The <u>adjoint semigroup</u> on E^* is given by the operators

$$T(t)^* := T(t)'_{|E^*} \quad , \; t \geq 0 \; .$$

Since $(T(t)^*)_{t\geq 0}$ is strongly continuous on E^* we call its generator $(A^*, D(A^*))$ the <u>adjoint generator</u>.

The above definition makes sense since E^* is norm-closed in E' and $(T(t)')$-invariant . The main point is that E^* is still reasonably large. In fact, since $\int_0^t T(s)'\phi \, ds$, understood in the weak sense, is contained in E^* for every $\phi \in E'$, $t \geq 0$ it follows that

$$\sup\{<f,\phi> : \phi \in E^* , \|\phi\| \leq 1\} \leq \|f\| \leq M\cdot\sup\{<f,\phi> : \phi \in E^* , \|\phi\| \leq 1\}$$

where $M := \limsup_{t\to 0} \|T(t)\|$ In particular, E^* separates E , i.e. E^* is $\sigma(E',E)$-dense in E' . In addition the estimate of $\|\cdot\|$ given above yields

$$\|T(t)^*\| \leq \|T(t)\| \leq M\|T(t)^*\| \quad \text{for all} \quad t \geq 0 \; .$$

In the following proposition we describe the relation between A^* and A' .

<u>Proposition</u>. For the adjoint generator A^* of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E the following assertions hold:

(i) E^* is the $\|\cdot\|$-closure of $D(A')$.

(ii) $D(A^*) = \{\phi \in D(A') : A'\phi \in E^*\}$.

(iii) A^* and A' coincide on $D(A^*)$.

<u>Proof</u>. (i) Take $\phi \in D(A')$ fixed. For every $f \in D(A)$ with $\|f\| \leq 1$ we define a continuously differentiable function

$$t \to \xi_f(t) := <T(t)f,\phi>$$

on $[0,1]$ with derivative $\xi_f'(t) = <T(t)Af,\phi> = <T(t)f,A'\phi>$.

Since $\{\xi_f'(t) : t \in [0,1], f \in D(A), \|f\| \leq 1\}$ is bounded it follows that the set

$$\{\xi_f : f \in D(A), \|f\| \leq 1\}$$

is equicontinuous at 0 , i.e. for every $\varepsilon > 0$ there exists $0 < t_o < 1$ such that

$$|\xi_f(s) - \xi_f(0)| = |<f,T(s)'\phi - \phi>| < \varepsilon$$

for every $0 \leq s \leq t_o$ and $f \in D(A)$, $\|f\| \leq 1$. But this implies $\|T(s)'\phi' - \phi'\| < \varepsilon$ and hence $\phi \in E^*$.

Conversely take $\psi \in E^*$. Then $\frac{1}{t}\int_0^t T(s)'\psi \, ds$, $t > 0$, belongs to $D(A')$ and norm converges to ψ as $t \to 0$, i.e. ψ belongs to the norm closure of $D(A')$.

(ii) and (iii): Since the weak* topology on E' is weaker than the norm topology it follows that A' is an extension of A^* .

Now take $\phi \in D(A')$ such that $A'\phi \in E^*$. As above define the func-