Green's formula,

Let $\beta \in L^\infty(\Omega)$. We define the Laplacian Δ^β with Robin boundary conditions as follows. Let

$$D(\Delta^\beta) := \{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega),$$
$$\partial_\nu u + \beta \operatorname{tr}(u) = 0 \}$$

$$\Delta^\beta u := \Delta u .$$

We call Δ^β briefly the Robin-Laplacian. Note that for $\beta = 0$, we obtain Neumann boundary conditions, and $\Delta^N := \Delta^0$ is the Neumann Laplacian.

The following result is valid.

<u>Theorem</u> 4.3. Assume that $\Omega \subset \mathbb{R}^d$ is bounded, open, connected with Lipschitz boundary and let $\beta \in L^\infty(\partial\Omega)$. Then Δ^β generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on $C(\bar{\Omega})$. Moreover, $T(t)$ is compact for all $t > 0$.

Irreducibility has strong consequences. One has $\sigma(\Delta^\beta) = \sigma_p(\Delta^\beta) \subset \mathbb{R}$. Denote by $s(\Delta^\beta)$ the spectral bound of Δ. Thus $s(\Delta^\beta)$ is the largest eigenvalue of Δ^β.

It is the unique eigenvalue with a positive eigenfunction $0 < u_0 \in D(\Delta^\beta)$.

The eigenfunction u_0 is strictly positive; i.e. there exists $\delta > 0$

such that $u(x) \geq \delta > 0$ for all $x \in \bar{\Omega}$.

The spectral bound $s(\Delta^{\beta})$ determines the asymptotic behavior of the semigroup $\mathcal{T}$. In fact, the following follows from B-III Proposition 3.5.

<u>Corollary</u>. 4.4. There exist a strictly positive Borel measure μ on $\bar{\Omega}$, $M \geq 0$ and $\varepsilon > 0$ such that

$$\langle \mu, u_0 \rangle = 1 \quad \text{and}$$

$$\| T(t) - e^{s(A)t} P \| \leq M e^{-\varepsilon t}$$

for all $t \geq 0$, where $P \in \mathcal{L}(C(\bar{\Omega}))$ is given by

$$Pf = \langle \mu, f \rangle u_0 .$$

The theorem says that the semigroup converges in the operator norm to the rank-1-projection T exponentially fast.

Elliptic operators in divergence form

The preceding results extend to elliptic operators in divergence form for bounded measurable coefficients.

Let $\Omega \subset \mathbb{R}^d$ be open and bounded.

Let $a_{k,\ell}, b_k, c_k, c_0 \in L^\infty(\Omega)$,

$k, \ell = 1, \dots, d$ such that for

some $\alpha > 0$

$$\sum_{k,\ell=1}^{d} a_{k,\ell}(x)\, \xi_k \overline{\xi_\ell} \geq \alpha |\xi|^2$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^d$, where $|\xi|^2 = \xi_1^2 + \dots + \xi_d^2$.

Let $H^1_{loc}(\Omega) := \{ v \in L^2_{loc}(\Omega) :$
$D_k v \in L^2_{loc}(\Omega),\ k = 1, \ldots d \}$.

Define $\mathcal{A} : H^1_{loc}(\Omega) \longrightarrow C_c^\infty(\Omega)'$

by

$$\langle \mathcal{A}u, v \rangle = \sum_{k,\ell=1}^{d} D_k (a_{k\ell} D_\ell u) + \sum_{k=1}^{d} D_k (b_k u)$$
$$+ \sum_{k=1}^{d} c_k D_k u v + c_0 u.$$

We define A_0 as the part of $\mathcal{A}$ in $C_0(\Omega)$; i.e.

$$D(A_0) := \{ u \in C_0(\Omega) \cap H^1_0(\Omega) : \mathcal{A}u \in C_0(\Omega) \}$$
$$A_0 u := \mathcal{A}u.$$

Then the Theorem 4.1 holds with Δ_0 replaced by A_0. It is remarkable that Dirichlet regularity of Ω is the right boundary condition again. This is due to fundamental results of Stampacchia and co-authors. We refer to

Arendt and Bénilan 1999 for

a proof of the following result.

Theorem 4.4. Assume that $\Omega \subset \mathbb{R}^d$ is a bounded, open, connected Dirichlet regular set. Then A_0 generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on $C_0(\Omega)$. Moreover, $T(t)$ is compact for all $t > 0$.

Also the results for Robin boundary conditions Theorem 4.3 and 4.4 can be extended for elliptic operators in divergence form on $C_0(\Omega)$; see

Elliptic operators in non-divergence form on $C_0(\Omega)$

ToDo

The Dirichlet-to-Neumann operator on $C(\partial\Omega)$

for this guy irreducibility is very surprising

References for Notes to B-II 2025.

W. Arendt, A.F.M. ter Elst, J. Glück: Strict positivity for the principal eigenfunction of elliptic operators with various boundary conditions. Adv. Nonlinear Stud. 2020; 20(3) 633–650

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