Before we can formulate the second lemma we have to fix some notation:

Definition 2.2.(a) Given $h \in C_o(X)$ such that $h(x) \neq 0$ for all $x \in X$ then the operator S_h is defined to be the multiplication operator with sign h, i.e.,

(2.3) $S_h f = h|h|^{-1} f \quad (f \in C_o(X))$.

(b) For $f \in C_o(X)$, $n \in \mathbb{Z}$ we define $f^{[n]} \in C_o(X)$ by

$$(2.4) \quad f^{[n]}(x) = \begin{cases} (f(x)/|f(x)|)^{n-1} \cdot f(x) & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

The following assertions are immediate consequences of the definition. They will be used frequently in the following.

(2.5) S_h is a linear isometry satisfying $|S_h f| = |f|$, its inverse being $S_{\bar{h}}$ where $\bar{h}$ is the complex conjugate of h.

(2.6) $f^{[1]} = f$, $f^{[0]} = |f|$, $f^{[-1]} = \bar{f}$,
$|f^{[n]}| = |f|$ for every $n \in \mathbb{Z}$.

(2.7) If $h(x) \neq 0$ for all $x \in X$, then $h^{[n]} = S_h^n |h| = S_h^{n-1} h$.

Lemma 2.3. Let T and R be bounded linear operators on $C_o(X)$ and assume that $h \in C_o(X)$ has no zeros. Suppose we have

(2.8) $Rh = h$, $T|h| = |h|$ and $|Rf| \leq T|f|$ for every $f \in C_o(X)$.

Then R and T are similar, more precisely, $T = S_h^{-1} R S_h$.
In particular, the spectra (and point spectra resp.) of T and R coincide.

Proof. We first note that the assertion $|Rf| \leq T|f|$ $(f \in E)$ implies that T is a positive operator. Therefore $T|h| = |h|$ implies that the principal ideal $E_h = \{f \in C_o(X) : |f| \leq n|h| \text{ for some } n \in \mathbb{N}\}$ is an invariant subspace for T and for R as well. E_h is isomorphic to $C^b(X) \cong C(\beta X)$ (βX denotes the Stone-Cech compactification of X), an isomorphism is given by $f \to f|h|$. Considering the restrictions $T|E_h$ and $R|E_h$ as operators on $C(\beta X)$ and denoting them $\tilde{T}$ and $\tilde{R}$ respectively, we have

(2.9) $\tilde{R}\tilde{h} = \tilde{h}$, $\tilde{T}1 = 1$, $\tilde{T} \geq 0$, $|\tilde{R}f| \leq \tilde{T}|f|$ for all f .