For a better understanding of the above definition we recall that to every direct sum decomposition $E = E_1 \oplus E_2$ there corresponds a continuous projection $P \in L(E)$ such that $PE = E_1$ and $P^{-1}(0) = E_2$. Moreover, the subspaces E_1, E_2 are T-invariant if and only if P commutes with the semigroup T, i.e. $T(t)P = PT(t)$ for every $t \geq 0$. In this case it follows that the domain $D(A)$ of the generator A splits analogously and $D(A) \cap E_i$ is the domain $D(A_i)$ of the generator A_i of the restricted semigroup T_i, $i = 1, 2$. We write

$$A = A_1 \oplus A_2 ,$$

say that " A commutes with P " and call P a spectral projection. In terms of the generator A this means that for $f \in D(A)$ we have $Pf \in D(A)$ and $APf = PAf$.
The existence of such projections is very helpful since it reduces the semigroup T into two (possibly simpler) semigroups T_1, T_2 such that

$$\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \quad \text{and} \quad \sigma(T(t)) = \sigma(T_1(t)) \cup \sigma(T_2(t)) .$$

For example, in some cases (see Theorem 3.3 below) it can be shown that one of the reduced semigroups has additional properties.

In order to achieve such decompositions we will assume that $\sigma(A)$ decomposes into sets σ_1 and σ_2 and will then try to find a corresponding spectral projection. Unfortunately such spectral decompositions do not exist in general.

Example 3.2. Take the rotation semigroup from A-I,2.4 on the Banach space $L^p(\Gamma)$, $1 \leq p < \infty$, $\tau = 2\pi$. It was stated in 2.4 and will be proved in Section 5 that its generator A has spectrum

$$\sigma(A) = P\sigma(A) = i\mathbb{Z} ,$$

where $\varepsilon_k(z) := z^k$ spans the eigenspace corresponding to ik, $k \in \mathbb{Z}$. Now, $\sigma(A)$ is the disjoint union of $\sigma_1 := \{0, i, 2i, \ldots\}$ and $\sigma_2 := \{-i, -2i, \ldots\}$. By a result of M. Riesz there is no projection $P \in L(L^1(\Gamma))$ satisfying $P\varepsilon_k = \varepsilon_k$ for $k \geq 0$, $P\varepsilon_k = 0$ for $k < 0$ (see Lindenstrauss-Tzafriri (1979), p.165), hence there is no spectral decomposition of $L^1(\Gamma)$ corresponding to σ_1, σ_2. On the other hand, for $L^p(\Gamma)$, $1 < p < \infty$, such a spectral projection exists (l.c., 2.c.15). As long as $p \neq 2$ we can always decompose $\sigma(A)$ into suitable subsets admitting no spectral decomposition (l.c., remark before 2.c.15). Clearly, for $p = 2$ such spectral decompositions always exist.