Theorem 1.1. If A is the generator of a positive semigroup on $E = C_o(X)$, then $s(A) \in \sigma(A)$ unless $s(A) = -\infty$. In case X is compact we always have $s(A) > -\infty$.

Proof. We suppose $\sigma(A) \neq \emptyset$ (i.e., $s(A) > -\infty$) and assume $s(A) \notin \sigma(A)$. Then there exist $\varepsilon > 0$ and α_o, $\beta_o \in \mathbb{R}$ such that

$$(1.1) \quad [s(A)-\varepsilon,\infty) \subset \rho(A)\ ,\ \mu_o := \alpha_o + i\beta_o \in \sigma(A) \ \text{ and } \ \alpha_o > s(A)-\varepsilon\ .$$

Now we choose $\lambda_o \in \mathbb{R}$ large enough such that

$$(1.2) \quad |\lambda_o - \mu_o| < \lambda_o - (s(A) - \varepsilon)\ ,$$

and, in addition, such that $\lambda_o > \omega(A)$. Then the resolvent $R(\lambda_o,A)$ is a positive bounded operator, hence its spectral radius $r(R(\lambda_o,A))$ is a spectral value. From A-III,Prop.2.5 it follows that

$$(1.3) \quad \lambda_o - r(R(\lambda_o,A))^{-1} \in \sigma(A) \ \text{ and } \ r(R(\lambda_o,A)) \geq |\lambda_o - \mu_o|^{-1}\ .$$

This and (1.2) implies that $\lambda_o - r(R(\lambda_o,A))^{-1}$ is a real spectral value which is greater than $s(A) - \varepsilon$. We have derived a contradiction to (1.1) and thus have proved the first statement of the theorem. To establish the second statement we recall that $\lim_{\lambda\to\infty}\lambda R(\lambda,A)f = f$ for every $f \in E$. In particular, for $f = 1_X$ we find a (large) $\lambda_o \in \mathbb{R}$ such that

$$(1.4) \quad \lambda_o R(\lambda_o,A)1_X \geq 1/2\cdot 1_X \quad \text{hence} \quad R(\lambda_o,A)1_X \geq (2\lambda_o)^{-1}\cdot 1_X\ .$$

We may assume $\lambda_o > \omega(A)$ then $R(\lambda_o,A) \geq 0$, and iterating (1.4) we obtain

$$(1.5) \quad R(\lambda_o,A)^n 1_X \geq (2\lambda_o)^{-n}\cdot 1_X > 0 \ \text{ for every } n \in \mathbb{N}\ .$$

It follows that $\|R(\lambda_o,A)^n\| \geq (2\lambda_o)^{-n}$ and therefore

$$(1.6) \quad r(R(\lambda_o,A)) = \lim_{n\to\infty}\|R(\lambda_o,A)^n\|^{1/n} \geq (2\lambda_o)^{-1} > 0\ .$$

Thus $\sigma(R(\lambda_o,A))$ contains non-zero spectral values which in view of A-III,Prop.2.5 is equivalent to $\sigma(A) \neq \emptyset$.

□

The following examples show that the spectrum may be empty in case X is not compact or if the semigroup is not positive.

Examples 1.2.(a) Consider $X = [0,1)$ and $(T(t))$ on $C_o(X)$ given by