Therefore the assumptions of C-III,Thm.3.8 are satisfied and formula C-III,(3.13) implies

(2.10) $\rho(A) = \rho(A) + i\alpha\mathbb{Z}$ and $\|R(\lambda,A)\| = \|R(\lambda+i\alpha k,A)\|$

for $\lambda \in \rho(A)$, $k \in \mathbb{Z}$.

Since 0 was supposed to be a pole of the resolvent we can decompose

$$\sigma(A) = \sigma_1 \cup \sigma_2 ,$$

where $\sigma_1 = i\alpha\mathbb{Z}$, $0 < \alpha \in \mathbb{R}$, and $\sup\{\mathrm{Re}\lambda : \lambda \in \sigma_2\} < 0$. Moreover, for small $\varepsilon > 0$, $\|R(-\varepsilon+i\lambda,A)\|$ is uniformly bounded for $\lambda \in \mathbb{R}$. Next, we construct a spectral decomposition of E and $\mathcal{T}$ corresponding to σ_1 and σ_2 (compare A-III,Sec.3).

Since 0 is an eigenvalue of A it follows that $\mathcal{T}$ has a quasi-interior fixed point $h \in E_+$ (use C-III,Prop.3.5(a)). Hence, $\{T(t)f : t \geq 0\}$ is contained in the weakly compact (see C-I,Sec.5) order interval $[-h,h]$ whenever $|f| \leq h$. Since h is a quasi-interior point and $\mathcal{T}$ is bounded it follows that $\mathcal{T}$ is relatively compact for the weak operator topology on $\mathcal{L}(E)$. Therefore the Jacobs-DeLeeuw-Glicksberg Splitting Theorem (see Krengel (1985), Chap.2,Thm.4.4 and 4.5) can be applied to (the weak closure of) $\mathcal{T}$ and we obtain a projection $Q \in \mathcal{L}(E)$ onto the closed subspace E_1 generated by the eigenvectors h_k of A corresponding to the eigenvalues $i\alpha k$, $k \in \mathbb{Z}$. Clearly, Q splits the semigroup $\mathcal{T}$ into the restricted semigroups $\mathcal{T}_1$ on $E_1 := QE$ and $\mathcal{T}_2$ on $E_2 := \ker Q$. We first describe $\mathcal{T}_1$ in more detail.

The projection Q is positive as an element of the weak closure of $\mathcal{T}$ and even strictly positive by the irreducibilitiy of $\mathcal{T}$. Its range E_1 is a closed sublattice of E (use Schaefer (1974),Prop.III.11.5) on which the semigroup $\mathcal{T}_1$ is periodic, irreducible and positive. In fact, $T(2\pi/\alpha)f = f$ for every $f = h_k$, $k \in \mathbb{Z}$, and hence for every $f \in E_1$, while irreducibility and positivity are inherited from $\mathcal{T}$.

It now follows from A-III,Lemma 5.2 that the generator $A_1 = A|_{E_1}$ of $\mathcal{T}_1$ has spectrum $\sigma(A_1) = i\alpha\mathbb{Z}$. Moreover in view of A-II,Prop.5.2 and Cor.5.3(ii) we have $\sigma(A_2) = \sigma(A) \setminus i\alpha\mathbb{Z}$. Therefore the decomposition $E = E_1 \oplus E_2$ is a spectral decomposition corresponding to σ_1 and σ_2 . This proves the first part of the following lemma.

Lemma 2.12. Under the above assumptions there exists a positive projection Q with range $E_1 := QE$ and kernel $E_2 := Q^{-1}(0)$ such that the following holds:

(a) $E = E_1 \oplus E_2$, $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ and $A = A_1 \oplus A_2$ is a spectral decomposition corresponding to the decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$ where $\sigma_1 = \sigma(A_1) = i\alpha\mathbb{Z}$ and $\sigma_2 = \sigma(A_2) = \sigma(A) \setminus i\alpha\mathbb{Z}$.