One can describe domination by an inequality for the generators in a manner analoguous to the characterization of positive semigroups in Section 1; however, no positive subeigenvectors are needed here.

Theorem 4.2. Let $(T(t))_{t\geq 0}$ be a positive semigroup with generator A and $(S(t))_{t\geq 0}$ a semigroup with generator B. The following assertions are equivalent.

(i) $|S(t)f| \leq T(t)|f|$ for all $f \in E$, $t \geq 0$.

(ii) $\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf,\phi\rangle \leq \langle|f|,A'\phi\rangle$ for all $f \in D(B)$, $\phi \in D(A')_+$.

Proof. (i) implies (ii). Let $f \in D(B)$, $\phi \in D(A')_+$. Then

$$\begin{aligned}\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf,\phi\rangle &= \mathrm{Re}\,\langle(\mathrm{sign}\,\bar{f})\lim_{t\downarrow 0}1/t(S(t)f-f),\phi\rangle\\ &= \langle\lim_{t\downarrow 0}1/t(\mathrm{Re}((\mathrm{sign}\,\bar{f})\ S(t)f)-|f|),\phi\rangle\\ &\leq \lim_{t\downarrow 0}\langle 1/t(|S(t)f|-|f|),\phi\rangle\\ &\leq \lim_{t\downarrow 0}\langle 1/t(T(t)|f|-|f|),\phi\rangle = \langle|f|,A'\phi\rangle .\end{aligned}$$

(ii) implies (i). Let $\lambda > \max\{\omega(A),\omega(B)\}$ and $g \in E$. We show that

$$|R(\lambda,B)g| \leq R(\lambda,A)|g| . \tag{4.3}$$

Let $\psi \in E'_+$. Then $\phi := R(\lambda,A)'\psi \in D(A')_+$.
Setting $f := R(\lambda,B)g \in D(B)$ we obtain by (ii)
$\langle|R(\lambda,B)g|,\psi\rangle = \langle|f|,(\lambda-A')\phi\rangle \leq \langle\lambda|f|,\phi\rangle - \mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf,\phi\rangle =$
$\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})(\lambda f-Bf),\phi\rangle = \mathrm{Re}\langle(\mathrm{sign}\,\bar{f})g,\phi\rangle \leq \langle|g|,\phi\rangle = \langle R(\lambda,A)|g|,\psi\rangle$.
Since $\psi \in E'_+$ is arbitrary (4.3) follows.

□

In order to deduce that (ii) implies (i) in Theorem 4.2, it is not necessary to assume that B is a generator. Merely a range condition is sufficient. The precise formulation is the following.

Theorem 4.3. Let $(T(t))_{t\geq 0}$ be a positive semigroup with generator A. Let B be a densely defined operator such that

$$\begin{aligned}&\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf,\phi\rangle \leq \langle|f|,A'\phi\rangle\\ &\text{for all } f \in D(B),\ \phi \in D(A')_+ .\end{aligned}\tag{4.4}$$

Then B is closable. Moreover, if $(\lambda - B)D(B)$ is dense in E for some $\lambda > \max\{0,s(A)\}$, then $\bar{B}$ (the closure of B) generates a semigroup which is dominated by $(T(t))_{t\geq 0}$.

Proof. 1. We show that B is closable.
Let $u_n \in D(B)$ satisfy $u_n \to 0$ and $Bu_n \to v$ $(n \to \infty)$. We have to