<u>Proof</u>. Without loss of generality we may assume $r = 1$, hence $\alpha = 0$ and $T(\tau)h_o = h_o$.

(a) Defining

(1.14) $h := \int_0^\tau T(s)h_o\,ds$

then for $0 \leq t \leq \tau$ we have

$$T(t)h = \int_0^\tau T(s+t)h_o\,ds = \int_t^\tau T(s)h_o\,ds + \int_\tau^{\tau+t} T(s-\tau)T(\tau)h_o\,ds =$$
$$= \int_t^\tau T(s)h_o\,ds + \int_0^t T(s)T(\tau)h_o\,ds = h\,.$$

It follows that $Ah = \lim t^{-1}(T(t)h - h) = 0$. So far, positivity was not used. The point is that in general, h may be zero. But if $(T(t))$ is positive and $h_o \geq 0$, then $s \to (T(s)h_o)(x)$ is a continuous positive function, hence $0 < h_o(x_o) = (T(0)h_o)(x_o)$ implies $h(x_o) = \int_0^\tau (T(s)h_o)(x_o)\,ds > 0$.

(b) Defining $\phi := \int_0^\tau T(s)'\phi_o\,ds$, one can proceed as in (a) to obtain the desired result.

□

We use Prop.1.5 to prove an analogue of the famous Krein-Rutman result. For the sake of completeness we include the proof of this classical result, which states that the spectral radius of a positive operator T on $C(K)$ (or more generally on an order unit space) is an eigenvalue of the adjoint T' (see the Corollary of Thm.2.6 in the appendix of Schaefer (1966)).

<u>Theorem</u> 1.6. Suppose K is compact and $(T(t))_{t\geq 0}$ is a positive semigroup with generator A. Then there exists a positive probability measure $\phi \in D(A')$ such that $A'\phi = \omega(A)\phi$.

<u>Proof</u>. Consider $T := T(1)$, $r := r(T) = e^{\omega(A)}$. In view of Prop.1.5 it is enough to show that r is an eigenvalue of T' with a positive eigenvector. Given $\lambda \in \mathbb{C}$, $|\lambda| > r$ and $f \in C(K)$ we have $|R(\lambda,T)f| = |\sum_{n=0}^\infty \lambda^{-n-1}T^n f| \leq \sum_{n=0}^\infty |\lambda|^{-n-1}T^n|f| = R(|\lambda|,T)|f|$.
It follows that $\|R(\lambda,T)\| \leq \|R(|\lambda|,T)\|$ and therefore

(1.15) $\lim_{\lambda\downarrow r}\|R(\lambda,T)\| = \infty$.

By the uniform boundedness principle there exist a sequence (λ_n), $\lambda_n \downarrow r$ and a positive $\Psi \in M(K)$ such that $\|R(\lambda_n,T)'\Psi\| \to \infty$.
Defining $\Psi_n := \|R(\lambda_n,T)'\Psi\|^{-1}R(\lambda_n,T)'\Psi$ we have