of a semigroup on a Hilbert space H, then it is shown in A-III, Cor.7.11 that

$$(1.10) \qquad \omega(A) = \inf\{w : \|R(\lambda,A)\| \le M_w \text{ for } \operatorname{Re}\lambda > w\} .$$

Unfortunately, the identity (1.10) does not hold on arbitrary Banach spaces, but we will see in Section 1 of C-IV that for every positive semigroup on a Banach lattice the identity

$$(1.11) \qquad s(A) = \omega_1(A) = \inf\{w : \|R(\lambda,A)\| \le M_w \text{ for } \operatorname{Re}\lambda > w\}$$

is valid. Therefore, for positive semigroups with $s(A) = \omega_1(A) < \omega(A)$ (see Ex.1.2.(2)) the equation (1.10) is not true. However, we can prove the following theorem.

Theorem 1.9. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\ge0}$ on a Banach space E. If there are constants $a \ge 0$ and $q \ge s(A)$ and if there are $C \in \mathbb{R}_+$, $n \in \mathbb{N}$ such that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > q$ and $|\operatorname{Im}\lambda| > a$ we have $\|R(\lambda,A)\| \le C|\lambda|^{n-2}$, then $\sup\{\omega(f), f \in D(A^n)\} \le q$.

Proof. The hypothesis $\|R(\lambda,A)\| \le C|\lambda|^{n-2}$ is invariant under rescaling; i.e., the resolvent $R(\lambda,-b+A)$ of the generator $-b+A$ of the rescaled semigroup $e^{-bt}\,T(t)$ satisfies $\|R(\lambda,-b+A)\| \le \tilde{C}|\lambda|^{n-2}$ for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > q-b$ and $|\operatorname{Im}\lambda| > a+2b$ and a suitable constant $\tilde{C}$. Therefore we may assume that $b := \max(\omega(A),q) < 0$. Let $\omega(A) < p < 0$. Then, by the inversion formula for the Laplace transform for every $f \in D(A)$ and $p' = \max\{p,q\} < 0$,

$$(1.12) \qquad T(t)f = \frac{1}{2\pi i}\cdot\int_{p'-i\infty}^{p'+i\infty} e^{\lambda t}\, R(\lambda,A)f\, d\lambda .$$

(For a proof of the vector valued version of the inversion formula one may follow [Widder(1946),p.66]; also see the notes to this section.) By the resolvent equation we obtain

$$R(\lambda,A)R(0,A)^n = \sum_{k=1}^{n} (-1)^{k+1}\cdot\lambda^{-k}\, R(0,A)^{n+1-k} + (-1)^n\cdot\lambda^{-n}\, R(\lambda,A) .$$

Using that $\frac{1}{2\pi i}\cdot\int_{p'-i\infty}^{p'+i\infty} e^{\lambda t}\cdot\lambda^{-k}\, d\lambda = 0$ for $k \ge 1$, $p' < 0$ and $t > 0$ we obtain

$$(1.13) \qquad T(t)R(0,A)^n f = (-1)^n\cdot\frac{1}{2\pi i}\cdot\int_{p'-i\infty}^{p'+i\infty} e^{\lambda t}\cdot\lambda^{-n}\, R(\lambda,A)f\, d\lambda$$

for every $f \in E$ and $t > 0$.

If $q < p'$, then, by Cauchy's Integral Theorem and since $\|R(\lambda,A)\| \le C\cdot|\lambda|^{n-2}$ we see that the path of integration can be shifted to $\operatorname{Re}\lambda = q$;

i.e., $T(t)R(0,A)^n f = (-1)^n\cdot\frac{1}{2\pi i}\cdot\int_{q-i\infty}^{q+i\infty} e^{\lambda t}\cdot\lambda^{-n}\, R(\lambda,A)f\, d\lambda$.