The interrelation between these stability concepts is given by

$$\begin{array}{ccccc} & & (v) & & \\ & \nearrow & & \searrow & \\ (iv) & & & & (vii)\;. \\ & \searrow & & \nearrow & \\ & & (vi) & & \end{array}$$

If A is a bounded operator, i.e., if $D(A) = E$, then (iv) <=> (v) and (vi) <=> (vii). If A is unbounded then the stability notions may differ as we will see in the following examples.

Examples 1.2. (a) Let $E = c_o$. Then $A : (x_n)_{n\in\mathbb{N}} \to (-1/n\cdot x_n)_{n\in\mathbb{N}}$ generates the semigroup $T(t)(x_n) = (e^{-t/n}x_n)_{n\in\mathbb{N}}$. It is easy to see that $\|T(t)\| = 1$ and $\|T(t)f\| \to 0$ for every $f \in c_o$. Moreover, A is a bounded operator, hence $D(A) = E$. This gives an example for a (uniformly) stable but not exponentially stable semigroup. The translation semigroups generated by the first derivative on $C_o(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$ for $1 < p < \infty$ give further examples for (uniformly) stable but not exponentially stable semigroups. Moreover, as seen in A-II, Ex.1.14, the Laplacian Δ on $C_o(\mathbb{R}^n)$ generates a bounded holomorphic semigroup given by

$$T(t)f(x) = (4\pi t)^{-n/2}\cdot \int_{\mathbb{R}^n} e^{-(x-y)^2/4t}\, f(y)\, dy$$

which cannot be exponentially stable because $0 \in \sigma(\Delta)$ ($\mathrm{im}\Delta \neq C_o(\mathbb{R}^n)$), see Cor.1.5 below. By a straightforward $(2-\varepsilon)$-argument using $(4\pi t)^{-n/2}\int_{\mathbb{R}^n} \exp(-y^2/4t)\, dy = 1$ one can easily show that $\|T(t)f\| \to 0$ for all $f \in C_o(\mathbb{R}^n)$ (see also B-III,Ex.1.7).
Therefore, the Laplacian on $C_o(\mathbb{R}^n)$ (and also on $L^p(\mathbb{R}^n)$ for $1<p<\infty$, see Ex.1.15 below) generates a (uniformly) stable but not exponentially stable semigroup.

(b) Note that the condition $0 \leq \omega(A) = \inf\{\omega : \|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0\}$ does not exclude that the semigroup $(T(t))_{t\geq 0}$ is exponentially stable. In fact, as shown in A-III,1.3 the translation semigroup $(T(t))_{t\geq 0}$ on $E := C_o(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x dx)$ satisfies $\|T(t)\| = 1$, hence $\omega(A) = 0$, and for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > -1$ the resolvent of the generator is given as $R(\lambda,A)f = \int_o^\infty e^{\lambda t}\, T(t)f\, dt$ for every $f \in E$. From the equation A-I,3.2

$$T(t)f = e^{\lambda t}\left(f - \int_o^t e^{-\lambda s}\, T(s)(\lambda - A)f\, ds\right)$$

and the existence of $\lim_{t\to\infty} \int_o^t e^{-\lambda s}\, T(s)(\lambda - A)f\, ds$ it follows that $\|T(t)f\| \leq M\cdot e^{\lambda t}$ for every $f \in D(A)$ and some constant M depending