Corollary 4.10. If $(T(t))_{t\in\mathbb{R}}$ is a positive group on a space L^2 or $C_o(X)$ with generator A, then

(4.11) $\sigma(T(t))\cap\mathbb{R}_+ = \exp(t\sigma(A)\cap\mathbb{R})$ for every $t \geq 0$.

Proof. We borrow from the next chapter that for positive semigroups on spaces L^1, L^2 or $C_o(X)$ spectral bound and growth bound coincide (see C-IV,Thm.1.1).
We only have to show that $\exp(t\rho(A)\cap\mathbb{R}) \subset \rho(T(t))\cap\mathbb{R}_+$.
If we consider a positive semigroup on an L^2-space, Thm.4.8 can be applied directly: Given $\mu \in \rho(A)\cap\mathbb{R}$, then $E = I_\mu \oplus J_\mu$ according to Thm.4.8 . The result mentioned above implies $r(T(t)|I_\mu) < e^{\mu t}$ and $r(T(-t)|J_\mu) < e^{\mu t}$. Hence $\sigma(T(t)|I_\mu) \subset \{\lambda \in \mathbb{C} : |\lambda| < e^{\mu t}\}$ and $\sigma(T(t)|J_\mu) = \left(\sigma(T(-t)|J_\mu)\right)^{-1} \subset \{\lambda \in \mathbb{C} : |\lambda| > e^{\mu t}\}$.
Thus $\sigma(T(t)) = \sigma(T(t)|I_\mu) \cup \sigma(T(t)|J_\mu)$ does not contain $e^{\mu t}$.
In case $(T(t))$ is a positive group on $C_o(X)$ then the adjoint group $(T(t)')$ is a group of lattice homomorphisms on E' . It follows that E^* is a sublattice of $C_o(X)' \cong M_b(X)$ hence a L^1-space. The argument given for the L^2-space yields $\sigma(T(t)^*)\cap\mathbb{R}_+ = \exp(t\sigma(A^*)\cap\mathbb{R})$ for every $t \geq 0$. Thus the assertion follows from A-III,4.4 .

□

We conclude by describing a general situation where lattice semigroups occur. In Section 4 of B-III we constructed semigroups of lattice homomorphisms on $C_o(X)$ starting with a continuous (semi-)flow on the locally compact space X and a multiplication operator. One can perform similar constructions on spaces $L^p(\mu)$ for $1 \leq p < \infty$ under certain conditions on the flow. We consider an example which shows where the problems are.
Define the semiflow ϕ on $\mathbb{R}_+$ as follows: $\phi(t,x) := x-t$ for $x \geq t$ and $\phi(t,x) := 0$ for $x \leq t$. For $f \in L^p(\mu)$ one has difficulties to define $f\circ\phi_t$ properly since the preimage of the zero-set $\{0\}$ does not have measure zero. This problem does not arise in case every transformation ϕ_t is measure preserving, i.e. $\mu(\phi_t^{-1}(C)) = \mu(C)$ for every Borel set C . A more general criterion is stated in the following proposition.

Proposition 4.11. Let X be a locally compact space and let μ be a regular, positive Borel measure on X . Assume that the continuous semiflow $\phi : \mathbb{R}_+\times X \to X$ satisfies the following condition:

(4.12) $\phi_t^{-1}(K)$ is compact for every compact set $K \subset X$, $t \geq 0$.