with domain $D(A'_{max}) = BV[-1,0]$ (the space of all functions of bounded variation on $[-1,0]$). Here we identify $BV[-1,0] \subset L^1[-1,0]$ with a subspace of $C[-1,0]'$ by setting $\langle f,\phi\rangle = \int_{-1}^{0} f(x)\phi(x)\,dx$ ($f \in C[-1,0]$, $\phi \in BV[-1,0]$). For $\phi \in BV[-1,0]$, $d\phi$ denotes the linear form on $C[-1,0]$ given by $f \to \int_{-1}^{0} f(x)\,d\phi(x)$.
We now show (2.16). Let $f \in D(A_{max}) = C^1[-1,0]$, $\phi \in D(A'_{max}) = BV[-1,0]$. By Lemma 2.4 and the chain rule (Prop. 2.3) we have $|f(x)|'$ $(:= d^+/dt_{|t=x}\,|f(t)| = \mathrm{Re}[(\widehat{\mathrm{sign}}\,\bar{f})f'](x)$ (where $f'(x) = (\mathrm{Re} f)'(x) + i(\mathrm{Im} f)'(x)$ in the complex case). Thus
$\langle \mathrm{Re}[(\widehat{\mathrm{sign}}\,\bar{f})Af],\phi\rangle = \int_{-1}^{0} |f(x)|'\phi(x)\,dx = \int_{-1}^{0} \phi(x)\,d|f(x)| =$
$\phi(0)|f(0)| - \phi(-1)|f(-1)| - \int_{-1}^{0} |f(x)|\,d\phi(x) = \langle |f|,A'_{max}\phi\rangle$.

□

Example 2.13. Let A on (the real or complex) space $C[-1,0]$ be given by $Af = f'$ with domain $D(A) = \{f \in C^1[-1,0] : f'(0) = Lf\}$ where $L \in M[-1,0] = C[-1,0]'$. Then A is the generator of a lattice semigroup if and only if $L = \alpha\delta_0$ for some $\alpha \geq 0$.

Proof. Assume that A is the generator of a lattice semigroup $(T(t))_{t\geq 0}$. There exists $\mu \in M[-1,0]$ satisfying $\mu(\{0\}) = 0$ and $\alpha \in \mathbb{R}$ such that $L = \alpha\delta_0 + \mu$. We claim that

(2.18) $|\langle f,\mu\rangle| = \langle |f|,\mu\rangle$ for all $f \in D(A)$ satisfying $f(0) = 0$.

In fact, by the definition of A we have

(2.19) $\delta_0 \in D(A')$ and $A'\delta_0 = L$.

Moreover, by Thm. 2.5, A satisfies Kato's inequality (2.9). Since $f(0) = 0$ this implies
$|\langle f,\mu\rangle| = |f'(0)| = \mathrm{Re}[(\widehat{\mathrm{sign}}\, f)(f')](0)$
$= \langle \mathrm{Re}[(\widehat{\mathrm{sign}}\, f)(Af)],\delta_0\rangle = \langle |f|,A'\delta_0\rangle$ (by (2.9))
$= \langle |f|,\mu\rangle$.
Since $\phi(f) = f'(0) - \langle f,\mu\rangle$ defines a linear form on the space $F = \{f \in C^1[-1,0] : f(0) = 0\}$ which is discontinuous for the supremum norm, the space $D(A) = \ker \phi$ is dense in F and consequently dense in $C_0[-1,0)$. It follows that (2.18) holds for all $f \in C_0[-1,0)$. So by B-I,Sec.2, there exist $\beta \geq 0$ and $x \in [-1,0)$ such that $\mu = \beta\delta_x$. Assume that $\beta \neq 0$. It is easy to see that there exists a real function $f \in C^1[-1,0]$ satisfying $f'(0) = \alpha f(0) + \beta f(x)$ and $f(0)f(x) < 0$. Hence $f \in D(A)$ but $\langle \mathrm{Re}[(\widehat{\mathrm{sign}}\, f)(Af)],\delta_0\rangle =$
$(\mathrm{sign}\, f(0))f'(0) = (\mathrm{sign}\, f(0))(\alpha f(0) + \beta f(x)) =$