In the above example both subsets $\sigma_1$ , $\sigma_2$ of $\sigma(A)$ are unbounded. But as soon as one of these sets is bounded a corresponding spectral decomposition can always be found.

Theorem 3.3. Let $\mathcal{T}$ be a strongly continuous semigroup on a Banach space $E$ and assume that the spectrum $\sigma(A)$ of the generator $A$ can be decomposed into the disjoint union of two non-empty closed subsets $\sigma_1$ , $\sigma_2$ . If $\sigma_1$ is compact then there exists a unique corresponding spectral decomposition $E = E_1 \oplus E_2$ such that the restricted semigroup $\mathcal{T}_1$ has a bounded generator.

Proof. We assume the reader to be familiar with the spectral decomposition theorem for bounded operators (see e.g. [Dunford-Schwartz (1958), p.572]) and apply the "spectral mapping theorem" for the resolvent (A-III,Thm.2.5) in order to decompose $R(\lambda,A)$ instead of $A$ : For $\lambda_0 > \omega(\mathcal{T})$ it follows from A-III,Thm.2.5 that $\sigma(R(\lambda_0,A)) \setminus \{0\} = (\lambda_0 - \sigma(A))^{-1}$ . From $\sigma(A) = \sigma_1 \cup \sigma_2$ we obtain a decomposition of $\sigma(R(\lambda_0,A)) \setminus \{0\}$ into

$$\tau_1 := (\lambda_0-\sigma_1)^{-1} \quad , \quad \tau_2 := (\lambda_0-\sigma_2)^{-1} .$$

Since $\sigma_1$ is compact the set $\tau_1$ is compact and does not contain $0$ . Only in the case that $\sigma_2$ is unbounded the point $0$ will be an accumulation point of $\tau_2$ . Therefore $\sigma(R(\lambda_0,A)) \cup \{0\}$ is the disjoint union of the closed sets $\tau_1$ and $\tau_2 \cup \{0\}$ .
Take now $P$ to be the spectral projection of $R(\lambda_0,A)$ corresponding to this decomposition. Then $P$ commutes with $R(\lambda_0,A)$ (by definition), with $R(\lambda,A)$ for every $\lambda > \omega(\mathcal{T})$ (use the series representation of the resolvent), with $T(t)$ for each $t \geq 0$ (use A-II, Prop.1.10) and therefore with the generator $A$ (in the sense explained above). In particular, we obtain

$$R(\lambda_0,A)P = R(\lambda_0,A_1) \quad , \quad R(\lambda_0,A)(Id-P) = R(\lambda_0,A_2)$$

for the generator $A_1$ of $\mathcal{T}_1 = (T(t)P)_{t\geq 0}$ and $A_2$ of $\mathcal{T}_2 = (T(t)(Id-P))_{t\geq 0}$ . Applying the Spectral Mapping Theorem 2.5 we conclude

$$\sigma(A_1) = \sigma_1 \quad \text{and} \quad \sigma(A_2) = \sigma_2 ,$$

i.e., $P$ is a spectral projection corresponding to $\sigma_1$ , $\sigma_2$ .
Finally, the above spectral decomposition of $R(\lambda_0,A)$ is unique and satisfies $0 \notin \sigma(R(\lambda_0,A_1))$ . Therefore $R(\lambda_0,A_1)^{-1} = (\lambda_0-A_1)$ is bounded.

□