Chapter 1

F-I

1 Supplement

In this part of the new edition we put together results on positive semigroups which appeared after the first edition of the book. We will not follow the matrix scheme of the first part any more. For instance when we apply the results to elliptic operators it will be convenient to establish the semigroups on $L^p(\Omega)$ and on $C_0(\Omega)$ and to study their asymptotic behavior.

2 Countable spectrum

In this chapter we formulate the countable spectrum theorem (ABLV-Theorem) and give a series of consequences for positive semigroups.

Let E be a complex Banach space.

Theorem 2.1 (ABLV-Theorem). *Let $\mathcal{J} = (T(t))_{t\geqslant 0}$ be a bounded semigroup with generator A. Assume that*

(i) $\sigma(A) \cap i\mathbb{R}$ is countable and

(ii) $P\sigma(A') \cap i\mathbb{R} = \emptyset$.

Then the semigroup is stable; i.e.,

$$\lim_{t\to\infty} T(t)f = 0 \quad \textit{for all } f \in E.$$

We comment on the two conditions (i) and (ii).

Countability of the boundary spectrum $\sigma(A) \cap i\mathbb{R}$ is not necessary for stability. The shift semigroup on $L^p(\mathbb{R})$, $1 \leqslant p < \infty$, given by $(T(t)f)(x) = f(x-t)$ is stable but the boundary spectrum is $i\mathbb{R}$. However, if $E \subset \mathbb{R}$ is closed and uncountable, then there exists an unstable, bounded semigroup such that (i) is satisfied and $\sigma(A)\cap i\mathbb{R} \subset iE$.

Condition (ii) is necessary. In fact, if $s \in \mathbb{R}$ such that $is \in P\sigma(A')$, then there exists $0 \neq \phi \in D(A')$ such that $A'\phi = is\phi$. Consequently $T(t)'\phi = e^{ist}\phi$.

Let $f \in E$ such that $\langle \phi, f \rangle \neq 0$. Then

$$\langle \phi, T(t)f \rangle = e^{ist} \langle \phi, f \rangle$$

does not converge to 0 as $t \to \infty$.

Bibliography