CHAPTER A-III

SPECTRAL THEORY

by
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1. INTRODUCTION

In this chapter we start a systematic analysis of the spectrum of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on a complex Banach space E. By the spectrum of the semigroup we understand the spectrum $\sigma(A)$ of the generator A of $\mathcal{T}$. In particular we are interested in precise relations between $\sigma(A)$ and $\sigma(T(t))$. The heuristic formula

$$" \; T(t) = e^{tA} \; "$$

serves as a leitmotiv and suggests relations of the form

$$" \; \sigma(T(t)) = e^{t\sigma(A)} = \{ e^{t\lambda} : \lambda \in \sigma(A) \} \; " ,$$

called 'spectral mapping theorem'. These - or similar - relations will be of great use in Chapter IV and enable us to determine the asymptotic behavior of the semigroup $\mathcal{T}$ by the spectrum of the generator. As a motivation as well as a preliminary step we concentrate here on the spectral radius

$$r(T(t)) := \sup \{ |\lambda| : \lambda \in \sigma(T(t)) \} , \; t \geq 0 \tag{1.1}$$

and show how it is related to the spectral bound

$$s(A) := \sup \{ \operatorname{Re}\lambda : \lambda \in \sigma(A) \} \tag{1.2}$$

of the generator A and to the growth bound

$$\omega := \inf \{ w \in \mathbb{R} : \|T(t)\| \leq M_w \cdot e^{wt} \text{ for all } t \geq 0 \text{ and suitable } M_w \} \tag{1.3}$$

of the semigroup $\mathcal{T} = (T(t))_{t\geq 0}$. (Recall that we sometimes write $\omega(\mathcal{T})$ or $\omega(A)$ instead of ω). The Examples 1.3 and 1.4 below illustrate the main difficulties to be encountered.

Proposition 1.1. Let ω be the growth bound of the strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$. Then

$$r(T(t)) = e^{\omega t} \tag{1.4}$$

for every $t \geq 0$.

Proof. From A-I,(1.1) we know that

$$\omega(\mathcal{T}) = \lim_{t\to\infty} \tfrac{1}{t}\log\|T(t)\| \ .$$

Since the spectral radius of $T(t)$ is given as

$$r(T(t)) = \lim_{n\to\infty} \|T(nt)\|^{1/n}$$

we obtain for $t > 0$

$$\begin{aligned} r(T(t)) &= \lim_{n\to\infty} \exp(\ t(nt)^{-1} \log \|T(nt)\|\ ) \\ &= e^{\omega t} \ . \end{aligned}$$

□

It was shown in A-I,Prop.1.11 that the spectral bound $s(A)$ is always dominated by the growth bound ω and therefore $e^{s(A)t} \leq r(T(t))$. If the above mentioned spectral mapping theorem holds - as is the case for bounded generators (e.g., see Thm. VII.3.11 of Dunford-Schwartz (1958)) we obtain the equality

$$e^{s(A)t} = r(T(t)) = e^{\omega t}$$

hence $s(A) = \omega$. Therefore the following corollary is a consequence of the definitions of $s(A)$ and ω .

Corollary 1.2. Consider the semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ generated by some bounded linear operator $A \in \mathcal{L}(E)$. If $\mathrm{Re}\lambda < 0$ for each $\lambda \in \sigma(A)$ then $\lim_{t\to\infty}\|T(t)\| = 0$.

Through this corollary we have re-established a famous result of Liapunov which assures that the solutions of the linear Cauchy problem

$$\dot{x}(t) = Ax(t) \ , \ x(0) = x_o \in \mathbb{C}^n \quad \text{and} \quad A = (a_{ij})_{n\times n}$$

are 'stable' ; i.e., they converge to zero as $t \to \infty$, if the real parts of all eigenvalues of the matrix A are smaller than zero.
For unbounded generators the situation is much more difficult and $s(A)$ may differ drastically from ω .

Example 1.3. (Banach function space, Greiner-Voigt-Wolff (1981)) Consider the Banach space E of all complex valued continuous functions on $\mathbb{R}_+$ which vanish at infinity and are integrable for $e^x dx$, i.e.

$$E := C_o(\mathbb{R}_+) \cap L^1(\mathbb{R}_+,\ e^x dx)$$

endowed with the norm

$$\|f\| := \|f\|_\infty + \|f\|_1 = \sup\{\ |f(x)|\ :\ x \in \mathbb{R}_+\ \} + \int_0^\infty |f(x)|e^x\, dx \ .$$

The translation semigroup

$$T(t)f(x) := f(x+t)$$

is strongly continuous on E and one shows as in A-I,2.4 that its generator is given by

$$Af = f' \quad , \quad D(A) = \{ f \in E : f \in C^1(\mathbb{R}_+) , f' \in E \} .$$

First we observe that $\|T(t)\| = 1$ for every $t \geq 0$, hence $\omega(T) = 0$. Moreover it is clear that λ is an eigenvalue of A as soon as $\mathrm{Re}\lambda < -1$ (in fact : the function

$$x \to \varepsilon_\lambda(x) := e^{\lambda x}$$

belongs to $D(A)$ and is an eigenvector of A), hence $s(A) \geq -1$. For $f \in E$, $\mathrm{Re}\lambda > -1$,

$$\|\cdot\|_1\text{-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)f \, ds$$

exists since $\|T(s)f\|_1 \leq e^{-s}\|f\|_1$, $s \geq 0$, and

$$\|\cdot\|_\infty\text{-}\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)f \, ds$$

exists since $\int_0^\infty e^x |f(x)| \, dx < \infty$. Therefore $\int_0^\infty e^{-\lambda s} T(s)f \, ds$ exists in E for every $f \in E$, $\mathrm{Re}\lambda > -1$. As we observed in A-I,Prop.1.11 this implies $\lambda \in \rho(A)$. Therefore $T = (T(t))_{t\geq 0}$ is a semigroup having $s(A) = -1$ but $\omega(T) = 0$.

Example 1.4. (Hilbert space, Zabczyk (1975)) For every $n \in \mathbb{N}$ consider the n-dimensional Hilbert space $E_n := \mathbb{C}^n$ and operators $A_n \in L(E_n)$ defined by the matrices

$$A_n = \begin{pmatrix} 0 & 1 & \cdot & \cdot & 0 \\ \cdot & 0 & 1 & & \cdot \\ & \cdot & \cdot & \cdot & \\ \cdot & & & \cdot & 1 \\ 0 & \cdot & & \cdot & 0 \end{pmatrix}_{n\times n} .$$

These matrices are nilpotent and therefore $\sigma(A_n) = \{0\}$. The elements $x_n := n^{-1/2}(1, \ldots , 1) \in E_n$ satisfy the following properties :

(i) $\|x_n\| = 1$ for every $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \|A_n x_n - x_n\| = 0$,

(iii) $\lim_{n\to\infty} \|\exp(tA_n)x_n - e^t x_n\| = 0$.

Consider now the Hilbert space $E := \oplus_{n\in\mathbb{N}} E_n$ and the operator $A := (A_n + 2\pi i n)_{n\in\mathbb{N}}$ with maximal domain in E . Analogously we define a semigroup $T = (T(t))_{t\geq 0}$ by

$$T(t) := (e^{2\pi i n t}\exp(tA_n))_{n\in\mathbb{N}} .$$

Since $\|\exp(tA_n)\| \leq e^t$ for every $n \in \mathbb{N}$, $t \geq 0$, and since $t \to T(t)x$ is continuous on each component E_n it follows that $\mathcal{T}$ is strongly continuous. Its generator is the operator A as defined above.
For $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda > 0$, we have $\lim_{n\to\infty} \|R(\lambda-2\pi in, A_n)\| = 0$, hence

$$(R(\lambda, A_n+2\pi in))_{n\in\mathbb{N}} = (R(\lambda-2\pi in, A_n))_{n\in\mathbb{N}}$$

is a bounded operator on E representing the resolvent $R(\lambda,A)$. Therefore we obtain $s(A) \leq 0$. On the other hand, each $2\pi in$ is an eigenvalue of A, hence $s(A) = 0$.
Take now $x_n \in E_n$ as above and consider the sequence $(x_n)_{n\in\mathbb{N}}$. From (iii) it follows that for $t > 0$ the number e^t is an approximate eigenvalue of $T(t)$ with approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ (see Def.2.1 below). Therefore $e^t \leq r(T(t)) \leq \|T(t)\|$ and hence $\omega(\mathcal{T}) \geq 1$. On the other hand, it is easy to see that $\|T(t)\| = e^t$, hence $\omega(\mathcal{T}) = 1$.
Finally if we take $S(t) := e^{-t/2}\cdot T(t)$ we obtain a semigroup having spectral bound $-\frac{1}{2}$ but such that $\lim_{t\to\infty} \|S(t)\| = \infty$ in contrast with Cor. 1.2 .

These examples show that neither the conclusion of Cor.1.2 , i.e. '$s(A) < 0$ implies stability', nor the 'spectral mapping theorem'

$$\sigma(T(t)) = \exp(t\cdot\sigma(A))$$

is valid for arbitrary strongly continuous semigroups. A careful analysis of the general situation will be given in Section 6 below, but we will first develop systematically the necessary spectral theoretic tools for unbounded operators.

2. THE FINE STRUCTURE OF THE SPECTRUM

As usual, with a closed linear operator A with dense domain $D(A)$ in a Banach space E, we associate its spectrum $\sigma(A)$, its resolvent set $\rho(A)$ and its resolvent

$$\lambda \to R(\lambda,A) := (\lambda - A)^{-1}$$

which is a holomorphic map from $\rho(A)$ into $\mathcal{L}(E)$. In contrast to the finite dimensional situation, where a linear operator fails to be surjective if and only if it fails to be injective, we now have to distinguish different cases of 'non-invertibility' of $\lambda - A$. This distinction gives rise to a subdivision of $\sigma(A)$ into different subsets. We point out that these subsets need not be disjoint, but our definition seems to be justified by the fact that for each of the following subsets of $\sigma(A)$ there exist canonical constructions converting the corresponding spectral values into eigenvalues (see Prop. 2.2.ii and Prop. 4.5 below).

Definition 2.1. For a closed, densely defined, linear operator A with domain $D(A)$ in the Banach space E denote by the

(i) point spectrum $P\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not injective.

(ii) approximate point spectrum $A\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not injective or $(\lambda - A)D(A)$ is not closed in E.

(iii) residual spectrum $R\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $(\lambda - A)D(A)$ is not dense in E.

From these definitions it follows that $\lambda \in \mathbb{C}$ is an eigenvalue of A, i.e. $\lambda \in P\sigma(A)$, if and only if there exists an eigenvector $0 \neq f \in D(A)$ such that $Af = \lambda f$. It follows from the Open Mapping Theorem that $\lambda \in A\sigma(A)$ if and only if λ is an approximate eigenvalue, i.e. there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset D(A)$, called an approximate eigenvector, such that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|Af_n - \lambda f_n\| = 0$.
Clearly we have $P\sigma(A) \subset A\sigma(A)$ and $\sigma(A) = A\sigma(A) \cup R\sigma(A)$ where the union need not be disjoint.

The following proposition is a first indication that the subdivision we made implies nice properties.

Proposition 2.2. For a closed, densely defined, linear operator $(A,D(A))$ in a Banach space E the following holds:

(i) The topological boundary $\partial\sigma(A)$ of $\sigma(A)$ is contained in $A\sigma(A)$.

(ii) $R\sigma(A) = P\sigma(A')$ for the adjoint operator A' on E'.

Proof. (i) Take $\lambda_o \in \partial\sigma(A)$ and $\lambda_n \in \rho(A)$ such that $\lambda_n \to \lambda_o$. Since $\|R(\lambda_n,A)\| \geq (\mathrm{dist}(\lambda_n,\sigma(A)))^{-1}$ (see Prop. 2.5.(ii)), by the uniform boundedness principle we find $f \in E$ such that

$$\lim_{n\to\infty} \|R(\lambda_n,A)f\| = \infty .$$

Define $g_n \in D(A)$ by

$$g_n := \|R(\lambda_n,A)f\|^{-1} R(\lambda_n,A)f$$

and use the identity

$$(\lambda_o - A)g_n = (\lambda_o - \lambda_n)g_n + (\lambda_n - A)g_n$$

to show that $(g_n)_{n\in\mathbb{N}}$ is an approximate eigenvector corresponding to λ_o.

(ii) This is a simple consequence of the Hahn-Banach theorem.

□

In order to illuminate the above definitons we now return to the Standard Examples introduced in Section 2 of A-I and discuss the fine structure of the spectrum of these strongly continuous semigroups, i.e. of their generators and their semigroup operators.

2.3 The Spectrum of Multiplication Semigroups.

Take $E = C_o(X)$ for some locally compact space X and take a continuous function $q : X \to \mathbb{C}$ whose real part is bounded above. As observed in A-I,2.3 the multiplication operator

$$M_q : f \to q\cdot f$$

with maximal domain $D(M_q)$ generates the multiplication semigroup

$$T(t)f := e^{tq}\cdot f \quad , \ f \in E .$$

Since M_q is bounded if and only if q is bounded we conclude that M_q is invertible (with bounded inverse $M_{1/q}$) if and only if

$$0 \notin \overline{\{q(x) : x \in X\}} .$$

Therefore we obtain

$$\sigma(M_q) = \overline{q(X)} = \overline{\{q(x) : x \in X\}}$$

and

$$\sigma(T(t)) = \overline{\{\exp(tq(x)) : x \in X\}} .$$

In particular the following 'weak spectral mapping theorem' is valid:

$$\sigma(T(t)) = \overline{\exp(t\sigma(M_q))} .$$

In addition we observe that to each spectral value of A (resp. of $T(t)$) there exists an approximate eigenvector and hence

$$\sigma(A) = A\sigma(A) \qquad \text{and} \qquad \sigma(T(t)) = A\sigma(T(t)).$$

Since each Dirac functional is an eigenvector for the adjoint multiplication operator we obtain

$$q(X) \subset R\sigma(M_q) \quad \text{and} \quad e^{tq(X)} \subset R\sigma(T(t)) .$$

The eigenvalues of M_q can be characterized as follows:
$\lambda \in P\sigma(M_q)$ if and only if the set $\{x \in X : q(x) = \lambda\}$ has non empty interior (analogously for $P\sigma(T(t))$). For example, it follows that $P\sigma(M_q) = \emptyset$ for $E = C_o(\mathbb{R}_+)$ and $q(x) = -x$, $x \in \mathbb{R}_+$.

On $E = L^p(X,\Sigma,\mu)$ analogous results are valid, but their exact formulation - using the notion 'essential range', see Goldstein (1985a) - is left to the reader.

2.4 The Spectrum of Translation Semigroups.
First we consider the translation semigroup

$$T(t)f(x) := f(x+t)$$

on $E = C_o(\mathbb{R}_+)$ (or $L^p(\mathbb{R}_+)$, see A-I,2.4). Its generator A is the first derivative and for every $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda < 0$, the function $\varepsilon_\lambda : x \to e^{\lambda x}$ belongs to $D(A)$ and satisfies

$$A\varepsilon_\lambda = \lambda\varepsilon_\lambda ,$$

hence $\lambda \in P\sigma(A)$. Since $\mathcal{T} = (T(t))_{t\geq 0}$ is a contraction semigroup it follows that $\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\}$ and $i\mathbb{R} \subset A\sigma(A)$ (use Prop. 2.2.(i) or show directly that $f_n(x) = e^{i\alpha x}\cdot e^{-x/n}$ defines an approximate eigenvector for $i\alpha$, $\alpha \in \mathbb{R}$). Using the same functions one obtains

$$P\sigma(T(t)) = \{ e^{\lambda t} : \mathrm{Re}\lambda < 0 \} = \{ z \in \mathbb{C} : |z| < 1 \}$$

and $\sigma(T(t)) = \{ z \in \mathbb{C} : |z| \leq 1 \}$ for every $t > 0$.
In the case of the translation group on $E = C_o(\mathbb{R})$ one has that $\sigma(A) \subset i\mathbb{R}$. As above one obtains approximate eigenvectors for every $\alpha \in \mathbb{R}$ from $f_n(x) = e^{i\alpha x}.e^{-|x|/n}$, hence

$$\sigma(A) = A\sigma(A) = i\mathbb{R}.$$

The generator A of the nilpotent translation semigroup A-I,2.6 has empty spectrum by A-I,Prop.1.11 . The resolvent is given by
$R(\lambda,A)f(x) = e^{\lambda x}\int_x^\tau e^{-\lambda s} f(s)\, ds$ $(f \in L^p([0,\tau]$, $\lambda \in \mathbb{C})$.
Finally the generator of the periodic translation group from A-I,2.5 on $E = \{ f \in C[0,1] : f(0) = f(1) \}$ has point spectrum

$$P\sigma(A) = 2\pi i\mathbb{Z}$$

with eigenfunctions $\varepsilon_n(x) := \exp(2\pi i n x)$. In Section 5 we show that

$$\sigma(A) = 2\pi i\mathbb{Z} .$$

We now return to the general theory and recall from Corollary 1.2 that it is very useful (e.g., for stability theory) to be able to convert spectral values of the generator A into spectral values of the semigroup operator $T(t)$ and vice versa. As shown in Examples 1.3 and 1.4 this is not possible in general. Therefore we tackle first a much easier 'spectral mapping theorem': the relation between $\sigma(A)$ and $\sigma(R(\lambda_o))$, where $R(\lambda_o) := R(\lambda_o,A)$ for some $\lambda_o \in \rho(A)$.

Proposition 2.5. Let $(A,D(A))$ be a densely defined closed linear operator with non-empty resolvent set $\rho(A)$. For each $\lambda_o \in \rho(A)$ the following assertions hold :

(i) $\sigma(R(\lambda_o)) \setminus \{0\} = (\lambda_o - \sigma(A))^{-1}$,
In particular, $r(R(\lambda_o)) = \left(\operatorname{dist}(\lambda_o,\sigma(A))\right)^{-1}$.

(ii) Analogous statements hold for the point-, approximate point-, residual spectra of A and $R(\lambda_o,A)$.

(iii) The point α is isolated in $\sigma(A)$ if and only if $(\lambda_o-\alpha)^{-1}$ is isolated in $\sigma(R(\lambda_o))$. In that case the residues (resp., the pole orders) in α and in $(\lambda_o-\alpha)^{-1}$ coincide.

Proof. (i) is well known. It can be found for example in [Dunford-Schwartz (1958), VII.9.2].
(ii) We show that $\alpha \in A\sigma(A)$ if $(\lambda_o-\alpha)^{-1} \in A\sigma(R(\lambda_o))$ and leave the proof of the remaining statements to the reader. Take $(f_n)_{n\in\mathbb{N}} \subset E$ such that $\|f_n\| = 1$, $\|(\lambda_o-\alpha)^{-1}f_n - R(\lambda_o,A)f_n\| \to 0$ and $\|R(\lambda_o,A)f_n\| \geq \frac{1}{2}|\lambda_o - \alpha|^{-1}$. Define

$$g_n := \|R(\lambda_o,A)f_n\|^{-1}\cdot R(\lambda_o,A)f_n \in D(A)$$

and deduce from

$$\begin{aligned}(\alpha-A)g_n &= \|R(\lambda_o,A)f_n\|^{-1}\cdot[(\lambda_o-A) - (\lambda_o-\alpha)]R(\lambda_o,A)f_n \\ &= \|R(\lambda_o,A)f_n\|^{-1}\cdot(\lambda_o-\alpha)[(\lambda_o-\alpha)^{-1} - R(\lambda_o,A)]f_n\end{aligned}$$

that (g_n) is an approximate eigenvector of A to the eigenvalue α.
(iii) Take a circle Γ with center α and sufficiently small radius. Then the residue P of $R(.,A)$ at α is

$$\begin{aligned}P &= \frac{1}{2\pi i}\int_\Gamma R(z,A)\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma (\lambda_o-z)^{-2}R((\lambda_o-z)^{-1},R(\lambda_o,A))\,dz \\ &\quad - \frac{1}{2\pi i}\int_\Gamma(\lambda_o-z)^{-1}dz \text{ , (use \$\$).}\end{aligned}$$

If λ_o lies in the exterior of Γ the second integral is zero. The substitution $\tilde{z} := (\lambda_o - z)^{-1}$ yields a path $\tilde{\Gamma}$ around $(\lambda_o-\alpha)^{-1}$ and we obtain

$$P = \frac{1}{2\pi i} \int_{\Gamma} R(\tilde{z}, R(\lambda_o, A))\, d\tilde{z} ,$$

which is the residue of $R(.,R(\lambda_o,A))$ at $(\lambda_o-\alpha)^{-1}$. The final assertion on the pole order follows from the identities

$$V_{-n} = ((\lambda_o-\alpha)^{-1} R(\lambda_o,A))^{n-1} U_{-n} , \quad n \in \mathbb{N} ,$$

where U_n, resp. V_n stand for the n-th coefficient in the Laurent series of $R(.,A)$, resp. $R(.,R(\lambda_o,A))$ at α, resp. $(\lambda_o-\alpha)^{-1}$. This has already been proved for $n = 1$ and follows for $n > 1$ by induction, using the relations

$$U_{-n-1} = (A - \alpha) U_{-n} \quad \text{and} \quad V_{-n-1} = (R(\lambda_o,A) - (\lambda_o-\alpha)^{-1}) V_{-n} .$$

□

3. SPECTRAL DECOMPOSITION

In the next two sections we develop some important techniques for our further investigation of semigroups and their generators. Even though these methods are well known (compare, e.g. Section VII.3 of Dunford-Schwartz (1958)) or rather technical, it is useful to present them in a coherent way.

Our interest in this section is the following : Let E be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a strongly continuous semigroup with generator A. Suppose that the spectrum $\sigma(A)$ splits into the disjoint union of two closed subsets σ_1 and σ_2. Does there exist a corresponding decomposition of the space E and the semigroup $\mathcal{T}$?

In the following definition we explain what we understand by "corresponding decomposition".

Definition 3.1. Assume that $\sigma(A)$ is the disjoint union

$$\sigma(A) = \sigma_1 \cup \sigma_2$$

of two non-empty closed subsets σ_1, σ_2. A decomposition

$$E = E_1 \oplus E_2$$

of E into the direct sum of two non-trivial closed $\mathcal{T}$-invariant subspaces is called a spectral decomposition corresponding to $\sigma_1 \cup \sigma_2$ if the spectrum $\sigma(A_i)$ of the generator A_i of $\mathcal{T}_i := (T(t)_{|E_i})_{t\geq 0}$ coincides with σ_i for $i = 1, 2$.

For a better understanding of the above definition we recall that to every direct sum decomposition $E = E_1 \oplus E_2$ there corresponds a continuous projection $P \in \mathcal{L}(E)$ such that $PE = E_1$ and $P^{-1}(0) = E_2$. Moreover, the subspaces E_1, E_2 are $\mathcal{T}$-invariant if and only if P commutes with the semigroup $\mathcal{T}$, i.e. $T(t)P = PT(t)$ for every $t \geq 0$. In this case it follows that the domain $D(A)$ of the generator A splits analogously and $D(A) \cap E_i$ is the domain $D(A_i)$ of the generator A_i of the restricted semigroup $\mathcal{T}_i$, $i = 1, 2$. We write

$$A = A_1 \oplus A_2 ,$$

say that " A commutes with P " and call P a <u>spectral projection</u>. In terms of the generator A this means that for $f \in D(A)$ we have $Pf \in D(A)$ and $APf = PAf$.
The existence of such projections is very helpful since it reduces the semigroup $\mathcal{T}$ into two (possibly simpler) semigroups $\mathcal{T}_1$, $\mathcal{T}_2$ such that

$$\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \quad \text{and} \quad \sigma(T(t)) = \sigma(T_1(t)) \cup \sigma(T_2(t)) .$$

For example, in some cases (see Theorem 3.3 below) it can be shown that one of the reduced semigroups has additional properties.

In order to achieve such decompositions we will assume that $\sigma(A)$ decomposes into sets σ_1 and σ_2 and will then try to find a corresponding spectral projection. Unfortunately such spectral decompositions do not exist in general.

<u>Example</u> 3.2. Take the rotation semigroup from A-I,2.4 on the Banach space $L^p(\Gamma)$, $1 \leq p < \infty$, $\tau = 2\pi$. It was stated in 2.4 and will be proved in Section 5 that its generator A has spectrum

$$\sigma(A) = P\sigma(A) = i\mathbb{Z} ,$$

where $\varepsilon_k(z) := z^k$ spans the eigenspace corresponding to ik, $k \in \mathbb{Z}$. Now, $\sigma(A)$ is the disjoint union of $\sigma_1 := \{0,i,2i,\ldots\}$ and $\sigma_2 := \{-i,-2i,\ldots\}$. By a result of M. Riesz there is no projection $P \in \mathcal{L}(L^1(\Gamma))$ satisfying $P\varepsilon_k = \varepsilon_k$ for $k \geq 0$, $P\varepsilon_k = 0$ for $k < 0$ (see Lindenstrauss-Tzafriri (1979), p.165), hence there is no spectral decomposition of $L^1(\Gamma)$ corresponding to σ_1, σ_2. On the other hand, for $L^p(\Gamma)$, $1 < p < \infty$, such a spectral projection exists (l.c., 2.c.15). As long as $p \neq 2$ we can always decompose $\sigma(A)$ into suitable subsets admitting no spectral decomposition (l.c., remark before 2.c.15). Clearly, for $p = 2$ such spectral decompositions always exist.

In the above example both subsets σ_1 , σ_2 of $\sigma(A)$ are unbounded. But as soon as one of these sets is bounded a corresponding spectral decomposition can always be found.

Theorem 3.3. Let $\mathcal{T}$ be a strongly continuous semigroup on a Banach space E and assume that the spectrum $\sigma(A)$ of the generator A can be decomposed into the disjoint union of two non-empty closed subsets σ_1 , σ_2 . If σ_1 is compact then there exists a unique corresponding spectral decomposition $E = E_1 \oplus E_2$ such that the restricted semigroup $\mathcal{T}_1$ has a bounded generator.

Proof. We assume the reader to be familiar with the spectral decomposition theorem for bounded operators (see e.g. [Dunford-Schwartz (1958), p.572]) and apply the "spectral mapping theorem" for the resolvent (A-III,Thm.2.5) in order to decompose $R(\lambda,A)$ instead of A : For $\lambda_0 > \omega(\mathcal{T})$ it follows from A-III,Thm.2.5 that $\sigma(R(\lambda_0,A)) \setminus \{0\} = (\lambda_0 - \sigma(A))^{-1}$. From $\sigma(A) = \sigma_1 \cup \sigma_2$ we obtain a decomposition of $\sigma(R(\lambda_0,A)) \setminus \{0\}$ into

$$\tau_1 := (\lambda_0-\sigma_1)^{-1} \quad , \quad \tau_2 := (\lambda_0-\sigma_2)^{-1} .$$

Since σ_1 is compact the set τ_1 is compact and does not contain 0 . Only in the case that σ_2 is unbounded the point 0 will be an accumulation point of τ_2 . Therefore $\sigma(R(\lambda_0,A)) \cup \{0\}$ is the disjoint union of the closed sets τ_1 and $\tau_2 \cup \{0\}$.
Take now P to be the spectral projection of $R(\lambda_0,A)$ corresponding to this decomposition. Then P commutes with $R(\lambda_0,A)$ (by definition), with $R(\lambda,A)$ for every $\lambda > \omega(\mathcal{T})$ (use the series representation of the resolvent), with $T(t)$ for each $t \geq 0$ (use A-II, Prop.1.10) and therefore with the generator A (in the sense explained above). In particular, we obtain

$$R(\lambda_0,A)P = R(\lambda_0,A_1) \quad , \quad R(\lambda_0,A)(Id-P) = R(\lambda_0,A_2)$$

for the generator A_1 of $\mathcal{T}_1 = (T(t)P)_{t\geq 0}$ and A_2 of $\mathcal{T}_2 = (T(t)(Id-P))_{t\geq 0}$. Applying the Spectral Mapping Theorem 2.5 we conclude

$$\sigma(A_1) = \sigma_1 \quad \text{and} \quad \sigma(A_2) = \sigma_2 ,$$

i.e., P is a spectral projection corresponding to σ_1 , σ_2 .
Finally, the above spectral decomposition of $R(\lambda_0,A)$ is unique and satisfies $0 \notin \sigma(R(\lambda_0,A_1))$. Therefore $R(\lambda_0,A_1)^{-1} = (\lambda_0-A_1)$ is bounded.

$\square$

Example. If we do not require T_1 to be uniformly continuous the above spectral decomposition need not be unique :
Consider a decomposition $E = E_1 \oplus E_2$ and add a direct summand E_3 with a strongly continuous semigroup T_3 whose generator A_3 has empty spectrum (e.g. A-I,Example 2.6). Then still $\sigma(A) = \sigma_1 \cup \sigma_2$ but $E_1 \oplus (E_2 \oplus E_3)$ and $(E_1 \oplus E_3) \oplus E_2$ are two different spectral decompositions corresponding to σ_1, σ_2.

The importance of the above theorem stems from the fact that T_1 has a bounded generator and therefore is easy to deal with. In particular the asymptotic behavior of T_1 can be deduced from the location of σ_1.

Corollary 3.4. Assume that $\sigma(A)$ splits into non-empty closed sets σ_1, σ_2 where σ_1 is compact and consider the corresponding spectral decomposition $E = E_1 \oplus E_2$ for which T_1 is uniformly continuous. For all constants $v, w \in \mathbb{R}$ satisfying

$$v < \inf\{\mathrm{Re}\,\lambda : \lambda \in \sigma_1\} \quad \text{and} \quad \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma_1\} < w$$

there exist $m, M \geq 1$ such that

$$m \cdot e^{vt}\|f\| \leq \|T_1(t)f\| \leq M \cdot e^{wt}\|f\|$$

for every $f \in E_1$, $t \geq 0$.

Proof. Since the generator A_1 of T_1 is bounded we have $T_1(t) = \exp(tA_1)$ and $\sigma(T_1(t)) = \exp(t\sigma(A_1))$. Therefore by the remark following Prop.1.1 the spectral bound $s(A_1)$ coincides with the growth bound $\omega(T_1)$ and we have the upper estimate. The lower estimate is obtained by applying the same reasoning to $-A_1$ which generates the semigroup $(T_1(t)^{-1})_{t\geq 0}$ on E_1.

□

It is clear from Examples 1.3, 1.4 that no norm estimates for $(T_2(t))_{t\geq 0}$ can be obtained from the location of σ_2. Only by adding appropriate hypotheses we will achieve spectral decompositions admitting norm estimates on both components (see A-III,6.6).
Another way of obtaining such norm estimates is by constructing spectral decompositions starting from a semigroup operator $T(t_o)$ (instead of A resp. $R(\lambda,A)$, as in Thm.3.3).

Corollary 3.5. If $\sigma(T(t_0)) = \tau_1 \cup \tau_2$ for two non-empty, closed, disjoint sets τ_1, τ_2 and if P is the spectral projection corresponding to $T(t_o)$ and τ_1, τ_2, then $\sigma(A)$ splits into closed subsets σ_1, σ_2 and P is the corresponding spectral projection for $\mathcal{T}$ and σ_1, σ_2.

<u>Proof</u>. The spectral projection P of $T(t_o)$ is obtained by integrating $R(\lambda, T(t_o))$ (see e.g. [Dunford-Schwartz (1958), Section VII.3]). Since every $T(t)$, $t \geq 0$, commutes with $T(t_o)$ it must commute with $R(\lambda, T(t_o))$, hence with P. The statement on the decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$ follows from the Spectral Inclusion Theorem 6.2 below.

□

This decomposition can be applied as follows to the study of the asymptotic behavior of $\mathcal{T}$: In the situation of Cor.3.5 assume

$$\sup\{|\lambda| : \lambda \in \tau_2\} < \alpha < \inf\{|\lambda| : \lambda \in \tau_1\}.$$

If we set $\beta := (\log\alpha)/t_o$ and use [Pazy(1984),Chap.I,Thm.6.5] we obtain $\omega(\mathcal{T}_2) < \beta$ and $\omega(\mathcal{T}_1^{-1}) < \beta$ by Prop.1.1. Therefore we have constants m, $M \geq 1$ such that

$$\|T(t)f\| \leq M \cdot e^{\beta t}\|f\| \quad \text{for } f \in E_2,$$
$$\|T(t)f\| \geq m \cdot e^{\beta t}\|f\| \quad \text{for } f \in E_1.$$

As nice as they might look results of this type are unsatisfactory : we need information on the semigroup in order to estimate its asymptotic behavior. In Chapter IV we will try to obtain such results by exploiting information about the generator only.

3.6 <u>Isolated singularities and poles</u>.

In case that λ_o is an isolated point of $\sigma(A)$ the holomorphic function $\lambda \to R(\lambda, A)$ can be expanded as a Laurent series $R(\lambda, A) = \sum_{n=-\infty}^{+\infty} U_n(\lambda - \lambda_o)^n$ for $0 < |\lambda - \lambda_o| < \delta$ and some $\delta > 0$. The coefficients U_n are bounded linear operators given by

$$(3.1) \quad U_n = \frac{1}{2\pi i}\int_\Gamma (z-\lambda_o)^{-(n+1)} R(z,A)\, dz \quad , \; n \in \mathbb{Z},$$

where $\Gamma = \{z \in \mathbb{C} : |z - \lambda_o| = \delta/2\}$.

The coefficient U_{-1} is the spectral projection corresponding to the spectral set $\{\lambda_o\}$ (see Def.3.1), it is called the <u>residue</u> of $R(\cdot, A)$ at λ_o, and will be denoted by P. From (3.1) one deduces

$$(3.2) \quad U_{-(n+1)} = (A - \lambda_o)^n \circ P \quad \text{and}$$
$$U_{-(n+1)} \circ U_{-(m+1)} = U_{-(n+m+1)} \quad \text{for } n, m \geq 0.$$

If there exists $k > 0$ such that $U_{-k} \neq 0$ while $U_{-n} = 0$ for all $n > k$ the point λ_o is called a <u>pole of $R(\cdot,A)$ of order k</u>. In view of (3.2) this is true if $U_{-k} \neq 0$ and $U_{-(k+1)} = 0$. In this case one can retrieve U_{-k} as

(3.3) $U_{-k} = \lim_{\lambda\to\lambda_o} (\lambda - \lambda_o)^k R(\lambda,A)$.

The dimension of PE (i.e., the dimension of the spectral subspace corresponding to $\{\lambda_o\}$) is called <u>algebraic multiplicity</u> m_a of λ_o , while the <u>geometric multiplicity</u> is $m_g := \dim \ker(\lambda_o - A)$. In case $m_a = 1$ we call λ_o an <u>algebraically simple pole</u>.
If k is the pole order ($k = \infty$ in case of an essential singularity) we have

(3.4) $\max\{m_g,k\} \leq m_a \leq k\cdot m_g$,

where $\infty\cdot 0 = \infty$. These inequalities yield the following implications:

- $m_a < \infty$ if and only if λ_o is a pole with $m_g < \infty$,
- if λ_o is a pole with order k , then $\lambda_o \in P\sigma(A)$ and $PE = \ker(\lambda_o - A)^k$.

If A has compact resolvent then every point of $\sigma(A)$ is a pole of finite algebraic multiplicity. This is a consequence of Prop.2.5(iii) and the well-known Riesz-Schauder Theory for compact operators (see [Dunford-Schwartz (1958),VII.4.5]).

3.7. <u>The essential spectrum</u>.

For $T \in L(E)$ the <u>Fredholm domain</u> $\rho_F(T)$ is

$$\begin{aligned}(3.5)\quad \rho_F(T) &:= \{\lambda \in \mathbb{C} : \lambda - T \text{ is a Fredholm operator}\}\\ &= \{\lambda \in \mathbb{C} : \ker(\lambda - T) \text{ and } E/\mathrm{im}(\lambda - T) \text{ are finite dimensional}\} .\end{aligned}$$

An equivalent characterization of $\rho_F(T)$ is obtained through the Calkin algebra $L(E)/K(E)$, where $K(E)$ stands for the closed ideal of all compact operators. In fact, $\rho_F(T)$ coincides with the resolvent set of the canonical image of T in the Calkin algebra. The complement of $\rho_F(T)$ is called <u>essential spectrum</u> of T and denoted by $\sigma_{ess}(T)$. The corresponding spectral radius, called <u>essential spectral radius</u>, satisfies

(3.6) $r_{ess}(T) := \sup \{|\lambda| : \lambda \in \sigma_{ess}(T)\} = \lim_{n\to\infty} \|T^n\|_{ess}^{1/n}$,

where $\|T\|_{ess} = \mathrm{dist}(T,K(E)) := \inf \{\|T - K\| : K\in K(E)\}$ is the norm of T in $L(E)/K(E)$.
For every compact operator K we have $\|T - K\|_{ess} = \|T\|_{ess}$, hence

(3.7) $r_{ess}(T - K) = r_{ess}(T)$.

A detailed analysis of $\rho_F(T)$ can be found in Section IV.5.6 of Kato (1966). In particular we recall that the poles of $R(\cdot,T)$ with finite algebraic multiplicity belong to $\rho_F(T)$. Conversely, an element of the unbounded component of $\rho_F(T)$ either belongs to $\rho(T)$ or is a pole of finite algebraic multiplicity. Thus $r_{ess}(T)$ can be characterized as follows

(3.8) $r_{ess}(T)$ is the smallest $r \in \mathbb{R}_+$ such that every $\lambda \in \sigma(T)$, $|\lambda| > r$ is a pole of finite algebraic multiplicity.

Now, if $\mathcal{T} = (T(t))_{t\geq 0}$ is a strongly continuous semigroup then VIII.1, Lemma 4 of Dunford-Schwartz (1958) applied to the function $t \to \log \|T(t)\|_{ess}$ ensures that

(3.9) $$\omega_{ess}(\mathcal{T}) := \lim_{t\to\infty} \frac{1}{t} \log\|T(t)\|_{ess} = \inf \{\frac{1}{t} \log\|T(t)\|_{ess} : t > 0\}$$

is well defined (possibly $-\infty$). By the definition of $\omega_{ess}(\mathcal{T})$ and by (3.6) we have

(3.10) $$r_{ess}(T(t)) = \exp(t\cdot\omega_{ess}(\mathcal{T})) , \ t \geq 0 .$$

Obviously, $\omega_{ess} \leq \omega$ and equality occurs if and only if $r_{ess}(T(t)) = r(T(t))$ for $t \geq 0$.
If $\omega_{ess} < \omega$ there exists an eigenvalue λ of $T(t)$ satisfying $|\lambda| = r(T(t))$, hence by Theorem 6.3 below there exists $\lambda_1 \in P\sigma(A)$ such that $\operatorname{Re}\lambda_1 = \omega$. Thus $\omega_{ess} < \omega$ implies $s(A) = \omega(\mathcal{T})$, i.e., we have

(3.11) $$\omega(\mathcal{T}) = \max\{\omega_{ess}(\mathcal{T}), s(A)\} .$$

As a final observation we point out that

(3.12) $$\omega_{ess}(\mathcal{T}) = \omega_{ess}(\mathcal{S})$$

whenever $\mathcal{T}$ is generated by A and $\mathcal{S}$ is generated by $A + K$ for some compact operator K (see Prop.2.8 and Prop.2.9 of B-IV).

4. THE SPECTRUM OF INDUCED SEMIGROUPS

In the previous section we tried to decompose a semigroup into the direct sum of two, hopefully simpler objects. Here we present other methods to reduce the complexity of a semigroup and its generator. Forming subspace or quotient semigroups as in A-I,3.2, A-I,3.3 are such methods. But also the constructions of new semigroups on canonically associated spaces such as the dual space, see A-I,3.4, or the F-product, see A-I,3.6, might be helpful. We review these constructions under the spectral theoretical point of view and collect a number of technical properties for later use.

We start by studying the spectrum of subspace and quotient semigroups. To that purpose assume that the strongly continuous semigroup $T = (T(t))_{t\geq 0}$ leaves invariant some closed subspace N of the Banach space E. There are canonically induced semigroups $T_|$ on N, resp. $T_/$ on $E/_N$ and their generators $A_|$, resp. $A_/$ are canonically obtained from the generator A of T (see A-I,Section 3). The following example shows that the spectra of A, $A_|$ and $A_/$ may differ quite drastically.

Example 4.1. As in the example in A-I,3.3 we consider the translation semigroup on $E = L^1(\mathbb{R})$ and the invariant subspace $N := \{f \in E : f(x) = 0 \text{ for } x \geq 1\}$. Then $\sigma(A) = i\mathbb{R}$ but $\sigma(A_|) = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq 0\}$. Next we take the translation invariant subspace $M := \{f \in N : f(x) = 0 \text{ for } 0 \leq x \leq 1\}$ and obtain $\sigma(A_{|/}) = \emptyset$ for the generator $A_{|/}$ of the quotient semigroup $T_{|/}$ (use the fact that $T_{|/}$ is nilpotent).

In the next proposition we collect the information on $\sigma(A)$ which in general can be obtained from the 'subspace spectrum' $\sigma(A_|)$ and the 'quotient spectrum' $\sigma(A_/)$.

Proposition 4.2. Using the standard notations the following inclusions hold:

$$\rho_+(A) \underset{(iii)}{\subset} [\rho(A_|)\cap\rho(A_/)] \underset{(ii)}{\subset} \rho(A) \underset{(i)}{\subset} [\rho(A_|)\cap\rho(A_/)] \cup [\sigma(A_|)\cap\sigma(A_/)],$$

where $\rho_+(A)$ denotes the connected component of $\rho(A)$ which is unbounded to the right.

Proof. (i) Assume $\lambda \in \rho(A)$, i.e. $(\lambda-A)$ is a bijection from $D(A)$ onto E. Since N is T-invariant we have $D(A_|) = D(A) \cap N$ and $(\lambda-A)D(A_|) \subset N$. If $(\lambda-A)D(A_|) = N$ then $R(\lambda,A)N = D(A_|)$ and the induced operators $R(\lambda,A)_|$, resp. $R(\lambda,A)_/$ are the inverses of $(\lambda-A_|)$, resp. $(\lambda-A_/)$. If $(\lambda-A)D(A_|) \neq N$ then $\lambda \in \sigma(A_|)$. In addition there exists $f \in D(A)\backslash N$ such that $g := (\lambda-A)f \in N$. Hence for $\hat{f} := f+N$, $\hat{g} := g+N \in E_/$ it follows that $(\lambda-A_/)\hat{f} = \hat{g} = 0$, i.e. $\lambda \in \sigma(A_/)$.

(ii) Take $\lambda \in \rho(A_|) \cap \rho(A_/)$. Then $(\lambda-A)$ is injective: $(\lambda-A)f = 0$ implies $(\lambda-A_/)\hat{f}= 0$, hence $\hat{f} = 0$, i.e. $f \in N$ and therefore $f = 0$.

In addition, $(\lambda-A)$ is surjective: For $g \in E$ there exists $\hat{f} \in E_/$ such that $(\lambda-A_/)\hat{f} = \hat{g}$, i.e. there exists $h \in N$ such that $(\lambda-A)f - g = h = (\lambda-A)k$ for some $k \in D(A_|)$.
Therefore we obtain $(\lambda-A)(f-k) = g$.

(iii) The integral representation of the resolvent for $\lambda > \omega(T)$ (see A-I, Prop.1.11) shows that $R(\lambda,A)N \subset N$. By the power series expansion for holomorphic functions this extends to all $\lambda \in \rho_+(A)$. Therefore the restriction $R(\lambda,A)_|$ coincides with the resolvent $R(\lambda,A_|)$. On the other hand $R(\lambda,A)_/$ is well defined on $E_/$ and satisfies

$$R(\lambda,A)_/(f+N) = R(\lambda,A)f + N$$

(use again the integral representation). This proves that

$$R(\lambda,A)_/ = R(\lambda,A_/) .$$

□

Corollary 4.3. Under the above assumptions take a point μ in the closure of $\rho_+(A)$. Then

(i) $\mu \in \sigma(A)$ if and only if $\mu \in \sigma(A_|)$ or $\mu \in \sigma(A_/)$.

(ii) μ is a pole of $R(\cdot,A)$ if and only if μ is a pole of $R(\cdot,A_|)$ and of $R(\cdot,A_/)$. In that case,

$$\max(k_|,k_/) \leq k \leq k_| + k_/$$

for the respective pole orders.

Proof. (i) follows from Prop.4.2, inclusions (ii) and (iii).
(ii) By the previous assertion we may assume that for some $\delta > 0$ the pointed disc

$$\{\lambda \in \mathbb{C} : 0 < |\lambda-\mu| < \delta\}$$

is contained in $\rho(A) \cap \rho(A_|) \cap \rho(A_/)$. Call U_n the coefficients of the Laurent expansion of $R(\cdot,A)$. Since N is $R(\lambda,A)$-invariant for $\lambda \in \rho_+(A)$ the same holds for each U_n . With the obvious notations we have

$$R(\lambda,A) = \sum U_n(\lambda-\mu)^n , \quad R(\lambda,A)_| = \sum U_{n|}(\lambda-\mu)^n \quad \text{and} \quad R(\lambda,A)_/ = \sum U_{n/}(\lambda-\mu)^n$$

which shows $\max(k_|,k_/) \leq k$. If $R(\cdot,A)_|$ has a pole in μ of order ℓ , then $U_{-(\ell+1)|} = 0$, i.e. $U_{-(\ell+1)}N = \{0\}$. Similarly it follows that $U_{-(m+1)}E \subset N$ if $R(.,A)_/$ has a pole in μ of order m .

Therefore $U_{-(\ell+1)} \circ U_{-(m+1)} = 0$. The relations (3.2) imply $U_{-(m+l+1)} = 0$, hence the pole order of $R(.,A)$ is dominated by $\ell + m$.

□

4.4. Spectrum of the adjoint semigroup.

We recall from A-I,3.4 that to every strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ there corresponds a strongly continuous adjoint semigroup $\mathcal{T}^* = (T(t)^*)_{t\geq 0}$ on the semigroup dual

$$E^* = \{\phi \in E' : \lim_{t\to\infty} \|T(t)'\phi-\phi\| = 0\} .$$

Its generator A^* is the maximal restriction of the adjoint A' to E^* . For these operators the spectra coincide, or more precisely

(i) $\sigma(T(t)) = \sigma(T(t)') = \sigma(T(t)^*)$,

$R\sigma(T(t)) = P\sigma(T(t)') = P\sigma(T(t)^*)$.

(ii) $\sigma(A) = \sigma(A') = \sigma(A^*)$, $R\sigma(A) = P\sigma(A') = P\sigma(A^*)$.

(iii) $s(A) = s(A^*)$, $\omega(A) = \omega(A^*)$.

The left part of these equalities is either well known or has been stated in Prop.2.2(ii). The first statment of (iii) follows from (ii), while the second is an immediate consequence of the estimate $\|T(t)^*\| \leq \|T(t)\| \leq M\cdot\|T(t)^*\|$ given in A-I,3.4. As a sample for the remaining assertions we show that $0 \notin \sigma(A)$ if and only if $0 \notin \sigma(A^*)$: If A and therefore A' is invertible it follows from A-I,3.4 that A^* is a bijection from $D(A^*)$ onto E^* . Conversely assume that A^* is invertible. Then A' must be injective by the Proposition in A-I,3.4. Moreover $A'(D(A'))$ contains $A^*(D(A^*)) = E^*$ and is $\sigma(E',E)$-dense in E' . By standard duality arguments follows that A is injective with dense image. We show that $A(D(A))$ is closed: For $f \in D(A)$ choose $\phi \in D(A')$ such that $\|\phi\| \leq 1$ and $|\langle f,\phi\rangle| \geq \frac{1}{2}\|f\|$. Then

$$\|(A^*)^{-1}\| \, \|Af\| \geq \|(A^*)^{-1}\| \, |\langle Af,\phi\rangle| \geq |\langle Af,(A^*)^{-1}\phi\rangle|$$
$$= |\langle f,\phi\rangle| \geq \frac{1}{2}\|f\| ,$$

hence

$$\|Af\| \geq \frac{1}{2}\|(A^*)^{-1}\|^{-1} \|f\| ,$$

and $A(D(A))$ is closed since A is closed.

□

4.5 Spectrum of the $\mathcal{F}$-product semigroup.

As stated in A-I,3.6 the $\mathcal{F}$-product semigroup $\mathcal{T}_{\mathcal{F}} = (T_{\mathcal{F}}(t))_{t\geq 0}$ on $E_{\mathcal{F}}^{\mathcal{T}}$ of a strongly continuous semigroup $\mathcal{T}$ on E serves to convert sequences in E into points in $E_{\mathcal{F}}^{\mathcal{T}}$. In particular it can be used to convert approximate eigenvectors of the generator A into eigenvectors of A_F .

Proposition. Let A be the generator of a strongly continuous semigroup. Then the generator A_F of the F-product semigroup satisfies

(i) $A\sigma(A) = A\sigma(A_F) = P\sigma(A_F)$.

(ii) $\sigma(A) = \sigma(A_F)$.

Remark: In case A is bounded then A is a generator and $E_F^T = E_F$ (cf. A-I,3.6). Thus the proposition applies to bounded linear operators and their canonical extensions to the F-product E_F .

Proof of the proposition. (i) The inclusion $P\sigma(A_F) \subset A\sigma(A_F)$ holds trivially. We show that $A\sigma(A_F) \subset A\sigma(A)$: Take $\lambda \in A\sigma(A_F)$ and an associated approximate eigenvector $(\hat{f}^m)_{n\in\mathbb{N}}$, i.e. $\hat{f}^m \in D(A_F)$, $\|\hat{f}^m\| = 1$ and $(\lambda - A_F)\hat{f}^m \to 0$ as $m \to \infty$. By the considerations in A-I,3.6 we can represent each $\hat{f}^m$ as a normalized sequence $(f_n^m)_{n\in\mathbb{N}}$ in $D(A)$ such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|(\lambda - A)f_n^m\| = 0 .$$

Therefore we can find a sequence $g_k = f_k^{m(k)}$ satisfying

$$\lim_{k\to\infty} \|(\lambda - A)g_k\| = 0 ,$$

i.e. $\lambda \in A\sigma(A)$.

Finally we show $A\sigma(A) \subset P\sigma(A_F)$: For $\lambda \in A\sigma(A)$ take a corresponding approximate eigenvector (f_n) . By A-I,(3.2) we have

$$\|T(t)f_n - f_n\| \leq \|T(t)f_n - e^{\lambda t}f_n\| + |e^{\lambda t} - 1|$$
$$= \left\|\int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)f_n\, ds\right\| + |e^{\lambda t} - 1|$$

which converges to zero uniformly in n as $t \to 0$, i.e. $(f_n) \in m^T(E)$. By the characterization of $D(A_F)$ given in A-I,3.6 it follows that

$$\hat{f} := (f_n) + c_F(E) \in D(A_F) ,$$

and $A_F\hat{f} = \lambda\hat{f}$, i.e. $\lambda \in P\sigma(A_F)$.

(ii) The inclusion $\sigma(A) \subset \sigma(A_F)$ follows from (i) and the inclusion $R\sigma(A) \subset R\sigma(A_F)$: For $\lambda \in R\sigma(A)$ choose $f \in E$ such that $\|(\lambda - A)g - f\| \geq 1$ for every $g \in D(A)$. Then $\|(\lambda - A_F)\hat{g} - \hat{f}\| \geq 1$ for every $\hat{g} \in D(A_F)$ and $\hat{f} = (f,f,\ldots) + c_F(E)$. Therefore $\lambda \in R\sigma(A_F)$. We now show $\rho(A) \subset \rho(A_F)$: Assume $\lambda \in \rho(A)$. By (i) $(\lambda - A_F)$ has to be injective. Choose $\hat{f} = (f_1, f_2, \ldots) + c_F(E)$ such that $(f_n) \in m^T(E)$. Then $(R(\lambda,A)f_n) \in m^T(E)$ and $(\lambda - A_F)((R(\lambda,A)f_n) + c_F(E)) = (f_n) + c_F(E)$, i.e., $(\lambda - A_F)$ is surjective and $\lambda \in \rho(A_F)$. □

Applying the proposition to a single operator $T(t)$ we obtain $A\sigma(T(t)) = P\sigma(T(t)_F)$. Note that in general $A\sigma(T(t)) \neq P\sigma(T_F(t))$ (see the Examples 1.3 and 1.4 in combination with Theorem 6.3).

5. THE SPECTRUM OF PERIODIC SEMIGROUPS

In this section we determine the spectrum of a particularly simple class of strongly continuous semigroups and thereby achieve a rather complete description of the semigroup itself. Besides being nice and simple these semigroups gain their importance as building blocks for the general theory.

Definition 5.1. A strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on a Banach space E is called periodic if $T(t_o) = Id$ for some $t_o > 0$. The period τ of $\mathcal{T}$ is obtained as $\tau := \inf\{t_o > 0 : T(t_o) = Id\}$.

We immediately observe that periodic semigroups are groups with inverses $T(t)^{-1} = T(n\tau - t)$ for $0 \leq t \leq n\tau$, τ the period of $\mathcal{T}$. Moreover, they are bounded, hence the growth bound is zero and $\sigma(A) \subset i\mathbb{R}$.

Lemma 5.2. Let $\mathcal{T}$ be a strongly continuous semigroup with period $\tau > 0$ and generator A. Then

$$\sigma(A) \subset 2\pi i/\tau \cdot \mathbb{Z} \text{ , and}$$

$$R(\mu, A) = (1 - e^{-\mu\tau})^{-1} \int_0^\tau e^{-\mu s} T(s)\, ds \tag{5.1}$$

for $\mu \notin 2\pi i/\tau \cdot \mathbb{Z}$.

Proof. From the basic identities A-I,(3.1) and A-I,(3.2) for $t = \tau$, it follows that $(\mu - A)$ has a left and right inverse if $\mu \neq 2\pi i n/\tau$, $n \in \mathbb{Z}$, and that the inverse is given by the above expression.

□

The representation of $R(\mu, A)$ given in A-I,Prop.1.11 shows that the resolvent of the generator of a periodic semigroup is a meromorphic function having only poles of order one and the residues

$P_n := \lim_{\mu \to \mu_n} (\mu - \mu_n) R(\mu, A)$ in $\mu_n := 2\pi i n/\tau$, $n \in \mathbb{Z}$, are

$$P_n = \tau^{-1} \int_0^\tau \exp(-\mu_n s) T(s)\, ds \,. \tag{5.2}$$

Moreover, it follows that the spectrum of A consists of eigenvalues only and each P_n is the spectral projection belonging to μ_n (see

3.6). Another way of looking at P_n is given by saying that P_n is the n-th Fourier coefficient of the τ-periodic function $s \to T(s)$. From this it follows that no non-zero $\phi \in E'$ vanishes on all P_nE simultaneously. By the Hahn-Banach theorem we conclude that spann $\cup_{n\in\mathbb{Z}} P_nE$ is dense in E.
Since $P_nE \subset D(A)$ we obtain from A-I,(3.1) that

$$AP_nf = \mu_nP_nf \tag{5.3}$$

for every $f \in E$, $n \in \mathbb{Z}$. This and A-I,(3.2) imply

$$T(t)P_nf = \exp(\mu_nt)\cdot P_nf \tag{5.4}$$

for every $t \geq 0$. Therefore μ_n is an eigenvalue of A and $\exp(\mu_nt)$ is an eigenvalue of $T(t)$ if and only if $P_n \neq 0$. In that case, P_nE is the corresponding eigenspace and we have the following lemma.

<u>Lemma</u> 5.3. For a τ-periodic semigroup $\mathcal{T}$ we take $\mu_n := 2\pi in/\tau$, $n \in \mathbb{Z}$ and consider

$$P_n := \tau^{-1}\cdot\int_0^\tau \exp(-\mu_ns)T(s)\,ds .$$

Then the following assertions are equivalent:

(a) $P_n \neq 0$

(b) $\mu_n \in P\sigma(A)$

(c) $\exp(\mu_nt) \in P\sigma(T(t))$ for every $t > 0$.

The action of A, resp. $T(t)$ on the subspaces P_nE, $n \in \mathbb{Z}$, is determined by (5.3), resp. (5.4). Moreover,

$$\begin{aligned} P_mP_nf &= \tau^{-1}\cdot\int_0^\tau \exp(-\mu_ms)T(s)P_nf\,ds = \\ &= \tau^{-1}\cdot\int_0^\tau \exp((\mu_n-\mu_m)s)P_nf\,ds = 0 \end{aligned}$$

for $n \neq m$, i.e. the subspaces P_nE are "orthogonal". Since their union is total in E one expects to be able to extend the representations (5.3) and (5.4) of A and $T(t)$. This is possible if

$$\sum_{-\infty}^{+\infty} P_n = \mathrm{Id} ,$$

where the series should be summable for the strong operator topology. Unfortunately this is false in general since the family of projections

$$Q_H := \sum_{n\in H} P_n ,$$

where H runs through all finite subsets of $\mathbb{Z}$, may be unbounded (see the example below). Nevertheless the following is true.

Theorem 5.4. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a τ-periodic semigroup on a Banach space E with generator A and associated spectral projections

$P_n := \tau^{-1}\cdot\int_0^\tau \exp(-\mu_n s)T(s)\,ds$, $\mu_n := 2\pi in/\tau$, $n \in \mathbb{Z}$.

For every $f \in D(A)$ one has $f = \sum_{-\infty}^{+\infty} P_n f$ and therefore

(i) $\quad T(t)f = \sum_{-\infty}^{+\infty} \exp(\mu_n t)P_n f \qquad$ if $f \in D(A)$,

(ii) $\quad Af = \sum_{-\infty}^{+\infty} \mu_n P_n f \qquad$ if $f \in D(A^2)$.

Proof. It suffices to prove the first statement. Then (i) and (ii) follow by (5.3) and (5.4).

We assume $\tau = 2\pi$ and show first that $\sum_{-\infty}^{+\infty} P_n f$ is summable for $f \in D(A)$: For $g := Af$ we obtain $P_n g = P_n Af = AP_n f = inP_n f$. Take H to be a finite subset of $\mathbb{Z} \setminus \{0\}$ and $\phi \in E'$. Then

$$\begin{aligned}|\textstyle\sum_{n\in H} \langle P_n f,\phi\rangle| &= |\textstyle\sum_{n\in H} (in)^{-1} \langle P_n g,\phi\rangle| \\ &\leq (\textstyle\sum_{n\in H} n^{-2})^{1/2}(\textstyle\sum_{n\in H} |\langle P_n g,\phi\rangle|^2)^{1/2} .\end{aligned}$$

From Bessel's inequality we obtain for the second factor

$$\begin{aligned}\textstyle\sum_{n\in H} |\langle P_n g,\phi\rangle|^2 &\leq 1/2\pi\cdot\int_0^{2\pi} |\langle T(s)g,\phi\rangle|^2\,ds \\ &\leq \|\phi\|^2\cdot 1/2\pi\cdot \int_0^{2\pi} \|T(s)g\|^2\,ds .\end{aligned}$$

With the constant $c := (1/2\pi\cdot \int_0^{2\pi} \|T(s)g\|^2\,ds)^{1/2}$ we obtain

$$\|\textstyle\sum_{n\in H} P_n f\| \leq c(\textstyle\sum_{n\in H} n^{-2})^{1/2}$$

for every finite subset H of $\mathbb{Z}$, i.e. $\sum_{-\infty}^{+\infty} P_n f$ is summable.

Next we set $h := \sum_{-\infty}^{+\infty} P_n f$ and observe that for every $\phi' \in E'$ the Fourier coefficients of the continuous, τ-periodic functions

$$s \to \langle T(s)h,\phi\rangle \quad \text{and} \quad s \to \langle T(s)f,\phi\rangle$$

coincide. Therefore these functions are identical for $s \geq 0$ and in particular for $s = 0$, i.e. $\langle h,\phi\rangle = \langle f,\phi\rangle$. By the Hahn-Banach Theorem we obtain $f = h$.

□

The above theorem contains rather precise information on periodic semigroups. In particular, it characterizes periodic semigroups by the fact that $\sigma(A)$ is contained in $i\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$ and the eigenfunctions of A form a total subset of E .

If we suppose in addition that a periodic semigroup has a bounded generator it follows that the spectrum of its generator is bounded.

Therefore only a finite number of spectral projections P_n are distinct from 0 and we have the following characterization.

Corollary 5.5. Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a semigroup with bounded generator on some Banach space E . This semigroup has period τ/k for some $k \in \mathbb{N}$ if and only if there exist finitely many pairwise orthogonal projections P_n , $-m \leq n \leq m$, $P_{-m} \neq 0$ or $P_m \neq 0$, such that

(i) $\sum_{-m}^{+m} P_n = \mathrm{Id}$,

(ii) $T(t) = \sum_{-m}^{+m} \exp(2\pi i n t/\tau) P_n$,

(iii) $A = \sum_{-m}^{+m} (2\pi i n/\tau) P_n$.

Example 5.6. From A-I,2.5 we recall briefly the rotation group $R_\tau(t) f(z) := f(\exp(2\pi i n t/\tau)\cdot z)$ on $E = C(\Gamma)$, resp. $E = L^p(\Gamma, m)$ for $1 \leq p < \infty$. The spectrum of the generator

$$Af(z) = (2\pi i/\tau) z \cdot f'(z)$$

is

$$\sigma(A) = (2\pi i/\tau)\cdot \mathbb{Z} .$$

The eigenfunctions $\varepsilon_n(z) := z^n$ yield the projections

$$P_n = (1/2\pi i)\cdot \varepsilon_{-(n+1)} \otimes \varepsilon_n \text{ , i.e.}$$

$$P_n f(z) = (1/2\pi i)\cdot (\textstyle\int_\Gamma f(w) w^{-(n+1)}\, dw)\cdot z^n .$$

It is left as an exercise to compute the norms of $Q_m := \sum_{-m}^{+m} P_n$ in $L^p(\Gamma)$ for various p and then check the assertions of Theorem 5.4. Clearly, this proves some classical convergence theorems for Fourier series (compare Davies (1980), Chap.8.1).

6. SPECTRAL MAPPING THEOREMS

We now return to the question posed in the introduction to this chapter: In which form and under which conditions is it true that the spectrum $\sigma(T(t))$ of the semigroup operators is obtained - via the exponential map - from the spectrum $\sigma(A)$ of the generator, or briefly

$$\sigma(T(t)) = \exp(t\sigma(A)) \ ?$$

This and similar statements will be called spectral mapping theorems for the semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ and its generator A .

In addition, we saw in Prop.1.1 that the validity of such a spectral mapping theorem implies

$$s(A) = \omega(A)$$

for the spectral- and growth bounds and therefore guarantees that the location of the spectrum of A determines the asymptotic behavior of $\mathcal{T}$. As we have seen in Examples 1.3 and 1.4 the last statement does not hold in general. We therefore present a detailed analysis, where and why it fails and what additional assumptions are needed for its validity. Before doing so we have another look at the examples.

6.1 <u>The counterexamples revisited</u>.

(i) Take the nilpotent translation semigroup from A-I,2.6. Then $\sigma(A) = \emptyset$ and $\sigma(T(t)) = 0$ for every $t > 0$. By this trivial example and since $e^z \neq 0$ for every $z \in \mathbb{C}$, it is natural to read the 'spectral mapping theorem' modulo the addition of $\{0\}$, i.e.

$$\sigma(T(t)) \cup \{0\} = \exp(t\sigma(A)) \cup \{0\} \quad \text{for} \quad t \geq 0 .$$

(ii) The spectrum of the generator A of the τ-periodic rotation group $(R_\tau(t))_{t\geq 0}$ on $C(\Gamma)$ is $\sigma(A) = 2\pi i/\tau \cdot \mathbb{Z}$ and $\exp(2\pi i n t/\tau)$, $n \in \mathbb{Z}$, is an eigenvalue of $R_\tau(t)$ for every $t \geq 0$ (see Example 5.6). If t/τ is irrational these eigenvalues form a dense subset of Γ . Since the spectrum is closed we obtain $\sigma(T(t)) = \Gamma$ for these t . Therefore in this example the spectral mapping theorem is valid only in the following 'weak' form:

$$\sigma(T(t)) = \overline{\exp(t\sigma(A))} , \quad t \geq 0 .$$

(iii) By Example 1.3 there exists a semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ with generator A such that $s(A) = -1$ and $\omega(\mathcal{T}) = 0$. This implies that for preassigned real numbers $\alpha < \beta$ there exists a semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ with generator B such that $s(B) = \alpha$ and $\omega(\mathcal{S}) = \beta$: Take $S(t) := e^{\beta t}T((\beta - \alpha)t)$ and observe that $B = (\beta-\alpha)A + \beta \mathrm{Id}$. In that case $\exp(t\sigma(B))$ is contained in the circle about 0 with radius $e^{\alpha t}$ by Lemma 1.1; hence there must be points in $\sigma(S(t))$ which are not in the closure of $\exp(t\sigma(B))$.

(iv) The Example 1.3 can be strengthend in order to yield a semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ with generator A such that $\sigma(A) = \emptyset$ but $\|T(t)\| = r(T(t)) = 1$ for $t \geq 0$, i.e. $s(A) = -\infty$, $\omega = 0$ and $s(A) < \omega$:

Take the translation semigroup on the Banach space

$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^{x^2}dx)$

with $\|f\| := \sup\{ |f(x)| : x \in \mathbb{R}_+ \} + \int_0^\infty |f(x)| e^{x^2}\, dx$

(see Greiner-Voigt-Wolff (1981)).

(v) Another modification of Example 1.3 yields a group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ satisfying $s(A) < \omega$. Therefore the spectral mapping theorem does not hold (see Wolff (1981)).

The next few theorems form the core of this chapter. We show that only one part of the spectrum and one inclusion is responsible for the failure of the spectral mapping theorem. The usefulness of this detailed analysis will become clear in the subsequent chapter on stability and asymptotics.

6.2. Spectral Inclusion Theorem. Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on some Banach space E. Then

$$\exp(t\sigma(A)) \subset \sigma(T(t)) \quad \text{for } t \geq 0 .$$

More precisely we have the following inclusions:

$$\exp(t\cdot P\sigma(A)) \subset P\sigma(T(t)) , \tag{6.1}$$

$$\exp(t\cdot A\sigma(A)) \subset A\sigma(T(t)) , \tag{6.2}$$

$$\exp(t\cdot R\sigma(A)) \subset R\sigma(T(t)) . \tag{6.3}$$

Proof. Since $e^{\lambda t} - T(t) = (\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)\, ds$ (see A-I,(3.1)) it follows that $(e^{\lambda t} - T(t))$ is not bijective if $(\lambda - A)$ fails to be bijective, which proves the main inclusion.
The inclusion (6.1) becomes evident from the following proof of (6.2): Take $\lambda \in A\sigma(A)$ and an associated approximate eigenvector $(f_n) \subset D(A)$. Then

$$g_n := e^{\lambda t}f_n - T(t)f_n = \int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)f_n\, ds$$

converges to zero as $n \to \infty$. Consequently, $e^{\lambda t} \in A\sigma(T(t))$ and in fact, the same approximate eigenvector (f_n) does the job for all $t \geq 0$.
For the proof of (6.3) we take $\lambda \in R\sigma(A)$ and observe that

$$(e^{\lambda t} - T(t))f = (\lambda - A)\left(\int_0^t e^{\lambda(t-s)}T(s)f\, ds\right) \in (\lambda - A)D(A)$$

for every $f \in E$.

□

As we know from the Examples 6.1, the converse inclusions do not hold in general, i.e., not every spectral value of a semigroup operator $T(t)$ comes - via the exponential map - from a spectral value of the generator. But at least this is true for some important parts of the spectrum.

6.3 <u>Spectral Mapping Theorem for Point and Residual Spectrum</u>. Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$. Then

$$(6.4) \qquad \exp(t\cdot P\sigma(A)) = P\sigma(T(t)) \setminus \{0\} ,$$

$$(6.5) \qquad \exp(t\cdot R\sigma(A)) = R\sigma(T(t)) \setminus \{0\} , \text{ for } t \geq 0 .$$

<u>Proof</u>. For the proof of (6.4) take $t_o > 0$ and $0 \neq \lambda \in P\sigma(T(t_o))$. After rescaling the semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ to the semigroup $(\exp(-t\cdot\log\lambda/t_o)T(t))_{t\geq 0}$ we may assume $\lambda = 1$. Then the closed, $\mathcal{T}$-invariant subspace

$$G := \{ g \in E : T(t_o)g = g \}$$

is non trivial. The restricted semigroup $\mathcal{T}_|$ is periodic with period $\tau \leq t_o$ and the spectrum of its generator $A_|$ contains at least one eigenvalue $\mu = 2\pi in/t_o$ for some $n \in \mathbb{Z}$ (see Lemma 5.3). Since every eigenvalue of $A_|$ is an eigenvalue of A we obtain that $1 \in \exp(t_o\cdot P\sigma(A))$. The converse inclusion has been proved in (6.1). In fact, even more can be said: Let $g \in G$ be an eigenvector of $T(t_o)$ corresponding to the eigenvalue $\lambda = 1$. For each $n \in \mathbb{Z}$ define

$$g_n := P_n g = 1/t_o\cdot\int_0^{t_o} \exp(-2\pi ins/t_o)T(s)g \, ds \in G$$

as in Section 5. Then g_n is an eigenvector of $A_|$, hence of A with eigenvalue $2\pi in/t_o$ as soon as g_n is distinct from zero. Since $D(A_|)$ is dense in G it follows from Theorem 5.4 that this holds for at least one $n \in \mathbb{Z}$. From the proof of (6.1) we know that this g_n is in fact an eigenvector for each $T(t)$, $t \geq 0$.
Since $R\sigma(A) = P\sigma(A^*)$ and $R\sigma(T(t)) = P\sigma(T(t)^*)$ (see 4.4) the assertion (6.5) follows from (6.4).

□

Note that the proof is essentially an application of the structure theorem for periodic semigroups as given in Thm.5.4. The information gained there can be reformulated into statements on the eigenspaces of A and $T(t)$.

Corollary 6.4. For the eigenspaces of the generator A, resp. of the semigroup operators $T(t)$, $t > 0$, the following holds:

(i) $\ker(\mu - A) = \bigcap_{s \geq 0} \ker(e^{\mu s} - T(s))$,

(ii) $\ker(e^{\mu t} - T(t)) = \overline{\text{span}}_{n \in \mathbb{Z}} \ker(\mu + 2\pi i n/t - A)$, $\mu \in \mathbb{C}$.

Remark that analogous statements are valid for $\ker(\mu - A')$ and $\ker(e^{\mu t} - T(t)')$ if we take in (ii) the $\sigma(E',E)$-closure.

Without proof (see Greiner (1981), Prop.1.10) we add another corollary showing that poles of the resolvent of $T(t)$ correspond necessarily to poles of the resolvent of the generator. Again the converse is not true as shown by Example 5.6 .

Corollary 6.5. Assume that $e^{\mu t}$ is a pole of order k of $R(\cdot, T(t))$ with residue P and Q as the k-th coefficient of the Laurent series. Then

(i) $\mu + 2\pi i n/t$ is a pole of $R(\cdot, A)$ of order $\leq k$ for every $n \in \mathbb{Z}$,

(ii) the residues P_n in $\mu + 2\pi i n/t$ yield $PE = \overline{\text{span}}_{n \in \mathbb{Z}} P_n E$,

(iii) the k-th coefficient of the Laurent series of $R(\cdot, A)$ at $\mu + 2\pi i n/t$ is

$$Q_n = (t \cdot e^{\mu t})^{1-k} \cdot Q \circ (1/t) \int_0^t e^{-(\mu + 2\pi i n/t)s} T(s)\, ds \, .$$

From Theorem 6.2 and 6.3 it follows that the approximate point spectrum is the trouble maker in the sense that not every approximate eigenvalue of $T(t)$ corresponds to an approximate eigenvalue of the generator A. Since nothing more can be said in general we now look for additional hypotheses on the semigroup implying the spectral mapping theorem.
As a simple example we assume $T(t_o)$ to be compact for some $t_o > 0$. Then $\sigma(T(t)) \setminus \{0\} = P\sigma(T(t)) \setminus \{0\}$ for $t \geq t_o$ and the spectral mapping theorem is valid by (6.4). A different class of semigroups verifying the spectral mapping theorem is given by the uniformly continuous semigroups (compare Cor.1.2).
Both cases, and many more , are included in the following result.

6.6 <u>Spectral Mapping Theorem for Eventually Norm Continuous Semigroups</u>.

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be an eventually norm continuous semigroup with generator A. Then the spectral mapping theorem is valid, i.e.

$$\sigma(T(t)) \setminus \{0\} = e^{t\cdot\sigma(A)} \quad \text{for every} \quad t \geq 0. \tag{6.6}$$

<u>Proof</u>. By the previous considerations it suffices to show that $A\sigma(T(t)) \setminus \{0\} \subset e^{t\cdot\sigma(A)}$ for $t > 0$. This will be done by converting approximate eigenvectors into eigenvectors in the semigroup $\mathcal{F}$-product (see 4.5). The assertion then follows from (6.4) and assertion (ii) of the Proposition in 4.5.

Assume $t \to T(t)$ to be norm continuous for $t \geq t_o$. Moreover it suffices to consider $1 \in A\sigma(T(t_1))$ for some $t_1 > 0$, i.e. we have a normalized sequence $(f_n)_{n\in\mathbb{N}} \subset E$ such that

$$\lim_{n\to\infty} \|T(t_1)f_n - f_n\| = 0 .$$

Choose $k \in \mathbb{N}$ such that $kt_1 > t_o$ and define $g_n := T(kt_1)f_n$. Then

$$\lim_{n\to\infty}\|g_n\| = \lim_{n\to\infty}\|T(t_1)^k f_n\| = \lim_{n\to\infty}\|f_n\| = 1$$

and

$$\lim_{n\to\infty} \|T(t_1)g_n - g_n\| = 0 ,$$

i.e. $(g_n)_{n\in\mathbb{N}}$ yields an approximate eigenvector of $T(t_1)$ with approximate eigenvalue 1. But the semigroup $\mathcal{T}$ is uniformly continuous on sets of the form $T(t_o)V$, V bounded in E. In particular, it is uniformly continuous on the sequence $(g_n)_{n\in\mathbb{N}_{\mathcal{T}}}$, which therefore defines an element $\hat{g}$ in the semigroup $\mathcal{F}$-product $E_{\mathcal{F}}^{\mathcal{T}}$.

Obviously, $\hat{g}$ is an eigenvector of $T_{\mathcal{F}}(t_1)$ with eigenvalue 1 and by (6.4) we obtain an eigenvalue $2\pi in/t_1$ of $A_{\mathcal{F}}$ for some $n \in \mathbb{Z}$. The coincidence of $\sigma(A)$ and $\sigma(A_{\mathcal{F}})$ proves the assertion.

□

We point out that the above spectral mapping theorem implies the coincidence of spectral bound and growth bound for eventually norm continuous semigroups, hence we have generalized the Liapunov Stability Theorem 1.2 to a much larger class of semigroups. As mentioned before this will be of great use in many applications. Therefore we state explicitly the spectral mapping theorem for several important classes of semigroups all of which are eventually norm continuous (cf. the diagram preceding A-II,Ex.1.27).

<u>Corollary</u> 6.7. The spectral mapping theorem

$$\sigma(T(t)) \setminus \{0\} = e^{t\cdot\sigma(A)} , \quad t \geq 0 , \tag{6.6}$$

holds for each of the following classes of strongly continuous semigroups:

(i) eventually compact semigroups,
(ii) eventually differentiable semigroups,
(iii) holomorphic semigroups,
(iv) uniformly continuous semigroups.

Another application of the above spectral mapping theorem can be made to tensor product semigroups (see A-I,3.7). Let $\mathcal{T}_1 = (T_1(t))_{t\geq 0}$, $\mathcal{T}_2 = (T_2(t))_{t\geq 0}$ be strongly continuous semigroups on Banach spaces E_1 , E_2 with generator A_1 , A_2 . The tensor product semigroup $\mathcal{T} = T_1 \otimes \mathcal{T}_2$ on some (appropriate) tensor product $E := E_1 \tilde{\otimes} E_2$ has the generator $A = A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$, but in general the spectrum of A is not determined by the spectra of A_1 , A_2 . But with an additional hypothesis the following can be proved.

<u>Corollary</u> 6.8. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are eventually norm continuous then

$$\sigma(A) = \sigma(A_1) + \sigma(A_2) ,$$

where A is the generator of the tensor product semigroup

$$\mathcal{T}_1 \otimes \mathcal{T}_2 = (T_1(t) \otimes T_2(t))_{t\geq 0} .$$

<u>Proof</u>. Clearly, the tensor product semigroup is eventually norm continuous and hence the spectral mapping theorem 6.6 is valid for all three semigroups $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}$. Moreover the spectrum of the tensor product of bounded operators is the product of the spectra [Reed-Simon (1978), XIII.9]. Therefore

$$\sigma(T_1(t)\otimes T_2(t)) = \sigma(T_1(t))\cdot\sigma(T_2(t)) , \quad t \geq 0 .$$

Consequently we have the following identity for every $t \geq 0$:

$$\begin{aligned} e^{t\cdot\sigma(A)} &= \sigma(T_1(t)\otimes T_2(t)) \setminus \{0\} \\ &= \sigma(T_1(t))\cdot\sigma(T_2(t)) \setminus \{0\} \\ &= e^{t\cdot\sigma(A_1)}\cdot e^{t\cdot\sigma(A_2)} \\ &= e^{t(\sigma(A_1)+\sigma(A_2))}. \end{aligned}$$

From this identity we want to deduce $\sigma(A) = \sigma(A_1) + \sigma(A_2)$.
" $\subset$ ": Take $\xi \in \sigma(A)$. Then for every $t > 0$ there exist $\mu_t \in \sigma(A_1)$, $\lambda_t \in \sigma(A_2)$ and $n_t \in \mathbb{Z}$ such that $\xi = \mu_t + \lambda_t + 2\pi i n_t/t$.
Since the real parts of μ_t , λ_t are bounded above, they lie in some interval $[a,b]$. But

$$\sigma(A_i) \cap ([a,b] + i\mathbb{R})$$

is compact for $i = 1$, 2 , since A_i is the generator of an eventually norm continuous semigroup (see A-II, Thm.1.20) . By taking t

sufficiently small we conclude that $n_{t'} = 0$ for some $t' > 0$, i.e. $\xi = \mu_{t'} + \lambda_{t'}$.

" $\supset$ ": Choose $\mu \in \sigma(A_1)$, $\lambda \in \sigma(A_2)$. For every $t > 0$ there exist $\eta_t \in \sigma(A)$, $m_t \in \mathbb{Z}$ such that $\mu + \lambda = \eta_t + 2\pi i m_t/t$.
Since $\mathrm{Re}\ \mu + \mathrm{Re}\ \lambda = \mathrm{Re}\ \eta_t$ and $\{\mathrm{Im}\ \eta_t : t > 0\}$ is bounded - $\mathcal{T} = (T_1(t) \otimes T_2(t))_{t\geq 0}$ is eventually norm continuous - it follows that $m_{t'} = 0$ for some $t' > 0$.

□

7. WEAK SPECTRAL MAPPING THEOREMS

In the previous section we showed under which hypotheses a spectral mapping theorem of the form

$$(7.1) \qquad \sigma(T(t)) \setminus \{0\} = e^{t\cdot\sigma(A)}, \quad t \geq 0,$$

is valid for the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$.
Among the various examples showing that (7.1) does not hold in general we recall the following.
Take the Banach space $E = c_o$, the multiplication operator $A(x_n)_{n\in\mathbb{N}} = (inx_n)_{n\in\mathbb{N}}$ with maximal domain and the corresponding semigroup $T(t)(x_n)_{n\in\mathbb{N}} = (e^{int}x_n)_{n\in\mathbb{N}}$. Then $\sigma(A) = \{in : n \in \mathbb{N}\}$ and the spectral mapping theorem is valid only in the following weak form:

$$(7.2) \qquad \sigma(T(t)) = \overline{\exp(t\cdot\sigma(A))}, \quad t \geq 0.$$

In this section we prove similar weak spectral mapping theorems. We start with a generalization of the above example.
Consider the Banach space $E = C_o(X,\mathbb{C}^n)$ of all continuous $\mathbb{C}^n$-valued functions vanishing at infinity on some locally compact space X. In analogy to A-I,2.3 we associate to every continuous function $q : X \to M(n)$, where $M(n)$ denotes the space of all complex $n\times n$-matrices, a "multiplication operator"

$$M_q : f \to q\cdot f \quad \text{such that} \quad (q\cdot f)(x) = q(x)\cdot f(x), \ x \in X,$$

on the maximal domain $D(M_q) = \{f \in E : q\cdot f \in E\}$. If $\|e^{tq(x)}\|$ is uniformly bounded for $0 \leq t \leq 1$ and $x \in X$ it follows that M_q generates the multiplication semigroup

$$(T(t)f)(x) = e^{tq(x)}(f(x)), \quad f \in E, \ x \in X, \ t \geq 0.$$

Since M_q has a bounded inverse if and only if $q(x)^{-1}$ exists and is uniformly bounded for $x \in X$ it follows that the eigenvalues of each matrix $q(x)$ are always contained in $\sigma(M_q)$. In fact, much more can be said in case the function is bounded.

Lemma 7.1. The spectrum of the matrix valued multiplication operator M_p where $p : X \to M(n)$ is bounded is given by $\sigma(M_p) = \overline{\cup_{x\in X}\ \sigma(p(x))}$.

Proof. It remains to show that $0 \notin \overline{\cup_{x\in X}\ \sigma(p(x))}$ implies $0 \notin \sigma(M_p)$. Since $\det p(x)$ is the product of n eigenvalues (according to their multiplicity) of $p(x)$ the hypothesis implies that $d := \inf\{|\det p(x)| : x \in X\} > 0$. By Formula 4.12 in Chapter I of Kato (1966) we obtain

$$\|p(x)^{-1}\| \leq \gamma\cdot\|p(x)\|^{n-1}\cdot|\det p(x)|^{-1} \leq \gamma/\alpha\cdot\|M_p\|^{n-1}$$

for every $x \in X$ and a constant γ depending only on the norm chosen on $\mathbb{C}^n$. Therefore $x \to p(x)^{-1}$ defines a bounded continuous function on X which obviously yields the inverse of M_p , i.e., $0 \notin \sigma(M_p)$.

□

Theorem 7.2. Let $A = M_q$ be a matrix multiplication operator on $C_o(X,\mathbb{C}^n)$ generating a strongly continuous semigroup $(T(t))_{t\geq 0} = (e^{tq(\cdot)})_{t\geq 0}$. Then the Weak Spectral Mapping Theorem of the form

$$\sigma(T(t)) = \overline{\exp(t\cdot\sigma(A))} \tag{7.2}$$

is valid.

Proof. By the Spectral Inclusion Theorem 6.2 we always have $\exp(t\sigma(A)) \subset \sigma(T(t))$. Since $T(t)$ is a matrix multiplication operator with a bounded function we obtain from Lemma 7.1

$$\sigma(T(t)) = \overline{\cup_{x\in X}\ \sigma(\exp(tq(x)))} = \overline{\cup_{x\in X}\ \exp(t\sigma(q(x)))} \subset \overline{\exp(t\sigma(A))}\ ,$$

which proves the assertion.

□

Corollary 7.3. The growth bound $\omega(A)$ and the spectral bound $s(A)$ coincide for matrix multiplication semigroups.

Remark. The above results remain valid for other Banach spaces of $\mathbb{C}^n$-valued functions such as $L^p(X,\mathbb{C}^n)$, $1 \leq p < \infty$.

The example given at the beginning of this section can be generalized in a different way. In fact, $A(x_n) := (inx_n)$ on $E = c_o$ generates a bounded group, and we will show that this property too ensures that the Weak Spectral Mapping Theorem (7.2) holds. Without any boundedness assumption on $(T(t))_{t\in\mathbb{R}}$ this result cannot be true (see [Hille-Phillips (1957),Sec.23.16] or [Wolff(1981)]).

Theorem 7.4. Let $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ be a strongly continuous group on some Banach space E such that $\|T(t)\| \leq p(t)$ for some polynomial p and all $t \in \mathbb{R}$. Then the Weak Spectral Mapping Theorem holds, i.e.,

$$\sigma(T(t)) = \overline{\exp(t\cdot\sigma(A))} \quad \text{for all } t \in \mathbb{R}\,. \tag{7.2}$$

From the proof we isolate a series of lemmas for which we always assume the hypothesis made in Thm.7.4. Moreover we recall from Fourier analysis that the Fourier transformation $\phi \to \vec{\phi}$, $\vec{\phi}(\alpha) := \int_{-\infty}^{\infty} \phi(x)e^{-i\alpha x}\,dx$, and its inverse $\Psi \to \overleftarrow{\Psi}$, $\overleftarrow{\Psi}(x) := 1/2\pi\cdot\int_{-\infty}^{\infty} \Psi(\alpha)\cdot e^{i\alpha x}\,d\alpha$ are topological isomorphisms of the Schwartz space $\mathcal{S}$ ($= \mathcal{S}(\mathbb{R})$). Since the subspace $\mathcal{D}$ of all functions having compact support is dense in $\mathcal{S}$ it follows that $\{\phi \in \mathcal{S} : \overleftarrow{\phi} \in \mathcal{D}\}$ is dense in $\mathcal{S}$.

Lemma 7.5. For every function $\phi \in \mathcal{S}$ we obtain an operator $T(\phi) \in L(E)$ by

$$T(\phi)f := \int_{-\infty}^{\infty} \phi(s)T(s)f\,ds \quad , \quad f \in E\,.$$

This operator can be represented as

(7.3)

$$T(\phi)f = \lim_{\varepsilon\to 0} 1/2\pi\cdot\int_{-\infty}^{\infty} \overleftarrow{\phi}(\alpha)[R(\varepsilon-i\alpha,A)f - R(-\varepsilon-i\alpha,A)f]\,d\alpha \quad , \quad f \in E\,.$$

Proof. That $T(\phi)$ is well-defined follows from the polynomial boundedness of $(T(t))_{t\in\mathbb{R}}$. In fact, $\phi \to T(\phi)$ is continuous from $\mathcal{S}$ into $(L(E),\|\cdot\|)$.

We obtain

$$\begin{aligned}
T(\phi)f &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \phi(s)e^{-\varepsilon|s|}T(s)f\,ds \\
&= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} 1/2\pi \int_{-\infty}^{\infty} \overleftarrow{\phi}(\alpha)e^{i\alpha s}e^{-\varepsilon|s|}T(s)f\,d\alpha\,ds \\
&= \lim_{\varepsilon\to 0} 1/2\pi \int_{-\infty}^{\infty} \overleftarrow{\phi}(\alpha) \int_{-\infty}^{\infty} e^{i\alpha s}e^{-\varepsilon|s|}\,T(s)f\,ds\,d\alpha \\
&= \lim_{\varepsilon\to 0} 1/2\pi \int_{-\infty}^{\infty} \overleftarrow{\phi}(\alpha)[R(\varepsilon-i\alpha,A)f - R(-\varepsilon-i\alpha,A)f]\,d\alpha\,.
\end{aligned}$$

Here we used that polynomially bounded semigroups have growth bound 0, hence $\omega(A) = \omega(-A) = 0$. Hence the integral representation of $R(\varepsilon-i\alpha,A)$ (cf. A-I,Prop.1.11) exists for $\varepsilon \neq 0$.

□

Lemma 7.6. If $E \neq \{0\}$, then $\sigma(A) \neq \emptyset$.

Proof. If $\sigma(A) = \emptyset$ then (7.3) implies $T(\phi) = 0$ whenever $\overleftarrow{\phi}$ has compact support. Since these functions form a dense subspace of $\mathcal{S}$ we conclude that $T(\phi) = 0$ for all $\phi \in \mathcal{S}$.

Choosing an approximate identity $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ we obtain

$$f = T(0)f = \lim_{n\to\mathbb{N}} T(\psi_n)f = 0$$

for every $f \in E$.

□

Proof of Theorem 7.4 (1[st] part). By the Spectral Inclusion Theorem 6.2 we have to show that every spectral value of $T(t)$ can be approximated by exponentials of spectral values of A . In view of the rescaling procedure it suffices to prove this when $-1 \in \rho(T(\pi))$, provided that the following condition is satisfied.

(7.4) There exists $\varepsilon > 0$ such that $\bigcup_{k\in\mathbb{Z}} i[2k+1-2\varepsilon, 2k+1+2\varepsilon] \subset \rho(A)$.

Assume now that (7.4) holds. Then each of the sets $\sigma_k := \{i\alpha \in \sigma(A) : \alpha \in [2k-1, 2k+1]\}$ is a spectral set of A with corresponding spectral projection P_k . If we choose $\phi_o \in \mathcal{D}$ such that $\operatorname{supp} \phi_o \subset [-1+\varepsilon, 1-\varepsilon]$ and $\phi_o(x) = 1$ for $x \in [-1+2\varepsilon, 1-2\varepsilon]$ it follows from (7.3) and the integral representation of P_k (cf.(3.1)) that $P_o = T(\overset{\leftarrow}{\phi}_o)$. More generally, since $(e^{i2k\cdot}\, \overset{\leftarrow}{\phi}_o)^{\rightarrow}(\alpha) = \phi_o(\alpha-2k)$, the assertions (7.3) and (7.4) imply

$$(7.5) \qquad P_k = \int_{-\infty}^{\infty} e^{i2ks}\, \overset{\leftarrow}{\phi}_o(s)\, T(s)\, ds \quad \text{for } k \in \mathbb{Z} .$$

At this point we isolate another lemma.

Lemma 7.7. span $\bigcup_{k\in\mathbb{Z}} P_k E$ is dense in E .

Proof. The closure of span $\bigcup_{k\in\mathbb{Z}} P_k E$ is a $\mathcal{T}$-invariant subspace G of E . Consider the quotient group $(T(t)_/)_{t\in\mathbb{R}}$ induced on E/G . The spectrum of its generator $A_/$ is contained in $\sigma(A)$ by Prop.4.2.iii. Moreover the spectral projection corresponding to $\sigma(A_/) \cap \sigma_k$ is the quotient operator $P_{k/}$. Obviously $P_{k/} = 0$, therefore $\sigma(A_/) \cap \sigma_k = \emptyset$ for every $k \in \mathbb{Z}$ and $\sigma(A_/) = \emptyset$. By Lemma 7.6 this implies $E/G = \{0\}$, i.e. $G = E$.

□

Proof of Theorem 7.4 (2[nd] part). We return to the situation of the first part. Using (7.5) the spectral projection P_k can be transformed into

$$\begin{aligned} P_k &= \int_{-\infty}^{\infty} e^{i2ks}\, \overset{\leftarrow}{\phi}_o(s)\, T(s)\, ds \\ &= \sum_{m\in\mathbb{Z}} \int_{(m-1/2)\pi}^{(m+1/2)\pi} e^{i2ks}\, \overset{\leftarrow}{\phi}_o(s)\, T(s)\, ds \\ &= \int_{-\pi/2}^{\pi/2} e^{i2ks} \sum_{m\in\mathbb{Z}} \overset{\leftarrow}{\phi}_o(s+m\pi)\, T(s+m\pi)\, ds , \end{aligned}$$

i.e., $P_k f$ is the k-th Fourier coefficient of the π-periodic, continuous function $\xi_f : s \to \sum_{m\in\mathbb{Z}} \check{\phi}_o(s+m\pi)\; T(s+m\pi)f$, $f \in E$. Since the projections P_k are mutually orthogonal, i.e. $P_k P_m = 0$ for $k \neq m$, it follows that $g = \sum_{n\in\mathbb{Z}} P_n g$ for every $g \in \operatorname{span} \cup_{k\in\mathbb{Z}} P_k E$. In particular, the Fourier coefficients of the function ξ_g are absolutely summable, hence the Fourier series of ξ_g converges to ξ . For $s = 0$ we obtain
$g = \sum_{n\in\mathbb{Z}} P_n g\cdot e^{-in0} = \sum_{m\in\mathbb{Z}} \check{\phi}_o(0 + m\pi)\; T(0 + m\pi)g \quad (g \in \operatorname{span} \cup_{k\in\mathbb{Z}} P_k E)$.
Since $\operatorname{span} \cup_{k\in\mathbb{Z}} P_k E$ is dense (Lemma 7.7) we conclude that

$$\sum_{m\in\mathbb{Z}} \check{\phi}_o(m\pi)\; T(m\pi) = \mathrm{Id} . \tag{7.6}$$

As the final step we construct the inverse operator of $\mathrm{Id} + T(\pi)$ showing that $-1 \in \rho(T(\pi))$. We define $\Psi_o(\alpha) := \phi_o(\alpha)\cdot(1 + e^{i\pi\alpha})^{-1}$, $\alpha \in \mathbb{R}$. Then we have $\Psi_o \in S$ and $\Psi_o(1 + e^{i\pi\cdot}) = \phi_o$, hence $\check{\Psi}_o(x) + \check{\Psi}_o(x + \pi) = \check{\phi}_o(x)$ for all $x \in \mathbb{R}$. Then (7.6) implies

$$\begin{aligned}\mathrm{Id} &= \sum_{m\in\mathbb{Z}} \check{\phi}_o(m\pi)T(m\pi)\\ &= \sum_{m\in\mathbb{Z}} (\check{\Psi}_o(m\pi) + \check{\Psi}_o((m+1)\pi))T(m\pi)\\ &= [\sum_{m\in\mathbb{Z}} \check{\Psi}_o(m\pi)T(m\pi)](\mathrm{Id} + T(\pi)) .\end{aligned}$$

□

In the rest of this section we discuss the behavior of the single spectral values λ of $T(t)$, $t > 0$. The aim is a characterization of $\sigma(T(t))$ involving only properties of the generator. By the rescaling procedure A-I,3.1 we may assume $\lambda = 1$ and $t = 2\pi$.
From the Spectral Inclusion Theorem 6.2 we know that $1 \in \rho(T(2\pi))$ implies $i\mathbb{Z} \subset \rho(A)$. As seen in many examples the converse does not hold and we are now looking for additional conditions.
Henceforth we assume $i\mathbb{Z} \subset \rho(A)$ and define for $k \in \mathbb{Z}$

$$Q_k := 1/2\pi\int_0^{2\pi} e^{-iks}\; T(s)\; ds = 1/2\pi(1 - T(2\pi))R(ik,A) , \tag{7.7}$$

(cf. Formula A-I,(3.1)).

Our approach is based on Fejér's Theorem (for Banach space valued functions). Let us recall this result. Suppose $\xi : [0,2\pi] \to E$ is a continuous function and let $\xi_k := 1/2\pi\int_0^{2\pi} e^{-iks}\; \xi(s)\; ds$ be its k-th Fourier coefficient. Then the Fourier series is Césaro summable to ξ in every point $t \in (0,2\pi)$. Moreover one has

$$1/2(\xi(0) + \xi(2\pi)) = C_1\text{-}\sum_{k\in\mathbb{Z}} \xi_n := \lim_{N\to\infty} 1/N\cdot\sum_{n=0}^{N-1}(\sum_{k=-n}^{n} \xi_k) . \tag{7.8}$$

This result enables us to prove the following proposition:

Proposition 7.8. Let $(T(t))_{t\geq 0}$ be a semigroup on a Banach space E and denote its generator by A. Then the following conditions are equivalent:

(a) $1 \in \rho(T(2\pi))$,

(b) $i\mathbb{Z} \subset \rho(A)$ and the series $\sum_{k\in\mathbb{Z}} R(ik,A)f$ is Césaro-summable for every $f \in E$,

(c) $i\mathbb{Z} \subset \rho(A)$ and the series $\sum_{k\in\mathbb{Z}} R(ik,A)Q_k f$ is Césaro-summable for every $f \in E$.

Proof. (a) → (b): The Spectral Inclusion Theorem implies $i\mathbb{Z} \subset \rho(A)$. By (7.7) we have $R(ik,A) = 2\pi\cdot(1-T(2\pi))^{-1}Q_k$. Since $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable (towards $1/2(f + T(2\pi)f)$) (see (7.8)) it follows that $\sum_{k\in\mathbb{Z}} R(ik,A)f$ is Césaro-summable as well.

(b) <=> (c): If we use A-I,(3.1) and integrate by parts, we obtain

$$\begin{aligned} R(ik,A)Q_k f &= 1/2\pi\int_0^{2\pi} e^{-iks}\, T(s)R(ik,A)f\, ds \\ &= 1/2\pi\int_0^{2\pi} [R(ik,A)f - \int_0^s e^{-ikt}\, T(t)f\, dt]\, ds \\ &= R(ik,A)f - 1/2\pi\int_0^{2\pi} e^{-ikt}\, (2\pi-t)\, T(t)f\, dt . \end{aligned}$$

Fejer's theorem ensures that $\sum_{k\in\mathbb{Z}} (1/2\pi)\int_0^{2\pi} e^{-ikt}\, (2\pi-t)\, T(t)f\, dt$ is Césaro summable. Hence $\sum_{k\in\mathbb{Z}} R(ik,A)Q_k f$ is Césaro-summable if and only if $\sum_{k\in\mathbb{Z}} R(ik,A)f$ is.

(b) → (a): We have $Q_k = \frac{1}{2\pi}(1 - T(2\pi))R(ik,A)$. Furthermore we know by (7.7) and (7.8) that $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable towards $\frac{1}{2}(f + T(2\pi)f)$.

If we define $S : E \to E$ by $Sf := \frac{f}{2} + \frac{1}{2\pi}\cdot C_1\text{-}\sum R(ik,A)f$ then we have

$$\begin{aligned} (1 - T(2\pi))Sf &= \tfrac{1}{2}(f - T(2\pi)f) + \tfrac{1}{2\pi}\cdot C_1\text{-}\textstyle\sum (1 - T(2\pi))R(ik,A)f = \\ &= \tfrac{1}{2}(f - T(2\pi)f) + \tfrac{1}{2}(f + T(2\pi)f) = f . \end{aligned}$$

Since S commutes with $T(2\pi)$ it follows that S is the inverse of $(1 - T(2\pi))$ thus $1 \in \rho(T(2\pi))$.

□

Based on the equivalence of (a) and (b), one can state a characterization of the spectrum of $T(t)$ in terms of the generator and its resolvent only. However, in general it is difficult to verify the summability condition stated in (b).

In Hilbert spaces the boundedness of the resolvents will suffice (see Thm.7.10 below).

Lemma 7.9. Let $(T(t))_{t\geq 0}$ be a semigroup on some Hilbert space H and assume $i\mathbb{Z} \subset \rho(A)$ for the generator A. Then we have

(i) $(Q_k f)_{k\in\mathbb{Z}} \subset \ell^2(H)$ for every $f \in H$, and

(ii) if $\sup_{k\in\mathbb{Z}}\|R(ik,A)\| < \infty$, then $\sum_{k\in\mathbb{Z}} R(ik,A)f_k$ is summable whenever $(f_k)_{k\in\mathbb{Z}} \in \ell^2(H)$.

Proof. (i) We consider the Hilbert space $L^2([0,2\pi],H)$ and obtain

$$\begin{aligned}
0 &\le \|T(\cdot)f - \textstyle\sum_{k=-n}^{n} Q_k f\cdot e^{ik\cdot}\|^2 \\
&= \int_0^{2\pi} \|T(s)f\|^2\,ds - \int_0^{2\pi} \textstyle\sum_{k=-n}^{n} (T(s)f|e^{iks}Q_k f)\,ds - \\
&\quad \int_0^{2\pi} \textstyle\sum_{k=-n}^{n} (e^{iks}Q_k f|T(s)f)ds + \int_0^{2\pi} (\textstyle\sum_{k=-n}^{n} e^{iks}Q_k f|\textstyle\sum_{\ell-n}^{n} e^{i\ell s}Q_\ell f)\,ds \\
&= \int_0^{2\pi} \|T(s)f\|^2\,ds - 2\pi\textstyle\sum_{k=-n}^{n} \|Q_k f\|^2, \text{ (use (7.5))}.
\end{aligned}$$

It follows that $\sum_{k\in\mathbb{Z}} \|Q_k f\|^2 \le \frac{1}{2\pi}\cdot\int_0^{2\pi} \|T(s)f\|^2\,ds < \infty$.

(ii) Fix $\lambda > 0$ sufficiently large and set

$$g_k := (1 + \lambda R(ik,A))f_k,\ k \in \mathbb{Z}.$$

Using the resolvent equation and then (A-I,(3.1)) we obtain

$R(ik,A)f_k = R(\lambda + ik,A)g_k = [1 - e^{-2\pi\lambda}T(2\pi)]^{-1}\int_0^{2\pi} e^{-\lambda s}e^{-iks}T(s)g_k\,ds$.

This yields for every finite subset N of $\mathbb{Z}$

$$\begin{aligned}
\|\textstyle\sum_{k\in N} R(ik,A)f_k\| &\le \|(1 - e^{-2\pi\lambda}T(2\pi))^{-1}\|\cdot\int_0^{2\pi}\|T(s)\|\,\|\textstyle\sum_{k\in N} e^{-iks}g_k\|\,ds \le \\
&\le \|(1 - e^{-2\pi\lambda}T(2\pi))^{-1}\|\cdot(\int_0^{2\pi}\|T(s)\|^2\,ds)^{1/2}\cdot(\int_0^{2\pi}\|\textstyle\sum_{k\in N} e^{-iks}g_k\|^2\,dx)^{1/2} \\
&= c(\textstyle\sum_{k\in N}\|g_k\|^2)^{1/2} \le c(1 + \lambda M)(\textstyle\sum_{k\in N}\|f_k\|^2)^{1/2}.
\end{aligned}$$

Here $c := \|(1 - e^{-2\pi\lambda}T(2\pi))^{-1}\|\cdot(\int_0^{2\pi}\|T(s)\|^2\,ds)^{1/2}$ and $M := \sup_{k\in\mathbb{Z}}\|R(ik,A)\|$.

□

Theorem 7.10. Let A be the generator of a semigroup $(T(t))_{t\ge 0}$ on some Hilbert space H. Then the following form of the spectral mapping theorem is valid

$$\sigma(T(t))\setminus\{0\} = \{e^{\lambda t} : \text{either } \mu_k := \lambda + 2\pi ik/t \in \sigma(A) \text{ for some } k \in \mathbb{Z} \text{ or } (\|R(\mu_k,A)\|)_{k\in\mathbb{Z}} \text{ is unbounded}\}.$$

Proof. If $e^{\lambda t} \notin \sigma(T(t))$ it follows from the spectral inclusion theorem that $\mu_k \notin \sigma(A)$ for every $k \in \mathbb{Z}$ and from A-I,3.1, Formula (3.1), that $\|R(\mu_k,A)\|$ is bounded. For the converse inclusion it suffices to assume $t = 2\pi$ and $\lambda = 0$ (use the rescaling procedure A-I,3.1). Assuming that $i\mathbb{Z} \subset \rho(A)$ and $\|R(ik,A)\|$ is bounded then $\sum_{k\in\mathbb{Z}} R(ik,A)Q_k f$ is summable by Lemma 7.9. Since every summable series is Césaro-summable condition (c) of Prop. 7.8 is satisfied and we conclude $1 \in \rho(T(2\pi))$.

□

As an immediate consequence we obtain an interesting characterization of the growth bound ω of semigroups on Hilbert spaces.

Corollary 7.11. The growth bound of a semigroup $(T(t))_{t\geq 0}$ on a Hilbert space H satisfies

(7.9) $\omega = \inf\{\lambda \in \mathbb{R} : \lambda + i\mathbb{R} \subset \rho(A)$ and $\|R(\lambda+i\mu, A)\|$ is bounded for $\mu \in \mathbb{R}\}$.

The Example 1.3 above in combination with C-III,Cor.1.3 shows that (7.9) is not valid in arbitrary Banach spaces.

NOTES.

Section 1. It was already known to Hille-Phillips (1957) that for strongly continuous semigroups $(T(t))_{t\geq 0}$ with generator A the spectral mapping theorem "$\sigma(T(t)) = \exp(t\sigma(A))$" may be violated in various ways [l.c.,Sec.23.16]. The simple Examples 1.3 and 1.4 are due to Wolff (see Greiner-Voigt-Wolff (1981)) and Zabczyk (1975). A more sophisticated example of a positive group with "$s(A) < \omega(A)$" is given in Wolff (1981).

Section 2. In Definition 2.1 we define the residual spectrum of A in such a way that it coincides with the point spectrum of the adjoint A' (see Prop. 2.2.(ii)). It therefore differs slightly from the one used, e.g., by Schaefer (1974). The spectral mapping theorem for the resolvent (Thm.2.5) is well known and can, e.g., be deduced from Lemma 9.2 and Thm.3.11 of Chap.VII in Dunford-Schwartz (1958).

Section 3. The general theory of spectral decompositions can be found in [Kato (1966), Chap.III,§ 6.4]. Applications to isolated singularities like 3.6 are discussed extensively in [l.c., Chap.III, §6.5] and [Yosida (1965), Chap.VIII, Sec.8]. There are many ways to introduce an "essential spectrum" (see the footnote on page 243 of Kato (1966)). However each one yields the same "essential spectral radius".

Section 4. These results are taken from Derndinger (1980) and Greiner (1981).

Section 5. Periodic semigroups are studied explicitely in Bart (1977) but most of the results of this section seem to be well known.

Section 6. The Spectral Inclusion Theorem 6.2 and the Spectral Mapping Theorem 6.6 for eventually norm continuous semigroups date back to Hille-Phillips (1957). Davies (1980) gives an elegant proof using Banach algebra techniques. Tensor products of operators and their spectral theory have been studied by Ichinose and others (see Chap. XIII.9 of Reed-Simon (1978)). The spectral mapping theorem in Corollary 6.8 generalizes Thm.XIII.35 of Reed-Simon (1978) (see also Herbst (1982)).

Section 7. Matrix valued multiplication semigroups appear as solution, via Fourier transformation, of systems of partial differential equations. Kreiss initiated a systematic investigation (see, e.g., Kreiss (1958), Kreiss (1959), Miller-Strang

(1966)) and the Weak Spectral Mapping Theorem 7.2 must have been known to him. The direct proof of the Weak Spectral Mapping Theorem 7.4 for polynomially bounded groups seems to be new. The result can also be deduced from the theory of spectral subspaces of group representations (see, e.g., Combes-Delaroche (1978)), since the Arveson spectrum of a strongly continuous one-parameter group can be identified with the spectrum of the generator (see Evans (1976)). The final part of this section is taken from Greiner (1985) and yields a new approach to Gearhart's characterization of the spectrum of semigroups on Hilbert spaces [Gearhart (1978)]. Different proofs can be found in Herbst (1983), Howland (1984) and Prüß (1984).