1. STABILITY OF POSITIVE SEMIGROUPS ON BANACH LATTICES

by

Günther Greiner and Frank Neubrander

In Section 1 of B-IV we have seen that the growth bound of a positive semigroup on spaces $C_o(X)$ coincides with the spectral bound of the generator A, which is - for positive semigroups - an element of the spectrum of A. Now, using the results of A-III, A-IV, B-IV,Sec.1 and C-III, it can be shown that this is valid for positive semigroups on AM- , AL- and Hilbert spaces.

Theorem 1.1. Let A be the generator of a positive semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E such that $s(A) > -\infty$. Each of the subsequent conditions implies

$$s(A) = \omega_1(A) = \omega(A) \in \sigma(A) \ .$$

(a) Either E is an AM-space or an L^2-space or an L^1-space.

(b) There exist $\tau > 0$, $h \in E_+$ such that $T(\tau)E \subset E_h$.

(c) There exist $\tau > 0$, $\phi \in E'_+$ such that $\|T(\tau)f\| \leq \langle f,\phi\rangle$ for all $f \in E_+$.

Proof. We know that $s(A) \leq \omega_1(A) \leq \omega(A)$ (see A-IV,Cor.1.5) and $s(A) \in \sigma(A)$ (see C-III,Cor.1.4). Thus we have to show $s(A) = \omega(A)$.
(a) For AM-spaces the proof given in Section 1 of B-IV works (cf. B-IV,Rem.1.5.).
Since for positive semigroups we always have $\|R(\lambda,A)\| \leq \|R(\mathrm{Re}\lambda,A)\|$ ($\mathrm{Re}\,\lambda > s(A)$) (see C-III,Cor.1.3) the assertion for L^2-spaces follows from A-III,Cor.7.10.
If E is an L^1-space the assumptions of (c) are satisfied.
(b) We identify E_h according to the Kakutani-Krein Theorem with a space $C(K)$, K compact. Considering $T(\tau)$ as operator from E into $C(K)$, we denote it by T_o. Then T_o is positive hence continuous (see Schaefer (1974), II.Thm.5.3). Let $j : C(K) \cong E_h \to E$ be the canonical inclusion. The spectral radii of $T(\tau) = j\circ T_o$ and $T_o\circ j$ coincide and are given by $\rho := \exp(\tau\cdot\omega(A))$. By the Krein-Rutman Theorem (cf. the Corollary to Thm.2.6 in the Appendix of Schaefer (1966)) there exists $0 < \mu \in C(K)'$ such that $(T_o\circ j)'\mu = \rho\cdot\mu$. Then $\phi := T_o'\mu$ is an eigenvector of $(j\circ T_o)'$ with eigenvalue ρ. Thus $\rho \in R\sigma(T(\tau))$ and hence $s(A) \geq \omega(A)$ by A-III,Thm.6.2.