Chapter D-IV

ASYMPTOTICS OF POSITIVE SEMIGROUPS ON C*- AND W*-ALGEBRAS

1. STABILITY OF POSITIVE SEMIGROUPS

As explained in A-III, Section 1 , it is possible to deduce uniform exponential stability of strongly continuous semigroups from the location of the spectrum of its generator if the spectral bound $s(A)$ and the growth bound ω coincide. In this section we prove '$s(A) = \omega$' for positive semigroups on C*-algebras and preduals of W*-algebras. A more general discussion of the "$s(A) = \omega$" problem can be found in [Greiner-Voigt-Wolff (1981)]. For the results of this section the existence of a unit is essential.

Theorem 1.1. Let M be a C*-algebra with unit and $\mathcal{T} = (T(t))_{t\geq 0}$ a positive semigroup on M . Then

$$-\infty < s(A) = \omega \in \sigma(A) .$$

Proof. For every $t\geq 0$ there exists ϕ_t in the state space $\mathcal{S}(M)$ of M such that

$$T(t)'\phi_t = r(T(t))\phi_t = \exp(\omega t)\phi_t$$

(see, e.g., [Groh (1981), 2.1]). Let $n\in\mathbb{N}$ and

$$E_n := \{\phi\in\mathcal{S}(M) \; : \; T(2^{-n})'\phi = \exp(\omega 2^{-n})\phi \} .$$

Then $\emptyset \neq E_{n+1} \subseteq E_n$ $(n\in\mathbb{N})$. Since $\mathcal{S}(M)$ is $\sigma(M,M')$-compact there exists $\phi\in\bigcap_{n\in\mathbb{N}} E_n$ and $T(t)'\phi = \exp(\omega t)\phi$ follows because the adjoint semigroup $(T(t)')_{t\geq 0}$ is a weak*-semigroup on M' . Suppose $-\infty = \omega$. Then for $t > 0$ $r(T(t)) = 0$ (A-III,Prop.1.1) or $T(t)'\phi = 0$, in particular $\phi(T(t)1) = 0$. From this we obtain the contra-