The Hille-Yosida theorem

Given an operator A frequently it is easier to obtain information about its resolvent than to solve the Cauchy problem. Therefore the following theorem is central in the theory of one-parameter semigroups.

Theorem 1.7 (Hille-Yosida). Let A be an operator on a Banach space E. The following conditions are equivalent.

(i) A is the generator of a strongly continuous semigroup.
(ii) There exist $w, M \in \mathbb{R}$ such that $(w,\infty) \subset \rho(A)$ and $\|(\lambda-w)^n R(\lambda,A)^n\| \leq M$ for all $\lambda > w$ and $n \in \mathbb{N}$.

In general it is not easy to give an estimate for the powers of the resolvent which enables one to apply Theorem 1.7. However, there is an important case when it suffices to consider merely the resolvent.

Corollary 1.8. For an operator A on a Banach space E the following assertions are equivalent.

(i) A is the generator of a strongly continuous contraction semigroup.
(ii) $(0,\infty) \subset \rho(A)$ and $\|\lambda R(\lambda,A)\| \leq 1$ for all $\lambda > 0$.

The difficult part in the proof of Theorem 1.7. is to show that (ii) implies (i). One has to construct the semigroup out of the resolvent. We mention two formulas which can be used for the proof.

Proposition 1.9. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. For $\lambda > 0$ let $A(\lambda) = \lambda^2 R(\lambda,A) - \lambda \mathrm{Id}$ $(=\lambda AR(\lambda,A))$. Then

(1.2) $T(t)f = \lim_{\lambda\to\infty} e^{tA(\lambda)}f$ for all $f \in E$ and $t \geq 0$.

Yosida's proof consists in showing that the limit in (1.2) exists under the hypothesis (ii) of Theorem 1.2. (see [Davies (1980)], [Goldstein (1985b)] or [Pazy (1982)]).
The proof of Hille (see [Kato (1966)]) is inspired by the following formula.