formation F which leaves $S(\mathbb{R}^n)$ invariant and yields

$$F(\mu_t * f) = (2\pi)^{n/2} F(\mu_t) \cdot F(f) = (2\pi)^{n/2} \hat{\mu}_t \cdot \hat{f}$$

where $f \in S(\mathbb{R}^n)$, $\hat{f} = Ff \in S(\mathbb{R}^n)$.

In other words, F transforms $(T(t)|_{S(\mathbb{R}^n)})_{t \geq 0}$ into a multiplication semigroup on $S(\mathbb{R}^n)$ which is pointwise continuous for the usual topology of $S(\mathbb{R}^n)$. The generator, i.e. the right derivative at 0, of this semigroup is the multiplication operator

$$B\hat{f}(x) := -|x|^2 \hat{f}(x)$$

for every $f \in S(\mathbb{R}^n)$.

Applying the inverse Fourier transformation and observing that the topology of $S(\mathbb{R}^n)$ is finer than the topology induced from $L^p(\mathbb{R}^n)$, we obtain that $(T(t))_{t \geq 0}$ is a semigroup which is strongly continuous (use Remark 1.2,(3)) and its generator A coincides with

$$\Delta f(x) = \sum_{i=1}^n \frac{\delta^2}{\delta x^2} f(x_1, \dots, x_n)$$

for every $f \in S(\mathbb{R}^n)$.

Since $S(\mathbb{R}^n)$ is $(T(t))$-invariant we have determined the generator on a core of its domain (see Prop.1.9.ii).

In particular the above semigroup 'solves' the initial value problem for the 'heat equation'

$$\frac{\delta}{\delta t} f(x,t) = \Delta f(x,t) \ , \ f(x,0) = f_o(x) \ , \ x \in \mathbb{R}^n.$$

For the analogous discussion of the unitary group on $L^2(\mathbb{R}^n)$ generated by

$$C := i\Delta$$

we refer to Section IX.7 in Reed-Simon (1975).

Analogous examples to 2.7 are valid in $L^p[0,1]$, resp. to 2.8 in $C_o(\mathbb{R}^n)$.

3. STANDARD CONSTRUCTIONS

Starting with a semigroup $(T(t))_{t \geq 0}$ on a Banach space E it is possible to construct new semigroups on spaces naturally associated with E. Such constructions will be important technical devices in many of the subsequent proofs. Although most of these constructions are rather routine, we present in the sequel a systematic account of them for the convenience of the reader.

We always start with a semigroup $(T(t))_{t \geq 0}$ on a Banach space E, and denote its generator by A on the domain $D(A)$.

3.0. Similar Semigroups

There is an easy way how to obtain different (but isomorphic) semi-