<u>Corollary</u> 1.5. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E. Then

$$s(A) \leq \omega_1(A) \leq \omega(A) .$$

Example 1.2.(2) shows that the uniform exponential stability of the semigroup is not equivalent to $\sigma(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq q < 0\}$. In the following example we will see that not only the semigroup (i.e., all generalized solutions of (ACP)), but also the strong solutions can be unstable even when $s(A) < 0$. In fact, we will give an example of a semigroup with $s(A) < \omega_1(A) < \omega(A)$.

<u>Example</u> 1.6. In A-III,Ex.1.4 it was shown that the semigroup $(T(t))_{t\geq 0}$ on the Hilbert space $E = \{(x^1,x^2,\dots),\ x^n \in \mathbb{C}^n : \sum_{i=1}^{\infty} \|x^i\|^2 < \infty\}$ given by $T(t) := (e^{2\pi int}\cdot\exp(tA_n))_{n\in\mathbb{N}}$ with

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \cdot & \cdot & & & \\ \cdot & & \cdot & & 0 \\ \cdot & & & \cdot & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}_{n\times n}$$

has growth e^t ($\|T(t)\| = e^t$). Thus $\omega(A) = 1$ whereas the generator $A = (2\pi in+A_n)_{n\in\mathbb{N}}$ has spectral bound 0. We will show first that for this semigroup $\omega_1(A) = \omega(A)$ holds (we will use this to construct a semigroup with $s(A) < \omega_1(A) < \omega(A)$).

Let $e_n = n^{-1/2}\cdot(1,\dots,1) \in \mathbb{C}^n$. Then we have

$$\begin{aligned}
&\|\exp(tA_n)\cdot e_n\|^2 = \\
&\quad = \frac{1}{n}\cdot\left\|\left(1 + t + \dots + \frac{t^{n-1}}{(n-1)!},\ 1 + t + \dots + \frac{t^{n-2}}{(n-2)!},\dots,\ 1+t,\ 1\right)\right\|^2 = \\
&\quad = \frac{1}{n}\cdot\sum_{r=o}^{n-1}\left(\sum_{j=o}^{r}\frac{1}{j!}\cdot t^j\right)^2 = \\
&\quad = \frac{1}{n}\cdot\sum_{r=o}^{n-1}\left(\sum_{j,s=o}^{r}\frac{1}{j!s!}\cdot t^{j+s}\right) = \\
&\quad = \frac{1}{n}\cdot\sum_{r=o}^{n-1}\sum_{i=o}^{2r} t^i \sum_{j+s=i}\frac{1}{j!s!} = \\
&\quad = \frac{1}{n}\cdot\sum_{r=o}^{n-1}\sum_{i=o}^{2r}\frac{t^i}{i!}\sum_{j=o}^{i}\binom{i}{j} = \\
&\quad = \frac{1}{n}\cdot\sum_{r=o}^{n-1}\sum_{i=o}^{2r}\frac{1}{i!}(2t)^i \geq \\
&\quad \geq \frac{1}{n^2}\cdot\sum_{i=o}^{n-1}\frac{1}{i!}(2t)^i .
\end{aligned}$$

For $0 < q < 1$ we consider $x_q \in E$ defined as follows

$$x_q := (q\cdot e_1,\ 2q^2e_2,\ \dots,\ nq^ne_n,\ \dots) .$$