Theorem 3.7. Let $\Phi \in L(E,F)$ be positive and let B be the generator of a positive semigroup on F. The following implications are valid :

(a) If $s(B + \Phi_\lambda) < \lambda$, then $s(A) < \lambda$.

(b) If $s(B + \Phi_\lambda) = \lambda$, then $s(A) = \lambda$.

(c) Suppose that B has compact resolvent and there exist $\lambda' \in \mathbb{R}$ with $\sigma(B + \Phi_{\lambda'}) \neq \emptyset$. Then

$$(3.6) \qquad s(B + \Phi_\lambda) \lesseqgtr \lambda \quad \text{if and only if} \quad s(A) \lesseqgtr \lambda .$$

Proof. (a) If $\lambda > s(B + \Phi_\lambda)$, then $\mu > s(B + \Phi_\mu)$ for all $\mu \geq \lambda$ by Prop.3.6 . Therefore, $\mu \in \rho(B + \Phi_\mu)$ for all $\mu \geq \lambda$. By Prop.3.4 this implies $\mu \in \rho(A)$ for all $\mu \geq \lambda$. Since $s(A) \in \sigma(A)$ by [C-III, Thm.1.1.(a)] we obtain $\lambda > s(A)$.

(b) If $\lambda = s(B + \Phi_\lambda)$, then again $\lambda \in \sigma(B + \Phi_\lambda)$ whence we obtain from Prop.3.4 $\lambda \in \sigma(A)$ and therefore $\lambda \leq s(A)$. In the same way as in (a) we conclude that $\mu \in \rho(A)$ if $\mu > \lambda$; hence $\lambda = s(A)$.

(c) It suffices to prove that $s(A) > \lambda$ whenever $s(B + \Phi_\lambda) > \lambda$. Assume the latter inequality. According to Prop.3.6 there exists a unique μ satisfying $\mu = s(B + \Phi_\mu)$. Still by Prop.3.6 it follows that $\lambda < \mu$. Assertion (b) now completes the proof.

□

Remark. We call (3.5) the generalized characteristic equation corresponding to (RCP). A justification for this terminology will be given in a remark following Cor.3.8 of Chapter C-IV.

The characterization (3.6) of $s(A)$ uses the continuity of $\lambda \to s(B + \Phi_\lambda)$. In the general case we apply the following lemma which is due to W. Arendt.

Lemma. Let $\Phi \in L(E,F)$ be positive and assume that B generates a positive semigroup on F. If we define

$$\mu := \begin{cases} \sup\{\lambda \in \mathbb{R} : s(B+\Phi_\lambda) > \lambda\} & \text{if } \sigma(B+\Phi_\lambda) \neq \emptyset \text{ for some } \lambda \in \mathbb{R}, \\ -\infty & \text{otherwise,} \end{cases}$$

then $s(A) = \mu$.

Proof. If $\sigma(B+\Phi_\lambda) = \emptyset$ for all $\lambda \in \mathbb{R}$ then $\sigma(A) = \emptyset$ by Prop.3.4 and there is nothing to prove.

Take now $\lambda \in \mathbb{R}$ with $\sigma(B+\Phi_\lambda) \neq \emptyset$ and show $\mu \in \sigma(B+\Phi_\mu)$.

Case 1: If $\mu = s(B+\Phi_\mu)$ then $\mu \in \sigma(B+\Phi_\mu)$ by C-III,Thm.1.1 .

Case 2: If $\mu < s(B+\Phi_\mu)$ we show $r \in \sigma(B+\Phi_\mu)$ for every $r \in (\mu, s(B+\Phi_\mu)]$.