Then p_u is a strict half-norm on $C(K)$ (see A-II, Sec. 2). Note that

(1.5) $\quad p_u(f)u - f \geq 0 \qquad (f \in C(K))$.

For $x \in K$, define $\phi_x \in C(K)'$ by $\langle f,\phi_x\rangle = f(x)/u(x)$.
Let $f \in C(K)$ such that $-f$ is not strictly positive. Then there exists $x \in K$ such that $f(x)/u(x) = p_u(f)$. For such an x we have

(1.6) $\quad \phi_x \in dp_u(f)$

(see A-II,Sec.2 for the definition of the subdifferential dp_u).

Note that for $f \in C(K)$ one has $f \geq 0$ if and only if $p_u(-f) \leq 0$ (i.e., the half-norm p_u induces the given ordering on $C(K)$ (cf. A-II,Rem.2.8)). As a consequence, every p_u-contractive bounded operator T on $C(K)$ is positive.

<u>Proposition</u> 1.10. Let A be a densely defined operator on $C(K)$. Then there exists a strictly positive $u \in D(A)$. For any such u the following assertions are equivalent.
(i) A is p_u-dissipative.
(ii) $Au \leq 0$ and A satisfies (P).

<u>Proof</u>. Since $\{u \in C(K) : u \gg 0\}$ is open and non-empty and $D(A)$ is dense, there exists $0 \ll u \in D(A)$.
(i) implies (ii). One has $p_u(u) = 1$. Let $x \in K$. It follows from (1.6) that $\phi_x \in dp_u(u)$. Since $D(A)$ is dense, it follows from A-II, Thm. 2.7 that A is strictly p_u-dissipative. Hence $\langle Au,\phi_x\rangle \leq 0$. Thus $(Au)(x) \leq 0$. We now show (P). Let $0 \leq f \in D(A)$ and $x \in K$ such that $f(x) = 0$. We have to show that $(Af)(x) \geq 0$. Since $f(x) = 0$ and $p_u(-f) = 0$ we have by (1.6) $\phi_x \in dp_u(-f)$. Since A is strictly p_u-dissipative we conclude that $-u(x)^{-1}(Af)(x) = \langle A(-f),\phi_x\rangle \leq 0$. Hence $(Af)(x) \geq 0$.
(ii) implies (i). Let $f \in D(A)$. If $p_u(f) = 0$, then $\phi := 0 \in dp_u(f)$ and $\langle Af,\phi\rangle \leq 0$. If $p_u(f) > 0$, then there exists $x \in K$ such that $\phi_x \in dp_u(f)$. Hence, $0 \leq p_u(f)u - f$ and $(p_u(f)u - f)(x) = 0$. It follows from (P) that $p_u(f)(Au)(x) - (Af)(x) \geq 0$. Hence $(Af)(x) \leq p_u(f)(Au)(x) \leq 0$ (by (ii)); i.e., $\langle Af,\phi_x\rangle \leq 0$.

□

<u>Corollary</u> 1.11. Let A be a densely defined operator on $C(K)$. If A satisfies (P) then A is closable and the closure of A satisfies (P) as well.