Theorem 3.1. The operator A defined above is the generator of a semigroup $(T(t))_{t\geq 0}$ on E.
For every $f \in E$, $t \geq 0$ we have for a.e. $s \in [-1,0]$

$$(3.3) \qquad (T(t)f)(s) = \begin{cases} f(t+s) & \text{if } t+s \leq 0 \\ \Phi(T(t+s)f) & \text{if } t+s > 0 . \end{cases}$$

Moreover, if $f \in D(A)$ then the translation property (T) (see B-IV, Thm.3.1) is satisfied.

Proof. Consider $E_1 := D(A)$ endowed with the graph norm and $A_1 := A$ restricted to $D(A_1) := D(A^2)$. By (A-I,3.5) A_1 generates the semigroup $(T(t)_{|D(A)})_{t\geq 0}$. On E_1 point evaluation is a continuous mapping and therefore the translation property can be shown as in the proof of B-IV,Thm.3.1. Hence we obtain

$$(3.4) \quad (T(t)f)(s) = \begin{cases} f(t+s) & \text{if } t+s\leq 0 \\ \Phi(T(t+s)f) & \text{if } t+s>0 \end{cases} = \begin{cases} f(t+s) & \text{if } t+s\leq 0 \\ (T(t+s)f)(0) & \text{if } t+s>0; \end{cases}$$

i.e. (3.3) is valid for $f \in D(A)$. It remains to show (3.3) for all $f \in E$. Fix $t \in \mathbb{R}_+$ and $s \in [-t,0]$. For $t + s > 0$ the equality follows immediately by the continuity of Φ from (3.4). For the case $t + s \leq 0$ we consider $g \in L^\infty[-1,0]$ with supp $g \subset [-1,-t]$. Comparing (3.1) and (3.4) we see that $\langle (T(t)-T_o(t))f,g\rangle = 0$ for all $f \in D(A)$, and hence for all $f \in E$.
Consequently $(T(t)-T_o(t))f = 0$ a.e. on $[-1,-t]$ which shows $(T(t)f)(s) = f(t+s)$ for a.e. $s \in [-1,-t]$.

□

The following corollary corresponds to B-IV,Cor.3.2 and assures the well-posedness of (RE):

Corollary 3.2. For every $f \in E$ the function u defined by

$$(3.5) \qquad u(t) := \begin{cases} f(t) & \text{if } -1\leq t\leq 0 \\ \Phi(T(t)f) & \text{if } t>0 \end{cases}$$

is the unique solution of (RE), in particular (RE) is well-posed. If $f \in D(A)$ then $u(t) = T(t)f(0)$ for $t > 0$.

Proof. As in the proof of B-IV,Cor.3.2 we have $u_t = T(t)f$ for $t\geq 0$ since $u_t(s) = u(t+s) = (T(t)f)(s)$ by the definition of u and by formula (3.3). Thus $u(t) = \Phi(T(t)f) = \Phi(u_t)$ if $t \geq 0$.
Also by the definition of u we have $u_o = f$.
It remains to show uniqueness. Let w be a solution of (RE) with initial function $w_o = 0$. Then