unique p-periodic generalized solution

$$u(t) = T(t)f + \int_0^p T(p-s)F(s)\,ds$$

is asymptotically stable; i.e., for every generalized solution $v(\cdot)$ of (2.1) we have $\lim_{t\to\infty} \|v(t) - u(t)\| = 0$.

Example 2.5. Let E be the Banach space $C_o(\mathbb{R}_+)$; $A = \frac{d}{dx}$ with $D(A) = \{f \in E: f' \in C^1$ and $f' \in E\}$ is the generator of the uniformly stable translation semigroup $T(t)f(x) := f(t+x)$. Applying (1.14) we obtain that $\operatorname{im} A = \{f : \int_0^\infty f(x)\,dx$ exists$\}$ is dense in $C_o(\mathbb{R}_+)$. Let $r \in \operatorname{im} A$ and let $F(\cdot)$ be a p-periodic real-valued function.

We apply Theorem 2.4 to the initial value problem

$$(*) \qquad \frac{d}{dt} u(t,x) = \frac{d}{dx} u(t,x) + r(x)F(x+t) \quad , \quad u(0,\cdot) \in D(A) .$$

We may rewrite (*) as

$$(**) \qquad \dot{v}(t) = Av(t) + G(t)$$

where $v(t) = u(t,\cdot)$ and $G : \mathbb{R}_+ \to E$ is defined by

$$G(t)(x) = r(x)F(x+t) .$$

G is p-periodic with values in E and $h_o := \int_o^p T(p-t)G(t)\,dt$ is the function $x \to [\int_0^p T(p-t)G(t)\,dt](x) = F(x)\int_x^{x+p} r(s)\,ds$. For the function $f = \sum_{k=0}^\infty T(kp)h_0$, which is given by $x \to F(x)\int_x^\infty r(s)\,ds$, we clearly have $(Id - T(p))f = h_o$. Therefore (**) has a unique p-periodic generalized solution (Thm.2.4) although $i\mathbb{R} \in \sigma(A)$ (compare with Remark 2.3).

The unique p-periodic generalized solution $u(t,\cdot)$ is given by $u(t,x) = F(x+t)\int_{x+t}^\infty r(s)ds + F(x+t)\int_x^{x+t} r(s)ds = F(x+t)\int_x^\infty r(s)ds$. For every solution $v(t,\cdot)$ of (*) we have, by Thm.2.4:

$\sup\{|v(t,x) - F(x+t)\int_x^\infty r(s)\,ds| : x \in \mathbb{R}_+\} \to 0$ as $t \to 0$.

NOTES.

Section 1. The exponential growth bounds $\omega(f)$ and $\omega(A)$ as well as the characterizations (1.2), (1.6) and Theorem 1.3 (i) can be found in Hille-Phillips (1957). Growth bounds similar to $\omega_1(A)$ were considered first in [D'Jacenko (1976)] and in [Zabczyk (1979), Prop.2]. Example 1.2.(2) is taken from Wolff (1981); other 'counterexamples' can be found in Hille-Phillips (1957), Foias (1973), Triggiani (1975), Zabczyk (1975) and Greiner-Voigt-Wolff (1981). The statements (1.2), (1.6) and Theorem 1.3.(i) are semigroup versions of results of classical Laplace transform