Characterization of Positive

Since the positive cone of a C*-algebra has non-empty interior many results of Chapter B-II can be applied verbatim to the characterization of the generator of positive semigroups on C^*. Other hand a concrete and detailed representation of such generators has been found only in the uniformly continuous case (see Lindblad (1976)). A third area of active research has been the following: Which maps on C^*-algebras (in particular, which derivations) commuting with certain automorphism groups are automatically generators of strongly continuous positive semigroups. For more informations we refer to the survey article of Evans (1984).

1 Positive Semigroups on Properly Infinite W^*-Algebras

The aim of this section is to show that strongly continuous semigroups of schwarz maps on properly infinite W^*-algebras are already uniformly continuous. In particular, our theorem is applicable to such semigroups on $B(H)$. It is worthwhile to remark, that the result of Lotz (1985) on the uniformly continuity of every strongly continuous semigroup on L^m (see A-II, sec.3) does not extend to arbitrary W^* algebras. For example, take $M = B(H)$, H infinite dimensional, and choose a projection $p \in M$ such that Mp is topologically isomorphic to H. Therefore $M = H \oplus M_o$, where $M_o = ker(x \to xp)$. Next take a strongly, but not uniformly continuous, semigroup S on H and consider the strongly continuous semigroup $S \oplus \mathrm{Id}$ on M. Our approach we refer to [Sakai (1971), 2.2] and [Takesaki (1979), V.1].

Theorem 1.1. *Every strongly continuous one-parameter semigroup of Schwarz type on a properly infinite W^*-algebra M is uniformly continuous.*

Proof. Let $T = (T(t)_*, t \geq 0)$ be strongly continuous on M and suppose T not to be uniformly continuous. Then there exists a sequence $(T_n) \subset T$ and $\varepsilon > 0$ such that $\|T_n - \mathrm{Id}\| \geq \varepsilon$ but $T_n \to \mathrm{Id}$ in the strong operator topology. We claim that for every sequence (p_k) of mutually orthogonal projections and all bounded sequences (x_k) in M

$$\lim_n \|(T_n - \mathrm{Id})(p_k x_k p_k)\| = 0$$

uniformly in $k \in \mathbb{N}$. This follows from an application of the Lemma of Phillips and the fact that the sequence $(p_k x_k p_k)$ is sumable in the $\sigma^*(M, M_*)$-topology (compare Elliot (1972)).

Let (p_k) be a sequence of mutually orthogonal projections in M such that every p_k is equivalent to 1 via some $u_k \in M$ [Sakai (1971), 2.2]. Without loss of generality we may assume $\|(T_n - \mathrm{Id})(u_n)\| \leq n^{-1}$ since the semigroup T is strongly continuous. Thus we obtained the following:

(i) $\lim_n \|(T_n - \mathrm{Id})(p_k x_k p_k)\| = 0$ uniformly in $k \in \mathbb{N}$ for every bounded sequence (x_k) in M.

(ii) Every projection p_k is equivalent to 1 via some $u_k \in M$.

(iii) $\|(T_n - \mathrm{Id})u_n\| \leq n^{-1}$ for all $n \in \mathbb{N}$.

For the following construction see A-I, 3.6 and D-II, Sec.2. Let $\hat{M}$ be an ultrapower of M, let $p := (p_k)^\wedge \in \hat{M}$, $T := (T_n)^\wedge \in (\hat{M})$ and $u := (u_k)^\wedge \in \hat{M}$. Then T is identity preserving and of schwarz type on $\hat{M}$. Since $u^*u = p$ and $uu^* = 1$, it follows $pu^* = u^*$ and $(uu^*)x(uu^*) = x$ for all $x \in \hat{M}$. Finally, $T(pxp) = pxp$ for all $x \in \hat{M}$, which follows from (1), and $T(u^*) = T(pu^*) = pu^* = u^*$ and $T(u) = u$, which follows from (3). Using the schwarz inequality we obtain

$$T(uu^*) = T(1) \leq 1 = uu^* = T(u)T(u)^*.$$

Using D-II, Lemma 1.1 we conclude $T(ux) = uT(x)$ and $T(xu^*) = T(x)u^*$ for all $x \in \hat{M}$. Hence $T(x) = T(uu^*xuu^*) = uT(u^*xu)u^* = uT(pu^*xup)u^* = upu^*xuu^* = uu^*xuu^* = x$ for all $x \in \hat{M}$. From this we obtain that for every bounded sequence (x_k) in M: $\lim_n \|(T_n - \mathrm{Id})(x_k)\| = 0$ uniformly in k and independently of the x_k's. This conflicts with our assumption at the beginning, hence the theorem is proved. □