__Definition__ 2.8. Let A be the generator of a positive semigroup $(T(t))_{t\geq 0}$ with spectral bound $s(A) > -\infty$. The resolvent is said to __grow slowly__ if one of the following (equivalent) conditions is satisfied:

(2.15a) $\{(\lambda - s(A))R(\lambda,A) : \lambda > s(A)\}$ is bounded in $L(E)$.

(2.15b) $\{\frac{1}{t}\int_0^t \exp(-\tau s(A))T(\tau)\, d\tau : t > 0\}$ is bounded in $L(E)$.

Before proving the equivalence of the two assertions we make some remarks.

(a) Since one always has $\lambda R(\lambda,A) \to Id$ for $\lambda \to \infty$ $\{(\lambda - s(A))R(\lambda,A) : \lambda > s(A)+\varepsilon\}$ is bounded for every $\varepsilon > 0$. Thus in (2.15a) the essential fact is boundedness near $s(A)$. On the other hand, $\{\frac{1}{t}\int_0^t \exp(-\tau s(A))T(\tau)\, d\tau : 0 \leq t \leq T\}$ is bounded for every $T \geq 0$, hence in (2.15b) the boundedness for $t \to \infty$ is important.

(b) By Cor.1.4 we have $\|R(\lambda,A)\| \geq r(R(\lambda,A)) = (\lambda - s(A))^{-1}$. Hence $\|R(\lambda,A)\|$ grows at least as fast as $(\lambda - s(A))^{-1}$. Thus if (2.15a) is satisfied the resolvent grows as slowly as it possibly can for $\lambda \downarrow s(A)$.

(c) We do not assume in Def.2.8 that spectral bound and growth bound coincide. A slight modification of A-III,Example 1.3 shows that there are semigroups with slowly growing resolvent and $s(A) < \omega(A)$.

To prove equivalence of (2.15a) and (2.15b) we assume $s(A) = 0$ and write $C(t) := \frac{1}{t}\int_0^t T(\tau)\, d\tau$.

(2.15a)$\Rightarrow$(2.15b): Consider $\lambda > 0$, $t > 0$ such that $\lambda t = 1$.
Then we have

$$\lambda\cdot\exp(-\lambda s) \geq \begin{cases} (et)^{-1} & \text{if } s \leq t \\ 0 & \text{if } s > t \end{cases}$$

Now (1.1) implies $\lambda R(\lambda,A) = \int_0^\infty \lambda\exp(-\lambda s)T(s)\, ds \geq e^{-1}C(t) = e^{-1}\cdot C(\frac{1}{\lambda}) \geq 0$. Thus $C(t)$ is bounded for $t \to \infty$ whenever $\lambda R(\lambda,A)$ is bounded for $\lambda \downarrow 0$.

(2.15b)$\Rightarrow$(2.15a): Let $M := \sup\{\|C(t)\| : t > 0\}$. Given $f \in E$, $\lambda > 0$, $r > 0$ then integration by parts yields:

$$\lambda\int_0^r e^{-\lambda s}T(s)f\, ds = \lambda e^{-\lambda r}\int_0^r T(\sigma)f\, d\sigma + \lambda^2\int_0^r se^{-\lambda s}\left(\tfrac{1}{s}\int_0^s T(\sigma)f\, d\sigma\right)ds$$

It follows that $\|\lambda\int_0^r e^{-\lambda s}T(s)f\, ds\| \leq \left(r\lambda e^{-r} + \lambda^2\int_0^r se^{-\lambda s}\, ds\right)M\|f\|$ $= (1 - e^{-\lambda r})M\|f\|$. Letting $r \to \infty$ we obtain by (1.1) $\|\lambda R(\lambda,A)f\| \leq M\|f\|$ $(f \in E,\ \lambda > 0)$ hence $\|\lambda R(\lambda,A)\| \leq M$

□