Dirichlet problem; i.e. $v_{|\partial\Omega} = g$ and $\Delta v = 0$ in $\Omega$. Then $u := w - v \in D(\Delta_0)$ and $\Delta u = f$.

We have shown that $\Delta_0$ satisfies condition (b) of Theorem 4.2. Thus $\Delta_0$ generates a positive, contractive $C_0$-semigroup $(T(t))_{t \geq 0}$ on $C_0(\Omega)$ and $s(\Delta_0) < 0$.

Since by C-IV Theorem 1.1 (i) $s(\Delta_0) = \omega_0(\Delta_0)$, it is exponentially stable.

We refer to Arendt and Bénilan [ArBe 99] for the proof of (b) ⇒ (a).

$\square$

We want to add two further comments on the Dirichlet Laplacian $\Delta_0$ on $C_0(\Omega)$.
The first concerns its domain

$$D(\Delta_0) = \{ u \in C_0(\Omega) : \Delta u \in C_0(\Omega) \}$$

This distributional domain is not contained in $C^2(\Omega)$ for any open set $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, see Arendt - Urban [AU 24, Theorem 6.60].

Our second comment concerns the proof of holomorphy. It can be given via Gaussian estimates (see the Extended Notes for C-II)
In our context, a short proof based on Kato's inequality of C-II, Section 2 is

more appealing (see Arendt - Batty [ArBa 92]).

Finally, we comment on irreducibility. On $C_0(\Omega)$ it is a strong property. By C-III, Theorem 3.2 (ii) it means that for $0 \leq f \in C_0(\Omega)$, $f \neq 0$,

$$(T(t)f)(x) > 0 \quad \text{for all } x \in \Omega,\ t > 0.$$

On $L^2(\Omega)$ irreducibility is much weaker (meaning that $(T(t)f)(x) > 0$ $x$-a.e.), but easy to prove (see the Extended Notes to C-II).

In the paper Arendt, ter Elst, Glück [AEG 20] an argument based on Banach lattice techniques shows how irreducibility on $L^2(\Omega)$ can be carried over to $C_0(\Omega)$ or even to $C(\bar{\Omega})$ in the case of Robin boundary conditions which we consider now.

By $H^1(\Omega) := \{ u \in L^2(\Omega) : \partial_j u \in L^2(\Omega)$ for $j = 1, \dots, n \}$

we denote the first Sobolev space.

We assume that $\Omega$
has Lipschitz boundary. Then
there exists a unique bounded
operator $\quad tr : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$
such that $\quad tr\, u = u_{|\partial\Omega}$ for all
$u \in C^1(\overline{\Omega})$. It is called the
<u>trace operator</u>.

Here the space $L^2(\partial\Omega)$
is defined with
respect to the surface measure
(i.e. the $(d-1)$-dimensional
Hausdorff measure) on $\partial\Omega$.

The <u>normal derivative</u> $\partial_\nu u$ of $u$ is defined as follows.
Let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Let $h \in L^2(\partial\Omega)$.
We say that $h$ is the <u>(outer) normal derivative</u> of $u$ and write $\partial_\nu u = h$ if

$$\int_\Omega \Delta u \, v + \int_\Omega \nabla u \nabla v = \int_{\partial\Omega} h v$$

for all $v \in C^1(\overline{\Omega})$.
If $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$ we say $\partial_\nu u \in L^2(\partial\Omega)$ if there exist $h \in L^2(\partial\Omega)$ such that $\partial_\nu u = h$.

<u>Remark</u>: Since $\Omega$ has Lipschitz boundary the outer normal $\nu(z)$ exists for almost all $z \in \partial\Omega$ and $\nu \in L^\infty(\partial\Omega)$. But we do not use this outer normal and rather define $\partial_\nu u$ weakly by the validity of

Green's formula.

Let $\beta \in L^\infty(\Omega)$. We define the Laplacian $\Delta^\beta$ with Robin boundary conditions as follows

$$D(\Delta^\beta) := \{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \partial_\nu u + \beta \operatorname{tr}(u) = 0 \}$$

$$\Delta^\beta u := \Delta u .$$

We call $\Delta^\beta$ briefly the <u>Robin-Laplacian</u>. Note that for $\beta = 0$, we obtain <u>Neumann boundary conditions</u>, and $\Delta^N := \Delta^0$ is the <u>Neumann Laplacian</u>.

The following result is valid.

<u>Theorem</u> 4.3. Assume that $\Omega \subset \mathbb{R}^d$ is bounded, open, connected with Lipschitz boundary and let $\beta \in L^\infty(\partial\Omega)$. Then $\Delta^\beta$ generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on $C(\bar{\Omega})$.
Moreover, $T(t)$ is compact for all $t > 0$.

The generation property on $C(\bar{\Omega})$ is due to Nittka [Ni11]. A major point is to show that the resolvent of the corresponding operator on $L^2(\Omega)$ leaves $C(\bar{\Omega})$ invariant. Given $f \in C(\bar{\Omega})$, $u \in H^1(\Omega)$ such that

$$u - \Delta u = f \;, \quad \partial_\nu u + \beta u_{|\partial\Omega} = 0.$$