closure $\bar{A} = \bar{B} + \mu$ of A generates a positive semigroup of type $\omega(\bar{A}) \leq \lambda$.
If $(\lambda - A)$ is surjective, then $A = \bar{A}$.

□

The proof of Theorem 1.13 yields estimates for the growth bound of a positive semigroup (see A-III,(1.3)) which we state explicitly in the next corollary.

Corollary 1.14. Let A be the generator of a strongly continuous positive semigroup on $C(K)$. Then

(1.7) $\quad -\infty < s(A) = \omega(A) \in \sigma(A)$. Moreover,

(1.8) $\quad s(A) = \inf\{\lambda \in \mathbb{R} : Au \leq \lambda u \text{ for some } 0 \ll u \in D(A)\}$; and

(1.9) $\quad s(A) \geq \sup\{\mu \in \mathbb{R} : Af \geq \mu f \text{ for some } 0 < f \in D(A)\}$.

Proof. Let $s = \inf\{\lambda \in \mathbb{R} : [\lambda,\infty) \subset \rho(A)\}$. Clearly, $s \leq s(A)$. Moreover, by Remark 1.7, $R(\lambda,A) \geq 0$ for all $\lambda > s$. Hence it follows from (1.3) that $\omega(A) \leq s$. Consequently, $s = s(A) = \omega(A)$.
Next we prove (1.9). Let $0 < f \in D(A)$ such that $Af \geq \mu f$. Assume that $\mu > s(A)$. Then $R(\mu,A) \geq 0$. Hence, $f = R(\mu,A)(\mu-A)f \leq 0$, a contradiction.
Since $D(A)$ is dense, there exists a strictly positive $u \in D(A)$. Then $Au \geq \mu u$ for some $\mu \in \mathbb{R}$. Hence, $-\infty < \mu \leq s(A)$ by (1.9).
Since $s(A) = s$ it is clear that $s(A) \in \sigma(A)$.
It remains to show (1.8). Let $\lambda > s(A)$ and $u = R(\lambda,A)1$. Then u is strictly positive (by Rem. 1.7) and $Au = \lambda u - 1 \leq \lambda u$. This proves one inequality in (1.8). Assume now that $u \in D(A)$ is strictly positive such that $Au \leq \lambda u$. Then by the proof of Thm. 1.13 we have $\omega(A) \leq \lambda$. This proves the other inequality in (1.8).

□

Remark 1.15. If A has compact resolvent, then by the Krein-Rutmann theorem there exists a positive eigenvector u of A corresponding to a real eigenvalue. So the equality is valid in (1.9) and the supremum is a maximum. If in addition the semigroup is irreducible (see B-III,Sec.3 below), then u is strictly positive and in (1.8) the infimum is attained as well.
Conversely, if in (1.8) the infimum is attained, then $s(A)$ is an eigenvalue.