3. INDUCTION AND REDUCTION

(a) If E is a Banach space and $\mathcal{S} \subseteq \mathcal{L}(E)$ a semigroup of bounded operators, then a closed subspace F is called $\mathcal{S}$-invariant, if $SF \subseteq F$ for all $S \in \mathcal{S}$. We call the semigroup $\mathcal{S}_| := \{S_{|F} : S \in \mathcal{S}\}$ the reduced semigroup. Note that for a one-parameter semigroup $\mathcal{T}$ (resp., pseudo-resolvent R) the reduced semigroup is again strongly continuous (resp. $R_|$ is again a pseudo-resolvent) (compare the construction in A-I,3.2).

(b) Let M be a W*-algebra, $p \in M$ a projection and $S \in \mathcal{L}(M)$ such that $S(p^\perp M) \subseteq p^\perp M$ and $S(Mp^\perp) \subseteq Mp^\perp$, where $p^\perp := 1-p$. Since for all $x \in M$:

$$p[S(x) - S(pxp)] = p[S(p^\perp xp) + S(xp^\perp)]p = 0 ,$$

we obtain $p(Sx)p = p(S(pxp))p$. Therefore the map

$$S_p := (x \to p(Sx)p) : pMp \to pMp$$

is well defined. We call S_p the induced map. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$.

If $\mathcal{T} = (T(t))_{t \geq 0}$ is a weak*-semigroup on M which is of Schwarz type and if $T(t)(p^\perp) \leq p^\perp$ for all $t \in \mathbb{R}_+$, then $\mathcal{T}$ leaves $p^\perp M$ and $Mp^\perp$ invariant. It is easy to see that the induced semigroup $\mathcal{T}_p = (T(t)_p)_{t \geq 0}$ is again a weak*-semigroup.

If R is an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in M such that $R(\mu)p^\perp \leq p^\perp$ for some $\mu \in \mathbb{R}_+$ then $p^\perp M$ and $Mp^\perp$ are R-invariant. Again, the induced pseudo-resolvent R_p is of Schwarz type and identity preserving.

(c) Let ϕ be a positive normal linear functional on a W*-algebra M such that $T_* \phi = \phi$ for some identity preserving Schwarz map T on M with preadjoint $T_* \in \mathcal{L}(M_*)$. Then $T(s(\phi)^\perp) \leq s(\phi)^\perp$ where $s(\phi)$ is the support projection of ϕ. To see this let $L_\phi := \{x \in M : \phi(xx^*) = 0\}$ and $M_\phi := L_\phi \cap L_\phi{}^*$. Since ϕ is T_*-invariant, and T is a Schwarz map, the subspaces L_ϕ and M_ϕ are T-invariant. From $M_\phi = s(\phi)^\perp M s(\phi)^\perp$ and $T(s(\phi)^\perp) \leq 1$ it follows that $T(s(\phi)^\perp) \leq s(\phi)^\perp$.