$$= \frac{1}{2}\left(\int_0^\infty\int_0^\infty e^{-\mu(r+t)}((S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^*)\,drdt\right.$$

$$\leq \frac{1}{2}\left(\int_0^\infty\int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*)\,drdt\right.$$

$$= \left(\int_0^\infty e^{-\mu s}ds\right)\left(\int_0^\infty e^{-\mu t}T(t)xx^*dt\right) = \mu^{-1}R(\mu,A)xx^*$$

where the handling of the integral is justified by [Bourbaki (1955), §8, n$^{\mathrm{o}}$ 4, Proposition 9].

□

<u>Corollary</u> 2.2. Let $\mathcal{T}$ be a semigroup of Schwarz maps (resp., weak*-semigroup of Schwarz maps). Then for all $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$:

$$(R(\lambda,A)x)(R(\lambda,A)x)^* \leq (\mathrm{Re}\lambda)^{-1}\, R(\mathrm{Re}\lambda,A)xx^* \quad , \quad x\in M .$$

In particular for all $(\mu,\alpha)\in\mathbb{R}_+\times\mathbb{R}$, $x\in M$:

$$(\mu R(\mu+i\alpha,A)x)(\mu R(\mu+i\alpha,A)x)^* \leq \mu R(\mu,A)(xx^*) .$$

<u>Proof</u>. Let $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Then the semigroup

$$\mathcal{S} := (e^{-i\mathrm{Im}(\lambda)t}T(t))_{t\geq 0}$$

fulfils the assumption of Thm 2.1. and $B := A - i\lambda$ is the generator of $\mathcal{S}$. Consequently $R(\lambda,A) = R(\mathrm{Re}\lambda,B)$ and the corollary follows from Theorem 2.1.

□

As in Section C-III the following notion will be an important tool for the spectral theory of semigroups.

<u>Definition</u> 2.3. Let E be a Banach space and $\emptyset \neq D$ an open subset of $\mathbb{C}$. A family $R\colon D \to L(E)$ is called a <u>pseudo-resolvent on</u> D <u>with values in</u> E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu)$$

for all λ and μ in D .