of Lunardi's results which were obtained under the more restrictive assumption of a C^2-boundary. Positivity and irreducibility are a consequence of the Alexandrov maximum principle. For C^2-boundary also results on $L^p(\Omega)$ spaces are obtained by Denk, Hieber and Prüss [DHP03] whose main interest lies in proving maximal regularity and establishing a bounded H^∞-calculus.

However, in the situation of Theorem 4.6, without assuming merely the uniform exterior cone condition on Ω, it seems not to be known whether the semigroup extends to a strongly continuous semigroup on $L^p(\Omega)$ for some $p \in [1, \infty)$.

Theorem 4.6 is extended by Arendt and Schätzle [AS 25] to unbounded open sets which statisfy the locally uniform exterior cone condition. However, in the case of unbounded Ω the semigroup converges merely strongly to 0 (and not exponentially fast).

The Dirichlet-to-Neumann operator on $C(\partial\Omega)$.

Let Ω be a bounded, open, connected subset of $\mathbb{R}^n$ with Lipschitz boundary and let $V \in L^\infty(\Omega)$. We consider the Dirichlet-to-Neumann operator with respect to $\Delta - V$ on the space $C(\partial\Omega)$. For that we first establish well-posedness of the Dirichlet Problem.

We assume throughout this subsection that

$$u \in C_0(\Omega),\ \Delta u - Vu = 0 \text{ implies } u = 0 \qquad (4.2)$$

This is exactly the condition that the solutions of the Dirichlet problem with respect to $\Delta - V$ formulated in Proposition 4.7 are unique. An equivalent condition is

$$u \in H_0^1(\Omega),\ \Delta u - Vu = 0 \text{ implies } u = 0; \qquad (4.3)$$

(which means that $0 \notin \sigma(\Delta_D - V)$ where Δ_D is the Dirichlet Laplacian on $L^2(\Omega)$).

<u>Proposition</u> 4.7. Assume (4.2).

Let $g \in C(\partial\Omega)$.

Then there exists a unique $u_g \in C(\bar{\Omega})$ such that

$$(\Delta - V)u_g = 0 \quad \text{and} \quad u_g|_{\partial\Omega} = g.$$

Thus, u_g is a harmonic function with respect to to $\Delta - V$ which has to be understood in the sense of distributions; i.e.,

$$\int_\Omega u_g \Delta\varphi - \int_\Omega V u_g \varphi = 0$$

for all $\varphi \in C_c^\infty(\Omega)$.

For a simple proof of Proposition 4.7 we refer to [AtE19].

Next we define the Dirichlet-to-Neumann operator N_V with respect to $\Delta - V$ on $C(\partial\Omega)$ as follows.

$$D(N_V) := \{ g \in C(\partial\Omega) : u_g \in H^1(\Omega), \text{ and } \partial_\nu u_g \in C(\partial\Omega) \}$$

$$N_V g := -\partial_\nu u_g .$$

Recall that $\partial_\nu u_g \in C(\partial\Omega)$ means that there exists $h \in C(\partial\Omega)$ such that

$$\int_\Omega \Delta u_g \cdot \varphi + \int_\Omega \nabla u_g \nabla \varphi = \int_{\partial\Omega} h\varphi$$

for all $\varphi \in C^1(\bar{\Omega})$. Then we put

$\partial_\nu u_g := h$.

We will need the hypothesis that $-\Delta_D + V$ is <u>form-positive</u>, i.e.

$$\int_\Omega \left(|\nabla u|^2 + V|u|^2 \right) \geq 0 \quad (4.4)$$

for all $u \in H_0^1(\Omega)$.

Theorem 4.8. Assume (4.2) and (4.4). Then $\Delta - V$ generates a positive, irreducible semigroup on $C(\partial\Omega)$. If $V \geq 0$, then the semigroup is contractive.

If Ω is of class C^∞ similar results have been obtained by Escher [Es 94] and Engel [En 03]. Under the very general conditions here, Theorem 4.8 is due to Arendt and ter Elst [AtE 20]. There it is shown that N_V is resolventpositive and that the domain is dense (which is the main difficulty). Then by B-II, Theorem 1.8 N_V generates a positive semigroup. Irreducibility is surprising. In fact, even though Ω is supposed to be connected, $\partial\Omega$ might not be connected (consider a ring for example).

The fact that the semigroup is irreducible shows that the operator N_V is non-local in quite a dramatic way.

A first result on irreducibili-
ty (on $L^2(\partial\Omega)$) was obtained
by Arendt and Mazzeo
[AM12].

It is not known so far whether the semigroup generated by N_V is holomorphic if Ω has Lipschitz boundary. If the boundary is of class $C^{1+\varkappa}$ with $\varkappa > 0$, then it is holomorphic of angle $\pi/2$. This is due to ter Elst and Ouhabaz [tEO 19].

The operator N_V is also called <u>voltage-to-current map</u> and has physical meaning. One version of the famous Calderon-Problem is the question whether for $V_1, V_2 \in L^\infty(\Omega)$, such that $0 \notin \sigma(\Delta_{V_1}) \cup \sigma(\Delta_{V_2})$, $N_{V_1} = N_{V_2}$ implies $V_1 = V_2$.

This is true under the only

assumption that Ω has Lip-

schitz boundary; see Theorem 1.1

by Krypchyk and Uhlemann

[KU14].

Finally we mention that N_V

may generate a positive semigroup

even if (4.4) is violated.
This and other surprising
phenomena were discovered

by Daners [Da14]. and led