convert approximate eigenvectors of the generator A into eigenvectors of A_F .

Proposition. Let A be the generator of a strongly continuous semigroup. Then the generator A_F of the F-product semigroup satisfies

(i) $A\sigma(A) = A\sigma(A_F) = P\sigma(A_F)$.

(ii) $\sigma(A) = \sigma(A_F)$.

Remark: In case A is bounded then A is a generator and $E_F^T = E_F$ (cf. A-I,3.6). Thus the proposition applies to bounded linear operators and their canonical extensions to the F-product E_F .

Proof of the proposition. (i) The inclusion $P\sigma(A_F) \subset A\sigma(A_F)$ holds trivially. We show that $A\sigma(A_F) \subset A\sigma(A)$: Take $\lambda \in A\sigma(A_F)$ and an associated approximate eigenvector $(\hat{f}^m)_{n\in\mathbb{N}}$, i.e. $\hat{f}^m \in D(A_F)$, $\|\hat{f}^m\| = 1$ and $(\lambda - A_F)\hat{f}^m \to 0$ as $m \to \infty$. By the considerations in A-I,3.6 we can represent each $\hat{f}^m$ as a normalized sequence $(f_n^m)_{n\in\mathbb{N}}$ in $D(A)$ such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|(\lambda-A)f_n^m\| = 0 .$$

Therefore we can find a sequence $g_k = f_k^{m(k)}$ satisfying

$$\lim_{k\to\infty} \|(\lambda-A)g_k\| = 0 ,$$

i.e. $\lambda \in A\sigma(A)$.

Finally we show $A\sigma(A) \subset P\sigma(A_F)$: For $\lambda \in A\sigma(A)$ take a corresponding approximate eigenvector (f_n) . By A-I,(3.2) we have

$$\begin{aligned} \|T(t)f_n - f_n\| &\le \|T(t)f_n - e^{\lambda t}f_n\| + |e^{\lambda t} - 1| \\ &= \left\|\int_0^t e^{\lambda(t-s)}T(s)(\lambda-A)f_n\,ds\right\| + |e^{\lambda t} - 1| \end{aligned}$$

which converges to zero uniformly in n as $t \to 0$, i.e. $(f_n) \in m^T(E)$. By the characterization of $D(A_F)$ given in A-I,3.6 it follows that

$$\hat{f} := (f_n) + c_F(E) \in D(A_F) ,$$

and $A_F\hat{f} = \lambda\hat{f}$, i.e. $\lambda \in P\sigma(A_F)$.

(ii) The inclusion $\sigma(A) \subset \sigma(A_F)$ follows from (i) and the inclusion $R\sigma(A) \subset R\sigma(A_F)$: For $\lambda \in R\sigma(A)$ choose $f \in E$ such that $\|(\lambda-A)g - f\| \ge 1$ for every $g \in D(A)$. Then $\|(\lambda-A_F)\hat{g} - \hat{f}\| \ge 1$ for every $\hat{g} \in D(A_F)$ and $\hat{f} = (f,f,\dots) + c_F(E)$. Therefore $\lambda \in R\sigma(A_F)$. We now show $\rho(A) \subset \rho(A_F)$: Assume $\lambda \in \rho(A)$. By (i) $(\lambda-A_F)$ has to be injective. Choose $\hat{f} = (f_1,f_2,\dots) + c_F(E)$ such that $(f_n) \in m^T(E)$. Then $(R(\lambda,A)f_n) \in m^T(E)$ and $(\lambda-A_F)((R(\lambda,A)f_n)+c_F(E)) = (f_n) + c_F(E)$, i.e., $(\lambda-A_F)$ is surjective and $\lambda \in \rho(A_F)$. □