Proof. (i)$\Rightarrow$(ii) holds true because $-A$ is a generator of a semigroup. (ii)$\Rightarrow$(i): We have to show that one (hence each) operator $T(t)$, $t \geq 0$ is invertible. Obviously this is true if ϕ is bijective. At first we assume that ϕ is surjective, that is, $K = K_\infty$. By Thm.4.4 we have that $\phi|_{K_\infty}$ is injective if (ii) is true. Thus ϕ is bijective. Now we assume that ϕ is injective. We have to show that $K = K_\infty$. By Thm.4.4 we have $K_\infty = K_s$ for some s, whenever (ii) is true. Given $x \in K$ then by Lemma 4.2(b) there exists $y \in K_\infty$ such that $\phi(s,x) = \phi(s,y)$. If ϕ is injective we have $x = y \in K_\infty$.

□

In the following example we consider semiflows related to ordinary differential equations on $\mathbb{R}^n$. In case there exists a corresponding global flow, it induces a group on $C_o(\mathbb{R}^n)$ in a canonical way. Even if there is no global flow, one can construct semigroups governed by a semiflow, and apply Thm.4.4(a) in order to describe the spectrum. These examples can be easily extended to differential equations on manifolds (see Sec.18.2 of Dieudonné (1971)).

Example 4.6. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. We denote the maximal flow corresponding to the differential equation $y' = F(y)$ by ϕ_o. In general, ϕ_o is only defined on an open subset of $\mathbb{R} \times \mathbb{R}^n$ which contains $\{0\} \times \mathbb{R}^n$. For $x \in \mathbb{R}^n$ there exist $\underline{t}_x$ and $\bar{t}_x$ such that

$$(4.11)\qquad \begin{aligned} &-\infty \leq \underline{t}_x < 0 < \bar{t}_x \leq \infty\ ; \\ &\phi_o(t,x) \text{ is defined if } \underline{t}_x < t < \bar{t}_x\ ; \\ &\text{if } \bar{t}_x < \infty \ (\underline{t}_x > -\infty) \text{ then } |\phi_o(t,x)| \to \infty \text{ as } t\uparrow\bar{t}_x \ (t\downarrow\underline{t}_x)\ . \end{aligned}$$

For details see Sect.18.2 of Dieudonné (1971)

(a) If ϕ_o is a global flow, i.e., if ϕ_o is defined on $\mathbb{R} \times \mathbb{R}^n$, then one has a corresponding (semi-)group on $C_o(\mathbb{R}^n)$. If F is differentiable, its generator is the closure of A_1 which is defined as follows (cf. B-II,Ex.3.15):

$$(4.12)\qquad \begin{aligned} &A_1 f = (F|\operatorname{grad} f) := \sum F_i \cdot \partial_i f \text{ with domain} \\ &D(A_1) := \{f \in C^1 : \operatorname{supp} f \text{ is compact}\}\ . \end{aligned}$$

ϕ_o can be uniquely extended to a flow $\tilde{\phi}_o$ on $\mathbb{R}^n \cup \{\infty\}$ by defining $\tilde{\phi}_o(t,\infty) := \infty$ for all $t \in \mathbb{R}$. ϕ_o and $\tilde{\phi}_o$ satisfy condition (c) of Thm.4.4 .