(d) Let $E = L^1([0,1]\times[-1,1])$ and consider the semigroup $(T(t))_{t\geq 0}$ defined as follows:

$$(3.3)\quad (T(t)f)(x,v) := \begin{cases} f(x-vt,v) & \text{for } 0 \leq x-vt \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$(T(t))_{t\geq 0}$ is a positive semigroup on E and

$$D_o := \{f \in C^1([0,1]\times[-1,1]) : \begin{array}{l} f(0,v) = f_x(0,v) = 0 \text{ if } v \geq 0 , \\ f(1,v) = f_x(1,v) = 0 \text{ if } v \leq 0 \} \end{array}$$

is a core for its generator A (cf. A-I,Cor.1.34). We have

$$(3.4)\quad (Af)(x,v) = -v\cdot\frac{\partial f}{\partial x}(x,v) \qquad (f \in D_o) .$$

The Laplace transform of $(T(t))$ is the resolvent of A. An explicit calculation yields:

$$(3.5)\quad (R(\lambda,A)f)(x,v) = \int_0^1 r_\lambda(x,x',v)f(x',v)\,dx' \quad (\lambda > 0)$$

where $r_\lambda : [0,1]\times[0,1]\times[-1,1] \to \mathbb{R}$ is given by

$$r_\lambda(x,x',v) = \begin{cases} |v|^{-1}\exp(-\lambda(x-x')v^{-1}) & \text{if either } v>0 \text{ and } x'\leq x \text{ or } v<0 \text{ and } x'\geq x ; \\ 0 & \text{otherwise .} \end{cases}$$

Let $\sigma : [0,1]\times[-1,1] \to \mathbb{R}_+$ and $\kappa : [0,1]\times[-1,1]\times[-1,1] \to \mathbb{R}_+$ be measurable functions and consider the operators M and K given by

$$(3.6)\quad Mf := \sigma f \quad , \quad Kf := \int_{-1}^{1} \kappa(.,.,v')f(.,v')\,dv' .$$

Then $B := A - M + K$ with domain $D(B) := D(A)$ is the generator of a positive semigroup.
Using Prop.3.3 we can prove the following irreducibility criterion for the semigroup $(S(t))_{t\geq 0}$ generated by B :

(3.7) If κ is strictly positive then $(S(t))_{t\geq 0}$ is irreducible.

Actually, in view of Prop.3.3 we have to show that a closed ideal which is invariant under $R(\lambda,A)$ and K has to be $\{0\}$ or E .
We recall that closed ideals of E are uniquely determined (up to sets of measure zero) by measurable subsets Y of $[0,1]\times[-1,1]$; i.e., every closed ideal has the form
$I_Y = \{f \in E : f \text{ vanishes (a.e.) on } [0,1]\times[-1,1] \setminus Y\}$.
Since we assumed that κ is strictly positive, I_Y is K-invariant if and only if $Y = X\times[-1,1]$ for some measurable set $X \subset [0,1]$. If we assume that X has positive measure and define $\alpha := \sup\{x \in [0,1] : \int_0^x 1_X(s)\,ds = 0 \}$ and $\beta := \inf\{x \in [0,1] : \int_x^1 1_X(s)\,ds = 0 \}$ then we have $\alpha < \beta$ and the support of the function $h := R(\lambda,A)1_Y$