A-II, Thm.2.6 that $p(e^{-\lambda t}T(t)f) \leq p(f)$ $(f \in E, t \geq 0)$. Hence,

(3.5) $\langle (T(t)f)^+, \phi\rangle \leq e^{\lambda t} \langle f^+, \phi\rangle$ $(f \in E, t \geq 0)$.

Now let $t > 0$ and $f \leq 0$; then $f^+ = 0$. It follows from (3.5) that $\langle (T(t)f)^+,\phi\rangle \leq 0$. Since $\phi \in M'$ is arbitrary and M' is strictly positive, it follows that $(T(t)f)^+ = 0$; i.e., $T(t)f \leq 0$. This implies that $T(t) \geq 0$.

□

Remark 3.11. a) The proof of Theorem 3.8 shows the following. If A is the generator of a positive semigroup and E' contains strictly positive linear forms, then there exist a continuous half-norm p on E and $w \in \mathbb{R}$ such that $A - w$ is p-dissipative. We stress that p cannot be replaced by the norm (or by N^+), since in general none of the semigroups $(e^{-wt}T(t))_{t\geq 0}$ $(w \in \mathbb{R})$ is contractive for the norm (cf. Derndinger (1984) and Batty-Davies (1982)).
b) Using Proposition 3.10 one can show with the help of the proof of A-II, Prop.2.9 that a densely defined operator is closable whenever there exists a strictly positive set M' of subeigenvectors of A' such that (K) holds for all $f \in D(A)$ and $\phi \in M'$.

Remark 3.12 In Theorem 3.8 and Corollary 3.9 one can replace inequality (K) by the inequality

(3.6) $$\langle P_{(f^+)}Af,\phi\rangle \leq \langle f^+,A'\phi\rangle ,$$

(with the notation of Prop.3.10).

Indeed, (3.6) for $-f$ yields $\langle -P_{(f^-)}Af,\phi\rangle \leq \langle f^-,A'\phi\rangle$. Adding up both inequalities one obtains $\langle (\text{sign } f)Af,\phi\rangle \leq \langle |f|,A'\phi\rangle$.
On the other hand, if A generates a positive semigroup, one sees by the obvious alterations in the proof of Theorem 2.4 that (3.6) holds for all $f \in D(A)$ and $\phi \in D(A')_+$.
Next we formulate the result for the space $C_o(X)$, where X is a locally compact space (concerning the notation cf. Thm.2.6 and Sec.2 of B-II).

Theorem 3.13. Let A be the generator of a semigroup on $C_o(X)$. The semigroup is positive if and only if there exists a core D_o of A and a strictly positive set M' of subeigenvectors of A' such that

(K) $\langle (\text{sign } f)Af,\phi\rangle \leq \langle |f|,A'\phi\rangle$ for all $f \in D_o$, $\phi \in M'$.