The Abstract Cauchy Problem

Let A be a closed operator on a Banach space E and consider the abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \dot{u}(t) = Au(t) \quad (t \geq 0) \\ u(0) = f \, . \end{cases}$$

By a <u>solution</u> of (ACP) for the initial value $f \in D(A)$ we understand a continuously differentiable function $u : [0,\infty) \to E$ satisfying $u(0) = f$ and $u(t) \in D(A)$ for all $t \geq 0$ such that $\dot{u}(t) = Au(t)$ for $t \geq 0$.

By A-I,Thm.1.7 there exists a unique solution of (ACP) for all initial values in the domain $D(A)$ whenever A is the generator of a strongly continuous semigroup. The converse does not hold (see Example 1.4. below). However, for the operator A_1 on the Banach space $E_1 = D(A)$ (see A-I,3.5) with domain $D(A_1) = D(A^2)$ given by $A_1 f = Af$ $(f \in D(A_1))$ the following holds.

<u>Theorem</u> 1.1. The following assertions are equivalent.

(i) For every $f \in D(A)$ there exists a unique solution of (ACP).

(ii) A_1 is the generator of a strongly continuous semigroup.

<u>Proof</u>. (i) implies (ii).
Assume that (i) holds; i.e., for every $f \in D(A)$ there exists a unique solution $u(\cdot,f) \in C^1([0,\infty),E)$ of (ACP). For $f \in E_1$ define $T_1(t)f := u(t,f)$ $(t \geq 0)$. By the uniqueness of the solutions it follows that $T_1(t)$ is a linear operator on E_1 and $T_1(s+t) = T_1(s)T_1(t)$. Moreover, since $u(\cdot,f) \in C^1$, it follows that $t \to T_1(t)f$ is continuous from $[0,\infty)$ into E_1. We show that $T_1(t)$ is a continuous operator for all $t>0$.
Let $t>0$. Consider the mapping $\eta : E_1 \to C([0,t],E_1)$ given by $\eta(f) = T_1(\cdot)f = u(\cdot,f)$. We show that η has a closed graph.
In fact, let $f_n \to f$ in E_1 and $\eta(f_n) = u(\cdot,f_n) \to v$ in $C([0,t],E_1)$. Then $u(s,f_n) = f_n + \int_0^s Au(r,f_n)\,dr$. Letting $n\to\infty$ we obtain $v(s) = f + \int_0^s Av(r)\,dr$ for $0 \leq s \leq t$. Let
$\tilde{v}(s) = T_1(s-t)v(t)$ for $s > t$, and $\tilde{v}(s) = v(s)$ for $0 \leq s \leq t$.