Part A
Positive Semigroups on $C^\star$- and $W^\star$-Algebras

Chapter A-I
Basic Results on Semigroups and Operator Algebras

This is not a systematic introduction into the theory of strongly continuous semigroups on C*- and W*-algebras. For that we refer to [2], [3] and the survey article of [7]. We only prepare for the subsequent chapters on spectral theory and asymptotics by fixing the notations and introducing some standard constructions.

1 Notations

1. By M we shall denote a C*-algebra with unit 1. $M^{sa} := \{x \in M : x^* = x\}$ is the self-adjoint part of M and $M_+ := \{x^*x : x \in M\}$ the positive cone in M. If M' is the dual of M, then $M'_+ := \{\psi \in M' : \psi(x) \geqslant 0, x \in M_+\}$ is a weak*-closed generating cone in M'. $S(M) := \{\psi \in M'_+ : \psi(1) = 1\}$ is called the state space of M. For the theory of C*-algebras and related notions we refer to [8].

M is called a W*-algebra, if there exists a Banach space M_*, such that its dual $(M_*)'$ is (isomorphic to) M. We call M_* the predual of M and $\psi \in M_*$ a normal linear functional. It is known that M_* is unique [9, 1.13.3]. For further properties of M_* we refer to [10, Chapter III].

2. A map $T \in L(M)$ is called positive (in symbols $T \geqslant 0$) if $T(M_+) \subseteq M_+$. $T \in L(M)$ is called n-positive ($n \in \mathbb{N}$) if $T \otimes \mathrm{Id}_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where Id_n is the identity map on the C*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive. We call $T \in L(M)$ a Schwarz map if T satisfies the inequality

$$T(x)T(x)^* \leqslant T(xx^*)\,,\ x \in M.$$

Note that such T is necessarily a contraction. It is well known that every n-positive contraction, $n \geqslant 2$ and that every positive contraction on a commutative C*-algebra is a Schwarz map [10, Corollary IV. 3.8.]. As we shall see, the Schwarz inequality is crucial for our investigations.

3. If M is a C*-algebra we assume $T = (T(t))_{t \geqslant 0}$ to be a strongly continuous semigroup (abbreviated semigroup) while on W*-algebras we consider weak*-semigroups, i.e. the mapping $(t \mapsto T(t)x)$ is continuous from $\mathbb{R}_+$ into $(M, \sigma(M, M_*))$, M_* the predual of M, and every $T(t) \in T$ is $\sigma(M, M_*)$-continuous. Note that the preadjoint semigroup

$$T_* = \{T(t)_* : T(t) \in T\}$$

is weakly, hence strongly continuous on M_* (see e.g., Davies (1980), Prop.1.23). We call T identity preserving if $T(t)1 = 1$ and of Schwarz type if every $T(t) \in T$ is a Schwarz map.

For the notations concerning one-parameter semigroups we refer to Part A. In addition we recommend to compare the results of this section of the book with the corresponding results for commutative C*-algebras, i.e. for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ (see Part B).

2 A Fundamental Inequality for the Resolvent

If $T = (T(t))_{t \geqslant 0}$ is a strongly continuous semigroup of Schwarz maps on a C*-algebra M (resp. a weak*-semigroup of Schwarz type on a W*-algebra M) with generator A, then the spectral bound $s(A) \leqslant 0$. Then for $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$, there exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t \, , \; x \in M$$

where the integral exists in the norm topology.

In [2] it is shown that T is a semigroup of Schwarz type if and only if $\mu R(\mu, A)$ is a Schwarz map for every $\mu \in \mathbb{R}_+$. Here we relate the domination of two semigroups to an inequality for the corresponding resolvent operator. This inequality will be needed later.

Theorem 2.1 *Let $T = (T(t))_{t \geqslant 0}$ be a semigroup of Schwarz type and $T = (S(t))_{t \geqslant 0}$ a semigroup on a C*-algebra M with generators A and B, respectively. If*

$$(*) \qquad (S(t)x)(S(t)x)^* \leqslant T(t)xx^*$$

for all $x \in M$ and $t \in \mathbb{R}_+$, then

$$(\mu R(\mu, B)x)(\mu R(\mu, B)x)^* \leqslant \mu R(\mu, A)xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$. The same result holds if T is a weak-semigroup of Schwarz type and S is a weak*-semigroup on a W*-algebra M such that $(*)$ is fulfilled.*

Proof From the assumption $(*)$ it follows that

$$
\begin{aligned}
&0 \leqslant (S(r)x - S(t)x)(S(r)x - S(t)x)^* = \\
&\quad = (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \\
&\qquad\qquad \leqslant T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^*
\end{aligned}
$$

for every $r, t \in \mathbb{R}_+$. Hence

$$(S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^* \leqslant T(r)xx^* + T(t)xx^*.$$

Obviously, $\|S(t)\| \leqslant 1$ for all $t \in \mathbb{R}_+$. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$:

$$
\begin{aligned}
(R(\mu, B)x)(R(\mu, B)x)^* &= \left(\int_0^\infty e^{-\mu r} S(r)x \, dr\right)\left(\int_0^\infty e^{-\mu t} S(t)x \, \mathrm{d}t\right)^* \\
&= \left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)} (S(r)x)(S(t)x)^* \, dr \, \mathrm{d}t\right) \\
&\leqslant \int_0^\infty \int_0^\infty e^{-\mu(r+t)} (T(r)xx^* + T(t)xx^*)/2 \, dr \, \mathrm{d}t \\
&= \int_0^\infty e^{-\mu r} T(r)xx^* \, dr \int_0^\infty e^{-\mu t} \, \mathrm{d}t \\
&= \mu^{-1} \int_0^\infty e^{-\mu r} T(r)xx^* \, dr = R(\mu, A)xx^*.
\end{aligned}
$$

Here we used the inequality derived above in the first step. The second step follows from $S(t)$ being a contraction semigroup and the third step is achieved by integration. □

Remark 2.2 The assumption that T is a semigroup of Schwarz type cannot be weakened in general to T being a positive contraction semigroup. This is shown by examples in [4] where $S(t)x$ is given by $e^{tB}x$ for a skew-adjoint generator B and $T(t)x \equiv x$.

Corollary 2.3 *Let $T = (T(t))_{t \geqslant 0}$ be a semigroup of Schwarz type on a $C^\star$-algebra M with generator A. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$:*

$$(\mu R(\mu, A)x)(\mu R(\mu, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof Just set $S = T$ in Theorem 2.1.

$$= \frac{1}{2}\Big(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}\big((S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^*\big)\,dr\,dt$$

$$\leqslant \frac{1}{2}\Big(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*)\,dr\,dt$$

$$= \Big(\int_0^\infty e^{-\mu s}\,ds\Big)\Big(\int_0^\infty e^{-\mu t}T(t)xx^*\,dt\Big) = \mu^{-1}R(\mu, A)xx^*$$

where the handling of the integral is justified by [1, §8, n° 4, Proposition 9]. ◻

Corollary 2.4 *Let T be a semigroup of Schwarz maps (resp., weak*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$:*

$$(R(\lambda, A)x)(R(\lambda, A)x)^* \leqslant (Re\lambda)^{-1}R(Re\lambda, A)xx^*\,,\ x \in M.$$

In particular for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$, $x \in M$:

$$(\mu R(\mu + i\alpha, A)x)(\mu R(\mu + i\alpha, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Then the semigroup

$$S := (e^{-i\mathrm{Im}(\lambda)t}T(t))_{t\geqslant 0}$$

fulfils the assumption of Thm 2.1 and $B := A - i\lambda$ is the generator of S. Consequently $R(\lambda, A) = R(\mathrm{Re}\lambda, B)$ and the corollary follows from Theorem 2.1.

As in Section C-III the following notion will be an important tool for the spectral theory of semigroups.

Definition 2.5 Let E be a Banach space and $\emptyset \neq D$ an open subset of $\mathbb{C}$. A family $R : D \to L(E)$ is called a pseudo-resolvent on D with values in E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu)$$

for all λ and μ in D.

If R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in a C*- or W*-algebra, then R is called of Schwarz type if

$$(R(\lambda)x)(R(\lambda)x)^* \leqslant (\mathrm{Re}\lambda)^{-1}R(\mathrm{Re}\lambda)xx^*$$

for all $\lambda \in D$ and $x \in M$. R is called identity preserving if $\lambda R(\lambda)1 = 1$ for all $\lambda \in D$.

For examples and properties of a pseudo-resolvent see C-III, 2.5. We state what will be used without further reference.

(a) If $\alpha \in \mathbb{C}$ and $x \in E$ such that $(\alpha - \lambda)R(\lambda)x = x$ for some $\lambda \in D$, then $(\alpha - \mu)R(\mu)x = x$ for all $\mu \in D$ (use the “resolvent equation”).

(b) If F is a closed subspace of E such that $R(\lambda)F \subseteq F$ for some $\lambda \in D$, then $R(\mu)F \subseteq F$ for all μ in a neighbourhood of λ. This follows from the fact that for all $\mu \in D$ near λ the pseudo-resolvent in μ is given by

$$R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}.$$

Definition 2.6 We call a semigroup T on the predual M_* of a W*-algebra M identity preserving and of Schwarz type, if its adjoint weak*-semigroup has these properties. Likewise, a pseudo-resolvent R on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in M_* is called identity preserving and of Schwarz type, if R' has these properties.

Since for a semigroup of contractions on a Banach space

$$\mathrm{Fix}(T) = \bigcap_{t \geqslant 0} \ker(\mathrm{Id} - T(t)) =$$

$$= \ker(\mathrm{Id} - \lambda R(\lambda, A)) = \mathrm{Fix}(\lambda R(\lambda, A))$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, it follows that a semigroup of contractions on M is identity preserving if and only if the (pseudo)-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ given by

$$R(\lambda) := R(\lambda, A)|_D$$

is identity preserving. By Corollary 2.3 an analogous statement holds for "Schwarz type".

3 Induction and Reduction

1. If E is a Banach space and $\mathcal{S} \subseteq L(E)$ a semigroup of bounded operators, then a closed subspace F is called $\mathcal{S}$-invariant, if $SF \subseteq F$ for all $S \in \mathcal{S}$. We call the semigroup $\mathcal{S}|_F := \{S|_F : S \in \mathcal{S}\}$ the reduced semigroup. Note that for a one-parameter semigroup T (resp., pseudo-resolvent R) the reduced semigroup is again strongly continuous (resp. $R|_F$ is again a pseudo-resolvent) (compare the construction in A-I,3.2).

2. Let M be a W*-algebra, $p \in M$ a projection and $S \in L(M)$ such that $S(p^\perp M) \subseteq p^\perp M$ and $S(Mp^\perp) \subseteq Mp^\perp$, where $p^\perp := 1 - p$. Since for all $x \in M$:

$$p[S(x) - S(pxp)] = p[S(p^\perp xp) + S(xp^\perp)]p = 0,$$

we obtain $p(Sx)p = p(S(pxp))p$. Therefore the map

$$S_p := (x \mapsto p(Sx)p) : pMp \to pMp$$

is well defined. We call S_p the induced map. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$.

If $T = (T(t))_{t \geqslant 0}$ is a weak*-semigroup on M which is of Schwarz type and if $T(t)(p^\perp) \leqslant p^\perp$ for all $t \in \mathbb{R}_+$, then T leaves $p^\perp M$ and $Mp^\perp$ invariant. It is easy to see that the induced semigroup $T_p = (T(t)_p)_{t \geqslant 0}$ is again a weak*-semigroup.

If R is an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$ with values in M such that $R(\mu)p^\perp \leqslant p^\perp$ for some $\mu \in \mathbb{R}_+$, then $p^\perp M$ and $Mp^\perp$ are R-invariant. Again, the induced pseudo-resolvent R_p is of Schwarz type and identity preserving.

3. Let φ be a positive normal linear functional on a W*-algebra M such that $T_*\varphi = \varphi$ for some identity preserving Schwarz map T on M with preadjoint $T_* \in L(M_*)$. Then $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$ where $s(\varphi)$ is the support projection of φ.

To see this let $L_\varphi := \{x \in M : \varphi(xx^*) = 0\}$ and $M_\varphi := L_\varphi \cap L_\varphi^*$. Since φ is T_*-invariant, and T is a Schwarz map, the subspaces L_φ and M_φ are T-invariant. From $M_\varphi = s(\varphi)^\perp M s(\varphi)^\perp$ and $T(s(\varphi)^\perp) \leqslant 1$ it follows that $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$.

Let $T_{s(\varphi)}$ be the induced map on $M_{s(\varphi)}$. If

$$s(\varphi)M_*s(\varphi) := \{\psi \in M_* : \psi = s(\varphi)\psi s(\varphi)\}$$

where $\langle s(\varphi)\psi s(\varphi), x\rangle := \langle \psi, s(\varphi)xs(\varphi)\rangle$ $(x \in M)$, and if $\psi \in s(\varphi)M_*s(\varphi)$, then for all $x \in M$:

$$(T_*\psi)(x) = \psi(Tx) = \langle \psi, s(\varphi)(Tx)s(\varphi)\rangle =$$

$$= \langle \psi, s(\varphi)(T(s(\varphi)xs(\varphi)))s(\varphi)\rangle = \langle T_*\psi, s(\varphi)xs(\varphi)\rangle,$$

hence $T_*\psi \in s(\varphi)M_*s(\varphi)$. Since the dual of $s(\varphi)M_*s(\varphi)$ is $M_{s(\varphi)}$, it follows that the adjoint of the reduced map $T_*|$ is identity preserving and of Schwarz type.

For example, if T is an identity preserving semigroup of Schwarz type on M_* such that $\varphi \in \operatorname{Fix}(T)$, then the semigroup $T|(s(\varphi)M_*s(\varphi))$ is again identity preserving and of Schwarz type. Furthermore, if R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$ with values in M_* which is identity preserving and of Schwarz type such that $R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$, then $R|s(\varphi)M_*s(\varphi)$ has the same properties.

Chapter A-II
Characterization of Positive Semigroups on W*-Algebras

Since the positive cone of a C*-algebra has non-empty interior many results of Chapter B-II can be applied verbatim to the characterization of the generator of positive semigroups on C*-algebras. On the other hand a concrete and detailed representation of such generators has been found only in the uniformly continuous case (see Lindblad (1976)). A third area of active research has been the following: Which maps on C*-algebras (in particular, which derivations) commuting with certain automorphism groups are automatically generators of strongly continuous positive semigroups. For more information we refer to the survey article of [5].

1 Semigroups on Properly Infinite W*-Algebras

The aim of this section is to show that strongly continuous semigroups of Schwarz maps on properly infinite W*-algebras are already uniformly continuous. In particular, our theorem is applicable to such semigroups on $B(H)$.

It is worthwhile to remark, that the result of [6] on the uniform continuity of every strongly continuous semigroup on L^∞ (see A-II, Sec.3) does not extend to arbitrary W*-algebras.

Example A-II.1 Take $M = B(H)$, H infinite dimensional, and choose a projection $p \in M$ such that Mp is topologically isomorphic to H. Therefore $M = H \oplus M_0$, where $M_0 = \ker(x \mapsto xp)$. Next take a strongly, but not uniformly continuous, semigroup S on H and consider the strongly continuous semigroup $S \oplus \mathrm{Id}$ on M.

For results from the classification theory of W*-algebras needed in our approach we refer to [9, 2.2] and [10, V.1].

Theorem 1.1 *Every strongly continuous one-parameter semigroup of Schwarz type on a properly infinite W*-algebra M is uniformly continuous.*

Proof Let $T = (T(t)_{t\geqslant 0})$ be strongly continuous on M and suppose T not to be uniformly continuous. Then there exists a sequence $(T_n) \subset T$ and $\epsilon > 0$ such that

$\|T_n - \mathrm{Id}\| \geqslant \epsilon$ but $T_n \longrightarrow \mathrm{Id}$ in the strong operator topology. We claim that for every sequence (P_k) of mutually orthogonal projections and all bounded sequences (x_k) in M

$$\lim_n \|(T_n - \mathrm{Id})(P_k x_k P_k)\| = 0$$

uniformly in $k \in \mathbb{N}$. This follows from an application of the *Lemma of Phillips* and the fact that the sequence $(P_k x_k P_k)$ is summable in the $s^*(M, M_*)$-topology (compare Elliot (1972)).

Let (P_k) be a sequence of mutually orthogonal projections in M such that every P_k is equivalent to 1 via some $u_k \in M$ [9, 2.2]. Without loss of generality we may assume $\|(T_n - \mathrm{Id})(u_n)\| \leqslant n^{-1}$ since the semigroup T is strongly continuous. Thus we obtained the following:

(i) $\lim_n \|(T_n - \mathrm{Id})(P_k x_k P_k)\| = 0$ uniformly in $k \in \mathbb{N}$ for every bounded sequence (x_k) in M.
(ii) Every projection P_k is equivalent to 1 via some $u_k \in M$.
(iii) $\|(T_n - \mathrm{Id})u_n\| \leqslant n^{-1}$ for all $n \in \mathbb{N}$.

For the following construction see A-I,3.6 and D-II,Sec.2. Let

(i) $\widehat{M}$ be an ultrapower of M,
(ii) let $p := \widehat{(P_k)} \in \widehat{M}$,
(iii) $T := \widehat{(T_n)} \in L(\widehat{M})$
(iv) and $u := \widehat{(u_k)} \in \widehat{M}$.

Then T is identity preserving and of Schwarz type on $\widehat{M}$. Since $u^*u = p$ and $uu^* = 1$ it follows $pu^* = u^*$ and $(uu^*)x(uu^*) = x$ for all $x \in \widehat{M}$. Finally, $T(pxp) = pxp$ for all $x \in \widehat{M}$, which follows from (i), and $T(u^*) = T(pu^*) = pu^* = u^*$ and $T(u) = u$, which follows from (iii). Using the Schwarz inequality we obtain

$$T(uu^*) = T(1) \leqslant 1 = uu^* = T(u)T(u)^*.$$

Using D-III, Lemma 1.1. we conclude $T(ux) = uT(x)$ and $T(xu^*) = T(x)u^*$ for all $x \in \widehat{M}$. Hence

$$\begin{aligned} T(x) &= T(uu^*xuu^*) = uT(u^*xu)u^* = uT(pu^*xup)u^* \\ &= upu^*xupu^* = uu^*xuu^* = x \end{aligned}$$

for all $x \in \widehat{M}$. From this we obtain that for every bounded sequence (x_k) in M

$$\lim_m \|T_m x_m - x_m\| = 0$$

for some subsequence of the T_n's and of the x_k's. This conflicts with our assumption at the beginning, hence the theorem is proved. □

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