We have shown that $\ker A \cap E_{\mathbb{R}}$ is totally ordered, hence at most one-dimensional (see Prop.3.4 of Schaefer (1974)).

□

In arbitrary Banach lattices it is no longer true that an irreducible semigroup has necessarily nonvoid spectrum. We indicate how an irreducible semigroup having empty spectrum can be constructed.

Example 3.6. Consider the Banach lattice $E = L^p(\Gamma\times\Gamma)$.
For (fixed) positive numbers α,β such that $\frac{\alpha}{\beta}$ is irrational we define a positive semigroup $(T_o(t))_{t\geq 0}$ as follows:

$$(3.9)\quad (T_o(t)f)(z,w) := f(z\cdot e^{i\alpha t}, w\cdot e^{i\beta t}) \qquad (z,w \in \Gamma = \{\xi\in\mathbb{C} : |\xi|=1\})$$

Next we define for a positive function $m : \Gamma\times\Gamma \to \mathbb{R}$ which is continuous on $\Gamma\times\Gamma\setminus(1,1)$ functions m_t , $t\geq 0$, as follows:

$$(3.10)\quad m_t(z,w) := \exp\left(-\int_0^t m(z\cdot e^{i\alpha s}, w\cdot e^{i\beta s})\, ds\right)$$

Then $(T(t))_{t\geq 0}$ defined by

$$(3.11)\quad T(t)f := m_t\cdot(T_o(t)f)$$

is a strongly continuous semigroup of positive contractions on E. Since $\frac{\alpha}{\beta}$ is irrational the semigroup $(T_o(t))$ is irreducible. Moreover, each m_t is strictly positive (i.e., $m_t > 0$ a.e.) thus $(T(t))$ is irreducible as well. If one chooses m such that $m(z,w)$ tends to $+\infty$ sufficiently fast as $(z,w) \to (1,1)$, one can get
$\|T(t)\| = \|m_t\|_\infty \leq \exp(-t^2)$ for all $t \geq 0$.
Obviously such an estimate of $\|T(t)\|$ implies $\omega(A) = -\infty$, hence $\sigma(A) = \emptyset$.

In the following theorem we collect some hypotheses which in combination with irreducibility guarantee that $\sigma(A) \neq \emptyset$. For the sake of completeness we include B-III,Prop.3.5(a).

Theorem 3.7. Suppose that $(T(t))_{t\geq 0}$ is an irreducible, positive semigroup on the Banach lattice E . Each of the following conditions on E and $(T(t))$, respectively, implies that $\sigma(A) \neq \emptyset$.

(a) $E = C_o(X)$ where X is locally compact;

(b) $E = \ell^p$ $(1 \leq p < \infty)$ (more generally, E contains atoms);

(c) either $T(t_o)$ is compact for some t_o or $R(\lambda_o,A)$ is compact for some $\lambda_o \in \rho(A)$;