Now let A be the generator of a strongly continuous positive semigroup $(T(t))_{t\geq 0}$. The positivity of the semigroup is equivalent to

$$(2.7)\qquad |T(t)f| \leq T(t)|f| \qquad (t \geq 0\ ,\ f \in E)\ .$$

In order to deduce from (2.7) a property for the generator A it is natural trying to differentiate (2.7) in $t = 0$. Let us assume for a moment that $E = L^p(X,\Sigma,\mu)$ (as in Ex.2.3).
Let $f \in D(A)$ and $0 \leq \phi \in D(A')$. Then by (2.7),

$$(2.8)\qquad \langle |T(t)f|,\phi\rangle \leq \langle T(t)|f|,\phi\rangle \qquad (t \geq 0)$$

where the equality holds for $t = 0$. Hence the inequality remains valid if we differentiate in 0 on both sides of (2.8).
Since $\phi \in D(A')$ we obtain $d/dt_{|t=0} \langle T(t)|f|,\phi\rangle = \langle |f|,A'\phi\rangle$ on the right side. By (2.6) and the chain rule B-II,Prop.2.3 one obtains $d/dt_{|t=0} |T(t)f| = \mathrm{Re}((\widehat{\mathrm{sign}}\ \bar{f})Af)$ on the left side.
Since $\mathrm{Re}((\mathrm{sign}\ \bar{f})Af) \leq \mathrm{Re}((\widehat{\mathrm{sign}}\ \bar{f})Af)$, this finally gives

$$(K)\qquad \mathrm{Re}\langle(\mathrm{sign}\ \bar{f})Af,\phi\rangle \leq \langle|f|,A'\phi\rangle \qquad (f \in D(A)\ ,\ 0 \leq \phi \in D(A')).$$

We refer to this as <u>Kato's inequality</u>, since it represents an abstract version of the classical inequality proved by Kato for the Laplacian (see Example 2.5).
We will see in the next Section that, together with an additional condition, this inequality is characteristic for the positivity of the semigroup.

By a different proof, we now show that Kato's inequality holds for generators of positive semigroups in general.

<u>Theorem</u> 2.4. The generator A of a strongly continuous positive semigroup $(T(t))_{t\geq 0}$ on a σ-order complete (real or complex) Banach lattice E satisfies Kato's inequality; i.e.,

$$(K)\qquad \mathrm{Re}\langle(\mathrm{sign}\ \bar{f})Af,\phi\rangle \leq \langle|f|,\ A'\phi\rangle \qquad (f \in D(A),\ 0 \leq \phi \in D(A'))\ .$$

<u>Proof</u>. Let $f \in D(A)$, $0 \leq \phi \in D(A')$. Then

$$\begin{aligned}\langle(\mathrm{sign}\ \bar{f})\ Af,\ \phi\rangle &= \lim_{t\to 0} 1/t\ \langle(\mathrm{sign}\ \bar{f})(T(t)f - f),\ \phi\rangle\\ &= \lim_{t\to 0} 1/t\ \langle(\mathrm{sign}\ \bar{f})T(t)f - |f|,\ \phi\rangle\\ &\leq \lim_{t\to 0} 1/t\ \langle|T(t)f| - |f|,\ \phi\ \rangle\end{aligned}$$