i.e., $P_k f$ is the k-th Fourier coefficient of the π-periodic, continuous function $\xi_f : s \to \sum_{m\in\mathbb{Z}} \check{\phi}_o(s+m\pi)\ T(s+m\pi)f$, $f \in E$. Since the projections P_k are mutually orthogonal, i.e. $P_k P_m = 0$ for $k \neq m$, it follows that $g = \sum_{n\in\mathbb{Z}} P_n g$ for every $g \in \operatorname{span} \cup_{k\in\mathbb{Z}} P_k E$. In particular, the Fourier coefficients of the function ξ_g are absolutely summable, hence the Fourier series of ξ_g converges to ξ . For $s = 0$ we obtain

$g = \sum_{n\in\mathbb{Z}} P_n g\cdot e^{-in0} = \sum_{m\in\mathbb{Z}} \check{\phi}_o(0 + m\pi)\ T(0 + m\pi)g \quad (g \in \operatorname{span} \cup_{k\in\mathbb{Z}} P_k E)$.

Since $\operatorname{span} \cup_{k\in\mathbb{Z}} P_k E$ is dense (Lemma 7.7) we conclude that

$$\sum_{m\in\mathbb{Z}} \check{\phi}_o(m\pi)\ T(m\pi) = \mathrm{Id} . \tag{7.6}$$

As the final step we construct the inverse operator of $\mathrm{Id} + T(\pi)$ showing that $-1 \in \rho(T(\pi))$. We define $\Psi_o(\alpha) := \phi_o(\alpha)\cdot(1 + e^{i\pi\alpha})^{-1}$, $\alpha \in \mathbb{R}$. Then we have $\Psi_o \in S$ and $\Psi_o(1 + e^{i\pi\cdot}) = \phi_o$, hence $\check{\Psi}_o(x) + \check{\Psi}_o(x + \pi) = \check{\phi}_o(x)$ for all $x \in \mathbb{R}$. Then (7.6) implies

$$\begin{aligned} \mathrm{Id} &= \textstyle\sum_{m\in\mathbb{Z}} \check{\phi}_o(m\pi)T(m\pi) \\ &= \textstyle\sum_{m\in\mathbb{Z}} (\check{\Psi}_o(m\pi) + \check{\Psi}_o((m+1)\pi))T(m\pi) \\ &= [\textstyle\sum_{m\in\mathbb{Z}} \check{\Psi}_o(m\pi)T(m\pi)](\mathrm{Id} + T(\pi)) . \end{aligned}$$

□

In the rest of this section we discuss the behavior of the single spectral values λ of $T(t)$, $t > 0$. The aim is a characterization of $\sigma(T(t))$ involving only properties of the generator. By the rescaling procedure A-I,3.1 we may assume $\lambda = 1$ and $t = 2\pi$.
From the Spectral Inclusion Theorem 6.2 we know that $1 \in \rho(T(2\pi))$ implies $i\mathbb{Z} \subset \rho(A)$. As seen in many examples the converse does not hold and we are now looking for additional conditions.
Henceforth we assume $i\mathbb{Z} \subset \rho(A)$ and define for $k \in \mathbb{Z}$

$$Q_k := 1/2\pi\int_0^{2\pi} e^{-iks}\ T(s)\ ds = 1/2\pi(1 - T(2\pi))R(ik,A) , \tag{7.7}$$

(cf. Formula A-I,(3.1)).

Our approach is based on Fejér's Theorem (for Banach space valued functions). Let us recall this result. Suppose $\xi : [0,2\pi] \to E$ is a continuous function and let $\xi_k := 1/2\pi\int_0^{2\pi} e^{-iks}\ \xi(s)\ ds$ be its k-th Fourier coefficient. Then the Fourier series is Césaro summable to ξ in every point $t \in (0,2\pi)$. Moreover one has

$$1/2(\xi(0) + \xi(2\pi)) = C_1\text{-}\sum_{k\in\mathbb{Z}} \xi_n := \lim_{N\to\infty} 1/N\cdot\sum_{n=0}^{N-1}(\sum_{k=-n}^{n} \xi_k) . \tag{7.8}$$

This result enables us to prove the following proposition: