This follows from Lemma 1.6 because $\mathrm{Fix}(\lambda R(\lambda+i\alpha)) = \ker(i\alpha-A)$ .

Proposition 1.8. Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type with generator $A$ on the predual of a W*-algebra and suppose that there exists a faithful family $\Psi$ of $\mathcal{T}$-invariant states. Then for all $\alpha\in\mathbb{R}$ we have

$$\dim\ker(i\alpha - A) = \dim\ker(i\alpha - A')$$

and

$$P\sigma(A) \cap i\mathbb{R} = P\sigma(A') \cap i\mathbb{R} .$$

Proof. The inequality $\dim\ker(i\alpha - A) \leq \dim\ker(i\alpha - A')$ follows from Corollary 1.7.

Let $D = \{\lambda\in\mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and $R$ the pseudo-resolvent induced by $R(\lambda,A)$ on $D$ . Then $R$ is identity preserving and of Schwarz type. Take $i\alpha\in P\sigma(A)$ $(\alpha\in\mathbb{R})$ and choose $0 < \mu\in\mathbb{R}$ . If $\psi_\alpha\in M_*$ is of norm one with polar decomposition $\psi_\alpha = u_\alpha|\psi_\alpha|$ such that $\psi_\alpha = (\mu - i\alpha)R(\mu)\psi_\alpha$ then $\mu R(\mu)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4.a). Since

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leq 1 - s(|\psi_\alpha|)$$

we obtain $\mu R(\mu)'s(|\psi_\alpha|) = s(|\psi_\alpha|)$ by the faithfulness of $\Psi$ . Hence the maps $S := (\mu - i\alpha)R(\mu)'$ and $T := \mu R(\mu)'$ fulfil the assumptions of Lemma 1.2.b. Therefore $Su_\alpha^* = u_\alpha^*$ or $(\mu-i\alpha)R(\mu)'u_\alpha^* = u_\alpha^*$ which implies $u_\alpha^*\in D(A')$ and $A'u_\alpha^* = i\alpha u_\alpha^*$ .

If $i\alpha\in P\sigma(A')$ , $\alpha\in\mathbb{R}$ , choose $0 \neq v_\alpha$ such that

$$v_\alpha = (\mu - i\alpha)R(\mu)'v_\alpha \quad (\mu\in\mathbb{R}_+)$$

and $\psi\in\Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$ . Since

$$0 \leq v_\alpha v_\alpha^* = ((\mu - i\alpha)R(\mu)'v_\alpha)((\mu - i\alpha)R(\mu)'v_\alpha)^* \leq$$

$$\leq \mu R(\mu)'(v_\alpha v_\alpha^*) ,$$

we obtain $\mu R(\mu)'(v_\alpha v_\alpha^*) = v_\alpha v_\alpha^*$ because $\Psi$ is faithful .