8. THE SIGNUM OPERATOR

We discuss in some detail how a mapping of the form

$$g \to (\text{sign } f)g$$

which has an obvious meaning, depending on f, in spaces $C(K)$, can be defined in an abstract complex Banach lattice. We prove the following:

Let E be a complex Banach lattice and let $f \in E$. If either E is order-complete or $|f|$ is a quasi-interior point in E_+, then there exists a unique linear mapping S_f, called the signum operator with respect to f, with the following properties:

(i) $S_f\bar{f} = |f|$, where $\bar{f} = \mathrm{Re}\, f - i\cdot \mathrm{Im} f$
(ii) $|S_f g| \leq |g|$ for every g in E
(iii) $S_f g = 0$ for every g in E orthogonal to f.

In fact, if $E = C(K)$ and if $|f|$ is a quasi-interior point in E, then $|f|$ is a strictly positive function and multiplication with the function $\text{sign } f = f\cdot|f|^{-1}$ has the desired properties. Uniqueness follows from Zaanen (1983) Chap. 20. We reduce the general situation to the case just considered in the following way:

1. If $|f|$ is quasi-interior to E_+, then $E_{|f|}$ is a dense subspace of E isomorphic to a space $C(K)$, and with the above arguments one gets a uniquely determined operator S_o on $E_{|f|}$ with the desired properties. Since (ii) implies the continuity of S_o for the norm induced by E, S_o can be extended to E.
2. If f is arbitrary, then as above one gets an operator S_o on $E_{|f|}$ with (i) - (ii). If E is order complete, an extension S_f of S_o to E satisfying (i) - (iii) is possible as soon as S_o can be extended to the band $\{x\}^{dd}$ of E : On the complementary band $\{x\}^d$ one has necessarily the values $\equiv 0$ for S_f. The extension to $\{x\}^{dd}$ is obtained as follows:
If S_o is positive (which means $f \geq 0$) then

$$S_f h = \sup \{ S_f g : g \in [0,h] \cap E_{|f|} \} \quad (h \geq 0)$$

will do. In general, the problem can be reduced to this situation by decomposing S_o into a sum of the form $S_o = (S_1 - S_2) + i(S_3 - S_4)$