(iii) There exists a closed subgroup G of K such that

$$N \cong L^{\infty}({}^{K}/_{G}, dm_{/}) \otimes R \ ,$$

where R is as in (ii) and $dm_{/}$ the normalized Haar measure on ${}^{K}/_{G}$ [l.c., Theorem 5.15].

So far we have studied weak*-semigroups on general W*-algebras. We now want to apply the results of this section to weak*-semigroup on $B(H)$. This is of interest in view of the results in [Davies (1976)]. To do this we call a triple (M,Φ,T) a W*-dynamical system if M is a W*-algebra, a weak*-semigroup of identity preserving Schwarz maps on M and Φ a faithful family of T-invariant normal states. We call (M,Φ,T) irreducible, if the preadjoint semigroup is irreducible (alternatively, if the fixed space of T is one dimensional).

Proposition 3.7. Let $(B(H),\Phi,T)$ be a W*-dynamical system on the W*-algebra $B(H)$ of all bounded linear operators on a Hilbert space H . Then the following assertions are equivalent:

(a) $P\sigma(A) \cap i\mathbb{R} = \{0\}$.

(b) $\lim_{s\to\infty} T(s)_* = P_*$ in the strong operator topology on $L(B(H)_*)$.

Proof. Obviously (b) implies (a). Suppose that (a) is fulfilled. Then the ergodic projection P_* of the preadjoint semigroup is equal to the associated semigroup projection. Consequently there exists an ultrafilter U on $\mathbb{R}_+$ such that $\lim_U T(t) = P$ in the weak operator topology. We claim that the convergence holds even in the strong operator topology. Taking this for granted it follows, since for every $t\in\mathbb{R}_+$ $T(t)$ is a contraction, that

$$\lim_{t\to\infty} \|T(t)_*\phi\| = 0$$

for all $\phi\in\ker(P_*)$. Since $T(t)_*\psi = \psi$ for every $\psi\in\mathrm{im}(P_*)$ and because

$$B(H)_* = \mathrm{im}(P_*) \oplus \ker(P_*)$$

the assertion is proved.

It remains to show that $\lim_U T(t)_* = P_*$ in the strong operator topology. Choose $0\le\phi\in B(H)_*$, $\|\phi\|\le 1$, let $\phi_t := T(t)_*\phi$ $(t>0)$,