(a) By the definition given in (4.4) there exists a $\tau > 0$ such that $\underline{c}_t(h,\phi) \geq \underline{c}(h,\phi) - \varepsilon$ for all $t \geq \tau$. It follows that

(4.5) $h_t(x) \geq e^{(\alpha+\varepsilon)t}$ whenever $t \geq \tau$, $x \in K$, $\alpha < \underline{c}(h,\phi)-2\varepsilon$.

Now we fix $\lambda = \alpha + i\beta \in H_{2\varepsilon}$ $(\alpha,\beta \in \mathbb{R})$ and construct an approximate eigenvector (g_n) of A corresponding to λ . For $n \leq \tau + 1$ we choose an arbitrary function $g_n \neq 0$. Now suppose $n > \tau + 1$. We choose $x_n \in K_{n+1/2} \setminus K_{n+1}$ (cf. Lemma 4.2(a)), then there exists $y_n \in K$ such that $\phi(n+1/2,y_n) = x_n$. We have $\phi([0,n+1/2],y_n) \cap K_{n+1} = \emptyset$ and the mapping $t \to \phi(t,y_n)$ is a continuous injection, hence a homeomorphism from $[0,n+1/2]$ into K (this is true because $\phi(n+1/2,y_n) \notin K_{n+1}$). By Tietze's Theorem there is $f_n \in C(K)$ such that

(4.6) $\|f_n\| \leq 1$, $f_n|_{K_{n+1}} = 0$,

$$f_n(\phi(t,y_n)) = 0 \quad \text{for} \quad 0 \leq t \leq n-(1+\delta) \quad \text{and} \quad n+\delta \leq t \leq n+1 ,$$

$$f_n(\phi(t,y_n)) = e^{i\beta t} \quad \text{for} \quad n-1 \leq t \leq n .$$

The constant $\delta \in (0,1/2)$ will be determined later.
Considering $g_n := \int_0^{n+1} e^{-\lambda t}\, T(t)f_n\, dt$, then $g_n \in D(A)$ and

(4.7) $$(\lambda - A)g_n = (1 - e^{-\lambda(n+1)}T(n+1))f_n = f_n - e^{-\lambda(n+1)}\cdot h_{n+1}\cdot f_n \circ \phi_{n+1} = f_n .$$

Moreover,
$\|g_n\| \geq |g_n(y_n)| = |\int_0^{n+1} e^{-\lambda t}\, h_t(y_n) f_n(\phi(t,y_n))\, dt| \geq$
$|\int_{n-1}^{n} e^{-\lambda t}\, h_t(y_n)\, e^{i\beta t}\, dt| - [\int_{n-(1+\delta)}^{n-1} + \int_n^{n+\delta} |e^{-\lambda t} h_t(y_n) f_n(\phi(t,y_n)|dt]$
$\geq \int_{n-1}^{n} e^{-\alpha t}\, e^{(\alpha+\varepsilon)t}\, dt - [\int_{n-(1+\delta)}^{n-1} + \int_n^{n+\delta} e^{-\alpha t}|h_t(y_n)|\, dt]$
$= 1/\varepsilon\cdot(e^{\varepsilon n} - e^{\varepsilon(n-1)}) - [\int_{n-(1+\delta)}^{n-1} + \int_n^{n+\delta} e^{-\alpha t}|h_t(y_n)|\, dt]$.
The constant δ can be chosen such that

(4.8) $\|g_n\| \geq 1/2\varepsilon\cdot(e^{\varepsilon n} - e^{\varepsilon(n-1)}) \to \infty$ for $n \to \infty$.

It follows from (4.8) and (4.7) that $g_n/\|g_n\|$ is an approximate eigenvector of A corresponding to λ . Thus (a) is proved. The proofs of (b) and (c) will be handled simultaneously. First we show that we can restrict attention to the case where $K = K_\infty$.
Indeed, K_∞ is a ϕ-invariant subset, hence $I_\infty := \{f \in C(K) : f|_{K_\infty} = 0\}$ is a $\mathcal{T}$-invariant ideal. Identifying $C(K)/I_\infty$ with $C(K_\infty)$ (cf. B-I, Sec.1), then $(T(t)/I_\infty)$ is the semigroup governed by $\phi|_{K_\infty}$ and $h|_{K_\infty}$. Since one always has $R\sigma(A_/) \subseteq R\sigma(A)$, assertion (b) is proved