<u>Proof</u>. We will show that each of the conditions (a), (b), (c) implies that $\ker(1 - T(s))$ is a Banach lattice (not necessarily a sublattice of E) for every $s \geq 0$. Then one argues as follows: Given $i\alpha \in P\sigma(A)$, $\alpha \in \mathbb{R}$ then $T(t)g = e^{i\alpha t}g$ for suitable $g \neq 0$. For $\tau := 2\pi|\alpha|^{-1}$ we have $g \in F := \ker(1 - T(\tau))$. Then the restriction $(T(t)_{|F})_{t\geq 0}$ is a τ-periodic positive semigroup on F. Since $T(t)_|^{-1} = T(n\tau - t)_| \geq 0$ it follows that $(T(t)_|)$ is a semigroup of lattice isomorphisms. Since $g \in F$ we have $i\alpha \in P\sigma(A_|)$ hence $i\alpha\mathbb{Z} \in P\sigma(A_|) \subset P\sigma(A)$ by Thm.4.2 .

Now we show that $\ker(1 - T(s))$ is a vector lattice for the induced order and a Banach lattice for an equivalent norm.
In case (c), $\ker(1 - T(s))$ is even a sublattice of E. Indeed, assume $T(t)'\phi = \phi$ and $\phi >> 0$ $(t \geq 0)$ then $T(s)f = f$ implies $T(s)|f| \geq |f|$. Thus from $\langle T(s)|f| - |f|, \phi\rangle = \langle |f|, T(s)'\phi - \phi\rangle = 0$ it follows that $T(s)|f| = |f|$.
Now we assume that E is weakly sequentially complete, which is equivalent to (cf. Sec.5 of C-I):

(4.5) Every increasing norm-bounded net of E_+ converges.

We fix $s > 0$ and define $F := \ker(1 - T(s))$, $T := T(s)$. Obviously $f \in F$ implies $\bar{f} \in F$ hence $F = F\cap E_{\mathbb{R}} + iF\cap E_{\mathbb{R}}$. Thus we have to show that $F_{\mathbb{R}} = F\cap E_{\mathbb{R}}$ is a sublattice. Given $f \in F_{\mathbb{R}}$ then $Tf = f$ hence $|f| \leq T|f|$. Iterating this inequality we obtain $|f| \leq T|f| \leq T^2|f| \leq T^3|f| \leq \ldots$. By (4.5) $|f|_o := \lim_{n\to\infty} T^n|f|$ exists and we have $T|f|_o = \lim_{n\to\infty} T^{n+1}|f| = |f|_o$, i.e. $|f|_o \in F_{\mathbb{R}}$. For $g \in F_{\mathbb{R}}$ satisfying $\pm f \leq g$ we have $|f|_o \leq g$ thus $|f|_o = \sup_F\{f,-f\}$. Moreover, $\|f\|_o := \||f|_o\|$ $(f \in F)$ is an equivalent norm on F such that $(F, \|\cdot\|_o)$ is a Banach lattice.
(b) If $T(s)$ is mean-ergodic then we have $\ker(1 - T(s)) = PE$ where P is the mean-ergodic projection, i.e. $Pf = \lim_{n\to\infty}\sum_{k=0}^{n-1} T(s)^k f$. Obviously P is positive, hence II.11.5 of Schaefer (1974) implies that PE is a Banach lattice (for the induced order and an equivalent norm).

□

The assumptions made in Cor.4.3 can be weakened slightly (cf. Greiner (1982)). However, one cannot prove cyclicity of $P\sigma_b(A)$ for arbitrary positive semigroups.

<u>Example</u> 4.4. At first we recall Ex.2.13 of Chapter B-III. There we constructed a bounded semigroup on the space $C(\Gamma)\times C_o(\mathbb{R})$ such that $P\sigma_b(A) = \{ik : k \in \mathbb{Z}, k \neq 0\}$.