Example. If we do not require $T_1$ to be uniformly continuous the above spectral decomposition need not be unique :
Consider a decomposition $E = E_1 \oplus E_2$ and add a direct summand $E_3$ with a strongly continuous semigroup $T_3$ whose generator $A_3$ has empty spectrum (e.g. A-I,Example 2.6). Then still $\sigma(A) = \sigma_1 \cup \sigma_2$ but $E_1 \oplus (E_2 \oplus E_3)$ and $(E_1 \oplus E_3) \oplus E_2$ are two different spectral decompositions corresponding to $\sigma_1$, $\sigma_2$.

The importance of the above theorem stems from the fact that $T_1$ has a bounded generator and therefore is easy to deal with. In particular the asymptotic behavior of $T_1$ can be deduced from the location of $\sigma_1$.

Corollary 3.4. Assume that $\sigma(A)$ splits into non-empty closed sets $\sigma_1$, $\sigma_2$ where $\sigma_1$ is compact and consider the corresponding spectral decomposition $E = E_1 \oplus E_2$ for which $T_1$ is uniformly continuous. For all constants $v, w \in \mathbb{R}$ satisfying

$$v < \inf \{\mathrm{Re}\,\lambda : \lambda \in \sigma_1\} \quad \text{and} \quad \sup \{\mathrm{Re}\,\lambda : \lambda \in \sigma_1\} < w$$

there exist $m, M \geq 1$ such that

$$m \cdot e^{vt} \|f\| \leq \|T_1(t) f\| \leq M \cdot e^{wt} \|f\|$$

for every $f \in E_1$, $t \geq 0$.

Proof. Since the generator $A_1$ of $T_1$ is bounded we have $T_1(t) = \exp(tA_1)$ and $\sigma(T_1(t)) = \exp(t\sigma(A_1))$. Therefore by the remark following Prop.1.1 the spectral bound $s(A_1)$ coincides with the growth bound $\omega(T_1)$ and we have the upper estimate. The lower estimate is obtained by applying the same reasoning to $-A_1$ which generates the semigroup $(T_1(t)^{-1})_{t \geq 0}$ on $E_1$.

□

It is clear from Examples 1.3, 1.4 that no norm estimates for $(T_2(t))_{t \geq 0}$ can be obtained from the location of $\sigma_2$. Only by adding appropriate hypotheses we will achieve spectral decompositions admitting norm estimates on both components (see A-III,6.6).
Another way of obtaining such norm estimates is by constructing spectral decompositions starting from a semigroup operator $T(t_o)$ (instead of $A$ resp. $R(\lambda, A)$, as in Thm.3.3).

Corollary 3.5. If $\sigma(T(t_0)) = \tau_1 \cup \tau_2$ for two non-empty, closed, disjoint sets $\tau_1$, $\tau_2$ and if $P$ is the spectral projection correspon-