on f . This yields $\omega_1(A) \leq -1 < 0 = \omega(A)$. Thus we have a semigroup which is exponentially, but not uniformly exponentially stable.

(c) Rescaling this semigroup (see A-I,3.1) we obtain a semigroup with $-1/2 = \omega_1(A)$ and $1/2 = \omega(A)$. Therefore there are exponentially stable and hence stable semigroups which are not bounded and hence not uniformly stable. We conclude from this example that there may be an essential difference between the long term behavior of the semigroup $(T(t))_{t\geq 0}$ (i.e. of the set of all mild solutions) and the long term behavior of the strong solutions $\{T(\cdot)f : f \in D(A)\}$ of (ACP).

In the following we characterize the exponential growth bounds $\omega(f)$, $\omega_1(A)$ and $\omega(A)$ by certain abscissas of simple or absolute convergence of the Laplace transform of $T(\cdot)f$. These characterizations will be the basic tool in showing that for certain semigroups the growth bounds $\omega(A)$ and/or $\omega_1(A)$ coincide with the spectral bound $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$.

We remark first that $s(A)$ can be regarded as the abscissa of holomorphy of the Laplace transform $\lambda \to \int_0^\infty e^{\lambda t}\, T(t)\, dt$ of the semigroup $(T(t))_{t\geq 0}$.
Next we recall that the Laplace transform exists for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \operatorname{Re}\mu$ as soon as it exists for μ . This follows from the equation

$$(1.1) \qquad \int_0^t e^{-\lambda s} f(s)\, ds = e^{-(\lambda-\mu)t}\cdot\int_0^t e^{-\mu s} f(s)\, ds + (\lambda-\mu)\int_0^t e^{-(\lambda-\mu)s}\int_0^s e^{-\mu r} f(r)\, dr\, ds .$$

Note that even boundedness of $\int_0^t e^{-\mu s} f(s)\, ds$ implies the existence of the Laplace transform for $\operatorname{Re}\lambda > \operatorname{Re}\mu$. Therefore the subset of $\mathbb{C}$ for which the Laplace transform exists is always a half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \gamma\} \cup H$, where H is a subset of the line $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \gamma\}$.

In the subsequent theorem we show that the bound of the half-plane for which the Laplace transform of $T(\cdot)f$ $(f \in E)$ exists absolutely and the bound of the half-plane for which the Laplace transform of $T(\cdot)Af$ $(f \in D(A))$ exists strongly coincide with the growth bound $\omega(f) = \inf\{\omega : \|T(t)f\| \leq Me^{\omega t}$ for all $t \geq 0\}$.