Corollary 3.9. Assume in addition that E' contains a strictly positive functional. Then the semigroup is positive if and only if there exists a core D_o of A and a strictly positive subeigenvector ϕ of A' such that

(K) $\quad \langle(\operatorname{sign} f)\, Af,\phi\rangle \leq \langle|f|,A'\phi\rangle \quad$ for all $f \in D_o$.

From the proof of Theorem 3.8 we isolate the following

Proposition 3.10. Let B be a densely defined operator on E and D_o be a core of B . Suppose that $\phi \in D(B')_+$ is such that $B'\phi \leq 0$. Denote by p the sublinear functional given by $p(f) = \langle f^+,\phi\rangle$. If

(K) $\quad \langle(\operatorname{sign} f)\, Bf,\phi\rangle \leq \langle|f|,B'\phi\rangle \quad (f \in D_o)$,

then B is p-dissipative.

Proof. Let $f \in D_o$. Set $P_+ := P_{f^+}$, $P_- := P_{f^-}$ and let $P := Id - P_+ - P_-$, $Q = P_+ + 1/2\, P$ and $\psi = Q'\phi$. We show that

(3.2) $\quad \psi \in dp(f)$.

Let $g \in E$. Since $0 \leq Q \leq Id$ we have $\langle g,\psi\rangle = \langle Qg,\phi\rangle \leq \langle Qg^+,\phi\rangle \leq \langle g^+,\phi\rangle = p(g)$. Moreover, $\langle f, \psi\rangle = \langle Qf,\phi\rangle = \langle P_+f + 1/2\, Pf,\phi\rangle = \langle f^+,\phi\rangle = p(f^+)$. So (3.2) follows by the definition of $dp(f)$. We show that

(3.3) $\quad \langle Bf,\psi\rangle \leq 0$.

One has trivially

(3.4) $\quad \langle(P_+ + P_- + P)\, Bf,\phi\rangle = \langle f,B'\phi\rangle$.

Addition of (3.4) and (K) gives
$\langle(2P_+ + P)Bf,\phi\rangle \leq \langle 2f^+,B'\phi\rangle \leq 0$. Hence $\langle Bf,\psi\rangle = \langle QBf,\phi\rangle \leq 0$.
Thus we have proved that $B_{|D_o}$ is p-dissipative. Hence B is p-dissipative as well (by A-II,Cor.2.5).

□

Proof of Theorem 3.8. Proposition 3.5 and Theorem 2.4 yield one implication. In order to show the other assume that the condition in Theorem 3.8 is satisfied. We have to show that $T(t) \geq 0$ for all $t \geq 0$.
Let $\phi \in M'$. Consider the half-norm $p(f) = \langle f^+,\phi\rangle$ and the operator $B = A - \lambda$, where $\lambda \in \mathbb{R}$ is such that $A'\phi \leq \lambda\phi$. Then B satisfies $B'\phi \leq 0$ and (K) as well. So it follows from Proposition 3.10 that B is p-dissipative.
Since B generates the semigroup $(e^{-\lambda t}T(t))_{t\geq 0}$ we obtain from