But this implies $s(\hat{\psi}_n) \leq \hat{p}$ in $\hat{M}''$. Since $\hat{M}_1^+$ is $\sigma(\hat{M}'',\hat{M}')$-dense in $(\hat{M}'')_1^+$ (Kaplanskys density theorem [Sakai (1971), 1.9.1] in combination with [Sakai (1971), 1.8.9 and 1.8.12]), there exists for all $n \in \mathbb{N}$ a net $(\hat{z}_{n,\gamma})$ in $\hat{M}_1^+$ such that

$$\sigma(\hat{M}'',\hat{M}')\text{-}\lim_\gamma \hat{z}_{n,\gamma} = s(\hat{\psi}_n) .$$

From [Sakai (1971), 1.7.8] and the considerations above we obtain that the net $(\hat{p}\hat{z}_{n,\gamma}\hat{p})$ converges to $s(\hat{\psi}_n)$ in the $\sigma(\hat{M}'',\hat{M}')$-topology. Therefore we may assume $\hat{z}_{n,\gamma} \in (\hat{M_p})_1^+$. In the following we denote by $\hat{\phi}$ the canonical image of ϕ in $(M_*)^\wedge$.

Since the projections $s(\hat{\psi}_n)$ are mutually orthogonal, there exists a real sequence (r_n), $0<r_n<1$, $\lim_n r_n = 0$ and $\hat{\phi}(s(\hat{\psi}_n)) \leq \frac{1}{2} r_n$. For all $n \in \mathbb{N}$ choose $\hat{z}_n \in (\hat{M_p})_1^+$ such that

$$|\langle\hat{\phi}, s(\hat{\psi}_n) - \hat{z}_n\rangle| \leq \frac{1}{2} r_n ,$$

$$|\langle\hat{\psi}_n, s(\hat{\psi}_n) - \hat{z}_n\rangle| \leq \frac{1}{2} r_n .$$

Hence $\hat{\phi}(\hat{z}_n) \leq r_n$ and $\hat{\psi}_n(\hat{z}_n) \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ let $(z_{n,k}) \in \hat{z}_n$ be a representing sequence in $(M_p)_1^+ = p(M_1^+)p$ (note that $\hat{M_p} = (M_p)^\wedge$) and fix $\mu \in \mathbb{R}_+$. Since $\mu R(\mu)^\wedge \hat{\psi}_n = \hat{\psi}_n$, $\hat{\phi}(\hat{z}_n) \leq r_n$ and $\hat{\psi}_n(\hat{z}_n) \geq \frac{1}{2}$ there exists for all $n \in \mathbb{N}$ an element $U_n \in \mathcal{U}$ such that for all $k \in U_n$:

(i)' $\phi(z_{n,k}) \leq r_n$,

(ii)' $\|(\mathrm{Id} - \mu R(\mu))\psi_{n,k}\| \leq r_n$,

(iii)' $\psi_{n,k}(z_{n,k}) \geq \frac{1}{2}$.

Inductively we find a sequence (z_n) in $(M_p)_1^+$ and a sequence of states (ϕ_n) in M_* such that for all $n \in \mathbb{N}$:

(i)'' $\lim_n \phi_n(z_n) = 0$,

(ii)'' $\lim_n \|(\mathrm{Id} - \mu R(\mu))\phi_n\| = 0$,

(iii)'' $\phi_n(z_n) \geq \frac{1}{2}$.

Since ϕ is faithful on M_p, condition (i)'' implies that $\lim_n z_n = 0$ in the $s^*(M_p,(M_p)_*)$-topology [Takesaki (1979), Proposition III.5.4].