implies $|\lambda| \leq r < 1$ for some r and each $\lambda \in A\sigma(T(t_o))$. Consequently, $\exp(\omega(A)\cdot t_o = r(T(t_o)) \leq \max\{\exp(t_o\cdot s(A)),r)\} < 1$ or $\omega(A) < 0$. This proves "(b) $\to$ (a)". For a proof of the equivalence of (a) and (c) we refer to Datko (1972) or Pazy (1983),Thm.4.4.1.

□

Rescaling a given semigroup $(T(t))_{t\geq 0}$ one obtains the following corollary from (1.1) and statement (c) of the above theorem.

<u>Corollary</u> 1.12. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space E. Then the set of complex numbers λ for which $\int_o^\infty \|e^{-\lambda t}\, T(t)f\|\, dt$ exists for every $f \in E$ is an open right half-plane.

In the next theorem we give two necessary conditions for stability of $(T(t))_{t\geq 0}$ in terms of the generator A. We will see in Chapter C-IV that for positive semigroups a condition similar to statement (ii) below is even sufficient for stability of the semigroup. We emphasize that stable semigroups need not be uniformly bounded (see Ex.1.2(3)) and that $s(A) = \omega(A) = 0$ does not imply boundedness or even stability of the semigroup (see also A-I, Ex.1.4.(i)).

<u>Theorem</u> 1.13. Let A be the generator of a stable semigroup $(T(t))_{t\geq 0}$ on a Banach space E. Then the following assertions hold:

(i) $s(A) \leq 0$ and $\mathrm{Re}\,\lambda < 0$ for every $\lambda \in P\sigma(A) \cup R\sigma(A)$.

(ii) $\lim_{\lambda\to 0+} \lambda R(\lambda,A)f$ exists for every $f \in D(A)$.

<u>Proof</u>. (i) If $(T(t))_{t\geq 0}$ is stable, then $\|T(t)f\| \leq M_f$ for every $f \in D(A)$. Therefore $s(A) \leq \omega_1(A) \leq 0$.
Assume there is $\lambda \in P\sigma(A)$ with $\mathrm{Re}\,\lambda = 0$. Then by A-III,Cor.6.4 there is $g \neq 0$ such that $T(t)g = e^{\lambda t}g$ for all $t \geq 0$. Since $|e^{\lambda t}| = 1$ this contradicts the stability of the semigroup.
Assume there is $\lambda \in R\sigma(A) = P\sigma(A')$ with $\mathrm{Re}\,\lambda = 0$. Then there is $0 \neq \Phi \in E'$ with $T(t)'\Phi = \exp(\lambda t)\cdot\Phi$ for all $t \geq 0$. Choose $f \in D(A)$ such that $\langle f,\Phi\rangle \neq 0$. Then $|\langle T(t)f,\Phi\rangle| = |\langle f,\Phi\rangle| > 0$ for every $t \geq 0$ which contradicts the stability of the semigroup.

(ii) From the stability of the semigroup and the identity $\int_o^t T(s)Af\, ds = T(t)f - f$ we see that $\int_o^\infty T(s)Af\, ds$ exists for every $f \in D(A)$. But $\omega_1(A) \leq 0$ and hence $R(\lambda,A)Af = \int_o^\infty e^{-\lambda s}\, T(s)Af\, ds$ for every $\lambda > 0$ (see Thm.1.4). By a classical theorem of Laplace transform theory (for a proof of the vector valued version one may