Spectral Theory of Positive Operators
January 17, 2025

Worksheet 4
Ultraproducts II

There are moments when everything works out for us. No need to be frightened: it will pass.

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Ultraproducts of C*- and W*-Algebras

This is a first draft that needs to be elaborated. Perhaps someone is interested in this.

Ultraproducts of C*- and W*-Algebras

1. For C*-algebras we have

Proposition 1 *If $\mathfrak{A}_\omega$ is the ultraproduct of a $C^\star$-algebra $\mathfrak{A}$, then:*

(i) *If a is self-adjoint in $\mathfrak{A}_\omega$, then there exists a sequence (a_j) in $\mathfrak{A}_h$ with $a = (a_j)_\omega$.*

(ii) *If a is positive in $\mathfrak{A}_\omega$, then there exists a sequence (a_j) in $\mathfrak{A}_+$ with $a = (a_j)_\omega$.*

(iii) *If p is a projection in $\mathfrak{A}_\omega$, then there exists a sequence (p_j) of projections in $\mathfrak{A}$ with $p = (p_j)_\omega$.*

(iv) *If u is unitary in $\mathfrak{A}_\omega$, then there exists a sequence (u_j) of unitary elements in $\mathfrak{A}$ with $u = (u_j)_\omega$.*

(v) *If $p = (p_j)_\omega$ and $q = (q_j)_\omega$ are projections in $\mathfrak{A}_\omega$ and v is a partial isometry with $v^*v = p$, $vv^* = q$, then there exist v_j in $\mathfrak{A}$ with*

$$v_j^* v_j = p_j \quad \text{and} \quad v_j v_j^* = q_j$$

along ω.

(vi) *If $p = (p_j)_\omega$ is a projection, $p \leqslant q$ for some projection q, then there exist projections q_j in $\mathfrak{A}$ with*

$$p_j \leqslant q_j \quad \text{and} \quad q = (q_j)_\omega$$

Proof: Literature citation

2. If $E = L^1$, then E is the predual of a W*-algebra and the ultraproduct of E is again an L^1 and thus again a predual. This also holds for general W*-algebras (see GROH [1]). If $\mathfrak{M}$ is a W*-algebra, then its ultraproduct is in general not a W*-algebra, since the kernel of the seminorm p_ω need not be a σ^*-closed ideal in the W*-algebra $\ell^\infty(\mathfrak{M})$. However, we have:

Proposition 2 *The ultraproduct of the predual of a $W^\star$-algebra is the predual of a $W^\star$-algebra.*

Let $\mathfrak{M}$ be a W*-algebra with predual $\mathfrak{M}_*$, then the mapping

$$\tau \colon (\mathfrak{M}_*)_\omega \mapsto (\mathfrak{M}_\omega)'$$

with

$$< \tau(\varphi_\omega), x_\omega >= \lim_\omega \varphi_j(x_j)$$

for $(x_j) \in x_\omega$ and $(\varphi_j) \in \varphi_\omega$ is an isometry. Let $\mathsf{M}(\hat{\mathfrak{A}})$ be the closed subspace $\tau((\mathfrak{M}_*)_\omega)$.

Show: This subspace is invariant under multiplication from the right and from the left with elements of $\mathfrak{M}_\omega$ and $\mathfrak{M}_\omega''$. Here one only needs to note that for $\varphi \in \mathfrak{M}_*$ and $x \in \mathfrak{M}$ we always have $x.\varphi$ and $\varphi.x$ are again elements of the predual. Thus the polar of $\mathsf{M}(\hat{\mathfrak{A}})$ in $\mathfrak{M}_\omega''$ is a σ^*-closed ideal and there exists a central projection z in $\mathfrak{M}_\omega''$ with

$$\mathsf{M}(\hat{\mathfrak{A}}) = \mathfrak{M}_\omega'.z = [(\mathfrak{M}_\omega)''.z]_* \ .$$

Proposition 3 *For the mapping τ defined above, we have for all $\varphi_\omega \in (\mathfrak{M}_*)_\omega$*

$$\tau(|\varphi_\omega|) = |\tau(\varphi_\omega)| \, .$$

The proof is based on TAKESAKI [2, III.4.10] and remains as $\boxed{\text{Exercise}}$.

Example 4 Example where $\mathfrak{M}_\omega$ is not a W*-algebra ->

Compatibility with Positive Maps

3. We have:

   (i) T is positive if and only if $\hat{T}$ is positive.

   (ii) T is a Schwarz operator if and only if $\hat{T}$ is a Schwarz operator.

   (iii) T is an n-positive operator if and only if $\hat{T}$ is an n-positive operator.

For the proof of the last claim one must show that we always have

$$M_n(A)_\omega = M_n(A_\omega)$$

References

[1] U. GROH: *Uniformly ergodic theorems for identity preserving Schwarz maps on W*-algebras.* J. Operat. Theor. **11** (1984), 395–404. (Cited on p. 1).

[2] M. TAKESAKI: *Theory of Operator Algebras I.* Springer (1979) (cited on p. 2).