Let $\alpha \in (0,\pi/2]$. A semigroup $(T(t))_{t\geq 0}$ is called <u>holomorphic of angle</u> α if it possesses an extension $T : S(\alpha) \to L(E)$ for some $\alpha \in (0,\pi/2]$ which satisfies all the requirements of Definition 1.11 except that it is not required to be bounded on any sector $S(\alpha_1)$.

<u>Theorem</u> 1.14. A densely defined operator A is the generator of a holomorphic semigroup if and only if there exist $M > 0$ and $r \geq 0$ such that $\lambda \in \rho(A)$ and $\|R(\lambda,A)\| \leq M/|\lambda|$ whenever $\operatorname{Re}\lambda > 0$, $|\lambda| \geq r$.

<u>Proof</u>. It is not difficult to show that A generates a holomorphic semigroup of angle α if and only if for every $\alpha_1 \in (0,\alpha)$ there exists $w \in \mathbb{R}$ such that $A-w$ generates a bounded holomorphic semigroup of angle α_1 (cf.[Reed-Simon (1978b),p.252]). As a consequence one obtains the following. A densely defined operator A generates a holomorphic semigroup of angle $\alpha \in (0,\pi/2]$ if and only if for every $\alpha_1 \in [0,\alpha)$ there exist a constant $M \geq 0$ and $r \geq 0$ such that

$$S(\alpha_1+\pi/2)\backslash B(r) \subset \rho(A) \quad (\text{where } B(r) = \{z \in \mathbb{C} : |z| \leq r\})$$

and

$$\|R(\lambda,A)\| \leq M/|\lambda| \quad \text{for all } \lambda \in S(\alpha_1)\backslash B(r).$$

This shows that the condition of the theorem is necessary. Conversely, assume that the condition holds. Since $\|R(\lambda,A)\| \to \infty$ when λ approaches $\sigma(A)$ (cf. Lemma 1.21 below), it follows that $\lambda \in \rho(A)$ and $\|R(\lambda,A)\| \leq M/|\lambda|$ if $\operatorname{Re}\lambda = 0$ and $|\lambda| > r$ as well.
Let $c = 1/2M$. If ξ, $\eta \in \mathbb{R}$ satisfy $|\xi| \leq c|\eta|$, $|\eta| \geq r$, then $\|\xi R(i\eta,A)\| \leq \xi\cdot M/|\eta| \leq c\cdot M = 1/2$.
Hence $R(\xi+i\eta,A) = \sum_{n=0}^{\infty} (-\xi)^n R(i\eta,A)^{n+1}$ exists and

$$\begin{aligned}\|R(\xi+i\eta,A)\| &\leq (|\xi+i\eta|)^{-1}\cdot|\xi+i\eta|\cdot\sum_{n=0}^{\infty} |\xi|^n M^{n+1}/|\eta|^{n+1} \\ &\leq (|\xi+i\eta|)^{-1}\cdot M\cdot(|\xi|^2+|\eta|^2)^{-1/2}/|\eta|\cdot\sum_{n=0}^{\infty} M^n c^n \\ &\leq (2M\cdot(c^2+1)^{-1/2})/|\xi+i\eta| \\ &= N/|\xi+i\eta| .\end{aligned}$$

This together with the assumption implies that there exist $N' \geq 0$ and $\alpha \in (0,\pi/2]$ such that $\lambda \in \rho(A)$ and $\|R(\lambda,A)\| \leq N'/|\lambda|$ for all $\lambda \in S(\alpha+\pi/2)$.

□

Compared with the Hille-Yosida theorem, Theorem 1.14 gives a very simple criterion for an operator to be the generator of a (holomorphic) semigroup. Merely the resolvent and not its powers have to be