$\langle \mathrm{Re}[(\mathrm{si\hat{g}n}\ \bar{f})(Af)],\phi\rangle = d/dt_{|t=0} \langle |T(t)f|,\phi\rangle = d/dt_{|t=0} \langle T(t)|f|,\phi\rangle = \langle |f|,A'\phi\rangle$.

Conversely, assume that (2.9) holds. Let $t > 0$, $f \in C_o(X)$. We have to show that $|T(t)f| = T(t)|f|$. Since $D(A)$ is dense in $C_o(X)$, we can assume that $f \in D(A)$. Moreover, since $D(A')$ is $\sigma(M(X),C_o(X))$-dense in $M(X)$, it suffices to show that

$$(2.13) \qquad \langle |T(t)f|,\phi\rangle = \langle T(t)|f|,\phi\rangle$$

for all $\phi \in D(A')$.

Let $\phi \in D(A')$ and define the function $k(s) = \langle T(t-s)|T(s)f|,\phi\rangle$ ($s \in [0,t]$). We claim that k is right-sided differentiable with derivative $k'(s) = 0$ for all $s \in [0,t]$. This implies that $k(0) = k(t)$ which is (2.13).

Since $\phi \in D(A')$ we have

$$(2.14) \qquad \lim_{h\downarrow o} 1/h \langle g,(T(t-(s+h)) - T(t-s))'\phi\rangle = - \langle g,A'T(t-s)'\phi\rangle$$

for all $g \in C_o(X)$. Consequently,

$$\overline{\lim}_{h\downarrow o} 1/h \,\|(T(t-(s+h)) - T(t-s))'\phi\| < \infty$$

by the uniform boundedness principle. Hence, since $\lim_{h\downarrow o} |T(s+h)f| = |T(s)f|$, (2.14) implies that

$$(2.15) \qquad \lim_{h\downarrow o} 1/h \langle |T(s+h)f|,(T(t-(s+h)) - T(t-s))'\phi\rangle = - \langle |T(s)f|,A'T(t-s)'\phi\rangle .$$

Using this we obtain

$\lim_{h\downarrow o} 1/h\ (k(s+h) - k(s))$

$= \lim_{h\downarrow o} 1/h\ (\langle T(t-(s+h))|T(s+h)f|,\phi\rangle - \langle T(t-s)|T(s+h)f|,\phi\rangle +$
$\qquad \lim_{h\downarrow o} 1/h\ (\langle T(t-s))|T(s+h)f| - T(t-s)|T(s)f|,\phi\rangle$

$= - \langle |T(s)f|,A'T(t-s)'\phi\rangle + \lim_{h\downarrow o} 1/h \langle (|T(s+h)f|-|T(s)f|,T(t-s)'\phi\rangle$.

By Lemma 2.6 the last term is

$- \langle |T(s)f|,A'T(t-s)'\phi\rangle + \langle \mathrm{Re}[(\mathrm{si\hat{g}n}\ \overline{T(s)f})(AT(s)f)],T(t-s)'\phi\rangle$,

and this is 0 by hypothesis.

□

<u>Remark</u> 2.7. We will see in Chapter C-II that the inequality $|T(t)f| \leq T(t)|f|$, which holds precisely for positive semigroups, implies the inequality corresponding to (2.9). For $A = \Delta$ (the Laplacian) this is a version of the classical Kato's inequality.