(ii) The generator of the translation (semi)group on $E = L^p(\mathbb{R}_{(+)})$, $1 \leq p < \infty$, is

$$Af := \frac{d}{dx} f = f' ,$$
$$D(A) := \{f \in E : f \text{ absolutely continuous}, f' \in E\} .$$

<u>Proof</u>. Take $f \in D(A)$ such that $\lim_{h\to 0} \frac{1}{h}(T(h)f - f) = g \in E$. Since integration is continuous we obtain for every $a, b \in \mathbb{R}_{(+)}$ that

$$(*) \quad \frac{1}{h}\int_b^{b+h} f(x)\,dx - \frac{1}{h}\int_a^{a+h} f(x)\,dx = \int_a^b \frac{f(x+h) - f(x)}{h}\,dx$$

converges to $\int_a^b g(x)\,dx$ as $h \to 0+$. But for almost all a, b the left hand side of $(*)$ converges to $f(b) - f(a)$. By redefining f on a nullset we obtain

$$f(y) = \int_a^y g(x)dx + f(a) \quad , \quad y \in \mathbb{R}_{(+)} ,$$

which is an absolutely continuous function whose derivative is (almost everywhere) equal to g.

On the other hand, let f be absolutely continuous such that $f' \in L^p$. Then

$$\lim_{h\to 0} \int \left|\frac{f(x+h) - f(x)}{h} - f'(x)\right|^p dx$$

$$= \lim_{h\to 0} \int \frac{1}{h}\left|\int_0^h (f'(x+s) - f'(x))\,ds\right|^p dx$$

$$= \lim_{h\to 0} \int \left|\int_0^1 (f'(x+uh) - f'(x))\,du\right|^p dx$$

$$\leq \lim_{h\to 0} \int \int_0^1 |f'(x+uh) - f'(x)|^p\,du\,dx$$

$$= \int_0^1 \lim_{h\to 0} \int |f'(x+uh) - f'(x)|^p\,dx\,du = 0 \text{ , hence } f \in D(A) .$$

□

2.5. <u>Rotation Groups</u>

On $E = C(\Gamma)$, resp. $E = L^p(\Gamma,m)$, $1 \leq p < \infty$, m Lebesgue measure we have canonical groups defined by rotations of the unit circle Γ with a certain period, i.e. for $0 < \tau \in \mathbb{R}$ the operators

$$R_\tau(t)f(z) := f(e^{2\pi it/\tau}\cdot z)$$

yield a group $(R_\tau(t))_{t\in\mathbb{R}}$ having period τ, i.e. $R_\tau(\tau) = Id$. As in Example 2.4 one shows that its generator has the form