Proof. Let N be the W*-subalgebra of $M = B(H)$ generated by the eigenvectors of A pertaining to the eigenvalues on $i\mathbb{R}$ and let Q be the faithful normal conditional expectation from M onto N (Theorem 3.7.). Since M is atomic, N is atomic [Størmer (1972)]. N is finite since there exists a finite, faithful normal trace on N. In particular the center of N is isomorphic to ℓ^∞. Let S be the restriction of T to the center. Then S is a weak*-semigroup such that every $S(t) \in S$ is $\sigma(\ell^\infty,\ell^1)$-continuous and a *-automorphism. From this it follows that $S(t)$ is induced by some continuous flow $\kappa_t\colon \mathbb{N} \to \mathbb{N}$. Indeed, if $\delta_n((\xi_m)) = \xi_n$ $(n \in \mathbb{N},\ (\xi_m) \in \ell^\infty)$, then $\delta_n \circ S(t)$ is a normal scalar valued *-homomorphism hence of the form δ_m for some $m = \kappa_t(n)$. But the function $(t \to \kappa_t)$ is continuous from $\mathbb{R}$ into $\mathbb{N}$, whence constant. Hence $S(t) = Id$. But the semigroup S is weak*-irreducible on the center. Consequently the center is one dimensional. Using [Takesaki, Theorem V.1.27] we obtain $N = B(H_n)$ where H_n is a finite dimensional Hilbert space. But if $0 \neq i\alpha \in P\sigma(A) \cap i\mathbb{R}$ then $i\alpha\mathbb{Z} \subseteq P\sigma(A)$ by D-III,Thm.1.10, whence N must be infinite dimensional. Therefore $P\sigma(A) \cap i\mathbb{R} = \{0\}$ as desired.

$\square$

An immediate and interesting consequence of Theorem 3.8 and Proposition 3.7 is the following.

Corollary 3.9. If $(B(H),\phi,T)$ is an irreducible W*-dynamical system, then

$$\lim_{s\to\infty} T(s) = 1 \otimes \phi$$

in the strong operator topology on $L(B(H)_*)$, where ϕ is the unique normal state generating the fixed space of T_*.

We are now going to discuss the asymptotic behavior of positive semigroups whose generator has boundary point spectrum different from 0. The standard example is the following:

If Γ is the unit circle, m the normalized Haar measure on Γ and $0 < \tau \in \mathbb{R}$, then we define the maps $R_\tau(t)$, $t \in \mathbb{R}_+$, on $L^1(\Gamma,m)$ by

$$(R_\tau(t)f)(\xi) = f(\xi \exp(\tfrac{2\pi i}{\tau}\, t)) \qquad (f \in L^1(\Gamma,m),\ \xi \in \Gamma) .$$

Then $R := (R_\tau(t))_{t\geq 0}$ forms a strongly continuous one parameter semigroup which is identity preserving and of Schwarz type. Since R