Consider the operator B given by $Bf = f'$ with domain $D(B) = \{f \in C^1([-1,0],\mathbb{C}) : f'(0) = \beta f(0) + \langle f,\nu\rangle\}$. We claim that

(4.13) B is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$. Moreover, $(S(t))_{t\geq 0}$ is dominated by $(T(t))_{t\geq 0}$ if and only if $\mathrm{Re}\beta \leq \alpha$ and $|\nu| \leq \mu$.

<u>Remark</u>. It is of interest to find a condition on B which implies that the semigroup $(S(t))_{t\geq 0}$ is stable (see A-IV,Sec.1).
Using the positivity of $(T(t))_{t\geq 0}$ it is shown in B-IV,Ex.3.9, that $(T(t))_{t\geq 0}$ is stable if and only if $\|\mu\| + \alpha < 0$. Since a semigroup which is dominated by a stable semigroup is itself stable we obtain from (4.13) that $(S(t))_{t\geq 0}$ is stable if $\|\nu\| + \mathrm{Re}\beta < 0$.

<u>Proof of (4.13)</u>. We first assume that $\alpha := \mathrm{Re}\,\beta$ and $\mu = |\nu|$. We show that (4.12) is satisfied. Consider the operator A_{max} on $C[-1,0]$ given by $A_{max}f = f'$ with domain $D(A_{max}) = C^1[-1,0]$. We know by B-II,Example 2.12 that $\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af,\phi\rangle \leq \mathrm{Re}\langle(\mathrm{sign}\,\bar{f})(Af),\phi\rangle = \langle|f|,(A_{max})'\phi\rangle$ for all $f \in D(A_{max})$, $0 \leq \phi \in D((A_{max})')$.
In particular

(4.14) $\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf,\phi\rangle \leq \langle|f|,A'\phi\rangle$

holds for all $f \in D(B)$, $0 \leq \phi \in D((A_{max})')$. It is not difficult to see that $D(A') = D((A_{max})') + \mathbb{C}\delta_o$, and since $D((A_{max})') = BV[-1,0]$ (see B-II,Example 2.12) this is an order direct sum.
Thus, in view of (4.14), it remains to show that

(4.15) $\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf,\delta_o\rangle \leq \langle|f|,A'\delta_o\rangle$

for all $f \in D(B)$. By the definition of A , $\delta_o \in D(A')$ and $A'\delta_o = \alpha\delta_o + \mu$. Hence for $f \in D(B)$,
$\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf,\delta_o\rangle = \mathrm{Re}((\mathrm{sign}\,\bar{f})f')(0) = \mathrm{Re}((\mathrm{sign}\,\overline{f(0)})\cdot(\beta f(0) + \langle f,\nu\rangle)) \leq \mathrm{Re}\beta\,|f(0)| + |\langle f,\nu\rangle| \leq \alpha|f(0)| + \langle|f|,\mu\rangle = \langle|f|,A'\delta_o\rangle$.
Thus (4.15) and so also (4.12) are proved.
As in the proof in Example 2.14 one shows that $\lambda - B$ is surjective for large real λ . Hence by Theorem 4.14, B is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ which is dominated by $(T(t))_{t\geq 0}$. This proves the first assertion of (4.13) and the sufficiency of the second.
Now we assume that the semigroup $(S(t))_{t\geq 0}$ is dominated by $(T(t))_{t\geq 0}$. We have to show that $\mathrm{Re}\beta \leq \alpha$ and $|\nu| \leq \mu$. Since $\delta_o \in D(A') \cap D(B')$ we have for all $f \in C[-1,0]_+$ satisfying $f(0) = 0$, $|\langle f,\nu\rangle| = |\langle f,B'\delta_o\rangle| = \lim_{t\to 0+} 1/t\,|\langle S(t)f - f,\delta_o\rangle|$
$= \lim_{t\to 0+} 1/t\,|(S(t)f)(0)|$