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One-parameter Semigroups of Positive Operators

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This Latex version of the book "One-Parameter Semigroups of Positive Operators" is dedicated to the memory of our co-authors, Heinrich P. Lotz, Ulrich Moustakas, and Ulf Schlotterbeck. Their contributions to the first edition remain an inspiration to us all. We miss their presence and remain grateful for the legacy they have left in this work.

Preface

As early as 1948 in the first edition of his fundamental treatise on Semigroups and Functional Analysis, E. Hille expressed the need for “developing an adequate theory of transformation semigroups operating in partially ordered spaces” (l.c., Foreword). In the meantime the theory of one-parameter semigroups of positive linear operators has grown continuously. Motivated by problems in probability theory and partial differential equations W. Feller (1952) and R. S. Phillips (1962) laid the first cornerstones by characterizing the generators of special positive semigroups. In the 60’s and 70’s the theory of positive operators on ordered Banach spaces was built systematically and is well documented in the monographs of H. H. Schaefer (1974) and A. C. Zaanen (1983). But in this process the original ties with the applications and, in particular, with initial value problems were at times obscured. Only in recent years an adequate and up-to-date theory emerged, largely based on the techniques developed for positive operators and thus recombining the functional analytic theory with the investigation of Cauchy problems having positive solutions to each positive initial value. Even though this development — in particular with respect to applications to concrete evolution equations in transport theory, mathematical biology, and physics — is far from being complete, the present volume is a first attempt to shape the multitude of available results into a coherent theory of one-parameter semigroups of positive linear operators on ordered Banach spaces.

The book is organized as follows. We concentrate our attention on three subjects of semigroup theory: *characterization*, *spectral theory* and *asymptotic behavior*. By *characterization*, we understand the problem of describing special properties of a semigroup, such as positivity, through the generator. By *spectral theory* we mean the investigation of the spectrum of a generator. *Asymptotic behavior* refers to the orbits of the initial values under a given semigroup and phenomena such as stability.

This program (characterization, spectral theory, asymptotic behavior) is worked out on four different types of underlying spaces:

(A) On Banach spaces — Here we present the background for the subsequent discussions related to order.

(B) On spaces $C_0(X)$ (X locally compact), which constitute an important class of ordered Banach spaces and where our results can be presented in a form which makes them accessible also for the non-expert in order-theory.
(C) On Banach lattices, which admit a rich theory and are still sufficiently general as to include many concrete spaces appearing in analysis; e.g., $C_0(X)$, $\mathcal{L}^p(k)$ or l^p.
(D) On non-commutative operator algebras such as C^*- or W^*-algebras, which are not lattice ordered but still possess an interesting order structure of great importance in mathematical physics.

In each of these cases we start with a short collection of basic results and notations, so that the contents of the book may be visualized in the form of a 4×4 matrix in a way which will allow "row readers" (interested in semigroups on certain types of spaces) and "column readers" (interested in certain aspects) to find a path through the book corresponding to their interest.

We display this matrix, together with the names of the authors contributing to the subjects defined through this scheme:

	I Basic Results	II Characterization	III Spectral Theory	IV Asymptotics
A. Banach Spaces	R. Nagel U. Schlotterbeck	W. Arendt H. P. Lotz	G. Greiner R. Nagel	F. Neubrander
B. $C_0(X)$	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas F. Neubrander
C. Banach Lattices	R. Nagel U. Schlotterbeck	G. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas R. Nagel F. Neubrander
D. Operator Algebras	U. Groh	U. Groh	U. Groh	U. Groh

This "matrix of contents" has been an indispensable guide line in our discussions on the scope and the spirit of the various contributions. However, we would not have succeeded in completing this manuscript, as a collection of independent contributions (personally accounted for by the authors), under less favorable conditions than we have actually met. For one thing, Rainer Nagel was an unfaltering and undisputed spiritus rector from the very beginning of the project. On the other hand we gratefully acknowledge the influence of Helmut H. Schaefer and his pioneering work on order structures in analysis. It was the team spirit produced by this common mathematical background which, with a little help from our friends, made it possible to overcome most difficulties.

We have prepared the manuscript with the aid of a word processor, but we confess that without the assistance of Klaus Kuhn the pitfalls of such a system would have been greater than its benefits.

The authors

Contents

Part A
One-parameter Semigroups on Banach Spaces

Chapter A-I
Basic Results on Semigroups on Banach Spaces

Since the basic theory of one-parameter semigroups can be found in several excellent books (e.g. Davies (1980), Goldstein (1985a), Pazy (1983) or Hille-Phillips (1957)) we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, to collect briefly some important definitions and results (Section 1), to present a list of standard examples (Section 2) and to discuss standard constructions of new semigroups from a given one (Section 3). In the entire chapter we denote by E a (real or) complex Banach space and consider one-parameter semigroups of bounded linear operators $T(t)$ on E. By this we understand a subset $\{T(t)\colon t \in \mathbb{R}_+\}$ of $L(E)$, usually written as $(T(t))_{t\geqslant 0}$, such that

$$\begin{aligned} T(0) &= Id \\ T(s+t) &= T(s)\cdot T(t) \text{ for all } s,t \in \mathbb{R}_+. \end{aligned}$$

In more abstract terms this means that the map $t \mapsto T(t)$ is a homomorphism from the additive semigroup $\mathbb{R}_+$ into the multiplicative semigroup $(L(E),\cdot)$. Similarly, a one-parameter group $(T(t))_{t\in\mathbb{R}}$ will be a homomorphic image of the group $(\mathbb{R},+)$ in $(L(E),\cdot)$.

1 Standard Definitions and Results

We consider a one-parameter semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E and observe that the domain $\mathbb{R}_+$ and the range $L(E)$ of the (semigroup) homomorphism $\tau\colon t \mapsto T(t)$ are topological semigroups for the natural topology on $\mathbb{R}_+$ and any one of the standard operator topologies on $L(E)$. We single out the strong operator topology on $L(E)$ and require τ to be continuous.

Definition 1.1 A one-parameter semigroup $(T(t))_{t\geqslant 0}$ is called strongly continuous if the map $t \mapsto T(t)$ is continuous for the strong operator topology on $L(E)$, i.e., $\lim_{t\to t_0} \|T(t)f - T(t_0)f\| = 0$ for every $f \in E$ and $t, t_0 \geqslant 0$.

Clearly one defines in a similar way weakly continuous, resp. uniformly continuous (compare A-II, Def. 1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology: From now on 'semigroup' always means strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

Remark 1.2 (i) A one-parameter semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E is strongly continuous if and only if for any $f \in E$ it is true that $T(t)f \to f$ as $t \to 0$.

(ii) For every strongly continuous semigroup $(T(t))_{t\geqslant 0}$ there exist constants $M \geqslant 1$, $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant M \cdot e^{\omega t}$ for every $t \geqslant 0$.

(iii) If $(T(t))_{t\geqslant 0}$ is a one-parameter semigroup such that $\|T(t)\|$ is bounded for $0 \leqslant t \leqslant \delta$ then it is strongly continuous if and only if $\lim_{t\to 0} T(t)f = f$ for every f in a total subset of E.

Definition 1.3 By the growth bound (or type) of the semigroup $(T(t))_{t\geqslant 0}$ we understand the number

$$\begin{aligned}\omega_0 &:= \inf\{w \in \mathbb{R}\colon \text{There exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leqslant Me^{wt} \text{ for } t \geqslant 0\} \qquad (*)\\ &= \lim_{t\to\infty} \frac{1}{t} \cdot \log \|T(t)\| = \inf_{t>0} \frac{1}{t} \cdot \log \|T(t)\|\end{aligned}$$

Particularly important are semigroups such that for every $t \geqslant 0$ we have $\|T(t)\| \leqslant M$ (bounded semigroups) or $\|T(t)\| \leqslant 1$ (contraction semigroups). In both cases we have $\omega_0 \leqslant 0$.

It follows from the subsequent examples and from 3.1 that ω_0 may be any number $-\infty \leqslant \omega_0 < +\infty$. Moreover the reader should observe that the infimum in $(*)$ need not be attained and that M may be larger than 1 even for bounded semigroups.

Example 1.4 (i) Take $E = \mathbb{C}^2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then for the 1-norm on E we obtain $\|T(t)\| = 1+t$, hence $(T(t))_{t\geqslant 0}$ is an unbounded semigroup having growth bound $\omega_0 = 0$.

(ii) Take $E = L^1(\mathbb{R})$ and for $f \in E$ and $t \geqslant 0$ define

$$T(t)f(x) := \begin{cases} 2 \cdot f(x+t) & \text{if } x \in [-t, 0] \\ f(x+t) & \text{otherwise.} \end{cases}$$

Each $T(t)$, $t > 0$, satisfies $\|T(t)\| = 2$ as can be seen by taking $f := 1_{[0,t]}$. Therefore $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup which is bounded, hence has $\omega_0 = 0$, but the constant M in $(*)$ cannot be chosen to be 1.

Definition 1.5 To every semigroup $(T(t))_{t\geqslant 0}$ there belongs an operator $(A, D(A))$, called the generator and defined on the domain

$$D(A) := \left\{ f \in E\colon \lim_{h\to 0} \frac{T(h)f - f}{h} \text{ exists in } E \right\}$$

by $Af := \lim_{h\to 0} \frac{T(h)f-f}{h}$ for $f \in D(A)$.

Clearly, $D(A)$ is a linear subspace of E and A is linear from $D(A)$ into E. Only in certain special cases (see 2.1) the generator is everywhere defined and therefore bounded (use Prop. 1.9(i)). In general the precise extent of the domain $D(A)$ is essential for the characterisation of the generator. But since the domain is canonically associated to the generator of a semigroup we shall write in most cases A instead of $(A, D(A))$.

Proposition 1.6 *For the generator A of a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E the following assertions hold:*

(i) If $f \in D(A)$ then $T(t)f \in D(A)$ for every $t \geqslant 0$.
(ii) The map $t \mapsto T(t)f$ is differentiable on $\mathbb{R}_+$ if and only if $f \in D(A)$. In that case one has

$$\frac{d}{dt}T(t)f = AT(t)f = T(t)Af. \tag{A-I.1}$$

(iii) Every $f \in E$ one has $\int_0^t T(s)f\,ds \in D(A)$ and

$$A\int_0^t T(s)f\,ds = T(t)f - f. \tag{A-I.2}$$

(iv) If $f \in D(A)$ then

$$T(t)f = f + \int_0^t AT(s)f\,ds = f + \int_0^t T(s)Af\,ds.$$

(v) The domain $D(A)$ is dense in E.

Theorem 1.7 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on the Banach space E. Then the 'abstract Cauchy problem' (ACP)*

$$\frac{d}{dt}\xi(t) = A\xi(t), \quad \xi(0) = f_0,$$

has a unique solution $\xi\colon \mathbb{R}_+ \to D(A)$ in $C^1(\mathbb{R}_+, E)$ for every $f_0 \in D(A)$. In fact, this solution is given by $\xi(t) := T(t)f_0$.

For the important relation of semigroups to abstract Cauchy problems we refer to A-II, Section 1. Here we only point out that the above theorem implies that a semigroup is uniquely determined by its generator. While the generator is bounded only for uniformly continuous semigroups (see 2.1 below), it always enjoys a weaker but useful property.

Definition 1.8 An operator B with domain $D(B)$ on a Banach space E is called closed if $D(B)$ endowed with the graph norm

$$\|f\|_B := \|f\| + \|Bf\|$$

becomes a Banach space. Equivalently, $(B, D(B))$ is closed if and only if its graph $\{(f, Bf) \colon f \in D(B)\}$ is closed in $E \times E$, i.e.

$$f_n \in D(B),\ f_n \to f \text{ and } Bf_n \to g \text{ implies } f \in D(B) \text{ and } Bf = g. \qquad \text{(A-I.3)}$$

It is clear from this definition that the 'closedness' of an operator B depends very much on the size of the domain $D(B)$. For example, a bounded and densely defined operator $(B, D(B))$ is closed if and only if $D(B) = E$. On the other hand it may happen that $(B, D(B))$ is not closed but has a closed extension $(C, D(C))$, i.e., $D(B) \subset D(C)$ and $Bf = Cf$ for every $f \in D(B)$. In that case, B is called closable, a property which is equivalent to the following:

$$f_n \in D(B),\ f_n \to 0 \text{ and } Bf_n \to g \text{ implies } g = 0. \qquad \text{(A-I.4)}$$

The smallest closed extension of $(B, D(B))$ will be called the closure $\bar{B}$ with domain $D(\bar{B})$. In other words, the graph of $\bar{B}$ is the closure of $\{(f, Bf) \colon f \in D(B)\}$ in $E \times E$. Finally we call a subset D_0 of $D(B)$ a core for B if D_0 is $\|\cdot\|_B$-dense in $D(B)$. This means that a closed operator is determined (via closure) by its restriction to a core in its domain.

Proposition 1.9 *For the generator A of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$ the following holds:*

(i) The generator A is a closed operator.
(ii) If a subspace D_0 of the domain $D(A)$ is dense in E and $(T(t))$-invariant, then it is a core for A.
(iii) Define $D(A^n) := \{f \in D(A^{n-1}) \colon Af \in D(A^{n-1})\}$, $D(A^1) = D(A)$. Then $D(A^\infty) := \bigcap_{n \in \mathbb{N}} D(A^n)$ is dense in E and a core for A.

Example 1.10 Property (iii) above does not hold for general densely defined closed operators. Take $E = C[0,1]$, $D(B) = C^1[0,1]$ and $Bf = q \cdot f$ for some nowhere differentiable function $q \in C[0,1]$. Then B is closed, but $D(B^2) = (0)$.

Proposition 1.11 *For the generator A of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$ on a Banach space E the following holds. If $\int_0^\infty e^{-\lambda t} T(t) f\, dt$ exists for every $f \in E$ and some $\lambda \in \mathbb{C}$, then $\lambda \in \varrho(A)$ and $R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f\, dt$. In particular,*

$$R(\lambda, A)^{n+1} f = \frac{(-1)^n}{n!} \left(\frac{d}{d\lambda}\right)^n R(\lambda, A) f = \int_0^\infty e^{-\lambda t} \frac{t^n}{n!} T(t) f\, dt \qquad \text{(A-I.5)}$$

for $n \in \mathbb{N}$, $f \in E$ and all λ with $\operatorname{Re} \lambda > \omega$.

Remark 1.12 (i) For continuous Banach space valued functions such as $t \mapsto T(t)f$ we consider the Riemann integral and define $\int_0^\infty T(t) f\, dt$ as $\lim_{t \to \infty} \int_0^t T(s) f\, ds$. Sometimes such integrals for strongly continuous semigroups $(T(t))_{t \geqslant 0}$ are written as $\int_a^b T(t)\, dt$ and understood in the strong sense.

(ii) Since the generator $(A, D(A))$ determines the semigroup $(T(t))_{t>0}$ uniquely, we will speak occasionally of the growth bound of the generator instead of the semigroup, i.e., we write $\omega = \omega(A) = \omega((T(t))_{t\geqslant 0})$.

(iii) For one-parameter groups it might seem to be more natural to define the generator as the 'derivative' rather than just the 'right derivative' at $t = 0$. This yields the same operator as the following result shows: The strongly continuous semigroup $(T(t))_{t\geqslant 0}$ with generator A can be extended to a strongly continuous one-parameter group $(U(t))_{t\in\mathbb{R}}$ if and only if $-A$ generates a semigroup $(S(t))_{t\geqslant 0}$. In that case $(U(t))_{t\in\mathbb{R}}$ is obtained as

$$U(t) := \begin{cases} T(t) & \text{for } t \geqslant 0 \\ S(-t) & \text{for } t \leqslant 0 \end{cases}$$

We refer to [Davies (1980), Prop. 1.14] for the details.

2 Standard Examples

In this section we list and discuss briefly the most basic examples of semigroups together with their generators. These semigroups will reappear throughout this book and will be used to illustrate the theory. We start with the class of semigroups mentioned after Definition 1.1.

2.1 Uniformly Continuous Semigroups

It follows from elementary operator theory that for every bounded operator $A \in L(E)$ the sum exists and determines a unique uniformly continuous (semi)group $(e^{tA})_{t\in\mathbb{R}}$ having A as its generator.

$$\sum_{n=0}^{m} \frac{t^n A^n}{n!} =: e^{tA}$$

Conversely, any uniformly continuous semigroup is of this form: If the semigroup $(T(t))_{t\geqslant 0}$ is uniformly continuous, then $\frac{1}{t}\int_0^t T(s)ds$ uniformly converges to $T(0) = Id$ as $t \longrightarrow 0$. Therefore for some $t' \neq 0$ the operator $\frac{1}{t}\int_0^t T(s)ds$ is invertible and every $f \in E$ is of the form $f = \frac{1}{t}\int_0^{t'} T(s)g ds$ for some $g \in E$. But these elements belong to $D(A)$ by (1.3), hence $D(A) = E$. Since the generator A is closed and everywhere defined it must be bounded. Remark that bounded operators are always generators of groups, not just semigroups. Moreover the growth bound ω satisfies $|\omega| \leqslant \|A\|$ in this situation.

The above characterization of the generators of uniformly continuous semigroups as the bounded operators shows that these semigroups are - at least in many aspects - rather simple objects.

2.2 Matrix Semigroups

The above considerations especially apply in the situation $E = \mathbb{C}^n$. If $n = 2$ and $A = (a_{ij})_{2\times 2}$ the following explicit formulas for e^{tA} might be of interest: Set $s :=$ trace A, $d :=$ det A and $D := (s^2 - 4d)^{1/2}$. Then

$$e^{tA} = e^{ts/2} \cdot [D^{-1} 2 \sinh(tD/2) \cdot A + (\cosh(tD/2) - sD^{-1} \sinh(tD/2)) \cdot Id]$$

if $D \neq 0$ and

$$e^{tA} = e^{ts/2} \cdot [tA + (1 - ts/2) \cdot Id]$$

if $D = \emptyset$ resp.

2.3 Multiplication Semigroups

Many Banach spaces appearing in applications are Banach spaces of (real or) complex valued functions over a set X. As the most standard of these "function spaces", we mention the space $C_0(X)$ of all continuous complex valued functions vanishing at infinity on a locally compact space X, or the space $L^p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, of all (equivalence classes of) p-integrable functions on a σ-finite measure space (X, Σ, μ).

On these function spaces $E = C_0(X)$, resp. $E = L^p(X, \Sigma, \mu)$, there is a simple way to define "multiplication operators": Take a continuous, resp. measurable function $q \colon X \to \mathbb{C}$ and define

$$M_q f := q \cdot f, \text{ i.e. } M_q f(x) := q(x) \cdot f(x) \text{ for } x \in X,$$

for every f in the "maximal" domain $D(M_q) := \{g \in E \colon q \cdot g \in E\}$.

For such multiplication operators many properties can be checked quite directly. For example, the following statements are equivalent:

(a) M_q is bounded.
(b) q is (p-essentially) bounded.

One has $\|M_q\| = \|q\|_\infty$ in this situation.

Observe that on spaces $C(K)$, K compact, there are no densely defined, unbounded multiplication operators.

By defining the multiplication semigroups

$$T(t)f(x) := \exp(t \cdot q(x)) f(x), x \in X, f \in E,$$

one obtains the following characterizations.

Proposition 2.13 *Let M_q be a multiplication operator on $E = C_0(X)$ or $E = L^p(X, \Sigma, \mu)$, $1 \leqslant p < \infty$. Then the properties (a) and (b), resp. (a') and (b'), are equivalent:*

(a) *M_q generates a strongly continuous semigroup.*
(b) $\sup\{\mathit{Re}\ q(x) \colon x \in X\} < \infty$.

(a′) *M_q generates a uniformly continuous semigroup.*
(b′) $\sup\{|q(x)| \colon x \in X\} < \infty$.

As a consequence one computes the growth bound of a multiplication semigroup as follows:

$$\omega = \sup\{\mathrm{Re}\ q(x) \colon x \in X\} \text{ in the case } E = C_0(X),$$

$$\omega = \text{ess-sup}\{\mathrm{Re}\ q(x) \colon x \in X\} \text{ in the case } E = L^p(\mu).$$

It is a nice exercise to characterize those multiplication operators which generate strongly continuous groups.

2.4 Translation (Semi)Groups

Let E be one of the following function spaces $C_0(\mathbb{R}_+)$, $C_0(\mathbb{R})$ or $L^p(\mathbb{R}_+)$, $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$. Define $T(t)$ to be the (left) translation operator

$$T(t)f(x) := f(x+t)$$

for $x,\ t \in \mathbb{R}_+$, resp. $x, t \in \mathbb{R}$ and $f \in E$. Then $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup, resp. group of contractions on E and its generator is the first derivative $\frac{d}{dx}$ with 'maximal' domain. In order to be more precise we have to distinguish the cases $E = C_0$ and $E = L^p$:

(i) The generator of the translation (semi)group on $E = C_0(\mathbb{R}_+)$ is

$$Af := \frac{d}{dx} f = f',$$

with domain

$$D(A) := \{f \in E \colon f \text{ differentiable and } f' \in E\}$$

Proof For $f \in D(A)$ it follows that for every $x \in \mathbb{R}_{(+)}$

$$\lim_{h\to 0} \frac{T(h)f(x) - f(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

(uniformly in x) and coincides with $Af(x)$. Therefore f is differentiable and $f' \in E$. On the other hand, take $f \in E$ differentiable such that $f' \in E$. Then

$$\left|\frac{f(x+h)-f(x)}{h}-f'(x)\right| \leqslant \frac{1}{h}\int_x^{x+h} |f'(y)-f'(x)|dy$$

where the last expression tends to zero uniformly in x as $h \to 0$. Thus $f \in D(A)$ and $f' = Af$. □

(ii) The generator of the translation (semi)group on $E = L^p(\mathbb{R}_{(+)})$, $1 \leqslant p < \infty$, is

$$Af := \frac{d}{dx}f = f',$$

with domain

$$D(A) := \{f \in E\colon f \text{ absolutely continuous}, f' \in E\}$$

2.5 Rotation Groups

On $E = C(\mathbb{T})$, resp. $E = L^p(\mathbb{T}, m)$, $1 \leqslant p < \infty$, m Lebesgue measure we have canonical groups defined by rotations of the unit circle $\mathbb{T}$ with a certain period. For $0 < \tau \in \mathbb{R}$ the operators

$$R_\tau(t)f(z) := f(e^{2\pi i t/\tau} \cdot z)$$

yield a group $(R_\tau(t))_{t\in\mathbb{R}}$ having period τ, i.e. $R_\tau(\tau) = Id$. As in Example 2.4 one shows that its generator has the form

$$D(A) = \{f \in E\colon f \text{ absolutely continuous}, f' \in E\},$$

$$Af(z) = (2\pi i/\tau) \cdot z \cdot f'(z).$$

An isomorphic copy of the group $(R_\tau(t))_{t\in\mathbb{R}}$ is obtained if we consider $E = \{f \in C[0,1]\colon f(0) = f(1)\}$, resp. $E = L^p([0,1])$ and the group of 'periodic translations'

$$T(t)f(x) := f(y) \text{ for } y \in [0,1],\, y = x + t \bmod 1$$

with generator

$$D(A) := \{f \in E\colon f \text{ absolutely continuous}, f' \in E\},$$

$$Af := f'.$$

2.6 Nilpotent Translation Semigroups

Take $E = L^p([0,\tau], m)$ for $1 \leqslant p < \infty$ and define

$$T(t)f(x) := \begin{cases} f(x+t) & \text{if } x+t \leqslant \tau \\ 0 & \text{otherwise} \end{cases}$$

Then $(T(t))_{t \geqslant 0}$ is a semigroup satisfying $T(t) = 0$ for $t \geqslant \tau$. Its generator is still the first derivative $A = \frac{d}{dx}$, but its domain is

$$D(A) = \{f \in L^p([0,\tau]) \colon f \text{ absolutely continuous}, f' \in L^p([0,\tau]), f(\tau) = 0\}.$$

In fact, if $f \in D(A)$ then f is absolutely continuous with $f' \in E$. By Prop.1.6.i it follows that $T(t)f$ is absolutely continuous and hence $f(\tau) = 0$.

2.7 One-dimensional Diffusion Semigroup

For the second derivative

$$Bf(x) := \frac{d^2}{dx^2} f(x) = f''(x)$$

we take the domain

$$D(B) := \{f \in C^2[0,1] \colon f'(0) = f'(1) = 0\}$$

in the Banach space $E = C[0,1]$. Then $D(B)$ is dense in $C[0,1]$, but closed for the graph norm. Obviously, each function

$$e_n(x) := \cos \pi n x, \quad n \in \mathbb{Z},$$

is contained in $D(B)$ and an eigenfunction of B pertaining to the eigenvalue $\lambda_n := -\pi^2 n^2$. The linear hull

$$\operatorname{span}\{e_n \colon n \in \mathbb{Z}\} =: E_0$$

forms a subalgebra of $D(B)$ which by the Stone-Weierstrass theorem is dense in E.

We now use e_n to define bounded linear operators

$$e_n \otimes e_n \colon f \to \left(\int_0^1 f(x) e_n(x) dx \right) e_n = \langle f, e_n \rangle e_n$$

satisfying $\|e_n \otimes e_n\| \leqslant 1$ and $(e_n \otimes e_n)(e_m \otimes e_m) = \delta_{n,m}(e_n \otimes e_n)$ for $n \in \mathbb{Z}$.

For $t > 0$ we define

$$\begin{aligned} T(t) &:= \sum_{n \in \mathbb{Z}} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n \\ &= e_0 \otimes e_0 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n \end{aligned}$$

or

$$T(t)f(x) = \int_0^1 k_t(x,y)f(y)dy$$
$$\text{where } k_t(x,y) = 1 + 2\sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t)\cos \pi n x \cos \pi n y.$$

Die Jacobi-Identität

$$\begin{aligned} w_t(x) &:= 1/(4\pi t)^{\frac{1}{2}} \sum_{m\in\mathbb{Z}} \exp(-(x+2m)^2/4t) \\ &= \frac{1}{2} + \sum_{n\in\mathbb{N}} \exp(-\pi^2 n^2 t)\cos \pi n x \end{aligned}$$

und trigonometrische Beziehungen zeigen, dass

$$k_t(x,y) = w_t(x+y) + w_t(x-y)$$

welches eine positive Funktion auf $[0,1]^2$ ist. Daher ist $T(t)$ ein beschränkter Operator auf $C[0,1]$ mit

$$\|T(t)\| = \|T(t)1\| = \sup_{x\in[0,1]} \int_0^1 k_t(x,y)dy = 1.$$

2.8 n-dimensional Diffusion Semigroup

On $E = L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, the operators

$$\begin{aligned} T(t)f(x) &:= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|x-y|^2/4t) f(y)dy \\ &:= \psi_t * f(x) \end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$ and $\psi_t(x) := (4\pi t)^{-n/2}\exp(-|x|^2/4t)$ form a strongly continuous semigroup.

In fact the integral exists for every $f \in L^p(\mathbb{R}^n)$, since ψ_t is an element of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of all rapidly decreasing smooth functions on $\mathbb{R}^n$.

Moreover,

$$\|T(t)f\|_p \leqslant \|\psi_t\|_1 \|f\|_p = \|f\|_p$$

by Young's inequality [Reed-Simon (1975), p.28], hence $\|T(t)\| \leqslant 1$ for every $t > 0$.

Next we observe that $\mathcal{S}(\mathbb{R}^n)$ is dense in E and invariant under each $T(t)$. Therefore we can apply the Fourier transformation $\mathcal{F}$ which leaves $\mathcal{S}(\mathbb{R}^n)$ invariant and yields

$$\mathcal{F}(\psi_t * f) = (2\pi)^{n/2}\mathcal{F}(\psi_t)\cdot\mathcal{F}(f) = (2\pi)^{n/2}\hat{\psi}_t \cdot \hat{f}$$

where $f \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f} = \mathcal{F} f \in \mathcal{S}(\mathbb{R}^n)$.

In other words, $\mathcal{F}$ transforms $(T(t)|_{\mathcal{S}(\mathbb{R}^n)})_{t\geqslant 0}$ into a multiplication semigroup on $\mathcal{S}(\mathbb{R}^n)$ which is pointwise continuous for the usual topology of $\mathcal{S}(\mathbb{R}^n)$. The generator, i.e. the right derivative at 0, of this semigroup is the multiplication operator

$$B\hat{f}(x) := -|x|^2 \hat{f}(x)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Applying the inverse Fourier transformation and observing that the topology of $\mathcal{S}(\mathbb{R}^n)$ is finer than the topology induced from $L^p(\mathbb{R}^n)$, we obtain that $(T(t))_{t\geqslant 0}$ is a semigroup which is strongly continuous (use Remark 1.2, (3)) and its generator A coincides with

$$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x_1, \ldots, x_n)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Since $\mathcal{S}(\mathbb{R}^n)$ is $(T(t))$-invariant we have determined the generator on a core of its domain (see Prop.1.9.ii).

In particular the above semigroup 'solves' the initial value problem for the "heat equation"

$$\frac{\partial}{\partial t} f(x,t) = \Delta f(x,t), \quad f(x,0) = f_0(x), \quad x \in \mathbb{R}^n.$$

For the analogous discussion of the unitary group on $L^2(\mathbb{R}^n)$ generated by –

$$C := i\Delta$$

we refer to Section IX.7 in Reed-Simon (1975).

3 Standard Constructions

Starting with a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E it is possible to construct new semigroups on spaces naturally associated with E. Such constructions will be important technical devices in many of the subsequent proofs. Although most of these constructions are rather routine, we present in the sequel a systematic account of them for the convenience of the reader.

We always start with a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E, and denote its generator by A on the domain $D(A)$.

3.1 Similar Semigroups

There is an easy way how to obtain different (but isomorphic) semigroups: Take any isomorphism $S \in L(E, F)$ between two Banach spaces E and F. Then

$$U(t) := ST(t)S^{-1}, \quad t \geqslant 0,$$

defines a semigroup on F which is strongly continuous if and only if $(T(t))_{t\geqslant 0}$ has this property. In this case the generator B of $(U(t))_{t\geqslant 0}$ is given by

$$B = SAS^{-1} \text{ with } D(B) = SD(A).$$

Chapter A-II
Characterization of Semigroups on Banach Spaces

In this chapter two different problems are treated:

(i) to characterize generators of strongly continuous semigroups;
(ii) to characterize various properties of strongly continuous semigroups in terms of their generators.

In Section 1 the first problem is solved by finding conditions on the Cauchy problem associated with A and also by finding conditions on the resolvent of A. The second problem is treated for a hierarchy of smoothness properties of the semigroup.

Contraction semigroups are considered in Section 2. Here, the first problem has a simple and extremely useful solution: A densely defined operator A is generator of a contraction semigroup if and only if A is dissipative and satisfies a range condition.

Our approach is quite general. We do not only consider contractions with respect to the norm but also with respect to "half-norms". This will allow us to obtain results on positive contraction semigroups simultaneously by choosing a suitable half-norm (cf. C-II,Sec.1).

The last section contains a surprising result: on certain Banach spaces (e.g., L^∞) only bounded operators are generators of strongly continuous semigroups.

1 The Abstract Cauchy Problem, Special Semigroups and Perturbation

Linear differential equations in Banach spaces are intimately connected with the theory of one-parameter semigroups. In fact, given a closed linear operator A with dense domain $D(A)$ the following statement is true (with some reservation regarding a technical detail): The abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) \quad (t \geqslant 0) \\ u(0) &= f \end{aligned}$$

has a unique solution for every $f \in D(A)$ if and only if A is the generator of a strongly continuous semigroup.

This is one characterization of generators which illustrates their important role for applications. The fundamental Hille-Yosida theorem gives a different characterization in terms of the resolvent and yields a powerful tool for actually proving that a given operator is the generator of a semigroup.

Another problem we will treat here is how diverse properties of a semigroup can be described in terms of its generator. This is a reasonable question from the theoretical point of view (since the generator uniquely determines the semigroup). It is of interest from the practical point of view as well: the generator is the given object, defined by the differential equation. It is useful to dispose of conditions of the generator itself giving information on the solutions (which might not be known explicitly). We discuss smoothness properties such as analyticity, differentiability, norm continuity and compactness of the semigroup.

A frequent method to obtain new generators out of a given one is by perturbation. We will have a brief look at this circle of problems at the end of this section.

The results are explained and illustrated by examples. Proofs are only given when new aspects are presented which are not yet contained in the literature, otherwise we refer to the recent monographs Davies (1980), Goldstein (1985a), Pazy (1983).

1.1 The Abstract Cauchy Problem

Let A be a closed operator on a Banach space E and consider the abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \dot{u}(t) &= Au(t) \quad (t \geqslant 0) \\ u(0) &= f. \end{cases}$$

By a solution of (ACP) for the initial value $f \in D(A)$ we understand a continuously differentiable function $u : [0, \infty) \to E$ satisfying $u(0) = f$ and $u(t) \in D(A)$ for all $t \geqslant 0$ such that $\dot{u}(t) = Au(t)$ for $t \geqslant 0$.

By A-I,Thm.1.7 there exists a unique solution of (ACP) for all initial values in the domain $D(A)$ whenever A is the generator of a strongly continuous semigroup. The converse does not hold (see Example 1.4. below). However, for the operator A_1 on the Banach space $E_1 = D(A)$ (see A-I,3.5) with domain $D(A_1) = D(A^2)$ given by $A_1 f = Af$ $(f \in D(A_1))$ the following holds.

Theorem 1.14 *The following assertions are equivalent.*

(a) For every $f \in D(A)$ there exists a unique solution of (ACP).
(b) A_1 is the generator of a strongly continuous semigroup.

Proof (i) implies (ii). Assume that (i) holds; i.e., for every $f \in D(A)$ there exists a unique solution $u(\cdot, f) \in C^1([0, \infty), E)$ of (ACP). For $f \in E_1$ define $T_1(t)f := u(t, f)$ $(t \geqslant 0)$. By the uniqueness of the solutions it follows that $T_1(t)$ is a linear operator on E_1 and $T_1(s+t) = T_1(s)T_1(t)$. Moreover, since $u(\cdot, f) \in C^1$, it follows that $t \mapsto T_1(t)f$ is continuous from $[0,\infty)$ into E_1. We show that $T_1(t)$ is a continuous operator for all $t > 0$.

Let $t > 0$. Consider the mapping $\eta : E_1 \to C([0,t], E_1)$ given by $\eta(f) = T_1(\cdot)f = u(\cdot, f)$. We show that η has a closed graph. In fact, let $f_n \to f$ in E_1 and $\eta(f_n) = u(\cdot, f_n) \to v$ in $C([0,t], E_1)$. Then $u(s, f_n) = f_n + \int_0^s Au(r, f_n)dr$. Letting $n \to \infty$ we obtain $v(s) = f + \int_0^s Av(r)dr$ for $0 \leqslant s \leqslant t$. Let $\tilde{v}(s) = T_1(s-t)v(t)$ for $s > t$, and $\tilde{v}(s) = v(s)$ for $0 \leqslant s \leqslant t$.

Then $\tilde{v}$ is a solution of (ACP). It follows that $\tilde{v}(s) = T_1(s)f$ for all $s \geqslant 0$. Hence $v = \eta(f)$. We have shown that η has a closed graph and so η is continuous. This implies the continuity of $T_1(t)$. It remains to show that A_1 is the generator of $(T_1(t))_{t\geqslant 0}$.

We first show that for $f \in D(A^2)$ one has

$$AT_1(t)f = T_1(t)Af. \tag{1.1}$$

In fact, let $v(t) = f + \int_0^t u(s, Af)ds$. Then $\dot{v}(t) = u(t, Af) = Af + \int_0^t Au(s, Af)ds = A(f + \int_0^t u(s, Af)ds) = Av(t)$. Since $v(0) = f$, it follows that $v(t) = u(t, f)$. Hence $Au(t, f) = Av(t) = \dot{v}(t) = u(t, Af)$. This is (1.1).

Now denote by B the generator of $(T_1(t))_{t\geqslant 0}$. For $f \in D(A^2)$ we have

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af$$

and by (1.1),

$$\lim_{t\to 0} A\frac{T_1(t)f - f}{t} = \lim_{t\to 0} \frac{T_1(t)Af - Af}{t} = A^2 f \text{ in the norm of } E.$$

Hence $\lim_{t\to 0} \frac{T_1(t)f-f}{t} = Af$ in the norm of E_1.

This shows that $A_1 \subset B$. In order to show the converse, let $f \in D(B)$. Then $\lim_{t\to 0} A\frac{T_1(t)f-f}{t}$ exists in the norm of E. Since $\lim_{t\to 0} \frac{T_1(t)f-f}{t} = Af$ in the norm of E, it follows that $Af \in D(A)$, since A is closed. Thus $f \in D(A^2) = D(A_1)$. We have shown that $B = A_1$.

(ii) implies (i). Assume that A_1 is the generator of a strongly continuous semigroup $(T_1(t))_{t\geqslant 0}$ on E_1. Let $f \in D(A)$ and set $u(t) = T_1(t)f$. Then $u \in C([0,\infty), E)$ and $Au(\cdot) \in C([0,\infty), E)$. Moreover, $\int_0^t u(s)ds = \int_0^t T_1(s)f ds \in D(A_1) = D(A^2)$ and $A\int_0^t u(s)ds = u(t) - u(0) = u(t) - f$ (by A-I, (1.3)). Consequently, $u(t) = f + A\int_0^t u(s)ds = f + \int_0^t Au(s)ds$. Hence $u \in C^1([0,\infty), E)$ and $\dot{u}(t) = Au(t)$. Thus u is a solution of (ACP).

In order to show uniqueness, assume that u is a solution of (ACP) with initial value 0. We have to show that $u \equiv 0$. Let $v(t) = \int_0^t u(s)ds$. Then $v(t) \in D(A)$ and $Av(t) = \int_0^t Au(s)ds = \int_0^t \dot{u}(s)ds = u(t) \in D(A)$. Consequently, $v(t) \in D(A^2)$ for all $t \geqslant 0$. Moreover, $\dot{v}(t) = u(t) = Av(t)$ and $\frac{d}{dt}Av(t) = Au(t) = A\dot{v}(t) = A^2 v(t)$.

Thus $v \in C^1([0,\infty), E_1)$ and $\dot{v}(t) = A_1 v(t)$. Since $v(0) = 0$, it follows that $v \equiv 0$. Thus $u \equiv v \equiv 0$. □

If (ACP) has a unique solution for every initial value in $D(A)$, then A is the generator of a strongly continuous semigroup only if some additional assumptions on the solutions (continuous dependence from the initial value) or on A ($\varrho(A) \neq \emptyset$) are made.

Corollary 1.15 *Let A be a closed operator. Consider the following existence and uniqueness condition.*

(EU) For every $f \in D(A)$ there exists a unique solution $u(\cdot, f) \in C^1([0,\infty), E)$ of the Cauchy problem associated with A having the initial value $u(0, f) = f$.

The following assertions are equivalent.

(a) A is the generator of a strongly continuous semigroup.
(b) A satisfies (EU) and $\varrho(A) \neq \emptyset$.
(c) A satisfies (EU) and for every $\mu \in \mathbb{R}$ there exists $\lambda > \mu$ such that $(\lambda - A)D(A) = E$.
(d) A satisfies (EU), has dense domain and for every sequence (f_n) in $D(A)$ satisfying $\lim_{n\to\infty} f_n = 0$ one has $\lim_{n\to\infty} u(t, f_n) = 0$ uniformly in $t \in [0, 1]$.

Proof It is clear that (i) implies the remaining assertions. So assume that A satisfy (EU). Then by Theorem 1.1., A_1 is a generator. If there exists $\lambda \in \varrho(A)$, then $(\lambda - A)$ is an isomorphism from E_1 onto E and A is similar to A_1 via this isomorphism [since $D(A_1) = \{(\lambda - A)^{-1} f : f \in D(A)\}$ and $Af = (\lambda - A)A_1(\lambda - A)^{-1} f$ for all $f \in D(A)$, see A-I,3.0]. Thus A is a generator on E and we have shown that (ii) implies (i).

If (iii) holds, then there exists $\lambda > s(A_1)$ such that $(\lambda - A)D(A) = E$. We show that $(\lambda - A)$ is injective. Then $\lambda \in \varrho(A)$ since A is closed. Assume that $Af = \lambda f$ for some $f \in D(A)$. Then $f \in D(A^2) = D(A_1)$, and so $f = 0$ since $\lambda \in \varrho(A_1)$. This proves that (iii) implies (ii). We have shown existence.

It remains to show that (iv) implies (i).

Assertion (iv) implies that for all $t \geqslant 0$ there exist bounded operators $T(t) \in \mathcal{L}(E)$ such that $u(t, f) = T(t)f$ if $f \in D(A)$. Moreover, $\sup_{0\leqslant t\leqslant 1} \|T(t)\| < \infty$. It follows that $T(\cdot)f$ is strongly continuous for all $f \in E$ (since it is so for $f \in D(A)$ and $D(A)$ is dense). Let $t > 1$. There exist unique $n \in \mathbb{N}$ and $s \in [0, 1)$ such that $t = n + s$. Let $T(t) := T(1)^n T(s)$. From the uniqueness of the solutions it follows that $T(t)f = u(t, f)$ for all $t \geqslant 0$ as well as $T(t + s)f = T(s)T(t)f$ for all $f \in D(A)$ and $s, t \geqslant 0$. Thus $(T(t))_{t\geqslant 0}$ is a semigroup. Denote by B its generator. It follows from the definition that $A \subset B$. Moreover, $D(A)$ is invariant under the semigroup. So by A-I,Prop.1.9.(ii) $D(A)$ is a core of B. Since A is closed this implies that $A = B$. □

Remark 1.16 It is surprising that from condition (ii) and (iii) in the corollary it follows automatically that $D(A)$ is dense. On the other hand this condition cannot be omitted in (iv). In fact, consider any generator $\tilde{A}$ and its restriction A with domain $D(A) = \{0\}$. Then A satisfies the remaining conditions in (iv) but is not a generator (if $\dim E > 0$).

Example 1.17 We give a densely defined closed operator A, such that there exists a unique solution of (ACP) for all initial values in $D(A)$, but A is not a generator. Let B be a densely defined unbounded closed operator on a Banach space F. Consider $E = F \oplus F$ and A on E given by

$$A = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

with domain $F \times D(B)$.

Then $D(A^2) = \{(f, g) \in F \times D(B) : Bg \in F\} = D(A)$ and so $A_1 \in \mathcal{L}(E_1)$. In particular, A_1 is a generator. But A is not. For instance condition (ii) in Corollary 1.2. does not hold, since for each $\lambda \in \mathbb{C}$,

$$(\lambda - A)D(A) = \{(\lambda f - Bg, \lambda g) : f \in F, g \in D(B)\} \subset F \times D(B) \neq F \times F = E.$$

So $\varrho(A) = \emptyset$.

As a further illustration, we note that the solution of the corresponding abstract Cauchy problem for the initial value $(f, g) \in F \times D(B)$ is given by $u(t) = (f + tBg, g)$. Since B is unbounded, condition (iv) of Corollary 1.2. is clearly violated.

Chapter A-III
Spectral Theory

1 Introduction

In this chapter we start a systematic analysis of the spectrum of a strongly continuous semigroup $T = (T(t))_{t \geqslant 0}$ on a complex Banach space E. By the spectrum of the semigroup we understand the spectrum $\sigma(A)$ of the generator A of T. In particular we are interested in precise relations between $\sigma(A)$ and $\sigma(T(t))$. The heuristic formula

$$\text{“}T(t) = e^{tA}\text{”}$$

serves as a leitmotiv and suggests relations of the form

$$\text{“}\sigma(T(t)) = e^{t\sigma(A)} = \{e^{t\lambda} : \lambda \in \sigma(A)\}\text{”},$$

called “spectral mapping theorem”. These - or similar - relations will be of great use in Chapter IV and enable us to determine the asymptotic behavior of the semigroup T by the spectrum of the generator.

As a motivation as well as a preliminary step we concentrate here on the spectral radius

$$r(T(t)) := \sup\{|\lambda| : \lambda \in \sigma(T(t))\}, \quad t \geqslant 0$$

and show how it is related to the spectral bound

$$s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\}$$

of the generator A and to the growth bound

$$\omega := \inf\{\omega \in \mathbb{R} : \|T(t)\| \leqslant M_\omega \cdot e^{\omega t} \text{ for all } t \geqslant 0 \text{ and suitable } M_\omega\}$$

of the semigroup $T = (T(t))_{t \geqslant 0}$. (Recall that we sometimes write $\omega(T)$ or $\omega(A)$ instead of ω). The Examples 1.3 and 1.4 below illustrate the main difficulties to be encountered.

Proposition 1.18 *Let ω be the growth bound of the strongly continuous semigroup $T = (T(t))_{t \geqslant 0}$. Then*

$$r(T(t)) = e^{\omega t}$$

for every $t \geqslant 0$.

Chapter A-IV
Asymptotics of Semigroups on Banach Spaces

In this chapter we study the asymptotic behavior of the solutions of the initial value problem

$$\dot{u}(t) = Au(t) + F(t), \quad u(0) = f$$

with respect to time $t \geqslant 0$. Here A will be a generator of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$ on a Banach space E and $F(\cdot)$ is a function from $\mathbb{R}_+$ with values in E.

In Section 1 we investigate whether and how fast a solution $T(\cdot)f$ of the homogeneous problem tends to the zero solution as $t \to \infty$. In Section 2 we consider the long term behavior of the solutions of $(*)$ for different classes of forcing terms F.

1 Stability: Homogeneous Case

Let $(T(t))_{t \geqslant 0}$ be a semigroup on E with generator A. An initial value $f \in D(A)$ is called *stable* if the solution $t \mapsto T(t)f$ of

$$\dot{u}(t) = Au(t), \quad u(0) = f$$

converges to zero as t tends to infinity. The semigroup is called *stable* if every solution converges to zero; i.e., if every initial value $f \in D(A)$ is stable.

If the space E is finite dimensional, then the stability of the semigroup implies that the decay is exponential. More precisely, the statements

(a) $\|T(t)f\| \to 0$ for every $f \in \mathbb{C}^n$,
(b) $\|T(t)\| \leqslant Me^{-\omega t}$ for some $\omega > 0$

Part B
Positive Semigroups on Spaces $C_0(X)$

Chapter B-I

Basic Results on $C_0(X)$

Chapter B-II

Characterization of Positive Semigroups on $C_0(X)$

Chapter B-III

Spectral Theory of Positive Semigroups on $C_0(X)$

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Part C
Positive Semigroups on Banach Lattices

Chapter C-I
Basic Results on Banach Lattices and Positive Operators

Chapter C-II

Characterization of Positive Semigroups on Banach Lattices and Positive Operators

Chapter C-III
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Chapter C-IV
Asymptotics of Positive Semigroups on Banach Lattices

Part D
Positive Semigroups on C*- and W*-Algebras

Chapter D-I
Basic Results on Semigroups and Operator Algebras

This is not a systematic introduction into the theory of strongly continuous semigroups on C*- and W*-algebras. For that we refer to [2], [3] and the survey article of [7]. We only prepare for the subsequent chapters on spectral theory and asymptotics by fixing the notations and introducing some standard constructions.

1 Notations

1. By M we shall denote a C*-algebra with unit 1. $M^{sa} := \{x \in M : x^* = x\}$ is the self-adjoint part of M and $M_+ := \{x^*x : x \in M\}$ the positive cone in M. If M' is the dual of M, then $M'_+ := \{\psi \in M' : \psi(x) \geqslant 0, x \in M_+\}$ is a weak*-closed generating cone in M'. $S(M) := \{\psi \in M'_+ : \psi(1) = 1\}$ is called the state space of M. For the theory of C*-algebras and related notions we refer to [8].

M is called a W*-algebra, if there exists a Banach space M_*, such that its dual $(M_*)'$ is (isomorphic to) M. We call M_* the predual of M and $\psi \in M_*$ a normal linear functional. It is known that M_* is unique [9, 1.13.3]. For further properties of M_* we refer to [10, Chapter III].

2. A map $T \in L(M)$ is called positive (in symbols $T \geqslant 0$) if $T(M_+) \subseteq M_+$. $T \in L(M)$ is called n-positive ($n \in \mathbb{N}$) if $T \otimes \mathrm{Id}_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where Id_n is the identity map on the C*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive. We call $T \in L(M)$ a Schwarz map if T satisfies the inequality

$$T(x)T(x)^* \leqslant T(xx^*)\,,\; x \in M.$$

Note that such T is necessarily a contraction. It is well known that every n-positive contraction, $n \geqslant 2$ and that every positive contraction on a commutative C*-algebra is a Schwarz map [10, Corollary IV. 3.8.]. As we shall see, the Schwarz inequality is crucial for our investigations.

3. If M is a C*-algebra we assume $T = (T(t))_{t\geqslant 0}$ to be a strongly continuous semigroup (abbreviated semigroup) while on W*-algebras we consider weak*-semigroups, i.e. the mapping $(t \mapsto T(t)x)$ is continuous from $\mathbb{R}_+$ into $(M, \sigma(M, M_*))$, M_* the predual of M, and every $T(t) \in T$ is $\sigma(M, M_*)$-continuous. Note that the preadjoint semigroup

$$T_* = \{T(t)_* : T(t) \in T\}$$

is weakly, hence strongly continuous on M_* (see e.g., Davies (1980), Prop.1.23). We call T identity preserving if $T(t)1 = 1$ and of Schwarz type if every $T(t) \in T$ is a Schwarz map.

For the notations concerning one-parameter semigroups we refer to Part A. In addition we recommend to compare the results of this section of the book with the corresponding results for commutative C*-algebras, i.e. for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ (see Part B).

2 A Fundamental Inequality for the Resolvent

If $T = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup of Schwarz maps on a C*-algebra M (resp. a weak*-semigroup of Schwarz type on a W*-algebra M) with generator A, then the spectral bound $s(A) \leqslant 0$. Then for $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$, there exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \,\mathrm{d}t\,,\; x \in M$$

where the integral exists in the norm topology.

In [2] it is shown that T is a semigroup of Schwarz type if and only if $\mu R(\mu, A)$ is a Schwarz map for every $\mu \in \mathbb{R}_+$. Here we relate the domination of two semigroups to an inequality for the corresponding resolvent operator. This inequality will be needed later.

Theorem 2.19 *Let $T = (T(t))_{t\geqslant 0}$ be a semigroup of Schwarz type and $T = (S(t))_{t\geqslant 0}$ a semigroup on a C*-algebra M with generators A and B, respectively. If*

$$(*) \qquad (S(t)x)(S(t)x)^* \leqslant T(t)xx^*$$

for all $x \in M$ and $t \in \mathbb{R}_+$, then

$$(\mu R(\mu, B)x)(\mu R(\mu, B)x)^* \leqslant \mu R(\mu, A)xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$. The same result holds if T is a weak-semigroup of Schwarz type and S is a weak*-semigroup on a W*-algebra M such that $(*)$ is fulfilled.*

Proof From the assumption $(*)$ it follows that

$$\begin{aligned} 0 &\leqslant (S(r)x - S(t)x)(S(r)x - S(t)x)^* = \\ &= (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \\ &\qquad \leqslant T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* \end{aligned}$$

for every $r, t \in \mathbb{R}_+$. Hence

$$(S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^* \leqslant T(r)xx^* + T(t)xx^*.$$

Obviously, $\|S(t)\| \leqslant 1$ for all $t \in \mathbb{R}_+$. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$:

$$\begin{aligned} (R(\mu, B)x)(R(\mu, B)x)^* &= (\int_0^\infty e^{-\mu r} S(r)x\,dr)(\int_0^\infty e^{-\mu t} S(t)x\ \mathrm{d}t)^* \\ &= (\int_0^\infty \int_0^\infty e^{-\mu(r+t)} (S(r)x)(S(t)x)^*\,dr\ \mathrm{d}t) \\ &\leqslant \int_0^\infty \int_0^\infty e^{-\mu(r+t)} (T(r)xx^* + T(t)xx^*)/2\,dr\ \mathrm{d}t \\ &= \int_0^\infty e^{-\mu r} T(r)xx^*\,dr \int_0^\infty e^{-\mu t}\ \mathrm{d}t \\ &= \mu^{-1} \int_0^\infty e^{-\mu r} T(r)xx^*\,dr = R(\mu, A)xx^*. \end{aligned}$$

Here we used the inequality derived above in the first step. The second step follows from $S(t)$ being a contraction semigroup and the third step is achieved by integration. □

Remark 2.20 The assumption that T is a semigroup of Schwarz type cannot be weakened in general to T being a positive contraction semigroup. This is shown by examples in [4] where $S(t)x$ is given by $e^{tB}x$ for a skew-adjoint generator B and $T(t)x \equiv x$.

Corollary 2.21 *Let $T = (T(t))_{t \geqslant 0}$ be a semigroup of Schwarz type on a $C^\star$-algebra M with generator A. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$:*

$$(\mu R(\mu, A)x)(\mu R(\mu, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof Just set $S = T$ in Theorem 2.19.

$$= \frac{1}{2}\Big(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}\big((S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^*\big)\,dr\,dt$$

$$\leqslant \frac{1}{2}\Big(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*)\,dr\,dt$$

$$= \Big(\int_0^\infty e^{-\mu s}\,ds\Big)\Big(\int_0^\infty e^{-\mu t}T(t)xx^*\,dt\Big) = \mu^{-1}R(\mu, A)xx^*$$

where the handling of the integral is justified by [1, §8, n° 4, Proposition 9]. □

Corollary 2.22 *Let T be a semigroup of Schwarz maps (resp., weak*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$:*

$$(R(\lambda, A)x)(R(\lambda, A)x)^* \leqslant (Re\lambda)^{-1}R(Re\lambda, A)xx^*\,,\ x \in M.$$

In particular for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$, $x \in M$:

$$(\mu R(\mu + i\alpha, A)x)(\mu R(\mu + i\alpha, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Then the semigroup

$$S := (e^{-i\mathrm{Im}(\lambda)t}T(t))_{t\geqslant 0}$$

fulfils the assumption of Thm 2.19 and $B := A - i\lambda$ is the generator of S. Consequently $R(\lambda, A) = R(\mathrm{Re}\lambda, B)$ and the corollary follows from Theorem 2.19.

As in Section C-III the following notion will be an important tool for the spectral theory of semigroups.

Definition 2.23 Let E be a Banach space and $\emptyset \neq D$ an open subset of $\mathbb{C}$. A family $R : D \to L(E)$ is called a pseudo-resolvent on D with values in E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu)$$

for all λ and μ in D.

If R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in a C*- or W*-algebra, then R is called of Schwarz type if

$$(R(\lambda)x)(R(\lambda)x)^* \leqslant (\mathrm{Re}\lambda)^{-1}R(\mathrm{Re}\lambda)xx^*$$

for all $\lambda \in D$ and $x \in M$. R is called identity preserving if $\lambda R(\lambda)1 = 1$ for all $\lambda \in D$.

For examples and properties of a pseudo-resolvent see C-III, 2.5. We state what will be used without further reference.

(a) If $\alpha \in \mathbb{C}$ and $x \in E$ such that $(\alpha - \lambda)R(\lambda)x = x$ for some $\lambda \in D$, then $(\alpha - \mu)R(\mu)x = x$ for all $\mu \in D$ (use the "resolvent equation").

(b) If F is a closed subspace of E such that $R(\lambda)F \subseteq F$ for some $\lambda \in D$, then $R(\mu)F \subseteq F$ for all μ in a neighbourhood of λ. This follows from the fact that for all $\mu \in D$ near λ the pseudo-resolvent in μ is given by

$$R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}.$$

Definition 2.24 We call a semigroup T on the predual M_* of a W*-algebra M identity preserving and of Schwarz type, if its adjoint weak*-semigroup has these properties. Likewise, a pseudo-resolvent R on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in M_* is called identity preserving and of Schwarz type, if R' has these properties.

Since for a semigroup of contractions on a Banach space

$$\begin{aligned}\mathrm{Fix}(T) &= \bigcap_{t \geqslant 0} \ker(\mathrm{Id} - T(t)) \\ &= \ker(\mathrm{Id} - \lambda R(\lambda, A)) = \mathrm{Fix}(\lambda R(\lambda, A))\end{aligned}$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, it follows that a semigroup of contractions on M is identity preserving if and only if the (pseudo)-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ given by

$$R(\lambda) := R(\lambda, A)|_D$$

is identity preserving. By Corollary 2.21 an analogous statement holds for "Schwarz type".

3 Induction and Reduction

1. If E is a Banach space and $\mathcal{S} \subseteq L(E)$ a semigroup of bounded operators, then a closed subspace F is called $\mathcal{S}$-invariant, if $SF \subseteq F$ for all $S \in \mathcal{S}$. We call the semigroup $\mathcal{S}|_F := \{S|_F : S \in \mathcal{S}\}$ the reduced semigroup. Note that for a one-parameter semigroup T (resp., pseudo-resolvent R) the reduced semigroup is again strongly continuous (resp. $R|_F$ is again a pseudo-resolvent) (compare the construction in A-I,3.2).

2. Let M be a W*-algebra, $p \in M$ a projection and $S \in L(M)$ such that $S(p^\perp M) \subseteq p^\perp M$ and $S(Mp^\perp) \subseteq Mp^\perp$, where $p^\perp := 1 - p$. Since for all $x \in M$:

$$p[S(x) - S(pxp)] = p[S(p^\perp xp) + S(xp^\perp)]p = 0,$$

we obtain $p(Sx)p = p(S(pxp))p$. Therefore the map

$$S_p := (x \mapsto p(Sx)p) : pMp \to pMp$$

is well defined. We call S_p the induced map. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$.

If $T = (T(t))_{t\geqslant 0}$ is a weak*-semigroup on M which is of Schwarz type and if $T(t)(p^{\perp}) \leqslant p^{\perp}$ for all $t \in \mathbb{R}_{+}$, then T leaves $p^{\perp}M$ and $Mp^{\perp}$ invariant. It is easy to see that the induced semigroup $T_p = (T(t)_p)_{t\geqslant 0}$ is again a weak*-semigroup.

If R is an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$ with values in M such that $R(\mu)p^{\perp} \leqslant p^{\perp}$ for some $\mu \in \mathbb{R}_{+}$, then $p^{\perp}M$ and $Mp^{\perp}$ are R-invariant. Again, the induced pseudo-resolvent R_p is of Schwarz type and identity preserving.

3. Let φ be a positive normal linear functional on a W*-algebra M such that $T_*\varphi = \varphi$ for some identity preserving Schwarz map T on M with preadjoint $T_* \in L(M_*)$. Then $T(s(\varphi)^{\perp}) \leqslant s(\varphi)^{\perp}$ where $s(\varphi)$ is the support projection of φ.

To see this let $L_\varphi := \{x \in M : \varphi(xx^*) = 0\}$ and $M_\varphi := L_\varphi \cap L_\varphi^*$. Since φ is T_*-invariant, and T is a Schwarz map, the subspaces L_φ and M_φ are T-invariant. From $M_\varphi = s(\varphi)^{\perp} M s(\varphi)^{\perp}$ and $T(s(\varphi)^{\perp}) \leqslant 1$ it follows that $T(s(\varphi)^{\perp}) \leqslant s(\varphi)^{\perp}$.

Let $T_{s(\varphi)}$ be the induced map on $M_{s(\varphi)}$. If

$$s(\varphi)M_*s(\varphi) := \{\psi \in M_* : \psi = s(\varphi)\psi s(\varphi)\}$$

where $\langle s(\varphi)\psi s(\varphi), x\rangle := \langle \psi, s(\varphi)xs(\varphi)\rangle$ $(x \in M)$, and if $\psi \in s(\varphi)M_*s(\varphi)$, then for all $x \in M$:

$$(T_*\psi)(x) = \psi(Tx) = \langle \psi, s(\varphi)(Tx)s(\varphi)\rangle =$$
$$= \langle \psi, s(\varphi)(T(s(\varphi)xs(\varphi)))s(\varphi)\rangle = \langle T_*\psi, s(\varphi)xs(\varphi)\rangle,$$

hence $T_*\psi \in s(\varphi)M_*s(\varphi)$. Since the dual of $s(\varphi)M_*s(\varphi)$ is $M_{s(\varphi)}$, it follows that the adjoint of the reduced map $T_*|$ is identity preserving and of Schwarz type.

For example, if T is an identity preserving semigroup of Schwarz type on M_* such that $\varphi \in \operatorname{Fix}(T)$, then the semigroup $T|(s(\varphi)M_*s(\varphi))$ is again identity preserving and of Schwarz type. Furthermore, if R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$ with values in M_* which is identity preserving and of Schwarz type such that $R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_{+}$, then $R|s(\varphi)M_*s(\varphi)$ has the same properties.

Chapter D-II
Characterization of Positive Semigroups on W*-Algebras

Since the positive cone of a C*-algebra has non-empty interior many results of Chapter B-II can be applied verbatim to the characterization of the generator of positive semigroups on C*-algebras. On the other hand a concrete and detailed representation of such generators has been found only in the uniformly continuous case (see Lindblad (1976)). A third area of active research has been the following: Which maps on C*-algebras (in particular, which derivations) commuting with certain automorphism groups are automatically generators of strongly continuous positive semigroups. For more information we refer to the survey article of [5].

1 Semigroups on Properly Infinite W*-Algebras

The aim of this section is to show that strongly continuous semigroups of Schwarz maps on properly infinite W*-algebras are already uniformly continuous. In particular, our theorem is applicable to such semigroups on $B(H)$.

It is worthwhile to remark, that the result of [6] on the uniform continuity of every strongly continuous semigroup on L^∞ (see A-II, Sec.3) does not extend to arbitrary W*-algebras.

Example 1.25 Take $M = B(H)$, H infinite dimensional, and choose a projection $p \in M$ such that Mp is topologically isomorphic to H. Therefore $M = H \oplus M_0$, where $M_0 = \ker(x \mapsto xp)$. Next take a strongly, but not uniformly continuous, semigroup S on H and consider the strongly continuous semigroup $S \oplus \mathrm{Id}$ on M.

For results from the classification theory of W⋆-algebras needed in our approach we refer to [9, 2.2] and [10, V.1].

Theorem 1.26 *Every strongly continuous one-parameter semigroup of Schwarz type on a properly infinite W*-algebra M is uniformly continuous.*

Proof Let $T = (T(t)_{t\geqslant 0})$ be strongly continuous on M and suppose T not to be uniformly continuous. Then there exists a sequence $(T_n) \subset T$ and $\epsilon > 0$ such that

$\|T_n - \mathrm{Id}\| \geqslant \epsilon$ but $T_n \longrightarrow \mathrm{Id}$ in the strong operator topology. We claim that for every sequence (P_k) of mutually orthogonal projections and all bounded sequences (x_k) in M

$$\lim_n \|(T_n - \mathrm{Id})(P_k x_k P_k)\| = 0$$

uniformly in $k \in \mathbb{N}$. This follows from an application of the *Lemma of Phillips* and the fact that the sequence $(P_k x_k P_k)$ is summable in the $s^*(M, M_*)$-topology (compare Elliot (1972)).

Let (P_k) be a sequence of mutually orthogonal projections in M such that every P_k is equivalent to 1 via some $u_k \in M$ [9, 2.2]. Without loss of generality we may assume $\|(T_n - \mathrm{Id})(u_n)\| \leqslant n^{-1}$ since the semigroup T is strongly continuous. Thus we obtained the following:

(i) $\lim_n \|(T_n - \mathrm{Id})(P_k x_k P_k)\| = 0$ uniformly in $k \in \mathbb{N}$ for every bounded sequence (x_k) in M.
(ii) Every projection P_k is equivalent to 1 via some $u_k \in M$.
(iii) $\|(T_n - \mathrm{Id})u_n\| \leqslant n^{-1}$ for all $n \in \mathbb{N}$.

For the following construction see A-I,3.6 and D-II,Sec.2. Let

(i) $\widehat{M}$ be an ultrapower of M,
(ii) let $p := \widehat{(P_k)} \in \widehat{M}$,
(iii) $T := \widehat{(T_n)} \in L(\widehat{M})$
(iv) and $u := \widehat{(u_k)} \in \widehat{M}$.

Then T is identity preserving and of Schwarz type on $\widehat{M}$. Since $u^*u = p$ and $uu^* = 1$ it follows $pu^* = u^*$ and $(uu^*)x(uu^*) = x$ for all $x \in \widehat{M}$. Finally, $T(pxp) = pxp$ for all $x \in \widehat{M}$, which follows from (i), and $T(u^*) = T(pu^*) = pu^* = u^*$ and $T(u) = u$, which follows from (iii). Using the Schwarz inequality we obtain

$$T(uu^*) = T(1) \leqslant 1 = uu^* = T(u)T(u)^*.$$

Using D-III, Lemma 1.1. we conclude $T(ux) = uT(x)$ and $T(xu^*) = T(x)u^*$ for all $x \in \widehat{M}$. Hence

$$\begin{aligned} T(x) &= T(uu^*xuu^*) = uT(u^*xu)u^* = uT(pu^*xup)u^* \\ &= upu^*xupu^* = uu^*xuu^* = x \end{aligned}$$

for all $x \in \widehat{M}$. From this we obtain that for every bounded sequence (x_k) in M

$$\lim_m \|T_m x_m - x_m\| = 0$$

for some subsequence of the T_n's and of the x_k's. This conflicts with our assumption at the beginning, hence the theorem is proved. □

Chapter D-III
Spectral Theory of Positive Semigroups on W*-Algebras and their Preduals

Motivated by the classical results of Perron and Frobenius one expects the following spectral properties for the generator A of a positive semigroup: The spectral bound $s(A) := \sup\{\Re(\lambda) : \lambda \in \sigma(A)\}$ belongs to the spectrum $\sigma(A)$ and the boundary spectrum

$$\sigma_b(A) := \sigma(A) \cap \{s(A) + i\mathbb{R}\}$$

possesses a certain symmetric structure, called cyclicity.

Results of this type have been proved in Chapter B-III for positive semigroups on commutative C*-algebras, but in the non-commutative case the situation is more complicated. While "$s(A) \in \sigma(A)$" still holds (see [Greiner-Voigt-Wolff (1980)]) the cyclicity of the boundary spectrum $\sigma_b(A)$ is true only under additional assumptions on the semigroup (e.g., irreducibility, see Section 1 below).

For technical reasons we consider mostly strongly continuous semigroups on the predual of a W*-algebra M or its adjoint semigroup which is a weak*-continuous semigroup on M.

1 Spectral Theory for Positive Semigroups on Preduals

The aim of this section is to develop a Perron-Frobenius theory for identity preserving semigroups of Schwarz type on W*-algebras. But as we will show in the example preceding Theorem 1.1 below the boundary spectrum is no longer cyclic. The appropriate hypothesis on the semigroup implying the desired results seems to be the concept of irreducibility.

Let us first recall some facts on normal linear functionals. If φ is a normal linear functional on a W*-algebra M then there exists a partial isometry $u \in M$ and a positive linear functional $|\varphi| \in M_*$ such that

$$\varphi(x) = |\varphi|(xu) =: (u|\varphi|)(x), x \in M$$

$$u^*u = s(|\varphi|),$$

where $s(|\varphi|)$ denotes the support projection of $|\varphi|$ in M. We refer to this as the *polar decomposition* of φ [Takesaki (1979), Theorem III.4.2]. In addition, $|\varphi|$ is uniquely determined by the following two conditions [Takesaki (1979), Proposition III.4.6]:

$$\|\varphi\| = \||\varphi|\|,$$

(∗)

$$|\varphi(x)|^2 \leqslant |\varphi|(xx^*) \quad (x \in M).$$

For the polar decomposition of φ^*, where $\varphi^*(x) = \varphi(x^*)^*$, we obtain

$$\varphi^* = u^*|\varphi^*|, \quad |\varphi^*| = u|\varphi|u^* \quad \text{and} \quad uu^* = s(|\varphi^*|).$$

It is easy to see that $u^* \in s(|\varphi|)M$.

If Ψ is a subset of the state space of a C*-algebra M, then Ψ is called *faithful* if $0 \leqslant x \in M$ and $\psi(x) = 0$ for all $\psi \in \Psi$ implies $x = 0$. Ψ is called *subinvariant* for a positive map $T \in \mathcal{L}(M)$ (resp., positive semigroup T) if $T'\psi \leqslant \psi$ for all $\psi \in \Psi$ (resp., $T(t)'\psi \leqslant \psi$ for all $T(t) \in T$ and $\psi \in \Psi$). Recall that for every positive map $T \in \mathcal{L}(M)$ there exists a state φ on M such that $T'\varphi = r(T)\varphi$ [Groh (1981), Theorem 2.1], where $r(T)$ denotes the spectral radius of T.

Let us start our investigation with two lemmas. Recall that $\operatorname{Fix}(()T)$ is the fixed space of T, i.e. the set $\{x \in M : Tx = x\}$.

Lemma 1.27 *Suppose M to be a C*-algebra and $T \in \mathcal{L}(M)$ an identity preserving Schwarz map.*

(i) Let $b : M \times M \to M$ be a sesquilinear map such that for all $z \in M$ $b(z,z) \geqslant 0$. Then $b(x,x) = 0$ for some $x \in M$ if and only if $b(x,y) = 0$ and $b(y,x) = 0$ for all $y \in M$.
(ii) If there exists a faithful family Ψ of subinvariant states for T on M, then $\operatorname{Fix}(()T)$ *is a $C^\star$-subalgebra of M and $T(xy) = xT(y)$ for all $x \in$* $\operatorname{Fix}(()T)$ *and $y \in M$.*

Proof (i) Take $0 \leqslant \psi \in M^*$ and consider $f := \psi \circ b$. Then f is a positive semidefinite sesquilinear form on M with values in $\mathbb{C}$. From the Cauchy-Schwarz inequality it follows that $f(x,x) = 0$ for some $x \in M$ if and only if $f(x,y) = 0$ and $f(y,x) = 0$ for all $y \in M$. Since the positive cone M_+^* is generating, assertion (a) is proved.

(ii) Since T is positive it follows $T(x)^* = T(x^*)$ for all $x \in M$. Hence $\operatorname{Fix}(()T)$ is a self adjoint subspace of M, i.e. invariant under the involution on M. For every $x, y \in M$ let

$$b(x,y) := T(xy^*) - T(x)T(y)^*.$$

Then b satisfies the assumptions of (i).

If $x \in \operatorname{Fix}(()T)$ then

$$0 \leqslant xx^* = (Tx)(Tx)^* \leqslant T(xx^*),$$

hence

$$0 \leqslant \psi(T(xx^*) - xx^*) \leqslant 0 \quad \text{for all } \psi \in \Psi.$$

But this implies $T(xx^*) = T(x)T(x)^* = xx^*$. Consequently, $b(x,x) = 0$. Hence $T(xy^*) = xT(y)^*$ for all $y \in M$ and (ii) is proved. □

Lemma 1.28 *Let M be a $W^\star$-algebra, T an identity preserving Schwarz map on M and $S \in \mathcal{L}(M)$ such that $S(x)(Sx)^* \leqslant T(xx^*)$ for every $x \in M$.*

(a) If $v \in M$ such that $S(v^*) = v^*$ and $T(v^*v) = v^*v$, then $T(xv) = S(x)v$ for all $x \in M$.
(b) Suppose there exists $\varphi \in M_*$ with polar decomposition $\varphi = u|\varphi|$ such that $S_*\varphi = \varphi$ and $T_*|\varphi| = |\varphi|$. If the closed subspace $s(|\varphi|)M$ is T-invariant, then $Su^* = u^*$ and $T(u^*u) = u^*u$.

Proof (a) Define a positive semidefinite sesquilinear map $b : M \times M \to M$ by

$$b(x, y) := T(xy^*) - S(x)S(y)^* \quad (x, y \in M).$$

Since $b(v^*, v^*) = 0$ we obtain $b(x, v^*) = 0$ for all $x \in M$ (Lemma 1.1.a), hence $T(xv) = S(x)v$.

(a) Since $s(|\varphi|)M$ is a closed right ideal, the closed face $F := s(|\varphi|)(M_+)s(|\varphi|)$ determines $s(|\varphi|)M$ uniquely, i.e.,

$$s(|\varphi|)M = \{x \in M : xx^* \in F\}$$

[Pedersen (1979), Theorem 1.5.2]. Since T is a Schwarz map and $s(|\varphi|)M$ is T-invariant, it follows $TF \subseteq F$. On the other hand, if $x \in s(|\varphi|)M$ then $xx^* \in F$. Consequently,

$$0 \leqslant S(x)S(x)^* \leqslant T(xx^*) \in F,$$

whence $S(x) \in s(|\varphi|)M$.
Next we show $T(u^*u) = u^*u$ and $Su^* = u^* \in s(|\varphi|)M$. First of all

$$0 \leqslant (Su^* - u^*)(Su^* - u^*)^* \leqslant$$
$$\leqslant T(u^*u) - u^*S(u^*)^* - (Su^*)u + u^*u.$$

Since $S_*\varphi = \varphi$, $T_*|\varphi| = |\varphi|$ and $\varphi = u|\varphi|$ it follows

$$0 \leqslant |\varphi|((Su^* - u^*)(Su^* - u^*)^*) \leqslant$$
$$\leqslant 2|\varphi|(u^*u) - |\varphi|(S(u^*)u)^* - |\varphi|(S(u^*)u) =$$
$$= 2|\varphi|(uu^*) - \varphi(u^*)^* - \varphi(u^*) =$$
$$= 2(|\varphi|(u^*u) - |\varphi|(u^*u)) = 0.$$

Since $(Su^* - u^*)(Su^* - u^*) \in F$ and $|\varphi|$ is faithful on F we obtain $Su^* = u^*$. Consequently,

$$0 \leqslant u^*u = (Su^*)(Su^*)^* \leqslant T(u^*u).$$

Hence $T(u^*u) = u^*u$ by the faithfulness and T-invariance of $|\varphi|$.

Remark 1.29 Take S and T as in Lemma 1.2 (b). If V_{u^*} (resp. V_u) is the map $(x \mapsto xu^*)$ (resp. $(x \mapsto xu)$) on M, then V_{u^*} is a continuous bijection from $Ms(|\varphi|)$ onto $Ms(|\varphi^*|)$ with inverse V_u (because $V_u \circ V_{u^*} = \mathrm{Id}_{Ms(|\varphi|)}$ and $V_{u^*} \circ V_u = \mathrm{Id}_{Ms(|\varphi^*|)}$). Let $x \in M$. From $T(xu) = S(x)u$ we obtain $T(xu)u^* = S(x)uu^*$. In particular, if $Ms(|\varphi^*|)$ is S-invariant, then

$$(V_{u^*} \circ T \circ V_u)(x) = T(xu)u^* = S(x)$$

for every $x \in Ms(|\varphi^*|)$. Let $T|$ (resp. $S|$) be the restriction of T to $Ms(|\varphi|)$ (resp. of S to $Ms(|\varphi^*|)$). Then the following diagram is commutative:

$$\begin{array}{ccc} Ms(|\varphi|) & \xrightarrow{T|} & Ms(|\varphi|) \\ {\scriptstyle V_u}\big\downarrow & & \big\downarrow{\scriptstyle V_{u^*}} \\ Ms(|\varphi^*|) & \xrightarrow[S|]{} & Ms(|\varphi^*|) \end{array}$$

In particular, $\sigma(S|) = \sigma(T|)$. Therefore we may deduce spectral properties of $S|$ from $T|$ and vice versa. More concrete applications of Lemma 1.2 will follow.

We now investigate the fixed space $\mathrm{Fix}(R) := \mathrm{Fix}(\lambda R(\lambda))$, $\lambda \in D$, of a pseudo-resolvent R with values in the predual of a W*-algebra M.

Proposition 1.30 *Let R be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in the predual M_* of a W*-algebra M and suppose R to be identity preserving and of Schwarz type.*

(a) If $a \in \mathbb{R}$ and $\psi \in M_*$ such that $(\gamma - ia)R(\gamma)\psi = \psi$ for some $\gamma \in D$, then $\lambda R(\lambda)|\psi| = |\psi|$ and $\lambda R(\lambda)|\psi^*| = |\psi^*|$ for all $\lambda \in D$.
(b) $\mathrm{Fix}(R)$ is invariant under the involution in M_*. If $\psi \in \mathrm{Fix}(R)$ is self adjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of $\mathrm{Fix}(R)$.

Proof If $(\gamma - i\alpha)R(\gamma)\psi = \psi$ then $(\lambda - i\alpha)R(\lambda)\psi = \psi$ for all $\lambda \in D$. In particular, $\mu R(\mu + i\alpha)\psi = \psi$ $(\mu \in \mathbb{R}_+)$. For all $x \in M$ we obtain

$$|\psi(x)|^2 = | < \mu R(\mu + i\alpha)'x, \psi > |^2 \leqslant$$

$$\leqslant \|\psi\| < (\mu R(\mu + i\alpha)'x)(\mu R(\mu + i\alpha)'x)^*, \psi > \leqslant$$

$$\leqslant \|\psi\| < \mu R(\mu)'(xx^*), |\psi| >$$

(D-I, Corollary 2.2). Since

$$\|\psi\| = \||\psi|\| = |\psi|(1) =$$

$$= < \mu R(\mu)'1, |\psi| > = \|\mu R(\mu)|\psi|\|,$$

we obtain $\mu R(\mu)|\psi| = |\psi|$ by the uniqueness theorem (*) mentioned at the beginning. Therefore $|\psi| \in \mathrm{Fix}(R)$. Since

$$0 \leqslant (\mu R(\mu)'x)(\mu R(\mu)'x)^* \leqslant \mu R(\mu)'xx^*,$$

the map $R(\mu)$ is positive. Consequently $(\mu + i\alpha)R(\mu)\psi^* = \psi^*$ from which $|\psi^*| \in \mathrm{Fix}(R)$ follows. If $\varphi \in \mathrm{Fix}(R)$ is selfadjoint with Jordan decomposition $\varphi = \varphi^+ - \varphi^-$, then $|\varphi| = \varphi^+ + \varphi^-$ [Takesaki (1979), Theorem III.4.2.]. From this we obtain that φ^+ and φ^- are in $\mathrm{Fix}(R)$. □

Corollary 1.31 *Let T be an identity preserving semigroup of Schwarz type on M_* with generator A and suppose $P\sigma(A) \cap i\mathbb{R} \neq \emptyset$.*

(i) If $\alpha \in \mathbb{R}$ and $\psi \in \ker(i\alpha - A)$, then $|\psi|$ and $|\psi^*|$ are elements of $\mathrm{Fix}(T) = \ker(A)$.
(ii) $\mathrm{Fix}(T)$ *is invariant under the involution of M_*. If $\psi \in$* $\mathrm{Fix}(T)$ *is self adjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of* $\mathrm{Fix}(T)$.

The proof follows immediately from D-I, Corollary 2.2 and the fact that $\ker(A) = \mathrm{Fix}(\lambda R(\lambda, A))$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$.

If T is the semigroup of translations on $L^1(\mathbb{R})$ and A' the genenerator of the adjoint weak*-semigroup, then $P_\sigma(A) \cap i\mathbb{R} = \emptyset$, while $P_\sigma(A') \cap i\mathbb{R} = i\mathbb{R}$.

For that reason we cannot expect a simple connection between these two sets.

But as we shall see below, if a semigroup on the predual of a W*-algebra has sufficiently many invariant states, then the point spectra of A and A' contained in $i\mathbb{R}$ are identical.

Helpful for these investigations will be the next lemma.

Lemma 1.32 *Let R be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\}$ with values in a Banach space E such that $\|R(\mu + i\alpha)\| \leqslant 1$ for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$.*

Then

$$\dim \mathrm{Fix}(()\lambda R(\lambda + i\alpha)) \leqslant \dim \mathrm{Fix}(()\lambda R(\lambda + i\alpha)')$$

for all $\lambda \in D$.

Proof Let $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $S := \mu R(\mu + i\alpha)$. Since S is a contraction, its adjoint S' maps the dual unit ball E_1' into itself.

Let U be a free ultrafilter on $[1, \infty)$ which converges to 1. Since E_1' is $\sigma(E', E)$-compact,

$$\psi_o := \lim_U (\lambda - 1)R(\lambda, S)'\psi$$

exists for all $\psi \in E_1'$. Since S' is $\sigma(E', E)$-continuous and since $S'R(\lambda, S)' = \lambda R(\lambda, S') - \mathrm{Id}$ we conclude $\psi_o \in \mathrm{Fix}(()S')$.

Take now $0 \neq x_o \in \mathrm{Fix}(()S)$ and choose $\psi \in E_1'$ such that $\psi(x_o)$ is different from zero.

From the considerations above it follows

$$\psi_o(x_o) = \lim_U (\lambda - 1)\psi(R(\lambda, S)x_o) = \psi(x_o) \neq 0$$

hence $0 \neq \psi_o \in \mathrm{Fix}(()S)$.

Therefore $\mathrm{Fix}(()S')$ separates the points of $\mathrm{Fix}(()S)$.

From this it follows that

$$\dim \operatorname{Fix}(()S) \leqslant \dim \operatorname{Fix}(()S')$$

Since R and R' are pseudo-resolvents, the assertion is proved. □

Corollary 1.33 *Let T be a semigroup of contractions on a Banach space E with generator A. Then*

$$\dim \ker(i\alpha - A) \leqslant \dim \ker(i\alpha - A')$$

for all $\alpha \in \mathbb{R}$.

This follows from Lemma 1.32 because $\operatorname{Fix}(()\lambda R(\lambda + i\alpha)) = \ker(i\alpha - A)$.

Proposition 1.34 *Let T be an identity preserving semigroup of Schwarz type with generator A on the predual of a W*-algebra and suppose that there exists a faithful family Ψ of T-invariant states.*

Then for all $\alpha \in \mathbb{R}$ we have

$$\dim \ker(i\alpha - A) = \dim \ker(i\alpha - A')$$

and

$$P_\sigma(A) \cap i\mathbb{R} = P_\sigma(A') \cap i\mathbb{R}$$

Proof The inequality $\dim \ker(i\alpha - A) \leqslant \dim \ker(i\alpha - A')$ follows from Corollary 1.33.

Let $D = \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\}$ and R the pseudo-resolvent induced by $R(\lambda, A)$ on D.

Then R is identity preserving and of Schwarz type.

Take $i\alpha \in P_\sigma(A)$ $(\alpha \in \mathbb{R})$ and choose $0 < \mu \in \mathbb{R}$.

If $\psi_\alpha \in M_*$ is of norm one with polar decomposition $\psi_\alpha = u_\alpha|\psi_\alpha|$ such that $\psi_\alpha = (\mu - i\alpha)R(\mu)\psi_\alpha$ then $\mu R(\mu)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4.a).

Since

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant 1 - s(|\psi_\alpha|)$$

we obtain $\mu R(\mu)' s(|\psi_\alpha|) = s(|\psi_\alpha|)$ by the faithfulness of Ψ.

Hence the maps $S := (\mu - i\alpha)R(\mu)'$ and $T := \mu R(\mu)'$ fulfil the assumptions of Lemma 1.2.b.

Therefore $Su_\alpha^* = u_\alpha^*$ or $(\mu - i\alpha)R(\mu)'u_\alpha^* = u_\alpha^*$ which implies $u_\alpha^* \in D(A')$ and $A'u_\alpha^* = i\alpha u_\alpha^*$.

If $i\alpha \in P_\sigma(A')$, $\alpha \in \mathbb{R}$, choose $0 \neq v_\alpha$ such that

$$v_\alpha = (\mu - i\alpha)R(\mu)'v_\alpha \quad (\mu \in \mathbb{R}_+)$$

and $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$.

Since

$$0 \leqslant v_\alpha v_\alpha^* = ((\mu - i\alpha)R(\mu)'v_\alpha)((\mu - i\alpha)R(\mu)'v_\alpha)^* \leqslant \mu R(\mu)'(v_\alpha v_\alpha^*)$$

we obtain $\mu R(\mu)'(v_\alpha v_\alpha^*) = v_\alpha v_\alpha^*$ because Ψ is faithful.

Hence from Lemma 1.28, Teil (a) it follows

$$\mu R(\mu)'(xv_\alpha^*) = ((\mu - i\alpha)R(\mu)'x)v_\alpha^*$$

for all $x \in M$.

Let ψ_α be the normal linear functional $(x \mapsto \psi(xv_\alpha^*))$ on M and note that $\psi_\alpha(v_\alpha) \neq 0$.

Then

$$\begin{aligned}\langle x, (\mu - i\alpha)R(\mu)\psi_\alpha\rangle &= \langle((\mu - i\alpha)R(\mu)'x)v_\alpha^*, \psi\rangle \\ &= \langle \mu R(\mu)'(xv_\alpha^*), \psi\rangle = \psi(xv_\alpha^*) = \psi_\alpha(x)\end{aligned}$$

for all $x \in M$.

Consequently $i\alpha \in P_\sigma(A)$ and

$$\dim\ker(i\alpha - A') \leqslant \dim\ker(i\alpha - A)$$

which proves the assertion. □

Remark 1.35 From the above proof we obtain the following: If $0 \neq \psi_\alpha \in \ker(i\alpha - A)$ with polar decomposition $\psi_\alpha = u_\alpha|\psi_\alpha|$ $(\alpha \in \mathbb{R})$ then $A'u_\alpha = i\alpha u_\alpha$.

Conversely, if $0 \neq v_\alpha \in \ker(i\alpha - A')$, then there exists $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$ and the normal linear form

$$\psi_\alpha := (x \mapsto \psi(xv_\alpha^*))$$

is an eigenvector of A pertaining to the eigenvalue $i\alpha$.

If T is a C_0-semigroup of Markov operators on a commutative C*-algebra with generator A, it has been shown in B-III, that the boundary spectrum $\sigma(A) \cap i\mathbb{R}$ of its generator is additively cyclic. This is no longer true in the non commutative case:

For $0 \neq \lambda \in i\mathbb{R}$ and $t \in \mathbb{R}$ let

$$u_t := \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda t}\end{pmatrix} \in M_2(\mathbb{C})$$

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The semigroup of *-automorphisms $(x \mapsto u_t x u_t^*)$ on $M_2(\mathbb{C})$ is identity preserving and of Schwarz type but the spectrum of its generator is $\{0, \lambda, \lambda^*\}$ hence is not additively cyclic.

It turns out that, in order to obtain a non commutative analogue of the Perron-Frobenius theorems, one has to consider semigroups which are irreducible.

Recall that a semigroup S of positive operators on an ordered Banach space (E, E_+) is called *irreducible* if no closed face of E_+, different from $\{0\}$ and E_+, is invariant under S.

Here a face F in E is a subcone of E_+ such that the conditions $0 \leqslant x \leqslant y$, $x \in E$, $y \in F$ imply $x \in F$ (compare Definitions 3.1 in B-III and C-III).

In the context of W*-algebras M we call a semigroup S of positive maps on M *weak*-irreducible*, if no $\sigma(M, M_*)$-closed face of M_+ is S-invariant.

Since the norm closed faces of M_* and the $\sigma(M, M_*)$-closed faces of M are related by formation of polars with respect to the dual system $\langle M, M_*\rangle$ (see [Pedersen (1979), Theorem 3.6.11 and Theorem 3.10.7.]) a semigroup S is (norm) irreducible on M_* if and only if its adjoint semigroup is weak*-irreducible.

Theorem 1.36 *Let T be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a W*-algebra and suppose $P_\sigma(A) \cap i\mathbb{R} \neq \emptyset$.*

(a) The fixed space of T is one dimensional and spanned by a faithful normal state.
(b) $P_\sigma(A) \cap i\mathbb{R}$ is an additive subgroup of $i\mathbb{R}$,

$$\sigma(A) = \sigma(A) + (P_\sigma(A) \cap i\mathbb{R})$$

and every eigenvalue in $i\mathbb{R}$ is simple.

(i) The fixed space of the adjoint weak-semigroup T' is one-dimensional.*
(ii) $P_\sigma(A') \cap i\mathbb{R} = P_\sigma(A) \cap i\mathbb{R}$ for the generator A' of the adjoint semigroup, and every $\gamma \in P_\sigma(A') \cap i\mathbb{R}$ is simple.

Proof Since $P_\sigma(A) \cap i\mathbb{R} \neq \emptyset$ there exists $\psi \in \operatorname{Fix}(()T)_+$ of norm one (Corollary 1.5).

If $F := \{x \in M_+ : \psi(x) = 0\}$ then F is a $\sigma(M, M_*)$-closed, T'-invariant face in M, hence $F = \{0\}$.

Therefore every $0 \neq \psi \in \operatorname{Fix}(()T)_+$ is faithful.

Let $\psi_1, \psi_2 \in \operatorname{Fix}(()T)_+$ be states such that

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$f := \psi_1 - \psi_2$ is different from zero.

If $f = f^+ - f^-$ is the Jordan decomposition of f, then f^+ and f^- are elements of $\operatorname{Fix}(()T)$, whence faithful.

Since the support projections of these two normal linear functionals are orthogonal, we obtain $f^+ = 0$ or $f^- = 0$ which implies $\psi_1 \leqslant \psi_2$ or $\psi_2 \leqslant \psi_1$.

Consequently $\psi_2 = \psi_1$.

Since $\operatorname{Fix}(()T)$ is positively generated (Corollary 1.5), $\operatorname{Fix}(()T) = \mathbb{C}\varphi$ for some faithful normal state φ.

Let $\mu \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ such that $i\alpha \in P_\sigma(A)$.

If $\psi_\alpha = u_\alpha|\psi_\alpha|$ is a normalized eigenvector of A pertaining to $i\alpha$, then $\varphi = |\psi_\alpha| = |\psi_\alpha^*|$ by Corollary 1.5 and the above considerations.

Hence $u_\alpha u_\alpha^* = u_\alpha^* u_\alpha = s(\varphi) = 1$.

Since

$$(\mu - i\alpha)R(\mu, A)\psi_\alpha = \psi_\alpha$$

and

$$\mu R(\mu, A)|\psi_\alpha| = |\psi_\alpha|$$

we obtain by Lemma 1.2.b that

$$(1) \quad \mu R(\mu, A) = V_\alpha \circ \mu R(\mu + i\alpha, A) \circ V_\alpha^{-1}$$

where V_α is the map $(x \mapsto xu_\alpha)$ on M.

Similarly for $i\beta \in P_\sigma(A)$, we find V_β such that $1 = u_\beta u_\beta^* = u_\beta u_\beta^*$ and

$$(2) \quad \mu R(\mu, A) = V_\beta \circ \mu R(\mu + i\beta, A) \circ V_\beta^{-1}$$

Hence

$$(3) \quad \mu R(\mu, A) = V_{\alpha\beta} \circ \mu R(\mu + i(\alpha + \beta), A) \circ V_{\alpha\beta}^{-1}$$

where $V_{\alpha\beta} := V_\alpha \circ V_\beta$.

Since u_α is unitary in M, it follows from (1) that $i\alpha$ is an eigenvalue which is simple because $\mathrm{Fix}(()T) = \mathrm{Fix}(()\mu R(\mu, A))$ is one dimensional.

From (3) it follows that $i(\alpha + \beta) \in P_\sigma(A)$ since $0 \in P_\sigma(A)$ and $V_{\alpha\beta}$ is bijective.

From the identity (1) we conclude that $\sigma(R(\mu, A)) = \sigma(R(\mu + i\alpha))$, which proves

$$\sigma(A) + (P_\sigma(A) \cap i\mathbb{R}) \subseteq \sigma(A)$$

The other inclusion is trivial since $0 \in P_\sigma(A)$. □

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Remarks (a) Let φ be the normal state on M such that $\mathrm{Fix}(()T) = \mathbb{C}\varphi$ and let $H := P_\sigma(A) \cap i\mathbb{R}$.
From the proof of Theorem 1.10 it follows that there exists a family $\{u_\eta : \eta \in H\}$ of unitaries in M such that $A'u_\eta = -\eta u_\eta$ and $A(u_\eta\varphi) = \eta(u_\eta\varphi)$ for all $\eta \in H$.

(b) If the group H is generated by a single element, i.e., $H = i\gamma\mathbb{Z}$ for some $\gamma \in \mathbb{R}$ then the family $\{u_\gamma^k : k \in \mathbb{Z}\}$ is a complete family of eigenvectors pertaining to the eigenvalues in H, where $u_\gamma \in M$ is unitary such that $A'u_\gamma = i\gamma u_\gamma$. □

Proposition 1.38 *Suppose that T and M satisfy the assumptions of Theorem 1.10, and let N_* be the closed linear subspace of M_* generated by the eigenvectors of A pertaining to the eigenvalues in $i\mathbb{R}$.*

Denote by T_o the restriction of T to N_*.

Then

(a) $G := (T_o)^- \subseteq L_s(N_*)$ is a compact, Abelian group.
(b) $\mathrm{Id}\,|N_* \in \{T_o(t) : t > s\}^- \subseteq L_s(N_*)$ for all $0 < s \in \mathbb{R}$.

Proof For $\eta \in H := P_\sigma(A) \cap i\mathbb{R}$ let

$$U(\eta) := \{\psi \in D(A) : A\psi = \eta\psi\}$$

and $U = \{U(\eta) : \eta \in H\}$.

Then $(U)^- = N_*$.

For each $\psi \in U$ there exists $\eta \in H$ such that

$$\{T_o(t)\psi : t \in \mathbb{R}_+\} = \{e^{-\eta t}\psi : t \in \mathbb{R}_+\}$$

Consequently this set is relatively compact in $L_s(N_*)$.

From [Schaefer (1966),III.4.5] we obtain that G is compact.

Next choose $\psi_1, ..., \psi_n \in U$, $0 < s \in \mathbb{R}$ and $\delta > 0$.

Since $T_o(t)\psi_i = e^{\eta_i t}\psi_i$ $(1 \leqslant i \leqslant n)$ for some $\eta_i \in H$, it follows from a theorem of Kronek-ker (see, [Jacobs (1976), Satz 6.1., p.77]) that there exists $s < t$ such that

$$|(1, 1, ..., 1) - (e^{\eta_1 t}, e^{\eta_2 t}, ..., e^{\eta_n t})| < \delta$$

hence

$$\sup\{\|\psi_i - T_o(t)\psi_i\| : 1 \leqslant i \leqslant n\} < \delta$$

or $\mathrm{Id}\,|N_* \in \{T_o(t) : t > s\}^- \subseteq L_s(N_*)$.

Finally we prove the group property of G.

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{R}$ such that $\lim_{\mathcal{U}} T_o(t) = \mathrm{Id}$ in the strong operator topology.

For positive $s \in \mathbb{R}$ let $S := \lim_{\mathcal{U}} T(t - s)$.

Then $ST_o(s) = T_o(s)S = \mathrm{Id}$, hence $T_o(s)^{-1}$ exists in G for all $s \in \mathbb{R}_+$.

From this it follows that G is a group. □

Remark 1.39 (i) Let $\kappa : \mathbb{R} \to G$ be given by

$$\kappa(t) = \begin{cases} T_o(t) & \text{if } 0 \leqslant t \\ T_o(t)^{-1} & \text{if } t \leqslant 0 \end{cases}$$

Then κ is a continuous homomorphism with dense range, i.e. (G, κ) is solenoidal (see [Hewitt-Ross (1963)]).

(ii) The compact group G and the discret group $P_\sigma(A) \cap i\mathbb{R}$ are dual in the sense of locally compact Abelian groups.

(iii) Let (G, κ) be a solenoidal compact group and let $N_* = L^1(G)$. Then the induced lattice semigroup $T = (\kappa(t))_{t\geqslant 0}$ fulfils the assertions of Theorem 1.10. For example, if G is the dual of $\mathbb{R}_d$, then $P_\sigma(A) \cap i\mathbb{R} = i\mathbb{R}$. Since the fixed space of $\kappa(t)$ is given by

$$\mathrm{Fix}(()\kappa(t)) = (\bigcup_{k\in\mathbb{Z}} \ker(\frac{2\pi i k}{t} - A))^{--}$$

no $T(t) \in T$ is irreducible.

(iv) If T is the irreducible semigroup of Schwarz type on the predual of $B(H)$ given in [Evans (1977)] then $P_\sigma(A) \cap i\mathbb{R} = \emptyset$.

2 Spectral Properties of Uniformly Ergodic Semigoups

The aim of this section is the study of spectral properties of semigroups which are uniformly ergodic, identity preserving and of Schwarz type. For the basic theory of uniformly ergodic semigroups on Banach spaces we refer to [Dunford-Schwartz (1958)].

Our first result yields an estimate for the dimension of the eigenspaces pertaining to eigenvalues of a pseudo-resolvent.

Proposition 2.40 *Let R be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\}$ with values in the predual of a W*-algebra M.*
If $\mathrm{Fix}(()\lambda R(\lambda))$ *is finite dimensional for some $\lambda \in D$, then*

$$\dim \mathrm{Fix}(()(\gamma - i\alpha)R(\gamma)) \leqslant \dim \mathrm{Fix}(()\lambda R(\lambda))$$

for all $\gamma \in D$ and $\alpha \in \mathbb{R}$.

Proof By D-IV, Remark 3.2.c we may assume without loss of generality that there exists a faithful family of R-invariant normal states on M.

In particular the fixed space N of the adjoint pseudo-resolvent R' is a W*-subalgebra of M with $1 \in N$ (by Lemma 1.1.b).

Since N is finite dimensional there exist a natural number n and a set $P := \{p_1, .., p_n\}$ of minimal, mutually orthogonal projections in N such that $\sum_{k=1}^{n} p_k = 1$.

These projections are also mutually orthogonal in M with sum 1.

Let R_j be the $\sigma(M, M_*)$-closed right ideal $p_j M$ and L_j the closed left invariant subspace $M_* p_j$ $(1 \leqslant j \leqslant n)$.

The map $\mu R(\mu)'$, $\mu \in \mathbb{R}_+$ is an identity preserving Schwarz map.

From Lemma 1.1.b we therefore obtain that for all $x \in N$ and $y \in M$,

$$\mu R(\mu)'(xy) = x(\mu R'(\mu)y)$$

In particular, R_j, resp., L_j are invariant under R', respectively, R.

Furthermore, if $\psi \in L_j$ with polar decomposition $\psi = u|\psi|$, then $u^*u \leqslant s(|\psi|) \leqslant p_j$.

Consequently, $|\psi| \in L_j$.

Let now $\alpha \in \mathbb{R}$ and suppose that there exists $\psi_\alpha \in L_j$ of norm 1, $\psi_\alpha = u_\alpha|\psi_\alpha|$, such that

$$\psi_\alpha \in \mathrm{Fix}(()(\lambda - i\alpha)R(\lambda)), \lambda \in D$$

Since $\lambda R(\lambda)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4), we obtain

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant (1 - s(|\psi_\alpha|), \mu \in \mathbb{R}_+$$

From the existence of a faithful family of R-invariant normal states and since R' is identity preserving it follows that

$$\mu R(\mu)' s(|\psi_\alpha|) = s(|\psi_\alpha|)$$

Thus $s(|\psi_\alpha|) \leqslant p_j$ and even $s(|\psi_\alpha|) = p_j$ by the minimality property of p_j.

On the other hand, $\psi_\alpha^* \in \mathrm{Fix}(()(\lambda + i\alpha)R(\lambda))$.

As above we obtain

$$\mu R(\mu)' s(|\psi_\alpha^*|) = s(|\psi_\alpha^*|)$$

Consequently, the closed left ideals $Ms(|\psi_\alpha^*|)$ and $Ms(|\psi_\alpha|)$ are R'-invariant.

Next fix $\mu \in \mathbb{R}_+$, let $S := (\mu - i\alpha)R(\mu)'$ and $T = \mu R(\mu)'$. Then $(Sx)(Sx)^* \leqslant T(xx^*)$, $S_*(\psi_\alpha^*) = \psi_\alpha^*$, $T_*(|\psi_\alpha^*|) = |\psi_\alpha^*|$, and T is an identity preserving Schwarz map.

Since $s(|\psi_\alpha^*|)M$ is T-invariant, the assumptions of Lemma 1.2 are fulfilled and we obtain for every $x \in M$

$$S(x)u_\alpha^* = T(xu_\alpha^*)$$

Since the closed left ideal Mp_j is S-invariant it follows

$$S(x) = T(xu_\alpha^*)u_\alpha \, , \; x \in Mp_j$$

(see Remark 1.3). Since u_α does not depend on $\mu \in \mathbb{R}_+$ we obtain for all $\mu \in \mathbb{R}_+$

$$\mu R(\mu + i\alpha)'x = \mu R(\mu)'(xu_\alpha^*)u_\alpha$$

Consequently, the holomorphic functions $(\mu \mapsto \mu R(\mu)'(xu_\alpha)u_\alpha^*)$ and $(\mu \mapsto \mu R(\mu + i\alpha)'x)$ coincide on $\mathbb{R}_+$ from which we conclude

$$\lambda R(\lambda + i\alpha)'x = \lambda R(\lambda)'(xu_\alpha^*)u_\alpha$$

for every $\lambda \in D$ and all $x \in Mp_j$.

Since the map $(y \mapsto yu_\alpha)$ is a continuous bijection from $M(u_\alpha u_\alpha^*)$ onto Mp_j and its inverse is the map $(y \mapsto yu_\alpha^*)$, we can deduce that

$$\dim \operatorname{Fix}(()(\lambda - i\alpha)R(\lambda)'|Mp_j) = \dim \operatorname{Fix}(()\lambda R(\lambda)')|M(u_\alpha u_\alpha^*) \leqslant \dim \operatorname{Fix}(()R')$$

Since $\bigoplus_{j=1}^n Mp_j = M$ and $\bigoplus_{j=1}^n L_j = M_*$ we obtain

$$\begin{aligned}\dim \operatorname{Fix}(()(\lambda - i\alpha)R(\lambda)')) &= \dim \operatorname{Fix}(()\lambda R(\lambda)') = \\ &= \dim \operatorname{Fix}(()\lambda R(\lambda))\end{aligned}$$

and the assertion follows from Lemma 1.6. □

Before going on let us recall the basic facts of the *ultrapower* $\hat{E}$ of a Banach space E with respect to some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (compare A-I,3.6).

If $\ell^\infty(E)$ is the Banach space of all bounded functions on $\mathbb{N}$ with values in E, then

$$c_{\mathcal{U}}(E) := \{(x_n) \in \ell^\infty(E) : \lim_{\mathcal{U}} \|x_n\| = 0\}$$

is a closed subspace of $\ell^\infty(E)$ and equal to the kernel of the seminorm

$$\|(x_n)\| := \lim_{\mathcal{U}} \|x_n\| \, , \; (x_n) \in \ell^\infty(E)$$

By the ultrapower $\hat{E}$ we understand the quotient space $\ell^\infty(E)/c_{\mathcal{U}}(E)$ with norm

$$\|\hat{x}\| = \lim_{\mathcal{U}} \|x_n\| \, , \; (x_n) \in \hat{x} \in \hat{E}$$

Moreover, for a bounded linear operator $T \in L(E)$, we denote by $\hat{T}$ the well defined operator $\hat{T}\hat{x} := (Tx_n) + c_{\mathcal{U}}(E)$, $(x_n) \in \hat{x}$.

It is clear by virtue of $(x \mapsto (x, x, ..) + c_{\mathcal{U}}(E))$ that each $x \in E$ defines an element $\hat{x} \in \hat{E}$.

This isometric embedding as well as the operator map $(T \mapsto \hat{T})$ are called canonical.

In particular, if $R : (D \to L(E))$ is a pseudo-resolvent, then

$$\hat{R} := (\lambda \mapsto R(\lambda)^{\wedge}) : D \to L(\hat{E})$$

is a pseudo-resolvent, too.

Recall that the approximative point spectrum $A_\sigma(T)$ is equal to the point spectrum $P_\sigma(\hat{T})$ (see, e.g., [Schaefer (1974), Chapter V, §1]).

This construction gives us the possibility to characterize uniformly ergodic semigroups with finite dimensional fixed space.

Lemma 2.41 *Let R be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\}$ such that $\|R(\mu + i\alpha)\| \leqslant 1$ for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and suppose*

$$0 < \dim \operatorname{Fix}(()(\lambda - i\alpha)\hat{R}(\lambda)) < \infty \quad \textit{for some} \quad \lambda \in D\,,\ \alpha \in \mathbb{R}$$

and the canonical extension $\hat{R}$ on some ultrapower $\hat{E}$.

Then the following assertions hold:

(i) $(\lambda - i\alpha)^{-1}$ *is a pole of the resolvent $R(., R(\lambda))$ for all $\lambda \in D$.*
(ii) $\dim \operatorname{Fix}(()(\lambda - i\alpha)R(\lambda)) = \dim \operatorname{Fix}(()(\lambda - i\alpha)\hat{R}(\lambda))$ *for all $\lambda \in D$.*
(iii) $i\alpha$ is a pole of the pseudo-resolvent R and the residue of R and $R(., R(\lambda))$ in $i\alpha$ respectively $(\lambda - i\alpha)^{-1}$ are identical.

Proof Take a normalized sequence (x_n) in E with

$$\lim_n \|(\lambda - i\alpha)R(\lambda)x_n - x_n\| = 0$$

The existence of such a sequence follows from the fact that the fixed space of $(\lambda - i\alpha)\hat{R}(\lambda)$ is non trivial.

Suppose (x_n) is not relatively compact.

Then we may assume that there exists $\delta > 0$ such that

$$\|x_n - x_m\| > \delta \quad \text{for} \quad n \neq m$$

Take $k \in \mathbb{N}$ and let $\hat{x}_k$ be the image of (x_{n+k}) in $\hat{E}$.

Since

$$\lim_n \|(\lambda - i\alpha)R(\lambda)x_{n+k} - x_{n+k}\| = 0$$

the so defined $\hat{x}_k$'s belong to $\operatorname{Fix}(()(\lambda - i\alpha)\hat{R}(\lambda))$.

Since this space is finite dimensional there exist $j < \ell$ such that

$$\|\hat{x}_j - \hat{x}_\ell\| \leqslant \frac{\delta}{2}$$

From the definition of the norm in $\hat{E}$ it follows that there are natural numbers $n < m$ such that

$$\|x_n - x_m\| \leqslant \frac{\delta}{2}$$

which leads to a contradiction.

Therefore every approximate eigenvector of $(\lambda - i\alpha)R(\lambda)$ pertaining to α is relatively compact. In particular it has a convergent subsequence from which it follows that the fixed space of $(\lambda - i\alpha)R(\lambda)$ is non trivial.

Obviously

$$\dim \operatorname{Fix}(()(\lambda - i\alpha)R(\lambda)) \leqslant \dim \operatorname{Fix}(()(\lambda - i\alpha)\hat{R}(\lambda))$$

If the last inequality is strict, then there exists $\gamma > 0$ and a normalized $\hat{x} \in \operatorname{Fix}(()(\lambda - i\alpha)\hat{R}(\lambda))$ such that

$$\gamma \leqslant \|\hat{y} - \hat{x}\|$$

for all $y \in \operatorname{Fix}(()(\lambda - i\alpha)R(\lambda))$.

Take a normalized sequence $(x_n) \in \hat{x}$.

Then (x_n) has a convergent subsequence whence we may assume that $\lim_n x_n = z$ exists in E.

Thus $0 \neq z \in \operatorname{Fix}(()(\lambda - i\alpha)R(\lambda))$.

From this we obtain the contradiction

$$\gamma \leqslant \|\hat{z} - \hat{x}\| = \lim \|z - x_n\| = 0$$

Consequently

$$\dim \operatorname{Fix}(()(\lambda - i\alpha)R(\lambda)) = \dim \operatorname{Fix}(()(\lambda - i\alpha)\hat{R}(\lambda))$$

Let $\{x_1, \ldots, x_n\}$ be a base of $\operatorname{Fix}(()(\lambda - i\alpha)R(\lambda))$ and choose $\{\varphi_1, \ldots, \varphi_n\}$ in $\operatorname{Fix}(()(\lambda - i\alpha)R(\lambda)')$ such that $\varphi_k(x_j) = \delta_{k,j}$ (Lemma 1.6).

Then

$$E = \operatorname{Fix}(()(\lambda - i\alpha)R(\lambda)) \oplus (\bigcap_{j=1}^{n} \ker \varphi_j)$$

where both subspaces on the right are $(\lambda - i\alpha)R(\lambda)$-invariant and 1 is a pole of $(\lambda - i\alpha)R(\lambda)|_{\operatorname{Fix}(()(\lambda - i\alpha)R(\lambda))}$ by the finite dimensionality of $\operatorname{Fix}(()(\lambda - i\alpha)R(\lambda))$.

Suppose 1 belongs to the spectrum of S where S is the restriction of $(\lambda - i\alpha)R(\lambda)$ to $\bigcap_{j=1}^{n} \ker \varphi_j$.

Then there exists a normalized sequence (y_n) in $\bigcap_{j=1}^{n} \ker \varphi_j$ such that

$$\lim_n \|(\lambda - i\alpha)R(\lambda)y_n - y_n\| = 0$$

Therefore (y_n) has an accumulation point different from zero in

$$\mathrm{Fix}(()(\lambda - i\alpha)R(\lambda)) \cap (\bigcap_{j=1}^{n} \ker \varphi_j)$$

This contradiction implies that 1 does not belong to the spectrum of S.

Since $\mathrm{Fix}(()(\lambda - i\alpha)R(\lambda))$ is finite dimensional, it follows from general spectral theory that $(\lambda - i\alpha)^{-1}$ is a pole of $R(., R(\lambda))$ for every λ.

Thus (a) and (b) are proved.

Assertion (c) follows from the resolvent equality as in the proof of [Greiner (1981), Proposition 1.2].

Proposition 2.42 *Let T be a semigroup of contractions on a Banach space E with generator A. Then the following assertions are equivalent:*

(a) Each $i\alpha$, $\alpha \in \mathbb{R}$, is a pole of the resolvent $R(., A)$ such that the corresponding residue has finite rank.
(b) dim $\mathrm{Fix}(()(\lambda - i\alpha)\hat{R}(\lambda, A)) < \infty$ *for some (hence all) $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$ and the canonical extensions $\hat{R}(\lambda, A)$ of $R(\lambda, A)$ to some ultrapower.*

Proof Let P_α be the residue of the resolvent $R(., A)$ in $i\alpha$. Then $P_\alpha = \lim_{\lambda \to i\alpha}(\lambda - i\alpha)R(\lambda, A)$ in the operator norm of $L(E)$. Since the canonical map $(T \mapsto \hat{T})$ is isometric and since $\hat{E}$ is an ultrapower, we obtain

$$\hat{P}_\alpha = \lim_{\lambda \to i\alpha} (\lambda - i\alpha)\hat{R}(\lambda, A)$$

in $L(\hat{E})$ and Rang (P_α) = Rang $(\hat{P}_\alpha)$. Because of

$$\hat{P}_\alpha(\hat{E}) = \mathrm{Fix}(()(\lambda - i\alpha)\hat{R}(\lambda))$$

one part of the corollary is proved. The other follows from Lemma 2.2. □

Remarks (a) By the results in [Lin (1974)] a semigroup of contractions on a Banach space is uniformly ergodic if and only if 0 is a pole of the generator with order $\leqslant$ 1. The residue of the resolvent in 0 and the associated ergodic projection are identical.

(b) Let M be a W*-algebra with predual M_*, $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$ and $\hat{M}$ (resp. $(M_*)^\wedge$) the ultrapower of M (resp. M_*) with respect to $\mathcal{U}$. Then it is easy to see that $c_{\mathcal{U}}(M)$ is a two sided ideal in $\ell^\infty(M)$ hence $\hat{M}$ is a C*-algebra, but in general not a W*-algebra. Note that the unit of $\hat{M}$ is the canonical image of 1. For $\hat{x} \in \hat{M}$ and $\hat{\varphi} \in (M_*)^\wedge$ let $J : (M_*)^\wedge \to \hat{M}'$ be defined by

$$\langle x, J(\hat{\varphi}) \rangle := \lim_{\mathcal{U}} \varphi_n(x_n)\,,\ (x_n) \in \hat{x}\,,\ (\varphi_n) \in \hat{\varphi}$$

J is well defined and is an isometric embedding. It turns out that $J((M_*)^\wedge)$ is a translation invariant subspace of $(\hat{M}')^\wedge$. Hence there exists a central projection $z \in \hat{M}''$ such that $J((M_*)^\wedge) = \hat{M}''z$ [Groh (1984), Proposition 2.2]. □

Below we identify $(M_*)^\wedge$ via J with this translation invariant subspace. From the construction the following is obvious: If T is an identity preserving Schwarz map with preadjoint $T_* \in L(M_*)$, then $\hat{T}$ is an identity preserving Schwarz map on $\hat{M}$ such that $(T_*)^\wedge = \hat{T}'|(M_*)^\wedge$.

Theorem 2.44 *Let T be an identity preserving semigroup of Schwarz type with generator A on the predual of a W*-algebra M. If T is uniformly ergodic with finite dimensional fixed space, then every $\gamma \in \sigma(A) \cap i\mathbb{R}$ is a pole of the resolvent $R(., A)$ and* $\dim \ker(\gamma - A) \leqslant \dim \operatorname{Fix}(()T)$.

Proof Let $D = \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\}$ and R the M_*-valued pseudo-resolvent of Schwarz type induced by $R(., A)$ on D. Then

$$P = \lim_{\mu \downarrow 0} \mu R(\mu)$$

exists in the uniform operator topology and Rang $(P) = \dim \operatorname{Fix}(()T) < \infty$. From this we obtain Rang $(P) =$ Rang $(\hat{P}) < \infty$ where $\hat{P}$ is the canonical extension of P onto $(M_*)^\wedge$. Since $\hat{P} = \lim_{\mu \downarrow 0} \mu R(\mu)^\wedge$ it follows that

$$\dim \operatorname{Fix}(()(\lambda - i\alpha)\hat{R}(\lambda)) \leqslant \text{Rang}\,(\hat{P}) < \infty$$

(Proposition 2.1) for all $\alpha \in \mathbb{R}$. Therefore the assertion follows from Lemma 2.2. □

The consequences of this result for the asymptotic behavior of one-parameter semigroups will be discussed in D-IV, Section 4.

Notes

Section 1: The Perron-Frobenius theory for a single positive operator on a non-commutative operator algebra is worked out in [Albeverio-Höegh-Krohn (1978)] and [Groh (1981)].

The limitations of the theory (in the continuous as in the discrete case) are explained by the example following Remark 1.9 (see also [Groh (1982a)]). Therefore we concentrate on irreducible semigroups. Our main result (Theorem 1.10) extends B-III, Thm.3.6 to the non-commutative setting.

Section 2: Theorem 2.5 has its roots in the Niiro-Sawashima Theorem for a single irreducible positive operator on a Banach lattice (see [Schaefer (1974), V.5.4]). The analogous semigroup result on Banach lattices is due to [Greiner (1982)]. The ultrapower technique in our proof is developed in [Groh (1984b)].

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