Proof. If $s(A)$ is not strictly dominant, then we have by Thm.2.9 and A-III,Cor.6.5 that $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda > s(A) - r\}$ contains infinitely many eigenvalues for every $r > 0$. From A-III,Cor.6.4 it follows that $\{\lambda \in \sigma(T(t)) : |\lambda| > r\}$ contains infinitely many eigenvalues (counted according to their multiplicities) for every $r < \exp(s(A)t) = r(T(t))$. This contradicts the assumption $r_{ess}(T(t)) < r(T(t))$ (see A-III,3.7) .

□

Corollary 2.12. Suppose A has compact resolvent and non-empty spectrum. If the corresponding semigroup is eventually norm continuous (e.g., if it is holomorphic or differentiable), then there is a strictly dominant eigenvalue admitting a positive eigenfunction.

Proof. Since $(T(t))_{t\geq 0}$ is eventually norm continuous, $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geq s(A)-r\}$ is compact for every $r > 0$ (see A-II,Thm.1.20) and this set does not have accumulation points because A has compact resolvent. In other words, it is a finite set. The assertion now follows from Thm.2.9 and Cor.1.4 .

□

We now consider some examples. The first one shows that there are positive semigroups with $P\sigma_b(A)$ being not cyclic. It is unknown if there are semigroups where $\sigma_b(A)$ is not cyclic.

Example 2.13. Consider $E = C(\Gamma) \times C_o(\mathbb{R})$ $\left(\cong C_o(\Gamma \dot{\cup} \mathbb{R}) \right)$. We fix a positive function $k \in C_o(\mathbb{R})$ with compact support. The operator A given by

$$(2.20) \qquad \begin{aligned} A(f,g) &:= (f', g' + \tfrac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \cdot k) \\ D(A) &:= \{(f,g) \in E : f,g \in C^1 ,\ g' \in C_o(\mathbb{R})\} \end{aligned}$$

generates a semigroup $(T(t))_{t\geq 0}$ which is given by

$$(2.21) \qquad \begin{aligned} T(t)(f,g) &= (f_t, g_t) \quad \text{with} \quad f_t(\theta) := f(\theta+t) , \\ g_t(x) &:= g(x+t) + \tfrac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \cdot \int_x^{x+t} k(u)\, du \end{aligned}$$

Then $(T(t))_{t\geq 0}$ is a positive semigroup and $\|T(t)\| \leq (1 + \|k\|_1)$. In particular, $s(A) \leq \omega(A) \leq 0$. It is easy to see that 0 is not an eigenvalue of A , while all ik , $k \in \mathbb{Z}$, $k \neq 0$ are eigenvalues, the corresponding eigenfunctions being $(e_k, 0)$ with $e_k(\theta) = e^{ik\theta}$.