whenever there exists a fixed vector which is a quasi-interior point of E_+. Indeed, if k is the order of the pole at $s(A) = 0$ then we have $0 \neq A^{k-1}P = \lim_{\lambda\to 0} \lambda^k R(\lambda,A)$. Thus $A^{k-1}P$ is a positive operator. Assuming $k > 1$ and denoting the quasi-interior fixed vector by u we have $Au = 0$ hence $A^{k-1}Pu = PA^{k-1}u = 0$. Since $A^{k-1}P$ is positive it vanishes on the principal ideal generated by u. Since this ideal is dense we obtain $A^{k-1}P = 0$ which is a contradiction.

(d) If $\mathcal{T} = (T(t))_{t\geq 0}$ is an irreducible semigroup with $s(A) = 0$, then quasi-compactness implies boundedness of $\mathcal{T}$ (This follows from (c) and C-III,Prop.3.5). Moreover, in this case the projection P has the form $P = \phi \otimes h$ where u is a quasi-interior point of E_+ and ϕ is a strictly positive linear form on E. This also is a consequence of C-III,Prop.3.5 .

(e) If $\mathcal{T} = (T(t))_{t\geq 0}$ is irreducible and eventually compact then the rescaled semigroup $(\exp(-\omega(\mathcal{T})t)T(t))$ satisfies the assumptions of Thm.2.1. Indeed, by C-III,Thm.3.7 we know that $\omega(\mathcal{T}) > -\infty$, while $\omega_{ess}(\mathcal{T}) = -\infty$. It follows that the rescaled semigroup is quasi-compact hence (d) is applicable.

The following example has a biological background, and the semigroup considered describes the time evolution of an age-structured population. For more details we refer to Greiner (1984a) or Webb (1984).

Example 2.3. On the Banach lattice $E = L^1([0,\infty))$ we consider the operator A defined by

$$Af := -f' - \mu f \quad \text{with domain}$$

$$(2.2)\quad D(A) := \{f \in E : f \text{ absolutely continuous, } f' \in E, \; f(0) = \int_0^\infty \beta(a)f(a)\,da \} .$$

Here we assume that μ and β are positive, measurable, bounded functions on $[0,\infty)$. One can show that A generates a strongly continuous semigroup $\mathcal{T}$ of positive operators. Assuming that $\mu(\infty) := \lim_{a\to\infty}\mu(a)$ exists we obtain $\omega_{ess}(\mathcal{T}) = -\mu(\infty)$. We suppose in addition that β and μ satisfy

$$(2.3)\quad \int_0^\infty \beta(a)\left(\exp\left(-\int_0^a \mu(x)\,dx\right)\right)da = 1 \quad \text{and} \quad \mu(\infty) > 0 .$$

The function h with $h(a) := \exp\left(-\int_0^a \mu(s)\,ds\right)$ is differentiable, $h \in E$ and $h' = -\mu h$. Moreover, (2.3) implies $\int_0^\infty \beta(a)h(a)\,da = 1 = h(0)$. Thus $h \in D(A)$ and $Ah = 0$. It follows that $s(A) = 0$. Indeed, since $s(A)$ is a pole of the resolvent there exists a positive eigenvector w of A' corresponding to $s(A)$. Since h is