(2.1) $|(\text{sign } f)g| \le |g|$ $(g \in E)$

(2.2) $(\text{sign } f)g = 0$ if $\inf\{|f|,|g|\} = 0$

(2.3) $(\text{sign } \bar{f})f = |f|$ (where $\bar{f} := \text{Re}f - i\text{Im}f$).

The operator $(\hat{\text{sign}}\ f)$ (which is non-linear in general) is defined by

(2.4) $(\hat{\text{sign}}\ f)g = (\text{sign } f)g + (\text{Id} - P_{|f|})|g|$

where for $h \in E_+$ we denote by P_h the band projection onto the band $\{h\}^{dd}$ generated by h.

If E is a real σ-order complete Banach lattice, then

(2.5) $\text{sign } f = P_{(f^+)} - P_{(f^-)}$.

Example 2.2. Let $f \in E := L^p(X,\Sigma,\mu)$ (real or complex) where (X,Σ,μ) is a σ-finite measure space and $1 \le p \le \infty$. Define

$$m(x) = \begin{cases} f(x)/|f(x)| & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 . \end{cases}$$

Then sign f is the multiplication operator defined by m; i.e.,

$$(\text{sign } f)g = m\cdot g \qquad (g \in E),$$

Moreover,

$$(\hat{\text{sign}}\ f)g = m\cdot g + 1_{[f(x)=0]}|g| \qquad (g \in E).$$

The operator $\hat{\text{sign}}\ f$ is related to the Gateaux-derivative (B-II, Definition 3.2) of the modulus. We explain this by an example.

Example 2.3. Let E be the real or complex space $L^p(X,\Sigma,\mu)$ where (X,Σ,μ) is a σ-finite measure space and $1 \le p < \infty$. Let $f,g \in E$ and $x \in X$. Then by B-II, Lemma 2.4

$$\lim_{t\downarrow 0} 1/t(|f(x)+tg(x)|-|f(x)|) = \begin{cases} \text{Re}(\text{sign } \overline{f(x)})g(x) & \text{if } f(x) \neq 0 \\ |g(x)| & \text{if } f(x) = 0. \end{cases}$$

If $\Theta : E \to E_+$ denotes the modulus function given by $\Theta(h) = |h|$, then it follows from the dominated convergence theorem that Θ is right-sided Gateaux-differentiable and

(2.6) $$D_g\Theta(f) = \text{Re}(\hat{\text{sign}}\ \bar{f})g.$$

Later we will see that (2.6) holds in every Banach lattice with order continuous norm.