$f := \psi_1 - \psi_2$ is different from zero. If $f = f^+ - f^-$ is the Jordan decomposition of f, then f^+ and f^- are elements of $\mathrm{Fix}(\mathcal{T})$, whence faithful. Since the support projections of these two normal linear functionals are orthogonal, we obtain $f^+ = 0$ or $f^- = 0$ which implies $\psi_1 \leq \psi_2$ or $\psi_2 \leq \psi_1$. Consequently $\psi_2 = \psi_1$. Since $\mathrm{Fix}(\mathcal{T})$ is positively generated (Corollary 1.5), $\mathrm{Fix}(\mathcal{T}) = \mathbb{C}\phi$ for some faithful normal state ϕ.

Let $\mu \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ such that $i\alpha \in P\sigma(A)$. If $\psi_\alpha = u_\alpha|\psi_\alpha|$ is a normalized eigenvector of A pertaining to $i\alpha$, then $\phi = |\psi_\alpha| = |\psi_\alpha{}^*|$ by Corollary 1.5 and the above considerations. Hence $u_\alpha u_\alpha{}^* = u_\alpha{}^* u_\alpha = s(\phi) = 1$. Since

$$(\mu - i\alpha)R(\mu,A)\psi_\alpha = \psi_\alpha$$

and

$$\mu R(\mu,A)|\psi_\alpha| = |\psi_\alpha|$$

we obtain by Lemma 1.2.b that

$$(1) \quad \mu R(\mu,A) = V_\alpha \circ \mu R(\mu+i\alpha,A) \circ V_\alpha^{-1} ,$$

where V_α is the map $(x \to xu_\alpha)$ on M. Similarly for $i\beta \in P\sigma(A)$, we find V_β such that $1 = u_\beta u_\beta{}^* = u_\beta u_\beta{}^*$ and

$$(2) \quad \mu R(\mu,A) = V_\beta \circ \mu R(\mu+i\beta,A) \circ V_\beta^{-1} .$$

Hence

$$(3) \quad \mu R(\mu,A) = V_{\alpha\beta} \circ \mu R(\mu+i(\alpha+\beta),A) \circ V_{\alpha\beta}^{-1} ,$$

where $V_{\alpha\beta} := V_\alpha \circ V_\beta$. Since u_α is unitary in M, it follows from (1) that $i\alpha$ is an eigenvalue which is simple because $\mathrm{Fix}(\mathcal{T}) = \mathrm{Fix}(\mu R(\mu,A))$ is one dimensional. From (3) it follows that $i(\alpha+\beta) \in P\sigma(A)$ since $0 \in P\sigma(A)$ and $V_{\alpha\beta}$ is bijective. From the identity (1) we conclude that $\sigma(R(\mu,A)) = \sigma(R(\mu+i\alpha))$, which proves

$$\sigma(A) + (P\sigma(A) \cap i\mathbb{R}) \subseteq \sigma(A) .$$

The other inclusion is trivial since $0 \in P\sigma(A)$.

□