CHAPTER C-II

CHARACTERIZATION OF POSITIVE SEMIGROUPS ON BANACH LATTICES

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In this chapter our first goal is to find conditions on a generator A of a semigroup $(T(t))_{t\geq 0}$ which are equivalent to the positivity of the semigroup. After the preparations in A-II,Sec.2 this is easy if in addition we ask that the semigroup be contractive: $T(t)$ is a positive contraction for all $t \geq 0$ if and only if A is dispersive (Section 1). For arbitrary (not necessarily contractive) semigroups a condition on the generator had been found in the case when $E = C(K)$ (K compact), namely the positive minimum principle (P) (see B-II). One may easily reformulate this condition in arbitrary Banach lattices and show its necessity. However, only in special cases (for example if A is bounded (see Section 1)) the positive minimum principle is sufficient for the positivity of the semigroup. In fact, on $L^2(\mathbb{R})$ there exists a non-positive semigroup whose generator satisfies (P) (Section 3).

Looking for another condition we consider the Laplacian Δ as a prototype. Defined on a suitable domain, Δ generates a positive semigroup on $L^p(\mathbb{R}^n)$. Kato proved the following distributional inequality for the Laplacian:

$$(\text{sign } \bar{f})\ \Delta f \leq \Delta |f|$$

for all $f \in L^1_{loc}$ such that $\Delta f \in L^1_{loc}$. In Section 3 we will show that an abstract version of Kato's inequality for a generator A together with an additional condition is equivalent to the positivity of the semigroup generated by A .

Domination of one semigroup by another can be characterized by an analoguous condition for the generators (Section 4). The results will be applied to Schrödinger operators on $L^p(\mathbb{R}^n)$.