$$(1.7) \qquad (T(t)f)(x) := \begin{cases} f(x+t) & \text{if} \quad x+t < 1 \\ 0 & \text{if} \quad x+t \geq 1 \ . \end{cases}$$

Then $(T(t))_{t\geq 0}$ is nilpotent (we have $T(t) = 0$ for $t \geq 1$).
It follows that $\sigma(T(t)) = \{0\}$ for all $t > 0$ and by A-III,Thm.6.2 we have $\sigma(A) = \emptyset$.

(b) The operator A on $E := C_o[0,\infty)$ given by

$$(1.8) \quad (Af)(x) = f'(x) - xf(x) \quad , \ D(A) = \{f \in E : f \in C^1, \ Af \in E\}$$

has empty spectrum. It is the generator of a positive non-nilpotent semigroup which is given by

$$(1.9) \quad (T(t)f)(x) = \exp(-(t^2/2) - xt)\cdot f(x+t) \ .$$

(c) Taking into account that $C_o([0,1))$ as well as $C_o([0,\infty))$ both are topologically (but not isometrically) isomorphic to $C([0,1])$ (see Semadeni (1971), Sec.21.5), one obtains from (a) and (b) (non-positive) semigroups on $C([0,1])$ whose generators have empty spectrum.

The proof of Thm.1.1 given above is based on the fact that the spectral radius of a bounded positive operator is an element of the spectrum. A direct proof not using this fact is given in C-III,Cor.1.4.

<u>Corollary</u> 1.3. Suppose $\lambda_o \in \rho(A)$. Then $R(\lambda_o,A)$ is a positive operator if and only if $\lambda_o > s(A)$.
For $\lambda > s(A)$ we have $r(R(\lambda,A)) = (\lambda - s(A))^{-1}$.

<u>Proof</u>. The second statement is an immediate consequence of Thm.1.1 and A-III,Prop.2.5 .
Given $\lambda_o > s(A)$ we choose $\lambda_1 > \max\{\lambda_o, \omega(A)\}$. Since $|\lambda_1 - \lambda_o| < |\lambda_1 - s(A)| = r(R(\lambda_1,A))^{-1}$ we have

$$(1.10) \quad R(\lambda_o,A) = \textstyle\sum_{n=0}^{\infty} (\lambda_1 - \lambda_o)^n \cdot R(\lambda_1,A)^{n+1} \ .$$

Since $R(\lambda_1,A)$ is positive, it follows that $R(\lambda_o,A)$ is positive as well.
On the other hand, assuming that $R(\lambda_o,A)$ is a positive operator, then λ_o has to be a real number (note that for $g \geq 0$ we have $f := R(\lambda_o,A)g \geq 0$ hence $\lambda_o f - Af = g = \bar{g} = \overline{(\lambda_o - A)f} = \bar{\lambda}_o f - Af$).
As we have shown above $R(\lambda,A)$ is positive for $\lambda > \max\{\lambda_o, s(A)\}$ hence an application of the resolvent equation yields:

$$(1.11) \quad R(\lambda_o,A) = R(\lambda,A) + (\lambda - \lambda_o)R(\lambda,A)R(\lambda_o,A) \geq R(\lambda,A) \geq 0 \ .$$