Next we describe special norm continuous semigroups.

Compact semigroups

Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup and $t_o > 0$. If $T(t_o)$ is compact, then it follows from the semigroup property that $T(t)$ is compact for all $t \geq t_o$. Moreover, $t \to T(t)$ is norm continuous in every $t > t_o$
[In fact, since $T(h) \to Id$ strongly with $h\downarrow 0$, it follows that $\lim_{h\downarrow 0} T(h)f = f$ uniformly on every compact subset K of E. Now let $t \geq t_o$. Then $K = \overline{T(t)U}$ is compact (where U denotes the unit ball of E). Hence $\lim_{h\downarrow 0} T(h+t)f = \lim_{h\downarrow 0} T(h)T(t)f$ uniformly for $f \in U$. So the semigroup is right-sided norm continuous on $[t_o,\infty)$ and so norm continuous on (t_o,∞).]

__Definition__ 1.22. A strongly continuous semigroup $(T(t))_{t\geq 0}$ is called __compact__ if $T(t)$ is compact for all $t > 0$; the semigroup is called __eventually compact__ if there exists $t_o > 0$ such that $T(t_o)$ is compact (and hence $T(t)$ is compact for all $t \geq t_o$).

We want to find a relation between the compactness of the semigroup and the compactness of the resolvent of its generator.

__Definition__ 1.23. Let A be an operator and $\rho(A)\neq\emptyset$. We say, A __has a compact resolvent__ if $R(\lambda,A)$ is compact for one (and hence all) $\lambda \in \rho(A)$.

__Proposition__ 1.24. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup and assume that its generator has a compact resolvent. If $t \to T(t)$ is norm continuous in t_o, then $T(t)$ is compact for all $t \geq t_o$.

__Proof__. Considering $(e^{-wt}T(t))_{t\geq 0}$ for some $w > 0$ if necessary, we can assume that $s(A) < 0$. Let $S(t) \in L(E)$ be given by
$S(t)f = \int_0^t T(s)f\,ds$ $(t\geq 0)$. Then $AS(t)f = T(t)f - f$ for all $f \in E$, and so $S(t) = R(0,A)(Id-T(t))$ is compact for all $t \geq 0$.
Since $t \to T(t)$ is norm continuous for $t \geq t_o$, one has
$\lim_{h\downarrow 0} \frac{1}{h}(S(t_o+h)-S(t_o)) = T(t_o)$ in the operator norm. Thus $T(t_o)$ is compact as limit of compact operators.

□