there exists $M \geq 0$ such that $\|R(\lambda_o+i\eta,A)\| \leq M/|\eta|$ for all $\eta \in \mathbb{R}$. Consequently, $\|R(\lambda_o+i\eta,A+B)\| \leq \|(Id-BR(\lambda_o+i\eta,A)^{-1}\|\cdot M/|\eta| \leq 2M/|\eta|$ for all $\eta \in \mathbb{R}$. Thus $A + B$ generates a holomorphic semigroup by the corollary of Thm.1.14. Moreover, it follows from (1.12) that $R(\lambda,A+B)$ is compact whenever $R(\lambda,A)$ is compact. Consequently by Theorem 1.25 and the assertion proved above, $(S(t))_{t\geq 0}$ is compact whenever $(T(t))_{t\geq 0}$ is compact.

Finally assume that B is compact and $t_o \geq 0$ such that $(T(t))_{t\geq 0}$ is norm continuous for $t > t_o$. Fix $t > t_o$. Denote by U the unit ball of E and fix $s \in (0,t]$. Then $\lim_{h\to 0} (T(t+s-h) - T(t-s))f = 0$ for all $f \in \overline{BS(s)U} =: K$.

Since K is compact it follows that the limit exists uniformly in $f \in K$; i.e. $\lim_{h\to 0} \|(T(t+s-h) - T(t-s))BS(s)\| = 0$. It follows from the dominated convergence theorem that

$$(1.13) \qquad \lim_{h\to 0} \int_0^t \|(T(t+s-h) - T(t-s))BS(s)\| ds = 0.$$

Using (1.9) we obtain $\|S(t+h) - S(t)\|$

$$\leq \|T(t+h) - T(t)\| + \|\int_0^{t+h} T(t+h-s)BS(s)ds - \int_0^t T(t-s)BS(s)ds\|$$

$$\leq \|T(t+h) - T(t)\| + \|\int_t^{t+h} T(t+h-s)BS(s)ds\|$$

$$+ \int_0^t \|(T(t+h-s) - T(t-s))BS(s)\|ds \to 0 \quad \text{when} \quad h\to 0.$$

□

In C-IV,Ex.2.15 a generator A of an eventually differentiable and eventually compact semigroup and a bounded operator B will be given such that the semigroup generated by $A+B$ is not eventually norm continuous.

Using Theorem 1.29 we now prove a perturbation result due to Desch-Schappacher(1984). Instead of assuming that $B \in L(E)$ we assume that $B \in L(D(A))$. The short proof given below is due to G.Greiner.

<u>Theorem</u> 1.31. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator A. Assume that $B : D(A) \to D(A)$ is linear and continuous for the graph norm on $D(A)$.

Then $A + B$ with domain $D(A + B) = D(A)$ is the generator of a strongly continuous semigroup. Moreover, there exists a <u>bounded</u> operator C on E such that $A + B$ is similar to $A + C$.

<u>Proof</u>. We first show that $(Id - BR(\lambda,A))$ is invertible for some $\lambda \in \mathbb{C}$. Choose $\lambda_o \in \rho(A)$. Then $S := (\lambda_o-A)BR(\lambda_o,A) \in L(E)$. Let $\lambda > s(A)$ be so large such that $\|SR(\lambda,A)\| < 1$.