Remarks 1.11. (a) Let ϕ be the normal state on M such that $\mathrm{Fix}(T) = \mathbb{C}\phi$ and let $H := P\sigma(A) \cap i\mathbb{R}$. From the proof of Theorem 1.10 it follows that there exists a family $\{u_\eta : \eta \in H\}$ of unitaries in M such that $A'u_\eta = -\eta u_\eta$ and $A(u_\eta \phi) = \eta(u_\eta \phi)$ for all $\eta \in H$.

(b) If the group H is generated by a single element, i.e., $H = i\gamma\mathbb{Z}$ for some $\gamma \in \mathbb{R}$ then the family $\{u_\gamma^{\,k} : k \in \mathbb{Z}\}$ is a complete family of eigenvectors pertaining to the eigenvalues in H , where $u_\gamma \in M$ is unitary such that $A'u_\gamma = i\gamma u_\gamma$.

Proposition 1.12. Suppose that T and M satisfy the assumptions of Theorem 1.10, and let N_* be the closed linear subspace of M_* generated by the eigenvectors of A pertaining to the eigenvalues in $i\mathbb{R}$. Denote by T_o the restriction of T to N_* . Then

(a) $G := (T_o)^- \subseteq L_s(N_*)$ is a compact, Abelian group.

(b) $\mathrm{Id}|N_* \in \{T_o(t) : t>s\}^- \subseteq L_s(N_*)$ for all $0<s\in\mathbb{R}$.

Proof. For $\eta \in H := P\sigma(A) \cap i\mathbb{R}$ let

$$U(\eta) := \{\psi \in D(A) : A\psi = \eta\psi\}$$

and $U = \{U(\eta) : \eta \in H\}$. Then $(\mathrm{span}\ U)^- = N_*$. For each $\psi \in U$ there exists $\eta \in H$ such that

$$\{T_o(t)\psi : t \in \mathbb{R}_+\} = \{e^{-\eta t}\psi : t \in \mathbb{R}_+\} .$$

Consequently this set is relatively compact in $L_s(N_*)$. From [Schaefer (1966), III.4.5] we obtain that G is compact.

Next choose $\psi_1, \ldots, \psi_n \in U$, $0<s\in\mathbb{R}$ and $\delta > 0$. Since $T_o(t)\psi_i = e^{\eta_i t}\psi_i$ $(1 \le i \le n)$ for some $\eta_i \in H$, it follows from a theorem of Kronekker (see, [Jacobs (1976), Satz 6.1., p.77]) that there exists $s < t$ such that

$$|(1,1, \ldots, 1) - (e^{\eta_1 t}, e^{\eta_2 t}, \ldots, e^{\eta_n t})| < \delta ,$$

hence

$$\sup\{\|\psi_i - T_o(t)\psi_i\| : 1 \le i \le n\} < \delta$$

or $\mathrm{Id}|N_* \in \{T_o(t) : t>s\}^- \subseteq L_s(N_*)$.