(b) If there exists a faithful family Ψ of subinvariant states for T on M, then $\mathrm{Fix}(T)$ is a C*-subalgebra of M and $T(xy) = xT(y)$ for all $x\in\mathrm{Fix}(T)$ and $y\in M$.

Proof. (a) Take $0 \leq \psi\in M^*$ and consider $f := \psi\circ b$. Then f is a positive semidefinite sesquilinear form on M with values in $\mathbb{C}$. From the Cauchy-Schwarz inequality it follows that $f(x,x) = 0$ for some $x\in M$ if and only if $f(x,y) = 0$ and $f(y,x) = 0$ for all $y\in M$. Since the positive cone M^*_+ is generating, assertion (a) is proved.

(b) Since T is positive it follows $T(x)^* = T(x^*)$ for all $x\in M$. Hence $\mathrm{Fix}(T)$ is a self adjoint subspace of M, i.e. invariant under the involution on M. For every $x,y\in M$ let

$$b(x,y) := T(xy^*) - T(x)T(y)^* .$$

Then b satisfies the assumptions of (a). If $x\in\mathrm{Fix}(T)$ then

$$0 \leq xx^* = (Tx)(Tx)^* \leq T(xx^*) ,$$

hence

$$0 \leq \psi(T(xx^*) - xx^*) \leq 0 \quad \text{for all} \quad \psi\in\Psi .$$

But this implies $T(xx^*) = T(x)T(x)^* = xx^*$. Consequently, $b(x,x) = 0$. Hence $T(xy^*) = xT(y)^*$ for all $y\in M$ and (b) is proved.

□

Lemma 1.2. Let M be a W*-algebra, T an identity preserving Schwarz map on M and $S\in\mathcal{L}(M)$ such that $S(x)(Sx)^* \leq T(xx^*)$ for every $x\in M$.

(a) If $v\in M$ such that $S(v^*) = v^*$ and $T(v^*v) = v^*v$, then $T(xv) = S(x)v$ for all $x\in M$.

(b) Suppose there exists $\phi\in M_*$ with polar decomposition $\phi = u|\phi|$ such that $S_*\phi = \phi$ and $T_*|\phi| = |\phi|$. If the closed subspace $s(|\phi|)M$ is T-invariant, then $Su^* = u^*$ and $T(u^*u) = u^*u$.

Proof. (a) Define a positive semidefinite sesquilinear map $b\colon M\times M \to M$ by

$$b(x,y) := T(xy^*) - S(x)S(y)^* \quad (x,y\in M) .$$