(2.11) Given $\lambda_o \in D$ then for $\mu \in D$ with $|\mu-\lambda_o| < \|R(\lambda_o)\|^{-1}$ we have $R(\mu) = \sum_{n=0}^{\infty} (\lambda_o-\mu)^n R(\lambda_o)^{n+1}$

(2.12) $\lambda \to R(\lambda)$ is a locally holomorphic function defined on $D \subseteq \mathbb{C}$ with values in $L(G)$.

We only sketch the proof of these assertions: (2.12) follows from (2.11) and the latter is a consequence of (2.10). The identity stated in (2.10) can be rewritten as follows:
$\left(1-(\lambda_o-\mu)R(\lambda_o)\right)\left(1-(\mu-\lambda_o)S\right) = 1 = \left(1-(\mu-\lambda_o)S\right)\left(1-(\lambda_o-\mu)R(\lambda_o)\right)$
Thus $S = (\mu-\lambda_o)^{-1}\left(1 - (1-(\lambda_o-\mu)R(\lambda_o))^{-1}\right)$ has to be unique.
It follows from (2.11) and (2.12) that every pseudo-resolvent has a unique maximal extension.
Further properties of pseudo-resovents are given in the following two propositions.

Proposition 2.6. Suppose G is a Banach space, $D \subseteq \mathbb{C}$ and $R : D \to L(G)$ is a pseudo-resolvent.

(a) Given $\alpha \in \mathbb{C}$, $x \in G$ one has $(\lambda-\alpha)R(\lambda)x = x$ either for all $\lambda \in D$ or for none.

(b) Suppose $\mu \in \bar{D}\setminus D$. Then R can be extended to an open set containing μ if and only if there exists a sequence $(\lambda_n) \subset D$ converging to μ such that $\|R(\lambda_n)\|$ is bounded.

Proof. (a) Suppose that $(\lambda-\alpha)R(\lambda)x = x$ for some fixed $\lambda \in D$, $x \in G$. Then using (2.8) we have for $\mu \in D$: $(\mu-\lambda)R(\mu)x = (\lambda-\alpha)(\mu-\lambda)R(\mu)R(\lambda)x$ $= (\lambda-\alpha)\left(R(\lambda)x - R(\mu)x\right) = x - (\lambda-\alpha)R(\mu)x$.
It follows that $(\mu-\alpha)R(\mu)x = x$ for all $\mu \in D$.
(b) If there exists an extension, then $\|R(\lambda_n)\|$ is bounded for every sequence (λ_n) converging to μ by (2.12). On the other hand assuming that $\|R(\lambda_n)\|$ is bounded by M for a fixed sequence $(\lambda_n) \subset D$ with $\lambda_n \to \mu$ $(M \geq 0)$, we have
$\|R(\lambda_n) - R(\lambda_m)\| = |\lambda_n-\lambda_m|\,\|R(\lambda_n)R(\lambda_m)\| \leq M^2|\lambda_n-\lambda_m|$
which shows that $(R(\lambda_n))$ is a Cauchy sequence in $L(G)$, hence $S := \lim_{n\to\infty} R(\lambda_n)$ exists. The assertion now follows from (2.10) and (2.11) .

□

In the next proposition we consider a positive pseudo-resolvent R on a Banach lattice E ; i.e., we assume that the domain D of R contains the positive real axis and that $R(\mu)$ is a positive operator for every $\mu > 0$. Applying Pringsheim's Theorem (see Thm.2.1 in the appen-