Then B is the generator of a unitary group $(U(t))_{t\in\mathbb{R}}$. In particular, B is skew-adjoint, i.e. $B' = -B$.

Moreover, we claim that

(3.18) B' has a strictly positive subeigenvector ϕ .

Proof. Let $\lambda > 0$ and $\phi \in C^3(\mathbb{R})$ such that $\phi(x) = e^{-|x|}$ for $|x| \geq 1$, $\phi(x) > 0$ for all $x \in \mathbb{R}$, $\phi(0) = 1$ and $\phi'(0) = \phi''(0) = 0$. Then $\phi \in D(B')$. Moreover, $-\phi^{(3)}(x) \leq \phi(x)$ for $|x| \geq 1$. Hence there exists $\mu \geq 1$ such that $B'\phi = -\phi^{(3)} \leq \mu\phi$.

□

But the semigroup $(U(t))_{t\geq 0}$ is not positive. In fact, we show that there exists $f \in D(B)$ such that

(3.19) $$\langle(\operatorname{sign} f)Bf,\phi\rangle > \langle|f|,B'\phi\rangle .$$

Proof. Let $f \in D(B)$ be such that $f(x) = e^{-x}\sin x$ in a neighborhood of 0, while $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$. Then

$$\langle(\operatorname{sign} f)Bf,\phi\rangle = -\int_{-\infty}^{0} f^{(3)}(x)\phi(x)\,dx + \int_{0}^{\infty} f^{(3)}(x)\phi(x)\,dx.$$

Hence,
$$\begin{aligned}\langle|f|,B'\phi\rangle &= \int_{-\infty}^{0} (-f(x))(-\phi^{(3)}(x))\,dx + \int_{0}^{\infty} f(x)(-\phi^{(3)}(x))\,dx \\ &= -\int_{-\infty}^{0} f^{(3)}(x)\phi(x)\,dx + \int_{0}^{\infty} f^{(3)}(x)\phi(x)\,dx \\ &\quad + [f''\phi]\big|_{-\infty}^{0} - [f''\phi]\big|_{0}^{\infty} \qquad (\text{since } \phi''(0)=\phi'(0)=0) \\ &= \langle(\operatorname{sign} f)\,Bf,\phi\rangle + 2f''(0)\phi(0) \\ &< \langle(\operatorname{sign} f)\,Bf,\phi\rangle \qquad (\text{since } f''(0)\phi(0) = f''(0) = -2).\end{aligned}$$

□

We now show that B satisfies Kato's inequality for positive elements, though; i.e.,

(3.20) $$P_f\, Bf \leq Bf \quad \text{for all } f \in D(B)_+ .$$

In fact, more is true. B is local, i.e.

(3.21) $$f \perp g \text{ implies } Bf \perp g \quad \text{for all } f \in D(B),\ g \in L^2(\mathbb{R}) .$$

Proof. Let A be the generator of the translation group which, in particular, is a lattice semigroup (see Section 5). We obtain from Proposition 5.4 below that A is local. Hence $B = A^3$ is local as well. □