Therefore $\|T(t)R(0,A)^n f\| \le C' e^{qt}\|f\| \int_{-\infty}^{\infty} (q^2+s^2)^{-1}\, ds = M\cdot e^{qt}\cdot \|f\|$
or $\|T(t)f\| \le M\cdot e^{qt}\|A^n f\|$ for $f \in D(A^n)$.

□

In view of the characterizations given in Section 1 of A-II, the semigroups occuring in the theorem are holomorphic if $n = 1$. In this case one may apply (1.7) to obtain the stronger statement (1.8).

Instead of making assumptions on the resolvent of A we now take a different view and characterize the property "$\omega(A) < 0$" in terms of the semigroup $(T(t))_{t\ge 0}$ directly.

Proposition 1.10. Let A be the generator of the strongly continuous semigroup $(T(t))_{t\ge 0}$. Then the following statements are equivalent:

(a) $\omega(A) < 0$

(b) $\lim_{t\to\infty} \|T(t)\| = 0$

(c) $\|T(t')\| < 1$ for some $t' > 0$.

Proof. The only nontrivial implication (c) $\to$ (a) follows from
$\omega(A) = \lim_{t\to\infty} \frac{1}{t}\log\|T(t)\|$ (see A-I,(1.1)) and
$\frac{\log\|T(t)\|}{t} \le \frac{\log\|T(t')\|}{t'} + \frac{\log\|T(t)\|}{nt'+s}$ for $t = nt'+s$, $s \in [0,t')$.

□

Other less obvious characterizations of the property "$\omega(A) < 0$" are given in the next theorem. The equivalence of (a) and (c) is known as Datko's Theorem.

Theorem 1.11. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\ge 0}$ on a Banach space E . Then the following statements are equivalent:

(a) $\omega(A) < 0$.

(b) $s(A) < 0$ and there is $t_o > 0$ such that $|\lambda| < 1$ for every $\lambda \in A\sigma(T(t_o))$.

(c) For every (some) $p\ge 1$ exists $\int_o^\infty \|T(t)f\|^p\, dt$ for every $f \in E$.

Proof. The implication "(a) $\to$ (b)" follows from $r(T(t)) = e^{\omega(A)t} < 1$ and $s(A) \le \omega(A) < 0$. For the point and residual spectrum the spectral mapping theorem is valid (see A-III,Thm.6.3). The approximate point spectrum is closed, hence the additional information in (ii)