Let us perform the same construction on the Hilbert space $H := L^2(\Gamma)\times L^2(\mathbb{R})$. For a fixed positive, non-zero function $k \in C_c(\mathbb{R})$ we define $T(t)$ on H as follows:

$$
\begin{aligned}
&T(t)(f,g) = (f_t,g_t) \quad \text{with} \\
(4.6)\quad &f_t(z) := f(z\cdot e^{it}) \quad (z \in \Gamma) \quad \text{and} \\
&g_t(x) := g(x+t) + \tfrac{1}{2\pi}\cdot\int_0^{2\pi} f(z\cdot e^{is})\, ds\cdot\int_x^{x+t} k(u)\, du .
\end{aligned}
$$

Then $(T(t))_{t\geq 0}$ is a positve semigroup on H and for the spectrum of the generator we obtain $\sigma(A) = i\mathbb{R}$, $P\sigma(A) = i\mathbb{Z}\setminus\{0\}$. In view of Cor.4.3(a) the semigroup cannot be bounded. (The explicit representation (4.6) only allows the estimate $\|T(t)\| \leq \sqrt{2} + t\cdot\|k\|_2$ $(t\geq 0)$.)

In the next proposition we show that for semigroups of lattice homomorphisms on L^1-spaces there is a spectral mapping theorem for the real part of the spectrum.

<u>Proposition</u> 4.5. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of lattice homomorphisms on an L^1-space and denote by A its generator. Then we have

$$(4.7)\quad \exp(t\sigma(A)\cap\mathbb{R}) = \sigma(T(t))\cap(0,\infty) \quad \text{for every} \quad t \geq 0 .$$

<u>Proof</u>. In view of A-III,6.2 it is enough to prove that the left hand side contains the set on the right.
Fix $t > 0$ and assume $r \in \sigma(T(t))$, $r > 0$ and let $\alpha := \frac{1}{t}\log r$.
At first we assume $r \in R(\sigma(T(t))$. Then by A-III,Thm.6.3 there exists $\beta \in \mathbb{R}$ such that $\alpha+i\beta \in R\sigma(A)$. By Lemma 4.1 either $\alpha+i\beta\mathbb{Z} \subset R\sigma(A)$ or $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \alpha\} \subset R\sigma(A)$. In both cases we have $\alpha \in \sigma(A)$.
Now we assume $r \in A\sigma(T(t))$. Then there exists a normalized sequence $(f_n) \subset E$ such that $\lim_{n\to\infty}\|T(t)f_n - rf_n\| = 0$. Since we have $|T(t)|f| - r|f|| = ||T(t)f| - r|f|| \leq |T(t)f - rf|$ $(f \in E)$ we may assume that (f_n) is a sequence in E_+.
Defining $g_n := \int_0^t e^{-\alpha s}T(s)f_n\, ds$ we have $g_n \in D(A)$ and $(A-\alpha)g_n = (A-\alpha)\int_0^t e^{-\alpha s}T(s)f_n\, ds = e^{-\alpha t}T(t)f_n - f_n = \frac{1}{r}(T(t)f_n - rf_n)$.
It follows that $\lim_{n\to\infty}\|(A-\alpha)g_n\| = 0$ and it remains to show that $\liminf_{n\to\infty}\|g_n\| > 0$. The latter assertion is a consequence of the following facts:

- Since f_n is positive and the norm is additive on E_+, we have $\|g_n\| = \int_0^t e^{-\alpha s}\|T(s)f_n\|\, ds$.
- The inequality $\|T(t)f\| \leq \|T(t-s)\|\|T(s)f\|$ implies $\|T(s)f\| \geq M^{-1}\|T(t)f\|$ for $0\leq s\leq t$, $f\in E$ and $M := \sup_{0\leq s\leq t}\|T(s)\|$.