sufficiently small we conclude that $\eta_{t'} = 0$ for some $t' > 0$, i.e. $\xi = \mu_{t'} + \lambda_{t'}$.

" $\supset$ ": Choose $\mu \in \sigma(A_1)$, $\lambda \in \sigma(A_2)$. For every $t > 0$ there exist $\eta_t \in \sigma(A)$, $m_t \in \mathbb{Z}$ such that $\mu + \lambda = \eta_t + 2\pi i m_t/t$.
Since $\mathrm{Re}\,\mu + \mathrm{Re}\,\lambda = \mathrm{Re}\,\eta_t$ and $\{\mathrm{Im}\,\eta_t : t > 0\}$ is bounded – $\mathcal{T} = (T_1(t) \otimes T_2(t))_{t\geq 0}$ is eventually norm continuous – it follows that $m_{t'} = 0$ for some $t' > 0$.

□

7. WEAK SPECTRAL MAPPING THEOREMS

In the previous section we showed under which hypotheses a spectral mapping theorem of the form

(7.1) $$\sigma(T(t)) \setminus \{0\} = e^{t\cdot\sigma(A)}, \quad t \geq 0,$$

is valid for the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$.
Among the various examples showing that (7.1) does not hold in general we recall the following.
Take the Banach space $E = c_o$, the multiplication operator $A(x_n)_{n\in\mathbb{N}} = (inx_n)_{n\in\mathbb{N}}$ with maximal domain and the corresponding semigroup $T(t)(x_n)_{n\in\mathbb{N}} = (e^{int}x_n)_{n\in\mathbb{N}}$. Then $\sigma(A) = \{in : n \in \mathbb{N}\}$ and the spectral mapping theorem is valid only in the following weak form:

(7.2) $$\sigma(T(t)) = \overline{\exp(t\cdot\sigma(A))}, \quad t \geq 0.$$

In this section we prove similar weak spectral mapping theorems. We start with a generalization of the above example.
Consider the Banach space $E = C_o(X,\mathbb{C}^n)$ of all continuous $\mathbb{C}^n$-valued functions vanishing at infinity on some locally compact space X. In analogy to A-I,2.3 we associate to every continuous function $q : X \to M(n)$, where $M(n)$ denotes the space of all complex $n\times n$-matrices, a "multiplication operator"

$$M_q : f \to q\cdot f \quad \text{such that} \quad (q\cdot f)(x) = q(x)\cdot f(x), \quad x \in X,$$

on the maximal domain $D(M_q) = \{f \in E : q\cdot f \in E\}$. If $\|e^{tq(x)}\|$ is uniformly bounded for $0 \leq t \leq 1$ and $x \in X$ it follows that M_q generates the multiplication semigroup

$$(T(t)f)(x) = e^{tq(x)}(f(x)), \quad f \in E, \ x \in X, \ t \geq 0.$$

Since M_q has a bounded inverse if and only if $q(x)^{-1}$ exists and is uniformly bounded for $x \in X$ it follows that the eigenvalues of each matrix $q(x)$ are always contained in $\sigma(M_q)$. In fact, much more can be said in case the function is bounded.