As we know from the Examples 6.1, the converse inclusions do not hold in general, i.e., not every spectral value of a semigroup operator $T(t)$ comes - via the exponential map - from a spectral value of the generator. But at least this is true for some important parts of the spectrum.

6.3 Spectral Mapping Theorem for Point and Residual Spectrum. Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geq 0}$. Then

(6.4) $\exp(t \cdot P\sigma(A)) = P\sigma(T(t)) \setminus \{0\}$,

(6.5) $\exp(t \cdot R\sigma(A)) = R\sigma(T(t)) \setminus \{0\}$, for $t \geq 0$.

Proof. For the proof of (6.4) take $t_o > 0$ and $0 \neq \lambda \in P\sigma(T(t_o))$. After rescaling the semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ to the semigroup $(\exp(-t \cdot \log\lambda / t_o) T(t))_{t \geq 0}$ we may assume $\lambda = 1$. Then the closed, $\mathcal{T}$-invariant subspace

$$G := \{ g \in E : T(t_o) g = g \}$$

is non trivial. The restricted semigroup $\mathcal{T}_|$ is periodic with period $\tau \leq t_o$ and the spectrum of its generator $A_|$ contains at least one eigenvalue $\mu = 2\pi i n / t_o$ for some $n \in \mathbb{Z}$ (see Lemma 5.3). Since every eigenvalue of $A_|$ is an eigenvalue of A we obtain that $1 \in \exp(t_o \cdot P\sigma(A))$. The converse inclusion has been proved in (6.1). In fact, even more can be said: Let $g \in G$ be an eigenvector of $T(t_o)$ corresponding to the eigenvalue $\lambda = 1$. For each $n \in \mathbb{Z}$ define

$$g_n := P_n g = 1/t_o \cdot \int_0^{t_o} \exp(-2\pi i n s / t_o) T(s) g \, ds \in G$$

as in Section 5. Then g_n is an eigenvector of $A_|$, hence of A with eigenvalue $2\pi i n / t_o$ as soon as g_n is distinct from zero. Since $D(A_|)$ is dense in G it follows from Theorem 5.4 that this holds for at least one $n \in \mathbb{Z}$. From the proof of (6.1) we know that this g_n is in fact an eigenvector for each $T(t)$, $t \geq 0$.
Since $R\sigma(A) = P\sigma(A^*)$ and $R\sigma(T(t)) = P\sigma(T(t)^*)$ (see 4.4) the assertion (6.5) follows from (6.4).

□

Note that the proof is essentially an application of the structure theorem for periodic semigroups as given in Thm.5.4. The information gained there can be reformulated into statements on the eigenspaces of A and $T(t)$.