Thm.2.4). There are several sufficient conditions on the generator A, the forcing term $F(\cdot)$ or the space E such that every mild solution is a strong solution of (2.1) (see Travis (1979) or Pazy (1983) Sec.4.2).

It is our aim in this section to study the asymptotic behavior of the solutions of (2.1) as $t \to \infty$. To that purpose we consider absolutely integrable or periodic forcing terms $F(\cdot)$, and assume the semigroup to be uniformly stable.
Similar results for integrable and convergent forcing terms $F(\cdot)$ can be obtained if the semigroup is supposed to be uniformly exponentially stable (see Pazy (1983), p.119 or Neubrander (1985b)). However, if the semigroup is positive, these results even hold for stable semigroups (see Section C-IV). From Theorem 1.13.(i) we recall that for stable semigroups $\operatorname{im} A$ is dense in E.

<u>Theorem</u> 2.1. Let A be the generator of a uniformly stable semigroup $(T(t))_{t\geq 0}$ on a Banach space E. If there is $g \in \operatorname{im} A$ such that $\int_0^\infty \|F(s) - g\|\, ds$ exists, then every generalized solution $u(\cdot)$ of (2.1) converges as $t \to \infty$ and $\lim_{t\to\infty} u(t) = -h$ where $h \in D(A)$ with $Ah = g$.

<u>Proof</u>. If $u(\cdot)$ is a generalized solution of (2.1), then, by (2.2), $u(t) = T(t)f + \int_0^t T(s)g\, ds + \int_0^t T(t-s)(F(s)-g)\, ds$. By the uniform stability and Thm.1.14 we see that the first term converges to zero and that the second one converges to $-h$. We have to show that the third term converges to zero. Take $\varepsilon > 0$ and $G(s) := F(s)-g$. Then

$$\begin{aligned}\left\|\int_0^t T(t-s)G(s)\, ds\right\| &\leq \left\|\int_0^{t'} T(t-t'+t'-s)G(s)\, ds\right\| + \left\|\int_{t'}^t T(t-s)G(s)\, ds\right\| \\ &\leq \left\|T(t-t')\int_0^{t'} T(t'-s)G(s)\, ds\right\| + M\int_{t'}^\infty \|G(s)\|\, ds .\end{aligned}$$

Since the semigroup is uniformly stable we obtain
$T(t-t')\int_0^{t'} T(t'-s)G(s)\, ds \to 0$ as $t \to 0$ for every $t' \geq 0$.
Therefore $\left\|\int_0^t T(t-s)G(s)\, ds\right\| \leq \varepsilon$ for all sufficiently large t.

□

In the subsequent theorem we see that if A is the generator of a uniformly stable semigroup, if the forcing term $F(\cdot)$ is p-periodic and if $\int_0^p T(p-s)F(s)\, ds \in \operatorname{im}(Id - T(p))$ (notice that, by Thm.1.13. (i) and A-III,Lemma 5.3, $\overline{\operatorname{im}(Id - T(p))} = E$), then (2.1) admits a unique p-periodic, asymptotically stable mild solution.