So it follows from (5.9) and the dominated convergence theorem that

$$\langle D_h\theta(f),\phi\rangle = \lim_{t\to 0+} 1/t \langle(|f + th| - |f|),\phi\rangle$$
$$= \int_K \mathrm{Re}(\mathrm{sign}(\bar{f}(x))h(x))\, d\mu(x) = \langle \mathrm{Re}((\mathrm{sign}\ \bar{f})h),\phi\rangle$$

(the last identity holds since by the definition of $\mathrm{sign}\ \bar{f} \in L(E)$, we have $(\mathrm{sign}\ \bar{f})h \in E_{|f|} = C(K)$ whenever $h \in C(K)$ and $((\mathrm{sign}\ \bar{f})h)(x) = (\mathrm{sign}\ \overline{f(x)})h(x)$ (see C-I,Sec.8)).
Consequently, $D_h\theta(f) = \mathrm{Re}(\mathrm{sign}\ \bar{f})h$ whenever $h \in E_{|f|}$. Since $D_h\theta(f)$ is continuous in h (in fact, $|D_h\theta(f) - D_k\theta(f)| \leq |h - k|$ for all $h,k \in E$) and $E_{|f|}$ is dense in $\{f\}^{dd}$, it follows that (5.8) holds for all $h \in \{f\}^{dd}$.

□

Remark 5.7. a) By the same argument as given in the proof one sees that θ is left-sided Gateaux differentiable and

$$D_g^-\theta(f) = \mathrm{Re}((\mathrm{sign}\ \bar{f})g) - P_f^d|g|$$

for all $f, g \in E$, where $D_g^-\theta(f) = \lim_{t\uparrow 0} 1/t(\theta(f + tg) - \theta(f))$ and P_f^d denotes the band projection onto $\{f\}^d$. In particular,

$$D_g^+\theta(f) = D_g^-\theta(f) \quad \text{whenever} \quad g \in \{f\}^{dd} . \tag{5.10}$$

b) The proof of Prop.5.6 shows that every convex function $\theta : E \to E_{\mathbb{R}}$ (where E is a Banach lattice with order continuous norm) is right- (and left-) sided Gateaux differentiable (cf. Arendt (1982)).

Proof of Theorem 5.5. Assume that (i) holds. Let $f \in D(B)$. Then $S(t)f$ is differentiable in t . By the chain rule B-II,Prop.2.3, $T(t)|f| = |S(t)f|$ is also differentiable and $d/dt_{|t=0}\ T(t)|f| = d/dt_{|t=0}\ |S(t)f| = \mathrm{Re}(\mathrm{si\hat{g}n}\ \bar{f})Bf$ (by Prop.5.6). Hence $|f| \in D(A)$ and $A|f| = \mathrm{Re}(\mathrm{si\hat{g}n}\ \bar{f})Bf$.
Conversely, assume that (ii) holds. Let $s > 0$, $f \in E$. We show that $|S(s)f| = T(s)|f|$. This implies that $S(s)$ is disjointness preserving and $|S(s)| = T(s)$ (by Proposition 5.1)). Since $D(B)$ is dense we can assume that $f \in D(B)$. Let $\xi(t) = T(s-t)|S(t)f|$ $(t \in [0,s])$. Since by assumption $|S(t)f| \in D(A)$ one obtains

$$d/dt\ \xi(t) = -AT(s-t)|S(t)f| + T(s-t)d/dr_{|r=t}\ |S(r)f|$$
$$= -AT(s-t)|S(t)f| + T(s-t)(\mathrm{Re}(\mathrm{si\hat{g}n}\ \overline{S(t)f})BS(t)f)$$

(by Prop.5.6 and the chain rule B-II,Prop.2.3)

$$= 0 \quad \text{by the assumption (ii).}$$

Hence $\xi(0) = \xi(s)$; i.e., $|S(s)f| = T(s)|f|$.

□