group) homomorphism $\tau : t \to T(t)$ are topological semigroups for the natural topology on $\mathbb{R}_+$ and any one of the standard operator topologies on $\mathcal{L}(E)$. We single out the strong operator topology on $\mathcal{L}(E)$ and require τ to be continuous.

<u>Definition</u> 1.1. A one-parameter semigroup $(T(t))_{t\geq 0}$ is called <u>strongly continuous</u> if the map $t \to T(t)$ is continuous for the strong operator topology on $\mathcal{L}(E)$, i.e. $\lim_{t\to t_0} \|T(t)f - T(t_0)f\| = 0$ for every $f \in E$ and $t, t_0 \geq 0$.

Clearly one defines in a similar way <u>weakly continuous</u>, resp. <u>uniformly continuous</u> (compare A-II, Def..1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology:

> If not stated otherwise, a <u>s e m i g r o u p</u> is a strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

<u>Remarks</u> 1.2. (1) A one-parameter semigroup $(T(t))_{t\geq 0}$ on a Banach space E is strongly continuous if and only if for any $f \in E$ it is true that $T(t)f \to f$ as $t \to 0$.

(2) For every strongly continuous semigroup $(T(t))_{t\geq 0}$ there exist constants $M \geq 1$, $w \in \mathbb{R}$ such that $\|T(t)\| \leq M\cdot e^{wt}$ for every $t \geq 0$.

(3) If $(T(t))_{t\geq 0}$ is a one-parameter semigroup such that $\|T(t)\|$ is bounded for $0 < t \leq \delta$ then it is strongly continuous if and only if $\lim_{t\to 0} T(t)f = f$ for every f in a <u>total subset</u> of E.

The exponential estimate from Remark 1.2,(2) for the growth of $\|T(t)\|$ can be used to define an important characteristic of the semigroup.

<u>Definition</u> 1.3. By the <u>growth bound</u> (or <u>type</u>) of the semigroup $(T(t))_{t\geq 0}$ we understand the number

$$\begin{aligned}\omega &:= \inf\{w \in \mathbb{R} : \text{There exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leq Me^{wt} \text{ for } t \geq 0\} \\ &= \lim_{t\to\infty} 1/t\cdot\log\|T(t)\| = \inf_{t>0} 1/t\cdot\log\|T(t)\| .\end{aligned} \tag{1.1}$$