CHAPTER A-III

SPECTRAL THEORY

by

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1. INTRODUCTION

In this chapter we start a systematic analysis of the spectrum of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on a complex Banach space E. By the spectrum of the semigroup we understand the spectrum $\sigma(A)$ of the generator A of $\mathcal{T}$. In particular we are interested in precise relations between $\sigma(A)$ and $\sigma(T(t))$. The heuristic formula

$$\text{" } T(t) = e^{tA} \text{ "}$$

serves as a leitmotiv and suggests relations of the form

$$\text{" } \sigma(T(t)) = e^{t\sigma(A)} = \{ e^{t\lambda} : \lambda \in \sigma(A) \} \text{ "},$$

called 'spectral mapping theorem'. These - or similar - relations will be of great use in Chapter IV and enable us to determine the asymptotic behavior of the semigroup $\mathcal{T}$ by the spectrum of the generator. As a motivation as well as a preliminary step we concentrate here on the spectral radius

(1.1) $$r(T(t)) := \sup \{ |\lambda| : \lambda \in \sigma(T(t)) \}, \quad t \geq 0$$

and show how it is related to the spectral bound

(1.2) $$s(A) := \sup \{ \mathrm{Re}\lambda : \lambda \in \sigma(A) \}$$

of the generator A and to the growth bound

(1.3) $$\omega := \inf \{ w \in \mathbb{R} : \|T(t)\| \leq M_w \cdot e^{wt} \text{ for all } t \geq 0 \text{ and suitable } M_w \}$$

of the semigroup $\mathcal{T} = (T(t))_{t\geq 0}$. (Recall that we sometimes write $\omega(\mathcal{T})$ or $\omega(A)$ instead of ω). The Examples 1.3 and 1.4 below illustrate the main difficulties to be encountered.

Proposition 1.1. Let ω be the growth bound of the strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$. Then

(1.4) $$r(T(t)) = e^{\omega t}$$

for every $t \geq 0$.