The case when $S(t) = T(t)$ $(t \geq 0)$ is of special interest: it yields a characterization of generators of lattice semigroups.
Recall that if a semigroup $(T(t))_{t\geq 0}$ is positive, i.e., if

$$(5.13) \qquad |T(t)f| \leq T(t)|f| \qquad (f \in E) ,$$

then its generator A satisfies Kato's inequality. We now obtain from Theorem 5.5: the semigroup consists of lattice homomorphisms (i.e., the equality holds in (5.13)) if and only if A satisfies Kato's equality. The precise statement is the following.

Corollary 5.8. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E with order continuous norm. The following assertions are equivalent.

(i) $(T(t))_{t\geq 0}$ is a lattice semigroup.

(ii) $f \in D(A)$ implies $|f| \in D(A)$ and $\mathrm{Re}((\widehat{\mathrm{sign}}\, \bar{f})Af) = A|f|$.

(iii) $f \in D(A)$ implies $|f|$, $\bar{f} \in D(A)$ and $\mathrm{Re}((\mathrm{sign}\, \bar{f})Af) = A|f|$ (Kato's equality).

Proof. The equivalence of (i) and (ii) follows directly from Thm. 5.5. If (i) holds, then A is local by Prop. 5.4.
Thus $(\mathrm{sign}\, \bar{f})Af = (\widehat{\mathrm{sign}}\, \bar{f})Af$ for all $f \in D(A)$ and so (iii) holds since (ii) is valid.
Assume now that (iii) holds. Then Kato's equality implies that $Af \in \{f\}^{dd}$ whenever $f \in D(A)_+$. Since $D(A)$ is a sublattice of E by hypothesis, this implies that A is local. Thus (ii) follows from (iii).

□

In the case when E is real this result can be reformulated.

Corollary 5.9. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a real Banach lattice E with order continuous norm. The following assertions are equivalent.

(i) $(T(t))_{t\geq 0}$ is a lattice semigroup.

(ii) $D(A)$ is a sublattice and A is local.

Proof. Assume that (ii) holds. Let $f \in D(A)$, and set $P_+ := P_{f^+}$ and $P_- := P_{f^-}$.
Then $(P_+)Af^- = (P_-)Af^+ = 0$ since A is local. Hence
$(\mathrm{sign}\, f)Af = (P_+ - P_-)Af = (P_+ - P_-)(Af^+ - Af^-) = (P_+)Af^+ + (P_-)Af^- = Af^+ + Af^- = A|f|$.