Chapter 1

Basic Results on Semigroups on Banach Spaces

Since the basic theory of one-parameter semigroups can be found in several excellent books (e.g. [Davies (1980)], [Goldstein (1985a)], [Pazy (1983)] or [Hille-Phillips (1957)]) we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, to collect briefly some important definitions and results (Section 1), to present a list of standard examples (Section 2) and to discuss standard constructions of new semigroups from a given one (Section 3).

In the entire chapter we denote by E a (real or) complex Banach space and consider one-parameter semigroups of bounded linear operators $T(t)$ on E. By this we understand a subset $\{T(t)\colon t \in \mathbb{R}_+\}$ of $L(E)$, usually written as $(T(t))_{t\geq 0}$, such that

$$\begin{aligned} T(0) &= \mathrm{Id}, \\ T(s+t) &= T(s)\cdot T(t) \text{ for all } s,t \in \mathbb{R}_+ \end{aligned}$$

In more abstract terms this means that the map $t \to T(t)$ is a homomorphism from the additive semigroup $(\mathbb{R}_+, +)$ into the multiplicative semigroup $(L(E),\cdot)$. Similarly, a one-parameter group $(T(t))_{t\in\mathbb{R}}$ will be a homomorphic image of the group $(\mathbb{R}, +)$ in $(L(E),\cdot)$.

1.1 Standard Definitions and Results

We consider a one-parameter semigroup $(T(t))_{t\geq 0}$ on a Banach space E

and observe that the domain $\mathbb{R}_+$ and the range $L(E)$ of the (semi- Group) homomorphism $\tau\colon t \to T(t)$ are topological semigroups for the natural topology on $\mathbb{R}_+$ and any one of the standard operator topologies on $L(E)$. We single out the strong operator topology on $L(E)$ and require τ to be continuous.

Definition 1.1. A one-parameter semigroup $(T(t))_{t\geq 0}$ is called strongly continuous if the map $t \to T(t)$ is continuous for the strong operator topology on $L(E)$, i.e.

$$\lim_{t\to t_0} \|T(t)f - T(t_0)f\| = 0$$

for every $f \in E$ and $t, t_0 \geq 0$.

Clearly one defines in a similar way weakly continuous, resp. uniformly continuous (compare A-II, Def. 1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology:

> If not stated otherwise, a *semigroup* is a strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

Remarks 1.2. (i) A one-parameter semigroup $(T(t))_{t\geq 0}$ on a Banach space E is strongly continuous if and only if for any $f \in E$ it is true that $T(t)f \to f$ as $t \to 0$.

(ii) For every strongly continuous semigroup there exist constants $M \geq 1, w \in \mathbb{R}$ such that $\|T(t)\| \leq M \cdot e^{wt}$ for every $t \geq 0$.

(iii) If $(T(t))_{t\geq 0}$ is a one-parameter semigroup such that $\|T(t)\|$ is bounded for $0 < t \leq \delta$ then it is strongly continuous if and only if $\lim_{t\to 0} T(t)f = f$ for every f in a total subset of E.

The exponential estimate from Remark **??** (ii) for the growth of $\|T(t)\|$ can be used to define an important characteristic of the semigroup.

Definition 1.3. By the growth bound (or type) of the semigroup $(T(t))_{t\geq 0}$ we understand the number

$$\begin{aligned}\omega_0 &\coloneqq \inf\{w \in \mathbb{R}\colon \text{There exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leq Me^{wt} \text{ for } t \geq 0\} \\ &= \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\| = \inf_{t>0} \frac{1}{t} \log \|T(t)\|\end{aligned}$$

Particularily important are semigroups such that for every $t \geq 0$ we have $\|T(t)\| \leq M$ (bounded semigroups) or $\|T(t)\| \leq 1$ (contraction semigroups). In both cases we have $\omega_0 \leq 0$.

It follows from the subsequent examples and from Definition **??** that ω_0 may be any number $-\infty \leq \omega < +\infty$. Moreover the reader should observe that the infimum in (1.3) need not be attained and that M may be larger than 1 even for bounded semigroups.

Examples 1.4. (i) Take $E = \mathbb{C}^2$,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then for the ℓ^1-norm on E we obtain $\|T(t)\| = 1+t$, hence $(T(t))_{t\geq 0}$ is an unbounded semigroup having growth bound $\omega_0 = 0$.

(ii) Take $E = L^1(\mathbb{R})$ and for $f \in E$, $t \geq 0$ define

$$T(t)f(x) := \begin{cases} 2 \cdot f(x+t) & \text{if } x \in [-t, 0] \\ f(x+t) & \text{otherwise.} \end{cases}$$

Each $T(t)$, $t > 0$, satisfies $\|T(t)\| = 2$ as can be seen by taking $f := \chi_{[0,t]}$. Therefore $(T(t))_{t\geq 0}$ is a strongly continuous semigroup which is bounded, hence has $\omega_0 = 0$, but the constant M in (1.3) cannot be chosen to be 1.

The most important object associated to a strongly continuous semigroup $(T(t))_{t\geq 0}$ is its *generator* which is obtained as the (right)derivative of the map $t \to T(t)$ at $t = 0$. Since for strongly continuous semigroups the functions $t \to T(t)f$, $f \in E$, are continuous but not always differentiable we have to restrict our attention to those $f \in E$ for which the desired derivative exists. We then obtain the *generator* as a not necessarily everywhere defined operator.

Definition 1.5. To every semigroup $(T(t))_{t\geq 0}$ there belongs an operator $(A, D(A))$, called the generator and defined on the domain

$$D(A) := \{f \in E\colon \lim_{h\to 0} \frac{T(h)f - f}{h} \text{ exists in } E\}$$

by

$$Af := \lim_{h\to 0} \frac{T(h)f - f}{h} \quad (f \in D(A)).$$

Clearly, $D(A)$ is a linear subspace of E and A is linear from $D(A)$ into E. Only in certain special cases (see 2.1) the generator