Lemma 2.2. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E and let $F(\cdot)$ be a p-periodic, locally integrable function, $p > 0$. Then the following statements are equivalent:

(a) $\dot{u}(t) = Au(t) + F(t)$ admits a (unique) generalized p-periodic solution.

(b) There exists a (unique) $f \in E$ such that $(Id - T(p))f = \int_0^p T(p-s)F(s)\,ds$.

Proof. "(a) → (b)". Let $f := u(0)$ be the initial value for which (2.1) has the p-periodic solution. Then we have

$$\begin{aligned} u(t) = u(t+p) &= T(t)T(p)f + \int_0^p T(t+p-s)F(s)\,ds + \int_p^{t+p} T(t+p-s)F(s)ds \\ &= T(t)[T(p)f + \int_0^p T(p-s)F(s)\,ds] + \int_0^t T(t-s)F(s)\,ds \end{aligned}$$

for every $t \geq 0$. Therefore $f = u(0) = T(p)f + \int_0^p T(p-s)F(s)\,ds$. Clearly, if $u(\cdot)$ is a unique periodic solution with $u(0) = f$, then f is the unique element for which $f = T(p)f + \int_0^p T(p-s)F(s)\,ds$ holds.

"(b) → (a)". Define $u(\cdot)$ as in (2.2). Then

$$u(t+p) = T(t)[T(p)f + \int_0^p T(p-s)F(s)\,ds] + \int_0^t T(t-s)F(s)\,ds = u(t).$$

If f is unique, then, by the considerations above, the solution is unique.

□

Remark 2.3. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ for which the spectral mapping theorem (see A-III, Sec.6) is valid and let F be a p-periodic forcing term.
If $\frac{2\pi i n}{p} \in \rho(A)$ for every $n \in \mathbb{Z}$, then, by Lemma 2.2,
$\dot{u}(t) = Au(t) + F(t)$ has a unique p-periodic solution with initial value $(Id - T(p))^{-1}\left(\int_0^p T(p-s)F(s)\,ds\right)$.

As a consequence of Thm.1.13 and A-III,Cor.6.4, for a uniformly stable semigroup there exists at most one $f \in E$ such that $(Id-T(p))f = \int_0^p T(p-s)F(s)\,ds$. This and Lemma 2.2 is used to prove the following theorem.

Theorem 2.4. Let A be the generator of a uniformly stable semigroup $(T(t))_{t\geq 0}$ and let $F(\cdot)$ be a p-periodic locally integrable function such that $(Id - T(p))f = \int_0^p T(p-s)F(s)\,ds$ for some $f \in E$. Then the