Example 3.12. Let $F := \mathbb{C}^n$, $E := L^1([0,r],F) = \Pi_{k=1}^n F_k$ , $F_k = L^1[0,r]$. and define $\Phi : E \to \mathbb{C}^n$ by $\Phi = (\nu_{ij})_{n\times n}$ where $\langle \nu_{ij}, f\rangle = \int_0^r \beta_{ij}(a)f(a)da$ for $f \in L^1[0,r]$ and $0 \leq \beta_{ij} \in L^\infty[0,r]$. As $\Phi_\lambda$ we obtain the scalar matrix,

$$\Phi_\lambda = \begin{pmatrix} (\int_0^r \beta_{11}(a)e^{-\lambda a}da) & (\int_0^r \beta_{12}(a)e^{-\lambda a}da) & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ (\int_0^r \beta_{n1}(a)e^{-\lambda a}da) & & & (\int_0^r \beta_{nn}(a)e^{-\lambda a}da) \end{pmatrix}.$$

Suppose additionally that $\Phi_\lambda$ is irreducible for each $\lambda$, which is, for example, satisfied if $\beta_{ij}(a) > 0$ for every $a \in [0,r]$ and $1 \leq i,j \leq n$ (see also [Bellman-Cooke (1963),p.257ff]).
Since $\Phi$ has finite dimensional range and hence is compact it follows that the function $h : \lambda \to s(\Phi_\lambda)$ is continuous. Moreover one shows that $h$ is strictly decreasing by using the same arguments as in Example 3.10 and by using the fact that $\Phi_\lambda$ is irreducible.
The system of differential equations corresponding to $A$ is

$$\frac{\partial}{\partial t}u_i(t,\alpha) = -\frac{\partial}{\partial \alpha}u_i(t,\alpha) \quad (i=1,\dots,n) \quad \text{for } t\in\mathbb{R}_+ ,\ \alpha\in[0,r]$$

with initial condition

$$u_i(0,\alpha) = v_i(\alpha) \quad (i=1,\dots,n) \quad \text{for } \alpha\in[0,r] \tag{3.13}$$

and boundary condition

$$u_i(t,0) = \int_0^r [\textstyle\sum_{j=1}^n \beta_{ij}(\alpha)u_j(t,\alpha)]d\alpha \quad (i=1,\dots,n) \quad \text{for } t\in\mathbb{R}_+.$$

This system has a solution for every $(v_1,\dots,v_n) \in D(A)$ and the asymptotic behavior is determined by the identity

$$0=\det\begin{pmatrix} (1-\int_0^r \beta_{11}(a)e^{-\lambda a}da) & (-\int_0^r \beta_{12}(a)e^{-\lambda a}da)\dots & & (-\int_0^r \beta_{1n}(a)e^{-\lambda a}da) \\ (-\int_0^r \beta_{21}(a)e^{-\lambda a}da) & & & \\ & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & (-\int_0^r \beta_{n-1,n}(a)e^{-\lambda a}da) \\ (-\int_0^r \beta_{n1}(a)e^{-\lambda a}da)\dots & & & (1-\int_0^r \beta_{nn}(a)e^{-\lambda a}da) \end{pmatrix},$$

whose unique real solution $\lambda$ is $s(A)$.