(d) If $T(1)'\phi_o = \phi_o$ for some $\phi_o \in E'_+$ then there exists $\phi \in D(A^*)_+$ with $\{f \in E : \langle |f|,\phi\rangle = 0\} \subseteq \{f \in E : \langle |f|,\phi_o\rangle = 0\}$ such that $A^*\phi = 0$.

The fact that $s(A)$ is always an eigenvalue of the adjoint (cf. B-III Thm.1.6) is characteristic for spaces $C(K)$, K compact, as can be seen considering the Laplacian on $L^p(\mathbb{R}^n)$ where $1 < p < \infty$ or on $C_o(\mathbb{R}^n)$ (see B-III,Ex.1.7). Another result which cannot be extended to arbitrary Banach lattices is that spectral bound and growth bound coincide (cf. B-IV,Thm.1.4); an example is given in A-III,Ex.1.3. Despite of this the resolvent $R(\lambda,A)$ of a positive semigroup is given as the Laplace transform of the semigroup in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > s(A)\}$ (even in case that $\omega(A) > s(A)$). Note however that the integral exists only as an improper Riemann integral. By Datko's Theorem (A-IV,Thm.1.11) the function $t \to e^{-\lambda t}\cdot T(t)f$ cannot be Bochner integrable for all $f \in E$ in case $\operatorname{Re}\lambda \leq \omega(A)$.

<u>Theorem</u> 1.2. Suppose A is the generator of a positive semigroup $(T(t))_{t\geq 0}$. For $\operatorname{Re}\lambda > s(A)$ we have:

$$(1.1)\quad R(\lambda,A)f = \lim_{t\to\infty} \int_0^t e^{-\lambda s}\, T(s)f\, ds \quad \text{for all } f \in E .$$

Moreover, the operators $\int_0^t e^{-\lambda s}\, T(s)\, ds$ tend to $R(\lambda,A)$ with respect to the operator norm as $t \to \infty$.

<u>Proof</u>. We fix $\lambda_o > \omega(A)$. Then by A-I,Prop.1.11

$$(1.2)\quad R(\lambda_o,A)^{n+1}f = \frac{1}{n!}\int_0^\infty s^n \exp(-\lambda_o s)\, T(s)f\, ds \quad (n \in \mathbb{N}_o\ ,\ f \in E)$$

Given μ such that $s(A) < \mu < \lambda_o$, $f \in E_+$, $\phi \in E'_+$ then

$$\begin{aligned}(1.3)\quad \langle R(\mu,A)f,\phi\rangle &= \sum_{n=0}^{\infty} (\lambda_o-\mu)^n \langle R(\lambda_o,A)^{n+1}f,\phi\rangle = \\ &= \sum_{n=0}^{\infty}\int_0^\infty \frac{1}{n!}((\lambda_o-\mu)s)^n \exp(-\lambda_o s)\langle T(s)f,\phi\rangle ds = \\ &= \int_0^\infty \sum_{n=0}^{\infty} \frac{1}{n!}((\lambda_o-\mu)s)^n \exp(-\lambda_o s)\langle T(s)f,\phi\rangle ds = \\ &= \int_0^\infty \exp((\lambda_o-\mu)s)\exp(-\lambda_o s)\langle T(s)f,\phi\rangle ds = \\ &= \int_0^\infty \exp(-\mu s)\langle T(s)f,\phi\rangle ds = \lim_{t\to\infty}\langle \int_0^t \exp(-\mu s)T(s)f ds,\phi\rangle\end{aligned}$$

Note that one can interchange summation and integration because all the integrands are positive functions.
It follows from (1.3) that the net $(\int_0^r \exp(-\mu s)T(s)\ f\ ds)_{r\geq 0}$ converges weakly to $R(\mu,A)f$ for $r \to \infty$. Because it is monotone increasing ($f \geq 0$), we have strong convergence (see the corollary to II.Thm.5.9 in Schaefer (1974)).