Theorem 1.3. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E. Then, for every $f \in E$,

(1.2) $$\omega(f) = \limsup_{t\to\infty} 1/t\cdot\log\|T(t)f\| ,$$

and

(i) $\omega(f) = \inf\{\mathrm{Re}\,\lambda : \int_0^\infty \|e^{-\lambda t} T(t)f\|\,dt \text{ exists}\}$.

If $\ker A = \{0\}$, then for every $f \in D(A)$ we have

(ii) $\omega(f) = \inf\{\mathrm{Re}\,\lambda : \int_0^\infty e^{-\lambda t} T(t)Af\,dt$ exists as an improper Riemann integral$\}$.

Proof. The proof of (1.2) is omitted (see Hille-Phillips (1957), p.306). In order to prove (i) and (ii) we need the following lemma.

Lemma. Let $F \in C(\mathbb{R}^+,\mathbb{R}^+)$ be such that $\int_0^\infty F(t)\,dt$ exists. If there is a positive number m and an interval $[0,n]$ such that $F(t + s) \leq m\cdot F(s)$ for all $s \geq 0$ and $t \in [0,n]$, then $\lim_{s\to\infty} F(s) = 0$.

Proof of the lemma. For all $\varepsilon > 0$ there exists $a > 0$ such that $A(a) := \int_a^\infty F(s)\,ds \leq \frac{n}{m}\cdot\varepsilon$. For all $t > a+n$ there exists $r \in [t-n,t]$ such that $F(r) \leq \frac{1}{n}\cdot A(a)$.
Therefore, $F(t) = F(t-r+r) \leq m\cdot F(r) \leq \frac{m}{n}\cdot A(a) \leq \varepsilon$.

□

In order to prove (i) we define $b := \inf\{\mathrm{Re}\lambda : \int_0^\infty \|e^{-\lambda t} T(t)f\|\,dt$ exists$\}$. A straightforward application of the lemma shows that $\omega(f) \leq b$. The definition of $\omega(f)$ gives the reverse inequality.
It remains to prove statement (ii) of Thm.1.3.
Assume that $\ker A = \{0\}$ and let $f \in D(A)$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \omega(f)$.
From the equation

$$\int_0^t e^{-\lambda s} T(s)Af\,ds = e^{-\lambda t} T(t)f - f + \lambda\int_0^t e^{-\lambda s} T(s)f\,ds$$

it follows that $\int_0^\infty e^{-\lambda s} T(s)Af\,ds$ exists.
Therefore $b := \inf\{\mathrm{Re}\,\lambda : \int_0^\infty e^{-\lambda t} T(t)Af\,dt \text{ exists}\} \leq \omega(f)$.

Next we show that $b < 0$ implies $b \leq \omega(f)$. Suppose $b < 0$. Then, by (1.1), $\int_0^\infty T(s)Af\,ds$ exists. By $\int_0^r T(s)Af\,ds = T(r)f - f$ we see that $\lim_{r\to\infty} T(r)f =: g$ exists. But, for every $t \geq 0$, $T(t)g = g$ and therefore $g \in \mathrm{Ker}\,A$ or $g = 0$. Hence $\int_t^\infty T(s)Af\,ds = -T(t)f$.
Then choosing r, $b < r < 0$, and integrating by parts we obtain