$$\langle h,\phi\rangle = \int h(x)\, d\phi(x)$$

for every bounded Borel function h on X and every $\phi \in M(X)$.

After these preparations we now can show that the lattice property $|T(t)f| = T(t)|f|$ of the semigroup corresponds to the identity (2.9) below for the generator, which we call Kato's equality (cf. Remark 2.7).

<u>Theorem</u> 2.5. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on $C_o(X)$ is a lattice semigroup if and only if its generator A satisfies

$$\langle \operatorname{Re}[(\widehat{\operatorname{sign}}\,\bar{f})(Af)],\phi\rangle = \langle |f|,A'\phi\rangle \quad \text{for all } f \in D(A),\ \phi \in D(A') \qquad \text{(Kato's equality)}. \tag{2.9}$$

From the proof of the theorem we isolate the following lemma.

<u>Lemma</u> 2.6. Let $(T(t))_{t\geq 0}$ be a semigroup on $C_o(X)$ with generator A . Then for every $f \in D(A)$, $\phi \in M(X)$,

$$d/dt_{|t=0} \langle |T(t)f|,\phi\rangle = \langle \operatorname{Re}[(\widehat{\operatorname{sign}}\,\bar{f})(Af)],\phi\rangle . \tag{2.10}$$

<u>Proof</u>. Let $f \in D(A)$ and $x \in X$. Define the function $k(t) = (T(t)f)(x)$ $(t \geq 0)$. Then k is right-sided differentiable in 0 with derivative $k'(0) = (Af)(x)$. It follows from the chain rule Prop. 2.3 that

$$d/dt_{|t=0} |(T(t)f)(x)| = \operatorname{Re}[(\widehat{\operatorname{sign}}\,\bar{f})(Af)](x) . \tag{2.11}$$

Moreover, $1/t\ ||T(t)f| - |f|| \leq 1/t\ |T(t)f - f|$. Thus $\sup_{1\geq t>o} 1/t\ \||T(t)f| - |f|\| < \infty$; i.e., the functions $k_t \in C_o(X)$ given by

$$k_t(x) = 1/t\ (|(T(t)f)(x)| - |f(x)|) \qquad (x \in X) \tag{2.12}$$

$(t > 0)$ are uniformly dominated by a constant. The dominated convergence theorem and (2.11) imply that
$d/dt_{|t=0} \langle |T(t)f|,\phi\rangle = \lim_{t\downarrow o} \langle k_t,\phi\rangle = \langle \operatorname{Re}[(\widehat{\operatorname{sign}}\,\bar{f})(Af)],\phi\rangle$.

□

<u>Proof of Theorem</u> 2.5. Assume that $(T(t))_{t\geq 0}$ is a lattice semigroup. Let $f \in D(A)$, $\phi \in D(A')$. It follows from the preceding lemma that