Thus Kato's equality holds and it follows from Corollary 5.8 that $(T(t))_{t\geq 0}$ is a lattice semigroup. The other implication follows directly from Corollary 5.8.

□

Example 5.10. Let $E = L^p(X,\mu)$ (where (X,μ) is a σ-finite measure space and $1 \leq p < \infty$) and let A_o be the generator of a semigroup of lattice homomorphisms. Let $h \in L^\infty$ and $B = A_o + h$ (i.e., B is given by $Bf = A_o f + h\cdot f$ for $f \in D(B) = D(A_o)$). Let $A = A_o + \operatorname{Re} h$. Since A_o generates a semigroup of lattice homomorphisms, we have $|f| \in D(A_o)$ whenever $f \in D(A_o)$ and $\operatorname{Re}((\widehat{\operatorname{sign}}\ \bar{f})A_o f) = A_o|f|$. Hence $\operatorname{Re}((\widehat{\operatorname{sign}}\ \bar{f})Bf) = \operatorname{Re}((\widehat{\operatorname{sign}}\ \bar{f})A_o f) + (\operatorname{Re} h)\cdot|f|) =$ $A_o|f| + (\operatorname{Re} h)\cdot|f| = A|f|$ for all $f \in D(B)$. Thus it follows from Theorem 5.5 that B generates a disjointness preserving semigroup whose modulus semigroup is generated by A.

Next we describe when a disjointness preserving semigroup is positive.

Proposition 5.11. Let E be a complex Banach lattice with order continuous norm and B be the generator of a disjointness preserving semigroup $(S(t))_{t\geq 0}$. The semigroup is positive if and only if B is real and $\operatorname{span} D(B)_+ = D(B)$.

Proof. The conditions are clearly necessary. In order to prove sufficiency, we can assume that E is real. Denote by A the generator of $(T(t))_{t\geq 0}$, where $T(t) = |S(t)|$. Let $f \in D(B)_+$. Since B is local we have $Bf = P_f\, Bf = (\operatorname{sign} f)Bf = A|f| = Af$. By assumption, $\operatorname{span} D(B)_+ = D(B)$. Thus it follows that $B \subset A$. This implies that $B = A$ since $\rho(B) \cap \rho(A) \neq \emptyset$.

□

Remark 5.12. If B is the generator of a disjointness preserving semigroup $(S(t))_{t\geq 0}$ on a real Banach lattice E with order continuous norm then Kato's inequality holds in the reverse sense; i.e.,

$$\langle(\operatorname{sign} f)Bf,\phi\rangle \geq \langle|f|,B'\phi\rangle \quad \text{for all } f \in D(B),\ \phi \in D(B')_+ .$$

(cf. (3.9) for a concrete example). In fact, let $T(t) = |S(t)|$ and denote by A the generator of $(T(t))_{t\geq 0}$. Let $f \in D(B)$, $\phi \in D(B')_+$. Then $\langle(\operatorname{sign} f)Bf,\phi\rangle = \langle A|f|,\phi\rangle = \lim_{t\to 0} (1/t)\ \langle T(t)|f| - |f|,\ \phi\rangle \geq$ $\lim_{t\to 0} 1/t\ \langle S(t)|f| - |f|,\ \phi\rangle = \langle|f|,\ B'\phi\rangle$.