If A is a generator, then the positivity of the resolvent $R(\lambda,A)$ for large real λ implies the positivity of the semigroup (by Prop. 1.1). On $C(K)$ much more is true. Even if A is not supposed to be a generator, the existence and positivity of $R(\lambda,A)$ for large real λ implies that A is a generator (of a positive semigroup). This is surprising, because it means that in the case when the resolvent is positive, the norm condition on the resolvent $\sup\{\|(\lambda-w)^n R(\lambda,A)^n\| : n \in \mathbb{N}, \lambda \geq 0\} < \infty$ which appears in the Hille-Yosida theorem (A-II, Thm.1.7) is automatically fulfilled.

Theorem 1.8. Let K be compact and A be a densely defined operator on $C(K)$. Suppose that there exists $w \in \mathbb{R}$ such that $[w,\infty) \subset \rho(A)$ and $R(\lambda,A) \geq 0$ for all $\lambda \geq w$. Then A is the generator of a strongly continuous positive semigroup. Moreover,

$$\omega(A) \leq w \,. \tag{1.3}$$

Proof. a) Assume that $w < 0$. Denote by 1 the constant-1-function. Let $u = R(0,A)1$. We claim that $u \gg 0$. If not, then there exists $x \in K$ such that $u(x) = 0$. Let $f \in C(K)$. Then $|f| \leq \|f\|1$. Consequently, $|R(0,A)f| \leq R(0,A)|f| \leq \|f\|R(0,A)1 = \|f\|u$. Hence $(R(0,A)f)(x) = 0$ for all $f \in C(K)$. Since $D(A) = R(0,A)C(K)$, it follows that $D(A)$ is not dense, a contradiction. Define $\|f\|_o = \inf\{\lambda > o : |f| \leq \lambda u\} = \|f/u\|_\infty$. Then $\|\ \|_o$ is an equivalent norm on $C(K)$. Moreover, $\|f\|_o \leq 1$ if and only if $f \in [-u,u]$. By the resolvent equation we have
$\lambda R(\lambda,A)u = \lambda R(\lambda,A)R(0,A)1 = R(0,A)1 - R(\lambda,A)1 \leq R(0,A)1 = u$ for all $\lambda \geq 0$. This implies that $\lambda R(\lambda,A)$ is contractive for the norm $\|\ \|_o$. Thus by the Hille-Yosida Theorem A is the generator of a semigroup which is contractive with respect to the norm $\|\ \|_o$, and so is bounded with respect to the supremum norm on $C(K)$.
b) If w is arbitrary, let $\lambda > w$ and consider $A - \lambda$. Then $[w-\lambda,\infty) \subset \rho(A-\lambda)$ and $R(\mu,A-\lambda) = R(\mu+\lambda,A) \geq 0$ for all $\mu \in [w-\lambda,\infty)$. Thus by a), $A - \lambda$ is the generator of a bounded positive semigroup. Consequently, A is a generator as well and $\omega(A) \leq \lambda$.

□

In Theorem 1.8 it is enough to assume that $R(\lambda_n,A) \geq 0$ for some sequence $(\lambda_n) \subset \rho(A) \cap \mathbb{R}$ satisfying $\lim_{n\to\infty} \lambda_n = \infty$. This follows from the following lemma.