that $2\|T(t) - Id\| - \|(T(t) - Id)^2\| \leq \|T(2t) - Id\|$. Hence $2s \leq \limsup_{t\downarrow 0} \|T(2t) - Id\|$. Obviously, $\limsup_{t\downarrow 0} \|T(2t) - Id\| = s$ and so, $2s \leq s$. Consequently, $s = 0$.

□

Remarks. 1. If in Lemma 3.1 $\mathcal{T} = (T(t))_{t\geq 0}$ is strongly continuous, in which case $s < \infty$, one can replace $\lim_{t\to 0} \|(T(t) - Id)^2\| = 0$ by the weaker condition $\limsup r(T(t) - Id) < 1$ [Lotz (1985), Lemma 2] where r denotes the spectral radius.

2. The condition $s < \infty$ in Lemma 3.1 is essential as the following example shows:

Let $f : \mathbb{R} \to \mathbb{R}$ be non-continuous with $f(s+t) = f(s) + f(t)$ for all $s, t \in \mathbb{R}$ (see [Hamel (1905)]). Then $\{(t,f(t)) : t \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$. Hence for the semigroups $\mathcal{T} = (T(t))_{t\geq 0}$ on $\mathbb{R}^2$ with

$$T(t) = \begin{pmatrix} 1 & f(t) \\ 0 & 1 \end{pmatrix} \quad \text{for } t \geq 0$$

we have $s = \infty$. Therefore $\mathcal{T}$ is not uniformly continuous. However, $(T(t) - Id)^2 = 0$ for all $t \geq 0$.

Lemma 3.2. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a one-parameter semigroup of operators on a Banach space E . Then the following assertions are equivalent:

(a) $\mathcal{T}' = (T(t)')_{t\geq 0}$ is a strongly continuous semigroup on the dual E' .

(b) $((T(t) - Id)x_n)$ converges weakly to zero for every bounded sequence (x_n) in E and every sequence (t_n) in $[0,\infty)$ with $\lim t_n = 0$.

Moreover, (a) implies

(c) $\mathcal{T}$ is strongly continuous.

Proof. Let $x' \in E'$ and $t_n \geq 0$ be given. Then $\lim \|(T(t_n) - Id)'x'\| = 0$ if and only if $\lim \langle x_n, (T(t_n) - Id)'x'\rangle = 0$ for every bounded sequence (x_n) in E . This easily implies the equivalence of (a) and (b). In particular, (a) implies that $((T(t_n) - Id)x)$ converges weakly to zero for every sequence (t_n) in $[0,\infty)$ with $\lim t_n = 0$ and every $x \in E$. Hence $\mathcal{T}$ is strongly continuous by Propositon 1.23 in [Davies (1980)].

□

We recall that a Banach space E is called a Grothendieck space if every weak* convergent sequence in E' converges weakly.