Remark. In Theorem 1.15 the assumption that $\pm A$ are generators can be relaxed. In fact, the proof shows the following. If A is a densely defined operator such that $\{\lambda \in \mathbb{C} \ \ \mathrm{Re}\,\lambda > 0\} \subset \rho(\pm A - w)$ and $\|R(\lambda, \pm A - w)\| \leq M/\mathrm{Re}\lambda$ for some $M \geq 0$, $w \geq 0$, then A^2 generates a holomorphic semigroup of angle $\pi/2$.

Next we consider semigroups satisfying a less restrictive smoothness condition.

Differentiable semigroups

Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator A. Let $t_o \geq 0$ and $f \in E$. Then the function $t \to T(t)f$ is right sided differentiable in t_o if and only if $T(t_o)f \in D(A)$; and in that case it is differentiable in every $s > t_o$ and the derivative is $AT(s)f$ (this follows from A-I,Prop.1.6(ii)).

Definition 1.16. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E is called eventually differentiable if there exists $t_o \geq 0$ such that the function $t \to T(t)f$ from (t_o,∞) into E is differentiable for every $f \in E$. The semigroup is called differentiable if t_o can be chosen 0.

It is not difficult to see that if $(T(t))_{t\geq 0}$ is differentiable for $t > t_o$, then it is n-times differentiable in all $s > nt_o$ and $T(s)E \subset D(A^n)$ $(n \in \mathbb{N})$. If $(T(t))_{t\geq 0}$ is differentiable, then the function $t \to T(t)f$ from $(0,\infty)$ into E is infinitely often differentiable for every $f \in E$.

Generators of (eventually) differentiable semigroups can be characterized similarly as those of holomorphic semigroups by the spectral behavior of the resolvent. Whereas the spectrum of the generator of a holomorphic semigroup is included in a sector, the spectrum of the generator of an eventually differentiable semigroup is limited by a function of exponential growth (instead of a line).