If $P\sigma(A) \cap i\mathbb{R}$ is non-empty, then the following assertions are true:

(a) $P\sigma(A) \cap i\mathbb{R}$ is a (additive) subgroup of $i\mathbb{R}$.

(b) The eigenspaces corresponding to $\lambda \in P\sigma(A) \cap i\mathbb{R}$ are one-dimensional.

(c) If $Ah = i\alpha h$ $(h \neq 0 , \alpha \in \mathbb{R})$ then $|h|$ is a quasi-interior point and the following holds:

(3.13) $S_h(D(A)) = D(A)$ and $S_h^{-1} \circ A \circ S_h = (A + i\alpha)$.

(d) 0 is the only eigenvalue of A admitting a positive eigenvector.

One can apply the theorem in order to prove that the rotation semigroup on Γ (cf. A-I,2.5) is the only positive periodic semigroup which is irreducible.

<u>Corollary</u> 3.9. Let $(T(t))_{t\geq 0}$ be a positive irreducible semigroup on a Banach lattice E which is periodic of period τ .
Assume that $\dim E > 1$. Then there exist continuous lattice homomorphisms $i : C(\Gamma) \to E$ and $j : E \to L^1(\Gamma)$, both injective with dense range, such that the diagramm commutes for all $t \geq 0$. Moreover, $j \circ i$ is the canonical inclusion of $C(\Gamma)$ in $L^1(\Gamma)$.

$$\begin{tikzcd}
C(\Gamma) \arrow[r, "i"] & E \arrow[r, "j"] & L^1(\Gamma) \\
C(\Gamma) \arrow[r, "i"'] \arrow[u, "R_\tau(t)"] & E \arrow[r, "j"'] \arrow[u, "T(t)"'] & L^1(\Gamma) \arrow[u, "R_\tau(t)"']
\end{tikzcd}$$

<u>Proof</u>. By Thm.3.8 and A-III,Thm.5.4 we have $R\sigma(A) = P\sigma(A) = \sigma(A) = i\alpha\mathbb{Z}$ with $\alpha := \frac{2\pi}{n\tau}$ for suitable $n \in \mathbb{N}$. We fix $h \in \ker(i\alpha - A)$, $h \neq 0$. Then $|h| \in \ker A$ and there exists $\phi \in \ker A'$ such that $\langle |h|, \phi\rangle = 1$. According to the Kakutani-Krein Theorem we identify $E_{|h|}$ with $C(K)$. Then h is a unimodular function onto Γ (use the argument given in the proof of B-III,Thm.3.6(c)).
We define $i : C(\Gamma) \to E$ by $i(f) := f \circ h \in C(K) \cong E_h \subset E$, then i is injective. For the monomials $e_n(z) := z^n$ $(n \in \mathbb{Z})$ we have $i(e_n) = h^{[n]}$ thus i has dense image in E (by A-III,Thm.5.4). Moreover, $2\pi \cdot \delta_{n0} = \langle h^{[n]}, \phi\rangle = \langle i(e_n), \phi\rangle = \int_0^{2\pi} e_n(e^{it})\, dt$ for all $n \in T$, hence $\int_0^{2\pi} f(e^{it})\, dt = \langle i(f), \phi\rangle$ for all $f \in C(\Gamma)$. It follows that $(E, \phi) \cong L^1(\Gamma)$ and we define j to be the canonical mapping from E into $(E, \phi) \cong L^1(\Gamma)$ (see C-I,Sec.4) . Then j has dense image and is injective since ϕ is strictly positive (cf. Prop.3.5(b)). One easily verifies that the diagramm commutes.

□