(1.5) $\omega_1(A) = \inf\{\mathrm{Re}\,\lambda : \text{weak-}\lim_{t\to\infty} \int_0^t e^{-\lambda s}\, T(s)\, ds \ \text{ exists}\}$
$= \inf\{\mathrm{Re}\,\lambda : \text{uniform-}\lim_{t\to\infty} \int_0^t e^{-\lambda s}\, T(s)\, ds \ \text{ exists}\}$.

(b) In the equations (1.4) and (1.5) the term "Re λ" may be replaced by "$\lambda \in \mathbb{R}$" (use (1.1)).

Proof of Thm.1.4. The equality $\omega_1(A) = \inf\{\mathrm{Re}\,\lambda : \int_0^\infty \|e^{-\lambda t}T(t)f\|\, dt$ exists for all $f \in D(A)\}$ follows from the definition of $\omega_1(A)$ and the lemma used in the proof of Thm.1.3 .
We prove $\omega_1(A) = \inf\{\mathrm{Re}\,\lambda : \int_0^\infty e^{-\lambda s}T(s)f\, ds$ exists for every $f \in E\}$.
The identity

$$T(t)f = e^{\lambda t}(f - \int_0^t e^{-\lambda s}\, T(s)(\lambda - A)f\, ds)$$

yields
$\omega_1(A) \le \inf\{\mathrm{Re}\,\lambda : \int_0^\infty e^{-\lambda t}\, T(t)f\, dt$ exists for every $f \in \mathrm{im}(\lambda - A)\}$.
Therefore
$\omega_1(A) \le \inf\{\mathrm{Re}\,\lambda : \int_0^\infty e^{-\lambda t}\, T(t)f\, dt$ exists for every $f \in E\} =: b$.
Take $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \omega_1(A)$. Then $\int_0^\infty e^{-\lambda t}\, T(t)f\, dt$ exists for every $f \in D(A)$. Define $g := \int_0^1 e^{-\lambda t}\, T(t)f\, dt$. Then $g \in D(A)$ and $\int_0^n e^{-\lambda t}\, T(t)f\, dt = \sum_{k=0}^{n-1} e^{-\lambda k}\, T(k)g$. Since $\mathrm{Re}\,\lambda > \omega_1(A)$ it follows that the sum converges for every $g \in D(A)$. Therefore the integral converges as $n \to \infty$ $(n \in \mathbb{N})$ for every $f \in E$. For every $t \in \mathbb{R}_+$ define a bounded operator T_t by $f \mapsto \int_0^t e^{-\lambda s}\, T(s)f\, ds$. As seen above, $T_n f$ converges as $n \to \infty$ $(n \in \mathbb{N})$ for every $f \in E$. It follows from the Uniform Boundedness Principle that the family $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded.
For every $t \in \mathbb{R}_+$ there exist $n \in \mathbb{N}$ and $t' \in [0,1)$ such that $T_t = T_{t'} + e^{-\lambda t'}T(t')T_n$. Since the operator families on the right side of the equation are uniformly bounded the same is true for $(T_t)_{t\ge 0}$. Since $(T_t f)_{t\ge 0}$ converges for every $f \in D(A)$ it follows that $(T_t f)_{t\ge 0}$ converges for every $f \in E$. Thus $b \le \omega_1(A)$.

□

The inequality

$$\omega(A) \ge \inf\{\mathrm{Re}\,\lambda : \int_0^\infty \|e^{-\lambda t}\, T(t)f\|\, dt \ \text{ exists for every } f \in E\}$$

in combination with the lemma of Thm.1.3 implies that the growth bound $\omega(A)$ coincides with the abscissa of absolute convergence of the Laplace transform of $(T(t))_{t\ge 0}$; i.e.,

(1.6) $\omega(A) = \inf\{\mathrm{Re}\,\lambda : \int_0^\infty \|e^{-\lambda t}\, T(t)f\|\, dt$ exists for every $f \in E\}$.

As seen in A-I,Prop.1.11, if $\int_0^\infty e^{-\lambda t}\, T(t)f\, dt$ exists for every $f \in E$, then $\lambda \in \rho(A)$ and $R(\lambda,A)f = \int_0^\infty e^{-\lambda t}\, T(t)f\, dt$. This and Thm.1.4 yield the following corollary.