(d) For the semigroup on $C(\Gamma\times\Gamma)$ given by
$(T(t)f)(z,w) = f(z\cdot e^{i\alpha t}, w\cdot e^{i\beta t})$ $(f \in C(\Gamma\times\Gamma)\ ,\ (z,w) \in \Gamma\times\Gamma)$
we have $P\sigma(A) = M_4$. If α/β is irrational, then this is a dense subset of $i\mathbb{R}$ and $\sigma_b(A) = \sigma(A) = i\mathbb{R}$.

(e) Consider $D := \{z \in \mathbb{C} : |z| \leq 1\} = \{r\cdot e^{i\omega} : r \in [0,1]\ ,\ \omega \in \mathbb{R}\}$, and a strictly positive function $\kappa \in C[0,1]$. The flow on D governed by the differential equation $\dot{r} = 0$, $\dot{\omega} = \kappa(r)$ induces a strongly continuous semigroup on $C(D)$ (which is given by $(T(t)f)(z) = f(z\cdot e^{i\kappa(|z|)t}))$. The boundary spectrum is M_6 with $\alpha := \inf \kappa(r)$, $\beta := \sup \kappa(r)$. In particular, for $\kappa(r) = 1 + r$ we obtain as boundary spectrum the set M_5 .

(f) Suppose M is a closed cyclic subset of $i\mathbb{R}$, $M = \bigcup_{\alpha\in S} i\alpha\mathbb{Z}$ for a suitable $S \subset \mathbb{R}$ (e.g. $S = M$) .
The space $E_1 := \{(f_\alpha)_{\alpha\in S} : f_\alpha \in C(\Gamma)\ ,\ \sup \|f_\alpha\| < \infty\}$ is a Banach space under the norm $\|(f_\alpha)\| := \sup \|f_\alpha\|$. The closure of the linear subspace $E_o := \{(f_\alpha) \in E_1 : f_\alpha \neq 0$ only for finitely many $\alpha \in S\}$ is isomorphic to $C_o(X)$ where X is the topological sum of $|S|$ copies of Γ .
Let $(T_\alpha(t))_{t\geq 0}$ denote the rotation semigroup on $C(\Gamma)$ with period $2\pi/\alpha$, then we define a semigroup $(T(t))_{t\geq 0}$ on $E := C_o(X)$ as follows:

$$(T(t)(f_\alpha)) := (T_\alpha(t)f_\alpha) \quad ((f_\alpha)_{\alpha\in S} \in E) .$$

This is a positive semigroup on $E = C_o(X)$ whose boundary spectrum is precisely the given closed cyclic set M . We leave the verification as an excercise.

Our first result concerns cyclicity of the eigenvalues contained in the boundary spectrum, i.e., of the set
$P\sigma_b(A) := P\sigma(A)\cap\sigma_b(A) = \{\lambda \in P\sigma(A) : \mathrm{Re}\,\lambda = s(A)\}$.
It is almost a straightforward consequence of Thm.2.4 .

<u>Proposition</u> 2.7. Assume that for some $t_o > 0$ there is a strictly positive measure ϕ such that $T(t_o)'\phi = \exp(s(A)t_o)\cdot\phi$.
Then $P\sigma_b(A)$ is imaginary additively cyclic.

<u>Proof</u>. If $P\sigma_b(A)$ is empty there is nothing to prove. Otherwise we have $s(A) > -\infty$. In view of the rescaling procedure we may assume $s(A) = 0$. By Prop.1.5(b) there exists $\Psi \gg 0$ such that $T(t)'\Psi = \Psi$ for all $t \geq 0$. Given $i\alpha \in P\sigma_b(A)$ then there is $h \in C_o(X)$, $h \neq 0$ such that $Ah = i\alpha h$ or $T(t)h = e^{i\alpha t}h$ for all t (A-III,Cor.6.4).