Since $b(v^*,v^*) = 0$ we obtain $b(x,v^*) = 0$ for all $x \in M$ (Lemma 1.1.a), hence $T(xv) = S(x)v$.

(b) Since $s(|\phi|)M$ is a closed right ideal, the closed face $F := s(|\phi|)(M_+)s(|\phi|)$ determines $s(|\phi|)M$ uniquely, i.e.,

$$s(|\phi|)M = \{x \in M : xx^* \in F\}$$

[Pedersen (1979), Theorem 1.5.2]. Since T is a Schwarz map and $s(|\phi|)M$ is T-invariant, it follows $TF \subseteq F$. On the other hand, if $x \in s(|\phi|)M$ then $xx^* \in F$. Consequently,

$$0 \leq S(x)S(x)^* \leq T(xx^*) \in F ,$$

whence $S(x) \in s(|\phi|)M$.

Next we show $T(u^*u) = u^*u$ and $Su^* = u^*$. For this recall that $u^* \in s(|\phi|)M$. First of all

$$0 \leq (Su^* - u^*)(Su^* - u^*)^* \leq$$

$$\leq T(u^*u) - u^*S(u^*)^* - (Su^*)u + u^*u .$$

Since $S_*\phi = \phi$, $T_*|\phi| = |\phi|$ and $\phi = u|\phi|$ it follows

$$0 \leq |\phi|((Su^* - u^*)(Su^* - u^*)^*) \leq$$

$$\leq 2|\phi|(u^*u) - |\phi|(S(u^*)u)^* - |\phi|(S(u^*)u) =$$

$$= 2|\phi|(uu^*) - \phi(u^*)^* - \phi(u^*) =$$

$$= 2(|\phi|(u^*u) - |\phi|(u^*u)) = 0 .$$

Since $(Su^* - u^*)(Su^* - u^*) \in F$ and $|\phi|$ is faithful on F we obtain $Su^* = u^*$. Consequently,

$$0 \leq u^*u = (Su^*)(Su^*)^* \leq T(u^*u) .$$

Hence $T(u^*u) = u^*u$ by the faithfulness and T-invariance of $|\phi|$. $\square$