Proof. The solution of (1.6) is given by

$$(1.8)\quad u(t) = T(t)u_o + \int_0^t T(s)f_o\,ds + \int_0^t T(s)(F(t-s)-f_o)\,ds$$

The first term tends to zero by Cor.1.4. The second term tends to $R(0,A)f_o = -A^{-1}f_o$ by C-III,Thm.1.2. By assumption we have $\lim_{s\to\infty}\|A(F(s)-f_o)\| = 0$ and from Thm.1.3 and A-IV,(1.3) we deduce that $\|T(s)R(0,A)\| \le Me^{-\varepsilon s}$ for $s \ge 0$ and suitable constants $M \ge 1$, $\varepsilon > 0$. Thus for the third term we have

$$\left\|\int_0^t T(s)(F(t-s)-f_o)\,ds\right\| \le \int_0^t \|T(s)R(0,A)\|\,\|A(F(t-s)-f_o)\|\,ds =$$
$$= \int_0^{t/2} \ldots\,ds + \int_{t/2}^t \ldots\,ds\,.$$

The first integral can be estimated by $\sup\{\|A(F(s)-f_o)\| : s \in [\frac{t}{2},t]\}\cdot\int_0^\infty M\cdot e^{-\varepsilon s}\,ds$ while the second integral can be estimated by $\sup\{\|A(F(s)-f_o)\| : s \ge 0\}\cdot\int_{t/2}^t M\cdot e^{-\varepsilon s}\,ds$.
It follows that the third term in (1.8) tends to zero.

□

2. CONVERGENCE OF POSITIVE SEMIGROUPS

by

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The considerations in this section are motivated by the following guideline:

> The asymptotic behavior of a strongly continuous semigroup $(T(t))_{t\ge 0}$ is determined by the (structure, location of the) spectrum $\sigma(A)$ of the generator A .

Unfortunately this principle does not hold in general, e.g., there are semigroups with spectral bound less than zero and growth bound greater than zero (see A-III,Ex.1.3 & 1.4). In order to prove results in the above direction we have to assume additional hypotheses on the semigroup. Positivity may serve to this purpose. For example, the norm convergence to zero, i.e. $\lim_{t\to\infty}\|T(t)\| = 0$, for a positive semigroup on certain Banach lattices is characterized by the condition $s(A) < 0$ (see Thm.1.1). Thus in this case the location of the spectrum determines the norm convergence of the semigroup.
Here we concentrate on the case $s(A) = 0$. At first we observe that