<u>Proof</u>. If $\sigma(A) = \emptyset$ there is nothing to prove, thus we can assume that $s(A) = 0$. In view of the lemma and A-III,Prop.4.3(i) we can assume that $s(A)$ is a first order pole with strictly positive residue, which we call Q . We have $AQ = QA = s(A)A = 0$ (see A-III, 3.6), hence

(2.18) $QT(t) = T(t)Q = Q$ for all $t \geq 0$.

If $Ah = i\alpha h$ for some $\alpha \in \mathbb{R}$, $h \neq 0$, then $T(t)h = e^{i\alpha t}h$ (by A-III, Cor.6.4). Hence $|h| = |e^{i\alpha t}h| = |T(t)h| \leq T(t)|h|$, or equivalently, $T(t)|h| - |h| \geq 0$. By (2.18) we have $Q(T(t)|h| - |h|) = 0$.
Since Q is strictly positive, it follows that $T(t)|h| = |h|$ or $A|h| = 0$. Now we can apply Thm.2.4 and obtain $Ah^{[n]} = in\alpha h^{[n]}$ for every $n \in \mathbb{Z}$. This shows that $P\sigma_b(A) = \sigma(A) \cap i\mathbb{R}$ is cyclic.

□

If A has compact resolvent then every point of $\sigma(A)$ is a pole of the resolvent (see A-III,3.6) hence we have:

<u>Corollary</u> 2.10. If A has compact resolvent, then $P\sigma_b(A) = \sigma_b(A)$ is cyclic.

If it is known that the boundary spectrum of a generator is cyclic and nonvoid, the following alternative holds:

(2.19) Either $\sigma_b(A) = \{s(A)\}$ or else $\sigma_b(A)$ is an infinite unbounded set.

If one can exclude the second alternative, then there is a unique spectral value having maximal real part. A real spectral value λ_o of a generator A is called a <u>dominant</u> provided that $\mathrm{Re}\,\lambda < \lambda_o$ for every $\lambda \in \sigma(A)$, it is called <u>strictly dominant</u> if for some $\delta > 0$ one has $\mathrm{Re}\,\lambda \leq \lambda_o - \delta$ for every $\lambda \in \sigma(A)$, $\lambda \neq \lambda_o$.
The assumptions of Cor.2.10 do not imply that $s(A)$ is dominant, the rotation semigroup (A-III,Ex.5.6) is a counterexample.

<u>Corollary</u> 2.11. Assume that for some $t_o > 0$ (hence for all $t > 0$) one has $r_{ess}(T(t_o)) < r(T(t_o))$, e.g., that $T(t_o)$ is compact and $r(T(t_o)) > 0$ (see A-III,3.7).
Then $s(A)$ is a strictly dominant eigenvalue.