CHAPTER A-IV

A S Y M P T O T I C S   O F   S E M I G R O U P S

O N   B A N A C H   S P A C E S

by

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In this chapter we study the asymptotic behavior of the solutions of
the initial value problem

$$(*) \qquad \qquad \dot{u}(t) = Au(t) + F(t) \ , \ u(0) = f$$

with respect to time $t \geq 0$. Here A will be a generator of a
strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space E and
$F(\cdot)$ is a function from $\mathbb{R}_+$ with values in E .
In Section 1 we investigate whether and how fast a solution $T(\cdot)f$ of
the homogeneous problem tends to the zero solution as $t \to \infty$; in
Section 2 we consider the long term behavior of the solutions of $(*)$
for different classes of forcing terms F .

1. STABILITY : HOMOGENEOUS CASE

Let $(T(t))_{t \geq 0}$ be a semigroup on E with generator A . An initial
value $f \in D(A)$ is called <u>stable</u> if the solution $t \to T(t)f$ of

$$(ACP) \qquad \qquad \dot{u}(t) = Au(t) \ , \ u(0) = f$$

converges to zero as t tends to infinity. The semigroup is called
<u>stable</u> if every solution converges to zero; i.e., if every initial
value $f \in D(A)$ is stable.
If the space E is finite dimensional, then the stability of the
semigroup implies that the decay is exponential. More precisely, the
statements

(a) $\qquad \qquad \|T(t)f\| \to 0$ for every $f \in \mathbb{C}^n$,

(b) $\qquad \qquad \|T(t)\| \leq Me^{-\omega t}$ for some $\omega > 0$

are equivalent. Moreover, these statements hold if and only if

$$s(A) = \sup\{\text{Re } \lambda : \lambda \in \sigma(A)\} < 0 ,$$

see A-III,Cor.1.2 .

As already discussed in Chapter A-III the situation is far more difficult in the infinite dimensional case. Here, and for unbounded generators, we have to distinguish between strong and generalized (mild) solutions of $\dot{u}(t) = Au(t)$ and between various notions of stability. Recall that for $f \in D(A)$ the function $T(\cdot)f$ is a strong solution of (ACP) (see A-II,Cor.1.2.); for arbitrary $f \in E$ the function $T(\cdot)f$ is called a generalized or mild solution of (ACP). Next we introduce several constants characterizing the growth of the solutions of (ACP).

Definition 1.1 (1st part). Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space E .
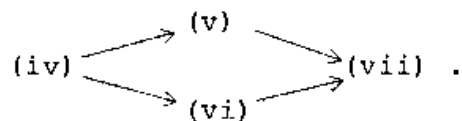Then

(i) $\omega(f) := \inf\{w : \|T(t)f\| \leq Me^{wt}$ for some M and every $t \geq 0\}$
        is called the (exponential) growth bound of $T(\cdot)f$.

(ii) $\omega_1(A) := \sup\{\omega(f) : f \in D(A)\}$ is called the (exponential) growth bound of the solutions of the Cauchy problem $\dot{u}(t) = Au(t)$.

(iii) $\omega(A) = \sup\{\omega(f) : f \in E\}$ is called the (exponential) growth bound of the mild solutions of the Cauchy problem $\dot{u}(t) = Au(t)$.

Note that, by the Principle of Uniform Boundedness, $\sup\{\omega(f) : f \in E\} = \inf\{\omega : \|T(t)\| \leq Me^{\omega t}$ for some M and every $t \geq 0\}$. Hence $\omega(A)$ coincides with the growth bound of the semigroup $(T(t))_{t \geq 0}$ as defined in A-I,1.3. With the constants defined above we obtain the following stability concepts.

Definition 1.1 (2nd part). The semigroup $(T(t))_{t \geq 0}$ is called

(iv)   uniformly exponentially stable if $\omega(A) < 0$;

(v)   exponentially stable if $\omega_1(A) < 0$;

(vi)   uniformly stable if $\|T(t)f\| \to 0$ (as $t \to \infty$) for every $f \in E$;

(vii)   stable if $\|T(t)f\| \to 0$ (as $t \to \infty$) for every $f \in D(A)$.

The interrelation between these stability concepts is given by

$$(iv) \begin{array}{c} \nearrow \ (v) \ \searrow \\ \searrow \ (vi) \ \nearrow \end{array} (vii) \ .$$

If A is a bounded operator, i.e., if $D(A) = E$, then (iv) $<=>$ (v) and (vi) $<=>$ (vii). If A is unbounded then the stability notions may differ as we will see in the following examples.

Examples 1.2. (a) Let $E = c_o$. Then $A : (x_n)_{n \in \mathbb{N}} \to (-1/n \cdot x_n)_{n \in \mathbb{N}}$ generates the semigroup $T(t)(x_n) = (e^{-t/n} x_n)_{n \in \mathbb{N}}$. It is easy to see that $\|T(t)\| = 1$ and $\|T(t)f\| \to 0$ for every $f \in c_o$. Moreover, A is a bounded operator, hence $D(A) = E$. This gives an example for a (uniformly) stable but not exponentially stable semigroup. The trans- lation semigroups generated by the first derivative on $C_o(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$ for $1 < p < \infty$ give further examples for (uniformly) stable but not exponentially stable semigroups. Moreover, as seen in A-II, Ex.1.14, the Laplacian Δ on $C_o(\mathbb{R}^n)$ generates a bounded holomorphic semigroup given by

$$T(t)f(x) = (4\pi t)^{-n/2} \cdot \int_{\mathbb{R}^n} e^{-(x-y)^2/4t} f(y) \, dy$$

which cannot be exponentially stable because $0 \in \sigma(\Delta)$ $(im\Delta \neq C_o(\mathbb{R}^n))$, see Cor.1.5 below. By a straightforward $(2-\epsilon)$-argument using $(4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-y^2/4t) \, dy = 1$ one can easily show that $\|T(t)f\| \to 0$ for all $f \in C_o(\mathbb{R}^n)$ (see also B-III,Ex.1.7).
Therefore, the Laplacian on $C_o(\mathbb{R}^n)$ (and also on $L^p(\mathbb{R}^n)$ for $1<p<\infty$, see Ex.1.15 below) generates a (uniformly) stable but not exponent- ially stable semigroup.

(b) Note that the condition $0 \leq \omega(A) = \inf\{\omega : \|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0\}$ does not exclude that the semigroup $(T(t))_{t \geq 0}$ is expo- nentially stable. In fact, as shown in A-III,1.3 the translation semigroup $(T(t))_{t \geq 0}$ on $E := C_o(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x dx)$ satisfies $\|T(t)\| = 1$, hence $\omega(A) = 0$, and for every $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > -1$ the resolvent of the generator is given as $R(\lambda,A)f = \int_o^\infty e^{\lambda t} T(t)f \, dt$ for every $f \in E$. From the equation A-I,3.2

$$T(t)f = e^{\lambda t}\left(f - \int_o^t e^{-\lambda s} T(s)(\lambda - A)f \, ds\right)$$

and the existence of $\lim_{t \to \infty} \int_o^t e^{-\lambda s} T(s)(\lambda - A)f \, ds$ it follows that $\|T(t)f\| \leq M \cdot e^{\lambda t}$ for every $f \in D(A)$ and some constant M depending

on f . This yields $\omega_1(A) \leq -1 < 0 = \omega(A)$. Thus we have a semigroup
which is exponentially, but not uniformly exponentially stable.

(c) Rescaling this semigroup (see A-I,3.1) we obtain a semigroup with
$-1/2 = \omega_1(A)$ and $1/2 = \omega(A)$. Therefore there are exponentially
stable and hence stable semigroups which are not bounded and hence not
uniformly stable. We conclude from this example that there may be an
essential difference between the long term behavior of the semigroup
$(T(t))_{t\geq 0}$ (i.e. of the set of all mild solutions) and the long term
behavior of the strong solutions $\{T(\cdot)f : f \in D(A)\}$ of (ACP).

In the following we characterize the exponential growth bounds $\omega(f)$,
$\omega_1(A)$ and $\omega(A)$ by certain abscissas of simple or absolute con-
vergence of the Laplace transform of $T(\cdot)f$. These characterizations
will be the basic tool in showing that for certain semigroups the
growth bounds $\omega(A)$ and/or $\omega_1(A)$ coincide with the spectral bound
$s(A) = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\}$.

We remark first that $s(A)$ can be regarded as the abscissa of holo-
morphy of the Laplace transform $\lambda \rightarrow \int_0^\infty e^{\lambda t} T(t) \, dt$ of the semigroup
$(T(t))_{t\geq 0}$.
Next we recall that the Laplace transform exists for every $\lambda \in \mathbb{C}$
with Re λ > Re μ as soon as it exists for μ . This follows from the
equation

(1.1) $\int_0^t e^{-\lambda s} f(s) \, ds = e^{-(\lambda-\mu)t} \cdot \int_0^t e^{-\mu s} f(s) \, ds$

                              $+ (\lambda - \mu)\int_0^t e^{-(\lambda-\mu)s} \int_0^s e^{-\mu r} f(r) \, dr \, ds$.

Note that even boundedness of $\int_0^t e^{-\mu s} f(s) \, ds$ implies the existence
of the Laplace transform for Re λ > Re μ . Therefore the subset of $\mathbb{C}$
for which the Laplace transform exists is always a half-plane $\{\lambda \in \mathbb{C} :$
Re λ > $\gamma\} \cup H$, where H is a subset of the line $\{\lambda \in \mathbb{C} : \text{Re } \lambda = \gamma\}$.

In the subsequent theorem we show that the bound of the half-plane for
which the Laplace transform of $T(\cdot)f$ $(f \in E)$ exists absolutely and
the bound of the half-plane for which the Laplace transform of $T(\cdot)Af$
$(f \in D(A))$ exists strongly coincide with the growth bound $\omega(f) =$
$\inf\{\omega : \|T(t)f\| \leq Me^{\omega t}$ for all $t \geq 0\}$.

Theorem 1.3. Let A be the generator of a strongly continuous semi-group $(T(t))_{t \geq 0}$ on a Banach space E . Then, for every $f \in E$,

(1.2) $\omega(f) = \lim \sup_{t \to \infty} 1/t \cdot \log \| T(t) f \|$,

and

(i) $\omega(f) = \inf \{ \mathrm{Re}\ \lambda : \int_0^\infty \| e^{-\lambda t} T(t) f \|\ dt\ \ exists \}$.

If ker A = {0}, then for every $f \in D(A)$ we have

(ii) $\omega(f) = \inf \{ \mathrm{Re}\ \lambda : \int_0^\infty e^{-\lambda t} T(t)Af\ dt$ exists as an improper

Riemann integral} .

Proof. The proof of (1.2) is omitted (see Hille-Phillips (1957), p.306). In order to prove (i) and (ii) we need the following lemma.

Lemma. Let $F \in C(\mathbb{R}^+, \mathbb{R}^+)$ be such that $\int_0^\infty F(t)\ dt$ exists. If there is a positive number m and an interval [0,n] such that $F(t + s) \leq m \cdot F(s)$ for all $s \geq 0$ and $t \in [0,n]$, then $\lim_{s \to \infty} F(s) = 0$.

Proof of the lemma. For all $\varepsilon > 0$ there exists $a > 0$ such that $A(a) := \int_a^\infty F(s)\ ds \leq \frac{n}{m} \cdot \varepsilon$. For all $t > a+n$ there exists $r \in [t-n,t]$ such that $F(r) \leq \frac{1}{n} \cdot A(a)$.
Therefore, $F(t) = F(t-r+r) \leq m \cdot F(r) \leq \frac{m}{n} \cdot A(a) \leq \varepsilon$.

□

In order to prove (i) we define $b := \inf \{ \mathrm{Re}\lambda : \int_0^\infty \| e^{-\lambda t} T(t) f \|\ dt$ exists}. A straightforward application of the lemma shows that $\omega(f) \leq b$. The definition of $\omega(f)$ gives the reverse inequality.
It remains to prove statement (ii) of Thm.1.3 .
Assume that ker A = {0} and let $f \in D(A)$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}\ \lambda > \omega(f)$.
From the equation

$$\int_0^t e^{-\lambda s} T(s)Af\ ds = e^{-\lambda t} T(t)f - f + \lambda \int_0^t e^{-\lambda s} T(s)f\ ds$$

it follows that $\int_0^\infty e^{-\lambda s} T(s)Af\ ds$ exists.
Therefore $b := \inf \{ \mathrm{Re}\ \lambda : \int_0^\infty e^{-\lambda t} T(t)Af\ dt$ exists} $\leq \omega(f)$.

Next we show that b < 0 implies $b \leq \omega(f)$. Suppose b < 0 . Then, by (1.1), $\int_0^\infty T(s)Af\ ds$ exists. By $\int_0^r T(s)Af\ ds = T(r)f - f$ we see that $\lim_{r \to \infty} T(r)f =: g$ exists. But, for every $t \geq 0$, $T(t)g = g$ and therefore $g \in \mathrm{Ker}\ A$ or g = 0 . Hence $\int_t^\infty T(s)Af\ ds = -T(t)f$. Then choosing r , b < r < 0, and integrating by parts we obtain

$$-T(t)f = \lim_{u \to \infty} \int_t^u e^{rs} e^{-rs} T(s) Af \, ds$$

$$= \lim_{u \to \infty} (e^{ru} \int_0^u e^{-rs} T(s) Af \, ds - e^{rt} \int_0^t e^{-rs} T(s) Af \, ds$$

$$- r \int_t^u e^{rs} \int_0^s e^{-rv} T(v) Af \, dv \, ds)$$

$$= -e^{rt} \int_0^t e^{-rs} T(s) Af \, ds - r \int_t^\infty e^{rs} \int_0^s e^{-rv} T(v) Af \, dv \, ds \ .$$

From $\left\| \int_0^t e^{-rs} T(s) Af \, ds \right\| \leq M$ for some M and every $t \geq 0$ we con-
clude that $\| T(t) f \| \leq \tilde{M} e^{rt}$ for all $t \geq 0$ and some constant $\tilde{M}$.
Hence $\omega(f) \leq r$ for every $b < r < 0$, i.e., $\omega(f) \leq b$.
If $b \geq 0$ and $w > b$, then $\left\| \int_0^t e^{-ws} T(s) Af \, ds \right\| \leq M$ for all $t \geq 0$.
By $T(t) f - f = \int_0^t e^{ws} e^{-ws} T(s) Af \, ds$

$$= e^{wt} \int_0^t e^{-ws} T(s) Af \, ds - w \int_0^t e^{ws} \int_0^s e^{-wr} T(r) Af \, dr \, ds$$

we obtain $\| T(t) f - f \| \leq M e^{wt} + M(e^{wt} - 1) \leq 2 M e^{wt}$. Hence $\omega(f) \leq w$
for every $w > b$, i.e., $\omega(f) \leq b$.

□

From (1.2) and the Uniform Boundedness Principle it follows that the
growth bound $\omega_1(A) = \sup\{\omega(f) : f \in D(A)\}$ satisfies

(1.3) $\omega_1(A) = \inf\{w :$ for every $f \in D(A)$ there exists a constant
M such that $\| T(t) f \| \leq M e^{wt}$ for every $t \geq 0\}$

$$= \limsup_{t \to \infty} 1/t \cdot \log \| T(t) R(\lambda, A) \| \qquad (\lambda \in \rho(A)) \ .$$

The subsequent theorem will be of particular importance in the stabil-
ity theory of positive semigroups. We show that the constant $\omega_1(A)$
coincides with the abscissa of simple convergence of the Laplace
transform of the semigroup and with the abscissa of absolute conver-
gence of the Laplace transform of the strong solutions of (ACP).

Theorem 1.4. Let A be the generator of a strongly continuous semi-
group $(T(t))_{t \geq 0}$ on a Banach space E . Then

(1.4) $\omega_1(A) = \inf\{\operatorname{Re} \lambda : \int_0^\infty e^{-\lambda t} T(t) f \, dt$ exists as an improper
Riemann integral for every $f \in E\}$

$$= \inf\{\operatorname{Re} \lambda : \int_0^\infty \| e^{-\lambda t} T(t) f \| dt \text{ exists for every}$$

$$f \in D(A)\} \ .$$

Remarks. (a) One can show that the abscissas of uniform, strong and
weak convergence of the Laplace transform coincide (see C-III,Thm.1.2,
last part of the proof). Therefore, by Thm.1.4,

$$(1.5) \quad \omega_1(A) = \inf\{\operatorname{Re} \lambda : \text{weak-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)\, ds \text{ exists}\}$$

$$= \inf\{\operatorname{Re} \lambda : \text{uniform-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)\, ds \text{ exists}\}.$$

(b) In the equations (1.4) and (1.5) the term "Re λ" may be replaced by "$\lambda \in \mathbb{R}$" (use (1.1)).

Proof of Thm.1.4. The equality $\omega_1(A) = \inf\{\operatorname{Re} \lambda : \int_0^\infty \|e^{-\lambda t} T(t) f\|\, dt$ exists for all $f \in D(A)\}$ follows from the definition of $\omega_1(A)$ and the lemma used in the proof of Thm.1.3.
We prove $\omega_1(A) = \inf\{\operatorname{Re} \lambda : \int_0^\infty e^{-\lambda s} T(s) f\, ds$ exists for every $f \in E\}$.
The identity

$$T(t) f = e^{\lambda t}(f - \int_0^t e^{-\lambda s} T(s)(\lambda - A) f\, ds)$$

yields

$\omega_1(A) \leq \inf\{\operatorname{Re} \lambda : \int_0^\infty e^{-\lambda t} T(t) f\, dt$ exists for every $f \in \operatorname{im}(\lambda - A)\}$.
Therefore

$\omega_1(A) \leq \inf\{\operatorname{Re} \lambda : \int_0^\infty e^{-\lambda t} T(t) f\, dt$ exists for every $f \in E\} =: b$.
Take $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega_1(A)$. Then $\int_0^\infty e^{-\lambda t} T(t) f\, dt$ exists for every $f \in D(A)$. Define $g := \int_0^1 e^{-\lambda t} T(t) f\, dt$. Then $g \in D(A)$ and $\int_0^n e^{-\lambda t} T(t) f\, dt = \sum_{k=0}^{n-1} e^{-\lambda k} T(k) g$. Since $\operatorname{Re} \lambda > \omega_1(A)$ it follows that the sum converges for every $g \in D(A)$. Therefore the integral converges as $n \to \infty$ ($n \in \mathbb{N}$) for every $f \in E$. For every $t \in \mathbb{R}_+$ define a bounded operator T_t by $f \mapsto \int_0^t e^{-\lambda s} T(s) f\, ds$. As seen above, $T_n f$ converges as $n \to \infty$ ($n \in \mathbb{N}$) for every $f \in E$. It follows from the Uniform Boundedness Principle that the family $(T_n)_{n \in \mathbb{N}}$ is uniformly bounded.
For every $t \in \mathbb{R}_+$ there exist $n \in \mathbb{N}$ and $t' \in [0,1)$ such that $T_t = T_{t'} + e^{-\lambda t'} T(t') T_n$. Since the operator families on the right side of the equation are uniformly bounded the same is true for $(T_t)_{t \geq 0}$. Since $(T_t f)_{t \geq 0}$ converges for every $f \in D(A)$ it follows that $(T_t f)_{t \geq 0}$ converges for every $f \in E$. Thus $b \leq \omega_1(A)$.

$\square$

The inequality

$\omega(A) \geq \inf\{\operatorname{Re} \lambda : \int_0^\infty \|e^{-\lambda t} T(t) f\|\, dt$ exists for every $f \in E\}$

in combination with the lemma of Thm.1.3 implies that the growth bound $\omega(A)$ coincides with the abscissa of absolute convergence of the Laplace transform of $(T(t))_{t \geq 0}$; i.e.,

$$(1.6) \quad \omega(A) = \inf\{\operatorname{Re} \lambda : \int_0^\infty \|e^{-\lambda t} T(t) f\|\, dt \text{ exists for every } f \in E\}.$$

As seen in A-I, Prop.1.11, if $\int_0^\infty e^{-\lambda t} T(t) f\, dt$ exists for every $f \in E$, then $\lambda \in \rho(A)$ and $R(\lambda, A) f = \int_0^\infty e^{-\lambda t} T(t) f\, dt$. This and Thm.1.4 yield the following corollary.

Corollary 1.5. Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space E . Then

$$s(A) \leq \omega_1(A) \leq \omega(A) .$$

Example 1.2.(2) shows that the uniform exponential stability of the semigroup is not equivalent to $\sigma(A) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq q < 0\}$. In the following example we will see that not only the semigroup (i.e., all generalized solutions of (ACP)), but also the strong solutions can be unstable even when $s(A) < 0$. In fact, we will give an example of a semigroup with $s(A) < \omega_1(A) < \omega(A)$.

Example 1.6. In A-III,Ex.1.4 it was shown that the semigroup $(T(t))_{t \geq 0}$ on the Hilbert space $E = \{(x^1, x^2, \ldots), x^n \in \mathbb{C}^n : \sum_{i=1}^{\infty} \|x^i\|^2 < \infty\}$ given by $T(t) := (e^{2\pi i n t} \cdot \exp(tA_n))_{n \in \mathbb{N}}$ with

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ & \cdot & \cdot & & \\ & \cdot & & \cdot & 0 \\ & \cdot & & & 1 \\ 0 & & \ldots & & 0 \end{pmatrix}_{n \times n}$$

has growth e^t $(\|T(t)\| = e^t)$. Thus $\omega(A) = 1$ whereas the generator $A = (2\pi i n + A_n)_{n \in \mathbb{N}}$ has spectral bound 0 . We will show first that for this semigroup $\omega_1(A) = \omega(A)$ holds (we will use this to construct a semigroup with $s(A) < \omega_1(A) < \omega(A)$).
Let $e_n = n^{-1/2} \cdot (1, \ldots, 1) \in \mathbb{C}^n$. Then we have

$\|\exp(tA_n) \cdot e_n\|^2 =$

$$= \frac{1}{n} \cdot \left\| (1 + t + \ldots + \frac{t^{n-1}}{(n-1)!}, 1 + t + \ldots + \frac{t^{n-2}}{(n-2)!}, \ldots, 1+t, 1) \right\|^2 =$$

$$= \frac{1}{n} \cdot \sum_{r=0}^{n-1} (\sum_{j=0}^{r} \frac{1}{j!} \cdot t^j)^2 =$$

$$= \frac{1}{n} \cdot \sum_{r=0}^{n-1} (\sum_{j,s=0}^{r} \frac{1}{j!s!} \cdot t^{j+s}) =$$

$$= \frac{1}{n} \cdot \sum_{r=0}^{n-1} \sum_{i=0}^{2r} t^i \sum_{j+s=i} \frac{1}{j!s!} =$$

$$= \frac{1}{n} \cdot \sum_{r=0}^{n-1} \sum_{i=0}^{2r} \frac{t^i}{i!} \sum_{j=0}^{i} \binom{i}{j} =$$

$$= \frac{1}{n} \cdot \sum_{r=0}^{n-1} \sum_{i=0}^{2r} \frac{1}{i!} (2t)^i \geq$$

$$\geq \frac{1}{n^2} \cdot \sum_{i=0}^{n-1} \frac{1}{i!} (2t)^i .$$

For $0 < q < 1$ we consider $x_q \in E$ defined as follows
$$x_q := (q \cdot e_1, 2q^2 e_2, \ldots, nq^n e_n, \ldots) .$$

Then $x_q \in D(A)$ and

$$\|T(t)x_q\|^2 = \sum_{n=1}^{\infty} n^2 q^{2n} \|\exp(tA_n)e_n\|^2 \geq$$

$$\geq \sum_{n=1}^{\infty} n^2 q^{2n} (\frac{1}{n^2} \cdot \sum_{i=0}^{n-1} \frac{1}{i!} (2t)^i)$$

$$= \sum_{i=0}^{\infty} \sum_{n=i+1}^{\infty} (q^{2n} \cdot \frac{1}{i!} (2t)^i)$$

$$= \sum_{i=0}^{\infty} q^{2i+2} \cdot (1-q^2)^{-1} \cdot \frac{1}{i!} (2t)^i =$$

$$= \frac{q^2}{1-q^2} \cdot \sum_{i=0}^{\infty} \frac{1}{i!} (2tq^2)^i = \frac{q^2}{1-q^2} \cdot e^{2tq^2} .$$

It follows that $\omega(x_q) \geq q^2$. Thus

$$1 = \sup\{\omega(x_q) : 0 < q < 1\} \leq \omega_1(A) \leq \omega(A) = 1 .$$

Rescaling the semigroup (i.e. looking at $e^{-3/2 \cdot t} T(t)$) we obtain a semigroup generator A on the Hilbert space E with $-3/2 = s(A)$ and $\omega_1(A) = \omega(A) = -1/2$. On the other hand, Example 1.2.(2) yields a semigroup on a Banach space F with generator B such that $-1 = s(B) = \omega_1(B)$ while $\omega(B) = 0$. Now the operator $C := A \oplus B$ on the Banach space $E \oplus F$ is a semigroup generator for which $s(C) = \max\{s(A),s(B)\} = -1$, $\omega_1(C) = \max\{\omega_1(A),\omega_1(B)\} = -1/2$ and $\omega(C) = \max\{\omega(A),\omega(B)\} = 0$.

(1.7) Important remark: For eventually norm continuous semigroups, in particular for compact, differentiable, holomorphic or nilpotent semi-groups the spectral mapping theorem $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$ holds, and therefore

(1.8) $s(A) = \omega_1(A) = \omega(A)$

is valid (Cor.1.5 and A-III,Cor.6.7).

Hence, if A is the generator of an eventually norm continuous semigroup, then the exponential growth bounds of the strong and the mild solutions of $\dot{u}(t) = Au(t)$ are determined by the spectral bound $s(A) = \sup\{Re\lambda : \lambda \in \sigma(A)\}$.

In general, the growth bound $\omega(A)$ can be obtained through the Hille-Yosida theorem (see A-II,Thm.1.7) as

(1.9) $\omega(A) = \inf\{w : \|R(\lambda,A)^n\| \leq M \cdot (Re \lambda - w)^{-n}$ for some M and
                        every $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $Re \lambda > w\}$.

In view of the difficulties in estimating all powers of the resolvent this equation is of little practical interest. If A is the generator

of a semigroup on a Hilbert space H, then it is shown in A-III, Cor.7.11 that

(1.10) $\omega(A) = \inf\{w : \|R(\lambda,A)\| \leq M_w \text{ for } \mathrm{Re}\ \lambda > w\}$.

Unfortunately, the identity (1.10) does not hold on arbitrary Banach spaces, but we will see in Section 1 of C-IV that for every positive semigroup on a Banach lattice the identity

(1.11) $s(A) = \omega_1(A) = \inf\{w : \|R(\lambda,A)\| \leq M_w \text{ for } \mathrm{Re}\ \lambda > w\}$

is valid. Therefore, for positive semigroups with $s(A) = \omega_1(A) < \omega(A)$ (see Ex.1.2.(2)) the equation (1.10) is not true. However, we can prove the following theorem.

Theorem 1.9. Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space E. If there are constants $a \geq 0$ and $q \geq s(A)$ and if there are $C \in \mathbb{R}_+$, $n \in \mathbb{N}$ such that, for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\ \lambda > q$ and $|\mathrm{Im}\ \lambda| > a$ we have $\|R(\lambda,A)\| \leq C|\lambda|^{n-2}$, then $\sup\{\omega(f), f \in D(A^n)\} \leq q$.

Proof. The hypothesis $\|R(\lambda,A)\| \leq C|\lambda|^{n-2}$ is invariant under rescaling; i.e., the resolvent $R(\lambda,-b+A)$ of the generator $-b + A$ of the rescaled semigroup $e^{-bt}\ T(t)$ satisfies $\|R(\lambda,-b+A)\| \leq \tilde{C}|\lambda|^{n-2}$ for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\ \lambda > q-b$ and $|\mathrm{Im}\ \lambda| > a+2b$ and a suitable constant $\tilde{C}$. Therefore we may assume that $b := \max(\omega(A),q) < 0$. Let $\omega(A) < p < 0$. Then, by the inversion formula for the Laplace transform for every $f \in D(A)$ and $p' = \max\{p,q\} < 0$,

(1.12) $T(t)f = \dfrac{1}{2\pi i} \cdot \int_{p'-i\infty}^{p'+i\infty} e^{\lambda t}\ R(\lambda,A)f\ d\lambda$.

(For a proof of the vector valued version of the inversion formula one may follow [Widder(1946),p.66]; also see the notes to this section.) By the resolvent equation we obtain

$$R(\lambda,A)R(0,A)^n = \sum_{k=1}^{n} (-1)^{k+1} \cdot \lambda^{-k}\ R(0,A)^{n+1-k} + (-1)^n \cdot \lambda^{-n}\ R(\lambda,A)$$.

Using that $\dfrac{1}{2\pi i} \cdot \int_{p'-i\infty}^{p'+i\infty} e^{\lambda t} \cdot \lambda^{-k}\ d\lambda = 0$ for $k \geq 1$, $p' < 0$ and $t > 0$ we obtain

(1.13) $T(t)R(0,A)^n f = (-1)^n \cdot \dfrac{1}{2\pi i} \cdot \int_{p'-i\infty}^{p'+i\infty} e^{\lambda t} \cdot \lambda^{-n}\ R(\lambda,A)f\ d\lambda$

for every $f \in E$ and $t > 0$.
If $q < p'$, then, by Cauchy's Integral Theorem and since $\|R(\lambda,A)\| \leq C \cdot |\lambda|^{n-2}$ we see that the path of integration can be shifted to $\mathrm{Re}\lambda = q$;
i.e., $T(t)R(0,A)^n f = (-1)^n \cdot \dfrac{1}{2\pi i} \cdot \int_{q-i\infty}^{q+i\infty} e^{\lambda t} \cdot \lambda^{-n}\ R(\lambda,A)f\ d\lambda$.

Therefore $\|T(t)R(0,A)^n f\| \le C'e^{qt}\|f\|\int_{-\infty}^{\infty} (q^2+s^2)^{-1} ds = M \cdot e^{qt} \cdot \|f\|$
or $\|T(t)f\| \le M \cdot e^{qt}\|A^n f\|$ for $f \in D(A^n)$.

<div align="right">☐</div>

In view of the characterizations given in Section 1 of A-II, the semigroups occuring in the theorem are holomorphic if $n = 1$. In this case one may apply (1.7) to obtain the stronger statement (1.8).

Instead of making assumptions on the resolvent of A we now take a different view and characterize the property "$\omega(A) < 0$" in terms of the semigroup $(T(t))_{t \ge 0}$ directly.

__Proposition__ 1.10. Let A be the generator of the strongly continuous semigroup $(T(t))_{t \ge 0}$. Then the following statements are equivalent:

(a) $\omega(A) < 0$

(b) $\lim_{t \to \infty} \|T(t)\| = 0$

(c) $\|T(t')\| < 1$ for some $t' > 0$.

__Proof.__ The only nontrivial implication (c) → (a) follows from

$\omega(A) = \lim_{t \to \infty} \frac{1}{t}\log\|T(t)\|$ (see A-I,(1.1)) and

$\frac{\log\|T(t)\|}{t} \le \frac{\log\|T(t')\|}{t'} + \frac{\log\|T(t)\|}{nt'+s}$ for $t = nt'+s$, $s \in [0,t')$.

<div align="right">☐</div>

Other less obvious characterizations of the property "$\omega(A) < 0$" are given in the next theorem. The equivalence of (a) and (c) is known as Datko's Theorem.

__Theorem__ 1.11. Let A be the generator of a strongly continuous semigroup $(T(t))_{t \ge 0}$ on a Banach space E . Then the following statements are equivalent:

(a) $\omega(A) < 0$.

(b) $s(A) < 0$ and there is $t_o > 0$ such that
     $|\lambda| < 1$ for every $\lambda \in A\sigma(T(t_o))$.

(c)   For every (some) $p \ge 1$ exists $\int_0^{\infty} \|T(t)f\|^p dt$ for every $f \in E$.

__Proof.__ The implication "(a) → (b)" follows from $r(T(t)) = e^{\omega(A)t} < 1$ and $s(A) \le \omega(A) < 0$. For the point and residual spectrum the spectral mapping theorem is valid (see A-III,Thm.6.3). The approximate point spectrum is closed, hence the additional information in (ii)

implies $|\lambda| \leq r < 1$ for some r and each $\lambda \in A\sigma(T(t_o))$. Consequently, $\exp(\omega(A) \cdot t_o) = r(T(t_o)) \leq \max\{\exp(t_o \cdot s(A)), r)\} < 1$ or $\omega(A) < 0$. This proves "(b) → (a)". For a proof of the equivalence of (a) and (c) we refer to Datko (1972) or Pazy (1983),Thm.4.4.1 .

$\square$

Rescaling a given semigroup $(T(t))_{t \geq 0}$ one obtains the following corollary from (1.1) and statement (c) of the above theorem.

<u>Corollary</u> 1.12. Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Banach space E . Then the set of complex numbers λ for which $\int_0^\infty \|e^{-\lambda t} T(t)f\| \, dt$ exists for every $f \in E$ is an open right half-plane.

In the next theorem we give two necessary conditions for stability of $(T(t))_{t \geq 0}$ in terms of the generator A . We will see in Chapter C-IV that for positive semigroups a condition similar to statement (ii) below is even sufficient for stability of the semigroup. We emphasize that stable semigroups need not be uniformly bounded (see Ex.1.2(3)) and that $s(A) = \omega(A) = 0$ does not imply boundedness or even stability of the semigroup (see also A-I, Ex.1.4.(i)).

<u>Theorem</u> 1.13. Let A be the generator of a stable semigroup $(T(t))_{t \geq 0}$ on a Banach space E . Then the following assertions hold:

(i) $s(A) \leq 0$ and $Re \, \lambda < 0$ for every $\lambda \in P\sigma(A) \cup R\sigma(A)$.

(ii) $\lim_{\lambda \to 0+} \lambda R(\lambda, A)f$ exists for every $f \in D(A)$.

<u>Proof.</u> (i) If $(T(t))_{t \geq 0}$ is stable, then $\|T(t)f\| \leq M_f$ for every $f \in D(A)$. Therefore $s(A) \leq \omega_1(A) \leq 0$.
Assume there is $\lambda \in P\sigma(A)$ with $Re \, \lambda = 0$. Then by A-III,Cor.6.4 there is $g \neq 0$ such that $T(t)g = e^{\lambda t}g$ for all $t \geq 0$. Since $|e^{\lambda t}| = 1$ this contradicts the stability of the semigroup.
Assume there is $\lambda \in R\sigma(A) = P\sigma(A')$ with $Re \, \lambda = 0$. Then there is $0 \neq \phi \in E'$ with $T(t)'\phi = \exp(\lambda t) \cdot \phi$ for all $t \geq 0$. Choose $f \in D(A)$ such that $<f, \phi> \neq 0$. Then $|<T(t)f, \phi>| = |<f, \phi>| > 0$ for every $t \geq 0$ which contradicts the stability of the semigroup.

(ii) From the stability of the semigroup and the identity $\int_0^t T(s)Af \, ds = T(t)f - f$ we see that $\int_0^\infty T(s)Af \, ds$ exists for every $f \in D(A)$. But $\omega_1(A) \leq 0$ and hence $R(\lambda, A)Af = \int_0^\infty e^{-\lambda s} T(s)Af \, ds$ for every $\lambda > 0$ (see Thm.1.4). By a classical theorem of Laplace transform theory (for a proof of the vector valued version one may

follow Widder (1971),p.196) we conclude that $\lim_{\lambda \to 0+} R(\lambda,A)Af$ exists and is equal to $\int_0^\infty T(s)Af\ ds$. The identity $R(\lambda,A)Af = \lambda R(\lambda,A)f - f$ yields the existence of $\lim_{\lambda \to 0+} \lambda R(\lambda,A)f$ for every $f \in D(A)$.

□

Bounded holomorphic semigroups (see A-II,Def.1.11) satisfy $\|AT(t)\| \leq m \cdot t^{-1}$ [Goldstein (1985a),p.33], hence $T(t)f \to 0$ as $t \to \infty$ whenever $f \in$ im A . If im A is dense (i.e., $0 \notin R_\sigma(A)$) we obtain uniform stability and the following corollary.

Corollary 1.14. Let A be the generator of a bounded holomorphic semigroup $(T(t))_{t \geq 0}$ on a Banach space E . Then the following statements are equivalent.

(a) $0 \notin P_\sigma(A) \cup R_\sigma(A)$.
(b) $(T(t))_{t \geq 0}$ is uniformly stable.

Example 1.15 The Laplacian Δ generates a bounded holomorphic semi-groups on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ (see the example proceeding Cor.1.13 of Chap. A-II). All solutions of $\Delta f = 0$ are either constant or unbounded, therefore $0 \notin P_\sigma(\Delta)$. If $1 < p < \infty$, then the adjoint of the Laplacian on $L^p(\mathbb{R}^n)$ is the Laplacian on $L^q(\mathbb{R}^n)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore $0 \notin R_\sigma(\Delta) \cup P_\sigma(\Delta')$ and we obtain by Cor.1.14 that Δ generates uniformly stable semigroups on the space $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ which are, by $\text{im}\Delta \neq L^p(\mathbb{R}^n)$ and Cor.1.5, not exponentially stable.

As seen in Thm.1.4, exponential stability can be defined by saying that the abscissa of convergence of the Laplace transform of $(T(t))_{t \geq 0}$ is less than zero. This should be compared to the assertion of our final theorem.

Theorem 1.16. Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space E . The following assertions are equivalent:

(a) $(T(t))_{t \geq 0}$ is stable.
(b)   ker A = {0} and $\int_0^\infty T(t)f\ dt$ exists for all $f \in$ im A .

Furthermore the following statements are equivalent:

(a') $(T(t))_{t \geq 0}$ is stable and bounded.

(b') $(T(t))_{t \geq 0}$ is uniformly stable.

(c') $(T(t))_{t \geq 0}$ is bounded and there is a dense subspace   D   such that $\int_0^\infty T(t) f\, dt$ exists for every $f \in D$.

Proof. If $(T(t))_{t \geq 0}$ is stable, then, by Thm.1.12, ker A = {0}   and $\int_0^t T(s) A f\, ds = T(t) f - f \to -f$ as $t \to \infty$. Therefore (a) implies (b). On the other hand, if $\int_0^t T(s) A f\, ds$ converges as $t \to \infty$,   then, by the equation above, $g := \lim_{t \to \infty} T(t) f$ exists. But   ker A = {0}   and therefore   g = 0.   This proves "(b) $\to$ (a)".

The implication   "(a') $\to$ (b')"   is obvious. If $T(t) f \to 0$ for every $f \in E$,   then $\|T(t)\| \leq M$ and $0 \notin R\sigma(A)$ (Thm.1.12).   Therefore $D := \text{im } A$ is dense and $\int_0^\infty T(t) f\, dt$ exists for every $f \in D$. This proves   "(b') $\to$ (c')".   We have to show that (c') implies (a'). Define $G := \{h \in E : h = \int_0^\infty T(t) g\, dt$ for some $g \in D\}$. We will show that   G   is dense in   E . First we notice that $g - T(s) g \in D$ whenever $g \in D$ and $s \in \mathbb{R}_+$.

Define $h_s = \frac{1}{s} \cdot \int_0^\infty T(t) (g - T(s) g)\, dt = \frac{1}{s} \cdot \int_0^s T(t) g\, dt$. Then $h_s \in G$ and $h_s \to g$ as $s \to 0$. Therefore $D \subset \bar{G}$ or $\bar{G} = E$. Now let $h \in G$. Then $T(t) h = T(t) \int_0^\infty T(s) g\, ds = \int_t^\infty T(s) g\, ds \to 0$ as $t \to \infty$. But $\|T(t)\| \leq M$ and therefore $T(t) f \to 0$ for every $f \in E$.

$\square$

Remarks 1.17. (a) If   A   is the generator of   a stable   semigroup $(T(t))_{t \geq 0}$ on a Banach space   E,   then, by the previous theorem,

$$\text{im } A \subset \{f \in E : \int_0^\infty T(t) f\, dt \text{ exists}\} =: H .$$

If $g \in H$, then $\int_0^\infty T(t) g\, dt \in D(A)$ and $A \int_0^\infty T(t) g\, dt = -g$. Therefore $g \in \text{im } A$ and we obtain that the dense subspace   im A   is given by

(1.14) $\text{im } A = \{f \in E : \int_0^\infty T(t) f\, dt \text{ exists}\}$

in case that   A   is the generator of a stable semigroup $(T(t))_{t \geq 0}$.

(b) If $\omega(f) < 0$ for every $f \in D(A)$, then $(T(t))$ is stable (but might not be exponentially stable if $0 = \omega_1(A) = \sup\{\omega(f) : f \in D(A)\}$. In this case it can be seen by a proof similar to the one of Thm.1.4, that $\sigma(A)$ has to be contained in the open left half-plane; i.e.   Re $\lambda < 0$ for $\lambda \in \sigma(A)$.

(c) If one defines a semigroup $(T(t))_{t \geq 0}$ to be <u>weakly stable</u> if $<T(t) f, \Phi> \to 0$ as $t \to \infty$ for every $f \in D(A)$ and $\Phi \in E'$ or as <u>weakly uniformly stable</u> if $<T(t) f, \Phi> \to 0$ as $t \to \infty$ for every $f \in E$ and $\Phi \in E'$, then Theorem 1.13 and 1.16 can be reformulated in a

weak form (i.e.; we replace stable by weakly stable and 'lim' by
'weak-lim'). The proofs require only some obvious modifications.

If A has a compact resolvent or if A is the generator of a bounded
holomorphic semigroup, then weak stability implies stability. In
general, this is no longer true; e.g., the translation semigroup on
$L^2(\mathbb{R})$ is weakly uniformly stable but not stable (see also
B-IV,Ex.1.2).

2. STABILITY: INHOMOGENEOUS CASE

Using the results of the first section, we now investigate the long
term behavior of the solutions of the inhomogeneous initial value
problem

(2.1) $\dot{u}(t) = Au(t) + F(t)$, $u(0) = f$

where A is the generator of a strongly continuous semigroup on a
Banach space E and $F(\cdot)$ is a locally integrable function from $\mathbb{R}_+$
into E called <u>forcing term</u>. A function $u(\cdot)$ is called a (<u>strong</u>)
<u>solution</u> of (2.1) if $u(\cdot)$: $\mathbb{R}_+ \to D(A)$, $u(\cdot) \in C^1(\mathbb{R}_+,E)$ and (2.1)
is satisfied for $t \geq 0$.
The assumption that A is the generator of a semigroup $(T(t))_{t \geq 0}$
yields the uniqueness of the solution of (2.1). If $u(\cdot)$ is a sol-
ution of (2.1), then the function $v(s) := T(t-s)u(s)$, $0 \leq s \leq t$, is
differentiable and $v'(s) = T(t-s)F(s)$. But $F(\cdot)$ is locally inte-
grable, and by $\int_0^t T(t-s)F(s) \, ds = v(t) - v(0) = u(t) - T(t)f$ we see
that the solution $u(t)$ of (2.1) is given by

(2.2) $u(t) = T(t)f + \int_0^t T(t-s)F(s) \, ds$.

<u>Example</u>. Let $(T(t))_{t \geq 0}$ be not eventually differentiable. Then there
exists $g \in E$ such that $t \to T(t)g$ is not differentiable on $(0,\infty)$.
The initial value problem $\dot{u}(t) = Au(t) + T(t)g$, $u(0) = 0$ has no
(strong) solution $u(\cdot)$ because otherwise
                        $u(t) = \int_0^t T(t-s)T(s)g \, ds = tT(t)g$
has to be differentiable on $\mathbb{R}_+$.

Whenever the expression (2.2) makes sense we call it a <u>generalized</u> (or
<u>mild</u>) <u>solution</u> of (2.1). If $F(\cdot)$ is continuous and $f \in D(A)$, then
the generalized solution of (2.1) is a strong solution if and only if
$v(t) := \int_0^t T(t-s)F(s) \, ds$ is differentiable (see Pazy (1983) Chap.4,

Thm.2.4). There are several sufficient conditions on the generator A,
the forcing term F(·) or the space E such that every mild solution
is a strong solution of (2.1) (see Travis (1979) or Pazy (1983)
Sec.4.2).

It is our aim in this section to study the asymptotic behavior of the
solutions of (2.1) as t → ∞ . To that purpose we consider absolutely
integrable or periodic forcing terms F(·) , and assume the semigroup
to be uniformly stable.
Similar results for integrable and convergent forcing terms F(·) can
be obtained if the semigroup is supposed to be uniformly exponentially
stable (see Pazy (1983), p.119 or Neubrander (1985b)). However, if the
semigroup is positive, these results even hold for stable semigroups
(see Section C-IV). From Theorem 1.13.(i) we recall that for stable
semigroups im A is dense in E .

<u>Theorem</u> 2.1. Let A be the generator of a uniformly stable semigroup
$(T(t))_{t \geq 0}$ on a Banach space E . If there is $g \in$ im A such that
$\int_0^\infty \|F(s) - g\|$ ds exists, then every generalized solution u(·) of
(2.1) converges as t → ∞ and $\lim_{t \to \infty} u(t) = -h$ where $h \in D(A)$ with
Ah = g.

<u>Proof.</u> If u(·) is a generalized solution of (2.1), then, by (2.2),
$u(t) = T(t)f + \int_0^t T(s)g \, ds + \int_0^t T(t-s)(F(s)-g) \, ds$. By the uniform
stability and Thm.1.14 we see that the first term converges to zero
and that the second one converges to -h . We have to show that the
third term converges to zero. Take ε > 0 and G(s) := F(s)-g . Then

$$\left\| \int_0^t T(t-s)G(s) \, ds \right\| \leq \left\| \int_0^{t'} T(t-t'+t'-s)G(s) \, ds \right\| + \left\| \int_{t'}^t T(t-s)G(s) \, ds \right\|$$

$$\leq \left\| T(t-t') \int_0^{t'} T(t'-s)G(s) \, ds \right\| + M \int_{t'}^\infty \|G(s)\| \, ds .$$

Since the semigroup is uniformly stable we obtain
$T(t-t') \int_0^{t'} T(t'-s)G(s) \, ds \to 0$ as t → 0 for every t' ≥ 0 .
Therefore $\left\| \int_0^t T(t-s)G(s) \, ds \right\| \leq \varepsilon$ for all sufficiently large t .

                                                                    □

In the subsequent theorem we see that if A is the generator of a
uniformly stable semigroup, if the forcing term F(·) is p-periodic
and if $\int_0^p T(p-s)F(s) \, ds \in$ im (Id - T(p)) (notice that, by
Thm.1.13. (i) and A-III,Lemma 5.3, $\overline{\text{im}(Id - T(p))} = E$), then (2.1)
admits a unique p-periodic, asymptotically stable mild solution.

<u>Lemma</u> 2.2. Let   A   be the generator of a strongly continuous semi-
group $(T(t))_{t \geq 0}$ on a Banach space   E   and let $F(\cdot)$ be a p-peri-
odic, locally integrable function,   p > 0 . Then the following state-
ments are equivalent:

(a) $\dot{u}(t) = Au(t) + F(t)$ admits a (unique) generalized p-periodic
      solution.

(b)   There exists a (unique) $f \in E$ such that $(Id - T(p))f = \int_0^p T(p-s)F(s) \, ds$.

<u>Proof.</u> "(a) → (b)". Let $f := u(0)$ be the initial value for which
(2.1) has the p-periodic solution. Then we have

$$u(t) = u(t+p) = T(t)T(p)f + \int_0^p T(t+p-s)F(s) \, ds + \int_p^{t+p} T(t+p-s)F(s) ds$$

$$= T(t)[T(p)f + \int_0^p T(p-s)F(s) \, ds] + \int_0^t T(t-s)F(s) \, ds$$

for every $t \geq 0$. Therefore $f = u(0) = T(p)f + \int_0^p T(p-s)F(s) \, ds$.
Clearly, if $u(\cdot)$ is a unique periodic solution with   u(0) = f , then
f   is the unique element for which $f = T(p)f + \int_0^p T(p-s)F(s) \, ds$
holds.

"(b) → (a)". Define $u(\cdot)$ as in (2.2). Then

$$u(t+p) = T(t)[T(p)f + \int_0^p T(p-s)F(s) \, ds] + \int_0^t T(t-s)F(s) \, ds = u(t) .$$

If   f   is unique, then, by the considerations above, the solution is
unique.

□

<u>Remark</u> 2.3. Let   A   be the generator of a strongly continuous semi-
group $(T(t))_{t \geq 0}$ for which the spectral mapping theorem (see A-III,
Sec.6) is valid and let   F   be a p-periodic forcing term.
If $\frac{2\pi i n}{p} \in \rho(A)$ for every $n \in \mathbb{Z}$, then, by Lemma 2.2,
$\dot{u}(t) = Au(t) + F(t)$ has a unique p-periodic solution with initial
value $(Id - T(p))^{-1} \{\int_0^p T(p-s)F(s) \, ds\}$.

As a consequence of Thm.1.13 and A-III,Cor.6.4, for a uniformly stable
semigroup there exists at most one $f \in E$ such that
$(Id-T(p))f = \int_0^p T(p-s)F(s) \, ds$. This and Lemma 2.2 is used to prove
the following theorem.

<u>Theorem</u> 2.4. Let   A   be the generator of a uniformly stable semigroup
$(T(t))_{t \geq 0}$ and let $F(\cdot)$ be a p-periodic locally integrable function
such that $(Id - T(p))f = \int_0^p T(p-s)F(s) \, ds$ for some $f \in E$. Then the

unique p-periodic generalized solution

$$u(t) = T(t)f + \int_0^p T(p-s)F(s)\,ds$$

is asymptotically stable; i.e., for every generalized solution $v(\cdot)$
of (2.1) we have $\lim_{t \to \infty} \| v(t) - u(t) \| = 0$.

Example 2.5. Let E be the Banach space $C_0(\mathbb{R}_+)$; $A = \frac{d}{dx}$ with
$D(A) = \{f \in E: f' \in C^1$ and $f' \in E\}$ is the generator of the uni-
formly stable translation semigroup $T(t)f(x) := f(t+x)$. Applying
(1.14) we obtain that $\text{im } A = \{f : \int_0^\infty f(x)\,dx$ exists$\}$ is dense in
$C_0(\mathbb{R}_+)$. Let $r \in \text{im } A$ and let $F(\cdot)$ be a p-periodic real-valued
function.

We apply Theorem 2.4 to the initial value problem

(*) $\frac{d}{dt} u(t,x) = \frac{d}{dx} u(t,x) + r(x)F(x+t)$, $u(0,\cdot) \in D(A)$.

We may rewrite (*) as

(**) $\dot{v}(t) = Av(t) + G(t)$

where $v(t) = u(t,\cdot)$ and $G : \mathbb{R}_+ \to E$ is defined by

$$G(t)(x) = r(x)F(x+t) .$$

G is p-periodic with values in E and $h_0 := \int_0^p T(p-t)G(t)\,dt$ is
the function $x \to [\int_0^p T(p-t)G(t)\,dt](x) = F(x)\int_x^{x+p} r(s)\,ds$. For the
function $f = \sum_{k=0}^\infty T(kp)h_0$, which is given by $x \to F(x)\int_x^\infty r(s)\,ds$,
we clearly have $(\text{Id} - T(p))f = h_0$. Therefore (**) has a unique
p-periodic generalized solution (Thm.2.4) although $i\mathbb{R} \in \sigma(A)$ (com-
pare with Remark 2.3).

The unique p-periodic generalized solution $u(t,\cdot)$ is given by
$u(t,x) = F(x+t)\int_{x+t}^\infty r(s)\,ds + F(x+t)\int_x^{x+t} r(s)\,ds = F(x+t)\int_x^\infty r(s)\,ds$. For
every solution $v(t,\cdot)$ of (*) we have, by Thm.2.4:
$\sup\{|v(t,x) - F(x+t)\int_x^\infty r(s)\,ds| : x \in \mathbb{R}_+\} \to 0$ as $t \to 0$.

NOTES.

Section 1. The exponential growth bounds $\omega(f)$ and $\omega(A)$ as well as the charac-
terizations (1.2), (1.6) and Theorem 1.3 (i) can be found in Hille-Phillips (1957).
Growth bounds similar to $\omega_1(A)$ were considered first in [D'Jacenko (1976)] and in
[Zabczyk (1979), Prop.2]. Example 1.2.(2) is taken from Wolff (1981); other 'coun-
terexamples' can be found in Hille-Phillips (1957), Foias (1973), Triggiani (1975),
Zabczyk (1975) and Greiner-Voigt-Wolff (1981). The statements (1.2), (1.6) and
Theorem 1.3.(i) are semigroup versions of results of classical Laplace transform

theory, see Hille-Phillips (1957) and Widder (1946). Theorem 1.3. (ii) is a semi-group version of Theorem 1.2.7 and 1.2.8 in Doetsch (1950). The lemma in the proof of Thm.1.3 is taken from Mil'stein (1975). Theorem 1.4 and Corollary 1.5 can be found in Neubrander (1985a). Example 1.6 follows Remark 2 in Zabczyk (1975). Statement (1.8) is sometimes called the 'spectrum determined growth assumption', see, for example, Triggiani (1975b). Theorem 1.9 is due to Slemrod (1976). The proof given here is based on a much sharper version of the inversion formula for the Laplace transform, than the one given by Hille-Phillips (1957), p.349. Using Widder (1946), p.66 or Doetsch (1950), p.212 one can show the following theorem (see Neubrander (1984b)).

Theorem. Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space E . For every $f \in D(A)$ and $p > \omega_1(A)$ we have

$$T(t)f = \frac{1}{2\pi i} \int_{p+i\infty}^{p+i\infty} e^{\mu t} R(\mu,A)f \, d\mu \ .$$

The equivalence of the statements (1.12), (1.13) and '$\omega(A) < 0$' were observed by many authors, see for example, Balakrishnan (1976), p.178 or Benchimol (1978). Theorem 1.11 is due to Datko (1970) and Delfour (1974); for a proof see Pazy (1983), p. 116. Theorems 1.13 and 1.16 can be found in Neubrander (1985b) and Corollary 1.14 is due to Komatsu (1969). An example of an unstable semigroup generator A with Re $\mu < 0$ for all $\mu \in \sigma(A)$ is given in Datko (1983).

Section 2. For a discussion of well-posedness of inhomogeneous Cauchy problems we refer to Goldstein (1985a), p.83 and Pazy (1983), p.105. Further results on the asymptotic behavior of the solutions of the inhomogeneous problem can be found in Rao-Hengartner (1974), Zaidman (1979), Pazy (1983), and Neubrander (1985b). Results similar to Lemma 2.2 and Theorem 2.4 are due to Prüß (1984). For a discussion of the asymptotic behavior of the solutions of $u'(t) = A(t)u(t) + F(t)$ see Datko (1972) and Pazy (1983),p.172 .