It follows that for all $\lambda > \max\{\lambda_o, s(A)\}$ we have

$$(1.12) \quad (\lambda - s(A))^{-1} = r(R(\lambda,A)) \leq \|R(\lambda,A)\| \leq \|R(\lambda_o,A)\| \ ,$$

which can be true only if λ_o is greater than $s(A)$.

□

Corollary 1.4. Suppose X is compact and A has compact resolvent. Then there exists a real eigenvalue λ_o admitting a positive eigenfunction such that $\mathrm{Re}\,\lambda \leq \lambda_o$ for every $\lambda \in \sigma(A)$.

Proof. By Thm.1.1. $\lambda_o := s(A)$ is a real number, contained in the spectrum and obviously $\mathrm{Re}\,\lambda \leq \lambda_o$ for every $\lambda \in \sigma(A)$. Since A has compact resolvent it follows that λ_o is a pole of the resolvent. Let k be its order, then the highest coefficient in the Laurent series is given by

$$(1.13) \quad Q := \lim_{\lambda \to s(A)} (\lambda - s(A))^k R(\lambda,A) \ .$$

It follows from Cor.1.3 that Q is a positive operator. Since $Q \neq 0$ there exists $g \geq 0$ such that $h := Qg > 0$. Moreover, by A-III,3.6. we have $(\lambda_o - A)h = (\lambda_o - A)Qg = 0$.

□

The example of the rotation semigroup (A-III,Ex.5.6) shows that the assumptions in Cor.1.4. do not imply that $s(A)$ is dominant. Additional hypotheses ensuring this stronger property will be given below (see Cor.2.11, 2.12).

The following result is elementary. However, positivity is the crucial point in its proof. Note that it is not just a consequence of the spectral mapping theorem for the point spectrum.

Proposition 1.5. Suppose A is the generator of the positive semigroup $(T(t))_{t\geq 0}$. Take $\tau > 0$, $r > 0$ and let $\alpha := \tau^{-1}\log(r)$.

(a) If r is an eigenvalue of $T(\tau)$ with positive eigenfunction h_o , then there is a positive $h \in D(A)$ such that $Ah = \alpha h$ and $\{x \in X : h_o(x) > 0\} \subset \{x \in X : h(x) > 0\}$.

(b) If r is an eigenvalue of $T(\tau)'$ with positive eigenvector ϕ_o , then there is a positive $\phi \in D(A^*)$ such that $A^*\phi = \alpha\phi$ and $\mathrm{supp}\,\phi_o \subset \mathrm{supp}\,\phi$.