Choosing an approximate identity $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ we obtain

$$f = T(0)f = \lim_{n\to\mathbb{N}} T(\psi_n)f = 0$$

for every $f \in E$.

□

<u>Proof of Theorem</u> 7.4 (1st part). By the Spectral Inclusion Theorem 6.2 we have to show that every spectral value of $T(t)$ can be approximated by exponentials of spectral values of A. In view of the rescaling procedure it suffices to prove this when $-1 \in \rho(T(\pi))$, provided that the following condition is satisfied.

(7.4) There exists $\varepsilon > 0$ such that $\bigcup_{k\in\mathbb{Z}} i[2k+1-2\varepsilon, 2k+1+2\varepsilon] \subset \rho(A)$.

Assume now that (7.4) holds. Then each of the sets $\sigma_k := \{i\alpha \in \sigma(A) : \alpha \in [2k-1, 2k+1]\}$ is a spectral set of A with corresponding spectral projection P_k. If we choose $\phi_o \in \mathcal{D}$ such that $\operatorname{supp} \phi_o \subset [-1+\varepsilon, 1-\varepsilon]$ and $\phi_o(x) = 1$ for $x \in [-1+2\varepsilon, 1-2\varepsilon]$ it follows from (7.3) and the integral representation of P_k (cf. (3.1)) that $P_o = T(\overset{\leftarrow}{\phi}_o)$. More generally, since $(e^{i2k\cdot} \overset{\leftarrow}{\phi}_o)^{\rightarrow}(\alpha) = \phi_o(\alpha - 2k)$, the assertions (7.3) and (7.4) imply

$$P_k = \int_{-\infty}^{\infty} e^{i2ks}\, \overset{\leftarrow}{\phi}_o(s)\, T(s)\, ds \quad \text{for } k \in \mathbb{Z}. \tag{7.5}$$

At this point we isolate another lemma.

<u>Lemma</u> 7.7. $\operatorname{span} \bigcup_{k\in\mathbb{Z}} P_k E$ is dense in E.

<u>Proof</u>. The closure of $\operatorname{span} \bigcup_{k\in\mathbb{Z}} P_k E$ is a $\mathcal{T}$-invariant subspace G of E. Consider the quotient group $(T(t)_/)_{t\in\mathbb{R}}$ induced on E/G. The spectrum of its generator $A_/$ is contained in $\sigma(A)$ by Prop.4.2.iii. Moreover the spectral projection corresponding to $\sigma(A_/) \cap \sigma_k$ is the quotient operator $P_{k/}$. Obviously $P_{k/} = 0$, therefore $\sigma(A_/) \cap \sigma_k = \emptyset$ for every $k \in \mathbb{Z}$ and $\sigma(A_/) = \emptyset$. By Lemma 7.6 this implies $E/G = \{0\}$, i.e. $G = E$.

□

<u>Proof of Theorem</u> 7.4 (2nd part). We return to the situation of the first part. Using (7.5) the spectral projection P_k can be transformed into

$$\begin{aligned} P_k &= \int_{-\infty}^{\infty} e^{i2ks}\, \overset{\leftarrow}{\phi}_o(s)\, T(s)\, ds \\ &= \sum_{m\in\mathbb{Z}} \int_{(m-1/2)\pi}^{(m+1/2)\pi} e^{i2ks}\, \overset{\leftarrow}{\phi}_o(s)\, T(s)\, ds \\ &= \int_{-\pi/2}^{\pi/2} e^{i2ks} \sum_{m\in\mathbb{Z}} \overset{\leftarrow}{\phi}_o(s+m\pi)\, T(s+m\pi)\, ds\,, \end{aligned}$$