Conversely, if F is a closed linear subspace of E with $A(D(A) \cap F) \subset F$ such that $A_|$ is a generator on F, then F is $(T(t))$-invariant.

An A-invariant subspace need not necessarily be $(T(t))$-invariant: Take for example the translation group with $T(t)f(x) = f(x+t)$ on $E = C_o(\mathbb{R})$ and $F := \{f \in E : f(x) = 0 \text{ for } x \leq 0\}$.

3.3. The Quotient Semigroup

Let F be a closed $(T(t))$-invariant subspace of E and consider the quotient space $E_/ := E_{/F}$ with quotient map $q : E \to E_/$. The quotient operators

$$T(t)_/ q(f) := q(T(t)f) \ , \ f \in E \ ,$$

are well defined and form a strongly continuous semigroup

$$(T(t)_/)_{t \geq 0}$$

on $E_/$. For the generator $(A_/, D(A_/))$ of $(T(t)_/)_{t \geq 0}$ the following holds:

$$D(A_/) = q(D(A)) \qquad \text{and} \qquad A_/ q(f) = q(Af)$$

for every $f \in D(A)$. Here we use the fact that every $\hat{f} := q(f) \in D(A_/)$ can be written as

$$\hat{f} = \int_0^\infty e^{-\lambda s} \hat{T}(s)_/ \hat{g} \, ds = \int_o^\infty e^{-\lambda s} q(T(s)g) \, ds = q(\int_o^\infty e^{-\lambda s} T(s)g \, ds) = q(h)$$

where $h \in D(A)$ and $\lambda > \omega$ (see (1.6)). In particular we point out that for every $\hat{f} \in D(A_/)$ there exist representatives $f \in \hat{f}$ belonging to $D(A)$.

Example. We start with the Banach space $E = L^1(\mathbb{R})$ and the translation semigroup $(T(t))_{t \geq 0}$ where $T(t)f(x) := f(x+t)$ (see Example 2.4). Then $L^1((-\infty,1])$ can be identified with the closed, $(T(t))$-invariant subspace

$$J := \{f \in E : f(x) = 0 \text{ for } 1 < x < \infty\}$$

and we obtain the subspace semigroup

$$T(t)_| f(x) = 1_{(-\infty,1]}(x) \cdot f(x+t) \ ,$$

where $f \in L^1((-\infty,1])$, $-\infty < x \leq 1$ and $t \geq 0$.

By 2.4 and 3.2 its generator is

$$A_| f := f'$$

for $f \in D(A_|) := \{f \in E : f \in AC \text{ with } f' \in E \text{ and } f(x) = 0 \text{ for } x \geq 1\}$.

Next we identify $L^1([0,1])$ with the quotient space $L^1((-\infty,1])_{/I}$ where

$$I := \{f \in L^1((-\infty,1]) : f(x) = 0 \text{ for } 0 \leq x \leq 1\} .$$

Again I is invariant for the restricted semigroup $(T(t)_|)$ and the