Example 3.11. The next example is of a more special form and occures as a model describing the cell cycle based on unequal division of cells, (see [Arino-Kimmel (1985)]). Let $F = L^1[0,1]$, $E = L^1([-1,0],F)$ and define an operator $\Phi : E \to F$ by

$\Phi(\psi)(x) := \int_0^1 k(x,x')\psi(q(x))(x')\,dx'$ for almost all $x \in [0,1]$.

Here q is a continuously differentiable function with strictly positive derivative satisfying $-1 \leq q(x) \leq \varepsilon < 0$ for all $x \in [0,1]$ and k is a bounded, measurable, strictly positive kernel.

Then (RE) has the form

$$(3.12)\qquad \begin{aligned} f(t)(x) &= \int_0^1 k(x,x')f(t+q(x))(x')\,dx', \quad t \geq 0, \\ f_o &= \psi \in E. \end{aligned}$$

It is easy to show that $\Phi \in L(E,F)$. If we define $K \in L(F)$ by $Kf(x) = \int_o^1 k(x,x')f(x')\,dx'$ we obtain $\Phi_\lambda f = e^{\lambda q(x)}Kf$ $(f \in F)$.

Again we have

$$s(A) \lesseqgtr \lambda \quad \text{if and only if} \quad s(\Phi_\lambda) \lesseqgtr 1 .$$

Proof. By Cor.3.8 it suffices to show that the map $h : \lambda \to s(\Phi_\lambda)$ is strictly decreasing and continuous.

Since k is bounded the operator K is weakly compact and so is Φ_λ. Since E has the Dunford-Pettis property $(\Phi_\lambda)^2$ is compact [Schaefer (1974),Thm.II.9.9] and this yields continuity of h.

Let $\lambda > \mu > 0$ and $0 < f \in F_+$.

Then $\Phi_\mu f(x) = e^{\mu q(x)}Kf(x) = e^{(\mu-\lambda)q(x)}e^{\lambda q(x)}Kf(x) = e^{(\mu-\lambda)q(x)}\Phi_\lambda f(x)$.

Since $q(x) \leq \varepsilon$ for all $x \in [0,1]$, we obtain, $\Phi_\mu f \geq e^{(\mu-\lambda)\varepsilon}\Phi_\lambda f$ and, moreover, $(\Phi_\mu)^n f \geq e^{n(\mu-\lambda)\varepsilon}\cdot(\Phi_\lambda)^n f$ for every $n \in \mathbb{N}$. Hence $\|(\Phi_\mu)^n\| > e^{n(\mu-\lambda)\varepsilon}\|(\Phi_\lambda)^n\|$ and consequently $r(\Phi_\mu) \geq e^{(\mu-\lambda)\varepsilon}r(\Phi_\lambda)$.

Now $(\mu-\lambda)\varepsilon > 0$ implies $r(\Phi_\mu) > r(\Phi_\lambda)$.

□

The theory developed so far can also be applied to certain population equations. We first notice that (ACP) is isomorphic (in an obvious manner) to the following Cauchy problem.

For some $r \in \mathbb{R}_+$ take $E := L^1([0,r],F)$ and let $A := -\frac{d}{dx}$ on the domain $D(A) := \{f \in AC([0,r],F) : f' \in E \text{ and } f(0) = \Phi(f)\}$ for some $\Phi \in L(E,F)$.

We adopt this setting and transform the above results; e.g., ε_λ has to be defined as $\varepsilon_\lambda(s) := e^{-\lambda s}$ instead of $e^{\lambda s}$. As a concrete example we consider the following.