Theorem 1.29. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and let $B \in L(E)$. Then $A + B$ with domain $D(A+B) = D(A)$ is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$.

It is possible to express the new semigroup $(S(t))_{t\geq 0}$ by known objects. The product formula

(1.8) $S(t)f = \lim_{n\to\infty} (T(t/n)e^{t/n\cdot B})^n f$

holds for all $t \geq 0$ and $f \in E$.

Moreover, $S(t)$ is the solution of the following integral equation

(1.9) $S(t)f = T(t)f + \int_0^t T(t-s)BS(s)f\,ds \quad (t \geq 0, f \in E)$.

Let $S_0(t) = T(t)$ and

(1.10) $S_n(t)f = \int_0^t T(t-s)BS_{n-1}(s)f\,ds \quad (f \in E)$ for $n \in \mathbb{N}$. Then

(1.11) $S(t) = \sum_{n=0}^{\infty} S_n(t)$,

where the series converges in the operator norm uniformly on bounded intervals. We refer to [Davies (1980),III.1], [Goldstein (1985a), I.6] or [Pazy (1983),Chap.3] for these results.

Several special properties discussed above are preserved by bounded perturbations.

Theorem 1.30. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator A. Let $B \in L(E)$. If $(T(t))_{t\geq 0}$ is holomorphic or norm continuous or compact, then so is the semigroup $(S(t))_{t\geq 0}$ generated by $A+B$.
If A has a compact resolvent then so has $A+B$.
Let $t_0 \geq 0$. If $(T(t))_{t\geq 0}$ is norm continuous for $t > t_0$ and if B is compact, then $(S(t))_{t\geq 0}$ is also norm continuous for $t > t_0$.

Proof. If $(T(t))_{t\geq 0}$ is norm continuous for $t > 0$, then $S_n(t)$ in (1.10) is norm continuous in $t > 0$ for every n. Thus $(S(t))_{t\geq 0}$ is norm continuous in $t > 0$ by (1.11). There exists $\lambda_0 \in \mathbb{R}$ such that $\|R(\lambda,A)\| \leq (2\|B\|)^{-1}$ for $\mathrm{Re}\lambda \geq \lambda_0$. Hence $(Id - BR(\lambda,A))^{-1}$ exists for $\mathrm{Re}\lambda \geq \lambda_0$. Since $(\lambda-(A+B))f = (Id-BR(\lambda,A))(\lambda-A)f$ for all $f \in D(A)$ it follows that $(\lambda-(A+B))^{-1}$ exists and is given by

(1.12) $R(\lambda,A+B) = R(\lambda,A)(Id-BR(\lambda,A))^{-1}$

whenever $\mathrm{Re}\lambda \geq \lambda_0$. Now if A generates a holomorphic semigroup,