The characteristic equation (in the classical sense) is:

$$(3.10)\qquad \lambda = p + e^{-\lambda\tau}q$$

We consider the case where the Cauchy problem without delay

$$\dot{u}(t) = (p + q)u(t)$$

is asymptotically stable, i.e. we choose $0 < p < 1$ and $q + p < 0$.

Claim. For every $0 < \lambda' < p$ there exists $\tau > 0$ such that $e^{\lambda' t}$ is a solution of $(3.9)_\tau$.

Consider the map $g : \mathbb{R}\times(\mathbb{R}_+\setminus\{0\}) \to \mathbb{R}$ defined by $g(\lambda,\tau) = p + e^{-\lambda\tau}q$. This function is continuous in λ and τ and increasing in λ. Furthermore $g(0,\tau) = p + q < 0$ for every $\tau > 0$ and $g(\lambda,\tau) \to p$ as $\tau \to \infty$ for every $\lambda \in \mathbb{R}_+$. For $0 < \lambda' < p$ fixed we can find $\tau > 0$ such that $g(\lambda',\tau) = \lambda'$.

Let $\Psi(t) = e^{\lambda' t}$ for $-\tau \leq t \leq 0$. If we define $u(t) := e^{\lambda' t}$ for $t \geq 0$ then the following is valid:

$$pu(t) + qu(t-\tau) = pe^{\lambda' t} + qe^{\lambda' t}e^{-\lambda'\tau} = (p+qe^{-\lambda'\tau})e^{\lambda' t} = \lambda' e^{\lambda' t} = \dot{u}(t).$$

Thus u is a solution of $(3.9)_\tau$ which is exponentially increasing as $t \to \infty$. In particular $(3.9)_\tau$ is not stable.

The precise region of stability in the scalar valued case is given, for example in [Hadeler (1978)] and [Hale (1977),107ff].

Remark. Consider the case $F = C(M)$ (M compact). Then $E = C([-1,0] \times M)$ and $(T(t))_{t\geq 0}$ is a positive semigroup on $C(K)$ where $K = [-1,0] \times M$ is compact. Thus spectral bound and growth bound of the semigroup generator coincide (B-IV,(1.1)). This yields a statement analogous to Cor.3.8 for uniform exponential stability.

We conclude this section with two examples fitting into the above framework.

Example 3.11. Consider the equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial^2 x}u(t,x) - d(x)u(t,x) + b(x)u(t-1,x) \qquad (t\geq 0, x\in[0,1])$$

with boundary condition

$$(3.11)\qquad \frac{\partial}{\partial x}u(t,x)\Big|_{x=0} = 0 = \frac{\partial}{\partial x}u(t,x)\Big|_{x=1} \qquad (t\geq 0)$$

and initial condition

$$u(s,x) = \psi(s,x) \qquad (s\in[-1,0], x\in[0,1]) .$$