For the proof of Theorem 1.20 we use the following lemma.

Lemma 1.21. Let A be an operator and $\lambda \in \rho(A)$. Then

$$\operatorname{dist}(\lambda,\sigma(A)) = r(R(\lambda,A))^{-1} .$$

Proof. One has $\{0\} \cup \sigma(R(\lambda,A)) = \{0\} \cup \{(\lambda-\mu)^{-1} : \mu \in \sigma(A)\}$ [Davies (1980), Lemma 2.11]. Hence $r(R(\lambda,A)) = \sup\{|\lambda-\mu|^{-1} : \mu \in \sigma(A)\} = [\inf\{|\lambda-\mu| : \mu \in \sigma(A)\}]^{-1} = \operatorname{dist}(\lambda,\sigma(A))^{-1}$.

□

Proof of Thm.1.20. It is enough to show the following. Let $a > \omega(A)$. Then for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $r_o \geq 0$ such that $\|R(a+ir,A)^n\|^{1/n} < \varepsilon$ for all $r \in \mathbb{R}$ satisfying $|r| \geq r_o$.
[In fact, then we have by the lemma,
$\operatorname{dist}(a+ir,\sigma(A)) = r(R(a+ir,A))^{-1} \geq \|R(a+ir,A)^n\|^{-1/n} > 1/\varepsilon$ whenever $|r| \geq r_o$].
So let $\varepsilon > 0$. If $\operatorname{Re}\lambda > \omega(A)$, then by A-I,Prop.1.11, $R(\lambda,A)^{n+1} = 1/n!\int_0^\infty e^{-\lambda t}t^nT(t)\,dt$ $(n \in \mathbb{N})$. Let $t' > 0$ such that $t \to T(t)$ is norm continuous on $[t',\infty)$. Let $w \in (\omega(A),a)$. There exists $M \geq 1$ such that $\|T(t)\| \leq Me^{wt}$ for all $t \geq 0$. Let $N := M\cdot\int_0^{t'} e^{-at}e^{wt}dt$. Since $\lim_{n\to\infty} c^n/n! = 0$ for all $c > 0$, there exists $n \in \mathbb{N}$ such that $N\cdot(t')^n/n! < \varepsilon^{n+1}/3$. Choose $T \geq t'$ such that $1/n!\cdot\int_T^\infty t^ne^{-at}\|T(t)\|dt < \varepsilon^{n+1}/3$.
Since $(T(t))_{t\geq 0}$ is norm continuous for $t \geq t'$, it follows from the Riemann-Lebesgue lemma that there exists $r_o \geq 0$ such that
$\|1/n!\cdot\int_{t'}^T t^ne^{-irt}e^{-at}T(t)dt\| < \varepsilon^{n+1}/3$ whenever $|r| \geq r_o$.
All together we obtain for $|r| \geq r_o$,

$$\begin{aligned}
\|R(a+ir,A)^{n+1}\| &= 1/n!\cdot\left\|\int_0^\infty e^{-(a+ir)t}t^nT(t)dt\right\| \\
&\leq 1/n!\cdot\int_0^{t'} e^{-at}t^n\|T(t)\|dt \\
&\quad + 1/n!\cdot\left\|\int_{t'}^T t^ne^{-irt}e^{-at}T(t)dt\right\| \\
&\quad + 1/n!\cdot\int_T^\infty e^{-at}t^n\|T(t)\|dt \\
&\leq 1/n!\cdot(t')^n\int_0^{t'} e^{-at}Me^{wt}dt + 2/3\cdot\varepsilon^{n+1} \\
&\leq N\cdot(t')^n/n! + 2/3\cdot\varepsilon^{n+1} \leq \varepsilon^{n+1}.
\end{aligned}$$

□

A complete characterization of eventually norm continuous semigroups in terms of their generator seems not to be known.
Eventually norm continuous semigroups are of particular interest in spectral theory (cf. A-III,Thm.6.6). Moreover their asymptotic behavior is easy to describe (see A-IV,(1.8)).