Proposition 3.21. Let $-\infty \leq a < b \leq \infty$. A mapping $\phi : \mathbb{R} \times (a,b) \to (a,b)$ defines a continuous flow if and only if there exists a finite or countable set of disjoint intervals $(a_n,b_n) \subset (a,b)$ $(n \in J)$ and for every $n \in J$ there exists a homeomorphism r_n from (a_n,b_n) onto $(-\infty,\infty)$ such that

$$\phi(t,x) = \begin{cases} x & \text{if } x \notin \bigcup_{n\in J} (a_n,b_n) \\ r_n^{-1}(r_n(x) + t) & \text{if } x \in (a_n,b_n),\ n \in J \end{cases}$$

for all $t \in \mathbb{R}$

Note: $J = \emptyset$ if and only if $\phi(t,x) = x$ for all $x \in (a,b)$ and $t \in \mathbb{R}$.

Proof. It is not difficult to see that the construction in the proposition defines a continuous flow on (a,b) . Now let ϕ be a continuous flow. The set $K = \{ x \in (a,b) : \phi(t,x) = x$ for all $t \in \mathbb{R} \}$ is closed in (a,b) . Thus $(a,b) \setminus K$ is the union of a finite or countable set of disjoint intervals (a_n,b_n) , $(n \in J)$. Pick $x_n \in (a_n,b_n)$, $(n \in J)$. Then $\alpha_n(t) := \phi(t,x_n)$ defines an injective mapping from $\mathbb{R}$ into (a_n,b_n) . Thus α_n is strictly monotonous. It is easy to see that $\lim_{t\to\infty} \phi(t,x_n)$ is an element of K whenever the limit exists in (a,b) ; similary for the limit as $t \to -\infty$. Consequently, $\alpha_n(\mathbb{R}) = (a_n,b_n)$; i.e., α_n is a homeomorphism from $\mathbb{R}$ onto (a_n,b_n) . Define r_n to be the inverse of α_n . Let $x \in (a_n,b_n)$. Then $\phi(t,x) = \phi(t,\alpha_n(r_n(x)))$ $= \phi(t,\phi(r_n(x),x_n)) = \phi(t + r_n(x),x_n) = \alpha_n(t + r_n(x)) = r_n^{-1}(r_n(x) + t)$ for all $t \in \mathbb{R}$. This proves that ϕ has the desired form.

□

If m is an admissible function on (a,b) , then $D(\delta)$ contains many differentiable functions. This can be expressed by two facts:

a) $C_c^1(a,b) := \{ f \in C^1(a,b) : f$ vanishes in a neighbourhood of a and $b \}$ is contained in $D(\delta_m)$ (this follows from the definition of δ_m) ; and

b) the set $D_o(\delta_m)$ of all differentiable functions in $D_o(\delta_m)$ is a core of δ_m (this follows from Theorem 3.17).

We will show below that these two properties are characteristic for the operators δ_m . For other generators of automorphism groups they can be violated dramatically as the following example shows.

Example 3.22. There exists a generator δ of an automorphism group on $C_o(\mathbb{R})$ such that $D(\delta) \cap C^1(\mathbb{R}) = \{0\}$. In fact, consider a strictly increasing continuous map q from $\mathbb{R}$ onto $\mathbb{R}$ such that