Proof. (a) Given $t > 0$ and $f > 0$ such that $T(t)f = 0$ and $\lambda > s(A)$ then we have $T(t)(R(\lambda,A)f) = R(\lambda,A)T(t)f = 0$. Since $R(\lambda,A)f$ is a quasi-interior point it follows that $T(t) = 0$. Thus for fixed $t \in \mathbb{R}_+$ we have either $T(t)$ is strictly positive or else $T(t) = 0$. Then strong continuity and $T(0) = Id \neq 0$ implies that there exists $\tau > 0$ such that $T(t)$ is strictly positive for $0 \leq t \leq \tau$. For arbitrary $t \in \mathbb{R}_+$ we find $n \in \mathbb{N}$ such that $\frac{t}{n} \leq \tau$. Then $T(t) = T(\frac{t}{n})^n$ is also strictly positive.

(b) We will prove that for an arbitrary holomorphic positive semigroup $(T(t))_{t \geq 0}$ the following holds:

Given $f > 0$, $\phi > 0$ then either $\langle T(t)f,\phi\rangle = 0$ for all $t \geq 0$ or $\langle T(t)f,\phi\rangle > 0$ for all $t > 0$.

Then it follows from Def.3.1(ii) that for irreducible semigroups always the second case occurs.

Assume that $\langle T(t_o)f,\phi\rangle = 0$ for some $t_o > 0$.

We consider a null sequence (t_n) , $0 < t_n < t_o$ such that $\|T(t_n)f - f\| \leq 2^{-n}$ and define $f_n := T(t_n)f$, $g_n := f - \sum_{k=n}^{\infty}(f-f_k)^+$.

Then we have $g_n \leq f$, $f = \lim_{n\to\infty} g_n$ and for $m \geq n$:

$g_n \leq f - (f-f_m)^+ = \inf\{f,f_m\} \leq f_m$.

For $n \in \mathbb{N}$ fixed and $m \geq n$ we obtain

$0 \leq \langle T(t_o-t_m)g_n^+,\phi\rangle \leq \langle T(t_o-t_m)f_m,\phi\rangle = \langle T(t_o)f,\phi\rangle = 0$.

Thus the function $t \to \langle T(t)g_n^+,\phi\rangle$ is identically zero by the uniqueness theorem for analytic functions. Since $f = \lim_{n\to\infty} g_n^+$ we have $\langle T(t)h,\phi\rangle = 0$ for all $t \in \mathbb{R}_+$.

□

The next result can be used to check irreducibility for a semigroup whose generator is a bounded perturbation of a known semigroup. It is a generalization (and an extension to Banach lattices) of B-III, Prop.3.3 .

Proposition 3.3. Suppose that A is the generator of a positive semigroup $\mathcal{T}$, further assume that K is a bounded positive operator and M is a bounded real multiplier (cf. C-I,Sec.8). Let $\mathcal{S}$ be the semigroup generated by $B := A + K + M$.

For a closed ideal $I \subset E$ the following assertions are equivalent:

(i) I is $\mathcal{S}$-invariant.

(ii) I is invariant both under $\mathcal{T}$ and K .

Proof. We recall that a closed subspace $I \subset E$ is invariant for a semigroup generated by C if and only if $C(D(C)\cap I) \subset I$ and the restriction $C_{|I}$ of C with domain $D_| := D(C)\cap I$ generates a semi-