2. STANDARD EXAMPLES

In this section we list and discuss briefly the most basic examples of semigroups together with their generators. These semigroups will reappear throughout this book and will be used to illustrate the theory. We start with the class of semigroups mentioned after Definition 1.1 .

2.1. Uniformly Continuous Semigroups

It follows from elementary operator theory that for every bounded operator $A \in L(E)$ the sum

$$\sum_{n=0}^{\infty} t^n A^n/n! \quad =: \quad e^{tA}$$

exists and determines a unique uniformly continuous (semi)group $(e^{tA})_{t\in\mathbb{R}}$ having A as its generator.

Conversely, any uniformly continuous semigroup is of this form: If the semigroup $(T(t))_{t\geq 0}$ is uniformly continuous, then $\frac{1}{t}\int_0^t T(s)\, ds$ uniformly converges to $T(0) = \mathrm{Id}$ as $t \to 0$. Therefore for some $t' > 0$ the operator $\frac{1}{t'}\int_0^{t'} T(s)\, ds$ is invertible and every $f \in E$ is of the form $f = \frac{1}{t'}\int_0^{t'} T(s)g\, ds$ for some $g \in E$. But these elements belong to $D(A)$ by (1.3) , hence $D(A) = E$. Since the generator A is closed and everywhere defined it must be bounded.

Remark that bounded operators are always generators of groups, not just semigroups. Moreover the growth bound ω satisfies $|\omega| \leq \|A\|$ in this situation.

The above characterization of the generators of uniformly continuous semigroups as the bounded operators shows that these semigroups are - at least in many aspects - rather simple objects.

2.2. Matrix Semigroups

The above considerations expecially apply in the situation $E = \mathbb{C}^n$. If $n = 2$ and $A = (a_{ij})_{2\times 2}$ the following explicit formulas for e^{tA} might be of interest:

Set $s :=$ trace A , $d := \det A$ and $D := (s^2 - 4d)^{1/2}$. Then

$$e^{tA} = e^{ts/2}\cdot[D^{-1}2\sinh(tD/2)\cdot A + (\cosh(tD/2) - sD^{-1}\sinh(tD/2))\cdot \mathrm{Id}]$$
$$\text{if } D \neq 0 \,,$$

$$e^{tA} = e^{ts/2}\cdot[tA + (1 - ts/2)\cdot \mathrm{Id}] \qquad \text{if } D = 0 \text{ , resp. .}$$

2.3. Multiplication Semigroups

Many Banach spaces appearing in applications are Banach spaces of (real or) complex valued functions over a set X . As the most