In order to solve (RE) we consider the differential operator $A := \frac{d}{dx}$ on $E = L^1([-1,0],F)$ with domain

$$D(A) := \{f \in AC([-1,0],F) : f' \in E \text{ and } f(0) = \Phi(f)\} .$$

We claim that $(A,D(A))$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E. To this end we first consider the operator $A_o f := f'$ with domain

$$D(A_o) := \{f \in E : f \in AC([-1,0],F) , f' \in E \text{ and } f(0) = 0\} .$$

Similarly to the special case where $F = \mathbb{R}$ (compare A-I,Ex.2.4.(ii)) it can be seen that the operator A_o generates a strongly continuous semigroup $(T_o(t))_{t\geq 0}$ given by

$$(3.1) \qquad (T_o(t)f)(s) = \begin{cases} f(t+s) & \text{if } t+s \leq 0 \\ 0 & \text{if } t+s > 0 . \end{cases}$$

Notice that $(T_o(t))_{t\geq 0}$ is a nilpotent semigroup.
Now consider the operators $S_\lambda : E \to E : f \to \varepsilon_\lambda \otimes \Phi(f)$, $\lambda > 0$, where ε_λ denotes the function $s \to e^{\lambda s}$ as an element of $L^1[-1,0]$ and $h\otimes x \in E$ is defined by $(h\otimes x)(s) := h(s)\cdot x$ for $h \in L^1[-1,0]$, $x \in F$ and $s \in [-1,0]$. Clearly $\|\varepsilon_\lambda\| = 1/\lambda\cdot(1 - e^{-\lambda}) \to 0$ as $\lambda \to \infty$ and we have $\|S_\lambda\| = \|\varepsilon_\lambda\| \cdot \|\Phi\| = 1/\lambda\cdot(1 - e^{-\lambda})\cdot\|\Phi\| \leq 1/\lambda\cdot\|\Phi\|$.
For every $\lambda > \|\Phi\|$, $(Id - S_\lambda)$ is an isomorphism of E and it is not difficult to see that it induces a bijection from $D(A)$ onto $D(A_o)$ such that

$$(3.2) \qquad (\lambda - A) = (\lambda - A_o)(Id - S_\lambda) .$$

Since A_o generates a semigroup of contractions $\lambda - A_o$ is invertible for each $\lambda > 0$. This yields the invertibility of $\lambda - A$ for each $\lambda \geq \|\phi\|$.
In order to obtain an estimate on $\|R(\lambda,A)\|$ we use Formula (3.2). Since $\|R(1,S_\lambda)\| = \|\sum_{n=0}^{\infty} S_\lambda^n\| \leq \sum_{n=0}^{\infty} \|\varepsilon_\lambda\|^n\cdot\|\Phi\|^n = (1-\|\varepsilon_\lambda\|\cdot\|\Phi\|)^{-1}$ and $\|R(\lambda,A_o)\| \leq 1/\lambda$ for $\lambda > 0$ we obtain for $\lambda \geq \|\Phi\|$:

$$\begin{aligned}\|R(\lambda,A)\| &\leq (1 - \|\varepsilon_\lambda\|\cdot\|\Phi\|)^{-1}\cdot 1/\lambda = (\lambda - \lambda\cdot\|\varepsilon_\lambda\|\cdot\|\Phi\|)^{-1} \\ &= (\lambda - (1 - e^{-\lambda})\cdot\|\Phi\|)^{-1} \leq (\lambda - \|\Phi\|)^{-1} .\end{aligned}$$

By using A-II,Cor.1.8 we thus have proved the first assertion of the following theorem: