphi)\psi s(\varphi)\}$$

where $\langle s(\varphi)\psi s(\varphi), x\rangle := \langle \psi, s(\varphi)xs(\varphi)\rangle$ $(x \in M)$. For any $\psi \in s(\varphi)M_s(\varphi)$ and all $x \in M$, the following equalities holds

$$\begin{aligned}(T_*\psi)(x) &= \psi(Tx) = \langle \psi, s(\varphi)(Tx)s(\varphi)\rangle \\ &= \langle \psi, s(\varphi)(T(s(\varphi)xs(\varphi)))s(\varphi)\rangle = \langle T_*\psi, s(\varphi)xs(\varphi)\rangle,\end{aligned}$$

hence $T_*\psi \in s(\varphi)M_*s(\varphi)$. Since the dual of $s(\varphi)M_*s(\varphi)$ is $M_{s(\varphi)}$, it follows that the adjoint of the reduced map $T_{*|}$ is identity preserving and of Schwarz type.
For example, if $\mathcal{T}$ is an identity preserving semigroup of Schwarz type on M_* such that $\varphi \in \mathrm{Fix}(T)$, then the semigroup $T_{|(s(\varphi)M_*s(\varphi))}$ is again identity preserving and of Schwarz type. Furthermore, if $\mathcal{R}$ is a pseudo-resolvent on

$$D = \{\lambda \in \mathbb{C} \colon \ \mathrm{Re}(\lambda) > 0\}$$

with values in M_* which is identity preserving and of Schwarz type such that $R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$, then $\mathcal{R}_{|s(\varphi)M_*s(\varphi)}$ has the same properties.

Notes

We refer to Bratteli and Robinson (1979), Davies (1976) and the survey article of Oseledets (1984).

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Chapter D-II
Characterization of Positive Semigroups on W*-Algebras

Since the positive cone of a C*-algebra has non-empty interior, many results of Chapter B-II can be applied verbatim to the characterization of the generator of positive semigroups on C*-algebras. On the other hand, a concrete and detailed representation of such generators has been found only in the uniformly continuous case (see Lindblad (1976)). A third area of active research has been the following: Which maps on C*-algebras (in particular, which derivations) commuting with certain automorphism groups are automatically generators of strongly continuous positive semigroups. For more information we refer to the survey article of Evans (1984).

1 Semigroups on Properly Infinite W*-Algebras

The aim of this section is to show that strongly continuous semigroups of Schwarz maps on properly infinite W*-algebras are already uniformly continuous. In particular, our theorem is applicable to such semigroups on $B(H)$.
It is worthwhile to remark that the result of Lotz (1985) on the uniform continuity of every strongly continuous semigroup on L^∞ (see A-II, Sec.3) does not extend to arbitrary W*-algebras.

Example 1.1 Take $M = \mathcal{B}(H)$, H infinite dimensional, and choose a projection $p \in M$ such that Mp is topologically isomorphic to H. Therefore $M = H \oplus M_0$, where $M_0 = \mathrm{Ker}(x \mapsto xp)$. Next, take a strongly, but not uniformly continuous semigroup $\mathcal{T}$ on H and consider the strongly continuous semigroup $\mathcal{T} \oplus \mathrm{Id}$ on M.

For results on the classification theory of W*-algebras needed in our approach we refer to Sakai (1971, 2.2) and Takesaki (1979, V.1).

Theorem 1.2 *Every strongly continuous one-parameter semigroup of Schwarz type on a properly infinite* W*-*algebra M is uniformly continuous.*

Proof Let $\mathcal{T} = (T(t)_{t\geqslant 0})$ be strongly continuous on M and suppose $\mathcal{T}$ not to be uniformly continuous. Then there exists a sequence (T_n) in $\mathcal{T}$ and $\varepsilon > 0$ such that $\|T_n - \mathrm{Id}\| \geqslant \varepsilon$, but $T_n \to \mathrm{Id}$ in the strong operator topology. We claim that for every sequence (p_k) of mutually orthogonal projections and all bounded sequences (x_k) in M

$$\lim_n \|(T_n - \mathrm{Id})(p_k x_k p_k)\| = 0$$

uniformly in $k \in \mathbb{N}$. This follows from the *Lemma of Phillips* (Schaefer (1974)) and the fact that the sequence $(p_k x_k p_k)$ is summable in the $s^*(M, M_*)$-topology (compare Elliot (1972), Lemma 2.).
Let (p_k) be a sequence of mutually orthogonal projections in M such that every p_k is equivalent to $\mathbb{1}$ via some $u_k \in M$ (Sakai, 1971, 2.2). Without loss of generality we may assume $\|(T_n - \mathrm{Id})(u_n)\| \leqslant n^{-1}$ since the semigroup T is strongly continuous. Thus we obtained the following.

(i) $\lim_n \|(T_n - \mathrm{Id})(p_k x_k p_k)\| = 0$ uniformly in $k \in \mathbb{N}$ for every bounded sequence (x_k) in M.
(ii) Every projection p_k is equivalent to 1 via some $u_k \in M$.
(iii) $\|(T_n - \mathrm{Id})u_n\| \leqslant n^{-1}$ for all $n \in \mathbb{N}$.

For the following construction see A-I,3.6 and D-II,Sec.2. Take

(i) $\widehat{M}$ be an ultrapower of M,
(ii) let $p := \widehat{(p_k)} \in \widehat{M}$,
(iii) let $T := \widehat{(T_n)} \in L(\widehat{M})$
(iv) and let $u := \widehat{(u_k)} \in \widehat{M}$.

Then T is identity preserving and of Schwarz type on $\widehat{M}$.
Since $u^*u = p$ and $uu^* = \mathbb{1}$, it follows $pu^* = u^*$ and $(uu^*)x(uu^*) = x$ for all $x \in \widehat{M}$. Finally, $T(pxp) = pxp$ for all $x \in \widehat{M}$ which follows from (i), and $T(u^*) = T(pu^*) = pu^* = u^*$ and $T(u) = u$, which follows from (iii). Using the Schwarz, inequality we obtain

$$T(uu^*) = T(\mathbb{1}) \leqslant \mathbb{1} = uu^* = T(u)T(u)^*.$$

From D-III, Lemma 1.1., we conclude $T(ux) = uT(x)$ and $T(xu^*) = T(x)u^*$ for all $x \in \widehat{M}$. Hence

$$\begin{aligned} T(x) &= T(uu^*xuu^*) = uT(u^*xu)u^* = uT(pu^*xup)u^* \\ &= upu^*xupu^* = uu^*xuu^* = x \end{aligned}$$

for all $x \in \widehat{M}$. From this we obtain that for every bounded sequence (x_k) in M

$$\lim_k \|T_k x_k - x_k\| = 0$$

for some subsequence of the T_k's and of the x_k's. This conflicts with our assumption at the beginning, hence the theorem is proved. □

Notes

Let M be a W*-algebra and H be an infinite-dimensional Hilbert-space. Then the W*-tensor product $N := M\overline{\otimes}\mathcal{B}(H)$ is a properly infinite W*-algebra (Sakai (1971, Thm. 2.6.6)). Let $\mathcal{S}$ be the semigroup

$$S(t) = T(t) \otimes \mathrm{Id}_H \quad (t \geqslant 0).$$

Then $S(t)$ is a Schwarz-map on N and contractive (Takesaki (1979, Prop. IV.5.13.)), hence the smigoup $\mathcal{S}$ is equicontinuous in LN.
Let $x \in M$ and $\xi \in H$. Since the norm on N is a cross-norm, we obtain

$$\lim_{t\to 0} \|(S(t) - \mathrm{Id})x \otimes \xi\| = \lim_{t\to 0} \|(S(t) - \mathrm{Id})x\|\|\xi\| = 0.$$

From Schaefer (1966, III.4.5) it follows that $\mathcal{S}$ is strongly-continuous, hence norm-continuous on N from which we conclude, that $\mathcal{T}$ is norm-continuous on M.

Remark 1.3 If M is a finite W*-algebra of Type I, then M is a Grothendieck space and has the Dunford-Pettis property. Hence we can apply the results of Lotz (1985). However, has W*-algebra have the Dunford-Pettis property iff it is finite and of Type I (Chu and Iochum (1990)). But is known that every W*-algebra is a Grothendick space (Pfitzner (1994).

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Chapter D-III
Spectral Theory of Positive Semigroups on W*-Algebras and their Preduals

Motivated by the classical results of Perron and Frobenius one expects the following spectral properties for the generator A of a positive semigroup on a C*-algebra.

> The spectral bound $s(A) := \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\}$ belongs to the spectrum $\sigma(A)$ and the boundary spectrum $\sigma_b(A) := \sigma(A) \cap \{s(A) + \mathrm{i}\mathbb{R}\}$ possesses a certain symmetric structure, called cyclicity.

Results of this type have been proved in Chapter B-III for positive semigroups on commutative C*-algebras, however in the non-commutative case the situation is more complicated. While "$s(A) \in \sigma(A)$" still holds (see Greiner et al. (1981) or the notes of this chapter), the cyclicity of the boundary spectrum $\sigma_b(A)$ is true only under additional assumptions on the semigroup (e.g., irreducibility, see Section 1 below).
For technical reasons we consider mostly strongly continuous semigroups on the predual of a W*-algebra M or its adjoint semigroup which is a weak*-continuous semigroup on M.

1 Spectral Theory for Positive Semigroups on Preduals

The aim of this section is to develop a Perron-Frobenius theory for identity preserving semigroups of Schwarz type on W*-algebras. However we will show in the example preceding Theorem 1.11 on page 126 below that the boundary spectrum is no longer cyclic. The appropriate hypothesis on the semigroup implying the desired results seems to be the concept of *irreducibility*.
Let us first recall some facts on normal linear functionals. If φ is a normal linear functional on a W*-algebra M, then there exists a partial isometry $u \in M$ and a positive linear functional $|\varphi| \in M_*$ such that

$$\begin{aligned} \varphi(x) = |\varphi|(xu) =: (u|\varphi|)(x) \quad (x \in M), \\ u^*u = s(|\varphi|), \end{aligned}$$

where $s(|\varphi|)$ denotes the support projection of $|\varphi|$ in M. We refer to this as the *polar decomposition* of φ. In addition, $|\varphi|$ is *uniquely determined* by the following two conditions.

$$\left.\begin{aligned} \|\varphi\| &= \||\varphi|\| \\ |\varphi(x)|^2 &\leqslant |\varphi|(xx^*) \quad (x \in M) \end{aligned}\right\} (*)$$

For the polar decomposition of the adjoint φ^*, where $\varphi^*(x) = \overline{\varphi(x^*)}$, we obtain

$$\varphi^* = u^*|\varphi^*|, \quad |\varphi^*| = u|\varphi|u^* \quad \text{and} \quad uu^* = s(|\varphi^*|).$$

It is easy to see that $u^* \in s(|\varphi|)M$ (Takesaki (1979, Theorem III.4.2 & Proposition III.4.6)).
If Ψ is a subset of the state space of a C*-algebra M, then Ψ is called *faithful* if $0 \leqslant x \in M$ and $\psi(x) = 0$ for all $\psi \in \Psi$ implies $x = 0$. Moreover Ψ is called *subinvariant* for a positive map $T \in \mathrm{L}M$ (resp.., positive semigroup $\mathcal{T}$) if $T'\psi \leqslant \psi$ for all $\psi \in \Psi$ (resp. $T(t)'\psi \leqslant \psi$ for all $T(t) \in \mathcal{T}$ and $\psi \in \Psi$). Recall that for every positive map $T \in \mathrm{L}M$ there exists a state φ on M such that $T'\varphi = r(T)\varphi$, where $r(T)$ denotes the spectral radius of T (Groh (1981, Theorem 2.1)).
Let us start our investigation with two lemmata where $\mathrm{Fix}\,(T)$ is the fixed space of T, i.e., the set $\{x \in M : Tx = x\}$.

Lemma 1.1 *Suppose M to be a* C**-algebra and $T \in LM$ an identity preserving Schwarz map.*

(i) Let $b\colon M \times M \to M$ be a sesquilinear map such that $b(z,z) \geqslant 0$ for all $z \in M$. Then $b(x,x) = 0$ for some $x \in M$ if and only if $b(x,y) = 0$ and $b(y,x) = 0$ for all $y \in M$.
(ii) If there exists a faithful family Ψ of subinvariant states for T on M, then $\mathrm{Fix}\,(T)$ is a C**-subalgebra of M and $T(xy) = xT(y)$ for all $x \in \mathrm{Fix}\,(T)$ and $y \in M$.*

Proof (i) Take $0 \leqslant \psi \in M^*$ and consider $f := \psi \circ b$. Then f is a positive semidefinite sesquilinear form on M with values in $\mathbb{C}$. From the Cauchy-Schwarz inequality it follows that $f(x,x) = 0$ for some $x \in M$ if and only if $f(x,y) = 0$ and $f(y,x) = 0$ for all $y \in M$. Since the positive cone M_+^* is generating, assertion (i) is proved.
(ii) Since T is positive, it follows that $T(x)^* = T(x^*)$ for all $x \in M$. Hence $\mathrm{Fix}\,(T)$ is a self adjoint subspace of M, i.e., invariant under the involution on M. For every $x, y \in M$ define

$$b(x,y) := T(xy^*) - T(x)T(y)^*.$$

Then b satisfies the assumptions of (i).
If $x \in \mathrm{Fix}\,(T)$, then

$$0 \leqslant xx^* = (Tx)(Tx)^* \leqslant T(xx^*),$$

hence

$$0 \leqslant \psi(T(xx^*) - xx^*) = 0 \quad \text{for all } \psi \in \Psi.$$

But this implies $T(xx^*) = T(x)T(x)^* = xx^*$ and consequently, $b(x, x) = 0$. Hence $T(xy^*) = xT(y)^*$ for all $y \in M$ and (ii) is proved. □

Lemma 1.2 *Let M be a* W**-algebra, T an identity preserving Schwarz map on M and $S \in LM$ such that $S(x)(Sx)^* \leqslant T(xx^*)$ for every $x \in M$.*

(i) If $v \in M$ such that $S(v^*) = v^*$ and $T(v^*v) = v^*v$, then $T(xv) = S(x)v$ for all $x \in M$.
(ii) Suppose there exists $\varphi \in M_*$ with polar decomposition $\varphi = u|\varphi|$ such that $S_*\varphi = \varphi$ and $T_*|\varphi| = |\varphi|$. If the closed subspace $s(|\varphi|)M$ is T-invariant, then $Su^* = u^*$ and $T(u^*u) = u^*u$.

Proof (i) Define a positive semidefinite sesquilinear map $b : M \times M \mapsto M$ by

$$b(x, y) := T(xy^*) - S(x)S(y)^* \quad (x, y \in M).$$

Since $b(v^*, v^*) = 0$ we obtain $b(x, v^*) = 0$ for all $x \in M$, hence $T(xv) = S(x)v$. (Lemma 1.1 (i))
(ii) Since $s(|\varphi|)M$ is a closed right ideal, the closed face $F := s(|\varphi|)(M_+)s(|\varphi|)$ determines $s(|\varphi|)M$ uniquely, i.e.,

$$s(|\varphi|)M = \{x \in M : xx^* \in F\}$$

(Pedersen (1979, Theorem 1.5.2)). Since T is a Schwarz map and $s(|\varphi|)M$ is T-invariant, it follows $TF \subseteq F$. On the other hand, if $x \in s(|\varphi|)M$, then $xx^* \in F$. Consequently,

$$0 \leqslant S(x)S(x)^* \leqslant T(xx^*) \in F,$$

whence $S(x) \in s(|\varphi|)M$.
Next we show $T(u^*u) = u^*u$ and $Su^* = u^* \in s(|\varphi|)M$. First of all

$$\begin{aligned} 0 &\leqslant (Su^* - u^*)(Su^* - u^*)^* \\ &\leqslant T(u^*u) - u^*S(u^*)^* - (Su^*)u + u^*u. \end{aligned}$$

Since $S_*\varphi = \varphi$, $T_*|\varphi| = |\varphi|$ and $\varphi = u|\varphi|$ it follows

$$\begin{aligned} 0 &\leqslant |\varphi|((Su^* - u^*)(Su^* - u^*)^*) \\ &\leqslant 2|\varphi|(u^*u) - |\varphi|(S(u^*)u)^* - |\varphi|(S(u^*)u) \\ &= 2|\varphi|(uu^*) - \varphi(u^*)^* - \varphi(u^*) \\ &= 2(|\varphi|(u^*u) - |\varphi|(u^*u)) = 0. \end{aligned}$$

But $(Su^* - u^*)(Su^* - u^*) \in F$ and $|\varphi|$ is faithful on F. Hence we obtain $Su^* = u^*$. Consequently,

$$0 \leqslant u^*u = (Su^*)(Su^*)^* \leqslant T(u^*u)$$

and $T(u^*u) = u^*u$ by the faithfulness and T-invariance of $|\varphi|$. □

Remark 1.3 Take S and T as in Lemma 1.2 (ii). If V_{u^*} (resp. V_u) is the map $(x \mapsto xu^*)$ (resp. $(x \mapsto xu)$) on M, then V_{u^*} is a continuous bijection from $Ms(|\varphi|)$ onto $Ms(|\varphi^*|)$ with inverse V_u (because $V_u \circ V_{u^*} = \mathrm{Id}_{Ms(|\varphi|)}$ and $V_{u^*} \circ V_u = \mathrm{Id}_{Ms(|\varphi^*|)}$). Let $x \in M$. From $T(xu) = S(x)u$ we obtain $T(xu)u^* = S(x)uu^*$. In particular, if $Ms(|\varphi^*|)$ is S-invariant, then

$$(V_{u^*} \circ T \circ V_u)(x) = T(xu)u^* = S(x)$$

for every $x \in Ms(|\varphi^*|)$. Let $T_|$ (resp. $S_|$) be the restriction of T to $Ms(|\varphi|)$ (resp. of $\mathcal{S}$ to $Ms(|\varphi^*|)$). Then the following diagram is commutative:

$$\begin{tikzcd}
Ms(|\varphi|) \arrow[r, "T_|"] \arrow[d, "V_u"'] & Ms(|\varphi|) \arrow[d, "V_{u^*}"] \\
Ms(|\varphi^*|) \arrow[r, "S_|"'] & Ms(|\varphi^*|)
\end{tikzcd}$$

In particular, $\sigma(S_|) = \sigma(T_|)$. Therefore we may deduce spectral properties of $S_|$ from $T_|$ and vice versa. More concrete applications of Lemma 1.2 will follow.

We now investigate the fixed space $\mathrm{Fix}\,(\mathcal{R}) := \mathrm{Fix}\,(\lambda R(\lambda))$, $\lambda \in D$, of a pseudo-resolvent $\mathcal{R}$ with values in the predual of a W*-algebra M.

Proposition 1.4 *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon \ \mathrm{Re}\,\lambda > 0\}$ with values in the predual M_* of a* W*-*algebra M and suppose $\mathcal{R}$ to be identity preserving and of Schwarz type.*

(i) If $\alpha \in \mathbb{R}$ and $\psi \in M_*$ such that $(\gamma - \mathrm{i}\alpha)R(\gamma)\psi = \psi$ for some $\gamma \in D$, then $\lambda R(\lambda)|\psi| = |\psi|$ and $\lambda R(\lambda)|\psi^*| = |\psi^*|$ for all $\lambda \in D$.

(ii) $\mathrm{Fix}\,(\mathcal{R})$ is invariant under the involution in M_*. If $\psi \in \mathrm{Fix}\,(\mathcal{R})$ is self adjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of $\mathrm{Fix}\,(\mathcal{R})$.

Proof If $(\gamma - \mathrm{i}\alpha)R(\gamma)\psi = \psi$ then $(\lambda - \mathrm{i}\alpha)R(\lambda)\psi = \psi$ for all $\lambda \in D$. In particular, $\mu R(\mu + \mathrm{i}\alpha)\psi = \psi$ $(\mu \in \mathbb{R}_+)$. For all $x \in M$ we obtain

$$\begin{aligned}
|\psi(x)|^2 &= |<\mu R(\mu + \mathrm{i}\alpha)'x, \psi>|^2 \leqslant \\
&\leqslant \|\psi\| < (\mu R(\mu + \mathrm{i}\alpha)'x)(\mu R(\mu + \mathrm{i}\alpha)'x)^*, \psi > \leqslant \\
&\leqslant \|\psi\| < \mu R(\mu)'(xx^*), |\psi| >
\end{aligned}$$

(D-I, Corollary 2.2). Since

$$\begin{aligned}
\|\psi\| &= \||\psi|\| = |\psi|(1) = \\
&= < \mu R(\mu)'1, |\psi| > = \|\mu R(\mu)|\psi|\|,
\end{aligned}$$

we obtain $\mu R(\mu)|\psi| = |\psi|$ by the uniqueness theorem (*) above for the absolut value— therefore $|\psi| \in \mathrm{Fix}\,(\mathcal{R})$. Since

$$0 \leqslant (\mu R(\mu)' x)(\mu R(\mu)' x)^* \leqslant \mu R(\mu)' x x^*,$$

the map $R(\mu)$ is positive. Consequently $(\mu + \mathrm{i}\alpha) R(\mu)\psi^* = \psi^*$ from which $|\psi^*| \in \operatorname{Fix}(\mathcal{R})$ follows. If $\varphi \in \operatorname{Fix}(\mathcal{R})$ is selfadjoint with Jordan decomposition $\varphi = \varphi^+ - \varphi^-$, then $|\varphi| = \varphi^+ + \varphi^-$ (Takesaki (1979, Theorem III.4.2.)). From this we obtain that φ^+ and φ^- are in $\operatorname{Fix}(\mathcal{R})$. □

Corollary 1.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on M_* with generator A and suppose $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.*

(i) If $\alpha \in \mathbb{R}$ and $\psi \in \ker(\mathrm{i}\alpha - A)$, then $|\psi|$ and $|\psi^*|$ are elements of $\operatorname{Fix}(\mathcal{T}) = \operatorname{Ker}(A)$.

(ii) $\operatorname{Fix}(\mathcal{T})$ *is invariant under the involution of M_*. If $\psi \in \operatorname{Fix}(\mathcal{T})$ is selfadjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of* $\operatorname{Fix}(\mathcal{T})$.

The proof follows immediately from Proposition 1.4 and the fact that $\operatorname{Ker}(A) = \operatorname{Fix}(\lambda R(\lambda, A))$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.
If $\mathcal{T}$ is the semigroup of translations on $L^1(\mathbb{R})$ and A' the generator of the adjoint weak*-semigroup, then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$, while $P\sigma(A') \cap \mathrm{i}\mathbb{R} = \mathrm{i}\mathbb{R}$. For that reason we cannot expect a simple connection between these two sets. But as we shall see below, if a semigroup on the predual of a W*-algebra has sufficiently many invariant states, then the point spectra contained in $\mathrm{i}\mathbb{R}$ of A and A' are identical.
Helpful for these investigations will be the next lemma.

Lemma 1.6 *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ with values in a Banach space E such that $\|R(\mu + \mathrm{i}\alpha)\| \leqslant 1$ for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$. Then*

$$\dim \operatorname{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)) \leqslant \dim \operatorname{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)')$$

for all $\lambda \in D$.

Proof Let $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $S := \mu R(\mu + \mathrm{i}\alpha)$. Since S is a contraction, its adjoint S' maps the dual unit ball E_1' into itself.
Let $\mathfrak{U}$ be a free ultrafilter on $[1, \infty[$ which converges to 1. Since E_1' is $\sigma(E', E)$-compact,

$$\psi_0 := \lim_{\mathfrak{U}} (\lambda - 1) R(\lambda, S)' \psi$$

exists for each $\psi \in E_1'$. Since S' is $\sigma(E', E)$-continuous and since $S' R(\lambda, S)' = \lambda R(\lambda, S') - \mathrm{Id}$ we conclude $\psi_0 \in \operatorname{Fix}(S')$.
Take now $0 \neq x_0 \in \operatorname{Fix}(S)$ and choose $\psi \in E_1'$ such that $\psi(x_0)$ is different from zero. From the considerations above it follows

$$\psi_0(x_0) = \lim_{\mathfrak{U}} (\lambda - 1) \psi(R(\lambda, S) x_0) = \psi(x_0) \neq 0$$

hence $0 \neq \psi_0 \in \operatorname{Fix}(S)$. Therefore $\operatorname{Fix}(S')$ separates the points of $\operatorname{Fix}(S)$.
From this it follows that

$$\dim \operatorname{Fix}(S) \leqslant \dim \operatorname{Fix}(S').$$

Since $\mathcal{R}$ and $\mathcal{R}'$ are pseudo-resolvents, the assertion is proved. □

Corollary 1.7 *Let $\mathcal{T}$ be a semigroup of contractions on a Banach space E with generator A. Then*

$$\dim \operatorname{Ker}(\mathrm{i}\alpha - A) \leqslant \dim \operatorname{Ker}(\mathrm{i}\alpha - A')$$

for all $\alpha \in \mathbb{R}$.

This follows from Lemma 1.6 on page 123 because $\operatorname{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)) = \operatorname{Ker}(\mathrm{i}\alpha - A)$.

Proposition 1.8 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra and suppose that there exists a faithful family Ψ of $\mathcal{T}$-invariant states. Then for all $\alpha \in \mathbb{R}$ we have*

$$\dim \operatorname{Ker}(\mathrm{i}\alpha - A) = \dim \operatorname{Ker}(\mathrm{i}\alpha - A')$$

and

$$P\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A') \cap \mathrm{i}\mathbb{R}.$$

Proof The inequality $\dim \operatorname{Ker}(\mathrm{i}\alpha - A) \leqslant \dim \operatorname{Ker}(\mathrm{i}\alpha - A')$ follows from Corollary 1.7.

Let $D = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ and $\mathcal{R}$ the pseudo-resolvent induced by $R(\lambda, A)$ on D. Then $\mathcal{R}$ is identity preserving and of Schwarz type. Take $\mathrm{i}\alpha \in P\sigma(A)$ ($\alpha \in \mathbb{R}$) and choose $0 < \mu \in \mathbb{R}$.

If $\psi_\alpha \in M_*$ is of norm one with polar decomposition $\psi_\alpha = u_\alpha|\psi_\alpha|$ such that $\psi_\alpha = (\mu - \mathrm{i}\alpha)R(\mu)\psi_\alpha$ then $\mu R(\mu)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4 (i) on page 122).

Since

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant 1 - s(|\psi_\alpha|),$$

we obtain $\mu R(\mu)' s(|\psi_\alpha|) = s(|\psi_\alpha|)$ by the faithfulness of Ψ. Hence the maps $S := (\mu - \mathrm{i}\alpha)R(\mu)'$ and $T := \mu R(\mu)'$ fulfill the assumptions of Lemma 1.2 (ii) on page 121. Therefore $Su_\alpha^* = u_\alpha^*$ or $(\mu - \mathrm{i}\alpha)R(\mu)'u_\alpha^* = u_\alpha^*$ which implies $u_\alpha^* \in D(A')$ and $A'u_\alpha^* = \mathrm{i}\alpha u_\alpha^*$.

If $\mathrm{i}\alpha \in P\sigma(A')$, $\alpha \in \mathbb{R}$, choose $0 \neq v_\alpha$ such that

$$v_\alpha = (\mu - \mathrm{i}\alpha)R(\mu)'v_\alpha \quad (\mu \in \mathbb{R}_+)$$

and $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$.

Since

$$0 \leqslant v_\alpha v_\alpha^* = ((\mu - \mathrm{i}\alpha)R(\mu)'v_\alpha)((\mu - \mathrm{i}\alpha)R(\mu)'v_\alpha)^* \leqslant \mu R(\mu)'(v_\alpha v_\alpha^*),$$

we obtain $\mu R(\mu)'(v_\alpha v_\alpha^*) = v_\alpha v_\alpha^*$ because Ψ is faithful.

Hence from Lemma 1.2 (i) on page 121 it follows that

$$\mu R(\mu)'(xv_\alpha^*) = ((\mu - \mathrm{i}\alpha)R(\mu)'x)v_\alpha^*$$

for all $x \in M$.
Let ψ_α be the normal linear functional $(x \mapsto \psi(xv_\alpha^*))$ on M and note that $\psi_\alpha(v_\alpha) \neq 0$. Then

$$\begin{aligned}\langle x, (\mu - \mathrm{i}\alpha)R(\mu)\psi_\alpha\rangle &= \langle((\mu - \mathrm{i}\alpha)R(\mu)'x)v_\alpha^*, \psi\rangle\\ &= \langle \mu R(\mu)'(xv_\alpha^*), \psi\rangle = \psi(xv_\alpha^*) = \psi_\alpha(x)\end{aligned}$$

for all $x \in M$. Consequently $\mathrm{i}\alpha \in P\sigma(A)$ and

$$\dim \operatorname{Ker}((\mathrm{i}\alpha - A')) \leqslant \dim \operatorname{Ker}((\mathrm{i}\alpha - A))$$

which proves the assertion. □

Remark 1.9 From the above proof we obtain the following: If $0 \neq \psi_\alpha \in \operatorname{Ker}(\mathrm{i}\alpha - A)$ for $\alpha \in \mathbb{R}$ with polar decomposition $\psi_\alpha = u_\alpha|\psi_\alpha|$ $(\alpha \in \mathbb{R})$, then $A'u_\alpha = \mathrm{i}\alpha u_\alpha$. Conversely, if $0 \neq v_\alpha \in \operatorname{Ker}(\mathrm{i}\alpha - A')$, then there exists $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$ and the normal linear form

$$\psi_\alpha := (x \mapsto \psi(xv_\alpha^*))$$

is an eigenvector of A pertaining to the eigenvalue $\mathrm{i}\alpha$.

If $\mathcal{T}$ is a C_0-semigroup of Markov operators on a commutative C*-algebra with generator A, it has been shown in B-III, that the boundary spectrum $\sigma(A) \cap \mathrm{i}\mathbb{R}$ of its generator is additively cyclic. This is no longer true in the non commutative case.

Example 1.10 For $0 \neq \lambda \in \mathrm{i}\mathbb{R}$ and $t \in \mathbb{R}$ let

$$u_t := \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda t}\end{pmatrix} \in M_2(\mathbb{C}).$$

The semigroup of *-automorphisms $(x \mapsto u_t x u_t^*)$ on $M_2(\mathbb{C})$ is identity preserving and of Schwarz type, but the spectrum of its generator is $\{0, \lambda, \lambda^*\}$ hence is not additively cyclic.

It turns out that, in order to obtain a non commutative analogue of the Perron-Frobenius theorems, one has to consider semigroups which are irreducible.
Recall that a semigroup $\mathcal{S}$ of positive operators on an ordered Banach space (E, E_+) is called *irreducible* if no closed face of E_+, different from $\{0\}$ and E_+, is invariant under $\mathcal{S}$. In the context of W*-algebras M we call a semigroup $\mathcal{S}$ of positive maps on M *weak*-irreducible* if no $\sigma(M, M_*)$-closed face of M_+ is $\mathcal{S}$-invariant.
Since the norm closed faces of M_* and the $\sigma(M, M_*)$-closed faces of M are related by formation of polars with respect to the dual system $\langle M, M_*\rangle$ (see Pedersen (1979, Theorem 3.6.11 and Theorem 3.10.7.)) a semigroup $\mathcal{S}$ is (norm) irreducible on M_* if and only if its adjoint semigroup is weak*-irreducible.

Theorem 1.11 *Let $\mathcal{T}$ be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra and suppose $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.*

(i) The fixed space of $\mathcal{T}$ is one dimensional and spanned by a faithful normal state.

(ii) $P\sigma(A) \cap \mathrm{i}\mathbb{R}$ is an additive subgroup of $\mathrm{i}\mathbb{R}$,

$$\sigma(A) = \sigma(A) + (P\sigma(A) \cap \mathrm{i}\mathbb{R})$$

and every eigenvalue in $\mathrm{i}\mathbb{R}$ *is simple.*

(iii) The fixed space of the adjoint weak-semigroup $\mathcal{T}'$ is one-dimensional.*

(iv) $P\sigma(A') \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R}$ for the generator A' of the adjoint semigroup, and every $\gamma \in P\sigma(A') \cap \mathrm{i}\mathbb{R}$ is simple.

Proof Since $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$, there exists $\psi \in \operatorname{Fix}(\mathcal{T})_+$ of norm one (Corollary 1.5). If $F := \{x \in M_+ : \psi(x) = 0\}$, then F is a $\sigma(M, M_*)$-closed, $\mathcal{T}'$-invariant face in M, hence $F = \{0\}$. Therefore every $0 \neq \psi \in \operatorname{Fix}(\mathcal{T})_+$ is faithful.
Let $\psi_1, \psi_2 \in \operatorname{Fix}(\mathcal{T})_+$ be states such that $f := \psi_1 - \psi_2$ is different from zero. If $f = f^+ - f^-$ is the Jordan decomposition of f, then f^+ and f^- are elements of $\operatorname{Fix}(\mathcal{T})$, whence faithful. Since the support projections of these two normal linear functionals are orthogonal, we obtain $f^+ = 0$ or $f^- = 0$ which implies $\psi_1 \leqslant \psi_2$ or $\psi_2 \leqslant \psi_1$. Consequently $\psi_2 = \psi_1$.
Since $\operatorname{Fix}(\mathcal{T})$ is positively generated (Corollary 1.5 on page 123), $\operatorname{Fix}(\mathcal{T}) = \{\lambda\varphi : \lambda \in \mathbb{C}\} =: \mathbb{C}.\varphi$ for some faithful normal state φ.
Let $\mu \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ such that $\mathrm{i}\alpha \in P\sigma(A)$. If $\psi_\alpha = u_\alpha|\psi_\alpha|$ is a normalized eigenvector of A pertaining to $\mathrm{i}\alpha$, then $\varphi = |\psi_\alpha| = |\psi_\alpha^*|$ (Corollary 1.5 and the above considerations). Hence $u_\alpha u_\alpha^* = u_\alpha^* u_\alpha = s(\varphi) = 1$.
Since

$$(\mu - \mathrm{i}\alpha)R(\mu, A)\psi_\alpha = \psi_\alpha$$

and

$$\mu R(\mu, A)|\psi_\alpha| = |\psi_\alpha|,$$

we obtain by Lemma 1.2 (ii) on page 121 that

$$\mu R(\mu, A) = V_\alpha \circ \mu R(\mu + \mathrm{i}\alpha, A) \circ V_\alpha^{-1} \quad (1)$$

where V_α is the map $(x \mapsto x u_\alpha)$ on M.
Similarly, for $i\beta \in P\sigma(A)$ we find V_β such that $1 = u_\beta u_\beta^* = u_\beta u_\beta^*$ and

$$\mu R(\mu, A) = V_\beta \circ \mu R(\mu + i\beta, A) \circ V_\beta^{-1}. \quad (2)$$

Hence

$$\mu R(\mu, A) = V_{\alpha\beta} \circ \mu R(\mu + i(\alpha + \beta), A) \circ V_{\alpha\beta}^{-1} \quad (3)$$

where $V_{\alpha\beta} := V_\alpha \circ V_\beta$.
Since u_α is unitary in M, it follows from (1) that $\mathrm{i}\alpha$ is an eigenvalue which is simple because $\operatorname{Fix}(T) = \operatorname{Fix}(\mu R(\mu, A))$ is one dimensional.

From (3) it follows that $i(\alpha+\beta) \in P\sigma(A)$ since $0 \in P\sigma(A)$ and $V_{\alpha\beta}$ is bijective. From the identity (1) we conclude that $\sigma(R(\mu, A)) = \sigma(R(\mu + \mathrm{i}\alpha))$, which proves

$$\sigma(A) + (P\sigma(A) \cap \mathrm{i}\mathbb{R}) \subseteq \sigma(A).$$

The other inclusion is trivial since $0 \in P\sigma(A)$. □

Remark 1.12 (i) Let φ be the normal state on M such that $\mathrm{Fix}\,(T) = \mathbb{C}.\varphi$ and let $H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$. From the proof of Theorem 1.10 it follows that there exists a family $\{u_\eta \colon \eta \in H\}$ of unitaries in M such that $A'u_\eta = -\eta u_\eta$ and $A(u_\eta \varphi) = \eta(u_\eta \varphi)$ for all $\eta \in H$.
(ii) If the group H is generated by a single element, i.e., $H = \mathrm{i}\gamma\mathbb{Z}$ for some $\gamma \in \mathbb{R}$, then $\{u_\gamma^k \colon k \in \mathbb{Z}\}$ is a complete family of eigenvectors pertaining to the eigenvalues in H, where $u_\gamma \in M$ is unitary such that $A'u_\gamma = \mathrm{i}\gamma u_\gamma$.

Proposition 1.13 *Suppose that $\mathcal{T}$ and M satisfy the assumptions of Theorem 1.10, and let N_* be the closed linear subspace of M_* generated by the eigenvectors of A pertaining to the eigenvalues in $\mathrm{i}\mathbb{R}$. Denote by $\mathcal{T}_0$ the restriction of $\mathcal{T}$ to N_*. Then*

(i) $G := (\mathcal{T}_0)^- \subseteq L_s(N_*)$ is a compact, Abelian group in the strong operator topology.
(ii) $\mathrm{Id}_{|N_*} \in \overline{\{T_0(t) \colon t > s\}} \subseteq L_s(N_*)$ for all $0 < s \in \mathbb{R}$.

Proof For $\eta \in H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$ let

$$U(\eta) := \{\psi \in D(A) : A\psi = \eta\psi\}$$

and $U = \{U(\eta) \colon \eta \in H\}$. Then $(U)^- = N_*$.
For each $\psi \in U$ there exists $\eta \in H$ such that

$$\{T_0(t)\psi \colon t \in \mathbb{R}_+\} = \{e^{-\eta t}\psi \colon t \in \mathbb{R}_+\}.$$

Consequently this set is relatively compact in $L_s(N_*)$. From [Schaefer (1966),III.4.5] we obtain that G is compact in the strong operator topology.
Next choose $\psi_1, \ldots, \psi_n \in U$, $0 < s \in \mathbb{R}$ and $\delta > 0$. Since $T_0(t)\psi_i = e^{\eta_i t}\psi_i$ $(1 \leqslant i \leqslant n)$ for some $\eta_i \in H$, it follows from a theorem of Kronecker (see, Jacobs (1972, Satz 6.1., p.77)) that there exists $s < t$ such that

$$|(1, 1, \ldots, 1) - (e^{\eta_1 t}, e^{\eta_2 t}, \ldots, e^{\eta_n t})| < \delta,$$

hence

$$\sup\{\|\psi_i - T_0(t)\psi_i\| \colon 1 \leqslant i \leqslant n\} < \delta$$

or $\mathrm{Id}_{|N_*} \in \overline{\{T_0(t) \colon t > s\}} \subseteq L_s(N_*)$.
Finally we prove the group property of G. Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{R}$ such that $\lim_{\mathfrak{U}} T_0(t) = \mathrm{Id}$ in the strong operator topology. For positive $s \in \mathbb{R}$ let $S := \lim_{\mathfrak{U}} T(t-s)$. Then $ST_0(s) = T_0(s)S = \mathrm{Id}$, hence $T_0(s)^{-1}$ exists in G for all $s \in \mathbb{R}_+$. From this it follows that G is a group. □

Remark 1.14 (i) Let $\kappa : \mathbb{R} \to G$ be given by

$$\kappa(t) = \begin{cases} T_0(t) & \text{if } 0 \leqslant t, \\ T_0(t)^{-1} & \text{if } t \leqslant 0. \end{cases}$$

Then κ is a continuous homomorphism with dense range, i.e., (G, κ) is solenoidal (see Hewitt and Ross (1963)).
(ii) The compact group G and the discret group $P\sigma(A) \cap \mathrm{i}\mathbb{R}$ are dual as locally compact Abelian groups.
(iii) Let (G, κ) be a solenoidal compact group and let $N_* = L^1(G)$. Then the induced lattice semigroup $T = (\kappa(t))_{t \geqslant 0}$ fulfils the assertions of Theorem 1.10. For example, if G is the dual of $\mathbb{R}_d$, then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\mathbb{R}$. Since the fixed space of $\kappa(t)$ is given by

$$\mathrm{Fix}\,(\kappa(t)) = \overline{\left(\bigcup_{k \in \mathbb{Z}} \mathrm{Ker}(\frac{2\pi i k}{t} - A) \right)},$$

however no $T(t) \in \mathcal{T}$ is irreducible.
(iv) If $\mathcal{T}$ is the irreducible semigroup of Schwarz type on the predual of $B(H)$ given in Evans (1977), then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$.

2 Spectral Properties of Uniformly Ergodic Semigoups

The aim of this section is the study of spectral properties of semigroups which are uniformly ergodic, identity preserving and of Schwarz type. For the basic theory of uniformly ergodic semigroups on Banach spaces we refer to Dunford and Schwartz (1958).
Our first result yields an estimate for the dimension of the eigenspaces pertaining to eigenvalues of a pseudo-resolvent.

Proposition 2.1 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \mathrm{Re}\,\lambda > 0\}$ with values in the predual of a* W*-*algebra M. If $Fix\lambda R(\lambda)$ is finite dimensional for some $\lambda \in D$, then*

$$\dim \mathrm{Fix}\,((\gamma - \mathrm{i}\alpha)\, R(\gamma)) \leqslant \dim \mathrm{Fix}\,(\lambda R(\lambda))$$

for all $\gamma \in D$ and $\alpha \in \mathbb{R}$.

Proof By D-IV, Remark 3.2.c, we may assume without loss of generality that there exists a faithful family of $\mathcal{R}$-invariant normal states on M. In particular the fixed space N of the adjoint pseudo-resolvent $\mathbb{R}\mathcal{R}'$ is a W*-subalgebra of M with $\mathbb{1} \in N$ (by Lemma 1.1 (ii)). Since N is finite dimensional, there exist a natural number n and a set $P := \{p_1, .., p_n\}$ of minimal, mutually orthogonal projections in N such that $\sum_{k=1}^{n} p_k = \mathbb{1}$. These projections are also mutually orthogonal in M with sum $\mathbb{1}$.

Let R_j be the $\sigma(M, M_*)$-closed right ideal $p_j M$ and L_j the closed left invariant subspace $M_* p_j$ for $(1 \leqslant j \leqslant n)$. Since the map $\mu R(\mu)'$, $\mu \in \mathbb{R}_+$ is an identity preserving Schwarz map, we obtain from Lemma 1.1.b that for all $x \in N$ and $y \in M$,

$$\mu R(\mu)'(xy) = x(\mu \mathcal{R}'(\mu) y).$$

In particular, R_j, resp. L_j are invariant under $\mathcal{R}'$, respectively, $\mathcal{R}$. Furthermore, if $\psi \in L_j$ with polar decomposition $\psi = u|\psi|$, then $u^* u \leqslant s(|\psi|) \leqslant p_j$. Consequently, $|\psi| \in L_j$.
Let now $\alpha \in \mathbb{R}$ and suppose that there exists $\psi_\alpha \in L_j$ of norm 1, $\psi_\alpha = u_\alpha |\psi_\alpha|$, such that

$$\psi_\alpha \in \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha) R(\lambda)\right), \lambda \in D.$$

Since $\lambda R(\lambda)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4 on page 122), we obtain

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant (1 - s(|\psi_\alpha|), \mu \in \mathbb{R}_+.$$

From the existence of a faithful family of $\mathcal{R}$-invariant normal states and since $\mathcal{R}'$ is identity preserving, it follows that

$$\mu R(\mu)' s(|\psi_\alpha|) = s(|\psi_\alpha|).$$

Thus $s(|\psi_\alpha|) \leqslant p_j$ and even $s(|\psi_\alpha|) = p_j$ by the minimality property of p_j.
On the other hand, $\psi_\alpha^* \in \operatorname{Fix}((\lambda + \mathrm{i}\alpha) R(\lambda))$. As above we obtain

$$\mu R(\mu)' s(|\psi_\alpha^*|) = s(|\psi_\alpha^*|).$$

Consequently, the closed left ideals $Ms(|\psi_\alpha^*|)$ and $Ms(|\psi_\alpha|)$ are $\mathcal{R}'$-invariant.
Next fix $\mu \in \mathbb{R}_+$, let $S := (\mu - \mathrm{i}\alpha) R(\mu)'$ and $T = \mu R(\mu)'$. Then

$$(Sx)(Sx)^* \leqslant T(xx^*),\ S_*(\psi_\alpha^*) = \psi_\alpha^*,\ T_*(|\psi_\alpha^*|) = |\psi_\alpha^*|,$$

and T is an identity preserving Schwarz map. Since $s(|\psi_\alpha^*|)M$ is T-invariant, the assumptions of Lemma 1.2 on page 121 are fulfilled and we obtain for every $x \in M$

$$S(x) u_\alpha^* = T(x u_\alpha^*).$$

The closed left ideal Mp_j is S-invariant, therefore it follows

$$S(x) = T(x u_\alpha^*) u_\alpha\,,\ x \in M p_j$$

(see Remark 1.3 on page 121). Since u_α does not depend on $\mu \in \mathbb{R}_+$, we obtain for all $\mu \in \mathbb{R}_+$

$$\mu R(\mu + \mathrm{i}\alpha)' x = \mu R(\mu)'(x u_\alpha^*) u_\alpha.$$

Consequently, the holomorphic functions

$$(\mu \mapsto \mu R(\mu)'(x u_\alpha) u_\alpha^*) \quad \text{and} \quad (\mu \mapsto \mu R(\mu + \mathrm{i}\alpha)' x)$$

coincide on $\mathbb{R}_+$ from which we conclude

$$\lambda R(\lambda + \mathrm{i}\alpha)' x = \lambda R(\lambda)'(x u_\alpha^*) u_\alpha$$

for every $\lambda \in D$ and all $x \in M p_j$.
Since the map $(y \mapsto y u_\alpha)$ is a continuous bijection from $M(u_\alpha u_\alpha^*)$ onto $M p_j$ with inverse $(y \mapsto y u_\alpha^*)$, we can deduce that

$$\begin{aligned} \dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha) R(\lambda)' | M p_j\right) &= \dim \operatorname{Fix}\left(\lambda R(\lambda)'\right) | M(u_\alpha u_\alpha^*) \\ &\leqslant \dim \operatorname{Fix}(\mathcal{R}'). \end{aligned}$$

Since $\bigoplus_{j=1}^n M p_j = M$ and $\bigoplus_{j=1}^n L_j = M_*$, we obtain

$$\begin{aligned} \dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha) R(\lambda)'\right) &= \dim \operatorname{Fix}\left(\lambda R(\lambda)'\right), \\ &= \dim \operatorname{Fix}\left(\lambda R(\lambda)\right), \end{aligned}$$

and the assertion follows from Lemma 1.6 on page 123. □

Before going on let us recall the basic facts of the *ultrapower* $\hat{E}$ of a Banach space E with respect to some free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ (compare A-I,3.6). If $\ell^\infty(E)$ is the Banach space of all bounded functions on $\mathbb{N}$ with values in E, then

$$c_{\mathfrak{U}}(E) := \{(x_n) \in \ell^\infty(E) \colon \lim_{\mathfrak{U}} \|x_n\| = 0\}$$

is a closed subspace of $\ell^\infty(E)$ and equal to the kernel of the seminorm

$$\|(x_n)\| := \lim_{\mathfrak{U}} \|x_n\|, \; (x_n) \in \ell^\infty(E).$$

By the *ultrapower* $\hat{E}$ we understand the quotient space $\ell^\infty(E)/c_{\mathfrak{U}}(E)$ with norm

$$\|\hat{x}\| = \lim_{\mathfrak{U}} \|x_n\|, \; (x_n) \in \hat{x} \in \hat{E}.$$

Moreover, for a bounded linear operator $T \in L(E)$, we denote by $\hat{T}$ the well defined operator $\hat{T}\hat{x} := (T x_n) + c_{\mathfrak{U}}(E)$, $(x_n) \in \hat{x}$.
It is clear by virtue of $(x \mapsto (x, x, ..) + c_{\mathfrak{U}}(E))$ that each $x \in E$ defines an element $\hat{x} \in \hat{E}$. This isometric embedding as well as the operator map $(T \mapsto \hat{T})$ are called canonical. In particular, if $\mathcal{R} \colon (D \to L(E))$ is a pseudo-resolvent, then

$$\hat{\mathcal{R}} := (\lambda \mapsto R(\lambda)^\wedge) : D \to L(\hat{E})$$

is a pseudo-resolvent, too. Recall that the approximative point spectrum $A\sigma(T)$ is equal to the point spectrum $P\sigma(\hat{T})$ (see, e.g., Schaefer (1974, Chapter V, §1)).
This construction gives us the possibility to characterize uniformly ergodic semigroups with finite dimensional fixed space.

Lemma 2.2 *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda > 0\}$ such that $\|R(\mu + \mathrm{i}\alpha)\| \leqslant 1$ for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and suppose*

$$0 < \dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right) < \infty \quad \textit{for some} \quad \lambda \in D\,,\ \alpha \in \mathbb{R}.$$

For the canonical extension $\hat{R}$ on some ultrapower $\hat{E}$, the following assertions hold.

(i) $(\lambda - \mathrm{i}\alpha)^{-1}$ *is a pole of the resolvent* $R(.,R(\lambda))$ *for all* $\lambda \in D$.
(ii) $\dim \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) = \dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right)$ *for all* $\lambda \in D$.
(iii) $\mathrm{i}\alpha$ *is a pole of the pseudo-resolvent* $\mathcal{R}$ *and the residue of* $\mathcal{R}$ *and* $R(.,R(\lambda))$ *in* $\mathrm{i}\alpha$ *respectively* $(\lambda - \mathrm{i}\alpha)^{-1}$ *are identical.*

Proof Take a normalized sequence (x_n) in E with

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)x_n - x_n\| = 0.$$

The existence of such a sequence follows from the fact that the fixed space of $(\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)$ is non trivial. Suppose (x_n) is not relatively compact. Then we may assume that there exists $\delta > 0$ such that

$$\|x_n - x_m\| > \delta \quad \text{for} \quad n \neq m.$$

Take $k \in \mathbb{N}$ and let $\hat{x}_k$ be the image of (x_{n+k}) in $\hat{E}$. Since

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)x_{n+k} - x_{n+k}\| = 0,$$

the so defined $\hat{x}_k$'s belong to $\operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right)$. Since this space is finite dimensional there exist $j < \ell$, such that

$$\|\hat{x}_j - \hat{x}_\ell\| \leqslant \frac{\delta}{2}.$$

From the definition of the norm in $\hat{E}$ it follows that there are natural numbers $n < m$ such that

$$\|x_n - x_m\| \leqslant \frac{\delta}{2},$$

leading to a contradiction.
Therefore every approximate eigenvector of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ pertaining to α is relatively compact. In particular, it has a convergent subsequence from which it follows that the fixed space of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ is non trivial.
Obviously

$$\dim \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) \leqslant \dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right).$$

If the last inequality is strict, then there exists $\gamma > 0$ and a normalized $\hat{x} \in \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right)$ such that

$$\gamma \leqslant \|\hat{y} - \hat{x}\|$$

for all $y \in \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$.
Take a normalized sequence $(x_n) \in \hat{x}$. Then (x_n) has a convergent subsequence, whence we may assume that $\lim_n x_n = z$ exists in E. Thus $0 \neq z \in \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$. From this we obtain the contradiction

$$0 \leqslant \gamma \leqslant \|\hat{z} - \hat{x}\| = \lim \|z - x_n\| = 0$$

Consequently,

$$\dim \operatorname{Fix}((\lambda - i\alpha)R(\lambda)) = \dim \operatorname{Fix}\left(\lambda - i\alpha)\hat{R}(\lambda)\right).$$

Let $\{x_1, ..., x_n\}$ be a base of $\operatorname{Fix}((\lambda - i\alpha)R(\lambda))$ and choose $\{\varphi_1, ..., \varphi_n\}$ in $\operatorname{Fix}((\lambda - i\alpha)R(\lambda)')$ such that $\varphi_k(x_j) = \delta_{k,j}$ (Lemma 1.6. Then

$$E = \operatorname{Fix}((\lambda - i\alpha)R(\lambda)) \oplus \bigcap_{j=1}^{n} \operatorname{Ker}(\varphi_j),$$

where both subspaces on the right are $(\lambda - i\alpha)R(\lambda)$-invariant and 1 is a pole of $(\lambda - i\alpha)R(\lambda)|_{\operatorname{Fix}(()(\lambda - i\alpha)R(\lambda))}$ by the finite dimensionality of $\operatorname{Fix}((\lambda - i\alpha)R(\lambda))$. Suppose 1 belongs to the spectrum of S where S is the restriction of $(\lambda - i\alpha)R(\lambda)$ to $\bigcap_{j=1}^{n} \ker \varphi_j$. Then there exists a normalized sequence (y_n) in $\bigcap_{j=1}^{n} \operatorname{Ker}(\varphi_j)$ such that

$$\lim_n \|(\lambda - i\alpha)R(\lambda)y_n - y_n\| = 0.$$

Therefore (y_n) has an accumulation point different from zero contained in

$$\operatorname{Fix}((\lambda - i\alpha)R(\lambda)) \cap (\bigcap_{j=1}^{n} \ker \varphi_j).$$

This contradiction implies that 1 does not belong to the spectrum of S.
Since $\operatorname{Fix}((\lambda - i\alpha)R(\lambda))$ is finite dimensional, it follows from general spectral theory that $(\lambda - i\alpha)^{-1}$ is a pole of $R(., R(\lambda))$ for every λ. Thus (i) and (ii) are proved and assertion (iii) follows from the resolvent equality as in the proof of Greiner (1981, Proposition 1.2). □

Proposition 2.3 *Let $\mathcal{T}$ be a semigroup of contractions on a Banach space E with generator A. Then the following assertions are equivalent.*

(a) *Each* $i\alpha$, $\alpha \in \mathbb{R}$, *is a pole of the resolvent* $R(., A)$ *such that the corresponding residue has finite rank.*
(b) $\dim \operatorname{Fix}\left((\lambda - i\alpha)\hat{R}(\lambda, A)\right) < \infty$ *for some (hence all)* $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$ *and the canonical extensions* $\hat{R}(\lambda, A)$ *of* $R(\lambda, A)$ *to some ultrapower.*

Proof Let P_α be the residue of the resolvent $R(., A)$ in $i\alpha$. Then $P_\alpha = \lim_{\lambda \to i\alpha}(\lambda - i\alpha)R(\lambda, A)$ in the operator norm of $L(E)$. Since the canonical map $(T \mapsto \hat{T})$ is isometric and since $\hat{E}$ is an ultrapower, we obtain

$$\hat{P}_\alpha = \lim_{\lambda \to i\alpha} (\lambda - i\alpha)\hat{R}(\lambda, A)$$

in $L(\hat{E})$ and $\operatorname{rank}(P_\alpha) = \operatorname{rank}(\hat{P}_\alpha)$. Because of

$$\hat{P}_\alpha(\hat{E}) = \operatorname{Fix}\left((\lambda - i\alpha)\hat{R}(\lambda)\right)$$

one part of the corollary is proved. The other follows from Lemma 2.2 on page 130. □

Remark 2.4 (i) By the results in Lin (1974) a semigroup of contractions on a Banach space is uniformly ergodic if and only if 0 is a pole of the generator with order $\leqslant 1$. The residue of the resolvent in 0 and the assocciated ergodic projection are identical.
(ii) Let M be a W^*-algebra with predual M_*, $\mathfrak{U}$ a free ultrafilter on $\mathbb{N}$ and $\widehat{M}$ (resp. $(M_*)^\wedge$) the ultrapower of M (resp. M_*) with respect to $\mathfrak{U}$. Then it is easy to see that $c_{\mathfrak{U}}(M)$ is a two sided ideal in $\ell^\infty(M)$ hence $\widehat{M}$ is a C^*-algebra, but in general not a W^*-algebra. Note that the unit of $\widehat{M}$ is the canonical image of 1. For $\hat{x} \in \hat{M}$ and $\hat{\varphi} \in (M_*)^\wedge$ let $J : (M_*)^\wedge \to \widehat{M}'$ be defined by

$$\langle x, J(\hat{\varphi})\rangle := \lim_{\mathfrak{U}} \varphi_n(x_n)\,,\ (x_n) \in \hat{x}\,,\ (\varphi_n) \in \hat{\varphi}.$$

Then J is well defined and an isometric embedding. It turns out that $J((M_*)^\wedge)$ is a translation invariant subspace of $\widehat{M'}$. Hence there exists a central projection $z \in \widehat{M''}$ such that $J((M_*)^\wedge) = \widehat{M''}z$ (Groh (1984, Proposition 2.2)).
Below we identify $(M_*)^\wedge$ via J with this translation invariant subspace. From the construction the following is obvious: If T is an identity preserving Schwarz map with preadjoint $T_* \in L(M_*)$, then $\widehat{T}$ is an identity preserving Schwarz map on $\widehat{M}$ such that $(T_*)^\wedge = \widehat{T}'|(M_*)^\wedge$.

Theorem 2.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type with generator A on the predual of a* W^**-algebra M. If $\mathcal{T}$ is uniformly ergodic with finite dimensional fixed space, then every $\gamma \in \sigma(A) \cap \mathrm{i}\mathbb{R}$ is a pole of the resolvent $R(.,A)$ and* $\dim \operatorname{Ker}(\gamma - A) \leqslant \dim \operatorname{Fix}(T)$.

Proof Let $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > 0\}$ and $\mathcal{R}$ the M_*-valued pseudo-resolvent of Schwarz type induced by $R(.,A)$ on D. Then

$$P = \lim_{\mu\downarrow 0} \mu R(\mu)$$

exists in the uniform operator topology. Since $P(E) = \operatorname{Fix}(T)$, we obtain $\hat{P}(\hat{E}) = \operatorname{Fix}(\hat{T})$ and $\dim \operatorname{Fix}(T) = \dim \operatorname{Fix}(\hat{T}) < \infty$, where $\hat{P}$ is the canonical extension of P onto $(M_*)^\wedge$. Since $\hat{P} = \lim_{\mu\downarrow 0} \mu R(\mu)^\wedge$ it follows that

$$\dim \operatorname{Fix}\big((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\big) \leqslant \dim \operatorname{Fix}\big(\hat{T}\big) < \infty$$

for all $\alpha \in \mathbb{R}$ (Proposition 2.1 on page 128). Therefore the assertion follows from Lemma 2.2 on page 130. □

The consequences of this result for the asymptotic behavior of one-parameter semigroups will be discussed in D-IV, Section 4.

Notes

Section 1: The Perron-Frobenius theory for a single positive operator on a non-commutative operator algebra is worked out in Albeverio and Hoegh-Krohn (1978) and Groh (1981). The limitations of the theory (in the continuous as in the discrete case) are explained by the example following Remark 1.9 on page 125 (see also Groh (1982).Therefore we concentrate on irreducible semigroups. Our main result Theorem 1.11 on page 126 extends B-III, Thm.3.6 to the non-commutative setting.

Section 2: Theorem 2.5 on page 133 has its roots in the Niiro-Sawashima Theorem for a single irreducible positive operator on a Banach lattice (see Schaefer (1974, V.5.4)). The analogous semigroup result on Banach lattices is due to Greiner (1982). The ultrapower technique in our proof is developed in Groh (1984).

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Chapter D-IV
Asymptotics of Positive Semigroups on C*- and W*-Algebras

1 Stability of Positive Semigroups

As explained in A-III, Section 1, it is possible to deduce uniform exponential stability of strongly continuous semigroups from the location of the spectrum of its generator if the spectral bound $s(A)$ and the growth bound ω_0 coincide. In this section we prove $s(A) = \omega_0$ for positive semigroups on C*-algebras and preduals of W*-algebras. A more general discussion of the "$s(A) = \omega_0$" problem can be found in Greiner et al. (1981). For the results of this section the existence of a unit is essential.

Theorem 1.1 *Let M be a* C*-*algebra with unit and $\mathcal{T} = (T(t))_{t\geqslant 0}$ a positive semigroup on M. Then*

$$-\infty < s(A) = \omega_0 \in \sigma(A).$$

Proof For every $t \geqslant 0$ there exists φ_t in the state space $S(M)$ of M such that

$$T(t)'\varphi_t = r(T(t))\varphi_t = \exp(\omega_0 t)\varphi_t$$

(see, e.g., Groh (1981, 2.1)).
Let $n \in \mathbb{N}$ and

$$E_n := \{\varphi \in S(M) \colon T(2^{-n})\varphi = \exp(\omega_0 2^{-n})\varphi\}.$$

Then $\emptyset \neq E_{n+1} \subseteq E_n$, $(n \in \mathbb{N})$. Since $S(M)$ is $\sigma(M, M')$-compact, there exists $\varphi \in \bigcap_{n\in\mathbb{N}} E_n$. Then $T(t)'\varphi = \exp(\omega_0 t)\varphi$ follows for all $0 \leqslant t$ because the adjoint semigroup $(T(t)')_{t\geqslant 0}$ is a weak*-semigroup on M'.
Suppose $-\infty = \omega_0$. Then for $t > 0$ either $r(T(t)) = 0$ (A-III,Prop.1.1) or $T(t)'\varphi = 0$, in particular $\varphi(T(t)\mathbb{1}) = 0$. From this we obtain the contradiction $\varphi(\mathbb{1}) = 0$.
Hence $-\infty < \omega_0$ and $\exp(\omega_0 t) \in \varrho(T(t)')$ for every $t \in \mathbb{R}_+$. Thus $\omega_0 \in \sigma(A)$ or $\omega_0 = s(A)$. ◻

Remark 1.2 (i) If we consider the nilpotent translation semigroup on the C*-algebra $C_0([0,1[)$, then $\sigma(A) = \emptyset$ and $\omega_0 = -\infty$. This shows that the existence of a unit is essential.
(ii) The equality $s(A) = \omega_0$ still holds for positive semigroups on commutative C*-algebras without unit (see B-IV, Rem.1.2.b).

Theorem 1.3 *Let M be a* W*-*algebra with predual M_* and let $(T(t))_{t\geqslant 0}$ be a positive semigroup on M_*. Then $s(A) = \omega_0$.*

Proof For all $\lambda > s(A)$ and $\varphi \in M_*$

$$R(\lambda, A)\varphi = \int_0^\infty e^{-\lambda t} T(s)\varphi ds$$

which follows as in C-III, Section 1 or Greiner et al. (1981, Theorem 3). Since $\|\varphi\| = \varphi(\mathbb{1})$ for every $\varphi \in M_*^+$ and since the norm is additive on the positive cone of M_*, the integral

$$\int_0^\infty e^{\lambda t} \|T(s)\varphi\| ds$$

exists for all $\varphi \in M_*$ and all $\lambda > s(A)$. From this the assumption follows by A-IV,Thm.1.11. □

Corollary 1.4 *Let M be a* C*-*algebra and $(T(t))_{t\geqslant 0}$ a positive semigroup on M'. Then $s(A) = \omega_0$ holds.*

This follows from the fact that the bidual of a C*-algebra is a W*-algebra (see Takesaki (1979, Theorem III.2.4.)).

Remark 1.5 A simple modification of A-III, Example 1.4 (take c_0 instead of ℓ^2) shows that Theorem 1.3 is no longer true for non-positive semigroups (for details see Groh and Neubrander (1981, Beispiel 2.5)).
While the growth bound ω_0 characterizes uniform exponential stability of the semigroup there are other (and weaker) stability concepts (cf. A-IV, Section 1).

Definition 1.6 Let E be a Banach space and $(T(t))_{t\geqslant 0}$ a semigroup on E. We call the semigroup

(i) *uniformly exponentially stable* if $\|T(t)\| \leqslant Me^{-\omega t}$ for some $\omega, M > 0$ and all $t \geqslant 0$.
(ii) *uniformly stable* if $\lim_{t\to\infty} T(t) = 0$ in the strong operator topology.
(iii) *weakly stable* if $\lim_{t\to\infty} T(t) = 0$ in the weak operator topology.

Surprisingly all these properties coincide for positive semigroups on C*-algebras with unit.

Theorem 1.7 *Let M be a C*-algebra with unit and $(T(t))_{t\geqslant 0}$ a positive semigroup on M. Then the following assertion are equivalent.*

(a) $s(A) < 0$.

(b) The semigroup $(T(t))_{t\geqslant 0}$ is uniformly exponentially stable.
(c) The semigroup $(T(t))_{t\geqslant 0}$ is uniformly stable.
(d) The semigroup $(T(t))_{t\geqslant 0}$ is weakly stable.

Proof Since $s(A) = \omega_0$ by Theorem 1.3, it suffices to show that (d) implies (a). For $t > 0$ there exists $\varphi \in S(M)$ such that

$$T(t)'\varphi = r(T(t))\varphi.$$

Then for $x \in M$

$$\varphi(T(t)^n x) = (r(T(t)))^n \varphi(x) \to 0$$

as $n \to \infty$. Therefore $r(T(t)) < 1$ or $\omega_0 < 0$. Since $s(A) \leqslant \omega_0$ the assertion follows. □

Remark 1.8 Consider the translation semigroup $(T(t))_{t\geqslant 0}$ on $C_0(\mathbb{R}_+)$. Then $\|T(t)\| = 1$, hence $s(A) = 1$, but $(T(t))_{t\geqslant 0}$ is strongly stable. The same holds for the translation semigroup on $L^1(\mathbb{R}_+)$. Thus Theorem 1.7 is not true for semigroups on C*-algebras without unit or on preduals of W*-algebras. For the discussion of the commutative situation we refer to B-IV, Section 1.

2 Stability of Implemented Semigroups

Let H be a Hilbert space, $\mathcal{U} = (U(t))_{t\geqslant 0}$ a strongly continuous semigroup on H with generator B and $M \subseteq \mathcal{B}(H)$ a W*-algebra, where $\mathcal{B}(H)$ is the W*-algebra of all bounded linear operators on H. Suppose $\mathcal{U}(t)^* M U(t) \subseteq M$. Then one can define a weak*-continuous semigroup $\mathcal{T}$ on M by

$$T(t)x := U(t)^* x U(t) \quad (t \in \mathbb{R}_+, x \in M).$$

We call $\mathcal{T}$ an *implemented semigroup*. Every map $T(t) \in \mathcal{T}$ of an implemented semigroup is weak*-continuous and n-positive for every $n \in \mathbb{N}$.

Remark 2.1 (i) Because of

$$\|T(t)\| = \|T(t)\mathbb{1}\| = \|U(t)^* U(t)\| = \|U(t)\|^2$$

it follows that $\omega_0(\mathcal{T}) = 2\,\omega_0(\mathcal{U})$.
(ii) If $\mathcal{T}$ is an implemented semigroup, then the preadjoint semigroup is strongly continuous on M_*. Therefore $s(A) = \omega_0$ for $\mathcal{T}$ by Theorem 1.3.
(iii) Since $\mathcal{U}$ is a strongly continuous semigroup on H, the same is true for the adjoint semigroup $\mathcal{U}^* = \{U(t)^* : U(t) \in \mathcal{U}\}$ and its generator is given by B^*. In analogy to Bratteli and Robinson (1979, 3.2.55) the following assertions for $x \in M$ are equivalent.

(a) $x \in D(A)$, A the generator of $\mathcal{T}$.

(b) For $\xi \in D(B)$ it follows $x\xi \in D(B^*)$ and the linear mapping

$$(\xi \mapsto x(B\xi) + B^*(x\xi)) : D(B) \to H \tag{*}$$

has a continuous extension to H.

Then for A is given as the continuous extension of (*), i.e., $Ax = xB + B^*x$ for $x \in D(A)$

In the next theorem we give some equivalent conditions for the uniform exponential stability of an implemented semigroup. As we shall see, the operator equality

$$yB + B^*y = -x \quad (x, y \in M_+)$$

is necessary and sufficient, which is in complete analogy to the classical Liapunov stability result.

Theorem 2.2 *Let M be a* W*-*algebra on a Hilbert space H and let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a weak*-semigroup on M with generator A implemented by the semigroup $\mathcal{U}$ on H with generator B. Then the following assertions are equivalent.*

(a) $\omega_0(\mathcal{T}) = s(A) < 0$.
(b) The semigroup $(U(t))_{t\geqslant 0}$ is uniformly exponentially stable.
(c) There exists $0 \leqslant x \in D(A)$ such that $Ax = -\mathbb{1}$.
(d) There exists $0 \leqslant x \in D(A)$ such that $x(D(B)) \subseteq D(B^*)$ and $xB + B^*x = -\mathbb{1}$.
(e) For every $0 \leqslant x \in D(A)$ there exists $0 \leqslant y \in D(A)$ such that $Ay = -x$.
(f) For every $0 \leqslant x \in D(A)$ there exists $0 \leqslant y \in D(A)$ such that $y(D(B)) \subseteq D(B^*)$ and $yB + B^*y = -x$.
(g) $\int_0^\infty \|U(s)\xi\|^2 ds$ exists for all $\xi \in H$.
(h) $\int_0^\infty |(T(s)x)\xi|\zeta) ds$ exists for all $\xi, \zeta \in H$ and all $x \in M$.

Proof The equivalence of (a) and (b) follows from Remark 2.1 (i), whereas (c) and (d)), resp. (e) and (f) are equivalent by the Remark 2.1 (iii)

(a) $\Longrightarrow$ (c): Since $s(A) < 0$ the resolvent $R(0, A)$ exists and is a positive map on M. Therefore $R(0, A)1 \in D(A)_+$ or $Ax = -\mathbb{1}$ for some $x \in D(A)_+$.

(c) $\Longrightarrow$ (e): Let $x \in D(A)_+$ such that $Ax = -\mathbb{1}$. Then

$$T(t)x - x = \int_0^t T(s)Ax\,\mathrm{d}s = -\int_0^t T(s)\mathbb{1}\,\mathrm{d}s \quad (t \geqslant 0),$$

hence

$$0 \leqslant \int_0^t T(s)\mathbb{1}\,\mathrm{d}s \leqslant x \quad (t \in \mathbb{R}_+).$$

Since the family $(\int_0^t T(s)\mathbb{1}\,\mathrm{d}s)_{t\geqslant 0}$ is increasing and bounded,

$$\lim_{t\to\infty} \int_0^t T(s)\mathbb{1}\,\mathrm{d}s$$

exists in the weak operator topology on $\mathcal{B}(H)$.

Since on bounded sets of M, the weak operator topology is equivalent to the $\sigma(M, M_*)$-topology, for every $\varphi \in M_*$ the integral $\int_0^\infty \varphi(T(s)\mathbb{1})\,\mathrm{d}s$ exists (Sakai (1971, 1.15.2.)). Take $x \in M_+$ and $\varphi \in M_*^+$. Then $x \leqslant \|x\|\mathbb{1}$ and therefore

$$\varphi(T(s)x) \leqslant \|x\|\varphi(T(s)\mathbb{1}) \quad (s \in \mathbb{R}_+).$$

Hence $\int_0^\infty \varphi(T(s)x)ds$ exists. Since the positive cones of M and M_* are generating, $\int_0^\infty \varphi(T(s)x)ds$ exists for every $x \in M$ and $\varphi \in M_*$. Therefore $R(0, A)$ exists and is positive which proves (e).
(c) $\implies$ (g): From the last paragraph we obtain that for all $\xi \in H$

$$\int_0^\infty \|U(s)\|^2 ds = \int_0^\infty (T(s)1\xi|\xi)ds$$

exists.
(g) $\implies$ (h): It follows from the polarization identity that the integral

$$\int_0^\infty (U(s)\xi|U(s)\zeta)ds$$

exists for all $\xi, \zeta \in H$. Using Takesaki (1979, Theorem III.4.2 and Theorem II.2.6), we conclude as in the implication from (c) to (e) that for all $\xi, \zeta \in H$ the integral

$$\int_0^\infty ((T(s)x)\xi|\zeta)ds \quad (x \in M)$$

is finite.
(g) $\implies$ (a): Since the vector states are dense in the predual of M and since the preadjoint semigroup of $\mathcal{T}$ is strongly continuous, it is easy to see that the integral

$$\int_0^\infty \varphi(T(s)x)ds$$

exists for all $x \in M$ and $\varphi \in M_*$ (Takesaki (1979, Theorem II.2.6)). Therefore, the resolvent $R(0, A)$ exists and is positive, hence $s(A) < 0$. □

3 Convergence of Positive Semigroups

In this section the asymptotic behavior of positive semigroups $(T(t))_{t\geqslant 0}$ on W*-algebras will be described in more detail. Essentially we distinguish three cases.

(i) The Cesàro means $\frac{1}{s}\int_0^s T(t)dt$ converge strongly to a projection P onto the fixed space of $(T(t))_{t\geqslant 0}$ (see Proposition 3.3 & 3.4).
(ii) The maps $T(t)$ converge strongly to P (see Proposition 3.7, **??** &. **??**).
(iii) The maps $T(t)$ behave asymptotically as a periodic group (Theorem 3.11).

Much of the following is based on the theory of weakly compact operator semigroups. Therefore the following compactness criterium is quite useful.

Proposition 3.1 *Let M be a* W**-algebra, $\mathcal{T}$ a bounded semigroup of positive maps on M_* and suppose that there exists a faithful family Φ of $\mathcal{T}$-subinvariant states in M_*. Then $\mathcal{T}$ is relatively compact in the weak operator topology of LM_*. In particular, $\mathcal{T}$ is strongly ergodic, i.e.,*

$$\lim_{s\to\infty} \frac{1}{s} \int_0^s T(t)x \,\mathrm{d}t$$

exists for every x in M and yields a projection onto $\operatorname{Fix}(\mathcal{T})$.

Proof Since the positive cone of M_* is generating, it is enough to show that for every $0 \leqslant \varphi \in M_*$ the orbit $\{T(t)\varphi\colon t \in \mathbb{R}_+\}$ is relatively weak compact. For this we use Takesaki (1979, Theorem III.5.4.(iii)).
Let $(p_n)_{n\in\mathbb{N}}$ be a decreasing sequence of projections in M such that $\inf_n p_n = 0$. Then $\lim_n \varphi(p_n) = 0$ for every $\varphi \in \Phi$. Since

$$(T(t)p_n)^2 \leqslant T(t)p_n, \quad t \in \mathbb{R}_+,$$

we obtain by a classical *inequality of Kadison* that

$$0 \leqslant \varphi((T(t)p_n)^2) \leqslant \varphi(T(t)p_n) \leqslant \varphi(p_n),$$

hence $\lim_n \varphi(T(t)p_n) = 0$ uniformly in $t \in \mathbb{R}_+$. Since the family Φ is faithful on M, it follows from Takesaki (1979, Proposition III.5.3) that $(T(t)p_n)$ converges to zero in the $s(M, M_*)$-topology uniformly in $t \in \mathbb{R}_+$. Since this topology is finer than the weak*-topology on M, we obtain the relative compactness of $\mathcal{T}$ which implies the strong ergodicity. □

Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a W*-algebra M. We call

$$p_r := \sup\{s(|\varphi|)\colon \varphi \in \operatorname{Fix}(T)\}$$

the recurrent projection associated with $\mathcal{T}$. For a motivation of this definition compare, e.g., Davies (1976, Section 6.3).
Since $T(t)|\varphi| = |\varphi|$ for all $\varphi \in \operatorname{Fix}(T)$ (D-III, Cor. 1.5), we obtain $T(t)'p_r \geqslant p_r$ (see D-I,Sec.3.(c)). Let $\mathcal{T}^{(r)}$ be the reduced semigroup on $p_r M_* p_r$ with generator $A^{(r)}$. Then $\mathcal{T}^{(r)}$ is identity preserving and of Schwarz type. Similarly, if $\mathcal{R}$ is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$ with values in M_* such that $\mathcal{R}$ is identity preserving and of Schwarz type, then the recurrent projection associated with $\mathcal{R}$ is defined using $\operatorname{Fix}(\mathcal{R})$.

Remark 3.2 (i) Let $\varphi \in M_*$ and $\alpha \in \mathbb{R}$ such that $(\mu - \mathrm{i}\alpha)R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$. Since $s(|\varphi|)$ and $s(|\varphi^*|)$ are majorized by p_r (D-III,Prop.1.4), it follows that φ and φ^* are in $p_r M_* p_r$.

(ii) From (i) and the observation that the family $\{|\varphi|\colon \varphi \in \mathrm{Fix}(\mathcal{T})\}$ is faithful on $p_r M p_r$ and consists of $\mathcal{T}^{(r)}$-invariant elements, it follows that

- $P\sigma(A) \cap \mathrm{i}\mathbb{R} = P_\sigma(A^{(r)}) \cap \mathrm{i}\mathbb{R}$.
- $\mathrm{Ker}((\mathrm{i}\alpha - A)) \subset p_r M_* p_r$ for all $\alpha \in \mathbb{R}$.
- The semigroup $\mathcal{T}^{(r)}$ is relatively compact in the weak operator topology and therefore strongly ergodic.

(iii) Similarly, let $\mathcal{R}$ be an identity preserving pseudo-resolvent with values in M_* on $D = \{\lambda \in \mathbb{C}\colon \mathrm{Re}(\lambda) > 0\}$ which is of Schwarz type. It follows as in (b) that $\mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$ is contained in $p_r M_* p_r$ for all $\lambda \in D$ and $\alpha \in \mathbb{R}$, where p_r is the associated recurrent projection.

We now give a characterization of strong ergodicity of semigroups which are identity preserving and of Schwarz type. For this we need that the Cesàro means

$$C(s)x = \frac{1}{s}\int_0^s T(t)x\,dt \quad (x \in M, 0 \leqslant s \in \mathbb{R})$$

are Schwarz maps. We omit the simple calculation (compare D-I,Thm.2.1).

Proposition 3.3 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W**-algebra M. Then the following assertions are equivalent.*

(a) $\mathcal{T}$ is strongly ergodic on M_*.
(b) $\sigma(M, M_*)$- $\lim_{s\to\infty} C(s)' p_r = 1$.
(c) $s^*(M, M_*)$- $\lim_{s\to\infty} C(s)' p_r = 1$.

Proof Suppose that (a) holds. Since $\mathrm{Fix}\,(T)$ separates $\mathrm{Fix}(T')$ (see Krengel (1985, Chap.2,Thm.1.4)), the fixed space of $\mathcal{T}'$ is non trivial, hence $p_r \neq 0$. Let $0 \leqslant \psi \in M_*$, then $\psi_0 := \lim_{s\to\infty} C(s)\psi \in \mathrm{Fix}\,(T)$ and $s(\psi_0) \leqslant p_r$. Therefore

$$\begin{aligned}\lim_{s\to\infty} \psi(C(s)' p_r) &= \lim_{s\to\infty} (C(s)\psi)(p_r) = \psi_0(p_r)\\ &= \psi_0(1) = \lim_{s\to\infty} (C(s)\psi)(1) = \psi(1)\end{aligned}$$

which proves (b).
Suppose that (b) is satisfied. Since $C(s)' p_r \leqslant 1$ for all $s \in \mathbb{R}_+$, we obtain (c). (Use that for $(x_\alpha) \in M_+$ we have $\lim_\alpha x_\alpha = 0$ in the weak*-topology if and only if $\lim_\alpha x_\alpha = 0$ in the $s^*(M, M_*)$-topology.)
Suppose that (c) holds. Since each $C(s)'$ is an identity preserving Schwarz map, we obtain for all $x \in M$

$$\begin{aligned}(C(s)'((1-p_r)x))(C(s)'((1-p_r)x)^*) &\leqslant C(s)'((1-p_r)xx^*(1-p_r))\\ &\leqslant \|x\|^2 C(s)'(1-p_r),\end{aligned}$$

hence

$$s^*(M, M_*)\text{-} \lim_{s\to\infty} C(s)'((1-p_r)x) = 0.$$

In particular, we obtain for all $x \in \mathrm{Fix}(\mathcal{T}')$ that $x = \sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)'x = \sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)'(p_r x)$.
Especially for $0 \neq x \in \mathrm{Fix}\,(\mathcal{T})$ we obtain $p_r x p_r \neq 0$. Since the W*-algebra $p_r M p_r$ is the dual of $p_r M_* p_r$ and since $\mathcal{T}^{(r)}$ is strongly ergodic, it follows that the fixed space of $\mathcal{T}$ separates the points of $\mathrm{Fix}\,(\mathcal{T}')$. Thus $\mathcal{T}$ is strongly ergodic (Krengel (1985, Chap. 2, Thm. 1.4)). □

It follows from the result above that the semigroup in Evans (1977) cannot be strongly ergodic on $\mathcal{B}(H)_*$ since the associated recurrent projection is zero. But for irreducible semigroups we have the following result.

Proposition 3.4 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W*-*algebra M. Then the following assertions are equivalent.*

(a) $\mathcal{T}$ is irreducible and $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.
(b) $\mathcal{T}$ is relatively compact in the weak operator topology and the fixed space of $\mathcal{T}$ is generated by a faithful state.
(c) $\mathcal{T}$ is strongly ergodic and the fixed space of $\mathcal{T}$ is generated by a faithful state.
(d) The fixed space of $\mathcal{T}$ is generated by a faithful state.

Proof Suppose (a) is satisfied. Since $\mathrm{Fix}\,(\mathcal{T}) \neq \{0\}$, there exists a faithful normal state φ on M such that $\mathrm{Fix}\,(\mathcal{T}) = \mathbb{C}\,\varphi$ (D-III, Thm.1.10.). Therefore $\mathcal{T}$ is relatively compact in the weak operator topology by Proposition 3.1., whence (b) holds and the implications from (b) to (c) and (c) to (d) are obvious.
Suppose that (d) holds. Let φ be a faithful normal state on M such that $\mathrm{Fix}\,(T) = \varphi\mathbb{C}$. By Proposition 3.1 the semigroup $\mathcal{T}$ is strongly ergodic. Therefore the fixed space of $\mathcal{T}$ separates the points of $\mathrm{Fix}(T')$. Consequently $\mathrm{Fix}(T') = \mathbb{C}1$. Thus the ergodic projection associated with $\mathcal{T}$ is given by $P = 1 \otimes \varphi$, i.e., $P\psi = \psi(1)\varphi$ for all $\psi \in M_*$. Let F be a closed $\mathcal{T}$-invariant face of M_*^+. If $0 \neq \psi \in F$ then

$$\lim_{s\to\infty} C(s)\psi = \psi(1)\varphi \in F.$$

Hence $\varphi \in F$ and therefore $F = M_*^+$ by the faithfulness of φ which proves (a). □

The next theorem is an extension of D-III, Thm.1.10 and shows the usefulness of the theory of semitopological semigroups. Assume $\mathcal{T} \subseteq \mathrm{L}M_*$ to be relatively compact in the weak operator topology. Since $\mathcal{T}$ is commutative its closure $\mathcal{S} = (\mathcal{T})^- \subseteq L_w(M_*)$ contains a unique minimal ideal $\mathcal{K}$, called the kernel of $\mathcal{S}$, which is a compact Abelian group (DeLeeuw and Glicksberg (1961), Junghenn (1971) & Krengel (1985, § 2.4)). The identity Q of $\mathcal{K}$ is a projection onto the closed linear span of all eigenvectors of A pertaining to the eigenvalues in $\mathrm{i}\mathbb{R}$.
Moreover, the dual group of $\mathcal{K}$ can be identified with the subgroup of $\mathrm{i}\mathbb{R}$ generated by $P\sigma(A) \cap \mathrm{i}\mathbb{R}$. We call Q the semigroup projection associated with $\mathcal{T}$. On the other hand, $\mathcal{T}$ is always strongly ergodic with projection P onto $\mathrm{Fix}\,(\mathcal{T})$. Obviously, the relation

$$0 \leqslant P \leqslant Q \leqslant \mathrm{Id}$$

holds, where the order relation is defined by the inclusion of the range spaces.

There are two extreme cases. First, $Q = \mathrm{Id}$ and $\operatorname{rank}(P)$. This corresponds to the Halmos-von Neumann Theorem in commutative ergodic theory and is discussed, at least for irreducible semigroups, in Olesen et al. (1980).
Second, $\mathrm{Id} > Q = P$, in particular $\operatorname{rank}(P) = 1$. This latter case will be investigated in detail for $M = \mathcal{B}(H)$, the W*-algebra of all bounded linear operators on a Hilbert space H. But we first need some preparations.

Theorem 3.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W**-algebra M and suppose there exists a faithful family of $\mathcal{T}$-invariant states on M. Let N be the $\sigma(M, M_*)$-closed linear span of all eigenvectors of A' pertaining to the eigenvalues in* $\mathrm{i}\mathbb{R}$. *If Q is the semigroup projection associated with $\mathcal{T}$, then the following holds.*

(i) The adjoint of Q is a faithful normal conditional expectation from M onto the W**-subalgebra N.*
*(ii) The restriction of T' to N can be embedded into a $\sigma(M, M_*)$-continuous, one-parameter group of *-automorphisms.*
(iii) If, in addition, $\mathcal{T}$ is irreducible and if φ is the normal state generating the fixed space of $\mathcal{T}$, then $\varphi|_N$ is a faithful normal trace.

Proof Consider $H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$ which is not empty by assumptions. From Proposition 3.1 it follows that $\mathcal{T}$ is relatively compact in the weak operator topology. Let K be the semigroup kernel of $\overline{\mathcal{T}w} \subset L(M_*)$ and Q the unit of K. Recall that $Q\psi_n = \psi_n$ for all $\psi_n \in M_*$ such that $A\psi_n = n\psi_n$ $(n \in H)$. Let $\mathcal{E}$ be the family of all eigenvectors of A' pertaining to the eigenvalues in H.
Then $\mathcal{E}$ is closed with respect to the multiplication in M and the formation of adjoints. Thus N is a W*-subalgebra of M, Sakai (1971, Corollary 1.7.9.), and $\mathcal{T}_0(t)' := T(t)'_{|N}$ is multiplicative (for this see D-III, Lemma 1.1).
Since $Q \in \overline{\mathcal{T}w} \subseteq L_w(M_*)$, there exists an ultrafilter $\mathfrak{U}$ on $\mathbb{R}_+$ such that

$$\lim_{\mathfrak{U}} \langle T(t)\psi, x\rangle = \langle Q\psi, x\rangle$$

for all $x \in M$ and $\psi \in M_*$. If $n \in H$ and $\psi_n \in M_*$ such that $A\psi_n = n\psi_n$, then for all $x \in M$ we obtain

$$\langle \psi_n, x\rangle = \langle Q\psi_n, x\rangle = \lim_{\mathfrak{U}} \langle T(t)\psi_n, x\rangle = (\lim_{\mathfrak{U}} e^{nt})\langle \psi_n, x\rangle,$$

hence $\lim_{\mathfrak{U}} e^{nt} = 1$. From this it follows that for all $\psi \in M_*$ we have

$$\langle \psi, Q'(u_n)\rangle = \lim_{\mathfrak{U}} \langle \psi, T(t)'u_n\rangle = (\lim_{\mathfrak{U}} e^{nt})\langle \psi, u_n\rangle = \langle \psi, u_n\rangle.$$

Hence $N \subseteq Q'(M)$.
For γ in the dual group of K and $x \in M$ we define x_γ by

$$\psi(x_\gamma) := \int_K \langle S\psi, x\rangle \langle S, \gamma\rangle^* \,\mathrm{d}m(S) \quad (\psi \in M_*^+).$$

Then $x_\gamma \in M$ and $T(t)'x_\gamma = \langle QT(t), \gamma\rangle x_\gamma$. Therefore $x_\gamma \in N$. Thus the inclusion $Q'M \subseteq N$ is proved if we can show that $Q'M$ belongs to the $\sigma(M, M_*)$-closed linear span of $\{x_\gamma : \gamma \in K, x \in M\}$. For this it is enough to show that every linear form $\psi \in M_*$ such that $\psi(x_\gamma) = 0$ for all $\gamma \in K$ satisfies $\psi(Qx) = 0$ for all $x \in M$. But if $\psi(x_\gamma) = 0$, then

$$\int_K \langle S\psi, x\rangle\langle S, \gamma\rangle^* \,\mathrm{d}m(S) = 0, \gamma \in K.$$

Since the map $(S \mapsto \psi(Sx))$ is continuous on K and since the elements of K form a complete orthonormal basis in $L^2(K, \mathrm{d}m)$, we obtain $\psi(Sx) = 0$ for all $S \in K$, in particular $\psi(Qx) = 0$ as desired.
Since the range of Q' is a W*-subalgebra of M it follows from Takesaki (1979, Theorem III.3.4) that Q' is a completely positive, normal conditional expectation. This Q' is faithful, i.e., $\mathrm{Ker}(Q') \cap M_+ = \{0\}$ since $Q\varphi = \varphi$ for the faithful linear form φ.
Let φ be the faithful normal state generating $\mathrm{Fix}\,(T)$ and let $\mathcal{U}$ be a family of unitary eigenvectors of A' pertaining to the eigenvalues in H (see D-III, Remark 1.11). If $u_1, u_2 \in U$, then

$$\varphi(u_1u_2^*) = \varphi(T_0(t)'(u_1u_2^*)) = e^{(n_1-n_2)t}\varphi(u_1u_2^*).$$

Therefore

$$\varphi(u_1u_2^*) = \begin{cases} 0 & \text{if } n_1 \neq n_2, \\ 1 & \text{if } n_1 = n_2. \end{cases}$$

Hence $\varphi(u_1u_2^*) = \varphi(u_2^*u_1)$ from which it follows that $\tau := \varphi|_N$ is a faithful normal trace. □

Remark 3.6 (i) Since $QM_* = N_*$ and $Q'M = N$, where N_* is as in D-III, Proposition 1.12, it follows from general duality theory that $(N_*)' = N$.
(ii) If $\psi \in N_*$, then $|\psi| \in N_*$. To see this, note that $Q\psi = \psi$ and Q is an identity preserving Schwarz map. Then the assertion follows from D-III, Proposition 1.4.
(iii) If $\psi \in N_*$, then $|T_0(t)\psi| = T_0(t)|\psi|$ for all $t \in \mathbb{R}$. This follows immediately from the fact that $\mathcal{T}_0(t)'$ is a *-automorphismus on N.
(iv) Let us add a few words concerning the structure of N: If $\mathcal{T}$ is irreducible and K is the semigroup kernel of $\mathcal{T}^- \subseteq L_w(M_*)$, then $(S \mapsto S') : K \to L((N, \sigma(N, N_*)))$ is a representation of the compact, Abelian group K as group of *-automorphism such that the fixed space is one dimensional. Therefore we are able to apply the results of Olesen et al. (1980). There are three possibilities for N.

1. $N = L^\infty(K, \mathrm{d}m)$ and $\mathcal{T}|_N$ is the translation group on N.
2. $N \cong R$ where $\mathcal{R}$ is the (unique) hyperfinite factor of type II_1. In that case (the image of) K is approximately inner on $\mathcal{R}$ [l.c., Theorem 5.8].
3. There exists a closed subgroup G of K such that

$$N = L^\infty(K/G, \mathrm{d}m) \otimes R$$

where R is as in (ii) and dm the normalized Haar measure on K/G [l.c., Theorem 5.15].

So far we have studied weak*-semigroups on general W*-algebras. We apply now these results to weak*-semigroup on $\mathcal{B}(H)$. To do this we call a triple $(M, \varphi, \mathcal{T})$ a W*-dynamical system if M is a W*-algebra, $\mathcal{T}$ a weak*-semigroup of identity preserving Schwarz maps on M and φ a faithful family of $\mathcal{T}$-invariant normal states. We call $(M, \varphi, \mathcal{T})$ irreducible, if the preadjoint semigroup is irreducible (alternatively, if the fixed space of $\mathcal{T}$ is one dimensional).

Proposition 3.7 *Let $(\mathcal{B}(H), \varphi, \mathcal{T})$ be a* W*-*dynamical system on the* W*-*algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H. Then the following assertions are equivalent:*

(a) $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$,
(b) $\lim_{s\to\infty} T(s)_* = P_*$ *in the strong operator topology on* $L\mathcal{B}(H)_*$.

Proof Obviously (b) implies (a). Suppose that (a) is fulfilled. Then the ergodic projection P_* of the preadjoint semigroup is equal to the associated semigroup projection. Consequently there exists an ultrafilter $\mathfrak{U}$ on $\mathbb{R}_+$ such that $\lim_{\mathfrak{U}} T(t) = P$ in the weak operator topology. We claim that the convergence holds even in the strong operator topology. Taking this for granted it follows, since for every $t \in \mathbb{R}_+$ $T(t)$ is a contraction, that

$$\lim_{t\to\infty} \|T(t)_*\varphi\| = 0$$

for all $\varphi \in \mathrm{Ker}(()P_*)$. Since $T(t)_*\psi = \psi$ for every $\psi \in \mathrm{im}(P_*)$ and

$$\mathcal{B}(H)_* = \mathrm{im}(P_*) \oplus \mathrm{Ker}(()P_*)$$

the assertion is proved.
It remains to show that $\lim_{\mathfrak{U}} T(t)_* = P_*$ in the strong operator topology. Choose $0 \leqslant \varphi \in \mathcal{B}(H)_*$, $\|\varphi\| \leqslant 1$ and let $\varphi_t := T(t)_*\varphi$ $(t > 0)$. $\varphi_0 := P_*\varphi$ and let $\{p_i : i \in A\}$ be an increasing net of projections of finite rank in $\mathcal{B}(H)$ with strong limit 1. Since the set $K := \{\varphi_t : t \geqslant 0\}$ is relatively compact in the $\sigma(\mathcal{B}(H)_*, \mathcal{B}(H))$-topology, there exists for every $\delta > 0$ an index $i_0 \in A$ such that

$$\|(1 - p_i)\psi(1 - p_i)\| \leqslant \delta$$

for every $\psi \in K$ and $i \geqslant i_0$ (Takesaki (1979, Theorem III.5.4.(vi))). In particular

$$|\psi(1 - p_i)| \leqslant \delta, \quad \psi \in K,\ i(0) \leqslant i.$$

Let $p := p_{i(0)}$. Then for all x in the unit ball of M it follows that

$$\begin{aligned} &|(\varphi_t - \varphi_0)(x)| \leqslant \\ &|(\varphi_t - \varphi_0)(pxp)| + |(\varphi_t - \varphi_0)((1 - p)xp)| \\ &+|(\varphi_t - \varphi_0)(x(1 - p))| \leqslant\leqslant |(\varphi_t - \varphi_0)(pxp)| + 4\sqrt{\delta}. \end{aligned}$$

Since the W*-algebra $p\mathcal{B}(H)p$ is finite dimensional, there exists $U \in \mathfrak{U}$ such that

$$\|(\varphi_t - \varphi_0)|_{p\mathcal{B}(H)p}\| \leqslant \delta.$$

for all $t \in U$. Consequently

$$\|(\varphi_t - \varphi_0)\| \leqslant (\delta + 4\sqrt{\delta})$$

for all $t \in U$. Therefore $\lim_{\mathfrak{U}} T(t)_* \varphi = P_* \varphi$ in the strong operator topology. Since the positive cone of $\mathcal{B}(H)_*$ is generating, the assertion is proved. □

We show next, that for irreducible W*-dynamical systems on $\mathcal{B}(H)$ the above properties always hold.

Theorem 3.8 *Let $(\mathcal{B}(H), \varphi, \mathcal{T})$ be an irreducible* W*-*dynamical system. Then*

$$P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}.$$

Proof Let N be the W*-subalgebra of $M = \mathcal{B}(H)$ generated by the eigenvectors of A pertaining to the eigenvalues on $\mathrm{i}\mathbb{R}$ and let Q be the faithful normal conditional expectation from M onto N (Proposition 3.7). Since M is atomic, N is atomic (Størmer (1972)). N is finite since there exists a finite, faithful normal trace on N. In particular the center of N is isomorphic to ℓ^∞.
Let $\mathcal{S}$ be the restriction of $\mathcal{T}$ to the center. Then $\mathcal{S}$ is a weak*-semigroup such that every $S(t) \in \mathcal{S}$ is $\sigma(\ell^\infty, \ell^1)$-continuous and a *-automorphism. From this it follows that $S(t)$ is induced by some continuous flow $\kappa_t : \mathbb{N} \to \mathbb{N}$. Indeed, if $\delta_n((\xi_m)) = \xi_n$ $(n \in \mathbb{N}, (\xi_m) \in \ell^\infty)$, then $\delta_n \circ S(t)$ is a normal scalar valued *-homomorphism hence of the form δ_m for some $m = \kappa_t(n)$. But the function $t \mapsto \kappa_t$ is continuous from $\mathbb{R}$ into $\mathbb{N}$, whence constant. Hence $S(t) = \mathrm{Id}$. But the semigroup S is weak*-irreducible on the center. Consequently, the center is one dimensional. Using [Takesaki, Theorem V.1.27] we obtain $N = B(H_n)$ where H_n is a finite dimensional Hilbert space. But if $0 \neq \mathrm{i}\alpha \in P\sigma(A) \cap \mathrm{i}\mathbb{R}$ then $\mathrm{i}\alpha\mathbb{Z} \subset P\sigma(A)$ by D-III,Thm.1.10, whence N must be infinite dimensional. Therefore $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$ as desired. □

Corollary 3.9 *If $(\mathcal{B}(H), \varphi, T)$ is an irreducible* W*-*dynamical system, then*

$$\lim_{s \to \infty} T(s) = 1 \otimes \varphi$$

in the strong operator topology on $L(\mathcal{B}(H)_*)$, where φ is the unique normal state generating the fixed space of T_*.

We are now going to discuss the asymptotic behavior of positive semigroups whose generator has boundary point spectrum different from 0. The standard example is the following. If Γ is the unit circle, $\mathrm{d}m$ the normalized Haar measure on Γ and $0 < \tau \in \mathbb{R}$, then we define the maps $T_\tau(t)$, $t \in \mathbb{R}_+$, on $L^1(\Gamma, m)$ by

$$(T_\tau(t)f)(\xi) = f(\xi \exp(\frac{2\pi i}{\tau} t)) \quad (f \in L^1(\Gamma, \mathrm{d}m), \xi \in \Gamma).$$

Then $\mathcal{T} := (T_\tau(t))_{t\geqslant 0}$ forms a strongly continuous one parameter semigroup which is identity preserving and of Schwarz type. Since $\mathcal{T}$ is periodic of period τ, it follows that 0 is a pole of the resolvent of its generator B with residuum $P = 1 \otimes 1$ and $\{\frac{2\pi \mathrm{i}}{\tau} \cdot k : k \in \mathbb{Z}\} = \sigma(B)$. Thus $\mathcal{T}$ is irreducible and uniformly ergodic on $L^1(\Gamma, \mathrm{d}m)$ (see A-II, Section 5).
Now let $\mathcal{T}$ be a semigroup on a predual M_* of a von Neumann-algebra M. It is called *partially periodic*, if there exists a projection $Q \in L(M_*)$ reducing T such that $Q(M_*) \cong L^1(\Gamma, \mathrm{d}m)$ and $T_{|\mathrm{im}(Q)}$ is conjugate to a periodic semigroup on $L^1(\Gamma, \mathrm{d}m)$.
In the main result we present a non commutative version of Nagel (1984) showing that certain dynamical systems are partially periodic semigroups.

Proposition 3.10 *Let $\mathcal{T}$ be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra M.*
If $\mathcal{T}$ is uniformly ergodic, then $\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$.
If additionally $\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \{0\}$, there exists a strictly positive projection Q on M_* which is identity preserving and completely positive such that

(i) Q reduces $\mathcal{T}$ and $Q(M_*) \cong L^1(\Gamma)$, Γ being the one dimensional torus.
(ii) The restriction T_0 of $\mathcal{T}$ to im(Q) is irreducible and conjugate to a rotation semigroup of period $\tau = \frac{2\pi}{\alpha}$ on Γ.
(iii) The spectral bound $s(A_{|\ \mathrm{Ker}(()Q)})$ is strictly smaller than 0*.*

Proof By D-III, Thm.1.11 and D-III, Thm.2.5 it follows that

$$\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z}$$

for some $\alpha \in \mathbb{R}$. Suppose $\alpha \neq 0$. Since $\sigma(A) + \mathrm{i}\alpha\mathbb{Z} = \sigma(A)$ and since every $n \in \mathrm{i}\alpha\mathbb{Z}$ is isolated, it follows that there exists $\delta > 0$ such that

$$\sigma(A) \setminus \mathrm{i}\alpha\mathbb{Z} \subseteq \{\lambda \in \mathbb{C} \colon \mathrm{Re}(\lambda) \leqslant \delta\}.$$

Let $\{u_\alpha^k : k \in \mathbb{Z}\}$ be a family of unitary eigenvectors of A' pertaining to the eigenvalues in $\mathrm{i}\mathbb{R}$. Then $Q'(M)$ is a commutative W*-algebra. For $\tau := \frac{2\pi}{\alpha}$, we obtain $T(\tau)u_\alpha^k = u_\alpha^k$, hence $T|_{\mathrm{im}(Q)}$ is periodic. From the Halmos-von Neumann theorem (see Schaefer (1974, Thm. III.7.11)) it follows that $T|_{\mathrm{im}(Q)}$ is conjugate to the rotation semigroup of period τ on $L^1(\Gamma, m)$. □

Using this proposition we obtain the following theorem.

Theorem 3.11 *Let $T = (T(t))_{t\geqslant 0}$ be a uniformly ergodic, identity preserving semigroup of Schwarz type on the predual of a* W*-*algebra M and suppose*

$$\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \{0\}.$$

Then there exists a partially periodic, identity preserving semigroup $S = (S(t))_{t\geqslant 0}$ of Schwarz type on M_* such that

$$\lim_{t\to\infty} (T(t) - S(t)) = 0$$

in the strong operator topology.

Proof Let φ be the normal state on M generating the fixed space of $\mathcal{T}$. Let $\mathcal{S} = (S(t))_{t\geqslant 0}$ where $S(t) := T(t) \circ Q$ and Q is as in 2.6. Obviously, $\mathcal{S}$ is partially periodic and $\varphi \in \operatorname{Fix}(\mathcal{S})$. Let H_φ be the GNS-Hilbert space pertaining to φ. Since φ is fixed under $\mathcal{T}$, $\mathcal{S}$ and Q, these objects have a canonical extension to H_φ (in the following denoted by the same symbols). If $H_0 := \operatorname{Ker}(()Q) \subseteq H_\varphi$, then it is easy to see that H_0 is invariant under the extension to H_φ and for the multiplication maps we defined in D-III, Remark 1.3.
Consequently, using the results in Groh and Kümmerer (1982), it follows that there exists $c \in \mathbb{R}$ such that for all γ near 0 and all $\beta \in \mathbb{R}$:

$$\|R(\gamma + \mathrm{i}\beta A_0)\| \leqslant c, \tag{*}$$

where $A_0 := A_{|\operatorname{Ker}(()Q)}$ (the norm taken in $L(H_\varphi)$). Using the result in A-III,Cor.7.11 it follows that

$$\lim_{t\to\infty} \|T(t)|_{H_0}\| = 0.$$

Since the $s(M, M_*)$-topology on the unit ball of M is nothing else than the restriction of the norm topology on H_φ, we obtain

$$s(M, M_*)\text{-}\lim_{t\to\infty} (T(t)' - S(t)')(x) = 0$$

uniformly on M_1. From this the assertion follows. □

4 Uniform Ergodic Theorems

As we have seen, uniformly ergodic semigroups have strong spectral properties. In this section we study sufficient conditions which imply uniform ergodicity thereby generalizing results of Groh (1984a). We first need some preparations.

Lemma 4.1 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$ with values in the predual of a* W*-*algebra* M. *If the fixed space of $\mathcal{R}$ is infinite dimensional, then there exists a sequence of states in* $\operatorname{Fix}(\mathcal{R})$ *such that the corresponding support projections are mutually orthogonal in M.*

Proof Let $\Phi = \{\varphi \in \operatorname{Fix}(\mathcal{R}) : \varphi \text{ state on } M\}$ and let $p = \sup\{s(\varphi) : \varphi \in \Phi\}$. Since $\lambda R(\lambda)\varphi = \varphi$ for all $\varphi \in \Phi$ and $\lambda \in D$, it follows $\mu R(\mu)(\mathbb{1} - s(\varphi)) = (\mathbb{1} - s(\varphi))$. Hence $\mu R(\mu)(\mathbb{1} - p) = (\mathbb{1} - p)$ for all $\mu \in \mathbb{R}_+$.
Let $\mathcal{R}_1$ be the induced pseudo-resolvent on pM_*p (D-I, Section 3.(c)). Then the family Φ is faithful on M_p and contained in the fixed space of $\mathcal{R}_1$. The adjoint $\mu R_1(\mu)'$ is an identity preserving Schwarz map. Consequently it follows from D-III, Lemma 1.1.(b) and, the $\sigma(M_p, (M_p)_*)$-continuity of $\mu R_1(\mu)'$ that $\operatorname{Fix}\left(R_1'\right)$ is a W*-subalgebra of M_p and by D-III, Lemma 1.5, $\dim \operatorname{Fix}(\mathcal{R}) \leqslant \dim \operatorname{Fix}\left(R_1'\right)$.

If $\mathrm{Fix}\,(\mathcal{R})$ is infinite dimensional, let (p_n) be a sequence of mutually orthogonal projections in $\mathrm{Fix}(R_1') \subseteq M_p$ and choose a sequence (φ_n) in Φ such that $\varphi_n(p_n) \neq 0$. For $n \in \mathbb{N}$ let ψ_n be the normal state

$$\psi_n(x) = \varphi_n(p_n)^{-1}\varphi_n(p_n x p_n)$$

on M. Because of $s(\psi_n) \leqslant p_n \leqslant p$, the support projections of the ψ_n's are mutually orthogonal in M. For $\mu \in \mathbb{R}_+$ and $x \in M$ we obtain

$$\begin{aligned}\langle x, \mu R(\mu)\psi_n\rangle &= \varphi_n(p_n)^{-1}\langle \mu p_n(R(\mu)'x)p_n, \varphi_n\rangle = \\ &= \varphi_n(p_n)^{-1}\langle \mu p_n p(R(\mu)p'x)p_n, \varphi_n\rangle = \\ &= \varphi_n(p_n)^{-1}\langle \mu p_n(pR_1(\mu)'xp)p_n, \varphi_n\rangle = \\ &= \varphi_n(p_n)^{-1}\langle \mu(p_n R_1(\mu)'xp_n), \varphi_n\rangle = \\ &= \varphi_n(p_n)^{-1}\varphi_n(x) = \psi_n(x).\end{aligned}$$

Therefore $\psi_n \in \mathrm{Fix}\,(\mathcal{R})$ for all $n \in \mathbb{N}$. □

Remark 4.2 (i) If $\dim \mathrm{Fix}\,(\mathcal{R}) \geqslant 2$ then the Jordan decomposition of self adjoint linear functionals implies that at least two states in $\mathrm{Fix}\,(\mathcal{R})$ have orthogonal support (compare D-III, Theorem 1.10.(a)).
(ii) If $\mathcal{R}$ is a pseudo-resolvent with values in a W*-algebra such that $\mathrm{Fix}\,(\mathcal{R}')$ is contained in M_*, then by D-III, Lemma 1.2, there exists a sequence of normal states in $\mathrm{Fix}\,(\mathcal{R}')$ with orthogonal supports in M.

Lemma 4.3 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}(\lambda) > 0\}$ with values in the predual of a* W**-algebra M. If the fixed space of the canonical extension $\widehat{\mathcal{R}}$ of $\mathcal{R}$ to some ultrapower of M_* is infinite dimensional, then there exists a sequence (z_n) in M_1^+ and a sequence of states (φ_n) in M_* such that*

(i) $\lim_n z_n = 0$ *in the $s^*(M, M_*)$-topology,*
(ii) $\lim_n \|(Id - \lambda R(\lambda))\varphi_n\| = 0$ *for all $\lambda \in D$,*
(iii) $\varphi_n(z_n) \geqslant \frac{1}{2}$ *for all $n \in \mathbb{N}$.*

Proof Let $(M_*)^\wedge$ be the ultrapower of M_* with respect to some free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$. Since $(M_*)^\wedge$ is the predual of a W*-subalgebra of $\widehat{M}$ (see D-III, Remark 2.4.(b)), there exists a sequence of states $(\hat{\psi}_n)$ in $\mathrm{Fix}\left(\widehat{\mathcal{R}}\right)$ such that the corresponding support projections are mutually orthogonal in $\widehat{M}$ (Lemma 4.1). For every $n \in \mathbb{N}$ let $(\psi_{n,k})$ be a representing sequence of states,

$$\varphi := \sum_{n,k} 2^{-(n+k+1)}\psi_{n,k}$$

and

$$p := \sup\{s(\psi_{n,k}) : n, k = 1, \ldots\}$$

in M. Then φ is a normal state on M which is faithful on the W*-algebra M_p. Since

$$1 = \langle \psi_{n,k}, s(\psi_{n,k}) \rangle = \psi_{n,k}(p) \quad (n, k \in \mathbb{N}),$$

it follows $\hat{\psi}_n(\hat{p}) = 1$ where $\hat{p}$ is the canonical image of p in $\widehat{M}$. But this implies $s(\hat{\psi}_n) \leqslant \hat{p}$ in $\widehat{M}$. Since $\widehat{M}_1^+$ is $\sigma(\widehat{M}, \widehat{M}')$-dense in $(\widehat{M}'')_1^+$ (Kaplansky's density theorem Sakai (1971, 1.9.1) with Sakai (1971, 1.8.9 and 1.8.12)), there exists for all $n \in \mathbb{N}$ a net $(z_{n,\gamma})$ in $\widehat{M}_1^+$ such that

$$\sigma(\widehat{M}'', \widehat{M}')\text{-}\lim_{\gamma} \hat{z}_{n,\gamma} = s(\hat{\psi}_n).$$

From Sakai (1971, 1.7.8) and the above considerations, we obtain that the net $(p\hat{z}_{n,\gamma}\hat{p})$ converges to $s(\hat{\psi}_n)$ in the $\sigma(\widehat{M}'', \widehat{M}')$-topology. Therefore we may assume $\hat{z}_{n,\gamma} \in (\widehat{M}'_p)_1^+$.
In the following we denote by $\hat{\varphi}$ the canonical image of φ in $(M_*)^\wedge$.
Since the projections $s(\hat{\psi}_n)$ are mutually orthogonal, there exists a real sequence (r_n), $0 < r_n < 1$, $\lim_n r_n = 0$ and $\hat{\varphi}(s(\hat{\psi}_n)) \leqslant \frac{1}{2} r_n$. For all $n \in \mathbb{N}$ choose $\hat{z}_n \in (\widehat{M}'_p)_1^+$ such that

$$|\langle \hat{\varphi}, s(\hat{\psi}_n) - \hat{z}_n \rangle| \leqslant \frac{1}{2} r_n,$$
$$|\langle \hat{\psi}_n, s(\hat{\psi}_n) - \hat{z}_n \rangle| \leqslant \frac{1}{2} r_n.$$

Hence $\hat{\varphi}(\hat{z}_n) \leqslant r_n$ and $\hat{\psi}_n(\hat{z}_n) \geqslant \frac{1}{2}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ let $(z_{n,k}) \in \hat{z}_n$ be a representing sequence in $(M_p)_1^+ = p(M_1^+)p$ (note that $M_{\hat{p}} = \widehat{M_p}$) and fix $\mu \in \mathbb{R}_+$. Since $\mu R(\mu)' \hat{\psi}_n = \hat{\psi}_n$, $\hat{\varphi}(\hat{z}_n) \leqslant r_n$ and $\hat{\psi}_n(\hat{z}_n) \geqslant \frac{1}{2}$, there exists for all $n \in \mathbb{N}$ an element $U_n \in \mathfrak{U}$ such that for all $k \in U_n$ and we obtain

(i′) $\varphi(z_{n,k}) \leqslant r_n$,
(ii′) $\|(Id - \mu R(\mu))\psi_{n,k}\| \leqslant r_n$,
(iii′) $\psi_{n,k}(z_{n,k}) \geqslant \frac{1}{2}$.

Inductively we find a sequence (z_n) in $(M_p)_1^+$ and a sequence of states (φ_n) in M_* such that for all $n \in \mathbb{N}$

(i″) $\lim_n \varphi_n(z_n) = 0$,
(ii″) $\lim_n \|(Id - \mu R(\mu))\varphi_n\| = 0$,
(iii″) $\varphi_n(z_n) \geqslant \frac{1}{2}$.

But φ is faithful on M_p. Therefore condition (ii″) implies that $\lim_n z_n = 0$ in the $s^*(M_p, (M_p)_*)$-topology (Takesaki (1979, Proposition III.5.4)). Since

$$s^*(M_p, (M_p)_*) = s^*(M, M_*)|_{M_p},$$

(i) follows immediately from (ii″). Using the resolvent equation for $\mathcal{R}$ it is easy to see that (ii″) implies

$$\lim_n \|(Id - \lambda R(\lambda))\varphi_n\| = 0$$

for all $\lambda \in D$ and the proof is complete. □

Without further comments, we will use following facts in this section.

(1) A sequence (φ_n) in M'_+ converges in the $\sigma(M', M)$-topology if and only if it converges in $\sigma(M', M'')$-topology (Akemann et al. (1972)).
(2) We can decompose $\varphi \in M'_+$ into its normal and singular part $\varphi = \varphi^{(n)} + \varphi^{(s)}$, $0 \leqslant \varphi^{(n)} \in M_*$, $0 \leqslant \varphi^{(s)} \in M_*^\perp$ and $\|\varphi\| = \|\varphi^{(n)}\| + \|\varphi^{(s)}\|$ (Takesaki (1979, Theorem III.2.14)).
(3) If (φ_k) is a sequence in M_* convergeing to zero in the $\sigma(M_*, M)$-topology and if (x_n) is a sequence in M converging to zero in the $s^*(M, M_*)$-topology, then $\lim_n \varphi_k(x_n) = 0$ uniformly in $k \in \mathbb{N}$ (Takesaki (1979, Lemma III.5.5)).

Theorem 4.4 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent on*

$$D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}(\lambda) > 0\}$$

with values in a W**-algebra M which is of Schwarz type and let $\mathcal{R}'$ br its adjoint pseudo-resolvent. Any one of the following conditions implies* dim Fix $\left(\widehat{\mathcal{R}}\right) < \infty$ *in some ultrapower of M.*

(i) The fixed space of $\mathcal{R}'$ is finite dimensional.
(ii) $\lim_{\mu\to 0} \mu R(\mu) = P$ *exists in the strong operator topology and rank$(P) < \infty$.*
(iii) The fixed space of $\mathcal{R}'$ is contained in M_*.
(iv) Every map $\mu R(\mu)$, $\mu \in \mathbb{R}_+$ is irreducible on M.

Proof Suppose that the dimension of the fixed space of $\widehat{\mathcal{R}'}$ in some ultrapower $\widehat{(M`)}$ of $M'$ is infinite dimensional. Since $\widehat{(M`)}$ is the predual of the W*-algebra $\widehat{M}$ and $\mathcal{R}'$ is identity preserving (since $R'\mathbb{1} = R\mathbb{1} = \mathbb{1}$) and of Schwarz type (because $\mu R''(\mu) = (\mu R(\mu))''$ is a Schwarz map for all $\mu \in \mathbb{R}_+$), we may apply Lemma 4.3. Suppose that the fixed space of the canonical extension of $\mathcal{R}'$ to some ultrapower of M' is infinite dimensional. Thus we may choose a sequence of states (φ_k) in M' and a sequence (z_k) in $(M'')_1$, $0 \leqslant z_k$, satisfying (i)–(ii) of Lemma 4.3. Remark (3) above implies that no subsequence of (φ_k) can converge in the $\sigma(M', M'')$-topology.

(i) If φ is a $\sigma(M', M)$-accumulation point of (φ_k), then $\varphi \in$ Fix $(\mathcal{R}')$. Since Fix $(\mathcal{R}')$ is finite dimensional, the set of accumulation points of the sequence (φ_k) is metrizable in the $\sigma(M', M)$-topology. Hence there exists a sequence $(k(n))$ of natural numbers such that $\sigma(M', M)$- $\lim_n \varphi_{k(n)} = \varphi$. Consequently, by Remark (1) above , $\varphi = \sigma(M', M'')$- $\lim_n \varphi_{k(n)}$. But this leads to a contradiction proving(i).
(ii) Since dim Fix $(\mathcal{R})$ = dim Fix $(\mathcal{R}')$ = rank $(P) < \infty$, (ii) follows from (i).
(iii) Suppose that the fixed space of R' is infinite dimensional. Since Fix $(\mathcal{R}') \subseteq M_*$, there exists a sequence of states (ψ_n) in Fix $(\mathcal{R}')$ with mutually orthogonal support projections in M (Lemma 4.1). Since every $\sigma(M', M)$-accumulation point

of the ψ_n's belongs to $\mathrm{Fix}\,(\mathcal{R}')$, hence is normal, the sequence (ψ_n) is relatively $\sigma(M_*, M)$-compact.
By Eberlein's theorem, we may assume that this sequence is weakly convergent (Schaefer (1966)). By the orthogonality of the $s(\psi_n)$'s this sequence converges to zero in the $s^*(M, M_*)$-topology, hence $\lim_n \psi_k(s(\psi_n)) = 0$ uniformly in $k \in \mathbb{N}$, a contradiction. Consequently $\dim \mathrm{Fix}\,(\mathcal{R}) < \infty$ and (i) is proved.
(iv) We prove $\dim \mathrm{Fix}\,(\mathcal{R}') = 1$ and apply (i) once again and need the following observation: If ψ is a faithful state on M, then the normal part is faithful too. Indeed, if $0 \neq x \in M$ such that $\psi^{(n)}(x) = 0$, choose a projection $0 \neq p \in M$ such that $\psi^{(n)}(p) = \psi^{(s)}(p) = 0$ (use Takesaki (1979, Theorem III.3.8)). Hence $\psi(p) = 0$ which conflicts with the faithfulness of ψ.
If $2 \leqslant \dim \mathrm{Fix}\,(\mathcal{R}')$ there are states ψ_1 and ψ_2 in $\mathrm{Fix}\,(\mathcal{R}')$ such that the corresponding support projections are orthogonal in M'' (Remark 4.2). Since every $\mathcal{R}'$-invariant state ψ is faithful on M, $\psi_i^{(n)} \neq 0$ (otherwise the norm closed face $\{\psi(x) = 0 : x \in M_+\}$ would be non trivial and $\mu R(\mu)$-invariant). The support projections of the $\psi_i^{(n)}$'s in M'' are orthogonal (since $\psi_1^{(n)} \leqslant \psi_i$) and different from zero. Let (z_γ) be a net in M_1^+ such that

$$\sigma(M'', M')\text{-}\lim_{\gamma} z_\gamma = s(\psi_1^{(n)}).$$

Then $\lim_\gamma \psi_1^{(n)}(z_\gamma) = 1$ but $\lim_\gamma \psi_2^{(n)}(z_\gamma) = 0$. Let z be a $\sigma(M, M_*)$-accumulation point of (z_γ) in M_+. Since every $\psi_i^{(n)}$ is normal, $\psi_1^{(n)}(z) = 1$ but $\psi_2^{(n)}(z) = 0$. The first condition implies $z \neq 0$ while the second shows that $\psi_2^{(n)}$ cannot be faithful. This is a contradiction and it implies $\dim \mathrm{Fix}\,(\mathcal{R}') = 1$, hence (iv). □

The next corollary is an easy application of Theorem 4.4 and of D-III, Proposition 2.3.

Corollary 4.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W*-algebra M. Then the following assertions are equivalent.*

(a) $\mathcal{T}$ is uniformly ergodic with finite dimensional fixed space.
(b) The adjoint weak-semigroup is strongly ergodic with finite dimensional fixed space.*
(c) Every $\mathcal{T}''$-invariant state is normal.

Proof If (a) is fulfilled, then the semigroup $\mathcal{T}$ is strongly ergodic on M_*. Since

$$\dim \mathrm{Fix}\,(\mathcal{T}) = \dim \mathrm{Fix}\,(\mathcal{T}') < \infty,$$

there exist normal states $\varphi_1, \ldots, \varphi_n$ in $\mathrm{Fix}\,(\mathcal{T})$ and $x_1, \ldots, x_k$ in $\mathrm{Fix}(\mathcal{T}')$ such that $\varphi_n(x_m) = \delta_{n,m}$ $(1 \leqslant n, m \leqslant k)$. Then

$$P = \sum_{i=1}^{k} \varphi_i \otimes x_i$$

is the associated ergodic projection. If $(C(s))_{s>0}$ is the family of Cesàro means of $\mathcal{T}$, then

$$\lim_{s\to\infty} C(s)''(\psi) = \sum_{i=1}^{k} \varphi_i(\psi)x_i \in M_*$$

for every $\psi \in M'$. Hence $\operatorname{Fix}(\mathcal{T}'') \subseteq M_*$ which implies (c).
If (c) is fulfilled, then $\operatorname{Fix}(\mathcal{T}') = \operatorname{Fix}(\mathcal{T}'')$. Therefore the fixed space of $\mathcal{T}'$ separates the points of $\operatorname{Fix}(\mathcal{T}'')$, hence $\mathcal{T}'$ is strongly ergodic on M (Krengel (1985, Chap.2, Thm.1.4)).
If (b) holds, then

$$P = \lim_{\mu\to 0} \mu R(\mu, A')$$

exists in the strong operator topology with A' is the generator of $\mathcal{T}'$. Therefore $\dim \operatorname{Fix}\left(\widehat{\mu R(\mu)}\right) < \infty$ in some ultrapower of M (Theorem 4.4). It follows from D-III, Proposition 2.3 that 0 is a pole of the resolvent of $R(\cdot, A)$. Therefore $\mathcal{T}$ is uniformly ergodic. □

Notes

Section 1: The stability concepts appearing in Theorem 1.7 coincide not only for positive semigroups on C*-algebras but on any order unit Banach space. We refer to Batty and Robinson (1984) for this more general setting and to B-IV, Section 1 for the analogous results on $C_0(X)$.

Section 2: Theorem 2.2 generalizes the Liapunov stability theorem from the matrix algebra $B(\mathbb{C}^n)$ to arbitrary W*-algebras. For the algebra $\mathcal{B}(H)$ it is due to Mil'stein (1975) and in the general form to Groh and Neubrander (1981).

Section 3: From the many papers dealing more or less explicitly with the asymptotic behavior of semigroups on operator algebras we quote Frigerio and Verri (1982) and Watanabe (1982). The background for our ergodic theorems (Proposition 3.3 & 3.4) can be found best in Krengel (1985). The "automatic" convergence theorem for an irreducible W*-dynamical system on $\mathcal{B}(H)$ stated in Corollary 3.9 is the continuous version of a result in Groh (1984b). Finally, the characterization of convergence towards a periodic semigroup through spectral properties of the generator—Theorem 3.11—is due to Nagel (1984) in the commutative case, i.e., in $L^1(\mu)$ (see also C-IV, Thm.2.14).

Section 4: Again we refer to Krengel (1985) for the (uniform) ergodic theory for a single operator or a one-parameter semigroup on a Banach space. The characterization given in Corollary 4.5 for positive semigroups on W*-algebras is based on a sophisticated use of ultrapower techniques and has its discrete forerunners in Lotz (1981) and Groh (1984a).

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