Chapter 1

Spectral Theory

1.1 Introduction

In this chapter we start a systematic analysis of the spectrum of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on a complex Banach space E. By the spectrum of the semigroup we understand the spectrum $\sigma(A)$ of the generator A of T. In particular we are interested in precise relations between $\sigma(A)$ and $\sigma(T(t))$. The heuristic formula

$$T(t) = e^{tA}$$

serves as a leitmotiv and suggests relations of the form

$$\text{“}\sigma(T(t)) = e^{t\sigma(A)} = \{e^{t\lambda} : \lambda \in \sigma(A)\}\text{”},$$

called *spectral mapping theorem*. These - or similar - relations will be of great use in Chapter IV and enable us to determine the asymptotic behavior of the semigroup T by the spectrum of the generator.

As a motivation as well as a preliminary step we concentrate here on the spectral radius

$$r(T(t)) := \sup\{|\lambda| : \lambda \in \sigma(T(t))\}, \quad t \geq 0$$

and show how it is related to the spectral bound

$$s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\}$$

of the generator A and to the growth bound

$$\omega := \inf\{\omega \in \mathbb{R} : \|T(t)\| \leq M_\omega \cdot e^{\omega t} \text{ for all } t \geq 0 \text{ and suitable } M_\omega\}$$

of the semigroup $T = (T(t))_{t\geq 0}$. (Recall that we sometimes write $\omega(T)$ or $\omega(A)$ instead of ω). The Examples 1.3 and 1.4 below illustrate the main difficulties to be encountered.

Proposition 1.1. *Let ω be the growth bound of the strongly continuous semigroup $T = (T(t))_{t\geq 0}$. Then*

$$r(T(t)) = e^{\omega t}$$

for every $t \geq 0$.

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Proof. From A-I, (1.1) we know that

$$\omega(T) = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|$$

Since the spectral radius of $T(t)$ is given as

$$r(T(t)) = \lim_{n\to\infty} \|T(nt)\|^{1/n}$$

we obtain for $t > 0$

$$r(T(t)) = \lim_{n\to\infty} \exp(t(nt)^{-1} \log \|T(nt)\|) = e^{\omega t}$$

□

It was shown in A-I, Prop.1.11 that the spectral bound $s(A)$ is always dominated by the growth bound ω and therefore $e^{s(A)t} \leq r(T(t))$. If the above mentioned spectral mapping theorem holds - as is the case for bounded generators (e.g., see Thm. VII.3.11 of Dunford-Schwartz (1958)) we obtain the equality

$$e^{s(A)t} = r(T(t)) = e^{\omega t}$$

hence $s(A) = \omega$. Therefore the following corollary is a consequence of the definitions of $s(A)$ and ω.

Corollary 1.2. *Consider the semigroup $T = (T(t))_{t\geq 0}$ generated by some bounded linear operator $A \in L(E)$. If $\Re\lambda < 0$ for each $\lambda \in \sigma(A)$ then $\lim_{t\to\infty} \|T(t)\| = 0$.*

Through this corollary we have re-established a famous result of Liapunov which assures that the solutions of the linear Cauchy problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in \mathbb{C}^n \quad \text{and} \quad A = (a_{ij})_{n\times n}$$

are *stable* i.e., they converge to zero as $t \to \infty$, if the real parts of all eigenvalues of the matrix A are smaller than zero.

For unbounded generators the situation is much more difficult and $s(A)$ may differ drastically from ω.

Example 1.3. (Banach function space, Greiner-Voigt-Wolff (1981)) Consider the Banach space E of all complex valued continuous functions on $\mathbb{R}_+$ which vanish at infinity and are integrable for $e^x dx$, i.e.

$$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x dx)$$

endowed with the norm

$$\|f\| := \|f\|_\infty + \|f\|_1 = \sup\{|f(x)| : x \in \mathbb{R}_+\} + \int_0^\infty |f(x)|e^x dx$$

The translation semigroup

$$T(t)f(x) := f(x+t)$$

is strongly continuous on E and one shows as in A-I,2.4 that its generator is given by

$$Af = f', \quad D(A) = \{f \in E : f \in C^1(\mathbb{R}_+), f' \in E\}$$

First we observe that $\|T(t)\| = 1$ for every $t \geq 0$, hence $\omega(T) = 0$. Moreover it is clear that λ is an eigenvalue of A as soon as $\Re\lambda < -1$ (in fact: the function

$$x \mapsto e_\lambda(x) := e^{\lambda x}$$

belongs to $D(A)$ and is an eigenvector of A), hence $s(A) \geq -1$. For $f \in E$, $\Re\lambda > -1$,

$$\|\cdot\|_1\text{-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s) f \, ds$$

exists since $\|T(s)f\|_1 \leq e^{-s}\|f\|_1$, $s \geq 0$, and

$$\|\cdot\|_\infty\text{-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s) f \, ds$$

exists since $\int_0^\infty e^x |f(x)| \, dx < \infty$. Therefore $\int_0^\infty e^{-\lambda s} T(s) f \, ds$ exists in E for every $f \in E$, $\Re\lambda > -1$. As we observed in A-I,Prop.1.11 this implies $\lambda \in \rho(A)$. Therefore $T = (T(t))_{t\geq 0}$ is a semigroup having $s(A) = -1$ but $\omega(T) = 0$.

Example 1.4. (Hilbert space, Zabczyk (1975)) For every $n \in \mathbb{N}$ consider the n-dimensional Hilbert space $E_n := \mathbb{C}^n$ and operators $A_n \in L(E_n)$ defined by the matrices

$$A_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdot & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & 0 \end{pmatrix}_{n\times n}$$

These matrices are nilpotent and therefore $\sigma(A_n) = \{0\}$. The elements $x_n := n^{-1/2}(1, \ldots, 1) \in E_n$ satisfy the following properties:

(i) $\|x_n\| = 1$ for every $n \in \mathbb{N}$

(ii) $\lim_{n\to\infty} \|A_n x_n - x_n\| = 0$

(iii) $\lim_{n\to\infty} \|\exp(tA_n)x_n - e^t x_n\| = 0$

Consider now the Hilbert space $E := \bigoplus_{n\in\mathbb{N}} E_n$ and the operator $A := (A_n + 2\pi i n)_{n\in\mathbb{N}}$ with maximal domain in E. Analogously we define a semigroup $T = (T(t))_{t\geq 0}$ by

$$T(t) := (e^{2\pi i n t} \exp(tA_n))_{n\in\mathbb{N}}$$

Since $\|\exp(tA_n)\| \leq e^t$ for every $n \in \mathbb{N}, t \geq 0$, and since $t \mapsto T(t)x$ is continuous on each component E_n it follows that T is strongly continuous. Its generator is the operator A as defined above.

For $\lambda \in \mathbb{C}$, $\Re\lambda > 0$, we have $\lim_{n\to\infty} \|R(\lambda - 2\pi in, A_n)\| = 0$, hence

$$(R(\lambda, A_n + 2\pi in))_{n\in\mathbb{N}} = (R(\lambda - 2\pi in, A_n))_{n\in\mathbb{N}}$$

is a bounded operator on E representing the resolvent $R(\lambda, A)$. Therefore we obtain $s(A) \leq 0$. On the other hand, each $2\pi in$ is an eigenvalue of A, hence $s(A) = 0$.

Take now $x_n \in E_n$ as above and consider the sequence $(x_n)_{n\in\mathbb{N}}$. From (iii) it follows that for $t > 0$ the number e^t is an approximate eigenvalue of $T(t)$ with approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ (see Def.2.1 below). Therefore $e^t \leq r(T(t)) \leq \|T(t)\|$ and hence $\omega(T) \geq 1$. On the other hand, it is easy to see that $\|T(t)\| = e^t$, hence $\omega(T) = 1$.

Finally if we take $S(t) := e^{-t/2}T(t)$ we obtain a semigroup having spectral bound $-\frac{1}{2}$ but such that $\lim_{t\to\infty} \|S(t)\| = \infty$ in contrast with Cor. 1.2.

These examples show that neither the conclusion of Cor.1.2, i.e. '$s(A) < 0$ implies stability', nor the 'spectral mapping theorem'

$$\sigma(T(t)) = \exp(t \cdot \sigma(A))$$

is valid for arbitrary strongly continuous semigroups. A careful analysis of the general situation will be given in Section 6 below, but we will first develop systematically the necessary spectral theoretic tools for unbounded operators.

1.3 The Fine Structure of the Spectrum

As usual, with a closed linear operator A with dense domain $D(A)$ in a Banach space E, we associate its spectrum $\sigma(A)$, its resolvent set $\rho(A)$ and its resolvent

$$\lambda \mapsto R(\lambda, A) := (\lambda - A)^{-1}$$

which is a holomorphic map from $\rho(A)$ into $L(E)$. In contrast to the finite dimensional situation, where a linear operator fails to be surjective if and only if it fails to be injective, we now have to distinguish different cases of 'non-invertibility' of $\lambda - A$. This distinction gives rise to a subdivision of $\sigma(A)$ into different subsets. We point out that these subsets need not be disjoint, but our defini-