$\|T(t)(Id-P)\| \le \|T(t)_{|\ker P}\| \cdot \|Id-P\| \le \|Id-P\| \cdot C_o \cdot \exp(\mathrm{Re}\,\lambda_m \cdot t)$.
We define $R_m(t) := T(t)(Id-P)$, $T_n(t) := T(t)P_n$ $(n \in \mathbb{N})$.
Then $R_m(t)$ satisfies the estimate stated in (2.1) and we have $T(t) = \sum_{n=1}^m T_n(t) + R_m(t)$ because $P = \sum_{n=1}^m P_n$ by A-III,Cor.6.5(ii) . The family of projections $Id-P$, P_1 , P_2 , .. , P_m reduces the semigroup. Thus in order to prove the representation of $T_n(t)$ stated in (2.1) we only have to ~~consider~~ elements $f \in P_nE = \ker(\lambda_n - A)$. That is we can assume $E = P_nE$, $\sigma(A) = \{\lambda_n\}$, $P_n = Id$ and for simplification we drop the index n , i.e., $\lambda = \lambda_n$, $k = k(n)$. Then A is a bounded operator satisfying $(\lambda - A)^k = 0$ and its resolvent is given by $R(\mu,A) = (\mu-\lambda)^{-1}\sum_{j=0}^{k-1}(\mu-\lambda)^{-j}(A-\lambda)^j$ for $\mu \neq \lambda$. It follows that $R(\mu,A)^i = (\mu-\lambda)^{-i}\sum_{j=0}^{k-1}\binom{j+i-1}{i-1}(\mu-\lambda)^{-j}(A-\lambda)^j$. Hence we have $(\frac{i}{t}R(\frac{i}{t},A))^i = (1-\lambda\frac{t}{i})^{-i}\sum_{j=0}^{k-1}\binom{j+i-1}{i-1}(i-\lambda t)^{-j}t^j(A-\lambda)^j$ for every $i \in \mathbb{N}$. Since $\lim_{i\to\infty}(1-\lambda\frac{t}{i})^{-i} = e^{\lambda t}$ and $\lim_{i\to\infty}\binom{j+i-1}{i-1}(i-\lambda t)^{-j} = \frac{1}{j!}$ for every $j \in \mathbb{N}$ the assertion follows from formula A-II,(1.3).

□

Combining Thm.2.1 with the results of Chapter B-III one can describe the behavior of $T(t)$ as $t \to \infty$ provided that $(T(t))_{t\ge0}$ is a positive semigroup. We give a typical example.

Corollary 2.2. Let $(T(t))_{t\ge0}$ be a positive semigroup on a space $C_o(X)$ which is irreducible and eventually compact. Then there exist a unique real number $r \in \mathbb{R}$, a strictly positive function h and a strictly positive bounded Borel measure ν such that for suitable constants $\delta > 0$, $M \ge 1$ one has

$$(2.2) \qquad \|\exp(-rt)\cdot T(t) - \nu\otimes h\| \le M\cdot e^{-\delta t} \quad \text{for all } t \ge 0 .$$

In particular, for every $f \in C_o(X)$ and $t \ge 0$ one has

$$(2.3) \qquad e^{rt}(|\textstyle\int f d\nu| - M\cdot e^{-\delta t}\|f\|) \le \|T(t)f\| \le e^{rt}(|\textstyle\int f d\nu| + M\cdot e^{-\delta t}\|f\|) .$$

Proof. We take $r := s(A)$. By B-III,Prop.3.5(a) we have $r > -\infty$. Moreover, by assertion (e) of the same proposition we know that r is an algebraically simple pole and the corresponding residue P has the form $P = \nu\otimes h$ for strictly positive eigenvectors ν and h of A and A', respectively. Without loss of generality we may assume $\|h\| = 1$. Corollary 2.11 of Chapter B-III implies that r is strictly dominant, i.e., enumerating the eigenvalues as described in Thm.2.1 we have $\mathrm{Re}\,\lambda_2 < \lambda_1 = r$. Now (2.2) follows from (2.1) for $m = 1$.