$$\begin{aligned}\|w(t)\|_F &= \|\Phi(w_t)\|_F \leq \|\Phi\|\cdot\|w_t\|_E = \|\Phi\|\cdot\int_{-1}^{0} \|w_t(s)\|_F\, ds\\ &= \|\Phi\|\cdot\int_{-1}^{0} \|w(t+s)\|_F ds = \|\Phi\|\cdot\int_{t-1}^{t} \|w(s)\|_F\, ds\\ &\leq \|\Phi\|\cdot\int_{-1}^{t} \|w(s)\|_F ds = \|\Phi\|\cdot\int_{0}^{t} \|w(s)\|_F ds \quad \text{for} \quad t \geq 0.\end{aligned}$$

By Gronwall's lemma $\|w(t)\|_F = 0$, thus $w(t) = 0$.

□

Now we turn to the aspect of positivity in (RE). We assume F to be a Banach lattice and let E inherit the canonical ordering from F making it a Banach lattice. Additionally, let Φ be positive.
The first observation is that A generates a positive semigroup. Indeed, it follows from equation (3.2) that $R(\lambda,A) = R(1,S_\lambda)R(\lambda,A_o)$ for $\lambda > \|\Phi\|$. Since S_λ is a positive operator we have $R(1,S_\lambda) \geq 0$. The semigroup $(T_o(t))_{t\geq 0}$ generated by A_o is positive (use (3.1)), hence $R(\lambda,A_o) \geq 0$. It follows that $R(\lambda,A) \geq 0$ which is equivalent to the positivity of $(T(t))_{t\geq 0}$ (see C-II,Prop.4.1).

Proposition 3.3. If $\Phi \in L(E,F)$ is a positive operator, then the solution semigroup $(T(t))_{t\geq 0}$ corresponding to (RE) is positive.

For the following considerations concerning spectral poperties of the semigroup $(T(t))_{t\geq 0}$ we always suppose Φ to be positive. Furthermore we define operators $\Phi_\lambda \in L(F)$, $\lambda \in \mathbb{R}$, by

(3.6) $\quad \Phi_\lambda x := \Phi(\varepsilon_\lambda \otimes x)$, $x \in F$.

Evidently, each Φ_λ is a positive operator on F and $\lambda \leq \mu$ implies $\Phi_\lambda \geq \Phi_\mu$. From this relation it follows that the spectral bound $s(\Phi_\lambda)$ which coincides with the spectral radius $r(\Phi_\lambda)$ is a decreasing function in λ.

Furthermore, we shall need the following properties.

Lemma 3.4. The map $h: \mathbb{R} \to \mathbb{R}_+ : \lambda \to s(\Phi_\lambda)$ is continuous from the left. If Φ_λ is compact for all $\lambda \in \mathbb{R}$, then h is continuous.

Proof. As indicated above, h is decreasing. Hence continuity from the left follows from the upper semicontinuity of the spectrum (see [Kato (1976), Chap.IV,Thm.3.1]).
Now take $\lambda \in \mathbb{R}$ with $s(\Phi_\lambda) > 0$ (if $s(\Phi_\lambda) = r(\Phi_\lambda) = 0$, then continuity in λ is trivial by the continuity from the left and the monotonicity). Since Φ_λ is positive and bounded we know that