Proof. By Formula (3.2) we have $R(\lambda,A) = S_\lambda^{-1}R(\lambda,A_o)$ for $\lambda > \|\Phi\|+w$, (where $S_\lambda f = f - \varepsilon_\lambda \otimes R(\lambda,B)\Phi f$ for $f \in E$). Thus the fact that $R(\lambda,A_o)$ is positive (C-II,Prop.4.1) reduces the problem to showing that S_λ^{-1} is a positive operator for $\lambda > \|\Phi\|+w$.
Since $S_\lambda = Id - \varepsilon_\lambda \otimes R(\lambda,B)\Phi$ and $\|\varepsilon_\lambda \otimes R(\lambda,B)\Phi\| \leq (\lambda-w)^{-1}\cdot\|\Phi\| < 1$ we see that $S_\lambda^{-1} = \sum_{n=0}^{\infty} (\varepsilon_\lambda \otimes R(\lambda,B)\Phi)^n$ is positive. Hence $(T(t))_{t\geq 0}$ is a positive semigroup again by C-II,Prop.4.1 .

□

Remark. Suppose that Φ has no mass in zero (i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\Phi f\| \leq \varepsilon\|f\|$ for all $f \in E$, $supp(f) \subset [-\delta,0]$). Then the positivity hypotheses in the above proposition are necessary in order to obtain positivity of $(T(t))_{t\geq 0}$ (cf. B-II,1.22 for the case $\dim F < \infty$ and [Kerscher (1986)] for the general case).

Proposition 3.6. Let $\Phi \in L(E,F)$ be positive and assume that B generates a positive semigroup on F. The "spectral bound function" $\lambda \to s(B + \Phi_\lambda)$ is decreasing and continuous from the left on $\mathbb{R}$.
If, additionally, B has compact resolvent and there exists $\lambda' \in \mathbb{R}$ with $\sigma(B + \Phi_{\lambda'}) \neq \emptyset$, then $\lambda \to s(B + \Phi_\lambda)$ is continuous and the spectral bound $s(A)$ is the unique solution of the equation

$$\lambda = s(B + \Phi_\lambda) \quad . \tag{3.5}$$

Proof (cf. also C-IV,Lemma 3.4). For $\lambda \leq \mu$ we have $0 \leq \Phi_\mu \leq \Phi_\lambda$ and hence $0 \leq R_\mu(t) \leq R_\lambda(t)$, $t \geq 0$, for the respective semigroups generated by $B + \Phi_\mu$ and $B + \Phi_\lambda$ (see A-II,Sec.1). This implies $s(B + \Phi_\mu) \leq s(B + \Phi_\lambda)$. The left-continuity follows by the semicontinuity of the spectrum (see [Kato (1976) Chap.IV, Thm.3.1]).
If B has compact resolvent then $B + \Phi_\lambda$ has compact resolvent as well. Now C-III,Thm.1.1.(a) shows that $s(B + \Phi_\lambda)$ belongs to $\sigma(B + \Phi_\lambda)$ and, by A-III,3.6 is a pole with residue of finite rank. This completes the proof since spectral points of compact operators depend continuously on smooth perturbations (see [Dunford-Schwartz (1958), VII,6.Thm.9]).

□

If $\sigma(B) \neq \emptyset$, then $-\infty < s(B) \leq s(B + \Phi_\lambda)$ for all $\lambda \in \mathbb{R}$ which implies $\sigma(B + \Phi_\lambda) \neq \emptyset$. On the other hand, if $\sigma(B + \Phi_\lambda) = \emptyset$ for all $\lambda \in \mathbb{R}$ then $\sigma(A) = \emptyset$ by Prop.3.4 .

We are now able to characterize the spectral bound of the generator A in E through spectral bounds of generators in F .