

# 1 ONE-PARAMETER SEMIGROUPS ON BANACH SPACES

## CHAPTER A-I

### BASIC RESULTS ON SEMIGROUPS ON BANACH SPACES

Since the basic theory of one-parameter semigroups can be found in several excellent books (e.g. Davies (1980), Goldstein (1985a), Pazy (1983) or Hille-Phillips (1957)) we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, to collect briefly some important definitions and results (Section 1), to present a list of standard examples (Section 2) and to discuss standard constructions of new semigroups from a given one (Section 3). In the entire chapter we denote by  $E$  a (real or) complex Banach space and consider one-parameter semigroups of bounded linear operators  $T(t)$  on  $E$ . By this we understand a subset  $\{T(t) : t \in \mathbb{R}_+\}$  of  $L(E)$ , usually written as  $(T(t))_{t \geq 0}$ , such that  $T(0) = Id$   $T(s+t) = T(s) \cdot T(t)$  for all  $s, t \in \mathbb{R}_+$ . In more abstract terms this means that the map  $t \mapsto T(t)$  is a homomorphism from the additive semigroup  $\mathbb{R}_+$  into the multiplicative semigroup  $(L(E), \cdot)$ . Similarly, a one-parameter group  $(T(t))_{t \in \mathbb{R}}$  will be a homomorphic image of the group  $(\mathbb{R}, +)$  in  $(L(E), \cdot)$ .

## 2 STANDARD DEFINITIONS AND RESULTS

We consider a one-parameter semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  and observe that the domain  $\mathbb{R}_+$  and the range  $L(E)$  of the (semigroup) homomorphism  $\tau : t \mapsto T(t)$  are topological semigroups for the natural topology on  $\mathbb{R}_+$  and any one of the standard operator topologies on  $L(E)$ . We single out the strong operator topology on  $L(E)$  and require  $\tau$  to be continuous.

**Definition 1.** *A one-parameter semigroup  $(T(t))_{t \geq 0}$  is called strongly continuous if the map  $t \mapsto T(t)$  is continuous for the strong operator topology on  $L(E)$ , i.e.,  $\lim_{t \rightarrow t_0} \|T(t)f - T(t_0)f\| = 0$  for every  $f \in E$  and  $t, t_0 \geq 0$ .*

Clearly one defines in a similar way weakly continuous, resp. uniformly continuous (compare A-II, Def. 1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology: From now on 'semigroup' always means strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

**Remark 1.** (i) *A one-parameter semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  is strongly continuous if and only if for any  $f \in E$  it is true that  $T(t)f \rightarrow f$  as  $t \rightarrow 0$ .*

(ii) *For every strongly continuous semigroup  $(T(t))_{t \geq 0}$  there exist constants  $M \geq 1$ ,  $\omega \in \mathbb{R}$  such that  $\|T(t)\| \leq M \cdot e^{\omega t}$  for every  $t \geq 0$ .*

(iii) If  $(T(t))_{t \geq 0}$  is a one-parameter semigroup such that  $\|T(t)\|$  is bounded for  $0 \leq t \leq \delta$  then it is strongly continuous if and only if  $\lim_{t \rightarrow 0} T(t)f = f$  for every  $f$  in a total subset of  $E$ .

**Definition 2.** By the growth bound (or type) of the semigroup  $(T(t))_{t \geq 0}$  we understand the number

$$\begin{aligned} \omega &:= \inf\{w \in \mathbb{R} : \text{There exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leq Me^{wt} \text{ for } t \geq 0\} \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \cdot \log \|T(t)\| = \inf_{t > 0} \frac{1}{t} \cdot \log \|T(t)\| \end{aligned} \quad (1)$$

Particularly important are semigroups such that for every  $t \geq 0$  we have  $\|T(t)\| \leq M$  (bounded semigroups) or  $\|T(t)\| \leq 1$  (contraction semigroups). In both cases we have  $\omega \leq 0$ . It follows from the subsequent examples and from 3.1 that  $\omega$  may be any number  $-\infty \leq \omega < +\infty$ . Moreover the reader should observe that the infimum in (1.1) need not be attained and that  $M$  may be larger than 1 even for bounded semigroups.

**Example 1.** (i) Take  $E = \mathbb{C}^2$ ,  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ . Then for the 1-norm on  $E$  we obtain  $\|T(t)\| = 1 + t$ , hence  $(T(t))_{t \geq 0}$  is an unbounded semigroup having growth bound  $\omega = 0$ .

(ii) Take  $E = L^1(\mathbb{R})$  and for  $f \in E$ ,  $t \geq 0$  define

$$T(t)f(x) := \begin{cases} 2 \cdot f(x+t) & \text{if } x \in [-t, 0] \\ f(x+t) & \text{otherwise.} \end{cases}$$

Each  $T(t)$ ,  $t > 0$ , satisfies  $\|T(t)\| = 2$  as can be seen by taking  $f := 1_{[0,t]}$ . Therefore  $(T(t))_{t \geq 0}$  is a strongly continuous semigroup which is bounded, hence has  $\omega = 0$ , but the constant  $M$  in (1.1) cannot be chosen to be 1.

**Definition 3.** To every semigroup  $(T(t))_{t \geq 0}$  there belongs an operator  $(A, D(A))$ , called the generator and defined on the domain

$$D(A) := \left\{ f \in E : \lim_{h \rightarrow 0} \frac{T(h)f - f}{h} \text{ exists in } E \right\}$$

by  $Af := \lim_{h \rightarrow 0} \frac{T(h)f - f}{h}$  for  $f \in D(A)$ .

Clearly,  $D(A)$  is a linear subspace of  $E$  and  $A$  is linear from  $D(A)$  into  $E$ . Only in certain special cases (see 2.1) the generator is everywhere defined and therefore bounded (use Prop. 1.9(i)). In general the precise extent of the domain  $D(A)$  is essential for the characterization of the generator. But since the domain is canonically associated to the generator of a semigroup we shall write in most cases  $A$  instead of  $(A, D(A))$ .

**Proposition 1.** For the generator  $A$  of a semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  the following assertions hold: (i) If  $f \in D(A)$  then  $T(t)f \in D(A)$  for

every  $t \geq 0$ . (ii) The map  $t \mapsto T(t)f$  is differentiable on  $\mathbb{R}_+$  if and only if  $f \in D(A)$ . In that case one has

$$\frac{d}{dt}T(t)f = AT(t)f = T(t)Af. \quad (2)$$

(iii) For every  $f \in E$  one has  $\int_0^t T(s)f ds \in D(A)$  and

$$A \int_0^t T(s)f ds = T(t)f - f. \quad (3)$$

(iv) If  $f \in D(A)$  then

$$T(t)f = f + \int_0^t AT(s)f ds = f + \int_0^t T(s)A f ds.$$

(v) The domain  $D(A)$  is dense in  $E$ .

**Theorem 1.** Let  $(A, D(A))$  be the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on the Banach space  $E$ . Then the 'abstract Cauchy problem' (ACP)

$$\frac{d}{dt}\xi(t) = A\xi(t), \quad \xi(0) = f_0,$$

has a unique solution  $\xi : \mathbb{R}_+ \rightarrow D(A)$  in  $C^1(\mathbb{R}_+, E)$  for every  $f_0 \in D(A)$ . In fact, this solution is given by  $\xi(t) := T(t)f_0$ .

For the important relation of semigroups to abstract Cauchy problems we refer to A-II, Section 1. Here we only point out that the above theorem implies that a semigroup is uniquely determined by its generator. While the generator is bounded only for uniformly continuous semigroups (see 2.1 below), it always enjoys a weaker but useful property.

**Definition 4.** An operator  $B$  with domain  $D(B)$  on a Banach space  $E$  is called closed if  $D(B)$  endowed with the graph norm

$$\|f\|_B := \|f\| + \|Bf\|$$

becomes a Banach space. Equivalently,  $(B, D(B))$  is closed if and only if its graph  $\{(f, Bf) : f \in D(B)\}$  is closed in  $E \times E$ , i.e.

$$f_n \in D(B), f_n \rightarrow f \text{ and } Bf_n \rightarrow g \text{ implies } f \in D(B) \text{ and } Bf = g. \quad (4)$$

It is clear from this definition that the 'closedness' of an operator  $B$  depends very much on the size of the domain  $D(B)$ . For example, a bounded and densely defined operator  $(B, D(B))$  is closed if and only if  $D(B) = E$ . On the other hand it may happen that  $(B, D(B))$  is not closed but has a closed extension  $(C, D(C))$ , i.e.,  $D(B) \subset D(C)$  and  $Bf = Cf$  for every  $f \in D(B)$ . In that case,  $B$  is called closable, a property which is equivalent to the following:

$$f_n \in D(B), f_n \rightarrow 0 \text{ and } Bf_n \rightarrow g \text{ implies } g = 0. \quad (5)$$

The smallest closed extension of  $(B, D(B))$  will be called the closure  $\bar{B}$  with domain  $D(\bar{B})$ . In other words, the graph of  $\bar{B}$  is the closure of  $\{(f, Bf) : f \in D(B)\}$  in  $E \times E$ . Finally we call a subset  $D_0$  of  $D(B)$  a core for  $B$  if  $D_0$  is  $\|\cdot\|_B$ -dense in  $D(B)$ . This means that a closed operator is determined (via closure) by its restriction to a core in its domain.

**Proposition 2.** *For the generator  $A$  of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  the following holds: (i) The generator  $A$  is a closed operator. (ii) If a subspace  $D_0$  of the domain  $D(A)$  is dense in  $E$  and  $(T(t))$ -invariant, then it is a core for  $A$ . (iii) Define  $D(A^n) := \{f \in D(A^{n-1}) : Af \in D(A^{n-1})\}$ ,  $D(A^1) = D(A)$ . Then  $D(A^\infty) := \bigcap_{n \in \mathbb{N}} D(A^n)$  is dense in  $E$  and a core for  $A$ .*

**Example 2.** *Property (iii) above does not hold for general densely defined closed operators. Take  $E = C[0, 1]$ ,  $D(B) = C^1[0, 1]$  and  $Bf = q \cdot f$  for some nowhere differentiable function  $q \in C[0, 1]$ . Then  $B$  is closed, but  $D(B^2) = (0)$ .*

**Proposition 3.** *For the generator  $A$  of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  the following holds. If  $\int_0^\infty e^{-\lambda t} T(t) f dt$  exists for every  $f \in E$  and some  $\lambda \in \mathbb{C}$ , then  $\lambda \in \rho(A)$  and  $R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f dt$ . In particular,*

$$R(\lambda, A)^{n+1} f = \frac{(-1)^n}{n!} \left( \frac{d}{d\lambda} \right)^n R(\lambda, A) f = \int_0^\infty e^{-\lambda t} \frac{t^n}{n!} T(t) f dt \quad (6)$$

for  $n \in \mathbb{N}$ ,  $f \in E$  and all  $\lambda$  with  $\operatorname{Re} \lambda > \omega$ .

**Remark 2.** (1) For continuous Banach space valued functions such as  $t \mapsto T(t)f$  we consider the Riemann integral and define  $\int_0^\infty T(t) f dt$  as  $\lim_{t \rightarrow \infty} \int_0^t T(s) f ds$ . Sometimes such integrals for strongly continuous semigroups  $(T(t))_{t \geq 0}$  are written as  $\int_a^b T(t) dt$  and understood in the strong sense. (2) Since the generator  $(A, D(A))$  determines the semigroup  $(T(t))_{t \geq 0}$  uniquely, we will speak occasionally of the growth bound of the generator instead of the semigroup, i.e., we write  $\omega = \omega(A) = \omega((T(t))_{t \geq 0})$ . (3) For one-parameter groups it might seem to be more natural to define the generator as the ‘derivative’ rather than just the ‘right derivative’ at  $t = 0$ . This yields the same operator as the following result shows: The strongly continuous semigroup  $(T(t))_{t \geq 0}$  with generator  $A$  can be extended to a strongly continuous one-parameter group  $(U(t))_{t \in \mathbb{R}}$  if and only if  $-A$  generates a semigroup  $(S(t))_{t \geq 0}$ . In that case  $(U(t))_{t \in \mathbb{R}}$  is obtained as

$$U(t) := \begin{cases} T(t) & \text{for } t \geq 0 \\ S(-t) & \text{for } t \leq 0 \end{cases}$$

We refer to [Davies (1980), Prop. 1.14] for the details.