one has to show that $u \in C(\overline{\Omega})$.
Wittka extends u to an open set $\tilde{\Omega}$ containing $\overline{\Omega}$ by reflecting u along the graph. Then $\tilde{u}$ becomes the solution of an elliptic problem on $\tilde{\Omega}$. Continuity on $\tilde{\Omega}$, and hence on $\overline{\Omega}$, follows from the De Giorgi – Nash Theorem.

Irreducibility is due to Arendt, ter Elst and Glück [AEG 20, Theorem 4.5].

Since the semigroup is holomorphic, by C-II, Theorem 3.2 (ii), it implies that

$$\inf_{x \in \overline{\Omega}} (T(t)f)(x) > 0 \qquad (4.1)$$

for all $t > 0$ and $0 \leq f \in C(\overline{\Omega})$, $f \neq 0$.

Denote by $s(\Delta^\beta)$ the spectral bound of Δ^β.
By C-III, Theorem 3.8 (iv), $s(\Delta^\beta)$ is the unique eigenvalue with a positive eigenfunction $u_0 \geqslant 0$, $u_0 \neq 0$. It follows from (4.1) that u_0 is strictly positive; i.e.,

$$\inf_{x \in \overline{\Omega}} u_0(x) > 0,$$

a remarkable property, which has important applications to semi-linear problems, see Arendt–Daners [ArDa 25].

The spectral bound $s(\Delta^\beta)$ determines the asymptotic behavior of the semigroup $\mathcal{T}$. In fact, the following follows from B-III Proposition 3.5.

<u>Corollary 4.4.</u> There exist a strictly positive Borel measure μ on $\overline{\Omega}$, $M \geq 0$ and $\varepsilon > 0$ such that

$$\langle \mu, u_0 \rangle = 1 \quad \text{and}$$

$$\| T(t) - e^{s(A)t} P \| \leq M e^{-\varepsilon t}$$

for all $t \geq 0$, where $P \in \mathcal{L}(C(\overline{\Omega}))$ is given by

$$Pf = \langle \mu, f \rangle u_0 .$$

The theorem says that the rescaled semigroup $\left(e^{-s(\Delta^{\beta})t}\,T(t)\right)_{t\geq 0}$ converges in the operator norm to the rank-1-projection P exponentially fast.

Elliptic operators in divergence form

The preceding results extend to elliptic operators in divergence form with bounded measurable coefficients.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

Let $a_{k,l}, b_k, c_k, c_0 \in L^\infty(\Omega)$,

$k, l = 1, \dots, n$ such that for

some $\alpha > 0$

$$\sum_{k,l=1}^{n} a_{k,l}(x) \xi_k \overline{\xi_l} \geq \alpha |\xi|^2$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$, where

$|\xi|^2 = \xi_1^2 + \dots + \xi_d^2$.

Let $H^1_{loc}(\Omega) := \{ v \in L^2_{loc}(\Omega) :$

$\partial_k v \in L^2_{loc}(\Omega),\ k = 1, \dots, n \}$

Define $\mathcal{A} : H^1_{loc}(\Omega) \longrightarrow C_c^\infty(\Omega)'$

by

$$\langle \mathcal{A}u, v \rangle = \sum_{k,l=1}^{d} \partial_k (a_{kl}\, \partial_l u) + \sum_{k=1}^{d} \partial_k (b_k u)$$
$$+ \sum_{k=1}^{d} c_k\, \partial_k v + c_0 u .$$

We define A_0 as the part of $\mathcal{A}$ in $C_0(\Omega)$; i.e.

$$D(A_0) := \{ u \in C_0(\Omega) \cap H^1_0(\Omega) : \mathcal{A}u \in C_0(\Omega) \}$$
$$A_0 u := \mathcal{A}u.$$

Then Theorem 4.1 holds with Δ_0 replaced by A_0. It is remarkable that Dirichlet regularity of Ω is the right regularity condition at the boundary, a discovery due to Stampacchia. We refer to

Arendt and Bénilan [ArBe 99, Section 4]

for a proof of the following result.

<u>Theorem</u> 4.5 Assume that $\Omega \subset \mathbb{R}^n$ is a bounded, open, connected, Dirichlet regular set. Then A_0 generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on $C_0(\Omega)$. Moreover, $T(t)$ is compact for all $t > 0$.

Remark. The proof of holomorphy depends on Gaussian estimates, which in [ArBe 99] were merely known if the $b_k \in W^{1,\infty}(\Omega)$. Later, it was shown by Daners [Da 00] that they always hold.

Also the results for Robin boundary conditions Theorem 4.3 and 4.4 can be extended to elliptic operators in divergence form on $C_0(\Omega)$; see Theorem 4.5 in Arendt, ter Elst, Glück [AEG20]. It uses results of Nittka [Ni11].

Elliptic operators in non-divergence forms

The techniques for elliptic operators in non-divergence form are quite different than those used in the divergence-case form. But the results are similar.

Let $\Omega \subset \mathbb{R}^n$ be open and connected. We assume that Ω satisfies the <u>uniform exterior cone condition</u>. This means the following. There exists a finite, right circular cone V such that for each $x \in \partial\Omega$ there exists a cone V_x which is congruent to V such that $V_x \cap \overline{\Omega} = \{x\}$.

Let $a_{ke} = a_{ek} \in C(\bar{\Omega})$, $b_k \in L^\infty(\Omega)$
$c \in L^\infty(\Omega)$, $c \leq 0$ such that

$$\sum_{k,l=1}^{n} a_{kl}(x)\, \xi_k \xi_l \geq \mu |\xi|^2$$

for all $x \in \bar{\Omega}$, $\xi \in \mathbb{R}^n$ and some $\mu > 0$.

For $u \in W^{2,n}_{loc}(\Omega)$ we define

$$Au = \sum \partial_k a_{ke} \partial_e u + \sum b_k \partial_k u + cu$$

Thus $A : W^{2,n}_{loc}(\Omega) \longrightarrow L^n_{loc}(\Omega)$ is linear. Here

$$W^{2,n}_{loc}(\Omega) := \{ u \in L^n_{loc}(\Omega) : \partial_k u \in L^n_{loc}(\Omega),\ \partial_k \partial_e u \in L^n_{loc}(\Omega) \text{ for all } k, l = 1, \dots, n \}.$$

We consider the operator A on $C_0(\Omega)$ defined by

$$D(A) := \{ u \in C_0(\Omega) \cap W^{2,n}_{loc}(\Omega) : \mathcal{A}u \in C_0(\Omega) \}$$

$$Au := \mathcal{A}u.$$

Then the following holds.

Theorem 4.6. The operator A generates a positive, irreducible, contractive holomorphic semigroup $(T(t))_{t\geq 0}$ on $C_0(\Omega)$. Moreover

$$\| T(t) \| \leq M e^{-\varepsilon t} \qquad (t \geq 0)$$

for some $\varepsilon > 0$, $M \geq 1$.

The resolvent of A is compact.

This result is proved by Arendt and Schätzle [AS 14, Proposition 4.7]. The monograph of Lunardi [Lu 95] is devoted to the study of holomorphic semigroups generated by elliptic operators in non-divergence form and Theorem 4.6 is a generalization