(3.17) $\hat{\phi}(|\hat{f}|) - |\hat{f}| \in I$ for every $\hat{f} \in M$.

This implies that $M \cap I = \{0\}$. Hence the canonical image $M_/$ of M in the quotient space $E_{\mathcal{U}}/I$ is infinite dimensional as well. By (3.16) and (3.17) the absolute value of an element $\tilde{f} \in M_/$ is a scalar multiple of $\tilde{u} := \hat{u} + I$. This is a contradiction by Lemma 3.11.

□

In view of A-III,Prop.4.2 the result above has consequences for semigroups which can be reduced (by considering restrictions to invariant ideals or quotients) to semigroups which satisfy the hypothesis of Thm.3.12. Semigroups having this property are precisely those for which $s(A)$ is a pole of the resolvent of finite algebraic multiplicity. The latter claim is a consequence of Prop.2.11 and the following lemma.

Lemma 3.13. Suppose that $\mathcal{T} = (T(t))_{t \geq 0}$ is a positive semigroup such that $s(A)$ is a first order pole of the resolvent. Moreover assume that the corresponding residue is a strictly positive operator of finite rank.
Then there are closed $(T(t))$-invariant ideals $J_1, J_2, \ldots J_m$ which are mutually orthogonal such that the following is true:
(We denote the restriction of $\mathcal{T}$ to J_k by $\mathcal{T}_k$ and set $J := J_1 \oplus J_2 \oplus \ldots \oplus J_m$)
(a) $\mathcal{T}_k$ is irreducible and has spectral bound $s(A)$;
(b) $s(A_{/J}) < s(A)$.

Proof. We assume $s(A) = 0$. Since P is a strictly positive projection $PE = \ker A$ is a sublattice of E . Actually, if $x \in PE$ i.e., $Px = x$, then $P|x| \geq |Px| = |x|$. Hence $P(|P|x|-|x||) = P^2|x| - P|x| = 0$ which implies that $P|x| - |x| = 0$ or $|x| \in PE$. Thus we know that $\ker A$ is a finite dimensional sublattice of E hence it is isomorphic to a space $\mathbb{C}^m$ for some $m \in \mathbb{N}$ (see Sec.II.4 of Schaefer (1974)). Then there exist vectors $e_j \in E_+$ $(1 \leq j \leq m)$ such that the following holds:

$$\begin{aligned}(3.18)\quad &\ker A = \operatorname{span}\{e_1, e_2, \ldots, e_m\} \quad \text{and} \\ &\inf\{e_i, e_j\} = 0 \quad \text{for} \quad i \neq j .\end{aligned}$$

We have $T(t)e_k = e_k$ hence the closed ideal generated by e_k is $\mathcal{T}$-invariant. We denote this ideal J_k and define $J := J_1 \oplus J_2 \oplus \ldots \oplus J_m$. J is closed (see [Schaefer (1974),III.Thm.1.2]), $\mathcal{T}$-invariant and we have $PE = \ker A \subset J$. Then $P_{/J} = 0$ hence the spectral bound $s(A_{/J})$