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One-parameter Semigroups of Positive Operators

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This Latex version of the book "One-Parameter Semigroups of Positive Operators" is dedicated to the memory of our co-authors, Heinrich P. Lotz and Ulf Schlotterbeck. Their contributions to the first edition remain an inspiration to us all. We miss their presence and remain grateful for the legacy they have left in this work.

Preface

As early as 1948 in the first edition of his fundamental treatise on *Semigroups and Functional Analysis*, E. Hille expressed the need for

> ... *developing an adequate theory of transformation semigroups operating in partially ordered spaces* (l.c., Foreword).

In the meantime the theory of one-parameter semigroups of positive linear operators has grown continuously. Motivated by problems in probability theory and partial differential equations W. Feller (1952) and R. S. Phillips (1962) laid the first cornerstones by characterizing the generators of special positive semigroups. In the 60's and 70's the theory of positive operators on ordered Banach spaces was built systematically and is well documented in the monographs of H. H. Schaefer (1974) and A. C. Zaanen (1983). But in this process the original ties with the applications and, in particular, with initial value problems were at times obscured. Only in recent years an adequate and up-to-date theory emerged, largely based on the techniques developed for positive operators and thus recombining the functional analytic theory with the investigation of Cauchy problems having positive solutions to each positive initial value. Even though this development — in particular with respect to applications to concrete evolution equations in transport theory, mathematical biology, and physics — is far from being complete, the present volume is a first attempt to shape the multitude of available results into a coherent theory of one-parameter semigroups of positive linear operators on ordered Banach spaces.

The book is organized as follows. We concentrate our attention on three subjects of semigroup theory: *characterization*, *spectral theory* and *asymptotic behavior*. By *characterization*, we understand the problem of describing special properties of a semigroup, such as positivity, through the generator. By *spectral theory* we mean the investigation of the spectrum of a generator. *Asymptotic behavior* refers to the orbits of the initial values under a given semigroup and phenomena such as stability.

This program (characterization, spectral theory, asymptotic behavior) is worked out on four different types of underlying spaces.

(A) On Banach spaces—Here we present the background for the subsequent discussions related to order.
(B) On spaces $C_0(X)$ (X locally compact), which constitute an important class of ordered Banach spaces and where our results can be presented in a form which makes them accessible also for the non-expert in order-theory.
(C) On Banach lattices, which admit a rich theory and are still sufficiently general as to include many concrete spaces appearing in analysis; e.g., $C_0(X)$, $\mathcal{L}^p(k)$ or l^p.
(D) On non-commutative operator algebras such as C*- or W*-algebras, which are not lattice ordered but still possess an interesting order structure of great importance in mathematical physics.

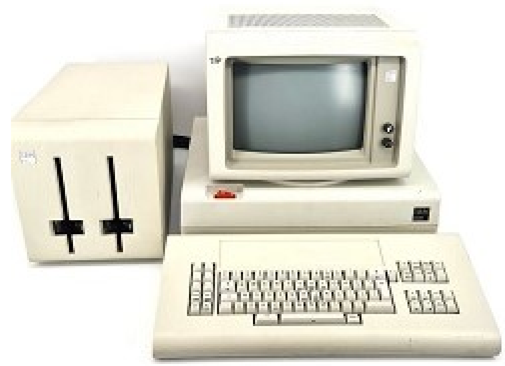
In each of these cases we start with a short collection of basic results and notations, so that the contents of the book may be visualized in the form of a 4×4 matrix in a way which will allow "row readers" (interested in semigroups on certain types of spaces) and "column readers" (interested in certain aspects) to find a path through the book corresponding to their interest.

We display this matrix, together with the names of the authors contributing to the subjects defined through this scheme.

	I Basic Results	II Characterization	III Spectral Theory	IV Asymptotics
A. Banach Spaces	R. Nagel U. Schlotterbeck	W. Arendt H. P. Lotz	G. Greiner R. Nagel	F. Neubrander
B. $C_0(X)$	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas F. Neubrander
C. Banach Lattices	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas R. Nagel F. Neubrander
D. Operator Algebras	U. Groh	U. Groh	U. Groh	U. Groh

This "matrix of contents" has been an indispensable guide line in our discussions on the scope and the spirit of the various contributions. However, we would not have succeeded in completing this manuscript, as a collection of independent contributions (personally accounted for by the authors), under less favorable conditions than we have actually met. For one thing, Rainer Nagel was an unfaltering and undisputed spiritus rector from the very beginning of the project. On the other hand we gratefully acknowledge the influence of Helmut H. Schaefer and his pioneering work on order structures in analysis. It was the team spirit produced by this common mathematical background which, with a little help from our friends, made it possible to overcome most difficulties.

We have prepared the manuscript with the aid of a word processor, but we confess that without the assistance of Klaus Kuhn the pitfalls of such a system would have been greater than its benefits.



The authors

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Acronyms

$E_{\mathbb{R}}$, $E_{\mathbb{C}}$ real, complex Banach lattice
E_+ positive cone
E' dual Banach space
E^* semigroup dual
E_F^T $\mathcal{F}$-product of E with respect to semigroup $\mathcal{T}$
E_F $\mathcal{F}$-product of E
E_f see C-I,4
(E,φ) see C-I,4
$E \otimes F$ tensor product
$\mathcal{L}(E)$ Banach space of bounded linear operators on E
$\mathcal{Z}(E)$ center of E
E_n n-th Sobolev space
$B(H)$ W*-algebra of bounded linear operators on a Hilbert space H

Part A
One-parameter Semigroups on Banach Spaces

Chapter A-I
Basic Results on Semigroups on Banach Spaces

Since the basic theory of one-parameter semigroups can be found in several excellent books (e.g., **?**], **?**], **?**] or **?**]), we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, to collect briefly some important definitions and results (Section 1), to present a list of *standard examples* in Section 2 and to discuss standard constructions of new semigroups from a given one in Section 3 on p. 15.

In the entire chapter we denote by E a (real or) complex Banach space and consider one-parameter semigroups of bounded linear operators $T(t)$ on E. By this we understand a subset $\{T(t)\colon t \in \mathbb{R}_+\}$ of $\mathcal{L}(E)$, usually written as $(T(t))_{t\geqslant 0}$, such that

$$\begin{aligned} T(0) &= \mathrm{Id}, \\ T(s+t) &= T(s)\cdot T(t) \text{ for all } s, t \in \mathbb{R}_+. \end{aligned}$$

In more abstract terms this means that the map $t \mapsto T(t)$ is a homomorphism from the additive semigroup $(\mathbb{R}_+, +)$ into the multiplicative semigroup $(\mathcal{L}(E), \cdot)$. Similarly, a one-parameter group $(T(t))_{t\in\mathbb{R}}$ will be a homomorphic image of the group $(\mathbb{R}, +)$ in $(\mathcal{L}(E), \cdot)$.

1 Standard Definitions and Results

We consider a one-parameter semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E and observe that the domain $\mathbb{R}_+$ and the range $\mathcal{L}(E)$ of the (semigroup) homomorphism $\tau\colon t \mapsto T(t)$ are topological semigroups for the natural topology on $\mathbb{R}_+$ and any one of the standard operator topologies on $\mathcal{L}(E)$. We single out the strong operator topology on $\mathcal{L}(E)$ and require τ to be continuous.

Definition 1.1 A one-parameter semigroup $(T(t))_{t\geqslant 0}$ is called *strongly continuous* if the map $t \mapsto T(t)$ is continuous for the strong operator topology on $\mathcal{L}(E)$, e.g.,

$$\lim_{t \to t_0} \|T(t)f - T(t_0)f\| = 0$$

for every $f \in E$ and $t, t_0 \geqslant 0$.

Clearly one defines in a similar way *weakly continuous*, resp. *uniformly continuous* (compare A-II, Def. 1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology.

If not stated otherwise, a *semigroup* is a strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

Remarks (1) A one-parameter semigroup $(T(t))_{t \geqslant 0}$ on a Banach space E is strongly continuous if and only if for any $f \in E$ it is true that $T(t)f \to f$ if $t \to 0$.
(2) For every strongly continuous semigroup there exist constants $M \geqslant 1$, $w \in \mathbb{R}$ such that $\|T(t)\| \leqslant M \cdot e^{wt}$ for every $t \geqslant 0$.
(3) If $(T(t))_{t \geqslant 0}$ is a one-parameter semigroup such that $\|T(t)\|$ is bounded for $0 < t \leqslant \delta$ then it is strongly continuous if and only if $\lim_{t \to 0} T(t)f = f$ for every f in a total subset of E.

The exponential estimate from Remark 1 (ii) for the growth of $\|T(t)\|$ can be used to define an important characteristic of the semigroup.

Definition 1.2 By the growth bound (or type) of the semigroup $(T(t))_{t \geqslant 0}$ we understand the number

$$\begin{aligned}\omega_0 &:= \inf\{w \in \mathbb{R} \colon \text{There exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leqslant M e^{wt} \text{ for } t \geqslant 0\} \\ &= \lim_{t \to \infty} \frac{1}{t} \log \|T(t)\| = \inf_{t > 0} \frac{1}{t} \log \|T(t)\|.\end{aligned}$$

Particularily important are semigroups such that for every $t \geqslant 0$ we have $\|T(t)\| \leqslant M$ (*bounded semigroups*) or $\|T(t)\| \leqslant 1$ (*contraction semigroups*). In both cases we have $\omega_0 \leqslant 0$.

It follows from the subsequent examples and from Def. 1.2 that ω_0 may be any number $-\infty \leqslant \omega < +\infty$. Moreover the reader should observe that the infimum in Def. 1.2 need not be attained and that M may be larger than 1 even for bounded semigroups.

Examples 1.3 (i) Take $E = \mathbb{C}^2$,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then for the ℓ^1-norm on E we obtain $\|T(t)\| = 1 + t$, hence $(T(t))_{t \geqslant 0}$ is an unbounded semigroup having growth bound $\omega_0 = 0$.

(ii) Take $E = L^1(\mathbb{R})$ and for $f \in E$, $t \geqslant 0$ define

$$T(t)f(x) := \begin{cases} 2 \cdot f(x+t) & \text{if } x \in [-t, 0] \\ f(x+t) & \text{otherwise.} \end{cases}$$

Each $T(t)$, $t > 0$, satisfies $\|T(t)\| = 2$ as can be seen by taking $f := \chi_{[0,t]}$. Therefore $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup which is bounded, hence has $\omega_0 = 0$, but the constant M in (1.3) cannot be chosen to be 1.

The most important object associated to a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is its *generator* which is obtained as the (right)derivative of the map $t \mapsto T(t)$ at $t = 0$. Since for strongly continuous semigroups the functions $t \mapsto T(t)f$, $f \in E$, are continuous but not always differentiable, we have to restrict our attention to those $f \in E$ for which the desired derivative exists. We then obtain the *generator* as a not necessarily everywhere defined operator.

Definition 1.4 To every semigroup $(T(t))_{t\geqslant 0}$ there belongs an operator $(A, D(A))$, called the *generator* and defined on the *domain*

$$D(A) := \{f \in E\colon \lim_{h\to 0} \frac{T(h)f - f}{h} \text{ exists in } E\} \text{ by}$$
$$Af := \lim_{h\to 0} \frac{T(h)f - f}{h} \quad (f \in D(A)).$$

Clearly, $D(A)$ is a linear subspace of E and A is linear from $D(A)$ into E. Only in certain special cases (see 2.1) the generator is everywhere defined and therefore bounded (use Prop. 1.8 (ii)) on p. 6). In general, the precise extent of the domain $D(A)$ is essential for the characterization of the generator. But since the domain is canonically associated to the generator of a semigroup, we shall write in most cases A instead of $(A, D(A))$.

As a first result we collect some information on the domain of the generator.

Proposition 1.5 *For the generator A of a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E the following assertions hold.*

(i) If $f \in D(A)$, then $T(t)f \in D(A)$ for every $t \geqslant 0$.
(ii) The map $t \mapsto T(t)f$ is differentiable on $\mathbb{R}_+$ if and only if $f \in D(A)$. In that case one has

$$\frac{d}{dt}T(t)f = AT(t)f = T(t)Af. \tag{1.1}$$

(iii) For every $f \in E$ and $t > 0$ the element $\int_0^t T(s)f\,\mathrm{d}s$ belongs to $D(A)$ and one has

$$A\int_0^t T(s)f\,\mathrm{d}s = T(t)f - f. \tag{1.2}$$

(iv) If $f \in D(A)$, then

$$\int_0^t T(s)Af\,\mathrm{d}s = T(t)f - f. \tag{1.3}$$

(v) The domain $D(A)$ is dense in E.

The identity (1.1) is of great importance and shows how semigroups are related to certain Cauchy problems. We state this explicitly in the following theorem.

Theorem 1.6 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on the Banach space E. Then the* abstract Cauchy problem

$$\frac{d}{dt}\xi(t) = A\xi(t), \quad \xi(0) = f_0 \tag{1.4}$$

has a unique solution $\xi\colon \mathbb{R}_+ \to D(A)$ in $C^1(\mathbb{R}_+, E)$ for every $f_0 \in D(A)$. In fact, this solution is given by $\xi(t) := T(t)f_0$.

For more on the relation of semigroups to abstract Cauchy problems we refer to A-II,Section 1. Here we only point out that the above theorem implies that a semigroup is uniquely determined by its generator.

While the generator is bounded only for uniformly continuous semigroups (see Sec. 2 below), it always enjoys a weaker but useful property.

Definition 1.7 An operator B with domain $D(B)$ on a Banach space E is called *closed* if $D(B)$ endowed with the *graph norm*

$$\|f\|_B := \|f\| + \|Bf\|$$

becomes a Banach space. Equivalently, $(B, D(B))$ is closed if and only if its *graph* $\{(f, Bf)\colon f \in D(B)\}$ is closed in $E \times E$, i.e.,

$$f_n \in D(B), f_n \to f \text{ and } Bf_n \to g \text{ implies } f \in D(B) \text{ and } Bf = g.$$

It is clear from this definition that the *closedness* of an operator B depends very much on the size of the domain $D(B)$. For example, a bounded and densely defined operator $(B, D(B))$ is closed if and only if $D(B) = E$.

On the other hand it may happen that $(B, D(B))$ is not closed but has a closed *extension* $(C, D(C))$, i.e., $D(B) \subseteq D(C)$ and $Bf = Cf$ for every $f \in D(B)$. In that case, B is called *closable*, a property which is equivalent to

$$f_n \in D(B), f_n \to 0 \text{ and } Bf_n \to g \text{ implies } g = 0.$$

The smallest closed extension of $(B, D(B))$ will be called the *closure* $\overline{B}$ with domain $D(\overline{B})$. In other words, the graph of $\overline{B}$ is the closure of $\{(f, Bf)\colon f \in D(B)\}$ in $E \times E$.

Finally we call a subset D_0 of $D(B)$ a *core* for B if D_0 is $\|\cdot\|_B$-dense in $D(B)$. This means that a closed operator is determined (via closure) by its restriction to a core in its domain.

We now collect the fundamental topological properties of semigroup generators, their domains (see also A-II,Cor.1.34) and their resolvents.

Proposition 1.8 *For the generator A of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ the following hold.*

(i) The generator A is a closed operator.
(ii) If a subspace D_0 of the domain $D(A)$ is dense in E and $(T(t))$-invariant, then it is a core for A.
(iii) Define $D(A^n) := \{f \in D(A^{n-1})\colon Af \in D(A^{n-1})\}$, $D(A^1) = D(A)$. Then $D(A^\infty) := \bigcap_{n\in\mathbb{N}} D(A^n)$ is dense in E and a core for A.

Example 1.9 Property (iii) above does not hold for general densely defined closed operators. Take $E = C\,[0,1]$, $D(B) = C^1\,[0,1]$ and $Bf = q \cdot f'$ for some nowhere differentiable function $q \in C\,[0,1]$. Then B is closed, but $D(B^2) = \{0\}$.

Proposition 1.10 *For the generator A of a strongly continous semigroup on a Banach space E the following hold.*

If $\int_0^\infty e^{-\lambda t} T(t) f\, dt$ exists for every $f \in E$ and some $\lambda \in \mathbb{C}$, then $\lambda \in \varrho(A)$ and $R(\lambda, A) f = \int_0^\infty e^{-\lambda t} T(t) f\, dt$. In particular,

$$R(\lambda, A)^{n+1} f = \frac{(-1)^n}{n!} \left(\frac{d}{d\lambda}\right)^n R(\lambda, A) f = \int_0^\infty e^{-\lambda t} \frac{t^n}{n!} T(t) f\, dt \tag{1.5}$$

for every $f \in E$, $n \geqslant 0$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega$.

Remarks 1.11 (i) For continuous Banach space valued functions such as $t \mapsto T(t)f$ we consider the Riemann integral and define

$$\int_0^\infty T(t) f\ \mathrm{d}t \quad \text{as} \quad \lim_{t\to\infty} \int_0^t T(s) f\ \mathrm{d}s.$$

Sometimes such integrals for strongly continuous semigroups are written as $\int_a^b T(t)\, dt$ but understood in the strong sense.
(ii) Since the generator $(A, D(A))$ determines the semigroup uniquely, we will speak occasionally of the growth bound of the generator instead of the semigroup, i.e., we write $\omega_0 = \omega_0(A) = \omega_0((T(t))_{t\geqslant 0}$.
(iii) For one-parameter groups it might seem to be more natural to define the generator as the *derivative* rather than just the *right derivative* at $t = 0$. This yields the same operator as the following result shows.

The strongly continuous semigroup $(T(t))_{t\geqslant 0}$ with generator A can be extended to a strongly continuous one-parameter group $(U(t))_{t\in\mathbb{R}}$ if and only if $-A$ generates a semigroup $(S(t))_{t\geqslant 0}$.

In that case $(U(t))_{t\in\mathbb{R}}$ is obtained as

$$U(t) = \begin{cases} T(t) & \text{for } t \geqslant 0, \\ S(-t) & \text{for } t \leqslant 0. \end{cases}$$

We refer to **?** , Prop.1.14] for the details.

2 Standard Examples

In this section we list and discuss briefly the most basic examples of semigroups together with their generators. These semigroups will reappear throughout this book and will be used to illustrate the theory. We start with the class of semigroups mentioned after Definition 1.1 on p. 3.

2.1 Uniformly Continuous Semigroups

It follows from elementary operator theory that for every bounded operator A in $\mathcal{L}(E)$ the sum

$$\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} =: e^{tA}$$

exists and determines a unique uniformly continuous (semi)group $(e^{tA})_{t\in\mathbb{R}}$ having A as its generator. Conversely, any uniformly continuous semigroup is of this form.

If the semigroup $(T(t))_{t\geqslant 0}$ is uniformly continuous, then $\frac{1}{t}\int_0^t T(s)\,\mathrm{d}s$ uniformly converges to $T(0) = \mathrm{Id}$ as $t \to 0$. Therefore for some $t' > 0$ the operator $\frac{1}{t'}\int_0^{t'} T(s)\,\mathrm{d}s$ is invertible and every $f \in E$ is of the form $f = \frac{1}{t'}\int_0^{t'} T(s)g\,\mathrm{d}s$ for some $g \in E$. But these elements belong to $D(A)$ by (1.3), hence $D(A) = E$. Since the generator A is closed and everywhere defined, it must be bounded.

Remark that bounded operators are always generators of groups, not just semigroups. Moreover, the growth bound ω satisfies $|\omega| \leqslant \|A\|$ in this situation.

The above characterization of the generators of uniformly continuous semigroups as the bounded operators shows that these semigroups are—at least in many aspects—rather simple objects.

2.2 Matrix Semigroups

The above considerations expecially apply in the situation $E = \mathbb{C}^n$. If $n = 2$ and $A = (a_{ij})_{2\times 2}$ the following explicit formulas for e^{tA} might be of interest.

Set (i) $s :=$ trace A, (ii) $d :=$ det A (iii) and $D := (s^2 - 4d)^{1/2}$. Then if $D \neq 0$

$$e^{tA} = e^{ts/2} \cdot [D^{-1} 2 \sinh(tD/2) \cdot A + (\cosh(tD/2) - sD^{-1}\sinh(tD/2)) \cdot \mathrm{Id}]$$

and if $D = 0$

$$e^{ts/2} \cdot [tA + (1 - ts/2) \cdot \mathrm{Id}].$$

2.3 Multiplication Semigroups

Many Banach spaces appearing in applications are Banach spaces of (real or) complex valued functions over a set X. As the most standard examples of these "function spaces", we mention the space $C_0(X)$ of all continuous complex valued functions vanishing at infinity on a locally compact space X, or the spaces $L^p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, of all (equivalence classes of) p-integrable functions on a σ-finite measure space (X, Σ, μ).

On these function spaces $E = C_0(X)$, resp. $E = L^p(X, \Sigma, \mu)$, there is a simple way to define *multiplication operators*.

Take a continuous, resp. measurable function $q \colon X \to \mathbb{C}$ and define

$$M_q f := q \cdot f, \quad \text{i.e.,} \quad M_q f(x) := q(x) \cdot f(x) \quad \text{for } x \in X$$

and for every f in the *maximal* domain $D(M_q) := \{g \in E \colon q \cdot g \in E\}$.

This natural domain is a dense subspace of $C_0(X)$, resp. $L^p(X, \Sigma, \mu)$, for $1 \leqslant p < \infty$. Moreover, $(M_q, D(M_q))$ is a closed operator. This is easy in case $E = C_0(X)$.

For $E = L^p(X, \Sigma, \mu)$, $1 \leqslant p < \infty$, we consider a sequence $(f_n) \subset E$ such that $\lim_{n\to\infty} f_n = f \in E$ and $\lim_{n\to\infty} q f_n =\colon g \in E$. Choose a subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f_{n(k)}(x) = f(x)$ and $\lim_{k\to\infty} q(x) f_{n(k)}(x) = g(x)$ for μ-almost every $x \in X$. Then $g = q \cdot f$ and $f \in D(M_q)$, i.e., M_q is closed.

For such multiplication operators many properties can be checked quite directly. For example, the following statements are equivalent.

(a) M_q is bounded.
(b) q is (μ-essentially) bounded.

One has $\|M_q\| = \|q\|_\infty$ in this situation. Observe that on spaces $C(K)$, K compact, there are no densely defined, unbounded multiplication operators.

By defining the multiplication semigroups

$$T(t) f(x) := \exp(t \cdot q(x)) f(x), \quad x \in X, f \in E,$$

one obtains the following characterizations.

Proposition 2.1 *Let M_q be a multiplication operator on $E = C_0(X)$ or $E = L^p(X, \Sigma, \mu)$, $1 \leqslant p < \infty$. Then the properties (a) and (b), resp. (a') and (b'), are equivalent.*

(a) M_q generates a strongly continuous semigroup.
(b) $\sup\{\mathrm{Re}(q(x)) \colon x \in X\} < \infty$.
(a') M_q generates a uniformly continuous semigroup.
(b') $\sup\{|q(x)| \colon x \in X\} < \infty$.

As a consequence one computes the growth bound of a multiplication semigroup as

$$\omega_0 = \sup\{\mathrm{Re}(q(x)) \colon x \in X\}$$

in the case $E = C_0(X)$ and

$$\omega_0 = \mu\text{-ess-}\sup\{\mathrm{Re}(q(x)) : x \in X\}$$

in the case $E = L^p(\mu)$. It is a nice exercise to characterize those multiplication operators which generate strongly continuous groups.

We point out that the above results cover the cases of sequence spaces such as c_0 or ℓ^p, $1 \leqslant p < \infty$. An abstract characterization of generators of multiplication semigroups will be given in C-II,Thm.5.13.

2.4 Translation (Semi)Groups

Let E to be one of the following function spaces $C_0(\mathbb{R}_+)$, $C_0(\mathbb{R})$ or $L^p(\mathbb{R}_+)$, $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$. Define $T(t)$ to be the (left) translation operator

$$T(t)f(x) := f(x+t)$$

for x ,, $t \in \mathbb{R}_+$, resp. $x, t \in \mathbb{R}$ and $f \in E$. Then $(T(t))_{t\geqslant 0}$ is a strongly continous semigroup, resp. group of contractions on E and its generator is the first derivative $\frac{d}{dx}$ with *maximal* domain. In order to be more precise we have to distinguish the cases $E = C_0$ and $E = L^p$.

The generator of the translation (semi)group on $E = C_0(\mathbb{R}_+)$ is

$$\begin{aligned} Af &:= \frac{d}{\mathrm{d}x} f = f' \\ D(A) &:= \{f \in E : f \text{ differentiable and } f' \in E\}. \end{aligned}$$

Proof For $f \in D(A)$ it follows that for every $x \in \mathbb{R}_{(+)}$

$$\lim_{h\to 0} \frac{T(h)f(x) - f(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

(uniformly in x) and coincides with $Af(x)$. Therefore f is differentiable and $f' \in E$.

On the other hand, take $f \in E$ differentiable such that $f' \in E$. Then

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leqslant \frac{1}{h} \int_x^{x+h} |f'(y) - f'(x)| \, \mathrm{d}y,$$

where the last expression tends to zero uniformly in x as $h \to 0$. Thus $f \in D(A)$ and $f' = Af$. □

The generator of the translation (semi)group on $E = L^p(\mathbb{R}_+)$, $1 \leqslant p < \infty$, is

$$\begin{aligned} Af &:= \frac{d}{\mathrm{d}x} f = f', \\ D(A) &:= \{f \in E : f \text{ absolutely continuous}, f' \in E\}. \end{aligned}$$

Proof Take $f \in D(A)$ such that $\lim_{h\to 0} \frac{1}{h}(T(h)f - f) = g \in E$. Since integration is continuous, we obtain for every $a, b \in \mathbb{R}_{(+)}$ that

$$(*) \quad \frac{1}{h}\int_{b+h}^{b} f(x)\,\mathrm{d}x - \frac{1}{h}\int_{a+h}^{a} f(x)\,\mathrm{d}x = \int_a^b \frac{f(x+h)-f(x)}{h}\,\mathrm{d}x$$

converges to $\int_a^b g(x)\,\mathrm{d}x$ as $h \to 0+$. But for almost all a, b the left hand side of $(*)$ converges to $f(b) - f(a)$. By redefining f on a nullset we obtain

$$f(y) = \int_a^y g(x)\,\mathrm{d}x + f(a), \quad y \in \mathbb{R}_{(+)},$$

which is an absolutely continuous function whose derivative is (almost everywhere) equal to g.

On the other hand, let f be absolutely continuous such that $f' \in L^p$. Then

$$\begin{aligned}\lim_{h\to 0}\int \left|\frac{f(x+h)-f(x)}{h} - f'(x)\right|^p \mathrm{d}x &= \lim_{h\to 0}\int \left|\frac{1}{h}\int_0^h (f'(x+s)-f'(x))\,\mathrm{d}s\right|^p \mathrm{d}x\\ &= \lim_{h\to 0}\int \left|\int_0^1 (f'(x+uh)-f'(x))\,\mathrm{d}u\right|^p \mathrm{d}x\\ &\leqslant \lim_{h\to 0}\int\int_0^1 |f'(x+uh)-f'(x)|^p\,\mathrm{d}u\,\mathrm{d}x\\ &= \int_0^1 \lim_{h\to 0}\int |f'(x+uh)-f'(x)|^p\,\mathrm{d}x\,\mathrm{d}u = 0,\end{aligned}$$

hence $f \in D(A)$. □

2.5 Rotation Groups

On $E = C(\Gamma)$, resp. $E = L^p(\Gamma, m)$, $1 \leqslant p < \infty$, m Lebesgue measure we have canonical groups defined by rotations of the unit circle Γ with a certain period, i.e., for $0 < \tau \in \mathbb{R}$ the operators

$$R_\tau(t)f(z) := f(e^{2\pi i t/\tau}\cdot z), \quad z \in \Gamma$$

yield a group $(R_\tau(t))_{t\in\mathbb{R}}$ having period τ, i.e., $R_\tau(\tau) = \mathrm{Id}$. As in Example 2.4 one shows that its generator has the form

$$\begin{aligned} D(A) &= \{f \in E : f \text{ absolutely continuous}, f' \in E\},\\ Af(z) &= (2\pi i/\tau)\cdot z \cdot f'(z).\end{aligned}$$

An isomorphic copy of the group $(R_\tau(t))_{t\in\mathbb{R}}$ is obtained if we consider $E = \{f \in C[0,1] : f(0) = f(1)\}$, resp. $E = L^p([0,1])$ and the group of *periodic translations*

$$T(t)f(x) := f(y) \quad \text{for } y \in [0,1]\,, y = x + t \bmod 1$$

with generator

$$D(A) := \{f \in E \colon f \text{ absolutely continuous}, f' \in E\}, \quad Af := f'.$$

2.6 Nilpotent Translation Semigroups

Take $E = L^p([0,\tau], m)$ for $1 \leqslant p < \infty$ and define

$$T(t)f(x) := \begin{cases} f(x+t) & \text{if } x+t \leqslant \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then $(T(t))_{t\geqslant 0}$ is a semigroup satisfying $T(t) = 0$ for $t \geqslant \tau$. Its generator is still the first derivative $A = \frac{d}{\mathrm{d}x}$, but with domain is

$$D(A) = \{f \in E \colon f \text{ absolutely continuous}, f' \in E, f(\tau) = 0\}.$$

In fact, if $f \in D(A)$, then f is absolutely continuous with $f' \in E$. By Prop. 1.5 (i) on p. 5 it follows that $T(t)f$ is absolutely continuous and hence $f(\tau) = 0$.

2.7 One-dimensional Diffusion Semigroup

For the second derivative

$$Bf(x) := \frac{d^2}{dx^2} f(x) = f''(x)$$

we take the domain

$$D(B) := \{f \in C^2\,[0,1] \,:\, f'(0) = f'(1) = 0\}$$

in the Banach space $E = C\,[0,1]$. Then $D(B)$ is dense in $C\,[0,1]$, but closed for the graph norm. Obviously, each function

$$e_n(x) := \cos \pi n x, \quad n \in \mathbb{Z},$$

is contained in $D(B)$ and is an eigenfunction of B pertaining to the eigenvalue $\lambda_n := -\pi^2 n^2$. The linear hull span $\{e_n \colon n \in \mathbb{Z}\} =: E_0$ forms a subalgebra of $D(B)$ which by the Stone-Weierstrass theorem is dense in E.

We now use e_n to define bounded linear operators

$$e_n \otimes e_n \colon f \mapsto \left(\int_0^1 f(x) e_n(x)\, \mathrm{d}x \right) e_n = (f|e_n) e_n$$

satisfying $\|e_n \otimes e_n\| \leqslant 1$ and $(e_n \otimes e_n)(e_m \otimes e_m) = \delta_{n,m}(e_n \otimes e_n)$ for $n \in \mathbb{Z}$.

For $t > 0$ we define

$$\begin{aligned} T(t) &\coloneqq \sum_{n\in\mathbb{Z}} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n \\ &= e_0 \otimes e_0 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n, \end{aligned}$$

or

$$T(t) f(x) = \int_0^1 k_t(x, y) f(x) dy$$
$$\text{where } k_t(x, y) = 1 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cos \pi n x \cos \pi n y.$$

The Jacobi identity

$$\begin{aligned} w_t(x) &\coloneqq 1/(4\pi t)^{\frac{1}{2}} \sum_{m\in\mathbb{Z}} \exp(-(x + 2m)^2/4t) \\ &= \frac{1}{2} + \sum_{n\in\mathbb{N}} \exp(-\pi^2 n^2 t) \cos \pi n x \end{aligned}$$

and trigonometric relations show that

$$k_t(x, y) = w_t(x + y) + w_t(x - y)$$

which is a positive function on $[0, 1]^2$. Therefore $T(t)$ is a bounded operator on $C\,[0, 1]$ with

$$\|T(t)\| = \|T(t)\mathbb{1}\| = \sup_{x\in[0,1]} \int_0^1 k_t(x, y) dy = 1.$$

From the behavior of $T(t)$ on the dense subspace E_0 it follows that $(T(t))_{t\geqslant 0}$ with $T(0) = \mathrm{Id}$ is a strongly continuous semigroup on E and its generator A coincides with B on E_0. Finally, we observe that E_0 is a core for $(A, D(A))$ by Prop.1.9(ii).

Consequently, $(T(t))_{t\geqslant 0}$ is the semigroup generated by the closure of the second derivative with domain $D(B)$.

2.8 n-dimensional Diffusion Semigroup

On $E = L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, the operators

$$\begin{aligned} T(t)f(x) &:= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|x-y|^2/4t) f(y)dy \\ &= \mu_t * f(x) \end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$ and $\mu_t(x) := (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ form a strongly continuous semigroup:

In fact the integral exists for every $f \in L^p(\mathbb{R}^n)$ since μ_t is an element of the Schwartz space $S(\mathbb{R}^n)$ of all rapidly decreasing smooth functions on $\mathbb{R}^n$.

Moreover,

$$\|T(t)f\|_p \leqslant \|\mu_t\|_1 \|f\|_p = \|f\|_p$$

by Young's inequality, **?** , p.28], hence $\|T(t)\| \leqslant 1$ for every $t > 0$. Next we observe that $S(\mathbb{R}^n)$ is dense in E and invariant under each $T(t)$. Therefore we can apply the Fourier transformation F which leaves $S(\mathbb{R}^n)$ invariant and yields

$$F(\mu_t * f) = (2\pi)^{n/2} F(\mu_t) \cdot F(f) = (2\pi)^{n/2} \hat{\mu}_t \cdot \hat{f}$$

where $f \in S(\mathbb{R}^n)$, $\hat{f} = Ff \in S(\mathbb{R}^n)$.

In other words, F transforms $(T(t)|_{S(\mathbb{R}^n)})_{t \geqslant 0}$ into a multiplication semigroup on $S(\mathbb{R}^n)$ which is pointwise continuous for the usual topology of $S(\mathbb{R}^n)$. The generator, i.e., the right derivative at 0, of this semigroup is the multiplication operator

$$B\hat{f}(x) := -|x|^2 \hat{f}(x) \quad (x \in \mathbb{R}^n)$$

for every $f \in S(\mathbb{R}^n)$.

Applying the inverse Fourier transformation and observing that the topology of $S(\mathbb{R}^n)$ is finer than the topology induced from $L^p(\mathbb{R}^n)$, we obtain that $(T(t))_{t \geqslant 0}$ is a semigroup which is strongly continuous (use Rem. 1 (iii) on p. 4).

Its generator A coincides with

$$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x_1, \ldots, x_n)$$

for every $f \in S(\mathbb{R}^n)$. Since $S(\mathbb{R}^n)$ is $(T(t))$-invariant, we have determined the generator on a core of its domain (see Prop.1.9.ii). In particular, the above semigroup *solves* the initial value problem for the *heat equation*

$$\frac{\partial}{\partial t} f(x,t) = \Delta f(x,t), \quad f(x,0) = f_0(x), \quad x \in \mathbb{R}^n.$$

For the analogous discussion of the unitary group on $L^2(\mathbb{R}^n)$ generated by

$$C := \mathrm{i}\Delta$$

we refer to Section IX.7 in **?**].

Analogous examples to 2.7 are valid in L^p $[0,1]$, resp. to 2.8 in $C_0(\mathbb{R}^n)$.

3 Standard Constructions

Starting with a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E it is possible to construct new semigroups on spaces naturally associated with E. Such constructions will be important technical devices in many of the subsequent proofs. Although most of these constructions are rather routine, we present in the sequel a systematic account of them for the convenience of the reader.

We always start with a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E, and denote its generator by A on the domain $D(A)$.

3.1 Similar Semigroups

There is an easy way how to obtain different (but isomorphic) semigroups out of a given semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E.

Let V be an isomorphism from E onto E. Then $S(t) := VT(t)V^{-1}$, $t \geqslant 0$, defines a strongly continuous semigroup. If A is the generator of $(T(t))_{t\geqslant 0}$ then

$$B := VAV^{-1} \text{ with domain } D(B) := \{f \in E : V^{-1}f \in D(A)\}$$

is the generator of $(S(t))_{t\geqslant 0}$.

3.2 The Rescaled Semigroup

For fixed $\lambda \in \mathbb{C}$ and $\alpha > 0$ the operators

$$S(t) := \exp(\lambda t)T(\alpha t)$$

yield a new semigroup having generator

$$B := \alpha A + \lambda\,\mathrm{Id} \text{ with } D(B) = D(A).$$

This *rescaled semigroup* enjoys most of the properties of the original semigroup and the same is true for the corresponding generators. However, by using this procedure certain constants associated with $(T(t))_{t\geqslant 0}$ and A can be normalized. For example, by this rescaling we may in many cases suppose without loss of generality that the growth bound ω_0 is zero.

Another application is the following. For $\lambda \in \mathbb{C}$ and $S(t) := \exp(-\lambda t)T(t)$ the formulas *(1.3)* and *(1.4)* yield:

$$e^{-\lambda t}T(t)f - f = (\lambda - A)\int_0^t e^{-\lambda s}T(s)f\,\mathrm{d}s \text{ or}$$
$$(e^{\lambda t} - T(t))f = (\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)f\,\mathrm{d}s \quad \text{for } f \in E,$$

and

$$e^{-\lambda t}T(t)f - f = \int_0^t e^{-\lambda s}T(s)(\lambda - A)f\,\mathrm{d}s\text{or}$$

$$(e^{\lambda t} - T(t))f = \int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)f\,\mathrm{d}s \quad \text{for } f \in D(A).$$

3.3 The Subspace Semigroup

Assume F to be a closed $(T(t))$-invariant or, equivalently, $R(\lambda, A)$-invariant for $\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > \omega_0$, subspace of E. Then the semigroup $(T(t)_|)_{t\geqslant 0}$ of all restrictions $T(t)_| := T(t)_{|F}$ is strongly continuous on F. If $(A, D(A))$ denotes the generator of $(T(t))_{t\geqslant 0}$ it follows from the $(T(t))$-invariance and closedness of F that A maps $D(A) \cap F$ into F. Therefore

$$A_| := A_{|(D(A)\cap F)} \text{ with domain } D(A_|) := D(A) \cap F$$

is the generator of $(T(t)_|)$. Conversely, if F is a closed *linear subspace* of E with $A(D(A) \cap F) \subset F$ such that $A_|$ is a generator on F, then F is $(T(t))$-invariant.

An A-invariant subspace need not necessarily be $(T(t))$-invariant: Take for example the translation group with $T(t)f(x) = f(x + t)$ on $E = C_0(\mathbb{R})$ and $F := \{f \in E : f(x) = 0 \text{ for } x \leqslant 0\}$.

3.4 The Quotient Semigroup

Let F be a closed $(T(t))$-invariant subspace of E and consider the quotient space $E_/ := {}^{E}\!/_{F}$ with quotient map $q: E \to E_/$. The quotient operators

$$T(t)_/ q(f) := q(T(t)f), \quad f \in E,$$

are well defined and form a strongly continuous semigroup $(T(t)_/)_{t\geqslant 0}$ on $E_/$. For the generator $(A_/, D(A_/))$ of $(T(t)_/)_{t\geqslant 0}$ the following holds:

$$D(A_{/}) = q(D(A)) \quad \text{and} \quad A_{/}q(f) = q(Af)$$

for every $f \in D(A)$. Here we use the fact that every $\hat{f} := q(f) \in D(A_{/})$ can be written as

$$\hat{f} = \int_0^\infty e^{-\lambda s}\hat{T}(s)_{/}\hat{g}\,\mathrm{d}s = \int_0^\infty e^{-\lambda s} q(T(s)g)\,\mathrm{d}s = q\Big(\int_0^\infty e^{-\lambda s} T(s)g\,\mathrm{d}s\Big) = q(h)$$

where $h \in D(A)$ and $\lambda > \omega$ (see Prop. 1.5). In particular we point out that for every $\hat{f} \in D(A/)$ there exist representatives $f \in \hat{f}$ belonging to $D(A)$.

Example 3.1 We start with the Banach space $E = L^1(\mathbb{R})$ and the translation semigroup $(T(t))_{t\geqslant 0}$ where $T(t)f(x) := f(x+t)$ (see Example 2.4). Then $L^1((-\infty, 1])$ can be identified with the closed, $(T(t))$-invariant subspace

$$J := \{f \in E\colon f(x) = 0 \text{ for } 1 < x < \infty\}.$$

There we obtain the subspace semigroup

$$T(t)|_{(-\infty,1]}(x) \cdot f(x+t),$$

where $f \in L^1((-\infty, 1])$, $-\infty < x \leqslant 1$ and $t \geqslant 0$.

By 2.4 and 3.2 its generator is

$$A|f := f'$$

for $f \in D(A|) := \{f \in E\colon f \in \mathrm{AC} \text{ with } f' \in E \text{ and } f(x) = 0 \text{ for } x \geqslant 1\}$.

Next we identify $L^1([0, 1])$ with the quotient space $L^1((-\infty, 1])/I$ where

$$I := \{f \in L^1((-\infty, 1])\colon f(x) = 0 \text{ for } 0 \leqslant x \leqslant 1\}.$$

Again I is invariant for the restricted semigroup $(T(t)_{|})$ and the quotient semigroup $(T(t)|/)$ on $L^1([0, 1])$ is the nilpotent translation semigroup as in Example 2.6. In particular it follows that the domain of its generator is

$$D(A_{|/}) = \{f \in L^1([0, 1])\colon f \in \mathrm{AC} \text{ with } f' \in L^1([0, 1]) \text{ and } f(1) = 0\}.$$

3.5 The Adjoint Semigroup

The adjoint operators $(T(t)')_{t\geqslant 0}$ of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E form a semigroup on E' which need, however, not be strongly continuous.

Example 3.2 Take the translation operators $T(t)f(x) = f(x+t)$ on $E = L^1(\mathbb{R})$ (see Example 2.4) and their adjoints

$$T(t)'f(x) = f(x-t)$$

on $E' = L^\infty(\mathbb{R})$. Then $(T(t)')_{t\in\mathbb{R}}$ is a one-parameter group which is not strongly continuous on $L^\infty(\mathbb{R})$ (take any non-trivial characteristic function).

Since the semigroup $(T(t)')_{t\geqslant 0}$ is obviously *weak*-continuous* in the sense that $\lim_{t\to s}\langle f, (T(t)' - T(s)')\varphi\rangle = 0$ for every $f \in E$, $\varphi \in E'$ and $s, t \geqslant 0$, it is natural to associate $(T(t)')_{t\geqslant 0}$ its a *weak*-generator*

$$A'\varphi := \sigma(E', E)\text{-}\lim \frac{1}{h}(T(h)'\varphi - \varphi) \text{ for every } \varphi \text{ in the domain}$$

$$D(A') := \{\varphi \in E' : \sigma(E', E)\text{-}\lim \frac{1}{h}(T(h)'\varphi - \varphi) \text{ exists}\}.$$

This operator coincides with the *adjoint* of the generator $(A, D(A))$, i.e.,

$$D(A') = \{\varphi \in E' : \text{there exists } \psi \in E' \text{ such that } \langle f, \psi\rangle = \langle Af, \varphi\rangle \text{ for all } f \in D(A)\}$$

and $A'\varphi = \psi$. In particular, A' is a closed and $\sigma(E', E)$-densely defined operator in E'.

It follows from **?** , Thm.III.5.30] that the resolvent $R(\lambda, A')$ of A' is $R(\lambda, A)'$. In particular, the spectra $\sigma(A)$ and $\sigma(A')$ coincide.

However, it is still necessary in some situations to have strong continuity for the adjoint semigroup. In order to achieve this we restrict $T(t)'$ to an appropriate subspace of E'.

Definition 3.3 (**?**]) The *semigroup dual* of the Banach space E with respect to the strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is

$$E^* := \{\varphi \in E' : \|\cdot\|\text{-}\lim_{t\to 0} T(t)'\varphi = \varphi\}.$$

The adjoint semigroup on E^* is given by the operators

$$T(t)^* := T(t)'|_{E^*}, \quad t \geqslant 0.$$

Since $(T(t)^*)_{t\geqslant 0}$ is strongly continuous on E^* we call its generator $(A^*, D(A^*))$ the *adjoint generator*.

The above definition makes sense since E^* is norm-closed in E' and $(T(t)')$-invariant. The main point is that E^* is still reasonably large. In fact, since $\int_0^t T(s)'\varphi \, \mathrm{d}s$, understood in the weak sense, is contained in E^* for every $\varphi \in E'$ and $t \geqslant 0$, it follows that

$$\sup\{\langle f, \varphi\rangle : \varphi \in E^*, \|\varphi\| \leqslant 1\} \leqslant \|f\| \leqslant M \cdot \sup\{\langle f, \varphi\rangle : \varphi \in E^*, \|\varphi\| \leqslant 1\}$$

where $M := \limsup_{t\to 0}\|T(t)\|$. In particular, E^* separates E, i.e., E^* is $\sigma(E', E)$-dense in E'. In addition, the estimate of $\|\cdot\|$ given above yields

$$\|T(t)^*\| \leqslant \|T(t)\| \leqslant M\|T(t)^*\| \quad \text{for all } t \geqslant 0.$$

In the following proposition we describe the relation between A^* and A'.

Proposition 3.4 *For the adjoint generator A^* of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on E the following assertions hold.*

(i) E^* *is the* $\|\cdot\|$*-closure of* $D(A')$.
(ii) $D(A^*) = \{\varphi \in D(A') \colon A'\varphi \in E^*\}$.
(iii) A^* *and* A' *coincide on* $D(A^*)$.

Proof (i) Take $\varphi \in D(A')$ fixed. For every $f \in D(A)$ with $\|f\| \leqslant 1$ we define a continuously differentiable function

$$t \mapsto \xi_f(t) := \langle T(t)f, \varphi\rangle$$

on $[0, 1]$ with derivative $\xi_f'(t) = \langle T(t)Af, \varphi\rangle = \langle T(t)f, A'\varphi\rangle$.

Since $\{\xi_f'(t) \colon t \in [0, 1], f \in D(A), \|f\| \leqslant 1\}$ is bounded, it follows that the set

$$\{\xi_f \colon f \in D(A), \|f\| \leqslant 1\}$$

is equicontinuous at 0, i.e., for every $\varepsilon > 0$ there exists $0 < t_0 < 1$ such that

$$|\xi_f(s) - \xi_f(0)| = |\langle f, T(s)'\varphi - \varphi\rangle| < \varepsilon$$

for every $0 \leqslant s \leqslant t_0$ and $f \in D(A)$, $\|f\| \leqslant 1$. But this implies $\|T(s)'\varphi - \varphi\| < \varepsilon$ and hence $\varphi \in E^*$.

Conversely, take $\psi \in E^*$. Then $\frac{1}{t}\int_0^t T(s)'\psi \, \mathrm{d}s$, $t > 0$, belongs to $D(A')$ and norm converges to ψ as $t \to 0$, i.e., ψ belongs to the norm closure of $D(A')$.

(ii) and (iii): Since the weak* topology on E' is weaker than the norm topology, it follows that A' is an extension of A^*. Now take $\varphi \in D(A')$ such that $A'\varphi \in E^*$. As above define the functions ξ_f. The assumption on φ implies the set of all derivatives

$$\{\xi_f' \colon f \in D(A), \|f\| \leqslant 1\}$$

to be equicontinuous at $t = 0$. This means that for every $\varepsilon > 0$ there exists $0 < t_o < 1$ such that $|f_f'(0) - f_f'(s)| < \varepsilon$ for every $f \in D(A)$, $\|f\| \leqslant 1$ and $0 < s < t_o$. In particular,

$$\varepsilon > |f_f'(0) - \frac{1}{s}(\xi_f(s) - \xi_f(0))| = | < f, A'\varphi - \frac{1}{s}(T(s)'\varphi - \varphi) > |,$$

hence

$$\varepsilon > \|A'\varphi - \frac{1}{s}(T(s)'\varphi - \varphi)\|$$

for all $0 \leqslant s \leqslant t_o$. From this it follows that $\varphi \in D(A^*)$. □

On reflexive Banach spaces we have $A^* = A'$ by the above proposition. In other cases this construction is more interesting.

Example 3.5 (continued) The adjoints of the (left) translation $T(t)$ on $E = L^1(\mathbb{R})$ are the (right) translations $T(t)'$ on $E' = L^\infty(\mathbb{R})$. The largest subspace of $L^\infty(\mathbb{R})$ on which these translations form a strongly-continuos semigroup with respect to the sup-norm, is the space of all bounded uniformly continuous functions on $\mathbb{R}$, i.e., $E^* = C_{bu}(\mathbb{R})$.

Calculating $D(A')$ and $D(A^*)$ respectively, one obtains

$$D(A') = \{f \in L^\infty(\mathbb{R}) : f \in AC, f' \in L^\infty(\mathbb{R})\},$$
$$D(A^*) = \{f \in L^\infty(\mathbb{R}) : f \in C^1(\mathbb{R}), f' \in C_{bu}(\mathbb{R})\}.$$

Obviously, the function $x \mapsto |\sin x|$ belongs to $D(A')$, but not to $D(A^*)$.

3.6 The Associated Sobolev Semigroups

Since the generator A of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is closed, its domain $D(A)$ becomes a Banach space for the graph norm

$$\|f\|_1 := \|f\| + \|Af\|.$$

We denote this Banach space by E_1 and the continuous injection from E_1 into E by i_1. Since E_1 is invariant under $(T(t))_{t\geqslant 0}$, apply Prop. 1.5 (i), it makes sense to consider the semigroup $(T_1(t))_{t\geqslant 0}$ of all restrictions $T_1(t) := T(t)|_{E_1}$. The results of Prop. 1.5 imply that $T_1(t) \in \mathrm{L}E_1$ and $\|T_1(t)f - f\|_1 \to 0$ as $t \to 0$ for every $f \in E_1$. Thus $(T_1(t))_{t\geqslant 0}$ is a strongly continuous semigroup on E_1 and has a generator denoted by $(A_1, D(A_1))$.

Using 1.5 again we see that A_1 is the restriction of A to E_1 with maximal domain, i.e., $D(A_1) = \{f \in E_1 : Af \in E_1\} = D(A^2)$ and $A_1 f = Af$ for every $f \in D(A_1)$.

It is now possible to repeat this construction in order to obtain Banach spaces E_n and semigroups $(T_n(t))_{t\geqslant 0}$ with generators $(A_n, D(A_n))$ which are related as visualized in the following diagram.

$$
\begin{tikzcd}
E \arrow[r, "T(t)"] \arrow[d, "i_1"'] & E \arrow[d, "i_1"] \\
E_1 \arrow[r, "T_1(t)"] \arrow[ur, "A"] \arrow[d, "i_2"'] & E_1 = D(A) \arrow[d, "i_2"] \\
E_2 \arrow[r, "T_2(t)"] \arrow[ur, "A_1"] & E_2 = D(A_1) = D(A^2)
\end{tikzcd}
$$

For the translation semigroup on $L^p(\mathbb{R})$ (see 2.3) the above construction leads to the usual *Sobolev spaces*. Therefore we might call E_n the *n-th Sobolev space* and $(T_n(t))_{t\geqslant 0}$ the *n-th Sobolev semigroup* associated to E and $(T(t))_{t\geqslant 0}$.

Remark 3.6 For $\lambda \in \varrho(A)$ the operator $(\lambda - A)$ and the resolvent $R(\lambda, A)$ are isomorphisms from E_1 onto E, resp. from E onto E_1 (show that $\|\cdot\|_1$ and $\|\cdot\|_\lambda$ with $\|\cdot\|_\lambda := \|(\lambda - A)\cdot\|$ are equivalent). In addition, the following diagram commutes.

$$
\begin{tikzcd}
E \arrow[r, "T(t)"] \arrow[d, "\lambda - A"'] & E \arrow[d, "R(\lambda, A)"] \\
E_1 \arrow[r, "T_1(t)"] & E_1
\end{tikzcd}
$$

Therefore all Sobolev semigroups $(E_n, T_n(t))_{t\geqslant 0}$, $n \in \mathbb{N}$, are isomorphic.

Remark 3.7 For $\lambda \in \varrho(A)$ consider the norm

$$\|f\|_{-1} := \|R(\lambda, A)f\|$$

for every $f \in E$ and define E_{-1} as the completion of E for $\|\cdot\|_{-1}$.

Then $(T(t))_{t\geqslant 0}$ extends continuously to a strongly continuous semigroup $(T_{-1}(t))_{t\geqslant 0}$ on E_{-1} and the above diagram can be extended to the negative integers.

4 The $\mathcal{F}$-Product Semigroup

It is standard in functional analysis to consider a sequence of points in a certain space as a point in a new and larger space. In particular such a method can serve to convert an approximate eigenvector of a linear operator into an eigenvector. Occasionally we will need such a construction and refer to Section V.1 of **?**].

If we try to adapt this construction to strongly continuous semigroups we encounter the difficulty that the semigroup extended to the larger space will not remain strongly continuous. An idea already used in 3.4 will help to overcome this difficulty.

Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on the Banach space E. Denote by $m(E)$ the Banach space of all bounded E-valued sequences endowed with the norm

$$\|(f_n)_{n\in\mathbb{N}}\| := \sup\{\|f_n\| : n \in \mathbb{N}\}.$$

It is clear that every $T(t)$ extends canonically to a bounded linear operator

$$\hat{T}(t)(f_n) := (T(t)f_n)$$

on $m(E)$, but the semigroup $(\hat{T}(t))_{t\geqslant 0}$ is strongly continuous if and only if T has a bounded generator. Therefore we restrict our attention to the closed, $(\hat{T}(t))$-invariant subspace

$$m^{\mathcal{T}}(E) := \{(f_n) \in m(E) : \lim_{t\to 0} \|T(t)f_n - f_n\| = 0 \text{ uniformly for } n \in \mathbb{N}\}.$$

Then the restricted semigroup

$$\hat{T}(t)(f_n) = (T(t)f_n), \quad (f_n) \in m^{\mathcal{T}}(E)$$

is strongly continuous and we denote its generator by $(\hat{A}, D(\hat{A}))$.

The following lemma shows that $\hat{A}$ is obtained canonically from A.

Lemma 4.1 *For the generator $\hat{A}$ of $(\hat{T}(t))_{t\geqslant 0}$ on $m^{\mathcal{T}}(E)$ one has the following properties.*

(i) $D(\hat{A}) = \{(f_n) \in m^{\mathcal{T}}(E) : f_n \in D(A)$ *and* $(Af_n) \in m^{\mathcal{T}}(E)\}$,
(ii) $\hat{A}(f_n) = (Af_n)$ *for* $(f_n) \in D(\hat{A})$.

For the proof we refer to Lemma 1.4. of **?**].

Now let $\mathcal{F}$ be any filter on $\mathbb{N}$ finer than the Frechét filter (i.e., the filter of sets with finite complement. In most cases F will be either the Frechét filter or some free ultra filter.) The space of all $\mathcal{F}$-null sequences in $m(E)$, i.e.,

$$c_{\mathcal{F}}(E) := \{(f_n) \in m(E) : \mathcal{F}\text{-}\lim \|f_n\| = 0\}$$

is closed in $m(E)$ and invariant under $(\hat{T}(t))_{t\geqslant 0}$. We call the quotient spaces

$$E_{\mathcal{F}} := m(E)/c_{\mathcal{F}}(E) \quad \text{and} \quad E_{\mathcal{F}}^{T} := m^{\mathcal{T}}(E)/c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E)$$

the *$\mathcal{F}$-product of E* and the *$\mathcal{F}$-product of E with respect to the semigroup T*, respectively.

Thus $E_{\mathcal{F}}^{T}$ can be considered as a closed linear subspace of $E_{\mathcal{F}}$. We have $E_{\mathcal{F}}^{T} = E_{\mathcal{F}}$ if (and only if) T has a bounded generator.

The canonical quotient norm on $E_{\mathcal{F}}$ is given by

$$\|(f_n) + c_{\mathcal{F}}(E)\| = \mathcal{F}\text{-}\limsup \|f_n\|.$$

We can apply Subsec. 3.4 in order to define the *$\mathcal{F}$-product semigroup* $(T_{\mathcal{F}}(t))_{t\geqslant 0}$ on $E_{\mathcal{F}}^{T}$ by

$$T_{\mathcal{F}}(t)((f_n) + c_{\mathcal{F}}(E)) := (T(t)f_n) + c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E)$$

Thus $T_{\mathcal{F}}(t)$ is the restriction of $T(t)_F$ where $T(t)_F$ denotes the canonical extension of $T(t)$ to the $\mathcal{F}$-product $E_{\mathcal{F}}$. But note that $(T(t)_F)_{t\geqslant 0}$ is not strongly continuous unless T has a bounded generator.

With the canonical injection $j\colon f \mapsto (f, f, f, \ldots) + c_{\mathcal{F}}(E)$ from E into $E_{\mathcal{F}}^{T}$ the operators $T_{\mathcal{F}}(t)$ are extensions of $T(t)$ satisfying $\|T_{\mathcal{F}}(t)\| = \|T(t)\|$. The basic facts about the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of $(T_{\mathcal{F}}(t))_{t\geqslant 0}$ follow from 3.3 and are collected in the following proposition.

Proposition 4.2 *For the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of the $\mathcal{F}$-product semigroup the following holds.*

(i) $D(A_{\mathcal{F}}) = \{(f_n) + c_{\mathcal{F}}(E)\colon f_n \in D(A); (f_n), (Af_n) \in m^{\mathcal{T}}(E)\}$,
(ii) $A_{\mathcal{F}}((f_n) + c_{\mathcal{F}}(E)) = (Af_n) + c_{\mathcal{F}}(E)$.

In case A is a bounded operator then $D(A_{\mathcal{F}}) = E_{\mathcal{F}}^{T} = E_{\mathcal{F}}$ and $A_{\mathcal{F}}$ is the canonical extension of A to $E_{\mathcal{F}}$.

We will show in A-III,4.5 that the above construction preserves and even improves many spectral properties of the semigroup and its generator.

5 The Tensor Product Semigroup

Real- or complex-valued functions of two variables x, y are often limits of functions of the form $\sum_{i=1}^{n} f_i(x)g_i(y)$ which, to some extent, allows one to consider the variables x and y separately. Since algebraic manipulation with these latter functions is governed by the formal rules of a tensor product, it is customary to identify (for example) the function

$$(x, y) \mapsto f(x)g(y)$$

with the tensor product $f \otimes g$ and to consider limits of linear combinations of such functions as elements of a completed tensor product.

To be more precise, we briefly present the most important examples for this situation.

Examples 5.1 (i) Let (X, Σ, μ) and (Y, Ω, ν) be measure spaces. If we identify for $f_i \in L^p(\mu)$, $g_i \in L^p(\nu)$ the elements $\sum_{i=1}^{n} f_i \otimes g_i$ of the tensor product

$$L^p(\mu) \otimes L^p(\nu)$$

with the (class of $\mu \times \nu$-a.e.-defined) functions

$$(x, y) \mapsto \sum_{i=1}^{n} f_i(x) g_i(y),$$

then $L^p(\mu) \otimes L^p(\nu)$ becomes a dense subspace of $L^p(X \times Y, \Sigma \times \Omega, \mu \times \nu)$ for $1 \leqslant p < \infty$.

(ii) Similarly, let X,Y be compact spaces. Then $C(X) \otimes C(Y)$ becomes a dense subspace of $C(X \times Y)$ by identifying, for $f \in C(X)$ and $g \in C(Y)$, $f \otimes g$ with the function

$$(x, y) \mapsto f(x)g(y).$$

We do not intend to go deeper into the quite sophisticated problems related to normed tensor products of general Banach spaces, but will rather confine ourselves to the discussion of certain special cases. These will always be related to one of the following standard methods to define a norm on the tensor product of two Banach spaces E, F.

Let $u := \sum_{i=1}^{n} f_i \otimes g_i$ be an element of $E \otimes F$. Then

(i) $\|u\|_\pi := \inf\{\sum_{j=1}^{m} \|h_j\| \|k_j\| : u = \sum_{j=1}^{m} h_j \otimes k_j, h_j \in E, k_j \in F\}$ defines the *greatest cross norm* π on $E \otimes F$.

(ii) $\|u\|_\varepsilon := \sup\{\langle u, \varphi \otimes \psi\rangle : \varphi \in E', \psi \in F', \|\varphi\|, \|\psi\| \leqslant 1\}$ defines the *least cross norm* ε on $E \times F$. Here, $\langle u, \varphi \otimes \psi\rangle$ denotes the canonical bilinear form on $(E \otimes F) \times (E' \otimes F')$, i.e., $\langle \sum_{i=1}^{n} f_i \otimes g_i, \varphi \otimes \psi\rangle = \sum_{i=1}^{n} \langle f_i, \varphi\rangle\langle g_i, \psi\rangle$.

(iii) if E and F are Hilbert spaces, $\|u\|_h = (u|u)_h^{1/2}$, where the scalar product $(\cdot|\cdot)_h$ is defined as in (ii), defines the *Hilbert norm* h on $E \otimes F$.

In the following we write $E \otimes_\alpha F$ for the tensor product of E and F endowed—if applicable—with one of the norms π, ε, h just defined. In each case one has $\|f \otimes g\| = \|f\| \|g\|$ for $f \in E$, $g \in F$.

By $E \widetilde{\otimes}_\alpha F$ we mean the completion of $E \otimes_\alpha F$. Moreover we recall how examples (i) and (ii) above fit into this pattern

$$L^1(\mu \otimes \nu) = L^1(\mu) \widetilde{\otimes}_\pi L^1(\nu), \quad L^2(\mu \otimes \nu) = L^2(\mu) \widetilde{\otimes}_h L^2(\nu),$$

$$C(X \otimes Y) = C(X) \widetilde{\otimes}_\varepsilon C(Y).$$

Finally, we point out that for any $S \in \mathcal{L}(E)$, $T \in \mathrm{L}F$, the mapping

$$\sum_{i=1}^{n} f_i \otimes g_i \mapsto \sum_{i=1}^{n} S f_i \otimes T g_i$$

defined on $E \otimes F$ is linear and continuous on $E \otimes_\alpha F$, hence has a continuous extension to $E \widetilde{\otimes}_\alpha F$. This operator, as well as its continuous extension, will be denoted by $S \otimes T$ and satisfies $\|S \otimes T\| = \|S\| \|T\|$. The notation $A \otimes B$ will also be used in the obvious way if A and B are not necessarily bounded operators on E and F. We are now ready to consider semigroups induced on the tensor product.

Proposition 5.2 *Let $(S(t))_{t\geqslant 0}$ and $(T(t))_{t\geqslant 0}$ be strongly continuous semigroups on Banach spaces E, F, and let A, B be their generators. Then the family $(S(t)\otimes T(t))_{t\geqslant 0}$ is a strongly continuous semigroup on $E\widetilde{\otimes}_\alpha F$. The closure of $A\otimes \mathrm{Id}+\mathrm{Id}\otimes B$, defined on the core $D(A)\otimes D(B)$, is its generator.*

Proof It is immediately verified that $(S(t)\otimes T(t))_{t\geqslant 0}$ is in fact a semigroup of operators on $E\widetilde{\otimes}_\alpha F$. The strong continuity need only be verified at $t = 0$ and on elements of the form $u = f\otimes g \in E\otimes F$.

This verification being straightforward, there remains to show that the generator of $(S(t)\otimes T(t))_{t\geqslant 0}$ is obtained as the closure of

$$(A\otimes \mathrm{Id}+\mathrm{Id}\otimes B, D(A)\otimes D(B)).$$

To this end, let $f\in D(A)$ and $g\in D(B)$. Then

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}(T(h)\otimes S(h)(f\otimes g)-f\otimes g)\\
&=\lim_{h\to 0}\frac{1}{h}(T(h)f\otimes(S(h)g-g)+(T(h)f-f)\otimes g)\\
&=(f\otimes Bg)+(Af\otimes g).
\end{aligned}$$

Since the elements of the form $f\otimes g$, $f\in D(A)$, $g\in D(B)$, generate the linear subspace $D(A)\otimes D(B)$ of $E\otimes_\alpha F$, this subspace belongs to the domain of the generator. Moreover, $D(A)\otimes D(B)$ is dense in $E\widetilde{\otimes}_\alpha F$ and invariant under $(S(t)\otimes T(t))_{t\geqslant 0}$, hence it is a core of $A\otimes \mathrm{Id}+\mathrm{Id}\otimes B$ by Prop. 1.8 (ii). □

6 The Product of Commuting Semigroups

Let $(S(t))_{t\geqslant 0}$ and $(T(t))_{t\geqslant 0}$ be semigroups with generators A and B, respectively on some Banach space E. It is not difficult to see that the following assertions are equivalent.

(a) $S(t)T(t) = S(t)T(t)$ for all $t \geqslant 0$.
(b) $R(\mu, A)R(\mu, B) = R(\mu, B)R(\mu, A)$ for some $\mu \in \varrho(A)\cap\varrho(B)$.
(c) $R(\mu, A)R(\mu, B) = R(\mu, B)R(\mu, A)$ for all $\mu \in \varrho(A)\cap\varrho(B)$.

In that case $U(t) = S(t)T(t)$ $(t \geqslant 0)$ defines a semigroup $(U(t))_{t\geqslant 0}$. Using Prop. 1.8 (ii) on p. 6 one easily shows that $D_0 := D(A)\cap D(B)$ is a core for its generator C and $Cf = Af + Bf$ for all $f\in D_0$.

Notes

For more complete information on semigroup theory we refer the reader to **?**], to the monographs by **?**], **?**] and **?**], to the survey article by **?**], to the bibliography by **?**] and to **?**].

Chapter A-II
Characterization of Semigroups on Banach Spaces

In this chapter two different problems are treated:

(i) to characterize generators of strongly continuous semigroups;
(ii) to characterize various properties of strongly continuous semigroups in terms of their generators.

In Section 1 the first problem is solved by finding conditions on the Cauchy problem associated with A and also by finding conditions on the resolvent of A. The second problem is treated for a hierarchy of smoothness properties of the semigroup.

Contraction semigroups are considered in Section 2. Here, the first problem has a simple and extremely useful solution: A densely defined operator A is generator of a contraction semigroup if and only if A is dissipative and satisfies a range condition.

Our approach is quite general. We do not only consider contractions with respect to the norm but also with respect to *half-norms*. This will allow us to obtain results on positive contraction semigroups simultaneously by choosing a suitable half-norm (cf. C-II, Section 1).

The last section contains a surprising result: on certain Banach spaces (e.g., L^∞) only bounded operators are generators of strongly continuous semigroups.

1 The Abstract Cauchy Problem, Special Semigroups and Perturbation

Linear differential equations in Banach spaces are intimately connected with the theory of one-parameter semigroups. In fact, given a closed linear operator A with dense domain $D(A)$ the following statement is true (with some reservation regarding a technical detail): The abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) \quad (t \geqslant 0) \\ u(0) &= f \end{aligned}$$

has a unique solution for every $f \in D(A)$ if and only if A is the generator of a strongly continuous semigroup. This is one characterization of generators which illustrates their important role for applications. The fundamental Hille-Yosida theorem gives a different characterization in terms of the resolvent and yields a powerful tool for actually proving that a given operator is the generator of a semigroup.

Another problem we will treat here is how diverse properties of a semigroup can be described in terms of its generator. This is a reasonable question from the theoretical point of view (since the generator uniquely determines the semigroup). It is of interest from the practical point of view as well: the generator is the given object, defined by the differential equation. It is useful to dispose of conditions of the generator itself giving information on the solutions (which might not be known explicitely). We discuss smoothness properties such as analyticity, differentiability, norm continuity and compactness of the semigroup.

A frequent method to obtain new generators out of a given one is by perturbation. We will have a brief look at this circle of problems at the end of this section.

The results are explained and illustrated by examples. Proofs are only given when new aspects are presented which are not yet contained in the literature, otherwise we refer to the recent monographs **?**], **?**], **?**].

1.1 The abstract Cauchy problem

Let A be a closed operator on a Banach space E and consider the abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \dot{u}(t) & = Au(t) \quad (t \geqslant 0) \\ u(0) & = f \end{cases}$$

By a solution of (ACP) for the initial value $f \in D(A)$ we understand a continuously differentiable function $u\colon\ [0, \infty[\to E$ satisfying $u(0) = f$ and $u(t) \in D(A)$ for all $t \geqslant 0$ such that $u'(t) = Au(t)$ for $t \geqslant 0$.

By A-I,Thm.1.7 there exists a unique solution of (ACP) for all initial values in the domain $D(A)$ whenever A is the generator of a strongly continuous semigroup. The converse does not hold (see Example 1.4. below). However, for the operator A_1 on the Banach space $E_1 = D(A)$ (see A-I,3.5) with domain $D(A_1) = D(A^2)$ given by $A_1 f = Af$ $(f \in D(A_1))$ the following holds.

Theorem 1.1 *The following assertions are equivalent.*

(a) For every $f \in D(A)$ there exists a unique solution of (ACP).
(b) A_1 is the generator of a strongly continuous semigroup.

Proof (a) $\Longrightarrow$ (b). Assume that (a) holds; i.e., for every $f \in D(A)$ there exists a unique solution $u(\cdot, f) \in C^1([0, \infty), E)$ of (ACP). For $f \in E_1$ define $T_1(t)f = u(t, f)$. By the uniqueness of the solutions it follows that $T_1(t)$ is a linear operator on E_1 and $T_1(s + t) = T_1(s)T_1(t)$. Moreover, since $u(\cdot, f) \in C^1$, it follows that

$t \mapsto T_1(t)f$ is continuous from $[0, \infty)$ into E_1. We show that $T_1(t)$ is a continuous operator for all $t > 0$.

Let $t > 0$. Consider the mapping $\eta\colon E_1 \to C([0,t], E_1)$ given by $\eta(f) = T_1(\cdot)f = u(\cdot, f)$. We show that η has a closed graph.

In fact, let $f_n \to f$ in E_1 and $\eta(f_n) = u(\cdot, f_n) \to v$ in $C([0,t], E_1)$. Then $u(s, f_n) = f_n + \int_0^s Au(r, f_n)\,\mathrm{d}r$. Letting $n \to \infty$ we obtain $v(s) = f + \int_0^s Av(r)\,\mathrm{d}r$ for $0 \leqslant s \leqslant t$. Let $\tilde{v}(s) = T_1(s-t)v(t)$ for $s > t$, and $\tilde{v}(s) = v(s)$ for $0 \leqslant s \leqslant t$. Then $\tilde{v}$ is a solution of (ACP). It follows that $\tilde{v}(s) = T_1(s)f$ for all $s \geqslant 0$. Hence $v = \eta(f)$.

We have shown that η has a closed graph and so η is continuous. This implies the continuity of $T_1(t)$. It remains to show that A_1 is the generator of $(T_1(t))_{t\geqslant 0}$.

We first show that for $f \in D(A^2)$ one has

$$AT_1(t)f = T_1(t)Af. \tag{1.1}$$

In fact, let $v(t) = f + \int_0^t u(s, Af)\,\mathrm{d}s$. Then

$$\dot{v}(t) = u(t, Af) = Af + \int_0^t Au(s, Af)\,\mathrm{d}s = A(f + \int_0^t u(s, Af)\,\mathrm{d}s) = Av(t).$$

Since $v(0) = f$, it follows that $v(t) = u(t, f)$. Hence $Au(t, f) = Av(t) = \dot{v}(t) = u(t, Af)$. This is (1.1). Now denote by B the generator of $(T_1(t))_{t\geqslant 0}$. For $f \in D(A^2)$, we have

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af$$

and by (1.1),

$$\lim_{t\to 0} A\frac{T_1(t)f - f}{t} = \lim_{t\to 0} \frac{T_1(t)Af - Af}{t} = A^2 f$$

in the norm of E. Hence

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af$$

in the norm of E_1. This shows that $A_1 \subset B$.

In order to show the converse, let $f \in D(B)$. Then

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t}$$

exists in the norm of E. Since

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af$$

in the norm of E, it follows that $Af \in D(A)$, since A is closed. Thus $f \in D(A^2) = D(A_1)$. We have shown that $B = A_1$.

(b) $\Longrightarrow$ (a) . Assume that A_1 is the generator of a strongly continuous semigroup $(T_1(t))_{t\geqslant 0}$ on E_1. Let $f \in D(A)$ and set $u(t) = T_1(t)f$. Then $u \in C([0, \infty), E)$ and $Au(\cdot) \in C([0, \infty), E)$.

Moreover, $\int_0^t u(s)\,\mathrm{d}s = \int_0^t T_1(s)f\,\mathrm{d}s \in D(A_1) = D(A^2)$ and $A\int_0^t u(s)\,\mathrm{d}s = u(t)-u(0) = u(t)-f$ (by A-I, (1.3)). Consequently, $u(t) = f + \int_0^t Au(s)\,\mathrm{d}s$. Hence $u \in C^1([0,\infty), E)$ and $\dot{u}(t) = Au(t)$. Thus u is a solution of (ACP). We have shown existence.

In order to show uniqueness, assume that u is a solution of (ACP) with initial value 0. We have to show that $u \equiv 0$.

Let $v(t) = \int_0^t u(s)\,\mathrm{d}s$. Then $v(t) \in D(A)$ and $Av(t) = \int_0^t Au(s)\,\mathrm{d}s = \int_0^t \dot{u}(s)\,\mathrm{d}s = u(t) \in D(A)$. Consequently, $v(t) \in D(A^2)$ for all $t \geqslant 0$. Moreover, $\dot{v}(t) = u(t) = Av(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}Av(t) = Au(t) = A_1\dot{v}(t) = A^2v(t)$. Thus $v \in C^1([0,\infty), E_1)$ and $\dot{v}(t) = A_1v(t)$. Since $v(0) = 0$, it follows that $v \equiv 0$. Thus $u \equiv v \equiv 0$. □

If (ACP) has a unique solution for every initial value in $D(A)$, then A is the generator of a strongly continuous semigroup only if some additional assumptions on the solutions (continuous dependence from the initial value) or on A ($\varrho(A) \neq \emptyset$) are made.

Corollary 1.2 *Let A be a closed operator. Consider the following existence and uniqueness condition.*

(EU) For every $f \in D(A)$ there exists a unique solution $u(\cdot, f) \in C^1([0,\infty), E)$ of the Cauchy problem associated with A having the initial value $u(0, f) = f$.

The following assertions are equivalent.

(a) A is the generator of a strongly continuous semigroup.
(b) A satisfies (EU) and $\varrho(A) \neq \emptyset$.
(c) A satisfies (EU) and for every $\mu \in \mathbb{R}$ there exists $\lambda > \mu$ such that $(\lambda - A)D(A) = E$.
(d) A satisfies (EU), has dense domain and for every sequence (f_n) in $D(A)$ satisfying $\lim_{n\to\infty} f_n = 0$ one has $\lim_{n\to\infty} u(t, f_n) = 0$ uniformly in $t \in [0,1]$.

Proof It is clear that (a) implies the remaining assertions. So assume that A satisfy (EU). Then by Theorem 1.1, A_1 is a generator. If there exists $\lambda \in \varrho(A)$, then $(\lambda - A)$ is an isomorphism from E_1 onto E and A is similar to A_1 via this isomorphism since $D(A_1) = \{(\lambda - A)^{-1}f : f \in D(A)\}$ and $Af = (\lambda - A)A_1(\lambda - A)^{-1}f$ for all $f \in D(A)$, see A-I, 3.0. Thus A is a generator on E and we have shown that (b) implies (a),

If (c) holds, then there exists $\lambda > s(A_1)$ such that $(\lambda - A)D(A) = E$. We show that $(\lambda - A)$ is injective. Then $\lambda \in \varrho(A)$ since A is closed. Assume that $\lambda f = Af$ for some $f \in D(A)$. Then $f \in D(A^2) = D(A_1)$, and so $f = 0$ since $\lambda \in \varrho(A_1)$. This proves that (c) implies (b).

It remains to show that (d) implies (a) . Assertion (d) implies that for all $t \geqslant 0$ there exist bounded operators $T(t) \in \mathcal{L}(E)$ such that $u(t, f) = T(t)f$ if $f \in D(A)$. Moreover, $\sup_{0\leqslant t\leqslant 1} \|T(t)\| < \infty$. It follows that $T(\cdot)f$ is strongly continuous for all $f \in E$ (since it is so for $f \in D(A)$ and $D(A)$ is dense). Let $t > 1$. There exist unique $n \in \mathbb{N}$ and $s \in [0, 1)$ such that $t = n + s$. Let $T(t) := T(1)^n T(s)$. From the uniqueness of the solutions it follows that $T(t)f = u(t, f)$ for all $t \geqslant 0$ as well as $T(t+s)f = T(s)T(t)f$ for all $f \in D(A)$ and $s, t \geqslant 0$. Thus T is a semigroup.

Denote by B its generator. It follows from the definition that $A \subset B$. Moreover, $D(A)$ is invariant under the semigroup. So by A-I, Prop.1.9. (ii) $D(A)$ is a core of B. Since A is closed this implies that $A = B$. □

Remark 1.3 It is surprising that from condition (b) and (c) in the corollary it follows automatically that $D(A)$ is dense. On the other hand this condition cannot be omitted in (d). In fact, consider any generator $\tilde{A}$ and its restriction A with domain $D(A) = \{0\}$. Then $\overline{A}$ satisfies the remaining conditions in (d) but is not a generator (if $\dim E > 0$).

Example 1.4 We give a densely defined closed operator A, such that there exists a unique solution of (ACP) for all initial values in $D(A)$, but A is not a generator.

Let B be a densely defined unbounded closed operator on a Banach space F. Consider $E = F \oplus F$ and A on E given by:

$$A = \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix}$$

with domain $F \times D(B)$.

Then $D(A^2) = \{(f, g) \in F \times D(B) \colon Bg \in F\} = D(A)$ and so $A_1 \in \mathrm{L}E_1$. In particular, A_1 is a generator. But A is not. For instance condition (b) in Corollary 1.2 . does not hold, since for each $\lambda \in \mathbb{C}$:

$$(\lambda - A)D(A) = \{(\lambda f - Bg, \lambda g) \colon f \in F, g \in D(B)\} \subset F \times D(B) \neq F \times F = E$$

So $\varrho(A) = \emptyset$.

As a further illustration, we note that the solution of the corresponding abstract Cauchy problem for the initial value $(f, g) \in F \times D(B)$ is given by $u(t) = (f + tBg, g)$. Since B is unbounded, condition (d) of Corollary 1.2 is clearly violated.

Remark Frequently a generator A can be extended to a closed operator B. Then one can consider the abstract Cauchy problem ACP(B) associated with B. It also has a solution for every initial value in $D(B)$, but none of the solutions is unique unless $A = B$.

In fact, denote by $(T(t))_{t \geqslant 0}$ the semigroup generated by A. Let $f \in D(B)$. Let $\lambda > \omega(A)$. Then there exists $g \in D(A)$ such that $(\lambda - B)f = (\lambda - A)g$. Let $h = f - g$. Then $h \in \ker(\lambda - B)$. Define u by $u(t) = e^{\lambda t}h + T(t)g$. Then u is a solution ACP(B) with initial value f. It follows from Cor. 1.2 that there exists a non-trivial solution for the initial value 0. □

1.2 One-parameter groups

Generators of one-parameter groups can be characterized similarly by existence and uniqueness of the solutions of the associated Cauchy problem. However, here the assumption of continuous dependence on the initial values can be relaxed (in fact, one has no longer to assume that the continuous dependence is uniform in t).

Theorem 1.6 *Let A be a closed densely defined operator. The following assertions are equivalent.*

(a) A is generator of a strongly continuous one-parameter group.
(b) For every $f \in D(A)$ there exists a unique function $u(\cdot, f) \in C^1(\mathbb{R})$ satisfying $u(t, f) \in D(A)$ for all $t \in \mathbb{R}$ and $u(0, f) = f$ such that $\frac{d}{dt}u = Au(t, f)$, and if $f_n \in D(A)$ such that $\lim_{n\to\infty} f_n = 0$, then $\lim_{n\to\infty} u(t, f_n) = 0$ for all $t \in \mathbb{R}$.

Proof It is clear that (a) implies (b) . If (b) holds then there exist operators $T(t) \in \mathcal{L}(E)$ such that $u(t, f) = T(t)f$ $(t \in \mathbb{R}, f \in D(A))$. It follows from the uniqueness of the solutions that $T(t+s) = T(t)T(s)$ $(s, t \in \mathbb{R})$. Let $f \in E$. There exist $(f_n) \in D(A)$ such that $\lim_{n\to\infty} f_n = f$.

Then $\lim_{n\to\infty} T(t)f_n = T(t)f$ for all $t \in \mathbb{R}$. Since $T(\cdot)f$ is continuous, it follows that $T(\cdot)f$ is measureable. Hence by [Hille-Phillips (1975),10.2.1] $\sup_{t\in J} \|T(t)\| < \infty$ for every compact interval $J \subset (0, \infty)$. Because of the group property this implies that $T(\cdot)$ is norm bounded on bounded subsets of $\mathbb{R}$. $T(\cdot)f$ is continuous if $f \in D(A)$. Since $D(A)$ is dense this implies the strong continuity of $(T(t))_{t\in\mathbb{R}}$. □

1.3 The Hille-Yosida theorem

Given an operator A frequently it is easier to obtain information about its resolvent than to solve the Cauchy problem. Therefore the following theorem is central in the theory of one-parameter semigroups.

Theorem 1.7 (Hille-Yosida) *Let A be an operator on a Banach space E. The following conditions are equivalent.*

(a) A is the generator of a strongly continuous semigroup.
(b) There exist $w, M \in \mathbb{R}$ textsuch that $(w, \infty) \subset \varrho(A)$ textand

$$\|(\lambda - w)^n R(\lambda - A)^{-n}\| \leqslant M$$

for all $\lambda > w$ and $n \in \mathbb{N}$.

In general it is not easy to give an estimate for the powers of the resolvent which enables one to apply Theorem 1.7. However, there is an important case when it suffices to consider merely the resolvent.

Corollary 1.8 *For an operator A on a Banach space E the following assertions are equivalent.*

(a) A is the generator of a strongly continuous contraction semigroup.
(b) $(0, \infty) \subset \varrho(A)$ and $\|\lambda R(\lambda, A)\| \leqslant 1$ for all $\lambda > 0$.

The difficult part in the proof of Theorem 1.7 is to show that (b) implies (a) . One has to construct the semigroup out of the resolvent. We mention two formulas which can be used for the proof.

Proposition 1.9 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$. For $\lambda > 0$ let $A(\lambda) = \lambda^2 R(\lambda, A) - \lambda Id$ $(= \lambda AR(\lambda, A))$. Then*

$$T(t)f = \lim_{\lambda\to\infty} e^{tA(\lambda)} f \tag{1.2}$$

for all $f \in E$ and $t \geqslant 0$.

Yosida's proof consists in showing that the limit in (1.2) exists under the hypothesis (b) of Theorem 1.7 (see **?**], **?**] or **?**][Pazy (1982)].

The proof of Hille (see **?**]) is inspired by the following formula.

Proposition 1.10 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$. Then*

$$T(t)f = \lim_{n\to\infty} (Id - t/nA)^{-n} f = \lim_{n\to\infty} (n/t \cdot R(n/t, A))^n f \tag{1.3}$$

for all $f \in E$ and $t \geqslant 0$.

1.4 Holomorphic semigroups

We now describe a hierarchy of smoothness conditions on the semigroup, starting with the most restrictive class; namely, holomorphic semigroups. The generators of these semigroups can be characterized by a particularly simple condition.

For $\alpha \in (0, \pi]$ we define the sector $S(\alpha)$ in the complex plane by

$$S(\alpha) = \{re^{i\vartheta} : r \geqslant 0, \vartheta \in (-\alpha, \alpha)\}$$

Definition 1.11 Let $\alpha \in (0, \pi/2]$. A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is called a bounded holomorphic semigroup of angle α if $T(\cdot)$ is the restriction of a holomorphic function

$$T : S(\alpha) \to \mathcal{L}(E)$$

satisfying

$$T(z)T(z') = T(z + z') \quad (z, z' \in S(\alpha)), \tag{1.4}$$

for each $\alpha_1 \in (0, \alpha)$ the set $\{T(z) : z \in S(\alpha_1)\}$ is uniformly bounded and

$$\lim_{n\to\infty} T(z_n)f = f \text{ for every null-sequence } (z_n) \text{ in } S(\alpha_1) \text{ and every } f \in E. \tag{1.5}$$

Remark A function $T : S(\alpha) \to L(E)$ is holomorphic with respect to the operator norm if and only if it is strongly holomorphic if and only if it is weakly holomorphic [Yosida (1965);V.3]. □

Theorem 1.12 *Let A be a densely defined operator on a Banach space E and $\alpha \in (0, \pi/2]$. Then A is the generator of a bounded holomorphic semigroup of angle α if and only if*

$$S(\alpha + \pi/2) \subset \varrho(A)$$

and for every $\alpha_1 \in (0, \alpha)$ there exists a constant M such that

$$\|R(\lambda, A)\| \leqslant M/|\lambda| \quad (\lambda \in S(\alpha_1 + \pi/2). \tag{1.6}$$

For the proof we refer to **?**], for example.

Remark Let A be the generator of a bounded holomorphic semigroup $(T(t))_{t\geqslant 0}$ of angle α, and let $z_0 \in S(\alpha)$. Then $z_0 A$ generates a bounded semigroup $(S(t))_{t\geqslant 0}$ given by $S(t) = T(z_0 t)$ $(t \geqslant 0)$ (where again we denote by $\mathcal{T}$ the holomorphic extension of $(T(t))_{t\geqslant 0}$ on $S(\alpha)$). □

As an application of Theorem 1.12 we prove the following.

Corollary 1.13 *Let A be the generator of a bounded group. Then A^2 generates a bounded holomorphic semigroup of angle $\pi/2$.*

Proof Let $0 < \alpha_1 < \pi/2$ and $\lambda \in S(\alpha_1 + \pi/2)$. There exists $r > 0$ and $\beta \in (-\beta_1, \beta_1)$, where $\beta_1 := (\alpha_1 + \pi/2)/2$, such that $\lambda = r^2 e^{i2\beta}$. Then $(\lambda - A^2) = (re^{\mathrm{i}\beta} - A)(re^{\mathrm{i}\beta} + A)$; so it follows that $\lambda \in \varrho(A)$ and

$$R(\lambda, A^2) = R(re^{\mathrm{i}\beta}, A)R(re^{-\mathrm{i}\beta}, -A) \tag{1.7}$$

Since A generates a bounded group, there exists a constant $N \geqslant 0$ such that $\|R(\mu, A)\| \leqslant N/\mathrm{Re}\mu$, $\|R(\mu, -A)\| \leqslant N/\mathrm{Re}\mu$ for all $\mu \in S(\pi/2)$. Consequently, $\|R(\lambda, A^2)\| \leqslant N^2/r^2(\cos\beta)^2 \leqslant 1/r^2[N/\cos\beta]^2 = M/|\lambda|$. □

The corollary will be extended below. We first consider an example.

Example (The Laplacian on $E = C_0(\mathbb{R}^n)$ or $L^p(R^n)$ $(1 \leqslant p < \infty)$)

(i) Let $n = 1$. Then $(U(t)f)(x) = f(x + t)$ $(x \in \mathbb{R})$ defines an isometric group on E. Its generator A is given by $Af = f'$ with $D(A) = \{f \in C^1(\mathbb{R}) \cap C_0(\mathbb{R}) \colon f' \in C_0(\mathbb{R})\}$ in the case $E = C_0(\mathbb{R})$ and $D(A) = \{f \in E \cap AC(\mathbb{R}) \colon f' \in E\}$ in the case $E = L^p$ (see A-I, 2.4). Thus A^2 generates a bounded holomorphic semigroup by Corollary 1.13.

(ii) Let $E = C_0(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$. For $i \in \{1, \ldots, n\}$ denote by $(U_i)_{t\geqslant 0}$ the group given by $(U_i(t)f)(x) = f(x_1, \ldots, x_{i-1}, x_i + t, \ldots, x_n)$ $(x \in \mathbb{R}^n, t \in \mathbb{R})$ and by A_i its generator. Since $U_i(t)U_j(s) = U_j(s)U_i(t)$ $(s, t \in \mathbb{R}, i, j \in \{1, \ldots, n\})$ it follows that the resolvents of A_i commute. So the same is true for the resolvents of A_i^2 (cf. (1.7) and A-I, 3.8)

Denote by $(T_i(t))_{t\geqslant 0}$ the semigroup generated by A_i^2 $(i = 1, \ldots, n)$. Then for $z, z' \in S(\pi/2)$ one has $T_i(z)T_j(z') = T_j(z')T_i(z)$ $(i, j = 1, \ldots, n)$. Consequently, $T(t) := T_1(t) \circ \cdots \circ T_n(t)$ $(t \geqslant 0)$ defines a holomorphic semigroup of angle $\pi/2$. According to A-I,3.8 the domain of its generator A contains $D(A_1^2) \cap \cdots \cap D(A_n^2)$;

in particular $D_0 = \{f \in E \cap C^2(\mathbb{R}^n) \colon D^\alpha f \in E$ for every multiindex α with $|\alpha| \leqslant 2\} \subset D(A)$ and the generator is given by

$$Af = (A_1^2 + \cdots + A_n^2)f = \sum_{i=1}^{n} \frac{\partial^2}{(\partial x_i)^2} f = \Delta F \quad \text{for all} \quad f \in D_0.$$

Let $\alpha \in (0, \pi/2]$. A semigroup $(T(t))_{t \geqslant 0}$ is called *holomorphic of angle* α if it possesses an extension $T \colon S(\alpha) \to L(E)$ for some $\alpha \in (0, \pi/2]$ which satisfies all the requirements of Definition 1.11 except that it is not required to be bounded on any sector $S(\alpha_1)$.

Theorem 1.14 *A densely defined operator A is the generator of a holomorphic semigroup if and only if there exist $M > 0$ and $r \geqslant 0$ such that $\lambda \in \varrho(A)$ and $\|R(\lambda, A)\| \leqslant M/|\lambda|$ whenever* $\mathrm{Re}\lambda > 0$, $|\lambda| \geqslant r$.

Proof It is not difficult to show that A generates a holomorphic semigroup of angle α if and only if for every $\alpha_1 \in (0, \alpha)$ there exists $w \in \mathbb{R}$ such that $A - w$ generates a bounded holomorphic semigroup of angle α_1 (cf. [Reed-Simon (1978b), p.252]). As a consequence one obtains the following. A densely defined operator A generates a holomorphic semigroup of angle $\alpha \in (0, \pi/2]$ if and only if for every $\alpha_1 \in [0, \alpha[$ there exist a constant $M \geqslant 0$ and $r \geqslant 0$ such that

$$S(\alpha_1 + \pi/2) \setminus B(r) \subset \varrho(A) \quad (\text{where} \quad B(r) = \{z \in \mathbb{C} \colon |z| \leqslant r\})$$

and

$$\|R(\lambda, A)\| \leqslant M/|\lambda| \quad \text{for all} \quad \lambda \in S(\alpha_1 + \pi/2) \setminus B(r).$$

This shows that the condition of the theorem is necessary. Conversely, assume that the condition holds. Since $\|R(\lambda, A)\| \to \infty$ when λ approaches $\sigma(A)$ (cf. Lemma 1.21 below), it follows that $\lambda \in \varrho(A)$ and $\|R(\lambda, A)\| \leqslant M/|\lambda|$ if $\mathrm{Re}\lambda = 0$ and $|\lambda| > r$ as well.

Let $c = 1/(2M)$. If $\xi, \eta \in \mathbb{R}$ satisfy $|\xi| \leqslant c|\eta|, |\eta| \geqslant r$, then $\|\xi R(\mathrm{i}\eta, A)\| \leqslant |\xi| \cdot M/|\eta| \leqslant c \cdot M = 1/2$. Hence $R(\xi + \mathrm{i}\eta, A) = \sum_{n=0}^{\infty} (-\xi)^n R(\mathrm{i}\eta, A)^{n+1}$ exists and

$$\begin{aligned}
\|R(\xi + \mathrm{i}\eta, A)\| &\leqslant (|\xi + \mathrm{i}\eta|)^{-1} \cdot |\xi + \mathrm{i}\eta| \cdot \sum_{n=0}^{\infty} |\xi|^n M^{n+1}/|\eta|^{n+1} \\
&\leqslant (|\xi + \mathrm{i}\eta|)^{-1} \cdot 2M(|\xi|^2 + |\eta|^2)^{1/2}/|\eta| \cdot \sum_{n=0}^{\infty} M^n c^n \\
&\leqslant (4M \cdot (c^2 + 1)^{1/2})/|\xi + \mathrm{i}\eta| \\
&\leqslant N/|\xi + \mathrm{i}\eta|.
\end{aligned}$$

This together with the assumption implies that there exist $N' \geqslant 0$ and $\alpha \in]0, \pi/2]$ such that $\lambda \in \varrho(A)$ and $\|R(\lambda, A)\| \leqslant N'/|\lambda|$ for all $\lambda \in S(\alpha + \pi/2)$. ◻

Compared with the Hille-Yosida theorem, Theorem 1.14 gives a very simple criterion for an operator to be the generator of a (holomorphic) semigroup. Merely the resolvent and not its powers have to be estimated. However, the resolvent has to be known in the right half-plane instead of a right half-line.

On the other hand, given a strongly continuous semigroup, merely an estimate on a vertical line implies that the semigroup is holomorphic. More precisely, the following holds.

Corollary A strongly continuous semigroup with generator A is holomorphic if and only if there exist $w > \omega(A)$ and $M \geqslant 0$ such that one has

$$\|R(w + \mathrm{i}\eta, A)\| \leqslant \frac{M}{|\eta|} \text{ for all } \quad \eta \in \mathbb{R}.$$

Proof In fact, assume that the condition holds. Since $A - w$ is the generator of a bounded semigroup one has $\|R(\lambda, A - w)\| \leqslant N/\mathrm{Re}\lambda$ for some $N > 0$ and all $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}\lambda > 0$. Consequently, for every $\alpha \in (0, \pi/2)$, $\|R(\lambda, A - w)\| \leqslant (|\lambda|/\mathrm{Re}\lambda)N/|\lambda| \leqslant N(\cos\alpha)^{-1}/|\lambda|$ for all $\lambda \in S(\alpha)$. The remaining estimate for a sector around the imaginary axis is given by the proof of Theorem 1.14 and shows that $A - w$ generates a holomorphic semigroup. The reverse implication is clear. □

We now prove the following extension of Corollary 1.13

Theorem 1.15 *Let A be the generator of a strongly continuous group. Then A^2 generates a holomorphic semigroup of angle $\pi/2$.*

Proof There exists $w \geqslant 0$ such that $(\pm A - w)$ generates a bounded semigroup. Consequently, there exists $M \geqslant 0$ such that $\|R(\mu, \pm A - w)\| \leqslant M/\mathrm{Re}\mu$ whenever $\mathrm{Re}\mu > 0$.

Let $\alpha \in (0, \pi/2)$. There exist $r_0 \geqslant 0$ and $\beta \in (0, \pi/2)$ such that

$$S(\alpha + \pi/2) \setminus B(r_0) \subset \{z^2 \colon z \in S(\beta) + w\}.$$

[In fact, the line $\{w + r(\cos\beta + \mathrm{i}\sin\beta) \colon r \geqslant 0\}$ can be parameterized by $z(t) = w + t + \mathrm{i} \cdot t/\epsilon$ $(t \geqslant 0)$ (where $\epsilon > 0$ depends on β). Then $z(t)^2 = (w + t)^2 - t^2/\epsilon^2 + i2t(w + t)/\epsilon$. Thus $\lim_{t\to\infty} \mathrm{Im}z(t)^2/\mathrm{Re}z(t)^2 = 2\epsilon/(\epsilon^2 - 1)$. Choose $\beta \in (\pi/4, \pi/2)$ such that $\tan(\alpha + \pi/2) > 2\epsilon/(\epsilon^2 - 1)$.]

Now let $\lambda \in S(\alpha + \pi/2) \setminus B(r_0)$. Then there exist $\vartheta \in (-\beta, \beta)$ and $r \geqslant 0$ such that $\lambda = (re^{\mathrm{i}\vartheta} + \omega)^2$. Thus $(\lambda - A^2) = (re^{\mathrm{i}\vartheta} + w - A)(re^{\mathrm{i}\vartheta} + w + A)$. Hence $\lambda \in \varrho(A^2)$ and $R(\lambda, A^2) = R(re^{\mathrm{i}\vartheta}, A - w)R(re^{\mathrm{i}\vartheta}, -A - w)$. We conclude that $|\lambda| \cdot \|R(\lambda, A^2)\| \leqslant |\lambda| \cdot M^2/(\cos\vartheta)^2 r^2 \leqslant (|\lambda|/r^2) \cdot M^2/(\cos\beta)^2$. Thus $|\lambda| \cdot \|R(\lambda, A^2)\|$ is uniformly bounded for $\lambda \in S(\alpha + \pi/2) \setminus B(r_0)$. □

Remark In Theorem 1.15 the assumption that $\pm A$ are generators can be relaxed. In fact, the proof shows the following. If A is a densely defined operator such that $\{\lambda \in \mathbb{C} \colon \mathrm{Re}\lambda > 0\} \subset \varrho(\pm A - w)$ and $\|R(\lambda, \pm A - w)\| \leqslant M/\mathrm{Re}\lambda$ for some $M \geqslant 0$, $w \geqslant 0$, then A^2 generates a holomorphic semigroup of angle $\pi/2$. □

Next we consider semigroups satisfying a less restrictive smoothness condition.

1.5 Differentiable semigroups

Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator A. Let $t_0 \geqslant 0$ and $f \in E$. Then the function $t \mapsto T(t)f$ is right sided differentiable at t_0 if and only if $T(t_0)f \in D(A)$; and in that case it is differentiable at every $s > t_0$ and the derivative is $AT(s)f$ (this follows from A-I, Proposition 1.6(ii)).

Definition 1.16 A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E is called *eventually differentiable* if there exists $t_0 \geqslant 0$ such that the function $t \mapsto T(t)f$ from (t_0, ∞) into E is differentiable for every $f \in E$. The semigroup is called *differentiable* if t_0 can be chosen 0.

It is not difficult to see that if $(T(t))_{t\geqslant 0}$ is differentiable for $t > t_0$, then for $n \in \mathbb{N}$ it is n-times differentiable at all $s > nt_0$ and $T(s)E \subset D(A^n)$. If $(T(t))_{t\geqslant 0}$ is differentiable, then the function $t \mapsto T(t)f$ from $(0, \infty)$ into E is infinitely often differentiable for every $f \in E$.

Generators of (eventually) differentiable semigroups can be characterized similarly as those of holomorphic semigroups by the spectral behavior of the resolvent. Whereas the spectrum of the generator of a holomorphic semigroup is included in a sector, the spectrum of the generator of an eventually differentiable semigroup is limited by a function of exponential growth (instead of a line).

Theorem 1.17 *A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is eventually differentiable if and only if its generator A satisfies the following: there exist constants $c > 0$, $b > 0$, $M > 0$ such that*

$$\Sigma := \{\lambda \in \mathbb{C} \colon ce^{-b\cdot \mathrm{Re}\lambda} \leqslant |\,\mathrm{Im}\lambda|\} \subset \varrho(A)$$

and

$$\|R(\lambda, A)\| \leqslant M \cdot |\,\mathrm{Im}\lambda| \textit{ for all } \lambda \in \Sigma \textit{ satisfying } \mathrm{Re}\lambda \leqslant \omega(A).$$

Theorem 1.18 *A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is differentiable if and only if its generator A satisfies the following: for all $b > 0$ there exist $c > 0$, $M > 0$ such that*

$$\Sigma := \{\lambda \in \mathbb{C} \colon ce^{-b\cdot \mathrm{Re}\lambda} \leqslant |\,\mathrm{Im}\lambda|\} \subset \varrho(A)$$

and

$$\|R(\lambda, A)\| \leqslant M \cdot |\,\mathrm{Im}\lambda| \textit{ for all } \lambda \in \Sigma \textit{ satisfying } \mathrm{Re}\lambda \leqslant \omega(A).$$

For the proofs of these two theorems we refer to **?** , Chapter 3, Theorem 4.7 and 4.8].

1.6 Norm continuous semigroups

Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup and $t' > 0$. If $\lim_{t\to t'} \|T(t) - T(t')\| = 0$, then it follows from the semigroup property, that the function $t \mapsto T(t)$ is norm continuous on the whole half line (t', ∞).

Definition 1.19 A semigroup $(T(t))_{t\geqslant 0}$ is called *eventually norm continuous* if there exists $t' \geqslant 0$ such that the function $t \mapsto T(t)$ from (t', ∞) into $\mathcal{L}(E)$ is norm continuous. The semigroup is called *norm continuous* if t' can be chosen equal to 0.

The spectrum of generators of eventually norm continuous semigroups still is compact in every right half-plane.

Theorem 1.20 *Let A be the generator of an eventually norm continuous semigroup. Then for every $b \in \mathbb{R}$ the set*

$$\{\lambda \in \sigma(A) \colon \ \mathrm{Re}\lambda \geqslant b\}$$

is bounded.

For the proof of Theorem 1.20 we use the following lemma.

Lemma 1.21 *Let A be an operator and $\lambda \in \varrho(A)$. Then*

$$dist(\lambda, \sigma(A)) = r(R(\lambda, A))^{-1}.$$

Proof One has $\{0\} \cup \sigma(R(\lambda, A)) = \{0\} \cup \{(\lambda - \mu)^{-1} \colon \mu \in \sigma(A)\}$ **?** , Lemma 2.11]. Hence $r(R(\lambda, A)) = \sup\{|\lambda - \mu|^{-1} \colon \mu \in \sigma(A)\} = (\inf\{|\lambda - \mu| \colon \mu \in \sigma(A)\})^{-1} = \mathrm{dist}(\lambda, \sigma(A))^{-1}$. □

Proof (Proof of Theorem 1.20) It is enough to show the following. Let $a > \omega(A)$. Then for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and $r_0 \geqslant 0$ such that

$$\|R(a + ir, A)^n\|^{1/n} < \epsilon \qquad \text{for all } r \in \mathbb{R} \text{ satisfying } |r| \geqslant r_0.$$

[In fact, then we have by the lemma,

$$\mathrm{dist}(a + ir, \sigma(A)) = r(R(a + ir, A))^{-1} \geqslant \epsilon^{-1} \quad \text{whenever} \quad |r| \geqslant r_0.]$$

So let $\epsilon > 0$. If $\mathrm{Re}\lambda > \omega(A)$, then by A-I, Proposition 1.11,

$$R(\lambda, A)^{n+1} = \frac{1}{n!} \int_0^\infty e^{-\lambda t} t^n T(t)\, \mathrm{d}t (n \in \mathbb{N}).$$

Let $t' > 0$ such that $t \mapsto T(t)$ is norm continuous on (t', ∞). Let $w \in (\omega(A), a)$. There exists $M \geqslant 1$ such that $\|T(t)\| \leqslant Me^{wt}$ for all $t \geqslant 0$. Let $N := M \cdot \int_0^{t'} e^{-at} e^{wt}\, \mathrm{d}t$. Since $\lim_{n\to\infty} c^n/n! = 0$ for all $c > 0$, there exists $n \in \mathbb{N}$ such that $N \cdot (t')^n/n! < \epsilon^{n+1}/3$. Choose $T \geqslant t'$ such that $\frac{1}{n!}\int_T^\infty t^n e^{-at} \|T(t)\|\, \mathrm{d}t < \epsilon^{n+1}/3$.

Since $(T(t))_{t\geqslant 0}$ is norm continuous for $t \geqslant t'$, it follows from the Riemann-Lebesgue lemma that there exists $r_0 \geqslant 0$ such that $\|\frac{1}{n!}\int_{t'}^T t^n e^{-irt} e^{-at} T(t)\, \mathrm{d}t\| < \epsilon^{n+1}/3$ whenever $|r| \geqslant r_0$.

All together we obtain for $|r| \geqslant r_0$,

$$\begin{aligned}
\|R(a+ir,A)^{n+1}\| &= \frac{1}{n!}\cdot \Big\|\int_0^\infty e^{-(a+ir)t} t^n T(t)\,\mathrm{d}t\Big\| \\
&\leqslant \frac{1}{n!}\cdot \int_0^{t'} e^{-at} t^n \|T(t)\|\,\mathrm{d}t \\
&+ \frac{1}{n!}\cdot \Big\|\int_{t'}^{T} t^n e^{-irt} e^{-at} T(t)\,\mathrm{d}t\Big\| \\
&+ \frac{1}{n!}\cdot \int_T^\infty e^{-at} t^n \|T(t)\|\,\mathrm{d}t \\
&\leqslant \frac{1}{n!}\cdot (t')^n \int_0^{t'} e^{-at} M e^{wt}\,\mathrm{d}t + \frac{2}{3}\cdot \epsilon^{n+1} \\
&\leqslant N\cdot (t')^n/n! + \frac{2}{3}\cdot \epsilon^{n+1} \\
&\leqslant \epsilon^{n+1}.
\end{aligned}$$

A complete characterization of eventually norm continuous semigroups in terms of their generator seems not to be known.

Eventually norm continuous semigroups are of particular interest in spectral theory (cf. A-III, Theorem 6.6). Moreover their asymptotic behavior is easy to describe (see A-IV, (1.8)). Next we describe special norm continuous semigroups.

1.7 Compact semigroups

Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup and $t_0 > 0$. If $T(t_0)$ is compact, then it follows from the semigroup property that $T(t)$ is compact for all $t \geqslant t_0$. Moreover, $t \mapsto T(t)$ is norm continuous at every $t > t_0$.

[In fact, since $T(h) \to \mathrm{Id}$ strongly with $h \downarrow 0$, it follows that $\lim_{h\downarrow 0} T(h)f = f$ uniformly on every compact subset K of E. Now let $t \geqslant t_0$. Then $K = T(t)(U)$ is compact (where U denotes the unit ball of E). Hence $\lim\limits_{h\downarrow 0} T(h+t)f = \lim\limits_{h\downarrow 0} T(h)T(t)f$ uniformly for $f \in U$. So the semigroup is right-sided norm continuous on $[t_0,\infty)$ and so norm continuous on (t_0,∞).]

Definition 1.22 A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is called *compact* if $T(t)$ is compact for all $t > 0$; the semigroup is called *eventually compact* if there exists $t_0 > 0$ such that $T(t_0)$ is compact (and hence $T(t)$ is compact for all $t \geqslant t_0$).

We want to find a relation between the compactness of the semigroup and the compactness of the resolvent of its generator.

Definition 1.23 Let A be an operator and $\varrho(A) \neq \emptyset$. We say, A has a compact resolvent if $R(\lambda, A)$ is compact for one (and hence all) $\lambda \in \varrho(A)$.

Proposition 1.24 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup and assume that its generator has a compact resolvent. If $t \mapsto T(t)$ is norm continuous at t_0, then $T(t)$ is compact for all $t \geqslant t_0$.*

Proof Considering $(e^{-wt}T(t))_{t\geqslant 0}$ for some $w > 0$ if necessary, we can assume that $\sigma(A) < 0$. Let $S(t) \in \mathcal{L}(E)$ be given by $S(t)f = \int_0^t T(s)f\,\mathrm{d}s$ $(t \geqslant 0)$. Then $AS(t)f = T(t)f - f$ for all $f \in E$, and so $S(t) = R(0, A)(Id - T(t))$ is compact for all $t \geqslant 0$.

Since $t \mapsto T(t)$ is norm continuous for $t \geqslant t_0$, one has $\lim_{h\downarrow 0} \frac{1}{h}(S(t_0+h)-S(t_0)) = T(t_0)$ in the operator norm. Thus $T(t_0)$ is compact as limit of compact operators. □

Theorem 1.25 *A strongly continuous semigroup* $(T(t))_{t\geqslant 0}$ *is compact if and only if it is norm continuous and its generator* A *has compact resolvent.*

Proof Assume that $(T(t))_{t\geqslant 0}$ is compact. Then $T(\cdot)$ is norm continuous on $(0, \infty)$, and so

$$\int_0^t e^{-ws}T(s)\,\mathrm{d}s$$

is compact as the norm limmit of linear combinations of compact operators, where $w > \omega A$. Since

$$R(w, A) = \lim_{t\to\infty} \int_0^t e^{-ws}T(s)\,\mathrm{d}s$$

in the operator norm, it follows that $R(w, A)$ is compact. This proves one implication. The other follows from Proposition 1.24 . □

Remark 1.26 (i) Generators of eventually compact semigroups do not necessarily have compact resolvent. Consider the nilpotent translation semigroup $(T(t))_{t\geqslant 0}$ on $F := L^1([0, 1])$ (see A-I, Example 2.6). Let $E = F\widetilde{\otimes}_\pi F = L^1([0, 1] \times [0, 1])$ and $S(t) = T(t)\otimes \mathrm{Id}$ $(t \geqslant 0)$. Then $(S(t))_{t\geqslant 0}$ is a strongly continuous semigroup (see A-I, 3.7). Denote by B its generator. $(S(t))_{t\geqslant 0}$ is a nilpotent semigroup (so it is eventually compact), but $R(\lambda, B) = R(\lambda, A) \otimes \mathrm{Id}$ is not compact.

(ii) It is obvious that a group $(T(t))_{t\in\mathbb{R}}$ is eventually norm continuous if and only if it is norm continuous in 0; i.e., its generator is bounded.

On the other hand, the generator of the rotation group (A-I, Example 2.5) has a compact resolvent. Hence this condition does not imply any smoothness property of the semigroup.

Positive eventually compact semigroups have remarkable properties in the setting of the Perron-Frobenius theory (see e.g., B-III, Corollary 2.12).

The following scheme indicates the relation between the different classes of semigroups defined so far:

$$\begin{array}{ccccc}
\text{holomorphic} & \longrightarrow & \text{differentiable} & \longrightarrow & \text{eventually differentiable} \\
 & & \downarrow & & \downarrow \\
 & & \text{norm continuous} & \longrightarrow & \text{eventually norm continuous} \\
 & & \uparrow & & \uparrow \\
 & & \text{compact} & \longrightarrow & \text{eventually compact}
\end{array}$$

All these classes are different. This is shown by the following examples.

Example 1.27 The nilpotent shift semigroup (A-I, 2.6) is obviously eventually differentiable, eventually compact and eventually norm continuous. But it is not norm continuous and consequently not differentiable or compact.

Example 1.28 We consider multiplication semigroups (see A-I, 2.3). Let $E = C_0(X)$, where X is a locally compact space, or $E = L^p(X, \Sigma, \mu)$, where $1 \leqslant p < \infty$ and (X, Σ, μ) is a σ-finite measure space. Let $m\colon X \to \mathbb{R}$ be continuous [resp., measurable] such that [ess]- $\sup_{x\in X} \operatorname{Re}(m(x)) < \infty$.

Then $Af = m \cdot f$ with domain $D(A) = \{f \in E : m \cdot f \in E\}$ is the generator of the semigroup $(T(t))_{t\geqslant 0}$ given by

$$(T(t)f)(x) = e^{tm(x)} f(x), \quad (t \geqslant 0, x \in X, f \in E)$$

Observe that $\sigma(A) = \overline{m(X)}$ in case $E = C_0(X)$ and $\sigma(A) =$ [ess]-image$(m) := \{\lambda \in \mathbb{C} : \mu(\{x \in X : |m(x) - \lambda| < \varepsilon\}) \neq 0 \text{ for all } \varepsilon > 0\}$ if $E = L^p$ (see A-II, 2.3). Consequently, $s(A) = \omega(A) =$ [ess]- $\sup_{x\in X} \operatorname{Re}(m(x))$.

(i) The semigroup is norm continuous for $t > 0$ if and only if it is eventually norm continuous if and only if $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geqslant b\}$ is bounded for every $b \in \mathbb{R}$. Thus the property proved in Theorem 1.20 characterizes generators of eventually norm continuous semigroups in the case that the semigroup consists of multiplication operators.

Proof Assume that $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geqslant b\}$ is bounded for every $b \in \mathbb{R}$. Let $t' > 0$. We show that the semigroup is norm continuous at t'. Let $\varepsilon > 0$. Let $b \in \mathbb{R}$ such that $2e^{(t'+1)b} < \varepsilon$.

If $\operatorname{Re}(m(x)) \leqslant b$, then

$$|e^{tm(x)} - e^{t'm(x)}| \leqslant e^{t \operatorname{Re}(m(x))} + e^{t' \operatorname{Re}(m(x))} \leqslant 2e^{(t'+1)b} < \varepsilon$$

whenever $|t - t'| \leqslant 1$.

By hypothesis, the set $H := \{m(x) : x \in X, \operatorname{Re}(m(x)) \geqslant b\}$ in the case $E = C_0(X)$ and $H := \{m(x) : \operatorname{Re}\lambda \geqslant b \text{ and for all } \eta > 0, \mu(\{x \in X : |m(x) - \lambda| < \eta\}) \neq 0\}$ in the case $E = L^p$ is a bounded subset of $\mathbb{C}$. Thus $\lim_{t\to t'} |e^{tz} - e^{t'z}| = 0$ uniformly for $z \in H$. Hence there exists $\delta \in]0, 1]$ such that [ess]- $\sup\{|e^{tm(x)} - e^{t'm(x)}| : x \in X, \operatorname{Re}(m(x)) > b\} < \varepsilon$ whenever $|t - t'| < \delta$. Together with the inequality above, we obtain that $\|T(t) - T(t')\| =$ [ess]- $\sup\{|e^{tm(x)} - e^{t'm(x)}| : x \in X\} < \varepsilon$ whenever $|t - t'| < \delta$. We have shown that the semigroup is norm continuous for $t > 0$ whenever $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geqslant b\}$ is bounded for all $b \in \mathbb{R}$. □

(ii) The semigroup is right-sided differentiable at a fixed point $t > 0$ if and only if there exists $c > 0$ such that $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| > c \cdot e^{-t \operatorname{Re}\lambda}\} \subset \varrho(A)$.

Proof The semigroup is right-sided differentiable at t if and only if $T(t)E \subset D(A)$ if and only if $e^{tm} \cdot f \cdot m \in E$ for all $f \in E$ if and only if $e^{tm} \cdot m$ is [essentially] bounded if and only if $e^{t \operatorname{Re} m} \cdot \operatorname{Im} m$ is [essentially] bounded if and only if there exists $c > 0$ such that [ess]-image$(m) \subset \{\lambda \in \mathbb{C} : e^{t \operatorname{Re}\lambda} |\operatorname{Im}\lambda| \leqslant c\}$ if and only if there exists $c > 0$ such that $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| > c \cdot e^{-t \operatorname{Re}\lambda}\} \subset \varrho(A)$. □

(iii) $(T(t))_{t\geqslant 0}$ is a bounded holomorphic semigroup of angle ϑ if and only if $S(\vartheta+\pi/2)\subset\varrho(A)$.

Proof The condition is necessary by Theorem 1.12. Conversely, if $S(\vartheta+\pi/2)\subset\varrho(A)$, then one verifies directly that

$$(T(z)f)(x)=e^{z\cdot m(x)}f(x)\quad (f\in E, x\in X)$$

defines a family $(T(z))_{z\in S(\vartheta)}$ of bounded operators satisfying conditions (1.4) and (1.5). □

(iv) Choosing $X=\mathbb{N}$ and the counting measure we have $E=c_0$ or ℓ^p. Then A has a compact resolvent if $\lim_{n\to\infty}|m(n)|=\infty$. [In fact, let $\lambda>s(A)$. Then $(R(\lambda,A)f)(n)=(\lambda-m(n))^{-1}f(n)$. Hence $R(\lambda,A)$ is compact if and only if $((\lambda-m(n))^{-1})_{n\in\mathbb{N}}\in c_0$.]

The semigroup is compact if and only if it is eventually compact if and only if $\lim_{n\to\infty}\mathrm{Re}(m(n))=-\infty$.

(v) Now it is easy to give concrete examples. Again let $X=\mathbb{N}$, so that $E=c_0$ or ℓ^p. Let $m(n)=-n+i\cdot\exp(n^2)$. Then the semigroup is compact and (consequently) norm continuous for $t>0$, but it is not eventually differentiable. Let $m(n)=-n+ie^{t'n}$. Then the semigroup is differentiable for $t>t'$ but not differentiable at $t\in[0,t')$. If $m(n)=-n+i\cdot n^2$, then the semigroup is differentiable but not holomorphic.

1.8 Perturbation of Generators

A useful way to construct new semigroups out of a given one is by additive perturbation.

Theorem 1.29 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ and let $B\in\mathcal{L}(E)$. Then $A+B$ with domain $D(A+B)=D(A)$ is the generator of a strongly continuous semigroup $(S(t))_{t\geqslant 0}$.*

It is possible to express the new semigroup $(S(t))_{t\geqslant 0}$ by known objects. The product formula

$$S(t)f=\lim_{n\to\infty}(T(t/n)e^{t/n\cdot B})^n f \tag{1.8}$$

holds for all $t\geqslant 0$ and $f\in E$.

Moreover, $S(t)$ is the solution of the following integral equation

$$S(t)f=T(t)f+\int_0^t T(t-s)BS(s)f\,\mathrm{d}s\quad (t\geqslant 0, f\in E). \tag{1.9}$$

Let $S_0(t)=T(t)$ and

$$S_n(t)f=\int_0^t T(t-s)BS_{n-1}(s)f\,\mathrm{d}s\quad (f\in E) \tag{1.10}$$

for $n \in \mathbb{N}$. Then

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \tag{1.11}$$

where the series converges in the operator norm uniformly on bounded intervals. We refer to **?** , III.1], **?** , I.6] or **?** , Chapter 3] for these results.

Several special properties discussed above are preserved by bounded perturbastions.

Theorem 1.30 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator A. Let $B \in \mathcal{L}(E)$. If $(T(t))_{t\geqslant 0}$ is holomorphic or norm continuous or compact, then so is the semigroup $(S(t))_{t\geqslant 0}$ generated by $A + B$.*

If A has a compact resolvent then so has $A + B$.

Let $t_0 \geqslant 0$. If $(T(t))_{t\geqslant t_0}$ is norm continuous for $t > t_0$ and if B is compact, then $(S(t))_{t\geqslant t_0}$ is also norm continuous for $t > t_0$.

Proof If $(T(t))_{t\geqslant 0}$ is norm continuous for $t > 0$, then $S_n(t)$ in (1.10) is norm continuous at $t > 0$ for every n. Thus $(S(t))_{t\geqslant 0}$ is norm continuous at $t > 0$ by (1.11). There exists $\lambda_0 \in \mathbb{R}$ such that $\|R(\lambda, A)\| \leqslant (2\|B\|)^{-1}$ for $\mathrm{Re}\lambda \geqslant \lambda_0$. Hence $(Id - BR(\lambda, A))^{-1}$ exists for $\mathrm{Re}\lambda \geqslant \lambda_0$. Since $(\lambda - (A+B))f = (Id - BR(\lambda, A))(\lambda - A)f$ for all $f \in D(A)$ it follows that $(\lambda - (A + B))^{-1}$ exists and is given by

$$R(\lambda, A + B) = R(\lambda, A)(Id - BR(\lambda, A))^{-1} \tag{1.12}$$

whenever $\mathrm{Re}\lambda \geqslant \lambda_0$. Now if A generates a holomorphic semigroup, there exists $M \geqslant 0$ such that $\|R(\lambda_0 + \mathrm{i}\eta, A)\| \leqslant M/|\eta|$ for all $\eta \in \mathbb{R}$. Consequently, $\|R(\lambda_0 + \mathrm{i}\eta, A+B)\| \leqslant \|(Id-BR(\lambda_0+i\eta, A))^{-1}\|\cdot 2M/|\eta| \leqslant 2M/|\eta|$ for all $\eta \in \mathbb{R}$. Thus $A+B$ generates a holomorphic semigroup by the corollary of Theorem 1.14. Moreover, it follows from (1.12) that $R(\lambda, A + B)$ is compact whenever $R(\lambda, A)$ is compact. Consequently by Theorem 1.25 and the assertion proved above, $(S(t))_{t\geqslant 0}$ is compact whenever $(T(t))_{t\geqslant 0}$ is compact.

Finally assume that B is compact and $t_0 \geqslant 0$ such that $(T(t))_{t\geqslant 0}$ is norm continuous for $t > t_0$. Fix $t > t_0$. Denote by U the unit ball of E and fix $s \in (0, t]$. Then

$$\lim_{h\to 0}(T(t + s - h) - T(t - s))f = 0$$

for all $f \in \overline{BS(s)U} =: K$.

Since K is compact it follows that the limit exists uniformly in $f \in K$; i.e. $\lim_{h\to 0} \|(T(t + s - h) - T(t - s))BS(s)\| = 0$. It follows from the dominated convergence theorem that

$$\lim_{h\to 0} \int_0^t \|(T(t + s - h) - T(t - s))BS(s)\| \, \mathrm{d}s = 0. \tag{1.13}$$

Using (1.9) we obtain $\|S(t + h) - S(t)\|$

$$\begin{aligned}
&\leqslant \|T(t+h) - T(t)\| + \|\int_0^{t+h} T(t+h-s)BS(s)\,\mathrm{d}s \\
&\qquad\qquad - \int_0^t (T(t-s)BS(s)\,\mathrm{d}s\| \\
&\leqslant \|T(t+h) - T(t)\| + \|\int_t^{t+h} T(t+h-s)BS(s)\,\mathrm{d}s\| \\
&\qquad\qquad + \int_0^t \|(T(t+h-s) - T(t-s))BS(s)\|\,\mathrm{d}s \to 0 \\
&\qquad\qquad \text{when} \quad h \to 0.
\end{aligned}$$

In C-IV, Example 2.15 a generator A of an eventually differentiable and eventually compact semigroup and a bounded operator B will be given such that the semigroup generated by $A + B$ is not eventually norm continuous.

Using Theorem 1.29 we now prove a perturbation result due to
?]. Instead of assuming that $B \in \mathcal{L}(E)$ we assume that $B \in \mathcal{L}(D(A))$. The short proof given below is due to G.Greiner.

Theorem 1.31 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator A. Assume that $B\colon D(A) \to D(A)$ is linear and continuous for the graph norm on $D(A)$.*

Then $A + B$ with domain $D(A + B) = D(A)$ is the generator of a strongly continuous semigroup. Moreover, there exists a bounded operator C on E such that $A + B$ is similar to $A + C$.

Proof We first show that $(Id - BR(\lambda, A))$ is invertible for some $\lambda \in \mathbb{C}$. Choose $\lambda_0 \in \varrho(A)$. Then $S := (\lambda_0 - A)BR(\lambda_0, A) \in \mathcal{L}(E)$. Let $\lambda > s(A)$ be so large such that $\|SR(\lambda, A)\| < 1$. Then $(1 - (\lambda_0 - A)BR(\lambda_0, A)R(\lambda, A))^{-1} = (1 - SR(\lambda, A)$ is invertible. Consequently, also $(1 - BR(\lambda, A))^{-1})$ exists (since $\sigma(TR(\lambda_0, A)) \setminus \{0\} = \sigma(R(\lambda_0, A)T) \setminus \{0\}, \quad T = (\lambda_0 - A)BR(\lambda, A))$.

Let $C = (A - \lambda)B(A - \lambda)^{-1} \in \mathcal{L}(E)$. Then $A + C$ is the generator of a strongly continuous semigroup by Theorem 1.29 . We show that $A + B$ is similar to $A + C$. In fact, let $U = (1 - BR(\lambda, A))$. Then U is an isomorphism on E such that $U(D(A)) = D(A)$. Moreover,

$$\begin{aligned}
U(A + C)U^{-1} &= U(A - \lambda + C)U^{-1} + \lambda \\
&= U[(A - \lambda) - (A - \lambda)BR(\lambda, A)]U^{-1} + \lambda \\
&= U(A - \lambda)[1 - BR(\lambda, A)]U^{-1} + \lambda \\
&= U(A - \lambda) + \lambda \\
&= A - \lambda + B + \lambda \\
&= A + B.
\end{aligned}$$

Corollary 1.32 *Keeping the hypotheses and notations of Theorem 1.31 denote by $(S(t))_{t\geqslant 0}$ the semigroup generated by $A + B$. If $(T(t))_{t\geqslant 0}$ is norm continuous or* *compact or holomorphic, then* $(S(t))_{t\geqslant 0}$ *has the corresponding properties. If* B *is compact as an operator on* $D(A)$ *endowed with the graph norm and if* $(T(t))_{t\geqslant 0}$ *is eventually norm continuous then so is* $(S(t))_{t\geqslant 0}$.

Proof This follows from Theorem 1.30 since $(US(t)U^{-1})_{t\geqslant 0}$ has $A+C$ as generator. □

1.9 Domains of Uniqueness

Given a semigroup $(T(t))_{t\geqslant 0}$ it is frequently difficult to determine the precise domain of its generator A. So it is important to know which (possibly strict) subspaces of $D(A)$ determine the semigroup uniquely. This can be formulated more precisely in the following way. Let D_0 be a subspace of $D(A)$ and consider the restriction A_0 of A to D_0. Under which condition on D_0 is A the only extension of A_0 which is a generator? One obvious condition is that D_0 is a core. [In fact, in that case, A is the closure of A_0. Since every generator B extending A_0 is closed, it follows that $A \subset B$ and hence $A = B$ since $\varrho(A) \cap \varrho(B) \neq \emptyset$].

We now show that cores are the only domains of uniqueness.

Theorem 1.33 *Let* A *be the generator of a semigroup and* D_0 *a subspace of* $D(A)$. *Consider the restriction* A_0 *of* A *to* D_0. *If* D_0 *is not a core of* A, *then there exists an infinite number of extensions of* A_0 *which are generators.*

Proof If D_0 is not dense in $D(A)$ with respect to the graph norm, then there exists a non-zero linear form φ on $D(A)$ which is continuous for the graph norm such that $\varphi(f) = 0$ for all $f \in D_0$. Let $u \in D(A)$ and $B\colon D(A) \to D(A)$ be given by $Bf = \varphi(f)u$ for all $f \in D(A)$. Then B is continuous for the graph norm. So by Theorem 1.31 the operator $A + B$ with domain $D(A)$ is a generator. Clearly, $A + B \neq A$ if $u \neq 0$ but $Af + Bf = Af$ for all $f \in D_0$. It is obvious that an infinite number of generators can be constructed in that way. □

Corollary 1.34 *Let* $(T(t))_{t\geqslant 0}$ *be a strongly continuous semigroup with generator* A. *Let* D_0 *be a dense subspace of* E. *Assume that* $D_0 \subset D(A)$ *and* $T(t)D_0 \subset D_0$ *for all* $t \geqslant 0$. *Then* D_0 *is a core.*

Proof Let $(S(t))_{t\geqslant 0}$ be a semigroup with generator B such that $B|_{D_0} = A|_{D_0}$. Let $f \in D_0$. Then $u(t) := T(t)f$ satisfies $u(0) = f$ and $\dot{u}(t) = AT(t)f = BT(t)f = Bu(t)$ $(t \geqslant 0)$. Since $v(t) = S(t)f$ $(t \geqslant 0)$ also is a solution of the Cauchy problem defined by B with initial value f it follows that $S(t)f = T(t)f$ $(t \geqslant 0)$. Since D_0 is dense in E, it follows that $S(t) = T(t)$ $(t \geqslant 0)$. □

2 Contraction Semigroups and Dissipative Operators

by
Wolfgang Arendt

The Hille-Yosida theorem gives a characterization of generators in terms of the resolvent of the operator. However, given an operator A, frequently it is difficult to compute the resolvent (and its powers). So it is desirable to find conditions more immanent on A. This is possible for generators of contraction semigroups. For later purposes (see B-II and C-II) it will be useful not only to consider semigroups which are contractive with respect to the norm but to consider more general sublinear functionals than the norm as well.

So our setting is the following. By E we denote a real Banach space throughout, and $p\colon E \to \mathbb{R}$ is a continuous sublinear function; i.e., p satisfies

$$p(f+g) \leqslant p(f) + p(g) \quad (f, g \in E), \tag{2.1}$$
$$p(\lambda f) = \lambda p(f) \quad (f \in E, \lambda \geqslant 0). \tag{2.2}$$

The continuity of p implies that there exists a constant $c > 0$ such that

$$|p(f)| \leqslant c\|f\| \quad (f \in E). \tag{2.3}$$

Moreover, it follows from (2.1) and (2.2) that

$$p(f) + p(-f) \geqslant p(0) = 0 \quad (f \in E). \tag{2.4}$$

A bounded operator T on E is called *p-contractive* if $p(Tf) \leqslant p(f)$ for all $f \in E$. Similarly, a semigroup $(T(t))_{t\geqslant 0}$ is called *p-contractive* if $T(t)$ is p-contractive for all $t \geqslant 0$.

Of course, the most important case we have in mind in this section is the case when p is the norm function N given by $N(f) = \|f\|$ $(f \in E)$. An N-contractive operator is just a contraction in the usual sense.

Remark However in Chapter B-II and C-II it will be important to dispose of a variety of sublinear functionals other than N. For example, we will consider N^+ on $C[0,1]$ given by $N^+(f) = \sup_{x\in[0,1]} f(x)$. Then a bounded operator T is N^+-contractive if and only if T is positive and $\|T\| \leqslant 1$. □

We first want to solve the following problem. Given the generator A of a semigroup $(T(t))_{t\geqslant 0}$ find a condition on A which is equivalent to $T(t)$ being p-contractive for all $t \geqslant 0$.

The *subdifferential* dp of p at f is defined by

$$\mathrm{d}p(f) = \{\varphi \in E'\colon \langle g, \varphi\rangle \leqslant p(g) \text{ for all } g \in E, \langle f, \varphi\rangle = p(f)\}. \tag{2.5}$$

It follows from the Hahn-Banach theorem that $\mathrm{d}p(f) \neq \emptyset$ for all $f \in E$.

Definition 2.1 An operator A on E is called *p-dissipative* if for all $f \in D(A)$ there exists $\varphi \in \mathrm{d}p(f)$ such that $\langle Af, \varphi\rangle \leqslant 0$; A is called *strictly p-dissipative* if for all $f \in D(A)$ the inequality $\langle Af, \varphi\rangle \leqslant 0$ holds for all $\varphi \in \mathrm{d}p(f)$.

For convenience we want to have a distinctive name for the norm function. So we denote by $N\colon E \to \mathbb{R}$ the function given by $N(f) = \|f\|$ throughout. Then (2.5) can be written in the form

$$\mathrm{d}N(f) = \{\varphi \in E'\colon \|\varphi\| \leqslant 1, \langle f, \varphi\rangle = \|f\|\}. \tag{2.6}$$

A [strictly] N-dissipative operator is simply called [strictly] *dissipative* (which is in accordance with the usual nomenclature).

Example 2.2 a) Let $E = C[0,1]$, $f \in E$. Then there exists $x \in [0,1]$ such that $|f(x)| = \|f\|_\infty$. Define $\varphi \in E'$ by $\langle g, \varphi\rangle = (\operatorname{sign} f(x))g(x)$. Then $\varphi \in \mathrm{d}N(f)$. Note that $\mathrm{d}N(f)$ may be an infinite set.

b) Let H be a Hilbert space, $f \in H$, $f \neq 0$. Then $\mathrm{d}N(f) = \{\varphi_f\}$ where $\langle g, \varphi_f\rangle = 1/\|f\|\,(g|f)$.

c) $A - \|A\|\,\mathrm{Id}$ is strictly dissipative for every bounded operator A.

Proposition 2.3 *Let A be an operator on E. Then A is p-dissipative if and only if*

$$p(f) \leqslant p(f - tAf) \text{ for all } f \in D(A), t > 0. \tag{2.7}$$

If in particular $(w, \infty) \subset \varrho(A)$ for some $w \in \mathbb{R}$, then A is p-dissipative if and only if

$$p(\lambda R(\lambda, A)f) \leqslant p(f) \text{ for all } f \in E, \lambda > w. \tag{2.8}$$

Proof Assume that A is p-dissipative. Let $f \in D(A)$, $t > 0$. There exists $\varphi \in \mathrm{d}p(f)$ such that $\langle Af, \varphi\rangle \leqslant 0$. Hence, $p(f) = \langle f, \varphi\rangle = \langle f - tAf + tAf, \varphi\rangle \leqslant \langle f - tAf, \varphi\rangle \leqslant p(f - tAf)$. So (2.7) holds.

Converse, let $f \in D(A)$. For every $t > 0$ choose $\varphi_t \in \mathrm{d}p(f - tAf)$. Then $\pm\langle g, \varphi_t\rangle \leqslant p(\pm g) \leqslant c\|g\|$ for all $g \in E$, $t > 0$. Thus the net $(\varphi_t)_{t>0}$ is bounded. Consequently it posseses a $\sigma(E', E)$-limit point φ as $t \to 0$. We show that $\varphi \in \mathrm{d}p(f)$ and $\langle Af, \varphi\rangle \leqslant 0$.

Since $\langle g, \varphi_t\rangle \leqslant p(g)$ for all $t > 0$ it follows that $\langle g, \varphi\rangle \leqslant p(g)$ $(g \in E)$. Moreover, $\langle f, \varphi_t\rangle - t\langle Af, \varphi_t\rangle = p(f - tAf)$ $(t > 0)$. Letting $t \to 0$ yields $\langle f, \varphi\rangle = p(f)$.

We have proved that $\varphi \in \mathrm{d}p(f)$. By hypothesis we have for all $t > 0$, $p(f) \leqslant p(f - tAf) = \langle f - tAf, \varphi_t\rangle = \langle f, \varphi_t\rangle - t\langle Af, \varphi_t\rangle \leqslant p(f) - t\langle Af, \varphi_t\rangle$. Consequently $\langle Af, \varphi_t\rangle \leqslant 0$ for all $t > 0$. Thus $\langle Af, \varphi\rangle \leqslant 0$. □

Remark 2.4 The function p is convex. So the one-sided Gateaux-derivatives

$$D_g^+ p(f) = \lim_{t\downarrow 0} 1/t(p(f + tg) - p(f)) \quad \text{and}$$
$$D_g^- p(f) = \lim_{t\uparrow 0} 1/t(p(f + tg) - p(f))$$

exist and satisfy $D_g^- p(f) \leqslant D_g^+ p(f)$ for all $f, g \in E$ (cf. **?**]). Moreover,

$$D_g^+ p(f) = \sup\{\langle g, \varphi\rangle\colon \varphi \in \mathrm{d}p(f)\}, \tag{2.9}$$

$$D_g^- p(f) = \inf\{\langle g, \varphi\rangle : \varphi \in \mathrm{d}p(f)\}. \tag{2.10}$$

Thus A is p-dissipative if and only if $D_{Af}^- p(f) \leqslant 0$, and A is strictly p-dissipative if and only if $D_{Af}^+ p(f) \leqslant 0$ for all $f \in D(A)$.

Corollary 2.5 *Let A be a closable operator. If A is p-dissipative, then so is its closure.*

Theorem 2.6 *Let p be a continuous sublinear functional on a real Banach space E. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$. The following assertions are equivalent.*

(a) $p(T(t)f) \leqslant p(f)$ for all $t \geqslant 0$, $f \in E$.
(b) A is strictly p-dissipative.
(c) There exists a core D of A such that $A|_D$ is p-dissipative.

Proof Assume that (a) holds. Let $f \in D(A)$, $\varphi \in \mathrm{d}p(f)$. Then

$$\begin{aligned}\langle Af, \varphi\rangle &= \lim_{t\to 0} 1/t(\langle T(t)f, \varphi\rangle - \langle f, \varphi\rangle) \\ &= \lim_{t\to 0} 1/t(\langle T(t)f, \varphi\rangle - p(f)) \\ &\leqslant \limsup_{t\to 0} 1/t(p(T(t)f) - p(f)) \leqslant 0.\end{aligned}$$

This proves (b) .

It is trivial that (b) implies (c). So let us assume (c) . Then it follows from Corollary 2.5 that A is p-dissipative. Hence by (2.8) $p(\lambda R(\lambda, A)g) \leqslant p(g)$ for all $g \in E$, $\lambda > \omega(A)$. Hence $\lambda R(\lambda, A)$ is p-contractive for $\lambda > \omega(A)$. This implies that $T(t)$ is p-contractive by the formula (1.3) :

$$T(t) = \lim_{t\to 0}(n/t\, R(n/t, A))^n \text{ (strongly) for } t \geqslant 0.$$

We have shown that for generators, p-dissipativity is equivalent to p-contractivity of the semigroup. Now we will consider a p-dissipative operator A (which is not a generator a priori) and investigate under which additional hypotheses A is the generator of a (necessarily contractive) semigroup. At first we present some consequences of p-dissipativity.

Theorem 2.7 *Let A be a p-dissipative operator. If $D(A)$ is dense, then A is strictly p-dissipative.*

Proof Let $f \in D(A)$, $\varphi \in \mathrm{d}p(f)$. Then for every $t > 0$ and $g \in D(A)$ we have

$$\begin{aligned}
\langle Af, \varphi\rangle &= \frac{1}{t}(\langle f + tAf, \varphi\rangle - \langle f, \varphi\rangle) \leqslant \frac{1}{t}(p(f + tAf) - p(f)) \\
&\leqslant \frac{1}{t}(p(f + tg) + tp(Af - g) - p(f)) \\
&\leqslant \frac{1}{t}(p((Id - tA)(f + tg)) + tp(Af - g) - p(f)) \quad \text{(by (2.7))} \\
&\leqslant \frac{1}{t}(p(f) + tp(g - Af) + t^2 p(-Ag) + tp(Af - g) - p(f)) \\
&\leqslant \frac{1}{t}(2tc\|g - Af\| + t^2 c\|Ag\|) \quad \text{(by (2.3))} \\
&= 2c\|g - Af\| + tc\|Ag\|.
\end{aligned}$$

Letting $t \to 0$ we obtain $\langle Af, \varphi\rangle \geqslant 2c\|g - Af\|$ for all $g \in D(A)$. Since $D(A)$ is dense in E, this implies that $\langle Af, \varphi\rangle \geqslant 0$. □

We now impose stronger conditions on p. A continuous sublinear function $p\colon E \to \mathbb{R}$ is called *half-norm* if

$$p(f) + p(-f) > 0 \text{ whenever } f \neq 0; \tag{2.11}$$

and p is called a *strict half-norm* if in addition there exists some constant $d > 0$ such that

$$p(f) + p(-f) \geqslant d\|f\| \text{ for all } f \in E. \tag{2.12}$$

If p is a half-norm, then

$$\|f\|_p = p(f) + p(-f) \quad (f \in E) \tag{2.13}$$

defines a norm on E which is equivalent to the given norm if and only if p is strict.

Remark 2.8 Every half-norm p induces a closed proper cone $E_p :=$ $\{f \in E\colon p(-f) \geqslant 0\}$ on E. Any p-contractive operator T on E leaves the cone E_p invariant (i.e. T is positive for the corresponding ordering).

Conversely, given a closed proper cone E_+ on E, then $p(f) := \text{dist}(-f, E_+) = \inf\{\|f + g\|\colon g \in E_+\}$ defines a half-norm on E such that $E_+ = E_p$. This half-norm is called the *canonical half-norm* on the ordered Banach space (E, E_+). The canonical half-norm is strict if and only if the cone E_+ is *normal* (this is equivalent to the fact that for every $\varphi \in E'$ there exist positive linear forms φ_1 and φ_2 on E such that $\varphi = \varphi_1 - \varphi_2$ (see **?**] or **?** , Chap. V]).

Proposition 2.9 *Let A be a p-dissipative operator where p is a half-norm. If $D(A)$ is dense, then A is closable (and the closure of A is p-dissipative as well (by Corollary 2.5)).*

Proof Let $f_n \in D(A)$, $\lim_{n\to\infty} f_n = 0$, $\lim_{n\to\infty} Af_n = g$. We have to show that $g = 0$. To this end let $h \in D(A)$. Then (2.7) gives $p(f_n + th) \leqslant p(f_n + th - tA(f_n + th))$ $(t > 0)$. Letting $n \to \infty$ we obtain $p(th) \leqslant p(th - tg - t^2 Ah) \quad (t > 0)$. Hence $p(h) \leqslant p((h - g) - tAh)$ $(t > 0)$ by positive homogeneity. Letting $t \downarrow 0$ finally we obtain $p(h) \leqslant p(h-g)$ for all $h \in D(A)$. Since $D(A)$ is dense by hypothesis, we can approximate g by $h \in D(A)$ and conclude that $p(g) \leqslant p(0) = 0$. Since $\lim_{n\to\infty} A(-f_n) = -g$, we have $p(-g) \leqslant 0$ by symmetry. Hence $p(g) + p(-g) \leqslant 0$ which implies $g = 0$ by (2.11) . □

Lemma 2.10 *Let p be a half-norm and A a p-dissipative operator. Then*

$$\lambda \|f\|_p \leqslant \|(\lambda - A)f\|_p \text{ for all } f \in D(A), \lambda > 0. \tag{2.14}$$

In particular, $(\lambda - A)$ is injective for all $\lambda > 0$. If p is strict and A is closed, then im$(\lambda - A)$ is closed for all $\lambda > 0$.

Proof Let $\lambda > 0$, $f \in D(A)$. Then by (2.7) , $\lambda p(\pm f) \leqslant p((\lambda - A)(\pm f))$. Hence $\lambda \|f\|_p = \lambda p(f) + \lambda p(-f) \leqslant p((\lambda - A)f) + p(-(\lambda - A)f) = \|(\lambda - A)f\|_p$. Thus (2.14) is proved. Now suppose that p is strict. Then $\|\cdot\|_p$ is equivalent to the given norm. Let $\lambda > 0$ and $g \in \overline{Im(\lambda - A)}$. Then $g = \lim_{n\to\infty}(\lambda - A)f_n$ for some sequence $(f_n)_{n\in\mathbb{N}} \subset D(A)$. It follows from (2.14) that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $f = \lim_{n\to\infty} f_n$. Then $\lim_{n\to\infty} Af_n = \lambda \lim_{n\to\infty} f_n - \lim_{n\to\infty}(\lambda - A)f_n = \lambda f - g$ exists. If A is closed, this implies that $f \in D(A)$ and $Af = \lambda f - g$. Hence $g = (\lambda - A)f \in \mathrm{im}(\lambda - A)$. We have shown that $\mathrm{im}(\lambda - A)$ is closed. □

The following is the main theorem of this section.

Theorem 2.11 *Let p be a strict half-norm and A an operator on E. The following assertions are equivalent.*

(a) A is the generator of a p-contraction semigroup.
(b) $D(A)$ is dense, A is p-dissipative and im$(\lambda - A) = E$ for some $\lambda > 0$.

Proof Since p is a strict half-norm we can assume that $\|f\| = \|f\|_p$ for all $f \in E$. By Theorem 2.6, condition (a) implies (b) .

Now suppose that (b) holds. Then it follows from Lemma 2.10 that $\mu \in \varrho(A)$ and $\|\mu R(\mu, A)\| \leqslant 1$ whenever $\mu > 0$ such that $\mathrm{im}(\mu - A) = E$. So by hypothesis $\lambda \in \varrho(A)$ and $\mathrm{dist}(\lambda, \sigma(A)) \geqslant \|R(\lambda, A)\|^{-1} \geqslant \lambda$. Hence $(0, 2\lambda) \subset \varrho(A)$. Iterating this argument we see that $(0, \infty) \subset \varrho(A)$. It follows from the Hille-Yosida theorem that A generates a contraction semigroup $(T(t))_{t\geqslant 0}$. Finally, from 2.6 it follows that $(T(t))_{t\geqslant 0}$ is p-contractive. □

Of course, the norm function N given by $N(f) = \|f\|$ is a strict half-norm. In the case when $p = N$, Theorem 2.11 is due to **?**]. It turns out to be extremely useful in showing that a concrete operator is a generator. Because of its importance we state this special case explicitly below (including the complex case). Before that let us formulate Theorem 2.11 for the case when the operator is merely given on a core.

Corollary 2.12 *Let p be a strict half-norm and A be a densely defined operator. If A is p-dissipative and $(\lambda - A)$ has dense range for some $\lambda > 0$, then A is closable and the closure $\overline{A}$ of A generates a p-contraction semigroup.*

Proof It follows from Proposition 2.9 that A is closable and the closure $\overline{A}$ is p-dissipative. Lemma 2.10 implies that $(\lambda - \overline{A})D(\overline{A}) = E$. So Theorem 2.11 yields the desired conclusion. ◻

We conclude this section indicating the results for the complex case.

Let E be a complex Banach space and $p\colon E \to \mathbb{R}_+$ be a seminorm on E (i.e., $p(f+g) \leqslant p(f) + p(g)$ and $p(\lambda f) = |\lambda| p(f)$ holds for all $f, g \in E$, $\lambda \in \mathbb{C}$). The subdifferential $\mathrm{d}p(f)$ of p in $f \in E$ is defined by

$$\mathrm{d}p(f) = \{\varphi \in E'\colon \operatorname{Re}\langle g, \varphi\rangle \leqslant p(g) \text{ for all } g \in E \quad \text{and } \langle f, \varphi\rangle = p(f)\}. \quad (2.15)$$

We assume in addition that p is continuous. Then it follows from the Hahn-Banach theorem that $\mathrm{d}p(f) \neq \emptyset$ for any $f \in E$. A linear operator A on E is called *p-dissipative* if for all $f \in D(A)$ there exists $\varphi \in \mathrm{d}p(f)$ such that $\operatorname{Re}\langle Af, \varphi\rangle \leqslant 0$.

The arguments given above show that also in the situation considered here A is p-dissipative if and only if

$$p((1 - tA)f) \geqslant p(f)$$

for all $f \in D(A)$, $t \geqslant 0$.

The results of this section carry over if they are appropriately modified. We explicitly state the most important result for the case when p is the norm. A linear operator A is simply called *dissipative* if it is N-dissipative where $N(f) = \|f\|$ $(f \in E)$.

Theorem 2.13 (Lumer-Phillips) *Let A be a densely defined operator on a complex Banach space E. The following assertions are equivalent.*

(a) A is closable and the closure of A is the generator of a contraction semigroup.
(b) A is dissipative and $(\lambda - A)$ has dense range for some $\lambda > 0$.

3 Semigroups on L^∞ and H^∞

In this section we shall prove that on L^∞, on H^∞, and on some other classical Banach spaces every strongly continuous semigroup of operators is uniformly continuous.

Lemma 3.1 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a one-parameter semigroup of operators on a Banach space E. Suppose that $s = \limsup_{t\to 0} \|T(t) - \mathrm{Id}\|$ is finite. If $\lim_{t\to 0} \|(T(t) - \mathrm{Id})^2\| = 0$, then $\mathcal{T}$ is uniformly continuous.*

Proof The identity $2(T(t) - \mathrm{Id}) = T(2t) - \mathrm{Id} - (T(t) - \mathrm{Id})^2$ shows that $2\|T(t) - \mathrm{Id}\| - \|(T(t) - \mathrm{Id})^2\| \leqslant \|T(2t) - \mathrm{Id}\|$. Hence $2s \leqslant \limsup_{t\downarrow 0} \|T(2t) - \mathrm{Id}\|$. Obviously, $\limsup_{t\downarrow 0} \|T(2t) - \mathrm{Id}\| = s$ and so, $2s \leqslant s$. Consequently, $s = 0$. ◻

Remarks (i) If in Lemma 3.1 $\mathcal{T} = (T(t))_{t\geqslant 0}$ is strongly continuous, in which case $s < \infty$, one can replace $\lim_{t\to 0} \|(T(t) - \mathrm{Id})^2\| = 0$ by the weaker condition $\limsup r(T(t) - \mathrm{Id}) < 1$ see **?** , Lemma 2] where r denotes the spectral radius.

(ii) The condition $s < \infty$ in Lemma 3.1 is essential as the following example shows: Let $f\colon \mathbb{R} \to \mathbb{R}$ be non-continuous with $f(s+t) = f(s) + f(t)$ for all $s, t \in \mathbb{R}$ (see **?**]). Then $\{(t, f(t)) : t \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$. Hence for the semigroups $\mathcal{T} = (T(t))_{t \geqslant 0}$ on $\mathbb{R}^2$ with

$$T(t) = \begin{pmatrix} 1 & f(t) \\ 0 & 1 \end{pmatrix} \quad \text{for } t \geqslant 0$$

we have $s = \infty$. Therefore $\mathcal{T}$ is not uniformly continuous. However, $(T(t) - \mathrm{Id})^2 = 0$ for all $t \geqslant 0$.

Lemma 3.2 *Let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a one-parameter semigroup of operators on a Banach space E. Then the following assertions are equivalent:*

(a) $\mathcal{T}' = (T(t)')_{t \geqslant 0}$ is a strongly continuous semigroup on the dual E'
(b) $((T(t) - \mathrm{Id})x_n)$ converges weakly to zero for every bounded sequence (x_n) in E and every sequence (t_n) in $[0, \infty)$ with $\lim t_n = 0$

Moreover, (a) implies

(c) $\mathcal{T}$ is strongly continuous

Proof Let $x' \in E'$ and $t_n \geqslant 0$ be given. Then $\lim \|(T(t_n) - \mathrm{Id})'x'\| = 0$ if and only if $\lim\langle x_n, (T(t_n) - \mathrm{Id})'x'\rangle = 0$ for every bounded sequence (x_n) in E. This implies the equivalence of (a) and (b) . In particular, (a) implies that $((T(t_n) - \mathrm{Id})x)$ converges weakly to zero for every sequence (t_n) in $[0, \infty)$ with $\lim t_n = 0$ and every $x \in E$. Hence $\mathcal{T}$ is strongly continuous by Proposition 1.23 in **?**]. $\square$

We recall that a Banach space E is called a *Grothendieck space* if every weak* convergent sequence in E' converges weakly.

Theorem 3.3 *Let E be a Grothendieck space. If $\mathcal{T} = (T(t))_{t \geqslant 0}$ is a strongly continuous semigroup in E then $\mathcal{T}'' = (T(t)'')_{t \geqslant 0}$ is strongly continuous in E''.*

Proof Suppose that (x_n') is a bounded sequence in E' and that $t_n \geqslant 0$ with $\lim t_n = 0$. Put $V_n = T(t_n) - \mathrm{Id}$. Then $\lim \|V_n x\| = 0$ and therefore $\lim\langle x, V_n' x_n'\rangle = 0$ for every $x \in E$. Hence $(V_n' x_n')$ w*-converges to zero. Since E is a Grothendieck space $(V_n' x_n')$ converges weakly to zero. Now Lemma 3.2 implies that $(T(t)'')$ is strongly continuous. $\square$

Recall now that a Banach space E is said to have the *Dunford-Pettis property* if $\lim\langle x_n, x_n'\rangle = 0$ whenever (x_n) in E and (x_n') in E' converge weakly to zero.

Theorem 3.4 *Let E be a Banach space with the Dunford-Pettis property and let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a one-parameter semigroup of operators on E. If $\mathcal{T}'' = (T(t)'')_{t \geqslant 0}$ is strongly continuous in E'', then $\mathcal{T}$ is uniformly continuous.*

Proof Suppose that $\mathcal{T}''$ is a strongly continuous semigroup. Then Lemma 3.2 implies that $\mathcal{T}'$ and $\mathcal{T}$ are strongly continuous. Hence by the uniform boundedness principle, $\limsup_{t\to 0}\|(T(t)-Id)\|$ is finite. By Lemma 3.1 it suffices to show that $\lim_{t\to 0}\|(T(t)-Id)^2\| = 0$. Let $t_n \geqslant 0$ with $\lim t_n = 0$ be given. Then there exists a bounded sequence (x_n) in E and a bounded sequence (x'_n) in E' such that $\|(T(t_n)-Id)^2\| = \langle (T(t_n)-Id)x_n, (T(t_n)-Id)'x'_n\rangle$. Since $\mathcal{T}'$ and $\mathcal{T}''$ are strongly continuous, Lemma 3.2 implies that $((T(t_n)-Id)x_n)$ and $((T(t_n)-Id)'x'_n)$ converge weakly to zero. Since E has the Dunford-Pettis property, $\lim\|(T(t_n)-Id)^2\| = 0$. Consequently, $\lim_{t\to 0}\|(T(t)-Id)^2\| = 0$. □

An immediate consequence of Theorem 3.3 and Theorem 3.4 is the following.

Theorem 3.5 *Let E be a Grothendieck space with the Dunford-Pettis property. Then every strongly continuous semigroup of operators on E is uniformly continuous.*

A compact Hausdorff space is called an *F-space* if the closures of two disjoint open F_σ-sets are disjoint and is called a *Stonean* (res., σ-*Stonean*) space if the closure of every open set (res., open F_σ-set) is open. Every σ-Stonean space is an F-space.

Theorem 3.6 *Every strongly continuous semigroups of operators on one of the following Banach spaces is uniformly continuous:*

(i) $C(K)$, where K is a compact F-space.
(ii) $L^\infty(S,\Sigma,\mu)$ for any measure space (S,Σ,μ).
(iii) The Banach space $B(S,\Sigma)$ of all bounded Σ-measurable functions on S if Σ is a σ-algebra of subsets of S.
(iv) The Banach space $\mathcal{H}(O)$ of all bounded continuous solutions of

$$\sum_{1\leqslant i\leqslant n}(\partial^2 f/\partial x_i^2) = 0$$

on an open subset O of $\mathbb{R}^n$.
(v) The Banach space $\mathcal{W}(O)$ of all bounded continuous solutions of

$$\sum_{1\leqslant i\leqslant n}(\partial^2 f/\partial x_i^2) = (\partial f/\partial x_{n+1})$$

on an open subset O of $\mathbb{R}^{n+1}$.
(vi) The Banach space $H^\infty(O)$ of bounded analytic functions on a finitely connected domain O of the complex plane.

Proof By Theorem 3.5 it suffices to show that the space listed above are Grothendieck spaces with the Dunford-Pettis property.

(i) If K is compact, then $C(K)$ has the Dunford-Pettis property **?** , Théorème 4]. If K is a compact F-space, then $C(K)$ is a Grothendieck space [Seever (1958) Theorem 2.5]; the special cases for Stonean and σ-Stonean spaces are due to **?** , Théorème 9] and **?**] respectively.

(ii) and (iii) It is well known that every σ-order complete AM-space with unit is isometric to a space $C(K)$ where K is a compact σ-Stonean space. Obviously, the spaces under (ii) and (iii) are σ-order complete AM-spaces with unit and therefore by proof of (i) Grothendieck spaces with the Dunford-Pettis property.

(iv) and (v) These spaces are order complete vector lattices. This follows from **?** , pp.18-22, Standardbeispiele 1 and 2 p.55]. Since these spaces contain the constant functions on O they are complete for the supremum-norm. Indeed if (f_n) is a Cauchy-sequence for this norm, it is easily seen that (f_n) converges in norm to $\inf_n \sup(f_k : n < k)$. Therefore these spaces are σ-order complete AM-spaces with unit and so as before Grothendieck spaces with the Dunford-Pettis property.

(vi) **?** , Corollary 3] proves that $H^\infty(D)$ has the Dunford-Pettis property and in [Bourgain (1984), Proposition III.1], that $H^\infty(D)$ is a Grothendieck space, where D is the open unit disc $\{z : |z| < 1\}$. If O is a finitely connected domain and H^∞ does not only contain the constant functions, then $H^\infty(O)$ is isomorphic to a finite direct sum of copies of $H^\infty(D)$. (Note that $H^\infty(D)$ is isomorphic to $\{f \in H^\infty(D) : f(0) = 0\}$ via the map $f \mapsto zf$. Then use **?** , p.77 and Proposition 4.4.1]). Hence $H^\infty(O)$ is a Grothendieck space with the Dunford-Pettis property. □

Remark (Final) It follows from Theorem 3.6 that on L^∞ the infinitesimal generator of a strongly continuous semigroup is necessarily bounded. It is not obvious that on $L^\infty([0,1])$ there exist closed densely defined unbounded operators. To see this let A be a closed densely defined unbounded operator form ℓ^2 into $L^\infty([0,1])$ with domain D (such operators can easily be constructed). By the Khintchine inequality, the map $R : (a_n) \mapsto \sum a_n r_n$ where r_n denotes the n^{th} Rademacher function, from ℓ^2 into $L^1([0,1])$ is a topological isomorphism. Hence $T = R'$ maps $L^\infty([0,1])$ onto ℓ^2. Banach's homomorphism theorem implies that $T^{-1}(D)$ is dense in $L^\infty([0,1])$ and that AT is a closed densely defined unbounded operator on $L^\infty([0,1])$ with domain $T^{-1}(D)$. This solves a problem raised by R.Kaufman.

H. Porta and the author have shown that if a Banach space E has an infinite dimensional separable quotient space and F is an infinite dimensional Banach space then there always exists a closed densely defined unbounded operator from E into F. □

Notes

Section 1: The abstract Cauchy problem is treated systematically in the monographs of **?**] and **?**]. We refer to these books for more details and historical notes. One implication of Theorem 1.1 is proved in **?** , Theorem 2.11].

The Hille-Yosida Theorem has been proved independently by **?**] and **?**] for contraction semigroups. The extension to arbitrary strongly continuous semigroups is independently due to **?**], **?**] and **?**]. Thus our terminology is slightly incorrect, some authors refer to the general version as the Hille-Yosida-Phillips theorem which is slightly more correct.

Holomorphic semigroups belong to the standard material of the theory of one-parameter semigroups. Our Theorem 1.14 deviates from the usual presentation since the condition on the resolvent is merely required on a half-plane.

Differentiable semigroups are treated in detail in the book of **?**] who discovered Theorem 1.17 and 1.18 . The spectral property of eventually norm continuous semigroups given in Theorem 1.20 is contained in **?** , Theorem 16.4.2] with a proof depending on Gelfand theory. For norm continuous semigroups it is contained in **?**] with a simpler proof. The elementary proof we give here is due to G. Greiner.

Theorem 1.29 on the perturbation by bounded operators is due to **?**] who also investigated permanence of smoothness properties by this kind of perturbation. We also refer to **?** , Section 3.1].

The observation that eventually norm continuity is preserved by perturbation by a compact operator (see Theorem 1.30) seems to be new.

The perturbation by continuous operators on the graph of the generator is due to **?**]. The short proof we give here is due to G. Greiner and has the advantage to yield the same permanence for smoothness properties as in the classical case, see Corollary 1.32 .

The characterization of a core as "domain of uniqueness" given in Theorem 1.33 seems to be new.

In this section we have presented part of the standard theory of one-parameter semigroups including some new aspects. A very elegant brief introduction to one-parameter semigroups is given in the treatise of **?**] where one can also find all the results on perturbation theory going beyond the elementary facts we discuss here. A complete information on the general theory can be obtained by consulting the books of **?**], **?**] and **?**]. The monograph of **?**] contains a variety of examples and applications.

Section 2: Dissipative operators were introduced by **?**]. The analogous notion of dispersiveness is due to **?**]. Our approach follows closely **?**] where half-norms were introduced. Related previous results were obtained by **?**], **?**], **?**], **?**] and **?**], where the two last consider non-linear semigroups. A further investigation of half-norms can be found in **?**] who consider ordered Banach spaces other than Banach lattices in great detail. We also refer to the historical notes given there.

Section 3: It had been proved by **?**] that every generator of a positive semigroup on L^∞ is bounded. That every strongly continuous semigroup on L^∞ is uniformly continuous was first shown by **?**], **?**] and **?**]. The proof of Lemma 3.1 was communicated to the author by T. Coulhon, who independently obtained a particular case (**?**]) of Lotz's result..

Chapter A-III
Spectral Theory

1 Introduction

In this chapter, we start a systematic analysis of the spectrum of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ on a complex Banach space E. By the spectrum of the semigroup we understand the spectrum $\sigma(A)$ of the generator A of $\mathcal{T}$. In particular, we are interested in the precise relations between $\sigma(A)$ and $\sigma(T(t))$. The heuristic formula

$$T(t) = \mathrm{e}^{tA}$$

serves as a leitmotiv and suggests relations of the form

$$\sigma(T(t)) = \mathrm{e}^{t\sigma(A)} = \{\mathrm{e}^{t\lambda} \colon \lambda \in \sigma(A)\},$$

called *spectral mapping theorem*. These—or similar—relations will be of great use in Chapter IV and enable us to determine the asymptotic behavior of the semigroup $\mathcal{T}$ by the spectrum of its generator.

As motivation and also as a preliminary step, we concentrate here on the *spectral radius*

$$r(T(t)) := \sup\{|\lambda| : \lambda \in \sigma(T(t))\}, \quad t \geqslant 0, \tag{1.1}$$

and show how it is related to the *spectral bound*

$$s(A) := \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} \tag{1.2}$$

of the generator A and to the *growth bound*

$$\omega_0 := \inf\{\omega \in R : \|T(t)\| \leqslant M_\omega \cdot \mathrm{e}^{\omega t} \text{ for all } t \geqslant 0 \text{ and suitable } M_\omega\} \tag{1.3}$$

of the semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$. (Recall that sometimes we write $\omega_0(\mathcal{T})$ or $\omega_0(A)$ instead of ω_0). The Examples 1.3 and 1.4 below illustrate the main difficulties to be encountered.

Proposition 1.1 *Let ω_0 be the growth bound of the strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$. Then*

$$r(T(t)) = \mathrm{e}^{\omega_0 t} \tag{1.4}$$

for every $t \geqslant 0$.

Proof From A-I, (1.1) we know that

$$\omega_0(\mathcal{T}) = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|$$

Since the spectral radius of $T(t)$ is given as

$$r(T(t)) = \lim_{n\to\infty} \|T(nt)\|^{1/n} ,$$

we obtain for $t > 0$

$$r(T(t)) = \lim_{n\to\infty} \exp\left(\frac{t}{nt} \log \|T(nt)\|\right) = \mathrm{e}^{\omega_0 t} .$$

It was shown in A-I, Proposition 1.11 that the spectral bound $s(A)$ is always dominated by the growth bound ω_0 and therefore $\mathrm{e}^{s(A)t} \leqslant r(T(t))$. If the above mentioned spectral mapping theorem holds — as is the case for bounded generators (e.g., see Theorem VII.3.11 of **?**]) — we obtain the equality

$$\mathrm{e}^{s(A)t} = r(T(t)) = \mathrm{e}^{\omega_0(\mathcal{T})t} ,$$

hence $s(A) = \omega_0(\mathcal{T})$. Therefore, the following corollary is a consequence of the definitions of $s(A)$ and $\omega_0(\mathcal{T})$.

Corollary 1.2 *Consider the semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ generated by some bounded linear operator $A \in LE$. If* $\operatorname{Re}\lambda < 0$ *for each $\lambda \in \sigma(A)$, then* $\lim_{t\to\infty} \|T(t)\| = 0$.

Through this corollary we have re-established a famous result of Liapunov which assures that the solutions of the linear Cauchy problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in \mathbb{C}^n \quad \text{and} \quad A = (a_{ij})_{n\times n}$$

are *stable*, i.e., they converge to zero as $t \to \infty$ if the real parts of all eigenvalues of the matrix A are smaller than zero.

For unbounded generators the situation is much more difficult and $s(A)$ may differ drastically from $\omega_0(\mathcal{T})$.

Example 1.3 (Banach function space, **?**]) Consider the Banach space E of all complex valued continuous functions on $\mathbb{R}_+$ which vanish at infinity and are integrable for $\mathrm{e}^x\,\mathrm{d}x$, i.e.,

$$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathrm{e}^x\,\mathrm{d}x)$$

endowed with the norm

$$\|f\| := \|f\|_\infty + \|f\|_1 = \sup\{|f(x)| : x \in \mathbb{R}_+\} + \int_0^\infty |f(x)|\mathrm{e}^x\,\mathrm{d}x .$$

The translation semigroup

$$T(t)f(x) := f(x+t)$$

is strongly continuous on E and one shows as in A-I,2.4 that its generator is given by

$$Af = f', \quad D(A) = \{f \in E\colon f \in C^1(\mathbb{R}_+), f' \in E\}\,.$$

First we observe that $\|T(t)\| = 1$ for every $t \geqslant 0$, hence $\omega_0(\mathcal{T}) = 0$. Moreover it is clear that λ is an eigenvalue of A as soon as $\operatorname{Re}\lambda < -1$ (in fact: the function

$$x \mapsto e_\lambda(x) := \mathrm{e}^{\lambda x}$$

belongs to $D(A)$ and is an eigenvector of A), hence $s(A) \geqslant -1$. For $f \in E$, $\operatorname{Re}\lambda > -1$,

$$\|\cdot\|_1\text{-}\lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s} T(s)f \;\mathrm{d}s$$

exists since $\|T(s)f\|_1 \leqslant \mathrm{e}^{-s}\|f\|_1$, $s \geqslant 0$, and

$$\|\cdot\|_\infty\text{-}\lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s} T(s)f \;\mathrm{d}s$$

exists since $\int_0^\infty \mathrm{e}^x |f(x)| \;\mathrm{d}x < \infty$. Therefore $\int_0^\infty \mathrm{e}^{-\lambda s} T(s)f \;\mathrm{d}s$ exists in E for every $f \in E$, $\operatorname{Re}\lambda > -1$. As we observed in A-I, Proposition 1.11, this implies $\lambda \in \varrho(A)$. Therefore $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a semigroup having $s(A) = -1$, but $\omega_0(\mathcal{T}) = 0$.

Example 1.4 (Hilbert space, **?**]) For every $n \in \mathbb{N}$ consider the n-dimensional Hilbert space $E_n := \mathbb{C}^n$ and operators $A_n \in \mathcal{L}(E_n)$ defined by the matrices

$$A_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdot & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & 0 \end{pmatrix}_{n\times n} .$$

These matrices are nilpotent and therefore $\sigma(A_n) = \{0\}$. The elements

$$x_n := n^{-1/2}(1, \ldots, 1) \in E_n$$

satisfy the following properties.

(i) $\|x_n\| = 1$ for every $n \in \mathbb{N}$,
(ii) $\lim_{n\to\infty} \|A_n x_n - x_n\| = 0$,
(iii) $\lim_{n\to\infty} \|\exp(tA_n)x_n - \mathrm{e}^t x_n\| = 0$.

Consider now the Hilbert space $E := \bigoplus_{n\in\mathbb{N}} E_n$ and the operator $A := (A_n + 2\pi i n)_{n\in\mathbb{N}}$ with maximal domain in E. Analogously we define a semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ by

$$T(t) := (\mathrm{e}^{2\pi i n t} \exp(tA_n))_{n\in\mathbb{N}}\,.$$

Since $\|\exp(tA_n)\| \leqslant \mathrm{e}^t$ for every $n \in \mathbb{N}$, $t \geqslant 0$, and since $t \mapsto T(t)x$ is continuous on each component E_n, it follows that $\mathcal{T}$ is strongly continuous. Its generator is the operator A as defined above.

For $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$, we have $\lim_{n\to\infty} \|R(\lambda - 2\pi\mathrm{i}n, A_n)\| = 0$, hence

$$(R(\lambda, A_n + 2\pi\mathrm{i}n))_{n\in\mathbb{N}} = (R(\lambda - 2\pi\mathrm{i}n, A_n))_{n\in\mathbb{N}}$$

is a bounded operator on E representing the resolvent $R(\lambda, A)$. Therefore we obtain $s(A) \leqslant 0$. On the other hand, each $2\pi\mathrm{i}n$ is an eigenvalue of A, hence $s(A) = 0$.

Take now $x_n \in E_n$ as above and consider the sequence $(x_n)_{n\in\mathbb{N}}$. From (iii) it follows that for $t > 0$ the number e^t is an approximate eigenvalue of $T(t)$ with approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ (see Definition 2.1 below). Therefore $\mathrm{e}^t \leqslant r(T(t)) \leqslant \|T(t)\|$ and hence $\omega_0(\mathcal{T}) \geqslant 1$. On the other hand, it is easy to see that $\|T(t)\| = \mathrm{e}^t$, hence $\omega_0(\mathcal{T}) = 1$.

Finally, if we take $S(t) := \mathrm{e}^{-t/2}T(t)$, we obtain a semigroup $\mathcal{S}$ having spectral bound $-\frac{1}{2}$ but satisfying $\lim_{t\to\infty} \|S(t)\| = \infty$ in contrast with Corollary 1.2.

These examples show that neither the conclusion of Corollary 1.2, i.e., "$s(A) < 0$ implies stability", nor the "spectral mapping theorem"

$$\sigma(T(t)) = \exp(t \cdot \sigma(A))$$

is valid for arbitrary strongly continuous semigroups. A careful analysis of the general situation will be given in Section 6 below, but we will first develop systematically the necessary spectral theoretic tools for unbounded operators.

2 The Fine Structure of the Spectrum

As usual, with a closed linear operator A with dense domain $D(A)$ in a Banach space E, we associate its spectrum $\sigma(A)$, its resolvent set $\varrho(A)$ and its resolvent

$$\lambda \mapsto R(\lambda, A) := (\lambda - A)^{-1}$$

which is a holomorphic map from $\varrho(A)$ into $\mathcal{L}(E)$. In contrast to the finite dimensional situation, where a linear operator fails to be surjective if and only if it fails to be injective, we now have to distinguish different cases of *non-invertibility* of $\lambda - A$. This distinction gives rise to a subdivision of $\sigma(A)$ into different subsets. We point out that these subsets need not be disjoint. Our definitions are justified by the fact that for each of the following subsets of $\sigma(A)$ there exist canonical constructions converting the corresponding spectral values into eigenvalues (see Proposition 2.2.(ii) and Proposition 4.4 below).

Definition 2.1 For a closed, densely defined, linear operator A with domain $D(A)$ in the Banach space E denote by the

(i) *point spectrum* $P\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not injective.

(ii) *approximate point spectrum* $A\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not injective or $(A - \lambda)D(A)$ is not closed in E.

(iii) *residual spectrum* $R\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $(A - \lambda)D(A)$ is not dense in E.

From these definitions it follows that $\lambda \in P\sigma(A)$ if and only if there exists a non-zero *eigenvector* $f \in D(A)$ such that $Af = \lambda f$, i.e., λ is an *eigenvalue*. It follows from the *Open Mapping Theorem* that $\lambda \in A\sigma(A)$ if and only if λ is an *approximate eigenvalue*, i.e., there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset D(A)$, called an *approximate eigenvector*, such that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|Af_n - \lambda f_n\| = 0$.

Clearly we have $P\sigma(A) \subset A\sigma(A)$ and $\sigma(A) = A\sigma(A) \cup R\sigma(A)$ where the union need not be disjoint.

The following proposition is a first indication that the subdivision we made yields nice properties.

Proposition 2.2 *For a closed, densely defined, linear operator* $(A, D(A))$ *in a Banach space* E *the following holds.*

(i) The topological boundary $\partial\sigma(A)$ *of* $\sigma(A)$ *is contained in* $A\sigma(A)$.

(ii) $R\sigma(A) = P\sigma(A')$ *for the adjoint operator* A' *on* E'.

Proof (i) Take $\lambda_0 \in \partial\sigma(A)$ and $\lambda_n \in \varrho(A)$ such that $\lambda_n \to \lambda_0$.

Since $\|R(\lambda_n, A)\| \geqslant r(R(\lambda_n, A)) = (\mathrm{dist}(x, \sigma(A)))^{-1}$ (see Proposition 2.5.(ii)), by the uniform boundedness principle we find $f \in E$ such that

$$\lim_{n\to\infty} \|R(\lambda_n, A)f\| = \infty\,.$$

Define $g_n \in D(A)$ by

$$g_n := \|R(\lambda_n, A)f\|^{-1} R(\lambda_n, A)f$$

and use the identity

$$(\lambda_0 - A)g_n = (\lambda_0 - \lambda_n)g_n + (\lambda_n - A)g_n$$

to show that $(g_n)_{n\in\mathbb{N}}$ is an approximate eigenvector corresponding to λ_0.

(ii) This is a simple consequence of the Hahn-Banach theorem. □

In order to illuminate the above definitons we now return to the Standard Examples introduced in Section 2 of A-I and discuss the fine structure of the spectrum of these strongly continuous semigroups, i.e., of their generators and their semigroup operators.

Example 2.3 (The Spectrum of Multiplication Semigroups)

Take $E = C_0(X)$ for some locally compact space X and take a continuous function $q : X \mapsto \mathbb{C}$ whose real part is bounded above. As observed in A-I,2.3 the multiplication operator

$$M_q : f \mapsto q \cdot f$$

with maximal domain $D(M_q)$ generates the multiplication semigroup

$$T(t)f := \mathrm{e}^{tq} \cdot f\,,\ f \in E\,.$$

Since M_q is bounded if and only if q is bounded, we conclude that M_q is invertible (with bounded inverse $M_{1/q}$) if and only if

$$0 \notin \overline{\{q(x)\colon x \in X\}}\,.$$

Therefore we obtain

$$\sigma(M_q) = \overline{q(X)} = \overline{\{q(x)\colon x \in X\}}\,,$$

and

$$\sigma(T(t)) = \overline{\{\exp(tq(x))\colon x \in X\}}\,.$$

In particular the following *weak spectral mapping theorem* is valid

$$\sigma(T(t)) = \overline{\exp(t\sigma(M_q))}\,.$$

In addition, we observe that to each spectral value of A (resp. of $T(t)$) there exists an approximate eigenvector and hence

$$\sigma(A) = A\sigma(A) \text{ and } \sigma(T(t)) = A\sigma(T(t))\,.$$

Since each Dirac functional is an eigenvector for the adjoint multiplication operator, we obtain

$$q(X) \subset R\sigma(M_q) \text{ and } \mathrm{e}^{tq(X)} \subset R\sigma(T(t))\,.$$

The eigenvalues of M_q can be characterized as follows.

> $\lambda \in P\sigma(M_q)$ if and only if the set $\{x \in X\colon q(x) = \lambda\}$ has non empty interior (analogously for $P\sigma(T(t))$).

For example, it follows that $P\sigma(M_q) = \emptyset$ for $E = C_0(\mathbb{R}_+)$ and $q(x) = -x$, $x \in \mathbb{R}_+$.

On $E = L^p(X, \Sigma, \mu)$ analogous results are valid, but their exact formulation—using the notion *essential range*, see **?**]—is left to the reader.

Example 2.4 (The Spectrum of Translation Semigroups)

We consider the translation semigroup

$$T(t)f(x) := f(x+t)$$

on $E = C_0(\mathbb{R}_+)$ (or $L^p(\mathbb{R}_+)$, see A-I,2.4). Its generator A is the first derivative and for every $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda < 0$, the function $\varepsilon_\lambda\colon x \mapsto \mathrm{e}^{\lambda x}$ belongs to $D(A)$ and satisfies

$$A\varepsilon_\lambda = \lambda\varepsilon_\lambda\,,$$

hence $\lambda \in P\sigma(A)$. Since $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a contraction semigroup it follows that $\sigma(A) = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant 0\}$ and $\mathrm{i}\mathbb{R} \subset A\sigma(A)$ (use Proposition 2.2.(i)) or show directly that $f_n(x) = \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{-x/n}$ defines an approximate eigenvector for $\mathrm{i}\alpha$, $\alpha \in \mathbb{R}$). Using the same functions one obtains

$$\begin{aligned} P\sigma(T(t)) &= \{\mathrm{e}^{\lambda t}\colon \operatorname{Re}\lambda < 0\} = \{z \in \mathbb{C}\colon |z| < 1\}\,, \\ \sigma(T(t)) &= \{z \in \mathbb{C}\colon |z| \leqslant 1\} \text{ for every } t > 0\,. \end{aligned}$$

In the case of the translation group on $E = C_0(\mathbb{R})$ one has $\sigma(A) \subset \mathrm{i}\mathbb{R}$. As above one obtains approximate eigenvectors for every $\alpha \in \mathbb{R}$ from $f_n(x) = \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{-|x|/n}$, hence

$$\sigma(A) = A\sigma(A) = \mathrm{i}\mathbb{R}\,.$$

The generator A of the nilpotent translation semigroup A-I,2.6 has empty spectrum by A-I, Proposition 1.11. The resolvent is given by

$$R(\lambda, A)f(x) = \mathrm{e}^{\lambda x}\int_x^\infty \mathrm{e}^{-\lambda s} f(s)\,\mathrm{d}s \quad (f \in L^p([0,\tau]), \lambda \in \mathbb{C})\,.$$

Finally, the generator of the periodic translation group from A-I,2.5 on $E = \{f \in C[0,1]\colon f(0) = f(1)\}$ has point spectrum

$$P\sigma(A) = 2\pi\mathrm{i}\mathbb{Z}$$

with eigenfunctions $\varepsilon_n(x) := \exp(2\pi\mathrm{i}nx)$. In Section 5 we show that $\sigma(A) = 2\pi\mathrm{i}\mathbb{Z}$.

We now return to the general theory and recall from Corollary 1.2 that it is very useful (e.g., for stability theory) to be able to convert spectral values of the generator A into spectral values of the semigroup operator $T(t)$ and vice versa. As shown in Examples 1.3 and 1.4 this is not possible in general. Therefore we tackle first a much easier *spectral mapping theorem*: the relation between $\sigma(A)$ and $\sigma(R(\lambda_0))$, where $R(\lambda_0) := R(\lambda_0, A)$ for some $\lambda_0 \in \varrho(A)$.

Proposition 2.5 *Let $(A, D(A))$ be a densely defined closed linear operator with non-empty resolvent set $\varrho(A)$. For each $\lambda_0 \in \varrho(A)$ the following assertions hold.*

(i) $\sigma(R(\lambda_0))\setminus\{0\} = (\lambda_0 - \sigma(A))^{-1}$, in particular, $r(R(\lambda_0)) = (\operatorname{dist}(\lambda_0, \sigma(A)))^{-1}$.
(ii) Analogous statements hold for the point-, approximate point-, residual spectra of A and $R(\lambda_0, A)$.
(iii) The point α is isolated in $\sigma(A)$ if and only if $(\lambda_0 - \alpha)^{-1}$ is isolated in $\sigma(R(\lambda_0))$. In that case the residues (resp. the pole orders) in α and in $(\lambda_0 - \alpha)^{-1}$ coincide.

Proof (i) is well known. It can be found for example in **?** , VII.9.2].

(ii) We show that $\alpha \in A\sigma(A)$ if $(\lambda_0 - \alpha)^{-1} \in A\sigma(R(\lambda_0))$ and leave the proof of the remaining statements to the reader.

Take $(f_n)_{n\in\mathbb{N}} \subset E$ such that $\|f_n\| = 1$, $\|(\lambda_0 - \alpha)^{-1}f_n - R(\lambda_0, A)f_n\| \to 0$ and $\|R(\lambda_0, A)f_n\| \geqslant \frac{1}{2}|\lambda_0 - \alpha|^{-1}$. Define

$$g_n := \|R(\lambda_0, A)f_n\|^{-1}R(\lambda_0, A)f_n \in D(A)$$

and deduce from

$$\begin{aligned}(\alpha - A)g_n &= \|R(\lambda_0, A)f_n\|^{-1} \cdot [(\lambda_0 - A) - (\lambda_0 - \alpha)]R(\lambda_0, A)f_n \\ &= \|R(\lambda_0, A)f_n\|^{-1} \cdot (\lambda_0 - \alpha)[(\lambda_0 - \alpha)^{-1} - R(\lambda_0, A)]f_n\end{aligned}$$

that (g_n) is an approximate eigenvector of A to the eigenvalue α.

(iii) First we recall the wellknown *resolvent equation*. For any $z, \lambda_0 \in \varrho(A)$ we have $R(\lambda_0, A) - R(z, A)) = -(\lambda_0 - z)R(\lambda_0, A)R(z, A))$. From this it follows that $(\lambda_0 - z)^2 \cdot R(z, A) = \big((\lambda_0 - z)^{-1} - R(\lambda_0, A)\big)^{-1} - (\lambda_0 - z)$. If we now take a circle Γ with center α and sufficiently small radius. Then the residue P of $R(\cdot, A)$ at α is

$$\begin{aligned}P &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma R(z, A)\,\mathrm{d}z = \\ &= \frac{1}{2\pi \mathrm{i}} \left[\int_\Gamma (\lambda_0 - z)^{-2} R((\lambda_0 - z)^{-1}\, R(\lambda_0, A))\;\mathrm{d}z - \int_\Gamma (\lambda_0 - z)^{-1}\,\mathrm{d}z\right].\end{aligned}$$

If λ_0 lies in the exterior of Γ, the second integral is zero. The substitution $\tilde{z} := (\lambda_0 - z)^{-1}$ yields a path $\tilde{\Gamma}$ around $(\lambda_0 - \alpha)^{-1}$ and we obtain

$$P = \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\Gamma}} R(\tilde{z}, R(\lambda_0, A))\,\mathrm{d}\tilde{z}$$

which is the residue of $R(\cdot, R(\lambda_0, A))$ at $(\lambda_0 - \alpha)^{-1}$. The final assertion on the pole order follows from the identities

$$V_{-n} = ((\lambda_0 - \alpha)^{-1} R(\lambda_0, A))^{n-1} U_{-n}, \quad n \in \mathbb{N},$$

where U_n, resp. V_n stand for the n-th coefficient in the Laurent series of $R(\cdot, A)$, resp. $R(\cdot, R(\lambda_0, A))$ at α, resp. $(\lambda_0 - \alpha)^{-1}$. This has already been proved for $n = 1$ and follows for $n > 1$ by induction using the relations

$$U_{-n-1} = (A - \alpha)U_{-n} \quad \text{and} \quad V_{-n-1} = \Big(R(\lambda_0, A) - (\lambda_0 - \alpha)^{-1}\Big) V_{-n}\,.$$

3 Spectral Decomposition

In the next two sections we develop some important techniques for our further investigation of semigroups and their generators. Even though these methods are well known (compare, e.g. Section VII.3 of **?**]) or rather technical, it is useful to present them in a coherent way.

Our interest in this section is the following: Let E be a Banach space and $\mathcal{T} = (T(t))_{t \geqslant 0}$ a strongly continuous semigroup with generator A. Suppose that the spectrum $\sigma(A)$ splits into the disjoint union of two closed subsets σ_1 and σ_2. Does there exist a corresponding decomposition of the space E and the semigroup $\mathcal{T}$?

In the following definition, we explain what we understand by "corresponding decomposition".

Definition 3.1 Assume that $\sigma(A)$ is the disjoint union

$$\sigma(A) = \sigma_1 \cup \sigma_2$$

of two non-empty closed subsets σ_1, σ_2. A decomposition

$$E = E_1 \oplus E_2$$

of E into the direct sum of two non-trivial closed $\mathcal{T}$-invariant subspaces is called a *spectral decomposition* corresponding to $\sigma_1 \cup \sigma_2$ if the spectrum $\sigma(A_i)$ of the generator A_i of $\mathcal{T}_i := (T(t)_{|E_i})_{t\geqslant 0}$ coincides with σ_i for $i = 1, 2$.

For a better understanding of the above definition we recall that to every direct sum decomposition $E = E_1 \oplus E_2$ there corresponds a continuous projection $P \in \mathcal{L}(E)$ such that $PE = E_1$ and $P^{-1}(0) = E_2$. Moreover, the subspaces E_1, E_2 are $\mathcal{T}$-invariant if and only if P commutes with the semigroup $\mathcal{T}$, i.e., $T(t)P = PT(t)$ for every $t \geqslant 0$. In this case it follows that the domain $D(A)$ of the generator A splits analogously and $D(A) \cap E_i$ is the domain $D(A_i)$ of the generator A_i of the restricted semigroup $\mathcal{T}_i$, $i = 1, 2$. We write

$$A = A_1 \oplus A_2 .$$

and say that A *commutes with* P and call P a *spectral projection*. In terms of the generator A this means that for $f \in D(A)$ we have $Pf \in D(A)$ and $APf = PAf$.

The existence of such projections reduces the semigroup $\mathcal{T}$ into two (possibly simpler) semigroups $\mathcal{T}_1$, $\mathcal{T}_2$ such that

$$\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \quad \text{and} \quad \sigma(T(t)) = \sigma(T_1(t)) \cup \sigma(T_2(t)) .$$

For example, in some cases (see Theorem 3.3 below) it can be shown that one of the reduced semigroups has additional properties.

In order to achieve such decompositions we will assume that $\sigma(A)$ decomposes into sets σ_1 and σ_2 and will then try to find a corresponding spectral projection. Unfortunately such spectral decompositions do not exist in general.

Example 3.2 Take the rotation semigroup from A-I,2.4 on the Banach space $L^p(\Gamma)$, $1 \leqslant p < \infty$, $\tau = 2\pi$. It was stated in Example 2.4 and will be proved in Section 5 that its generator A has spectrum

$$\sigma(A) = P\sigma(A) = \mathrm{i}\mathbb{Z}$$

where $\varepsilon_k(z) := z^k$ spans the eigenspace corresponding to $\mathrm{i}k$, $k \in \mathbb{Z}$.

Now, $\sigma(A)$ is the disjoint union of $\sigma_1 := \{0, \mathrm{i}, 2\mathrm{i}, \ldots\}$ and $\sigma_2 := \{-\mathrm{i}, -2\mathrm{i}, \ldots\}$. By a result of M. Riesz there is no projection $P \in \mathrm{L}L^1(\Gamma)$ satisfying $P\varepsilon_k = \varepsilon_k$ for $k \geqslant 0$, $P\varepsilon_k = 0$ for $k < 0$, hence there is no spectral decomposition of $L^1(\Gamma)$ corresponding to σ_1, σ_2 (**?** , p.165]).

On the other hand, for $L^p(\Gamma)$, $1 < p < \infty$, such a spectral projection exists (l.c., 2.c.15). As long as $p \neq 2$ we can always decompose $\sigma(A)$ into suitable subsets admitting no spectral decomposition (l.c., remark before 2.c.15). Clearly, for $p = 2$ such spectral decompositions always exist.

In the above example both subsets σ_1, σ_2 of $\sigma(A)$ are unbounded. But as soon as one of these sets is bounded a corresponding spectral decomposition can always be found.

Theorem 3.3 *Let $\mathcal{T}$ be a strongly continuous semigroup on a Banach space E and assume that the spectrum $\sigma(A)$ of the generator A can be decomposed into the disjoint union of two non-empty closed subsets σ_1, σ_2.*

If σ_1 is compact, then there exists a unique corresponding spectral decomposition $E = E_1 \oplus E_2$ such that the restricted semigroup $\mathcal{T}_1$ has a bounded generator.

Proof We assume the reader to be familiar with the spectral decomposition theorem for bounded operators (see, e.g., **?** , p.572]) and apply the spectral mapping theorem for the resolvent (Proposition 2.5.(i)) in order to decompose $R(\lambda, A)$ instead of A.

For $\lambda_0 > \omega_0(\mathcal{T})$ it follows from Proposition 2.5 that $\sigma(R(\lambda_0, A)) \setminus \{0\} = (\lambda_0 - \sigma(A))^{-1}$. From $\sigma(A) = \sigma_1 \cup \sigma_2$ we obtain a decomposition of $\sigma(R(\lambda_0, A)) \setminus \{0\}$ into

$$\tau_1 := (\lambda_0 - \sigma_1)^{-1}, \quad \tau_2 := (\lambda_0 - \sigma_2)^{-1}.$$

Since σ_1 is compact, the set τ_1 is compact and does not contain 0. Only in the case that σ_2 is unbounded the point 0 will be an accumulation point of τ_2. Therefore $\sigma(R(\lambda_0, A)) \cup \{0\}$ is the disjoint union of the closed sets τ_1 and $\tau_2 \cup \{0\}$.

Take now P to be the spectral projection of $R(\lambda_0, A)$ corresponding to this decomposition. Then P commutes with $R(\lambda_0, A)$ (by definition), with $R(\lambda, A)$ for every $\lambda > \omega_0(\mathcal{T})$ (use the series representation of the resolvent), with $T(t)$ for each $t \geqslant 0$ (use A-II, Proposition 1.10) and therefore with the generator A (in the sense explained above). In particular, we obtain

$$R(\lambda_0, A)P = R(\lambda_0, A_1), \quad R(\lambda_0, A)(Id - P) = R(\lambda_0, A_2)$$

for the generator A_1 of $T_1 = (T(t)P)_{t \geqslant 0}$ and A_2 of $T_2 = (T(t)(Id - P))_{t \geqslant 0}$. Applying the Spectral Mapping Theorem 2.5 we conclude

$$\sigma(A_1) = \sigma_1 \text{ and } \sigma(A_2) = \sigma_2\,,$$

i.e., P is a spectral projection corresponding to σ_1, σ_2. Finally, the above spectral decomposition of $R(\lambda_0, A)$ is unique and satisfies $0 \notin \sigma(R(\lambda_0, A_1))$. Therefore $R(\lambda_0, A_1)^{-1} = (\lambda_0 - A_1)$ is bounded. □

If we do not require $\mathcal{T}_1$ to be uniformly continuous, the above spectral decomposition need not be unique, as can be seen from the following example.

Consider a decomposition $E = E_1 \oplus E_2$ and add a direct summand E_3 with a strongly continuous semigroup T_3 whose generator A_3 has empty spectrum (e.g., A-I,Example 2.6). Then still $\sigma(A) = \sigma_1 \cup \sigma_2$, but $E_1 \oplus (E_2 \oplus E_3)$ and $(E_1 \oplus E_3) \oplus E_2$ are two different spectral decompositions corresponding to σ_1, σ_2.

The importance of the above theorem stems from the fact that $\mathcal{T}_1$ has a bounded generator and therefore is easy to deal with. In particular the asymptotic behavior of $\mathcal{T}_1$ can be deduced from the location of σ_1.

Corollary 3.4 *Assume that $\sigma(A)$ splits into non-empty closed sets σ_1, σ_2 where σ_1 is compact and consider the corresponding spectral decomposition $E = E_1 \oplus E_2$ for which $\mathcal{T}_1$ is uniformly continuous.*

For all constants ν, $\omega \in \mathbb{R}$ satisfying

$$\nu < \inf\{\operatorname{Re}\lambda \colon \lambda \in \sigma_1\} \leqslant \sup\{\operatorname{Re}\lambda \colon \lambda \in \sigma_1\} < \omega$$

there exist $m \leqslant 1$, $M \geqslant 1$ such that

$$m \cdot \mathrm{e}^{\nu t}\|f\| \leqslant \|T_1(t)f\| \leqslant M \cdot \mathrm{e}^{\omega t}\|f\|$$

for every $f \in E_1$, $t \geqslant 0$.

Proof Since the generator A_1 of $\mathcal{T}_1$ is bounded, we have $T_1(t) = \exp(tA_1)$ and $\sigma(T_1(t)) = \exp(t\sigma(A_1))$. Therefore by the remark following Proposition 1.1, the spectral bound $s(A_1)$ coincides with the growth bound $\omega_0(T_1)$ and we have the upper estimate. The lower estimate is obtained by applying the same reasoning to $-A_1$ which generates the semigroup $(T_1(t)^{-1})_{t\geqslant 0}$ on E_1. □

It is clear from Examples 1.3 and 1.4 on page 59 that no norm estimates for $(T_2(t))_{t\geqslant 0}$ can be obtained from the location of σ_2. Only by adding appropriate hypotheses we will achieve spectral decompositions admitting norm estimates on both components (see Theorem 6.6 below).

Another way of obtaining such norm estimates is by constructing spectral decompositions starting from a semigroup operator $T(t_0)$ (instead of A, and $R(\lambda, A)$ resp., as in Theorem 3.3).

Corollary 3.5 *If $\sigma(T(t_0)) = \tau_1 \cup \tau_2$ for two non-empty, closed, disjoint sets τ_1, τ_2 and if P is the spectral projection corresponding to $T(t_0)$ and τ_1, τ_2, then $\sigma(A)$ splits into closed subsets σ_1, σ_2 and P is the corresponding spectral projection for $\mathcal{T}$ and σ_1, σ_2.*

Proof The spectral projection P of $T(t_0)$ is obtained by integrating $R(\lambda, T(t_0))$ (see, e.g., **?** , Section VII.3]). Since every $T(t)$, $t \geqslant 0$, commutes with $T(t_0)$, it must commute with $R(\lambda, T(t_0))$, hence with P. The statement on the decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$ follows from the Spectral Inclusion Theorem 6.2 below. □

This decomposition can be applied to the study of the asymptotic behavior of $\mathcal{T}$. In the situation of Corollary 3.5 assume

$$\sup\{|\lambda| \colon \lambda \in \tau_2\} < \alpha < \inf\{|\lambda| \colon \lambda \in \tau_1\}\,.$$

for some $\alpha > 0$. If we set $\beta := (\log \alpha)/t_0$ and use **?** , Chap.I, Theorem 6.5] we obtain $\omega_0(\mathcal{T}_2) < \beta$ and $\omega_0(\mathcal{T}_1^{-1}) < \beta$ by Proposition 1.1. Therefore we have constants m, M with $m \leq 1 \leq M$ such that

$$\|T(t)f\| \leqslant M \cdot e^{\beta t} \|f\| \quad \text{for } f \in E_2\,,$$
$$\|T(t)f\| \geqslant m \cdot e^{-\beta t} \|f\| \quad \text{for } f \in E_1\,.$$

As nice as they might look, results of this type are unsatisfactory. We need information on the semigroup in order to estimate its asymptotic behavior. In Chapter IV we will try to obtain such results by exploiting information about the generator only.

Example 3.6 **(Isolated singularities and poles)**

In case that λ_0 is an isolated point of $\sigma(A)$ the holomorphic function $\lambda \mapsto R(\lambda, A)$ can be expanded as a Laurent series

$$R(\lambda, A) = \sum_{n=-\infty}^{+\infty} U_n (\lambda - \lambda_0)^n \text{ for } 0 < |\lambda - \lambda_0| < \delta \text{ and some } \delta > 0\,.$$

The coefficients U_n are bounded linear operators given by

$$U_n = \frac{1}{2\pi i} \int_\Gamma (z - \lambda_0)^{-(n+1)} R(z, A)\, dz\,,\ n \in \mathbb{Z}\,, \tag{3.1}$$

where $\Gamma = \{z \in \mathbb{C} \colon |z - \lambda_0| = \delta/2\}$. The coefficient U_{-1} is the spectral projection corresponding to the spectral set $\{\lambda_0\}$ (see Definition 3.1) is called the *residue* of $R(\cdot, A)$ at λ_0, and will be denoted by P. From (3.1) one deduces

$$U_{-(n+1)} = (A - \lambda_0)^n \circ P \quad \text{and} \quad U_{-(n+1)} \circ U_{-(m+1)} = U_{-(n+m+1)} \text{ for } n, m \geqslant 0\,. \tag{3.2}$$

If there exists $k > 0$ such that $U_{-k} \neq 0$ while $U_{-n} = 0$ for all $n > k$, the point λ_0 is called a *pole of* $R(\cdot, A)$ *of order* k. In view of (3.2) this is true if $U_{-k} \neq 0$ and $U_{-(k+1)} = 0$. In this case one can retrieve U_{-k} as

$$U_{-k} = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^k R(\lambda, A)\,. \tag{3.3}$$

The dimension of PE (i.e., the dimension of the spectral subspace corresponding to $\{\lambda_0\}$) is called the *algebraic multiplicity* m_a of λ_0, while the *geometric multiplicity* is $m_g := \dim \ker(\lambda_0 - A)$. In case $m_a = 1$, we call λ_0 an *algebraically simple pole*.

If k is the pole order ($k = \infty$ in case of an essential singularity), we have

$$\max\{m_g, k\} \leqslant m_a \leqslant k \cdot m_g \tag{3.4}$$

where $\infty \cdot 0 = \infty$.

These inequalities yield the following implications.

(i) $m_a < \infty$ if and only if λ_0 is a pole with $m_g < \infty$,
(ii) if λ_0 is a pole with order k, then $\lambda_0 \in P\sigma(A)$ and $PE = \ker(\lambda_0 - A)^k$.

If A has compact resolvent, then every point of $\sigma(A)$ is a pole of finite algebraic multiplicity. This is a consequence of Proposition 2.5.(iii) and the well-known Riesz-Schauder Theory for compact operators (see **?** , VII.4.5]).

Example 3.7 **(The essential spectrum)**

For an operator $T \in \mathcal{L}(E)$ the *Fredholm domain* $\varrho_F(T)$ is

$$\begin{aligned} \varrho_F(T) &:= \{\lambda \in \mathbb{C} : \lambda - T \text{ is a Fredholm operator}\} \\ &= \{\lambda \in \mathbb{C} : \ker(\lambda - T) \text{ and } E/\mathrm{im}(\lambda - T) \text{ are finite dimensional}\}. \end{aligned} \tag{3.5}$$

An equivalent characterization of $\varrho_F(T)$ is obtained through the Calkin algebra $\mathcal{L}(E)/\mathcal{K}(E)$, where $\mathcal{K}(E)$ stands for the closed ideal of all compact operators. In fact, $\varrho_F(T)$ coincides with the resolvent set of the canonical image of T in the Calkin algebra. The complement of $\varrho_F(T)$ is called *essential spectrum* of T and denoted by $\sigma_{\mathrm{ess}}(T)$. The corresponding spectral radius, called *essential spectral radius*, satisfies

$$r_{\mathrm{ess}}(T) := \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{ess}}(T)\} = \lim_{n\to\infty} \|T^n\|_{\mathrm{ess}}^{1/n}, \tag{3.6}$$

where $\|T\|_{\mathrm{ess}} = \mathrm{dist}(T, \mathcal{K}(E)) := \inf\{\|T - K\| : K \in \mathcal{K}(E)\}$ is the norm of T in $\mathcal{L}(E)/\mathcal{K}(E)$.

For every compact operator K we have $\|T - K\|_{\mathrm{ess}} = \|T\|_{\mathrm{ess}}$, hence

$$r_{\mathrm{ess}}(T - K) = r_{\mathrm{ess}}(T). \tag{3.7}$$

A detailed analysis of $\varrho_F(T)$ can be found in Section IV.5.6 of **?**]. In particular we recall that the poles of $R(\cdot, T)$ with finite algebraic multiplicity belong to $\varrho_F(T)$. Conversely, an element of the unbounded component of $\varrho_F(T)$ either belongs to $\varrho(T)$ or is a pole of finite algebraic multiplicity. Thus $r_{\mathrm{ess}}(T)$ can be characterized as

$$\begin{aligned} &r_{\mathrm{ess}}(T) \text{ is the smallest } r \in \mathbb{R}_+ \text{ such that every } \lambda \in \sigma(T), |\lambda| > r \\ &\qquad\qquad\qquad \text{is a pole of finite algebraic multiplicity.} \end{aligned} \tag{3.8}$$

Now, if $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup, then VIII.1, Lemma 4 of **?**] applied to the function $t \mapsto \log \|T(t)\|_{\mathrm{ess}}$ ensures that

$$\omega_{\mathrm{ess}}(\mathcal{T}) := \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|_{\mathrm{ess}} = \inf\{\frac{1}{t} \log \|T(t)\|_{\mathrm{ess}} : t > 0\} \tag{3.9}$$

is well defined (possibly $-\infty$). By the definition of $\omega_{\mathrm{ess}}(\mathcal{T})$ and (3.6) we have

$$r_{\mathrm{ess}}(T(t)) = \exp(t\omega_{\mathrm{ess}}(\mathcal{T})), \quad t \geqslant 0. \tag{3.10}$$

Obviously, $\omega_{\mathrm{ess}} \leqslant \omega_0$ and equality occurs if and only if $r_{\mathrm{ess}}(T(t)) = r(T(t))$ for $t \geqslant 0$.

If $\omega_{\mathrm{ess}} < \omega_0$, there exists an eigenvalue λ of $T(t)$ satisfying $|\lambda| = r(T(t))$, hence by Theorem 6.3 below there exists $\lambda_1 \in P\sigma(A)$ such that $\mathrm{Re}\,\lambda_1 = \omega_0$. Thus $\omega_{\mathrm{ess}} < \omega_0$ implies $s(A) = \omega_0(\mathcal{T})$, i.e., we have

$$\omega_0(\mathcal{T}) = \max\{\omega_{\mathrm{ess}}(\mathcal{T}), s(A)\}. \tag{3.11}$$

As a final observation we point out that

$$\omega_{\mathrm{ess}}(\mathcal{T}) = \omega_{\mathrm{ess}}(\mathcal{S})\,, \tag{3.12}$$

whenever $\mathcal{T}$ is generated by A and $\mathcal{S}$ is generated by $A + K$ for some compact operator K see Proposition 2.8 and Proposition 2.9 of B-IV).

4 The Spectrum of Induced Semigroups

In the previous section we tried to decompose a semigroup into the direct sum of two, hopefully simpler objects. Here we present other methods to reduce the complexity of a semigroup and its generator. Forming subspace or quotient semigroups as in A-I,3.2, A-I,3.3 are such methods. But also the constructions of new semigroups on canonically associated spaces such as the dual space, see A-I,3.4, or the $\mathcal{F}$-product, see A-I,3.6, might be helpful. We review these constructions under the spectral theoretical point of view and collect a number of technical properties for later use.

We start by studying the spectrum of subspace and quotient semigroups. To that purpose assume that the strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ leaves invariant some closed subspace N of the Banach space E. There are canonically induced semigroups $\mathcal{T}_|$ on N, resp. $\mathcal{T}_/$ on E/N and their generators $A_|$, resp. $A_/$ are canonically obtained from the generator A of $\mathcal{T}$ (see A-I, Section 3). The following example shows that the spectra of A, $A_|$ and $A_/$ may differ quite drastically.

Example 4.1 As in the example in A-I,3.3 we consider the translation semigroup on $E = L^1(\mathbb{R})$ and the invariant subspace $N := \{f \in E\colon f(x) = 0 \text{ for } x \geqslant 1\}$. Then $\sigma(A) = \mathrm{i}\mathbb{R}$ but $\sigma(A_|) = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant 0\}$. Next we take the translation invariant subspace $M := \{f \in N\colon f(x) = 0 \text{ for } 0 \leqslant x \leqslant 1\}$ and obtain $\sigma(A_{|/}) = \emptyset$ for the generator $A_{|/}$ of the quotient semigroup $\mathcal{T}_{|/}$ (use the fact that $\mathcal{T}_{|/}$ is nilpotent).

In the next proposition we collect the information on $\sigma(A)$ which in general can be obtained from the "subspace spectrum" $\sigma(A_|)$ and the "quotient spectrum" $\sigma(A_/)$.

Proposition 4.2 *Using the standard notations the following inclusions hold.*

(i) $\varrho(A) \subset [\varrho(A_|) \cap \varrho(A_/)] \cup [\sigma(A_|) \cap \sigma(A_/)]$,
(ii) $[\varrho(A_|) \cap \varrho(A_/)] \subset \varrho(A)$,
(iii) $\varrho_+(A) \subset [\varrho(A_|) \cap \varrho(A_/)]$,

where $\varrho_+(A)$ denotes the connected component of $\varrho(A)$ which is unbounded to the right.

Proof (i) Assume $\lambda \in \varrho(A)$, i.e., $(\lambda - A)$ is a bijection from $D(A)$ onto E. Since N is T-invariant, we have $D(A_|) = D(A) \cap N$ and $(\lambda - A)D(A_|) \subset N$. If $(\lambda - A)D(A_|) = N$, then $R(\lambda, A)N = D(A_|)$ and the induced operators $R(\lambda, A)_|$, resp. $R(\lambda, A)_/$ are the inverses of $(\lambda - A_|)$, resp. $(\lambda - A_/)$. If $(\lambda - A)D(A_|) \neq N$, then $\lambda \in \sigma(A_|)$.

In addition there exists $f \in D(A)\backslash N$ such that $g := (\lambda - A)f \in N$. Hence for $\hat{f} := f + N$, $\hat{g} := g + N \in E_/$ it follows that $(\lambda - A_/)\hat{f} = \hat{g} = 0$, i.e., $\lambda \in \sigma(A_/)$

(ii) Take $\lambda \in \varrho(A_|) \cap \varrho(A_/)$. Then $(\lambda - A)$ is injective, since $(\lambda - A)f = 0$ implies $(\lambda - A_/)\hat{f} = 0$, hence $\hat{f} = 0$, i.e., $f \in N$ and therefore $f = 0$. In addition, $(\lambda - A)$ is surjective: For $g \in E$ there exists $\hat{f} \in E_/$ such that $(\lambda - A_/)\hat{f} = \hat{g}$, i.e., there exists $h \in N$ such that $(\lambda - A)f - g = h = (\lambda - A)k$ for some $k \in D(A_|)$. Therefore we obtain $(\lambda - A)(f - k) = g$.

(iii) The integral representation of the resolvent for $\lambda > \omega_0(\mathcal{T})$ (see A-I, Proposition 1.11) shows that $R(\lambda, A)N \subset N$. By the power series expansion for holomorphic functions this extends to all $\lambda \in \varrho_+(A)$. Therefore the restriction $R(\lambda, A)_|$ coincides with the resolvent $R(\lambda, A_|)$. On the other hand $R(\lambda, A)_/$ is well defined on $E_/$ and satisfies

$$R(\lambda, A)_/(f + N) = R(\lambda, A)f + N$$

(use again the integral representation). This proves that $R(\lambda, A)_/ = R(\lambda, A_/)$. □

Corollary 4.3 *Under the above assumptions take a point μ in the closure of $\varrho_+(A)$. Then*

(i) $\mu \in \sigma(A)$ if and only if $\mu \in \sigma(A_|)$ or $\mu \in \sigma(A_/)$.
(ii) μ is a pole of $R(\cdot, A)$ if and only if μ is a pole of $R(\cdot, A_|)$ and of $R(\cdot, A_/)$. In that case,

$$\max\{k_|, k_/\} \leqslant k \leqslant k_| + k_/$$

for the respective pole orders. Note that hereby pole orders 0 *are allowed.*

Proof (i) follows from Proposition 4.2, inclusions (ii) and (iii).

(ii) By the previous assertion we may assume that for some $\delta > 0$ the pointed disc

$$\{\lambda \in \mathbb{C} \colon 0 < |\lambda - \mu| < \delta\}$$

is contained in $\varrho(A) \cap \varrho(A_|) \cap \varrho(A_/)$.

Call U_n the coefficients of the Laurent expansion of $R(\cdot, A)$. Since N is $R(\lambda, A)$-invariant for $\lambda \in \varrho_+(A)$, the same holds for each U_n. With the obvious notations we have $R(\lambda, A) = \sum_n U_n(\lambda - \mu)^n$, $R(\lambda, A)_| = \sum U_{n|}(\lambda - \mu)^n$ and $R(\lambda, A)_/ = \sum U_{n/}(\lambda - \mu)^n$ which shows $\max\{k_|, k_/\} \leqslant k$.

If $R(\cdot, A)_|$ has a pole in μ of order ℓ, then $U_{-(\ell+1)|} = 0$, i.e., $U_{-(\ell+1)}N = \{0\}$. Similarly it follows that $U_{-(m+1)}E \subset N$ if $R(\cdot, A)_/$ has a pole in μ of order m. Therefore $U_{-(\ell+1)} \circ U_{-(m+1)} = 0$.

The relations (3.2) imply $U_{-(m+\ell+1)} = 0$, hence the pole order of $R(\cdot, A)$ is dominated by $\ell + m$. □

4.1 Spectrum of the adjoint semigroup

We recall from A-I,3.4 that to every strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ there corresponds a strongly continuous adjoint semigroup $\mathcal{T}^* = (T(t)^*)_{t \geqslant 0}$ on the semigroup dual

$$E^* = \{\varphi \in E' : \lim_{t\to\infty} \|T(t)'\varphi - \varphi\| = 0\}.$$

Its generator A^* is the maximal restriction of the adjoint A' to E^*. For these operators the spectra coincide, or more precisely.

(i) $\sigma(T(t)) = \sigma(T(t)') = \sigma(T(t)^*)$,
$R\sigma(T(t)) = P\sigma(T(t)') = P\sigma(T(t)^*)$,
(ii) $\sigma(A) = \sigma(A') = \sigma(A^*)$, $R\sigma(A) = P\sigma(A') = P\sigma(A^*)$,
(iii) $s(A) = s(A^*)$, $\omega_0(A) = \omega_0(A^*)$.

Proof The left part of these equalities is either well known or has been stated in 2.2(ii). The first statement of (iii) follows from (ii), while the second is an immediate consequence of the estimate $\|T(t)^*\| \leqslant \|T(t)\| \leqslant M \cdot \|T(t)^*\|$ given in A-I,3.4.

As a sample for the remaining assertions we show that $0 \notin \sigma(A)$ if and only if $0 \notin \sigma(A^*)$: If A and therefore A' is invertible, it follows from A-I,3.4 that A^* is a bijection from $D(A^*)$ onto e*.

Conversely assume that A^* is invertible. Then A' must be injective by the Proposition in A-I,3.4. Moreover $A'(D(A'))$ contains $A^*(D(A^*)) =$ e* and is $\sigma(E', E)$-dense in E'. By standard duality arguments it follows that A is injective with dense image. Next we show that $A(D(A))$ is closed: For $f \in D(A)$ choose $\varphi \in D(A')$ such that $\|\varphi\| \leqslant 1$ and $|\langle f, \varphi\rangle| \geqslant \frac{1}{2}\|f\|$. Then

$$\begin{aligned}\|(A^*)^{-1}\|\|Af\| &\geqslant \|(A^*)^{-1}\||\langle Af, \varphi\rangle| \geqslant |\langle Af, (A^*)^{-1}\varphi\rangle| \\ &= |\langle f, \varphi\rangle| \geqslant \frac{1}{2}\|f\|,\end{aligned}$$

hence

$$\|Af\| \geqslant \frac{1}{2}\|(A^*)^{-1}\|^{-1}\|f\|,$$

and $A(D(A))$ is closed since A is closed. □

4.2 Spectrum of the

As stated in A-I,3.6 the $\mathcal{F}$-product semigroup $\mathcal{T}_{\mathcal{F}} = (T_{\mathcal{F}}(t))_{t\geqslant 0}$ on $E_{\mathcal{F}}^{\mathcal{T}}$ of a strongly continuous semigroup $\mathcal{T}$ on E serves to convert sequences in E into points in $E_{\mathcal{F}}^{\mathcal{T}}$. In particular it can be used to convert approximate eigenvectors of the generator A into eigenvectors of $A_{\mathcal{F}}$.

Proposition 4.4 *Let A be the generator of a strongly continuous semigroup. Then the generator $A_{\mathcal{F}}$ of the $\mathcal{F}$-product semigroup satisfies.*

(i) $A\sigma(A) = A\sigma(A_{\mathcal{F}}) = P\sigma(A_{\mathcal{F}})$,
(ii) $\sigma(A) = \sigma(A_{\mathcal{F}})$.

Remark 4.5 In case A is bounded, then the canonical extension $A_{\mathcal{F}}$ is a generator and $E_{\mathcal{F}}^{\mathcal{T}} = E_{\mathcal{F}}$ (cf. A-I,3.6). Thus the proposition applies to bounded linear operators and their canonical extensions to the $\mathcal{F}$-product $E_{\mathcal{F}}$.

Proof (Proof of the proposition) (i) The inclusion $P\sigma(A_{\mathcal{F}}) \subset A\sigma(A_{\mathcal{F}})$ holds trivially.

We show that $A\sigma(A_{\mathcal{F}}) \subset A\sigma(A)$: Take $\lambda \in A\sigma(A_{\mathcal{F}})$ and an associated approximate eigenvector $(\hat{f}^m)_{n\in\mathbb{N}}$, i.e., $\hat{f}^m \in D(A_{\mathcal{F}})$, $\|\hat{f}^m\| = 1$ and $(\lambda - A_{\mathcal{F}})\hat{f}^m \to 0$ as $m \to \infty$.

By the considerations in A-I,3.6 we can represent each $\hat{f}^m$ as a normalized sequence $(f_n^m)_{n\in\mathbb{N}}$ in $D(A)$ such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|(\lambda - A) f_n^m\| = 0\,.$$

Therefore we can find a sequence $g_k = f_k^{m(k)}$ satisfying

$$\lim_{k\to\infty} \|(\lambda - A) g_k\| = 0\,,$$

i.e., $\lambda \in A\sigma(A)$.

Finally we show $A\sigma(A) \subset P\sigma(A_{\mathcal{F}})$: For $\lambda \in A\sigma(A)$ take a corresponding approximate eigenvector (f_n). By A-I,(3.2) we have

$$\begin{aligned}\|T(t)f_n - f_n\| &\leqslant \|T(t)f_n - \mathrm{e}^{\lambda t} f_n\| + |\mathrm{e}^{\lambda t} - 1|\\ &= \left\|\int_0^t \mathrm{e}^{\lambda(t-s)} T(s)(\lambda - A) f_n \,\mathrm{d}s\right\| + |\mathrm{e}^{\lambda t} - 1|\end{aligned}$$

which converges to zero uniformly in n as $t \to 0$, i.e., $(f_n) \in m^{\mathcal{T}}(E)$. From the characterization of $D(A_{\mathcal{F}})$ given in A-I,3.6 it follows that

$$\hat{f} := (f_n) + c_F(E) \in D(A_{\mathcal{F}})$$

and $A_{\mathcal{F}}\hat{f} = \lambda\hat{f}$, i.e., $\lambda \in P\sigma(A_{\mathcal{F}})$.

(ii) The inclusion $A\sigma(A) \subset \sigma(A_{\mathcal{F}})$ follows from (i). Now we show $R\sigma(A) \subset R\sigma(A_{\mathcal{F}})$: For $\lambda \in R\sigma(A)$ choose $f \in E$ such that $\|(\lambda - A)g - f\| \geqslant 1$ for every $g \in D(A)$. Then $\|(\lambda - A_{\mathcal{F}})g - \hat{f}\| \geqslant 1$ for every $\hat{g} \in D(A_{\mathcal{F}})$ and $\hat{f} = (f, f, \ldots) + c_F(E)$. Therefore $\lambda \in R\sigma(A_{\mathcal{F}})$.

We now show $\varrho(A) \subset \varrho(A_{\mathcal{F}})$: Assume $\lambda \in \varrho(A)$. By (i) $(\lambda - A_{\mathcal{F}})$ has to be injective. Choose $\hat{f} = (f_1, f_2, \ldots) + c_{\mathcal{F}}(E)$ such that $(f_n) \in m^{\mathcal{T}}(E)$. Then $(R(\lambda, A)f_n) \in m^{\mathcal{T}}(E)$ and $(\lambda - A_{\mathcal{F}})((R(\lambda, A)f_n) + c_{\mathcal{F}}(E)) = (f_n) + c_{\mathcal{F}}(E)$, i.e., $(\lambda - A_{\mathcal{F}})$ is surjective and $\lambda \in \varrho(A_{\mathcal{F}})$. □

Applying this proposition to a single operator $T(t)$, we obtain

$$A\sigma(T(t)) = P\sigma(T(t)_{\mathcal{F}}).$$

Note that in general $A\sigma(T(t)) \neq P\sigma(T_{\mathcal{F}}(t))$ (see the Examples 1.3 and 1.4 in combination with Theorem 6.3).

5 The Spectrum of Periodic Semigroups

In this section we determine the spectrum of a particularly simple class of strongly continuous semigroups and thereby achieve a rather complete description of the semigroup itself. Besides being nice and simple these semigroups gain their importance as building blocks for the general theory.

Definition 5.1 A strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ on a Banach space E is called *periodic* if $T(t_0) = \mathrm{Id}$ for some $t_0 > 0$.

The *period* τ of $\mathcal{T}$ is obtained as

$$\tau := \inf\{t_0 > 0 \colon T(t_0) = \mathrm{Id}\}\,.$$

We immediately observe that periodic semigroups are groups with inverses $T(t)^{-1} = T(n\tau - t)$ for $0 \leqslant t \leqslant n\tau$, τ the period of $\mathcal{T}$. Moreover, they are bounded, hence the growth bound is zero and $\sigma(A) \subset \mathrm{i}\mathbb{R}$.

Lemma 5.2 *Let T be a strongly continuous semigroup with period $\tau > 0$ and generator A. Then*

$$\sigma(A) \subset 2\pi\mathrm{i}/\tau \cdot \mathbb{Z}$$

and

$$R(\mu, A) = (1 - \mathrm{e}^{-\mu\tau})^{-1} \int_0^\tau \mathrm{e}^{-\mu s} T(s)\, \mathrm{d}s \tag{5.1}$$

for $\mu \notin 2\pi\mathrm{i}/\tau \cdot \mathbb{Z}$.

Proof From the basic identities A-I,(3.1) and A-I,(3.2) for $t = \tau$, it follows that $(\mu - A)$ has a left and right inverse if $\mu \neq 2\pi\mathrm{i}n/\tau$, $n \in \mathbb{Z}$, and that the inverse is given by the above expression. □

The representation of $R(\mu, A)$ given in A-I, Proposition 1.11 shows that the resolvent of the generator of a periodic semigroup is a meromorphic function having only poles of order one and the residues

$$P_n := \lim_{\mu \to \mu_n} (\mu - \mu_n) R(\mu, A) \quad \text{in} \quad \mu_n := 2\pi\mathrm{i}n/\tau, \quad n \in \mathbb{Z}\,,$$

are

$$P_n = \frac{1}{\tau} \int_0^\tau \exp(-\mu_n s) T(s)\, \mathrm{d}s\,. \tag{5.2}$$

Moreover, it follows that the spectrum of A consists of eigenvalues only and each P_n is the spectral projection belonging to μ_n (see 3.6). Another way of looking at P_n is given by saying that P_n is the n-th Fourier coefficient of the τ-periodic function $s \mapsto T(s)$. From this it follows that no non-zero $\varphi \in E'$ vanishes on all $P_n E$

simultaneously. By the Hahn-Banach theorem we conclude that span $\cup_{n\in\mathbb{Z}} P_n E$ is dense in E.

Since $P_n E \subset D(A)$, we obtain from A-I,(3.1) that

$$AP_n f = \mu_n P_n f \tag{5.3}$$

for every $f \in E$, $n \in \mathbb{Z}$. This and A-I,(3.2) imply

$$T(t)P_n f = \exp(\mu_n t) \cdot P_n f \tag{5.4}$$

for every $t \geqslant 0$. Therefore μ_n is an eigenvalue of A and $\exp(\mu_n t)$ is an eigenvalue of $T(t)$ if and only if $P_n \neq 0$. In that case, $P_n E$ is the corresponding eigenspace and we have the following lemma.

Lemma 5.3 *For a τ-periodic semigroup $\mathcal{T}$ we take $\mu_n := 2\pi i n/\tau$, $n \in \mathbb{Z}$, and consider*

$$P_n := \frac{1}{\tau}\int_0^\tau \exp(-\mu_n s)T(s)\, \mathrm{d}s\,.$$

Then the following assertions are equivalent.

(a) $P_n \neq 0\,,$
(b) $\mu_n \in P\sigma(A)\,,$
(c) $\exp(\mu_n t) \in P\sigma(T(t))$ *for every* $t > 0\,.$

The action of A, resp. $T(t)$ in the subspaces $P_n E$, $n \in \mathbb{Z}$, is determined by (5.3) and (5.4) resp.. Moreover,

$$P_m P_n f = \tfrac{1}{\tau}\textstyle\int_0^\tau \exp(-\mu_m s)T(s)P_n f\, \mathrm{d}s = \tfrac{1}{\tau}\int_0^\tau \exp((\mu_n - \mu_m)s)P_n f\, \mathrm{d}s = 0$$

for $n \neq m$, i.e., the subspaces $P_n E$ are "orthogonal". Since their union is total in E, one expects to be able to extend the representations (5.3) and (5.4) of A and $T(t)$. This is possible if

$$\sum_{-\infty}^{+\infty} P_n = \mathrm{Id}\,,$$

where the series should be summable for the strong operator topology.

Unfortunately this is false in general since the family of projections

$$Q_H := \sum_{n\in H} P_n\,,$$

where H runs through all finite subsets of $\mathbb{Z}$, may be unbounded (see the example below). Nevertheless the following is true.

Theorem 5.4 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a τ-periodic semigroup on a Banach space E with generator A and associated spectral projections*

$$P_n := \frac{1}{\tau}\int_0^\tau \exp(-\mu_n s)T(s)\, \mathrm{d}s, \quad \mu_n := 2\pi i n/\tau, \quad n \in \mathbb{Z}\,.$$

For every $f \in D(A)$ one has $f = \sum_{-\infty}^{+\infty} P_n f$ and therefore

(i) $T(t)f = \sum_{-\infty}^{+\infty} \exp(\mu_n t) P_n f \quad$ *if* $f \in D(A)$,
(ii) $Af = \sum_{-\infty}^{+\infty} \mu_n P_n f \quad$ *if* $f \in D(A^2)$.

Proof It suffices to prove the first statement. Then (i) and (ii) follow by (5.3) and (5.4).

We assume $\tau = 2\pi$ and show first that $\sum_{-\infty}^{+\infty} P_n f$ is summable for $f \in D(A)$: For $g := Af$ we obtain $P_n g = P_n A f = A P_n f = in P_n f$. Take H to be a finite subset of $\mathbb{Z} \setminus \{0\}$ and $\varphi \in E'$. Then

$$|\sum_{n\in H} \langle P_n f, \varphi\rangle| = \left|\sum_{n\in H} \tfrac{1}{in} \langle P_n g, \varphi\rangle\right| \leqslant \left(\sum_{n\in H} \tfrac{1}{n^2}\right)^{1/2} \left(\sum_{n\in H} |\langle P_n g, \varphi\rangle|^2\right)^{1/2}$$

From Bessel's inequality we obtain for the second factor

$$\sum_{n\in H} |\langle P_n g, \varphi\rangle|^2 \leqslant \tfrac{1}{2\pi} \cdot \int_0^{2\pi} |\langle T(s)g, \varphi\rangle|^2 \, \mathrm{d}s \leqslant \|\varphi\|^2 \cdot \tfrac{1}{2\pi} \cdot \int_0^{2\pi} \|T(s)g\|^2 \, \mathrm{d}s\,.$$

With the constant $c := \left(\frac{1}{2\pi} \cdot \int_0^{2\pi} \|T(s)g\|^2 \, \mathrm{d}s\right)^{1/2}$ we obtain

$$\|\sum_{n\in H} P_n f\| \leqslant c \left(\sum_{n\in H} n^{-2}\right)^{1/2}$$

for every finite subset H of $\mathbb{Z}$, i.e., $\sum_{-\infty}^{+\infty} P_n f$ is summable.

Next we set $h := \sum_{-\infty}^{+\infty} P_n f$ and observe that for every $\varphi' \in E'$ the Fourier coefficients of the continuous, τ-periodic functions $s \mapsto \langle T(s)h, \varphi\rangle$ and $s \mapsto \langle T(s)f, \varphi\rangle$ coincide. Therefore these functions are identical for $s \geqslant 0$ and in particular for $s = 0$, i.e., $\langle h, \varphi\rangle = \langle f, \varphi\rangle$. By the Hahn-Banach Theorem we obtain $f = h$. □

The above theorem contains rather precise information on periodic semigroups. In particular, it characterizes periodic semigroups by the fact that $\sigma(A)$ is contained in $\mathrm{i}\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$ and the eigenfunctions of A form a total subset of E.

For a periodic semigroup with bounded generator only a finite number of spectral projections P_n are distinct from 0 and we have the following characterization.

Corollary 5.5 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a semigroup with bounded generator on some Banach space E.*

This semigroup has period τ/k for some $k \in \mathbb{N}$ if and only if there exist finitely many pairwise orthogonal projections P_n, $-m \leqslant n \leqslant m$, $P_{-m} \neq 0$ or $P_m \neq 0$, such that

(i) $\sum_{-m}^{+m} P_n = \mathrm{Id}$,
(ii) $T(t) = \sum_{-m}^{+m} \exp(2\pi \mathrm{i} nt/\tau) P_n$,
(iii) $A = \sum_{-m}^{+m} (2\pi \mathrm{i} n/\tau) P_n$.

Example 5.6 From A-I,2.5 we recall briefly the rotation group

$$R_\tau(t) f(z) := f(\exp(2\pi \mathrm{i} nt/\tau) \cdot z)$$

on $E = C(\Gamma)$, resp. $E = L^p(\Gamma, m)$ for $1 \leqslant p < \infty$. The spectrum of the generator $Af(z) = (2\pi\mathrm{i}/\tau)z \cdot f'(z)$ is $\sigma(A) = (2\pi\mathrm{i}/\tau)\cdot\mathbb{Z}$. The eigenfunctions $\varepsilon_n(z) := z^n$ yield the projections

$$P_n = (1/2\pi\mathrm{i}) \cdot \varepsilon_{-(n+1)} \otimes \varepsilon_n, \text{ i.e.,}$$

$$P_n f(z) = (1/2\pi\mathrm{i}) \cdot (\int_\Gamma f(w) w^{-(n+1)} \, dw) \cdot z^n .$$

It is left as an exercise to compute the norms of $Q_m := \sum_{-m}^{+m} P_n$ in $L^p(\Gamma, m)$ for various p and then check the assertions of Theorem 5.4.

Clearly, this proves some classical convergence theorems for Fourier series (compare **?** , Chap.8.1]).

6 Spectral Mapping Theorems

We now return to the question posed in the introduction to this chapter: In which form and under which conditions is it true that the spectrum $\sigma(T(t))$ of the semigroup operators is obtained—via the exponential map—from the spectrum $\sigma(A)$ of the generator, or briefly

Do we have $\sigma(T(t)) = \exp(t\sigma(A))$ *or at least* $\sigma(T(t)) = \overline{\exp(t\sigma(A))}$?

This and similar statements will be called *spectral mapping theorems* for the semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ and its generator A. In addition, we saw in Proposition 1.1 that the validity of such a spectral mapping theorem implies

$$s(A) = \omega_0(A)$$

for the spectral- and growth bounds and therefore guarantees that the location of the spectrum of A determines the asymptotic behavior of $\mathcal{T}$. As we have seen in Examples 1.3 and 1.4 the last statement does not hold in general. We therefore present a detailed analysis, where and why it fails and what additional assumptions are needed for its validity. Before doing so, we have another look at the examples.

Example 6.1 (The counterexamples revisited)

(i) Take the nilpotent translation semigroup from A-I,2.6. Then $\sigma(A) = \emptyset$ and $\sigma(T(t)) = 0$ for every $t > 0$. By this trivial example and since $\mathrm{e}^z \neq 0$ for every $z \in \mathbb{C}$, it is natural to read the spectral mapping theorem modulo the addition of $\{0\}$, i.e.,

$$\sigma(T(t)) \setminus \{0\} = \exp(t\sigma(A)) \text{ for } t \geqslant 0 .$$

(ii) The spectrum of the generator A of the τ-periodic rotation group $(R_\tau(t))_{t\geqslant 0}$ on $C(\Gamma)$ is $\sigma(A) = 2\pi\mathrm{i}/\tau \cdot \mathbb{Z}$ and $\exp(2\pi\mathrm{i}nt/\tau)$, $n \in \mathbb{Z}$, is an eigenvalue of $R_\tau(t)$ for every $t \geqslant 0$ (see Example 5.6. If t/τ is irrational, these eigenvalues form a dense subset of Γ. Since the spectrum is closed, we obtain $\sigma(T(t)) = \Gamma$ for these t. Therefore in this example the spectral mapping theorem is valid only in the following "weak" form

$$\sigma(T(t)) = \overline{\exp(t\sigma(A))}, \quad t \geqslant 0 .$$

(iii) By Example 1.3 there exists a semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ with generator A such that $s(A) = -1$ and $\omega_0(\mathcal{T}) = 0$. This implies that for preassigned real numbers $\alpha < \beta$ there exists a semigroup $\mathcal{S} = (S(t))_{t\geqslant 0}$ with generator B such that $s(B) = \alpha$ and $\omega_0(\mathcal{S}) = \beta$. Indeed, take $S(t) = \mathrm{e}^{\beta t}T((\beta-\alpha)t)$ and observe that $B = (\beta-\alpha)A + \beta\,\mathrm{Id}$. In that case $\exp(t\sigma(B))$ is contained in the circle about 0 with radius $\mathrm{e}^{\alpha t}$ while $\sigma(S(t))$ has spectral values satisfying $|\lambda| = r(S(t)) = \mathrm{e}^{\beta t} > \mathrm{e}^{\alpha t}$.

(iv) The Example 1.3 can be strengthend in order to yield a semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ with generator A such that $\sigma(A) = \emptyset$, but $\|T(t)\| = r(T(t)) = 1$ for $t \geqslant 0$, i.e., $s(A) = -\infty$, $\omega_0 = 0$ and $s(A) < \omega_0$ take the translation semigroup on the Banach space

$$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathrm{e}^{x^2}\,\mathrm{d}x)$$

with $\|f\| := \sup\{|f(x)| : x \in \mathbb{R}_+\} + \int_0^\infty |f(x)|\mathrm{e}^{x^2}\,\mathrm{d}x$ (see **?**]).

(v) Another modification of Example 1.3 yields a group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ satisfying $s(A) < \omega_0$. Therefore the spectral mapping theorem does not hold in the setting of groups (see Wolff (1981)).

The next few theorems form the core of this chapter. We show that only one part of the spectrum and one inclusion is responsible for the failure of the spectral mapping theorem. The usefulness of this detailed analysis will become clear in the subsequent chapters on stability and asymptotics.

Proposition 6.2 (Spectral Inclusion)

Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ on some Banach space E. Then

$$\exp(t\sigma(A)) \subset \sigma(T(t)) \text{ for } t \geqslant 0 .$$

More precisely we have the following inclusions.

$$\exp(t\cdot P\sigma(A)) \subset P\sigma(T(t)) , \tag{6.1}$$
$$\exp(t\cdot A\sigma(A)) \subset A\sigma(T(t)) , \tag{6.2}$$
$$\exp(t\cdot R\sigma(A)) \subset R\sigma(T(t)) . \tag{6.3}$$

Proof Since $\mathrm{e}^{\lambda t} - T(t) = (\lambda - A)\int_0^t \mathrm{e}^{\lambda(t-s)}T(s)\,\mathrm{d}s$ (see A-I,(3.1)), it follows that $(\mathrm{e}^{\lambda t} - T(t))$ is not bijective if $(\lambda - A)$ fails to be bijective proving the main inclusion.

The inclusion (6.1) becomes evident from the following proof of (6.2). Take $\lambda \in A\sigma(A)$ and an associated approximate eigenvector $(f_n) \subset D(A)$. Then

$$g_n := \mathrm{e}^{\lambda t}f_n - T(t)f_n = \int_0^t \mathrm{e}^{\lambda(t-s)}T(s)(\lambda - A)f_n\,\mathrm{d}s$$

converges to zero as $n \to \infty$. Consequently, $e^{\lambda t} \in A\sigma(T(t))$ and, in fact, the same approximate eigenvector (f_n) does the job for all $t \geqslant 0$.

For the proof of (6.3) we take $\lambda \in R\sigma(A)$ and observe that $(e^{\lambda t} - T(t))f = (\lambda - A)(\int_0^t e^{\lambda(t-s)}T(s)f \, ds) \in (\lambda - A)D(A)$ for every $f \in E$. ◻

As we know from the Examples 6.1, the converse inclusions do not hold in general, i.e., not every spectral value of a semigroup operator $T(t)$ comes - via the exponential map - from a spectral value of the generator. But at least this is true for some important parts of the spectrum.

Theorem 6.3 (Spectral Mapping Theorem for Point and Residual Spectrum) *Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$. Then*

$$\exp(t \cdot P\sigma(A)) = P\sigma(T(t)) \setminus \{0\}\,, \tag{6.4}$$

$$\exp(t \cdot R\sigma(A)) = R\sigma(T(t)) \setminus \{0\} \text{ for } t \geqslant 0\,. \tag{6.5}$$

Proof For the proof of (6.4), take $t_0 > 0$ and $0 \neq \lambda \in P\sigma(T(t_0))$.

After rescaling the semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ to the semigroup

$$(\exp(-t \cdot \log \lambda / t_0) T(t))_{t\geqslant 0}\,,$$

we may assume $\lambda = 1$. Then the closed, $\mathcal{T}$-invariant subspace

$$G := \{g \in E : T(t_0)g = g\}$$

is non trivial. The restricted semigroup $T_|$ is periodic with period $\tau \leqslant t_0$ and the spectrum of its generator $A_|$ contains at least one eigenvalue $\mu = 2\pi i n/t_0$ for some $n \in \mathbb{Z}$ (see Lemma 5.3). Since every eigenvalue of $A_|$ is an eigenvalue of A, we obtain that $1 \in \exp(t_0 \cdot P\sigma(A))$. The converse inclusion has been proved in (6.1).

In fact, even more can be said: Let $g \in G$ be an eigenvector of $T(t_0)$ corresponding to the eigenvalue $\lambda = 1$. For each $n \in \mathbb{Z}$ define

$$g_n := P_n g = 1/t_0 \cdot \int_0^{t_0} \exp(-2\pi i n s/t_0) T(s) g \, ds \in G$$

as in Section 5. If $g_n \neq 0$, then G_n is an eigenvector of $A_|$, hence of A with eigenvalue $2\pi i n/t_0$ as soon as g_n is distinct from zero. Since $D(A_|)$ is dense in G it follows from Theorem 5.4 that this holds for at least one $n \in \mathbb{Z}$. And from the proof of (6.1) we know that this g_n is in fact an eigenvector for each $T(t)$, $t \geqslant 0$.

Since $R\sigma(A) = P\sigma(A^*)$ and $R\sigma(T(t)) = P\sigma(T(t)^*)$ (see Proposition 4.4) the assertion (6.5) follows from (6.4) . ◻

Note that the proof is essentially an application of the structure theorem for periodic semigroups as given in Theorem 5.4. The information gained there can be reformulated into statements on the eigenspaces of A and $T(t)$.

Corollary 6.4 *For the eigenspaces of the generator A, resp. of the semigroup operators $T(t)$, $t > 0$, the following holds for $\mu \in \mathbb{C}$.*

(i) $\ker(\mu - A) = \bigcap_{s\geqslant 0} \ker(\mathrm{e}^{\mu s} - T(s))$,
(ii) $\ker(\mathrm{e}^{\mu t} - T(t)) = \overline{\mathrm{span}_{n\in\mathbb{Z}}\{\ker(\mu + 2\pi\mathrm{i}n/t - A)\}}$.

We note that an analogous statements is valid for $\ker(\mu - A')$ and $\ker(\mathrm{e}^{\mu t} - T(t)')$ if we take in (ii) the $\sigma(E', E)$-closure.

Without proof (see **?** , Proposition 1.10]) we add another corollary showing that poles of the resolvent of $T(t)$ correspond necessarily to poles of the resolvent of the generator. Again the converse is not true as shown by Example 5.6.

Corollary 6.5 *Assume that* $\mathrm{e}^{\mu t}$ *is a pole of order* k *of* $R(\cdot, T(t))$ *with residue* P *and* Q *as the* k*-th coefficient of the Laurent series. Then*

(i) $\mu + 2\pi\mathrm{i}n/t$ *is a pole of* $R(\cdot, A)$ *of order* $\leqslant k$ *for every* $n \in \mathbb{Z}$,
(ii) the residues P_n *in* $\mu + 2\pi\mathrm{i}n/t$ *yield* $PE = \overline{\mathit{span}_{n\in\mathbb{Z}}\{P_n E\}}$,
(iii) the k*-th coefficient of the Laurent series of* $R(\cdot, A)$ *at* $\mu + 2\pi\mathrm{i}n/t$ *is*

$$Q_n = (t \cdot \mathrm{e}^{\mu t})^{1-k} \cdot Q \circ (1/t) \int_0^t \mathrm{e}^{-(\mu + 2\pi\mathrm{i}n/t)s} T(s)\,\mathrm{d}s\,.$$

From Proposition 6.2 and Theorem 6.3 it follows that the approximate point spectrum is the trouble maker in the sense that not every approximate eigenvalue of $T(t)$ corresponds to an approximate eigenvalue of the generator A. Since nothing more can be said in general, we now look for additional hypotheses on the semigroup implying the spectral mapping theorem.

As a simple example we assume $T(t_0)$ to be compact for some $t_0 > 0$. Then $\sigma(T(t)) \setminus \{0\} = P\sigma(T(t)) \setminus \{0\}$ for $t \geqslant t_0$ and the spectral mapping theorem is valid by (6.4). A different class of semigroups verifying the spectral mapping theorem is given by the uniformly continuous semigroups (compare Corollary 1.2).

Both cases, and many more, are included in the following result.

Theorem 6.6 (Spectral Mapping Theorem for Eventually Continuous Semigroups) *Let* $\mathcal{T} = (T(t))_{t\geqslant 0}$ *be an eventually norm continuous semigroup with generator* A*. Then the spectral mapping theorem is valid, i.e.,*

$$\sigma(T(t)) \setminus \{0\} = \mathrm{e}^{t\cdot\sigma(A)} \text{ for every } t \geqslant 0\,. \tag{6.6}$$

Proof By the previous considerations it suffices to show that $A\sigma(T(t)) \setminus \{0\} \subset \mathrm{e}^{t\cdot\sigma(A)}$ for $t > 0$. This will be done by converting approximate eigenvectors into eigenvectors in the semigroup $\mathcal{F}$-product (see subsection 4.2). The assertion then follows from (6.4) and Proposition 4.4.(ii).

Assume $t \mapsto T(t)$ to be norm continuous for $t \geqslant t_0$. Moreover it suffices to consider $1 \in A\sigma(T(t_1))$ for some $t_1 > 0$, i.e., we have a normalized sequence $(f_n)_{n\in\mathbb{N}} \subset E$ such that

$$\lim_{n\to\infty} \|T(t_1)f_n - f_n\| = 0\,.$$

Choose $k \in \mathbb{N}$ such that $kt_1 > t_0$ and define $g_n := T(kt_1)f_n$. Then

$$\lim_{n\to\infty} \|g_n\| = \lim_{n\to\infty} \|T(t_1)^k f_n\| = \lim_{n\to\infty} \|f_n\| = 1$$

and

$$\lim_{n\to\infty} \|T(t_1)g_n - g_n\| = 0\,,$$

i.e., $(g_n)_{n\in\mathbb{N}}$ yields an approximate eigenvector of $T(t_1)$ with approximate eigenvalue 1. But the semigroup $\mathcal{T}$ is uniformly continuous on sets of the form $T(t_0)V$, V bounded in E. In particular, it is uniformly continuous on the sequence $(g_n)_{n\in\mathbb{N}}$, which therefore defines an element g in the semigroup $\mathcal{F}$-product $E_{\mathcal{F}}$.

Obviously, g is an eigenvector of $T_{\mathcal{F}}(t_1)$ with eigenvalue 1 and by (6.4) we obtain an eigenvalue $2\pi i n/t_1$ of $A_{\mathcal{F}}$ for some $n \in \mathbb{Z}$. The coincidence of $\sigma(A)$ and $\sigma(A_{\mathcal{F}})$ proves the assertion. □

We point out that the above spectral mapping theorem implies the coincidence of spectral bound and growth bound for eventually norm continuous semigroups, hence we have generalized the Liapunov Stability Theorem (see 1.2) to a much larger class of semigroups. As mentioned before, this will be of great use in many applications. Therefore we state explicitly the spectral mapping theorem for several important classes of semigroups all of which are eventually norm continuous (cf. the diagram preceding A-II, Example 1.27).

Corollary 6.7 *The spectral mapping theorem 6.6 holds for each of the following classes of strongly continuous semigroups.*

(i) eventually compact semigroups,
(ii) eventually differentiable semigroups,
(iii) holomorphic semigroups,
(iv) uniformly continuous semigroups.

Another application of the above spectral mapping theorem can be made to tensor product semigroups (see A-I,3.7). Let $\mathcal{T}_1 = (T_1(t))_{t\geqslant 0}$, $\mathcal{T}_2 = (T_2(t))_{t\geqslant 0}$ be strongly continuous semigroups on Banach spaces E_1, E_2 with generator A_1, A_2. The tensor product semigroup $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$ on some (appropriate) tensor product $E := E_1 \otimes E_2$ has the generator $A = A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$, but in general the spectrum of A is not determined by the spectra of A_1, A_2. But with an additional hypothesis the following can be proved.

Corollary 6.8 *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are eventually norm continuous, then*

$$\sigma(A) = \sigma(A_1) + \sigma(A_2)\,,$$

where A is the generator of the tensor product semigroup

$$\mathcal{T}_1 \otimes \mathcal{T}_2 = (T_1(t) \otimes T_2(t))_{t\geqslant 0}\,.$$

Proof Clearly, the tensor product semigroup is eventually norm continuous and hence the Spectral Mapping Theorem 6.6 is valid for all three semigroups $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}$. Moreover the spectrum of the tensor product of bounded operators is the product of the spectra **?** , XIII.9]. Therefore

$$\sigma(T_1(t) \otimes T_2(t)) = \sigma(T_1(t)) \cdot \sigma(T_2(t)), \quad t \geqslant 0\,.$$

Consequently we have the following identity for every $t \geqslant 0$

$$\begin{aligned} e^{t \cdot \sigma(A)} &= \sigma(T_1(t) \otimes T_2(t)) \setminus \{0\} \\ &= \sigma(T_1(t)) \cdot \sigma(T_2(t)) \setminus \{0\} \\ &= e^{t \cdot \sigma(A_1)} \cdot e^{t \cdot \sigma(A_2)} \\ &= e^{t(\sigma(A_1)+\sigma(A_2))} . \end{aligned}$$

From this identity we want to deduce $\sigma(A) = \sigma(A_1) + \sigma(A_2)$.

"$\subset$" Take $\xi \in \sigma(A)$. Then for every $t > 0$ there exist $\mu_t \in \sigma(A_1)$, $\lambda_t \in \sigma(A_2)$ and $n_t \in \mathbb{Z}$ such that $\xi = \mu_t + \lambda_t + 2\pi i n_t / t$.

Since the real parts of μ_t, λ_t are bounded above, they lie in some interval $[a, b]$. But $\sigma(A_i) \cap ([a, b] + i\mathbb{R})$ is compact for $i = 1, 2$ since A_i is the generator of an eventually norm continuous semigroup (see A-II, Theorem 1.20). By taking t sufficiently small, we conclude that $n_{t'} = 0$ for some $t' > 0$, i.e., $\xi = \mu_{t'} + \lambda_{t'}$.

"$\supset$" Choose $\mu \in \sigma(A_1)$, $\lambda \in \sigma(A_2)$. For every $t > 0$ there exist $\eta_t \in \sigma(A)$, $m_t \in \mathbb{Z}$ such that $\mu + \lambda = \eta_t + 2\pi i m_t / t$. Since $\operatorname{Re} \mu + \operatorname{Re} \lambda = \operatorname{Re} \eta_t$ and $\{\operatorname{Im} \eta_t : t > 0\}$ is bounded - $\mathcal{T} = (T_1(t) \otimes T_2(t))_{t \geqslant 0}$ is eventually norm continuous - it follows that $m_{t'} = 0$ for some $t' > 0$. □

7 Weak Spectral Mapping Theorems

In the previous section we showed under which hypotheses a spectral mapping theorem of the form

$$\sigma(T(t)) \setminus \{0\} = e^{t \cdot \sigma(A)}, \quad t \geqslant 0, \tag{7.1}$$

is valid for the generator A of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$.

Among the various examples showing that (7.1) does not hold in general we recall the following. Take the Banach space $E = c_0$, the multiplication operator $A(x_n)_{n \in \mathbb{N}} = (i n x_n)_{n \in \mathbb{N}}$ with maximal domain and the corresponding semigroup $T(t)(x_n)_{n \in \mathbb{N}} = (e^{int} x_n)_{n \in \mathbb{N}}$. Then $\sigma(A) = \{in \colon n \in \mathbb{N}\}$ and the spectral mapping theorem is valid only in the following weak form

$$\sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))}, \quad t \geqslant 0 . \tag{7.2}$$

In this section we prove similar weak spectral mapping theorems. We start with a generalization of the above example.

Consider the Banach space $E = C_0(X, \mathbb{C}^n)$ of all continuous $\mathbb{C}^n$-valued functions vanishing at infinity on some locally compact space X. In analogy to A-I,2.3 we associate to every continuous function $q \colon X \to M(n)$, where $M(n)$ denotes the space of all complex $n \times n$-matrices, a "multiplication operator" $M_q \colon f \to q \cdot f$ such that $(q \cdot f)(x) = q(x) \cdot f(x), x \in X$, on the maximal domain $D(M_q) = \{f \in E \colon q \cdot f \in E\}$. If $\|e^{tq(x)}\|$ is uniformly bounded for $0 \leqslant t \leqslant 1$ and $x \in X$, it follows that M_q generates the multiplication semigroup

$$(T(t)f)(x) = \mathrm{e}^{tq(x)} \cdot f(x), \quad f \in E, \quad x \in X, \quad t \geqslant 0\,.$$

Since M_q has a bounded inverse if and only if $q(x)^{-1}$ exists and is uniformly bounded for $x \in X$, it follows that the eigenvalues of each matrix $q(x)$ are always contained in $\sigma(M_q)$. In fact, much more can be said, in case the function is bounded.

Lemma 7.1 *The spectrum of the matrix valued multiplication operator M_q, where $q\colon X \to M(n)$, is bounded, is given by $\sigma(M_q) = \overline{\bigcup_{x\in X}\sigma(q(x))}$.*

Proof It remains to show that $0 \notin \overline{\bigcup_{x\in X}\sigma(q(x))}$ implies $0 \notin \sigma(M_q)$. Since $\det q(x)$ is the product of n eigenvalues (according to their multiplicity) of $q(x)$, the hypothesis implies that $d := \inf\{|\det q(x)|\colon x \in X\} > 0$. By Formula 4.12 in Chapter I of **?**] we obtain

$$\|q(x)^{-1}\| \leqslant \gamma \cdot \|q(x)\|^{n-1} \cdot |\det q(x)|^{-1} \leqslant \gamma/d \cdot \|M_q\|^{n-1}$$

for every $x \in X$ and a constant γ depending only on the norm chosen on $\mathbb{C}^n$. Therefore, $x \mapsto q(x)^{-1}$ defines a bounded continuous function on X, which obviously yields the inverse of M_q, i.e., $0 \notin \sigma(M_q)$. □

Theorem 7.2 *Let $A = M_q$ be a matrix multiplication operator on $C_0(X, \mathbb{C}^n)$ generating a strongly continuous semigroup $(T(t))_{t\geqslant 0}$, $T(t) = M_{\mathrm{e}^{tq}}$ for $t \geqslant 0$. Then the Weak Spectral Mapping Theorem 7.2 holds true, i.e.,*

$$\sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))}\,.$$

Proof By the Spectral Inclusion Proposition 6.2 we always have $\exp(t\sigma(A)) \subset \sigma(T(t))$. Since $T(t)$ is a matrix multiplication operator with a bounded function, we obtain from Lemma 7.1

$$\sigma(T(t)) = \overline{\bigcup_{x\in X}\sigma(\exp(tq(x)))} = \overline{\bigcup_{x\in X}\exp(t\sigma(q(x)))} \subset \overline{\exp(t\sigma(A))}$$

which proves the assertion. □

Corollary 7.3 *The growth bound $\omega_0(A)$ and the spectral bound $s(A)$ coincide for matrix multiplication semigroups.*

The above results remain valid for other Banach spaces of $\mathbb{C}^n$-valued functions such as $L^p(X, \mathbb{C}^n)$, $1 \leqslant p < \infty$.

The example given at the beginning of this section can be generalized in a different way. In fact, $A(x_n) := (\mathrm{i}nx_n)$ on $E = c_0$ generates a bounded group, and we will show that this property too ensures that the Weak Spectral Mapping Theorem 7.2 holds. Without any boundedness assumption on $(T(t))_{t\in\mathbb{R}}$ this result cannot be true (see **?** , Sec.23.16] or **?**]).

Theorem 7.4 *Let $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ be a strongly continuous group on some Banach space E such that $\|T(t)\| \leqslant p(t)$ for some polynomial p and all $t \in \mathbb{R}$. Then the Weak Spectral Mapping Theorem 7.2 holds, i.e.,*

$$\sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))} \text{ for all } t \in \mathbb{R}\,.$$

From the proof we isolate a series of lemmas for which we always assume the hypothesis made in Theorem 7.4. Moreover we recall from Fourier analysis that the Fourier transformation $\varphi \mapsto \hat{\varphi}$,

$$\hat{\varphi}(\alpha) := \int_{-\infty}^{\infty} \varphi(x)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}x$$

and its inverse $\Psi \mapsto \check{\Psi}$,

$$\check{\Psi}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\alpha)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}\alpha$$

are topological isomorphisms of the Schwartz space $\mathcal{S}(=\mathcal{S}(\mathbb{R}))$. Since the subspace $\mathcal{D}$ of all functions having compact support is dense in $\mathcal{S}$, it follows that $\{\varphi \in \mathcal{S} : \check{\varphi} \in \mathcal{D}\}$ is also dense in $\mathcal{S}$.

Lemma 7.5 *For every function $\varphi \in \mathcal{S}$ we obtain an operator $T(\varphi) \in \mathcal{L}(E)$ by*

$$T(\varphi)f := \int_{-\infty}^{\infty} \varphi(s)T(s)f\,\mathrm{d}s, \quad f \in E\,.$$

This operator can be represented as

$$T(\varphi)f = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} \check{\varphi}(\alpha)[R(\varepsilon-\mathrm{i}\alpha, A)f - R(-\varepsilon-\mathrm{i}\alpha, A)f]d\alpha, \quad f \in E\,. \tag{7.3}$$

Proof That $T(\varphi)$ is well-defined follows from the polynomial boundedness of $(T(t))_{t\in\mathbb{R}}$. In fact, $\varphi \to T(\varphi)$ is continuous from $\mathcal{S}$ into $(\mathcal{L}(E), \|\cdot\|)$. We obtain

$$\begin{aligned}
T(\varphi)f &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \varphi(s)\mathrm{e}^{-\varepsilon|s|}T(s)f\,\mathrm{d}s\\
&= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \check{\varphi}(\alpha)\mathrm{e}^{\mathrm{i}\alpha s}\mathrm{e}^{-\varepsilon|s|}T(s)f\,d\alpha\,\mathrm{d}s\\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} \check{\varphi}(\alpha)\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\alpha s}\mathrm{e}^{-\varepsilon|s|}T(s)f\,\mathrm{d}s d\alpha\\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} \check{\varphi}(\alpha)[R(\varepsilon-\mathrm{i}\alpha, A)f - R(-\varepsilon-\mathrm{i}\alpha, A)f]d\alpha\,.
\end{aligned}$$

Here we used that polynomially bounded groups have growth bound 0, hence $\omega_0(A) = \omega_0(-A) = 0$. Therefore the integral representation of $R(\varepsilon-\mathrm{i}\alpha, A)$ (cf. A-I, Proposition 1.11) exists for $\varepsilon \neq 0$. □

Lemma 7.6 *If $E \neq \{0\}$, then $\sigma(A) \neq \emptyset$.*

Proof If $\sigma(A) = \emptyset$, then (7.3) implies $T(\varphi) = 0$ whenever $\check{\varphi}$ has compact support. Since these functions form a dense subspace of $\mathcal{S}$, we conclude that $T(\varphi) = 0$ for all $\varphi \in \mathcal{S}$. Choosing an approximate identity $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{D}$, we obtain

$$f = T(0)f = \lim_{n\to\infty} T(\psi_n)f = 0$$

for every $f \in E$. □

Proof (Proof of Theorem 7.4 (1st part)) By the Spectral Inclusion (see Proposition 6.2), we have to show that every spectral value of $T(t)$ can be approximated by exponentials of spectral values of A. In view of the rescaling procedure it suffices to prove this when $-1 \in \varrho(T(\pi))$, provided that the following condition is satisfied.

$$\text{There exists } \varepsilon > 0 \text{ such that } \bigcup_{k\in\mathbb{Z}} i[2k+1-2\varepsilon, 2k+1+2\varepsilon] \subset \varrho(A)\,. \tag{7.4}$$

Assume now that (7.4) holds. Then each of the sets

$$\sigma_k := \{\mathrm{i}\alpha \in \sigma(A) \colon \alpha \in [2k-1, 2k+1]\}$$

is a spectral set of A with corresponding spectral projection P_k. If we choose $\varphi_0 \in \mathcal{D}$ such that

$$\operatorname{supp}(\varphi_0) \subset [-1+\varepsilon, 1-\varepsilon] \text{ and } \varphi_0(x) = 1 \text{ for } x \in [-1+2\varepsilon, 1-2\varepsilon]\,,$$

it follows from (7.3) and the integral representation of P_k (cf. (3.1)) that $P_0 = T(\check{\varphi}_0)$. More generally, since $\left(\mathrm{e}^{i2k\cdot}\check{\varphi}_0\right)\hat{}(\alpha) = \varphi_0(\alpha - 2k)$, the assertions (7.3) and (7.4) imply

$$P_k = \int_{-\infty}^{\infty} \mathrm{e}^{i2ks}\check{\varphi}_0(s)T(s)\,\mathrm{d}s \text{ for } k \in \mathbb{Z}\,. \tag{7.5}$$

At this point we isolate another lemma.

Lemma 7.7 span $\bigcup_{k\in\mathbb{Z}} P_k E$ *is dense in E.*

Proof The closure of span $\bigcup_{k\in\mathbb{Z}} P_k E$ is a $\mathcal{T}$-invariant subspace G of E. Consider the quotient group $(T(t)_/)_{t\in\mathbb{R}}$ induced on E/G. The spectrum of its generator $A_/$ is contained in $\sigma(A)$ by Proposition 4.2.(ii). Moreover, the spectral projection corresponding to $\sigma(A_/) \cap \sigma_k$ is the quotient operator $P_{k/}$. Obviously $P_{k/} = 0$, hence $\sigma(A_/) \cap \sigma_k = \emptyset$ for every $k \in \mathbb{Z}$ and $\sigma(A_/) = \emptyset$. By Lemma 7.6 this implies $E/G = \{0\}$, i.e., $G = E$. □

Proof (Proof of Theorem 7.4 (2nd part)) We return to the situation of the first part of the proof. Using (7.5) the spectral projection P_k can be transformed into

$$\begin{aligned}
P_k &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2ks} \check{\varphi}_0(s) T(s)\,\mathrm{d}s \\
&= \sum_{m\in\mathbb{Z}} \int_{(m-1/2)\pi}^{(m+1/2)\pi} \mathrm{e}^{\mathrm{i}2ks} \check{\varphi}_0(s) T(s)\,\mathrm{d}s \\
&= \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}2ks} \sum_{m\in\mathbb{Z}} \check{\varphi}_0(s+m\pi) T(s+m\pi)\,\mathrm{d}s\,,
\end{aligned}$$

i.e., $P_k f$ is the k-th Fourier coefficient of the π-periodic, continuous function $\xi_f\colon s \mapsto \sum_{m\in\mathbb{Z}} \check{\varphi}_0(s+m\pi)T(s+m\pi)f$, $f \in E$. Since the projections P_k are mutually orthogonal, i.e., $P_k P_m = 0$ for $k \neq m$, it follows that $g = \sum_{n\in\mathbb{Z}} P_n g$ for every $g \in \operatorname{span} \bigcup_{k\in\mathbb{Z}} P_k E$. In particular, the Fourier coefficients of the function ξ_g are absolutely summable, hence the Fourier series of ξ_g converges to ξ.

For $s = 0$ we obtain

$$g = \sum_{n\in\mathbb{Z}} P_n g \cdot \mathrm{e}^{-\mathrm{i}n0} = \sum_{m\in\mathbb{Z}} \check{\varphi}_0(0+m\pi)T(0+m\pi)g \qquad \left(g \in \operatorname{span} \bigcup_{k\in\mathbb{Z}} P_k E\right).$$

Since $\operatorname{span} \bigcup_{k\in\mathbb{Z}} P_k E$ is dense (Lemma 7.7), we conclude that

$$\sum_{m\in\mathbb{Z}} \varphi_0(m\pi)T(m\pi) = \mathrm{Id}\,. \tag{7.6}$$

As the final step we construct the inverse operator of $\mathrm{Id} + T(\pi)$ showing that $-1 \in \varrho(T(\pi))$. We define $\psi_0(\alpha) := \varphi_0(\alpha)\cdot(1+\mathrm{e}^{\mathrm{i}\pi\alpha})^{-1}$, $\alpha \in \mathbb{R}$. Then we have $\psi_0 \in \mathcal{S}$ and $\psi_0 \cdot (1+\mathrm{e}^{\mathrm{i}\pi\cdot}) = \varphi_0$, hence $\check{\psi}_0(x) + \check{\psi}_0(x+\pi) = \check{\varphi}_0(x)$ for all $x \in \mathbb{R}$. Then (7.6) implies

$$\begin{aligned}
\mathrm{Id} &= \sum_{m\in\mathbb{Z}} \check{\varphi}_0(m\pi)T(m\pi) \\
&= \sum_{m\in\mathbb{Z}} (\check{\psi}_0(m\pi) + \check{\psi}_0((m+1)\pi))T(m\pi) \\
&= [\sum_{m\in\mathbb{Z}} \check{\psi}_0(m\pi)T(m\pi)](\mathrm{Id} + T(\pi))\,.
\end{aligned}$$

In the rest of this section we discuss the behavior of the single spectral values λ of $T(t)$, $t > 0$. The goal is a characterization of $\sigma(T(t))$ involving only properties of the generator. By the rescaling procedure A-I,3.1 we may assume $\lambda = 1$ and $t = 2\pi$.

From the Spectral Inclusion (see)Proposition 6.2) we know that $1 \in \varrho(T(2\pi))$ implies $\mathrm{i}\mathbb{Z} \subset \varrho(A)$. As seen in many examples the converse does not hold and we are now looking for additional conditions. Henceforth we assume $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ and define for $k \in \mathbb{Z}$

$$Q_k := \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ks} T(s)\,\mathrm{d}s = \frac{1}{2\pi}(1 - T(2\pi))R(\mathrm{i}k, A) \tag{7.7}$$

(cf. Formula A-I, (3.1)).

Our approach is based on Fejér's Theorem (for Banach space valued functions). Let us recall this result. Suppose $\xi\colon [0, 2\pi] \to E$ is a continuous function and let $\xi_k := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ks}\xi(s)\,\mathrm{d}s$ be its k-th Fourier coefficient. Then the Fourier series is Césaro summable to ξ in every point $t \in (0, 2\pi)$. Moreover one has

$$\frac{1}{2}(\xi(0) + \xi(2\pi)) = C_1\text{-}\sum_{k\in\mathbb{Z}} \xi_n := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (\sum_{k=-n}^{n} \xi_k)\,. \tag{7.8}$$

This result enables us to prove the following proposition.

Proposition 7.8 *Let $(T(t))_{t\geqslant 0}$ be a semigroup on a Banach space E and denote its generator by A. Then the following conditions are equivalent*

(a) $1 \in \varrho(T(2\pi))$,
(b) $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ *and the series* $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)f$ *is Césaro-summable for every* $f \in E$,
(c) $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ *and the series* $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)Q_k f$ *is Césaro-summable for every* $f \in E$.

Proof (a) $\Rightarrow$ (b) The Spectral Inclusion (see)Proposition 6.2) implies $\mathrm{i}\mathbb{Z} \subset \varrho(A)$. By (7.7) we have $R(\mathrm{i}k, A) = 2\pi \cdot (1 - T(2\pi))^{-1} Q_k$. Since $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable (towards $\frac{1}{2}(f + T(2\pi)f)$) (see (7.8), it follows that $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)f$ is Césaro-summable as well.

(b) $\Leftrightarrow$ (c) If we use A-I,(3.1) and integrate by parts, we obtain

$$\begin{aligned} R(\mathrm{i}k, A)Q_k f &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ks} T(s)R(\mathrm{i}k, A)f\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left[R(\mathrm{i}k, A)f - \int_0^s \mathrm{e}^{-\mathrm{i}kt} T(t)f\,\mathrm{d}t\right] \mathrm{d}s \\ &= R(\mathrm{i}k, A)f - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt}(2\pi - t)T(t)f\,\mathrm{d}t\,. \end{aligned}$$

Fejér's theorem ensures that $\sum_{k\in\mathbb{Z}} (1/2\pi) \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt}(2\pi - t)T(t)f\,\mathrm{d}t$ is Césaro summable. Hence $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)Q_k f$ is Césaro-summable if and only if $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)f$ is.

(b) $\Rightarrow$ (a) We have $Q_k = \frac{1}{2\pi}(1 - T(2\pi))R(\mathrm{i}k, A)$. Furthermore we know by (7.7) and (7.8) that $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable towards $\frac{1}{2}(f + T(2\pi)f)$. If we define $S\colon E \to E$ by $Sf := \frac{1}{2}f + \frac{1}{2\pi} \cdot C_1\text{-}\sum_k R(\mathrm{i}k, A)f$, then we have

$$\begin{aligned} (1 - T(2\pi))Sf &= \frac{1}{2}(f - T(2\pi)f) + \frac{1}{2\pi} \cdot C_1\text{-}\sum_k (1 - T(2\pi))R(\mathrm{i}k, A)f \\ &= \frac{1}{2}(f - T(2\pi)f) + \frac{1}{2}(f + T(2\pi)f) = f\,. \end{aligned}$$

Since S commutes with $T(2\pi)$, it follows that S is the inverse of $(1 - T(2\pi))$ thus $1 \in \varrho(T(2\pi))$. □

Based on the equivalence of (a) and (b), one can state a characterization of the spectrum of $T(t)$ in terms of the generator and its resolvent only. However, in general it is difficult to verify the summability condition stated in (b).

In Hilbert spaces the boundedness of the resolvents will suffice (see Theorem 7.10 below).

Lemma 7.9 *Let $(T(t))_{t\geqslant 0}$ be a semigroup on some Hilbert space H and assume $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ for the generator A. Then we have*

(i) $(Q_k f)_{k\in\mathbb{Z}} \subset \ell^2(H)$ for every $f \in H$, and
(ii) if $\sup_{k\in\mathbb{Z}} \|R(\mathrm{i}k, A)\| < \infty$, then $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A) f_k$ is summable whenever $(f_k)_{k\in\mathbb{Z}} \in \ell^2(H)$.

Proof (i) We consider the Hilbert space $L^2([0, 2\pi], H)$ and obtain

$$\begin{aligned}
0 &\leqslant \left\| T(\cdot)f - \sum_{k=-n}^{n} Q_k f \cdot \mathrm{e}^{\mathrm{i}k\cdot} \right\|^2 \\
&= \int_0^{2\pi} \|T(s)f\|^2 \,\mathrm{d}s - \int_0^{2\pi} \sum_{k=-n}^{n} (T(s)f|\mathrm{e}^{\mathrm{i}ks} Q_k f)\,\mathrm{d}s - \\
&\quad \int_0^{2\pi} \sum_{k=-n}^{n} (\mathrm{e}^{\mathrm{i}ks} Q_k f|T(s)f)\,\mathrm{d}s + \int_0^{2\pi} \left(\sum_{k=-n}^{n} \mathrm{e}^{\mathrm{i}ks} Q_k f | \sum_{\ell=-n}^{n} \mathrm{e}^{i\ell s} Q_\ell f \right) \mathrm{d}s \\
&= \int_0^{2\pi} \|T(s)f\|^2 \,\mathrm{d}s - 2\pi \sum_{k=-n}^{n} \|Q_k f\|^2 \ (\text{use } (7.5))\ .
\end{aligned}$$

It follows that $\sum_{k\in\mathbb{Z}} \|Q_k f\|^2 \leqslant \frac{1}{2\pi} \int_0^{2\pi} \|T(s)f\|^2 \,\mathrm{d}s < \infty$.

(ii) Fix $\lambda > 0$ sufficiently large and set

$$g_k := (1 + \lambda R(\mathrm{i}k, A)) f_k,\ k \in \mathbb{Z}\,.$$

Using the resolvent equation and then (A-I,(3.1)), we obtain

$$R(\mathrm{i}k, A) f_k = R(\lambda + \mathrm{i}k, A) g_k = [1 - \mathrm{e}^{-2\pi\lambda} T(2\pi)]^{-1} \int_0^{2\pi} \mathrm{e}^{-\lambda s} \mathrm{e}^{-\mathrm{i}ks} T(s) g_k \,\mathrm{d}s\,.$$

This yields for every finite subset N of $\mathbb{Z}$ that

$$\left\| \sum_{k\in N} R(\mathrm{i}k, A) f_k \right\| \leqslant \|(1 - \mathrm{e}^{-2\pi\lambda} T(2\pi))^{-1}\| \cdot \int_0^{2\pi} \|T(s)\| \left\| \sum_{k\in N} \mathrm{e}^{-\mathrm{i}ks} g_k \right\| \mathrm{d}s$$

$$\leqslant \|(1 - \mathrm{e}^{-2\pi\lambda} T(2\pi))^{-1}\| \cdot \left(\int_0^{2\pi} \|T(s)\|^2 \,\mathrm{d}s \right)^{1/2} \cdot \left(\int_0^{2\pi} \left\| \sum_{k\in N} \mathrm{e}^{-\mathrm{i}ks} g_k \right\|^2 \mathrm{d}x \right)^{1/2}$$

$$= c(\sum_{k\in N} \|g_k\|^2)^{1/2} \leqslant c(1+\lambda M)\left(\sum_{k\in N} \|f_k\|^2\right)^{1/2} .$$

Here $c := \|(1-\mathrm{e}^{-2\pi\lambda}T(2\pi))^{-1}\|\cdot\left(\int_0^{2\pi} \|T(s)\|^2\,\mathrm{d}s\right)^{1/2}$ and $M := \sup_{k\in\mathbb{Z}} \|R(\mathrm{i}k, A)\|$.□

Theorem 7.10 *Let A be the generator of a semigroup $(T(t))_{t\geqslant 0}$ on some Hilbert space H. Then the following form of the spectral mapping theorem is valid.*

$$\sigma(T(t)) \setminus \{0\} = \{\mathrm{e}^{\lambda t}\colon \text{ either } \mu_k := \lambda + 2\pi\mathrm{i}k/t \in \sigma(A) \text{ for some } k \in \mathbb{Z} \\ \text{or } (\|R(\mu_k, A)\|)_{k\in\mathbb{Z}} \text{ is unbounded}\} .$$

Proof If $\mathrm{e}^{\lambda t} \notin \sigma(T(t))$, it follows from the spectral inclusion theorem that $\mu_k \notin \sigma(A)$ for every $k \in \mathbb{Z}$ and from Formula (3.1) in A-I, that $\|R(\mu_k, A)\|$ is bounded. For the converse inclusion it suffices to assume $t = 2\pi$ and $\lambda = 0$ (use the rescaling procedure A-I,3.1). Assuming that $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ and $\|R(\mathrm{i}k, A)\|$ is bounded, then $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)Q_k f$ is summable by Lemma 7.9. Since every summable series is Cesàro-summable, condition (c) of Proposition 7.8 is satisfied and we conclude $1 \in \varrho(T(2\pi))$. □

As an immediate consequence we obtain an interesting characterization of the growth bound ω_0 of semigroups on Hilbert spaces.

Corollary 7.11 *The growth bound of a semigroup $(T(t))_{t\geqslant 0}$ on a Hilbert space H satisfies*

$$\omega_0 = \inf\{\lambda \in \mathbb{R}\colon \lambda + \mathrm{i}\mathbb{R} \subset \varrho(A) \text{ and } \|R(\lambda + \mathrm{i}\mu, A)\| \text{ is bounded for } \mu \in \mathbb{R}\} . \quad (7.9)$$

The Example 1.3 above in combination with C-III,Corollary 1.3 shows that (7.9) is not valid in arbitrary Banach spaces.

NOTES

Section 1: It was already known to **?**] that for strongly continuous semigroups $(T(t))_{t\geqslant 0}$ with generator A the spectral mapping theorem "$\sigma(T(t)) = \exp(t\sigma(A))$" may be violated in various ways [l.c.,Sec.23.16]. The simple Examples 1.3 and 1.4 are due to Wolff (see **?**]) and **?**]. A more sophisticated example of a positive group with "$s(A) < \omega_0(A)$" is given in **?**].

Section 2: In Definition 2.1 we define the residual spectrum of A in such a way that it coincides with the point spectrum of the adjoint A' (see Proposition 2.2.(ii)). It therefore differs slightly from the one used, e.g., by **?**]. The spectral mapping theorem for the resolvent (Theorem 2.5) is well known and can, e.g., be deduced from Lemma 9.2 and Theorem 3.11 of Chap.VII in **?**].

Section 3: The general theory of spectral decompositions can be found in **?**], Chap.III,§ 6.4]. Applications to isolated singularities like 3.6 are discussed extensively in [l.c., Chap. III, §6.5] and **?** , Chap.VIII, Sec.8]. There are many ways to introduce an "essential spectrum" (see the footnote on page 243 of **?**]). However each one yields the same "essential spectral radius".

Section 4: These results are taken from **?**] and **?**].

Section 5: Periodic semigroups are studied explicitly in **?**], but most of the results of this section seem to be well known.

Section 6: The Spectral Inclusion (see Proposition 6.2) and the Spectral Mapping Theorem 6.6 for eventually norm continuous semigroups date back to **?**]. **?**] gives an elegant proof using Banach algebra techniques. Tensor products of operators and their spectral theory have been studied by Ichinose and others (see Chap. XIII.9 of **?**]). The spectral mapping theorem in Corollary 6.8 generalizes Theorem XIII.35 of **?**] (see also **?**]).

Section 7: Matrix valued multiplication semigroups appear as solution, via Fourier transformation, of systems of partial differential equations. Kreiss initiated a systematic investigation (see, e.g., **?**], **?**], **?**]) and the Weak Spectral Mapping Theorem 7.2 must have been known to him. The direct proof of the Weak Spectral Mapping Theorem 7.4 for polynomially bounded groups seems to be new. The result can also be deduced from the theory of spectral subspaces of group representations (see, e.g., **?**]), since the Arveson spectrum of a strongly continuous one-parameter group can be identified with the spectrum of the generator (see **?**]). The final part of this section is taken from **?**] and yields a new approach to Gearhart's characterization of the spectrum of semigroups on Hilbert spaces **?**]. Different proofs can be found in **?**], **?**] and **?**].

Part B
Positive Semigroups on Spaces $C_0(X)$

Chapter B-I
Basic Results on $C_0(X)$

This part of the book is devoted to one-parameter semigroups of operators on spaces of continuous functions of type $C_0(X)$. Such spaces are Banach lattices of a very special kind. We treat this case separately since we feel that an intermingling with the abstract Banach lattice situation considered in Part C would have made it difficult to appreciate the easy accessibility and the pilot function of methods and results available here. In this chapter we want to fix the notation we are going to use and to collect some basic facts about the spaces we are considering.

If X is a locally compact topological space, then $C_0(X)$ denotes the space of all continuous complex-valued functions on X which vanish at infinity, endowed with the supremum-norm. If X is compact, then any continuous function on X "vanishes at infinity" and $C_0(X)$ is the space of all continuous functions on X. We often write $C(X)$ instead of $C_0(X)$ in this situation.

We sometimes study real-valued functions and write the corresponding real spaces as $C_0(X,\mathbb{R})$ and $C(X,\mathbb{R})$, and the notations $C_0(X,\mathbb{C})$ and $C(X,\mathbb{C})$ are used if there might be confusion between both cases.

1 Algebraic and Order-Structure: Ideals and Quotients

Any space $C_0(X)$ is a commutative C*-algebra under the sup-norm and the pointwise multiplication, and by the *Gelfand Representation Theorem* any commutative C*-algebra can, on the other hand, be canonically represented as an algebra $C_0(X)$ on a suitable locally compact space X. The algebraic notions of ideal, quotient, homomorphism are well known and need not be explained further.

Another natural and important structure of $C_0(X)$ is the *pointwise* ordering, under which $C_0(X,\mathbb{R})$ is a (real) Banach lattice and $C_0(X,\mathbb{C})$ a complex Banach lattice in the sense explained in Chapter C-I. Concerning the order structure of $C_0(X)$ we use the following notations. For a function f in $C_0(X,\mathbb{R})$

1. A function f is called *positive*, $f \geqslant 0$, if $f(t) \geqslant 0$ for all $t \in X$,
2. We write $f > 0$ if f is positive but does not vanish identically,

3. We call f *strictly positive* if $f(t) > 0$ for all $t \in X$.

The notion of an order ideal explained in Chapter C-I applies of course to the Banach lattices $C_0(X)$ and might give rise to confusion with the corresponding algebraic notion.

However, since we are mainly considering closed ideals and since a closed linear subspace I of $C_0(X)$ is a lattice ideal if and only if I is an algebraic ideal, we may and will simply speak of closed ideals without specifying whether we think of the algebraic or the order theoretic meaning of this word.

An important and frequently used characterization of these objects is the following: A subspace I of $C_0(X)$ is a closed ideal if and only if there exists a closed subset A of X such that a function f belongs to I if and only if f vanishes on A. The set A is of course uniquely determined by I and is called the *support* of I. If $I = I_A$ is a closed ideal with support A, then I_A is naturally isomorphic to $C_0(X \backslash A)$ and the quotient $C_0(X)/I$ (under the natural quotient structure) is again a Banach algebra and a Banach lattice that can be identified canonically (via the map $f + I \to f_{|A}$) with $C_0(A)$.

2 Linear Forms and Duality

The *Riesz Representation Theorem* asserts that the dual of $C_0(X)$ can be identified in a natural way with the space of bounded regular Borel measures on X. While there is no natural algebra structure on this dual, the dual ordering (see Chapter C-I) makes $C_0(X)'$ into a Banach lattice. We will occasionally make use of the order structure of $C_0(X)'$, but since at least its measure theoretic interpretation is supposed to be well-known, it may suffice here to refer to Chapter C-I, Sections 3 and 7, for a more detailed discussion and to recall only some basic notations here.

If μ is a linear form on $C_0(X, \mathbb{R})$, then

$\mu \geqslant 0$ means $\mu(f) \geqslant 0$ for all $f \geqslant 0$; μ is then called *positive*,
$\mu > 0$ means that μ is positive but does not vanish identically,
$\mu \gg 0$ means that $\mu(f) > 0$ for any $f > 0$; μ is then called *strictly positive*.

If μ is a linear form on $C_0(X, \mathbb{C})$, then μ can be written uniquely as $\mu = \mu_1 + i\mu_2$ where μ_1 and μ_2 map $C_0(X, \mathbb{R})$ into $\mathbb{R}$ (decomposition into *real* and *imaginary parts*). We call μ positive (strictly positive) and use the above notations if $\mu_2 = 0$ and μ_1 is positive (strictly positive). We point out that strictly positive linear forms need not exist on $C_0(X)$, but if X is separable, then a strictly positive linear form is obtained by summing a suitable sequence of point measures.

The coincidence of the notions of a closed algebraic and a closed lattice ideal in $C_0(X)$ has of course its effect on the algebraic and the lattice theoretic notions of a homomorphism. The case of a homomorphism into another space $C_0(Y)$ will be discussed below.

As for homomorphisms into the scalar field, we have essentially coincidence between the algebraic and the order theoretic meaning of this word, more exactly:

A linear form $\mu \neq 0$ on $C_0(X)$ is a lattice homomorphism if and only if μ is, up to normalization, an algebra homomorphism (algebra homomorphisms $\neq 0$ must necessarily have norm 1). Since the algebra homomorphisms $\neq 0$ on $C_0(X)$ are known to be the point measures (denoted by δ_t) on X and since on the other hand μ is a lattice homomorphism if and only if $|\mu(f)|$ equals $\mu(|f|)$ for all f, it follows that this latter condition on μ is equivalent to $\mu = \alpha\delta_t$ for a suitable t in X and a positive real number α.

This can be summarized by saying that X can be canonically identified, via the map $t \longrightarrow \delta_t$, with the subset of the dual $C_0(X)'$ consisting of the non-zero algebra homomorphisms, which is also the set of all normalized lattice homomorphisms. This identification is a topological isomorphisms with respect to the weak*-topology of $C_0(X)'$.

3 Linear Operators

A linear mapping T from $C_0(X,\mathbb{R})$ into $C_0(Y,\mathbb{R})$ is called

positive (notation: $T \geqslant 0$) if Tf is a positive whenever f is positive,
lattice homomorphism if $|Tf| = T|f|$ all f,
Markov-operator if X and Y are compact and T is a positive operator mapping 1_X to 1_Y.

In the case of complex scalars, T can be decomposed into real and imaginary parts. We call T positive in this situation if the imaginary part of T is $= 0$ and the real part is positive. The terms *Markov operator* and *lattice homomorphism* are defined as above. Note that a complex lattice homomorphism is necessarily positive, and that the *complexification* of a real lattice homomorphism is a complex lattice homomorphism. Positive Operators are always continuous.

Note that the adjoint of a Markov operator T maps positive normalized measures into positive normalized measures while the adjoint of an algebra homomorphism (lattice homomorphism) maps point measures into (multiples of) point measures. Therefore the adjoint of a Markov lattice homomorphism as well as the adjoint of an algebra homomorphism induces a continuous map φ from Y (viewed as a subset of the weak dual $C(Y)'$) into X (viewed as a subset of $C(X)'$).

This mapping φ determines T in a natural and unique way, so that the following are equivalent assertions on a linear mapping T from a space $C(X)$ into a space $C(Y)$.

(a) T is a Markov lattice homomorphism.
(b) T is a Markov algebra homomorphism.
(c) There exists a continuous map φ from Y into X such that $Tf = f \circ \varphi$ for all $f \in C(X)$.

If T is, in addition, bijective, then the mapping φ in (c) is a homeomorphism from Y onto X. This characterization of homomorphisms carries over mutatis mutandis to situations where the conditions on X, Y or T are less restrictive. For later reference we explicitly state the following.

(i) Let K be compact. Then $T \in \mathrm{L}C(K)$ is a lattice homomorphism if and only if there is a mapping φ from K into K and a function $h \in C(K)$ such that $Tf(s) = h(s)f(\varphi(s))$ holds for all $s \in K$. The mapping φ is continuous in every point t with $h(t) \neq 0$.

(ii) Let X be locally compact and $T \in \mathrm{L}C_0(X)$. Then T is a lattice isomorphism if and only if there is a homeomorphism φ from X onto X and a bounded continuous function h on X such that $h(s) \geqslant \delta > 0$ for all s and $Tf(s) = h(s)f(\varphi(s))$ $(s \in X)$. Moreover, T is an algebraic $*$-isomorphism if and only if T is a lattice isomorphism and the function h above is $\equiv 1$.

Notes

For the representation theory of commutative C*-algebras we refer to **?**]. This and the other mentioned properties like algebraic ideals, their connections with closed sets, the representation of latice or algebraic homomorpism etc. we refer to the excellent book **?**].

Chapter B-II
Positive Semigroups on $C_0(X)$

It lies in the very nature of the theory of one-parameter semigroups that frequently an operator A is known to be a generator but the semigroup is not known explicitly. Thus, since the semigroup is uniquely determined by the generator, it is a central task in the theory to express properties of the semigroup in terms of its generator. In this chapter we do this for two properties. We characterize generators of positive semigroups and generators of lattice semigroups.

In Section 1 we consider a semigroup $(T(t))_{t\geqslant 0}$ on the real space $C(K)$ (K compact). It is shown that the semigroup consists of positive operators if and only if its generator satisfies a positive minimum principle (P). Even without assuming a priori that A is a generator the positive minimum principle has strong consequences. Together with a range condition it implies that A is a generator (of a positive semigroup). Moreover, we show that for a densely defined operator A to be the generator of a positive semigroup it is already sufficient that the resolvent $R(\lambda, A)$ of A exists and is positive for all sufficiently large real λ. For all these results it is essential to assume that K is compact. Concerning the characterization of positive semigroups on $C_0(X)$ (X locally compact, non-compact) we follow a completely different line and will treat this case in the context of general Banach lattices in Chapter C-II.

A special class of positive semigroups are lattice semigroups; i.e., semigroups of lattice homomorphisms. We show in Section 2 that $(T(t))_{t\geqslant 0}$ is a lattice semigroup if and only if its generator A satisfies an identity (K), the so-called Kato's Equality (Theorem 2.5).

We refer to Chapter C-II for a discussion of this identity and classical analogs for the Laplacian due to **?**].

After the abstract characterization in Section 2 we show that every continuous semiflow on X together with a cocycle defines a lattice semigroup in a canonical way, and on $C(K)$, every lattice semigroup can be so represented. This furnishes a wide class of examples. Furthermore, positive one-parameter groups on $C_0(X)$ (which form a particular type of lattice semigroups) are discussed. Their generators are similar to a derivation perturbated by a multiplication operator (Section 3).

1 Generators of Positive Semigroups on $C(K)$

Let X be a locally compact space. Throughout this section we denote by $C_0(X)$ the space of all real-valued continuous functions on $C_0(X)$ which vanish in infinity. Recall that a semigroup $(T(t))_{t\geqslant 0}$ on $C_0(X)$ is called *positive* if $T(t) \geqslant 0$ for all $t \geqslant 0$. It is easy to describe the positivity of $(T(t))_{t\geqslant 0}$ in terms of the resolvent $R(\lambda, A)$ of its generator A because of the close relation between these two objects. In fact, the resolvent is expressed by the semigroup by

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)\,\mathrm{d}t \quad (\lambda > \omega(A)); \tag{1.1}$$

and conversely, the semigroup by the resolvent via the formula

$$T(t) = \lim_{n\to\infty} (n/tR(n/t, A))^n \quad \text{strongly} \tag{1.2}$$

(cf. A-II, Proposition 1.10). So we obtain the following.

Proposition 1.1 *Let $(T(t))_{t\geqslant 0}$ be a semigroup with generator A. The semigroup is positive if and only if $R(\lambda, A) \geqslant 0$ for all sufficiently large real λ.*

It is more difficult and more interesting to characterize the positivity of the semigroup by intrinsic conditions on the generator. This is the purpose of this section. As a first orientation we consider bounded generators. We need the following lemma.

Lemma 1.2 *Let X be a locally compact space, $x \in X$ and μ a regular bounded Borel measure on X such that $\mu(\{x\}) = 0$. Then $\mu \geqslant 0$ if and only if $\langle f, \mu\rangle \geqslant 0$ for all $f \in C_0(X)_+$ satisfying $f(x) = 0$.*

We omit the easy proof.

Theorem 1.3 *Let X be locally compact and A be a bounded operator on $C_0(X)$. The following assertions are equivalent.*

(a) $e^{tA} \geqslant 0$ $(t \geqslant 0)$.
(b) For $0 \leqslant f \in C_0(X)$ *and* $x \in X$, $f(x) = 0$ *implies* $(Af)(x) \geqslant 0$.
(c) $A + \|A\| Id \geqslant 0$.

Proof (a) implies (b) . Let $f \in C_0(X)_+$ and $x \in X$ such that $f(x) = 0$. Then

$$\begin{aligned}(Af)(x) &= \lim_{t\to 0} 1/t((e^{tA} f(x) - f(x)) \\ &= \lim_{t\to 0} 1/t((e^{tA} f(x)) \geqslant 0\end{aligned}$$

(b) implies (c) . Let $x \in X$. We have to show that $(Af)(x) + \|A\| f(x) \geqslant 0$ for all $f \in C_0(X)$. Let $A'\delta_x = \mu + c\delta_x$ where $\mu \in M(X)$ such that $\mu(\{x\}) = 0$ and $c \in \mathbb{R}$. We claim that $\mu \geqslant 0$. Let $0 \leqslant f \in C_0(X)$ such that $f(x) = 0$.

Then $\langle f, \mu\rangle = \langle f, A'\delta_x\rangle = (Af)(x) \geqslant 0$ by (b) . Thus $\mu \geqslant 0$ by Lemma 1.2. Moreover, $|c| = \|c\delta_x\| \leqslant \|c\delta_x + \mu\| = \|A'\delta_x\| \leqslant \|A\|$. Hence, for $f \in C_0(X)_+$, $(Af)(x) + \|A\| f(x) = \langle f, A'\delta_x + \|A\|\delta_x\rangle = \langle f, \mu + (c + \|A\|)\delta_x\rangle \geqslant 0$. This shows (b) to hold.

(c) implies (a). We have $e^{tA} = e^{-t\|A\|}e^{t(A+\|A\|)} \geqslant e^{-t\|A\|}Id$ for all $t \geqslant 0$. □

Example 1.4 a) Let B be a positive operator on $C_0(X)$ and $m\colon X \longrightarrow \mathbb{R}$ be a continuous and bounded mapping. Let $Af = Bf - m \cdot f$ $(f \in C_0(X))$. Then $e^{tA} \geqslant 0$ for all $t \geqslant 0$.

b) Let A be an $n \times n$ - matrix. Then $e^{tA} \geqslant 0$ for all $t \geqslant 0$ if and only if $a_{ij} \geqslant 0$ for $i \neq j$. This is the linear version of Kamke's theorem (see **?**]).

Now we come to the actual subject of this section, the characterization of strongly continuous positive semigroups on $C(K)$. Here K denotes a compact space and $C(K)$ the space of all real-valued continuous functions on K. It will be essential that K is compact for all what follows since it will be needed that the positive cone of $C(K)$ has interior points.

We reformulate condition (b) of Theorem 1.3 for unbounded operators.

Definition 1.5 An (unbounded) operator A on $C(K)$ is said to satisfy the *positive minimum principle* if

(P) for every $0 \leqslant f \in D(A)$ and $x \in K$,
$f(x) = 0$ implies $(Af)(x) \geqslant 0$

Our next theorem shows that the positive minimum principle characterizes the positivity of the semigroup; and in fact, the proof is very elementary. Using more involved arguments we will later prove a much stronger result (Theorem 1.13).

Theorem 1.6 *Let A be the generator of a strongly continuous semigroup on $C(K)$. Then the semigroup is positive if and only if the generator A satisfies the positive minimum principle (P).*

Proof The necessity of the condition is proved as "(a) implies (b) " in Theorem 1.3. Assume that (P) holds. We claim that $R(\lambda, A) \geqslant 0$ for sufficiently large real λ. (This implies the positivity of the semigroup by Proposition 1.1). Let $s := \inf\{\lambda \in \mathbb{R} : [\lambda, \infty) \subset \varrho(A)\}$. Then $s \leqslant \omega(A) < \infty$. Let $0 \ll u \in C(K)$. Then $\lambda_0 := \inf\{\lambda > s : R(\mu, A)u \gg 0 \text{ for all } \mu \in (\lambda, \infty)\} < \infty$ since $\lim_{\mu\to\infty} \mu R(\mu, A)u = u$.

We claim that $\lambda_0 = s$.

In fact, if this is not true, then $[\lambda_0, \infty) \subset \varrho(A)$ and $R(\lambda_0, A)u \geqslant 0$ but $R(\lambda_0, A)u$ is not strictly positive. Consequently there exists $x \in K$ such that $(R(\lambda_0, A)u)(x) = 0$. Then (P) implies that $A(R(\lambda_0, A)u)(x) \geqslant 0$. Hence, $0 < u(x) = \lambda_0(R(\lambda_0, A)u)(x) - A(R(\lambda_0, A)u)(x) \leqslant 0$, a contradiction. We have shown that $R(\lambda, A)u \gg 0$ for all $u \gg 0$ and $\lambda > s$. Since $\{u \in C(K) : u \gg 0\}$ is dense in $C(K)_+$, it follows that $R(\lambda, A) \geqslant 0$ for all $\lambda > s$. □

Remark 1.7 The proof of Theorem 1.6 shows that for the generator A of a positive semigroup on $C(K)$, $R(\lambda, A)u \gg 0$ whenever $0 \ll u \in C(K)$ and $[\lambda, \infty) \subset \varrho(A)$. In particular, $R(\lambda, A) \geqslant 0$ whenever $[\lambda, \infty) \subset \varrho(A)$.

If A is a generator, then the positivity of the resolvent $R(\lambda, A)$ for large real λ implies the positivity of the semigroup (by Proposition. 1.1. On $C(K)$ much more is true. Even if A is not supposed to be a generator, the existence and positivity of $R(\lambda, A)$ for large real λ implies that A is a generator (of a positive semigroup). This is surprising, because it means that in the case when the resolvent is positive, the norm condition on the resolvent $\sup\{|(\lambda-w)^n R(\lambda, A)^n| : n \in \mathbb{N}, \lambda \geqslant 0\} < \infty$ which appears in the Hille-Yosida Theorem (A-II, Theorem 1.7) is automatically fulfilled.

Theorem 1.8 *Let K be compact and A be a densely defined operator on $C(K)$. Suppose that there exists $w \in \mathbb{R}$ such that $[w, \infty) \subset \varrho(A)$ and $R(\lambda, A) \geqslant 0$ for all $\lambda \geqslant w$. Then A is the generator of a strongly continuous positive semigroup. Moreover,*

$$\omega(A) \leqslant w. \tag{1.3}$$

Proof a) Assume that $w < 0$. Denote by 1 the constant-1-function. Let $u = R(0, A)1$. We claim that $u \gg 0$. If not, then there exists $x \in K$ such that $u(x) = 0$. Let $f \in C(K)$. Then $|f| \leqslant \|f\|1$. Consequently, $|R(0, A)f| \leqslant R(0, A)|f| \leqslant \|f\|R(0, A)1 = \|f\|u$. Hence $(R(0, A)f)(x) = 0$ for all $f \in C(K)$. Since $D(A) = R(0, A)C(K)$, it follows that $D(A)$ is not dense, a contradiction. Define $\|f\|_0 = \inf\{\lambda > 0 : |f| \leqslant \lambda u\} = \|f/u\|_\infty$. Then $\|\cdot\|_0$ is an equivalent norm on $C(K)$. Moreover, $\|f\|_0 \leqslant 1$ if and only if $f \in [-u, u]$. By the resolvent equation we have $\lambda R(\lambda, A)u = \lambda R(\lambda, A)R(0, A)1 = R(0, A)1 - R(\lambda, A)1 \leqslant R(0, A)1 = u$ for all $\lambda \geqslant 0$. This implies that $\lambda R(\lambda, A)$ is contractive for the norm $\|\cdot\|_0$. Thus by the Hille-Yosida Theorem A is the generator of a semigroup which is contractive with respect to the norm $\|\cdot\|_0$ and so is bounded with respect to the supremum norm on $C(K)$.

b) If w is arbitrary, let $\lambda > w$ and consider $A - \lambda$. Then $[w - \lambda, \infty) \subset \varrho(A - \lambda)$ and $R(\mu, A - \lambda) = R(\mu + \lambda, A) \geqslant 0$ for all $\mu \in [w - \lambda, \infty)$. Thus by a), $A - \lambda$ is the generator of a bounded positive semigroup. Consequently, A is a generator as well and $\omega(A) \leqslant \lambda$. □

In Theorem 1.8 it is enough to assume that $R(\lambda_n, A) \geqslant 0$ for some sequence $(\lambda_n) \subset \varrho(A) \cap \mathbb{R}$ satisfying $\lim_{n\to\infty} \lambda_n = \infty$. This follows from the following lemma.

Lemma 1.9 *Let B be an operator on $C(K)$ (more generally, on a Banach lattice). If $\mu_1, \mu_2 \in \varrho(B) \cap \mathbb{R}$ such that $0 \leqslant R(\mu_1, B)$, $0 \leqslant R(\mu_2, B)$ and $\mu_1 < \mu_2$, then $[\mu_1, \mu_2] \subset \varrho(B)$ and*

$$0 \leqslant R(\mu_2, B) \leqslant R(\mu, B) \leqslant R(\mu_1, B) \quad \textit{for all } \mu \in [\mu_1, \mu_2]$$

Proof Let $M := \{\mu \in \varrho(B) \cap [\mu_1, \mu_2] : [\mu, \mu_2] \subset \varrho(B)$ and $R(\lambda, B) \geqslant 0$ for all $\lambda \in [\mu, \mu_2]\}$.

a) The set M is open. In fact, let $\mu \in M$. Then for small $h > 0$ one has $R(\mu - h, B) = \sum_{n=0}^{\infty} h^n R(\mu, B)^{n+1} \geqslant 0$.

b) M is closed. In fact, by the resolvent equation one has for $\mu \in M$, $R(\mu_1, B) - R(\mu, B) = (\mu - \mu_1)R(\mu_1, B)R(\mu, B) \geqslant 0$, hence $R(\mu, B) \leqslant R(\mu_1, B)$. Consequently,

$\operatorname{dist}(\mu, \sigma(B)) \geqslant 1/\|R(\mu, B)\| \geqslant 1/\|R(\mu_1, B)\|$ for all $\mu \in M$. This implies that M is closed.

The assertions a) and b) imply that $M = [\mu_1, \mu_2]$. □

Remark a) The lemma shows in particular that the resolvent of the generator A of a positive semigroup is decreasing on $(s(A), \infty)$.
b) There exists a linear operator B on $\mathbb{R}^n$ such that $R(\mu, B) \geqslant 0$ on some interval $[\mu_1, \mu_2] \subset \varrho(B) \cap \mathbb{R}$ but $(e^{tB})_{t\geqslant 0}$ is not positive (see **?**]).

Remark Theorem 1.8 does not hold in $C_0(X)$, in general. In fact, the operator A on $C_0(0,1]$ given by $Af(x) = f'(x) + \alpha/x f(x)$ $(x \in (0,1])$ with domain $D(A) = \{f \in C^1[0,1] \colon f'(0) = f(0) = 0\}$ where $\alpha \in (0,1)$ satisfies the following: $\varrho(A) = \mathbb{C}$, $R(\lambda, A) \geqslant 0$ for all $\lambda \in \mathbb{R}$. But A is not the generator of a semigroup (even if more general classes than C_0-semigroups are admitted). See **?**] for this example and a general theory of resolvent positive operators. Another example is given by **?**]. □

Next we investigate consequences of the positive minimum principle for a densely defined operator which is not a priori assumed to be a generator. For that we will make use of the theory of half-norms developed in A-II, Section 2.

For $0 \ll u \in C(K)$ let

$$p_u(f) = \inf\{\lambda \in \mathbb{R}_+ \colon f \leqslant \lambda u\} = \sup_{x\in K} \frac{f^+(x)}{u(x)}. \tag{1.4}$$

Then p_u is a strict half-norm on $C(K)$ (see A-II, Section 2). Note that

$$p_u(f)u - f \geqslant 0 \quad (f \in C(K)). \tag{1.5}$$

For $x \in K$, define $\varphi_x \in C(K)'$ by $\langle f, \varphi_x\rangle = f(x)/u(x)$. Let $f \in C(K)$ such that $-f$ is not strictly positive. Then there exists $x \in K$ such that $f(x)/u(x) = p_u(f)$. For such an x we have

$$\varphi_x \in \mathrm{d}p_u(f) \tag{1.6}$$

(see A-II, Section 2 for the definition of the subdifferential $\mathrm{d}p_u$).

Note that for $f \in C(K)$ one has $f \geqslant 0$ if and only if $p_u(-f) \leqslant 0$ (i.e., the half-norm p_u induces the given ordering on $C(K)$ (cf. A-II, Remark 2.8)). As a consequence, every p_u-contractive bounded operator T on $C(K)$ is positive.

Proposition 1.10 *Let A be a densely defined operator on $C(K)$. Then there exists a strictly positive $u \in D(A)$. For any such u the following assertions are equivalent.*

(a) A is p_u-dissipative.
(b) $Au \leqslant 0$ and A satisfies (P).

Proof Since $\{u \in C(K) \colon u \gg 0\}$ is open and non-empty and $D(A)$ is dense, there exists $0 \ll u \in D(A)$.

(a) implies (b). One has $p_u(u) = 1$. Let $x \in K$. It follows from (1.6) that $\varphi_x \in \mathrm{d}p_u(u)$. Since $D(A)$ is dense, it follows from A-II, Theorem 2.7 that A is strictly p_u-dissipative. Hence $\langle Au, \varphi_x \rangle \leqslant 0$. Thus $(Au)(x) \leqslant 0$. We now show (P). Let $0 \leqslant f \in D(A)$ and $x \in K$ such that $f(x) = 0$. We have to show that $(Af)(x) \geqslant 0$. Since $f(x) = 0$ and $p_u(-f) = 0$ we have by (1.6) $\varphi_x \in \mathrm{d}p_u(-f)$. Since A is strictly p_u-dissipative we conclude that $-u(x)(Af)(x) = \langle A(-f), \varphi_x \rangle \leqslant 0$. Hence $(Af)(x) \geqslant 0$.

(b) implies (a). Let $f \in D(A)$. If $p_u(f) = 0$ then $\varphi := 0 \in \mathrm{d}p_u(f)$ and $\langle Af, \varphi \rangle \leqslant 0$. If $p_u(f) > 0$, then there exists $x \in K$ such that $\varphi_x \in \mathrm{d}p_u(f)$. Hence, $0 \leqslant p_u(f)u - f$ and $(p_u(f)u - f)(x) = 0$. It follows from (P) that $p_u(f)(Au)(x) - (Af)(x) \geqslant 0$. Hence $(Af)(x) \leqslant p_u(f)(Au)(x) \leqslant 0$ (by (b); i.e., $\langle Af, \varphi_x \rangle \leqslant 0$. ◻

Corollary 1.11 *Let A be a densely defined operator on $C(K)$. If A satisfies (P) then A is closable and the closure of A satisfies (P) as well.*

Proof (Proof of Corollary 1.11) Let $u \in D(A)$ be strictly positive. Then there exists $\lambda \in \mathbb{R}$ such that $Au \leqslant \lambda u$. The operator $B = A - \lambda$ satisfies (P) as well and $Bu \leqslant 0$. Then by Proposition 1.10, B is p_u-dissipative. Hence B is closable and the closure $\overline{B}$ of B is p_u-dissipative as well (by A-II Proposition 2.9). Then by Proposition 1.10 $\overline{B}$ satisfies (P). Thus A is closable and its closure $\overline{A} = \overline{B} + \lambda$ satisfies (P) as well. ◻

Corollary 1.12 *Let $A\colon C(K) \to C(K)$ be linear. If A satisfies (P) then A is bounded and $A + \|A\| Id \geqslant 0$.*

Proof It follows from Corollary 1.11 that A is closed. Hence A is bounded. Since A satisfies (P), it follows from Theorem 1.3 that $A + \|A\|\mathrm{Id} \geqslant 0$. ◻

The next result is a strengthened form of Theorem 1.6. It is somewhat similar to the Lumer-Phillips Theorem (A-II, Theorem 2.13). Note that, however, in contrast to the condition of dissipativity, $A + w$ satisfies (P) for any $w \in \mathbb{R}$ whenever (P) holds for A. Thus (P) is not a "metric" condition; that is, it does not imply any norm estimate for the semigroup. We also point out that, if $(T(t))_{t \geqslant 0}$ is a positive semigroup on $C(K)$, then in general none of the semigroups $(e^{-wt}T(t))_{t \geqslant 0}$ $(w \in \mathbb{R})$ is contractive (see **?**] or **?**]).

Theorem 1.13 *Let A be a densely defined operator on $C(K)$ which satisfies (P). Then*

$$\lambda_0 := \inf\{\lambda \in \mathbb{R} \colon Au \leqslant \lambda u \text{ for some } 0 \ll u \in D(A)\} < \infty.$$

(i) If $(\lambda - A)D(A)$ is dense for some $\lambda > \lambda_0$, then A is closable and the closure $\overline{A}$ of A is the generator of a positive semigroup.
(ii) If $\lambda - A$ is surjective for some $\lambda > \lambda_0$, then A is the generator of a positive semigroup.

Proof It follows from Proposition 1.10 that $\lambda_0 < \infty$.

Assume that $(\lambda - A)D(A)$ is dense, where $\lambda > \lambda_0$. Let $\lambda_0 < \mu < \lambda$ and $B = A - \mu$. Then B satisfies (P) and $Bu \leqslant 0$ for some strictly positive $u \in D(B) = D(A)$. Thus B is p_u-dissipative by Proposition 1.10 . Moreover, $((\lambda - \mu) - B)D(B)$ is dense. Thus by A-II, Corollary 2.12 the closure $\overline{B}$ of B generates a p_u-contraction semigroup. Hence the closure $\overline{A} = \overline{B} + \mu$ of A generates a positive semigroup of type $\omega(\overline{A}) \leqslant \lambda$. If $(\lambda - A)$ is surjective, then $A = \overline{A}$. ◻

The proof of Theorem 1.13 yields estimates for the growth bound of a positive semigroup (see A-III, (1.3)) which we state explicitly in the next corollary.

Corollary 1.14 *Let A be the generator of a strongly continuous positive semigroup on $C(K)$. Then*

$$-\infty < s(A) = \omega(A) \in \sigma(A). \tag{1.7}$$

Moreover,

$$s(A) = \inf\{\lambda \in \mathbb{R} \colon Au \leqslant \lambda u \text{ for some } 0 \ll u \in D(A)\} \tag{1.8}$$

and

$$s(A) \geqslant \sup\{\mu \in \mathbb{R} \colon Af \geqslant \mu f \text{ for some } 0 < f \in D(A)\}. \tag{1.9}$$

Proof Let $s := \inf\{\lambda \in \mathbb{R} \colon [\lambda, \infty) \subset \varrho(A)\}$. Clearly, $s \leqslant s(A)$. Moreover, by Remark 1.7, $R(\lambda, A) \geqslant 0$ for all $\lambda > s$. Hence it follows from (1.3) that $\omega(A) \leqslant s$. Consequently, $s = s(A) = \omega(A)$.

Next we prove (1.9). Let $0 < f \in D(A)$ such that $Af \geqslant \mu f$. Assume that $\mu > s(A)$. Then $R(\mu, A) \geqslant 0$. Hence, $f = R(\mu, A)(\mu - A)f \leqslant 0$, a contradiction.

Since $D(A)$ is dense, there exists a strictly positive $u \in D(A)$. Then $Au \geqslant \mu u$ for some $\mu \in \mathbb{R}$. Hence, $-\infty < s(A)$ by (1.9). Since $s(A) = s$ it is clear that $s(A) \in \sigma(A)$.

It remains to show (1.8). Let $\lambda > s(A)$ and $u = R(\lambda, A)1$. Then u is strictly positive (by Remark 1.7) and $Au = \lambda u - 1 \leqslant \lambda u$. This proves one inequality in (1.8). Assume now that $u \in D(A)$ is strictly positive such that $Au \leqslant \lambda u$. Then by the proof of Theorem 1.13 we have $\omega(A) \leqslant \lambda$. This proves the other inequality in (1.8). ◻

Remark 1.15 If A has compact resolvent, then by the Krein-Rutmann Theorem there exists a positive eigenvector u of A corresponding to a real eigenvalue. So the equality is valid in (1.9) and the supremum is a maximum. If in addition the semigroup is irreducible (see B-III, Section 3 below), then u is strictly positive and in (1.8) the infimum is attained as well.

Conversely, if in (1.8) the infimum is attained, then $s(A)$ is an eigenvalue.

Example 1.16 Let $A = (a_{ij})$ be an $n \times n$-matrix such that $a_{ij} \geqslant 0$ whenever $i \neq j$ (see Example 1.4b). Then by Corollary 1.14

$$\begin{aligned} s(A) &= \inf\{\lambda \in \mathbb{R} \colon Au \leqslant \lambda u \text{ for some strictly positive } u\} \\ &= \inf_{u \gg 0} \inf\{\lambda \in \mathbb{R} \colon Au \leqslant \lambda u\} \\ &= \inf\{\sup_i \sum_{j=1}^{n} a_{ij} u_j / u_i \colon u \gg 0\}. \end{aligned}$$

This formula is due to **?**] (see also **?** , Chapter I, Exercise 20] and **?**]).

Corollary 1.17 *Let $(T(t))_{t \geqslant 0}$ be a strongly continuous positive semigroup on $C(K)$. Then $T(t)u \gg 0$ for all $u \gg 0$, $t \geqslant 0$.*

Proof Denote by A the generator of $(T(t))_{t\geqslant 0}$. Then by the proof of Theorem 1.13 there exist $u \gg 0$ and $\lambda \in \mathbb{R}$ such that $A - \lambda$ is p_u-dissipative. This implies that $p_u(T(t)f) \leqslant e^{\lambda t} p_u(f)$. Observing that $f \gg 0$ if and only if $p_u(-f) < 0$ the claim follows. □

Remark 1.18 Corollaries 1.14 and 1.17 do not hold on $C_0(X)$. For example, let $X = [0, 1)$ and

$$(T(t)f)(x) = \begin{cases} f(x+t) & \text{if } x+t \leqslant 1 \\ 0 & \text{if } x+t > 0 \end{cases}.$$

Then $(T(t))_{t\geqslant 0}$ is a positive semigroup on $C_0(X)$ and $T(t) = 0$ for all $t \geqslant 1$. The generator A of $(T(t))_{t\geqslant 0}$ has empty spectrum, so that (1.7) is violated. However, it is still true that $s(A) = \omega(A)$ for generators of positive semigroups on $C_0(X)$ (see B-IV, Theorem 1.4).

Remark 1.19 So far, the results of this section do not depend on the lattice structure of $C(K)$. They also hold on an ordered Banach space E with normal cone E_+ which has non-empty interior. We refer to **?**] [Arendt-Chernoff-Kato (1982)] and to **?**] for this more general setting.

Next we apply Theorem 1.13 to prove a result on the multiplicative perturbation of a generator A which is due to **?**] in the case when A is dissipative.

Theorem 1.20 *Let A be the generator of a positive semigroup on $C(K)$ and $m \in C(K)$ be strictly positive. Then the operator $m \cdot A$ given by $(m \cdot A)f = m \cdot (Af)$ on the domain $D(m \cdot A) = D(A)$ is the generator of a positive semigroup. Moreover,*

$$\|m^{-1}\|_\infty^{-1}\, \omega(A) \leqslant \omega(m \cdot A) \leqslant \|m\|_\infty\, \omega(A). \tag{1.10}$$

Proof We can assume that $\|m\|_\infty \leqslant 1$ (in fact, if $B := (m/\|m\|_\infty) \cdot A$ is the generator of a positive semigroup, then by A-I, 3.1 $m \cdot A = \|m\|_\infty B$ also generates a positive semigroup). The assertion of the theorem holds for A if and only if it is valid for $A - w$ ($w \in \mathbb{R}$). So by the proof of Theorem 1.13 we can assume that there exists $0 \ll u \in C(K)$ such that A is p_u-dissipative. We first show the following. Let $0 \ll q \in C(K)$.

$$\text{If } B \text{ is a } p_u\text{-dissipative operator, then } q \cdot B \text{ is } p_u\text{-dissipative.} \tag{1.11}$$

Let $f \in D(q \cdot B) = D(B)$. There exists $x \in K$ such that $\varphi_x \in \mathrm{d}p_u(f)$ (by (1.6)). Hence $\langle Bf, \varphi_x\rangle \leqslant 0$. Consequently, $\langle q \cdot Bf, \varphi_x\rangle = q(x)\langle Bf, \varphi_x\rangle \leqslant 0$.

Next we show,

$$\begin{gathered}\text{if } B \text{ is the generator of a } p_u\text{-contraction semigroup and} \\ 1 \geqslant q \in C(K)_+ \text{ is such that } \|1 - q\|_\infty < 1/2, \\ \text{then } q \cdot B \text{ generates a } p_u\text{-contraction semigroup.}\end{gathered} \tag{1.12}$$

Because of (1.11) we only have to show that $(I - q \cdot B)$ is surjective. Note that $1 \in \varrho(B)$. We have $(Id - q\cdot B) = (Id - B - (q-1)B) = (Id - (q-1)BR(1, B))(Id - B)$.

Thus it suffices to show that $Id - (q-1)BR(1,B)$ is invertible. The norm $\|f\|_u = \max\{p_u(f), p_u(-f)\} = \sup_{x\in K} |f(x)|/u(x)$ is equivalent to the supremum norm. Denote by $\|T\|_u$ the operator norm corresponding to $\|\cdot\|_u$ $(T \in \mathcal{L}(E))$. Then it is enough to show that $\|(q-1)BR(1,B)\|_u = \|(q-1)(R(1,B)-I)\|_u < 1$.

For $r \in C(K)_+$ denote by M_r the multiplication operator given by $M_r f = r \cdot f$. Then $\|M_r\|_u = \sup\{\|r \cdot f\|_u : \|f\|_u \leqslant 1\} = \sup\{\sup_{x\in K} r(x)|f(x)|/u(x) : \|f\|_u \leqslant 1\} \leqslant \|r\|_\infty$. Since B is p_u-dissipative we have $\|R(1,B)\|_u \leqslant 1$ (by A-II, Lemma 2.10). This gives $\|(q-1)(R(1,B)-I)\|_u \leqslant \|M_{(1-q)}\|_u(\|R(1,B)\|_u + 1) \leqslant 2\|1-q\|_\infty < 1$.

The proof of (1.12) is complete.

There exists $k \in \mathbb{N}$ such that $\|1-m^{1/k}\|_\infty < 1/2$. Applying now (1.12) succesively to $B = m^{1/k} \cdot A$ and $q = m^{1/k}$ $(l = 1, \ldots, k-1)$ we obtain that $m \cdot A$ generates a p_u-contraction semigroup (which in particular is positive).

Finally we show (1.10) to hold. Let $0 \ll u \in D(A) = D(m \cdot A)$ and $Au \leqslant \lambda u$. Then $m \cdot Au \leqslant \|m\|_\infty \lambda u$. So (1.8) implies that $\omega(m \cdot A) \leqslant \|m\|_\infty \omega(A)$. This is one part of (1.10). The other part follows from this since $\omega(A) = \omega(m^{-1} \cdot m \cdot A) \leqslant \|m^{-1}\|_\infty \omega(m \cdot A)$. □

In the following lemma a condition (P') is introduced which is dual to the positive minimum principle.

Lemma 1.21 *Let A be the generator of a strongly continuous positive semigroup on $C(K)$. Then*

(P') $f \in C(K)_+$, $0 \leqslant \mu \in D(A')$, $\langle f, \mu\rangle = 0$ implies $\langle f, A'\mu\rangle \geqslant 0$.

Proof $\langle f, A'\mu\rangle = \lim_{t\to 0} \frac{1}{t}\langle T(t)f - f, \mu\rangle = \lim_{t\to 0} \frac{1}{t}\langle T(t)f, \mu\rangle \geqslant 0$. □

Example 1.22 Let $K = [-1,0]$. Let $\alpha \in \mathbb{R}$ and μ be a measure on $[-1,0]$ such that $\mu(\{0\}) = 0$. Define the operator A on $C[-1,0]$ by $Af = f'$ with domain $D(A) = \{f \in C^1[-1,0] : f'(0) = \alpha f(0) + \langle f, \mu\rangle\}$.

Claim: A is the generator of a positive semigroup if and only if $\mu \geqslant 0$.

Proof (Proof of the claim) Assume that A generates a positive semigroup. By the definition of A one has $\delta_0 \in D(A')$ and $A'\delta_0 = \alpha\delta_0 + \mu$. So it follows from (P') that $\langle f, \mu\rangle = \langle f, A'\delta_0\rangle \geqslant 0$ for all $f \in C[-1,0]_+$ such that $f(0) = 0$. By Lemma 1.2 this implies that $\mu \geqslant 0$.

In order to show the converse assume that $\mu \geqslant 0$.

a) We show that A is densely defined. Consider the normed space $F = C^1[-1,0]$ with the supremum norm. Then $\psi : F \to \mathbb{R}$ given by $\psi(f) = f'(0) - \alpha f(0) - \langle f, \mu\rangle$ is a discontinuous linear form on F. Consequently $D(A) = \ker\psi$ is dense in F. Since F is dense in $C[-1,0]$, $D(A)$ is dense in $C[-1,0]$ as well.

b) A satisfies (P) (see Definition 1.5). In fact, let $f \in D(A)_+$ and $x \in [-1,0]$ such that $f(x) = 0$. It is clear that $Af(x) = f'(x) \geqslant 0$ if $x < 0$. But if $f(0) = 0$, then $Af(0) = f'(0) = \langle f, \mu\rangle \geqslant 0$ since $f \in D(A)$.

c) We show that $(\lambda - A)$ is bijective for $\lambda > \alpha + \|\mu\|$. Let $g \in C[-1,0]$. The solutions of the equation $\lambda f - f' = g$ $(f \in C[-1,0])$ are given by $f(x) = e^{\lambda x}[\int_x^0 e^{-\lambda y} g(y)\,\mathrm{d}y + c]$ where $c \in \mathbb{R}$. Moreover, $f \in D(A)$ if and only if

$$c(\lambda-\alpha-\int_{-1}^{0} e^{\lambda x}\,\mathrm{d}\mu(x)) = g(0) + \int_{-1}^{0} e^{\lambda x}\int_{x}^{0} e^{-\lambda y} g(y)\,\mathrm{d}y\,\mathrm{d}\mu(x). \qquad (1.13)$$

If $\lambda > \alpha + \|\mu\|$, then $\lambda - \alpha - \int_{-1}^{0} e^{\lambda x} d\mu(x) \neq 0$ and there exists exactly one $c \in \mathbb{R}$ satisfying (1.13). We have shown that $(\lambda - A)$ is bijective for $\lambda > \alpha + \|\mu\|$.

By Theorem 1.13, it follows from a), b) and c) that A generates a positive semigroup. □

Let us mention in addition that it follows from a) in the proof that $(\alpha + \|\mu\|, \infty) \subset \varrho(A)$, since A is closed. By Remark 1.7 we thus have

$$s(A) \leqslant \alpha + \|\mu\|. \qquad (1.14)$$

Example 1.23 Let $E = C([-1,0],\mathbb{R}^n)$. Then $u \in E$ is given by $u = (u_1,\ldots,u_n)$ where $u_i \in C[-1,0]$ $(i = 1,\ldots,n)$. Let A be defined by $Au = u' = (u_1',\ldots,u_n')$ with domain $D(A) = \{u \in C^1([-1,0],\mathbb{R}^n) : u'(0) = Lu\}$.

Here L is defined by

$$Lu = \begin{pmatrix} L_{11}u_1 + \ldots + L_{1n}u_n \\ \vdots \\ L_{n1}u_1 + \ldots + L_{nn}u_n \end{pmatrix}$$

where $L_{ij} \in C[-1,0]'$ $(1 \leqslant i,j \leqslant n)$. Let $L_{ii} = \alpha_i\delta_0 + \mu_i$ with $\mu_i(\{0\}) = 0$ $(i = 1,\ldots,n)$. Then A generates a positive semigroup if and only if

$$L_{ij} \geqslant 0 \text{ for } i \neq j \text{ and } \mu_i \geqslant 0 \quad (i,j = 1,\ldots,n).$$

bib:1984!a This can be proved in a similar way as the claim in Example 1.22 (see **?**]a).

Example 1.24 Let A on $C[0,1]$ be given by $Af = f''$ with domain $D(A) = \{f \in C^2[0,1] : f'(0) + \alpha f(0) = 0, f'(1) + \beta f(1) = 0\}$, where $\alpha, \beta \in \mathbb{R}$. Then A is the generator of positive semigroup.

Proof The operator A satisfies (P). In fact, let $0 \leqslant f \in D(A)$ and $f(a) = 0$ where $a \in [0,1]$. If $a \in (0,1)$ then $f''(a) \geqslant 0$ since f has a minimum in a. If $a = 0$ then $f'(0) = f'(0) + \alpha f(0) = 0$ since $f \in D(A)$. Consequently, $f(x) = \int_0^x (x-y) f''(y)\,\mathrm{d}y \geqslant 0$ for all $x \geqslant 0$. This implies $f''(0) \geqslant 0$. The argument for $a = 1$ is analoguous. It remains to show that $\mu - A$ is surjective for large real μ. Let $g \in C[0,1]$. Let $\lambda > 0$ and $k = 1/2\lambda[e^{\lambda x}\int_x^1 e^{-\lambda y} g(y)\,\mathrm{d}y - e^{-\lambda x}\int_x^1 e^{\lambda y} g(y)\,\mathrm{d}y]$. Then $k \in C^2[0,1]$ and $\lambda^2 k - k'' = g$. Let $h = ae^{\lambda x} + be^{-\lambda x}$, where $a, b \in \mathbb{R}$. Then $h \in C^2[0,1]$ and $\lambda^2 h - h'' = 0$. Let $f = k + h$. Then $\lambda^2 f - f'' = g$. The condition that $f \in D(A)$ leads to two linear equations in a and b, and it is easy to see that they have a solution $(a,b) \in \mathbb{R}^2$ if $(\lambda+\alpha)(\beta-\lambda) + (\lambda-\alpha)(\lambda+\beta)\exp(\lambda^2) \neq 0$. Thus there exists a solution if λ is large enough, and $(\lambda^2 - A)$ is surjective. □

2 Lattice Semigroups on $C_0(X)$

Throughout this section X denotes a locally compact space and $C_0(X,\mathbb{R})$ (resp., $C_0(X,\mathbb{C})$) the space of all real-valued (resp., complex-valued) continuous functions on X which vanish at infinity. If we do not want to specify the field we simply write $C_0(X)$.

Recall from B-I, Section 3 that a linear bounded operator T on $C_0(X)$ is positive if and only if

$$|Tf| \leqslant T|f| \quad \text{for all } f \in C_0(X). \tag{2.1}$$

The operator T is a lattice homomorphism if and only if in (2.1) equality holds; i.e.,

$$|Tf| = T|f| \quad \text{for all } f \in C_0(X). \tag{2.2}$$

Remark 2.1 If T is a lattice homomorphism on $C_0(X,\mathbb{C})$ then T leaves $C_0(X,\mathbb{R})$ invariant and the restriction $T_{\mathbb{R}}$ of T to $C_0(X,\mathbb{R})$ is a lattice homomorphism. Conversely, the linear extension T of a lattice homomorphism $T_{\mathbb{R}}$ on $C_0(X,\mathbb{R})$ to $C_0(X,\mathbb{C})$ is a lattice homomorphism (see B-I, Section 3).

A semigroup $(T(t))_{t\geqslant 0}$ is called *lattice semigroup* if $T(t)$ is a lattice homomorphism for all $t \geqslant 0$. In Section 3 we will give a concrete representation of lattice-semigroups which shows that there is a large variety of examples. This section is devoted to the characterization of lattice semigroups in terms of their generators. The heuristic idea is the following. Let $(T(t))_{t\geqslant 0}$ be a lattice semigroup with generator A. Let $f \in D(A)$ and assume that the modulus function ϑ given by $\vartheta(g) = |g|$ is differentiable at f (in some sense which has to be made precise). Then one expects that a chain rule holds so that $\vartheta(T(t)f) = |T(t)f|$ is differentiable at $t = 0$. Since $|T(t)f| = T(t)|f|$, this implies $|f| \in D(A)$ and $A|f| = \mathrm{d}/\mathrm{d}t_{|t=0}\vartheta(T(t)f) = D_{Af}\vartheta(f)\,\mathrm{d}/\,\mathrm{d}t_{|t=0}T(t)f = (D_{Af}\vartheta(f)Af)$ (where the precise meaning of $(D_{Af}\vartheta(f))Af$ depends on the chain rule which we will have to establish). So we obtain an identity for the generator A which corresponds exactly to the lattice property $|T(t)f| = T(t)|f|$ of the semigroup. We will see in C-II, Section 5 that in a Banach lattice with order continuous norm the above argument is rigorous (for all $f \in D(A)$). On $C_0(X)$ we have to use a weak form of the argument and $|f| \in D(A)$ only holds for special $f \in D(A)$ (see Corollary 2.8).

We start by investigating differentiability of the modulus and by establishing a chain rule. For later use we formulate the following definition and proposition for a general Banach space G even though only $G = \mathbb{C}$ will be considered in this section.

Definition 2.2 Let G be a Banach space and $\vartheta\colon G \to G$ a mapping. Let $f \in G$, $u \in G$. Then ϑ is called *right-sided Gateaux differentiable at f in direction u* if

$$D_u\vartheta(f) := \lim_{t\to 0} 1/t(\vartheta(f+tu) - \vartheta(f)) \text{ exists.} \tag{2.3}$$

The mapping ϑ is *right-sided Gateaux differentiable at f* if $D_u\vartheta(f)$ exists for all directions $u \in G$; and if ϑ is right-sided Gateaux-differentiable at every point $f \in G$ then we call ϑ *right-sided Gateaux differentiable*.

Proposition 2.3 *(chain rule). Let G be a Banach space and $k\colon \mathbb{R} \to G$ be right-sided differentiable at $a \in \mathbb{R}$ (with right derivative $k'(a)$). Suppose that $\vartheta\colon G \to G$ is a Lipschitz continuous mapping. If ϑ is right-sided Gateaux-differentiable at $k(a)$ in the direction of $k'(a)$, then $\vartheta \circ k\colon \mathbb{R} \to G$ is right-sided differentiable at a and has a right derivative*

$$(\vartheta \circ k)'(a) = D_{k'(a)}\vartheta(k(a)). \tag{2.4}$$

Proof There exists $L \geqslant 0$ such that $\|\vartheta(x) - \vartheta(y)\| \leqslant L\|x - y\|$ for all $x, y \in G$. Then

$$\begin{aligned}
\lim\nolimits_{t\downarrow 0}\ & \|1/t(\vartheta(k(a+t)) - \vartheta(k(a))) - D_{k'(a)}\vartheta(k(a))\| \\
&\leqslant \limsup\nolimits_{t\downarrow 0} \|1/t(\vartheta(k(a+t)) - \vartheta(k(a) + tk'(a)))\| \\
&\quad + \limsup\nolimits_{t\downarrow 0} \|1/t[\vartheta(k(a) + tk'(a)) - \vartheta(k(a)) - D_{k'(a)}\vartheta(k(a))]\| \\
&\leqslant \limsup\nolimits_{t\downarrow 0} L \cdot \|1/t(k(a+t) - k(a) - tk'(a))\| + 0 \\
&= 0.
\end{aligned}$$

For $z \in \mathbb{C}$ we let

$$\operatorname{sign} z = \begin{cases} z/|z| & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \tag{2.5}$$

Lemma 2.4 *The function $\vartheta\colon \mathbb{C} \to \mathbb{C}$ given by $\vartheta(z) = |z|$ is right-sided Gateaux differentiable and*

$$D_u\vartheta(z) = \begin{cases} \operatorname{Re}[(\operatorname{sign} \bar{z}) \cdot u] & \textit{if } z \neq 0 \\ |u| & \textit{if } z = 0 \end{cases} \tag{2.6}$$

Proof If $z = 0$, relation (2.6) is obvious from the definiton. Let $z = (x_0 + iy_0) \neq 0$. We identify $\mathbb{C}$ and $\mathbb{R}^2$. Then $\vartheta(x, y) = (x^2 + y^2)^{1/2}$ is differentiable in z and

$$D_u\vartheta(z) = (\vartheta(x_0, y_0)|u) = 1/|z|\,((x_0, y_0)|(u_1, u_2)) =$$
$$1/|z|\,(x_0u_1 + y_0u_2) = 1/|z|\,\operatorname{Re}((x_0 - iy_0) \cdot (u_1 + iu_2)) = \operatorname{Re}[(\operatorname{sign} \bar{z}) \cdot u],$$

where $u = u_1 + iu_2 = (u_1, u_2) \in \mathbb{C} = \mathbb{R}^2$ and $(v|u)$ denotes the canonical scalar product of $v, u \in \mathbb{R}^2$. ◻

Let $f, g \in C_0(X)$. We denote by $(\widehat{\operatorname{sign}} f)(g)$ the bounded Borel function given by

$$[(\widehat{\operatorname{sign}} f)(g)](x) = \begin{cases} (\operatorname{sign} f(x)) \cdot g(x) & \text{if } f(x) \neq 0 \\ |g(x)| & \text{if } f(x) = 0 \end{cases} \tag{2.7}$$

Similarly, $(\operatorname{sign} f)(g)$ is defined by

$$[(\operatorname{sign} f)(g)](x) = (\widehat{\operatorname{sign}} f(x)) \cdot g(x). \tag{2.8}$$

We identify the dual space of $C_0(X)$ with $M(X)$, the space of all bounded regular Borel measures on X. We extend the duality by setting

$$\langle h, \varphi \rangle = \int h(x)\, d\varphi(x)$$

for every bounded Borel function h on X and every $\varphi \in M(X)$.

After these preparations we now can show that the lattice property $|T(t)f| = T(t)|f|$ of the semigroup corresponds to the identity (2.9) below for the generator, which we call Kato's equality (cf. Remark 2.7).

Theorem 2.5 *A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on $C_0(X)$ is a lattice semigroup if and only if its generator A satisfies*

$$\begin{cases} \langle \mathrm{Re}[(\widehat{\mathrm{sign}}\bar{f})(Af)], \varphi\rangle = \langle |f|, A'\varphi\rangle \\ \textit{for all } f \in D(A), \varphi \in D(A') \end{cases} \textit{(Kato's equality)}. \tag{2.9}$$

From the proof of the theorem we isolate the following lemma.

Lemma 2.6 *Let $(T(t))_{t\geqslant 0}$ be a semigroup on $C_0(X)$ with generator A. Then for every $f \in D(A)$, $\varphi \in M(X)$,*

$$\frac{d}{dt}|_{t=0}\langle |T(t)f|, \varphi\rangle = \langle \mathrm{Re}[(\widehat{\mathrm{sign}}\bar{f})(Af)], \varphi\rangle. \tag{2.10}$$

Proof Let $f \in D(A)$ and $x \in X$. Define the function $k(t) = (T(t)f)(x)$ for all $t \geqslant 0$. Then k is right-sided differentiable in 0 with derivative $k'(0) = (Af)(x)$. It follows from the chain rule Proposition 2.3 that

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\, |(T(t)f)(x)| = \mathrm{Re}[(\widehat{\mathrm{sign}}\bar{f})(Af)](x) \tag{2.11}$$

Moreover, $1/t\,|\,|T(t)f| - |f|\,| \leqslant 1/t\,|T(t)f - f|$. Thus $\sup_{0<t\leqslant 1} 1/t\,\|\,|T(t)f| - |f|\,\| < \infty$; i.e., the functions $k_t \in C_0(X)$ given by

$$k_t(x) = 1/t\,(|(T(t)f)(x)| - |f(x)|) \quad (x \in X) \tag{2.12}$$

are uniformly on $(0, 1]$ dominated by a constant. The dominated convergence theorem and (2.11) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\,\langle |T(t)f|, \varphi\rangle = \lim_{t\downarrow 0}\langle k_t, \varphi\rangle = \langle \mathrm{Re}[(\widehat{\mathrm{sign}}\bar{f})(Af)], \varphi\rangle$$

Proof (Proof of Theorem 2.5) Assume that $(T(t))_{t\geqslant 0}$ is a lattice semigroup. Let $f \in D(A)$, $\varphi \in D(A')$. It follows from the preceding lemma that

$$\langle \mathrm{Re}[(\widehat{\mathrm{sign}}\bar{f})(Af)], \varphi\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle |T(t)f|, \varphi\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle T(t)|f|, \varphi\rangle = \langle |f|, A'\varphi\rangle$$

Conversely, assume that (2.9) holds. Let $t > 0$, $f \in C_0(X)$. We have to show that $|T(t)f| = T(t)|f|$. Since $D(A)$ is dense in $C_0(X)$, we can assume that $f \in D(A)$. Moreover, since $D(A')$ is $\sigma(M(X), C_0(X))$-dense in $M(X)$, it suffices to show that

$$\langle |T(t)f|, \varphi\rangle = \langle T(t)|f|, \varphi\rangle \tag{2.13}$$

for all $\varphi \in D(A')$.

Let $\varphi \in D(A')$ and define the function $k(s) = \langle T(t-s)|T(s)f|, \varphi\rangle$ $(s \in [0,t])$. We claim that k is right-sided differentiable with derivative $k'(s) = 0$ for all $s \in [0,t]$. This implies that $k(0) = k(t)$ which is (2.13).

Since $\varphi \in D(A')$ we have

$$\lim_{h\downarrow 0} 1/h\, \langle g, (T(t-(s+h)) - T(t-s))\varphi\rangle = -\langle g, A'T(t-s)'\varphi\rangle \tag{2.14}$$

for all $g \in C_0(X)$. Consequently,

$$\overline{\lim}_{h\downarrow 0}\, 1/h\, \|(T(t-(s+h)) - T(t-s))'\varphi\| < \infty$$

by the uniform boundedness principle. Hence, since $\lim_{h\downarrow 0} |T(s+h)f| = |T(s)f|$, (2.14) implies that

$$\begin{aligned}\lim_{h\downarrow 0} 1/h\, \langle |T(s+h)f|, (T(t-(s+h)) - T(t-s))'\varphi\rangle& \\ = -\langle |T(s)f|, A'T(t-s)'\varphi\rangle&\end{aligned} \tag{2.15}$$

Using this we obtain

$$\begin{aligned}&\lim_{h\downarrow 0} 1/h\,(k(s+h) - k(s))\\ &\quad = \lim_{h\downarrow 0} 1/h\,(\langle (T(t-(s+h))|T(s+h)f|, \varphi\rangle - \langle T(t-s)|T(s+h)f|, \varphi\rangle)\\ &\qquad + \lim_{h\downarrow 0} 1/h \langle T(t-s)|T(s+h)f| - T(t-s)|T(s)f|, \varphi\rangle\\ &\quad = -\langle |T(s)f|, A'T(t-s)'\varphi\rangle + \lim_{t\downarrow 0} 1/h\,\langle (|T(s+h)f| - |T(s)f|, T(t-s)'\varphi\rangle.\end{aligned}$$

By Lemma 2.6 the last term is

$$-\langle |T(s)f|, A'T(t-s)'\varphi\rangle + \langle \mathrm{Re}[(\hat{\mathrm{sign}}\overline{T(s)f})(AT(s)f)], T(t-s)'\varphi\rangle,$$

and this is 0 by hypothesis. □

Remark 2.7 We will see in Chapter C-II that the inequality $|T(t)f| \leqslant T(t)|f|$, which holds precisely for positive semigroups, implies the inequality corresponding to (2.9). For $A = \Delta$ (the Laplacian) this is a version of the classical Kato's inequality.

Corollary 2.8 *Let $(T(t))_{t\geqslant 0}$ be a lattice semigroup on $C_0(X)$ with generator A. If $f \in D(A)$ and $f(x) \neq 0$ for all $x \in X$, then $|f| \in D(A)$ and* $\mathrm{Re}[(\mathrm{sign}\,\bar{f})(Af)] = A|f|$.

Proof If $f \in D(A)$ and $f(x) \neq 0$ for all $x \in X$, then $(\mathrm{sign}\,\bar{f})(Af) = (\mathrm{sign}\,\bar{f})(Af) \in C_0(X)$. Hence by (2.9), $\langle \mathrm{Re}[(\mathrm{sign}\,\bar{f})(Af)], \varphi\rangle = \langle |f|, A'\varphi\rangle$ for all $\varphi \in D(A')$. So the assertion follows from the following lemma. □

Lemma 2.9 *Let A be a densely defined closed operator on a (real or complex) Banach space G. Let $f, g \in G$ such that $\langle f, \varphi\rangle = \langle g, A'\varphi\rangle$ for all $\varphi \in D(A')$. Then $g \in D(A)$ and $Ag = f$.*

Proof Denote by $G(A) := \{(h, Ah) : h \in D(A)\} \subset G \times G$ the graph of A. Assume that $(g, f) \notin G(A)$. Since $G(A)$ is closed, it follows from the Hahn-Banach theorem that there exists $(\psi_1, \psi_2) \in G' \times G'$ such that $\langle h, \psi_1\rangle + \langle Ah, \psi_2\rangle = 0$ for all $h \in D(A)$ and $\langle g, \psi_1\rangle + \langle f, \psi_2\rangle \neq 0$. By the definition of A' this implies that $\psi_2 \in D(A')$ and $A'\psi_2 = -\psi_1$. Hence $\langle f, \psi_2\rangle \neq -\langle g, \psi_1\rangle = \langle g, A'\psi_2\rangle$. So the condition in the lemma is violated. □

Next we prove a converse of Corollary 2.8.

Theorem 2.10 *Let A be the generator of a real semigroup $(T(t))_{t\geqslant 0}$ on $C(K, \mathbb{C})$, where K is compact. Then $(T(t))_{t\geqslant 0}$ is a lattice semigroup if and only if $f \in D(A)$, $f(x) \neq 0$ for all $x \in K$ implies $|f| \in D(A)$ and $A|f| = \mathrm{Re}((\mathrm{sign}\, \bar{f})Af)$.*

Remark Although we assume that $(T(t))_{t\geqslant 0}$ is a real semigroup (i.e., $T(t)C(K, \mathbb{R}) \subset C(K, \mathbb{R})$ for all $t \geqslant 0$), it is important for the proof that we consider the space of all complex-valued continuous functions on K. In fact, if K is connected, the condition in the theorem is always trivially satisfied for all $f \in C(K, \mathbb{R})$. □

Proof It follows from Corollary 2.8 that the condition is necessary. So assume that the condition is satisfied. Since $(T(t))_{t\geqslant 0}$ is real, the restriction $T_{\mathbb{R}}(t)$ of $T(t)$ to $C(K, \mathbb{R})$ $(t \geqslant 0)$ defines a strongly continuous semigroup. Its generator $A_{\mathbb{R}}$ is a restriction of A. Since $D(A_{\mathbb{R}})$ is dense in $C(K, \mathbb{R})$, there exists a strictly positive $u \in D(A_{\mathbb{R}})$. Moreover, $\lim_{t\to 0} T(t)u = u$ uniformly. Thus there exists $t_0 > 0$ such that $T(t)u$ is strictly positive for all $t \in [0, t_0]$.

Let $f \in D(A_{\mathbb{R}})$. For $\epsilon > 0$ let $f_\epsilon := f + i\epsilon u$. Then $T(t)f_\epsilon \in D(A)$ and $|T(t)f_\epsilon|$ is strictly positive for all $t \in [0, t_0]$. By hypothesis, $|T(t)f_\epsilon| \in D(A)$ and $\mathrm{Re}[(\mathrm{sign}\,\overline{T(t)f_\epsilon})AT(t)f_\epsilon] = A|T(t)f_\epsilon|$ for all $t \in [0, t_0]$. One sees as in the proof of Theorem 2.5 that this implies that $|T(t)f_\epsilon| = T(t)|f_\epsilon|$ for all $t \in [0, t_0]$. Letting $\epsilon \to 0$ one obtains that $|T(t)f| = T(t)|f|$ $(t \in [0, t_0])$. Since $D(A)$ is dense in $C(K, \mathbb{R})$ we conclude that $|T(t)f| = T(t)|f|$ for all $f \in C(K, \mathbb{R})$ and all $t \in [0, t_0]$. Let $s > t_0$. Then $s/n \leqslant t_0$ for some $n \in \mathbb{N}$. Hence $|T(s)f| = |T(s/n)^n f| = T(s/n)^n|f| = T(s)|f|$ for all $f \in C(K, \mathbb{R})$. We have shown that $T_{\mathbb{R}}(t)$ is a lattice homomorphism for all $t \geqslant 0$; hence $T(t)$ is so as well (cf. Remark 2.1). □

Corollary 2.11 *Let A be the generator of a lattice semigroup on $C(K, \mathbb{C})$ (K compact). Assume that $m \in C(K)$ is strictly positive. Then $m \cdot A$ with domain $D(m \cdot A) = D(A)$ generates a lattice semigroup.*

Proof By Theorem 1.20 $m \cdot A$ is the generator of a strongly continuous semigroup. It remains to show that it is a lattice semigroup. We use Theorem 2.10. Let $f \in D(m \cdot A) = D(A)$ such that $f(x) \neq 0$ for all $x \in K$. Then $\mathrm{Re}[(\mathrm{sign}\, \bar{f})m \cdot Af] = m \cdot \mathrm{Re}[(\mathrm{sign}\, \bar{f})Af] = m \cdot A|f|$. □

Example 2.12 The operator $A_{\max}$ on the (real or complex space) $C[-1,0]$ given by $A_{\max}f = f'$ with domain $D(A_{\max}) = C^1[-1,0]$ satisfies Kato's equality; i.e.,

$$\langle \mathrm{Re}[(\operatorname{sign}\bar{f})(A_{\max}f)], \varphi\rangle = \langle |f|, A'_{\max}\varphi\rangle \tag{2.16}$$

$(f \in D(A_{\max}), \varphi \in D(A'_{\max}))$.

Moreover, $(\lambda - A_{\max})$ is surjective for $\lambda \geqslant 0$ (cf. Example 1.22). Thus, since $\ker(\lambda - A_{\max}) = \mathbb{C}e_\lambda$ $(e_\lambda(x) = e^{\lambda x})$, Kato's equality does not have as strong consequences as the positive minimum principle (which by Theorem 1.13 would imply that $A_{\max}$ is a generator).

Proof It is not difficult to prove that the adjoint $A'_{\max}$ of $A_{\max}$ is given by

$$A'_{\max}\varphi = \varphi(0)\delta_0 - \varphi(-1)\delta_{-1} - d\varphi \tag{2.17}$$

with domain $D(A'_{\max}) = BV[-1,0]$ (the space of all functions of bounded variation on $[-1,0]$). Here we identify $BV[-1,0] \subset L^1[-1,0]$ with a subspace of $C[-1,0]'$ by setting $\langle f, \varphi\rangle = \int_{-1}^0 f(x)\varphi(x)\,\mathrm{d}x$ $(f \in C[-1,0], \varphi \in BV[-1,0])$. For $\varphi \in BV[-1,0]$, $\mathrm{d}\varphi$ denotes the linear form on $C[-1,0]$ given by $f \mapsto \int_{-1}^0 f(x)\,\mathrm{d}\varphi(x)$.

We now show (2.16). Let $f \in D(A_{\max}) = C^1[-1,0]$, $\varphi \in D(A'_{max}) = BV[-1,0]$. By Lemma 2.4 and the chain rule (Proposition 2.3 we have $|f(x)|' := \mathrm{d}^+/\mathrm{d}t_{|t=x}|f(t)| = \mathrm{Re}[(\operatorname{sign}\bar{f})f'](x)$ (where $f'(x) = (\mathrm{Re}f)'(x) + i(\mathrm{Im}f)'(x)$ in the complex case). Thus

$$\begin{aligned}\langle \mathrm{Re}[(\operatorname{sign}\bar{f})Af], \varphi\rangle &= \int_{-1}^0 |f(x)|'\varphi(x)\,\mathrm{d}x = \int_0^1 \varphi(x)\,\mathrm{d}|f(x)| = \\ &= \varphi(0)|f(0)| - \varphi(-1)|f(-1)| - \int_{-1}^0 |f(x)|\,\mathrm{d}\varphi(x) \\ &= \langle |f|, A'_{\max}\varphi\rangle.\end{aligned}$$

Example 2.13 Let A on (the real or complex) space $C[-1,0]$ be given by $Af = f'$ with domain $D(A) = \{f \in C^1[-1,0] : f'(0) = Lf\}$ where $L \in M[-1,0] = C[-1,0]'$. Then A is the generator of a lattice semigroup if and only if $L = \alpha\delta_0$ for some $\alpha \geqslant 0$.

Proof Assume that A is the generator of a lattice semigroup $(T(t))_{t\geqslant 0}$. There exists $\mu \in M[-1,0]$ satisfying $\mu(\{0\}) = 0$ and $\alpha \in \mathbb{R}$ such that $L = \alpha\delta_0 + \mu$. We claim that

$$|\langle f, \mu\rangle| = \langle |f|, \mu\rangle \quad \text{for all } f \in D(A) \text{ satisfying } f(0) = 0. \tag{2.18}$$

In fact, by the definition of A we have

$$\delta_0 \in D(A') \text{ and } A'\delta_0 = L. \tag{2.19}$$

Moreover, by Theorem 2.5, A satisfies Kato's Equality (2.9). Since $f(0) = 0$ this implies

$$\begin{aligned}|\langle f,\mu\rangle| &= |f'(0)| = \operatorname{Re}[(\widehat{\operatorname{sign}} f)(f')](0)\\ &= \langle \operatorname{Re}[(\widehat{\operatorname{sign}} f)(Af)],\delta_0\rangle = \langle |f|, A'\delta_0\rangle \quad \text{(by (2.9))}\\ &= \langle |f|,\mu\rangle.\end{aligned}$$

Since $\varphi(f) = f'(0) - \langle f,\mu\rangle$ defines a linear form on the space $F = \{f \in C^1[-1,0] : f(0) = 0\}$ which is discontinuous for the supremum norm, the space $D(A) = \ker\varphi$ is dense in F and consequently dense in $C_0[-1,0)$. It follows that (2.18) holds for all $f \in C_0[-1,0)$. So by B-I, Section 2, there exist $\beta \geqslant 0$ and $x \in [-1,0)$ such that $\mu = \beta\delta_x$. Assume that $\beta \neq 0$. It is easy to see that there exists a real function $f \in C^1[-1,0]$ satisfying $f'(0) = \alpha f(0)+\beta f(x)$ and $f(0)f(x) < 0$. Hence $f \in D(A)$ but $\langle \operatorname{Re}[(\operatorname{sign} f)(Af)],\delta_0\rangle = (\operatorname{sign} f(0))f'(0) = (\operatorname{sign} f(0))(\alpha f(0) + \beta f(x))\ \alpha|f(0)| + \beta(\operatorname{sign} f(0))f(x) \neq \alpha|f(0)| + \beta|f(x)| = \langle |f|, \alpha\delta_0 + \beta\delta_x\rangle = \langle |f|, A'\delta_0\rangle$. This contradicts (2.9). We have shown that $\beta = 0$; i.e., $L = \alpha\delta_0$.

The converse can be shown by using Theorem 2.5 again. However, if $L = \alpha\delta_0$, then it is easy to see that A generates the semigroup $(T(t))_{t\geqslant 0}$ given by

$$(T(t)f)(x) = \begin{cases} f(x+t) & \text{if } x+t < 0\\ e^{t\alpha} f(0) & \text{if } x+t \geqslant 0\end{cases}$$

So $(T(t))_{t\geqslant 0}$ is clearly a lattice semigroup. □

3 Semiflows, Flows and Positive Groups

In this section we establish a relation between generators of lattice homomorphisms and derivations. On the space $C_0(\mathbb{R})$, for example, this will enable us, to give a detailed description of all generators of positive groups.

At first we consider a compact space K and denote by $C(K) = C(K,\mathbb{R})$ the space of all real valued continuous functions on K. The extension of the subsequent results to the complex space is obvious.

A lattice homomorphism T on $C(K)$ is an algebra homomorphism if and only if $T\mathbb{1} = \mathbb{1}$ (see B-I, Section 3). We start by describing semigroups of algebra homomorphisms on $C(K)$.

Definition 3.1 A mapping $\varphi\colon\ [0,\infty)\times K \to K$ is called *semiflow* if for each $t \geqslant 0$ the mapping φ_t given by $\varphi_t(x) = \varphi(t,x)$ is continuous and

$$\varphi_s \circ \varphi_t = \varphi_{s+t} \text{ for all } s,t \geqslant 0 \tag{3.1}$$

$$\varphi_0(x) = x \quad (x \in K) \tag{3.2}$$

A semiflow φ on K induces a family $(T(t))_{t\geqslant 0}$ of algebra homomorphisms on $C(K)$ by

$$T(t)f = f \circ \varphi_t. \tag{3.3}$$

Then clearly $T(t)T(s) = T(t+s)$ $(t, s \geqslant 0)$; i.e., $(T(t))_{t \geqslant 0}$ has the semigroup property. Conditions for strong continuity are given in the following lemma.

Lemma 3.2 *The following assertions are equivalent:*

(a) The mapping $\varphi\colon \mathbb{R}_+ \times K \to K$ *is continuous (where* $\mathbb{R} \times K$ *carries the product topology).*
(b) The mapping φ *is separately continuous.*
(c) $(T(t))_{t \geqslant 0}$ *is a strongly continuous semigroup on* $C(K)$.

Proof (a) trivially implies (b). If (b) holds, then $t \mapsto T(t)f$ is weakly continuous for every $f \in C(K)$ (by the theorem of dominated convergence). This implies strong continuity (see for example **?** , Proposition 1.23]. It remains to show that (c) implies (a). Because of (3.1) it suffices to show that the restriction $\tilde{\varphi}$ of φ to $[0, 1] \times K$ is continuous. By hypothesis, the mapping $W\colon f \mapsto (t \mapsto T(t)f)$ from $C(K)$ into $C([0, 1], C(K))$ is continuous. Identifying $C([0, 1], C(K))$ canonically with $C([0, 1] \times K)$ the mapping W obtains the form $f \mapsto f \circ \tilde{\varphi}$. Since W is continuous, $\tilde{\varphi}$ is continuous as well. ◻

A semiflow is called *continuous* if it satisfies the equivalent conditions of Lemma 3.2.

Definition 3.3 An operator δ on $C(K)$ is called *derivation* if $D(\delta)$ is a subalgebra of $C(K)$ such that

$$\delta(f \cdot g) = (\delta f)g + f(\delta g) \quad \text{for all } f, g \in D(\delta). \tag{3.4}$$

$$\mathbb{1} \in D(\delta) \tag{3.5}$$

Note that (3.4) implies $\delta\mathbb{1} = 0$.

A lattice semigroup $(T(t))_{t \geqslant 0}$ on $C(K)$ is called *Markovian* if $T(t)\mathbb{1} = \mathbb{1}$ for all $t \geqslant 0$.

Theorem 3.4 *Let* $(T(t))_{t \geqslant 0}$ *be a semigroup on* $C(K)$ *with generator* A. *The following assertions are equivalent.*

(a) $(T(t))_{t \geqslant 0}$ *is a Markovian lattice semigroup.*
(b) $T(t)$ *is an algebra homomorphism for every* $t \geqslant 0$.
(c) There exists a continuous semiflow φ *on* K *such that* $T(t)f = f \circ \varphi_t$ $(t \geqslant 0)$.
(d) A *is a derivation.*

Proof (a) and (b) are equivalent by the remark at the beginning of this section. Assume that (b) holds. Then there exists a continuous mapping $\varphi_t\colon K \to K$ such that $T(t)f = f \circ \varphi_t$ for all $f \in C(K)$ (see B-I, Section 3). The semigroup property implies that $(\varphi_t)_{t \geqslant 0}$ is a continuous semiflow. This shows (c) to hold.

If (c) holds, then $T(t)\mathbb{1} = \mathbb{1}$ for all $t \geqslant 0$. Hence $\mathbb{1} \in D(A)$ and $A\mathbb{1} = 0$. Let $f, g \in D(A)$. Then $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} T(t)(f \cdot g) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (T(t)f) \cdot (T(t)g) = (Af) \cdot g + f \cdot (Ag)$. Thus $f \cdot g \in D(A)$ and (3.4) holds. Hence A is a derivation.

Finally assume that (d) holds. We prove (b), i.e., we have to show that $T(t)(f \cdot g) = T(t)f \cdot T(t)g$ for $t > 0$. Since $D(A)$ is a dense subalgebra, we can assume that

$f, g \in D(A)$. Define $\eta\colon [0,t] \to C(K)$ by $\eta(s) := T(t-s)[T(s)f \cdot T(s)g]$. Then $\eta(0) = T(t)(f \cdot g)$ and $\eta(t) = T(t)f \cdot T(t)g$. Since A is a derivation, $\eta'(s) = 0$ for $s \in [0,t]$. Hence $\eta(0) = \eta(t)$. This shows (b) to hold. $\square$

If δ is the generator of a semigroup $(T(t))_{t\geqslant 0}$ given by $T(t)f = f \circ \varphi_t$, then we call φ given by $\varphi(t,x) = \varphi_t(x)$ the *semiflow associated with* $(T(t))_{t\geqslant 0}$ (or *associated with* δ). We now can describe the generator of any lattice semigroup as a perturbation of a derivation. If $\mathbb{1}$ is in the domain of the generator, an additive perturbation (by a multiplication operator) suffices; in general a similarity transformation has to be applied in addition. This is the assertion of the following two theorems.

Theorem 3.5 *Let A be a generator of a semigroup $(T(t))_{t\geqslant 0}$ on $C(K)$. Suppose that $\mathbb{1} \in D(A)$. Then the following assertions are equivalent.*

(a) $(T(t))_{t\geqslant 0}$ is a lattice semigroup.
(b) There exist a derivation δ (generating a semigroup of algebra homomorphisms) and a multiplier $h \in C(K)$ such that $A = \delta + h$ (i.e., $D(A) = D(\delta)$ and $Af = \delta f + h \cdot f$ for $f \in D(A)$).

Moreover, if (b) holds, then $(T(t))_{t\geqslant 0}$ is given by

$$(T(t)f)(x) = \exp\left(\int_0^t h(\varphi(s,x))\ \mathrm{d}s\right) \cdot f(\varphi(t,x)) \tag{3.6}$$

where φ is the semiflow associated with δ.

Proof Let $h = A\mathbb{1}$ and $\delta = A - h$. Then the semigroup $(T_0(t))_{t\geqslant 0}$ generated by δ is a lattice semigroup if and only if $(T(t))_{t\geqslant 0}$ is a lattice semigroup [since $T_0(t)f = \lim_{n\to\infty}(e^{-t/n\cdot h} \cdot T(\frac{t}{n}))^n f$ and $T(t)f = \lim_{n\to\infty}(e^{t/n\cdot h} \cdot T_0(\frac{t}{n}))^n f$ for all $t \geqslant 0$, $f \in C(K)$ by A-II, (1.8). Since $0 \cdot 1 = 0$ one has $T_0(t)1 = 1$ for all $t \geqslant 0$ and the equivalence of (a) and (b) follows with the help of Theorem 3.4.

Now assume that (a) and (b) hold. Let

$$(S(t)f)(x) = \exp\int_0^t h(\varphi(s,x))ds \cdot f(\varphi(t,x))$$

$x \in K$ $f \in C(K)$ $t \geqslant 0$. Then one easily shows that $(S(t))_{t\geqslant 0}$ is a strongly continuous semigroup. Denote by B its generator. For $f \in D(\delta)$, $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}S(t)f = h \cdot f + \delta f$. Hence $\delta + h \subset B$. Since $\delta + h$ also is a generator, it follows that $\delta + h = B$. $\square$

Theorem 3.6 *An operator A is generator of a lattice semigroup on $C(K)$ if and only if there exists a derivation δ which is a generator, a function $h \in C(K)$ and a strictly positive function $p \in C(K)$ such that*

$$A = M\delta M^{-1} + h \tag{3.7}$$

where $M \in L(C(K))$ is given by $Mf = p \cdot f$.

Proof In order to show the non-trivial implication assume that A generates a lattice semigroup. Since $D(A)$ is dense in $C(K)$ there exists $0 \ll p \in D(A)$. Let $h(x) = (Ap)(x)/p(x)$. The operator given by $Mf = f \cdot p$ is a lattice isomorphism. Thus $\delta := M^{-1}(A-h)M$ generates a lattice semigroup. Since $M\mathbb{1} = p \in D(A)$ one has $\mathbb{1} \in D(\delta)$ and $\delta\mathbb{1} = M^{-1}(A-h)p = 0$. Thus δ is the generator of a semigroup of algebra homomorphisms, hence a derivation by Theorem 3.4. ◻

At the end of this section we will show that any derivation on $C[0,1]$ which generates a group is similar to a differential operator of first order. This in connection with Theorem 3.6 will enable us to describe all generators of positive groups as perturbations of a differential operator.

In Section 1 we had obtained a very simple condition describing generators of positive semigroups on $C(K)$ by the positive minimum principle and a range condition. This result yields a characterization of generators of automorphism groups by "locality" and a range condition. By an *automorphism* we understand an algebra isomorphism of $C(K)$ onto itself.

Theorem 3.7 *Let A be a densely defined operator on $C(K)$. The following assertions are equivalent.*

(a) A is the generator of an automorphism group.
(b) $\mathbb{1} \in D(A)$ and $A\mathbb{1} = 0$; $(\pm\mathbb{1} - A)D(A) = C(K)$ and A is local, in the sense that for $0 \leqslant f \in D(A)$, $f(x) = 0$ implies $(Af)(x) = 0$ $(x \in K)$.

Proof An invertible operator T such that $T \geqslant 0$ and $T^{-1} \geqslant 0$ is an automorphism if and only if $T\mathbb{1} = \mathbb{1}$. Hence A is the generator of an automorphism group if and only if A and $-A$ generate a positive group, $\mathbb{1} \in D(A)$ and $A\mathbb{1} = 0$. Thus Theorem 3.7 follows from Theorem 1.13. ◻

Remark It is remarkable that from locality, the range condition and $\mathbb{1} \in D(A)$, $A\mathbb{1} = 0$ it follows that $D(A)$ actually is a subalgebra of $C(K)$ and A is a derivation. The "order-theoretical" property of locality is in some aspects stronger than the algebraic property of being a derivation. For example a local, densely defined operator is closable (by (LNM1184: Proposition) Corollary 1.11); but there exist derivations on $C[0,1]$ which are not closable (see **?**]). ◻

Remark (an excursion to C*-algebras) Theorem 3.7 also holds for non-commutative C*-algebras. More precisely: Let $\mathcal{A}$ be a C*-algebra with unit $\mathbb{1}$ and let $\mathcal{A}_h$ be the real Banach space of all hermitian elements in $\mathcal{A}$. Then $\mathcal{A}_h$ is a real ordered Banach space and $\mathbb{1}$ is an interior point of $(\mathcal{A}_h)_+$. Let A be a densely defined operator on $\mathcal{A}_h$. Then A is the generator of an automorphism group if and only if $\mathbb{1} \in D(A)$ and $A\mathbb{1} = 0$; $(\pm\mathbb{1} - A)(D(A)) = \mathcal{A}_h$ and A is local in the sense that for $0 \leqslant x \in D(A)$, $0 \leqslant \varphi \in (\mathcal{A}_h)'$, $\varphi(x) = 0$ implies $\varphi(Ax) = 0$.

The proof of Theorem 3.7 can be carried over to this case if one notices the following. A strongly continuous group $T(t)_{t\in\mathbb{R}}$ on $\mathcal{A}_h$ is an automorphism group if and only if it is positive and $T(t)\mathbb{1} = \mathbb{1}$ for all $t \in \mathbb{R}$ [see **?** , Corollary 3.2.21]]. ◻

Now we let X be a locally compact space and consider positive groups on $C_0(X) = C_0(X, \mathbb{R})$, the space of all continuous real-valued functions on X which vanish at infinity. Our aim is to describe their generators as perturbations of generators of automorphism groups; i.e., we will extend Theorem 3.6 by allowing X to be noncompact but restrict ourselves to positive groups (or equivalently semigroups of lattice isomorphisms). And in fact, it is not difficult to show by an example that the corresponding result is wrong for lattice semigroups in general.

A mapping $\varphi\colon \mathbb{R}\times X \to X$ is called a *flow* on X if the partial maps $\varphi_t : X \to X$ given by $\varphi_t(x) = \varphi(t,x)$ are continuous and satisfy

$$\varphi_0(x) = x \quad (x \in X) \tag{3.8}$$

$$\varphi_s \circ \varphi_t = \varphi_{s+t} \quad (s,t \in \mathbb{R}) \tag{3.9}$$

It follows from the definition that each φ_t is a homeomorphism on X and $\varphi_{-t} = (\varphi_t)^{-1}$.

A flow φ is called *continuous* if it is continuous with respect to the product topology on $\mathbb{R}\times X$.

Given a flow φ a family $(h_t)_{t\in\mathbb{R}} \subset C^b(X)$ is called a *cocycle* of φ if

$$h_0 = \mathbb{1} \tag{3.10}$$

$$h_{t+s} = h_t \cdot (h_s \circ \varphi_t) \quad (s,t \in \mathbb{R}) \tag{3.11}$$

It follows from (3.10) and (3.11) that $h_t(x) \neq 0$ for all $x \in X$ and $1/h_t(x) = h_{-t}(\varphi_t(x))$ $(t \in \mathbb{R})$. The cocycle is called *continuous* if the mapping $(t,x) \mapsto h_t(x)$ from $\mathbb{R}\times X$ into $\mathbb{R}$ is continuous with respect to the product topology on $\mathbb{R}\times X$.

Let φ be a flow and $(h_t)_{t\in\mathbb{R}}$ a cocycle of φ. Then

$$T(t)f = h_t \cdot f \circ \varphi_t \tag{3.12}$$

defines a bounded operator $T(t)$ on $C_0(X)$ $(t \in \mathbb{R})$. Clearly $T(s+t) = T(s)T(t)$ for all $s,t \in \mathbb{R}$.

Proposition 3.8 *Let $\varphi\colon \mathbb{R}\times X \to X$ be a flow and $(h_t)_{t\in\mathbb{R}}$ a cocycle of φ. If for every $x \in X$ the mappings $t \mapsto \varphi_t(x)$ and $t \mapsto h_t(x)$ are continuous, then (3.12) defines a strongly continuous group.*

Proof We first note that $\|T(t)\|$ is bounded on compact intervals of $\mathbb{R}$. This follows from **?** , 7.4.1] since $q(t) = \log \|T(t)\|$ defines a subadditive, measurable function from $\mathbb{R}$ into $\mathbb{R}$ [In fact, $\|T(t)\| = \sup_{x\in X} |h_t(x)|$ for $t \in \mathbb{R}$. So it follows from the assumption that $t \mapsto \|T(t)\|$ is lower semicontinuous and hence measurable]. If $f \in C_0(X)$, then by hypothesis the function $t \mapsto h_t(x)f(\varphi(t,x)) = (T(t)f)(x)$ is continuous on $\mathbb{R}$. It follows from the dominated convergence theorem that $T(.)f$ is weakly continuous. Hence $(T(t))_{t\in\mathbb{R}}$ is strongly continuous (see e.g., **?** , Proposition 1.23]). □

The group defined by (3.12) is positive whenever $(h_t)_{t\in\mathbb{R}} \subset C^b(X)_+$. We now show that every positive group on $C_0(X)$ is of the form (3.12).

Proposition 3.9 *Let $(T(t))_{t\in\mathbb{R}}$ be a strongly continuous group of positive operators on $C_0(X)$. Then there exist a continuous flow on X and a continuous cocycle $(h_t)_{t\in\mathbb{R}}$ of Φ such that (3.12) holds.*

Proof Since $T(t)$ and $T(t)^{-1} = T(-t)$ are positive operators, $T(t)$ actually is a lattice isomorphism. Then there exist a homeomorphism Φ_t on X and $h_t \in C^b(X)_+$ such that $T(t)f = h_t \cdot f \circ \Phi_t$ for all $f \in C_0(X)$ $(t \in \mathbb{R})$. The group property of $(T(t))_{t\in\mathbb{R}}$ then implies that $\Phi(t,x) := \Phi_t(x)$ defines a flow on X and that $(h_t)_{t\in\mathbb{R}}$ is a cocycle of Φ. It remains to show that Φ and $(h_t)_{t\in\mathbb{R}}$ are continuous.

At first we consider the flow. Since we have $\Phi_{t+s} = \Phi_t \circ \Phi_s$ and every Φ_t is a homeomorphism on X, it is enough to establish continuity of Φ at points $(0,x_0) \in \mathbb{R}\times X$. Given a compact neighbourhood V of $x_0 = \Phi(0,x_0)$, there exists a continuous function $f\colon X \to [0,1]$ satisfying $f(x_0) = 1$ and $\operatorname{supp} f \subset V$. There exists $t_0 > 0$ such that $\|T(t)f - f\| < \frac{1}{2}$ for $|t| \leqslant t_0$. Let $W\colon \{x \in X\colon |f(x)| > \frac{1}{2}\}$; then for $|t| \leqslant t_0$ and $x \in W$ we have $|h_t(x)\cdot f(\Phi(t,x)) - f(x)| < \frac{1}{2}$ and $|f(x)| > \frac{1}{2}$; hence $f(\varphi(t,x)) > 0$. This implies that $\varphi(t,x) \in V$ whenever $|t| \leqslant t_0$ and $x \in W$.

To prove continuity of the cocycle we first remark that by strong continuity of $(T(t))_{t\in\mathbb{R}}$ the mapping $(t,x) \mapsto (T(t)f)(x)$ is continuous on $\mathbb{R}\times X$ for every fixed $f \in C_0(X)$. Given compact subsets $A \subset \mathbb{R}, B \subset X$, the set $C := \varphi(A\times B)$ is compact; hence there exists $f \in C_0(X)$ such that $f|_C = 1$. For $(t,x) \in A\times B$ we have $h_t(x) = (T(t)f)(x)$. Thus $(t,x) \mapsto h_t(x)$ is continuous on $A\times B$. □

Corollary 3.10 *Let φ be a separately continuous flow. Then φ is continuous. If $(h_t)_{t\in\mathbb{R}}$ is a cocycle of φ such that $t \mapsto h_t(x)$ is continuous for every $x \in X$, then $(h_t)_{t\in\mathbb{R}}$ is continuous.*

This follows from Proposition 3.8 and Proposition 3.9 .

Example 3.11 Let φ be a continuous flow on X.

a) Let p be a continuous function on X such that $\inf_{x\in X} p(x) > 0$ and $\sup_{x\in X} p(x) < \infty$. Then

$$p_t := p/(p\circ\varphi_t) \quad (t \in \mathbb{R}) \tag{3.13}$$

defines a continuous cocycle of φ.

b) For $h \in C^b(X)$ define

$$h_t(x) := \exp\Big(\int_0^t h(\varphi(s,x))\,\mathrm{d}s\Big). \tag{3.14}$$

Then $(h_t)_{t\in\mathbb{R}}$ is a continuous cocycle of φ (compare (3.6)).

Cocycles as defined by (3.13) are always globally bounded. In general this is false for cocycles of the second type. On the other hand, a cocycle described by (3.14) is differentiable with respect to t. This is not satisfied by cocycles of the first type in general. Thus neither (3.13) nor (3.14) gives a description of arbitrary cocycles. However every positive cocyle is a product of cocycles of the form (3.13) and (3.14). More precisely, we have the following lemma.

Lemma 3.12 *Let φ be a continuous flow on X and $(k_t)_{t\in\mathbb{R}} \subset C^b(X)_+$ a continuous cocycle of φ. Then there exist $p \in C^b(X)$ satisfying $\inf_{x\in X} p(x) > 0$ and $h \in C^b(X)$ such that*

$$k_t(x) = (p(x)/p(\varphi(t,x))) \cdot \exp\Big(\int_0^t h(\varphi(s,x))\,\mathrm{d}s\Big) \tag{3.15}$$

for all $t \in \mathbb{R}$, $x \in X$.

Proof We first note that there exist constants $M, \omega \geqslant 1$ such that

$$(Me^{(\omega-1)|t|})^{-1} \leqslant k_t(x) \leqslant Me^{(\omega-1)|t|} \quad \text{for all } t \in \mathbb{R}, x \in X. \tag{3.16}$$

In fact, let $(T(t))_{t\in\mathbb{R}}$ be the group given by $T(t)f = k_t \cdot f \circ \varphi_t$ $(t \in \mathbb{R}, f \in C_0(X))$. Then there exist constants $M, \omega \geqslant 1$ such that

$$\|T(t)\| \leqslant Me^{(\omega-1)|t|} \tag{3.17}$$

for all $t \in \mathbb{R}$. Since $\|T(t)\| = \sup_{x\in X} k_t(x)$ the right inequality of (3.16) follows. Moreover, $k_{-t} = 1/(k_t \circ \varphi_{-t})$. Hence $\|T(-t)\| = \sup_{x\in X} 1/k_t(x) = [\inf_{x\in X} k_t(x)]^{-1}$. So (3.17) also implies the first inequality in (3.16).

Now we define p and h by

$$p(x) := \int_0^\infty e^{-ws} k_s(x)\,\mathrm{d}s, \quad h(x) = w - 1/p(x) \quad (x \in X).$$

Then p is a continuous function and we have

$$\begin{aligned}(M(2w-1))^{-1} &= \int_0^\infty e^{-ws}(Me^{(w-1)s})^{-1}\,\mathrm{d}s \\ &\leqslant p(x) \\ &\leqslant \int_0^\infty e^{-ws} M e^{(w-1)s}\,\mathrm{d}s \\ &= M \text{ for all } x \in X.\end{aligned}$$

In particular, it follows that $h \in C^b(X)$.

For all $x \in X$, $t \in \mathbb{R}$ we have $k_t(x) \cdot p(\varphi(t,x)) = \int_0^\infty e^{-ws} k_{t+s}(x)\,\mathrm{d}s = e^{wt} \int_t^\infty e^{-ws} k_s(x)\,\mathrm{d}s$.

Now fix $x \in X$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := k_t(x)p(\varphi(t,x))/p(x) = [e^{wt}/p(x)] \cdot \int_t^\infty e^{-ws} k_s(x)\,\mathrm{d}s$.

The function f is differentiable and satisfies the following differential equation $f'(t) = wf(t) - k_t(t)/p(x) = h(\varphi(t,x))f(t)$. Moreover $f(0) = 1$. Hence $f(t) = \exp(\int_0^t h(\varphi(s,x))\,\mathrm{d}s)$ for every $t \in \mathbb{R}$. This is (3.15). □

As before we call a group $(T_0(t))_{t\in\mathbb{R}}$ on $C_0(X)$ an *automorphism group* if each $T_0(t)$ is an algebra isomorphism on $C_0(X)$. Analogously an operator δ on $C_0(X)$ is called a *derivation* if $D(\delta)$ is a subalgebra of $C_0(X)$ and $\delta(f \cdot g) = (\delta f) \cdot g + f \cdot (\delta g)$ for all $f, g \in D(\delta)$.

Proposition 3.13 *Let $(T_0(t))_{t\in\mathbb{R}}$ be a group on $C_0(X)$. The following assertions are equivalent:*

(a) $(T_0(t))_{t\in\mathbb{R}}$ is an automorphism group.
(b) There exists a continuous flow φ on X such that $T_0(t)f = f\circ\varphi_t$ for all $f \in C_0(X)$, and all $t \in \mathbb{R}$.
(c) The generator of $(T_0(t))_{t\in\mathbb{R}}$ is a derivation.

Proof Every automorphism group is positive. So by Proposition 3.9 it is defined via (3.12) by some continuous flow and cocycle. It is easy to see that the cocycle is identically 1. Thus (a) implies (b). One shows as in Theorem 3.4 that (b) implies (c) and (c) implies (a). □

If $(T_0(t))_{t\in\mathbb{R}}$ is an automorphism group with generator δ we call φ in (b) of Proposition 3.13 *the flow associated with* $(T_0(t))_{t\in\mathbb{R}}$ (or *associated with* δ).

Now we can show that every generator of a positive group is a perturbation of a derivation.

Theorem 3.14 *An operator A on $C_0(X)$ is the generator of a positive group $(T(t))_{t\in\mathbb{R}}$ if and only if there exist a derivation δ on $C_0(X)$ which is the generator of a group, a function $h \in C^b(X)$ and $p \in C^b(X)$ satisfying $\inf_{x\in X} p(x) > 0$ such that*

$$A = V\delta V^{-1} + h \tag{3.18}$$

where $V\colon C_0(X) \to C_0(X)$ is given by $Vf = p\cdot f$. In that case one has

$$(T(t)f)(x) = [p(x)/p(\varphi_t(x))]\cdot(\exp\int_0^t h(\varphi(s,x))\,\mathrm{d}s)\cdot f(\varphi_t(x)) \tag{3.19}$$

for all $f \in C_0(X)$, $t \in \mathbb{R}$, $x \in X$.

Note: (3.18) means that $D(A) = \{f : V^{-1}f \in D(\delta)\}$ and $Af = V\delta V^{-1}f + hf$.

Proof Assume that A is given by (3.18). Since V is a lattice isomorphism, it is clear that $V^{-1}\delta V$ generates a positive group; and consequently, A does so as well (cf. the proof of Theorem 3.5). Conversely, let $(T(t))_{t\in\mathbb{R}}$ be a positive group with generator A. By Proposition 3.9 and Lemma 3.12 we know that there exist a continuous flow φ, $0 \ll p \in C^b(X)$ and $h \in C^b(X)$ such that (3.19) holds. Let δ be the generator of the automorphism group defined by φ. We have to show that (3.18) holds. As in Theorem 3.5 one sees that $\delta + h$ generates the group $(S(t))_{t\in\mathbb{R}}$ given by $(S(t)f)(x) = \exp(\int_0^t h(\varphi(s,x))\,\mathrm{d}s)\cdot f(\varphi_t(x))$. Hence $V\delta V^{-1} + h = V(\delta + h)V^{-1}$ generates $(VS(t)V^{-1})_{t\in\mathbb{R}} = (T(t))_{t\in\mathbb{R}}$. This is (3.18). □

Since every generator of a positive group is the perturbation of a derivation, we now look for examples of derivations which generate a group.

Example 3.15 Let $X = \mathbb{R}^n$. Consider a function $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $\sup_{x\in\mathbb{R}^n} \|DF(x)\| < \infty$ where $DF(x) \in \mathrm{L}(\mathbb{R}^n)$ denotes the derivative of F at x. Then there exists a continuous flow φ on $\mathbb{R}^n$ such that

$$\frac{\partial}{\partial t}\varphi(t,x) = F(\varphi(t,x)) \text{ for all } t \in \mathbb{R}, x \in \mathbb{R}^n \tag{3.20}$$

Consider the automorphism group $(T_o(t))_{t\in\mathbb{R}}$ given by $T_o(t)f = f \circ \varphi_t$ and denote by δ its generator. Then

$$D_o = \{f \in C_0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n) \colon \lim_{\|x\|\to\infty} \|(grad\ f)(x)\| = 0\}$$

is a core of δ and

$$(\delta f)(x) = ((grad\ f)(x)|F(x)) \text{ for all } f \in D_o, x \in \mathbb{R}^n, \tag{3.21}$$

where $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^n$

Proof Let $f \in D_o$. Then $g = f - (grad\ f|F) \in C_0(\mathbb{R}^n)$ and $(R(1,\delta)g)(x) = \int_0^\infty e^{-t} f(\varphi(x,t))\,\mathrm{d}t - \int_0^\infty e^{-t}((gradf)(\varphi(t,x))|F(\varphi(t,x)))\,\mathrm{d}t = f(x)$ by integrating by parts. Hence $f \in D(\delta)$ and $f - \delta f = g$; i.e. $\delta f = (grad\ f|F)$. This proves (3.21). Next we show $T_o(t)D_o \subset D_o$ for all $t \geqslant 0$, which implies that D_o is a core of δ by A-I, Theorem 1.9 (or A-II, Corollary 1.34). Since $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, it follows that $\varphi \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ (see e.g., **?** , 15.2]). Moreover for each $x \in \mathbb{R}^n$, $(D\varphi_t(x))' = DF(\varphi_t(x)) \cdot D\varphi_t(x)$ and $D\varphi_0(x) = Id$, (see **?** , p. 300]; here $Id \in \mathrm{L}(\mathbb{R}^n)$ denotes the identity operator. Hence $D\varphi_t(x) = Id + \int_0^t DF(\varphi_s(x)) \cdot D\varphi_s(x)\,\mathrm{d}s$. Consequently $\|D\varphi_t(x)\| \leqslant 1 + \int_0^t M \cdot \|D\varphi_s(x)\|\,\mathrm{d}s$ for all $t \geqslant 0$ and $x \in \mathbb{R}^n$; where $M := \sup_{x\in\mathbb{R}^n} \|DF(x)\| < \infty$ by hypothesis. Hence by Gronwall's inequality, $\|D\varphi_t(x)\| \leqslant e^{Mt}$ $(t \geqslant 0)$ for all $x \in \mathbb{R}^n$. Now let $f \in D_o$, $t \geqslant 0$. Then $[grad\ (f \circ \varphi_t)](x) = [(grad\ f)(\varphi_t(x))] \cdot D\varphi_t(x)$. Hence $\|[grad\ (f \circ \varphi_t)](x)\| \leqslant e^{Mt}\|(grad\ f)(\varphi_t(x))\|$, and so $\lim_{\|x\|\to\infty} \|[grad\ (f \circ \varphi_t)](x)\| \leqslant e^{Mt} \lim_{\|x\|\to\infty} \|(grad\ f)(\varphi_t(x))\| = 0$. Thus $f \circ \varphi_t \in D_o$ for all $t \geqslant 0$. □

As a second class of examples we consider derivations on $C_0(a,b)$. Eventually we will determine all derivations on $C_0(a,b)$, which are generators of a group. We start by looking at differential operators of first order. Let $-\infty \leqslant a < b \leqslant \infty$ and let $m\colon (a,b) \to \mathbb{R}$ be a continuous function. We consider the operator δ_m on $C_0(a,b)$ given by

$$(\delta_m f)(x) = \begin{cases} m(x)f'(x) & \text{if } m(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

with domain
$D(\delta_m) = \{f \in C_0(a,b) \colon f$ is differentiable at x if $m(x) \neq 0$ and $\delta_m f \in C_0(a,b)\}$
Note that δ_m is a derivation on $C_0(a,b)$.

Definition 3.16 A function $m\colon (a,b) \to \mathbb{R}$ is *admissible* if it is continuous and the following holds. Whenever $a \leqslant c < d \leqslant b$ such that $m(x) \neq 0$ for $x \in (c,d)$ and $m(c) = 0$ or $c = a = -\infty$ and $m(d) = 0$ or $d = b = +\infty$, then

$$\int_c^z 1/|m(x)|\,\mathrm{d}x = \int_z^d 1/|m(x)|\,\mathrm{d}x = \infty \text{ for } z \in (c,d).$$

Note: If m is admissible and $a > -\infty$, then $m(a) = 0$; similary, if $b < \infty$, then $m(b) = 0$. Moreover every Lipschitz continuous function is admissible.

Theorem 3.17 *Let $m\colon (a,b) \to \mathbb{R}$ be a continuous function. The operator δ_m is generator of an automorphism group on $C_0(a,b)$ if and only if m is admissible. In that case $D_o(\delta_m) := \{f \in D(\delta_m)\colon f$ is differentiable on $(a,b)\}$ is a core of δ_m.*

Additional properties: *If m is admissible, then the flow φ defining the group generated by δ_m can be described explicitely: The set $\{x \in (a,b)\colon m(x) \neq 0\}$ is the union of a finite or countable number of disjoint intervals (a_n, b_n) $(n \in J)$. Let $c_n \in (a_n, b_n)$ and $g_n(x) := \int_{c_n}^x 1/m(y)\,\mathrm{d}y$ $(x \in (a_n, b_n), n \in J)$. Since m is admissible, g_n is a homeomorphism from (a_n, b_n) onto $\mathbb{R}$. Now the flow φ is defined by*

$$\varphi(t,x) = \begin{cases} x & \text{if } m(x) = 0 \\ g_n^{-1}(g_n(x) + t) & \text{if } x \in (a_n, b_n) \end{cases} \tag{3.22}$$

for all $t \in \mathbb{R}$.

We first prove a special case of Theorem 3.17.

Proposition 3.18 *Suppose that $m(x) \neq 0$ for all $x \in (a,b)$. Then δ_m is the generator of a group on $C_0(a,b)$ if and only if m is admissible. In that case the group generated by δ_m is similar to the translation group on $C_0(\mathbb{R})$.*

Proof Let $q \in C^1(a,b)$ such that $q'(x) = 1/m(x)$ for all $x \in (a,b)$. Then q is a C^1-diffeomorphism from (a,b) onto an interval (a', b'). By $Vf = f \circ q$ one defines an isomorphism from $C_0(a',b')$ onto $C_0(a,b)$. Let B on $C_0(a',b')$ be given by $B = V^{-1}\delta_m V$. Then

$$\begin{aligned} D(B) &= \{g \in C_0(a',b')\colon g \circ q \in D(\delta_m)\} \\ &= \{g \in C_0(a',b') \cap C^1(a',b')\colon g' \circ q = m \cdot (g \circ q)' \in C_0(a,b)\} \\ &= \{g \in C_0(a',b') \cap C^1(a',b')\colon g' \in C_0(a',b')\} \end{aligned}$$

and $Bg = V^{-1}\delta_m V = V^{-1}(m(g \circ q)') = V^{-1}(g' \circ q) = g'$. Now observe that m is admissible if and only if $a' = -\infty$ and $b' = \infty$. If $a' = -\infty$ and $b' = \infty$, then B is the generator of the translation group on $C_0(\mathbb{R})$. Hence also δ_m is the generator of a group $(T(t))_{t\in\mathbb{R}}$ on $C_0(a,b)$. Conversely, assume that B generates a group $(T(t))_{t\in\mathbb{R}}$. Assume that $a' > -\infty$. Then $C_0(a',b')$ is a closed subspace of $C_0[a',b')$. Let

$$(T_1(t)f)(x) = \begin{cases} f(x+t) & \text{for } x + t < b' \\ 0 & \text{for } x + t \geqslant b' \end{cases}$$

for all $f \in C_0\,[a',b')$, $x \in [a',b')$, $t \geqslant 0$. Then $(T_1(t))_{t\geqslant 0}$ is a semigroup on $C_0[a',b')$ with generator B_1 given by $B_1 f = f'$ with domain $D(B_1) = \{f \in C_0\,[a',b') \cap C^1(a',b)\colon \lim_{x\to b'} f'(x) = 0\}$. If we consider B as an operator on $C_0\,[a',b')$, then $B \subset B_1$. Let $f \in D(B)$. Then $u(t) := T(t)f \in D(B) \subset D(B_1)$ for all $t \geqslant 0$; and $\dot{u}(t) = Bu(t) = B_1 u(t)$; $u(0) = f$. It follows from A-I, Theorem 1.7.

(or A-II, Corollary 1.2.) that $T_1(t)f = u(t)$. Hence $T_1(t)f \in C_0(a', b')$ for all $t \geqslant 0$ and $f \in D(B)$. This is impossible since $a' > -\infty$. Similary one shows that $b' = \infty$.□

Proof (Proof of Theorem 3.17) Suppose that m is admissible. It is easy to see that (3.22) then defines a continuous flow on (a, b). Moreover, for every $x \in (a, b)$ the function $\varphi(\cdot, x)$ is differentiable and satisfies

$$\frac{\partial}{\partial t}\varphi(t, x) = m(\varphi(t, x)) \quad (x \in (a, b), t \in \mathbb{R}). \tag{3.23}$$

Denote by $(T(t))_{t\in\mathbb{R}}$ the group on $C_0(a, b)$ given by $T(t)f = f \circ \varphi_t$ $(t \in \mathbb{R}, f \in C_0(a, b))$ and let A be its generator. Take $g \in C_0(a, b)$ and $f = R(1, A)g$. Then $f(x) = \int_0^\infty e^{-t} g(\varphi(t, x))\,\mathrm{d}t$, $x \in (a, b)$. If $m(x) = 0$ then $f(x) = \int_0^\infty e^{-t} g(x)\,\mathrm{d}t = g(x)$. If $x \in (a_n, b_n)$ $(n \in J)$, then $f(x) = \int_0^\infty e^{-t} g(q_n^{-1}(q_n(x) + t))\,\mathrm{d}t = e^{q_n(x)} \int_{q_n(x)}^\infty e^{-s} g(q_n^{-1}(s))\,\mathrm{d}s$. Thus f is differentiable at x and $f'(x) = (1/m(x))(f(x) - g(x))$. Consequently $f \in D(\delta_m)$ and $\delta_m f = f - g$. This shows that $A \subset \delta_m$. In order to show the converse inclusion, let $f \in D(\delta_m)$ and $g = f - \delta_m(f) \in C_0(a, b)$. Then $R(1, A)g(x) = f(x)$ if $m(x) = 0$ and $R(1, A)g(x) = \int_0^\infty e^{-t} f(\varphi(t, x))\,\mathrm{d}t - \int_0^\infty e^{-t} m(\varphi(t, x)) f'(\varphi(t, x))\,\mathrm{d}t = \int_0^\infty e^{-t} f(\varphi(t, x))\,\mathrm{d}t - \int_0^\infty e^{-t} \frac{\partial}{\partial t} f(\varphi(t, x))\,\mathrm{d}t$ (by (3.23)) $= f(x)$ by integrating by parts. This shows that $f = R(1, A)g \in D(A)$ and that δ_m is the generator of the group $(T(t))_{t\in\mathbb{R}}$. Finally we show that $T(t)D_o(\delta_m) \subset D_o(\delta_m)$, which implies that $D_o(\delta_m)$ is a core (by A-II, Corollary 1.34.). Let $t \in \mathbb{R}$. The function $\varphi_t(\cdot)$ is differentiable on (a, b) and $m(x)\frac{\partial}{\partial x}\varphi(t, x) = m(\varphi(t, x))$ for all $x \in (a, b)$. Let $f \in D_o(\delta_m) = D(\delta_m) \cap C^1$, $t \in \mathbb{R}$. Then $T(t)f = f \circ \varphi_t$ is differentiable and so in $D_o(\delta_m)$.

Conversely, assume that δ_m is generator of a group $(T(t))_{t\in\mathbb{R}}$ on $C_0(a, b)$. Since δ_m is a derivation, there exists a continuous flow $(\varphi_t)_{t\in\mathbb{R}}$ on (a, b) such that $T(t)f = f \circ \varphi_t$ for all $f \in C_0(a, b), t \in \mathbb{R}$. In order to show that m is admissible let $a \leqslant c < d \leqslant b$ such that $m(x) \neq 0$ for all $x \in (c, d)$ and $m(c) = 0$ or $a = c = -\infty$ and $m(d) = 0$ or $d = b = \infty$. If $a < c$ then $m(c) = 0$; consequently $(\delta_m f)(c) = 0$ for all $f \in D(\delta_m)$. Thus $(T(t)f)(c) = f(c)$ for all $f \in D(\delta_m)$ and $t \in \mathbb{R}$. This shows that $\varphi(t, c) = c$ for all $t \in \mathbb{R}$. Consequently $\varphi_t(a, c) \subset (a, c)$ for all $t \in \mathbb{R}$. Similary $\varphi_t(d, b) \subset (d, b)$ for all $t \in \mathbb{R}$. Thus the space $E_o := \{f \in C_0(a, b) \colon f \text{ vanishes off } (c, d)\}$ is invariant under the group $(T(t))_{t\in\mathbb{R}}$. We denote the group restricted to E_o by $(T_o(t))_{t\in\mathbb{R}}$ and by A_o its generator. Then $D(A_o) = \{f \in E_o \cap D(\delta_m) \colon \delta_m f \in E_o\}$. Identifying E_o with $C_0(c, d)$ we obtain $A_o = \delta_{m'}$, where m' denotes the restriction of m to (c, d). So it follows from Proposition 3.18 that m' is admissible. □

Remark 3.19 If φ is a flow on (a, b), a point $x \in (a, b)$ is called *stationary* if $\varphi(t, x) = x$ for all $t \in \mathbb{R}$. Let δ be the generator of the group $(T(t))_{t\in\mathbb{R}}$ associated with φ. Then $x \in (a, b)$ is a stationary point if and only if $(\delta f)(x) = 0$ for all $f \in D(\delta)$. If m is an admissible function on (a, b) then we have seen that $x \in (a, b)$ is a stationary point of the flow associated with δ_m if and only if $m(x) = 0$. This does no longer hold for functions which are not admissible as the following example shows.

Example 3.20 Consider the flow $\varphi(t,x) = (x^{1/3}+t)^3$ on $\mathbb{R}$ and the group $(T(t))_{t\in\mathbb{R}}$ induced by this flow on $C_0(\mathbb{R})$. One can easily see that the generator δ of $(T(t))_{t\in\mathbb{R}}$ is the following operator. Let $m(x) = 3x^{2/3}$. Then $(\delta f)(x) = m(x)f'(x)$ for $x \neq 0$ and $D(\delta) = \{f \in C_0(\mathbb{R})\colon f$ is differentiable in $x \neq 0$ and $m(x)f'(x)$ has a continuous extension in $C_0(\mathbb{R})\}$. However the function m is not admissible. And in fact $m(0) = 0$ but 0 is not a stationary point of φ. In particular, there exists a function $f \in D(\delta)$ such that $(\delta f)(0) \neq 0$.

Next we describe an arbitrary continuous flow on an open interval.

Proposition 3.21 *Let $-\infty \leqslant a < b \leqslant \infty$. A mapping $\varphi\colon \mathbb{R}\times(a,b) \to (a,b)$ defines a continuous flow if and only if there exists a finite or countable set of disjoint intervals $(a_n,b_n) \subset (a,b)$ $(n \in J)$ and for every $n \in J$ there exists a homeomorphism r_n from (a_n,b_n) onto $(-\infty,\infty)$ such that*

$$\varphi(t,x) = \begin{cases} x & \text{if } x \notin \bigcup_{n\in J}(a_n,b_n) \\ r_n^{-1}(r_n(x)+t) & \text{if } x \in (a_n,b_n), n \in J \end{cases}$$

for all $t \in \mathbb{R}$

Note: $J = \emptyset$ if and only if $\varphi(t,x) = x$ for all $x \in (a,b)$ and $t \in \mathbb{R}$.

Proof It is not difficult to see that the construction in the proposition defines a continuous flow on (a,b). Now let φ be a continuous flow. The set $K = \{x \in (a,b)\colon \varphi(t,x) = x$ for all $t \in \mathbb{R}\}$ is closed in (a,b). Thus $(a,b) \setminus K$ is the union of a finite or countable set of disjoint intervals (a_n,b_n) $(n \in J)$. Pick $x_n \in (a_n,b_n)$ $(n \in J)$. Then $\alpha_n(t) := \varphi(t,x_n)$ defines an injective mapping from $\mathbb{R}$ into (a_n,b_n). Thus α_n is strictly monotonous. It is easy to see that $\lim_{t\to\infty}\varphi(t,x_n)$ is an element of K whenever the limit exists in (a,b); similary for the limit as $t \to -\infty$. Consequently, $\alpha_n(\mathbb{R}) = (a_n,b_n)$ i.e., α_n is a homeomorphism from $\mathbb{R}$ onto (a_n,b_n). Define r_n to be the inverse of α_n. Let $x \in (a_n,b_n)$. Then $\varphi(t,x) = \varphi(t,\alpha_n(r_n(x))) = \varphi(t,\varphi(r_n(x),x_n)) = \varphi(t+r_n(x),x_n) = \alpha_n(t+r_n(x)) = r_n^{-1}(r_n(x)+t)$ for all $t \in \mathbb{R}$. This proves that φ has the desired form. $\square$

If m is an admissible function on (a,b), then $D(\delta)$ contains many differentiable functions. This can be expressed by two facts:

(i) $C_c^1(a,b) := \{f \in C^1(a,b)\colon f$ vanishes in a neighbourhood of a and $b\}$ is contained in $D(\delta_m)$ (this follows from the definition of δ_m); and
(ii) the set $D_o(\delta_m)$ of all differentiable functions in $D_o(\delta_m)$ is a core of δ_m (this follows from Theorem 3.17).

We will show below that these two properties are characteristic for the operators δ_m. For other generators of automorphism groups they can be violated dramatically as the following example shows.

Example 3.22 There exists a generator δ of an automorphism group on $C_0(\mathbb{R})$ such that $D(\delta) \cap C^1(\mathbb{R}) = \{0\}$. In fact, consider a strictly increasing continuous map q from $\mathbb{R}$ onto $\mathbb{R}$ such that $q'(x) = 0$ a.e. Then $V\colon C_0(\mathbb{R}) \to C_0(\mathbb{R})$ given by $Vf = f \circ q$ is an algebra isomorphism. Let A be the generator of the translation group on $C_0(\mathbb{R})$ and $\delta = V^{-1}AV$. Then $D(\delta) = \{f \in C_0(\mathbb{R})\colon Vf \in D(A)\} = \{f \in C_0(\mathbb{R})\colon f \circ q \in C^1(\mathbb{R}), (f \circ q)' \in C_0(\mathbb{R})\}$. Let $f \in C^1(\mathbb{R}) \cap D(\delta)$. If $f \neq 0$, then f is not constant. Hence there exists $x_o \in \mathbb{R}$ such that $f'(x_o) \neq 0$. Then f has a continuously differentiable inverse in some open neighbourhood of x_o. Since $f \circ q \in C^1(\mathbb{R})$, it follows that q is continuously differentiable in some neighborhood of $q^{-1}(x_o)$. This is a contradiction since $q'(y) = 0$ a.e.

Theorem 3.23 *Let δ be the generator of an automorphism group on $C_0((a, b))$, where $-\infty \leqslant a < b \leqslant \infty$. The following assertions are equivalent.*

(a) There exists a continuous admissible function $m\colon (a, b) \to \mathbb{R}$ such that $\delta = \delta_m$.
(b) $C_c^1(a, b) \subset D(\delta)$ and $D_o(\delta) = \{f \in D(\delta)\colon f$ is differentiable $\}$ is a core of δ.

Proof We have already pointed out that as a consequence of Theorem 3.17, (a) implies (b). So assume that (b) holds. Let $(T(t))_{t\in\mathbb{R}}$ be the group generated by δ and φ the continuous flow associated with the group. We can assume that φ is of the form given in Proposition 3.21. Let $n \in J$. We show that $r_n^{-1}\colon \mathbb{R} \to (a_n, b_n)$ is continuously differentiable. Let $x_o \in (a_n, b_n)$. There exists $f \in C_c^1(a, b)$ such that $f(x) = x$ in a neighborhood of x_o. Then $r_n^{-1}(r_n(x_o) + t) = f(\varphi(t, x_o)) = (T(t)f)(x_o)$ for all t in some neighborhood of 0. Since $f \in D(\delta)$ it follows that the function $t \mapsto r_n^{-1}(r_n(x_o) + t)$ is continuously differentiable in some neighborhood of 0 and so r_n^{-1} is continuously differentiable in $r_n(x_o)$. Since $r_n\colon (a_n, b_n) \to \mathbb{R}$ is surjective this proves the claim. Next we show $(r_n^{-1})'(t) \neq 0$ for all $t \in \mathbb{R}$. In fact, let $x_o \in (a_n, b_n)$ and assume that $(r_n^{-1})'(r_n(x_o)) = 0$. Then for all $f \in D_o(\delta)$ one has $(\delta f)(x_o) = \frac{\partial}{\partial t}|_{t=0} f(r_n^{-1}(r_n(x_o) + t)) = f'(x_o)(r_n^{-1})'(r_n(x_o)) = 0$. Since $D_o(\delta)$ is a core of δ this implies that $\varphi(t, x_o) = x_o$ for all $t \in \mathbb{R}$. Hence $x_o \in K$, a contradiction. It follows that $r_n\colon (a_n, b_n) \to \mathbb{R}$ is a C^1-diffeomorphism for all $n \in J$.

Define $m\colon (a, b) \to \mathbb{R}$ by

$$m(x) = \begin{cases} 0 & \text{if } x \in K \\ 1/r_n'(x) & \text{if } x \in (a_n, b_n), n \in J \end{cases}$$

Then m is continuous and admissible. The given flow coincides with the one constructed from m in Theorem 3.17. Thus $\delta = \delta_m$. □

Remark Let $m\colon (a, b) \to \mathbb{R}$ be continuous. Then m is admissible if and only if the initial value problem

$$\begin{aligned} \dot{y}(t) &= m(y(t)) \quad (t \in \mathbb{R}) \\ y(0) &= x \end{aligned}$$

has a unique solution $y \in C^1(\mathbb{R}, (a, b))$ which depends continuously on the initial value x (i.e., if $x_n \to x$ in (a, b) then the solution $y_n \in C^1(\mathbb{R}, (a, b))$ with initial value $y_n(0) = x_n$ satisfies $y_n(t) \to y(t)$ $(n \to \infty)$ for all $t \in \mathbb{R}$). This is not difficult to see. □

As we have seen above the operators δ_m, where m is an admissible function, do not exhaust all generators of automorphism groups. But one can obtain every such generator by a similarity transformation (see A-I,3.0) from some δ_m.

Theorem 3.24 *Let $-\infty \leqslant a < b \leqslant \infty$. An operator δ on $C_0(a,b)$ is the generator of an automorphism group on $C_0(a,b)$ if and only if there exists an algebra isomorphism V from $C_0(a,b)$ onto $C_0(a,b)$ and an admissible function $m\colon (a,b) \to \mathbb{R}$ such that $\delta = V^{-1}\delta_m V$.*

Proof In order to prove the non-trivial implication let $(T(t))_{t\in\mathbb{R}}$ be an automorphism group on $C_0(a,b)$ with generator δ. Let φ be the continuous flow on (a,b) such that $T(t)f = f \circ \varphi_t$ $(f \in C_0(a,b), t \in \mathbb{R})$. Then φ is of the form given in Proposition 3.21. For every $n \in J$ choose a C^1-diffeomorphism q_n from (a_n, b_n) onto $(-\infty, \infty)$ satisfying $q_n'(x) > 0$ for all $x \in (a_n, b_n)$ in the case when r_n is increasing and $q_n'(x) < 0$ for all $x \in (a_n, b_n)$ in the case when r_n is decreasing. Then $\beta_n := r_n^{-1} \circ q_n$ is a homeomorphism from (a_n, b_n) onto itself satisfying $\lim_{x\downarrow a_n} \beta_n(x) = a_n$ and $\lim_{x\uparrow b_n} \beta_n(x) = b_n$. Let $\beta\colon (a,b) \to (a,b)$ be defined by

$$\beta(x) = \begin{cases} x & \text{if } x \in K \\ \beta_n(x) & \text{if } x \in (a_n, b_n), n \in J \end{cases}$$

Then β is a homeomorphism from (a,b) onto (a,b) and $\psi_t := \beta^{-1} \circ \varphi_t \circ \beta$ $(t \in \mathbb{R})$ defines a continuous flow on (a,b). Define $m\colon (a,b) \to \mathbb{R}$ by

$$m(x) = \begin{cases} 0 & \text{if } x \in K \\ 1/q_n'(x) & \text{if } x \in (a_n, b_n) \end{cases}$$

Then m is continuous and admissible and the flow ψ coincides with the flow constructed from m in Theorem 3.17. Hence δ_m is the generator of the group $(S(t))_{t\in\mathbb{R}}$ given by $S(t)f = f \circ \psi_t = f \circ \beta^{-1} \circ \varphi_t \circ \beta = VT(t)V^{-1}f$, where V is the isomorphism on $C_0(a,b)$ given by $Vf = f \circ \beta$. Consequently, $\delta = V^{-1}\delta_m V$. □

Now we are able to describe arbitrary generators of positive groups on $C_0(a,b)$.

Theorem 3.25 *Let $-\infty \leqslant a < b \leqslant \infty$. An operator A generates a positive group on $C_0(a,b)$ if and only if there exist*

i) a lattice isomorphism V on $C_0(a,b)$,
ii) an admissible function m on (a,b),
iii) a bounded continuous function $h\colon (a,b) \to \mathbb{R}$ such that

$$A = V^{-1}\delta_m V + h. \tag{3.24}$$

Proof Let A be the generator of a positive group on $C_0(a,b)$. By Theorem 3.14 there exist a continuous bounded function $p\colon (a,b) \to \mathbb{R}$ such that $\inf_{x\in(a,b)} p(x) > 0$ and $h \in C^b(a,b)$ and the generator δ of an automorphism group such that $A = M\delta M^{-1} + h$ where $M \in \mathrm{L}(C_0(a,b))$ is given by $Mf = p \cdot f$. By Theorem 3.24 there exist an admissible continuous function $m\colon (a,b) \to \mathbb{R}$ and a lattice isomorphism $U \in \mathrm{L}(C_0(a,b))$ such that $\delta = U\delta_m U^{-1}$. Setting $V = MU$ we obtain $A = V\delta_m V^{-1} + h$. □

Finally we consider compact intervals. Let $-\infty < a < b < \infty$ and φ be a continuous flow on $[a,b]$. Then it is easy to see that $\varphi(a,t) = a$ and $\varphi(b,t) = b$ for all $t \in \mathbb{R}$. So the restriction φ_0 of φ to (a,b) is a continuous flow on (a,b). Conversely, if φ_0 is a continuous flow on (a,b) the extension φ_0 to $\varphi\colon \mathbb{R} \times [a,b] \to [a,b]$ by setting $\varphi(t,a) = a$; $\varphi(t,b) = b$ for all $t \in \mathbb{R}$ defines a continuous flow on $[a,b]$. This consideration allows us to extend easily the preceding results to the space $C[a,b]$. Let $m\colon (a,b) \to \mathbb{R}$ be a continuous function. We define the operator $\tilde{\delta}_m$ on $C[a,b]$ by $\tilde{\delta}_m f = g$ such that

$$g(x) = \begin{cases} m(x)f'(x) & \text{if } m(x) \neq 0 \\ 0 & \text{if } m(x) = 0 \end{cases} \tag{3.25}$$

for all $x \in (a,b)$ and

$$D(\tilde{\delta}_m) = \left\{ f \in C[a,b] \colon \begin{array}{l} f \text{ is differentiable at } x \in (a,b) \text{ whenever } m(x) \neq 0 \\ \text{and there exists a (necessarily unique) } g \in C[a,b] \\ \text{such that (3.25) holds} \end{array} \right\}.$$

Theorem 3.26 *Let m be a continuous function on (a,b). The operator $\tilde{\delta}_m$ is generator of an automorphism group on $C[a,b]$ if and only if m is admissible.*

Proof If $\tilde{\delta}_m$ generates an automorphism group $(T(t))_{t\in\mathbb{R}}$ then by the remark above $T(t)C_0(a,b) = C_0(a,b)$ $(t \in \mathbb{R})$. The generator of the restricted group has the domain $\{f \in C_0(a,b) \cap D(\tilde{\delta}_m)\colon \tilde{\delta}_m f \in C_0(a,b)\} = D(\delta_m)$. Hence δ_m is a generator and so m is admissible by Theorem 3.17. Conversely, if m is admissible, then δ_m generates a group on $C_0(a,b)$ given by a flow φ_0 on (a,b). Extending φ_0 to $[a,b]$ as above one obtains a continuous flow φ on $[a,b]$ which defines a group $(T(t))_{t\in\mathbb{R}}$. It is easy to verify, that the generator of this group is $\tilde{\delta}_m$. □

Theorem 3.27 *Let δ be the generator of an automorphism group on $C[a,b]$. Then there exists an admissible function $m\colon (a,b) \to \mathbb{R}$ and an algebra isomorphism V from $C[a,b]$ onto $C[a,b]$ such that $\delta = V^{-1}\tilde{\delta}_m V$.*

Proof The restriction δ_o of δ to $C_0(a,b)$ is the generator of an automorphism group. Thus by Theorem 3.24 there exists a continuous admissible function $m\colon (a,b) \to \mathbb{R}$ and an algebra isomorphism V_o from $C_0(a,b)$ onto $C_0(a,b)$ such that $\delta_o = V_o^{-1}\delta_m V_o$. Let V be the unique algebra isomorphism on $C[a,b]$ which extends V_o. Then it is easy to see that $\delta = V^{-1}\tilde{\delta}_m V$. □

Theorem 3.28 *An operator A on $C[a,b]$ is the generator of a positive group on $C[a,b]$ if and only if there exist*

- *a lattice isomorphism V on $C[a,b]$*
- *an admissible function $m\colon (a,b) \to \mathbb{R}$*

– *and a function* $h \in C[a, b]$ *such that* $A = V^{-1}\delta_m V + h$.

The proof follows from Theorem 3.14 via Theorem 3.27 in the same way as Theorem 3.25 (via Theorem 3.24).

NOTES

Section 1

Concerning bounded generators of positive semigroups and the positive minimum principle we refer to the corresponding notes in Chapter C-II. Theorem 1.6 and 1.8 are due to **?**], but we give a more direct proof here. Theorem 1.13 and its corollary are from the same source. In the case when A is dissipative Theorem 1.20 is due to **?**]. We use precisely Dorroh's arguments to verify the range condition. Other !!Jahr 74/75 extensions of Dorroh's result have been given by **?**] and **?**].

Section 2

A characterization of generators of lattice semigroups by Kato's equality is due to **?**] if the underlying space has order continuous norm (see C-II, Section 5), for general Banach spaces and $C_0(X)$ in particular the problem has been considered in **?**]. Theorem 2.10 is due to **?**].

Section 3

The characterization of generators of lattice semigroups as perturbation of a derivation (Theorem 3.5 and 3.6) is due to **?**]. The corresponding result for positive groups Lin: cite wie on $C_0(X)$ (Theorem 3.14) was obtained by **?**]. **?**] [Lin-Montgomery-Sine (1977)]

consider multiplicative perturbations of a generator of an automorphism group on $C(K)$ (K compact) by a function m which has a finite number of zeros. The function m is assumed to satisfy the "generalized Osgood condition" which is similar to being *admissable* (in our sense) but in addition the given flow is involved in the definition.

?] determined all densely defined derivations δ on $C[0, 1]$ which are well-behaved (i.e., $\pm\delta$ is dispersive) by a representation similar to Theorem 3.24. In contrast to Batty, here we assume that δ is the generator of a group. This simplifies the matter considerably since all continuous flows on an interval are easy to determine (Proposition 3.21). Our approach is inspired by **?**] to whom Theorem 3.24 is due.

For simplicity we confined ourselves to groups. **?**] determined all semi-flows on an interval.

In the sequel of Batty's work (loc.cit.) a characterization of all densely defined closed derivations on $C[0,1]$ has been obtained by [Kurose] in a series of papers (1981), (1982), (1983) [**?**], **?**], **?**]]. wie??

Chapter B-III

Spectral Theory of Positive Semigroups on $C_0(X)$

It is known that for a single operator $T \in \mathrm{L}C_0(X)$ the positivity of T has influence on the spectrum of T, mainly on the peripheral spectrum, i.e., the part of the spectrum containing all spectral values of maximal absolute value. This part of the spectrum is of interest because it determines the asymptotic behavior of the iterates T^n for large $n \in \mathbb{N}$. The spectral properties indicated above were first proved by **?**] and **?**] for positive square matrices, i.e., for positive operators on the Banach lattice $\mathbb{C}^n$. Later these results were extended to the infinite dimensional setting; important contributions are due to Jentzsch, Karlin, Krein, Krasnoselski'i, Lotz, Rota, Rutman, Schaefer and others (see Chapt.V of **?**]).

In this chapter we investigate the spectrum $\sigma(A)$ of the generator A of a positive semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ on the Banach space $C_0(X)$. Throughout this chapter we assume that $C_0(X)$ is the space of all *complex-valued* functions on the locally compact space X. In case we restrict to compact spaces we write K instead of X.

1 The Spectral Bound

One of the basic results on the spectrum of a positive operator is the fact that its spectral radius is an element of the spectrum (**?** , V.Proposition 4.1]). We begin the investigation of the spectrum of positive semigroups with the analogous result. To that purpose we recall that the spectral bound $s(A)$ of a generator A is defined as the least upper bound of the real parts of all spectral values (cf. A-III,(1.2)).

Theorem 1.1 *If A is the generator of a positive semigroup on $E = C_0(X)$, then $s(A) \in \sigma(A)$ unless $s(A) = -\infty$. In case X is compact, we always have $s(A) > -\infty$.*

Proof We suppose $\sigma(A) \neq \emptyset$ (i.e., $s(A) > -\infty$) and assume $s(A) \notin \sigma(A)$. Then there exist $\varepsilon > 0$ and $\alpha_0, \beta_0 \in \mathbb{R}$ such that

$$[s(A) - \varepsilon, \infty) \subset \varrho(A), \quad \mu_0 := \alpha_0 + \mathrm{i}\beta_0 \in \sigma(A) \quad \text{and} \quad \alpha_0 > s(A) - \varepsilon\,. \tag{1.1}$$

Now we choose $\lambda_0 \in \mathbb{R}$ large enough such that

$$|\lambda_0 - \mu_0| < \lambda_0 - (s(A) - \varepsilon) \tag{1.2}$$

and, in addition, such that $\lambda_0 > \omega_0(A)$. Then the resolvent $R(\lambda_0, A)$ is a positive bounded operator, hence its spectral radius $r(R(\lambda_0, A))$ is a spectral value. From A-III, Proposition 2.5 it follows that

$$\lambda_0 - r(R(\lambda_0, A))^{-1} \in \sigma(A) \quad \text{and} \quad r(R(\lambda_0, A)) \geqslant |\lambda_0 - \mu_0|^{-1} \tag{1.3}$$

This and (1.2) implies that $\lambda_0 - r(R(\lambda_0, A))^{-1}$ is a real spectral value which is greater than $s(A) - \varepsilon$. We have derived a contradiction to (1.1) and thus have proved the first statement of the theorem.

To establish the second statement we recall that $\lim_{\lambda\to\infty} \lambda R(\lambda, A)f = f$ for every $f \in E$. In particular, for $f = \mathbb{1}_X$ we find a (large) $\lambda_0 \in \mathbb{R}$ such that

$$\lambda_0 R(\lambda_0, A)\mathbb{1}_X \geqslant 1/2 \cdot \mathbb{1}_X \quad \text{hence} \quad R(\lambda_0, A)\mathbb{1}_X \geqslant (2\lambda_0)^{-1} \cdot \mathbb{1}_X \tag{1.4}$$

We may assume $\lambda_0 > \omega_0(A)$ then $R(\lambda_0, A) \geqslant 0$, and iterating (1.4) we obtain

$$R(\lambda_0, A)^n \mathbb{1}_X \geqslant (2\lambda_0)^{-n} \cdot \mathbb{1}_X > 0 \quad \text{for every } n \in \mathbb{N}\,. \tag{1.5}$$

It follows that $\|R(\lambda_0, A)^n\| \geqslant (2\lambda_0)^{-n}$ and therefore

$$r(R(\lambda_0, A)) = \lim_{n\to\infty} \|R(\lambda_0, A)^n\|^{1/n} \geqslant (2\lambda_0)^{-1} > 0\,. \tag{1.6}$$

Thus $\sigma(R(\lambda_0, A))$ contains non-zero spectral values which in view of A-III, Proposition 2.5 is equivalent to $\sigma(A) \neq \emptyset$. □

The following examples show that the spectrum may be empty in case X is not compact or if the semigroup is not positive.

Examples 1.2 (i) Consider $X = [0, 1)$ and $(T(t))$ on $C_0(X)$ given by

$$(T(t)f)(x) := \begin{cases} f(x+t) & \text{if } x + t < 1\,, \\ 0 & \text{if } x + t \geqslant 1\,. \end{cases} \tag{1.7}$$

Then $(T(t))_{t\geqslant 0}$ is nilpotent (we have $T(t) = 0$ for $t \geqslant 1$). It follows that $\sigma(T(t)) = \{0\}$ for all $t > 0$ and by A-III, Theorem 6.2 we have $\sigma(A) = \emptyset$.

(ii) The operator A on $E := C_0[0, \infty)$ given by

$$(Af)(x) = f'(x) - xf(x), D(A) = \{f \in E : f \in C^1, Af \in E\} \tag{1.8}$$

has empty spectrum. It is the generator of a positive non-nilpotent semigroup which is given by

$$(T(t)f)(x) = \exp(-(t^2/2) - xt) \cdot f(x+t)\,. \tag{1.9}$$

(iii) Taking into account that $C_0([0, 1[)$ as well as $C_0([0, \infty[)$ both are topologically (but not isometrically) isomorphic to $C([0, 1])$ (see **?** , Section 21.5]), one obtains from (i) and (ii) (non-positive) semigroups on $C([0,1])$ whose generators have empty spectrum.

The proof of Theorem 1.1 given above is based on the fact that the spectral radius of a bounded positive operator is an element of the spectrum. A direct proof not using this fact is given in C-III,Corollary 1.4.

Corollary 1.3 *Suppose $\lambda_0 \in \varrho(A)$. Then $R(\lambda_0, A)$ is a positive operator if and only if $\lambda_0 > s(A)$. For $\lambda > s(A)$ we have $r(R(\lambda, A)) = (\lambda - s(A))^{-1}$.*

Proof The second statement is an immediate consequence of Theorem 1.1 and A-III, Proposition 2.5.

Given $\lambda_0 > s(A)$ we choose $\lambda_1 > \max\{\lambda_0, \omega_0(A)\}$. Since

$$|\lambda_1 - \lambda_0| < |\lambda_1 - s(A)| = r(R(\lambda_1, A))^{-1}$$

we have

$$R(\lambda_0, A) = \sum_{n=0}^{\infty} (\lambda_1 - \lambda_0)^n \cdot R(\lambda_1, A)^{n+1} . \tag{1.10}$$

Since $R(\lambda_1, A)$ is positive, it follows that $R(\lambda_0, A)$ is positive as well. On the other hand, assuming that $R(\lambda_0, A)$ is a positive operator, then λ_0 has to be a real number (note that for $g \geqslant 0$ we have $f := R(\lambda_0, A)g \geqslant 0$ hence $\lambda_0 f - Af = g = \overline{g} = \overline{(\lambda_0 - A)f} = \overline{\lambda_0} f - Af$). As we have shown above, $R(\lambda, A)$ is positive for $\lambda > \max\{\lambda_0, s(A)\}$ hence an application of the resolvent equation yields

$$R(\lambda_0, A) = R(\lambda, A) + (\lambda - \lambda_0) R(\lambda, A) R(\lambda_0, A) \geqslant R(\lambda, A) \geqslant 0 \tag{1.11}$$

It follows that for all $\lambda > \max\{\lambda_0, s(A)\}$ we have

$$(\lambda - s(A))^{-1} = r(R(\lambda, A)) \leqslant \|R(\lambda, A)\| \leqslant \|R(\lambda_0, A)\| \tag{1.12}$$

which can be true only if λ_0 is greater than $s(A)$. □

Corollary 1.4 *Suppose X is compact and A has compact resolvent. Then there exists a real eigenvalue λ_0 admitting a positive eigenfunction such that* $\operatorname{Re}\lambda \leqslant \lambda_0$ *for every $\lambda \in \sigma(A)$.*

Proof From Theorem 1.1 we conclude that $\lambda_0 := s(A)$ is a real number, contained in the spectrum and obviously $\operatorname{Re}\lambda \leqslant \lambda_0$ for every $\lambda \in \sigma(A)$. Since A has compact resolvent it follows that λ_0 is a pole of the resolvent. Let k be its order, then the highest coefficient in the Laurent series is given by

$$Q := \lim_{\lambda \to s(A)} (\lambda - s(A))^k R(\lambda, A) . \tag{1.13}$$

It follows from Corollary 1.3 that Q is a positive operator. Since $Q \neq 0$, there exists $g \geqslant 0$ such that $h := Qg > 0$. Moreover, by A-III,3.6., we have

$$(\lambda_0 - A)h = (\lambda_0 - A)Qg = 0.$$

The example of the rotation semigroup (A-III,Example 5.6) shows that the assumptions in Corollary 1.4 do not imply that $s(A)$ is dominant. Additional hypotheses ensuring this stronger property will be given below (see Corollary 2.11 and 2.12).

The following result is elementary. However, positivity is the crucial point in its proof. Note that it is not just a consequence of the spectral mapping theorem for the point spectrum.

Proposition 1.5 *Suppose A is the generator of the positive semigroup $(T(t))_{t\geqslant 0}$. Take $\tau > 0$, $r > 0$ and let $\alpha := \tau^{-1}\log(r)$.*

(i) If r is an eigenvalue of $T(\tau)$ with positive eigenfunction h_0, then there is a positive $h \in D(A)$ such that $Ah = \alpha h$ and $\{x \in X : h_0(x) > 0\} \subset \{x \in X : h(x) > 0\}$.

(ii) If r is an eigenvalue of $T(\tau)'$ with positive eigenvector φ_0, then there is a positive $\varphi \in D(A^*)$ such that $A^*\varphi = \alpha\varphi$ and $\operatorname{supp}\varphi_0 \subset \operatorname{supp}\varphi$.

Proof Without loss of generality we may assume $r = 1$, i.e., $\alpha = 0$ and $T(\tau)h_0 = h_0$.

(i) Defining

$$h := \int_0^\tau T(s)h_0 \,\mathrm{d}s\,, \tag{1.14}$$

then for $0 \leqslant t \leqslant \tau$ we have

$$\begin{aligned} T(t)h &= \int_0^\tau T(s+t)h_0 \,\mathrm{d}s = \int_t^\tau T(s)h_0 \,\mathrm{d}s + \int_\tau^{\tau+t} T(s-\tau)T(\tau)h_0 \,\mathrm{d}s \\ &= \int_t^\tau T(s)h_0 \,\mathrm{d}s + \int_0^t T(s)T(\tau)h_0 \,\mathrm{d}s = h \end{aligned}$$

It follows that $Ah = \lim_{t\to 0} t^{-1}(T(t)h - h) = 0$. So far, positivity was not used. The point is that in general, h may be zero. But if $(T(t))$ is positive and $h_0 \geqslant 0$, then $s \mapsto (T(s)h_0)(x)$ is a continuous positive function, hence $0 < h_0(x_0) = (T(0)h_0)(x_0)$ implies $h(x_0) = \int_0^\tau (T(s)h_0)(x_0)\,\mathrm{d}s > 0$.

(ii) Defining $\varphi := \int_0^\tau T(s)'\varphi_0 \,\mathrm{d}s$, one can proceed as in (i) to obtain the desired result. $\square$

We use Proposition 1.5 to prove an analogue of the famous Krein-Rutman result. For the sake of completeness we include the proof of this classical result, stating that the spectral radius of a positive operator T on $C(K)$ (or more generally on an order unit space) is an eigenvalue of the adjoint T' (see the Corollary of Theorem 2.6 in the appendix of **?**]).

Theorem 1.6 *Suppose K is compact and $(T(t))_{t\geqslant 0}$ is a positive semigroup on $C(K)$ with generator A. Then there exists a positive probability measure $\varphi \in D(A')$ such that $A'\varphi = \omega_0(A)\varphi$.*

Proof Consider $T := T(1)$, $r := r(T) = \mathrm{e}^{\omega_0(A)}$. In view of Proposition 1.5 it is enough to show that r is an eigenvalue of T' with a positive eigenvector. Given $\lambda \in \mathbb{C}$, $|\lambda| > r$ and $f \in C(K)$ we have

$$|R(\lambda,T)f| = \left|\sum_{n=0}^{\infty} \lambda^{-n-1} T^n f\right| \leqslant \sum_{n=0}^{\infty} |\lambda|^{-n-1} T^n |f| = R(|\lambda|,T)|f|\,.$$

It follows that $\|R(\lambda,T)\| \leqslant \|R(|\lambda|,T)\|$ and therefore

$$\lim_{\lambda \downarrow r} \|R(\lambda,T)\| = \infty\,. \tag{1.15}$$

By the uniform boundedness principle there exist a sequence (λ_n), $\lambda_n \downarrow r$ and a positive $\psi \in M(K)$ such that $\|R(\lambda_n,T)'\psi\| \to \infty$. Defining $\psi_n := \|R(\lambda_n,T)'\psi\|^{-1} R(\lambda_n,T)'\psi$ we have

$$\begin{aligned}(r - T')\psi_n &= \|R(\lambda_n,T)'\psi\|^{-1} \cdot ((r-\lambda_n) + (\lambda_n - T'))R(\lambda_n,T)'\psi \\ &= (r-\lambda_n)\psi_n + \|R(\lambda_n,T)'\psi\|^{-1}\psi \to 0\,.\end{aligned} \tag{1.16}$$

Since $(r-T')$ is $\sigma(M(K),C(K))$-continuous, (1.16) implies that every $\sigma(M(K),C(K))$ cluster point of (ψ_n) is a positive eigenvector, provided that it is non-zero. Because K is compact we have

$$\{\varphi \in M(K) \colon \varphi \geqslant 0, \|\varphi\| = 1\} = \{\varphi \in M(K) \colon \varphi \geqslant 0, \langle \varphi, 1\rangle = 1\},$$

which shows that the set of probability measures is $\sigma(M(K),C(K))$-compact. Therefore the sequence (ψ_n) has non-zero cluster points. ◻

This theorem implies that for positive semigroups on $C(K)$ the growth and spectral bounds coincide (cf. A-III,4.4). Actually, this is true for locally compact spaces as well and can be proved directly (see B-IV, Theorem 1.4). Using this result one can prove Theorem 1.6 by applying the classical Krein-Rutman theorem to any resolvent operator $R(\lambda,A)$ for $\lambda \in \mathbb{R}$ sufficiently large.

The theorem ensures that A' always has eigenvalues, but the generator itself may have no eigenvalue at all. Multiplication operators have no eigenvalues unless the multiplier is constant on an open subset. Theorem 1.6 fails to be true for locally compact spaces as the following example shows.

Example 1.7 Consider $E = C_0(\mathbb{R}^n)$ and the semigroup $(T(t))_{t\geqslant 0}$ generated by the Laplacian (cf. A-I,2.8). From the explicit representation of $T(t)$,

$$(T(t)f)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-(x-y)^2/4t) \cdot f(y)\,\mathrm{d}y\,, \tag{1.17}$$

it follows that $\lim_{t\to\infty} T(t)f = 0$ for every $f \in C_0(\mathbb{R}^n)$ (Note that $\|T(t)f\| \leqslant (4\pi t)^{-n/2} \int_{\mathbb{R}^n} |f(y)|\,\mathrm{d}y \to 0$ provided that f has compact support and that $\|T(t)\| = 1$ for all $t \geqslant 0$).

If φ is an eigenvector of A' corresponding to $s(A) = \omega_0(A) = 0$, we have $T(t)'\varphi = \varphi$ for all $t \geqslant 0$, hence $\langle \varphi, f\rangle = \lim_{t\to\infty}\langle T(t)f, \varphi\rangle = 0$ for every f, i.e., $\varphi = 0$.

2 The Boundary Spectrum

In this section we restrict our attention to the *boundary spectrum* $\sigma_b(A)$ of a generator A, which, by definition, is the intersection of $\sigma(A)$ with the line $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda = s(A)\}$. Thus $\sigma_b(A)$ contains all spectral values of A which have maximal real part. Note that in general the boundary spectrum is a proper subset of the topological boundary of $\sigma(A)$. Our aim is to prove results ensuring that $\sigma_b(A)$ is a cyclic set (see Definition 2.5).

While most of the results of the preceding section were obtained by transforming the problem to a resolvent operator $R(\lambda, A)$ ($\lambda \in \mathbb{R}$ large enough), this procedure fails here. The reason is that there is no one-to-one correspondence between the boundary spectrum of A and the peripheral spectrum of $R(\lambda, A)$. Actually, from Theorem 1.1 and A-III, Proposition 2.5 it follows that the peripheral spectrum of $R(\lambda, A)$ (i.e., the set of spectral values having maximal absolute value) is trivial, since it only contains the spectral radius $r(R(\lambda, A)) = (\lambda - s(A))^{-1}$.

We begin our discussion with two lemmas.

Lemma 2.1 *Suppose K, L are compact and $T\colon C(K) \to C(L)$ is a linear operator satisfying $T\mathbb{1}_K = \mathbb{1}_L$. Then we have $T \geqslant 0$ if and only if $\|T\| \leqslant 1$.*

Proof If T is positive, then

$$|Tf| \leqslant T|f| \leqslant T(\|f\| \cdot \mathbb{1}_K) = \|f\| \cdot T(\mathbb{1}_K), \quad f \in C(K)\,, \tag{2.1}$$

hence $\|T\| = \|T\mathbb{1}_K\|$ whenever T is positive. This shows that $T \geqslant 0$ implies $\|T\| \leqslant 1$ whenever $T\mathbb{1}_K = \mathbb{1}_L$.

To prove the reverse direction, we first observe that for complex numbers and hence for complex-valued functions the following equivalence holds.

$$-1 \leqslant f \leqslant 1 \text{ if and only if } \|f - \mathrm{i} \cdot r \cdot 1\| \leqslant \varrho_r := (1 + r^2)^{1/2} \text{ for every } r \in \mathbb{R}\,. \tag{2.2}$$

Now suppose $f \in C(K)$, $0 \leqslant f \leqslant 2 \cdot \mathbb{1}_K$. Then we have $-\mathbb{1}_K \leqslant f - \mathbb{1}_K \leqslant \mathbb{1}_K$ hence by (2.2) $\|f - \mathbb{1}_K - \mathrm{i} \cdot r \cdot \mathbb{1}_K\| \leqslant \varrho_r$ for every $r \in \mathbb{R}$. From $T\mathbb{1}_K = \mathbb{1}_L$ and $\|T\| \leqslant 1$ it follows that $\|Tf - \mathbb{1}_L - \mathrm{i} \cdot r \cdot \mathbb{1}_L\| \leqslant \varrho_r$ for every $r \in \mathbb{R}$. Using (2.2) once again, we obtain $-\mathbb{1}_L \leqslant Tf - \mathbb{1}_L \leqslant \mathbb{1}_L$ or $0 \leqslant Tf \leqslant 2 \cdot \mathbb{1}_L$. □

Before we can formulate the second lemma we have to fix some notation.

Definition 2.2 (i) Given $h \in C_0(X)$ such that $h(x) \neq 0$ for all $x \in X$ then the operator S_h is defined to be the multiplication operator with sign h, i.e.,

$$S_h f = h|h|^{-1} f \quad (f \in C_0(X))\,. \tag{2.3}$$

(ii) For $f \in C_0(X)$, $n \in \mathbb{Z}$ we define $f^{[n]} \in C_0(X)$ by

$$f^{[n]}(x) = \begin{cases} (f(x)/|f(x)|)^{n-1} \cdot f(x) & \text{if } f(x) \neq 0\,, \\ 0 & \text{if } f(x) = 0\,. \end{cases} \tag{2.4}$$

The following assertions are immediate consequences of the definition. They will be used frequently in the following.

$$\begin{aligned}&S_h \text{ is a linear isometry satisfying } |S_h f| = |f|,\\ &\text{its inverse being } S_{\overline{h}} \text{ where } \overline{h} \text{ is the complex conjugate of } h,\end{aligned} \tag{2.5}$$

$$f^{[1]} = f, f^{[0]} = |f|, f^{[-1]} = \overline{f}, |f^{[n]}| = |f| \text{ for every } n \in \mathbb{Z}, \tag{2.6}$$

$$\text{If } h(x) \neq 0 \text{ for all } x \in X, \text{ then } h^{[n]} = S_h^n|h| = S_h^{n-1}h \text{ for every } n \in \mathbb{Z}. \tag{2.7}$$

Lemma 2.3 *Let T and R be bounded linear operators on $C_0(X)$ and assume that $h \in C_0(X)$ has no zeros. Suppose we have*

$$Rh = h, T|h| = |h| \text{ and } |Rf| \leqslant T|f| \text{ for every } f \in C_0(X). \tag{2.8}$$

Then R and T are similar, more precisely, $T = S_h^{-1}RS_h$. In particular, the spectra (and point spectra, resp.) of T and R coincide.

Proof We first note that the assertion $|Rf| \leqslant T|f|$ $(f \in E)$ implies that T is a positive operator. Therefore $T|h| = |h|$ implies that the principal ideal $E_h = \{f \in C_0(X) \colon |f| \leqslant n|h| \text{ for some } n \in \mathbb{N}\}$ is an invariant subspace for T and for R as well. E_h is isomorphic to $C^b(X) \cong C(\beta X)$ (βX denotes the Stone-Čech compactification of X), an isomorphism is given by $f \mapsto f|h|$. Considering the restrictions $T_{|E_h}$ and $R_{|E_h}$ as operators on $C(\beta X)$ and denoting them $\tilde{T}$ and $\tilde{R}$ respectively, we have

$$\tilde{R}\tilde{h} = \tilde{h}, \quad \tilde{T}\mathbb{1} = \mathbb{1}, \quad \tilde{T} \geqslant 0, \quad |\tilde{R}f| \leqslant \tilde{T}|f| \text{ for all } f\,. \tag{2.9}$$

Here $\tilde{h}$ denotes the continuous extension of $h/|h|$ to βX.

Defining $T_1 := M_{\tilde{h}}^{-1}\tilde{R}M_{\tilde{h}}$ we have by (2.9)

$$T_1 1 = M_{\tilde{h}}^{-1}\tilde{R}\tilde{h} = 1 \text{ and} \tag{2.10}$$

$$|T_1 f| = |M_{\tilde{h}}^{-1}\tilde{R}M_{\tilde{h}}f| = |\tilde{R}M_{\tilde{h}}f| \leqslant \tilde{T}|M_{\tilde{h}}f| = \tilde{T}|f| \text{ for all } f. \tag{2.11}$$

Hence we have $\|T_1\| \leqslant \|\tilde{T}\| = 1$ (by (2.11), (2.9), (2.1). Then it follows from Lemma 2.1 that T_1 is a positive operator. Thus (2.11) implies that $0 \leqslant T_1 \leqslant \tilde{T}$ and therefore $\|\tilde{T} - T_1\| = \|(\tilde{T} - T_1)1\| = 0$ (by (2.10), (2.9), (2.1). □

We are now able to prove a result which in some sense is the key to cyclicity results for the spectrum. These general results will be proved by reducing the problem in such a way that the following theorem can be applied.

Theorem 2.4 *(i) Let T be a positive linear operator on $C_0(X)$, let $h \in C_0(X)$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$. If $Th = \lambda h$ and $T|h| = |h|$, then we have $Th^{[n]} = \lambda^n h^{[n]}$ for every $n \in \mathbb{Z}$ (cf. (2.4)). If h does not have zeros in X, then*

$$\lambda T = S_h^{-1}TS_h\,.$$

(ii) Suppose A is the generator of a positive semigroup, $h \in C_0(X)$, $\alpha, \beta \in \mathbb{R}$ such that $Ah = (\alpha + \mathrm{i}\beta)h$ and $A|h| = \alpha|h|$. Then we have $Ah^{[n]} = (\alpha + \mathrm{i}n\beta)h^{[n]}$ for every $n \in \mathbb{Z}$. If h does not have zeros, then

$$S_h D(A) = D(A) \text{ and } \mathrm{i}\beta + A = S_h^{-1} A S_h \,.$$

Proof (i) The closed principal ideal $\overline{E_h}$, which is canonically isomorphic to $C_0(X_1)$ with $X_1 = \{x \in X\colon\ h(x) \neq 0\}$, is T-invariant. We give an object a tilde when we consider it as an element of $\overline{E_h} \cong C_0(X_1)$. Defining $\tilde{R} := \lambda\tilde{T}$, then $\tilde{T}$, $\tilde{R}$, $\tilde{h}$ satisfy (2.8), hence we have

$$\tilde{T} = S_{\tilde{h}}^{-1} \circ \tilde{R} \circ S_{\tilde{h}} = \bar{\lambda} \cdot S_{\tilde{h}}^{-1} \circ \tilde{T} \circ S_{\tilde{h}} \tag{2.12}$$

which by iteration yields

$$\tilde{T} = \bar{\lambda}^n \cdot S_{\tilde{h}}^n \circ \tilde{T} \circ S_{\tilde{h}}^n \text{ for all } n \in \mathbb{Z}\,. \tag{2.13}$$

It follows that

$$\tilde{T}\tilde{h}^{[n]} = \tilde{T} \circ S_{\tilde{h}}^n |\tilde{h}| = \lambda^n \cdot S_{\tilde{h}}^n \circ \tilde{T}|\tilde{h}| = \lambda^n \cdot S_{\tilde{h}}^n \tilde{h} = \lambda^n \cdot \tilde{h}^{[n]}$$

(see (2.7) and (2.12)), which is precisely $Th^{[n]} = \lambda^n h^{[n]}$ for all $n \in \mathbb{Z}$.

If h does not have zeros, then $\overline{E_h} = E$, hence $T = \tilde{T}$, $h = \tilde{h}$ and the remaining assertion follows from (2.12).

(ii) This can be deduced easily from (i) as follows. If $Ah = (\alpha + \mathrm{i}\beta)h$, $A|h| = \alpha|h|$, then we have by A-III,Corollary 6.4

$$\mathrm{e}^{-\alpha t}T(t)h = \mathrm{e}^{\mathrm{i}\beta t}h \text{ and } \mathrm{e}^{-\alpha t}T(t)|h| = |h| \text{ for every } t \geqslant 0\,.$$

Hence by (a) $\mathrm{e}^{-\alpha t}T(t)h^{[n]} = \mathrm{e}^{\mathrm{i}n\beta t}h^{[n]}$ $(t \geqslant 0,\ n \in \mathbb{Z})$ which is equivalent to $Ah^{[n]} = (\alpha + \mathrm{i}n\beta)h^{[n]}$. If h does not have zeros, then $\mathrm{e}^{-\alpha t}T(t) = \mathrm{e}^{-\alpha t}\mathrm{e}^{\mathrm{i}\beta t}S_h^{-1}T(t)S_h$ for every $t \geqslant 0$ which is equivalent to the final statement of (ii). □

Before we state a first cyclicity result we give the definition and illustrate it by some examples.

Definition 2.5 A subset $M \subset \mathbb{C}$ is called *imaginary additively cyclic* (or simply *cyclic*), if it satisfies the following condition.

$\alpha + \mathrm{i}\beta \in M$, $\alpha, \beta \in \mathbb{R}$ implies that $\alpha + \mathrm{i}k\beta \in M$ for every $k \in \mathbb{Z}$.

Every subset of $\mathbb{R}$ is cyclic. On the other hand, if M is cyclic and $M \not\subset \mathbb{R}$, then M has to be unbounded.

For a subset M of $\mathrm{i}\mathbb{R}$ we give the following equivalent conditions.

(a) M is imaginary additively cyclic,
(b) M is the union of (additive) subgroups of $\mathrm{i}\mathbb{R}$,
(c) $M = \cup_{\alpha \in S}\mathrm{i}\alpha\mathbb{Z}$ for some set $S \subset \mathbb{R}$.

Here are some concrete cyclic subsets of $\mathrm{i}\mathbb{R}$.

$$M_1 = \{0\}, \quad M_2 = \mathrm{i}\mathbb{R}, \quad M_3 = \mathrm{i}\alpha\mathbb{Z} \quad (\alpha > 0),$$
$$M_4 = \mathrm{i}\alpha\mathbb{Z} + \mathrm{i}\beta\mathbb{Z} = \{\mathrm{i}n\alpha + \mathrm{i}\beta\colon n, m \in \mathbb{Z}\} \quad (\alpha, \beta \in \mathbb{R}),$$
$$M_5 = \{0\} \cup \{\mathrm{i}\lambda\colon \lambda \in \mathbb{R}, |\lambda| \geqslant 1\},$$
$$M_6 = \cup_{n=0}^{\infty}\{\mathrm{i}\lambda\colon \lambda \in \mathbb{R}, n\alpha \leqslant |\lambda| \leqslant n\beta\} \quad (0 < \alpha \leqslant \beta, \alpha, \beta \in \mathbb{R})$$

In the following we consider the boundary spectrum of several semigroups. The notation M_i refers to the sets just defined.

Examples 2.6 (a) For the Laplacian Δ on $\mathbb{R}^n$ or the second derivative on $[0,1]$ with Neumann boundary conditions the boundary spectrum is M_1.

(b) The first derivative on $\mathbb{R}$ or $\mathbb{R}_+$ is an example where the boundary spectrum of the generator is M_2.

(c) The rotation semigroup on $C(\Gamma)$ (see A-III,Example 5.6) with period $2\pi/\alpha$ has boundary spectrum M_3.

(d) For the semigroup on $C(\Gamma \times \Gamma)$ given by

$$(T(t)f)(z, w) = f(z \cdot \mathrm{e}^{\mathrm{i}\alpha t}, w \cdot \mathrm{e}^{\mathrm{i}\beta t}) \quad (f \in C(\Gamma \times \Gamma), (z, w) \in \Gamma \times \Gamma)$$

we have $P\sigma(A) = M_4$. If α/β is irrational, then this is a dense subset of $\mathrm{i}\mathbb{R}$ and $\sigma_b(A) = \sigma(A) = \mathrm{i}\mathbb{R}$.

(e) Consider $D := \{z \in \mathbb{C}\colon |z| \leqslant 1\} = \{r \cdot \mathrm{e}^{\mathrm{i}\omega}\colon r \in [0,1], \omega \in \mathbb{R}\}$, and a strictly positive function $\kappa \in C[0,1]$. The flow on D governed by the differential equation $\dot{r} = 0$, $\dot{\omega} = \kappa(r)$ induces a strongly continuous semigroup on $C(D)$ (which is given by

$$(T(t)f)(z) = f(z \cdot \mathrm{e}^{\mathrm{i}\kappa(|z|)t}) .$$

The boundary spectrum is M_6 with $\alpha := \inf \kappa(r)$, $\beta := \sup \kappa(r)$. In particular, for $\kappa(r) = 1 + r$ we obtain as boundary spectrum the set M_5.

(f) Suppose M is a closed cyclic subset of $\mathrm{i}\mathbb{R}$, $M = \cup_{\alpha \in S}\mathrm{i}\alpha\mathbb{Z}$ for a suitable $S \subset \mathbb{R}$ (e.g., $S = M$).

The space $E_1 := \{(f_\alpha)_{\alpha \in S}\colon f_\alpha \in C(\Gamma), \sup \|f_\alpha\| < \infty\}$ is a Banach space under the norm $\|(f_\alpha)\| := \sup \|f_\alpha\|$. The closure of the linear subspace $E_0 := \{(f_\alpha) \in E_1\colon f_\alpha \neq 0$ only for finitely many $\alpha \in S\}$ is isomorphic to $C_0(X)$ where X is the topological sum of $|S|$ copies of Γ.

Let $(T_\alpha(t))_{t \geqslant 0}$ denote the rotation semigroup on $C(\Gamma)$ with period $2\pi/\alpha$, then we define a semigroup $(T(t))_{t \geqslant 0}$ on $E := C_0(X)$ as

$$(T(t)(f_\alpha)) := (T_\alpha(t) f_\alpha) \quad ((f_\alpha)_{\alpha \in S} \in E) .$$

This is a positive semigroup on $E = C_0(X)$ whose boundary spectrum is precisely the given closed cyclic set M. We leave the verification as an excercise.

Our first result concerns cyclicity of the eigenvalues contained in the boundary spectrum, i.e., of the set

$$P\sigma_b(A) := P\sigma(A) \cap \sigma_b(A) = \{\lambda \in P\sigma(A)\colon \operatorname{Re}\lambda = s(A)\}$$

It is an almost straightforward consequence of Theorem 2.4.

Proposition 2.7 *Assume that for some $t_0 > 0$ there is a strictly positive measure φ such that $T(t_0)'\varphi = \exp(s(A)t_0) \cdot \varphi$.*

Then $P\sigma_b(A)$ is imaginary additively cyclic.

Proof If $P\sigma_b(A)$ is empty, there is nothing to prove. Otherwise we have $s(A) > -\infty$. In view of the rescaling procedure we may assume $s(A) = 0$. By Proposition 1.5(ii) there exists $\Psi \gg 0$ such that $T(t)'\Psi = \Psi$ for all $t \geqslant 0$. Given $\mathrm{i}\alpha \in P\sigma_b(A)$ then there is $h \in C_0(X)$, $h \neq 0$, such that $Ah = \mathrm{i}\alpha h$ or $T(t)h = \mathrm{e}^{\mathrm{i}\alpha t}h$ for all t (A-III, Corollary 6.4). Then we have

$$|h| = |\mathrm{e}^{\mathrm{i}\alpha t}h| = |T(t)h| \leqslant T(t)|h| \text{ or } T(t)|h| - |h| \geqslant 0 \tag{2.14}$$

$$\langle T(t)|h| - |h|, \Psi\rangle = \langle |h|, T(t)'\Psi\rangle - \langle |h|, \Psi\rangle = 0. \tag{2.15}$$

Since Ψ is strictly positive, (2.14) and (2.15) imply that $T(t)|h| = |h|$ for $t \geqslant 0$ or equivalently $A|h| = 0$.

Now Theorem 2.4 implies that $Ah^{[n]} = \mathrm{i}n\alpha h^{[n]}$ $(n \in \mathbb{Z})$. □

Concerning the hypothesis $T(t_0)'\varphi = \exp(s(A)t_0) \cdot \varphi \gg 0$ we recall that in case X is compact there are always positive linear forms such that $T(t)'\varphi = \mathrm{e}^{s(A)t}\varphi$ (see Theorem 1.6). If the semigroup is irreducible, then one also has $\varphi \gg 0$ (see Section 3 below).

In a second result we consider semigroups having compact resolvent. An important step of the proof is isolated as a lemma. Before stating it, we recall that given a closed ideal $I \subset C_0(X)$, then I as well as $C_0(X)/I$ are spaces of continuous functions on a locally compact space vanishing at infinity. More precisely, if $I = \{f \in C_0(X) \colon f|_M = 0\}$ for a suitable closed subset $M \subset X$, then $I \cong C_0(X\backslash M)$ and $C_0(X)/I \cong C_0(M)$ (cf. B-I). It follows that given another closed ideal $J = \{f \in C_0(X) \colon f|_N = 0\}$ such that $I \subset J$ i.e., $N \subset M$, then J/I can be identified with $C_0(M\backslash N)$.

We do not use this concrete representation of J/I. However, this shows that we stay within our setting of Banach spaces of continuous functions on locally compact spaces.

Lemma 2.8 *Suppose A is the generator of a positive semigroup $\mathcal{T}$ such that the spectral bound $s(A)$ is a pole of the resolvent of order k. Then there is a sequence*

$$I_{-1} := \{0\} \subset I_0 \subsetneq I_1 \subsetneq \ldots \subsetneq I_k := E \tag{2.16}$$

of $\mathcal{T}$-invariant closed ideals with the following properties. Denoting by A_n $(n = 0, 1, \ldots, k)$ the generator of the semigroup on I_n/I_{n-1} which is induced by $(T(t))$ we have

(i) $s(A_0) < s(A)$,

(ii) If $n \geqslant 1$, then $s(A_n) = s(A)$ is a first order pole of the resolvent $R(\cdot, A_n)$. The corresponding residue is a strictly positive operator.

Proof We can assume that $s(A) = 0$ and we will denote the negative coefficients of the Laurent series of $R(\cdot, A)$ at 0 by Q_n. Thus the following relations hold (see A-III,3.6),

$$\begin{aligned}
&Q_n = \frac{1}{2\pi i}\int_\gamma z^{n-1}R(z,A)\ \mathrm{d}z \quad (n \in \mathbb{N}),\\
&Q_n \neq 0 \text{ if } n \leqslant k \text{ and } Q_n = 0 \text{ for } n > k,\\
&Q_n = A^{n-1}Q_1 \quad (n \in \mathbb{N}); \quad Q_k = \lim_{z\to 0} z^k \cdot R(z,A).
\end{aligned} \tag{2.17}$$

We define I_n as follows $(n = 0, 1, \ldots, k-1)$.

$$I_n := \{f \in E \colon Q_{n+1}|f| = Q_{n+2}|f| = \ldots = Q_k|f| = 0\}$$

At first we restrict our attention to I_{k-1}.

Since $R(\lambda, A)$ is positive if $\lambda > 0$ (Corollary 1.3), it follows from (2.17) that Q_k is a positive bounded operator, hence $I_{k-1} = \{f \in E \colon Q_k|f| = 0\}$ is a closed ideal. Since Q_k commutes with $R(\lambda, A)$ (see (2.17), it follows that I_{k-1} is a T-invariant ideal. By A-III,Corollary 4.3 the generators $A|_{I_{k-1}}$ and A_k induced by A on I_{k-1} and E/I_{k-1}, respectively, have a pole at 0. The coefficients of the Laurent series are the operators induced by Q_n on E/I_{k-1} and I_{k-1}, respectively.

Suppose that the pole order of $R(\cdot, A_k)$ is greater than 1, say m. Then $Q_{m/} = \lim_{z\to 0} z^m R(z, A_k)$ is a positive non-zero operator, hence we find for every $x \in E_+$ an element $y \in I_{k-1}$ such that $Q_m x + y \geqslant 0$. Then we have

$$0 \leqslant Q_k|Q_m x + y| = Q_k Q_m x + Q_k y = Q_{k+m-1}x + Q_k y = 0 + Q_k y \leqslant Q_k|y| = 0$$

hence $Q_m x = (Q_m x + y) - y \in I_{k-1}$ $(x \in E_+)$. It follows that $Q_{m/} = 0$ which is a contradiction.

So far we know that the resolvent of A_k has a pole of order $\leqslant 1$. Moreover, since $Q_{k|I_{k-1}} = 0$, the resolvent of $A_{|I_{k-1}}$ has a pole of order $\leqslant k-1$. From A-III,Corollary 4.3 it follows that the pole order of A_k and $A_{|I_{k-1}}$ is precisely 1 and $k-1$, respectively. The residue $Q_{1/I_{k-1}} = \lim_{z\to 0} zR(z, A_k)$ is positive since $R(z, A_k) \geqslant 0$ for $z > 0$ (see Corollary 1.3). To prove that it is strictly positive we assume $Q_{1/I_{k-1}}(|x + I_{k-1}|) = 0$ which means $Q_1|x| \in I_{k-1}$ hence $Q_k|x| = A^{k-1}Q_1|x| = 0$, that is, $x \in I_{k-1}$ or $x + I_{k-1} = 0$.

Applying what we have proved so far to I_{k-1} and $A_{|I_{k-1}}$ we obtain I_{k-2}, A_{k-1}, and so on. After k steps $(n = 1)$ we conclude that I_0 is T-invariant and that the order of the pole of $R(\cdot, A_{|I_0})$ is 0, implying that $0 \in \varrho(A_{|I_0})$. Since $A_{|I_0}$ generates a positive semigroup and $R(\lambda, A_{|I_0}) = R(\lambda, A)_{|I_0}$ is positive for $\lambda > 0$, it follows from Corollary 1.3 that $s(A_0) = s(A_{|I_0}) < 0$. □

One can check the different steps of the proof by studying the following example Consider this matrix as generator on $\mathbb{C}^4$

$$\begin{pmatrix} -1 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{where } a, b, c, d, e, f \geqslant 0.$$

The result is summarized in the following table ($e_j := (\delta_{jk})$).

	pole order	I_0	I_2	I_2	I_3
$d > 0,\ f > 0,\ e \geq 0$	3	$\langle e_1 \rangle$	$\langle e_1, e_2 \rangle$	$\langle e_1, e_2, e_3 \rangle$	$\mathbb{C}^4$
$d = 0,\ f > 0,\ e \geq 0$	2	$\langle e_1 \rangle$	$\langle e_1, e_2, e_3 \rangle$	$\mathbb{C}^4$	
$d = 0,\ f = 0,\ e > 0$	2	$\langle e_1 \rangle$	$\langle e_1, e_2, e_3 \rangle$	$\mathbb{C}^4$	
$d > 0,\ f = 0,\ e > 0$	2	$\langle e_1 \rangle$	$\langle e_1, e_2 \rangle$	$\mathbb{C}^4$	
$d > 0,\ f = 0,\ e = 0$	2	$\langle e_1 \rangle$	$\langle e_1, e_2, e_4 \rangle$	$\mathbb{C}^4$	

This example also shows that the operators $Q_{k-1}, \ldots, Q_1$ are not necessarily positive (e.g., $a > 0$, $b = c = 0$, $d = e = f = 2$).

A more general (and more interesting) example is the following. Suppose that A_i ($i = 1, \ldots, n$) are generators of positive semigroups on $C_0(X)$ such that $s(A_i) = 0$ is a first order pole of the resolvent. And let A_{ij} ($1 \leqslant i < j \leqslant n$) be positive bounded operators on $C_0(X)$. Then

$$A := \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1n} \\ 0 & A_2 & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}$$

is the generator of a positive semigroup on

$$C_0(X, \mathbb{C}^n) \cong C_0(X) \times C_0(X) \times \cdots \times C_0(X),$$

and $s(A) = 0$ is a pole of the resolvent of order k where $1 \leqslant k \leqslant n$.

Theorem 2.9 *Suppose A is the generator of a positive semigroup on $C_0(X)$ such that every point of $\sigma_b(A)$ is a pole of the resolvent Then $P\sigma_b(A) = \sigma_b(A)$ is cyclic.*

Proof If $\sigma(A) = \emptyset$, there is nothing to prove, thus we can assume that $s(A) = 0$. In view of the lemma and A-III, Proposition 4.3(i) we can assume that $s(A)$ is a first order pole with strictly positive residue, which we call Q. We have $AQ = QA = s(A)A = 0$ (see A-III, 3.6), hence

$$QT(t) = T(t)Q = Q \text{ for all } t \geqslant 0\,. \tag{2.18}$$

If $Ah = \mathrm{i}\alpha h$ for some $\alpha \in \mathbb{R}$, $h \neq 0$, then $T(t)h = \mathrm{e}^{\mathrm{i}\alpha t}h$ (by A-III, Corollary 6.4). Hence $|h| = |\mathrm{e}^{\mathrm{i}\alpha t}h| = |T(t)h| \leqslant T(t)|h|$, or equivalently, $T(t)|h| - |h| \geqslant 0$. By (2.18) we have $Q(T(t)|h| - |h|) = 0$. Since Q is strictly positive, it follows that $T(t)|h| = |h|$

or $A|h| = 0$. Now we can apply Theorem 2.4 and obtain $Ah^{[n]} = \mathrm{i}n\alpha h^{[n]}$ for every $n \in \mathbb{Z}$. This shows that $P\sigma_b(A) = \sigma(A) \cap \mathrm{i}\mathbb{R}$ is cyclic. □

If A has compact resolvent then every point of $\sigma(A)$ is a pole of the resolvent (see A-III,3.6) hence we have

Corollary 2.10 *If A has compact resolvent, then $P\sigma_b(A) = \sigma_b(A)$ is cyclic.*

If it is known that the boundary spectrum of a generator is cyclic and nonvoid, the following alternative holds.

$$\textit{Either}\ \ \sigma_b(A) = s(A)\ \ \textit{or else}\ \ \sigma_b(A)\ \ \textit{is an infinite unbounded set.} \tag{2.19}$$

If one can exclude the second alternative, then there is a unique spectral value having maximal real part. A real spectral value λ_0 of a generator A is called a *dominant* provided that $\mathrm{Re}\,\lambda < \lambda_0$ for every $\lambda \in \sigma(A)$, it is called *strictly dominant* if for some $\delta > 0$ one has $\mathrm{Re}\,\lambda \leqslant \lambda_0 - \delta$ for every $\lambda \in \sigma(A)$, $\lambda \neq \lambda_0$.

The assumptions of Corollary 2.10 do not imply that $s(A)$ is dominant, the rotation semigroup (A-III,Example 5.6) is a counterexample.

Corollary 2.11 *Assume that for some $t_0 > 0$ (hence for all $t > 0$) one has $r_{ess}(T(t_0)) < r(T(t_0))$, e.g., that $T(t_0)$ is compact and $r(T(t_0)) > 0$ (see A-III,3.7). Then $s(A)$ is a strictly dominant eigenvalue.*

Proof If $s(A)$ is not strictly dominant, then we have by Theorem 2.9 and A-III,Corollary 6.5 that $\{\lambda \in \sigma(A) \colon \mathrm{Re}\,\lambda > s(A) - r\}$ contains infinitely many eigenvalues for every $r > 0$. From A-III,Corollary 6.4 it follows that $\{\lambda \in \sigma(T(t)) \colon |\lambda| > r\}$ contains infinitely many eigenvalues (counted according to their multiplicities) for every $r < \exp(s(A)t) = r(T(t))$. This contradicts the assumption $r_{\mathrm{ess}}(T(t)) < r(T(t))$ (see A-III,3.7). □

Corollary 2.12 *Suppose A has compact resolvent and non-empty spectrum. If the corresponding semigroup is eventually norm continuous (e.g., if it is holomorphic or differentiable), then there is a strictly dominant eigenvalue admitting a positive eigenfunction.*

Proof Since $(T(t))_{t\geqslant 0}$ is eventually norm continuous, $\{\lambda \in \sigma(A) \colon \mathrm{Re}\,\lambda \geqslant s(A) - r\}$ is compact for every $r > 0$ (see A-II, Theorem 1.20) and this set does not have accumulation points because A has compact resolvent. In other words, it is a finite set. The assertion now follows from Theorem 2.9 and Corollary 1.4. □

We now consider some examples. The first one shows that there are positive semigroups with $P\sigma_b(A)$ being not cyclic. It is unknown whether there are positive semigroups where $\sigma_b(A)$ is not cyclic.

Example 2.13 Consider $E = C(\Gamma) \times C_0(\mathbb{R})$ ($\cong C_0(\Gamma \dot{\cup} \mathbb{R})$). We fix a positive function $k \in C_0(\mathbb{R})$ with compact support. The operator A given by

$$
\begin{aligned}
A(f,g) &:= \left(f', g' + \frac{1}{2\pi}\int_0^{2\pi} f(\vartheta)\,\mathrm{d}\vartheta \cdot k\right) \\
D(A) &:= \{(f,g) \in E\colon f,g \in C^1, g' \in C_0(\mathbb{R})\}
\end{aligned}
\tag{2.20}
$$

generates a semigroup $(T(t))_{t\geqslant 0}$ which is given by

$$
\begin{aligned}
&T(t)(f,g) := (f_t, g_t) \text{ with} \\
&\quad f_t(\vartheta) := f(\vartheta + t), \\
&\quad g_t(x) := g(x+t) + \frac{1}{2\pi}\int_0^{2\pi} f(\vartheta)\,\mathrm{d}\vartheta \cdot \int_x^{x+t} k(u)\,\mathrm{d}u\,.
\end{aligned}
\tag{2.21}
$$

Then $(T(t))_{t\geqslant 0}$ is a positive semigroup and $\|T(t)\| \leqslant (1 + \|k\|_1)$. In particular, $s(A) \leqslant \omega_0(A) \leqslant 0$. It is easy to see that 0 is not an eigenvalue of A, while all $\mathrm{i}k$, $k \in \mathbb{Z}$, $k \neq 0$ are eigenvalues, the corresponding eigenfunctions being $(e_k, 0)$ with $e_k(\vartheta) = \mathrm{e}^{\mathrm{i}k\vartheta}$.

Example 2.14 (i) *One-dimensional Schrödinger operator.*

Let $X = \mathbb{R}$, $E = C_0(X)$ and $V\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\inf_x V(x) > -\infty$.

If we define

$$
\begin{aligned}
(Af)(x) &:= f''(x) - V(x)f(x), \\
D(A) &:= \{f \in C_0(X)\colon f \in C^2, Af \in C_0(X)\}\,.
\end{aligned}
\tag{2.22}
$$

then A is the generator of a positive semigroup.

In case $\lim_{|x|\to\infty} V(x) = \infty$, A has compact resolvent. Then, by Corollary 2.10, there exists a dominant real eigenvalue with corresponding positive eigenfunction. Actually, the eigenfunction f is strictly positive. (In fact, if $f \in C^2$, $f \geqslant 0$ and $f(x_0) = 0$ for some x_0, then $f'(x_0) = 0$. Therefore the uniqueness theorem for ordinary differential equations implies that f is identically zero).

(ii) *A retarded linear differential equation.*

Consider $E = C[-1,0]$ and define A_m and A_0 as follows,

$$
A_m f := f', \quad f \in D(A_m) = C^1[-1,0]\,, \tag{2.23}
$$

$$
A_0 f := f', \quad f \in D(A_0) = \{f \in C^1[-1,0]\colon f'(0) = 0\}\,. \tag{2.24}
$$

A_0 generates a contraction semigroup $(T_0(t))_{t\geqslant 0}$ which is given by

$$
(T_0(t)f)(x) = \begin{cases} f(x+t) & \text{if } x+t \leqslant 0\,, \\ f(0) & \text{if } x+t \geqslant 0\,. \end{cases} \tag{2.25}
$$

This semigroup is positive, eventually norm continuous ($T_0(t) = \delta_0 \otimes \mathbb{1}$ for $t \geqslant 1$) and has compact resolvent. Given a linear functional Ψ on $C[-1,0]$, we consider

$$
A_\Psi := A_{m|D(A_\Psi)} \text{ with } D(A_\Psi) := \{f \in C^1[-1,0]\colon f'(0) = \langle f, \Psi\rangle\}\,. \tag{2.26}
$$

Denoting the exponential function $x \mapsto e^{\lambda x}$ by e_λ, we have for real λ and $\lambda > \|\Psi\|$.

$$\begin{aligned} &\mathrm{Id} - 1/\lambda \cdot \Psi \otimes e_\lambda \text{ is a bijection of } D(A_\Psi) \text{ onto } D(A_0) \\ &\text{and} \quad \lambda - A_\Psi = (\lambda - A_0)(\mathrm{Id} - 1/\lambda \cdot \Psi \otimes e_\lambda)\,. \end{aligned} \tag{2.27}$$

Using the Neumann series expansion of $(\mathrm{Id}-1/\lambda\cdot\Psi\otimes e_\lambda)^{-1}$ one obtains the following estimate.

$$\|(\mathrm{Id} - 1/\lambda \cdot \Psi \otimes e_\lambda)^{-1}\| \leqslant \lambda/(\lambda - \|\Psi\|) \quad \text{if } \lambda > \|\Psi\|\,. \tag{2.28}$$

It follows from (2.25) and (2.26) that for $\lambda > \|\Psi\|$, the resolvent $R(\lambda, A_\Psi)$ exists and satisfies $\|R(\lambda, A_\Psi)\| \leqslant \lambda/(\lambda - \|\Psi\|) \cdot 1/\lambda = 1/(\lambda - \|\Psi\|)$. Then the Hille-Yosida Theorem (A-II, Theorem 1.7) implies that A_Ψ generates a semigroup $(T(t))$ satisfying $\|T(t)\| \leqslant \exp(\|\Psi\|t)$. Moreover, this semigroup is eventually norm continuous (see B-IV,Corollary 3.3).

By B-II,Example 1.22 we have the following equivalence.

$$A_\Psi \text{ generates a positive semigroup if and only if } \Psi + r\delta_0 \geqslant 0 \text{ for some } r \in \mathbb{R}\,. \tag{2.29}$$

Thus Corollary 2.12 is applicable if $\Psi + r\delta_0 \geqslant 0$ for some $r \in \mathbb{R}$. Since every eigenvalue of A_Ψ is an eigenvalue of A_m and since $\mathrm{Ker}(\lambda - A_m) = \{\alpha e_\lambda : a \in \mathbb{C}\}$, the spectral bound $s(A_\Psi)$ is determined by the (unique) real $\lambda \in \mathbb{R}$ such that $e_\lambda \in D(A_\Psi)$ or equivalently, λ is a solution of the so-called *characteristic equation*

$$\lambda = \Psi(e_\lambda), \quad \lambda \in \mathbb{R}\,. \tag{2.30}$$

(The assumption $\Psi + r\delta_0 \geqslant 0$ implies that the function $\lambda \mapsto \Psi(e_\lambda)$ is strictly decreasing and $\lim_{\lambda\to\infty}\langle e_\lambda, \Psi\rangle > -\infty$, $\lim_{\lambda\to-\infty}\langle e_\lambda, \Psi\rangle = \infty$ unless $\Psi = r_0\delta_0$ for some $r_0 \in \mathbb{R}$.)

We conclude this section with some additional remarks related to Theorem 2.9 and its corollaries.

Remarks 2.15 (i) If $s(A)$ is a pole of the resolvent, then for generators of positive semigroups one has the following equivalences.

(a) $s(A)$ is a first order pole.
(b) For every $0 < f \in \mathrm{Ker}(s(A) - A)$ there exists $0 \leqslant \Psi \in \mathrm{Ker}(s(A) - A')$ such that $\langle f, \Psi\rangle > 0$.
(c) For every $0 < \Psi \in \mathrm{Ker}(s(A) - A')$ there exists $0 \leqslant f \in \mathrm{Ker}(s(A) - A)$ such that $\langle f, \Psi\rangle > 0$.

In particular, if $\mathrm{Ker}(s(A) - A)$ contains a strictly positive function or if $\mathrm{Ker}(s(A) - A')$ contains a strictly positive measure, then $s(A)$ is a first order pole.

We sketch the proof of (a) ⇔ (b) assuming that $s(A) = 0$. If 0 is a first order pole, then the residue P is a positive projection satisfying $PE = \mathrm{Ker}(A)$, $P'E' = \mathrm{Ker}(A)'$ (see A-III,3.6). Thus given $0 < f \in \mathrm{Ker}(A)$ and any $0 \leqslant \varphi \in E'$ such that $\langle f, \varphi\rangle > 0$, we have for $\Psi := P'\varphi$: $\langle f, \Psi\rangle = \langle f, P'\varphi\rangle = \langle Pf, \varphi\rangle = \langle f, \varphi\rangle > 0$. To prove the reverse direction, we first observe that the highest coefficient Q_k of the Laurent expansion is a positive operator. Thus if 0 is a pole of order $k \geqslant 2$, we choose $0 < h \in E$ such that $f := Q_k h > 0$. Then $Af = AQ_k h = 0$ and for every $\Psi \in \operatorname{Ker}(A)'$ we have $\langle f, \Psi \rangle = \langle Q_k h, \Psi \rangle = \langle h, Q_k' \Psi \rangle = \langle h, Q_{k-1}' A' \Psi \rangle = 0$.

(ii) If a linear operator S on $C_0(X)$ is weakly compact, then S^2 is compact (see B-IV, Proposition 2.4(b)). Therefore every non-zero spectral value of a weakly compact operator is a pole of the resolvent. This shows that Theorem 2.9 is applicable if either $T(t_0)$ is weakly compact for some t_0 or $R(\lambda, A)$ is weakly compact for some $\lambda \in \varrho(A)$. We quote two criteria for weak compactness.

If $T \in \mathrm{L}(C(K))$, K compact, is positive, then it is weakly compact if and only if its biadjoint T'' maps bounded Borel functions into $C(K)$ (see B-IV, Proposition 2.4) (2.31)

A positive operator T on $C_0(X)$ which is dominated by a finite rank operator, is weakly compact. (2.32)

Actually, its adjoint T' is dominated by a finite rank operator as well, hence it maps the unit ball in an order interval. It follows that T' is weakly compact hence so is T.

(iii) Stronger results than Theorem 2.9 will be proved in Chapter C-III. Actually, assuming only that $s(A)$ is a pole of finite algebraic multiplicity one can show that $\sigma_b(A)$ contains only poles of finite multiplicity (C-III, Theorem 3.13). In C-III,Corollary 2.12 we will show that $\sigma_b(A)$ is cyclic whenever $s(A)$ is a pole of the resolvent.

(iv) Example 2.14(ii) can be extended to systems of functional differential equations even the infinite dimensional case. For details we refer to Section 3 of Chapter B-IV.

3 Irreducible Semigroups

In the case of matrices it is well known that considerably stronger results are available if one considers positive matrices which are irreducible. The concept of irreducibility can be extended to our setting and in many cases one can check easily whether a given semigroup has this property (see Example 3.4). We will show that irreducible semigroups have many interesting properties. For example, the spectrum $\sigma(A)$ is always non-empty, positive eigenfunctions are strictly positive and if $s(A)$ is a pole, it is algebraically (and geometrically) simple (see Proposition 3.5). Moreover, in certain cases irreducibility ensures that $\sigma_b(A)$ and $P\sigma_b(A)$ are not only cyclic subsets but *subgroups* (see Theorem 3.6 and Theorem 3.11 for details).

We start the discussion with several, mutually equivalent, definitions of irreducibility.

Definition 3.1 A positive semigroup $\mathcal{T} = (T(t))$ on $E = C_0(X)$, X locally compact, with generator A is called *irreducible* if one of the following mutually equivalent conditions is satisfied.

(a) There is no T-invariant closed ideal except $\{0\}$ and E.
(b) Given $0 < f \in E$, $0 < \varphi \in E'$, then $\langle T(t_0)f, \varphi\rangle > 0$ for some $t_0 \geqslant 0$.
(c) For every $f > 0$ we have $\bigcup_{t\geqslant 0}\{x \in X : (T(t)f)(x) > 0\} = X$.
(d) For some (every) $\lambda > s(A)$ there exists no closed ideal which is invariant under $R(\lambda, A)$ except $\{0\}$ and E.
(e) For some (every) $\lambda > s(A)$, we have $R(\lambda, A)f$ is strictly positive whenever $f > 0$.
(f) $\bigcup_{t\geqslant 0} \operatorname{supp} T(t)'\delta_x$ is dense in X for every $x \in X$.

That these six conditions are actually equivalent can be seen as follows.

(a) $\implies$ (b): Suppose there are $0 < f \in E$, $0 < \varphi \in E'$ such that $\langle T(t)f, \varphi\rangle = 0$ for every $t \geqslant 0$. Then the ideal I generated by $\{T(t)f : t \geqslant 0\}$ satisfies $\{0\} \neq I \subset \{g \in E : \varphi(|g|) = 0\} \neq E$. Obviously I is $\mathcal{T}$-invariant.

(b) $\implies$ (c): Given $0 < f \in E$, $x \in X$. By (b) there exists t_0 such that $(T(t_0)f)(x) = \langle T(t_0)f, \delta_x\rangle > 0\,.$

(c) $\implies$ (d): Suppose that $\bigcup_{t\geqslant 0} \operatorname{supp} T(t)'\delta_y$ is not dense for some $y \in X$. Then there exists $f_0 \in E$, $f_0 > 0$ such that $\operatorname{supp} f_0 \cap \operatorname{supp} T(t)'\delta_y = \emptyset$ for every $t \geqslant 0$. Hence $(T(t)f_0)(y) = \langle f_0, T(t)'\delta_y\rangle > 0$, that is, $y \notin \bigcup_{t\geqslant 0}\{x \in X : (T(t)f_0)(x) > 0\}$.

(d) $\implies$ (e): Given $0 < f \in E$, $\lambda > \omega_0(A)$, $y \in X$, there exists $t_0 \geqslant 0$ such that $\{x : f(x) > 0\} \cap \operatorname{supp} T(t_0)'\delta_y \neq \emptyset$. Hence, $(T(t_0)(f)(y) = \langle f, T(t_0)'\delta_y\rangle > 0$ and therefore

$$(R(\lambda, A)f)(y) = \int_0^\infty \mathrm{e}^{-\lambda t}(T(t)f)(y)\,\mathrm{d}t > 0\,.$$

Since $\lambda \mapsto R(\lambda, A)f$ is decreasing in the interval $(s(A), \infty)$ (use the resolvent equation and the fact that $R(\lambda, A)$ is positive) we have $R(\lambda, A)f \gg 0$ for all $\lambda > s(A)$.

(e) $\implies$ (f): If I is a $R(\lambda, A)$-invariant ideal and $0 < f \in I$, , then $g := R(\lambda, A)f \in I$. By (e), g is strictly positive, thus I has to be dense (it contains all functions of compact support).

(f) $\implies$ (a): At first we recall that a closed linear subspace which is invariant for $R(\lambda_0, A) (\lambda_0 \in \varrho(A))$, is invariant for $R(\lambda, A)$ whenever λ and λ_0 belong to the same component of $\varrho((A)$. Hence by A-I,3.2 every $R(\lambda_0, A)$-invariant subspace where $\lambda_0 \in \varrho_+(A)$ is $\mathcal{T}$-invariant and vice versa.

Remark 3.2 Obviously, irreducibility of a semigroup $(T(t))_{t\geqslant 0}$ is implied by the following condition.

(g) $T(t)f \gg 0$ whenever $f > 0$ and $t > 0$.

The rotation semigroup (see A-I,2.5) is irreducible but it does not satisfy condition (g). However, assuming that the semigroup $(T(t))$ is holomorphic, then (g) is equivalent to irreducibility. We will give a proof of this result in the more general situation of Banach lattices (see C-III, Theorem 3.2(b)).

A semigroup $(T(t))_{t\geq 0}$ is irreducible if and only if $(\mathrm{e}^{-\alpha t}T(t))_{t\geq 0}$, $\alpha \in \mathbb{R}$ is. More generally, irreducibility is invariant under perturbations by multiplication operators. In fact, we have the following result.

Proposition 3.3 *Suppose A generates a positive semigroup $\mathcal{T}$ on $C_0(X)$ and let h be a continuous, bounded real-valued function on X. Then the semigroup $\mathcal{S}$ generated by $B := A + M_h$ is irreducible if and only if $\mathcal{T}$ has this property.*

Proof Since every closed ideal is of the form $\{f \in E : f|_M = 0\}$ where $M \subset X$ is a closed subset (cf. Section 1 of B-I), it is clear that all closed ideals are invariant under the multiplication operator M_h and M_{-h}, respectively. Thus the assertion follows from the expansions which are true for λ sufficiently large

$$R(\lambda, B) = (1 - R(\lambda, A)M_h)^{-1}R(\lambda, A) = \sum_{n=0}^{\infty}(R(\lambda, A)M_h)^n R(\lambda, A)$$

$$R(\lambda, A) = (1 - R(\lambda, B)M_{-h})^{-1}R(\lambda, B) = \sum_{n=0}^{\infty}(R(\lambda, B)M_{-h})^n R(\lambda, B)$$

Examples 3.4 (a) (cf. B-II,Section 3). Suppose $(T(t))_{t\geqslant 0}$ is governed by a continuous semiflow $\varphi : \mathbb{R}_+ \times X \to X$, i.e., $T(t)f = f \circ \varphi_t$ $(f \in C_0(X))$. Then the following assertions are equivalent.

(i) $(T(t))_{t\geqslant 0}$ is irreducible.
(ii) There is no closed subset of X which is φ-invariant except $\emptyset$ and X.
(iii) Every orbit $\{\varphi(t, x) : t \in \mathbb{R}_+\}$ is dense in X.

More generally, these equivalences still hold if the semigroup $(T(t))$ is given by $T(t)f = h_t \cdot (f \circ \varphi_t)$ where h_t are suitable continuous, strictly positive, bounded functions on X.

(b) Suppose that the semigroup $(T(t))_{t\geqslant 0}$ has the following form. There exist a positive measure μ on X and a positive continuous function $k : (0, \infty) \times X \times X \to \mathbb{R}$ such that

$$(T(t)f)(x) = \int_X k(t, x, y)f(y)\,\mathrm{d}\mu(y) \quad (t > 0, f \in C_0(X), x \in X)\,. \tag{3.1}$$

Then $(T(t))_{t\geqslant 0}$ is irreducible if and only if $\bigcup_{t>0} \operatorname{supp}\{k(t, x, .)\}$ is dense in X for every $x \in X$.

(c) We consider the first derivative $Af = f'$ (cf. A-I,2.4). If $E = C_0(\mathbb{R})$, then the corresponding semigroup $(T(t))$ is not irreducible. Note however, that there is no closed invariant ideal I with $\{0\} \neq I \neq E$ which is invariant under the group $(T(t))_{t\in\mathbb{R}}$ generated by A. For $E = C_0[0, \infty)$ and $E = C_0(-\infty, 0)$ the corresponding semigroups are reducible (i.e., not irreducible) as well. If $E = C_{2\pi}(\mathbb{R})$ (i.e., the 2π-periodic functions), then $Af = f'$ generates an irreducible semigroup on E. It is (isomorphic to) the semigroup of rotations on the unit circle.

(d) (cf. Example 2.14(ii)) Consider $Af = f'$ on $E = C[-1, 0]$ with $D(A_\Psi) = \{f \in C^1 : f'(0) = \Psi(f)\}$ where the linear functional Ψ satisfies $\Psi + \alpha\delta_0 \geqslant 0$ for some $\alpha \in \mathbb{R}$ (see B-II,Example 1.22). The corresponding semigroup is irreducible if and only if $-1 \in \operatorname{supp}\Psi$.

(e) The second derivative $Af = f''$ generates an irreducible semigroup on $C_0(\mathbb{R})$ and on $C_0(0,1)$ (cf. A-I,2.7). With Neumann boundary conditions (or more generally, $f'(0) = \alpha_0 f(0)$, $f'(1) = \alpha_1 f(1)$ where $\alpha_0, \alpha_1 \in \mathbb{R}$) the second derivative generates an irreducible semigroup on $C[0,1]$ (cf. A-I,2.7).

The operator $Af = f'' - Vf$ on $C_0(\mathbb{R})$, where V is continuous, real-valued with $\inf V(x) > -\infty$ (see Example 2.14(i)) also generates an irreducible semigroup. This can be derived from the maximum principle as follows. For $\lambda > -\inf V(x)$, $f \in C_0(\mathbb{R})$, $g := R(\lambda, A)f$ we have $g \in C^2$ and $g'' - (\lambda + V)g = -f$. If $f > 0$, then $g > 0$, hence **?** , Chap.I, Theorem 3] implies that g is strictly positive.

(f) The Laplacian Δ generates an irreducible semigroup on $C_0(\mathbb{R}^n)$ as can be seen easily from A-I,2.8. More general elliptic operators will be discussed below (see Example 3.10(ii)).

We now return to the general situation and show that irreducible semigroups possess several interesting properties.

Proposition 3.5 *Suppose A is the generator of a strongly continuous semigroup on $C_0(X)$ which is irreducible. Then the following assertions are true.*

(i) $\sigma(A) \neq \emptyset$.
(ii) Every positive eigenfunction of A is strictly positive .
(iii) Every positive eigenvector of A' is strictly positive .
(iv) If $\mathrm{Ker}(s(A) - A')$ *contains a positive element (e.g., if X is compact (cf. Theorem 1.6), then* $\dim(\mathrm{Ker}(s(A) - A) \leqslant 1$.
(v) if $s(A)$ is a pole of the resolvent, then it is algebraically simple. The residue has the form $P = \varphi \otimes u$ where $\varphi \in E'$ and $u \in E$ are strictly positive eigenvectors of A' and A, respectively, satisfying $\langle u, \varphi \rangle = 1$.

Proof (i) Take any $f_0 \in C_0(X)$ which is positive and has compact support. If $\lambda > s(A)$, then $R(\lambda, A)f_0$ is strictly positive (by Definition 3.1(e)), hence there exists $\varepsilon > 0$ such that $R(\lambda, A)f_0 \geqslant \varepsilon f_0$. It follows that $R(\lambda, A)^n f_0 \geqslant \varepsilon^n f_0 \geqslant 0$ for all $n \in \mathbb{N}$ and therefore $r(R(\lambda, A)) = \lim_{n\to\infty} \|R(\lambda, A)^n\|^{1/n} \geqslant \varepsilon > 0$. The assertion now follows from A-III, Proposition 2.5.

(ii) Suppose $Ah = rh$ where $h \neq 0$ is positive. Then r has to be real and we have $T(t)h = \mathrm{e}^{rt}h$ (A-III,Corollary 6.4). For $|f| \leqslant n \cdot h$ ($n \in \mathbb{N}$) we have

$$|T(t)f| \leqslant T(t)|f| \leqslant n \cdot T(t)h = n \cdot \mathrm{e}^{rt}h\,. \tag{3.2}$$

This shows that the ideal generated by h is invariant, hence dense by irreducibility. This is true if and only if h is strictly positive.

(iii) Suppose $A'\varphi = r\varphi$ for some $0 < \varphi \in E'$. Again r has to be real and $T(t)'\varphi = \mathrm{e}^{rt}\varphi$ ($t \geqslant 0$). From

$$\langle |T(t)f|, \varphi\rangle \leqslant \langle T(t)|f|, \varphi\rangle = \langle |f|, \mathrm{e}^{rt}\varphi\rangle,\ f \in E \tag{3.3}$$

it follows that $I := \{f \in E : \varphi(|f|) = 0\}$ is an invariant ideal. We have $I \neq E$ (because $\varphi \neq 0$), hence the irreducibility implies $I = \{0\}$, i.e., φ is strictly positive.

(iv) By (i) we know that $s(A) > -\infty$ hence we can assume without loss of generality that $s(A) = 0$. By (iii) there exists a strictly positive $\Psi \in E'$ such that $A'\Psi = 0$. It follows from (2.14) and (2.15) that

$$h \in \operatorname{Ker}(A) \text{ implies } |h| \in \operatorname{Ker}(A) \tag{3.4}$$

Assuming $\dim(\operatorname{Ker}(A)) \geqslant 2$, then there is an eigenfunction $h \in \operatorname{Ker}(A)$, $h \neq 0$ which has at least one zero in X ($h := h_1(x_0) \cdot h_2 - h_2(x_0) \cdot h_1$, where h_1, h_2 are linearly independent, $x_0 \in X$). By (3.4) $|h|$ is a positive eigenfunction but not strictly positive. This is a contradiction with (ii).

(v) If $s(A)$ is a pole, then there exists a corresponding positive eigenfunction (see the proof of Corollary 1.4). By (ii) it is even strictly positive, thus $s(A)$ is a first order pole by Rem.2.15(a). The residue P is a positive operator satisfying $PE = \operatorname{Ker}(s(A) - A)$ and $P'E' = \operatorname{Ker}(s(A) - A')$, therefore the remaining assertion follows from (ii) and (iv). ◻

In the remainder of this section we focus our interest on the boundary spectrum of irreducible semigroups, more precisely, on the eigenvalues and the corresponding eigenfunctions of the boundary spectrum. In view of assertion (a) of Proposition 3.5 the assumption "$s(A) = 0$" is not crucial in the following theorem. However, it allows a simpler formulation.

Theorem 3.6 *Suppose $\mathcal{T} = (T(t))$ is an irreducible semigroup with generator A and spectral bound $s(A) = 0$. Assume that there exists a positive linear form $\Psi \neq 0$ such that $A'\Psi = 0$. (This is automatically satisfied whenever X is compact (see Theorem 1.6).) If $P\sigma(A) \cap \mathrm{i}\mathbb{R}$ is non-empty, then the following assertions are true.*

(i) *$P\sigma(A) \cap \mathrm{i}\mathbb{R}$ is a (additive) subgroup of $\mathrm{i}\mathbb{R}$.*
(ii) *The eigenspaces corresponding to $\lambda \in P\sigma(A) \cap \mathrm{i}\mathbb{R}$ are one-dimensional.*
(iii) *If $Ah = \mathrm{i}\alpha h$ ($h \neq 0, \alpha \in \mathbb{R}$), then h has no zeros in X. In case $\alpha = 0$ then $h(x)/|h(x)|$ is constant; otherwise, $\{h(x)/|h(x)| : x \in X\}$ is a dense subset of Γ.*
(iv) *If $Ah = \mathrm{i}\alpha h$ ($h \neq 0, \alpha \in \mathbb{R}$), then*

$$S_h(D(A)) = D(A) \text{ and } S_h^{-1} \circ A \circ S_h = (A + \mathrm{i}\alpha)\,. \tag{3.5}$$

In particular, spectrum and point spectrum of A are invariant under translations by $\mathrm{i}\alpha$.

(v) *0 is the only eigenvalue admitting a positive eigenfunction.*

Proof By Proposition 3.5(iii) the invariant linear form Ψ is strictly positive and it satisfies $T(t)'\Psi = \Psi$ ($t \geqslant 0$).

(iv) Supposing $Ah = \mathrm{i}\alpha h$ ($h \neq 0, \alpha \in \mathbb{R}$) then $A|h| = 0$ by (2.14) and (2.15). By Proposition 3.5(ii) $|h|$ is strictly positive, thus Theorem 2.4(b) implies (3.5).

(ii) Assertion (iv) implies that S_h maps $\operatorname{Ker}(m + A)$ onto $\operatorname{Ker}(A)$ whenever $\mathrm{i}\alpha \in P\sigma(A) \cap \mathrm{i}\mathbb{R}$. Moreover, we have seen in the proof of (iv) that $\operatorname{Ker}(A) \neq \{0\}$ hence it is one-dimensional by Proposition 3.5(iv).

(i) Assume that $Ah = \mathrm{i}\alpha h$, $Ag = \mathrm{i}\beta g$ $(\alpha, \beta \in \mathbb{R}, h \neq 0, g \neq 0)$. By (3.5) we have $S_{\bar{g}} A S_g = A + \mathrm{i}\beta$ and $S_h A S_{\bar{h}} = A - \mathrm{i}\alpha$, therefore

$$A + i(\beta - \alpha) = S_h (A + \mathrm{i}\beta) S_{\bar{h}} = S_h S_{\bar{g}} A S_g S_h^{-1} \,. \tag{3.6}$$

It follows that $\mathrm{Ker}(A + i(\beta - \alpha)) = S_h S_{\bar{g}}(\mathrm{Ker}(A)) \neq \{0\}$, hence $\mathrm{i}(\beta - \alpha) \in P\sigma(A)$.

(v) If $Af = \lambda f$ where $f > 0$, then

$$\lambda \langle f, \Psi \rangle = \langle Af, \Psi \rangle = \langle f, A'\Psi \rangle = 0 \,. \tag{3.7}$$

Since Ψ is strictly positive we have $\langle f, \Psi \rangle > 0$ hence $\lambda = 0$.

(iii) We already know that $Ah = \mathrm{i}\alpha h$ implies that $A|h| = 0$. It follows from Proposition 3.5(ii) that h is strictly positive; i.e., h has no zeros in X. By Proposition 3.5(iv) $\mathrm{Ker}(A)$ is one-dimensional hence every eigenfunction corresponding to 0 is the scalar multiple of a strictly positive function. If $Ah = \mathrm{i}\alpha h$, $h \neq 0$, $\alpha \neq 0$, we consider $\tilde{h}(x) := h(x)/|h(x)|$. Assuming that $\tilde{h}(X)$ is not dense in Γ, there exists a sequence of polynomials $(p_n)_{n \in \mathbb{N}}$ such that

$$p_n(z) \to 1/z \text{ uniformly in } z \in \tilde{h}(X) \,. \tag{3.8}$$

It follows that $h(x) \cdot p_n(\tilde{h}(x)) \to |h|(x)$ uniformly in $x \in X$. Obviously, $h \cdot p_n(h)$ is a linear combination of $h^{[1]}$, $h^{[2]}$, $h^{[3]}$, ..., that is, it is an element of

$$\mathrm{span}\,\{\,\mathrm{Ker}(\mathrm{i}k\alpha - A) \colon\ k = 0, 1, 2, \ldots\}, \ \text{(cf. Theorem 2.4).}$$

By (3.7) the linear form Ψ vanishes on $\mathrm{Ker}(\lambda - A)$ whenever $\lambda \neq 0$. Therefore $\langle h \cdot P_n(h), \Psi \rangle = 0$ and we have $0 < \langle |h|, \Psi \rangle = \lim_{n \to \infty} \langle h \cdot P_n(h), \Psi \rangle = 0$ which is a contradiction. □

The group $P\sigma(A) \cap \mathrm{i}\mathbb{R}$ need not be discrete. For example, the semigroup described in Example 2.6(d) satisfies the assumptions of Theorem 3.6 if $\frac{\alpha}{\beta}$ is irrational. In this case $P\sigma(A) = \mathrm{i}\alpha\mathbb{Z} + \mathrm{i}\beta\mathbb{Z}$ is a dense subgroup of $\mathrm{i}\mathbb{R}$. Actually one can show that for every subgroup H of $\mathrm{i}\mathbb{R}$ there is an irreducible semigroup on $C(G)$, $G := (H_d)\tilde{}$, such that $P\sigma(A) = H$). Here $(H_d)\tilde{}$ denotes the dual of the abelian group H equipped with the discrete topology. For details see **?** , p.62].

An immediate consequence of assertion (iv) of Theorem 3.6 is the following corollary.

Corollary 3.7 *Suppose $\mathcal{T}$ satisfies the hypotheses of Theorem 3.6 and let A be its generator. If k is a bounded continuous real-valued function, M_k the corresponding multiplication operator, then for $B := A + M_k$ we have $\sigma(B) + P\sigma(A) \cap \mathrm{i}\mathbb{R} = \sigma(B)$. In particular, $s(B) + P\sigma(A) \cap \mathrm{i}\mathbb{R} \subset \sigma(B)$.*

The next two corollaries are essentially consequences of assertion (iii) of Theorem 3.6. The statement of the first one can be summarized as follows. In case there are non-real eigenvalues in the boundary spectrum, then the semigroup *contains* the semigroup of rotations on Γ.

Corollary 3.8 *Suppose that the hypotheses of Theorem 3.6 are satisfied and that there is an eigenvalue* $\mathrm{i}\alpha$ *of* A *with* $\alpha > 0$. *Let* $\tau := 2\pi/\alpha$. *Then there exists a continuous injective lattice homomorphism* $j : C(\Gamma) \to C_0(X)$ *such that the diagram commutes.*

$$\begin{array}{ccc} C(\Gamma) & \xrightarrow{R_\tau(t)} & C(\Gamma) \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ C_0(X) & \xrightarrow[T(t)]{} & C_0(X) \end{array}$$

Here, $(R_\tau(t))$ *denotes the rotation semigroup of period* τ *(see A-I,2.5). If* X *is compact, then* j *is a topological embedding.*

Proof Assume that $Ah = \mathrm{i}\alpha h$, $\alpha > 0$, and let $\tilde{h}(x) := h(x)/|h(x)|$. Then we define j by

$$j(f) := |h| \cdot f \circ \tilde{h} \quad (\text{i.e.,} (j(f))(x) = |h(x)| \cdot f(\tilde{h}(x)))\,. \tag{3.9}$$

Obviously, j is a lattice homomorphism and because h has no zeros and $\tilde{h}$ has a dense image in Γ (Theorem 3.6(c)), it follows that j is injective. For the functions $e_n \in C(\Gamma)$ given by $e_n(z) = z^n$ $(n \in \mathbb{Z})$ one has $j(e_n) = h^{[n]}$ $(n \in \mathbb{Z})$ and therefore $T(t) \circ j(e_n) = T(t)h^{[n]} = \mathrm{e}^{\mathrm{i}n\alpha t} \cdot h^{[n]}$ (cf. Theorem 2.4) and $j \circ R_\tau(t)(e_n) = j(\mathrm{e}^{\mathrm{i}n\alpha t} e_n) = \mathrm{e}^{\mathrm{i}n\alpha t} \cdot h^{[n]}$. Since $\{e_n : n \in \mathbb{Z}\}$ is a total subset of $C(\Gamma)$, we have $T(t) \circ j = j \circ R_\tau(t)$ for every $t > 0$.

If X is compact, then $\tilde{h}(X)$ is closed, hence $\tilde{h}$ is onto, moreover, $|h| \geqslant \varepsilon$ for some $\varepsilon > 0$, thus the definition of j implies that $\|j(f)\| > \varepsilon \|f\|$ for every $f \in C(\Gamma)$. ◻

A consequence of Corollary 3.8 is the following. If $s(A) \notin P\sigma(A) \cap \mathrm{i}\mathbb{R}$, then for every $\varepsilon > 0$ there exists $g > 0$ such that $T(t)g$ and $T(s)g$ have disjoint support whenever $|s - t| = \varepsilon$. Another consequence is that there exist positive functions f_1 and f_2 such that $T(t)f_1$ and $T(t)f_2$ have disjoint support for every $t \geqslant 0$ (consider the images under j of two disjoint functions on $C(\Gamma)$). This observation proves the following corollary.

Corollary 3.9 *Suppose that the hypotheses of Theorem 3.6 are satisfied and that for some* $t_0 > 0$ *we have* $T(t_0)f \gg 0$ *whenever* $f > 0$. *Then* $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$.

Corollary 3.9 can be applied if $T(t_0)$ is a kernel operator with strictly positive kernel. We give some examples.

Examples 3.10 (a) We assume that the semigroup $(T(t))$ satisfies the hypotheses of Theorem 3.6 and that it is given by

$$(T(t)f)(x) = \int_X k(t, x, y) f(y) \,\mathrm{d}\mu(y)\,,$$

where μ is a positive measure and k is a positive continuous function (see Example 3.4(ii)). We will show that $P\sigma(A) \cap (s(A) + \mathrm{i}\mathbb{R}) = \{s(A)\}$.

Assuming the contrary, by Theorem 3.6(d) there exist $\alpha \neq 0$, $h \in C_0(X)$ such that

$$S_h^{-1}T(t) \circ S_h = \mathrm{e}^{\mathrm{i}\alpha t} \cdot T(t) \text{ for all } t \geqslant 0\,. \tag{3.10}$$

This implies that k satisfies

$$\frac{\overline{h(x)}}{|h(x)|} \cdot \frac{h(y)}{|h(y)|} \cdot k(t,x,y) = \mathrm{e}^{\mathrm{i}\alpha t} k(t,x,y) \quad (t > 0, x, y \in X)\,. \tag{3.11}$$

It follows that for $0 < |s-t| < 2\pi/\alpha$ the kernel functions $k(t,.,.)$ and $k(s,.,.)$ have disjoint support. This is impossible if k is continuous.

(b) Let Ω be a domain in $\mathbb{R}^n$ and define L_0 as

$$L_0 f := \sum_{i,j=1}^{n} a_{ij} f_{ij}' + \sum_{i=1}^{n} b_i f_i' + cf\,,$$

with domain $D(L_0) := \{f \in C_0(\Omega) : f \text{ is } C^\infty, L_0 f \in C_0(\Omega)\}$. (Here f_i' stands for $\partial f/\partial x_i$, and $f_{ij}' = \partial^2 f/\partial x_i \partial x_j)$.

Suppose that L_0 is elliptic, a_{ij}, b_i, c are real-valued C^∞-functions with $\lambda_0 := \sup c < \infty$, assume further that the closure L of L_0 is the generator of a positive semigroup on $C_0(\Omega)$ which has compact resolvent. For example, this is true if $\partial\Omega$ is C^∞ and $a_{ij} \in C^\infty(\overline{\Omega})$ (cf. Theorem 4.8.3 of **?**]). We will show that $P\sigma(A) \cap (s(A) + \mathrm{i}\mathbb{R}) = \{s(A)\}$.

In order to apply Theorem 3.6, we have to show that the corresponding semigroup $(T(t))$ is irreducible. Given $0 < f \in E$, then there is $g \in D(L_0)$ such that $0 < g \leqslant f$. $h := R(\lambda, L)g$ is C^∞ (Weyl's Lemma) and satisfies $L_0 h - \lambda h = -g < 0$. Assuming that $\lambda > \lambda_0$, then h is positive, even strictly positive by the maximum principle **?** , Chap.2, Theorem 6]. It follows from $R(\lambda, L)f \geqslant R(\lambda, L)g = h \gg 0$ that $(T(t))$ is irreducible.

Next we apply Theorem 3.6(d) in order to show that the spectral bound is a dominant eigenvalue. We can assume that $s(L) = 0$. If $s(L)$ is not dominant, then by Theorem 3.6(d) we have

$$L_0 h = i\alpha h, L_0|h| = 0, L_0\bar{h} = -\mathrm{i}\alpha\bar{h} \text{ for some } h \neq 0, \alpha > 0\,. \tag{3.12}$$

If we define $u := |h|$ and $w := h/|h|$, then (3.12) reads

$$L_0(uw) = \mathrm{i}\alpha uw, \quad L_0(u) = 0, \quad L_0(u/w) = -\mathrm{i}\alpha \cdot u/w\,. \tag{3.13}$$

Explicit calculation of $L_0(uw)$ and $L_0(u/w)$ using the product formula yields

$$\begin{aligned} L_0(uw) &= wL_0(u) + u\sum_{i,j} a_{ij} w_{ij}' + \sum_i \left(ub_i + \sum_j a_{ij} u_j'\right) w_i'\,, \\ L_0(u/w) &= \frac{1}{w} L_0(u) + u\sum_{i,j} a_{ij} (1/w)_{ij}' + \sum_i \left(ub_i + \sum_j a_{ij} u_j'\right) (1/w)_i'\,. \end{aligned} \tag{3.14}$$

Observing that $(1/w)_i' = -w^{-2} \cdot w_i'$ and $(1/w)_{ij}' = w^{-3} \cdot (2w_i'w_j' - ww_{ij}')$, we obtain

$$L_0(uw) + w^2 L_0(u/w) = 2wL_0(u) + 2u/w \cdot \sum_{ij} a_{ij} w_i' w_j' \ . \tag{3.15}$$

This identity and (3.13) implies that $2u/w \cdot \sum_{ij} a_{ij} w_i' w_j' = 0$. Since u has no zeros and (a_{ij}) is positive definite, we have $\operatorname{grad} w = (w_i') = 0$ in Ω, hence $w = \text{const}$. Then, by (3.13), we have $\mathrm{i}\alpha uw = L_0(uw) = wL_0(u) = 0$, a contradiction.

The assumption that L_0 is elliptic, i.e., that (a_{ij}) is positive definite, is essential in order to show that there is only one eigenvalue in the boundary spectrum. In the following example (a_{ij}) is positive semi-definite and $P\sigma_b(A) = s(A) + \mathrm{i}\alpha\mathbb{Z}$.

(c) We consider $\Omega = \{(x, y) \in \mathbb{R}^2 : 1 < (x^2 + y^2)^{1/2} < 2\}$, and the second order differential operator L_0 given by

$$(L_0 f)(x, y) = 1/(x^2 + y^2) \cdot (x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}) + (x f_y - y f_x) \ .$$

The assertion concerning the boundary spectrum can be verified easily by using polar coordinates, $x = r \cdot \cos\omega$, $y = r \cdot \sin\omega$. Then L_0 becomes $L_0 f = f_{rr} + f_\omega$ on the space $C_0(1, 2) \otimes C_{2\pi}(\mathbb{R})$.

In this section we have seen that the eigenvalues in the boundary spectrum of an irreducible semigroup form a subgroup of $\mathrm{i}\mathbb{R}$ (provided that $s(A) = 0$). We conclude this section mentioning an analogous statement for the whole boundary spectrum of Markov semigroups on $C(K)$, K compact. It seems to be unknown if this is true for irreducible semigroups in general. To prove this result one uses the proof of the analogous result for a single operator (cf. **?** , Theorem 7]) as a guideline.

Theorem 3.11 *Suppose that $\mathcal{T}$ is an irreducible semigroup of Markov operators on $C(K)$, K compact. Then $\sigma_b(A)$ is a ()closed) subgroup of* $\mathrm{i}\mathbb{R}$. *Hence either $\sigma_b(A) = \{0\}$ or $= \mathrm{i}\mathbb{R}$ or $= \mathrm{i}\alpha\mathbb{Z}$ for some $\alpha > 0$.*

4 Semigroups of Lattice Homomorphisms

As we have seen in Section 2 the boundary spectrum of a many positive semigroups is a cyclic set. However, there are hardly any restrictions on the set

$$\lambda \in \sigma(A) : \operatorname{Re}\lambda < s(A)\} \ ,$$

except that it is symmetric with respect to the real axis. For semigroups of lattice homomorphisms the situation is quite different. We will show that the whole spectrum is an imaginary additively cyclic subset of $\mathbb{C}$ (see Definition 2.5). A complete proof of this results requires some facts of the theory of Banach lattices, therefore, we postpone it to Part C (see C-III, Theorem 4.2).

Theorem 4.1 *If A is the generator of a semigroup of lattice homomorphisms, then $\sigma(A)$, $A\sigma(A)$ and $P\sigma(A)$ are cyclic subsets of $\mathbb{C}$.*

Proof (1st part of the proof) We prove the assertion concerning $A\sigma(A)$ and $P\sigma(A)$. Assume that $Ah = (\alpha + \mathrm{i}\beta)h$, $\alpha, \beta \in \mathbb{R}$, $h \neq 0$, then $T(t)h = \mathrm{e}^{\alpha t}\mathrm{e}^{\mathrm{i}\beta t}h$ for all $t \geqslant 0$ (A-III,Corollary 6.4). Since $T(t)$ is a lattice homomorphism we have $T(t)|h| = |T(t)h| = \mathrm{e}^{\alpha t}|h|$ $(t \geqslant 0)$ or $A|h| = \alpha|h|$, hence $Ah^{[n]} = (\alpha + \mathrm{i}n\beta)h^{[n]}$ for all $n \in \mathbb{Z}$ by Theorem 2.4(ii). We have shown that $P\sigma(A)$ is cyclic.

To prove that $A\sigma(A)$ is cyclic as well, one considers a semigroup $\mathcal{F}$-product $E_{\mathcal{F}}^{\mathcal{T}}$ of E (see A-III,4.5). It is easy to see that $E_{\mathcal{F}}^{\mathcal{T}}$ is a Banach lattice and $(T_{\mathcal{F}}(t))$ is a semigroup of lattice homomorphisms. The proposition in A-III,3.5 implies $A\sigma(A) = P\sigma(A_{\mathcal{F}})$. Thus the assertion follows from the cyclicity of point spectrum.□

Performing a similar construction as in Example 2.6(f) one can show that every closed cyclic subset of $\mathbb{C}$ which is contained in a left halfplane is the spectrum of a suitable semigroup of lattice homomorphisms. For details see **?**]. In the following we restrict ourselves to the case of compact spaces. Then a semigroup of lattice homomorphisms can be described explicitly by a semi-flow φ, and real-valued functions h and p (see B-II, Thms.3.5 & 3.6). The function p has no influence on spectral properties (cf. B-II,(3.7)). Therefore we will assume that $(T(t))$ has the following form (cf. B-II, Theorem 3.5).

$$T(t)f = h_t \cdot f \circ \varphi_t \quad (t \geqslant 0, f \in C(K)) \tag{4.1}$$

where $\varphi = (\varphi_t) : \mathbb{R}_+ \times K \to K$ is a continuous semiflow and $h_t(x) = \exp \int_0^t h(\varphi(s,x))\,\mathrm{d}s$ $(t \geqslant 0, x \in K)$ for some continuous function $h : K \to \mathbb{R}$. In the following we will describe the spectrum of the semigroup given by (4.1) in terms of φ and h. At first we have to fix some notation. Let K, φ, h be as in (4.1). Then

$$K_t := \varphi_t(K) \quad (t < \infty), \quad K_\infty := \bigcap_{t<\infty} K_t\,. \tag{4.2}$$

Some properties of the sets K_t are listed in the following lemma. The proof is not difficult and is left as an excercise.

Lemma 4.2 *Every K_t $(0 \leqslant t \leqslant \infty)$ is a non-empty closed subset of K which is invariant under the semiflow φ. Moreover, the following assertions are true.*

(i) For $s > t$ we have $K_s \subset K_t$. In case that $K_s = K_t$ then $K_t = K_\infty$.
(ii) $\varphi_t(K_\infty) = K_\infty$ for all $t \geqslant 0$.
(iii) If one partial mapping φ_t, $t > 0$, is injective (surjective), then all mappings φ_s are injective (surjective).

We call a *semiflow φ injective (surjective)* if one and hence all of the partial mappings φ_t are injective (surjective). Studying the spectrum of the semigroup given by (4.1) we divide the complex plane into these three sets

$$\begin{aligned} &\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \underline{c}(h,\varphi)\}\,, \\ &\{\lambda \in \mathbb{C} : \underline{c}(h,\varphi) \leqslant \operatorname{Re}\lambda \leqslant \overline{c}(h,\varphi)\}\,, \\ &\{\lambda \in \mathbb{C} : \overline{c}(h,\varphi) < \operatorname{Re}\lambda\}\,. \end{aligned} \tag{4.3}$$

The quantities $\underline{c}(h,\varphi)$ and $\overline{c}(h,\varphi)$ are defined as follows.

$$\overline{c}(h,\varphi) := \lim_{t\to\infty} \overline{c}_t(h,\varphi) = \inf_{t>0} \overline{c}_t(h,\varphi) \text{ where } \overline{c}_t(h,\varphi) := \sup_{x\in K}\{\frac{1}{t}\int_0^t h(\varphi(s,x))\,\mathrm{d}s\},$$
$$\underline{c}(h,\varphi) := \lim_{t\to\infty} \underline{c}_t(h,\varphi) = \sup_{t>0} \underline{c}_t(h,\varphi) \text{ where } \underline{c}_t(h,\varphi) := \inf_{x\in K}\{\frac{1}{t}\int_0^t h(\varphi(s,x))\,\mathrm{d}s\}. \tag{4.4}$$

It is easy to see that $\overline{c}_t(h,\varphi) = 1/t\cdot\log\|T(t)\|$, hence in the definition of $\overline{c}(h,\varphi)$, both the limit and the infimum exist and coincide with the growth bound (see A-I,(1.1)). Furthermore, $\underline{c}_t(h,\varphi) = -\overline{c}_t(-h,\varphi)$. Therefore, $\underline{c}(h,\varphi)$ is well defined too.

First we will describe the part of $\sigma(A)$ which is contained in the left half-plane determined by $\underline{c}(h,\varphi)$. It turns out that either the whole half-plane is contained in $\sigma(A)$ or it has empty intersection with $\sigma(A)$. This depends only on properties of φ. Essentially there

are three different cases. Before we state the general result (see Theorem 4.4) we give some typical examples.

Examples 4.3 (a) Consider on $K = [0,\infty]$ the semiflow defined by $\varphi(t,x) := x+t$ ($\infty + t = \infty$). Then $K_t = [t,\infty]$ and $K_\infty = \{\infty\}$. The spectrum of the corresponding semigroup $T(t)f = f\circ\varphi_t$ is $\sigma(A) = A\sigma(A) = \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda \leqslant 0\}$.

(b) Consider again $K = [0,\infty]$ and define a semiflow by

$$\varphi(t,x) := \begin{cases} x-t & \text{if } x \geqslant t\,, \\ 0 & \text{if } x < t\,. \end{cases} \qquad (\infty - t = \infty)$$

Then $K_t = K$ for all t, hence $K_\infty = K$ and $\sigma(A) = \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda \leqslant 0\}$, $R\sigma(A) = \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda < 0\}\cup\{0\}$.

(c) Consider on $K_1 := [-1,\infty)$ the equivalence relation $\sim$ defined by "$x \sim y$ if and only if $x, y \geqslant 0$ and $x - y \in \mathbb{Z}$". The semiflow φ_1 on K_1 given by $\varphi_1(t,x) = x+t$ induces a semiflow φ on $K := K_1/\!\sim$. For $0 < t < 1$ we have $K \neq K_t \neq K_\infty$ ($K_\infty = [0,1]/\!\sim\, \cong \Gamma$). The spectrum is $\sigma(A) = 2\pi i\mathbb{Z}$.

(d) Consider on $K = [-1,1]$ the flow φ given by

$$\varphi(t,x) := \begin{cases} -1 & \text{if } x < 0 \text{ and } t > -\frac{x+1}{x}\,, \\ \frac{x}{1+tx} & \text{otherwise}\,. \end{cases}$$

Then $K_t = [-1, \frac{1}{1+t}]$, $K_\infty = [-1,0]$ and

$$\sigma(A) = \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda \leqslant 0\}, \quad \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda < 0\} \subsetneq A\sigma(A)\cap R\sigma(A)\,.$$

Theorem 4.4 *Suppose $\mathcal{T}$ is a semigroup of lattice homomorphisms given by (4.1) with generator A. Considering $H := \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda < \underline{c}(h,\varphi)\}$, where $\underline{c}(h,\varphi)$ is given by (4.4), then we have.*

(i) If $K_t \neq K_\infty$ for every $t < \infty$, then $H \subseteq A\sigma(A)$.
(ii) If $\varphi_{|K_\infty}$ is not injective, then $H \subseteq R\sigma(A)$.
(iii) If $K_s = K_\infty$ for some $s < \infty$ and $\varphi_{|K_\infty}$ is injective, then $H \cap \sigma(A) = \emptyset$.

Proof For $\varepsilon > 0$ we define $H_\varepsilon = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \underline{c}(h,\varphi) - \varepsilon\}$. Obviously it is enough to prove assertion (i), (ii) and (iii) respectively for $H_{2\varepsilon}$, $\varepsilon > 0$ arbitrary, instead of H. (i) By the definition given in (4.1) there exists a $\tau > 0$ such that $\underline{c}_t(h,\varphi) \geqslant \underline{c}(h,\varphi) - \varepsilon$ for all $t \geqslant \tau$. It follows that

$$h_t(x) \geqslant \mathrm{e}^{(\alpha+\varepsilon)t} \text{ whenever } t \geqslant \tau, x \in K, \alpha < \underline{c}(h,\varphi) - 2\varepsilon\,. \tag{4.5}$$

Now we fix $\lambda = a + \mathrm{i}\beta \in H_{2\varepsilon}$ $(a, \beta \in \mathbb{R})$ and construct an approximate eigenvector (g_n) of A corresponding to λ. For $n \leqslant \tau + 1$ we choose an arbitrary function $g_n \neq 0$. Now suppose $n > \tau+1$. We choose $x_n \in K_{n+1/2} \setminus K_{n+1}$ (cf. Lemma 4.2(i)), then there exists $y_n \in K$ such that $\varphi(n+1/2, y_n) = x_n$. We have $\varphi([0, n+1/2], y_n) \cap K_{n+1} = \emptyset$ and the mapping $t \mapsto \varphi(t, y_n)$ is a continuous injection, hence a homeomorphism from $[0, n+1/2]$ into K (this is true because $\varphi(n+1/2, y_n) \notin K_{n+1}$). By Tietze's Theorem there is $f_n \in C(K)$ such that

$$\begin{aligned} &\|f_n\| \leqslant 1, \quad f_{n|K_{n+1}} = 0\,, \\ &f_n(\varphi(t, y_n)) = 0 \text{ for } 0 \leqslant t \leqslant n - (1+\delta) \text{ and } n + \delta \leqslant t \leqslant n+1\,, \\ &f_n(\varphi(t, y_n)) = \mathrm{e}^{\mathrm{i}\beta t} \text{ for } n-1 \leqslant t \leqslant n\,. \end{aligned} \tag{4.6}$$

The constant $\delta \in (0, 1/2)$ will be determined later.

Considering $g_n := \int_0^{n+1} \mathrm{e}^{-\lambda t} T(t) f_n \,\mathrm{d}t$, then $g_n \in D(A)$ and

$$(A-\lambda)g_n = (1-\mathrm{e}^{-\lambda(n+1)}T(n+1))f_n = f_n - \mathrm{e}^{-\lambda(n+1)} \cdot h_{n+1} \cdot f_n \circ \varphi_{n+1} = f_n\,. \tag{4.7}$$

Moreover,

$$\|g_n\| \geqslant \left| g_n(y_n)\right| = \left| \int_0^{n+1} \mathrm{e}^{-\lambda t} h_t(y_n) f_n(\varphi(t, y_n))\,\mathrm{d}t \right| \geqslant$$

$$\left|\int_{n-1}^{n} \mathrm{e}^{-\lambda t} h_t(y_n) \mathrm{e}^{\mathrm{i}\beta t}\,\mathrm{d}t\right| - \left[\int_{n-(1+\delta)}^{n-1} + \int_n^{n+\delta} \left|\mathrm{e}^{-\lambda t}|h_t(y_n) f_n(\varphi(t, y_n))\right| \mathrm{d}t\right]$$

$$\geqslant \int_{n-1}^{n} \mathrm{e}^{-at} \mathrm{e}^{(a+\varepsilon)t}\,\mathrm{d}t - \left[\int_{n-(1+\delta)}^{n-1} + \int_n^{n+\delta} \mathrm{e}^{-at} |h_t(y_n)|\,\mathrm{d}t\right]$$

$$= 1/\varepsilon \cdot \left(\mathrm{e}^{\varepsilon n} - \mathrm{e}^{\varepsilon(n-1)}\right) - \left[\int_{n-(1+\delta)}^{n-1} + \int_n^{n+\delta} \mathrm{e}^{-at} |h_t(y_n)|\,\mathrm{d}t\right].$$

The constant δ can be chosen such that

$$\|g_n\| \geqslant 1/2\varepsilon \cdot \left(\mathrm{e}^{\varepsilon n} - \mathrm{e}^{\varepsilon(n-1)}\right) \to \infty \text{ for } n \to \infty. \tag{4.8}$$

It follows from (4.8) and (4.7) that $g_n/\|g_n\|$ is an approximate eigenvector of A corresponding to λ. Thus (a) is proved.

The proofs of (ii) and (iii) will be handled simultaneously. First we show that we can restrict attention to the case where $K = K_\infty$.

Indeed, K_∞ is a φ-invariant subset, hence $I_\infty := \{f \in C(K) : f|_{K_\infty} = 0\}$ is a T-invariant ideal. Identifying $C(K)/I_\infty$ with $C(K_\infty)$ (cf. B-I, Section 1), then $(T(t)_{/I_\infty})$ is the semigroup governed by $\varphi_{|K_\infty}$ and $h_{|K_\infty}$. Since one always has $R\sigma(A_/) \subseteq R\sigma(A)$, assertion (ii) is proved. when we can show that $H_{2\varepsilon} \subseteq R\sigma(A_/)$. In case (iii) one has $K_s = K_\infty$ for some $s < \infty$, which implies $T(s)_{|I_\infty} = 0$. Hence we have $\sigma(A_{|I_\infty}) = \emptyset$ and therefore $\sigma(A) = \sigma(A_{/I_\infty})$ by A-III, Proposition 4.2.

Henceforth we will assume that $K = K_\infty$, that is, φ is surjective (cf. Lemma 4.2(ii)).

We choose $\tau > 0$ such that (4.5) is true. Since φ is surjective, for every $f \in C(K)$ there is a $x_f \in K$ such that $\|f\| = \|f(\varphi(\tau, x_f))\|$ and we obtain for $\lambda \in H_{2\varepsilon}$, $\lambda = a + \mathrm{i}\beta$, $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned}
\|(\mathrm{e}^{\lambda\tau} - T(\tau))f\| &\geqslant |h_\tau(x_f)f(\varphi(\tau, x_f)) - \mathrm{e}^{\lambda\tau}f(x_f)| \\
&\geqslant h_\tau(x_f)\|f\| - \mathrm{e}^{a\tau}|f(x_f)| \\
&\geqslant \mathrm{e}^{(a+\varepsilon)\tau}\|f\| - \mathrm{e}^{a\tau}\|f\| \\
&= \mathrm{e}^{a\tau}(\mathrm{e}^{\varepsilon\tau} - 1)\|f\| .
\end{aligned} \tag{4.9}$$

It follows that the disc $D := \{\lambda \in \mathbb{C} : |\lambda| < \exp(\underline{c}(h, \varphi) - 2\varepsilon)\}$ has an empty intersection with $A\sigma(T(\tau))$ and therefore $H_{2\varepsilon} \cap A_\sigma(A) = \emptyset$ by A-III,6.2. Since every boundary point of the spectrum is an approximate eigenvalue (by A-III, Proposition 2.2(i)) we have the following alternative.

$$\begin{aligned}
&\text{Either } D \subseteq \varrho(T(\tau)) \text{ and } H_{2\varepsilon} \subseteq \varrho(A) \\
&\text{or } D \subseteq R\sigma(T(\tau)) \text{ and } H_{2\varepsilon} \subseteq R\sigma(A).
\end{aligned} \tag{4.10}$$

It is not difficult to see that $0 \in \varrho(T(\tau))$ whenever φ_t is bijective and that 0 is an eigenvalue of $T(\tau)'$ if φ_τ is not injective. Since we assumed that that φ is surjective, assertions (ii) and (iii) of the theorem are immediate consequences of (4.10). □

The Examples 4.3(a), (b) and (c) respectively are prototypes of the three different cases considered in Theorem 4.4. Example 4.34.3(c) also shows that there are semigroups whose spectrum is contained in a right half-plane, although they cannot be embedded in a group (compare Corollary 4.5 below). Example 4.3(d) shows that (a) and (b) do not exclude each other.

Corollary 4.5 *If φ is injective or surjective, then the following assertions are equivalent.*

(a) A is the generator of a strongly continuous group.
(b) $\sigma(A)$ is contained in a right half-plane.

Proof $(a) \implies (b)$: holds true because $-A$ is a generator of a semigroup.

$(b) \implies (a)$: We have to show that one (hence each) operator $T(t)$, $t > 0$ is invertible. Obviously this is true if φ is bijective. At first we assume that φ is surjective, that is, $K = K_\infty$. By Theorem 4.4 we have that $\varphi|_{K_\infty}$ is injective if (b) is true. Thus φ is bijective. Now we assume that φ is injective. We have to show that $K = K_\infty$. By Theorem 4.4 we have $K_\infty = K_s$ for some s, whenever (b) is true. Given

$x \in K$ then by Lemma 4.2(ii) there exists $y \in K_\infty$ such that $\varphi(s,x) = \varphi(s,y)$. If φ is injective we have $x = y \in K_\infty$. □

Example 4.6 Suppose $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. We denote the maximal flow corresponding to the differential equation $y' = F(y)$ by φ_0. In general, φ_0 is only defined on an open subset of $\mathbb{R}\times\mathbb{R}^n$ which contains $\{0\}\times\mathbb{R}^n$. For $x \in \mathbb{R}^n$ there exist $\underline{t}_x$ and $\overline{t}_x$ such that

$$\begin{aligned} &-\infty \leqslant \underline{t}_x < 0 < \overline{t}_x \leqslant \infty\,,\\ &\varphi_0(t,x) \text{ is defined if } \underline{t}_x < t < \overline{t}_x\,,\\ &\text{if } \overline{t}_x < \infty\ (\underline{t}_x > -\infty) \text{ then } |\varphi_0(t,x)| \to \infty \text{ as } t \uparrow \overline{t}_x\ (t \downarrow \underline{t}_x)\,. \end{aligned} \tag{4.11}$$

For details see Sect.18.2 of **?**].

(a) If φ_0 is a global flow, i.e., if φ_0 is defined on $\mathbb{R}\times\mathbb{R}^n$, then one has a corresponding (semi-)group on $C_0(\mathbb{R}^n)$. If F is differentiable, its generator is the closure of A_1 which is defined as follows (cf. B-II,Example 3.15).

$$\begin{aligned} A_1 f = (F|\mathrm{grad}\, f) &:= \sum F_i \cdot \partial_i f\,,\\ D(A_1) &:= \{f \in C^1 : \mathrm{supp}\, f \textit{ is compact}\}\,. \end{aligned} \tag{4.12}$$

Then φ_0 can be uniquely extended to a flow $\tilde{\varphi}_0$ on $\mathbb{R}^n\cup\{\infty\}$ by defining $\tilde{\varphi}_0(t,\infty) := \infty$ for all $t \in \mathbb{R}$. φ_0 and $\tilde{\varphi}_0$ satisfy condition (c) of Theorem 4.4. A global flow exists if F is globally Lipschitz continuous or uniformly bounded. For $\{x \in \mathbb{R}^n : (x|F(x)) > 0\}$ bounded in $\mathbb{R}^n$, a global semiflow always exists (cf. **?** , Section 5.2]).

(b) We do not asssume that φ_0 is globally defined. Instead we consider a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ such that $(F(x)|\nu(x)) > 0$ for every $x \in \partial\Omega$, where $\nu(x)$ denotes the outward normal vector.

Then for $x \in \overline{\Omega}$ we have $\underline{t}_x = -\infty$. Moreover, either $\varphi_0(t,x) \in \Omega$ for all $t \geqslant 0$ or else there exists a unique s_x with $0 \leqslant s_x < \overline{t}_x$ such that $\varphi_0(s_x,x) \in \partial\Omega$. In the first case we write $s_x := \infty$. Then we define $\varphi : \mathbb{R}_+ \times \overline{\Omega} \to \overline{\Omega}$ as

$$\varphi(t,x) := \begin{cases} \varphi_0(t,x) & \text{if } 0 \leqslant t < s_x\,,\\ \varphi_0(s_x,x) & \text{if } t \geqslant s_x\,. \end{cases}$$

Then φ is a continuous semiflow on the compact set $K := \overline{\Omega}$. We have $K_\infty = K$ and $\varphi_{|K_\infty}$ is not injective.

In case F is differentiable, the generator of the corresponding semigroup is the closure of the operator A_2 defined by

$$A_2 f := (F|\mathrm{grad}\, f), \quad D(A_2) := \{f \in C^1(\overline{\Omega}) : (F|\mathrm{grad}\, f) = 0 \text{ on } \partial\Omega\}\,.$$

(c) We consider Ω as in (b) and assume that $(F(x)|\nu(x)) \leqslant 0$ for every $x \in \partial\Omega$. Then for every $x \in \overline{\Omega}$ we have $\overline{t}_x = \infty$. Thus $\varphi := \varphi_{0|\mathbb{R}_+\times\overline{\Omega}}$ is a continuous semiflow on $K := \overline{\Omega}$.

If $(F(x)|\nu(x)) < 0$ for some $x \in \partial\Omega$, we have $K_t \subsetneq K_s$ whenever $t > s$ and $\varphi|_{K_\infty}$ is injective. For a differentiable vector field F, the generator of the corresponding semigroup is the closure of A_3 defined as

$$A_3 f := (F|\operatorname{grad} f), \quad D(A_3) := C^1(\overline{\Omega}) .$$

We conclude the discussion of semi-flows associated with ordinary differential equations by remarking that the ideas of (b) and (c) can be combined to obtain semigroups for more general subsets Ω.

We continue the discussion of the spectrum of semigroups of lattice homomorphisms on $C(K)$ given by (4.1). Theorem 4.4 gives a good description of the part contained in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \underline{c}(h,\varphi)\}$.

The half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \overline{c}(h,\varphi)\}$ is always a subset of the resolvent set (see Proposition 4.8(a) below). The description of the remaining part $\{\lambda \in \sigma(A) : \underline{c}(h,\varphi) \leqslant \operatorname{Re}\lambda \leqslant \overline{c}(h,\varphi)\}$ is more difficult. First we discuss some examples and then give a partial answer to this problem (see Proposition 4.8(ii)-(v)).

Example 4.7 (a) Consider the flow on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ defined by

$$\varphi(t,x) := \arctan(\tan x - t), x \in [-\pi/2, \pi/2], t \in \mathbb{R} .$$

It belongs to the differential equation $y' = -\cos^2 y$, and a continuous function $h : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ with $h(-\frac{\pi}{2}) \leqslant h(\frac{\pi}{2})$. Then we have $\underline{c}(h,\varphi) = h(-\frac{\pi}{2})$ and $\overline{c}(h,\varphi) = h(\frac{\pi}{2})$. The spectrum of the corresponding semigroup is given by

$$\sigma(A) = \{\lambda \in \mathbb{C} : h(-\pi/2) \leqslant \operatorname{Re}\lambda \leqslant h(\pi/2)\} .$$

(b) Consider $K = \{z \in \mathbb{C} : 1 \leqslant |z| \leqslant 2\} = \{r \cdot \mathrm{e}^{\mathrm{i}\omega} : \omega \in \mathbb{R}, 1 \leqslant r \leqslant 2\}$ and a continuous function $\kappa : [1,2] \to \mathbb{R}_+$.

Let $\tilde{\varphi}$ be the flow on K governed by the differential equation $\dot{\omega} = \kappa(r)$, $\dot{r} = 0$ (hence $\tilde{\varphi}(t, r \cdot \mathrm{e}^{\mathrm{i}\omega}) = r \cdot \mathrm{e}^{\mathrm{i}(\omega + \kappa(r)t)}$).

For a continuous function $h : K \to \mathbb{R}$ let $\hat{h}(r) := \frac{1}{2\pi}\int_0^{2\pi} h(r \cdot \mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t$ $(1 \leqslant r \leqslant 2)$. The spectrum of the semigroup corresponding to φ and h (cf. (4.1)) is given by

$$\sigma(A) = \{\hat{h}(r) + \mathrm{i}k\kappa(r) : k \in \mathbb{Z}, 1 \leqslant r \leqslant 2\} \cup \{h(z) : \kappa(|z|) = 0\}.$$

Proposition 4.8 *Suppose the semigroup $(T(t))_{t\geqslant 0}$ on $C(K)$ is given by (4.1) and let $\underline{c}(h,\varphi)$, $\overline{c}(h,\varphi)$ be defined as in (4.4). Then the following assertions hold.*

(i) $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \overline{c}(h,\varphi)\} \subset \varrho(A)$,
(ii) $\overline{c}(h,\varphi)$ *and* $\underline{c}(h,\varphi)$ *are spectral values* ,
(iii) If $\varphi(t,x_0) = x_0$ *for every* $t \geqslant 0$, *then* $h(x_0) \in R\sigma(A)$,
(iv) Assume x_0 has a finite orbit (i.e., $\varphi(\mathbb{R}_+, x_0) = \varphi([0,T], x_0)$ for some $T < \infty$) and $\tau := \inf\{t > 0 : \varphi(T+t, x_0) = \varphi(T, x_0)\} > 0$, then $\hat{h}(x_0) + \frac{2\pi}{\tau}\mathrm{i}\mathbb{Z} \subset R\sigma(A)$ where $\hat{h}(x_0) := \frac{1}{\tau}\int_T^{T+\tau} h(\varphi(s,x_0))\,\mathrm{d}s$,
(v) If x_0 has an infinite orbit and $\hat{h} := \lim_{t\to\infty} h(\varphi(t,x_0))$ exists, then $\hat{h} + \mathrm{i}\mathbb{R} \subset \sigma(A)$.

Proof (i) and (ii): A look at (4.4) shows that $\overline{c}_t(h, \varphi) = 1/t \cdot \log \|T(t)\|$ hence $\overline{c}(h, \varphi) = \omega_0(A)$ (cf. A-I, (1.1)). Consequently, we have $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \overline{c}(h, \varphi)\} \subset \varrho(A)$ and $\overline{c}(h, \varphi) \in \sigma(A)$ by Theorem 1.6.

To prove $\underline{c}(h, \varphi) \in \sigma(A)$, we can assume by Theorem 4.4 that $K_\infty = K_s$ for some s and that $\varphi_{|K_\infty}$ is injective. It is easy to see that $\underline{c}(h, \varphi) = \underline{c}(h_{|K_\infty}, \varphi_{|K_\infty})$, moreover, we have $\sigma(A_{|I_\infty}) = \emptyset$ hence $\sigma(A) = \sigma(A_{/I_\infty})$ by A-III, Proposition 4.2. This shows that we also can assume that $K = K_\infty$, i.e., φ is bijective or A is the generator of a group. Now the assertion follows from $\underline{c}(h, \varphi) = \underline{c}(h, \varphi^{-1}) = -\overline{c}(-h, \varphi^{-1}) = -s(-A)$.

(iii) and (iv): One can check easily that in case of (iii) the Dirac functional δ_{x_0} is an eigenvector of A' corresponding to $h(x_0)$. A little bit more calculation is necessary to check that in case of (iv) the functional Ψ_k defined by

$$\Psi_k(f) := \int_T^{T+\tau} \exp\left(-\mathrm{i} \cdot \frac{2\pi k}{\tau} \cdot t\right) \cdot h_t(x_0) \cdot f(\varphi(t, x_0))\, \mathrm{d}t \; (k \in \mathbb{Z}, f \in C(K))$$

is an eigenvector of A' corresponding to $\hat{h}(x_0) + \mathrm{i} \cdot \frac{2\pi k}{\tau}$.

(v) Given $\beta \in \mathbb{R}$ we will show that $\hat{h} + \mathrm{i}\beta \in A_\sigma(A') \subseteq \sigma(A)$. For $n, m \in \mathbb{N}$ we define a linear functional Ψ_{nm} as

$$\Psi_{nm}(f) := \frac{1}{n} \int_0^n \exp(-(h^\star + \mathrm{i}\beta)t) \cdot h_t(\varphi(m, x_0)) \cdot f(\varphi(m+t, x_0))\, \mathrm{d}t, \quad f \in C(K).$$

For $f \in D(A)$ we have

$$\begin{aligned}&\langle (\hat{h} + \mathrm{i}\beta - A) f, \Psi_{nm} \rangle = \\ &\frac{1}{n} \cdot \left(f(\varphi(m, x_0)) - \mathrm{e}^{-\mathrm{i}\beta n} \cdot \exp\left(\int_m^{m+n} (h(\varphi(s, x_0)) - \hat{h})\, \mathrm{d}s \right) f(\varphi(m+n, x_0)) \right).\end{aligned}$$

It follows that $\varphi_{nm} \in D(A')$ and, since $\lim_{t\to\infty} h(\varphi(t, x_0)) = \hat{h}$,

$$\limsup_{m\to\infty} \|(\hat{h} + \mathrm{i}\beta - A')\Psi_{nm}\| \leqslant 1/n \text{ for every } n \in \mathbb{N}. \tag{4.13}$$

Because the orbit is infinite we have

$$\begin{aligned}\|\Psi_{nm}\| &= \frac{1}{n} \int_0^n \left| \mathrm{e}^{-(\hat{h}+\mathrm{i}\beta)t} h_t(\varphi(m, x_0)) \right| \mathrm{d}t \\ &= \frac{1}{n} \int_0^n \exp\left(\int_m^{m+t} (h(\varphi(s, x_0)) - \hat{h})\, \mathrm{d}s \right)\end{aligned}$$

which shows that

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$$\lim_{m\to\infty} \|\Psi_{nm}\| = 1 \text{ for every } n \in \mathbb{N}. \tag{4.14}$$

In view of (4.13) and (4.14) it is not difficult to find a subsequence $k(n)$ of $\mathbb{N}$ such that $(\Psi_{n,k(n)})$ is an approximate eigenvector of A' corresponding to $\hat{h} + \mathrm{i}\beta$. □

We are now going to apply the results obtained so far to the special case where $h = 0$, i.e., we consider semigroups of lattice homomorphisms which are Markov operators.

Theorem 4.9 *Suppose $\mathcal{T}$ is a semigroup of Markov lattice homomorphisms on $C(K)$ governed by the semiflow φ.*

(i) If $\varphi_{|K_\infty}$ is not injective or if $K_t \neq K_\infty$ for every $t < \infty$, then $\sigma(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant 0\}$.
(ii) If $K_\infty = K_s$ for some s and $\varphi_{|K_\infty}$ is injective, then $\sigma(A)$ is a cyclic closed subset of $\mathrm{i}\mathbb{R}$. Moreover, we have $\sigma(A) \neq \mathrm{i}\mathbb{R}$ if and only if there is a $T < \infty$ such that every orbit of φ has length less than T (i.e., $\varphi(\mathbb{R}_+, x) = \varphi([0, T], x)$ for every $x \in K$).

Proof (i) This is an immediate consequence of Theorem 4.4 and Proposition 4.8.

(ii) The first assertion follows from Theorem 4.4 and Theorem 4.1. Moreover, as in the proof of Theorem 4.4(ii) and (iii) we can assume without loss of generality that $K = K_\infty$, hence φ is bijective. If there is no upper bound for the length of the orbits, then $\sigma(A) = \mathrm{i}\mathbb{R}$ by assertions (iv) and (v) of Proposition 4.8.

Now we assume that the lengths of the orbits are bounded by T. Because φ is bijective, for every $x \in K$ there exists a $r = r_x$ with $T/2 \leqslant r \leqslant T$ such that

$$\varphi(t, x) = \varphi(t + r, x) = \varphi(t + 2r, x) = \cdots = \varphi(t + kr, x) \quad (t \in \mathbb{R}_+, k \in \mathbb{N}).$$

Therefore we have for $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$, $f \in C(K)$, $x \in K$

$$\begin{aligned}(R(\lambda, A)f)(x) &= \int_0^\infty \mathrm{e}^{-\lambda t} f(\varphi(t, x))\,\mathrm{d}t = \\ &= \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda kr} \int_{kr}^{(k+1)r} \mathrm{e}^{-\lambda(t-kr)} f(\varphi(t - kr, x))\,\mathrm{d}t = \\ &= (1 - \mathrm{e}^{-\lambda r})^{-1} \cdot \int_0^r \mathrm{e}^{-\lambda t} f(\varphi(t, x))\,\mathrm{d}t\,. \end{aligned} \tag{4.15}$$

If $0 < \beta < 2\pi/T$, then the assumption $T/2 \leqslant r \leqslant T$ implies that there exists a neighborhood U of $\lambda_0 := \mathrm{i}\beta$ such that the functions $\lambda \mapsto (1 - \exp(-\lambda r_x))^{-1}$ are uniformly bounded on U, by M say. Then (4.15) implies that $\|R(\lambda, A)f\| \leqslant M\left(\int_0^r \left|\mathrm{e}^{-\lambda t}\right| \mathrm{d}t\right)\|f\|$ for $\lambda \in U$, $\operatorname{Re}\lambda > 0$, therefore $\lambda_0 = \mathrm{i}\beta \in \varrho(A)$. □

Remark 4.10 In case $\sigma(A) \neq \mathrm{i}\mathbb{R}$, then $\varphi_{|K_\infty}$ is bijective and has only finite orbits. Therefore every point $x \in K_\infty$ has a well-defined period $\tau_x := \inf\{\tau > 0 : \varphi(\tau, x) = x\}$. A more detailed analysis yields the following description

$$\sigma(A) = \overline{\{\mathrm{i} \cdot 2\pi k/\tau_x : k \in \mathbb{Z}, x \in K_\infty, \tau_x > 0\}} \cup \{0\}\,. \tag{4.16}$$

The inclusion "⊆" can be derived from Theorem 4.11 which is stated below. The reverse inclusion follows from Proposition 4.8(d).

In our detailed discussion of the spectrum of lattice homomorphisms we restricted ourselves to the case where the space K is compact. The main reason is that there is no description as given in (4.1) of the semigroups for locally compact spaces X. In general, it is difficult to define a semiflow on X because points may tend to infinity in a finite time. But even if one can find a flow on a suitable compactification of C, it may be impossible to find a multiplicator. This can be seen by studying the following example.

Suppose φ_1 is a semiflow on a compact space K_1 and K_0 is a closed φ_1-invariant subset, h a continuous function on K_1 . The semigroup $(T_1(t))$ on $C(K_1)$ corresponding to φ_1 and h leaves the ideal $I := \{f \in C(K_1) : f_{|K_0} = 0\}$ invariant and induces via restriction a semigroup $(T(t))$ on $I = C_0(X)$, where $X = K_1 \setminus K_0$. In this case one can construct semi-flows associated with $(T(t))$ on $X \cup \{\infty\}$ or on $\overline{X}$ (closure in K_1), but in general one cannot find a corresponding multiplicator which is defined on one of these compactifications.

The situation is much nicer when groups of lattice homomorphisms instead of semigroups are considered. In this case there is an analogue of (4.1) (cf. B-II, Theorem 3.14) and the spectrum can be described completely. For more details and the proof of the following result we refer to **?**].

Theorem 4.11 *Suppose X is a locally compact space and $(T(t))_{t\in\mathbb{R}}$ is a group of lattice homomorphisms governed by the flow φ and the multiplicator h. Then we have*

(i) $\sigma(A) = \sigma_1 \cup \sigma_2 \cup \sigma_3$ *where the sets* σ_i *are defined as follows.*

$$\sigma_1 := \overline{\{\hat{h}(x) + \mathrm{i}\cdot 2\pi k/\tau_x : x \in X, 0 < \tau_x < \infty\}}\,,$$
$$\sigma_2 := \overline{\{h(x) : x \in X, \tau_x = 0\}}\,,$$
$$\sigma_3 := \{\lambda \in \mathbb{C} : \lambda + \mathrm{i}\mathbb{R} \subseteq \sigma(A)\}$$

(ii) $\sigma(T(t)) = \overline{\exp(t\sigma(A))}$ *for every* $t \geqslant 0$.

(iii) Every isolated point of $\sigma(A)$ *is a first order pole of the resolvent.*

GG: $\hat{h}(x) =?$

Notes

Spectral theory for a single positive operator is an essential cornerstone for spectral theory of positive one-parameter semigroups. Many of the results we have presented in this chapter have analogues in the discrete case (i.e., for a single operator). This relation may serve as a guide. However, only in few cases can the continuous version be deduced directly from its discrete analogue. Therefore we will not try to trace back the roots of every result to the discrete version. Instead we refer to **?**] and the notes and references given there.

Many of the results we have presented in this chapter can be extended (more or less easily) to the more general setting of Banach lattices, which include the very important examples of L^p-spaces. Others are typical for $C_0(X)$ and allow no extension. We will discuss this fact in more detail in Chapter C-III. The more general setting considered there also allows us to prove results for $C_0(X)$ which we could not obtain staying within the framework of spaces of continuous functions.

Section 1: Theorem 1.1 was stated by **?**], but a complete proof is given in **?**]. Proposition 1.5 is taken from **?**] and Theorem 1.6 is (implicitely) contained in **?**]. A generalization to (non-lattice) ordered Banach spaces can be found in Section 2.4 of **?**].

Section 2: Lemma 2.3 dates back to Rota (see **?**]). Our approach follows **?**]. The notion "(imaginary) additively cyclic" was introduced by **?**] (and Schaefer (1980) respectively). Derndinger proves some cyclicity results for the boundary spectrum. A result similar to Proposition 2.7 is given in Section 7.4 of **?**]. Lemma 2.8 in combination with C-III,Lemma 3.13 can be used to characterize semigroups whose spectral bound is a pole of finite algebraic multiplicity (see C-III,(3.19)). The hypothesis of Theorem 2.9 can be weakened, one only needs that $s(A)$ is a pole of the resolvent (see C-III,Corollary 2.12). Further results on the cyclicity of the boundary spectrum will be given in Chapter C-III. In particular we refer to C-III, Theorems.2.10, 3.11 and 3.13. The dichotomy stated in (2.19) is probably the most interesting consequence of cyclicity results. It has far reaching consequences on the asymptotic behavior of positive semigroups. Example 2.13 is due to Davies (unpublished note). Example 3.4(ii) will be discussed in more detail and more generality in Section 3 of Chapter B-IV. We return to Remark 2.15(b) in Section 2 of B-IV.

Section 3: The concept of irreducibility as defined in 3.1 closely related to various other notions: In topological dynamics flows inducing irreducible semigroups are called "minimal flows" (cf. Example 3.4(i)). Moreover, "ergodicity" and "unique ergodicity" are closely related to irreducibility (see**?**] or **?**]). Irreducible semigroups are discussed to some extent in **?**]. E.g. he proves a special case of Theorem 3.6. Proposition 3.3 will be generalized in C-III, Proposition 3.3. Assertion (a) of Proposition 3.5 was proven by **?**] while Theorem 3.6 is taken from **?**]. Elliptic operators (more general than Example 3.4(ii)) as generators on spaces of continuous functions, were investigated by many people, e.g., **?**], **?**], [Roth (1976) & (1978)] and **?**].

Section 4: Theorem 4.1 is due to **?**]. The spectrum of semigroups of Markov lattice homomorphisms is investigated by **?**]. In particular they prove Theorem 4.4 for Markov semigroups. Earlier results are due to **?**]. We indicated briefly in Example 4.6 that there is a relationship between spectral properties of lattice semigroups and differentiable dynamics. For more details we refer to **?**] and **?**]. E.g., the "annular hull theorem" is a special case of Theorem 4.11(b). The general result 4.11 was proven by **?**].

Chapter B-IV
Asymptotics of Positive Semigroups on $C_0(X)$

In the following chapter, we examine the asymptotic behavior of positive semigroups in spaces of continuous functions. The first section is devoted to the various *growth constants* defined in Chapter A-IV and to their coincidence for positive semigroups. In the second section, we treat the asymptotic behavior of positive semigroups, which do not differ “too much” from compact semigroups. Properties such as eventual compactness or quasi-compactness allow us to describe the long term behavior of the semigroup by using the results from A-III and B-III on the spectrum of the generator. In the last section, we investigate differential delay equations by semigroup methods. In particular, we characterize the spectral bound of the solution semigroups, yielding simple conditions for stability. Numerous examples conclude the chapter.

1 Stability of Positive Semigroups on $C_0(X)$

In Chapter A-IV we have seen that the long term behavior of a semigroup $(T(t))_{t\geqslant 0}$ is strongly connected with the existence (and growth) of the resolvent of its generator A in a right halfplane. In particular, the exponential growth of certain semigroups is determined solely by the location of the spectrum (see A-IV,(1.7) and (1.8)). In these cases, spectral bound $s(A)$ and growth bound $\omega_0(A)$ coincide and the equality

$$s(A) = \omega_1(A) = \omega_0(A) \tag{1.1}$$

holds.

Unfortunately, (1.1) does not hold for positive semigroups in general. In A-IV, Example 1.2(2), we have seen that for the generator A of the (positive) translation semigroup on the Banach lattice $C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x\,\mathrm{d}x)$ the strict inequalitiy

$$\omega_1(A) < \omega_0(A)$$

is valid. However, for positive semigroups on certain nice Banach lattices, (1.1) is true. One of these nice Banach lattices is $C_0(X)$. This will be proved in Theorem 1.4.

For compact X, (1.1) was already proved in B-II, Corollary 1.14 and B-III, Theorem 1.6 respectively. Actually, much more is true and for positive semigroups on $C(K)$, K compact, all stability concepts mentioned in Chapter A-IV are mutually equivalent.

Theorem 1.1 *Let A be the generator of a positive semigroup $(T(t))_{t\geqslant 0}$ on $C(K)$, K compact. Then*

$$s(A) = \inf\{\lambda \in \mathbb{R}\colon\ Af \leqslant \lambda f \text{ for some } 0 \ll f \in D(A)\}\,. \tag{1.2}$$

Moreover, $s(A) = \omega_0(A) \in R\sigma(A) = P\sigma(A')$ and the following statements are mutually equivalent.

(a) $s(A) < 0$,
(b) $(T(t))_{t\geqslant 0}$ is uniformly exponentially stable,
(c) $(T(t))_{t\geqslant 0}$ is weakly stable; i.e., $\langle T(t)f, \mu\rangle \to 0$ as $t \to \infty$ for every $f \in D(A)$ and every $\mu \in C(X)'$.

Proof (1.2) follows directly from A-III,4.4 and the results from B-II and B-III mentioned above. It remains to show the implication $(c) \Rightarrow (a)$.

If $\langle T(t)f, \mu\rangle \to 0$ for every $\mu \in C(K)'$, then, by the Uniform Boundedness Principle, $\|T(t)f\| \leqslant M_f$ for every $f \in D(A)$. Using

$$s(A) \leqslant \sup\{\omega(f)\colon\ f \in D(A)\} = \omega_1(A) \text{ (see A-IV,Theorem 1.4)}$$

we obtain that $s(A) \leqslant 0$. Suppose $0 = s(A)$. From B-III,Theorem 1.6 it follows that $s(A) \in P\sigma(A')$, hence there is $0 < \mu \in C(K)'$ such that $T(t)'\mu = \mu$ for $t \geqslant 0$. Since $D(A)$ is dense, there exists $f \in D(A)$ such that $\langle f, \mu\rangle \neq 0$. Then $|\langle T(t)f, \mu\rangle| = |\langle f, \mu\rangle| \geq 0$ which contradicts the weak stability. Therefore, $s(A) < 0$. □

For spaces $C_0(X)$, X locally compact, the different stability concepts are no longer equivalent.

Example 1.2 (a) The left-translation semigroup on $C_0(\mathbb{R}_+)$ or the semigroup generated by the Laplacian on $C_0(\mathbb{R}^n)$, see B-III, Example 1.7, are uniformly stable but not exponentially stable.

(b) The left translations $T(t)f(x) = f(x+t)$ on $C_0(\mathbb{R})$ form a group of isometries. Hence $(T(t))_{t\geqslant 0}$ is not stable. However, $(T(t))_{t\geqslant 0}$ is weakly stable. Indeed, identifying $C_0(\mathbb{R})'$ with the space of all bounded Borel measures on $\mathbb{R}$, for $f \in C_0(\mathbb{R})$, $\mu \in C_0(\mathbb{R})'$ we have

$$\langle T(t)f, \mu\rangle = \int (T(t)f)(x)\ \mathrm{d}\mu(x)\,.$$

Obviously, $T(t)f$ tends pointwise to 0 as $t \to \infty$ and is dominated by the μ-integrable function $\|f\|_\infty \cdot 1$. Thus Lebesgue's Dominated Convergence Theorem implies

$$\lim_{t\to\infty} \langle T(t)f, \mu\rangle = 0.$$

(c) Finally we give an example of a positive semigroup on $C_0(X)$ which is not weakly stable, but satisfies $\mathrm{Re}\,(P\sigma(A) \cup R\sigma(A)) < 0$. (Compare with A-IV, Corollary 1.14).

Consider in the space $\mathbb{C}\backslash\{0\}$ a flow Φ having the following properties.

- The orbits starting at z with $|z| \neq 1$ spiral towards the unit circle Γ;
- 1 is a fixed point and $\Gamma\backslash\{1\}$ is a *homoclinic orbit* i.e., $\lim_{t\to+\infty} \Phi(t,z) = \lim_{t\to-\infty} \Phi(t,z) = 1$ for every $z \in \Gamma$.

A concrete example of this type is the flow governed by the following differential equations for the polar coordinates (i.e., $z = r \cdot e^{\mathrm{i}\omega}$)

$$\begin{aligned} \dot{r} &= 1 - r\,, \\ \dot{\omega} &= 1 + (r^2 - 2r \cdot \cos\omega)\,. \end{aligned}$$

The locally compact set $X := \{z \in \mathbb{C} \colon 0 < |z| < 2, z \neq 1\}$ is invariant under the flow Φ and we consider on the space $C_0(X)$ the semigroup $(T(t))_{t\geqslant 0}$ associated with Φ (i.e., $T(t)f = f \circ \Phi_t$, $f \in C_0(X)$). We claim that

(i) $(T(t))_{t\geqslant 0}$ is not weakly uniformly stable;
(ii) $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$;
(iii) $R\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$.

Proof of (i): Given $z \in X$, $|z| \neq 1$, there exist sequences (t_n), (s_n) both tending to ∞ such that $\Phi(t_n, z) \to 1$ and $\Phi(s_n, z) \to -1$. Hence for $f \in C_0(X)$ we have

$$\begin{aligned} \langle T(t_n)f, \delta_z\rangle &= f(\Phi(t_n, z)) \to 0\,, \\ \langle T(s_n)f, \delta_z\rangle &= f(\Phi(s_n, z)) \to f(-1)\,. \end{aligned}$$

Thus $\lim_{t\to\infty}\langle T(t)f, \delta_z\rangle$ does not exist for every $f \in C_0(X)$.

Proof of (ii): Assume that $T(t)f = e^{\mathrm{i}\alpha t} f$ for every $t \geqslant 0$ and some $\alpha \in \mathbb{R}$ (cf. A-III, Corollary 6.4). Given $z \in X$, there exists a sequence (t_n) such that $\Phi(t_n, z) \to 1$, hence $f(z) = (e^{-\mathrm{i}\alpha t_n} T(t_n) f)(z) = e^{-\mathrm{i}\alpha t_n} f(\Phi(t_n, z)) \to 0$. Thus $f = 0$.

Proof of (iii): At first we point out that for $f \in C_0(X)$ such that f vanishes on the unit circle Γ, we have $\lim_{t\to\infty} \|T(t)f\| = 0$. Assume that μ is a bounded Borel measure such that $T(t)'\mu = e^{\mathrm{i}\alpha t}\mu$ for every $t \geqslant 0$ and some $\alpha \in \mathbb{R}$. Then $\langle f, \mu\rangle = e^{\mathrm{i}\alpha t}\langle f, T(t)'\mu\rangle = e^{\mathrm{i}\alpha t}\langle T(t)f, \mu\rangle \to 0$ for every $f \in C_0(X)$ with $f|_\Gamma = 0$. It follows that the support of μ is contained in Γ. Since $\lim_{t\to\infty} \Phi(t, z) = 1$ for every $z \in \Gamma$, we obtain for arbitrary $f \in C_0(X)$ that $(T(t)f)(z) \to 0$ μ-a.e.. Lebesgue's Dominated Convergence Theorem implies $\langle f, \mu\rangle = e^{-\mathrm{i}\alpha t}\langle T(t)f, \mu\rangle \to 0$ as $t \to \infty$ for every $f \in C_0(X)$. Thus $\mu = 0$. ◻

Now we are going to prove the main result of this section. At first we note that the positive part of the domain of the adjoint operator is sufficiently large. In fact, we know that $\lambda R(\lambda, A) \to \mathrm{Id}$ strongly as $\lambda \to \infty$. It follows that $\lambda^2 R(\lambda, A)^2 \to \mathrm{Id}$ strongly, hence $\lambda^2 R(\lambda, A)'^2 \to \mathrm{Id}$ with respect to $\sigma(E', E)$-topology. If A generates a positive semigroup, then $\lambda^2 R(\lambda, A)'^2\mu \in D(A^*)_+ := D(A^*) \cap E'_+$ for $\mu \in E'_+$. (Note that $R(\lambda, A)'E' \subset D(A') \subset E^*$, thus $R(\lambda, A)'^2 E' \subset R(\lambda, A)'E^* = D(A^*)$.)

We summarize these observations in the following lemma.

Lemma 1.3 *Let A be the generator of a positive semigroup on a Banach lattice E. Then every $\mu \in E'_+$ is the $\sigma(E', E)$-limit of elements in $D(A^*)_+$; i.e., $\overline{D(A^*)_+}^{\sigma(E',E)} = E'_+$.*

Theorem 1.4 *Let A be the generator of a positive semigroup on $C_0(X)$. Then*

$$s(A) = \omega_1(A) = \omega_0(A) \in \sigma(A) .$$

Proof Rescaling the semigroup we may assume $\omega_0(A) = 0$, since in case $\omega_0(A) = -\infty$, then $\sigma(A) = \emptyset$, hence $s(A) = -\infty$.

Suppose $0 \notin \sigma(A) = \sigma(A^*)$. Then, by the holomorphy of the resolvent and by A-II, Proposition 1.11 we obtain

$$R(0, A^*)\Phi = \sum_{n=0}^{\infty} R(1, A^*)^{n+1}\Phi = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{n!} t^n e^{-t} T(t)^* \Phi \, \mathrm{d}t$$

for every $\Phi \in C_0(X)^*$. If $0 \leqslant \Phi \in C_0(X)^*$ and $0 \leqslant \varrho \in C_0(X)$" we can interchange integration and summation by the Monotone Convergence Theorem, i.e.,

$$\langle R(0, A^*)\Phi, \varrho\rangle = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{n!} t^n e^{-t} \langle T(t)^*\Phi, \varrho\rangle \, \mathrm{d}t = \int_0^{\infty} \langle T(t)^*\Phi, \varrho\rangle \, \mathrm{d}t \qquad (1.3)$$

Since every element of $C_0(X)^*$ and $C_0(X)$" is the difference of positive elements, the equation (1.3) holds for every $\Phi \in C_0(X)^*$, $\varrho \in C_0(X)$". This means that the net $(\int_0^r T(t)^*\Phi \, \mathrm{d}t)_{r>0}$ converges weakly to $R(0, A^*)\Phi$. But for positive Φ the net is monotone and therefore strongly convergent by Dini's Theorem (see **?**], II.Theorem 5.9). Hence $R(0, A^*)\Phi = \int_0^{\infty} T(t)^*\Phi \, \mathrm{d}t$ for every $\Phi \in C_0(X)^*$.

Now we make use of the special character of the space $C_0(X)$. For positive functions $f_1, f_2 \in C_0(X)$ we have $\sup(\|f_1\|, \|f_2\|) = \|\sup(f_1, f_2)\|$. Let $\mu_1, \mu_2 \in C_0(X)'_+$ and $\varepsilon > 0$. Then there are positive elements f, g in the unit ball of $C_0(X)$ such that $\langle f, \mu\rangle \geqslant \|\mu_1\| - \varepsilon$ and $\langle g, \mu_2\rangle \geqslant \|\mu_2\| - \varepsilon$. For $h := \sup(f, g)$ we obtain $\|h\| \leqslant 1$ and $\|\mu_1 + \mu_2\| \geqslant \langle h, \mu_1 + \mu_2\rangle \geqslant \langle f, \mu_1\rangle + \langle g, \mu_2\rangle \geqslant \|\mu_1\| + \|\mu_2\| - 2\varepsilon$. Hence $\|\mu_1 + \mu_2\| = \|\mu_1\| + \|\mu_2\|$ for $\mu_1, \mu_2 \in C_0(X)'_+$ (see also C-I).

Approximating the integral by Riemann sums one obtains

$$\| \int_0^r T(t)^*\mu \, \mathrm{d}t\| = \int_0^r \|T(t)^*\mu\| \, \mathrm{d}t \text{ for } \mu \in C_0(X)^*_+ , r > 0 .$$

and therefore, for $r \to \infty$,

$$R(0, A^*)\mu\| = \left\| \int_0^{\infty} T(t)^*\mu \, \mathrm{d}t \right\| = \int_0^{\infty} \|T(t)\mu\| \, \mathrm{d}t \ (\mu \in C_0(X)^*_+) .$$

Given $\mu \in C_0(X)'$ there is a sequence $\mu_n \in C_0(X)^*_+$ converging $\sigma(E', E)$ to $|\mu|$ (Lemma 1.3). From $|\langle f, T(t)'\mu\rangle| \leqslant \langle T(t)|f|, |\mu|\rangle = \lim_{n\to\infty}\langle T(t)|f|, \mu_n\rangle$ we con-

clude $|\langle f, T(t)'\mu\rangle| \leqslant \liminf_{n\to\infty} \|f\| \cdot \|T(t)^*\mu_n\|$ and
$\|T(t)'\mu\| \leqslant \liminf_{n\to\infty} \|T(t)^*\mu_n\|$ for $t \geqslant 0$. Applying Fatou's Lemma we obtain

$$\int_0^\infty \|T(t)'\mu\| \,\mathrm{d}t \leqslant \int_0^\infty (\liminf_{n\to\infty} \|T(t)'\mu_n\|) \,\mathrm{d}t \leqslant \liminf_{n\to\infty} \int_0^\infty \|T(t)'\mu_n\| \,\mathrm{d}t = \\ = \liminf_{n\to\infty} \|R(0, A^*)\mu_n\| \leqslant \|R(0, A^*)\| \cdot \liminf_{n\to\infty} \|\mu_n\| < \infty\,.$$

(observe that $t \to \|T(t)'\mu\| = \sup\{\langle T(t)f, \mu\rangle \colon \|f\| \leqslant 1\}$ is lower semi-continuous and hence measurable). Using A-IV,Theorem 1.10 we obtain $\omega_0(A^*) < 0$. But $\omega_0(A) = \omega_0(A^*)$ by A-III,4.4(iii), contradicting $\omega_0(A) = 0$. □

Stimmt (iii)?

Remark 1.5 If $(T(t))$ is a positive semigroup on an α-directed ordered Banach space E (see **?**], p.39), then the dual of E admits a reversion of the triangle inequality, i.e., $\sum \|\mu_i\| \leqslant \alpha \|\sum \mu_i\|$ for $\mu_i \in E'_+$, and Theorem 1.4 remains valid (see **?**]). The proof given above may be used with almost no modification.

At this point we close the discussion of the stability of positive semigroups on $C_0(X)$ and refer to Section 1 of C-IV and D-IV, respectively, where the stability of positive semigroups on arbitrary Banach lattices and on C*-algebras will be treated.

2 Compact and Quasi-Compact Semigroups

Using the Riesz-Schauder Theory for compact operators (see, e.g., Chapter VII.4 of **?**] or Section 26 of **?**]) and the results of Chapter A-III, one can easily describe the asymptotic behavior of eventually compact semigroups. Since no positivity is involved, we state the fundamental result for arbitrary Banach spaces.

Theorem 2.1 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space G which is eventually compact (i.e.,, there is $t_0 > 0$ such that $T(t_0)$ is a compact operator). Then the spectrum of the generator A is a countable set (possibly finite or empty) and contains only poles of finite algebraic multiplicity. Furthermore, the set $\{\mu \in \sigma(A)\colon \operatorname{Re}\mu \geqslant r\}$ is finite for every $r \in \mathbb{R}$. Thus $\sigma(A) = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ with $\operatorname{Re}\lambda_{n+1} \leqslant \operatorname{Re}\lambda_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\operatorname{Re}\lambda_n = -\infty$ if $\sigma(A)$ is infinite. Denoting the pole order at λ_n by $k(n)$ and the corresponding residue by P_n, we have for every $m \in \mathbb{N}$*

$$\begin{aligned}
T(t) &= T_1(t) + T_2(t) + \,..\, + T_m(t) + R_m(t)\,,\\
T_n(t) &= \exp(\lambda_n t)\cdot \sum_{j=0}^{k(n)-1} \frac{1}{j!} t^j (A-\lambda_n)^j \circ P_n \quad (t \geqslant 0)\,,\\
\|R_m(t)\| &\leqslant C\cdot \exp((\varepsilon + \operatorname{Re}\lambda_m)t) \text{ for } t \geqslant 0, \varepsilon > 0\\
&\qquad \text{for } t \geqslant 0, \varepsilon > 0 \text{ and a suitable constant } C = C(\varepsilon, m)\,.
\end{aligned} \tag{2.1}$$

Proof Fix $r \in \mathbb{R}$. By the Riesz-Schauder Theory we know that

$$\{z \in \sigma(T(t_0))\colon |z| \geqslant \exp(rt_0)\}$$

is a finite set and contains only poles of finite algebraic multiplicity. Thus the first assertion follows from A-III, Corollary 6.5. To prove the remaining assertion we fix $m \in \mathbb{N}$ and apply the spectral decomposition as described in Section 3 of Chapter A-III. For simplicity we assume $\operatorname{Re}\lambda_{m+1} < \operatorname{Re}\lambda_m$. Let P be the spectral projection of $T(t_0)$ corresponding to the spectral set $\{z \in \sigma(T(t_0)) : |z| \geqslant \exp(\operatorname{Re}\lambda_m \cdot t_0)\}$. Then P reduces the semigroup and we have $\sigma(T(t_0)|_{\ker P}) \subset \{z \in \mathbb{C}\colon |z| < \exp(\operatorname{Re}\lambda_m \cdot t_0)\}$. Hence the type of $(T(t_0)|_{\ker P})$ is less than $\operatorname{Re}\lambda_m$. Then there exists a constant C_0 such that

$$\|T(t)(\operatorname{Id} - P)\| \leqslant \|T(t)|_{\ker P}\| \cdot \|\operatorname{Id} - P\| \leqslant \|\operatorname{Id} - P\| \cdot C_0 \cdot \exp(\operatorname{Re}\lambda_m \cdot t)\,.$$

We define $R_m(t) := T(t)(\operatorname{Id} - P)$ and $T_n(t) := T(t)P_n$ $(n \in \mathbb{N})$. Then $R_m(t)$ satisfies the estimate stated in (2.1), and we have $T(t) = \sum_{n=1}^m T_n(t) + R_m(t)$ because $P = \sum_{n=1}^m P_n$ by A-III, Corollary 6.5(ii). The family of projections $\operatorname{Id} - P$, $P_1, P_2, \ldots, P_m$ reduces the semigroup. Thus in order to prove the representation of $T_n(t)$ stated in (2.1), we only have to consider elements $f \in P_n E = \ker(\lambda_n - A)$. Hence we can assume $E = P_n E$, $\sigma(A) = \{\lambda_n\}$, $P_n = \operatorname{Id}$ and for simplification we drop the index n, i.e.,, $\lambda = \lambda_n$, $k = k(n)$. Then A is a bounded operator satisfying $(\lambda - A)^k = 0$ and its resolvent is given by

$R(\nu, A) = (\nu - \lambda)^{-1} \sum_{j=0}^{k-1} (\nu - \lambda)^{-j} (A - \lambda)^j$ for $\nu \neq \lambda$. It follows that
$R(\nu, A)^i = (\nu - \lambda)^{-i} \sum_{j=0}^{k-1} (\binom{j+i-1}{i-1})(\nu - \lambda)^{-j} (A - \lambda)^j$. Hence we have
$(\frac{1}{t} R(\frac{1}{t}, A))^i = (1 - \lambda \frac{t}{1})^{-i} \sum_{j=0}^{k-1} (\binom{j+i-1}{i-1})(i - \lambda t)^{-j} t^j (A - \lambda)^j$ for every $i \in \mathbb{N}$.

Since $\lim_{i\to\infty} (1 - \lambda \frac{t}{i})^{-i} = e^{\lambda t}$ and $\lim_{i\to\infty} (\binom{j+i-1}{i-1})(i - \lambda t)^{-j} = \frac{1}{j!}$ for every $j \in \mathbb{N}$, the assertion follows from formula (1.3) of A-II. □

Combining Theorem 2.1 with the results of Chapter B-III one can describe the behavior of $T(t)$ as $t \to \infty$ provided that $(T(t))_{t\geqslant 0}$ is a positive semigroup. We give a typical example.

Corollary 2.2 *Let $(T(t))_{t\geqslant 0}$ be a positive semigroup on a space $C_0(X)$ which is irreducible and eventually compact. Then there exist a unique real number $r \in \mathbb{R}$, a strictly positive function h and a strictly positive bounded Borel measure ν such that for suitable constants $\delta > 0$, $M \geqslant 1$ one has*

$$\|e^{-rt} \cdot T(t) - \nu \otimes h\| \leqslant M \cdot e^{-\delta t} \textit{ for all } t \geqslant 0\,. \tag{2.2}$$

In particular, for every $f \in C_0(X)$ and $t \geqslant 0$ one has

$$\left(\left|\int f \,\mathrm{d}\nu\right| - M \cdot e^{-\delta t}\|f\|\right) \leqslant e^{-rt}\|T(t)f\| \leqslant \left(\left|\int f \,\mathrm{d}\nu\right| + M \cdot e^{-\delta t}\|f\|\right). \tag{2.3}$$

Proof We take $r := s(A)$. By B-III, Proposition 3.5(a) we have $r > -\infty$. Moreover, ?(a) oder (i)?
by assertion (e) of the same proposition we know that r is an algebraically simple pole and the corresponding residue P has the form $P = \nu \otimes h$ for strictly positive eigenvectors ν of A and h of A', respectively. Without loss of generality we may assume $\|h\| = 1$. Corollary 2.11 of Chapter B-III implies that r is strictly dominant, i.e.,, enumerating the eigenvalues as described in Theorem 2.1 we have $\mathrm{Re}\,\lambda_2 < \lambda_1 = r$. Now (2.2) follows from (2.1) for $m = 1$. Applying the triangle inequality to $T(t)f = e^{rt}(Pf + (e^{-rt}T(t)f - Pf))$ and using (2.2) one easily deduces (2.3). □

Let us point out the following consequence of Corollary 2.2. For every positive, non-zero initial value f the solution $T(\cdot)f$ of the abstract Cauchy problem $\dot{u} = Au$ decreases or increases exponentially in norm according to the sign of $r = s(A)$. If $s(A) = 0$, then $T(\cdot)f$ tends to an equilibrium state which is unique up to a constant and is non-zero whenever the initial value is positive and non-zero.

In order to apply Theorem 2.1 and its corollary to concrete problems one needs conditions which ensure that the semigroup is eventually compact. We discuss this problem for the spaces $C(K)$, K compact, in more detail. The crucial tool is the following characterization of weakly compact subsets in the dual space $M(K) = C(K)'$, due to **?**].

Proposition 2.3 *Let K be a compact space. For a subset $D \subset M(K) = C(K)'$ the following assertions are equivalent.*

(a) D is relatively compact for the weak topology $\sigma(M(K), M(K)')$.

(b) *For each weak null sequence* (f_n) *in* $C(K)$, $\lim_{n\to\infty}\langle f_n, \nu\rangle = 0$ *uniformly for* $\nu \in D$.
(c) *For each sequence* (U_n) *of disjoint open subsets of* K, $\lim_{n\to\infty}\nu(U_n) = 0$ *uniformly for* $\nu \in D$.

For a proof of this result, see e.g., II.9.8 in **?**]. We use this proposition in order to describe weakly compact operators on spaces $C(K)$. As usual we naturally identify the bounded Borel functions on K with a subspace $B(K)$ of $M(K)' = C(K)''$, note that in general, $C(K) \not\subset B(K) \not\subset C(K)''$.

Proposition 2.4 *Let* K *be a compact space,* G *be a Banach space, and let* $R: C(K) \to G$ *be a bounded linear operator.*

(i) *The following assertions are equivalent.*

  (a) R *is weakly compact,*
  (b) *for every bounded Borel function* g *on* K *we have* $R''g \in G$,
  (c) *for every Borel set* $C \subset K$ *we have* $R''(\mathbb{1}_C) \in G$.

  In case $G = C(K)$ *these conditions are equivalent to the following.*

  (d) *if* $(f_n) \subset C(K)$ *is a bounded sequence, then* (Rf_n) *has a subsequence which converges pointwise to a continuous function.*

(ii) *If* R *is weakly compact, then it maps weakly convergent sequences into norm convergent sequences. In particular, the square of a weakly compact operator* $T : C(K) \to C(K)$ *is a compact operator.*

Proof (i) $(a) \Rightarrow (b)$ follows from the following characterization of weakly compact operators (see e.g., II. Proposition 9.4 of **?**]).

> *An operator is weakly compact if and only if its second adjoint maps the bidual into the original space.*

$(b) \Rightarrow (c)$ is trivial and it remains to show that $(c) \Rightarrow (a)$.

On the Borel field $\mathcal{B}$ we define m by $m(C) := R''(\mathbb{1}_C)$. Then m is a G-valued additive set function. For $y' \in G'$ we have $y' \circ m = R'y' \in M(K)$. Hence $y' \circ m$ is a countable additive set function, for every $y' \in G'$ i.e., m is weakly countably additive. By Pettis' Theorem (see IV. Theorem 10.1 in **?**]) we have that m is countably additive with respect to the norm. In particular, for a sequence U_n of mutually disjoint Borel sets we have $\lim_{n\to\infty}\|m(U_n)\| = 0$. It follows that $\lim_{n\to\infty} y' \circ m(U_n) = 0$ uniformly for $y' \in G'$, $\|y'\| \leqslant 1$. Now condition (c) of Proposition 2.3 shows that $\{R'y' : y' \in G', \|y'\| \leqslant 1\}$ is relatively weakly compact, i.e.,, R' is weakly compact. Thus R is weakly compact as well.

In case $G = C(K)$ the equivalence of (a) and (d) is a consequence of two results. First, *Eberlein's Theorem* states that for the weak topology in any Banach space compactness and sequential compactness are equivalent. Second, *Lebesgue's Dominated Convergence Theorem* assures that a sequence $(f_n) \subset C(K)$ converges weakly to $f \in C(K)$ if and only if it is bounded and $f_n(x) \to f(x)$ for every $x \in K$.

(ii) Suppose that (f_n) is a sequence in $C(K)$ which converges to 0 for the weak topology. Since R is weakly compact, the same is true for the adjoint R', i.e., $\{R'y' : y' \in G', \|y'\| \leqslant 1\}$ is relatively weakly compact in $M(K)$. From Proposition 2.3 $(a) \Rightarrow (b)$ we obtain that $\langle Rf_n, y'\rangle = \langle f_n, R'y'\rangle \to 0$ as $n \to \infty$ uniformly for $y' \in G'$, $\|y'\| \leqslant 1$. That is $\lim_{n\to\infty} \|Rf_n\| = 0$. The final assertion follows from the first and the characterization of weakly compact operators stated in (d) of (i). □

The next result which is an immediate consequence of Theorem 2.1 and Proposition 2.4 is motivated by the theory of Markov processes. For a Markov operator (see B-I, Section 3) condition (b) of Proposition 2.4(i) is called the strong *Feller property*.

Theorem 2.5 *Let $(T(t))_{t\geqslant 0}$ be a semigroup of Markov operators on $C(K)$, K compact, such that one operator $T(t_0)$ has the strong Feller property. Then there exists a positive projection P of finite rank such that $\|T(t) - P\| \leqslant M \cdot e^{-\delta t}$ for suitable constants $\delta > 0$, $M \geqslant 1$.*

Proof By Proposition 2.4(i), $T(t_0)$ is weakly compact. Thus, by Proposition 2.4(ii), $T(2t_0)$ is compact, i.e.,, $(T(t))_{t\geqslant 0}$ is eventually compact. Moreover, by B-III, Corollary 2.11 $s(A) = 0$ is strictly dominant and a first order pole of the resolvent by B-III, Remark 2.15(a). The assertion now follows easily from Theorem 2.1. □ ?(a) or (i)?

We close the discussion of eventually compact semigroups by describing a situation where Theorem 2.5 can be applied. A more detailed description of the connection between Markov processes and positive semigroups on $C(K)$ is given in Chapter 2 of **?**].

Example 2.6 Let K be a compact space and $\{P_t : t > 0\}$ be a Markov transition function on K which satisfies the strong Feller property and which is stochastically continuous. That is, every P_t is a real-valued function defined on the product $K \times \mathcal{B}$, where $\mathcal{B}$ denotes the Borel field on K, such that

(i) for $x \in K$ and $t > 0$ fixed, $P_t(x, .)$ is probability measure,
(ii) for $C \in \mathcal{B}$ and $t > 0$ fixed, $P_t(., C)$ is a continuous function,
(iii) $P_{t+s}(x, C) = \int_K P_s(y, C) P_t(x, \mathrm{d}y)$ for all $s, t > 0$, $x \in K$, $C \in \mathcal{B}$,
(iv) $\lim_{t\downarrow 0} P_t(x, U) = 1$ for every open set U containing x.

Condition (ii) is the *strong Feller property*, (iii) is the *Chapman-Kolmogorov equation* and (iv) expresses stochastic continuity. Given $\{P_t\}$ as above, one defines for $f \in C(K)$, $x \in K$ and $t > 0$

$$(T(t)f)(x) := \int_K f(y) P_t(x, \mathrm{d}y)\,. \tag{2.4}$$

Then it is not difficult to verify that

- $T(t)f \in C(K)$,
- $T(t)$ is a Markov operator on $C(K)$,
- $(T(t))_{t\geqslant 0}$ — with $T(0) = \mathrm{Id}$ — is a one-parameter semigroup.

In fact, the first assertion is a consequence of (i) and (ii), the second follows from (i) and the semigroup property is implied by the Chapman-Kolmogorov equation (iii). Moreover, the semigroup $(T(t))_{t\geqslant 0}$ is strongly continuous. This can be seen as follows. In view of Proposition 1.23 in **?**] we only have to show that $\lim_{t\downarrow 0}\langle T(t)f - f, \nu\rangle = 0$ for every $f \in C(K)$, $\nu \in M(K)$. Due to Lebesgue's Dominated Convergence Theorem this is true whenever $\lim_{t\downarrow 0}(T(t)f)(x) = f(x)$ for every $f \in C(K)$, $x \in K$. Given f, x and $\varepsilon > 0$ there exists an open neighborhood U of x such that $|f(x) - f(y)| < \varepsilon$ for every $y \in U$. Then we have

$$\begin{aligned}(T(t)f)(x) - f(x) &= \int_K f(y)P_t(x, \mathrm{d}y) - \int_K f(x)P_t(x, \mathrm{d}y) = \\ &= \int_U (f(y) - f(x))P_t(x, \mathrm{d}y) + \int_{K\setminus U} (f(y) - f(x))P_t(x, \mathrm{d}y) \leqslant \\ &\leqslant \varepsilon \cdot P_t(x, U) + 2\|f\|_\infty \cdot P_t(x, K\setminus U)\,.\end{aligned}$$

Since $P_t(x, U) \leqslant 1$ and $\lim_{t\downarrow 0} P_t(x, U) = 1 = P_t(x, K)$, this estimate implies $\limsup_{t\downarrow 0}((T(t)f)(x) - f(x)) \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary we have pointwise convergence, hence strong continuity of the semigroup. Finally we observe that every operator $T(t)$ defined by (2.4) has the strong Feller property since $T(t)$"$\mathbb{1}_C = P_t(., C)$ for every Borel set $C \subset K$ (see Proposition 2.4(i)). Thus Theorem 2.5 can be applied in this situation.

We now turn our interest from eventually compact semigroups to *quasi-compact semigroups*. While "eventually compact" means that the operators $T(t)$ with $t \geqslant t_0$ have to be compact, "quasi-compactness" only means that $T(t)$ approaches the compact operators as $t \to \infty$. To make this precise we introduce the following notations. For a Banach space G, the ideal of all compact linear operators on G is denoted by $\mathcal{K}(G)$. For $T \in \mathcal{L}(G)$ we define $\operatorname{dist}(T, \mathcal{K}(G)) := \inf\{\|T - K\| : K \in \mathcal{K}(G)\}$.

Definition 2.7 A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space G is called *quasi-compact* if $\lim_{t\to\infty} \operatorname{dist}(T(t), \mathcal{K}(G)) = 0$.

Quasi-compactness can be characterized in many ways. Two of them are stated in the following proposition. The first one uses the notion of the essential growth bound $\omega_{\mathrm{ess}}(\mathcal{T})$ of a semigroup $\mathcal{T}$ as introduced in A-III,3.7.

Proposition 2.8 *For a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ on a Banach space G the following conditions are equivalent.*

(a) $\mathcal{T}$ is quasi-compact,
(b) $\omega_{ess}(\mathcal{T}) < 0$,
(c) There exist $t_0 > 0$, $K \in \mathcal{K}(G)$ such that $\|T(t_0) - K\| < 1$.

Proof $(a) \Rightarrow (c)$ is obvious by the definition of quasi-compactness.

$(c) \Rightarrow (b)$: Recalling the definition of the essential spectral radius from A-III, (3.6), assertion (c) implies $r_{\mathrm{ess}}(T(t_0)) \leqslant \|T(t_0)\|_{\mathrm{ess}} < 1$. Then $\omega_{\mathrm{ess}}(\mathcal{T}) < 0$ by A-III, (3.10).

$(b) \Rightarrow (a)$: By A-III, (3.10) we have $r_{\text{ess}}(T(1)) < 1$. Then A-III, (3.6) implies $\lim_{n\to\infty} \|T(n)\|_{\text{ess}}^{1/n} < 1$, where $\|T\|_{\text{ess}} = \text{dist}(T, \mathcal{K}(G))$. Thus for suitable $n_0 \in \mathbb{N}$, $a < 1$ we have $\|T(n)\|_{\text{ess}} < a^n$ for $n \geqslant n_0$.

Choosing a sequence $K_n \in \mathcal{K}(G)$ such that $\|T(n) - K_n\| < a^n$ for $n \geqslant n_0$ and defining $M := \sup_{0\leqslant s\leqslant 1} \|T(s)\|$ we obtain for $t \in [n, n+1]$ $(n \geqslant n_0)$

$$\|T(t) - T(t-n)K_n\| \leqslant \|T(t-n)\|\|T(n) - K_n\| \leqslant M \cdot a^n .$$

This implies that $\lim_{t\to\infty} \text{dist}(T(t), \mathcal{K}(G)) = 0$. □

A typical situation where quasi-compact semigroups occur is the following. If $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup with $\omega_{\text{ess}}(\mathcal{T}) < \omega_0(\mathcal{T})$, then the rescaled semigroup $(\exp(-\omega_0(\mathcal{T})t) \cdot T(t))_{t\geqslant 0}$ is quasi-compact. Obviously every semigroup with growth bound less than zero is quasi-compact. A more interesting situation is the following.

If $(T_0(t))_{t\geqslant 0}$ is a semigroup with growth bound less than zero and A_0 is its generator, then for every compact operator K the perturbed operator $A := A_0 + K$ generates a quasi-compact semigroup.

More generally we have the following result.

Proposition 2.9 *If $(T(t))_{t\geqslant 0}$ is a quasi-compact semigroup on a Banach space G with generator A and K is a compact operator, then $A+K$ generates a quasi-compact semigroup.*

Proof If $(T(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$ are the semigroups generated by A and $A+K$, respectively, we have $S(t) = T(t)+\int_0^t T(t-s)KS(s)\, \mathrm{d}s$. In view of Proposition 2.8(c) it is enough to show that $\int_0^t T(t-s)KS(s)\, \mathrm{d}s$ is a compact operator.

Since the mapping $(t, x) \mapsto T(t)x$ is jointly continuous on $\mathbb{R}_+ \times G$ and since K is compact, the set $M_t := \{T(s)Kx : 0 \leqslant s \leqslant t, \|x\| \leqslant 1\}$ is relatively compact in G. Having in mind that $\int_0^t T(t-s)KS(s)x\, \mathrm{d}s$ $(x \in G)$ is the norm limit of Riemann sums, one observes that $(ct)^{-1} \int_0^t T(t-s)KS(s)x\, \mathrm{d}s$ is an element of the closed convex hull $\overline{\text{co}(M_t)}$ of M_t, where $c := \sup\{\|S(s)\| : 0 \leqslant s \leqslant t\}$ and $\|x\| \leqslant 1$. Since $\overline{\text{co}(M_t)}$ is compact (see II.4.3 in **?**]) the assertion follows. □

We will now show that for quasi-compact semigroups the asymptotic behavior is similar to the one stated for eventually compact semigroups in Theorem 2.1. One obtains a representation as in (2.1) with a remainder of exponential decay. However, the rate of decay cannot be chosen arbitrarily large.

Theorem 2.10 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a quasi-compact semigroup on a Banach space G with generator A. Then $\{\lambda \in \sigma(A) : \text{Re}\,\lambda \geqslant 0\}$ is a finite set (possibly empty) and contains only poles of finite algebraic multiplicity. Denoting the eigenvalues with nonnegative real part by $\lambda_1, \lambda_2, \ldots, \lambda_m$, the corresponding residues $P_1, P_2, \ldots, P_m$ and the orders of the poles $k(1), k(2), \ldots, k(m)$ we have*

$$T(t) = T_1(t) + T_2(t) + \ldots + T_m(t) + R(t) \quad \textit{where}$$

$$T_n(t) = \exp(\lambda_n t) \cdot \sum_{j=0}^{k(n)-1} \frac{1}{j!} \cdot t^j (A - \lambda_n)^j \circ P_n \quad (t \geqslant 0) \quad \textit{and} \tag{2.5}$$

$$\|R(t)\| \leqslant C \cdot e^{-\varepsilon t} \quad \textit{for suitable constants} \quad \varepsilon > 0, C \geqslant 1\,.$$

Proof We have $\omega_{ess}(\mathcal{T}) < 0$, hence $r_{ess}(T(1)) < 1$ (see A-III,(3.10)).

Therefore $\{z \in \sigma(T(1)) \colon |z| \geqslant 1\}$ is a finite set and contains only poles of finite algebraic multiplicity (cf. A-III, (3.8)). Let P denote the spectral projection of $T(1)$ corresponding to $\{z \in \sigma(T(1)) \colon |z| \geqslant 1\}$. Then A-III, Corollary 6.5 implies that $\{\lambda \in \sigma(A) \colon \operatorname{Re}\lambda \geqslant 0\}$ is a finite set, contains only poles of $R(\cdot, A)$ of finite algebraic multiplicity and $P = P_1 + P_2 + \ldots + P_m$. One can now prove the representation of $T(t)$ stated in (2.5) in the same way as statement (2.1). □

In case we consider positive quasi-compact semigroups on $C_0(X)$ one can combine Theorem 2.10 with the results of B-III. For example, B-III, Corollary 2.11 assures that, in case there is at least one eigenvalue with nonnegative real part, the generator has a strictly dominant eigenvalue $r \in \mathbb{R}$. Thus in (2.5) the operators $T_j(t)$ belonging to $\lambda_j = r$ will determine the long-term behavior of $(T(t))$. More precisely, one has the following.

Corollary 2.11 *Let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a positive semigroup on $C_0(X)$ which is quasi-compact and let A be its generator.*

(i) Let r be an eigenvalue of A admitting a stricly positive eigenfunction and satisfying $\operatorname{Re}(r) \geqslant 0$. *Then $r = \omega_0(\mathcal{T}) = s(A)$ and there is a positive projection P of finite rank such that for suitable constants $\delta > 0$, $M \geqslant 1$ we have*

$$\|e^{-rt}T(t) - P\| \leqslant M \cdot e^{-\delta t} \quad \textit{for all} \quad t \geqslant 0. \tag{2.6}$$

(ii) If $(T(t))_{t \geqslant 0}$ is irreducible and $\omega_0(\mathcal{T}) \geqslant 0$, there exist a strictly positive function $h \in C_0(X)$ and a strictly positive bounded measure $\nu \in M(X)$ such that for suitable constants $\delta > 0$, $M \geqslant 1$ one has

$$\|\exp(-\omega_0(\mathcal{T})t) \cdot T(t) - \nu \otimes h\| \leqslant M \cdot e^{-\delta t} \quad \textit{for all} \quad t \geqslant 0. \tag{2.7}$$

In both cases (i) and (ii) the estimates (2.3) for $\|T(t)f\|$ hold true (in case (i) one has to replace $|\int f \, d\nu|$ by $\|Pf\|$).

Proof (i) By B-III, Corollary 2.11 we know that $s(A)$ is a strictly dominant eigenvalue of A. By Theorem 2.10 both $s := s(A)$ and r are poles of the resolvent. Moreover, there exists a positive measure ν such that $A^*\nu = s\nu$. Denoting the strictly positive eigenfunction corresponding to r by h we have $\langle h, \nu \rangle > 0$. Hence $s\langle h, \nu \rangle = \langle h, A^*\nu \rangle = \langle Ah, \nu \rangle = = r\langle h, \nu \rangle$ implies $r = s$. By B-III, Remark 2.15 we know that s is a first order pole of the resolvent. Since s is strictly dominant, (2.6) follows from (2.5).

(ii) can be proved in the same way as Corollary 2.2. We omit the details. □

Corollary 2.11 can be used to describe the asymptotic behavior as $t \to \infty$ of certain semigroups if only its generators is known. We explain this by discussing a concrete example.

Example 2.12 Let $X := [0, \infty)$ and define on $E := C_0(X)$ the operator A as

$$\begin{aligned} Af &:= -f' + mf \text{ with domain } D(A) \text{ given by} \\ D(A) &:= \{f \in C_0(X) \colon f \text{ is differentiable, } f' \in C_0(X) \\ &\qquad \text{and } f'(0) = \alpha f(0) - \int_0^\infty f(x)\, \mathrm{d}u(x)\}. \end{aligned} \tag{2.8}$$

Here α is a real number, ν is a bounded positive Borel measure with $\nu(\{0\}) = 0$ and m is a continuous function on X such that $m(\infty) := \lim_{x\to\infty} m(x)$ exists. It is not difficult to see that A generates a positive semigroup. Moreover, one can show that it is quasi-compact if (and only if) $m(\infty) < 0$. In order to find eigenvalues and eigenfunctions, one has to solve the ordinary differential equation $f' = mf - \lambda f$. Any solution has (up to a constant) the following form

$$g_\lambda(x) = \exp\left(\int_0^x (m(y) - \lambda)\, \mathrm{d}y\right) = e^{-\lambda x} \cdot \exp\left(\int_0^x m(y)\, \mathrm{d}y\right). \tag{2.9}$$

We assume that $m(\infty) < 0$ and $r \geqslant 0$. Then g_r is differentiable with $g_r, g_r' \in C_0(X)$. Thus, $g_r \in D(A)$ if and only if $g_r'(0) = \alpha g(0) - \int_0^\infty g_r(y)\, \mathrm{d}u(y)$. Inserting (2.9), then this condition becomes

$$m(0) - r = \alpha - \int_0^\infty e^{-ry} \cdot \exp\left(\int_0^y m(z)\, \mathrm{d}z\right) \mathrm{d}\nu(y)\,.$$

By monotonicity this equation has a unique solution $r \geqslant 0$ if and only if

$$m(0) + \int_0^\infty \exp\left(\int_0^y m(z)\, dz\right) \mathrm{d}u(y) \geqslant \alpha\,. \tag{2.10}$$

In case α, ν and m satisfy (2.10) and $m(\infty) < 0$, then g_r is a strictly positive eigenfunction of A corresponding to $r \geqslant 0$. Thus all assumptions of Corollary 2.11(i) are satisfied. In addition, the semigroup is irreducible if (and only if) the support of ν is an unbounded subset of $[0, \infty)$.

Similar examples will be discussed in the next section and in C-IV, Section 3.

We finally give a criterion for quasi-compactness of positive semigroups on spaces $C(K)$. It is based on a criterion given by Doeblin for operators on spaces of bounded measurable functions and can be easily deduced from Proposition 3 in **?**].

Proposition 2.13 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a semigroup of Markov operators on $C(K)$, K compact, satisfying the following condition.*

$$\begin{aligned} &\textit{There exist } t_0 > 0, 0 < \mu \in M(K) \textit{ and } \gamma \in \mathbb{R}, 0 < \gamma < 1\,, \\ &\textit{such that } T(t_0)f - \mu(f)\mathbb{1}_K \leqslant \gamma \cdot \mathbb{1}_K \textit{ for all } 0 \leqslant f \leqslant \mathbb{1}_K\,. \end{aligned} \tag{2.11}$$

Then $\mathcal{T}$ is quasi-compact.

3 A Semigroup Approach to Retarded Differential Equations

The aim of this section is to put into evidence the connection between retarded differential equations and one-parameter semigroups. Special emphasis will lie, as the general theme of this chapter suggests, on positive solutions of such equations and on their asymptotic behavior. Scalar examples were already considered in B-III, Example 2.14, B-II, Example 1.21, B-II,Example 1.23, B-II, Example 2.11 and B-IV, Example 2.12. In this section, we will treat retarded differential equations, also called ”delay differential equations”, with values in arbitrary Banach spaces. A slight modification of the methods used in the scalar case will also work in this setting. The main question is whether or how a time delay affects the qualitative behavior of the solution of an abstract Cauchy problem. In particular, we will show in Theorem 3.7, resp., Corollary 3.8 that under certain positivity assumptions the delay has no influence on the stability.

Let F be a Banach space, let $E = C([-1,0],F)$ be the Banach space of all continuous functions on $[-1,0]$ with values in F normed by the supremum norm, and let Φ be a bounded linear operator from E into F. For $u \in C([-1,\infty),F)$ and $t \geqslant 0$ we define the function $u_t \in E$ by $u_t(s) := u(t+s)$ for all $s \in [-1,0]$. This is the "history segment" of u with length 1 starting at $t-1$. Furthermore, let B be the generator of a strongly continuous semigroup on F such that $B - w$ generates a contraction semigroup for some $w \in \mathbb{R}_+$. This additional condition can always be satisfied by renorming the Banach space F (see, e.g., [Goldstein (1985a), Theorem 2.13]).

Using this framework throughout this section, it should be mentioned that in general $E = C([-1,0],F)$ is not a space of type $C(K)$ or even $C_0(X)$. However, the formal appearance justifies a treatment in this chapter. Moreover, if $F = C(L)$ (L compact) it is well known that E is isomorphic to $C([-1,0]\times L)$ and thus is a space of type $C(K)$ (K compact) as well.

With the above notations we consider the initial value problem

$$\begin{aligned} \dot{u}(t) &= Bu(t) + \Phi(u_t), \quad t \geqslant 0, \\ u_0 &= g \in E. \end{aligned} \tag{RCP}$$

We call (RCP) an abstract *retarded Cauchy problem*.

A function $u \in C([-1,\infty), F)$ is a *solution* of (RCP) if

(i) u is right-sided differentiable at 0 and continuously differentiable for $t > 0$,
(ii) $u(t) \in D(B)$ for $t \geqslant 0$,
(iii) (RCP) is satisfied for $t \geqslant 0$.

To (RCP) we associate the following operator A on the Banach space E. Let A be defined as

$$\begin{aligned} Af &:= f' \\ D(A) &:= \{f \in C^1([-1,0],F)\colon f(0) \in D(B), f'(0) = Bf(0) + \Phi f\}. \end{aligned} \tag{3.1}$$

First we show that A is a generator on E.

Theorem 3.1 *The operator A defined in (3.1) is the generator of a strongly continuous semigroup* $(T(t))_{t\geqslant 0}$ *on E satisfying the* translation property

$$T(t)f(s) = \begin{cases} f(t+s) & \textit{if } t+s \leqslant 0 \\ T(t+s)f(0) & \textit{if } t+s > 0 \end{cases}, \quad f \in E\,. \tag{T}$$

Proof We argue as in B-III, Example 2.14.(b) and consider the operator $A_0 f := f'$ on the domain

$$D(A_0) := \{f \in C^1([-1,0],F)\colon f(0) \in D(B), f'(0) = Bf(0)\}.$$

If $(S(t))_{t\geqslant 0}$ is the semigroup on F generated by B and ω_0 the growth bound of $(S(t))_{t\geqslant 0}$, then A_0 generates the semigroup $(T_0(t))_{t\geqslant 0}$ given by

$$T_0(t)f(s) = \begin{cases} f(t+s) & \text{if } t+s \leqslant 0 \\ S(t+s)f(0) & \text{if } t+s > 0 \end{cases}, \quad f \in E.$$

For $\lambda > \omega_0$ define the map $S_\lambda \in \mathcal{L}(E)$ by $S_\lambda f := f - \varepsilon_\lambda \otimes R(\lambda, B)\Phi f$ where $\varepsilon_\lambda(s) = e^{\lambda s}$ and $(h \otimes x)(s) := h(s) \cdot x$ for $h \in C[-1,0]$, $x \in F$ and $s \in [-1,0]$. Since $\|R(\lambda,B)\| \leqslant (\lambda - \omega_0)^{-1}$ it follows that S_λ is invertible for $\lambda > \|\Phi\| + \omega_0$ and that $\|S_\lambda^{-1}\| \leqslant (\lambda - \omega_0)\cdot(\lambda - \|\Phi\| - \omega_0)^{-1}$. Moreover, S_λ induces a bijection from $D(A)$ onto $D(A_0)$ such that

$$\begin{aligned} \lambda - A &= (\lambda - A_0)S_\lambda\,, \\ R(\lambda, A) &= S_\lambda^{-1} R(\lambda, A_0)\,. \end{aligned} \tag{3.2}$$

Proceeding as in the example mentioned above we obtain

$$\|R(\lambda,A)\| \leqslant (\lambda-\omega_0)\cdot(\lambda - \|\Phi\| - \omega_0)^{-1}\cdot(\lambda-\omega_0)^{-1} \leqslant (\lambda - \|\Phi\| - \omega_0)^{-1}.$$

Thus A is a generator by A-II, Theorem 1.7.

It suffices to show the translation property (T) for $f \in D(A)$ only. To that purpose, we treat two cases separately.

1. Let $t \geqslant 0$, $s \in [-1, 0]$ and $t + s > 0$. It suffices to prove $T(-s)g(s) = g(0)$ for $g := T(t + s)f$. For arbitrary $g \in D(A)$ we define the map

$$h : [-t, 0] \to F \quad \text{by} \quad h(r) := \delta_r T(-r)g,$$

where δ_r denotes the point evaluation $f \mapsto f(r)$ on E. For $\theta \neq 0$ we have

$$\begin{aligned} 1/\theta \cdot (h(r+\theta) - h(r)) &= 1/\theta \cdot (T(-r-\theta)g(r+\theta) - T(-r)g(r)) = \\ \text{(Term1)} \quad &= 1/\theta \cdot (T(-r-\theta)g(r) - T(-r)g(r)) \\ \text{(Term2)} \quad &\quad + 1/\theta \cdot (\delta_{r+\theta} - \delta_r)(T(-r-\theta)g - T(-r)g) \\ \text{(Term3)} \quad &\quad + 1/\theta \cdot (T(-r)g(r+\theta) - T(-r)g(r)). \end{aligned}$$

As $\theta \to 0$, (Term1) converges to $-A[T(-r)g](r)$, (Term2) converges to zero and (Term3) converges to $A[T(-r)g](r)$. Thus h is continuously differentiable with derivative zero, whence $h(r) = h(0)$ for all $r \in [-t, 0]$. Taking $r = s$ yields $T(-s)g(s) = g(0)$.

2. Let $t \geqslant 0$, $s \in [-1, 0]$ and $t + s \leqslant 0$. As in the first case we show that the map $k : [0, t] \to F, r \mapsto [T(r)f](t+s-r)$ is continuously differentiable with derivative zero. Thus $f(t + s) = k(0) = k(t) = T(t)f(s)$. □

The translation property (T) enables us to specify the correspondence between the semigroup $(T(t))_{t\geqslant 0}$ generated by the operator in (3.1) and the solution of the retarded Cauchy problem (RCP).

Corollary 3.2 *For $g \in D(A)$ define $u : [-1, \infty) \to F$ by*

$$u(t) := \begin{cases} g(t) & \text{if } -1 \leqslant t \leqslant 0 \\ T(t)g(0) & \text{if } 0 < t. \end{cases}$$

Then u is the unique solution of (RCP).

Proof Evidently $u \in C([-1, \infty), F)$ for $g \in D(A)$. since $\int_0^t T(s)g\,\mathrm{d}s \in D(A)$. From A-I, Proposition 1.6.(iii) and the definition of $D(A)$ we obtain, since $\int_0^t T(s)g\,\mathrm{d}s \in D(A)$, (iii) or (c)

$$\begin{aligned} T(t)g(0) - g(0) &= \left[A\left(\int_0^t T(s)g\,\mathrm{d}s\right)\right](0) = \\ &= B\left[\left(\int_0^t T(s)g\,\mathrm{d}s\right)(0)\right] + \Phi\left(\int_0^t T(s)g\,\mathrm{d}s\right) \\ &= B\left(\int_0^t T(s)g(0)\,\mathrm{d}s\right) + \int_0^t \Phi T(s)g\,\mathrm{d}s \\ &= B\left(\int_0^t u(s)\,\mathrm{d}s\right) + \int_0^t \Phi T(s)g\,\mathrm{d}s\,. \end{aligned}$$

Since $u(t) = (T(t)g)(0) \in D(B)$ for $t \geqslant 0$, the above calculation shows that u is right-sided differentiable at 0 and differentiable for $t > 0$, hence

$$\dot{u}(t) = Bu(t) + \Phi(T(t)g) .$$

By the translation property (T) we have $T(t)g = u_t$, indeed

$$u_t(s) = u(t+s) = \begin{cases} g(t+s) & \text{if } t+s \leqslant 0 \\ T(t+s)g(0) & \text{if } t+s > 0 \end{cases} = T(t)g(s) .$$

Therefore $\dot{u}(t) = Bu(t) + \Phi(u_t)$, i.e., u solves (RCP).

In order to show uniqueness of the solution, we take w to be a solution of (RCP) satisfying $w_0 = 0$. Let $x(t) := w_t$, $t \geqslant 0$. It is easy to see that $x(t) \in C^1([-1,0], F)$. Moreover, since $\dot{w}(0) = \dot{w}(t) = Bw(t) + \Phi(w_t)$, we obtain $x(t) \in D(A)$. By the definition of A we have $Ax(t) = \dot{w}_t$. On the other hand, $x(\cdot) \in C^1([0,\infty), E)$ and

$$(\dot{x}(t))(s) = \lim_{h\to 0} 1/h \cdot (w_{t+h}(s) - w_t(s))$$

$$= \lim_{h\to 0} 1/h \cdot (w_t(h+s) - w_t(s)) = \dot{w}_t(s), \text{ whence } \dot{x}(t) = \dot{w}_t .$$

Therefore we obtain $\dot{x}(t) = Ax(t)$. As $x(0) = w_0 = 0$, it follows, by the well-posedness of the abstract Cauchy problem corresponding to A, that $x(t) = 0$ for each $t \geqslant 0$. This proves $w \equiv 0$. □

Remarks (i) By similar arguments the following can be proved. If u is a solution of (RCP) such that $u_0 \in D(A)$, then x given by $x(t) := u_t$ is a solution of the abstract Cauchy problem associated with the operator A defined in (3.1). In this sense, (RCP) and the semigroup generated by A correspond to each other.

(ii) If, additionally to the assumptions of Corollary 3.2, $B \in \mathcal{L}(F)$, then u is a solution of (RCP) for every $g \in E$. [Indeed, a careful inspection shows that the proof of Corollary 3.2 can be generalized to this situation, since $u(t) = (T(t)g)(0) \in F = D(B)$ for all $g \in E$ and $t \geqslant 0$.]

(iii) For general $g \in E$ the retarded Cauchy problem (RCP) may not have a solution. Indeed, if u is a solution of (RCP), then the following is valid for $0 \leqslant s \leqslant t$:

$$\begin{aligned}\frac{d}{ds} S(t-s)u(s) &= -BS(t-s)u(s) + S(t-s)\dot{u}(s) = \\ &= -BS(t-s)u(s) + S(t-s)Bu(s) + S(t-s)\Phi(u_s) = \\ &= S(t-s)\Phi(u_s) .\end{aligned}$$

Hence

$$u(t) - S(t)u(0) = \int_0^t S(t-s)\Phi(u_s)\, \mathrm{d}s.$$

Let $(S(t))_{t\geqslant 0}$ be a strongly continuous semigroup which is not differentiable (for examples see A-II, 1.28). Define $g \in E$ by $g(s) := \tilde{g}$ for all $s \in [-1,0]$ where $\tilde{g} \in F$ is chosen such that $t \mapsto S(t)\tilde{g}$ is not differentiable in $t' \in \mathbb{R}_+$.

Assume that there exists a solution of (RCP). By the preceding considerations

$$u(t) = S(t)g(0) + \int_0^t S(t-s)\Phi(u_s)\,\mathrm{d}s = S(t)\tilde{g} + \int_0^t S(t-s)\Phi(u_s)\,\mathrm{d}s\,.$$

Thus u is not differentiable in t' and we have a contradiction.

Corollary 3.3 *Keep the above notation and let F be finite dimensional. Then the solution semigroup $(T(t))_{t\geqslant 0}$ in E corresponding to (RCP) is compact for each $t > 1$ and therefore is eventually norm continuous.*

Proof Let $t > 1$. By the translation property (T) we have $T(t)f(s) = T(t+s)f(0)$. Whenever $t+s > 0$, then Remark (ii) following Corollary 3.2 shows that $(T(t)f)(s) = (T(t+s)f)(0) = u(t+s)$ is differentiable from $s \in [-1, 0]$ for each $f \in E$.

Since $t > 1$, we thus have $T(t)f \in C^1([-1, 0], F)$ for all $f \in E$. The *Closed Graph Theorem* yields the continuity of $T(t)$ from E into C^1. Hence $T(t)$ maps the unit ball of E into a bounded set of $C^1([-1, 0], F)$. Again we use the assertion that $\dim F < \infty$ and obtain by the *Theorem of Arzela-Ascoli* that every bounded set of $C^1([-1, 0], F)$ is relatively compact in E. Thus $T(t)$ is compact for each $t > 1$. □

The assertion of Corollary 3.3 remains true if $(S(t))_{t\geqslant 0}$ is a compact semigroup on a (not necessarily finite dimensional) Banach space F (see **?**]).

In order to describe the asymptotic behavior of the solutions of (RCP) it is enough to examine the corresponding semigroup $(T(t))_{t\geqslant 0}$ on E. Indeed, Corollary 3.2 shows that the solutions u are given by $u(t) = T(t)g(0)$ for all $t > 0$ and thus the long term behavior of u can be deduced from that one of $(T(t))_{t\geqslant 0}$. Our approach uses the characterization of the stability of the semigroup $(T(t))_{t\geqslant 0}$ by the location of the spectrum $\sigma(A)$ of the generator A as developed in A-IV, Section 1, B-IV, Section 1 and C-IV, Section 1.

We define, for $\lambda \in \mathbb{C}$, operators $\Phi_\lambda \in \mathcal{L}(F)$ by

$$\Phi_\lambda x := \Phi(e_\lambda \otimes x)\,,\ x \in F\,. \tag{3.3}$$

Since Φ_λ is bounded, the operator $B + \Phi_\lambda$ is a generator on F. The spectrum of A can now be characterized in terms of the spectrum of the operators $B + \Phi_\lambda$.

Proposition 3.4 *Take the operators A, B and Φ as above. For every $\lambda \in \mathbb{C}$ the following equivalence holds.*

$$\lambda \in \sigma(A) \quad \textit{if and only if} \quad \lambda \in \sigma(B + \Phi_\lambda). \tag{3.4}$$

Proof By definition, $\lambda \in \varrho(A)$ if and only if for every $g \in E$ there exists a unique $f \in D(A)$ such that $\lambda f - f' = g$. This equality is satisfied if and only if there exists $x \in F$ such that

$$f(t) = \int_t^0 e^{\lambda(t-s)} g(s)\,\mathrm{d}s + e^{\lambda t}\cdot x \quad \text{for} \quad -1 \leqslant t \leqslant 0\,.$$

On the other hand, $f \in D(A)$ if and only if $x \in D(B)$ and

$$\lambda x - g(0) = Bx + \Phi H_\lambda g + \Phi_\lambda x \text{ where } H_\lambda g(t) := \int_t^0 e^{\lambda(t-s)} g(s)\, \mathrm{d}s\,.$$

Thus $\lambda \in \varrho(A)$ if and only if for every $g \in E$ there exists a unique $x \in D(B)$ such that $(\lambda - B - \Phi_\lambda)x = g(0) + \Phi H_\lambda g$. Notice that the map $x \mapsto x + \Phi H_\lambda(e_\mu \otimes x)$ $(x \in F)$ is surjective on F if μ is chosen so large that $\|\Phi H_\lambda(e_\mu \otimes x)\| \leqslant 1/2 \cdot \|x\|$ for all $x \in F$. Hence the map $g \mapsto g(0) + \Phi H_\lambda g$ is surjective from E onto F and this shows that $\lambda \in \varrho(A)$ if and only if $\lambda - B - \Phi_\lambda$ is invertible. □

An immediate consequence of the proof is the following corollary.

Corollary. *With the notations of the above proposition and A_0 as in the proof of Theorem 3.1 we have the following assertion.*

(i) $R(\lambda, A)g = \varepsilon_\lambda \otimes R(\lambda, B + \Phi_\lambda)(g(0) + \Phi H_\lambda g) + H_\lambda g$ *for* $\lambda \in \varrho(A), g \in E$.
(ii) $R(\lambda, A_0)g = \varepsilon_\lambda \otimes R(\lambda, B)g(0) + H_\lambda g$ *for* $\lambda \in \varrho(A_0), g \in E$.

We now turn to the aspect of positivity in (RCP) and its impact on the asymptotic behavior of the solution semigroup $(T(t))_{t\geqslant 0}$.

To this end, we let F be a Banach lattice, making $E = C([-1, 0], F)$ into a Banach lattice as well. Furthermore, let $(S(t))_{t\geqslant 0}$ be a positive semigroup with generator B and let $\Phi \in \mathcal{L}(E, F)$ be a positive operator. As before we restrict our attention to the case that $B - w$ generates a positive contraction semigroup for some $w \in \mathbb{R}$. Indeed, if B generates a bounded positive semigroup on F, then $\|x\| := \sup_{t\geqslant 0} \|S(t)|x|\|$ for $x \in F$ defines an equivalent lattice norm on F for which $(S(t))_{t\geqslant 0}$ is contractive.

Proposition 3.5 *If $\Phi \in \mathcal{L}(E, F)$ is a positive operator and if B generates a positive semigroup on F, then the semigroup $(T(t))_{t\geqslant 0}$ on E generated by $Af := f'$ with domain $D(A) := \{f \in C^1 : f(0) \in D(B), f'(0) = Bf(0) + \Phi f\}$ is positive.*

Proof By Formula (3.2) we have $R(\lambda, A) = S_\lambda^{-1} R(\lambda, A_0)$ for $\lambda > \|\Phi\| + w$, (where $S_\lambda f = f - \varepsilon_\lambda \otimes R(\lambda, B)\Phi f$ for $f \in E$). Thus the fact that $R(\lambda, A_0)$ is positive (C-II, Proposition4.1) reduces the problem to showing that S_λ^{-1} is a positive operator for $\lambda > \|\Phi\| + w$.

Since $S_\lambda = \mathrm{Id} - \varepsilon_\lambda \otimes R(\lambda, B)\Phi$ and $\|\varepsilon_\lambda \otimes R(\lambda, B)\Phi\| \leqslant (\lambda - w)^{-1} \cdot \|\Phi\| < 1$ we see that $S_\lambda^{-1} = \sum_{n=0}^{\infty} (\varepsilon_\lambda \otimes R(\lambda, B)\Phi)^n$ is positive. Hence $(T(t))_{t\geqslant 0}$ is a positive semigroup again by C-II, Proposition 4.1. □

Remark. Suppose that Φ has no mass in zero (i.e.,, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\Phi f\| \leqslant \varepsilon \|f\|$ for all $f \in E$, $\mathrm{supp}(f) \subset [-\delta, 0]$). Then the positivity hypotheses in the above proposition are necessary in order to obtain positivity of $(T(t))_{t\geqslant 0}$ (cf. B-II,1.22 for the case $\dim F < \infty$ and **?**] for the general case).

Proposition 3.6 *Let $\Phi \in \mathcal{L}(E, F)$ be positive and assume that B generates a positive semigroup on F. The* spectral bound function *$\lambda \mapsto s(B + \Phi_\lambda)$ is decreasing and continuous from the left on $\mathbb{R}$.*

If, additionally, B has compact resolvent and there exists $\lambda' \in \mathbb{R}$ with $\sigma(B+\Phi_{\lambda'}) \neq \emptyset$, then $\lambda \mapsto s(B + \Phi_\lambda)$ is continuous and the spectral bound $s(A)$ is the unique solution of the equation

$$\lambda = s(B + \Phi_\lambda)\,. \tag{3.5}$$

Proof (cf. also C-IV, Lemma 3.4). For $\lambda \leqslant \mu$ we have $0 \leqslant \Phi_\mu \leqslant \Phi_\lambda$ and hence $0 \leqslant R_\mu(t) \leqslant R_\lambda(t)$, $t \geqslant 0$, for the respective semigroups generated by $B + \Phi_\mu$ and $B + \Phi_\lambda$ (see A-II, Section 1). This implies $s(B + \Phi_\mu) \leqslant s(B + \Phi_\lambda)$. The left-continuity follows by the semicontinuity of the spectrum (see [Kato (1976), Chapter IV, Theorem 3.1]). **?** , Chapter IV, Theorem 3.1] If B has compact resolvent, then $B + \Phi_\lambda$ has compact resolvent as well. Now C-III, Theorem 1.1.(a) shows that $s(B + \Phi_\lambda)$ belongs to $\sigma(B + \Phi_\lambda)$ and is, by A-III,3.6 a pole with residue of finite rank. This completes the proof, since spectral points of compact operators depend continuously on smooth perturbations (see **?** , VII,6.Theorem 9]). □ Lit-Zitat !(a) oder (i)!

If $\sigma(B) \neq \emptyset$, then $-\infty < s(B) \leqslant s(B + \Phi_\lambda)$ for all $\lambda \in \mathbb{R}$ which implies $\sigma(B + \Phi_\lambda) \neq \emptyset$. On the other hand, if $\sigma(B + \Phi_\lambda) = \emptyset$ for all $\lambda \in \mathbb{R}$, then $\sigma(A) = \emptyset$ by Proposition 3.4.

We are now able to characterize the spectral bound of the generator A in E through spectral bounds of generators in F.

Theorem 3.7 *Let $\Phi \in \mathcal{L}(E, F)$ be positive and let B be the generator of a positive semigroup on F. The following implications are valid.*

(i) If $s(B + \Phi_\lambda) < \lambda$, then $s(A) < \lambda$.
(ii) If $s(B + \Phi_\lambda) = \lambda$, then $s(A) = \lambda$.
(iii) Suppose that B has compact resolvent and there exists $\lambda' \in \mathbb{R}$ with $\sigma(B+\Phi_{\lambda'}) \neq \emptyset$. Then

$$s(B + \Phi_\lambda) \gtreqless \lambda \quad \textit{if and only if} \quad s(A) \gtreqless \lambda. \tag{3.6}$$

Proof (i) If $\lambda > s(B+\Phi_\lambda)$, then $\mu > s(B+\Phi_\mu)$ for all $\mu \geqslant \lambda$ by Proposition 3.6. Therefore, $\mu \in \varrho(B + \Phi_\mu)$ for all $\mu \geqslant \lambda$. By Proposition 3.4 this implies $\mu \in \varrho(A)$ for all $\mu \geqslant \lambda$. Since $s(A) \in \sigma(A)$ by C-III, Theorem 1.1.(a), we obtain $\lambda > s(A)$. !(a) oder (i)!

(ii) If $\lambda = s(B + \Phi_\lambda)$, then again $\lambda \in \sigma(B + \Phi_\lambda)$ whence we obtain from Proposition 3.4 that $\lambda \in \sigma(A)$ and therefore $\lambda \leqslant s(A)$. As in (i) we conclude that $\mu \in \varrho(A)$ if $\mu > \lambda$; hence $\lambda = s(A)$.

(iii) It suffices to prove that $s(A) > \lambda$ whenever $s(B + \Phi_\lambda) > \lambda$. Assume the latter inequality. According to Proposition 3.6 there exists a unique μ satisfying $\mu = s(B + \Phi_\mu)$. Still by Proposition 3.6 it follows that $\lambda < \mu$. Assertion (ii) now completes the proof. □

Remark. We call (3.5) the *generalized characteristic equation* corresponding to (RCP). A justification for this terminology will be given in a remark following Corollary 3.8 of Chapter C-IV.

The characterization (3.6) of $s(A)$ uses the continuity of $\lambda \mapsto s(B + \Phi_\lambda)$. In the general case we apply the following lemma which is due to W. Arendt.

Lemma. *Let $\Phi \in \mathcal{L}(E, F)$ be positive and assume that B generates a positive semigroup on F. If we define*

$$\mu := \begin{cases} \sup\{\lambda \in \mathbb{R} \colon s(B + \Phi_\lambda) > \lambda\} & \text{if } \sigma(B + \Phi_\lambda) \neq \emptyset \text{ for some } \lambda \in \mathbb{R}, \\ -\infty & \text{otherwise,} \end{cases}$$

then $s(A) = \mu$.

Proof If $\sigma(B + \Phi_\lambda) = \emptyset$ for all $\lambda \in \mathbb{R}$, then $\sigma(A) = \emptyset$ by Proposition 3.4 and there is nothing to prove.

Take now $\lambda \in \mathbb{R}$ with $\sigma(B + \Phi_\lambda) \neq \emptyset$ and show $\mu \in \sigma(B + \Phi_\mu)$.

Case 1: If $\mu = s(B + \Phi_\mu)$, then $\mu \in \sigma(B + \Phi_\mu)$ by C-III, Theorem 1.1.

Case 2: If $\mu < s(B+\Phi_\mu)$,we show $r \in \sigma(B+\Phi_\mu)$ for every $r \in (\mu, s(B+\Phi_\mu)]$. Let $r \in (\mu, s(B + \Phi_\mu)]$ and assume $r \in \varrho(B + \Phi_\mu)$. By the definition of μ we have $r \in \varrho(B + \Phi_{\mu+\varepsilon})$ for all $\varepsilon > 0$. By C-III,Theorem 1.1 $R(r, B + \Phi_{\mu+\varepsilon}) \geqslant 0$ and by the assumption $R(r, B + \Phi_\mu) \geqslant 0$ as well. Now C-III,Theorem 1.1 implies $r > s(B + \Phi_\mu)$ which yields a contradiction to the choice of r. Thus $r \in \sigma(B + \Phi_\mu)$ for every $r \in (\mu, s(B + \Phi_\mu)]$ and hence $\mu \in \sigma(B + \Phi_\mu)$. Consequently $s(A) \geqslant \mu$.

Finally we assume $s(A) > \mu$. The definition of μ yields $s(A) > s(B + \Phi_{s(A)})$. Hence $s(A) \in \varrho(B + \Phi_{s(A)})$ and thus $s(A) \in \varrho(A)$ by Proposition 3.4. This yields a contradiction, since A generates a positive semigroup, hence $s(A) = \mu$. □

An immediate consequence of the preceding lemma is the following stability criterion.

Corollary 3.8 *Let $\Phi \in \mathcal{L}(E, F)$ be positive and let B be the generator of a positive semigroup. The following assertions are equivalent:*

(a) The semigroup generated by A is exponentially stable in E.
(b) The semigroup generated by $B + \Phi_0$ is exponentially stable in F.

Proof We can assume that there exists $\lambda \in \mathbb{R}$ with $\sigma(B + \Phi_\lambda) \neq \emptyset$.

The implication "$(a) \Rightarrow (b)$" follows immediately from Theorem 3.7(i).

To show "$(b) \Rightarrow (a)$", let $s(B+\Phi_0) < 0$. By the lemma and since $\lambda \mapsto s(B+\Phi_\lambda)$ is non-increasing we have $s(A) = \mu = \sup\{\lambda \in \mathbb{R} \colon s(B + \Phi_\lambda) > \lambda\} < 0$. Thus the semigroup generated by A is exponentially stable. □

Remark. In the situation of Theorem 3.7(iii) we have the stronger result that $s(A)$ and $s(B + \Phi_0)$ have the same sign.

Example 3.9 (see also C-II, Example 4.14). Take $E = C([-1, 0], \mathbb{C})$, $\alpha \in \mathbb{C}$ and $\mu \in M[-1, 0]_+$ such that $\mu(\{0\}) = 0$. Then the operator A given by $Af = f'$ on $D(A) = \{f \in C^1([-1, 0], \mathbb{C}) \colon f'(0) = \alpha f(0)+ < f, \mu >\}$ generates a strongly continuous semigroup $(T(t))_{t\geqslant 0}$. In fact, this follows from Theorem 3.1 if we put $F = \mathbb{C}$, $\Phi = \mu$ and B the multiplication by α. Moreover Φ_0 is the multiplication by $< f_0, \Phi >= \|\Phi\|$ (notice $\Phi \geqslant 0$) and $s(B + \Phi_0) = \alpha + \|\Phi\|$. Since $\omega_0(A) = s(A)$ by (1.1), we obtain from Corollary 3.8 that A generates a uniformly exponentially stable semigroup if and only if $\alpha + \|\Phi\| < 0$.

The preceding considerations remain true if we consider an (arbitrary) finite time delay τ where $0 < \tau < \infty$. Clearly, (RCP) can be treated as a differential equation with corresponding generator A (see (3.1) for the definition) in $C([-\tau, 0], F)$ (instead of $C([-1, 0], F)$).

Example 3.10 In order to illustrate the consequences of Corollary 3.8 we consider the Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Bu(t) + Su(t-\tau), t \geqslant 0, \\ u(t) &= \psi(t), -\tau \leqslant t \leqslant 0 \, (0 < \tau < \infty), \psi \in E, \end{aligned} \tag{3.7}$$

where B is the generator of a positive semigroup on F, $\sigma(B) \neq \emptyset$ and $S \in \mathcal{L}(F)$ is positive.

Using the above terminology, we have $\Phi f = S(f(-\tau))$ for all $f \in E$, hence $\Phi_0 = S$. By Corollary 3.8 the solution semigroup corresponding to the retarded differential equation (3.7) is exponentially stable if and only if the semigroup generated by $B+S$ is exponentially stable.

But the semigroup generated by $B+S$ is the solution semigroup of the "undelayed" Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Bu(t) + Su(t), t \geqslant 0, \\ u(0) &= x, \qquad\qquad x \in F. \end{aligned} \tag{3.8}$$

More precisely, we obtain the following corollary.

Corollary. *The solution of* (3.7) *is exponentially stable for every* $\tau > 0$ *if and only if the solution of* (3.8) *is exponentially stable.*

In other words, the corollary states that for this "positive-type" delay equations $((S(t))_{t\geqslant 0}$ and Φ positive) exponential stability is independent of the delay (see **?**] for a detailed analysis of this phenomenon).

This is a rather untypical behavior since even a scalar valued delay differential equation may be stable for "small" delays but unstable for "large" delays.

We give an example and show how a stable Cauchy problem with non-positive solutions (see the remark following Proposition 3.5) can be destabilized by an increase of the time lag τ.

Let $0 < \tau < \infty$ and $p, q \in \mathbb{R}$ and consider the (RCP)

$$\begin{aligned} \dot{u}(t) &= pu(t) + qu(t-\tau), \quad t \geqslant 0, \\ u(t) &= \Psi(t), \qquad\qquad -\tau \leqslant t \leqslant 0, \Psi \in C[-\tau, 0]. \end{aligned} \tag{3.9}$$

Its characteristic equation (in the classical sense) is

$$\lambda = p + e^{-\lambda\tau} q \, . \tag{3.10}$$

We consider the case where the Cauchy problem without delay

$$\dot{u}(t) = (p+q)u(t)$$

is asymptotically stable, i.e., we choose $0 < p < 1$ and $q + p < 0$.

Claim. *For every* $0 < \lambda' < p$ *there exists* $\tau > 0$ *such that* $e^{\lambda' t}$ *is a solution of* $(3.9)_\tau$.

?$(3.9)_\tau$?

Consider the map $g\colon \mathbb{R} \times (\mathbb{R}_+ \setminus \{0\}) \to \mathbb{R}$ defined by $g(\lambda, \tau) = p + e^{-\lambda\tau} q$. This function is continuous in λ and τ and increasing in λ. Furthermore $g(0,\tau) = p+q < 0$ for every $\tau > 0$ and $g(\lambda,\tau) \to p$ as $\tau \to \infty$ for every $\lambda \in \mathbb{R}_+$. For $0 < \lambda' < p$ fixed, we can find $\tau > 0$ such that $g(\lambda',\tau) = \lambda'$.

Let $\Psi(t) = e^{\lambda' t}$ for $-\tau \leqslant t \leqslant 0$. If we define $u(t) := e^{\lambda' t}$ for $t \geqslant 0$ then the following holds.

$$pu(t) + qu(t-\tau) = pe^{\lambda' t} + qe^{\lambda' t} e^{-\lambda'\tau} = (p + qe^{-\lambda'\tau}) e^{\lambda' t} = \lambda' e^{\lambda' t} = \dot{u}(t)\,.$$

Thus u is a solution of $(3.9)_\tau$ which is exponentially increasing as $t \to \infty$. In particular $(3.9)_\tau$ is not stable.

The precise region of stability in the scalar valued case is given, e.g., in **?**] and **?** , p.107ff].

Remark. Consider the case $F = C(L)$ (L compact). Then $E = C([-1,0] \times L)$ and $(T(t))_{t\geqslant 0}$ is a positive semigroup on $C(K)$ where $K = [-1,0] \times L$ is compact. Thus, spectral bound and growth bound of the semigroup generator coincide (see (1.1)). This yields a statement analogous to Corollary 3.8 for uniform exponential stability.

We conclude this section with two examples fitting into the above framework.

Example 3.11 Consider the equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) - d(x)u(t,x) + b(x)u(t-1,x) \quad (t \geqslant 0, x \in [0,1])$$

with boundary condition

$$\frac{\partial}{\partial x} u(t,x)\Big|_{x=0} = 0 = \frac{\partial}{\partial x} u(t,x)\Big|_{x=1} \quad (t \geqslant 0) \tag{3.11}$$

and initial condition

$$u(s,x) = \psi(s,x) \quad (s \in [-1,0], x \in [0,1]).$$

Let $F = C[0,1]$, $E = C([-1,0] \times [0,1])$ and let $\tilde{B}$ be defined by $\tilde{B}h = h''$ with domain $D(\tilde{B}) := \{h \in C^2[0,1] \colon h'(0) = h'(1) = 0\}$.

Denote by M_b and M_d the respective multiplication operators for $0 \leqslant b, d \in F$. Then (3.11) takes the abstract form

$$\begin{aligned} \dot{u}(t) &= \tilde{B}u(t) - M_d u(t) + M_b u(t-1)\,, \\ u_0 &= \psi \in E\,. \end{aligned}$$

It is well-known that $\tilde{B}$ generates a positive contraction semigroup and has compact resolvent (see A-I,2.7). The same is true for the operator $B := \tilde{B} - M_d$ (see A-II, Theorem 1.29 and Theorem 1.30). Thus, by the above results, the solution semigroup of (3.11) is positive and its asymptotic behavior can be investigated by the “undelayed” equation

$$\dot{u}(t) = (\tilde{B} + M_h)u(t), \text{ where } h := b - d.$$

Let $h(x) < 0$ for all $x \in [0,1]$. Then $s(\tilde{B} + M_h) \leqslant \max\{h(x) \colon x \in [0,1]\} < 0$. Hence the solutions of (3.11) are uniformly exponentially stable.

Interpretation. The solution u of (3.11) can be interpreted as the density of a population, distributed over an "area" $[0,1]$. The operator $\frac{\partial^2}{\partial x^2}$ describes the internal migration of the population and the functions b and d are the "place specific" birth- resp. death rates of the population members. The time delay 1 stands for the gestation period. The stability condition $h(x) < 0$ for all $x \in [0,1]$ means that the death rate has to majorize the birth rate in each spatial point to lead to extinction of the population, no matter whether the equation with or without delay is considered.

Example 3.12 An interesting example from cell biology is given by **?**]. They investigate a balance equation for the size distribution of a cell population which is structured by size. To point out the main ideas we will restrict the complex situation to the simple case of linear cell growth and refer to the original paper for details and the more general case.

Let $0 < r < 1$ and let $a = r$ be the minimal cell size. Furthermore let $F = L^1([a,1])$ and $E = C([-r,0],F)$. The retarded differential equation of interest is

$$\begin{aligned} \frac{d}{dt}u(t) &= Bu(t) + Lu(t-r) \\ u &= \Psi \in E. \end{aligned} \tag{3.12}$$

Here $Bf := -f'$ on $D(B) := \{f \in L^1[a,1] \colon f \in AC[a,1], f(a) = 0\}$ and $L : F \to F$ is defined by

$$Lf(x) := \begin{cases} k(x)f(2x-r) & \text{if } x \in [a, 1/2(r+1)], \\ 0 & \text{if } x \in (1/2(r+1), 1], \end{cases}$$

where $k \in C[a,1]$.

It is easy to verify that L is positive and bounded, and that B is the generator of the positive semigroup $(S(t))_{t\geqslant 0}$ defined by

$$[S(t)f](x) = \begin{cases} f(x-t) & \text{if } x-t \geqslant a \\ 0 & \text{if } x-t < a \end{cases} \quad (x \in [a,1]).$$

Furthermore B has compact resolvent. Define $\Phi f := L(f(-r))$ for $f \in E$ such that (3.12) can be written as retarded Cauchy problem (RCP).

As before (see Formula (3.3)) Φ_λ is defined by $\Phi_\lambda x := \Phi(e_\lambda \otimes x)$ for $x \in F$. Gyllenberg and Heijmans have shown that $s(B + \Phi_\lambda) > -\infty$. Thus we can apply Theorem 3.7 and obtain that $s(A) = \lambda$ if and only if $\lambda = s(B + \Phi_\lambda)$.

NOTES

Section 1.

The coincidence of spectral bound and growth bound for positive semigroups on $C(K)$ was first observed by **?**] and then generalized to $C_0(X)$ and non-commutative C*-algebras by **?**] and **?**]. The stability theorem 1.1 is a continuous version of a result of Choquet-Foias (see **?**], V.8.8).

Section 2.

For the Riesz-Schauder Theory of compact operators we refer to **?**], Section VII.4 and **?**], Section 26. Theorem 1.1 seems to be folklore. Proposition 2.3 is due to **?**] and can be found in Section II.9 of **?**]. Proposition 2.4 is due to Dieudonné (see §3 of **?**] and **?**], II.Exc.27). The notion *strong Feller property* used in Theorem 2.5 is due to Girsanov (see **?**]) while the theorem itself was proven by **?**]. It is well known that there is a close relationship between Markov processes and Markov semigroups. This relation more detailed than in Example2.6, can be found, e.g. in **?**], in Chapter 2 of **?**] or in Chapter 7 of **?**]. The notion *quasi-compact* for a single operator dates back to **?**] (see also **?**] and Section 26.4 of **?**]). Quasi-compactness for strongly continuous semigroups and its relationship to uniform ergodicity is investigated in **?**]. Proposition 2.9 is due to Voigt (1980), a special case was proven by Vidav (1970). Corollaries 2.2 and 2.11 can be found in [Greiner(1984)]. The criterion stated in (2.11) is known as *Doeblin's condition* (see, e.g., **?**]). It is sufficient and necessary for quasi-compactness of the semigroup. A new proof of this result is given in **?**].

Section 3.

The standard reference to retarded differential equations is **?**], where it is shown that the solutions of (RCP), with values in a finite dimensional space F, yield an operator semigroup. The extension to arbitrary Banach spaces F was first made by **?**]. **?**] showed the translation property (T) for the solution semigroup. Among the many papers pursuing this functional analytic investigation of partial differential equations with delay we quote **?**] and **?**].

Our approach is essentially due to W. Kerscher. We show that the first derivative with an appropriate domain is the generator of a one-parameter semigroup on an abstract function space. Due to the translation property this semigroup yields the solutions of (RCP).

The aspect of positivity in (RCP) and its influence on the stability of the solutions was first studied in Section 4 of **?**]. In **?**] this is pursued by showing how Theorem 3.7 in combination with the domination of semigroups (see C-II, Section 4) can be used to derive many of the known "stability independent of the delay" - results (e.g., **?**]).

Part C
Positive Semigroups on Banach Lattices

Chapter C-I
Basic Results on Banach Lattices and Positive Operators

This introductory chapter is intended to give a brief exposition of those results on Banach lattices and ordered Banach spaces which are indispensable for an understanding of the subsequent chapters. We do not give proofs of the results, since these can easily be found in the literature (e.g., in **?**]). We rather want to give the reader, who is unfamiliar with the results or the terminology used in this book, the necessary information for an intelligent reading of the main discussions. Since relatively few general results on ordered Banach spaces are needed, we will primarily talk about Banach lattices. The scalar field will be $\mathbb{R}$ except for the last three sections, where complex Banach lattices will be discussed.

The notion of a Banach lattice was devised to obtain a common abstract setting within which one could talk about phenomena related to positivity. This has previously been studied in various types of spaces of real-valued functions, such as the spaces $C(K)$ of continuous functions on a compact topological space K, the Lebesgue spaces $L^1(\mu)$ or more generally the spaces $L^p(\mu)$ constructed over a measure space (X, Σ, μ) for $1 \leqslant p \leqslant \infty$. Thus it is a good idea to think of such spaces first in order to get a feeling for the concrete meaning of the abstract notions we introduce. Later we will see that the connections between *abstract* Banach lattices and the *concrete* function lattices $C(K)$ and $L^1(\mu)$ are closer than one might think at first. We will use without further explanation the terms *order relation* (ordering), *ordered set*, *majorant*, *minorant*, *supremum*, *infimum*.

By definition, a Banach lattice is a Banach space $(E, \|\cdot\|)$ which is endowed with an order relation, usually written $\leqslant$, such that $(E, \leqslant)$ is a lattice and the ordering is compatible with the Banach space structure of E. We elaborate this in more detail now. The axioms of compatibility between the linear structure of E and the order are

$$f \leqslant g \text{ implies } f + h \leqslant g + h \text{ for all } f, g, h \text{ in } E, \tag{LO1}$$

$$f \geqslant 0 \text{ implies } \lambda f \geqslant 0 \text{ for all } f \text{ in } E \text{ and } \lambda \geqslant 0. \tag{LO2}$$

Any (real) vector space with an ordering satisfying (LO_1) and (LO_1) is called an *ordered vector space*. The property expressed in (LO_1) is sometimes called *translation invariance* and implies that the ordering of an ordered vector space E is completely determined by the positive part $E_+ = \{f \in E : f \geqslant 0\}$ of E. In fact, one has $f \leqslant g$ if and only if $g - f \in E_+$. (LO_1) together with (LO_2) furthermore imply that the positive part of E is a convex set and a cone with vertex 0 (often called the *positive cone* of E). It is easily verified that conversely any proper convex cone C with vertex 0 in E is the positive part of E with respect to a uniquely determined compatible ordering.

An ordered vector space E is called a *vector lattice* if any two elements f, g in E have a supremum, which is denoted by $\sup(f,g)$ or by $f \vee g$, and an infimum, denoted by $\inf(f,g)$ or by $f \wedge g$. It is obvious that the existence of, e.g., the supremum of any two elements in an ordered vector space implies the existence of the supremum of any finite number of elements in E and, since $f \leqslant g$ is equivalent to $-g \leqslant -f$ this automatically implies the existence of finite infima. However, suprema (infima) of infinite majorized subsets need not exist in a vector lattice. If they do, then the vector lattice is called *order complete* (*countably order complete* or *σ-order complete* if suprema of countable majorized subsets exist). At any rate, the binary relations *sup* and *inf* in a vector lattice automatically satisfy the (infinite) distributive laws

$$\begin{aligned}\inf(\sup_\alpha f_\alpha, h) &= \sup_\alpha(\inf(f_\alpha, h)),\\ \sup(\inf_\alpha f_\alpha, h) &= \inf_\alpha(\sup(f_\alpha, h)),\end{aligned}$$

whenever one side exists. This gives rise to the following definitions.

$$\begin{aligned}\sup(f,-f) &= |f| \text{ is called the } \textit{absolute value} \text{ of } f,\\ \sup(f,0) &= f^+ \text{ is called the } \textit{positive part} \text{ of } f,\\ \sup(-f,0) &= f^- \text{ is called the } \textit{negative part} \text{ of } f.\end{aligned}$$

Note that the negative part of f is positive. We call two elements f, g of a vector lattice *orthogonal* or *lattice disjoint* and write $f \perp g$ if $\inf(|f|,|g|) = 0$.

Apart from this, the above definitions allow us to formulate the axiom of compatibility between norm and order requested in a Banach lattice in the following short way.

$$|f| \leqslant |g| \text{ implies } \|f\| \leqslant \|g\|. \tag{LN}$$

A norm on a vector lattice is called a *lattice norm* if it satisfies (LN). With these notations we can now give the definition of a Banach lattice as follows.

A Banach lattice *is a Banach space E endowed with an ordering $\leqslant$ such that $(E, \leqslant)$ is a vector lattice and the norm on E is a lattice norm. By a* normed vector lattice *we understand a vector lattice endowed with a lattice norm.*

There is a number of elementary, but very important formulas valid in any vector lattice, such as

$$\begin{aligned} f &= f^+ - f^- \qquad & |f+g| &\leqslant |f| + |g|\\ |f| &= f^+ + f^- \qquad & f+g &= \sup(f,g) + \inf(f,g)\end{aligned}$$

(see, e.g., **?**]). Let us note in passing the following consequences.

(i) The lattice operations $(f, g) \mapsto \sup(f, g)$ and $(f, g) \mapsto \inf(f, g)$ and the mappings $f \mapsto f^+$, $f \mapsto f^-$, $f \mapsto |f|$ are uniformly continuous.
(ii) The positive cone is closed.
(iii) *Order intervals*, i.e., sets of the form

$$[f, g] = \{h \in E : f \leqslant h \leqslant g\}$$

are closed and bounded.

Instead of dwelling upon a detailed discussion of the above equalities and inequalities let us rather formulate the following principle, which allows us to verify any of them and to invent, prove or disprove new ones whenever necessary.

> *Any general formula relating a finite number of* variables *to each other by means of lattice operations and/or linear operations is valid in any Banach lattice as soon as it is valid in the real number system.*

In fact, we see below that any Banach lattice E is, as a vector lattice, *locally* of type $C(X)$, more exactly: Given any finite number $x_1, \ldots, x_n$ of elements in E, there is a compact topological space X and a vector sublattice J of E which is isomorphic to $C(X)$ and contains $x_1, \ldots, x_n$ (see Section. 4). The above principle is an easy consequence of the following: In a space $C(X)$ it is clear that a formula of the type considered need only be verified pointwise, i.e., in $\mathbb{R}$.

The reader may now be prepared to follow a concise presentation of the most basic facts on Banach lattices.

1 Sublattices, Ideals, Bands

The notion of a *vector sublattice* of a vector lattice E is self-explanatory, but it should be pointed out that a vector subspace F of E which is a vector lattice for the ordering induced by E need not be a vector sublattice of E (formation of suprema may differ in E and in F), and that a vector sublattice need not contain (or may lead to different) infinite suprema and infima. The following are necessary and sufficient conditions on a vector subspace G of E to be a vector sublattice.

(a) $|h| \in G$ for all $h \in G$,
(b) $h^+ \in G$ for all $h \in G$,
(c) $h^- \in G$ for all $h \in G$.

A subset B of a vector lattice is called *solid* if $f \in B$, $|g| \leqslant |f|$ implies $g \in B$. Thus a norm on a vector lattice is a lattice norm if and only if its unit ball is solid. A solid linear subspace is called an *ideal*. Ideals are automatically vector sublattices since $|\sup(f, g)| \leqslant |f| + |g|$. On the other hand, a vector sublattice F is an ideal in E if $g \in F$ and $0 \leqslant f \leqslant g$ imply $f \in F$. A *band* in a vector lattice E is an ideal which contains arbitrary suprema, or more exactly:

B is a band in E if B is an ideal in E and sup M *is contained in B whenever M is contained in B and has a supremum in E.*

Since the notions of sublattice, ideal, band are invariant under the formation of arbitrary intersections there exists, for any subset B of E, a uniquely determined smallest sublattice (ideal, band) of E containing B, i.e., the *sublattice* (*ideal*, *band*) *generated by B*.

If we denote by B^d the set $\{h \in E\colon \inf(|h|, |f|) = 0 \text{ for all } f \in B\}$, then B^d is a band for any subset B of E, and $(B^d)^d = B^{dd}$ is a band containing B. If E is a normed vector lattice (more generally, if E is archimedean ordered, see e.g., **?**]), then B^{dd} is the band generated by B.

If two ideals I, J of a vector lattice E have trivial intersection $\{0\}$, then I and J are *lattice disjoint*, i.e., $I \subset J^d$. Thus if E is the direct sum of two ideals I, J, then one has automatically $I = J^d$ and $J = I^d$, hence $I = I^{dd}$ and $J = J^{dd}$ must be bands in this situation. In general, an ideal I need not have a complementary ideal J even if $I = I^{dd}$ is a band. This amounts to the same as saying that even if $I = I^{dd}$ (which is always true if I is a band in a normed vector lattice) one need not necessarily have $E = I + I^d$.

An ideal I is called a *projection band* if it does have a complementary ideal, and in this case the projection of E onto I with kernel I^d is called the *band projection* belonging to I. An example of a band which is not a projection band is furnished by the subspace of $C([0, 1])$ consisting of the functions vanishing on $[0, 1/2]$.

The *Riesz Decomposition Theorem* asserts that in an order complete vector lattice every band is a projection band. As a consequence, if E is order complete and B is an arbitrary subset of E, then E is the direct sum of the complementary bands B^d and B^{dd}.

This theorem, which is quite easy to prove, is widely used in analysis and gives an abstract background to, e.g., the decomposition of a measure into atomic and diffuse parts (the atomic measures being those contained in the band generated by the point measures, the diffuse measures those disjoint to the latter). Or, more specifically, to the well-known decomposition of a measure on $[a, b]$ into an atomic part and a diffuse part, which latter can in turn be decomposed into the sum of a measure which is *absolutely continuous* (which means, contained in the band generated by Lebesgue measure) and a *singular measure* (which means, a diffuse measure disjoint to Lebesgue measure).

A band in a normed vector lattice is necessarily closed. By contrast, an ideal need not be closed, but the closure of an ideal is again an ideal. The situation, where every closed ideal is a band, will be briefly discussed in Section 5.

2 Order Units, Weak Order Units, Quasi-Interior Points

An element u in the positive cone of a vector lattice E is called an *order unit* if the ideal generated by u is all of E. If the band generated by u is all of E (which is equivalent to $\{u\}^d = 0$ whenever E is archimedean, hence in particular if E is a normed vector lattice), then u is called a *weak order unit* of E. If E is a Banach lattice, then any order unit in E is an interior point of the positive cone E_+, and conversely any interior point of E_+ must be an order unit of E. Every space $C(K)$ has order units (namely, the strictly positive functions), and conversely by the Kakutani-Krein Representation Theorem (see Section 4), every Banach lattice with an order unit is isomorphic to a space $C(K)$.

If an element u in the positive cone of a Banach lattice E has the property that the closed ideal generated by u is all of E, then u is called a *quasi-interior point* of E_+. Quasi-interior points of the positive cone exist, e.g., in any separable Banach lattice. If $E = C(K)$, then the quasi-interior points and the interior points of E_+ coincide, while the weak order units of E are the (positive) functions vanishing on a nowhere dense subset of K. If E is a space $L^p(\mu)$ with σ-finite μ and $1 \leqslant p < \infty$, then the weak order units and the quasi-interior points of E_+ coincide with the functions strictly positive μ-a.e., while E_+ does not contain any interior point.

3 Linear Forms and Duality

A linear functional φ on a vector lattice E is called

order-bounded if φ is bounded on order intervals of E,
positive if $\varphi(f) \geqslant 0$ for all $f \geqslant 0$,
strictly positive if $\varphi(f) > 0$ for all $f > 0$.

Any positive linear functional is order bounded, and the positive functionals form a proper convex cone with vertex 0 in the linear space $E^\#$ of all order bounded functionals, thus defining a natural ordering (given by $\varphi \leqslant \psi$ if and only if $\varphi(f) \leqslant \psi(f)$ for all $f \in E_+$) under which $E^\#$ is an order complete vector lattice. In particular, positive part, negative part and absolute value exist for any order bounded functional on E, the absolute value of $\varphi \in E^\#$ being given by

$$|\varphi|(f) = \sup\{\varphi(h)\colon |h| \leqslant f \text{ for } f \in E_+ \ .$$

As a consequence, one has $|\varphi(f)| \leqslant |\varphi|(|f|)$ for all f in E whenever φ is order bounded, and $|\varphi(f)| \leqslant \varphi(|f|)$ if and only if φ is positive. An order bounded linear functional φ is called *order-continuous* (*σ-order-continuous*) if both positive and negative part of φ have the property that they transform any decreasing net (any decreasing sequence) with infimum 0 into a net (sequence) converging to 0 in $\mathbb{R}$. The order-continuous (σ-order-continuous) functionals form a band in $E^\#$.

In general, a vector lattice E need not admit any non-zero order-bounded linear functional. However, if E is a normed lattice, then any continuous functional is order-bounded, and if E is a Banach lattice, then one has coincidence between $E^\#$ and E'. Still, order-continuous functionals $\neq 0$ need not exist on a Banach lattice. Situations where every order-bounded functional is order-continuous will be briefly discussed in Section 5.

If E is a Banach lattice, then the dual norm on $E' = E^{\#}$ is a lattice norm, hence E' is an order-complete Banach lattice under the natural norm and order. The evaluation map from E into the second dual E'' is a lattice homomorphism (for the definition see Section 6) into the band of order-continuous functionals on E'. In particular, every dual Banach lattice E admits sufficiently many order-continuous functionals to separate the points of E.

4 AM- and AL-Spaces

If the norm on a Banach lattice E satisfies

$$\|\sup(f, g)\| = \sup(\|f\|, \|g\|) \text{ for } f, g \in E_+, \tag{M}$$

then E is called an abstract M-space or an *AM-space*. If, in addition, the unit ball of E contains a largest element u, then u must be an order unit of E and E is then called an *(AM)-space with unit*. Condition (M) in E implies that in the dual of E one has

$$\|f + g\| = \|f\| + \|g\| \text{ for } f, g \in E'_+. \tag{L}$$

Any Banach lattice satisfying (L) is called an abstract L-space or an *AL-space*. Thus the dual of an AM-space is an AL-space.

It is quite easy to verify that, on the other hand, the dual of an AL-space is an AM-space with unit, the unit being the uniquely determined linear functional that coincides with the norm on the positive cone. Putting this together, one gets that the second dual of an AM-space E is an AM-space with unit. If E already has a unit u, then u is also the unit of E'', so that the ideal of E'' generated by E is all of E''. By contrast, if E is an AL-space, then E is an ideal (even a band) in E''. Infinite-dimensional AL- or AM-spaces are never reflexive.

The importance of AL- and AM-spaces in the general theory of Banach lattices is due to the fact that these spaces have very special concrete representations as function lattices and that, on the other hand, any general Banach lattice E is in a very intimate way connected to certain families of AL- and AM-spaces canonically associated with E. Let us first discuss the natural representations of AM- and AL-spaces.

If E is an AM-space with unit u, then the set K of lattice homomorphisms from E into $\mathbb{R}$ taking the value 1 on u is a non-empty, $\sigma(E', E)$-compact subset of E' and the natural evaluation map from E into $\mathbb{R}^K$ maps E isometrically onto the continuous real-valued functions on K (cf. Section 6). This is the *Kakutani-Krein Representation Theorem*, which is an order-theoretic counterpart to the Gelfand Representation Theorem in the theory of commutative C*-algebras. If E is an AM-space without unit, then the second dual of E has a unit and thus gives a representation of E as a closed sublattice of a space $C(K)$.

If E is an AL-space, then the representation of the dual of E as a space $C(K)$ leads to an interpretation of the elements of the bidual of E as Radon measures on K. If E_+ has a quasi-interior point h, then in this interpretation E consists exactly of the measures absolutely continuous with respect to (the measure corresponding to) h, thus by the *Radon-Nikodym-Theorem*, $E = L^1(K, h)$. In general, a similar argument leads to a representation of E as a space $L^1(X, \mu)$ constructed over a locally compact space X.

If E is an arbitrary Banach lattice and $f \in E_+$, then the ideal I generated by f in E (which is the union of the positive multiples of the interval $[-f, f]$) can be made into an AM-space with unit f by endowing it with the gauge function p_f of $[-f, f]$. We denote (I, p_f) by E_f. On the other hand, if f' is a positive linear functional on E, then the mapping $q_{f'}\colon f \mapsto \langle |f|, f' \rangle$ is a semi-norm on E. The kernel J of $q_{f'}$ is an ideal in E, and the completion of E/J with respect to the norm canonically derived from $q_{f'}$ becomes an AL-space which we denote by (E, x'). A good illustration for these constructions is the following.

If $E = C(K)$ and if μ is a positive linear form (Radon measure) on E, then (E, μ) is just $L^1(K, \mu)$; if $E = L^p(\mu)$ $(1 \leqslant p < \infty$, μ finite), then $E_{1_X} = L^\infty(\mu)$.

5 Special Connections Between Norm and Order

If an increasing net $(x_\alpha)_{\alpha \in A}$ in a normed vector lattice is convergent, then its limit must be the supremum as a consequence of the closedness of the positive cone. On the other hand, if $\{x_\alpha\colon \alpha \in A\}$ has a supremum, the net $(x_\alpha)_{\alpha \in A}$ need not converge. A Banach lattice is said to have *order-continuous norm* (*σ-order-continuous norm*) if any increasing net (sequence) which has a supremum is automatically convergent. This is of course equivalent to saying that any decreasing net (sequence) with an infimum is convergent. Since infimum and limit must coincide, the order continuity (σ-order continuity) of the norm in a Banach lattice is also equivalent to the propperty that any decreasing net (sequence) with infimum 0 converges to 0.

A Banach lattice with order-continuous norm must be order complete, but σ-order-continuity of the norm need not imply order completeness. At any rate, one has the following characterization.

A Banach lattice E has order-continuous norm if and only if any one of the following equivalent assertions holds.

(a) E is σ-order complete and has σ-order-continuous norm.
(b) Every order interval in E is weakly compact.
(c) E is (under evaluation) an ideal in E''.
(d) Every continuous linear form on E is order continuous.
(e) Every closed ideal in E is a projection band.

An even more stringent condition than order-continuity of the norm is that any increasing norm-bounded net be convergent. This condition is satisfied if and only if any one of the following equivalent assertions holds.

(a) E is (under evaluation) a band in E''.
(b) E is weakly sequentially complete.
(c) Every order-continuous linear form on E' belongs to E.

(d) No closed sublattice of E is isomorphic to c_0.

The most important examples of non-reflexive Banach lattices with this property are the AL-spaces.

6 Positive Operators, Lattice Homomorphisms

A linear mapping T from an ordered Banach space E into an ordered Banach space F is called *positive* (notation: $T \geqslant 0$) if $Tf \in F_+$ for all $f \in E_+$; T is called *strictly positive* if $T \geqslant 0$ and $\{f \in E : T|f| = 0\} = \{0\}$. The set of all positive linear mappings is a convex cone in the space LE, F of all linear mappings from E into F defining the *natural ordering* of LE, F. The linear subspace of LE, F generated by the positive maps (i.e. the space of linear maps that can be written as differences of positive maps) is denoted by $\mathcal{L}^r(E; F)$ and its elements are called *regular* mappings. If E and F are Banach lattices, then any regular mapping from E into F is continuous, but $\mathcal{L}^r(E; F)$ is in general a proper subspace of the space LE, F of all continuous linear mappings. One has coincidence of $\mathcal{L}^r(E; F)$ and LE, F, e.g., when $E = F$ is an order complete AM-space with unit or an AL-space. At any rate, if F is order complete, then $\mathcal{L}^r(E; F)$ under the natural ordering is an order-complete vector lattice, and a Banach lattice under the norm

$$T \mapsto \|T\|_r = \||T|\|,$$

the right hand side denoting the operator norm of the absolute value of T. The absolute value of $T \in \mathcal{L}^r(E; F)$, if it exists, is given by

$$|T|(f) := \sup\{Th : |h| \leqslant f,\ f \in E_+\}.$$

Thus T is positive if and only if $|Tf| \leqslant T|f|$ holds for any f in E.

An operator $T \in$ LE, F is called a *lattice homomorphism* if $|Tf| = T|f|$ holds for all $f \in E$. Lattice homomorphisms are alternatively characterized by any one of the following conditions holding for all f, and $g \in E$.

(i) $(Tf)^+ = T(f^+)$,
(ii) $(Tf)^- = T(f^-)$,
(iii) $T(f \vee g) = Tf \vee Tg$,
(iv) $T(f \wedge g) = Tf \wedge Tg$,
(v) $T(f^+) \wedge T(f^-) = 0$.

The kernel of a lattice homomorphism is an ideal. If T is bijective, then T is a lattice homomorphism if and only if T and T^{-1} are positive.

7 Complex Banach Lattices

Although the notion of a Banach lattice is intrinsically related to the real number system, it is possible and often desirable to extend discussions to complexifications of Banach lattices in such a way that the order-related terms introduced in the real situation essentially retain their meaning. Thus we define a *complex Banach lattice* E to be the complexification of a real Banach lattice $E_{\mathbb{R}}$ in the sense that

$$E = E_{\mathbb{R}} \oplus \mathrm{i}E_{\mathbb{R}}$$

which means more exactly $E = E_{\mathbb{R}} \times E_{\mathbb{R}}$ with scalar multiplication

$$(\alpha + \mathrm{i}\beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y).$$

The space $E_{\mathbb{R}}$ will sometimes be called the *underlying real Banach lattice* or the *real part* of E. The classical complex Banach spaces $C(X)$, $L^p(\mu)$ are complex Banach lattices in this sense, the underlying real Banach lattices being the corresponding (real) subspaces of real-valued functions. We want to extend the formation of absolute values, which is a priori defined only in the real part of E, in such a way that in the classical situation $E = C(X)$ or $E = L^p(\mu)$ the usual absolute value of a function is obtained. This is in fact possible by putting, for $h = f + ig$ in E,

$$|h| = \sup\{\mathrm{Re}\left(e^{\mathrm{i}\vartheta} h\right) : 0 \leqslant \vartheta \leqslant 2\pi\}.$$

The only problem with this definition being the existence of the right hand side without the assumption of order-completeness on $E_{\mathbb{R}}$. But for this we just have to observe that the set $M = \{\mathrm{Re}(e^{\mathrm{i}\vartheta} h) \colon 0 \leqslant \vartheta \leqslant 2\pi\}$ is contained and order bounded in the ideal generated in $E_{\mathbb{R}}$ by $|f| + |g|$, which in turn is by the Kakutani-Krein Representation Theorem isomorphic to a space $C_{\mathbb{R}}(X)$ under the pointwise ordering. Now the pointwise supremum of M in $\mathbb{R}^X$ is readily seen to be a continuous function (namely, the modulus of the complex valued continuous function corresponding to $f + \mathrm{i}g$), so that M has a supremum in $C_{\mathbb{R}}(X) = (E_{\mathbb{R}})_{|f|+|g|}$.

Since the mapping $f \mapsto |f|$ now has a meaning in E, the definition of an ideal can be extended formally unchanged to the complex situation. We observe that $|f + \mathrm{i}g| = |f - \mathrm{i}g| \leqslant |f| + |g|$ implies that any ideal J in a complex Banach lattice is conjugation invariant and itself the complexification of the ideal $J \cap E_{\mathbb{R}}$ of the real part of E.

Suffice it now to say that the meaning of most of the terms introduced for real Banach lattices can be extended to the complex situation under retention (mutatis mutandis) of the corresponding results valid in the real case by either using the complex modulus or else, if the formation of suprema or infima is involved, by relating them to real parts. For example $f \in E$ is called *positive* if $f = |f|$ which means that f is a positive element of $E_{\mathbb{R}}$, E is called order complete if $E_{\mathbb{R}}$ is order complete, and an ideal J is called a band if the real part of J is a band. We refer to

Chapter II, Section 11 of **?**] for a detailed discussion of this and restrict ourselves to a short discussion of linear mappings.

Let E and F be complex Banach lattices with real parts $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$. Then a linear mapping T from E into F is determined by its restriction T_0 to $E_{\mathbb{R}}$, and T_0 can be written in the form $T_0 = T_1 + iT_2$ with real-linear mappings T_j from $E_{\mathbb{R}}$ into $F_{\mathbb{R}}$. Thus $L(E, F)$ is the complexification of the real linear space $L(E_{\mathbb{R}}, F_{\mathbb{R}})$. With the above notation, T is called *real* if T_2 is $= 0$, *positive* if T is real and T_1 is positive, and a *lattice homomorphism* if T is real and T_1 is a lattice homomorphism. Lattice homomorphisms are characterized by the equality $|Th| = T|h|$ as in the real case.

8 The Signum Operator

We discuss in some detail how a mapping of the form

$$g \mapsto (\operatorname{sign} f)g$$

which has an obvious meaning, depending on f, in spaces $C(K)$, can be defined in an abstract complex Banach lattice. We prove the following

Let E be a complex Banach lattice and let $f \in E$. If either E is order-complete or $|f|$ is a quasi-interior point in E_+, then there exists a unique linear mapping S_f, called the *signum operator* with respect to f, with the following properties.

(i) $S_f \bar{f} = |f|$, where $\bar{f} = \operatorname{Re}(f) - \mathrm{i} \cdot \operatorname{Im}(f)$,
(ii) $|S_f g| \leqslant |g|$ for every g in E,
(iii) $S_f g = 0$ for every g in E orthogonal to f.

In fact, if $E = C(K)$ and if $|f|$ is a quasi-interior point in E, then $|f|$ is a strictly positive function and multiplication with the function sign $f = f \cdot |f|^{-1}$ has the desired properties. Uniqueness follows from **?** , Chap. 20]. We reduce the general situation to the case just considered in the following way.

— If $|f|$ is quasi-interior to E_+, then $E_{|f|}$ is a dense subspace of E isomorphic to a space $C(K)$, and with the above arguments one gets a uniquely determined operator S_0 on $E_{|f|}$ with the desired properties. Since (ii) implies the continuity of S_0 for the norm induced by E, S_0 can be extended to E.
— If f is arbitrary, then, as above, one gets an operator S_0 on $E_{|f|}$ with (i) and (ii). If E is order complete, an extension S_f of S_0 to E satisfying (i)–(iii) is possible as soon as S_0 can be extended to the band $\{x\}^{dd}$ of E.

  - On the complementary band $\{x\}^d$ one has necessarily the values = 0 for S_f.
  - The extension to $\{x\}^{dd}$ is obtained as follows: If S_0 is positive (which means $f \geqslant 0$), then

$$S_f h = \sup\{S_f g \colon g \in [0, h] \cap E_{|f|} \text{ for } h \geqslant 0\}$$

  will do.

In general, the problem can be reduced to this situation by decomposing S_0 into a sum of the form $S_0 = (S_1 - S_2) + i(S_3 - S_4)$ with positive operators S_j. Such a decomposition of S_0 exists since the order completeness of E implies the order completeness of $E_{|f|} = C(K)$ and since every continuous linear operator on a space $C(K)$ is necessarily order-bounded.

9 The Center of $\mathcal{L}(E)$

We give a short description of a special, but important class of operators.

Let E be a (complex) Banach lattice. For $T \in \mathcal{L}(E)$ the following conditions are equivalent.

(a) $f \perp g$ implies $Tf \perp g \quad (f, g \in E)$,
(b) $\pm T \leqslant \|T\|\mathrm{Id}$,
(c) $TJ \subseteq J$ for every ideal J in E.

If E is countably order complete, then this is equivalent to:

(d) $TJ \subseteq J$ for every projection band J in E.

The last assertion also means that T commutes with every band projection.

The set of all $T \in \mathcal{L}(E)$ satisfying the above conditions is called the *center* of $\mathcal{L}(E)$ and denoted $\mathcal{Z}(E)$. Under its natural ordering, the operator norm and the composition product is $\mathcal{Z}(E)$ isomorphic to a Banach lattice algebra $C(K)$ with K compact. The following examples may illustrate the situation and explain why the term *multiplication operator* is often used for operators in $\mathcal{Z}(E)$.

(i) If $E = L^p(X, \Sigma, \mu)$ $(1 \leqslant p \leqslant \infty)$ with σ-finite μ, then $Z(E)$ is isomorphic to $L^\infty(\mu)$ via the natural identification of a function $f \in L^\infty(\mu)$ with the multiplication operator $g \mapsto f \cdot g$ on E.
(ii) If X is locally compact, $E = C_0(X)$, then similarly $\mathcal{Z}(E) \cong C^b(X)$ via the identification of $f \in C^b(X)$ with the mapping $g \mapsto f \cdot g$ $(g \in C_0(X))$.

Chapter C-II
Characterization of Positive Semigroups on Banach Lattices and Positive Operators

Chapter C-III
Spectral Theory of Positive Semigroups on Banach Lattices

In Chapter B-III we have shown that positive semigroups on spaces $C_0(X)$ possess several interesting spectral properties. Now we are going to extend many of these results to the more general setting of Banach lattices. We will improve some of them considerably and give the complete proof of B-III,Theorem 4.1.

Throughout this chapter we will assume that $E \neq \{0\}$ is a complex Banach lattice.

1 The Spectral Bound

The fact that the spectral bound of a positive semigroup is always contained in the spectrum (provided that the spectrum is non-empty) is also true in the general setting of Banach lattices. The proof given in B-III, Theorem 1.1 for spaces $C_0(X)$ works in the general case too. Another proof is given below (cf. Corollary 1.4). Furthermore, Corollary 1.3 and Proposition 1.5 of B-III are true in the setting of Banach lattices and their proofs can be carried over to the general case. For the sake of completeness we summarize these results in the following theorem.

Theorem 1.1 *Let A be the generator of a positive semigroup $(T(t))_{t\geqslant 0}$ on a Banach lattice E.*

(i) $s(A) \in \sigma(A)$ *unless* $\sigma(A) = \emptyset$.
(ii) *For* $\lambda_0 \in \varrho(A)$ *we have* $R(\lambda_0, A)$ *is positive if and only if* $\lambda_0 > s(A)$. *In this case* $r(R(\lambda_0, A)) = (\lambda_0 - s(A))^{-1}$.
(iii) *If* $T(1)$ *has a positive fixed vector* h_0, *then* $\ker(A)$ *contains a positive element* h *such that* $h_0 \in E_{|h|}$.
(iv) *If* $T(1)'\varphi_0 = \varphi_0$ *for some* $\varphi_0 \in E'_+$, *then there exists* $\varphi \in D(A^*)_+$ *with* $\{f \in E : \langle |f|, \varphi\rangle = 0\} \subseteq \{f \in E : \langle |f|, \varphi_0\rangle = 0\}$ *such that* $A^*\varphi = 0$.

The fact that $s(A)$ is always an eigenvalue of the adjoint (cf. B-III Theorem 1.6) is characteristic for spaces $C(K)$, K compact, as can be seen byconsidering the Laplacian on $L^p(\mathbb{R}^n)$, where $1 < p < \infty$, or on $C_0(\mathbb{R}^n)$ (see B-III,Example 1.7).

Another result which cannot be extended to arbitrary Banach lattices is that spectral bound and growth bound coincide (cf. B-IV,Theorem 1.4); an example is given in A-III,Example 1.3. Despite of this the resolvent $R(\lambda, A)$ of a positive semigroup is given as the Laplace transform of the semigroup in the half-plane $\{z \in \mathbb{C}\colon \operatorname{Re} z > s(A)\}$ (even in case that $\omega_0(A) > s(A)$). Note however that the integral exists only as an improper Riemann integral. By Datko's Theorem (A-IV,Theorem 1.11) the function $t \mapsto \mathrm{e}^{-\lambda t}T(t)f$ cannot be Bochner integrable for all $f \in E$ in the case $\operatorname{Re}\lambda \leqslant \omega_0(A)$.

Theorem 1.2 *Suppose A is the generator of a positive semigroup $(T(t))_{t\geqslant 0}$.*
For $\operatorname{Re}\lambda > s(A)$ we have

$$R(\lambda, A)f = \lim_{t\to\infty}\int_0^t \mathrm{e}^{-\lambda s}T(s)f \,\mathrm{d}s \text{ for all } f \in E. \tag{1.1}$$

Moreover, the operators $\int_0^t \mathrm{e}^{-\lambda s}T(s)\,\mathrm{d}s$ tend to $R(\lambda, A)$ with respect to the operator norm as $t \to \infty$.

Proof We fix $\lambda_0 > \omega_0(A)$. Then, by A-I,Proposition 1.11,

$$R(\lambda_0, A)^{n+1}f = \frac{1}{n!}\int_0^\infty s^n \exp(-\lambda_0 s)T(s)f \,\mathrm{d}s \quad (n \in \mathbb{N}_0, f \in E)\,. \tag{1.2}$$

Given μ such that $s(A) < \mu < \lambda_0$, $f \in E_+$, $\varphi \in E'_+$, then

$$\begin{aligned}
\langle R(\mu, A)f, \varphi\rangle &= \sum_{n=0}^{\infty}(\lambda_0-\mu)^n\langle R(\lambda_0, A)^{n+1}f, \varphi\rangle = \\
&= \sum_{n=0}^{\infty}\int_0^\infty \frac{1}{n!}(\lambda_0-\mu)^n\exp(-\lambda_0 s)\langle T(s)f, \varphi\rangle\,\mathrm{d}s = \\
&= \int_0^\infty\sum_{n=0}^{\infty}\frac{1}{n!}(\lambda_0-\mu)^n\exp(-\lambda_0 s)\langle T(s)f, \varphi\rangle\,\mathrm{d}s = \\
&= \int_0^\infty \exp((\lambda_0-\mu)s)\exp(-\lambda_0 s)\langle T(s)f, \varphi\rangle\,\mathrm{d}s = \\
&= \int_0^\infty \exp(-\mu s)\langle T(s)f, \varphi\rangle\,\mathrm{d}s = \\
&= \lim_{t\to\infty}\left\langle\int_0^t \exp(-\mu s)T(s)f\,\mathrm{d}s, \varphi\right\rangle .
\end{aligned} \tag{1.3}$$

Note that one can interchange summation and integration because all the integrands are positive functions.

It follows from (1.3) at the net $(\int_0^t \exp(-\mu s)T(s)f \,\mathrm{d}s)_{t\geqslant 0}$ converges weakly to $R(\mu, A)f$ for $t \to \infty$. Because it is monotone increasing ($f \geqslant 0$), we have strong convergence (see the corollary to II.Theorem 5.9 in **?**]). If $\lambda = \mu + \mathrm{i}\nu$ with μ, ν real and $\mu > s(A)$, we have for arbitrary $f \in E$, $\varphi \in E'$

$$\left|\left\langle \int_r^t e^{-\lambda s} T(s) f \, ds, \varphi \right\rangle\right| \leqslant \int_r^t e^{-\mu s} \langle T(s)|f|, |\varphi|\rangle \, ds$$
$$\text{hence } \left\| \int_r^t e^{-\lambda s} T(s) f \, ds \right\| \leqslant \left\| \int_r^t e^{-\mu s} T(s) |f| \, ds \right\|$$

which shows that $\lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s) f \, ds$ exists. Thus $R(\lambda, A) f = \int_0^\infty e^{-\lambda s} T(s) f \, ds$ by A-I,Proposition 1.11.

It remains to prove that the net $\left(\int_0^t \exp(-\mu s) T(s) \, ds\right)_{t \geqslant 0}$ converges with respect to the operator norm. We fix μ such that $s(A) < \mu < \operatorname{Re} \lambda$. As we have seen above, the map $s \mapsto e^{-\mu s} \langle T(s) f, \varphi \rangle$ is Lebesgue integrable for every $(f, \varphi) \in E \times E'$, thus defining a bilinear map $b \colon E \times E' \to L^1(\mathbb{R}_+)$. Using the closed graph theorem, it is easy to see that b is separately continuous, hence jointly continuous by **?** , III.Theorem 5.1]. Thus there is a constant M such that

$$\int_0^\infty e^{-\mu s} |\langle T(s) f, \varphi \rangle| \, ds = \|b(f, \varphi)\| \leqslant M \|f\| \|\varphi\| \quad (f \in E, \varphi \in E') . \tag{1.4}$$

Given $0 \leqslant t < r$ and setting $\varepsilon := \operatorname{Re} \lambda - \mu$ we have

$$\begin{aligned} \left| \int_t^r e^{-\lambda s} \langle T(s) f, \varphi \rangle \, ds \right| &\leqslant \int_t^r \exp(-(\operatorname{Re} \lambda - \mu) s) e^{-\lambda s} |\langle T(s) f, \varphi \rangle| \, ds \\ &\leqslant e^{-\varepsilon t} \int_t^r e^{-\lambda s} |\langle T(s) f, \varphi \rangle| \, ds \leqslant e^{-\varepsilon t} M \|f\| \|\varphi\|. \end{aligned}$$

It follows that $\left\| \int_t^r e^{-\lambda s} T(s) \, ds \right\| \leqslant M e^{-\varepsilon t}$, hence $\left(\int_0^t e^{-\lambda s} T(s) \, ds\right)_{t \geqslant 0}$ is a Cauchy net with respect to the operator norm. □

Theorem 1.2 has many consequences. In particular, we can conclude that $s(A) \in \sigma(A)$ whenever $s(A) > -\infty$ (without using the analogous result for bounded operators, cf. Corollary 1.4 below).

In each of the following corollaries we assume that A is the generator of a positive semigroup on a Banach lattice E.

Corollary 1.3 *If* $\operatorname{Re} \lambda > s(A)$*, then we have*

$$|R(\lambda, A) f| \leqslant R(\operatorname{Re} \lambda, A) |f| \quad (f \in E). \tag{1.5}$$

The proof is an immediate consequence of Theorem 1.2.

Corollary 1.4 *We have* $s(A) \in \sigma(A)$ *unless* $s(A) = -\infty$.

Proof Assume that $s(A) > -\infty$ and $s(A) \notin \sigma(A)$, then it follows from (1.5) that $\{R(\lambda, A) \colon \operatorname{Re} \lambda > s(A)\}$ is uniformly bounded in $\mathcal{L}(E)$, by M say. Then $\{z \in \mathbb{C} \colon \operatorname{Re} z = s(A)\} \subseteq \varrho(A)$ and $\|R(z, A)\| \leqslant M$ for z with $\operatorname{Re} z = s(A)$. It follows that $\{z \in \mathbb{C} \colon |\operatorname{Re} z - s(A)| < M^{-1}\} \subseteq \varrho(A)$, which is absurd by the definition of $s(A)$. □

Corollary 1.5 *Suppose that $s(A)$ is a pole of order m of the resolvent $R(\lambda, A)$. Then m is a majorant for the order of any other pole on the line $s(A) + \mathrm{i}\mathbb{R}$.*

Proof Without loss of generality we may assume that $s(A) = 0$. By (1.5) we have $\|R(\varepsilon + \mathrm{i}\beta, A)\| \leqslant \|R(\varepsilon, A)\|$ for every $\beta \in \mathbb{R}$, $\varepsilon > 0$. Therefore $\lim_{\varepsilon\to 0} \|\varepsilon^k R(\varepsilon + \mathrm{i}\beta, A)\| \leqslant \lim_{\varepsilon\to 0} \|\varepsilon^k R(\varepsilon, A)\| = 0$ for $k > m$. □

The spectrum of a positive semigroup may be empty (see B-III, Example 1.2(a)) and the spectrum of a general group may be empty as well (see **?** , Section 23.16]). However, for positive groups this cannot occur. More precisely, we have the following result

Corollary 1.6 *If A is the generator of a positive group, then $\sigma(A) \cap \mathbb{R} \neq \emptyset$.*

Proof Both A and $-A$ are generators of positive semigroups, hence if $\sigma(A) = \emptyset$, then $s(A) = s(-A) = -\infty$ and (1.5) implies that

$$\{R(\lambda, A) \colon \operatorname{Re}\lambda \geqslant 0\} \cup \{R(\lambda, -A) \colon \operatorname{Re}\lambda \geqslant 0\}$$

is bounded in $\mathcal{L}(E)$, i.e., $\{R(\lambda, A) \colon \lambda \in \mathbb{C}\}$ is bounded. By Liouville's Theorem the function $\lambda \mapsto R(\lambda, A)$ is constant, hence identically zero because $\lim_{\lambda\to\infty} R(\lambda, A) = 0$. Thus we arrive at a contradiction. □

We conclude this section by indicating possible extensions and further consequences of the results stated above.

Remarks 1.7 (i) Many of the results of this section remain true for positive semigroups on ordered Banach spaces more general than Banach lattices. The interested reader is referred to **?**].

(ii) From Theorem 1.2 one can easily deduce that for positive semigroups on L^1-spaces spectral bound and growth bound coincide. To prove the analoguous result for L^2-spaces one makes use of Corollary 1.3. For details we refer to C-IV,Theorem 1.1.

2 The Boundary Spectrum

In Chapter B-III we have seen that under suitable assumptions the boundary spectrum $\sigma_b(A)$, consisting of all spectral values with maximal real part is a cyclic set (cf. B-III,Definition 2.5). In the main theorem of this section we prove a result which is more general and which is true for arbitrary Banach lattices.

We first want to extend some of the notions used in B-III to the more general setting considered here. If E is a Banach lattice and $f, g \in E$ such that $g \in E_{|f|}$, then $(\operatorname{sign} f)g$ is well-defined (cf. Section 8 of C-I). Thus the following definition makes sense

Definition 2.1 If E is a Banach lattice then for $f \in E$, $n \in \mathbb{Z}$ we define $f^{[n]}$ recursively as follows

$$\begin{aligned} f^{[0]} &:= |f|\,, \\ f^{[n]} &:= (\operatorname{sign} f) f^{[n-1]} \text{ if } n > 0\,, \\ f^{[n]} &:= (\operatorname{sign} \overline{f}) f^{[n+1]} \text{ if } n < 0\,. \end{aligned} \tag{2.1}$$

Obviously, for $E = C_0(X)$ this amounts to the same as B-III,Definition 2.2. Moreover, in case E is an L^p-space, then $f^{[n]}$ is the function given by

$$f^{[n]}(x) = \begin{cases} (f(x)/|f(x)|)^{n-1} f(x) & \text{if } f(x) \neq 0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The following properties are immediate consequences Definition 2.1

$$f^{[0]} = |f|, f^{[1]} = f, f^{[-1]} = \overline{f}, |f^{[n]}| = |f| \quad (n \in \mathbb{Z}) \tag{2.3}$$

$$f^{[n]} = (\operatorname{sign} f) f^{[n-1]} = (\operatorname{sign} \overline{f}) f^{[n+1]} \text{ for all } n \in \mathbb{Z}; \tag{2.4}$$

$$(\alpha f)^{[n]} = \alpha (\alpha/|\alpha|)^{n-1} f^{[n]} \text{ for } n \in \mathbb{Z}, \alpha \in \mathbb{C}, \alpha \neq 0. \tag{2.5}$$

Next we show that B-III,Theorem 2.4 is true for arbitrary Banach lattices. For definition and simple properties of the signum operator S_h see C-I,Section 8.

Theorem 2.2 *Let $(T(t))_{t \geqslant 0}$ be a positive semigroup on a Banach lattice E with generator A and suppose that for $h \in E$, $\alpha, \beta \in \mathbb{R}$ we have*

$$Ah = (\alpha + \mathrm{i}\beta)h, \quad A|h| = \alpha|h|. \tag{2.6}$$

Then the following holds true

$$Ah^{[n]} = (\alpha + \mathrm{i}n\beta)h^{[n]} \textit{ for all } n \in \mathbb{Z}. \tag{2.7}$$

In case $|h|$ is a quasi-interior point of E_+, then

$$S_h D(A) = D(A) \textit{ and } A + \mathrm{i}\beta\,\mathrm{Id} = S_h^{-1} A S_h\,.$$

Proof Without loss of generality we may assume that $\alpha = 0$. Then the assumption (2.6) implies that $T(t)|h| = |h|$ and $T(t)h = \mathrm{e}^{\mathrm{i}\beta t}h$ for $t \geqslant 0$ (see A-III,Corollary 6.4). In particular, the principal ideal $E_{|h|}$ is invariant under every operator $T(t)$. By the Kakutani-Krein Theorem (C-I,Section 4) we can identify $E_{|h|}$ with a space $C(K)$, K compact. Then the restrictions $\overline{T}(t) := T(t)|_{E_{|h|}}$ are positive operators on $C(K)$ satisfying $\overline{T}(t)|\overline{h}| = |\overline{h}|$ and $\overline{T}(t)\overline{h} = \mathrm{e}^{\mathrm{i}\beta t}\overline{h}$.

From B-III,Theorem 2.4(a) we conclude $\overline{T}(t)\overline{h}^{[n]} = \mathrm{e}^{\mathrm{i}\beta t}\overline{h}^{[n]}$ for all $t \geqslant 0$, $n \in \mathbb{Z}$. Translating this back to $T(t)$ and E this means precisely $T(t)h^{[n]} = \mathrm{e}^{in\beta}h^{[n]}$, hence $h^{[n]} \in D(A)$ and $Ah^{[n]} = \mathrm{i}n\beta h^{[n]}$.

Moreover, by B-III,Theorem 2.4(a) we have $\mathrm{e}^{\mathrm{i}\beta t}\overline{T}(t) = S_h^{-1}\overline{T}(t)S_h$. If $|h|$ is a quasi-interior point, this relation extends by continuity from the dense subspace $E_{|h|}$ to the whole space E, i.e., we have $\mathrm{e}^{\mathrm{i}\beta t}T(t) = S_h^{-1}T(t)S_h$ for all $t \geqslant 0$. □

In the proof above we could not apply assertion (b) of B-III,Theorem 2.4 because the semigroup $(\overline{T}(t))$ on $E_{|h|} \cong C(K)$ need not be strongly continuous with respect to the sup-norm.

As a first application of Theorem 2.2 we prove a cyclicity result for the point spectrum of contraction semigroups on a class of Banach lattices which includes the L^p-spaces.

Corollary 2.3 *Suppose E is a Banach lattice such that the norm is strictly monotone on E_+ (i.e., $0 \leqslant f < g \Rightarrow \|f\| < \|g\|$). If $(T(t))$ is a positive contraction semigroup on E with $s(A) = 0$, then $P\sigma_b(A) = P\sigma(A) \cap \mathrm{i}\mathbb{R}$ is imaginary additively cyclic.*

Proof Suppose that $Ah = \mathrm{i}\beta h$ $(\beta \in \mathbb{R}, h \in E)$. Then we have $T(t)h = \mathrm{e}^{\mathrm{i}\beta t}h$ $(t \geqslant 0)$ and $|h| = |T(t)h| \leqslant T(t)|h|$ since $T(t)$ is positive. Moreover, $\|h\| \leqslant \|T(t)|h|\| \leqslant \|h\|$ since $\|T(t)\| \leqslant 1$. The assumption on the norm of E implies that $T(t)|h| = |h|$ for all $t \geqslant 0$, equivalently $A|h| = 0$. Now we can apply Theorem 2.2 in order to obtain the desired result. □

A more general result on cyclicity of the eigenvalues in the boundary spectrum will be proved in Corollary 4.3. In the remaining part of this section we focus our interest on the entire boundary spectrum. We will prove that it is cyclic provided that the resolvent $R(\lambda, A)$ does not grow too fast as $\lambda \to s(A)$. We start with some preparations. An important tool in the proof are pseudo-resolvents.

Definition 2.4 Let D be an open (non-empty) subset of $\mathbb{C}$ and let G be a Banach space. A mapping $R\colon D \to \mathcal{L}(G)$ which satisfies

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu) \quad (\lambda, \mu \in D) \tag{2.8}$$

is called a *pseudo-resolvent* on G.

An equivalent (often quite useful) version of (2.8) is the following

$$(1 - (\lambda - \mu)R(\lambda))(1 - (\mu - \lambda)R(\mu)) = 1 \quad (\lambda, \mu \in D) \tag{2.9}$$

Obviously, the resolvent of a closed linear operator A on G is a pseudo-resolvent on $D = \varrho(A)$. In general a pseudo-resolvent need not be the resolvent of an operator. Further information can be found in **?**], **?**] or **?**]. For our purposes the following examples are of particular interest

Example 2.5 (a) Suppose A is a densely defined linear operator on G with $\varrho(A) \neq \emptyset$ and let $G_{\mathcal{F}}$ be an F-product of G (cf. A-I,3.6). Then the canonical extensions $R(\lambda, A)_{\mathcal{F}}$ of $R(\lambda, A)$ form a pseudo-resolvent $R_{\mathcal{F}}$ on $G_{\mathcal{F}}$ with $\varrho(A)$ as domain of definition. If A is unbounded, then $0 \in A\sigma(R(\lambda, A))$ hence $0 \in P\sigma(R_{\mathcal{F}}(\lambda, A))$ (cf. A-III, 4.5). It follows that $R_{\mathcal{F}}$ is not the resolvent of an operator on G.

(b) If $\{R(\lambda)\}_{\lambda \in D}$ is a pseudo-resolvent on G, then $\{R(\lambda)'\}_{\lambda \in D}$ is a pseudo-resolvent on G'. Moreover, if H is a closed linear subspace of G which is $\{R(\lambda)\}_{\lambda \in D}$-invariant ($R(\lambda)H \subset H$ for all $\lambda \in D$), then the operators on H and G/H induced by $R(\lambda)$ in the canonical way form a pseudo-resolvent on H and G/H, respectively.

In the following we list several simple properties. We assume that $R\colon D \to \mathcal{L}(G)$ is a pseudo-resolvent on a Banach space G

$$\begin{aligned}&\text{Given } \lambda_0 \in D, \mu \in \mathbb{C} \text{ there exists at most one operator } S \in \mathcal{L}(G)\\ &\text{such that} \quad R(\lambda_0) - S = -(\lambda_0 - \mu)R(\lambda_0)S = -(\lambda_0 - \mu)SR(\lambda_0)\,.\\ &\text{In case such an operator exists w,e have}\\ &R(\lambda) - S = -(\lambda - \mu)R(\lambda)S = -(\lambda - \mu)SR(\lambda) \text{ for all } \lambda \in D\,.\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\text{Given } \lambda_0 \in D, \text{ then for } \mu \in D \text{ with } |\mu - \lambda_0| < \|R(\lambda_0)\|^{-1}\\ &\text{we have } R(\mu) = \sum_{n=0}^{\infty}(\lambda_0 - \mu)^n R(\lambda_0)^{n+1}\ .\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\text{The map } \lambda \mapsto R(\lambda) \text{ is a locally holomorphic function defined on } D \subseteq \mathbb{C}\\ &\text{with values in } \mathcal{L}(G)\,.\end{aligned} \tag{2.12}$$

We only sketch the proof of these assertions. (2.12) follows from (2.11) and the latter is a consequence of (2.10). The identity stated in (2.10) can be rewritten as follows $(1-(\lambda_0-\mu)R(\lambda_0))(1-(\mu-\lambda_0)S) = 1 = (1-(\mu-\lambda_0)S)(1-(\lambda_0-\mu)R(\lambda_0))$. Thus $S = (\mu - \lambda_0)^{-1}(1 - (1 - (\lambda_0 - \mu)R(\lambda_0))^{-1})$ has to be unique.

It follows from (2.11) and (2.12) that every pseudo-resolvent has a unique maximal extension. Further properties of pseudo-resovents are given in the following two propositions.

Proposition 2.6 *Suppose G is a Banach space, $D \subseteq \mathbb{C}$ and $R\colon D \to \mathcal{L}(G)$ is a pseudo-resolvent.*

(i) Given $\alpha \in \mathbb{C}$, $x \in G$ one has $(\lambda - \alpha)R(\lambda)x = x$ either for all $\lambda \in D$ or for none.

(ii) Suppose $\mu \in \overline{D}\backslash D$. Then R can be extended to an open set containing μ if and only if there exists a sequence $(\lambda_n) \subset D$ converging to μ such that $\|R(\lambda_n)\|$ is bounded.

Proof (i) Suppose that $(\lambda - \alpha)R(\lambda)x = x$ for some fixed $\lambda \in D$, $x \in G$. Then using (2.8) we have for $\mu \in D$ $(\mu-\alpha)(\mu-\lambda)R(\mu)R(\lambda)x = (\lambda-\alpha)(R(\lambda)x-R(\mu)x) = x - (\lambda - \alpha)R(\mu)x$. It follows that $(\mu - \alpha)R(\mu)x = x$ for all $\mu \in D$.

(ii) If there exists an extension, then $\|R(\lambda_n)\|$ is bounded for every sequence (λ_n) converging to μ by (2.12). On the other hand assuming that $\|R(\lambda_n)\|$ is bounded by M for a fixed sequence $(\lambda_n) \subset D$ with $\lambda_n \to \mu$ $(M \geqslant 0)$, we have $\|R(\lambda_n) - R(\lambda_m)\| = |\lambda_n - \lambda_m|\|R(\lambda_n)R(\lambda_m)\| \leqslant M^2|\lambda_n - \lambda_m|$ which shows that $(R(\lambda_n))$ is a Cauchy sequence in $\mathcal{L}(G)$, hence $S := \lim_{n\to\infty} R(\lambda_n)$ exists. The a1ls now follows from (2.10) and (2.11). ◻

In the next proposition, we consider a positive pseudo-resolvent R on a Banach lattice E, i.e., we assume that the domain D of R contains the positive real axis and that $R(\mu)$ is a positive operator for every $\mu > 0$. Applying Pringsheim's Theorem (see Theorem 2.1 in the appendix of **?**] to the expansion given in (2.11) one can conclude that R has an extension to the halfplane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

This shows that without loss of generality one can assume that the domain of a positive pseudo-resolvent contains the halfplane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Proposition 2.7 *Suppose $R : \Delta \to \mathcal{L}(E)$ is a positive pseudo-resolvent on a Banach lattice E and $\Delta := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.*

If for some $\beta \in \mathbb{R}$, $h \in E$ we have $\lambda R(\lambda + \mathrm{i}\beta)h = h$ and $\lambda R(\lambda)|h| = |h|$ $(\lambda \in \Delta)$, then

$\lambda R(\lambda + in\beta)h^{[n]} = h^{[n]}$ for all $n \in \mathbb{Z}$, $\lambda \in \Delta$.

Proof At first we prove the following domination property which is the extension of (1.5) to pseudo-resolvents

$$|R(\lambda)f| \leqslant R(\operatorname{Re}\lambda)|f| \text{ for every } \lambda \in \Delta, f \in E\,. \tag{2.13}$$

To do this we fix $\lambda \in \Delta$. Then there exists $r_0 > 0$ such that $|r - \lambda| < r$ whenever $r > r_0$. Therefore $R(\lambda) = \sum_{n=0}^{\infty}(r-\lambda)^n R(r)^{n+1}$ for $r > r_0$, which implies for $f \in E$

$$\begin{aligned}|R(\lambda)f| &\leqslant \sum_{n=0}^{\infty} |r-\lambda|^n R(r)^{n+1}|f| = \\ &= \sum_{n=0}^{\infty}(r - (r - |r-\lambda|))^n R(r)^{n+1}|f| = R(r - |\lambda - r|)|f|\,.\end{aligned}$$

Since $\lim_{r\to\infty}(r - |\lambda - r|) = \operatorname{Re}\lambda$, we obtain (2.13).

As a consequence of (2.13) and the assumption $rR(r)|h| = |h|$ we have that the principal ideal $E_{|h|}$ is $\{R(\lambda)\}_{\lambda\in\Delta}$-invariant. Identifying, according to the Kakutani-Krein Theorem, $E_{|h|}$ with a space $C(K)$, K compact, and by restricting the operators $R(\lambda)$ to $E_{|h|} \cong C(K)$, we obtain a positive pseudo-resolvent $\tilde{R} : \Delta \to L(C(K))$. Then we have for every $\alpha > 0$ and $f \in E$

$$\alpha\tilde{R}(\alpha + \mathrm{i}\beta)h = h, \alpha\tilde{R}(\alpha)|h| = |h| = \mathbb{1}_K, \alpha|\tilde{R}(\alpha + \mathrm{i}\beta)f| \leqslant \alpha\tilde{R}(\alpha)|f|\,.$$

Applying B-III,Lemma 2.3 we obtain $\tilde{R}(\alpha) = S_h^{-1}\tilde{R}(\alpha + \mathrm{i}\beta)S_h$ for every $\alpha > 0$ and using the uniqueness theorem for holomorphic functions we obtain

$$\tilde{R}(z) = S_h^{-n}\tilde{R}(z + in\beta)S_h^n \text{ for all } z \in \Delta,\ n \in \mathbb{Z}\,. \tag{2.14}$$

In particular, $S_h^n|h| = S_h^{-n}z\tilde{R}(z)|h| = z\tilde{R}(z + in\beta)S_h^n|h|$.

In terms of the initial space this means precisely $h^{[n]} = zR(z + in\beta)h^{[n]}$, and the proposition is proved. ◻

We will prove cyclicity of the boundary spectrum under a *growth condition* which is stated in the following definition.

Definition 2.8 Let A be the generator of a positive semigroup $(T(t))_{t\geqslant 0}$ with spectral bound $s(A) > -\infty$. The resolvent is said to *grow slowly* if one of the following (equivalent) conditions is satisfied.

$$\begin{aligned} &\text{(i)} \quad \{(\lambda - s(A))R(\lambda, A) : \lambda > s(A)\} \text{ is bounded in } \mathcal{L}(E)\,. \\ &\text{(ii)} \quad \left\{\frac{1}{t}\int_0^t \exp(-\tau s(A))T(\tau)\,\mathrm{d}\tau : t > 0\right\} \text{ is bounded in } \mathcal{L}(E)\,. \end{aligned} \tag{2.15}$$

Before proving the equivalence of the two assertions we make some remarks.

(a) Since one always has $\lambda R(\lambda, A) \to \mathrm{Id}$ for $\lambda \to \infty$, $\{(\lambda - s(A))R(\lambda, A) : \lambda > s(A) + \varepsilon\}$ is bounded for every $\varepsilon > 0$. Thus in (2.15)(i) the essential fact is boundedness near $s(A)$. On the other hand, $\{\frac{1}{t}\int_0^t \exp(-\tau s(A))T(\tau)\,\mathrm{d}\tau : 0 \leqslant t \leqslant T\}$ is bounded for every $T \geqslant 0$, hence in (2.15)(ii) the boundedness for $t \to \infty$ is important.

(b) By Corollary 1.4 we have $\|R(\lambda, A)\| \geqslant r(R(\lambda, A)) = (\lambda - s(A))^{-1}$. Hence $\|R(\lambda, A)\|$ grows at least as fast as $(\lambda - s(A))^{-1}$. Thus if (2.15)(i) is satisfied, the resolvent grows as slowly as it possibly can for $\lambda \to s(A)$.

(c) We do not assume in Definition 2.8 that spectral bound and growth bound coincide. A slight modification of A-III,Example 1.3 shows that there are semigroups with slowly growing resolvent and $s(A) < \omega_0(A)$.

Proof To prove equivalence of (2.15)(i) and (2.15)(ii), we assume $s(A) = 0$ and write $C(t) := \frac{1}{t}\int_0^t T(\tau)\,\mathrm{d}\tau$.

(2.15)(i) $\Rightarrow$ (2.15)(ii) Consider $\lambda > 0$, $t > 0$ such that $\lambda t = 1$. Then we have

$$\lambda \cdot \mathrm{e}^{-\lambda s} \geqslant \begin{cases} (\mathrm{e}t)^{-1} & \text{if } s \leqslant t\,, \\ 0 & \text{if } s > t\,. \end{cases}$$

Now (1.1) implies

$$\lambda R(\lambda, A) = \int_0^\infty \lambda \exp(-\lambda s)T(s)\,\mathrm{d}s \geqslant \mathrm{e}^{-1}C(t) = \mathrm{e}^{-1} \cdot C(\frac{1}{\lambda}) \geqslant 0\,.$$

Thus $C(t)$ is bounded for $t \to \infty$ whenever $\lambda R(\lambda, A)$ is bounded for $\lambda \downarrow 0$.

(2.15)(ii) $\Rightarrow$ (2.15)(i) Let $M := \sup\{\|C(t)\| : t > 0\}$. Given $f \in E$, $\lambda > 0$ and $r > 0$ then integration by parts yields

$$\lambda \int_0^r \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s = \lambda\mathrm{e}^{-\lambda r}\int_0^r T(\sigma)f\,\mathrm{d}\sigma + \lambda^2\int_0^r s\mathrm{e}^{-\lambda s}\left(\frac{1}{s}\int_0^s T(\sigma)f\,\mathrm{d}\sigma\right)\mathrm{d}s$$

$$\begin{aligned} \left\|\lambda\int_0^r \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s\right\| &\leqslant \left(r\lambda\mathrm{e}^{-r} + \lambda^2\int_0^r s\mathrm{e}^{-\lambda s}\,\mathrm{d}s\right)M\|f\| \\ &= (1 - \mathrm{e}^{-\lambda r})M\|f\|\,. \end{aligned}$$

Letting $r \to \infty$, we obtain by (1.1) $\|\lambda R(\lambda, A)f\| \leqslant M\|f\|$ $(f \in E, \lambda > 0)$ hence $\|\lambda R(\lambda, A)\| \leqslant M$ □

Two sufficient conditions for a resolvent to grow slowly are stated in the following proposition. Its simple proof is omitted.

Proposition 2.9 *Suppose* $(T(t))_{t\geqslant 0}$ *is a positive semigroup with generator* A. *Each of the following conditions guarantees that the resolvent grows slowly.*

(i) $(T(t))_{t \geqslant 0}$ *is bounded and* $s(A) = 0$*;*
(ii) $s(A)$ *is a first order pole of the resolvent.*

In case $s(A)$ is a pole of order greater than 1, the resolvent does not grow slowly. We will treat this case in Corollary 2.12

Theorem 2.10 *The boundary spectrum of a positive semigroup with slowly growing resolvent is cyclic.*

Proof Without loss of generality we can assume that $s(A) = 0$.

Given $\mathrm{i}\beta \in \sigma(A)$ ($\beta \in \mathbb{R}$), then $\mathrm{i}\beta \in A\sigma(A)$ (A-III,Proposition 2.2) and $(\lambda - \mathrm{i}\beta)^{-1} \in A\sigma(R(\lambda, A))$ (A-III,Proposition 2.5). We consider an $\mathcal{F}$-product $E_{\mathcal{F}}$ of E and for convenience write E_1 instead of $E_{\mathcal{F}}$. The canonical extensions of $R(\lambda, A)$ to E_1 form a positive pseudo-resolvent $\{(R_1(\lambda)\}_{\mathrm{Re}\,\lambda>0}$ on E_1. By Proposition 2.6(i) and A-III,4.5 there exists $h_1 \in E_1$, $h_1 \neq 0$ such that

$$\lambda R_1(\lambda + \mathrm{i}\beta)h_1 = h_1 \textit{ for } \mathrm{Re}\,\lambda > 0\,. \tag{2.16}$$

By (2.13) we have

$$|h_1| = |\tau R_1(\tau + \mathrm{i}\beta)h_1| \leqslant \tau R_1(\tau)|h_1| \ (r > 0)\,. \tag{2.17}$$

Next, we choose any $\varphi \in E_1'$ such that $\langle h_1, \varphi\rangle \neq 0$. Since $\|R_1(\lambda)'\| = \|R_1(\lambda)\| = \|R(\lambda, A)\|$, the assumption of slow growth implies that $\{\lambda R_1(\lambda)'\varphi : \lambda > 0\}$ is bounded in E_1', hence $\sigma(E_1', E_1)$-relatively compact by *Alaoglu's Theorem*. Thus, there exist $\varphi_1 \in \cap_{\varepsilon>0}\{\tau R_1(\tau)'\varphi : 0 < \tau < \varepsilon\}^{-\sigma(E_1', E_1)}$.

Using the resolvent equation (2.8) we obtain for $r > 0$, $\varepsilon > 0$

$$(1 - \tau R_1(\tau)')\varepsilon R_1(\varepsilon)'\varphi = \varepsilon(\tau - \varepsilon)^{-1}(\tau R_1(\tau)'\varphi - \varepsilon R_1(\varepsilon)'\varphi)\,.$$

Since the right hand side tends to 0 as $\varepsilon \to 0$, we have $(1 - \tau R_1(\tau)')\varphi_1 = 0$ or

$$\lambda R_1(\lambda)'\varphi_1 = \varphi_1 \textit{ for } \mathrm{Re}\lambda > 0\,. \tag{2.18}$$

Moreover, from

$$0 < |\langle h_1, \varphi\rangle| \leqslant \langle |h_1|, |\varphi|\rangle \leqslant \langle R_1(\tau)|h_1|, |\varphi|\rangle = \langle |h_1|, \tau R_1(\tau)'\varphi\rangle$$

it follows that

$$\langle |h_1|, \varphi_1\rangle > 0\,. \tag{2.19}$$

for arbitrary $f_1 \in E_1$, $\mathrm{Re}\,\lambda > 0$ we have $\langle |R_1(\lambda)f_1|, \varphi_1\rangle \leqslant \langle R_1(\mathrm{Re}\,\lambda)|f_1|, \varphi_1\rangle = \langle |f_1|, R_1(\mathrm{Re}\,\lambda)'\varphi_1\rangle = (\mathrm{Re}\,\lambda)^{-1}\langle |f_1|, \varphi_1\rangle$.

Therefore the ideal $I := \{f_1 \in E_1 : \langle |f_1|, \varphi_1\rangle = 0\}$ is invariant under $\{(R_1(\lambda)\}_{\mathrm{Re}\,\lambda>0}$. Furthermore we have (see (2.17), (2.18))

$$\begin{aligned}\langle |rR_1(r)|h_1| - |h_1||, \varphi_1\rangle &= \langle rR_1(r)|h_1| - |h_1|, \varphi_1\rangle = \\ &= \langle |h_1|, rR_1(r)'\varphi_1 - \varphi_1\rangle = 0\end{aligned}$$

which implies

$$rR_1(r)|h_1| - |h_1| \in I \text{ for } r > 0)\,. \tag{2.20}$$

Denote by E_2 the quotient space E_1/I and by $\{(R_2(\lambda)\}_{\mathrm{Re}\,\lambda>0}$ the pseudo-resolvent on E_2 induced by $\{(R_1(\lambda)\}_{\mathrm{Re}\,\lambda>0}$ in the canonical way. Then $h_2 := h_1 + I \neq 0$ (by (2.19). Moreover, $\lambda R_2(\lambda + \mathrm{i}\beta)h_2 = h_2$ (by (2.16) and $\lambda R_2(\lambda)|h_2| = |h_2|$ (by (2.20) and Proposition 2.6(i)).

Now we apply Proposition 2.7(b) and obtain

$$\lambda R_2(\lambda + in\beta)h_2^{[n]} = h_2^{[n]} \text{ for } \mathrm{Re}\lambda > 0, n \in \mathbb{Z}\,. \tag{2.21}$$

In particular, we have $\|R_2(r + \mathrm{i}n\beta)\| \geqslant \frac{1}{r}$, thus

$$\|R(r + \mathrm{i}n\beta, A)\| = \|R_1(r + \mathrm{i}n\beta)\| \geqslant \|R_2(r + \mathrm{i}n\beta)\| \geqslant \frac{1}{r} \mathit{for}\; r > 0\,.$$

This finally implies that $\mathrm{i}n\beta \in \sigma(A)$ for $n \in \mathbb{Z}$. □

To prove cyclicity of the boundary spectrum in case $s(A)$ is a pole (of arbitrary order), one applies B-III,Lemma 2.8 to reduce the problem to the case of first order poles. Actually, B-III,Lemma 2.8 is true for arbitrary Banach lattices and the proof given in chapter B-III works in the general case as well. For completeness we recall this result.

Proposition 2.11 *Let A be the generator of a positive semigroup $\mathcal{T}$ on a Banach lattice E and suppose that the spectral bound $s(A)$ is a pole of the resolvent of order k. Then there is a sequence*

$$I_{-1} := \{0\} \subset I_0 \subsetneq I_1 \subsetneq \ldots \subsetneq I_k := E \tag{2.22}$$

of $\mathcal{T}$-invariant closed ideals with the following properties. If A_n is the generator of the semigroup induced by $\mathcal{T}$ on the quotient I_n/I_{n-1}, then we have

(i) $s(A_0) < s(A)$,
(ii) If $n \geqslant 1$, then $s(A_n) = s(A)$ is a first order pole of the resolvent $R(\cdot, A_n)$. The corresponding residue is a strictly positive operator.

Combining Theorem 2.10 and Proposition 2.11 one obtains the following generalization of B-III,Theorem 2.9. In contrast with the former result we do not assume that every point of $\sigma_b(A)$ is a pole.

Corollary 2.12 *If A is the generator of a positive semigroup such that $s(A)$ is a pole of the resolvent, then $\sigma_b(A)$ is cyclic.*

Proof Considering the sequence of ideals as described in Proposition 2.11 and the corresponding generators A_n $(0 \leqslant n \leqslant k)$, then we have by A-III,Proposition 4.2 that $\sigma_b(A) = \cup_{n=1}^{k} \sigma_b(A_n)$.

By Theorem 2.10 each set $\sigma_b(A_n)$ is cyclic, hence so is $\sigma_b(A)$. □

The proof of the preceding corollary indicates a possible generalization of Theorem 2.10. Instead of assuming that the resolvent grows slowly one merely needs that there exist a finite chain of closed $\mathcal{T}$-invariant ideals as described in (2.22) such that the semigroups induced on the corresponding quotient spaces have slowly growing resolvents.

In case that $\sigma_b(A)$ is cyclic one has the alternative (cf. B-III, (2.19))

Either $\sigma_b(A) = \{s(A)\}$ or else $\sigma_b(A)$ is an unbounded set.

Obviously one can use this fact to prove the existence of a dominant spectral value (cf. the explanation preceding B-III,Corollary 2.11). We give a typical example.

Corollary 2.13 *Let A be the generator of a positive, eventually norm-continuous semigroup. Suppose either that the resolvent grows slowly or that $s(A)$ is a pole of the resolvent. Then $s(A)$ is a dominant spectral value.*

Proof The boundary spectrum $\sigma_b(A)$ is cyclic (Theorem 2.10 and Corollary 2.12 resp.) and compact (A-II,Theorem 1.20). Hence $\sigma_b(A) = \{s(A)\}$. □

A situation in which Corollary 2.13 can be applied is described in the following example. For more details and proofs we refer to **?**].

Example 2.14 Let Ω be a bounded domain in $\mathbb{R}^n$ of class C^2.

Assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where Γ_0 and Γ_1 are disjoint open and closed subsets of $\partial\Omega$. On $E = L^p(\Omega)$ $(1 \leqslant p < \infty)$ we consider a differential operator $L_{p,o}$ which is defined as follows

$$\begin{aligned} L_{p,0} f &:= \sum_{i,j=1}^{n} a_{ij} f''_{ij} + \sum_{i=1}^{n} b_i f'_i + c f \text{ , with domain} \\ D(L_{p,0}) &:= \{ f \in C^2(\bar{\Omega}) : \ f(x) = 0 \text{ for } x \in \Gamma_0 \\ &\qquad \text{and } \partial f / \partial \nu(x) + \gamma(x) f(x) = 0 \text{ for } x \in \Gamma_1 \} \end{aligned} \tag{2.23}$$

Here f'_i stands for $\partial f/\partial x_i$, thus $f''_{ij} = \partial^2 f/\partial x_i \partial x_j$. We assume that $L_{p,0}$ is elliptic and that all coefficients are real-valued satisfying $a_{ij} \in C^2(\bar{\Omega})$, $b_i \in C^1(\bar{\Omega})$, $\gamma \in C^1(\bar{\Omega})$, $c \in C^1(\bar{\Omega})$.

Then $L_{p,0}$ is closable and its closure L_p is the generator of a holomorphic semigroup of positive operators. Moreover, the resolvent is compact. Thus Corollary 2.13 is applicable and one obtains that $s(A)$ is strictly dominant provided that $\sigma(A) \neq \emptyset$. Using the results of Section 3 one can show that $\sigma(A) \neq \emptyset$ and that $s(A)$ is an algebraically simple eigenvalue (see Theorem 3.7 and Proposition 3.5).

We conclude with some remarks.

Remarks 2.15 (a) In the proof of Theorem 2.10 we did not use the assumption that R is the resolvent of a semigroup. In fact one can state this theorem for closed operators having positive resolvent. In this case one has to assume that $\{(\lambda - s(A))R(\lambda, A) : s(A) < \lambda < s(A) + 1\}$ is bounded in $\mathcal{L}(E)$.

One can go even further and consider positive pseudo-resolvents $\{R(\lambda)\}_{\lambda \in D}$. This is also possible with respect to Corollary 2.12.

(b) If $s(A)$ is a pole, then the criteria stated in B-III,Remark 2.15(a) for $s(A)$ to be a first order pole are valid in the setting of arbitrary Banach lattices as well. In particular, one has a first order pole provided that $\ker(s(A) - A)$ contains a quasi-interior point or in case that $\ker(s(A) - A')$ contains a strictly positive linear form.

(c) It is not difficult to give examples of semigroups whose resolvent do not grow slowly or cannot be reduced by a finite chain of invariant ideals as described after Corollary 2.12. E.g., one can take a bounded positive operator A which is not nilpotent and satisfies $\sigma(A) = \{0\}$. However, the following question is still unanswered.

Does every positive semigroup have cyclic boundary spectrum?

3 Irreducible Semigroups

The concept of irreducibility is very natural in the general setting of Banach lattices. However, some of the (equivalent) assertions stated in B-III,Definition 3.1 do not make sense here, others need a slightly different formulation.

Definition 3.1 A positive semigroup $(T(t))_{t \geqslant 0}$ on a Banach lattice E with generator A is called *irreducible* if one of the following (mutually equivalent) conditions is satisfied

(a) There is no $(T(t))$-invariant closed ideal except $\{0\}$ and E.

(b) Given $f \in E$, $\varphi \in E'$ such that $f > 0$, $\varphi > 0$ then $\langle T(t_0)f, \varphi \rangle > 0$ for some $t_0 \geqslant 0$.

(c) For arbitrary $f, g \in E_+$, $f > 0$, $g > 0$ there exists t_0 such that $\inf\{T(t_0)f, g\} > 0$.

(d) For some (every) $\lambda > s(A)$ there is no closed ideal other than $\{0\}$ or E which is invariant under $R(\lambda, A)$.

(e) For some (every) $\lambda > s(A)$ we have $R(\lambda, A)f$ is a quasi-interior point of E_+ whenever $f > 0$.

Equivalence of the five conditions above is obtained by a slight modification of the arguments given in B-III,Definition 3.1. Since there are no difficulties we omit a detailed proof. Obviously, a semigroup is irreducible if one single operator $T(t_0)$ is irreducible. In general the converse does not hold (see p.65 in **?**]). The situation is different when holomorphic semigroups are considered. Then an even stronger assertion holds. In fact irreducibility of a holomorphic semigroup implies that every single operator maps the positive elements onto quasi-interior points. This is the second statement of the following theorem (see also B-III,Remark 3.2).

Theorem 3.2 *(i) If $(T(t))$ is an irreducible semigroup then every operator $T(t)$ is strictly positive, i.e., given $f > 0$, $t \geqslant 0$, then $T(t)f > 0$.*

(ii) Suppose $(T(t))_{t\geqslant 0}$ *is a holomorphic positive semigroup. If* $(T(t))$ *is irreducible, then* $T(t)f$ *is a quasi-interior point of* E_+ *whenever* $f > 0$ *and* $t > 0$*. Equivalently, given* $f \in E$, $\varphi \in E'$ *such that* $f > 0$, $\varphi > 0$*, then* $\langle T(t)f, \varphi\rangle > 0$ *for all* $t > 0$.

Proof (i) Given $t > 0$ and $f > 0$ such that $T(t)f = 0$ and $\lambda > s(A)$, then we have $T(t)(R(\lambda, A)f) = R(\lambda, A)T(t)f = 0$. Since $R(\lambda, A)f$ is a quasi-interior point, it follows that $T(t) = 0$. Thus for fixed $t \in \mathbb{R}_+$ we have either $T(t)$ is strictly positive or else $T(t) = 0$. Then strong continuity and $T(0) = \mathrm{Id} \neq 0$ implies that there exists $\tau > 0$ such that $T(t)$ is strictly positive for $0 \leqslant t \leqslant \tau$. For arbitrary $t \in \mathbb{R}_+$ we find $n \in \mathbb{N}$ such that $\frac{t}{n} \leqslant \tau$. Then $T(t) = T(\frac{t}{n})^n$ is also strictly positive.

(ii) We prove that for an arbitrary holomorphic positive semigroup $(T(t))_{t\geqslant 0}$ the following holds

Given $f > 0$, $\varphi > 0$ then either $\langle T(t)f, \varphi\rangle = 0$ for all $t \geqslant 0$ or $\langle T(t)f, \varphi\rangle > 0$ for all $t > 0$.

Then it follows from Definition 3.1(b) that for irreducible semigroups always the second case occurs.

Assume that $\langle T(t_0)f, \varphi\rangle = 0$ for some $t_0 > 0$.

We consider a null sequence (t_n), $0 < t_n < t_0$, such that $\|T(t_n)f - f\| \leqslant 2^{-n}$ and define $f_n := T(t_n)f$, $g_n := f - \sum_{k=n}^{\infty}(f - f_k)^+$.

Then for $g_n \leqslant f$, $f = \lim_{n\to\infty} g_n$ and $m \geqslant n$, we have

$g_n \leqslant f - (f - f_m)^+ = \inf\{f, f_m\} \leqslant f_m$.

For $n \in \mathbb{N}$ fixed and $m \geqslant n$ we obtain

$0 \leqslant \langle T(t_0 - t_m)g_n^+, \varphi\rangle \leqslant \langle T(t_0 - t_m)f_m, \varphi\rangle = \langle T(t_0)f, \varphi\rangle = 0.$

Thus the function $t \mapsto \langle T(t)g_n^+, \varphi\rangle$ is identically zero by the uniqueness theorem for analytic functions. Since $f = \lim_{n\to\infty} g_n^+$, we have $\langle T(t)h, \varphi\rangle = 0$ for all $t \in \mathbb{R}_+$. □

The next result can be used to check irreducibility for a semigroup whose generator is a bounded perturbation of a known semigroup. It is a generalization (and an extension to Banach lattices) of B-III, Proposition 3.3.

Proposition 3.3 *Suppose that* A *is the generator of a positive semigroup* $\mathcal{T}$*, further assume that* K *is a bounded positive operator and* M *is a bounded real multiplier (cf. C-I,Section 8). Let* $\mathcal{S}$ *be the semigroup generated by* $B := A + K + M$.

For a closed ideal $I \subset E$ *the following assertions are equivalent.*

(a) I *is* $\mathcal{S}$*-invariant.*
(b) I *is invariant both under* $\mathcal{T}$ *and* K.

Proof We recall that a closed subspace $I \subset E$ is invariant for a semigroup generated by C if and only if $C(D(C) \cap I) \subset I$ and the restriction $C_{|I}$ of C with domain $D_{|I} := D(C) \cap I$ generates a semigroup on I (see A-I,3.2). Now let I be a closed ideal of E. (b)$\Rightarrow$(a). If I is $\mathcal{T}$-invariant then $A_{|I}$ generates a semigroup on I. The restrictions $K_{|I}$ nd $M_{|I}$ of K and M respectively are bounded linear operators on I. Note that each closed ideal is invariant for M, cf. C-I,Section 8. Thus $B_{|I} = A_{|I} + M_{|I} + K_{|I}$ with domain $D(A_{|I}) = D(A) \cap I = D(B) \cap I$ is the generator of a semigroup on I. It follows that I is invariant for the semigroup generated by B.

(a)⇒(b). Without loss of generality we assume $M \geqslant 0$. Then we have $0 \leqslant T(t) \leqslant S(t)$ for all $t \geqslant 0$. It follows that I is T-invariant. Thus for $x \in D(A) \cap I = D(B) \cap I$, we have $Kx = Bx - Ax - Mx$. This shows that $K(D(B) \cap I) \subset I$. Since $B|_I$ is a generator $D(B) \cap I$ is dense in I. Then, by continuity, we have $KI \subset I$, i.e., I is K-invariant. □

Next we consider some concrete examples.

Examples 3.4 (a) Suppose that on $E = L^p(\mu)$ $(1 \leqslant p < \infty)$ the semigroup $(T(t))$ is given by

$$(T(t)f)(x) = \int_X k(t,x,y)f(y)\,\mathrm{d}\mu(y) \quad (x \in X,\ t > 0) \tag{3.1}$$

for some measurable function $k\colon \mathbb{R}_+ \times X \times X \to \mathbb{R}_+$. Then $(T(t))$ is irreducible if and only if for any two measurable sets M and N with $0 < \mu(M) < \infty$, $0 < \mu(N) < \infty$, $\mu(M \cap N) = 0$ there exist $t_0 > 0$ such that $\int_M \int_N k(t_0,x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) > 0$.

(b) Consider the first derivative on $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{R}_{2\pi} = \Gamma$ as operator on the corresponding L^p-space (with respect to the Lebesgue measure.) Then the statements made in B-III,Example 3.4(c) are true. The same is true for B-III,Example 3.5(e) and (f) (second order differential operator) when the corresponding L^p-spaces are considered. B-III,Ex3.5(e) checken

(c) Let $E = L^1[-1,0]$ and for $g \in L^\infty$ consider the operator A_g given by

$$A_g f := f', \quad D(A_g) := \{f \in W^1[-1,0]\colon f(0) = \int_{-1}^0 f(x)g(x)\,\mathrm{d}x\} \tag{3.2}$$

If $g \geqslant 0$ then A_g generates a positive semigroup. In case there exist $\varepsilon > 0$ such that g vanishes a.e. on $[-1,-1+\varepsilon]$, then $I := \{f \in L^1\colon f|_{[-1+\varepsilon,0]} = 0\}$ is a non-trivial closed ideal which is invariant under the semigroup. It is not difficult to see that the condition on g stated above is also necessary for $(T(t))$ to be reducible (i.e., not irreducible.)

(d) Let $E = L^1([0,1] \times [-1,1])$ and consider the semigroup $(T(t))_{t\geqslant 0}$ defined as follows

$$(T(t)f)(x,v) := \begin{cases} f(x-vt,v) & \text{for } 0 \leqslant x - vt \leqslant 1\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

$(T(t))_{t\geqslant 0}$ is a positive semigroup on E and

$$D_0 := \{f \in C^1([0,1] \times [-1,1])\colon f(0,v) = f_x(0,v) = 0 \text{ if } v \geqslant 0\,, \\ f(1,v) = f_x(1,v) = 0 \text{ if } v \leqslant 0\}$$

is a core for its generator A (cf. A-I,Corollary 1.34). We have

$$(Af)(x,v) = -v\frac{\partial f}{\partial x}(x,v) \quad (f \in D_0). \tag{3.4}$$

The Laplace transform of $(T(t))$ is the resolvent of A. An explicit calculation yields

$$(R(\lambda, A)f)(x, v) = \int_0^1 r_\lambda(x, x', v) f(x', v)\, \mathrm{d}x' \quad (\lambda > 0) \tag{3.5}$$

where $r_\lambda\colon [0,1] \times [0,1] \times [-1,1] \to \mathbb{R}$ is given by

$$r_\lambda(x, x', v) = \begin{cases} |v|^{-1} \exp(-\lambda(x - x')v^{-1}) & \begin{cases}\text{if either } v > 0 \text{ and } x' \leqslant x \\ \text{or } v < 0 \text{ and } x' \geqslant x, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\sigma\colon [0,1] \times [-1,1] \to \mathbb{R}_+$ and $\kappa\colon [0,1] \times [-1,1] \times [-1,1] \to \mathbb{R}_+$ be measurable functions and consider the operators M and K given by

$$Mf := \sigma f, \quad Kf := \int_{-1}^1 \kappa(\cdot, \cdot, v') f(\cdot, v')\, dv'. \tag{3.6}$$

Then $B := A - M + K$ with domain $D(B) := D(A)$ is the generator of a positive semigroup.

Using Proposition 3.3 we can prove the following irreducibility criterion for the semigroup $(S(t))_{t \geqslant 0}$ generated by B.

$$\text{If } \kappa \text{ is strictly positive, then } (S(t))_{t\geqslant 0} \text{ is irreducible.} \tag{3.7}$$

Actually, in view of Proposition 3.3, we have to show that a closed ideal which is invariant under $R(\lambda, A)$ and K has to be $\{0\}$ or E.

We recall that closed ideals of E are uniquely determined (up to sets of measure zero) by measurable subsets Y of $[0,1] \times [-1,1]$, i.e., every closed ideal has the form

$$I_Y = \{f \in E\colon f \text{ vanishes (a.e.) on } [0,1] \times [-1,1] \setminus Y\}.$$

Since we assumed that κ is strictly positive, I_Y is K-invariant if and only if $Y = X \times [-1,1]$ for some measurable set $X \subset [0,1]$. If we assume that X has positive measure and define

$$\alpha := \sup\{x \in [0,1]\colon \int_0^x \mathbb{1}_X(s)\, \mathrm{d}s = 0\}, \beta := \inf\{x \in [0,1]\colon \int_x^1 \mathbb{1}_X(s)\, \mathrm{d}s = 0\},$$

then we have $\alpha < \beta$ and the support of the function $h := R(\lambda, A)\mathbb{1}_Y$ ($Y := X \times [-1,1]$) is given by $\operatorname{supp} h = [\alpha, 1] \times [0,1] \cup [0, \beta] \times [-1, 0]$. Since we assumed that I_Y is $R(\lambda, A)$-invariant, we have $h \in I_Y$, i.e., $\operatorname{supp} h \subset Y = X \times [-1,1]$. Obviously, this is true only if $Y = [0,1] \times [-1,1]$ or $I_Y = E$.

A weaker condition than (3.7) entailing irreducibility is the following.

$$\begin{aligned}&\text{There exists } \delta > 0 \text{ such that } \kappa \text{ is strictly positive} \\ &\text{on the sets } [0,\delta] \times [-1,1] \text{ and } [1-\delta, 1] \times [-1,1]\,.\end{aligned} \tag{3.8}$$

For details we refer to **?**] . [Greiner (1984d)]

In the following proposition we list some properties which are consequences of irreducibility. This extends B-III,Proposition 3.5 to the setting of Banach lattices. The first assertion of the latter proposition is no longer true in the general setting (see Example 3.6 and Theorem 3.7).

Proposition 3.5 *Suppose A to be the generator of an irreducible, positive semigroup on a Banach lattice E. Then the following assertions are true.*

(i) Every positive eigenvector of A is a quasi-interior point.
(ii) Every positive eigenvector of A' is strictly positive.
(iii) $\ker(s(A) - A')$ *contains a positive element, then* $\dim(\ker(s(A) - A)) \leqslant 1$.
(iv) If $s(A)$ is a pole of the resolvent, then it has algebraic (and geometric) multiplicity 1. *The corresponding residue has the form $P = \varphi \otimes u$, where $\varphi \in E'$ is a positive eigenvector of A', $u \in E$ is a positive eigenvector of A and $\langle u, \varphi \rangle = 1$.*

Proof To prove (i), (ii) and (iv) one can proceed as in the case $C_0(X)$ (see B-III,Proposition 3.5). We only prove (iii) and assume $s(A) = 0$. By assumption and by assertion (i) there exists $\varphi \gg 0$ such that $T(t)'\varphi = \varphi$ $(t \geqslant 0)$. Given $f \in \ker A$, then $T(t)f = f$ hence $|f| = |T(t)f| \leqslant T(t)|f|$. Since φ is strictly positive and $\langle |f|, \varphi \rangle \leqslant \langle T(t)|f|, \varphi \rangle = \langle |f|, \varphi \rangle$, it follows that $|f| = T(t)|f|$. We have shown that $\ker A$ is a sublattice. Then for $f \in \ker A$, f real, i.e., $f = \overline{f}$, we have that f^+ and f^- are elements of $\ker A$. Hence the principal ideals generated by f^+ and f^- are T-invariant. Since these ideals are orthogonal, the irreducibility of T implies that either f^+ or f^- is zero. We have shown that $\ker A \cap E_{\mathbb{R}}$ is totally ordered, hence at most one-dimensional (see Proposition 3.4 of **?**]). □

In arbitrary Banach lattices it is no longer true that an irreducible semigroup has necessarily nonvoid spectrum. We indicate how an irreducible semigroup having empty spectrum can be constructed.

Example 3.6 Consider the Banach lattice $E = L^p(\Gamma \times \Gamma)$. For (fixed) positive numbers α, β such that $\frac{\alpha}{\beta}$ is irrational we define a positive semigroup $(T_0(t))_{t \geqslant 0}$ as follows

$$(T_0(t)f)(z, w) \coloneqq f(z \cdot \mathrm{e}^{\mathrm{i}\alpha t}, w \cdot \mathrm{e}^{\mathrm{i}\beta t}) \quad (z, w \in \Gamma = \{\xi \in \mathbb{C} : |\xi| = 1\}) \,. \tag{3.9}$$

Next we define for a positive function $m\colon \Gamma \times \Gamma \to \mathbb{R}$ which is continuous on $\Gamma \times \Gamma \setminus (1, 1)$ functions m_t, $t \geqslant 0$, as follows

$$m_t(z, w) \coloneqq \exp(-\int_0^t m(z \cdot \mathrm{e}^{\mathrm{i}\alpha s}, w \cdot \mathrm{e}^{\mathrm{i}\beta s}) \,\mathrm{d}s) \,. \tag{3.10}$$

Then $(T(t))_{t \geqslant 0}$ defined by

$$T(t)f \coloneqq m_t \cdot (T_0(t)f) \tag{3.11}$$

is a strongly continuous semigroup of positive contractions on E. Since $\frac{\alpha}{\beta}$ is irrational, the semigroup $(T_0(t))$ is irreducible. Moreover, each m_t is strictly positive (i.e., $m_t > 0$ a.e.) thus $(T(t))$ is irreducible as well. If one chooses m such that $m(z, w)$ tends to $+\infty$ sufficiently fast as $(z, w) \to (1, 1)$, one obtains $\|T(t)\| = \|m_t\|_\infty \leqslant \exp(-t^2)$ for all $t \geqslant 0$. Obviously such an estimate of $\|T(t)\|$ implies $\omega_0(A) = -\infty$, hence $\sigma(A) = \emptyset$.

Theorem 3.7 *Suppose that $(T(t))_{t\geqslant 0}$ is an irreducible, positive semigroup on the Banach lattice E. Each of the following conditions on E and $(T(t))$, respectively, implies that $\sigma(A) \neq \emptyset$*

(i) $E = C_0(X)$ where X is locally compact.
(ii) $E = \ell^p$ $(1 \leqslant p < \infty)$ (more generally, E contains atoms).
(iii) either $T(t_0)$ is compact for some t_0 or $R(\lambda_0, A)$ is compact for some $\lambda_0 \in \varrho(A)$.
(iv) E has order continuous norm and either $T(t_0)$ or $R(\lambda_0, A)$ is a kernel operator for some $t_0 \geqslant 0$ $(\lambda_0 \in \varrho(A))$. [1] .

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(v) E is reflexive and there exist $t_0 > 0$, $h \in E_+$ such that $T(t_0)E \subset E_h$.

Proof (i) is proved in B-III,Proposition 3.5(a).

Assertion (ii)-(f) will be proved utilizing the analoguous results for a single operator. In view of A-III,Proposition 2.5 we have to show that $r(R(\lambda, A)) > 0$ for some $\lambda \in \varrho(A)$. Moreover, from A-I,(3.1) we deduce

$$T(t)R(\lambda, A) = e^{\lambda t}R(\lambda, A) - e^{\lambda t}\int_0^t e^{-\lambda s}T(s)\,ds \leqslant e^{\lambda t}R(\lambda, A)\ (t \geqslant 0, \lambda > s(A))\,.$$

Since the spectral radius is an isotone function on the cone of positive operators, it is enough to show that

$$r(T(t)R(\lambda, A)) > 0 \text{ for some } t \geqslant 0,\ \lambda > s(A). \tag{3.12}$$

Using Theorem 3.2(i) it is easy to see that $T(t)R(\lambda, A) = R(\lambda, A)T(t)$ is irreducible.

The assertions (ii),(iv) and (v) now follow from the corresponding results for a single operator as presented in Sect.V.6 of **?**] (see Proposition 6.1, Theorem 6.5 Corollary and Theorem 6.5 l.c.). (iii) follows from the recent result of **?**] which ensures that every positive operator on a Banach lattice which is compact and irreducible has positive spectral radius. □

The theorem can be used to prove that elliptic operators as described in Example 2.14 have non-empty spectrum. It is shown in **?**] that these operators have compact resolvent and generate irreducible semigroups. Thus the assumption of (iii) is satisfied.

Concerning the eigenvalues of an irreducible semigroup which are contained in $\sigma_b(A)$ all statements established for spaces $C_0(X)$ in B-III,Theorem 3.6 are true in

[1] For a precise definition of a kernel operator we refer to Section IV.9 of **?**] or Chapter 13 of **?**].

the setting of Banach lattices. The proof can be translated without difficulties and will be omitted (see also **?** , Theorem 2.6]).

Theorem 3.8 *Suppose $\mathcal{T}$ is an irreducible semigroup on the Banach lattice E and let A be its generator. Assume that $s(A) = 0$ and that there exists a positive linear form $\psi \in D(A')$ with $A'\psi \leqslant 0$. If $P\sigma(A) \cap \mathrm{i}\mathbb{R}$ is non-empty, then the following assertions are true.*

(i) $P\sigma(A) \cap \mathrm{i}\mathbb{R}$ is a (additive) subgroup of $\mathrm{i}\mathbb{R}$.

(ii) The eigenspaces corresponding to $\lambda \in P\sigma(A) \cap \mathrm{i}\mathbb{R}$ are one-dimensional.

(iii) If $Ah = \mathrm{i}\alpha h$ ($h \neq 0$, $\alpha \in \mathbb{R}$) then $|h|$ is a quasi-interior point and the following holds

$$S_h(D(A)) = D(A) \text{ and } S_h^{-1} \circ A \circ S_h = (A + \mathrm{i}\alpha\,\mathrm{Id}). \tag{3.13}$$

(iv) 0 is the only eigenvalue of A admitting a positive eigenvector.

One can apply the theorem in order to prove that the rotation semigroup on Γ (cf. A-I,2.5) is the only positive periodic semigroup which is irreducible.

Corollary 3.9 *Let $(T(t))_{t\geqslant 0}$ be a positive irreducible semigroup on a Banach lattice E which is periodic of period τ. Assume that $\dim E > 1$. Then there exist continuous lattice homomorphisms $i\colon C(\Gamma) \to E$ and $j\colon E \to L^1(\Gamma)$, both injective with dense range, such that the diagram commutes for all $t \geqslant 0$. Hereby, $j \circ i$ is the canonical inclusion of $C(\Gamma)$ in $L^1(\Gamma)$.*

$$\begin{tikzcd}
C(\Gamma) \arrow[r, "i"] \arrow[d, "R_\tau(t)"'] & E \arrow[r, "j"] \arrow[d, "T(t)"'] & L^1(\Gamma) \arrow[d, "R_\tau(t)"] \\
C_(\Gamma) \arrow[r, "i"'] & E \arrow[r, "j"'] & L^1(\Gamma)
\end{tikzcd}$$

Proof By Theorem 3.8 and A-III,Theorem 5.4 we have $R\sigma(A) = P\sigma(A) = \sigma(A) = \mathrm{i}\alpha\mathbb{Z}$ with $\alpha := \frac{2\pi}{n\tau}$ for suitable $n \in \mathbb{N}$. We fix $h \in \ker(\mathrm{i}\alpha - A)$, $h \neq 0$. Then $|h| \in \ker A$ and there exists $\varphi \in \ker A'$ such that $\langle |h|, \varphi\rangle = 1$. According to the Kakutani-Krein Theorem we identify $E_{|h|}$ with $C(K)$. Then h is a unimodular function onto Γ (use the argument given in the proof of B-III,Theorem 3.6(c)).

We define $i\colon C(\Gamma) \to E$ by $i(f) := f \circ h \in C(K) \cong E_h \subset E$, then i is injective. For the monomials $e_n(z) := z^n$ ($n \in \mathbb{Z}$) we have $i(e_n) = h^{[n]}$ thus i has dense image in E (by A-III,Theorem 5.4). Moreover, GG: Was ist δ_{n0}

$$2\pi \cdot \delta_{n0} = \langle h^{[n]}, \varphi\rangle = \langle i(e_n), \varphi\rangle = \int_0^{2\pi} e_n(\mathrm{e}^{it})\,\mathrm{d}t \text{ for all } n \in \mathbb{Z},$$

hence $\int_0^{2\pi} f(\mathrm{e}^{it})\,\mathrm{d}t = \langle i(f), \varphi\rangle$ for all $f \in C(\Gamma)$. It follows that $(E, \varphi) \cong L^1(\Gamma)$, and we define j to be the canonical mapping from E into $(E, \varphi) \cong L^1(\Gamma)$ (see C-I,Section 4). Then j has dense image and is injective since φ is strictly positive (cf. Proposition 3.5(ii)). One easily verifies that the diagramm commutes. □

Now we are going to prove the main result of this section. As in the proof of Theorem 2.10 we will utilize pseudo-resolvents on a suitable $\mathcal{F}$-product of the Banach lattice. To simplify the proof we isolate two lemmas.

Lemma 3.10 *Let $\mathcal{F}$ be a filter on $\mathbb{N}$ which is finer than the Frechet filter and let $E_{\mathcal{F}}$ be the $\mathcal{F}$-product of the Banach lattice E. Given $R \in \mathcal{L}(E)$ and denoting its canonical extension to $E_{\mathcal{F}}$ by $R_{\mathcal{F}}$ the following is true.*

If $\alpha \in A\sigma(R) \setminus P\sigma(R)$, then $\ker(\alpha - R_{\mathcal{F}})$ is infinite dimensional.

Proof Let $(f_n)_{n\geqslant 1}$ be a normalized approximate eigenvector of R corresponding to α. Since every accumulation point of (f_n) is an eigenvector of R, the assumption $\alpha \notin P\sigma(A)$ implies that (f_n) does not have any accumulation points. Then there exist an $\varepsilon > 0$ and a subsequence (g_n) of (f_n) such that

$$\|g_n - g_m\| \geqslant \varepsilon \text{ whenever } n \neq m. \tag{3.14}$$

Obviously, (g_n) is a normalized approximate eigenvector of R and so is every subsequence of (g_n). In particular, for $k \in \mathbb{N}$ the sequence $(g_{n+k})_{n\geqslant 1}$ is a normalized approximate eigenvector of R. Then the elements $\hat{g}^k \in E_{\mathcal{F}}$ given by $\hat{g}^k := ((g_{n+k})_{n\geqslant 1} + c_{\mathcal{F}}(E))$ are normalized eigenvectors of $R_{\mathcal{F}}$ corresponding to α. As a consequence of (3.14) we obtain $\|\hat{g}^k - \hat{g}^m\| = \mathcal{F}\text{-}\limsup \|g_{n+k} - g_{n+m}\| \geqslant \varepsilon$ provided that $k \neq m$. This shows that the unit ball in $\ker(\alpha - R_{\mathcal{F}})$ is not relatively compact, hence $\ker(\alpha - R_{\mathcal{F}})$ has to be infinite dimensional. □

Lemma 3.11 *Let E be a Banach lattice and let M, L be two linear subspaces of E. Assume that $f \in M$ implies $|f| \in L$, then $\dim L \geqslant \dim M$.*

Proof Let $\{g_1, g_2, \dots, g_m\}$ $(m \geqslant 1)$ be any (finite) subset of M which is linearly independent. For $u := \sum_{n=1}^{m} |g_n|$ all vectors g_n are contained in the principal ideal E_u which (by the Kakutani-Krein Theorem) is isomorphic to a space $C(K)$. Considering $g_1, g_2, \dots, g_m$ as continuous functions on K, there exist points $x_1, x_2, \dots, x_m \in K$ and functions $h_1, h_2, \dots, h_m \in \operatorname{span}\{g_1, g_2, \dots, g_m\}$ such that $h_i(x_j) = \delta_{ij}$. Then $|h_i|(x_j) = \delta_{ij}$ hence $\{|h_j| : 1 \leqslant j \leqslant m\}$ is a subset of m linearly independent vectors which (by assumption) is contained in L. □

The surprising fact in the following theorem is the conclusion that every point in the boundary spectrum is a simple algebraic pole if only $s(A)$ is supposed to be a pole.

Theorem 3.12 *Let T be an irreducible semigroup on a Banach lattice and let A be its generator. If $s(A)$ is a pole of the resolvent, then there exists $\alpha \geqslant 0$ such that $\sigma_b(A) = s(A) + \mathrm{i}\alpha\mathbb{Z}$. Moreover, $\sigma_b(A)$ contains only algebraically simple poles.*

Proof We will assume that $s(A) = 0$. Assuming first that every element of $\sigma_b(A)$ is an eigenvalue of A, one can conclude the following. By Theorem 3.8(i) we know that $\sigma_b(A)$ is an additive subgroup of $\mathrm{i}\mathbb{R}$. Since it is a closed subset and 0 is an isolated point, it follows that $\sigma_b(A) = \mathrm{i}\alpha\mathbb{Z}$ for some $\alpha \geqslant 0$. Moreover, as a consequence of (3.13), for every $k \in \mathbb{Z}$ we obtain

$$R(\lambda + \mathrm{i}k\alpha, A) = S_h^{-k} \circ R(\lambda, A) \circ S_h^k \quad (\lambda \in \varrho(A),\ k \in \mathbb{Z}). \tag{3.15}$$

By Proposition 3.5(iv) 0 is an algebraically simple pole. Then (3.15) implies that every point $\mathrm{i}k\alpha$ has the same property.

We now show that every element $\mathrm{i}\beta$ is an eigenvalue of A. By Proposition 3.5(iv) the residue of $R(\cdot, A)$ in $\lambda = 0$ has the form $P = \varphi \otimes u$ with $\varphi(u) = 1$. Given an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ which is finer than the Frechet filter, then $\lim_{\mathcal{U}} \varphi(f_n)$ exists for every bounded sequence $(f_n) \subset E$. Using this fact it is easy to see that the canonical extension $P_{\mathcal{U}}$ of P to the $\mathcal{U}$-product $E_{\mathcal{U}}$ of E has the following form.

$$\begin{aligned} &P_{\mathcal{U}} = \hat{\varphi} \otimes \hat{u} \quad \textit{where } \hat{u} := (u, u, u, \ldots) + c_{\mathcal{U}}(E) \in E_{\mathcal{U}} \textit{ and } \hat{\varphi} \in (E_{\mathcal{U}})' \\ &\textit{is given by } \hat{\varphi}((f_n) + c_{\mathcal{U}}(E)) := \lim_{\mathcal{U}} \varphi(f_n)\ ((f_n) + c_{\mathcal{U}}(E) \in E_{\mathcal{U}})\,. \end{aligned} \tag{3.16}$$

Given $\mathrm{i}\beta \in \sigma_b(A)$ then $\mathrm{i}\beta \in A\sigma(A)$ hence $1 \in A\sigma(\lambda R(\lambda + \mathrm{i}\beta, A))$. Assuming $\mathrm{i}\beta \notin P\sigma(A)$, then $1 \notin P\sigma(\lambda R(\lambda + \mathrm{i}\beta, A))$. Then Lemma 3.10 implies that $M := \ker(1 - \lambda R(\lambda + \mathrm{i}\beta, A)_{\mathcal{U}})$ is infinite dimensional (and independent of λ by Proposition 2.6(i).) For $\hat{f} \in M$ we have $|\hat{f}| = \gamma R(\gamma + \mathrm{i}\beta, A)_{\mathcal{U}}|\hat{f}| \leqslant \gamma R(\gamma, A)_{\mathcal{U}}|\hat{f}|$ for every $\gamma > 0$. It follows that $\hat{\varphi}(|\hat{f}|) = P_{\mathcal{U}}|\hat{f}| = \lim_{\gamma\to 0} \gamma R(\gamma, A)_{\mathcal{U}}|\hat{f}| \geqslant |\hat{f}|$. Thus considering the closed ideal $I := \{\hat{f} \in E_{\mathcal{U}} \colon \hat{\varphi}(|\hat{f}|) = 0\}$ we have

$$\tilde{\varphi}(|\hat{f}|) - |\hat{f}| \in I \textit{ for every } \hat{f} \in M\,. \tag{3.17}$$

This implies that $M \cap I = \{0\}$. Hence the canonical image $M_{/}$ of M in the quotient space $E_{\mathcal{U}/I}$ is infinite dimensional as well. By (3.16) and (3.17) the absolute value of an element $\tilde{f} \in M_{/}$ is a scalar multiple of $\tilde{u} := \hat{u} + I$. This is a contradiction by Lemma 3.11. □

In view of A-III,Proposition 4.2 the result above has consequences for semigroups which can be reduced (by considering restrictions to invariant ideals or quotients) to semigroups which satisfy the hypothesis of Theorem 3.12. Semigroups having this property are precisely those for which $s(A)$ is a pole of the resolvent of finite algebraic multiplicity. The latter claim is a consequence of Proposition 2.11 and the following lemma.

Lemma 3.13 *Suppose that $\mathcal{T} = (T(t))_{t\geq 0}$ is a positive semigroup such that $s(A)$ is a first order pole of the resolvent. Moreover assume that the corresponding residue is a strictly positive operator of finite rank.*

Then there are closed $\mathcal{T}$-invariant ideals $J_1, J_2, ..J_m$ which are mutually orthogonal. For the restrictions $\mathcal{T}_k$ of $\mathcal{T}$ to J_k the following is true.

(i) $\mathcal{T}_k$ is irreducible with spectral bound $s(A_k) = s(A)$.
(ii) $s(A_{/J}) < s(A)$ where $J := J_1 \oplus J_2 \oplus .. \oplus J_m$.

Proof We assume $s(A) = 0$. Since P is a strictly positive projection $PE = \ker A$ is a sublattice of E. Actually, if $x \in PE$ i.e., $Px = x$, then $P|x| \geqslant |Px| = |x|$. Hence $P(|P|x| - |x||) = P^2|x| - P|x| = 0$ which implies that $P|x| - |x| = 0$ or $|x| \in PE$. Thus we know that ker A is a finite dimensional sublattice of E hence it is isomorphic to a space $\mathbb{C}^m$ for some $m \in \mathbb{N}$ (see Section II.4 of **?**]). Lit. is missing

Then there exist vectors $e_j \in E_+$ $(1 \leqslant j \leqslant m)$ such that the following holds.

$$\ker A = \operatorname{span}\{e_1, e_2, .., e_m\} \text{ and } \inf\{e_i, e_j\} = 0 \textit{ for } i \neq j\,. \tag{3.18}$$

[Schaefer (1974),III.Theorem 1.2]

We have $T(t)e_k = e_k$ hence the closed ideal generated by e_k is $\mathcal{T}$-invariant. We denote this ideal J_k and define $J := J_1 \oplus J_2 \oplus .. \oplus J_m$. The ideal J is closed (see **?** , III Theorem 1.2], $\mathcal{T}$-invariant and we have $PE = \ker A \subset J$. Then $P_{/J} = 0$ hence the spectral bound $s(A_{/J})$ is strictly less than zero (by Theorem 1.1(i)). Moreover, the residue corresponding to the resolvent of $\mathcal{T}_k$, we denote it P_k, is the restriction of P to J_k. Then P_k is strictly positive and $P(J_k) = \operatorname{span}\{e_k\}$. To show that T_k is irreducible, we consider an invariant ideal I. Then we have $R(\lambda, A_k)I \subset I$ for $\lambda > 0$ hence $P_k = \lim_{\lambda\to 0} \lambda R(\lambda, A_k)$ leaves I invariant. If $I \neq \{0\}$, then $P_k I \neq \{0\}$ since P_k is strictly positive. Then $e_k \in P_k J \subset I$ which implies that $J_k \subset I$. □

Combining the lemma with Proposition 2.11 one obtains the following.

If $s(A)$ is a pole of finite algebraic multiplicity, then there exists a finite chain of $\mathcal{T}$-invariant ideals $I_{-1} := \{0\} \subset I_0 \subset \ldots \subset I_N := E$ $(N \in \mathbb{N})$ such that the following is true.

For the semigroup $\mathcal{T}_n$ on I_n/I_{n-1} $(0 \leqslant n \leqslant N)$ which is induced by $\mathcal{T}$ we have either $s(A_n) = s(A)$ and $\mathcal{T}_n$ is irreducible or $s(A_n) < s(A)$. (3.19)

The following theorem is an immediate consequence of (3.19), Theorem 3.12 and A-III,Proposition 4.2.

Theorem 3.14 *Let $\mathcal{T}$ be a positive semigroup on a Banach lattice with generator A. If $s(A)$ is a pole of finite algebraic multiplicity, then $\sigma_b(A)$ is a finite union of discrete subgroups (i.e., $\sigma_b(A) = s(A) + \cup_{k=1}^{N} \mathrm{i}\alpha_k\mathbb{Z}$ with $\alpha_k \in \mathbb{R}$).*

Moreover, σ_b contains only poles of finite algebraic multiplicity.

Here the assumption that the multiplicity of $s(A)$ is finite is essential as can be seen from the following example.

Example 3.15 Consider $X := [0,1] \times V$, $V := \{v \in \mathbb{R} : v_1 < |v| < v_2\}$ $(0 < v_1 < v_2 < \infty)$. The flow in the phase space X which describes the free motion in the interval $[0,1]$ with velocities in V assuming that the particles are reflected at the endpoints generates a positive semigroup on $L^p(X,\mu)$ (μ the Lebesgue measure). For the spectrum of the generator A one obtains $\sigma(A) = \{\mathrm{i}y : v_1 \leqslant |y| \leqslant v_2 \text{ for some } n \in \mathbb{N}_0\}$ with $y_1 := \pi v_1^{-1}$, $y_2 := \pi v_2^{-1}$. Moreover, 0 is a first order pole of the resolvent, obviously the only pole in $\sigma_b(A) = \sigma(A)$. These statements can be verified by calculating the resolvent explicitely. This can be done using the integral representation. The semigroup is given as follows.

$$(T(t)f)(x,v) = \begin{cases} f(x - vt + k, v) & \textit{if } k-1 \leqslant vt - x \leqslant k \textit{ and } k \textit{ even,} \\ f(1-(x-vt+k), -v) & \textit{if } k-1 \leqslant vt - x \leqslant k \textit{ and } k \textit{ odd.} \end{cases} \tag{3.20}$$

Obviously one can apply Theorem 3.12 and Theorem 3.14, respectively, in order to prove existence of strictly dominant eigenvalues. We consider two typical cases in the following corollaries. The meaning of $r_{ess}(T(t))$ and $\omega_{ess}(\mathcal{T})$ is explained in A-III,3.7.

Corollary 3.16 *Suppose that $\mathcal{T}$ is a positive semigroup such that $\omega_{ess}(\mathcal{T}) < \omega_0(\mathcal{T})$. Then $s(A) = \omega_0(\mathcal{T})$ is a strictly dominant eigenvalue. If, in addition, there exists an eigenfunction which is a quasi-interior point of E_+ (e.g., if $\mathcal{T}$ is irreducible), then $s(A)$ is a first order pole of $R(.,A)$.*

Proof There exist $\varepsilon > 0$ such that for every $t > 0$ the set $\{\lambda \in \sigma(T(t)) : |\lambda| \geqslant \exp((s(A) - \varepsilon)t)\}$ contains only (finitely many) poles of $R(\cdot, T(t))$ each being of finite algebraic multiplicity. In view of A-III,Corollary 6.5 the set $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda > s(A) - \varepsilon\}$ is finite and contains only poles of $R(\cdot, A)$. Thus we can apply Theorem 3.14. It follows that $s(A)$ is strictly dominant. For the final assertion we refer to Remark 2.15(b). □

Corollary 3.17 *Suppose that $\mathcal{T}$ is an irreducible, eventually norm continuous semigroup having compact resolvent. Then $s(A) = \omega_0(\mathcal{T})$ is an algebraically simple pole and a strictly dominant eigenvalue.*

Proof By Theorem 3.7(iii) we know that $s(A) > -\infty$. Theorem 3.12 is applicable since we assumed that T is irreducible and has compact resolvent. Thus $s(A)$ is an algebraically simple pole and $\sigma_b(A) = s(A) + i\alpha\mathbb{Z}$ for some $\alpha \geqslant 0$. In addition $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geqslant -1\}$ is compact since T is eventually norm-continuous (see A-II,Theorem 1.20). It follows that $s(A)$ is strictly dominant. By A-III,Theorem 6.6 we have $s(A) = \omega_0(\mathcal{T})$. □

In the following proposition we give a condition which ensures that Theorem 3.14 can be applied to certain perturbations. Moreover, we state a criterion ensuring the existence of a dominant eigenvalue.

Proposition 3.18 *Suppose that A is the generator of a positive semigroup and that $K \in \mathcal{L}(E)$ is a positive linear operator. If K is A-compact (i.e., if $KR(\lambda_o, A)$ is compact for some $\lambda_o \in \varrho(A)$) and if $s(A+K) > s(A)$, then $B := A + K$ satisfies the assumptions of Theorem 3.14.*

If, in addition, K is irreducible, then $s(B)$ is a dominant eigenvalue and the semigroup generated by B is irreducible.

Proof The resolvent equation $R(\lambda, A) = R(\lambda_0, A)(1 - (\lambda - \lambda_0)R(\lambda, A))$ implies that $KR(\lambda, A)$ is a compact operator for every $\lambda \in \varrho(A)$. For $\lambda > s(A)$ we have $\lambda - B = (1 - KR(\lambda, A))(\lambda - A)$ and $(1 - KR(\lambda, A))^{-1}$ exists for $\lambda > s(B)$. Therefore Theorem 1.2,XIII.13 of **?**] implies that $R(\lambda, B) = R(\lambda, A)(1 - KR(\lambda, A))^{-1}$ has only poles of finite algebraic multiplicity in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > s(A)\}$. This proves the first claim. In order to prove the second, we denote the semigroup corresponding to A and B by $(T(t))$ and $(S(t))$, respectively. It follows from Proposition 3.3 that $(S(t))$ is irreducible and we have $S(t) = T(t) + \int_0^t T(t-s)KS(s)\, ds$ (see A-II,(1.9)). Iterating this identity we obtain for every $m \in \mathbb{N}$, $t \geqslant 0$ GG: Lit. is missing

$$S(t) = \sum_{n=0}^{m-1} T_n(t) + R_m(t) \text{ where}$$

$$T_0(t) := T(t),\ T_n(t) := \int_0^t T(t-s)KT_{n-1}(s)\ \mathrm{d}s \textit{ for } n \in \mathbb{N},$$

$$R_m(t) := \int_0^t \int_0^{t_1} .. \int_0^{t_{m-1}} T(t-t_1)KT(t_1-t_2)K..T(t_{m-1}-t_m)KS(t_m)\,\mathrm{d}t_m..\mathrm{d}t_1\,. \tag{3.21}$$

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We fix $0 < f \in E$, $0 < \varphi \in E'$, $t > 0$. By Theorem 3.2(i), $S(t)f > 0$. Since K is irreducible, there exists $m \in \mathbb{N}$ such that $\langle K^m S(t)f, \varphi\rangle > 0$. Thus the integrand appearing in the the representation (3.21) of $\langle R_m(t)f, \varphi\rangle$ is non-zero at $t_1 = t_2 = .. = t_{m-1} = t$, $t_m = t$.

Since the integrand is positive and continuous, we conclude

$$\langle S(t)f, \varphi\rangle \geqslant \langle R_m(t)f, \varphi\rangle > 0 \textit{ for } 0 < f, 0 < \varphi, t > 0\,. \tag{3.22}$$

It follows that $(\mathrm{e}^{-ts(B)}S(t))_{t\geqslant 0}$ cannot contain the rotation semigroup on Γ. On the other hand, assuming that $s(B)$ is not dominant, then $\dim(\ker((\mathrm{e}^{\tau\cdot s(B)} - S(\tau))) > 1$ for some $\tau > 0$. Hence, the restriction $(\mathrm{e}^{-ts(B)}S(t)_{|F})_{t\geqslant 0}$ where $F := \ker(\mathrm{e}^{\tau\cdot s(B)} - S(\tau))$, contains the rotation semigroup by Corollary 3.9. □

We conclude this section considering once again Example 3.4(d). The generator considered there is $B = (A - M) + K$, where K is positive linear. From (3.5) and (3.6) one deduces that

$$(KR(\lambda, A)f)(x, v) = \int_0^1 \int_{-1}^1 k(x, v, x', v')f(x', v')\ \mathrm{d}x'dv'$$

where the kernel k is given by $k(x, v, x', v') := \kappa(x, v, v')r_\lambda(x, x', v')$ (cf. (3.5), (3.7)). Using this representation of $KR(\lambda, A)$ it follows that K is A-compact. Moreover for λ sufficiently large one has $R(\lambda, A - M) = R(\lambda, A)(1 - MR(\lambda, A))^{-1}$ which shows that K is also $(A - M)$-compact. In order to apply Theorem 3.14 one needs $s(B) > s(A - M)$ (see Proposition 3.18) which is difficult to verify. In case the function σ is continuous one can state a sufficient condition as follows. There exist $r \in \mathbb{R}$ and $g \in L^1([0, 1]\times[-1, 1])$, $g > 0$ such that $r < \inf\{\sigma(x, 0) : x \in [0, 1]\}$ and $Bg \geqslant -rg$. The additional assumption made in the second part of Proposition 3.18 is not satisfied in this example. Nevertheless one can show that $s(B)$ is strictly dominant in this situation (provided that $s(B) > s(A)$). For details we refer to **?**] or **?**] where the linear transport equation in higher dimensional spaces is discussed.

[Greiner (1984d)]

4 Semigroups of Lattice Homomorphisms

In Section 2 we proved that the boundary spectrum of certain positive semigroups is a cyclic set. For semigroups of lattice homomorphisms much more can be said.

The whole spectrum is an imaginary additively cyclic subset of $\mathbb{C}$ (cf. Theorem 4.2). This result can be used to derive cyclicity results for the eigenvalues in the boundary spectrum of positive semigroups (cf. Corollary 4.3). In the last part of this section we discuss a spectral decomposition of positive groups (cf. Theorem 4.8).

Lemma 4.1 *Suppose that $(T(t))_{t\geq 0}$ is a semigroup of lattice homomorphisms on a Banach lattice E with generator A. In case $\mathrm{i}\alpha \in R\sigma(A)$, $\alpha \in \mathbb{R}$, then one of the following assertions are true*

(a) $\mathrm{i}\alpha\mathbb{Z} \subset R\sigma(A)$;
(b) $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \subset R\sigma(A)$.

Proof There exists $\varphi \in E'$, $\varphi \neq 0$ such that $T(t)'\varphi = \mathrm{e}^{\mathrm{i}\alpha t}\varphi$ $(t \geqslant 0)$. Then we have $|\varphi| = |T(t)'\varphi| \leqslant T(t)'|\varphi|$ $(t \geqslant 0)$. If we fix $r > \omega_0(\mathcal{T})$ and define $\psi := rR(r,A)'|\varphi|$, we have

$$T(t)'\psi \leqslant \mathrm{e}^{rt}\psi, T(t)'\psi \geqslant \psi \ (t \geqslant 0) \ \ \textit{and} \ \ |\varphi| \leqslant \psi\,. \tag{4.1}$$

In fact, A-I,(3.1) implies $(\mathrm{e}^{rt} - T(r))R(r,A) \geqslant 0$ hence $T(t)'\psi = rR(r,A)'T(t)'|\varphi| \leqslant r \cdot \mathrm{e}^{rt}R(r,A)'|\varphi| = \mathrm{e}^{rt}\psi$. Moreover, the inequality $T(t)'|\varphi| \geqslant |\varphi|$ $(t \geqslant 0)$ implies $T(t)'\psi = rR(r,A)'T(t)'|\varphi| \geqslant rR(r,A)'|\varphi| = \psi$ and $\psi = rR(r,A)'|\varphi| = r\int_0^\infty \mathrm{e}^{-rt}T(t)'|\varphi|\,\mathrm{d}t \geqslant r\int_0^\infty \mathrm{e}^{-rt}|\varphi|\,\mathrm{d}t = |\varphi|$. □

Now, considering the AL-space (E,ψ) (see C-I,Section 4) the first inequality of (4.1) implies that $(T(t))_{t\geqslant 0}$ induces a strongly continuous semigroup $(T_1(t))_{t\geqslant 0}$ on (E,ψ). That is we have

$$\begin{tikzcd} E \arrow[r, "T(t)"] \arrow[d, "q_\psi"'] & E \arrow[d, "q_\psi"] \\ (E,\psi) \arrow[r, "T_1(t)"'] & (E,\psi)) \end{tikzcd}$$

$$T_1(t) \circ q_\psi = q_\psi \circ T(t) \ (t \geqslant 0)\,. \tag{4.2}$$

Denoting by A_1 the generator of $(T_1(t))$ we have $R\sigma(A_1) \subset R\sigma(A)$. Indeed, $A_1^*x = \lambda x$ implies $T_1(t)'x = \mathrm{e}^{\lambda t}x$ hence by (4.2) $T(t)'q_\psi'(x) = \mathrm{e}^{\lambda t}q_\psi'(x)$ or equivalently $A^*(q_\psi'(x)) = q_\psi'(x)$. Thus it remains to show that either $\mathrm{i}\alpha\mathbb{Z}$ or $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ is contained in $R\sigma(A_1)$.

Obviously, $(T_1(t))$ is a semigroup of lattice homomorphisms as well. The second inequality of (4.1) implies

$$\|T_1(t)f\|_\psi = \langle |T_1(t)f|, \psi\rangle = \langle |f|, T_1(t)'\psi\rangle \geqslant \langle |f|, \psi\rangle = \|f\|_\psi\,. \tag{4.3}$$

Then for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ we have

$$\|(\mathrm{e}^{\lambda t} - T_1(t))f\|_\psi \geqslant \|T_1(t)f\|_\psi - \|\mathrm{e}^{\lambda t}f\|_\psi \geqslant (1 - |\mathrm{e}^{\lambda t}|)\|f\|_\psi \ (f \in (E,\psi))$$

and we obtain for the corresponding generator

$$\begin{aligned} \|(\lambda - A_1)f\|_\psi &= \lim_{t\to 0}\left\|\frac{1}{t}(\mathrm{e}^{-\lambda t}T_1(t)f - f)\right\|_\psi \geqslant \lim_{t\to 0}\frac{1}{t}(\mathrm{e}^{-t\operatorname{Re}\lambda} - 1)\|f\|_\psi \\ &= -\operatorname{Re}\lambda \cdot \|f\|_\psi \ \ \textit{for} \ \ \operatorname{Re}\lambda < 0 \ \textit{and} \ \ f \in (E,\psi)\,. \end{aligned} \tag{4.4}$$

It follows from (4.3) and (4.4) that $A\sigma(T_1(t)) \cap \{z \in \mathbb{C} : |z| < 1\} = \emptyset$ and $A\sigma(A_1) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} = \emptyset$. Since the toplogical boundary of the spectrum is always contained in the approximate point spectrum (see A-III,Proposition 2.2) and $R\sigma(T(t))\backslash\{0\} = \exp(tR\sigma(A))$ (see A-III, Theorem 6.3), precisely one of the following two cases occurs.

(A) $\{\lambda \in \mathbb{C}\,\mathrm{Re}\,\lambda < 0\} \subset \varrho(A_1)$ and $\{z \in \mathbb{C} : |z| < 1\} \subset \varrho(T_1(t))$;
(B) $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} \subset R\sigma(A_1)$ and $\{z \in \mathbb{C} : |z| < 1\} \subset R\sigma(T_1(t))$.

We mentioned above that $R\sigma(A_1) \subset R\sigma(A)$. Thus we only have to analyze case (A). In this case each operator $T_1(t)$ is an invertible lattice homomorphism hence a lattice isomorphism. It follows that $T_1(t)'$ is a lattice isomorphism as well. The third inequality in (4.1) implies that φ can be considered as an element of $(E, \psi)'$ and $T(t)'\varphi = \mathrm{e}^{\mathrm{i}\alpha t}\varphi$ $(t \geqslant 0)$ implies $T_1(t)'\varphi = \mathrm{e}^{\mathrm{i}\alpha t}\varphi$. Furthermore, we have $T_1(t)'|\varphi| = |T_1(t)'\varphi| = |\mathrm{e}^{\mathrm{i}\alpha t}\varphi|$ or equivalently $A_1^*|\varphi| = 0$. Now we can apply Theorem 2.2 and obtain $\mathrm{i}\alpha\mathbb{Z} \subset P\sigma(A_1^*) = R\sigma(A_1)$. □

Theorem 4.2 *Let A be the generator of a semigroup $(T(t))_{t\geqslant 0}$ of lattice homomorphisms on a Banach lattice E. Then $\sigma(A)$, $A\sigma(A)$ and $P\sigma(A)$ are imaginary additively cyclic subsets of $\mathbb{C}$.*

Proof We first consider the point spectrum. If $\lambda \in P\sigma(A)$, $\lambda = \alpha + \mathrm{i}\beta$ $(\alpha, \beta \in \mathbb{R})$, then there exists $f \in E$, $f \neq 0$ such that $Af = \lambda f$. It follows that $T(t)f = \mathrm{e}^{\lambda t}f$ $(t \geqslant 0)$ hence $T(t)|f| = |T(t)f| = \mathrm{e}^{\alpha t}|f|$ $(t \geqslant 0)$, or equivalently, $A|f| = \alpha|f|$. Now Theorem 2.2 is applicable and we obtain $A(f^{[n]}) = (\alpha + in\beta)f^{[n]}$ for all $n \in \mathbb{Z}$.

To prove the assertion for $A\sigma(A)$ we consider an $\mathcal{F}$-product semigroup in order to reduce the problem to the point spectrum. We use the notation of A-I,3.6. Obviously the space $m(E)$ is a Banach lattice and every operator $\hat{T}(t)$ is a lattice homomorphism. We have $|T(t)|f| - |f|| = ||T(t)f| - |f|| \leqslant |T(t)f - f|$ $(f \in E)$, hence $(|f_n|) \in m^{\mathcal{T}}(E)$ whenever $(f_n) \in m^{\mathcal{T}}(E)$. This proves that $m^{\mathcal{T}}(E)$ is a sublattice, hence a Banach lattice as well. Obviously, $c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E)$ is an order ideal. Thus $E_{\mathcal{F}}^{\mathcal{T}}$ is a Banach lattice and $(T_{\mathcal{F}}(t))$ is a semigroup of lattice homomorphisms. It follows that $P\sigma(A_{\mathcal{F}})$ is cyclic hence $A\sigma(A)$ is cyclic by A-III,4.5.

Cyclicity of the entire spectrum now follows from the cyclicity of $A\sigma(A)$ and Lemma 4.1. □

One can use Theorem 4.2 in order to prove cyclicity for the eigenvalues in the boundary spectrum of positive semigroups. We list some typical cases in the following corollary.

Corollary 4.3 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a positive semigroup on a Banach lattice E which is bounded. Each of the following conditions implies that $P\sigma(A) \cap i\mathbb{R}$ is imaginary additively cyclic.*

(i) E is weakly sequentially complete (e.g. $E = L^p(\mu)$, $1 \leqslant p < \infty$);
(ii) Every operator $T(t)$ is mean ergodic (i.e., the Césaro means $\frac{1}{n}\sum_{k=0}^{n-1} T(t)^k$ converge strongly as $n \to \infty$);
(iii) There is a strictly positive linear form which is $\mathcal{T}$-invariant.

Proof We will show that each of the conditions (i), (ii), (iii) implies that $\ker(1 - T(s))$ is a Banach lattice (not necessarily a sublattice of E) for every $s \geqslant 0$. Then one argues as follows. Given $i\alpha \in P\sigma(A)$, $\alpha \in \mathbb{R}$, then $T(t)g = e^{i\alpha t}g$ for suitable $g \neq 0$. For $\tau := 2\pi|\alpha|^{-1}$ we have $g \in F := \ker(1 - T(\tau))$. Then the restriction $(T(t)_{|F})_{t\geqslant 0}$ is a τ-periodic positive semigroup on F. Since $\big(T(t)_{|F}\big)^{-1} = T(n\tau - t)_{|F} \geqslant 0$, it follows that $(T(t)_{|F})$ is a semigroup of lattice isomorphisms. Since $g \in F$, we have $i\alpha \in P\sigma(A|)$ hence $i\alpha\mathbb{Z} \in P\sigma(A_{|F}) \subset P\sigma(A)$ by Theorem 4.2.

Now we show that $\ker(1 - T(s))$ is a vector lattice for the induced order and a Banach lattice for an equivalent norm.

In case (iii), $\ker(1 - T(s))$ is even a sublattice of E. Indeed, assume $T(t)'\varphi = \varphi$ and $\varphi \gg 0$, $(t \geqslant 0)$, then $T(s)f = f$ implies $T(s)|f| \geqslant |f|$. Thus from $< T(s)|f| - |f|, \varphi >=< |f|, T(s)'\varphi - \varphi >= 0$ it follows that $T(s)|f| = |f|$.

Now we assume that E is weakly sequentially complete, which is equivalent to (cf. Section 5 of C-I)

$$\textit{Every increasing norm-bounded net of } E_+ \textit{ converges}. \tag{4.5}$$

We fix $s > 0$ and define $F := \ker(1 - T(s))$, $T := T(s)$. Obviously $f \in F$ implies $\bar{f} \in F$ hence $F = F \cap E_{\mathbb{R}} + iF \cap E_{\mathbb{R}}$. Thus we have to show that $F_{\mathbb{R}} = F \cap \mathbb{E}_{\mathbb{R}}$ is a sublattice. Given $f \in F_{\mathbb{R}}$, then $Tf = f$ hence $|f| \leqslant T|f|$. Iterating this inequality we obtain $|f| \leqslant T|f| \leqslant T^2|f| \leqslant T^3|f| \leqslant \dots$. By (4.5) $|f|_o := \lim_{n\to\infty} T^n|f|$ exists and we have $T|f|_o = \lim_{n\to\infty} T^{n+1}|f| = |f|_o$, i.e., $|f|_o \in F_{\mathbb{R}}$. And, for $g \in F_{\mathbb{R}}$ satisfying $\pm f \leqslant g$, we have $|f|_o \leqslant g$ thus $|f|_o = \sup_F\{f, -f\}$. Moreover, $\|f\|_o := \||f|_o\|$ $(f \in F)$ is an equivalent norm on F such that $(F, \|\cdot\|_o)$ is a Banach lattice.

(ii) If $T(s)$ is mean-ergodic, then we have $\ker(1 - T(s)) = PE$ where P is the mean-ergodic projection, i.e., $Pf = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} T(s)^k f$. Obviously P is positive, hence II.11.5 of **?**] implies that PE is a Banach lattice (for the induced order and an equivalent norm). □

The assumptions made in Corollary 4.3 can be weakened slightly (cf. **?**]). However, one still cannot prove cyclicity of $P\sigma_b(A)$ for arbitrary positive semigroups.

Example 4.4 At first we recall Example 2.13 of Chapter B-III. There we constructed a bounded semigroup on the space $C(\Gamma)\times C_0(\mathbb{R})$ such that $P\sigma_b(A) = \{ik : k \in \mathbb{Z}, k \neq 0\}$. Let us perform the same construction on the Hilbert space $H := L^2(\Gamma) \times L^2(\mathbb{R})$. For a fixed positive, non-zero function $k \in C_c(\mathbb{R})$, we define $T(t)$ on H as follows.

$$\begin{aligned} T(t)([f,g]) &:= [f_t, g_t] \quad \textit{with} \\ &\quad f_t(z) := f(z\cdot e^{it}) \quad (z \in \Gamma) \quad \text{and} \\ &\quad g_t(x) := g(x+t) + \frac{1}{2\pi}\int_0^{2\pi} f(z\cdot e^{is})\,ds \cdot \int_x^{x+t} k(u)\,du\,. \end{aligned} \tag{4.6}$$

Then $(T(t))_{t\geqslant 0}$ is a positve semigroup on H and for the spectrum of the generator we obtain $\sigma(A) = i\mathbb{R}$, $P\sigma(A) = i\mathbb{R}\backslash\{0\}$. In view of Corollary 4.3(i) the semigroup cannot be bounded. (The explicit representation (4.6) only allows the estimate $\|T(t)\| \leqslant \sqrt{2} + t\cdot\|k\|_2$ $(t \geqslant 0)$.)

In the next proposition we show that for semigroups of lattice homomorphisms on L^1-spaces, there is a spectral mapping theorem for the real part of the spectrum.

Proposition 4.5 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup of lattice homomorphisms on an L^1-space and denote by A its generator. Then we have*

$$\exp(t\sigma(A)\cap\mathbb{R}) = \sigma(T(t))\cap(0,\infty) \quad \textit{for every} \quad t\geqslant 0\,. \tag{4.7}$$

Proof In view of A-III,6.2 it is enough to prove that the left hand side contains the set on the right.

Fix $t>0$ and assume $r\in\sigma(T(t))$, $r>0$ and let $\alpha := \frac{1}{t}\log r$. At first we assume $r\in R(\sigma(T(t)))$. Then by A-III,Theorem 6.3 there exists $\beta\in\mathbb{R}$ such that $\alpha+\mathrm{i}\beta\in R\sigma(A)$. By Lemma 4.1 either $\alpha+\mathrm{i}\beta\mathbb{Z}\subset R\sigma(A)$ or $\{\lambda\in\mathbb{C}\colon\ \mathrm{Re}\,\lambda<\alpha\}\subset R\sigma(A)$. In both cases we have $\alpha\in\sigma(A)$.

Now we assume $r\in A\sigma(T(t))$. Then there exists a normalized sequence $(f_n)\subset E$ such that $\lim_{n\to\infty}\|T(t)f_n - rf_n\| = 0$. Since we have

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$$|(T(t)|f|-r|f|)| = |(|T(t)f|-r|f|)| \leqslant |T(t)f - rf|,\ (f\in E)$$

we may assume that (f_n) is a sequence in E_+.

Defining $g_n := \int_0^t \mathrm{e}^{-\alpha s}T(s)f_n\ \mathrm{d}s$ we have $g_n\in D(A)$ and

$$(A-\alpha)g_n = (A-\alpha)\int_0^t \mathrm{e}^{-\alpha s}T(s)f_n\ \mathrm{d}s = \mathrm{e}^{-\alpha t}T(t)f_n - f_n = \frac{1}{r}(T(t)f_n - rf_n)\,.$$

Therefore $\lim_{n\to\infty}\|(A-\alpha)g_n\| = 0$. It remains to prove $\liminf_{n\to\infty}\|g_n\|>0$. The latter assertion is a consequence of the following facts.

- Since f_n is positive and the norm is additive on E_+, we have $\|g_n\| = \int_0^t \mathrm{e}^{-\alpha s}\|T(s)f_n\|\ \mathrm{d}s$.
- The inequality $\|T(t)f\|\leqslant\|T(t-s)\|\|T(s)f\|$ implies $\|T(s)f\|\geqslant M^{-1}\|T(t)f\|$ for $0\leqslant s\leqslant t$, $f\in E$ and $M := \sup_{0\leqslant s\leqslant t}\|T(s)\|$.
- Since $\lim_{n\to\infty}\|T(t)f_n - rf_n\| = 0$ and $\|f_n\| = 1$, we have $\lim_{n\to\infty}\|T(t)f_n\| = r>0$. □

For semigroups satisfying the assumption of Proposition 4.5, the spectrum $\sigma(A)$ is additively cyclic (by Theorem 4.2) and $\sigma(T(t))$ is multiplicatively cyclic (by **?** , V.Theorem 4.4]). Then the relation (4.7) implies that decompositions of the spectrum by vertical lines allow a spectral decomposition of the semigroup (cf. A-III,Definition 3.1). (One simply performs a spectral decomposition of a single operator $T(t)$). In the following we will show that for positive groups (on arbitrary Banach lattices) spectral decompositions of this type always exist. Moreover, it will turn out that the decomposition is compatible with the lattice structure. The proof of this result uses Kato's equality (see Section 5 of C-II). As a consequence of C-II,Corollary 5.8 we have the following.

Let E be a Banach lattice with order continuous norm and $(T(t))_{t\in\mathbb{R}}$ be a group of positive operators on E with generator A. Then the domain $D(A)$ is a sublattice of E and

$$A|f| = \mathrm{Re}[(\mathrm{sign}\,f)Af] \quad \text{for every} \quad f\in D(A)\,, \tag{4.8}$$

For real μ one has $\mu|f| = \mathrm{Re}[(\mathrm{sign}\ f)\mu f]$, hence

$$(\mu - A)|f| = \mathrm{Re}[(\mathrm{sign}\ f)(\mu - A)f] \quad \textit{for} \quad \mu \in \mathbb{R}, f \in D(A)\,.$$

The relations $f^+ = \frac{1}{2}(|f| + f),\ f^- = \frac{1}{2}(|f| - f)$ yield

$$\begin{aligned}(\mu - A)f^+ &= \frac{1}{2}[(\mathrm{sign}\ f)(\mu - A)f + (\mu - A)f] \quad \textit{and}\\ (\mu - A)f^- &= \frac{1}{2}[(\mathrm{sign}\ f)(\mu - A)f - (\mu - A)f]\,,\end{aligned}$$

in case f is contained in the underlying real Banach lattice $E_{\mathbb{R}}$. For $\mu \in \varrho(A) \cap \mathbb{R}$, we can apply $R(\mu, A)$ on both sides and the substitution $f = R(\mu, A)g$ finally leads to

$$\begin{aligned}(R(\mu, A)g)^+ &= \frac{1}{2}R(\mu, A)[(\mathrm{sign}\ R(\mu, A)g)g + g]\\ (R(\mu, A)g)^- &= \frac{1}{2}R(\mu, A)[(\mathrm{sign}\ R(\mu, A)g)g - g]\end{aligned} \tag{4.9}$$

for all $g \in E_{\mathbb{R}}$. If we set

$$g_1 := \frac{1}{2} \cdot (g + (\mathrm{sign}\ R(\mu, A)g)g) \ \ \textit{and} \ \ g_2 := \frac{1}{2} \cdot (g - (\mathrm{sign}\ R(\mu, A)g)g)\,,$$

then obviously $g = g_1 + g_2$. Moreover, g is positive if and only if both, g_1 and g_2 are positive. We summarize these considerations in the following lemma.

Lemma 4.6 *Let A be the generator of a positive group on a Banach lattice E which has order continuous norm. Given $\mu \in \varrho(A) \cap \mathbb{R}$, then every $g \in E_{\mathbb{R}}$ is representable as sum of two elements g_1 and g_2 such that*

(i) $g \geqslant 0$ *if and only if both g_1 and g_2 are positive,*
(ii) $R(\mu, A)g_1 = (R(\mu, A)g)^+$,
(iii) $R(\mu, A)g_2 = -(R(\mu, A)g)^-$.

We need another lemma. It can be formulated for arbitrary positive semigroups on Banach lattices.

Lemma 4.7 *Let $(T(t))_{t\geqslant 0}$ be a positive semigroup on a Banach lattice E with generator A. Given $\mu \in \varrho(A) \cap \mathbb{R}$ and $h \in E_+$, then the following assertions are equivalent.*

(a) $R(\mu, A)h \geqslant 0$,
(b) $\left\{\int_0^t \mathrm{e}^{-\mu s}T(s)h\ \mathrm{d}s : t \in \mathbb{R}_+\right\}$ *is bounded in E.*

Proof $(a) \Rightarrow (b)$: We have $\int_0^t \mathrm{e}^{-\mu s}T(s)h\ \mathrm{d}s = (Id - \mathrm{e}^{-\mu t}T(t))R(\mu, A)h$ (see A-I,(3.2)). Since $R(\mu, A)h \geqslant 0$ and $T(t)$ is a positive operator we obtain

$$\int_0^t \mathrm{e}^{-\mu s}T(s)h\ \mathrm{d}s = R(\mu, A)h - \mathrm{e}^{-\mu t}T(t)R(\mu, A)h \leqslant R(\mu, A)h$$

which implies assertion (b).

$(b) \Rightarrow (a)$: The assumption implies that

$$\int_0^\infty \mathrm{e}^{-\nu s} T(s)h \, \mathrm{d}s := \lim_{t\to\infty} \int_0^t \mathrm{e}^{-\nu s} T(s)h \, \mathrm{d}s \text{ exists for } \nu > \mu \, .$$

Using that A is a closed operator it follows that

$$(\nu - A)(\int_0^\infty \mathrm{e}^{-\nu s} T(s)h \, \mathrm{d}s) = h \, .$$

For ν sufficiently close to μ such that $\nu \in \varrho(A) \cap \mathbb{R}$ we have

$$R(\nu, A)h = \int_0^\infty \mathrm{e}^{-\nu s} T(s)h \, \mathrm{d}s \geqslant 0 \, .$$

By continuity we conclude $R(\mu, A)h \geqslant 0$. □

By now we are prepared to prove the spectral decomposition for positive groups. Before we formulate the theorem we recall the following consequence of Theorem 4.2. For any $\mu \in \varrho(A) \cap \mathbb{R}$ the line $\mu + i\mathbb{R}$ is a subset of the resolvent set and divides $\sigma(A)$ into disjoint sets. Both sets will be unbounded in general.

Theorem 4.8 *Let $(T(t))_{t\in\mathbb{R}}$ be strongly continuous group of positive operators on a Banach lattice E with order continuous norm. If A is the generator and $\mu \in \varrho(A)\cap\mathbb{R}$, then $I_\mu := \{f \in E : R(\mu, A)|f| \geqslant 0\}$ and $J_\mu := \{f \in E : R(\mu, A)|f| \leqslant 0\}$ are $(T(t))_{t\in\mathbb{R}}$-invariant projection bands, the direct sum of them is E, and the spectra of the restrictions satisfy*

$$\sigma(A_{|I_\mu}) = \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \mu\},$$
$$\sigma(A_{|J_\mu}) = \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \mu\}.$$

Proof At first we consider I_μ. Obviously it is a closed subset. From Lemma 4.7 we deduce that it is a lattice ideal. Moreover, I_μ is $R(\mu, A)$-invariant and $(T(t))_{t\in\mathbb{R}}$-invariant as well (use Lemma 4.7 again).

Since $-A$ is the generator of the positive group $(T(-t))_{t\in\mathbb{R}}$ and $J_\mu = \{f \in E : R(-\mu, -A)|f| \geqslant 0\}$, J_μ has the same properties.

If $f \in I_\mu \cap J_\mu$, then $R(\mu, A)|f| = 0$, hence $f = 0$ which shows that $I_\mu \cap J_\mu = \{0\}$. On the other hand, decomposing $0 \leqslant h = h_1 + h_2 \in E_+$ according to Lemma 4.6, then assertion (ii) of this lemma implies that $h_1 \in I_\mu$, while assertion (iii) ensures that $h_2 \in J_\mu$. Since the positive cone E_+ is generating, we have $E = I_\mu \oplus J_\mu$ and the first part of the theorem is proved.

Since I_μ is $R(\mu, A)$-invariant, we have $\mu \in \varrho(A_{|I_\mu})$ and $R(\mu, A_{|I_\mu}) = R(\mu, A)_{|I_\mu} \geqslant 0$. Theorem 1.1(ii) then implies $\sigma(A|_{I_\mu}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \mu\}$. The same argument applied to $-A$ and $-\mu$ yields $\sigma(A_{|J_\mu}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \mu\}$. Now the assertion follows from A-III,Proposition 4.2. □

The spectral projections corresponding to the spectral decomposition described above have the expected representation as an integral 'around' the spectral sets (see Corollary 3 in **?**]).

Corollary 4.9 *Assume that the assumptions of the theorem are satisfied, $\mu \in \varrho(A) \cap \mathbb{R}$, $\beta > s(A)$, $\alpha < -s(-A)$. If we denote the projections corresponding to the decomposition $E = I_\mu \oplus J_\mu$ by P_μ and Q_μ (i.e., $P_\mu E = \ker Q_\mu = I_\mu$, $Q_\mu E = \ker P_\mu = J_\mu$), then for $f \in D(A^2)$ we have*

$$\begin{aligned} P_\mu f &= \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} R(\mu + \mathrm{i}\tau, A) f \, \mathrm{d}\tau - \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} R(\alpha + \mathrm{i}\tau, A) f \, \mathrm{d}\tau, \\ Q_\mu f &= \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} R(\beta + \mathrm{i}\tau, A) f \, \mathrm{d}\tau - \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} R(\mu + \mathrm{i}\tau, A) f \, \mathrm{d}\tau. \end{aligned} \tag{4.10}$$

(The integrals appearing in (4.10) *are improper Riemann integrals.)*

We mention another consequence of Theorem 4.8. Just like Proposition 4.5 it is a spectral mapping theorem for the real part of the spectrum.

Corollary 4.10 *If $(T(t))_{t\in\mathbb{R}}$ is a positive group on a space L^2 or $C_0(X)$ with generator A, then*

$$\sigma(T(t)) \cap \mathbb{R}_+ = \exp(t\sigma(A) \cap \mathbb{R}) \quad \textit{for every} \quad t \geqslant 0. \tag{4.11}$$

Proof We borrow from the next chapter that for positive semigroups on spaces L^1, L^2 or $C_0(X)$ spectral bound and growth bound coincide (see C-IV,Theorem 1.1).

We only have to show that $\exp(t\varrho(A) \cap \mathbb{R}) \subset \varrho(T(t)) \cap \mathbb{R}_+$.

If we consider a positive semigroup on an L^2-space, Theorem 4.8 can be applied directly. Given $\mu \in \varrho(A) \cap \mathbb{R}$, then $E = I_\mu \oplus J_\mu$ according to Theorem 4.8. The result mentioned above implies $r(T(t)_{|I_\mu}) < \mathrm{e}^{\mu t}$ and $r(T(-t)_{|J_\mu}) < \mathrm{e}^{\mu t}$. Hence $\sigma(T(t)_{|I_\mu}) \subset \{\lambda \in \mathbb{C} : |\lambda| < \mathrm{e}^{\mu t}\}$ and $\sigma(T(t)_{|J_\mu}) = (\sigma(T(-t)_{|J_\mu}))^{-1} \subset \{\lambda \in \mathbb{C} : |\lambda| > \mathrm{e}^{\mu t}\}$. Thus $\sigma(T(t)) = \sigma(T(t)_{|I_\mu}) \cup \sigma(T(t)_{|J_\mu})$ does not contain $\mathrm{e}^{\mu t}$.

In case $(T(t))$ is a positive group on $C_0(X)$, then the adjoint group $(T(t)')$ is a group of lattice homomorphisms on E'. It follows that $E*$ is a sublattice of $C_0(X)'$ which is isomorphic to $M_b(X)$, hence an L^1-space. The argument given for the L^2-space yields $\sigma(T(t)*) \cap \mathbb{R}_+ = \exp(t\sigma(A*) \cap \mathbb{R})$ for every $t \geqslant 0$. Thus the assertion follows from A-III,4.4. □ □

We conclude by describing a general situation of lattice semigroups. In Section 4 of B-III we constructed semigroups of lattice homomorphisms on $C_0(X)$ starting with a continuous (semi-) flow on the locally compact space X and a multiplication operator. One can perform similar constructions on spaces $L^p(\mu)$ for $1 \leqslant p < \infty$ under certain conditions on the flow. We consider an example which shows where the problems are.

Define the semiflow φ on $\mathbb{R}_+$ as follows. $\varphi(t,x) := x - t$ for $x \geqslant t$ and $\varphi(t,x) := 0$ for $x < t$. For $f \in L^p(\mu)$ one has difficulties to define $f \circ \varphi_t$ properly since the preimage of the zero-set $\{0\}$ does not have measure zero. This problem does not arise in case every transformation φ_t is measure preserving, i.e. $\mu(\varphi_t^{-1}(C)) = \mu(C)$ for every Borel set C. A more general criterion is stated in the following proposition.

Proposition 4.11 *Let X be a locally compact space and let μ be a regular, positive Borel measure on X. Assume that the continuous semiflow $\varphi : \mathbb{R}_+ \times X \to X$ satisfies the condition*

$$\varphi_t^{-1}(K) \quad \text{is compact for every compact set} \quad K \subset X, t \geqslant 0\,. \tag{4.12}$$

(i) For every p, $1 \leqslant p < \infty$ the following assertions are equivalent.

(a) The operators $T(t)$ defined by $T(t)f := f \circ \varphi_t$ for $f \in L^p(\mu)$, $t \geqslant 0$, are well-defined as bounded linear operator on $L^p(\mu)$ and $(T(t))_{t \geqslant 0}$ is a strongly continuous semigroup.

(b) There exist constants $t_0 > 0$, $M > 0$ such that $\mu(\varphi_t^{-1}(C)) \leqslant M \cdot \mu(C)$ for every open (compact) set $C \subset X$ and every $t \leqslant t_0$.

(ii) In case the conditions (i) and (ii) are fulfilled, then $(T(t))_{t \geqslant 0}$ is a semigroup of lattice homomorphisms on $L^p(\mu)$ and $C_c(X) \cap D(A)$ is a core of the generator.

Proof (i) Since μ is assumed to be regular, the inequality stated in (b) holds true for all Borel sets provided it is true for all open sets (compact sets, respectively).

$(a) \Rightarrow (b)$: Assume that $(T(t))$ is a strongly continuous semigroup on $L^p(\mu)$, $1 \leqslant p < \infty$. For $t_0 > 0$ we define $M := (\sup\{\|T(t)\| : 0 \leqslant t \leqslant t_0\})^{1/p}$. Given a Borel set $C \subset X$ we write $C(t) := \varphi_t^{-1}(C)$. Then we have $T(t)\mathbb{1}_C = \mathbb{1}_{C(t)}$, hence $\mu(\varphi_t^{-1}(C)) = \|\mathbb{1}_{C(t)}\|_p^p = \|T(t)\mathbb{1}_C\|_p^p \leqslant M \cdot \|\mathbb{1}_C\|_p^p = M \cdot \mu(C)$.

$(b) \Rightarrow (a)$: Since the inequality in (b) holds for all Borel sets, $\varphi_t^{-1}(C)$ is a μ-null set whenever C is a μ-null set. Thus given Borel functions f, g such that $f = g$ μ-a.e., then $f \circ \varphi_t = g \circ \varphi_t$ μ-a.e.. Moreover, for $0 \leqslant f \in L^p(\mu)$, there exists an increasing sequence (h_n) of simple functions converging pointwise to f. Then $(h_n \circ \varphi_t)$ is a monotone sequence converging pointwise to $f \circ \varphi_t$. Using the fact that $\mathbb{1}_C \circ \varphi_t = \mathbb{1}_{C(t)}$, $C(t)$ as above, and the assumption $\mu(C(t)) \leqslant M \cdot \mu(C)$, it is easy to see that $\|h_n \circ \varphi_t\|_p^p \leqslant M \cdot \|h_n\|_p^p \leqslant M \cdot \|f\|_p^p$. Thus by the Monotone Convergence Theorem we have $f \circ \varphi_t \in L^p(\mu)$ and $\|f \circ \varphi_t\|_p \leqslant M^{1/p}\|f\|_p$. It follows that $T(t)$ is a bounded linear operator on $L^p(\mu)$ and $\|T(t)\| \leqslant M^{1/p}$ for $0 \leqslant t \leqslant t_0$. Since φ is a semiflow, we have $T(0) = \mathrm{Id}$ and $T(t+s) = T(s)T(t)$ $(0 \leqslant s, t < \infty)$. It remains to prove strong continuity. Since φ is continuous and (4.12) holds, we know that $T(t)(C_c(X)) \subset C_c(X)$ and that $T(t)f$ tends to f uniformly as $t \to 0$ provided that $f \in C_c(X)$. It follows that $\lim_{t \to 0} \|T(t)f - f\|_p = 0$ for $f \in C_c(X)$. Since $C_c(X)$ is dense in $L^p(\mu)$ and $\|T(t)\| \leqslant M^{1/p}$ for $0 \leqslant t \leqslant t_0$, the semigroup is strongly continuous.

(ii) Obviously every operator $T(t)$ defined in assertion (a) of (i) is a lattice homomorphism. Above we pointed out that $C_c(X)$ is invariant under $(T(t))$, then $D(A) \cap C_c(X)$ is invariant as well. It is dense because the elements of the form $\int_0^r T(s)f\,ds$, $f \in C_c(X)$, $r > 0$ belong to $C_c(X)$ and to $D(A)$. Hence $D(A) \cap C_c(X)$ is a core (by A-I,Corollary 1.34). ◻

Proposition 4.11 can be used to prove that flows corresponding to certain ordinary differential equations on $\mathbb{R}^n$ generate strongly continuous semigroups on $L^p(\mathbb{R}^n)$ (where $\mathbb{R}^n$ is equipped with the Lebesgue measure). One has to impose conditions on the corresponding vector field. Note that for continuous flows condition (4.12) is automatically satisfied because for a compact $K \subset X$ the set $\varphi_t^{-1}(K) = \varphi_{-t}(K)$ is compact as the continuous image of a compact set.

Example 4.12 Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-vector field and assume that the derivative DF is uniformly bounded on $\mathbb{R}^n$. Then the ordinary differential equation $y' = F(y)$ possesses a global flow $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which is C^1. Moreover, we have

$$\|D\varphi_t(x)\| \leqslant \mathrm{e}^{M|t|} \quad \text{for all} \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \text{ where } M := \sup\{\|DF(x)\| : x \in \mathbb{R}^n\}\,. \tag{4.13}$$

All these properties were proven in Example 3.15 of B-II. We will show that φ satisfies condition (ii) of Proposition 4.11(a). Hence it induces a strongly continuous (semi-)group of lattice homomorphisms on $L(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$ via $T(t)f = f \circ \varphi_t$. This is done using the change of variables formula as follows.

Let U be an open subset of $\mathbb{R}^n$, then $\varphi_t^{-1}(U) = \varphi_{-t}(U) =: U(-t)$. If λ denotes the Lebesgue measure, then

$$\begin{aligned} \lambda(\varphi_t^{-1}(U)) = \int_{U(-t)} 1 \,\mathrm{d}x = \int_U 1 \circ \varphi_{-t}(x) \cdot |\det D\varphi_{-t}(x)| \,\mathrm{d}x = \\ \int_U |\det D\varphi_{-t}(x)| \,\mathrm{d}x \leqslant \int_U \mathrm{e}^{nM|t|} \,\mathrm{d}x = \mathrm{e}^{nM|t|} \cdot \lambda(U). \end{aligned} \tag{4.14}$$

Here we used (4.13) and the fact that the determinant of an n×n-matrix B satisfies $|\det B| \leqslant \|B\|^n$.

In general, existence of a global flow does not ensure that one can associate a semigroup of bounded linear operators on $L^p(\mathbb{R}^n)$, even if the vector field is C^∞. For example the differential equation $y' = \sin(y^2)$ does not induce a semigroup on $L^p(\mathbb{R})$. There is another important class of differential equations, "Hamiltonian differential equations", which do induce semigroups of lattice homomorphisms on L^p-spaces. In fact, Liouville's Theorem states that the flow corresponding to a Hamiltonian vector field preserves the volume (see **?** , Section 3.3]). Thus assertion (b) of Proposition 4.11(i) is trivially satisfied. Further examples of flows which are measure preserving and therefore induce semigroups of lattice homomorphisms on L^p-spaces are billiard flows on compact convex subsets of $\mathbb{R}^n$ and geodesic flows on Riemannian manifolds (see **?**]).

Notes

Spectral theory for a single positive operator as developed in Chapter V of **?**] is an essential tool for this chapter. Various results on the spectral theory of positive one-parameter semigroups can be found in Chapter 7 of **?**] and in the second part of **?**].

Section 1: That the spectral bound is always an element of the spectrum was stated by **?**], but a valid proof was given by **?**]. This assertion as well as assertion (b) of Theorem 1.1 allow generalizations in various directions. They are valid for ordered Banach spaces (see **?**] and **?**]) and one only needs that A has positive resolvent (see **?**] or **?**]). Theorem 1.2 as well as its corollaries are also valid in ordered Banach spaces. For the analogue in the theory of the Laplace transform we refer to Section 10.5 in **?**] and **?**].

Section 2: Theorem 2.2 is the basis for the subsequent cyclicity results. Pseudoresolvents are discussed e.g. in **?**] or **?**]. For nonpositive semigroups the two assertions stated in Definition 2.8 are no longer equivalent. A special case of Theorem 2.10 was proven by**?**] while the general result is due to **?**]. Instead of pseudo-resolvents on the whole $\mathcal{F}$-product Derndinger works with the semigroup on the semigroup $\mathcal{F}$-product. Therefore he can only consider eigenvalues. Elliptic differential operators as generators of positive semigroups are discussed by many authors, e.g., **?**], **?**], **?**] or**?**].

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Section 3: There exist various notions which are (more or less closely) related to irreducibility, e.g. "positivity improving" in **?**], "u_0-positivity" in **?**] or "quasi-strictly positive" in **?**]). **?**] uses "non-support" instead of irreducible. She also discusses several modifications ("semi-non-support", "strictly non-support", "strongly positive") and the interrelation between these notions. The notion of irreducibility can be extended to the (non-lattice) ordered setting (see **?**]). Assertion (b) of Theorem 3.2 is due to **?**] while special cases can be found in Section XIII.12 of **?**] and in **?**]. Proposition 3.3 is due to **?**]. Retarded equations as dicussed in Example 3.4(c) will be discussed in more detail in Section 3 of C-IV. Example 3.4(d) is a one-dimensional version of the linear transport equation. The higher dimensional equation is more delicate but can be treated similarly (see e.g. **?**], **?**], or **?**]). A special case of Proposition 3.5 can be found in **?**]. Theorem 3.7 and Example 3.6 are taken from **?**]. The most interesting criterion of Theorem 3.7 seems to be condition (iii), since it gives a sufficient condition for the existence of eigenvalues for a sufficiently large class of semigroups. For semigroups induced by measure-preserving flows Theorem 3.8 and Corollary 3.9 are proven in **?**]. Corollary 3.9 is a special case of the Halmos-von Neumann Theorem which classifies irreducible semigroups having discrete spectrum (see **?**], **?**] and **?**] for the general result).

Lemma 3.10 is taken from **?**] and Theorems 3.12 and 3.14 can be found (with slightly different proofs) in **?**].

Section 4: It was **?**] who proved Theorem 4.2. In Corollary 4.3 one can replace boundedness of the semigroup by the assumption that the resolvent grows slowly (see **?**]). Example 4.4 is due to Davies and Proposition 4.5 to H. Kellermann (both unpublished). The spectral decomposition for positive groups as described in Theorem 4.8 is valid in arbitrary Banach lattices (see **?**] and **?**]). This also holds for Corollaries 4.9 and 4.10. Proposition 4.11 and Example 4.12 indicate the relationship of positive groups to dynamical systems. For example, the "Annular Hull Theorem" (see **?**]) is closely related to the results of this section.

Chapter C-IV
Asymptotics of Positive Semigroups on Banach Lattices

Part D
Positive Semigroups on C*- and W*-Algebras

Chapter D-I
Basic Results on Semigroups and Operator Algebras

This is not a systematic introduction to the theory of strongly continuous semigroups on C*- and W*-algebras. We only prepare for the following chapters on spectral and asymptotic theory by fixing the notations and introducing some standard constructions. For results on strongly continuous semigroups on Banach spaces, we refer to Chapter A-I.

1 Notations

1. Let M denote a C*-algebra with unit $\mathbb{1}$, where $M^{sa} := \{x \in M : x^* = x\}$ is the self-adjoint part of M and $M_+ := \{x^*x : x \in M\}$ is the positive cone in M. If M' is the dual of M, then $M'_+ := \{\varphi \in M' : \varphi(x) \geqslant 0, x \in M_+\}$ is a weak*-closed generating cone in M' and $S(M) := \{\varphi \in M'_+ : \varphi(\mathbb{1}) = 1\}$ is called the state space of M. For the theory of C*-algebras and related notions see Pedersen [5].
2. We say that M is a W*-algebra if there exists a Banach space M_* such that its dual $(M_*)'$ is (isomorphic to) M. We call M_* the *predual* of M and $\varphi \in M_*$ a *normal linear functional*. It is known that M_* is unique. For this and other properties of M_*, see Takesaki [6, Chapter III].
3. A map $T \in \mathrm{L}M$ is called *positive* (in symbols $T \geqslant 0$) if $T(M_+) \subseteq M_+$. It is called *n-positive* ($n \in \mathbb{N}$) if $T \otimes \mathrm{Id}_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where Id_n is the identity map on the C*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive.

We call a contraction $T \in \mathrm{L}M$ a *Schwarz map* if T satisfies the so called *Schwarz-inequality*

$$T(x)T(x)^* \leqslant T(xx^*)$$

for all $x \in M$. It is well known that every n-positive contraction, for $n \geqslant 2$ and every positive contraction on a commutative C*-algebra is a Schwarz map. (Takesaki [6, Chapter IV]) As we shall see, the Schwarz inequality is crucial for our investigations.

4. If M is a C*-algebra, we assume that $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup (abbreviated as semigroup), while for W*-algebras we consider weak*-semigroups, i.e. the mapping $(t \mapsto T(t)x)$ is continuous from $\mathbb{R}_+$ into $(M, \sigma(M, M_*))$, where M_* is the predual of M, and every $T(t) \in \mathcal{T}$ is $\sigma(M, M_*)$-continuous. Note that the preadjoint semigroup

$$\mathcal{T}_* = \{T(t)_* \colon T(t) \in \mathcal{T}\}$$

is weakly, hence strongly continuous on M_*. (Chapter A-I, **??**)
5. We call the semigroup $\mathcal{T}$ *identity preserving* if $T(t)\mathbb{1} = \mathbb{1}$ and of *Schwarz type* if every $T(t)$ is a Schwarz map.

For the notations concerning one-parameter semigroups we refer to Part A. In addition, we recommend to compare the results of this section with the corresponding results for commutative C*-algebras, i.e., for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ in Part B.

2 A Fundamental Inequality for the Resolvent

If $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup of Schwarz maps on a C*-algebra M (resp. a weak*-semigroup of Schwarz type on a W*-algebra M) with generator A, then the spectral bound satisfies $s(A) \leqslant 0$. The resolvent $R(\lambda, A)$ exists for $\mathrm{Re}(\lambda) > 0$ and is positive for $\lambda \in \mathbb{R}_+$. There exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \,\mathrm{d}t \quad (x \in M)$$

where the integral exists in the norm topology.

In Bratteli and Robinson [2, Theorem 5] it is shown, that $\mathcal{T}$ is a semigroup of Schwarz type if and only of $\mu R(\mu, A)$ is a Schwarz map for every $\mu \in \mathbb{R}_+$. Here we relate the domination of two semigroups to an inequality for the corresponding resolvent operators. This inequality will be needed later.

Theorem 2.1 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a semigroup of Schwarz type with generator A and $\mathcal{S} = (S(t))_{t\geqslant 0}$ a semigroup with generator B on a* C**-algebra M. If*

$$(S(t)x)(S(t)x)^* \leqslant T(t)(xx^*) \tag{*}$$

for all $x \in M$ and $t \in \mathbb{R}_+$. Then

$$(\mu R(\mu, B)x)\,(\mu R(\mu, B)x)^* \leqslant \mu R(\mu, A)xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$. The same result holds if $\mathcal{T}$ is a weak-semigroup of Schwarz type and $\mathcal{S}$ is a weak*-semigroup on a* W**-algebra M such that $(*)$ is fulfilled.*

Proof From the assumption $(*)$ it follows that

$$\begin{aligned}0 \leqslant (S(r)x - S(t)x)\,(S(r)x - S(t)x)^* \\ &= (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* \\ &\quad - (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \\ &\leqslant T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^*\end{aligned}$$

for every $r, t \in \mathbb{R}_+$ and therefore

$$(S(r)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \leqslant T(r)xx^* + T(t)xx^*.$$

Obviously, $\|S(t)\| \leqslant 1$ for all $t \in \mathbb{R}_+$. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$

$$\begin{aligned}(R(\mu, B)x)\,(R(\mu, B)x)^* &= \left(\int_0^\infty e^{-\mu r} S(r)x\,\mathrm{d}r\right)\left(\int_0^\infty e^{-\mu t} S(t)x\,\mathrm{d}t\right)^* \\ &= \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}((S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^*\,\mathrm{d}r\,\mathrm{d}t\right) \\ &\leqslant \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*)\,\mathrm{d}r\,\mathrm{d}t\right) \\ &= \left(\int_0^\infty e^{-\mu s} ds\right)\left(\int_0^\infty e^{-\mu t} T(t)xx^*\,\mathrm{d}t\right) = \mu^{-1} R(\mu, A)xx^*\end{aligned}$$

where the handling of the integral is justified by Bourbaki [1, Chap. V, §8, n° 4, Proposition 9]. The claim is obtained by multiplying both sides by μ^2. □

Corollary 2.2 *Let $\mathcal{T}$ be a semigroup of Schwarz maps (resp. weak*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with* Re $(\lambda) > 0$ *we have*

$$(R(\lambda, A)x)\,(R(\lambda, A)x)^* \leqslant \mathrm{Re}(\lambda)^{-1} R(\mathrm{Re}(\lambda), A)xx^*\,,\ x \in M.$$

In particular for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in M$

$$(\mu R(\mu + \mathrm{i}\alpha, A)x)(\mu R(\mu + \mathrm{i}\alpha, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Then the semigroup

$$S := \left(e^{-\mathrm{i}(\lambda)t} T(t)\right)_{t \geqslant 0}$$

fulfills the assumption of Theorem 2.1 and $B := A - \mathrm{i}\lambda$ is the generator of S. Consequently $R(\lambda, A) = R(\mathrm{Re}\lambda, B)$ and the corollary follows from Theorem 2.1. □

Remark 2.3 (Bratteli and Robinson [2, Theorem 5]) Since

$$T(t)x = \lim_n \left(\frac{n}{t} R\left(\frac{n}{t}, A\right)\right)^n x, \quad x \in M,$$

it follows from above, that $\mathcal{T}$ is a semigroup of Schwarz-type, if and only if $\mu R(\mu, A)$ is a Schwarz-operator for every $\mu \in \mathbb{R}_+$.

As in Section C-III the following notion will be an important tool for the spectral theory of semigroups on C*- and W*-algebras.

Definition 2.4 Let E be a Banach space and let D be a non-empty open subset of $\mathbb{C}$. A family $\mathcal{R}\colon D \mapsto L(E)$ is called a *pseudo-resolvent* on D with values in E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu) \qquad \text{(Resolvent Equation)}$$

for all λ, μ in D and $R \in \mathcal{R}$.

If $\mathcal{R}$ is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}(\lambda) > 0\}$ with values in a C*- or W*-algebra, then $\mathcal{R}$ is called of Schwarz type if

$$(R(\lambda)x)(R(\lambda)x)^* \leqslant (\mathrm{Re}\lambda)^{-1} R(\mathrm{Re}\lambda)xx^*$$

and *identity preserving* if $\lambda R(\lambda)\mathbb{1} = \mathbb{1}$ for all $\lambda \in D$ and $R \in \mathcal{R}$. For examples and properties of a pseudo-resolvent, see C-III, 2.5.

We state what will be used without further reference.

(i) If $\alpha \in \mathbb{C}$ and $x \in E$ such that $(\alpha - \lambda)R(\lambda)x = x$ for some $\lambda \in D$, then $(\alpha - \mu)R(\mu)x = x$ for all $\mu \in D$ (use the *resolvent equation*).
(ii) If F is a closed subspace of E such that $R(\lambda)F \subseteq F$ for some $\lambda \in D$, then $R(\mu)F \subseteq F$ for all μ in a neighborhood of λ. This follows from the fact that for all $\mu \in D$ near λ the pseudo-resolvent in μ is given by

$$R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}.$$

Definition 2.5 We call a semigroup $\mathcal{T}$ on the *predual* M_* of a W*-algebra M *identity preserving and of Schwarz type* if its adjoint weak*-semigroup has these properties. Similarly, a pseudo-resolvent $\mathcal{R}$ on $D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}(\lambda) > 0\}$ with values in M_* is said to be identity preserving and of Schwarz type if $\mathcal{R}'$ has these properties.

For a semigroup of contractions on a Banach space we have

$$\begin{aligned}\mathrm{Fix}\,(T) &= \bigcap_{t\geqslant 0} \ker(\mathrm{Id} - T(t)) \\ &= \ker(\mathrm{Id} - \lambda R(\lambda, A)) = \mathrm{Fix}\,((\lambda R(\lambda, A))\end{aligned}$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Therefore a semigroup of contractions on M is identity preserving, if and only if the pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}(\lambda) > 0\}$ given by

$$R(\lambda) \coloneqq R(\lambda, A)_{|D}$$

is identity preserving. By Corollary 2.2 an analogous statement holds for *Schwarz type*.

3 Induction and Reduction

1. If E is a Banach space and $\mathcal{S} \subseteq \mathcal{L}(E)$ is a semigroup of bounded operators, then a closed subspace F is called $\mathcal{S}$-invariant, if $SF \subseteq F$ for all $S \in \mathcal{S}$. We call the semigroup $\mathcal{S}_{|F} := \{S_{|F} : S \in \mathcal{S}\}$ the reduced semigroup. Note that for a one-parameter semigroup $\mathcal{T}$ (resp., pseudo-resolvent $\mathcal{R}$) the reduced semigroup is again strongly continuous (resp. $\mathcal{R}_{|F}$ is again a pseudo-resolvent). (Compare A-I, 3.2).
2. Let M be a W*-algebra, $p \in M$ a projection and $S \in \mathrm{L}M$ such that

$$S(p^\perp M) \subseteq p^\perp M \quad \text{and} \quad S(Mp^\perp) \subseteq Mp^\perp,$$

where $p^\perp := \mathbb{1} - p$. Since for all $x \in M$

$$p[S(x) - S(pxp)] = p[S(p^\perp xp) + S(xp^\perp)]p = 0,$$

we obtain $p(Sx)p = p(S(pxp))p$. Therefore, the map

$$S_p := (x \mapsto p(Sx)p) \colon pMp \to pMp$$

is well defined and we call S_p the *induced map*. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$.
3. If $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a weak*-semigroup on M which is of Schwarz type and if $T(t)(p^\perp) \leqslant p^\perp$ for all $t \in \mathbb{R}_+$, then T leaves $p^\perp M$ and $Mp^\perp$ invariant. One can verify that the induced semigroup $T_p = (T(t)p)_{t\geqslant 0}$ is again a weak*-semigroup.

If $\mathcal{R}$ is an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} \colon \mathrm{Re}(\lambda) > 0\}$ with values in M such that $R(\mu)p^\perp \leqslant p^\perp$ for some $\mu \in \mathbb{R}_+$, then $p^\perp M$ and $Mp^\perp$ are $\mathcal{R}$-invariant. It follows directly that the induced pseudo-resolvent $\mathcal{R}_p$ has both the Schwarz type property and is identity preservation.
4. Let φ be a positive normal linear functional on a W*-algebra M such that $T_*\varphi = \varphi$ for some identity preserving Schwarz map T on M with preadjoint $T_* \in L(M_*)$. Then $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$ where $s(\varphi)$ is the support projection of φ.

Let

$$L_\varphi := \{x \in M \colon \varphi(xx^*) = 0\} \quad \text{and} \quad M_\varphi := L_\varphi \cap L_\varphi^*.$$

Since φ is T_*-invariant and T is a Schwarz map, the subspaces L_φ and M_φ are T-invariant. From $M_\varphi = s(\varphi)^\perp M s(\varphi)^\perp$ and $T(s(\varphi)^\perp) \leqslant 1$ it follows that $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$.

Let $T_{s(\varphi)}$ be the induced map on $M_{s(\varphi)}$ and define

$$s(\varphi)M_*s(\varphi) := \{\psi \in M_* \colon \psi = s(\varphi)\psi s(\varphi)\}$$

where $\langle s(\varphi)\psi s(\varphi), x\rangle := \langle \psi, s(\varphi)xs(\varphi)\rangle$ $(x \in M)$. For any $\psi \in s(\varphi)M_s(\varphi)$ and all $x \in M$, the following equalities holds

$$\begin{aligned}(T_*\psi)(x) &= \psi(Tx) = \langle \psi, s(\varphi)(Tx)s(\varphi)\rangle \\ &= \langle \psi, s(\varphi)(T(s(\varphi)xs(\varphi)))s(\varphi)\rangle = \langle T_*\psi, s(\varphi)xs(\varphi)\rangle,\end{aligned}$$

hence $T_*\psi \in s(\varphi)M_*s(\varphi)$. Since the dual of $s(\varphi)M_*s(\varphi)$ is $M_{s(\varphi)}$, it follows that the adjoint of the reduced map $T_{*_|}$ is identity preserving and of Schwarz type.

For example, if $\mathcal{T}$ is an identity preserving semigroup of Schwarz type on M_* such that $\varphi \in \mathrm{Fix}(T)$, then the semigroup $T_{|(s(\varphi)M_*s(\varphi))}$ is again identity preserving and of Schwarz type. Furthermore, if $\mathcal{R}$ is a pseudo-resolvent on

$$D = \{\lambda \in \mathbb{C} \colon \ \mathrm{Re}(\lambda) > 0\}$$

with values in M_* which is identity preserving and of Schwarz type such that $R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$, then $\mathcal{R}_{|s(\varphi)M_*s(\varphi)}$ has the same properties.

Notes

We refer to Bratteli and Robinson [2], Davies [3] and the survey article of Oseledets [4].

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Chapter D-II
Characterization of Positive Semigroups on W*-Algebras

Since the positive cone of a C*-algebra has non-empty interior, many results of Chapter B-II can be applied verbatim to the characterization of the generator of positive semigroups on C*-algebras. On the other hand, a concrete and detailed representation of such generators has been found only in the uniformly continuous case (see Lindblad [4]). A third area of active research has been the following: Which maps on C*-algebras (in particular, which derivations) commuting with certain automorphism groups are automatically generators of strongly continuous positive semigroups. For more information we refer to the survey article of Evans [3].

1 Semigroups on Properly Infinite W*-Algebras

The aim of this section is to show that strongly continuous semigroups of Schwarz maps on properly infinite W*-algebras are already uniformly continuous. In particular, our theorem is applicable to such semigroups on $B(H)$.
It is worthwhile to remark that the result of Lotz [5] on the uniform continuity of every strongly continuous semigroup on L^∞ (see A-II, Sec.3) does not extend to arbitrary W*-algebras.

Example 1.1 Take $M = \mathcal{B}(H)$, H infinite dimensional, and choose a projection $p \in M$ such that Mp is topologically isomorphic to H. Therefore $M = H \oplus M_0$, where $M_0 = \mathrm{Ker}(x \mapsto xp)$. Next, take a strongly, but not uniformly continuous semigroup $\mathcal{T}$ on H and consider the strongly continuous semigroup $\mathcal{T} \oplus \mathrm{Id}$ on M.

For results on the classification theory of W*-algebras needed in our approach we refer to Sakai [7, 2.2] and Takesaki [10, V.1].

Theorem 1.2 *Every strongly continuous one-parameter semigroup of Schwarz type on a properly infinite* W* *-algebra M is uniformly continuous.*

Proof Let $\mathcal{T} = (T(t)_{t\geqslant 0})$ be strongly continuous on M and suppose $\mathcal{T}$ not to be uniformly continuous. Then there exists a sequence (T_n) in $\mathcal{T}$ and $\varepsilon > 0$ such that $\|T_n - \mathrm{Id}\| \geqslant \varepsilon$, but $T_n \to \mathrm{Id}$ in the strong operator topology. We claim that for every sequence (p_k) of mutually orthogonal projections and all bounded sequences (x_k) in M

$$\lim_n \|(T_n - \mathrm{Id})(p_k x_k p_k)\| = 0$$

uniformly in $k \in \mathbb{N}$. This follows from the *Lemma of Phillips* (Schaefer [9]) and the fact that the sequence $(p_k x_k p_k)$ is summable in the $s^*(M, M_*)$-topology (compare Elliot [2], Lemma 2.).
Let (p_k) be a sequence of mutually orthogonal projections in M such that every p_k is equivalent to $\mathbb{1}$ via some $u_k \in M$ [7, 2.2]. Without loss of generality we may assume $\|(T_n - \mathrm{Id})(u_n)\| \leqslant n^{-1}$ since the semigroup T is strongly continuous. Thus we obtained the following.

(i) $\lim_n \|(T_n - \mathrm{Id})(p_k x_k p_k)\| = 0$ uniformly in $k \in \mathbb{N}$ for every bounded sequence (x_k) in M.
(ii) Every projection p_k is equivalent to 1 via some $u_k \in M$.
(iii) $\|(T_n - \mathrm{Id})u_n\| \leqslant n^{-1}$ for all $n \in \mathbb{N}$.

For the following construction see A-I,3.6 and D-II,Sec.2. Take

(i) $\widehat{M}$ be an ultrapower of M,
(ii) let $p := \widehat{(p_k)} \in \widehat{M}$,
(iii) let $T := \widehat{(T_n)} \in L(\widehat{M})$
(iv) and let $u := \widehat{(u_k)} \in \widehat{M}$.

Then T is identity preserving and of Schwarz type on $\widehat{M}$.
Since $u^*u = p$ and $uu^* = \mathbb{1}$, it follows $pu^* = u^*$ and $(uu^*)x(uu^*) = x$ for all $x \in \widehat{M}$. Finally, $T(pxp) = pxp$ for all $x \in \widehat{M}$ which follows from (i), and $T(u^*) = T(pu^*) = pu^* = u^*$ and $T(u) = u$, which follows from (iii). Using the Schwarz, inequality we obtain

$$T(uu^*) = T(\mathbb{1}) \leqslant \mathbb{1} = uu^* = T(u)T(u)^*.$$

From D-III, Lemma 1.1., we conclude $T(ux) = uT(x)$ and $T(xu^*) = T(x)u^*$ for all $x \in \widehat{M}$. Hence

$$\begin{aligned} T(x) &= T(uu^*xuu^*) = uT(u^*xu)u^* = uT(pu^*xup)u^* \\ &= upu^*xupu^* = uu^*xuu^* = x \end{aligned}$$

for all $x \in \widehat{M}$. From this we obtain that for every bounded sequence (x_k) in M

$$\lim_k \|T_k x_k - x_k\| = 0$$

for some subsequence of the T_k's and of the x_k's. This conflicts with our assumption at the beginning, hence the theorem is proved. □

Notes

Let M be a W*-algebra and H be an infinite-dimensional Hilbert-space. Then the W*-tensor product $N := M\overline{\otimes}\mathcal{B}(H)$ is a properly infinite W*-algebra (Sakai [7, Thm. 2.6.6]). Let $\mathcal{S}$ be the semigroup

$$S(t) = T(t) \otimes \mathrm{Id}_H \quad (t \geqslant 0).$$

Then $S(t)$ is a Schwarz-map on N and contractive (Takesaki [10, Prop. IV.5.13.]), hence the smigoup $\mathcal{S}$ is equicontinuous in LN.
Let $x \in M$ and $\xi \in H$. Since the norm on N is a cross-norm, we obtain

$$\lim_{t\to 0} \|(S(t) - \mathrm{Id})x \otimes \xi\| = \lim_{t\to 0} \|(S(t) - \mathrm{Id})x\|\|\xi\| = 0.$$

From Schaefer [8, III.4.5] it follows that $\mathcal{S}$ is strongly-continuous, hence norm-continuous on N from which we conclude, that $\mathcal{T}$ is norm-continuous on M.

Remark 1.3 If M is a finite W*-algebra of Type I, then M is a Grothendieck space and has the Dunford-Pettis property. Hence we can apply the results of Lotz [5]. However, has W*-algebra have the Dunford-Pettis property iff it is finite and of Type I (Chu and Iochum [1]). But is known that every W*-algebra is a Grothendick space (Pfitzner [6].

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Chapter D-III
Spectral Theory of Positive Semigroups on W*-Algebras and their Preduals

Motivated by the classical results of Perron and Frobenius one expects the following spectral properties for the generator A of a positive semigroup on a C*-algebra.

> The spectral bound $s(A) := \sup\{\operatorname{Re}\lambda \colon \lambda \in \sigma(A)\}$ belongs to the spectrum $\sigma(A)$ and the boundary spectrum $\sigma_b(A) := \sigma(A) \cap \{s(A) + \mathrm{i}\mathbb{R}\}$ possesses a certain symmetric structure, called cyclicity.

Results of this type have been proved in Chapter B-III for positive semigroups on commutative C*-algebras, however in the non-commutative case the situation is more complicated. While "$s(A) \in \sigma(A)$" still holds (see Greiner et al. [6] or the notes of this chapter), the cyclicity of the boundary spectrum $\sigma_b(A)$ is true only under additional assumptions on the semigroup (e.g., irreducibility, see Section 1 below).
For technical reasons we consider mostly strongly continuous semigroups on the predual of a W*-algebra M or its adjoint semigroup which is a weak*-continuous semigroup on M.

1 Spectral Theory for Positive Semigroups on Preduals

The aim of this section is to develop a Perron-Frobenius theory for identity preserving semigroups of Schwarz type on W*-algebras. However we will show in the example preceding Theorem 1.11 on page 270 below that the boundary spectrum is no longer cyclic. The appropriate hypothesis on the semigroup implying the desired results seems to be the concept of *irreducibility*.
Let us first recall some facts on normal linear functionals. If φ is a normal linear functional on a W*-algebra M, then there exists a partial isometry $u \in M$ and a positive linear functional $|\varphi| \in M_*$ such that

$$\begin{aligned} \varphi(x) = |\varphi|(xu) &=: (u|\varphi|)(x) \quad (x \in M), \\ u^*u &= s(|\varphi|), \end{aligned}$$

where $s(|\varphi|)$ denotes the support projection of $|\varphi|$ in M. We refer to this as the *polar decomposition* of φ. In addition, $|\varphi|$ is *uniquely determined* by the following two conditions.

$$\left.\begin{aligned} \|\varphi\| &= \||\varphi|\| \\ |\varphi(x)|^2 &\leqslant |\varphi|(xx^*) \quad (x \in M) \end{aligned}\right\} (*)$$

For the polar decomposition of the adjoint φ^*, where $\varphi^*(x) = \overline{\varphi(x^*)}$, we obtain

$$\varphi^* = u^*|\varphi^*|, \quad |\varphi^*| = u|\varphi|u^* \quad \text{and} \quad uu^* = s(|\varphi^*|).$$

It is easy to see that $u^* \in s(|\varphi|)M$ (Takesaki [15, Theorem III.4.2 & Proposition III.4.6]).
If Ψ is a subset of the state space of a C*-algebra M, then Ψ is called *faithful* if $0 \leqslant x \in M$ and $\psi(x) = 0$ for all $\psi \in \Psi$ implies $x = 0$. Moreover Ψ is called *subinvariant* for a positive map $T \in \mathrm{L}M$ (resp.., positive semigroup $\mathcal{T}$) if $T'\psi \leqslant \psi$ for all $\psi \in \Psi$ (resp. $T(t)'\psi \leqslant \psi$ for all $T(t) \in \mathcal{T}$ and $\psi \in \Psi$). Recall that for every positive map $T \in \mathrm{L}M$ there exists a state φ on M such that $T'\varphi = r(T)\varphi$, where $r(T)$ denotes the spectral radius of T (Groh [7, Theorem 2.1]).
Let us start our investigation with two lemmata where $\mathrm{Fix}\,(T)$ is the fixed space of T, i.e., the set $\{x \in M : Tx = x\}$.

Lemma 1.1 *Suppose M to be a* C**-algebra and $T \in LM$ an identity preserving Schwarz map.*

(i) Let $b\colon M \times M \to M$ be a sesquilinear map such that $b(z,z) \geqslant 0$ for all $z \in M$. Then $b(x,x) = 0$ for some $x \in M$ if and only if $b(x,y) = 0$ and $b(y,x) = 0$ for all $y \in M$.

(ii) If there exists a faithful family Ψ of subinvariant states for T on M, then $\mathrm{Fix}\,(T)$ *is a* C**-subalgebra of M and $T(xy) = xT(y)$ for all $x \in$* $\mathrm{Fix}\,(T)$ *and $y \in M$.*

Proof (i) Take $0 \leqslant \psi \in M^*$ and consider $f := \psi \circ b$. Then f is a positive semidefinite sesquilinear form on M with values in $\mathbb{C}$. From the Cauchy-Schwarz inequality it follows that $f(x,x) = 0$ for some $x \in M$ if and only if $f(x,y) = 0$ and $f(y,x) = 0$ for all $y \in M$. Since the positive cone M^*_+ is generating, assertion (i) is proved.
(ii) Since T is positive, it follows that $T(x)^* = T(x^*)$ for all $x \in M$. Hence $\mathrm{Fix}\,(T)$ is a self adjoint subspace of M, i.e., invariant under the involution on M. For every $x, y \in M$ define

$$b(x,y) := T(xy^*) - T(x)T(y)^*.$$

Then b satisfies the assumptions of (i).
If $x \in \mathrm{Fix}\,(T)$, then

$$0 \leqslant xx^* = (Tx)(Tx)^* \leqslant T(xx^*),$$

hence

$$0 \leqslant \psi(T(xx^*) - xx^*) = 0 \quad \text{for all } \psi \in \Psi.$$

But this implies $T(xx^*) = T(x)T(x)^* = xx^*$ and consequently, $b(x,x) = 0$. Hence $T(xy^*) = xT(y)^*$ for all $y \in M$ and (ii) is proved. □

Lemma 1.2 *Let M be a* W**-algebra, T an identity preserving Schwarz map on M and $S \in LM$ such that $S(x)(Sx)^* \leqslant T(xx^*)$ for every $x \in M$.*

(i) If $v \in M$ such that $S(v^*) = v^*$ and $T(v^*v) = v^*v$, then $T(xv) = S(x)v$ for all $x \in M$.
(ii) Suppose there exists $\varphi \in M_*$ with polar decomposition $\varphi = u|\varphi|$ such that $S_*\varphi = \varphi$ and $T_*|\varphi| = |\varphi|$. If the closed subspace $s(|\varphi|)M$ is T-invariant, then $Su^* = u^*$ and $T(u^*u) = u^*u$.

Proof (i) Define a positive semidefinite sesquilinear map $b : M \times M \mapsto M$ by

$$b(x,y) := T(xy^*) - S(x)S(y)^* \quad (x, y \in M).$$

Since $b(v^*, v^*) = 0$ we obtain $b(x, v^*) = 0$ for all $x \in M$, hence $T(xv) = S(x)v$. (Lemma 1.1 (i))
(ii) Since $s(|\varphi|)M$ is a closed right ideal, the closed face $F := s(|\varphi|)(M_+)s(|\varphi|)$ determines $s(|\varphi|)M$ uniquely, i.e.,

$$s(|\varphi|)M = \{x \in M : xx^* \in F\}$$

(Pedersen [13, Theorem 1.5.2]). Since T is a Schwarz map and $s(|\varphi|)M$ is T-invariant, it follows $TF \subseteq F$. On the other hand, if $x \in s(|\varphi|)M$, then $xx^* \in F$. Consequently,

$$0 \leqslant S(x)S(x)^* \leqslant T(xx^*) \in F,$$

whence $S(x) \in s(|\varphi|)M$.
Next we show $T(u^*u) = u^*u$ and $Su^* = u^* \in s(|\varphi|)M$. First of all

$$\begin{aligned} 0 &\leqslant (Su^* - u^*)(Su^* - u^*)^* \\ &\leqslant T(u^*u) - u^*S(u^*)^* - (Su^*)u + u^*u. \end{aligned}$$

Since $S_*\varphi = \varphi$, $T_*|\varphi| = |\varphi|$ and $\varphi = u|\varphi|$ it follows

$$\begin{aligned} 0 &\leqslant |\varphi|((Su^* - u^*)(Su^* - u^*)^*) \\ &\leqslant 2|\varphi|(u^*u) - |\varphi|(S(u^*)u)^* - |\varphi|(S(u^*)u) \\ &= 2|\varphi|(uu^*) - \varphi(u^*)^* - \varphi(u^*) \\ &= 2(|\varphi|(u^*u) - |\varphi|(u^*u)) = 0. \end{aligned}$$

But $(Su^* - u^*)(Su^* - u^*) \in F$ and $|\varphi|$ is faithful on F. Hence we obtain $Su^* = u^*$. Consequently,

$$0 \leqslant u^*u = (Su^*)(Su^*)^* \leqslant T(u^*u)$$

and $T(u^*u) = u^*u$ by the faithfulness and T-invariance of $|\varphi|$. □

Remark 1.3 Take S and T as in Lemma 1.2 (ii). If V_{u^*} (resp. V_u) is the map $(x \mapsto xu^*)$ (resp. $(x \mapsto xu)$) on M, then V_{u^*} is a continuous bijection from $Ms(|\varphi|)$ onto $Ms(|\varphi^*|)$ with inverse V_u (because $V_u \circ V_{u^*} = \mathrm{Id}_{Ms(|\varphi|)}$ and $V_{u^*} \circ V_u = \mathrm{Id}_{Ms(|\varphi^*|)}$). Let $x \in M$. From $T(xu) = S(x)u$ we obtain $T(xu)u^* = S(x)uu^*$. In particular, if $Ms(|\varphi^*|)$ is S-invariant, then

$$(V_{u^*} \circ T \circ V_u)(x) = T(xu)u^* = S(x)$$

for every $x \in Ms(|\varphi^*|)$. Let $T_|$ (resp. $S_|$) be the restriction of T to $Ms(|\varphi|)$ (resp. of $\mathcal{S}$ to $Ms(|\varphi^*|)$). Then the following diagram is commutative:

$$\begin{tikzcd}
Ms(|\varphi|) \arrow[r, "T_|"] \arrow[d, "V_u"'] & Ms(|\varphi|) \arrow[d, "V_{u^*}"] \\
Ms(|\varphi^*|) \arrow[r, "S_|"'] & Ms(|\varphi^*|)
\end{tikzcd}$$

In particular, $\sigma(S_|) = \sigma(T_|)$. Therefore we may deduce spectral properties of $S_|$ from $T_|$ and vice versa. More concrete applications of Lemma 1.2 will follow.

We now investigate the fixed space $\mathrm{Fix}\,(\mathcal{R}) := \mathrm{Fix}\,(\lambda R(\lambda))$, $\lambda \in D$, of a pseudo-resolvent $\mathcal{R}$ with values in the predual of a W*-algebra M.

Proposition 1.4 *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \mathrm{Re}\,\lambda > 0\}$ with values in the predual M_* of a* W*-*algebra M and suppose $\mathcal{R}$ to be identity preserving and of Schwarz type.*

(i) If $\alpha \in \mathbb{R}$ and $\psi \in M_*$ such that $(\gamma - \mathrm{i}\alpha)R(\gamma)\psi = \psi$ for some $\gamma \in D$, then $\lambda R(\lambda)|\psi| = |\psi|$ and $\lambda R(\lambda)|\psi^*| = |\psi^*|$ for all $\lambda \in D$.
(ii) $\mathrm{Fix}\,(\mathcal{R})$ *is invariant under the involution in M_*. If $\psi \in \mathrm{Fix}\,(\mathcal{R})$ is self adjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of* $\mathrm{Fix}\,(\mathcal{R})$.

Proof If $(\gamma - \mathrm{i}\alpha)R(\gamma)\psi = \psi$ then $(\lambda - \mathrm{i}\alpha)R(\lambda)\psi = \psi$ for all $\lambda \in D$. In particular, $\mu R(\mu + \mathrm{i}\alpha)\psi = \psi$ $(\mu \in \mathbb{R}_+)$. For all $x \in M$ we obtain

$$\begin{aligned}
|\psi(x)|^2 &= |< \mu R(\mu + \mathrm{i}\alpha)'x, \psi >|^2 \leqslant \\
&\leqslant \|\psi\| < (\mu R(\mu + \mathrm{i}\alpha)'x)(\mu R(\mu + \mathrm{i}\alpha)'x)^*, \psi > \leqslant \\
&\leqslant \|\psi\| < \mu R(\mu)'(xx^*), |\psi| >
\end{aligned}$$

(D-I, Corollary 2.2). Since

$$\begin{aligned}
\|\psi\| &= \||\psi|\| = |\psi|(1) = \\
&= < \mu R(\mu)'1, |\psi| > = \|\mu R(\mu)|\psi|\|,
\end{aligned}$$

we obtain $\mu R(\mu)|\psi| = |\psi|$ by the uniqueness theorem (*) above for the absolut value— therefore $|\psi| \in \mathrm{Fix}\,(\mathcal{R})$. Since

$$0 \leqslant (\mu R(\mu)' x)(\mu R(\mu)' x)^* \leqslant \mu R(\mu)' x x^*,$$

the map $R(\mu)$ is positive. Consequently $(\mu + \mathrm{i}\alpha) R(\mu)\psi^* = \psi^*$ from which $|\psi^*| \in \operatorname{Fix}(\mathcal{R})$ follows. If $\varphi \in \operatorname{Fix}(\mathcal{R})$ is selfadjoint with Jordan decomposition $\varphi = \varphi^+ - \varphi^-$, then $|\varphi| = \varphi^+ + \varphi^-$ (Takesaki [15, Theorem III.4.2.]). From this we obtain that φ^+ and φ^- are in $\operatorname{Fix}(\mathcal{R})$. □

Corollary 1.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on M_* with generator A and suppose $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.*

(i) If $\alpha \in \mathbb{R}$ and $\psi \in \ker(\mathrm{i}\alpha - A)$, then $|\psi|$ and $|\psi^*|$ are elements of $\operatorname{Fix}(\mathcal{T}) = \operatorname{Ker}(A)$.
(ii) $\operatorname{Fix}(\mathcal{T})$ *is invariant under the involution of M_*. If $\psi \in \operatorname{Fix}(\mathcal{T})$ is selfadjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of* $\operatorname{Fix}(\mathcal{T})$.

The proof follows immediately from Proposition 1.4 and the fact that $\operatorname{Ker}(A) = \operatorname{Fix}(\lambda R(\lambda, A))$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.
If $\mathcal{T}$ is the semigroup of translations on $L^1(\mathbb{R})$ and A' the generator of the adjoint weak*-semigroup, then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$, while $P\sigma(A') \cap \mathrm{i}\mathbb{R} = \mathrm{i}\mathbb{R}$. For that reason we cannot expect a simple connection between these two sets. But as we shall see below, if a semigroup on the predual of a W*-algebra has sufficiently many invariant states, then the point spectra contained in $\mathrm{i}\mathbb{R}$ of A and A' are identical.
Helpful for these investigations will be the next lemma.

Lemma 1.6 *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda > 0\}$ with values in a Banach space E such that $\|R(\mu + \mathrm{i}\alpha)\| \leqslant 1$ for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$. Then*

$$\dim \operatorname{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)) \leqslant \dim \operatorname{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)')$$

for all $\lambda \in D$.

Proof Let $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $S := \mu R(\mu + \mathrm{i}\alpha)$. Since S is a contraction, its adjoint S' maps the dual unit ball E'_1 into itself.
Let $\mathfrak{U}$ be a free ultrafilter on $[1, \infty[$ which converges to 1. Since E'_1 is $\sigma(E', E)$-compact,

$$\psi_0 := \lim_{\mathfrak{U}} (\lambda - 1) R(\lambda, S)' \psi$$

exists for each $\psi \in E'_1$. Since S' is $\sigma(E', E)$-continuous and since $S' R(\lambda, S)' = \lambda R(\lambda, S') - \mathrm{Id}$ we conclude $\psi_0 \in \operatorname{Fix}(S')$.
Take now $0 \neq x_0 \in \operatorname{Fix}(S)$ and choose $\psi \in E'_1$ such that $\psi(x_0)$ is different from zero. From the considerations above it follows

$$\psi_0(x_0) = \lim_{\mathfrak{U}} (\lambda - 1) \psi(R(\lambda, S) x_0) = \psi(x_0) \neq 0$$

hence $0 \neq \psi_0 \in \operatorname{Fix}(S)$. Therefore $\operatorname{Fix}(S')$ separates the points of $\operatorname{Fix}(S)$.
From this it follows that

$$\dim \operatorname{Fix}(S) \leqslant \dim \operatorname{Fix}(S').$$

Since $\mathcal{R}$ and $\mathcal{R}'$ are pseudo-resolvents, the assertion is proved. □

Corollary 1.7 *Let $\mathcal{T}$ be a semigroup of contractions on a Banach space E with generator A. Then*

$$\dim \operatorname{Ker}(\mathrm{i}\alpha - A) \leqslant \dim \operatorname{Ker}(\mathrm{i}\alpha - A')$$

for all $\alpha \in \mathbb{R}$.

This follows from Lemma 1.6 on page 267 because $\operatorname{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)) = \operatorname{Ker}(\mathrm{i}\alpha - A)$.

Proposition 1.8 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra and suppose that there exists a faithful family Ψ of $\mathcal{T}$-invariant states. Then for all $\alpha \in \mathbb{R}$ we have*

$$\dim \operatorname{Ker}(\mathrm{i}\alpha - A) = \dim \operatorname{Ker}(\mathrm{i}\alpha - A')$$

and

$$P\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A') \cap \mathrm{i}\mathbb{R}.$$

Proof The inequality $\dim \operatorname{Ker}(\mathrm{i}\alpha - A) \leqslant \dim \operatorname{Ker}(\mathrm{i}\alpha - A')$ follows from Corollary 1.7.
Let $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > 0\}$ and $\mathcal{R}$ the pseudo-resolvent induced by $R(\lambda, A)$ on D. Then $\mathcal{R}$ is identity preserving and of Schwarz type. Take $\mathrm{i}\alpha \in P\sigma(A)$ ($\alpha \in \mathbb{R}$) and choose $0 < \mu \in \mathbb{R}$.
If $\psi_\alpha \in M_*$ is of norm one with polar decomposition $\psi_\alpha = u_\alpha|\psi_\alpha|$ such that $\psi_\alpha = (\mu - \mathrm{i}\alpha)R(\mu)\psi_\alpha$ then $\mu R(\mu)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4 (i) on page 266).
Since

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant 1 - s(|\psi_\alpha|),$$

we obtain $\mu R(\mu)' s(|\psi_\alpha|) = s(|\psi_\alpha|)$ by the faithfulness of Ψ. Hence the maps $S := (\mu - \mathrm{i}\alpha)R(\mu)'$ and $T := \mu R(\mu)'$ fulfill the assumptions of Lemma 1.2 (ii) on page 265. Therefore $Su_\alpha^* = u_\alpha^*$ or $(\mu - \mathrm{i}\alpha)R(\mu)' u_\alpha^* = u_\alpha^*$ which implies $u_\alpha^* \in D(A')$ and $A' u_\alpha^* = \mathrm{i}\alpha u_\alpha^*$.
If $\mathrm{i}\alpha \in P\sigma(A')$, $\alpha \in \mathbb{R}$, choose $0 \neq v_\alpha$ such that

$$v_\alpha = (\mu - \mathrm{i}\alpha)R(\mu)' v_\alpha \quad (\mu \in \mathbb{R}_+)$$

and $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$.
Since

$$0 \leqslant v_\alpha v_\alpha^* = ((\mu - \mathrm{i}\alpha)R(\mu)' v_\alpha)((\mu - \mathrm{i}\alpha)R(\mu)' v_\alpha)^* \leqslant \mu R(\mu)'(v_\alpha v_\alpha^*),$$

we obtain $\mu R(\mu)'(v_\alpha v_\alpha^*) = v_\alpha v_\alpha^*$ because Ψ is faithful.
Hence from Lemma 1.2 (i) on page 265 it follows that

$$\mu R(\mu)'(xv_\alpha^*) = ((\mu - \mathrm{i}\alpha)R(\mu)'x)v_\alpha^*$$

for all $x \in M$.
Let ψ_α be the normal linear functional $(x \mapsto \psi(xv_\alpha^*))$ on M and note that $\psi_\alpha(v_\alpha) \neq 0$. Then

$$\begin{aligned}\langle x, (\mu - \mathrm{i}\alpha)R(\mu)\psi_\alpha\rangle &= \langle((\mu - \mathrm{i}\alpha)R(\mu)'x)v_\alpha^*, \psi\rangle \\ &= \langle \mu R(\mu)'(xv_\alpha^*), \psi\rangle = \psi(xv_\alpha^*) = \psi_\alpha(x)\end{aligned}$$

for all $x \in M$. Consequently $\mathrm{i}\alpha \in P\sigma(A)$ and

$$\dim \operatorname{Ker}((\mathrm{i}\alpha - A')) \leqslant \dim \operatorname{Ker}((\mathrm{i}\alpha - A))$$

which proves the assertion. □

Remark 1.9 From the above proof we obtain the following: If $0 \neq \psi_\alpha \in \operatorname{Ker}(\mathrm{i}\alpha - A)$ for $\alpha \in \mathbb{R}$ with polar decomposition $\psi_\alpha = u_\alpha|\psi_\alpha|$ ($\alpha \in \mathbb{R}$), then $A'u_\alpha = \mathrm{i}\alpha u_\alpha$. Conversely, if $0 \neq v_\alpha \in \operatorname{Ker}(\mathrm{i}\alpha - A')$, then there exists $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$ and the normal linear form

$$\psi_\alpha := (x \mapsto \psi(xv_\alpha^*))$$

is an eigenvector of A pertaining to the eigenvalue $\mathrm{i}\alpha$.

If $\mathcal{T}$ is a C_0-semigroup of Markov operators on a commutative C*-algebra with generator A, it has been shown in B-III, that the boundary spectrum $\sigma(A) \cap \mathrm{i}\mathbb{R}$ of its generator is additively cyclic. This is no longer true in the non commutative case.

Example 1.10 For $0 \neq \lambda \in \mathrm{i}\mathbb{R}$ and $t \in \mathbb{R}$ let

$$u_t := \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \in M_2(\mathbb{C}).$$

The semigroup of *-automorphisms $(x \mapsto u_t x u_t^*)$ on $M_2(\mathbb{C})$ is identity preserving and of Schwarz type, but the spectrum of its generator is $\{0, \lambda, \lambda^*\}$ hence is not additively cyclic.

It turns out that, in order to obtain a non commutative analogue of the Perron-Frobenius theorems, one has to consider semigroups which are irreducible.
Recall that a semigroup $\mathcal{S}$ of positive operators on an ordered Banach space (E, E_+) is called *irreducible* if no closed face of E_+, different from $\{0\}$ and E_+, is invariant under $\mathcal{S}$. In the context of W*-algebras M we call a semigroup $\mathcal{S}$ of positive maps on M *weak*-irreducible* if no $\sigma(M, M_*)$-closed face of M_+ is $\mathcal{S}$-invariant.
Since the norm closed faces of M_* and the $\sigma(M, M_*)$-closed faces of M are related by formation of polars with respect to the dual system $\langle M, M_*\rangle$ (see Pedersen [13, Theorem 3.6.11 and Theorem 3.10.7.]) a semigroup $\mathcal{S}$ is (norm) irreducible on M_* if and only if its adjoint semigroup is weak*-irreducible.

Theorem 1.11 *Let $\mathcal{T}$ be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra and suppose $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.*

(i) The fixed space of $\mathcal{T}$ is one dimensional and spanned by a faithful normal state.

(ii) $P\sigma(A) \cap \mathrm{i}\mathbb{R}$ is an additive subgroup of $\mathrm{i}\mathbb{R}$,

$$\sigma(A) = \sigma(A) + (P\sigma(A) \cap \mathrm{i}\mathbb{R})$$

and every eigenvalue in $\mathrm{i}\mathbb{R}$ *is simple.*

(iii) The fixed space of the adjoint weak-semigroup $\mathcal{T}'$ is one-dimensional.*

(iv) $P\sigma(A') \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R}$ for the generator A' of the adjoint semigroup, and every $\gamma \in P\sigma(A') \cap \mathrm{i}\mathbb{R}$ is simple.

Proof Since $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$, there exists $\psi \in \operatorname{Fix}(\mathcal{T})_+$ of norm one (Corollary 1.5). If $F := \{x \in M_+ : \psi(x) = 0\}$, then F is a $\sigma(M, M_*)$-closed, $\mathcal{T}'$-invariant face in M, hence $F = \{0\}$. Therefore every $0 \neq \psi \in \operatorname{Fix}(\mathcal{T})_+$ is faithful.
Let $\psi_1, \psi_2 \in \operatorname{Fix}(\mathcal{T})_+$ be states such that $f := \psi_1 - \psi_2$ is different from zero. If $f = f^+ - f^-$ is the Jordan decomposition of f, then f^+ and f^- are elements of $\operatorname{Fix}(\mathcal{T})$, whence faithful. Since the support projections of these two normal linear functionals are orthogonal, we obtain $f^+ = 0$ or $f^- = 0$ which implies $\psi_1 \leqslant \psi_2$ or $\psi_2 \leqslant \psi_1$. Consequently $\psi_2 = \psi_1$.
Since $\operatorname{Fix}(\mathcal{T})$ is positively generated (Corollary 1.5 on page 267), $\operatorname{Fix}(\mathcal{T}) = \{\lambda\varphi : \lambda \in \mathbb{C}\} =: \mathbb{C}.\varphi$ for some faithful normal state φ.
Let $\mu \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ such that $\mathrm{i}\alpha \in P\sigma(A)$. If $\psi_\alpha = u_\alpha|\psi_\alpha|$ is a normalized eigenvector of A pertaining to $\mathrm{i}\alpha$, then $\varphi = |\psi_\alpha| = |\psi_\alpha^*|$ (Corollary 1.5 and the above considerations). Hence $u_\alpha u_\alpha^* = u_\alpha^* u_\alpha = s(\varphi) = 1$.
Since

$$(\mu - \mathrm{i}\alpha)R(\mu, A)\psi_\alpha = \psi_\alpha$$

and

$$\mu R(\mu, A)|\psi_\alpha| = |\psi_\alpha|,$$

we obtain by Lemma 1.2 (ii) on page 265 that

$$\mu R(\mu, A) = V_\alpha \circ \mu R(\mu + \mathrm{i}\alpha, A) \circ V_\alpha^{-1} \quad (1)$$

where V_α is the map $(x \mapsto x u_\alpha)$ on M.
Similarly, for $i\beta \in P\sigma(A)$ we find V_β such that $1 = u_\beta u_\beta^* = u_\beta u_\beta^*$ and

$$\mu R(\mu, A) = V_\beta \circ \mu R(\mu + i\beta, A) \circ V_\beta^{-1}. \quad (2)$$

Hence

$$\mu R(\mu, A) = V_{\alpha\beta} \circ \mu R(\mu + i(\alpha + \beta), A) \circ V_{\alpha\beta}^{-1} \quad (3)$$

where $V_{\alpha\beta} := V_\alpha \circ V_\beta$.
Since u_α is unitary in M, it follows from (1) that $\mathrm{i}\alpha$ is an eigenvalue which is simple because $\operatorname{Fix}(T) = \operatorname{Fix}(\mu R(\mu, A))$ is one dimensional.

From (3) it follows that $i(\alpha + \beta) \in P\sigma(A)$ since $0 \in P\sigma(A)$ and $V_{\alpha\beta}$ is bijective. From the identity (1) we conclude that $\sigma(R(\mu, A)) = \sigma(R(\mu + \mathrm{i}\alpha))$, which proves

$$\sigma(A) + (P\sigma(A) \cap \mathrm{i}\mathbb{R}) \subseteq \sigma(A).$$

The other inclusion is trivial since $0 \in P\sigma(A)$. □

Remarks 1.12 (i) Let φ be the normal state on M such that $\operatorname{Fix}(T) = \mathbb{C}.\varphi$ and let $H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$. From the proof of Theorem 1.10 it follows that there exists a family $\{u_\eta : \eta \in H\}$ of unitaries in M such that $A' u_\eta = -\eta u_\eta$ and $A(u_\eta \varphi) = \eta(u_\eta \varphi)$ for all $\eta \in H$.
(ii) If the group H is generated by a single element, i.e., $H = \mathrm{i}\gamma\mathbb{Z}$ for some $\gamma \in \mathbb{R}$, then $\{u_\gamma^k : k \in \mathbb{Z}\}$ is a complete family of eigenvectors pertaining to the eigenvalues in H, where $u_\gamma \in M$ is unitary such that $A' u_\gamma = \mathrm{i}\gamma u_\gamma$.

Proposition 1.13 *Suppose that $\mathcal{T}$ and M satisfy the assumptions of Theorem 1.10, and let N_* be the closed linear subspace of M_* generated by the eigenvectors of A pertaining to the eigenvalues in* $\mathrm{i}\mathbb{R}$. *Denote by $\mathcal{T}_0$ the restriction of $\mathcal{T}$ to N_*. Then*

(i) $G := (\mathcal{T}_0)^- \subseteq L_s(N_*)$ *is a compact, Abelian group in the strong operator topology.*
(ii) $\mathrm{Id}_{|N_*} \in \overline{\{T_0(t) : t > s\}} \subseteq L_s(N_*)$ *for all* $0 < s \in \mathbb{R}$.

Proof For $\eta \in H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$ let

$$U(\eta) := \{\psi \in D(A) : A\psi = \eta\psi\}$$

and $U = \{U(\eta) : \eta \in H\}$. Then $(U)^- = N_*$.
For each $\psi \in U$ there exists $\eta \in H$ such that

$$\{T_0(t)\psi : t \in \mathbb{R}_+\} = \{e^{-\eta t}\psi : t \in \mathbb{R}_+\}.$$

Consequently this set is relatively compact in $L_s(N_*)$. From [Schaefer (1966),III.4.5] we obtain that G is compact in the strong operator topology.
Next choose $\psi_1, \ldots, \psi_n \in U$, $0 < s \in \mathbb{R}$ and $\delta > 0$. Since $T_0(t)\psi_i = e^{\eta_i t}\psi_i$ $(1 \leqslant i \leqslant n)$ for some $\eta_i \in H$, it follows from a theorem of Kronecker (see, Jacobs [11, Satz 6.1., p.77]) that there exists $s < t$ such that

$$|(1, 1, \ldots, 1) - (e^{\eta_1 t}, e^{\eta_2 t}, \ldots, e^{\eta_n t})| < \delta,$$

hence

$$\sup\{\|\psi_i - T_0(t)\psi_i\| : 1 \leqslant i \leqslant n\} < \delta$$

or $\mathrm{Id}_{|N_*} \in \overline{\{T_0(t) : t > s\}} \subseteq L_s(N_*)$.
Finally we prove the group property of G. Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{R}$ such that $\lim_{\mathfrak{U}} T_0(t) = \mathrm{Id}$ in the strong operator topology. For positive $s \in \mathbb{R}$ let $S := \lim_{\mathfrak{U}} T(t - s)$. Then $ST_0(s) = T_0(s)S = \mathrm{Id}$, hence $T_0(s)^{-1}$ exists in G for all $s \in \mathbb{R}_+$. From this it follows that G is a group. □

Remark 1.14 (i) Let $\kappa : \mathbb{R} \to G$ be given by

$$\kappa(t) = \begin{cases} T_0(t) & \text{if } 0 \leqslant t, \\ T_0(t)^{-1} & \text{if } t \leqslant 0. \end{cases}$$

Then κ is a continuous homomorphism with dense range, i.e., (G, κ) is solenoidal (see Hewitt and Ross [10]).
(ii) The compact group G and the discret group $P\sigma(A) \cap \mathrm{i}\mathbb{R}$ are dual as locally compact Abelian groups.
(iii) Let (G, κ) be a solenoidal compact group and let $N_* = L^1(G)$. Then the induced lattice semigroup $T = (\kappa(t))_{t \geqslant 0}$ fulfils the assertions of Theorem 1.10. For example, if G is the dual of $\mathbb{R}_d$, then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\mathbb{R}$. Since the fixed space of $\kappa(t)$ is given by

$$\mathrm{Fix}\,(\kappa(t)) = \overline{\left(\bigcup_{k \in \mathbb{Z}} \mathrm{Ker}(\frac{2\pi i k}{t} - A)\right)},$$

however no $T(t) \in \mathcal{T}$ is irreducible.
(iv) If $\mathcal{T}$ is the irreducible semigroup of Schwarz type on the predual of $B(H)$ given in Evans [3], then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$.

2 Spectral Properties of Uniformly Ergodic Semigoups

The aim of this section is the study of spectral properties of semigroups which are uniformly ergodic, identity preserving and of Schwarz type. For the basic theory of uniformly ergodic semigroups on Banach spaces we refer to Dunford and Schwartz [2].
Our first result yields an estimate for the dimension of the eigenspaces pertaining to eigenvalues of a pseudo-resolvent.

Proposition 2.1 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \mathrm{Re}\,\lambda > 0\}$ with values in the predual of a* W**-algebra M. If $Fix \lambda R(\lambda)$ is finite dimensional for some $\lambda \in D$, then*

$$\dim \mathrm{Fix}\,((\gamma - \mathrm{i}\alpha)\, R(\gamma)) \leqslant \dim \mathrm{Fix}\,(\lambda R(\lambda))$$

for all $\gamma \in D$ and $\alpha \in \mathbb{R}$.

Proof By D-IV, Remark 3.2.c, we may assume without loss of generality that there exists a faithful family of $\mathcal{R}$-invariant normal states on M. In particular the fixed space N of the adjoint pseudo-resolvent $\mathbb{R}\mathcal{R}'$ is a W*-subalgebra of M with $\mathbb{1} \in N$ (by Lemma 1.1 (ii)). Since N is finite dimensional, there exist a natural number n and a set $P := \{p_1, .., p_n\}$ of minimal, mutually orthogonal projections in N such that $\sum_{k=1}^{n} p_k = \mathbb{1}$. These projections are also mutually orthogonal in M with sum $\mathbb{1}$.

Let R_j be the $\sigma(M, M_*)$-closed right ideal $p_j M$ and L_j the closed left invariant subspace $M_* p_j$ for $(1 \leqslant j \leqslant n)$. Since the map $\mu R(\mu)'$, $\mu \in \mathbb{R}_+$ is an identity preserving Schwarz map, we obtain from Lemma 1.1.b that for all $x \in N$ and $y \in M$,

$$\mu R(\mu)'(xy) = x(\mu \mathcal{R}'(\mu)y).$$

In particular, R_j, resp. L_j are invariant under $\mathcal{R}'$, respectively, $\mathcal{R}$. Furthermore, if $\psi \in L_j$ with polar decomposition $\psi = u|\psi|$, then $u^*u \leqslant s(|\psi|) \leqslant p_j$. Consequently, $|\psi| \in L_j$.
Let now $\alpha \in \mathbb{R}$ and suppose that there exists $\psi_\alpha \in L_j$ of norm 1, $\psi_\alpha = u_\alpha |\psi_\alpha|$, such that

$$\psi_\alpha \in \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)R(\lambda)\right), \lambda \in D.$$

Since $\lambda R(\lambda)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4 on page 266), we obtain

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant (1 - s(|\psi_\alpha|), \mu \in \mathbb{R}_+.$$

From the existence of a faithful family of $\mathcal{R}$-invariant normal states and since $\mathcal{R}'$ is identity preserving, it follows that

$$\mu R(\mu)' s(|\psi_\alpha|) = s(|\psi_\alpha|).$$

Thus $s(|\psi_\alpha|) \leqslant p_j$ and even $s(|\psi_\alpha|) = p_j$ by the minimality property of p_j.
On the other hand, $\psi_\alpha^* \in \operatorname{Fix}\left((\lambda + \mathrm{i}\alpha)R(\lambda)\right)$. As above we obtain

$$\mu R(\mu)' s(|\psi_\alpha^*|) = s(|\psi_\alpha^*|).$$

Consequently, the closed left ideals $Ms(|\psi_\alpha^*|)$ and $Ms(|\psi_\alpha|)$ are $\mathcal{R}'$-invariant.
Next fix $\mu \in \mathbb{R}_+$, let $S := (\mu - \mathrm{i}\alpha)R(\mu)'$ and $T = \mu R(\mu)'$. Then

$$(Sx)(Sx)^* \leqslant T(xx^*),\ S_*(\psi_\alpha^*) = \psi_\alpha^*,\ T_*(|\psi_\alpha^*|) = |\psi_\alpha^*|,$$

and T is an identity preserving Schwarz map. Since $s(|\psi_\alpha^*|)M$ is T-invariant, the assumptions of Lemma 1.2 on page 265 are fulfilled and we obtain for every $x \in M$

$$S(x)u_\alpha^* = T(xu_\alpha^*).$$

The closed left ideal Mp_j is S-invariant, therefore it follows

$$S(x) = T(xu_\alpha^*)u_\alpha\,,\ x \in Mp_j$$

(see Remark 1.3 on page 265). Since u_α does not depend on $\mu \in \mathbb{R}_+$, we obtain for all $\mu \in \mathbb{R}_+$

$$\mu R(\mu + \mathrm{i}\alpha)'x = \mu R(\mu)'(xu_\alpha^*)u_\alpha.$$

Consequently, the holomorphic functions

$$(\mu \mapsto \mu R(\mu)'(xu_\alpha)u_\alpha^*) \quad \text{and} \quad (\mu \mapsto \mu R(\mu + \mathrm{i}\alpha)'x)$$

coincide on $\mathbb{R}_+$ from which we conclude

$$\lambda R(\lambda + \mathrm{i}\alpha)' x = \lambda R(\lambda)'(x u_\alpha^*) u_\alpha$$

for every $\lambda \in D$ and all $x \in M p_j$.
Since the map $(y \mapsto y u_\alpha)$ is a continuous bijection from $M(u_\alpha u_\alpha^*)$ onto $M p_j$ with inverse $(y \mapsto y u_\alpha^*)$, we can deduce that

$$\begin{aligned}\dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha) R(\lambda)' | M p_j\right) &= \dim \operatorname{Fix}\left(\lambda R(\lambda)'\right) | M(u_\alpha u_\alpha^*)\\ &\leqslant \dim \operatorname{Fix}(\mathcal{R}')\,.\end{aligned}$$

Since $\bigoplus_{j=1}^n M p_j = M$ and $\bigoplus_{j=1}^n L_j = M_*$, we obtain

$$\begin{aligned}\dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha) R(\lambda)'\right) &= \dim \operatorname{Fix}\left(\lambda R(\lambda)'\right),\\ &= \dim \operatorname{Fix}\left(\lambda R(\lambda)\right),\end{aligned}$$

and the assertion follows from Lemma 1.6 on page 267. □

Before going on let us recall the basic facts of the *ultrapower* $\hat{E}$ of a Banach space E with respect to some free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ (compare A-I,3.6). If $\ell^\infty(E)$ is the Banach space of all bounded functions on $\mathbb{N}$ with values in E, then

$$c_{\mathfrak{U}}(E) := \{(x_n) \in \ell^\infty(E) :\ \lim_{\mathfrak{U}} \|x_n\| = 0\}$$

is a closed subspace of $\ell^\infty(E)$ and equal to the kernel of the seminorm

$$\|(x_n)\| := \lim_{\mathfrak{U}} \|x_n\|\,,\ (x_n) \in \ell^\infty(E).$$

By the *ultrapower* $\hat{E}$ we understand the quotient space $\ell^\infty(E)/c_{\mathfrak{U}}(E)$ with norm

$$\|\hat{x}\| = \lim_{\mathfrak{U}} \|x_n\|\,,\ (x_n) \in \hat{x} \in \hat{E}.$$

Moreover, for a bounded linear operator $T \in L(E)$, we denote by $\hat{T}$ the well defined operator $\hat{T}\hat{x} := (T x_n) + c_{\mathfrak{U}}(E)$, $(x_n) \in \hat{x}$.
It is clear by virtue of $(x \mapsto (x, x, ..) + c_{\mathfrak{U}}(E))$ that each $x \in E$ defines an element $\hat{x} \in \hat{E}$. This isometric embedding as well as the operator map $(T \mapsto \hat{T})$ are called canonical. In particular, if $\mathcal{R} : (D \to L(E))$ is a pseudo-resolvent, then

$$\hat{\mathcal{R}} := (\lambda \mapsto R(\lambda)^\wedge) : D \to L(\hat{E})$$

is a pseudo-resolvent, too. Recall that the approximative point spectrum $A\sigma(T)$ is equal to the point spectrum $P\sigma(\hat{T})$ (see, e.g., Schaefer [14, Chapter V, §1]).
This construction gives us the possibility to characterize uniformly ergodic semigroups with finite dimensional fixed space.

Lemma 2.2 *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} :\ \operatorname{Re}\lambda > 0\}$ such that $\|R(\mu + \mathrm{i}\alpha)\| \leqslant 1$ for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and suppose*

$$0 < \dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right) < \infty \quad \textit{for some} \quad \lambda \in D\,,\ \alpha \in \mathbb{R}.$$

For the canonical extension $\hat{R}$ on some ultrapower $\hat{E}$, the following assertions hold.

(i) $(\lambda - \mathrm{i}\alpha)^{-1}$ *is a pole of the resolvent* $R(., R(\lambda))$ *for all* $\lambda \in D$.
(ii) $\dim \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) = \dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right)$ *for all* $\lambda \in D$.
(iii) $\mathrm{i}\alpha$ *is a pole of the pseudo-resolvent* $\mathcal{R}$ *and the residue of* $\mathcal{R}$ *and* $R(., R(\lambda))$ *in* $\mathrm{i}\alpha$ *respectively* $(\lambda - \mathrm{i}\alpha)^{-1}$ *are identical.*

Proof Take a normalized sequence (x_n) in E with

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)x_n - x_n\| = 0.$$

The existence of such a sequence follows from the fact that the fixed space of $(\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)$ is non trivial. Suppose (x_n) is not relatively compact. Then we may assume that there exists $\delta > 0$ such that

$$\|x_n - x_m\| > \delta \quad \text{for} \quad n \neq m.$$

Take $k \in \mathbb{N}$ and let $\hat{x}_k$ be the image of (x_{n+k}) in $\hat{E}$. Since

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)x_{n+k} - x_{n+k}\| = 0,$$

the so defined $\hat{x}_k$'s belong to $\operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right)$. Since this space is finite dimensional there exist $j < \ell$, such that

$$\|\hat{x}_j - \hat{x}_\ell\| \leqslant \frac{\delta}{2}.$$

From the definition of the norm in $\hat{E}$ it follows that there are natural numbers $n < m$ such that

$$\|x_n - x_m\| \leqslant \frac{\delta}{2},$$

leading to a contradiction.

Therefore every approximate eigenvector of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ pertaining to α is relatively compact. In particular, it has a convergent subsequence from which it follows that the fixed space of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ is non trivial.

Obviously

$$\dim \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) \leqslant \dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right).$$

If the last inequality is strict, then there exists $\gamma > 0$ and a normalized $\hat{x} \in \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right)$ such that

$$\gamma \leqslant \|\hat{y} - \hat{x}\|$$

for all $y \in \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$.

Take a normalized sequence $(x_n) \in \hat{x}$. Then (x_n) has a convergent subsequence, whence we may assume that $\lim_n x_n = z$ exists in E. Thus $0 \neq z \in \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$. From this we obtain the contradiction

$$0 \leqslant \gamma \leqslant \|\hat{z} - \hat{x}\| = \lim \|z - x_n\| = 0$$

Consequently,

$$\dim \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) = \dim \operatorname{Fix}\left(\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right).$$

Let $\{x_1, ..., x_n\}$ be a base of $\operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$ and choose $\{\varphi_1, ..., \varphi_n\}$ in $\operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)')$ such that $\varphi_k(x_j) = \delta_{k,j}$ (Lemma 1.6. Then

$$E = \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) \oplus \bigcap_{j=1}^{n} \operatorname{Ker}(\varphi_j),$$

where both subspaces on the right are $(\lambda - \mathrm{i}\alpha)R(\lambda)$-invariant and 1 is a pole of $(\lambda - \mathrm{i}\alpha)R(\lambda)|_{\operatorname{Fix}(()(\lambda - \mathrm{i}\alpha)R(\lambda))}$ by the finite dimensionality of $\operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$. Suppose 1 belongs to the spectrum of S where S is the restriction of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ to $\bigcap_{j=1}^{n} \ker \varphi_j$. Then there exists a normalized sequence (y_n) in $\bigcap_{j=1}^{n} \operatorname{Ker}(\varphi_j)$ such that

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)y_n - y_n\| = 0.$$

Therefore (y_n) has an accumulation point different from zero contained in

$$\operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) \cap (\bigcap_{j=1}^{n} \ker \varphi_j).$$

This contradiction implies that 1 does not belong to the spectrum of S.
Since $\operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$ is finite dimensional, it follows from general spectral theory that $(\lambda - \mathrm{i}\alpha)^{-1}$ is a pole of $R(., R(\lambda))$ for every λ. Thus (i) and (ii) are proved and assertion (iii) follows from the resolvent equality as in the proof of Greiner [4, Proposition 1.2]. □

Proposition 2.3 *Let $\mathcal{T}$ be a semigroup of contractions on a Banach space E with generator A. Then the following assertions are equivalent.*

(a) Each $\mathrm{i}\alpha$, $\alpha \in \mathbb{R}$, *is a pole of the resolvent* $R(., A)$ *such that the corresponding residue has finite rank.*
(b) $\dim \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda, A)\right) < \infty$ *for some (hence all)* $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$ *and the canonical extensions* $\hat{R}(\lambda, A)$ *of* $R(\lambda, A)$ *to some ultrapower.*

Proof Let P_α be the residue of the resolvent $R(., A)$ in $\mathrm{i}\alpha$. Then $P_\alpha = \lim_{\lambda \to \mathrm{i}\alpha}(\lambda - \mathrm{i}\alpha)R(\lambda, A)$ in the operator norm of $L(E)$. Since the canonical map $(T \mapsto \hat{T})$ is isometric and since $\hat{E}$ is an ultrapower, we obtain

$$\hat{P}_\alpha = \lim_{\lambda \to \mathrm{i}\alpha} (\lambda - \mathrm{i}\alpha)\hat{R}(\lambda, A)$$

in $L(\hat{E})$ and $\operatorname{rank}(P_\alpha) = \operatorname{rank}(\hat{P}_\alpha)$. Because of

$$\hat{P}_\alpha(\hat{E}) = \operatorname{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right)$$

one part of the corollary is proved. The other follows from Lemma 2.2 on page 274. ◻

Remarks 2.4 (i) By the results in Lin [12] a semigroup of contractions on a Banach space is uniformly ergodic if and only if 0 is a pole of the generator with order $\leqslant$ 1. The residue of the resolvent in 0 and the assocciated ergodic projection are identical.
(ii) Let M be a W*-algebra with predual M_*, $\mathfrak{U}$ a free ultrafilter on $\mathbb{N}$ and $\widehat{M}$ (resp. $(M_*)^\wedge$) the ultrapower of M (resp. M_*) with respect to $\mathfrak{U}$. Then it is easy to see that $c_{\mathfrak{U}}(M)$ is a two sided ideal in $\ell^\infty(M)$ hence $\widehat{M}$ is a C*-algebra, but in general not a W*-algebra. Note that the unit of $\widehat{M}$ is the canonical image of 1. For $\hat{x} \in \hat{M}$ and $\hat{\varphi} \in (M_*)^\wedge$ let $J : (M_*)^\wedge \to \widehat{M}'$ be defined by

$$\langle x, J(\hat{\varphi})\rangle := \lim_{\mathfrak{U}} \varphi_n(x_n)\,,\ (x_n) \in \hat{x}\,,\ (\varphi_n) \in \hat{\varphi}.$$

Then J is well defined and an isometric embedding. It turns out that $J((M_*)^\wedge)$ is a translation invariant subspace of $\widehat{M'}$. Hence there exists a central projection $z \in \widehat{M''}$ such that $J((M_*)^\wedge) = \widehat{M''}z$ (Groh [9, Proposition 2.2]).
Below we identify $(M_*)^\wedge$ via J with this translation invariant subspace. From the construction the following is obvious: If T is an identity preserving Schwarz map with preadjoint $T_* \in L(M_*)$, then $\widehat{T}$ is an identity preserving Schwarz map on $\widehat{M}$ such that $(T_*)^\wedge = \widehat{T}'|(M_*)^\wedge$.

Theorem 2.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra M. If $\mathcal{T}$ is uniformly ergodic with finite dimensional fixed space, then every $\gamma \in \sigma(A) \cap \mathrm{i}\mathbb{R}$ is a pole of the resolvent $R(.,A)$ and* $\dim\ \mathrm{Ker}(\gamma - A) \leqslant \dim\ \mathrm{Fix}\,(T)$.

Proof Let $D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}\,\lambda > 0\}$ and $\mathcal{R}$ the M_*-valued pseudo-resolvent of Schwarz type induced by $R(.,A)$ on D. Then

$$P = \lim_{\mu\downarrow 0} \mu R(\mu)$$

exists in the uniform operator topology. Since $P(E) = \mathrm{Fix}\,(T)$, we obtain $\hat{P}(\hat{E}) = \mathrm{Fix}\,(\hat{T})$ and $\dim\ \mathrm{Fix}\,(T) = \dim\ \mathrm{Fix}\,(\hat{T}) < \infty$, where $\hat{P}$ is the canonical extension of P onto $(M_*)^\wedge$. Since $\hat{P} = \lim_{\mu\downarrow 0} \mu R(\mu)^\wedge$ it follows that

$$\dim\ \mathrm{Fix}\,\big((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\big) \leqslant \dim\ \mathrm{Fix}\,\big(\hat{T}\big) < \infty$$

for all $\alpha \in \mathbb{R}$ (Proposition 2.1 on page 272). Therefore the assertion follows from Lemma 2.2 on page 274. ◻

The consequences of this result for the asymptotic behavior of one-parameter semigroups will be discussed in D-IV, Section 4.

Notes

Section 1: The Perron-Frobenius theory for a single positive operator on a non-commutative operator algebra is worked out in Albeverio and Hoegh-Krohn [1] and Groh [7]. The limitations of the theory (in the continuous as in the discrete case) are explained by the example following Remark 1.9 on page 269 (see also Groh [8].Therefore we concentrate on irreducible semigroups. Our main result Theorem 1.11 on page 270 extends B-III, Thm.3.6 to the non-commutative setting.

Section 2: Theorem 2.5 on page 277 has its roots in the Niiro-Sawashima Theorem for a single irreducible positive operator on a Banach lattice (see Schaefer [14, V.5.4]). The analogous semigroup result on Banach lattices is due to Greiner [5]. The ultrapower technique in our proof is developed in Groh [9].

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Chapter D-IV
Asymptotics of Positive Semigroups on C*- and W*-Algebras

1 Stability of Positive Semigroups

As explained in A-III, Section 1, it is possible to deduce uniform exponential stability of strongly continuous semigroups from the location of the spectrum of its generator if the spectral bound $s(A)$ and the growth bound ω_0 coincide. In this section we prove $s(A) = \omega_0$ for positive semigroups on C*-algebras and preduals of W*-algebras. A more general discussion of the "$s(A) = \omega_0$" problem can be found in Greiner et al. [7]. For the results of this section the existence of a unit is essential.

Theorem 1.1 *Let M be a* C*-*algebra with unit and $\mathcal{T} = (T(t))_{t \geqslant 0}$ a positive semigroup on M. Then*

$$-\infty < s(A) = \omega_0 \in \sigma(A).$$

Proof For every $t \geqslant 0$ there exists φ_t in the state space $S(M)$ of M such that

$$T(t)'\varphi_t = r(T(t))\varphi_t = \exp(\omega_0 t)\varphi_t$$

(see, e.g., Groh [8, 2.1]).
Let $n \in \mathbb{N}$ and

$$E_n := \{\varphi \in S(M) : T(2^{-n})\varphi = \exp(\omega_0 2^{-n})\varphi\}.$$

Then $\emptyset \neq E_{n+1} \subseteq E_n$, $(n \in \mathbb{N})$. Since $S(M)$ is $\sigma(M, M')$-compact, there exists $\varphi \in \bigcap_{n \in \mathbb{N}} E_n$. Then $T(t)'\varphi = \exp(\omega_0 t)\varphi$ follows for all $0 \leqslant t$ because the adjoint semigroup $(T(t)')_{t \geqslant 0}$ is a weak*-semigroup on M'.
Suppose $-\infty = \omega_0$. Then for $t > 0$ either $r(T(t)) = 0$ (A-III,Prop.1.1) or $T(t)'\varphi = 0$, in particular $\varphi(T(t)\mathbb{1}) = 0$. From this we obtain the contradiction $\varphi(\mathbb{1}) = 0$. Hence $-\infty < \omega_0$ and $\exp(\omega_0 t) \in \varrho(T(t)')$ for every $t \in \mathbb{R}_+$. Thus $\omega_0 \in \sigma(A)$ or $\omega_0 = s(A)$. □

Remark 1.2 (i) If we consider the nilpotent translation semigroup on the C*-algebra $C_0([0,1[)$, then $\sigma(A) = \emptyset$ and $\omega_0 = -\infty$. This shows that the existence of a unit is essential.
(ii) The equality $s(A) = \omega_0$ still holds for positive semigroups on commutative C*-algebras without unit (see B-IV, Rem.1.2.b).

Theorem 1.3 *Let M be a* W*-*algebra with predual M_* and let $(T(t))_{t\geqslant 0}$ be a positive semigroup on M_*. Then $s(A) = \omega_0$.*

Proof For all $\lambda > s(A)$ and $\varphi \in M_*$

$$R(\lambda, A)\varphi = \int_0^\infty e^{-\lambda t} T(s)\varphi ds$$

which follows as in C-III, Section 1 or Greiner et al. [7, Theorem 3]. Since $\|\varphi\| = \varphi(\mathbb{1})$ for every $\varphi \in M_*^+$ and since the norm is additive on the positive cone of M_*, the integral

$$\int_0^\infty e^{\lambda t}\|T(s)\varphi\| ds$$

exists for all $\varphi \in M_*$ and all $\lambda > s(A)$. From this the assumption follows by A-IV,Thm.1.11. □

Corollary 1.4 *Let M be a* C*-*algebra and $(T(t))_{t\geqslant 0}$ a positive semigroup on M'. Then $s(A) = \omega_0$ holds.*

This follows from the fact that the bidual of a C*-algebra is a W*-algebra (see Takesaki [23, Theorem III.2.4.]).

Remark 1.5 A simple modification of A-III, Example 1.4 (take c_0 instead of ℓ^2) shows that Theorem 1.3 is no longer true for non-positive semigroups (for details see Groh and Neubrander [12, Beispiel 2.5]).
While the growth bound ω_0 characterizes uniform exponential stability of the semigroup there are other (and weaker) stability concepts (cf. A-IV, Section 1).

Definition 1.6 Let E be a Banach space and $(T(t))_{t\geqslant 0}$ a semigroup on E. We call the semigroup

(i) *uniformly exponentially stable* if $\|T(t)\| \leqslant Me^{-\omega t}$ for some $\omega, M > 0$ and all $t \geqslant 0$.
(ii) *uniformly stable* if $\lim_{t\to\infty} T(t) = 0$ in the strong operator topology.
(iii) *weakly stable* if $\lim_{t\to\infty} T(t) = 0$ in the weak operator topology.

Surprisingly all these properties coincide for positive semigroups on C*-algebras with unit.

Theorem 1.7 *Let M be a C*-algebra with unit and $(T(t))_{t\geqslant 0}$ a positive semigroup on M. Then the following assertion are equivalent.*

(a) $s(A) < 0$.

(b) The semigroup $(T(t))_{t\geqslant 0}$ is uniformly exponentially stable.
(c) The semigroup $(T(t))_{t\geqslant 0}$ is uniformly stable.
(d) The semigroup $(T(t))_{t\geqslant 0}$ is weakly stable.

Proof Since $s(A) = \omega_0$ by Theorem 1.3, it suffices to show that (d) implies (a). For $t > 0$ there exists $\varphi \in S(M)$ such that

$$T(t)'\varphi = r(T(t))\varphi.$$

Then for $x \in M$

$$\varphi(T(t)^n x) = (r(T(t)))^n \varphi(x) \to 0$$

as $n \to \infty$. Therefore $r(T(t)) < 1$ or $\omega_0 < 0$. Since $s(A) \leqslant \omega_0$ the assertion follows. □

Remark 1.8 Consider the translation semigroup $(T(t))_{t\geqslant 0}$ on $C_0(\mathbb{R}_+)$. Then $\|T(t)\| = 1$, hence $s(A) = 1$, but $(T(t))_{t\geqslant 0}$ is strongly stable. The same holds for the translation semigroup on $L^1(\mathbb{R}_+)$. Thus Theorem 1.7 is not true for semigroups on C*-algebras without unit or on preduals of W*-algebras. For the discussion of the commutative situation we refer to B-IV, Section 1.

2 Stability of Implemented Semigroups

Let H be a Hilbert space, $\mathcal{U} = (U(t))_{t\geqslant 0}$ a strongly continuous semigroup on H with generator B and $M \subseteq \mathcal{B}(H)$ a W*-algebra, where $\mathcal{B}(H)$ is the W*-algebra of all bounded linear operators on H. Suppose $\mathcal{U}(t)^* M U(t) \subseteq M$. Then one can define a weak*-continuous semigroup $\mathcal{T}$ on M by

$$T(t)x := U(t)^* x U(t) \quad (t \in \mathbb{R}_+, x \in M).$$

We call $\mathcal{T}$ an *implemented semigroup*. Every map $T(t) \in \mathcal{T}$ of an implemented semigroup is weak*-continuous and n-positive for every $n \in \mathbb{N}$.

Remarks 2.1 (i) Because of

$$\|T(t)\| = \|T(t)\mathbb{1}\| = \|U(t)^* U(t)\| = \|U(t)\|^2$$

it follows that $\omega_0(\mathcal{T}) = 2\,\omega_0(\mathcal{U})$.
(ii) If $\mathcal{T}$ is an implemented semigroup, then the preadjoint semigroup is strongly continuous on M_*. Therefore $s(A) = \omega_0$ for $\mathcal{T}$ by Theorem 1.3.
(iii) Since $\mathcal{U}$ is a strongly continuous semigroup on H, the same is true for the adjoint semigroup $\mathcal{U}^* = \{U(t)^* : U(t) \in \mathcal{U}\}$ and its generator is given by B^*. In analogy to Bratteli and Robinson [2, 3.2.55] the following assertions for $x \in M$ are equivalent.

(a) $x \in D(A)$, A the generator of $\mathcal{T}$.

(b) For $\xi \in D(B)$ it follows $x\xi \in D(B^*)$ and the linear mapping

$$(\xi \mapsto x(B\xi) + B^*(x\xi)) : D(B) \to H \tag{*}$$

has a continuous extension to H.

Then for A is given as the continuous extension of (*), i.e., $Ax = xB + B^*x$ for $x \in D(A)$

In the next theorem we give some equivalent conditions for the uniform exponential stability of an implemented semigroup. As we shall see, the operator equality

$$yB + B^*y = -x \quad (x, y \in M_+)$$

is necessary and sufficient, which is in complete analogy to the classical Liapunov stability result.

Theorem 2.2 *Let M be a* W*-*algebra on a Hilbert space H and let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a weak*-semigroup on M with generator A implemented by the semigroup $\mathcal{U}$ on H with generator B. Then the following assertions are equivalent.*

(a) $\omega_0(\mathcal{T}) = s(A) < 0$.
(b) The semigroup $(U(t))_{t\geqslant 0}$ is uniformly exponentially stable.
(c) There exists $0 \leqslant x \in D(A)$ such that $Ax = -\mathbb{1}$.
(d) There exists $0 \leqslant x \in D(A)$ such that $x(D(B)) \subseteq D(B^*)$ and $xB + B^*x = -\mathbb{1}$.
(e) For every $0 \leqslant x \in D(A)$ there exists $0 \leqslant y \in D(A)$ such that $Ay = -x$.
(f) For every $0 \leqslant x \in D(A)$ there exists $0 \leqslant y \in D(A)$ such that $y(D(B)) \subseteq D(B^*)$ and $yB + B^*y = -x$.
(g) $\int_0^\infty \|U(s)\xi\|^2 ds$ exists for all $\xi \in H$.
(h) $\int_0^\infty |(T(s)x)\xi|\zeta) ds$ exists for all $\xi, \zeta \in H$ and all $x \in M$.

Proof The equivalence of (a) and (b) follows from Remark 2.1 (i), whereas (c) and (d)), resp. (e) and (f) are equivalent by the Remark 2.1 (iii)

(a) $\implies$ (c): Since $s(A) < 0$ the resolvent $R(0, A)$ exists and is a positive map on M. Therefore $R(0, A)1 \in D(A)_+$ or $Ax = -\mathbb{1}$ for some $x \in D(A)_+$.

(c) $\implies$ (e): Let $x \in D(A)_+$ such that $Ax = -\mathbb{1}$. Then

$$T(t)x - x = \int_0^t T(s)Ax \, ds = -\int_0^t T(s)\mathbb{1} \, ds \quad (t \geqslant 0),$$

hence

$$0 \leqslant \int_0^t T(s)\mathbb{1} \, ds \leqslant x \quad (t \in \mathbb{R}_+).$$

Since the family $(\int_0^t T(s)\mathbb{1} \, ds)_{t\geqslant 0}$ is increasing and bounded,

$$\lim_{t\to\infty} \int_0^t T(s)\mathbb{1} \, ds$$

exists in the weak operator topology on $\mathcal{B}(H)$.

Since on bounded sets of M, the weak operator topology is equivalent to the $\sigma(M, M_*)$-topology, for every $\varphi \in M_*$ the integral $\int_0^\infty \varphi(T(s)\mathbb{1})\,\mathrm{d}s$ exists (Sakai [19, 1.15.2.]). Take $x \in M_+$ and $\varphi \in M_*^+$. Then $x \leqslant \|x\|\mathbb{1}$ and therefore

$$\varphi(T(s)x) \leqslant \|x\|\varphi(T(s)\mathbb{1}) \quad (s \in \mathbb{R}_+).$$

Hence $\int_0^\infty \varphi(T(s)x)ds$ exists. Since the positive cones of M and M_* are generating, $\int_0^\infty \varphi(T(s)x)ds$ exists for every $x \in M$ and $\varphi \in M_*$. Therefore $R(0, A)$ exists and is positive which proves (e).
(c) $\implies$ (g): From the last paragraph we obtain that for all $\xi \in H$

$$\int_0^\infty \|U(s)\|^2 ds = \int_0^\infty (T(s)1\xi|\xi)ds$$

exists.
(g) $\implies$ (h): It follows from the polarization identity that the integral

$$\int_0^\infty (U(s)\xi|U(s)\zeta)ds$$

exists for all $\xi, \zeta \in H$. Using Takesaki [23, Theorem III.4.2 and Theorem II.2.6], we conclude as in the implication from (c) to (e) that for all $\xi, \zeta \in H$ the integral

$$\int_0^\infty ((T(s)x)\xi|\zeta)ds \quad (x \in M)$$

is finite.
(g) $\implies$ (a): Since the vector states are dense in the predual of M and since the preadjoint semigroup of $\mathcal{T}$ is strongly continuous, it is easy to see that the integral

$$\int_0^\infty \varphi(T(s)x)ds$$

exists for all $x \in M$ and $\varphi \in M_*$ (Takesaki [23, Theorem II.2.6]). Therefore, the resolvent $R(0, A)$ exists and is positive, hence $s(A) < 0$. □

3 Convergence of Positive Semigroups

In this section the asymptotic behavior of positive semigroups $(T(t))_{t\geqslant 0}$ on W*-algebras will be described in more detail. Essentially we distinguish three cases.

(i) The Cesàro means $\frac{1}{s}\int_0^s T(t)dt$ converge strongly to a projection P onto the fixed space of $(T(t))_{t\geqslant 0}$ (see Proposition 3.3 & 3.4).
(ii) The maps $T(t)$ converge strongly to P (see Proposition 3.7, **??** &. **??**).
(iii) The maps $T(t)$ behave asymptotically as a periodic group (Theorem 3.11).

Much of the following is based on the theory of weakly compact operator semigroups. Therefore the following compactness criterium is quite useful.

Proposition 3.1 *Let M be a* W*-*-algebra, $\mathcal{T}$ a bounded semigroup of positive maps on M_* and suppose that there exists a faithful family Φ of $\mathcal{T}$-subinvariant states in M_*. Then $\mathcal{T}$ is relatively compact in the weak operator topology of LM_*. In particular, $\mathcal{T}$ is strongly ergodic, i.e.,*

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s T(t)x\,\mathrm{d}t$$

exists for every x in M and yields a projection onto $\operatorname{Fix}(\mathcal{T})$.

Proof Since the positive cone of M_* is generating, it is enough to show that for every $0 \leqslant \varphi \in M_*$ the orbit $\{T(t)\varphi \colon t \in \mathbb{R}_+\}$ is relatively weak compact. For this we use Takesaki [23, Theorem III.5.4.(iii)].
Let $(p_n)_{n\in\mathbb{N}}$ be a decreasing sequence of projections in M such that $\inf_n p_n = 0$. Then $\lim_n \varphi(p_n) = 0$ for every $\varphi \in \Phi$. Since

$$(T(t)p_n)^2 \leqslant T(t)p_n, \quad t \in \mathbb{R}_+,$$

we obtain by a classical *inequality of Kadison* that

$$0 \leqslant \varphi((T(t)p_n)^2) \leqslant \varphi(T(t)p_n) \leqslant \varphi(p_n),$$

hence $\lim_n \varphi(T(t)p_n) = 0$ uniformly in $t \in \mathbb{R}_+$. Since the family Φ is faithful on M, it follows from Takesaki [23, Proposition III.5.3] that $(T(t)p_n)$ converges to zero in the $s(M, M_*)$-topology uniformly in $t \in \mathbb{R}_+$. Since this topology is finer than the weak*-topology on M, we obtain the relative compactness of $\mathcal{T}$ which implies the strong ergodicity. □

Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a W*-algebra M. We call

$$p_r := \sup\{s(|\varphi|) \colon \varphi \in \operatorname{Fix}(T)\}$$

the recurrent projection associated with $\mathcal{T}$. For a motivation of this definition compare, e.g., Davies [3, Section 6.3].
Since $T(t)|\varphi| = |\varphi|$ for all $\varphi \in \operatorname{Fix}(T)$ (D-III, Cor. 1.5), we obtain $T(t)' p_r \geqslant p_r$ (see D-I,Sec.3.(c)). Let $\mathcal{T}^{(r)}$ be the reduced semigroup on $p_r M_* p_r$ with generator $A^{(r)}$. Then $\mathcal{T}^{(r)}$ is identity preserving and of Schwarz type. Similarly, if $\mathcal{R}$ is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \operatorname{Re}(\lambda) > 0\}$ with values in M_* such that $\mathcal{R}$ is identity preserving and of Schwarz type, then the recurrent projection associated with $\mathcal{R}$ is defined using $\operatorname{Fix}(\mathcal{R})$.

Remark 3.2 (i) Let $\varphi \in M_*$ and $\alpha \in \mathbb{R}$ such that $(\mu - \mathrm{i}\alpha)R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$. Since $s(|\varphi|)$ and $s(|\varphi^*|)$ are majorized by p_r (D-III,Prop.1.4), it follows that φ and φ^* are in $p_r M_* p_r$.

(ii) From (i) and the observation that the family $\{|\varphi|: \varphi \in \mathrm{Fix}(\mathcal{T})\}$ is faithful on $p_r M p_r$ and consists of $\mathcal{T}^{(r)}$-invariant elements, it follows that

- $P\sigma(A) \cap \mathrm{i}\mathbb{R} = P_\sigma(A^{(r)}) \cap \mathrm{i}\mathbb{R}$.
- $\mathrm{Ker}((\mathrm{i}\alpha - A)) \subset p_r M_* p_r$ for all $\alpha \in \mathbb{R}$.
- The semigroup $\mathcal{T}^{(r)}$ is relatively compact in the weak operator topology and therefore strongly ergodic.

(iii) Similarly, let $\mathcal{R}$ be an identity preserving pseudo-resolvent with values in M_* on $D = \{\lambda \in \mathbb{C}: \mathrm{Re}(\lambda) > 0\}$ which is of Schwarz type. It follows as in (b) that $\mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$ is contained in $p_r M_* p_r$ for all $\lambda \in D$ and $\alpha \in \mathbb{R}$, where p_r is the associated recurrent projection.

We now give a characterization of strong ergodicity of semigroups which are identity preserving and of Schwarz type. For this we need that the Cesàro means

$$C(s)x = \frac{1}{s}\int_0^s T(t)x\,dt \quad (x \in M, 0 \leqslant s \in \mathbb{R})$$

are Schwarz maps. We omit the simple calculation (compare D-I,Thm.2.1).

Proposition 3.3 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W*-*algebra M. Then the following assertions are equivalent.*

(a) $\mathcal{T}$ is strongly ergodic on M_*.
(b) $\sigma(M, M_*)$- $\lim_{s\to\infty} C(s)' p_r = 1$.
(c) $s^*(M, M_*)$- $\lim_{s\to\infty} C(s)' p_r = 1$.

Proof Suppose that (a) holds. Since Fix (T) separates $\mathrm{Fix}(T')$ (see Krengel [14, Chap.2,Thm.1.4]), the fixed space of $\mathcal{T}'$ is non trivial, hence $p_r \neq 0$. Let $0 \leqslant \psi \in M_*$, then $\psi_0 := \lim_{s\to\infty} C(s)\psi \in$ Fix (T) and $s(\psi_0) \leqslant p_r$. Therefore

$$\begin{aligned}\lim_{s\to\infty} \psi(C(s)' p_r) &= \lim_{s\to\infty} (C(s)\psi)(p_r) = \psi_0(p_r)\\ &= \psi_0(1) = \lim_{s\to\infty} (C(s)\psi)(1) = \psi(1)\end{aligned}$$

which proves (b).
Suppose that (b) is satisfied. Since $C(s)' p_r \leqslant 1$ for all $s \in \mathbb{R}_+$, we obtain (c). (Use that for $(x_\alpha) \in M_+$ we have $\lim_\alpha x_\alpha = 0$ in the weak*-topology if and only if $\lim_\alpha x_\alpha = 0$ in the $s^*(M, M_*)$-topology.)
Suppose that (c) holds. Since each $C(s)'$ is an identity preserving Schwarz map, we obtain for all $x \in M$

$$\begin{aligned}(C(s)'((1-p_r)x))(C(s)'((1-p_r)x)^*) &\leqslant C(s)'((1-p_r)xx^*(1-p_r))\\ &\leqslant \|x\|^2 C(s)'(1-p_r),\end{aligned}$$

hence

$$s^*(M, M_*)\text{-}\lim_{s\to\infty} C(s)'((1-p_r)x) = 0.$$

In particular, we obtain for all $x \in \mathrm{Fix}(\mathcal{T}')$ that $x = \sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)'x = \sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)'(p_r x)$.
Especially for $0 \neq x \in \mathrm{Fix}(\mathcal{T})$ we obtain $p_r x p_r \neq 0$. Since the W*-algebra $p_r M p_r$ is the dual of $p_r M_* p_r$ and since $\mathcal{T}^{(r)}$ is strongly ergodic, it follows that the fixed space of $\mathcal{T}$ separates the points of $\mathrm{Fix}(\mathcal{T}')$. Thus $\mathcal{T}$ is strongly ergodic (Krengel [14, Chap. 2, Thm. 1.4]). □

It follows from the result above that the semigroup in Evans [5] cannot be strongly ergodic on $\mathcal{B}(H)_*$ since the associated recurrent projection is zero. But for irreducible semigroups we have the following result.

Proposition 3.4 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W***-algebra M. Then the following assertions are equivalent.*

(a) $\mathcal{T}$ is irreducible and $P\sigma(A) \cap i\mathbb{R} \neq \emptyset$.
(b) $\mathcal{T}$ is relatively compact in the weak operator topology and the fixed space of $\mathcal{T}$ is generated by a faithful state.
(c) $\mathcal{T}$ is strongly ergodic and the fixed space of $\mathcal{T}$ is generated by a faithful state.
(d) The fixed space of $\mathcal{T}$ is generated by a faithful state.

Proof Suppose (a) is satisfied. Since $\mathrm{Fix}(\mathcal{T}) \neq \{0\}$, there exists a faithful normal state φ on M such that $\mathrm{Fix}(\mathcal{T}) = \mathbb{C}\varphi$ (D-III, Thm.1.10.). Therefore $\mathcal{T}$ is relatively compact in the weak operator topology by Proposition 3.1., whence (b) holds and the implications from (b) to (c) and (c) to (d) are obvious.
Suppose that (d) holds. Let φ be a faithful normal state on M such that $\mathrm{Fix}(T) = \varphi\mathbb{C}$. By Proposition 3.1 the semigroup $\mathcal{T}$ is strongly ergodic. Therefore the fixed space of $\mathcal{T}$ separates the points of $\mathrm{Fix}(T')$. Consequently $\mathrm{Fix}(T') = \mathbb{C}1$. Thus the ergodic projection associated with $\mathcal{T}$ is given by $P = 1 \otimes \varphi$, i.e., $P\psi = \psi(1)\varphi$ for all $\psi \in M_*$. Let F be a closed $\mathcal{T}$-invariant face of M_*^+. If $0 \neq \psi \in F$ then

$$\lim_{s\to\infty} C(s)\psi = \psi(1)\varphi \in F.$$

Hence $\varphi \in F$ and therefore $F = M_*^+$ by the faithfulness of φ which proves (a). □

The next theorem is an extension of D-III, Thm.1.10 and shows the usefulness of the theory of semitopological semigroups. Assume $\mathcal{T} \subseteq \mathrm{L}M_*$ to be relatively compact in the weak operator topology. Since $\mathcal{T}$ is commutative its closure $\mathcal{S} = (\mathcal{T})^- \subseteq L_w(M_*)$ contains a unique minimal ideal $\mathcal{K}$, called the kernel of $\mathcal{S}$, which is a compact Abelian group (DeLeeuw and Glicksberg [4], Junghenn [13] & Krengel [14, § 2.4]). The identity Q of $\mathcal{K}$ is a projection onto the closed linear span of all eigenvectors of A pertaining to the eigenvalues in $i\mathbb{R}$.
Moreover, the dual group of $\mathcal{K}$ can be identified with the subgroup of $i\mathbb{R}$ generated by $P\sigma(A) \cap i\mathbb{R}$. We call Q the semigroup projection associated with $\mathcal{T}$. On the other hand, $\mathcal{T}$ is always strongly ergodic with projection P onto $\mathrm{Fix}(\mathcal{T})$. Obviously, the relation

$$0 \leqslant P \leqslant Q \leqslant \mathrm{Id}$$

holds, where the order relation is defined by the inclusion of the range spaces.

There are two extreme cases. First, $Q = \mathrm{Id}$ and $\mathrm{rank}(P)$. This corresponds to the Halmos-von Neumann Theorem in commutative ergodic theory and is discussed, at least for irreducible semigroups, in Olesen et al. [18].
Second, $\mathrm{Id} > Q = P$, in particular rank $(P) = 1$. This latter case will be investigated in detail for $M = \mathcal{B}(H)$, the W*-algebra of all bounded linear operators on a Hilbert space H. But we first need some preparations.

Theorem 3.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W**-algebra M and suppose there exists a faithful family of $\mathcal{T}$-invariant states on M. Let N be the $\sigma(M, M_*)$-closed linear span of all eigenvectors of A' pertaining to the eigenvalues in* $\mathrm{i}\mathbb{R}$*. If Q is the semigroup projection associated with $\mathcal{T}$, then the following holds.*

(i) The adjoint of Q is a faithful normal conditional expectation from M onto the W**-subalgebra N.*
*(ii) The restriction of T' to N can be embedded into a $\sigma(M, M_*)$-continuous, one-parameter group of *-automorphisms.*
(iii) If, in addition, $\mathcal{T}$ is irreducible and if φ is the normal state generating the fixed space of $\mathcal{T}$, then $\varphi|_N$ is a faithful normal trace.

Proof Consider $H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$ which is not empty by assumptions. From Proposition 3.1 it follows that $\mathcal{T}$ is relatively compact in the weak operator topology. Let K be the semigroup kernel of $\overline{\mathcal{T}w} \subset L(M_*)$ and Q the unit of K. Recall that $Q\psi_n = \psi_n$ for all $\psi_n \in M_*$ such that $A\psi_n = n\psi_n$ $(n \in H)$. Let $\mathcal{E}$ be the family of all eigenvectors of A' pertaining to the eigenvalues in H.
Then $\mathcal{E}$ is closed with respect to the multiplication in M and the formation of adjoints. Thus N is a W*-subalgebra of M, Sakai [19, Corollary 1.7.9.], and $\mathcal{T}_0(t)' := T(t)'_{|N}$ is multiplicative (for this see D-III, Lemma 1.1).
Since $Q \in \overline{\mathcal{T}w} \subseteq L_w(M_*)$, there exists an ultrafilter $\mathfrak{U}$ on $\mathbb{R}_+$ such that

$$\lim_{\mathfrak{U}} \langle T(t)\psi, x\rangle = \langle Q\psi, x\rangle$$

for all $x \in M$ and $\psi \in M_*$. If $n \in H$ and $\psi_n \in M_*$ such that $A\psi_n = n\psi_n$, then for all $x \in M$ we obtain

$$\langle \psi_n, x\rangle = \langle Q\psi_n, x\rangle = \lim_{\mathfrak{U}} \langle T(t)\psi_n, x\rangle = (\lim_{\mathfrak{U}} e^{nt})\langle \psi_n, x\rangle,$$

hence $\lim_{\mathfrak{U}} e^{nt} = 1$. From this it follows that for all $\psi \in M_*$ we have

$$\langle \psi, Q'(u_n)\rangle = \lim_{\mathfrak{U}} \langle \psi, T(t)'u_n\rangle = (\lim_{\mathfrak{U}} e^{nt})\langle \psi, u_n\rangle = \langle \psi, u_n\rangle.$$

Hence $N \subseteq Q'(M)$.
For γ in the dual group of K and $x \in M$ we define x_γ by

$$\psi(x_\gamma) := \int_K \langle S\psi, x\rangle\langle S, \gamma\rangle^* \,\mathrm{d}m(S) \quad (\psi \in M_*^+).$$

Then $x_\gamma \in M$ and $T(t)'x_\gamma = \langle QT(t), \gamma\rangle x_\gamma$. Therefore $x_\gamma \in N$. Thus the inclusion $Q'M \subseteq N$ is proved if we can show that $Q'M$ belongs to the $\sigma(M, M_*)$-closed linear span of $\{x_\gamma : \gamma \in K, x \in M\}$. For this it is enough to show that every linear form $\psi \in M_*$ such that $\psi(x_\gamma) = 0$ for all $\gamma \in K$ satisfies $\psi(Qx) = 0$ for all $x \in M$. But if $\psi(x_\gamma) = 0$, then

$$\int_K \langle S\psi, x\rangle\langle S, \gamma\rangle^* \,\mathrm{d}m(S) = 0, \gamma \in K.$$

Since the map $(S \mapsto \psi(Sx))$ is continuous on K and since the elements of K form a complete orthonormal basis in $L^2(K, \mathrm{d}m)$, we obtain $\psi(Sx) = 0$ for all $S \in K$, in particular $\psi(Qx) = 0$ as desired.
Since the range of Q' is a W*-subalgebra of M it follows from Takesaki [23, Theorem III.3.4] that Q' is a completely positive, normal conditional expectation. This Q' is faithful, i.e., $\mathrm{Ker}(Q') \cap M_+ = \{0\}$ since $Q\varphi = \varphi$ for the faithful linear form φ.
Let φ be the faithful normal state generating $\mathrm{Fix}\,(T)$ and let $\mathcal{U}$ be a family of unitary eigenvectors of A' pertaining to the eigenvalues in H (see D-III, Remark 1.11). If $u_1, u_2 \in U$, then

$$\varphi(u_1u_2^*) = \varphi(T_0(t)'(u_1u_2^*)) = e^{(n_1-n_2)t}\varphi(u_1u_2^*).$$

Therefore

$$\varphi(u_1u_2^*) = \begin{cases} 0 & \text{if } n_1 \neq n_2, \\ 1 & \text{if } n_1 = n_2. \end{cases}$$

Hence $\varphi(u_1u_2^*) = \varphi(u_2^*u_1)$ from which it follows that $\tau := \varphi|_N$ is a faithful normal trace. □

Remarks 3.6 (i) Since $QM_* = N_*$ and $Q'M = N$, where N_* is as in D-III, Proposition 1.12, it follows from general duality theory that $(N_*)' = N$.
(ii) If $\psi \in N_*$, then $|\psi| \in N_*$. To see this, note that $Q\psi = \psi$ and Q is an identity preserving Schwarz map. Then the assertion follows from D-III, Proposition 1.4.
(iii) If $\psi \in N_*$, then $|T_0(t)\psi| = T_0(t)|\psi|$ for all $t \in \mathbb{R}$. This follows immediately from the fact that $\mathcal{T}_0(t)'$ is a *-automorphismus on N.
(iv) Let us add a few words concerning the structure of N: If $\mathcal{T}$ is irreducible and K is the semigroup kernel of $\mathcal{T}^- \subseteq L_w(M_*)$, then $(S \mapsto S') : K \to L((N, \sigma(N, N_*)))$ is a representation of the compact, Abelian group K as group of *-automorphism such that the fixed space is one dimensional. Therefore we are able to apply the results of Olesen et al. [18]. There are three possibilities for N.

1. $N = L^\infty(K, \mathrm{d}m)$ and $\mathcal{T}|_N$ is the translation group on N.
2. $N \cong R$ where $\mathcal{R}$ is the (unique) hyperfinite factor of type II_1. In that case (the image of) K is approximately inner on $\mathcal{R}$ [l.c., Theorem 5.8].
3. There exists a closed subgroup G of K such that

$$N = L^\infty(K/G, \mathrm{d}m) \otimes R$$

where R is as in (ii) and dm the normalized Haar measure on K/G [l.c., Theorem 5.15].

So far we have studied weak*-semigroups on general W*-algebras. We apply now these results to weak*-semigroup on $\mathcal{B}(H)$. To do this we call a triple $(M, \varphi, \mathcal{T})$ a W*-dynamical system if M is a W*-algebra, $\mathcal{T}$ a weak*-semigroup of identity preserving Schwarz maps on M and φ a faithful family of $\mathcal{T}$-invariant normal states. We call $(M, \varphi, \mathcal{T})$ irreducible, if the preadjoint semigroup is irreducible (alternatively, if the fixed space of $\mathcal{T}$ is one dimensional).

Proposition 3.7 *Let $(\mathcal{B}(H), \varphi, \mathcal{T})$ be a* W**-dynamical system on the* W**-algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H. Then the following assertions are equivalent:*

(a) $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$,
(b) $\lim_{s\to\infty} T(s)_* = P_*$ *in the strong operator topology on* $L\mathcal{B}(H)_*$.

Proof Obviously (b) implies (a). Suppose that (a) is fulfilled. Then the ergodic projection P_* of the preadjoint semigroup is equal to the associated semigroup projection. Consequently there exists an ultrafilter $\mathfrak{U}$ on $\mathbb{R}_+$ such that $\lim_{\mathfrak{U}} T(t) = P$ in the weak operator topology. We claim that the convergence holds even in the strong operator topology. Taking this for granted it follows, since for every $t \in \mathbb{R}_+$ $T(t)$ is a contraction, that

$$\lim_{t\to\infty} \|T(t)_*\varphi\| = 0$$

for all $\varphi \in$ Ker$(()P_*)$. Since $T(t)_*\psi = \psi$ for every $\psi \in \mathrm{im}(P_*)$ and

$$\mathcal{B}(H)_* = \mathrm{im}(P_*) \oplus \text{ Ker}(()P_*)$$

the assertion is proved.
It remains to show that $\lim_{\mathfrak{U}} T(t)_* = P_*$ in the strong operator topology. Choose $0 \leqslant \varphi \in \mathcal{B}(H)_*$, $\|\varphi\| \leqslant 1$ and let $\varphi_t := T(t)_*\varphi$ $(t > 0)$. $\varphi_0 := P_*\varphi$ and let $\{p_i : i \in A\}$ be an increasing net of projections of finite rank in $\mathcal{B}(H)$ with strong limit 1. Since the set $K := \{\varphi_t : t \geqslant 0\}$ is relatively compact in the $\sigma(\mathcal{B}(H)_*, \mathcal{B}(H))$-topology, there exists for every $\delta > 0$ an index $i_0 \in A$ such that

$$\|(1 - p_i)\psi(1 - p_i)\| \leqslant \delta$$

for every $\psi \in K$ and $i \geqslant i_0$ (Takesaki [23, Theorem III.5.4.(vi)]). In particular

$$|\psi(1 - p_i)| \leqslant \delta, \quad \psi \in K,\ i(0) \leqslant i.$$

Let $p := p_{i(0)}$. Then for all x in the unit ball of M it follows that

$$\begin{gathered} |(\varphi_t - \varphi_0)(x)| \leqslant \\ |(\varphi_t - \varphi_0)(pxp)| + |(\varphi_t - \varphi_0)((1 - p)xp)| \\ +|(\varphi_t - \varphi_0)(x(1 - p))| \leqslant\leqslant |(\varphi_t - \varphi_0)(pxp)| + 4\sqrt{\delta}. \end{gathered}$$

Since the W*-algebra $p\mathcal{B}(H)p$ is finite dimensional, there exists $U \in \mathfrak{U}$ such that

$$\|(\varphi_t - \varphi_0)|_{p\mathcal{B}(H)p}\| \leqslant \delta.$$

for all $t \in U$. Consequently

$$\|(\varphi_t - \varphi_0)\| \leqslant (\delta + 4\sqrt{\delta})$$

for all $t \in U$. Therefore $\lim_{\mathfrak{U}} T(t)_* \varphi = P_* \varphi$ in the strong operator topology. Since the positive cone of $\mathcal{B}(H)_*$ is generating, the assertion is proved. □

We show next, that for irreducible W*-dynamical systems on $\mathcal{B}(H)$ the above properties always hold.

Theorem 3.8 *Let $(\mathcal{B}(H), \varphi, \mathcal{T})$ be an irreducible* W*-*dynamical system. Then*

$$P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}.$$

Proof Let N be the W*-subalgebra of $M = \mathcal{B}(H)$ generated by the eigenvectors of A pertaining to the eigenvalues on $\mathrm{i}\mathbb{R}$ and let Q be the faithful normal conditional expectation from M onto N (Proposition 3.7). Since M is atomic, N is atomic (Størmer [22]). N is finite since there exists a finite, faithful normal trace on N. In particular the center of N is isomorphic to ℓ^∞.
Let $\mathcal{S}$ be the restriction of $\mathcal{T}$ to the center. Then $\mathcal{S}$ is a weak*-semigroup such that every $S(t) \in \mathcal{S}$ is $\sigma(\ell^\infty, \ell^1)$-continuous and a *-automorphism. From this it follows that $S(t)$ is induced by some continuous flow $\kappa_t : \mathbb{N} \to \mathbb{N}$. Indeed, if $\delta_n((\xi_m)) = \xi_n$ $(n \in \mathbb{N}, (\xi_m) \in \ell^\infty)$, then $\delta_n \circ S(t)$ is a normal scalar valued *-homomorphism hence of the form δ_m for some $m = \kappa_t(n)$. But the function $t \mapsto \kappa_t$ is continuous from $\mathbb{R}$ into $\mathbb{N}$, whence constant. Hence $S(t) = \mathrm{Id}$. But the semigroup S is weak*-irreducible on the center. Consequently, the center is one dimensional. Using [Takesaki, Theorem V.1.27] we obtain $N = B(H_n)$ where H_n is a finite dimensional Hilbert space. But if $0 \neq \mathrm{i}\alpha \in P\sigma(A) \cap \mathrm{i}\mathbb{R}$ then $\mathrm{i}\alpha\mathbb{Z} \subset P\sigma(A)$ by D-III,Thm.1.10, whence N must be infinite dimensional. Therefore $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$ as desired. □

Corollary 3.9 *If $(\mathcal{B}(H), \varphi, T)$ is an irreducible* W*-*dynamical system, then*

$$\lim_{s \to \infty} T(s) = 1 \otimes \varphi$$

in the strong operator topology on $L(\mathcal{B}(H)_*)$, where φ is the unique normal state generating the fixed space of T_*.

We are now going to discuss the asymptotic behavior of positive semigroups whose generator has boundary point spectrum different from 0. The standard example is the following. If Γ is the unit circle, dm the normalized Haar measure on Γ and $0 < \tau \in \mathbb{R}$, then we define the maps $T_\tau(t)$, $t \in \mathbb{R}_+$, on $L^1(\Gamma, m)$ by

$$(T_\tau(t)f)(\xi) = f(\xi \exp(\frac{2\pi i}{\tau} t)) \quad (f \in L^1(\Gamma, \mathrm{d}m), \xi \in \Gamma).$$

Then $\mathcal{T} := (T_\tau(t))_{t\geqslant 0}$ forms a strongly continuous one parameter semigroup which is identity preserving and of Schwarz type. Since $\mathcal{T}$ is periodic of period τ, it follows that 0 is a pole of the resolvent of its generator B with residuum $P = 1 \otimes 1$ and $\{\frac{2\pi \mathrm{i}}{\tau} \cdot k : k \in \mathbb{Z}\} = \sigma(B)$. Thus $\mathcal{T}$ is irreducible and uniformly ergodic on $L^1(\Gamma, \mathrm{d}m)$ (see A-II, Section 5).
Now let $\mathcal{T}$ be a semigroup on a predual M_* of a von Neumann-algebra M. It is called *partially periodic*, if there exists a projection $Q \in L(M_*)$ reducing T such that $Q(M_*) \cong L^1(\Gamma, \mathrm{d}m)$ and $T_{|\mathrm{im}(Q)}$ is conjugate to a periodic semigroup on $L^1(\Gamma, \mathrm{d}m)$.
In the main result we present a non commutative version of Nagel [17] showing that certain dynamical systems are partially periodic semigroups.

Proposition 3.10 *Let $\mathcal{T}$ be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra M.*
If $\mathcal{T}$ is uniformly ergodic, then $\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$.
If additionally $\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \{0\}$, there exists a strictly positive projection Q on M_* which is identity preserving and completely positive such that

(i) Q reduces $\mathcal{T}$ and $Q(M_*) \cong L^1(\Gamma)$, Γ being the one dimensional torus.
(ii) The restriction T_0 of $\mathcal{T}$ to im(Q) is irreducible and conjugate to a rotation semigroup of period $\tau = \frac{2\pi}{\alpha}$ on Γ.
(iii) The spectral bound $s(A_{|\ \mathrm{Ker}(()Q)})$ is strictly smaller than 0*.*

Proof By D-III, Thm.1.11 and D-III, Thm.2.5 it follows that

$$\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z}$$

for some $\alpha \in \mathbb{R}$. Suppose $\alpha \neq 0$. Since $\sigma(A) + \mathrm{i}\alpha\mathbb{Z} = \sigma(A)$ and since every $n \in \mathrm{i}\alpha\mathbb{Z}$ is isolated, it follows that there exists $\delta > 0$ such that

$$\sigma(A) \setminus \mathrm{i}\alpha\mathbb{Z} \subseteq \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}(\lambda) \leqslant \delta\}.$$

Let $\{u_\alpha^k : k \in \mathbb{Z}\}$ be a family of unitary eigenvectors of A' pertaining to the eigenvalues in $\mathrm{i}\mathbb{R}$. Then $Q'(M)$ is a commutative W*-algebra. For $\tau := \frac{2\pi}{\alpha}$, we obtain $T(\tau)u_\alpha^k = u_\alpha^k$, hence $T|_{\mathrm{im}(Q)}$ is periodic. From the Halmos-von Neumann theorem (see Schaefer [21, Thm. III.7.11]) it follows that $T|_{\mathrm{im}(Q)}$ is conjugate to the rotation semigroup of period τ on $L^1(\Gamma, m)$. □

Using this proposition we obtain the following theorem.

Theorem 3.11 *Let $T = (T(t))_{t\geqslant 0}$ be a uniformly ergodic, identity preserving semigroup of Schwarz type on the predual of a* W*-*algebra M and suppose*

$$\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \{0\}.$$

Then there exists a partially periodic, identity preserving semigroup $S = (S(t))_{t\geqslant 0}$ of Schwarz type on M_* such that

$$\lim_{t\to\infty} (T(t) - S(t)) = 0$$

in the strong operator topology.

Proof Let φ be the normal state on M generating the fixed space of $\mathcal{T}$. Let $\mathcal{S} = (S(t))_{t\geqslant 0}$ where $S(t) := T(t) \circ Q$ and Q is as in 2.6. Obviously, $\mathcal{S}$ is partially periodic and $\varphi \in \operatorname{Fix}(\mathcal{S})$. Let H_φ be the GNS-Hilbert space pertaining to φ. Since φ is fixed under $\mathcal{T}$, $\mathcal{S}$ and Q, these objects have a canonical extension to H_φ (in the following denoted by the same symbols). If $H_0 := \operatorname{Ker}(()Q) \subseteq H_\varphi$, then it is easy to see that H_0 is invariant under the extension to H_φ and for the multiplication maps we defined in D-III, Remark 1.3.
Consequently, using the results in Groh and Kümmerer [11], it follows that there exists $c \in \mathbb{R}$ such that for all γ near 0 and all $\beta \in \mathbb{R}$:

$$\|R(\gamma + \mathrm{i}\beta A_0)\| \leqslant c, \tag{*}$$

where $A_0 := A_{|\operatorname{Ker}(()Q)}$ (the norm taken in $L(H_\varphi)$). Using the result in A-III,Cor.7.11 it follows that

$$\lim_{t\to\infty} \|T(t)|_{H_0}\| = 0.$$

Since the $s(M, M_*)$-topology on the unit ball of M is nothing else than the restriction of the norm topology on H_φ, we obtain

$$s(M, M_*)\text{-}\lim_{t\to\infty} (T(t)' - S(t)')(x) = 0$$

uniformly on M_1. From this the assertion follows. □

4 Uniform Ergodic Theorems

As we have seen, uniformly ergodic semigroups have strong spectral properties. In this section we study sufficient conditions which imply uniform ergodicity thereby generalizing results of Groh [9]. We first need some preparations.

Lemma 4.1 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$ with values in the predual of a* W*-*algebra M. If the fixed space of $\mathcal{R}$ is infinite dimensional, then there exists a sequence of states in* $\operatorname{Fix}(\mathcal{R})$ *such that the corresponding support projections are mutually orthogonal in M.*

Proof Let $\Phi = \{\varphi \in \operatorname{Fix}(\mathcal{R}) : \varphi \text{ state on } M\}$ and let $p = \sup\{s(\varphi) : \varphi \in \Phi\}$. Since $\lambda R(\lambda)\varphi = \varphi$ for all $\varphi \in \Phi$ and $\lambda \in D$, it follows $\mu R(\mu)(\mathbb{1} - s(\varphi)) = (\mathbb{1} - s(\varphi))$. Hence $\mu R(\mu)(\mathbb{1} - p) = (\mathbb{1} - p)$ for all $\mu \in \mathbb{R}_+$.
Let $\mathcal{R}_1$ be the induced pseudo-resolvent on pM_*p (D-I, Section 3.(c)). Then the family Φ is faithful on M_p and contained in the fixed space of $\mathcal{R}_1$. The adjoint $\mu R_1(\mu)'$ is an identity preserving Schwarz map. Consequently it follows from D-III, Lemma 1.1.(b) and, the $\sigma(M_p, (M_p)_*)$-continuity of $\mu R_1(\mu)'$ that $\operatorname{Fix}\left(R_1'\right)$ is a W*-subalgebra of M_p and by D-III, Lemma 1.5, $\dim \operatorname{Fix}(\mathcal{R}) \leqslant \dim \operatorname{Fix}\left(R_1'\right)$.

If $\text{Fix}\,(\mathcal{R})$ is infinite dimensional, let (p_n) be a sequence of mutually orthogonal projections in $\text{Fix}(R_1') \subseteq M_p$ and choose a sequence (φ_n) in Φ such that $\varphi_n(p_n) \neq 0$. For $n \in \mathbb{N}$ let ψ_n be the normal state

$$\psi_n(x) = \varphi_n(p_n)^{-1}\varphi_n(p_n x p_n)$$

on M. Because of $s(\psi_n) \leqslant p_n \leqslant p$, the support projections of the ψ_n's are mutually orthogonal in M. For $\mu \in \mathbb{R}_+$ and $x \in M$ we obtain

$$\begin{aligned}
\langle x, \mu R(\mu)\psi_n\rangle &= \varphi_n(p_n)^{-1}\langle \mu p_n(R(\mu)'x)p_n, \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\langle \mu p_n p(R(\mu)p'x)p_n, \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\langle \mu p_n(pR_1(\mu)'xp)p_n, \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\langle \mu(p_n R_1(\mu)'xp_n), \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\varphi_n(x) = \psi_n(x).
\end{aligned}$$

Therefore $\psi_n \in \text{Fix}\,(\mathcal{R})$ for all $n \in \mathbb{N}$. □

Remark 4.2 (i) If $\dim \text{Fix}\,(\mathcal{R}) \geqslant 2$ then the Jordan decomposition of self adjoint linear functionals implies that at least two states in $\text{Fix}\,(\mathcal{R})$ have orthogonal support (compare D-III, Theorem 1.10.(a)).
(ii) If $\mathcal{R}$ is a pseudo-resolvent with values in a W*-algebra such that $\text{Fix}\,(\mathcal{R}')$ is contained in M_*, then by D-III, Lemma 1.2, there exists a sequence of normal states in $\text{Fix}\,(\mathcal{R}')$ with orthogonal supports in M.

Lemma 4.3 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \text{Re}(\lambda) > 0\}$ with values in the predual of a* W*-*algebra M. If the fixed space of the canonical extension $\widehat{\mathcal{R}}$ of $\mathcal{R}$ to some ultrapower of M_* is infinite dimensional, then there exists a sequence (z_n) in M_1^+ and a sequence of states (φ_n) in M_* such that*

(i) $\lim_n z_n = 0$ *in the $s^*(M, M_*)$-topology,*
(ii) $\lim_n \|(Id - \lambda R(\lambda))\varphi_n\| = 0$ *for all $\lambda \in D$,*
(iii) $\varphi_n(z_n) \geqslant \frac{1}{2}$ *for all $n \in \mathbb{N}$.*

Proof Let $(M_*)^\wedge$ be the ultrapower of M_* with respect to some free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$. Since $(M_*)^\wedge$ is the predual of a W*-subalgebra of $\widehat{M}$ (see D-III, Remark 2.4.(b)), there exists a sequence of states $(\hat{\psi}_n)$ in $\text{Fix}\left(\widehat{\mathcal{R}}\right)$ such that the corresponding support projections are mutually orthogonal in $\widehat{M}$ (Lemma 4.1). For every $n \in \mathbb{N}$ let $(\psi_{n,k})$ be a representing sequence of states,

$$\varphi := \sum_{n,k} 2^{-(n+k+1)}\psi_{n,k}$$

and

$$p := \sup\{s(\psi_{n,k}) : n, k = 1, \ldots\}$$

in M. Then φ is a normal state on M which is faithful on the W*-algebra M_p. Since

$$1 = \langle \psi_{n,k}, s(\psi_{n,k}) \rangle = \psi_{n,k}(p) \quad (n, k \in \mathbb{N}),$$

it follows $\hat{\psi}_n(\hat{p}) = 1$ where $\hat{p}$ is the canonical image of p in $\widehat{M}$. But this implies $s(\hat{\psi}_n) \leqslant \hat{p}$ in $\widehat{M}$. Since $\widehat{M}_1^+$ is $\sigma(\widehat{M}, \widehat{M}')$-dense in $(\widehat{M}'')_1^+$ (Kaplansky's density theorem Sakai [19, 1.9.1] with Sakai [19, 1.8.9 and 1.8.12]), there exists for all $n \in \mathbb{N}$ a net $(z_{n,\gamma})$ in $\widehat{M}_1^+$ such that

$$\sigma(\widehat{M}'', \widehat{M}')\text{-}\lim_\gamma \hat{z}_{n,\gamma} = s(\hat{\psi}_n).$$

From Sakai [19, 1.7.8] and the above considerations, we obtain that the net $(p\hat{z}_{n,\gamma}\hat{p})$ converges to $s(\hat{\psi}_n)$ in the $\sigma(\widehat{M}'', \widehat{M}')$-topology. Therefore we may assume $\hat{z}_{n,\gamma} \in (\widehat{M}'_p)_1^+$.
In the following we denote by $\hat{\varphi}$ the canonical image of φ in $(M_*)^\wedge$.
Since the projections $s(\hat{\psi}_n)$ are mutually orthogonal, there exists a real sequence (r_n), $0 < r_n < 1$, $\lim_n r_n = 0$ and $\hat{\varphi}(s(\hat{\psi}_n)) \leqslant \frac{1}{2} r_n$. For all $n \in \mathbb{N}$ choose $\hat{z}_n \in (\widehat{M}'_p)_1^+$ such that

$$|\langle \hat{\varphi}, s(\hat{\psi}_n) - \hat{z}_n \rangle| \leqslant \frac{1}{2} r_n,$$
$$|\langle \hat{\psi}_n, s(\hat{\psi}_n) - \hat{z}_n \rangle| \leqslant \frac{1}{2} r_n.$$

Hence $\hat{\varphi}(\hat{z}_n) \leqslant r_n$ and $\hat{\psi}_n(\hat{z}_n) \geqslant \frac{1}{2}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ let $(z_{n,k}) \in \hat{z}_n$ be a representing sequence in $(M_p)_1^+ = p(M_1^+)p$ (note that $M_{\hat{p}} = \widehat{M_p}$) and fix $\mu \in \mathbb{R}_+$. Since $\mu R(\mu)'\hat{\psi}_n = \hat{\psi}_n$, $\hat{\varphi}(\hat{z}_n) \leqslant r_n$ and $\hat{\psi}_n(\hat{z}_n) \geqslant \frac{1}{2}$, there exists for all $n \in \mathbb{N}$ an element $U_n \in \mathfrak{U}$ such that for all $k \in U_n$ and we obtain

(i′) $\varphi(z_{n,k}) \leqslant r_n$,
(ii′) $\|(Id - \mu R(\mu))\psi_{n,k}\| \leqslant r_n$,
(iii′) $\psi_{n,k}(z_{n,k}) \geqslant \frac{1}{2}$.

Inductively we find a sequence (z_n) in $(M_p)_1^+$ and a sequence of states (φ_n) in M_* such that for all $n \in \mathbb{N}$

(i″) $\lim_n \varphi_n(z_n) = 0$,
(ii″) $\lim_n \|(Id - \mu R(\mu))\varphi_n\| = 0$,
(iii″) $\varphi_n(z_n) \geqslant \frac{1}{2}$.

But φ is faithful on M_p. Therefore condition (ii″) implies that $\lim_n z_n = 0$ in the $s^*(M_p, (M_p)_*)$-topology (Takesaki [23, Proposition III.5.4]). Since

$$s^*(M_p, (M_p)_*) = s^*(M, M_*)|_{M_p},$$

(i) follows immediately from (ii″). Using the resolvent equation for $\mathcal{R}$ it is easy to see that (ii″) implies

$$\lim_n \|(Id - \lambda R(\lambda))\varphi_n\| = 0$$

for all $\lambda \in D$ and the proof is complete. ◻

Without further comments, we will use following facts in this section.

(1) A sequence (φ_n) in M'_+ converges in the $\sigma(M', M)$-topology if and only if it converges in $\sigma(M', M'')$-topology (**?**]).
(2) We can decompose $\varphi \in M'_+$ into its normal and singular part $\varphi = \varphi^{(n)} + \varphi^{(s)}$, $0 \leqslant \varphi^{(n)} \in M_*$, $0 \leqslant \varphi^{(s)} \in M_*^{\perp}$ and $\|\varphi\| = \|\varphi^{(n)}\| + \|\varphi^{(s)}\|$ (Takesaki [23, Theorem III.2.14]).
(3) If (φ_k) is a sequence in M_* convergeing to zero in the $\sigma(M_*, M)$-topology and if (x_n) is a sequence in M converging to zero in the $s^*(M, M_*)$-topology, then $\lim_n \varphi_k(x_n) = 0$ uniformly in $k \in \mathbb{N}$ (Takesaki [23, Lemma III.5.5]).

Theorem 4.4 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent on*

$$D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}(\lambda) > 0\}$$

with values in a W**-algebra M which is of Schwarz type and let $\mathcal{R}'$ br its adjoint pseudo-resolvent. Any one of the following conditions implies* dim Fix $\left(\widehat{\mathcal{R}}\right) < \infty$ *in some ultrapower of M.*

(i) The fixed space of $\mathcal{R}'$ is finite dimensional.
(ii) $\lim_{\mu\to 0} \mu R(\mu) = P$ exists in the strong operator topology and rank$(P) < \infty$.
(iii) The fixed space of $\mathcal{R}'$ is contained in M_*.
(iv) Every map $\mu R(\mu)$, $\mu \in \mathbb{R}_+$ is irreducible on M.

Proof Suppose that the dimension of the fixed space of $\widehat{\mathcal{R}'}$ in some ultrapower $\widehat{(M`)}$ of $M'$ is infinite dimensional. Since $\widehat{(M`)}$ is the predual of the W*-algebra $\widehat{M}$ and $\mathcal{R}'$ is identity preserving (since $R'\mathbb{1} = R\mathbb{1} = \mathbb{1}$) and of Schwarz type (because $\mu R''(\mu) = (\mu R(\mu))''$ is a Schwarz map for all $\mu \in \mathbb{R}_+$), we may apply Lemma 4.3. Suppose that the fixed space of the canonical extension of $\mathcal{R}'$ to some ultrapower of M' is infinite dimensional. Thus we may choose a sequence of states (φ_k) in M' and a sequence (z_k) in $(M'')_1$, $0 \leqslant z_k$, satisfying (i)–(ii) of Lemma 4.3. Remark (3) above implies that no subsequence of (φ_k) can converge in the $\sigma(M', M'')$-topology.

(i) If φ is a $\sigma(M', M)$-accumulation point of (φ_k), then $\varphi \in$ Fix $(\mathcal{R}')$. Since Fix $(\mathcal{R}')$ is finite dimensional, the set of accumulation points of the sequence (φ_k) is metrizable in the $\sigma(M', M)$-topology. Hence there exists a sequence $(k(n))$ of natural numbers such that $\sigma(M', M)$-$\lim_n \varphi_{k(n)} = \varphi$. Consequently, by Remark (1) above , $\varphi = \sigma(M', M'')$-$\lim_n \varphi_{k(n)}$. But this leads to a contradiction proving(i).
(ii) Since dim Fix $(\mathcal{R})$ = dim Fix $(\mathcal{R}')$ = rank $(P) < \infty$, (ii) follows from (i).
(iii) Suppose that the fixed space of R' is infinite dimensional. Since Fix $(\mathcal{R}') \subseteq M_*$, there exists a sequence of states (ψ_n) in Fix $(\mathcal{R}')$ with mutually orthogonal support projections in M (Lemma 4.1). Since every $\sigma(M', M)$-accumulation point

of the ψ_n's belongs to $\mathrm{Fix}\,(\mathcal{R}')$, hence is normal, the sequence (ψ_n) is relatively $\sigma(M_*, M)$-compact.
By Eberlein's theorem, we may assume that this sequence is weakly convergent (Schaefer [20]). By the orthogonality of the $s(\psi_n)$'s this sequence converges to zero in the $s^*(M, M_*)$-topology, hence $\lim_n \psi_k(s(\psi_n)) = 0$ uniformly in $k \in \mathbb{N}$, a contradiction. Consequently $\dim \mathrm{Fix}\,(\mathcal{R}) < \infty$ and (i) is proved.
(iv) We prove $\dim \mathrm{Fix}\,(\mathcal{R}') = 1$ and apply (i) once again and need the following observation: If ψ is a faithful state on M, then the normal part is faithful too. Indeed, if $0 \neq x \in M$ such that $\psi^{(n)}(x) = 0$, choose a projection $0 \neq p \in M$ such that $\psi^{(n)}(p) = \psi^{(s)}(p) = 0$ (use Takesaki [23, Theorem III.3.8]). Hence $\psi(p) = 0$ which conflicts with the faithfulness of ψ.
If $2 \leqslant \dim \mathrm{Fix}\,(\mathcal{R}')$ there are states ψ_1 and ψ_2 in $\mathrm{Fix}\,(\mathcal{R}')$ such that the corresponding support projections are orthogonal in M'' (Remark 4.2). Since every $\mathcal{R}'$-invariant state ψ is faithful on M, $\psi_i^{(n)} \neq 0$ (otherwise the norm closed face $\{\psi(x) = 0 : x \in M_+\}$ would be non trivial and $\mu R(\mu)$-invariant). The support projections of the $\psi_i^{(n)}$'s in M'' are orthogonal (since $\psi_1^{(n)} \leqslant \psi_i$) and different from zero. Let (z_γ) be a net in M_1^+ such that

$$\sigma(M'', M')\text{-}\lim_\gamma z_\gamma = s(\psi_1^{(n)}).$$

Then $\lim_\gamma \psi_1^{(n)}(z_\gamma) = 1$ but $\lim_\gamma \psi_2^{(n)}(z_\gamma) = 0$. Let z be a $\sigma(M, M_*)$-accumulation point of (z_γ) in M_+. Since every $\psi_i^{(n)}$ is normal, $\psi_1^{(n)}(z) = 1$ but $\psi_2^{(n)}(z) = 0$. The first condition implies $z \neq 0$ while the second shows that $\psi_2^{(n)}$ cannot be faithful. This is a contradiction and it implies $\dim \mathrm{Fix}\,(\mathcal{R}') = 1$, hence (iv). $\square$

The next corollary is an easy application of Theorem 4.4 and of D-III, Proposition 2.3.

Corollary 4.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W**-algebra M. Then the following assertions are equivalent.*

(a) $\mathcal{T}$ is uniformly ergodic with finite dimensional fixed space.
(b) The adjoint weak-semigroup is strongly ergodic with finite dimensional fixed space.*
(c) Every $\mathcal{T}''$-invariant state is normal.

Proof If (a) is fulfilled, then the semigroup $\mathcal{T}$ is strongly ergodic on M_*. Since

$$\dim \mathrm{Fix}\,(\mathcal{T}) = \dim \mathrm{Fix}\,(\mathcal{T}') < \infty,$$

there exist normal states $\varphi_1, \dots, \varphi_n$ in $\mathrm{Fix}\,(\mathcal{T})$ and $x_1, \dots, x_k$ in $\mathrm{Fix}(\mathcal{T}')$ such that $\varphi_n(x_m) = \delta_{n,m}$ $(1 \leqslant n, m \leqslant k)$. Then

$$P = \sum_{i=1}^{k} \varphi_i \otimes x_i$$

is the associated ergodic projection. If $(C(s))_{s>0}$ is the family of Cesàro means of $\mathcal{T}$, then

$$\lim_{s\to\infty} C(s)''(\psi) = \sum_{i=1}^{k} \varphi_i(\psi)x_i \in M_*$$

for every $\psi \in M'$. Hence $\operatorname{Fix}(\mathcal{T}'') \subseteq M_*$ which implies (c).
If (c) is fulfilled, then $\operatorname{Fix}(\mathcal{T}') = \operatorname{Fix}(\mathcal{T}'')$. Therefore the fixed space of $\mathcal{T}'$ separates the points of $\operatorname{Fix}(\mathcal{T}'')$, hence $\mathcal{T}'$ is strongly ergodic on M (Krengel [14, Chap.2, Thm.1.4]).
If (b) holds, then

$$P = \lim_{\mu\to 0} \mu R(\mu, A')$$

exists in the strong operator topology with A' is the generator of $\mathcal{T}'$. Therefore $\dim \operatorname{Fix}\left(\widehat{\mu R(\mu)}\right) < \infty$ in some ultrapower of M (Theorem 4.4). It follows from D-III, Proposition 2.3 that 0 is a pole of the resolvent of $R(\cdot, A)$. Therefore $\mathcal{T}$ is uniformly ergodic. □

Notes

Section 1: The stability concepts appearing in Theorem 1.7 coincide not only for positive semigroups on C*-algebras but on any order unit Banach space. We refer to Batty and Robinson [1] for this more general setting and to B-IV, Section 1 for the analogous results on $C_0(X)$.

Section 2: Theorem 2.2 generalizes the Liapunov stability theorem from the matrix algebra $B(\mathbb{C}^n)$ to arbitrary W*-algebras. For the algebra $\mathcal{B}(H)$ it is due to Mil'stein [16] and in the general form to Groh and Neubrander [12].

Section 3: From the many papers dealing more or less explicitly with the asymptotic behavior of semigroups on operator algebras we quote Frigerio and Verri [6] and Watanabe [24]. The background for our ergodic theorems (Proposition 3.3 & 3.4) can be found best in Krengel [14]. The "automatic" convergence theorem for an irreducible W*-dynamical system on $\mathcal{B}(H)$ stated in Corollary 3.9 is the continuous version of a result in Groh [10]. Finally, the characterization of convergence towards a periodic semigroup through spectral properties of the generator—Theorem 3.11—is due to Nagel [17] in the commutative case, i.e., in $L^1(\mu)$ (see also C-IV, Thm.2.14).

Section 4: Again we refer to Krengel [14] for the (uniform) ergodic theory for a single operator or a one-parameter semigroup on a Banach space. The characterization given in Corollary 4.5 for positive semigroups on W*-algebras is based on a sophisticated use of ultrapower techniques and has its discrete forerunners in Lotz [15] and Groh [9].

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