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# One-parameter Semigroups of Positive Operators

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**Part A**  
**Extended Notes**



## Chapter A-II

## Extended Notes: Chapter B-II

### 1 Positive semigroups generated by elliptic operators on spaces of continuous functions

Important examples of semigroups on  $C_0(\Omega)$  or  $C(\overline{\Omega})$ , where  $\Omega \subset \mathbb{R}^n$  is open and bounded, are generated by elliptic differential operators. In the following we put together a series of results starting with the Laplacian subject to Dirichlet and to Robin boundary conditions and ending with the Dirichlet-to-Neumann operator on  $C(\partial\Omega)$ . Each time we obtain a positive irreducible semigroup. We consider  $\mathbb{K} = \mathbb{R}$  throughout this section.

#### 1.1 The Laplacian

Let  $\Omega \subset \mathbb{R}^d$  be open and bounded. We say that  $\Omega$  is *Dirichlet-regular* if for every  $g \in C(\partial\Omega)$  there exists a (unique) function  $u \in C(\overline{\Omega}) \cap C^2(\Omega)$  such that

$$\begin{aligned}\Delta u &= 0 \quad \text{and} \\ u|_{\partial\Omega} &= g.\end{aligned}$$

This means that the Dirichlet problem is well-posed. This property is very well understood and precise characterizations in terms of barriers or of capacity are known. If  $\Omega$  has Lipschitz boundary, then  $\Omega$  is Dirichlet regular. In dimension  $d = 2$  it suffices that  $\Omega$  is simply connected.

We refer to Arendt and Urban [9, Section 6.9] or Gilbarg and Trudinger [19, Section 2.8] for further information on the Dirichlet Problem.

The Dirichlet Laplacian  $\Delta_0$  on  $C_0(\Omega)$  is defined by

$$\begin{aligned}\Delta_0 u &:= \Delta u \\ D(\Delta_0) &:= \{u \in C_0(\Omega) : \Delta u \in C_0(\Omega)\}.\end{aligned}$$

Here  $\Delta u$  is to be understood in the sense of distributions.

In this section we consider always real spaces. Then a semigroup is called *holomorphic* if its extension to the coresponding complexification (here  $C_0(\Omega, \mathbb{C})$ ) is holomorphic.

**Theorem 1.1** *The following are equivalent.*

- (a)  $\Omega$  is Dirichlet regular;
- (b)  $\Delta_0$  generates a positive semigroup  $\mathcal{T}$  on  $C_0(\Omega)$ .

*In that case the semigroup  $\mathcal{T}$  is holomorphic of angle  $\pi/2$ . Moreover  $T(t)$  is compact for all  $t > 0$ . If  $\Omega$  is connected, then the semigroup is irreducible. Moreover,*

$$\|T(t)\| \leq M e^{-\epsilon t} \quad (t \geq 0) \tag{1.1}$$

*for some  $\epsilon > 0$ ,  $M \geq 1$ .*

This result is due to Arendt-Bénilan [2] besides irreducibility on which we comment later. In Example C-II.1.5 (e), the generation result was obtained if  $\Omega$  has  $C^2$ -boundary.

The implication (a)  $\Rightarrow$  (b) of Theorem 4.1 is proved below in order to show how the Dirichlet problem comes into play and leads to a result with minimal regularity assumptions on the boundary of  $\Omega$ .

We use the following abstract generation result which is of independent interest. By C-II, Theorem 1.2 a densely defined operator  $A$  generates a contractive positive semigroup if and only if  $A$  is dispersive and  $(\lambda - A)$  is surjective for some  $\lambda > 0$ . We now describe the case  $\lambda = 0$ .

**Theorem 1.2** *Let  $A$  be a densely defined operator on a real or complex Banach lattice  $E$ . The following are equivalent.*

- (a)  $A$  generates a positive, contractive semigroup and  $s(A) \leq 0$ .
- (b)  $A$  is dispersive and surjective.

*In particular, (b) implies that  $A$  is closed.*

Dispersive operators are defined before C-II, Theorem 1.2. A densely defined operator  $A$  on  $C_0(\Omega)$  is dispersive iff for  $u \in D(A)$ ,  $x_0 \in \overline{\Omega}$ :

$$u(x_0) = \sup_{x \in \overline{\Omega}} u(x) > 0 \text{ implies } (Au)(x_0) \leq 0.$$



**Proof** (Theorem 4.2.) (b)  $\Rightarrow$  (a)

Consider the equivalent norm

$$\|u\|_1 := \|u^+\| + \|u^-\|$$

on  $E$ . Since  $A$  is dispersive it is dissipative with respect to this new norm as is easy to see. Now Theorem 4.5 of Arendt, Chalendar and Moletsare [11] implies that  $A$  generates a contraction semigroup  $\mathcal{T}$  and  $A$  is invertible. Since  $A$  is dispersive, it follows from C-II, Theorem 1.2 that  $\mathcal{T}$  is positive and contractive (with respect to the original norm). Since  $R(\lambda, A) \geq 0$  for  $\lambda > 0$ , it follows that  $-A^{-1} \geq 0$ . Now C-I, Theorem 1.1 (vi) implies that  $s(A) \leq 0$ .

(a)  $\Rightarrow$  (b) is obvious from C-II, Theorem 1.2.

**Proof** (Theorem 4.1.) (a)  $\Rightarrow$  (b)

The operator  $\Delta_0$  is dispersive by the maximum principle. If  $\Omega$  is Dirichlet regular, then  $\Delta_0$  is surjective. In fact, let  $f \in C_0(\Omega)$ . Extend  $f$  by 0 to  $\mathbb{R}^n$  and let  $w = \Gamma * f$ , where  $\Gamma$  is the fundamental solution of Laplace's equation (see Gilbarg and Trudinger [19, 2.12]). Then  $w \in C(\mathbb{R}^n)$  and  $\Delta w = f$  in the sense of distributions. Let  $g = w|_{\partial\Omega}$  and let  $v \in C^2(\overline{\Omega}) \cap C(\overline{\Omega})$  be the solution of the Dirichlet problem, i.e.,

$$v|_{\partial\Omega} = g \quad \text{and} \quad \Delta v = 0 \text{ in } \Omega.$$

Then  $u := w - v \in D(\Delta_0)$  and  $\Delta u = f$ .

We have shown that  $\Delta_0$  satisfies condition (b) of Theorem 4.2. Thus  $\Delta_0$  generates a positive, contractive  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $C_0(\Omega)$  and  $s(\Delta_0) \leq 0$ . Since by C-IV Theorem 1.1 (iv)  $s(\Delta_0) = \omega_0(\Delta_0)$ , it is exponentially stable.

We refer to Arendt and Bénylan [2] for the proof of (b)  $\Rightarrow$  (a). □

We want to add two further comments on the Dirichlet Laplacian  $\Delta_0$  on  $C_0(\Omega)$ . The first concerns its domain

$$D(\Delta_0) = \{u \in C_0(\Omega) : \Delta u \in C_0(\Omega)\}.$$

This distributional domain is not contained in  $C^2(\Omega)$  for any open set  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , see Arendt-Urban [9, Theorem 6.60].

Our second comment concerns the proof of holomorphy. It can be given via Gaussian estimates (see the Extended Notes for C-II). In our context, a short proof based on Kato's inequality of C-II, Section 2 is more appealing (see Arendt-Batty [1]).

Finally, we comment on irreducibility. On  $C_0(\Omega)$  it is a strong property. By C-III, Theorem 3.2 (ii) it means that for  $0 \leq f \in C_0(\Omega)$ ,  $f \neq 0$ ,

$$(T(t)f)(x) > 0 \text{ for all } x \in \Omega, t > 0.$$

On  $L^2(\Omega)$  irreducibility is much weaker (meaning that  $(T(t)f)(x) > 0$   $x$ -a.e.), but easy to prove (see the Extended Notes to C-I). In the paper Arendt, ter Elst, Glück [10] an argument based on Banach lattice technique shows how irreducibility on  $L^2(\Omega)$  can be carried over to  $C_0(\Omega)$  or even to  $C(\overline{\Omega})$  in the case of Robin boundary conditions which we consider now.

By

$$H^1(\Omega) := \{u \in L^2(\Omega) : \partial_j u \in L^2(\Omega) \text{ for } j = 1, \dots, n\}$$

we denote the first Sobolev space. We assume that  $\Omega$  has Lipschitz boundary. Then there exists a unique bounded operator

$$\text{tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega)$$

such that  $\text{tr}(u) = u|_{\partial\Omega}$  for all  $u \in C^1(\overline{\Omega})$ . It is called the *trace operator*.

Here the space  $L^2(\partial\Omega)$  is defined with respect to the surface measure (i.e. the  $(d-1)$ -dimensional Hausdorff measure) on  $\partial\Omega$ .

The normal derivative  $\partial_\nu u$  of  $u$  is defined as follows. Let  $u \in H^1(\Omega)$  such that  $\Delta u \in L^2(\Omega)$ . Let  $h \in L^2(\partial\Omega)$ . We say that  $h$  is the (*outer*) *normal derivative* of  $u$  and write  $\partial_\nu u = h$  if

$$\int_{\Omega} \Delta u v + \int_{\Omega} \nabla u \nabla v = \int_{\partial\Omega} h v$$

for all  $v \in C^1(\overline{\Omega})$ .

If  $u \in H^1(\Omega)$  such that  $\Delta u \in L^2(\Omega)$  we say  $\partial_\nu u \in L^2(\partial\Omega)$  if there exists  $h \in L^2(\partial\Omega)$  such that  $\partial_\nu u = h$ .

**Remark** Since  $\Omega$  has Lipschitz boundary the outer normal  $\nu(z)$  exists for almost all  $z \in \partial\Omega$  and  $\nu \in L^\infty(\partial\Omega)$ . But we do not use this outer normal and rather define  $\partial_\nu u$  weakly by the validity of Green's formula.

Let  $\beta \in L^\infty(\partial\Omega)$ . We define the Laplacian  $\Delta^\beta$  with Robin boundary conditions as follows:

$$D(\Delta^\beta) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \partial_\nu u + \beta \text{tr}(u) = 0\} \quad (1.2)$$

$$\Delta^\beta u := \Delta u. \quad (1.3)$$

We call  $\Delta^\beta$  briefly the Robin-Laplacian. Note that for  $\beta = 0$ , we obtain Neumann boundary conditions, and  $\Delta^0 =: \Delta^N$  is the Neumann Laplacian.

The following result is valid.

**Theorem 1.3 (4.3)** Assume that  $\Omega \subset \mathbb{R}^d$  is bounded, open, connected with Lipschitz boundary, and let  $\beta \in L^\infty(\partial\Omega)$ . Then  $\underline{\Delta}^\beta$  generates a positive, irreducible, holomorphic semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  on  $C(\overline{\Omega})$ . Moreover,  $T(t)$  is compact for all  $t > 0$ .

The generation property on  $C(\overline{\Omega})$  is due to Nittka [22]. A major point is to show that the resolvent of the corresponding operator on  $L^2(\Omega)$  leaves  $C(\overline{\Omega})$  invariant. Given  $f \in C(\overline{\Omega})$ ,  $u \in H^1(\Omega)$  such that  $u - \Delta u = f$ ,  $\partial_\nu u + \beta u|_{\partial\Omega} = 0$ .

**Theorem 1.4 (4.4)** *Assume (4.2) and (4.4). Then  $\Delta - V$  generates a positive, irreducible semigroup on  $C(\partial\Omega)$ . If  $V \geq 0$ , then the semigroup is contractive.*

If  $\Omega$  is of class  $C^\infty$  similar results have been obtained by Escher [18] and Engel [17]. Under the very general conditions here, Theorem 4.8 is due to Arendt and ter Elst [8]. There it is shown that  $N_V$  is resolvent-

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One has to show that  $u \in C(\overline{\Omega})$ . Nittka extends  $u$  to an open set  $\tilde{\Omega}$  containing  $\overline{\Omega}$  by reflecting  $u$  along the graph. Then  $u$  becomes the solution of an elliptic problem on  $\tilde{\Omega}$ . Continuity on  $\tilde{\Omega}$ , and hence on  $\overline{\Omega}$ , follows from the De Giorgi-Nash Theorem.

Irreducibility is due to Arendt, ter Elst and Glück [10], Theorem 4.5. These results have first been proved for  $\beta \geq 0$ . Daves [15] has shown how one can treat general  $\beta \in L^\infty(\partial\Omega)$ .

Since the semigroup is holomorphic, by C-II, Theorem 3.2 (ii), it implies that

$$\inf_{x \in \overline{\Omega}} (T(t)f)(x) > 0 \quad (4.1)$$

for all  $t > 0$  and  $0 \leq f \in C(\overline{\Omega})$ ,  $f \neq 0$ .

Denote by  $s(\Delta^\beta)$  the spectral bound of  $\Delta^\beta$ . By C-III, Theorem 3.8 (iv),  $s(\Delta^\beta)$  is the unique eigenvalue with a positive eigenfunction  $u_0 \geq 0$ ,  $u_0 \neq 0$ . It follows from (4.1) that  $u_0$  is strictly positive; i.e.,

$$\inf_{x \in \overline{\Omega}} u_0(x) > 0,$$

a remarkable property, which has important applications to semi-linear problems, see Arendt-Daners [3].

The spectral bound  $s(\Delta^\beta)$  determines the asymptotic behavior of the semigroup  $\mathcal{T}$ . In fact, the following corollary follows from B-III Proposition 3.5.

**Corollary 1.5** *There exist a strictly positive Borel measure  $\mu$  on  $\overline{\Omega}$ ,  $M \geq 0$  and  $\epsilon > 0$  such that  $\langle \mu, u_0 \rangle = 1$  and*

$$\|T(t) - e^{s(\Delta^\beta)t} P\| \leq M e^{-\epsilon t}$$

for all  $t \geq 0$ , where  $P \in \mathcal{L}(C(\overline{\Omega}))$  is given by

$$Pf = \langle \mu, f \rangle u_0.$$

The theorem says that the rescaled semigroup  $(e^{-s(\Delta^\beta)t}T(t))_{t \geq 0}$  converges in the operator norm to the rank-1-projection  $P$  exponentially fast.

## 1.2 Elliptic operators in divergence form

The preceding results extend to elliptic operators in divergence form with bounded measurable coefficients. Let  $\Omega \subset \mathbb{R}^n$  be open and bounded.

Let  $a_{k,\ell}, b_k, c_k, c_0 \in L^\infty(\Omega)$ ,  $k, \ell = 1, \dots, n$  such that for some  $\alpha > 0$

$$\sum_{k,\ell=1}^n a_{k,\ell}(x) \xi_k \xi_\ell \geq \alpha |\xi|^2$$

for all  $x \in \Omega$ ,  $\xi \in \mathbb{R}^n$ , where  $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$ . Let

$$H_{\text{loc}}^1(\Omega) := \{u \in L_{\text{loc}}^2(\Omega) : \partial_k u \in L_{\text{loc}}^2(\Omega), k = 1, \dots, n\}.$$

Define  $\mathcal{A} : H_{\text{loc}}^1(\Omega) \rightarrow C_c^\infty(\Omega)'$  by

$$\langle \mathcal{A}u, v \rangle = \int_{\Omega} \left( \sum_{k,\ell=1}^d \partial_k (a_{k\ell} \partial_\ell u) + \sum_{k=1}^d \partial_k (b_k u) + \sum_{k=1}^d c_k \partial_k u + c_0 u \right) dx.$$

We define  $A_0$  as the part of  $\mathcal{A}$  in  $C_0(\Omega)$ ; i.e.,

$$\begin{aligned} D(A_0) &:= \{u \in C_0(\Omega) \cap H_{\text{loc}}^1(\Omega) : \mathcal{A}u \in C_0(\Omega)\} \\ A_0 u &:= \mathcal{A}u. \end{aligned}$$

Then Theorem 4.1 holds with  $\Delta_0$  replaced by  $A_0$ . It is remarkable that Dirichlet regularity of  $\Omega$  is the right regularity condition at the boundary, a discovery due to Stampacchia. We refer to Arendt and B nilan [2], Section 4 for a proof of the following result.

**Theorem 1.6** *Assume that  $\Omega \subset \mathbb{R}^n$  is a bounded, open, connected, Dirichlet regular set. Then  $A_0$  generates a positive, irreducible, holomorphic semigroup  $\mathcal{T} = (T(t))_{t \geq 0}$  on  $C_0(\Omega)$ . Moreover,  $T(t)$  is compact for all  $t > 0$ .*

**Remark** The proof of holomorphy depends on Gaussian estimates, which in [2] were merely known if the  $b_k \in W^{1,\infty}(\Omega)$ . Later, it was shown by Davies [12] that they always hold.

Also the results for Robin boundary conditions Theorem 4.3 and 4.4 can be extended to elliptic operators in divergence form on  $C(\overline{\Omega})$ ; see Theorem 4.5 in Arendt, ter Elst, Glück [10]. It uses results of Nittka [22].

### 1.3 Elliptic operators in non-divergence forms

The techniques for elliptic operators in non-divergence form are quite different than those used in the divergence-case form. But the results are similar.

Let  $\Omega \subset \mathbb{R}^n$  be open and connected. We assume that  $\Omega$  satisfies the uniform exterior cone condition. This means the following. There exists a finite, right circular cone  $V$  such that for each  $x \in \partial\Omega$  there exists a cone  $V_x$  which is congruent to  $V$  such that  $V_x \cap \Omega = \{x\}$ .

Let  $a_{k\ell} = a_{\ell k} \in C(\overline{\Omega})$ ,  $b_k \in L^\infty(\Omega)$ ,  $c \in L^\infty(\Omega)$ ,  $c \leq 0$  such that

$$\sum_{k,\ell=1}^n a_{k\ell}(x) \xi_k \xi_\ell \geq \mu |\xi|^2$$

for all  $x \in \overline{\Omega}$ ,  $\xi \in \mathbb{R}^n$  and some  $\mu > 0$ .

For  $u \in W_{\text{loc}}^{2,n}(\Omega)$  we define

$$\mathcal{A}u = \sum_{k,\ell=1}^n \partial_k a_{k\ell} \partial_\ell u + \sum_{k=1}^n b_k \partial_k u + cu.$$

Thus  $\mathcal{A}: W_{\text{loc}}^{2,n}(\Omega) \rightarrow L_{\text{loc}}^n(\Omega)$  is linear. Here

$$W_{\text{loc}}^{2,n}(\Omega) := \{u \in L_{\text{loc}}^n(\Omega) : \partial_k u \in L_{\text{loc}}^n(\Omega), \partial_k \partial_\ell u \in L_{\text{loc}}^n(\Omega) \text{ for all } k, \ell = 1, \dots, n\}.$$

We consider the operator  $A$  on  $C_0(\Omega)$  defined by

$$D(A) := \{u \in C_0(\Omega) \cap W_{\text{loc}}^{2,n}(\Omega) : \mathcal{A}u \in C_0(\Omega)\}$$

$$Au := \mathcal{A}u.$$

Then the following holds.

**Theorem 1.7** *The operator  $A$  generates a positive, irreducible, contractive holomorphic semigroup  $(T(t))_{t \geq 0}$  on  $C_0(\Omega)$ . Moreover*

$$\|T(t)\| \leq M e^{-\epsilon t} \quad (t \geq 0)$$

for some  $\epsilon > 0$ ,  $M \geq 1$ . The resolvent of  $A$  is compact.

This result is proved by Arendt and Schätzle [5], Proposition 4.7.

Positivity and irreducibility are a consequence of the Alexandrov maximum principle. For  $C^2$ -boundary also results on  $L^p(\Omega)$  spaces are obtained by Denk, Hieber and Prüss [16] whose main interest lies in proving maximal regularity and establishing a bounded  $H^\infty$ -calculus.

However, in the situation of Theorem 4.6, without assuming merely the uniform exterior cone condition on  $\Omega$ , it seems not to be known whether the semigroup extends to a strongly continuous semigroup on  $L^p(\Omega)$  for some  $p \in [1, \infty)$ .

Theorem 4.6 is extended by Arendt and Schätzle [6] to unbounded open sets which satisfy the locally uniform exterior cone condition. However, in the case of unbounded  $\Omega$  the semigroup converges merely strongly to 0 (and not exponentially fast).

The monograph of Lunardi [21] is devoted to the study of holomorphic semigroups generated by elliptic operators in non-divergence form.

#### 1.4 The Dirichlet-to-Neumann operator on $C(\partial\Omega)$

Let  $\Omega$  be a bounded, open, connected subset of  $\mathbb{R}^n$  with Lipschitz boundary and let  $V \in L^\infty(\Omega)$ . We consider the Dirichlet-to-Neumann operator with respect to  $\Delta - V$  on the space  $C(\partial\Omega)$ . For that we first establish well-posedness of the Dirichlet Problem.

We assume throughout this subsection that

$$u \in C_0(\Omega), \Delta u - Vu = 0 \text{ implies } u = 0. \quad (4.2)$$

This is exactly the condition that the solutions of the Dirichlet problem with respect to  $\Delta - V$  formulated in Proposition 4.7 are unique. An equivalent condition is

$$u \in H_0^1(\Omega), \Delta u - Vu = 0 \text{ implies } u = 0; \quad (4.3)$$

(which means that  $0 \notin \sigma(\Delta_0 - V)$  where  $\Delta_0$  is the Dirichlet Laplacian on  $L^2(\Omega)$ ).

**Proposition 1.8** *Assume (4.2). Let  $g \in C(\partial\Omega)$ . Then there exists a unique  $u_g \in C(\overline{\Omega})$  such that*

$$(\Delta - V)u_g = 0 \quad \text{and} \quad u_g|_{\partial\Omega} = g.$$

*Thus,  $u_g$  is a harmonic function with respect to  $\Delta - V$  which has to be understood in the sense of distributions; i.e.,*

$$\int_{\Omega} u_g \Delta \phi - \int_{\Omega} V u_g \phi = 0$$

for all  $\phi \in C_c^\infty(\Omega)$ .

For a simple proof of Proposition 4.7 we refer to [7].

Next we define the Dirichlet-to-Neumann operator  $N_V$  with respect to  $\Delta - V$  on  $C(\partial\Omega)$  as follows.

$$D(N_V) := \{g \in C(\partial\Omega) : u_g \in H^1(\Omega), \text{ and } \partial_\nu u_g \in C(\partial\Omega)\} \quad (1.4)$$

$$N_V g := -\partial_\nu u_g. \quad (1.5)$$

Recall that  $\partial_\nu u_g \in C(\partial\Omega)$  means that there exists  $h \in C(\partial\Omega)$  such that

$$\int_{\Omega} \Delta u_g \phi + \int_{\Omega} \nabla u_g \nabla \phi = \int_{\partial\Omega} h \phi$$

for all  $\phi \in C^1(\overline{\Omega})$ . Then we put  $\partial_\nu u_g := h$ .

We will need the hypothesis that  $-\Delta_0 + V$  is form-positive i.e.

$$\int_{\Omega} (|\nabla u|^2 + V|u|^2) \geq 0 \quad (4.4)$$

for all  $u \in H_0^1(\Omega)$ .

**Theorem 1.9** *Assume (4.2) and (4.4). Then  $\Delta - N_V$  generates a positive, irreducible semigroup on  $C(\partial\Omega)$ . If  $V \geq 0$ , then the semigroup is contractive.*

If  $\Omega$  is of class  $C^\infty$  similar results have been obtained by Escher [18] and Engel [17]. Under the very general conditions here, Theorem 4.8 is due to Arendt and ter Elst [8]. There it is shown that  $N_V$  is resolvent-positive and that the domain is dense (which is the main difficulty). Then by B-II, Theorem 1.8  $N_V$  generates a positive semigroup. Irreducibility is surprising. In fact, even though  $\Omega$  is supposed to be connected,  $\partial\Omega$  might not be connected (consider a ring for example). The fact that the semigroup is irreducible shows that the operator  $N_V$  is non-local in quite a dramatic way.

A first result on irreducibility (on  $L^2(\partial\Omega)$ ) was obtained by Arendt and Mazzeo [4].

It is not known so far whether the semigroup generated by  $N_V$  is holomorphic if  $\partial\Omega$  has Lipschitz boundary. If the boundary is of class  $C^{n+\alpha}$  with  $\alpha > 0$ , then it is holomorphic of angle  $\pi/2$ . This is due to ter Elst and Ouhabaz [23].

The operator  $N_V$  is also called *voltage-to-current map* and has physical meaning. One version of the famous Calderón-Problem is the question whether for  $V_1, V_2 \in L^\infty(\Omega)$ , such that  $0 \notin \sigma(\Delta_{V_1}) \cup \sigma(\Delta_{V_2})$ ,

$$N_{V_1} = N_{V_2} \text{ implies } V_1 = V_2.$$

This is true under the only assumption that  $\Omega$  has Lipschitz boundary; see Theorem 1.1 by Krupchyk and Uhlmann [20].

Finally we mention that  $N_V$  may generate a positive semigroup even if (4.4) is violated. This and other surprising phenomena were discovered by Daners [13], and led to the new theory of eventually positive semigroups; see e.g. [14].



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