space of all $\mathcal{F}$-null sequences in $m(E)$

$$c_{\mathcal{F}}(E) := \{(f_n) \in m(E) : \mathcal{F}\text{-}\lim\|f_n\| = 0\}$$

is closed in $m(E)$ and invariant under $(\hat{T}(t))_{t\geq 0}$. We call the quotient spaces

$$E_{\mathcal{F}} := m(E)/_{c_{\mathcal{F}}(E)} \qquad \text{and} \qquad E_{\mathcal{F}}^{\mathcal{T}} := m^{\mathcal{T}}(E)/_{c_{\mathcal{F}}(E)\cap m^{\mathcal{T}}(E)}$$

the <u>$\mathcal{F}$-product of</u> $E$ and the <u>$\mathcal{F}$-product of</u> $E$ <u>with respect to the semigroup</u> $\mathcal{T}$, respectively. Thus $E_{\mathcal{F}}^{\mathcal{T}}$ can be considered as a closed linear subspace of $E_{\mathcal{F}}$. We have $E_{\mathcal{F}}^{\mathcal{T}} = E_{\mathcal{F}}$ if (and only if) $\mathcal{T}$ has a bounded generator.

The canonical quotient norm on $E_{\mathcal{F}}$ is given by

$$\|(f_n) + c_{\mathcal{F}}(E)\| = \mathcal{F}\text{-}\lim\sup \|f_n\| .$$

We can apply 3.3 in order to define the <u>$\mathcal{F}$-product semigroup</u> $(T_{\mathcal{F}}(t))_{t\geq 0}$ on $E_{\mathcal{F}}^{\mathcal{T}}$ by

$$T_{\mathcal{F}}(t)((f_n) + c_{\mathcal{F}}(E)) := (T(t)f_n) + c_{\mathcal{F}}(E)\cap m^{\mathcal{T}}(E) .$$

Thus $T_{\mathcal{F}}(t)$ is the restriction of $T(t)_{\mathcal{F}}$ where $T(t)_{\mathcal{F}}$ denotes the canonical extension of $T(t)$ to the $\mathcal{F}$-product $E_{\mathcal{F}}$. (Note that $(T(t)_{\mathcal{F}})_{t\geq 0}$ is not strongly continuous unless $\mathcal{T}$ has a bounded generator.)

With the canonical injection $j : f \to (f,f,f,..) + c_{\mathcal{F}}(E)$ from $E$ into $E_{\mathcal{F}}^{\mathcal{T}}$ the operators $T_{\mathcal{F}}(t)$ are extensions of $T(t)$ satisfying $\|T_{\mathcal{F}}(t)\| = \|T(t)\|$. The basic facts about the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of $(T_{\mathcal{F}}(t))_{t\geq 0}$ follow from 3.3 and are collected in the following proposition.

<u>Proposition</u>. For the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of the $\mathcal{F}$-product semigroup the following holds:

(i) $\quad D(A_{\mathcal{F}}) = \{(f_n) + c_{\mathcal{F}}(E) : f_n \in D(A);\ (f_n),\ (Af_n) \in m^{\mathcal{T}}(E)\}$ ,

(ii) $\quad A_{\mathcal{F}}((f_n) + c_{\mathcal{F}}(E)) = (Af_n) + c_{\mathcal{F}}(E)$ .

In case $A$ is a bounded operator then $D(A_{\mathcal{F}}) = E_{\mathcal{F}}^{\mathcal{T}} = E_{\mathcal{F}}$ and $A_{\mathcal{F}}$ is the canonical extension of $A$ to $E_{\mathcal{F}}$.

We will show in A-III,4.5 that the above construction preserves and even improves many spectral properties of the semigroup and its generator.

## 3.7. <u>The Tensor Product Semigroup</u>

Real- or complex-valued functions of two variables $x$, $y$ are often limits of functions of the form $\sum_{i=1}^{n} f_i(x)g_i(y)$ , which to some extent allows one to consider the variables $x$ and $y$ separately.