is a subset of m linearly independent vectors which (by assumption) is contained in L.

□

The surprising fact in the following theorem is the conclusion that every point in the boundary spectrum is a simple algebraic pole if only $s(A)$ is supposed to be a pole.

Theorem 3.12. Let T be an irreducible semigroup on a Banach lattice and let A be its generator.
If $s(A)$ is a pole of the resolvent then there exists $\alpha \geq 0$ such that $\sigma_b(A) = s(A) + i\alpha\mathbb{Z}$. Moreover, $\sigma_b(A)$ contains only algebraically simple poles.

Proof. We will assume that $s(A) = 0$. Assuming first that every element of $\sigma_b(A)$ is an eigenvalue of A one can conclude as follows: By Thm.3.8(a) we know that $\sigma_b(A)$ is an additive subgroup of $i\mathbb{R}$. Since it is a closed subset and 0 is an isolated point it follows that $\sigma_b(A) = i\alpha\mathbb{Z}$ for some $\alpha \geq 0$. Moreover as a consequence of (3.13), for every $k \in \mathbb{Z}$ we obtain

$$(3.15) \qquad R(\lambda+ik\alpha,A) = S_h^{-k}\circ R(\lambda,A)\circ S_h^k \qquad (\lambda \in \rho(A),\ k \in \mathbb{Z}).$$

By Prop.3.5(d) 0 is an algebraically simple pole. Then (3.15) implies that every point $ik\alpha$ has the same property.
We now show that every element $i\beta$ is an eigenvalue of A. By Prop.3.5(d) the residue of $R(.,A)$ in $\lambda = 0$ has the form $P = \phi\otimes u$ with $\phi(u) = 1$. Given an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ which is finer than the Frechet filter, then $\lim_{\mathcal{U}}\phi(f_n)$ exists for every bounded sequence $(f_n) \subset E$. Using this fact it is easy to see that the canonical extension $P_{\mathcal{U}}$ of P to the $\mathcal{U}$-product $E_{\mathcal{U}}$ of E has the following form:

$$(3.16) \qquad P_{\mathcal{U}} = \hat{\phi}\otimes\hat{u} \quad \text{where} \quad \hat{u} := (u,u,u,\,..\,)+c_{\mathcal{U}}(E) \in E_{\mathcal{U}} \quad \text{and} \quad \hat{\phi} \in (E_{\mathcal{U}})'$$
$$\text{is given by} \quad \hat{\phi}((f_n)+c_{\mathcal{U}}(E)) := \lim_{\mathcal{U}}\phi(f_n) \quad ((f_n)+c_{\mathcal{U}}(E) \in E_{\mathcal{U}}).$$

Given $i\beta \in \sigma_b(A)$ then $i\beta \in A\sigma(A)$ hence $1 \in A\sigma(\lambda R(\lambda+i\beta,A))$. Assuming $i\beta \notin P\sigma(A)$, then $1 \notin P\sigma(\lambda R(\lambda+i\beta,A))$. Then Lemma 3.10 implies that $M := \ker(1 - \lambda R(\lambda+i\beta,A)_{\mathcal{U}})$ is infinite dimensional (and independent of λ by Prop.2.6(a).) For $\hat{f} \in M$ we have
$|\hat{f}| = |\gamma R(\gamma+i\beta,A)_{\mathcal{U}}\hat{f}| \leq \gamma R(\gamma,A)_{\mathcal{U}}|\hat{f}|$ for every $\gamma > 0$.
It follows that $\hat{\phi}(|\hat{f}|) = P_{\mathcal{U}}|\hat{f}| = \lim_{\gamma\to 0}\gamma R(\gamma,A)_{\mathcal{U}}|\hat{f}| \geq |\hat{f}|$.
Thus considering the closed ideal $I := \{\hat{f} \in E_{\mathcal{U}} : \hat{\phi}(|\hat{f}|) = 0\}$ we have