If we define $u := |h|$ and $w := h/|h|$ , then (3.12) reads

$$(3.13)\qquad L_o(uw) = i\alpha uw\ ,\ L_o(u) = 0\ ,\ L_o(u/w) = -i\alpha\cdot u/w\ .$$

Explicit calculation of $L_o(uw)$ and $L_o(u/w)$ using the product formula for differentation yields (as above we write $f_i'$ instead of $\partial f/\partial x_i$):

$$(3.14)\qquad \begin{aligned} L_o(uw) &= wL_o(u) + u\sum_{i,j} a_{ij}w_{ij}'' + \sum_i \left(ub_i + \sum_j a_{ij}u_j'\right)w_i' \\ L_o(u/w) &= 1/w\cdot L_o(u) + u\sum_{i,j}a_{ij}(1/w)_{ij}'' + \sum_i\left(ub_i + \sum_j a_{ij}u_j'\right)(1/w)_i' \end{aligned}$$

Observing that $(1/w)_i' = -w^{-2}\cdot w_i'$ and $(1/w)_{ij}'' = w^{-3}\cdot(2w_i'w_j' - ww_{ij}'')$ we obtain:

$$(3.15)\qquad L_o(uw) + w^2L_o(u/w) = 2wL_o(u) + 2u/w\cdot\sum_{ij} a_{ij}w_i'w_j'\ .$$

This identity and (3.13) implies that $2u/w\cdot\sum_{ij} a_{ij}w_i'w_j' = 0$ . Since $u$ has no zeros and $(a_{ij})$ is positive definite, it follows that $\operatorname{grad} w = (w_i') = 0$ in $\Omega$ , hence $w = \text{const.}$ . Then by (3.13) we have $i\alpha uw = L_o(uw) = wL_o(u) = 0$ , a contradiction.

The assumption that $L_o$ is elliptic, i.e., that $(a_{ij})$ is positive definite, is essential in order to show that there is only one eigenvalue in the boundary spectrum. In the following example $(a_{ij})$ is positive semi-definite and $P\sigma_b(A) = s(A) + i\alpha\mathbb{Z}$ .

(c) We consider $\Omega = \{(x,y) \in \mathbb{R}^2 : 1 < (x^2 + y^2)^{1/2} < 2\}$ , and the second order differential operator $L_o$ given by
$(L_of)(x,y) = 1/(x^2 + y^2)\cdot(x^2f_{xx} + 2xyf_{xy} + y^2f_{yy}) + (xf_y - yf_x)$ .
The assertion concerning the boundary spectrum can be verified easily by using polar coordinates: $x = r\cdot\cos\omega$ , $y = r\cdot\sin\omega$ . Then $L_o$ becomes $L_of = f_{rr} + f_\omega$ on the space $C_o(1,2) \otimes C_{2\pi}(\mathbb{R})$ .

In this section we have seen that the eigenvalues in the boundary spectrum of an irreducible semigroup form a subgroup of $i\mathbb{R}$ (provided that $s(A) = 0$). We conclude this section mentioning an analogous statement for the whole boundary spectrum of Markov semigroups on $C(K)$ , $K$ compact. It seems to be unknown if this is true for irreducible semigroups in general. To prove this result one uses the proof of the analogous result for a single operator (cf. Schaefer (1968), Thm.7) as a guideline.

Theorem 3.11. Suppose that $\mathcal{T}$ is an irreducible semigroup of Markov operators on $C(K)$ , $K$ compact. Then $\sigma_b(A)$ is a subgroup of $i\mathbb{R}$ . Hence either $\sigma_b(A) = \{0\}$ or $= i\mathbb{R}$ or $= i\alpha\mathbb{Z}$ for some $\alpha > 0$ .