$D(A_1) = \{f \in E_1 : Af \in E_1\} = D(A^2)$ and

$A_1 f = Af$ for every $f \in D(A_1)$.

It is now possible to repeat this construction in order to obtain Banach spaces E_n and semigroups $(T_n(t))_{t\geq 0}$ with generators $(A_n, D(A_n))$ which are related as visualized in the following diagram:

$$\begin{tikzcd}
E \arrow[r, "T(t)"] & E & & & \\
E_1 \arrow[u, "i_1"] \arrow[ur, "A"] \arrow[r, "T_1(t)"] & E_1 \arrow[u, "i_1"'] & = & D(A) & & \\
E_2 \arrow[u, "i_2"] \arrow[ur, "A_1"] \arrow[r] & E_2 \arrow[u, "i_2"'] & = & D(A_1) & = & D(A^2) \\
{} \arrow[u, "i_3"] \arrow[ur] & {} \arrow[u, "i_3"'] & & & &
\end{tikzcd}$$

For the translation semigroup on $L^p(\mathbb{R})$ (see 2.3) the above construction leads to the usual 'Sobolev spaces'. Therefore we might call E_n the <u>n-th Sobolev space</u> and $(T_n(t))_{t\geq 0}$ the <u>n-th Sobolev semigroup</u> associated to E and $(T(t))_{t\geq 0}$.

Remarks: 1. For $\lambda \in \rho(A)$ the operator $(\lambda - A)$ and the resolvent $R(\lambda, A)$ are isomorphisms from E_1 onto E , resp. from E onto E_1 (show that $\|\cdot\|_1$ and $\|\cdot\|_\lambda$ with $\|\cdot\|_\lambda := \|(\lambda - A)\cdot\|$ are equivalent). In addition, the diagram

$$\begin{tikzcd}
E \arrow[r, "T(t)"] & E \\
E_1 \arrow[u, "\lambda - A"] \arrow[r, "T_1(t)"] & E_1 \arrow[u, "R(\lambda,A)"']
\end{tikzcd}$$

commutes. Therefore all Sobolev semigroups $(E_n, (T_n(t))_{t\geq 0})$, $n \in \mathbb{N}$, are isomorphic.

2. For $\lambda \in \rho(A)$ consider the norm

$\|f\|_{-1} := \|R(\lambda, A) f\|$

for every $f \in E$ and define E_{-1} as the completion of E for $\|\cdot\|_{-1}$.