spectral values of the generator A into spectral values of the semigroup operator $T(t)$ and vice versa. As shown in Examples 1.3 and 1.4 this is not possible in general. Therefore we tackle first a much easier 'spectral mapping theorem': the relation between $\sigma(A)$ and $\sigma(R(\lambda_o))$, where $R(\lambda_o) := R(\lambda_o,A)$ for some $\lambda_o \in \rho(A)$.

Proposition 2.5. Let $(A,D(A))$ be a densely defined closed linear operator with non-empty resolvent set $\rho(A)$. For each $\lambda_o \in \rho(A)$ the following assertions hold :

(i) $\sigma(R(\lambda_o)) \setminus \{0\} = (\lambda_o - \sigma(A))^{-1}$,
In particular, $r(R(\lambda_o)) = \left(\mathrm{dist}(\lambda_o,\sigma(A))\right)^{-1}$.

(ii) Analogous statements hold for the point-, approximate point-, residual spectra of A and $R(\lambda_o,A)$.

(iii) The point α is isolated in $\sigma(A)$ if and only if $(\lambda_o-\alpha)^{-1}$ is isolated in $\sigma(R(\lambda_o))$. In that case the residues (resp., the pole orders) in α and in $(\lambda_o-\alpha)^{-1}$ coincide.

Proof. (i) is well known. It can be found for example in [Dunford-Schwartz (1958), VII.9.2].
(ii) We show that $\alpha \in A\sigma(A)$ if $(\lambda_o-\alpha)^{-1} \in A\sigma(R(\lambda_o))$ and leave the proof of the remaining statements to the reader. Take $(f_n)_{n\in\mathbb{N}} \subset E$ such that $\|f_n\| = 1$, $\|(\lambda_o-\alpha)^{-1}f_n - R(\lambda_o,A)f_n\| \to 0$ and $\|R(\lambda_o,A)f_n\| \geq \frac{1}{2}|\lambda_o - \alpha|^{-1}$. Define

$$g_n := \|R(\lambda_o,A)f_n\|^{-1}\cdot R(\lambda_o,A)f_n \in D(A)$$

and deduce from

$$\begin{aligned}(\alpha-A)g_n &= \|R(\lambda_o,A)f_n\|^{-1}\cdot[(\lambda_o-A) - (\lambda_o-\alpha)]R(\lambda_o,A)f_n \\ &= \|R(\lambda_o,A)f_n\|^{-1}\cdot(\lambda_o-\alpha)[(\lambda_o-\alpha)^{-1} - R(\lambda_o,A)]f_n\end{aligned}$$

that (g_n) is an approximate eigenvector of A to the eigenvalue α.
(iii) Take a circle Γ with center α and sufficiently small radius. Then the residue P of $R(.,A)$ at α is

$$\begin{aligned}P &= \frac{1}{2\pi i}\int_\Gamma R(z,A)\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma (\lambda_o-z)^{-2}R((\lambda_o-z)^{-1},R(\lambda_o,A))\,dz \\ &\quad - \frac{1}{2\pi i}\int_\Gamma(\lambda_o-z)^{-1}dz\ , \quad \text{(use \$\$).}\end{aligned}$$

If λ_o lies in the exterior of Γ the second integral is zero. The