Chapter 3

Spectral Theory of Positive Semigroups on W^*-algebras and their Preduals

Motivated by the classical results of Perron and Frobenius one expects the following spectral properties for the generator A of a positive semigroup.

> The spectral bound $s(A) := \sup\{\Re\,\lambda\colon \lambda \in \sigma(A)\}$ belongs to the spectrum $\sigma(A)$ and the boundary spectrum $\sigma_b(A) := \sigma(A) \cap \{s(A) + i\mathbb{R}\}$ possesses a certain symmetric structure, called cyclicity.

Results of this type have been proved in Chapter B-III for positive semigroups on commutative C^*-algebras, but in the non-commutative case the situation is more complicated. While "$s(A) \in \sigma(A)$" still holds (see Greiner et al. (1981) or the notes of this chapter), the cyclicity of the boundary spectrum $\sigma_b(A)$ is true only under additional assumptions on the semigroup (e.g., irreducibility, see Section 1 below).

For technical reasons we consider mostly strongly continuous semigroups on the predual of a W^*-algebra M or its adjoint semigroup which is a weak*-continuous semigroup on M.

1 Spectral Theory for Positive Semigroups on Preduals

The aim of this section is to develop a Perron-Frobenius theory for identity preserving semigroups of Schwarz type on W^*-algebras. But as we will show in the example preceding Theorem 1.11 on page 7 below the boundary spectrum is no longer cyclic. The appropriate hypothesis on the semigroup implying the desired results seems to be the concept of irreducibility.

Let us first recall some facts on normal linear functionals. If φ is a normal linear functional on a W^*-algebra M then there exists a partial isometry $u \in M$ and a

positive linear functional $|\varphi| \in M_*$ such that

$$\varphi(x) = |\varphi|(xu) =: (u|\varphi|)(x) \quad (x \in M),$$
$$u^*u = s(|\varphi|),$$

where $s(|\varphi|)$ denotes the support projection of $|\varphi|$ in M. We refer to this as the *polar decomposition* of φ. In addition, $|\varphi|$ is *uniquely determined* by the following two conditions.

$$\left.\begin{array}{c} \|\varphi\| = \||\varphi|\| \\ |\varphi(x)|^2 \leqslant |\varphi|(xx^*) \quad (x \in M) \end{array}\right\} \ (*)$$

For the polar decomposition of φ^*, where $\varphi^*(x) = \overline{\varphi(x^*)}$, we obtain

$$\varphi^* = u^*|\varphi^*|, \quad |\varphi^*| = u|\varphi|u^* \quad \text{and} \quad uu^* = s(|\varphi^*|).$$

It is easy to see that $u^* \in s(|\varphi|)M$. (Takesaki (1979, Theorem III.4.2 & Proposition III.4.6))

If Ψ is a subset of the state space of a C*-algebra M, then Ψ is called *faithful* if $0 \leqslant x \in M$ and $\psi(x) = 0$ for all $\psi \in \Psi$ implies $x = 0$. Ψ is called *subinvariant* for a positive map $T \in \mathcal{L}(M)$ (resp.., positive semigroup $\mathcal{T}$) if $T'\psi \leqslant \psi$ for all $\psi \in \Psi$ (resp., $T(t)'\psi \leqslant \psi$ for all $T(t) \in \mathcal{T}$ and $\psi \in \Psi$). Recall that for every positive map $T \in \mathcal{L}(M)$ there exists a state φ on M such that $T'\varphi = r(T)\varphi$, where $r(T)$ denotes the spectral radius of T. (Groh (1981, Theorem 2.1))

Let us start our investigation with two lemmata, where $\mathrm{Fix}(T)$ is the fixed space of T, i.e., the set $\{x \in M : Tx = x\}$.

Lemma 1.1. *Suppose M to be a C*-algebra and $T \in \mathcal{L}(M)$ an identity preserving Schwarz map.*

(i) *Let $b \colon M \times M \to M$ be a sesquilinear map such that $b(z, z) \geqslant 0$ for all $z \in M$. Then $b(x, x) = 0$ for some $x \in M$ if and only if $b(x, y) = 0$ and $b(y, x) = 0$ for all $y \in M$.*

(ii) *If there exists a faithful family Ψ of subinvariant states for T on M, then $\mathrm{Fix}(T)$ is a C*-subalgebra of M and $T(xy) = xT(y)$ for all $x \in \mathrm{Fix}(T)$ and $y \in M$.*

Proof. (i) Take $0 \leqslant \psi \in M^*$ and consider $f := \psi \circ b$. Then f is a positive semidefinite sesquilinear form on M with values in $\mathbb{C}$. From the Cauchy-Schwarz inequality it follows that $f(x, x) = 0$ for some $x \in M$ if and only if $f(x, y) = 0$ and $f(y, x) = 0$ for all $y \in M$. Since the positive cone M_+^* is generating, assertion (i) is proved.

(ii) Since T is positive it follows $T(x)^* = T(x^*)$ for all $x \in M$. Hence $\mathrm{Fix}(T)$ is a self adjoint subspace of M, i.e., invariant under the involution on M. For every x, $y \in M$ let

$$b(x, y) := T(xy^*) - T(x)T(y)^*.$$

Then b satisfies the assumptions of (i).

If $x \in \mathrm{Fix}(T)$ then

$$0 \leqslant xx^* = (Tx)(Tx)^* \leqslant T(xx^*),$$

hence

$$0 \leqslant \psi(T(xx^*) - xx^*) = 0 \quad \text{for all } \psi \in \Psi.$$

But this implies $T(xx^*) = T(x)T(x)^* = xx^*$ and consequently, $b(x, x) = 0$. Hence $T(xy^*) = xT(y)^*$ for all $y \in M$ and (ii) is proved. □

Lemma 1.2. *Let M be a* W*-*algebra, T an identity preserving Schwarz map on M and $S \in \mathcal{L}(M)$ such that $S(x)(Sx)^* \leqslant T(xx^*)$ for every $x \in M$.*

- *(i) If $v \in M$ such that $S(v^*) = v^*$ and $T(v^*v) = v^*v$, then $T(xv) = S(x)v$ for all $x \in M$.*

- *(ii) Suppose there exists $\varphi \in M_*$ with polar decomposition $\varphi = u|\varphi|$ such that $S_*\varphi = \varphi$ and $T_*|\varphi| = |\varphi|$. If the closed subspace $s(|\varphi|)M$ is T-invariant, then $Su^* = u^*$ and $T(u^*u) = u^*u$.*

Proof. (i) Define a positive semidefinite sesquilinear map $b : M \times M \mapsto M$ by

$$b(x, y) := T(xy^*) - S(x)S(y)^* \quad (x, y \in M).$$

Since $b(v^*, v^*) = 0$ we obtain $b(x, v^*) = 0$ for all $x \in M$, hence $T(xv) = S(x)v$. (Lemma 1.1 (i))

(ii) Since $s(|\varphi|)M$ is a closed right ideal, the closed face $F := s(|\varphi|)(M_+)s(|\varphi|)$ determines $s(|\varphi|)M$ uniquely, i.e.,,

$$s(|\varphi|)M = \{x \in M : xx^* \in F\}$$

(Pedersen (1979, Theorem 1.5.2)). Since T is a Schwarz map and $s(|\varphi|)M$ is T-invariant, it follows $TF \subseteq F$. On the other hand, if $x \in s(|\varphi|)M$ then $xx^* \in F$. Consequently,

$$0 \leqslant S(x)S(x)^* \leqslant T(xx^*) \in F,$$

whence $S(x) \in s(|\varphi|)M$.

Next we show $T(u^*u) = u^*u$ and $Su^* = u^* \in s(|\varphi|)M$. First of all

$$\begin{aligned} 0 &\leqslant (Su^* - u^*)(Su^* - u^*)^* \\ &\leqslant T(u^*u) - u^*S(u^*)^* - (Su^*)u + u^*u. \end{aligned}$$

Since $S_*\varphi = \varphi$, $T_*|\varphi| = |\varphi|$ and $\varphi = u|\varphi|$ it follows

$$\begin{aligned} 0 &\leqslant |\varphi|((Su^* - u^*)(Su^* - u^*)^*) \\ &\leqslant 2|\varphi|(u^*u) - |\varphi|(S(u^*)u)^* - |\varphi|(S(u^*)u) \\ &= 2|\varphi|(uu^*) - \varphi(u^*)^* - \varphi(u^*) \\ &= 2(|\varphi|(u^*u) - |\varphi|(u^*u)) = 0. \end{aligned}$$

But $(Su^* - u^*)(Su^* - u^*) \in F$ and $|\varphi|$ is faithful on F. Hence we obtain $Su^* = u^*$. Consequently,

$$0 \leqslant u^*u = (Su^*)(Su^*)^* \leqslant T(u^*u)$$

and $T(u^*u) = u^*u$ by the faithfulness and T-invariance of $|\varphi|$. □

Remark 1.3. Take S and T as in Lemma 1.2 (ii). If V_{u^*} (resp. V_u) is the map $(x \mapsto xu^*)$ (resp. $(x \mapsto xu)$) on M, then V_{u^*} is a continuous bijection from $Ms(|\varphi|)$ onto $Ms(|\varphi^*|)$ with inverse V_u (because $V_u \circ V_{u^*} = \mathrm{Id}_{Ms(|\varphi|)}$ and $V_{u^*} \circ V_u = \mathrm{Id}_{Ms(|\varphi^*|)}$). Let $x \in M$. From $T(xu) = S(x)u$ we obtain $T(xu)u^* = S(x)uu^*$. In particular, if $Ms(|\varphi^*|)$ is S-invariant, then

$$(V_{u^*} \circ T \circ V_u)(x) = T(xu)u^* = S(x)$$

for every $x \in Ms(|\varphi^*|)$. Let $T_|$ (resp. $S_|$) be the restriction of T to $Ms(|\varphi|)$ (resp. of S to $Ms(|\varphi^*|)$). Then the following diagram is commutative:

$$
\begin{array}{ccc}
Ms(|\varphi|) & \xrightarrow{\ \ T_|\ \ } & Ms(|\varphi|) \\
V_u \downarrow & & \downarrow V_{u^*} \\
Ms(|\varphi^*|) & \xrightarrow{\ \ S_|\ \ } & Ms(|\varphi^*|)
\end{array}
$$

In particular, $\sigma(S_|) = \sigma(T_|)$. Therefore we may deduce spectral properties of $S_|$ from $T_|$ and vice versa. More concrete applications of Lemma 1.2 will follow.

We now investigate the fixed space $\mathrm{Fix}(R) := \mathrm{Fix}(\lambda R(\lambda))$, $\lambda \in D$, of a pseudo-resolvent $\mathcal{R}$ with values in the predual of a W*-algebra M.

Proposition 1.4. *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \Re\lambda > 0\}$ with values in the predual M_* of a W*-algebra M and suppose $\mathcal{R}$ to be identity preserving and of Schwarz type.*

(i) If $\alpha \in \mathbb{R}$ and $\psi \in M_*$ such that $(\gamma - \mathrm{i}\alpha)R(\gamma)\psi = \psi$ for some $\gamma \in D$, then $\lambda R(\lambda)|\psi| = |\psi|$ and $\lambda R(\lambda)|\psi^*| = |\psi^*|$ for all $\lambda \in D$.

(ii) $\mathrm{Fix}(\mathcal{R})$ is invariant under the involution in M_*. If $\psi \in \mathrm{Fix}(\mathcal{R})$ is self adjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of $\mathrm{Fix}(\mathcal{R})$.

Proof. If $(\gamma - \mathrm{i}\alpha)R(\gamma)\psi = \psi$ then $(\lambda - \mathrm{i}\alpha)R(\lambda)\psi = \psi$ for all $\lambda \in D$. In particular, $\mu R(\mu + \mathrm{i}\alpha)\psi = \psi$ $(\mu \in \mathbb{R}_+)$. For all $x \in M$ we obtain

$$
\begin{aligned}
|\psi(x)|^2 = |<\mu R(\mu + \mathrm{i}\alpha)'x, \psi>|^2 &\leqslant \\
&\leqslant \|\psi\| <(\mu R(\mu + \mathrm{i}\alpha)'x)(\mu R(\mu + \mathrm{i}\alpha)'x)^*, \psi> \leqslant \\
&\leqslant \|\psi\| <\mu R(\mu)'(xx^*), |\psi|>
\end{aligned}
$$

(D-I, Corollary 2.2). Since

$$
\begin{aligned}
\|\psi\| = \||\psi|\| = |\psi|(1) &= \\
&= <\mu R(\mu)'1, |\psi|> = \|\mu R(\mu)|\psi|\|,
\end{aligned}
$$

we obtain $\mu R(\mu)|\psi| = |\psi|$ by the uniqueness theorem (*) above for the absolut value—therefore $|\psi| \in \mathrm{Fix}(\mathcal{R})$. Since

$$0 \leqslant (\mu R(\mu)'x)(\mu R(\mu)'x)^* \leqslant \mu R(\mu)'xx^*,$$

the map $R(\mu)$ is positive. Consequently $(\mu + \mathrm{i}\alpha)R(\mu)\psi^* = \psi^*$ from which $|\psi^*| \in$ Fix$(\mathcal{R})$ follows. If $\varphi \in$ Fix$(\mathcal{R})$ is selfadjoint with Jordan decomposition $\varphi = \varphi^+ - \varphi^-$, then $|\varphi| = \varphi^+ + \varphi^-$. (Takesaki (1979, Theorem III.4.2.)) From this we obtain that φ^+ and φ^- are in Fix$(\mathcal{R})$. □

Corollary 1.5. *Let* $\mathcal{T}$ *be an identity preserving semigroup of Schwarz type on* M_* *with generator* A *and suppose* $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.

(i) *If* $\alpha \in \mathbb{R}$ *and* $\psi \in \ker(\mathrm{i}\alpha - A)$, *then* $|\psi|$ *and* $|\psi^*|$ *are elements of* Fix$(T) = \ker(A)$.

(ii) Fix$(\mathcal{T})$ *is invariant under the involution of* M_*. *If* $\psi \in$ Fix$(\mathcal{T})$ *is selfadjoint, then the positive part* ψ^+ *and the negative part* ψ^- *of* ψ *are elements of* Fix$(\mathcal{T})$.

The proof follows immediately from D-I, Corollary 2.2 and the fact that $\mathrm{Kern}(A) = \mathrm{Fix}(\lambda R(\lambda, A))$ for all $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$.

If $\mathcal{T}$ is the semigroup of translations on $L^1(\mathbb{R})$ and A' the generator of the adjoint weak*-semigroup, then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$, while $P\sigma(A') \cap \mathrm{i}\mathbb{R} = \mathrm{i}\mathbb{R}$. For that reason we cannot expect a simple connection between these two sets. But as we shall see below, if a semigroup on the predual of a W*-algebra has sufficiently many invariant states, then the point spectra of A and A' contained in $\mathrm{i}\mathbb{R}$ are identical.

Helpful for these investigations will be the next lemma.

Lemma 1.6. *Let* $\mathcal{R}$ *be a pseudo-resolvent on* $D = \{\lambda \in \mathbb{C} : \Re \lambda > 0\}$ *with values in a Banach space* E *such that* $\|R(\mu + \mathrm{i}\alpha)\| \leqslant 1$ *for all* $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$. *Then*

$$\dim \, \mathrm{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)) \leqslant \dim \, \mathrm{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)')$$

for all $\lambda \in D$.

Proof. Let $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $S := \mu R(\mu + \mathrm{i}\alpha)$. Since S is a contraction, its adjoint S' maps the dual unit ball E_1' into itself.

Let $\mathfrak{U}$ be a free ultrafilter on $[1, \infty[$ which converges to 1. Since E_1' is $\sigma(E', E)$-compact,

$$\psi_0 := \lim_{\mathfrak{U}} (\lambda - 1)R(\lambda, S)'\psi$$

exists for all $\psi \in E_1'$. Since S' is $\sigma(E', E)$-continuous and since $S'R(\lambda, S)' = \lambda R(\lambda, S') - \mathrm{Id}$ we conclude $\psi_0 \in \mathrm{Fix}(S')$.

Take now $0 \neq x_0 \in \mathrm{Fix}(S)$ and choose $\psi \in E_1'$ such that $\psi(x_0)$ is different from zero. From the considerations above it follows

$$\psi_0(x_0) = \lim_{\mathfrak{U}} (\lambda - 1)\psi(R(\lambda, S)x_0) = \psi(x_0) \neq 0$$

hence $0 \neq \psi_0 \in \mathrm{Fix}(S)$. Therefore $\mathrm{Fix}(S')$ separates the points of $\mathrm{Fix}(S)$.

From this it follows that

$$\dim \, \mathrm{Fix}(S) \leqslant \dim \, \mathrm{Fix}(S').$$

Since $\mathcal{R}$ and $\mathcal{R}'$ are pseudo-resolvents, the assertion is proved. □

Corollary 1.7. *Let* $\mathcal{T}$ *be a semigroup of contractions on a Banach space* E *with generator* A. *Then*

$$\dim \mathrm{Kern}(\mathrm{i}\alpha - A) \leqslant \dim \mathrm{Kern}(\mathrm{i}\alpha - A')$$

for all $\alpha \in \mathbb{R}$.

This follows from Lemma 1.6 on page 5 because $\mathrm{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)) = \mathrm{Kern}(\mathrm{i}\alpha - A)$.

Proposition 1.8. *Let* $\mathcal{T}$ *be an identity preserving semigroup of Schwarz type with generator* A *on the predual of a* W^**-algebra and suppose that there exists a faithful family* Ψ *of* $\mathcal{T}$*-invariant states. Then for all* $\alpha \in \mathbb{R}$ *we have*

$$\dim \mathrm{Kern}(\mathrm{i}\alpha - A) = \dim \mathrm{Kern}(\mathrm{i}\alpha - A')$$

and

$$P\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A') \cap \mathrm{i}\mathbb{R}$$

Proof. The inequality $\dim \mathrm{Kern}(\mathrm{i}\alpha - A) \leqslant \dim \mathrm{Kern}(\mathrm{i}\alpha - A')$ follows from Corollary 1.7.

Let $D = \{\lambda \in \mathbb{C} \colon \Re\,\lambda > 0\}$ and $\mathcal{R}$ the pseudo-resolvent induced by $R(\lambda, A)$ on D. Then $\mathcal{R}$ is identity preserving and of Schwarz type. Take $\mathrm{i}\alpha \in P\sigma(A)$ $(\alpha \in \mathbb{R})$ and choose $0 < \mu \in \mathbb{R}$.

If $\psi_\alpha \in M_*$ is of norm one with polar decomposition $\psi_\alpha = u_\alpha |\psi_\alpha|$ such that $\psi_\alpha = (\mu - \mathrm{i}\alpha)R(\mu)\psi_\alpha$ then $\mu R(\mu)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4 (i) on page 4).

Since

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant 1 - s(|\psi_\alpha|)$$

we obtain $\mu R(\mu)'s(|\psi_\alpha|) = s(|\psi_\alpha|)$ by the faithfulness of Ψ.

Hence the maps $S := (\mu - \mathrm{i}\alpha)R(\mu)'$ and $T := \mu R(\mu)'$ fulfil the assumptions of Lemma 1.2 (ii) on page 3.

Therefore $Su_\alpha^* = u_\alpha^*$ or $(\mu - \mathrm{i}\alpha)R(\mu)'u_\alpha^* = u_\alpha^*$ which implies $u_\alpha^* \in D(A')$ and $A'u_\alpha^* = \mathrm{i}\alpha u_\alpha^*$.

If $\mathrm{i}\alpha \in P\sigma(A')$, $\alpha \in \mathbb{R}$, choose $0 \neq v_\alpha$ such that

$$v_\alpha = (\mu - \mathrm{i}\alpha)R(\mu)'v_\alpha \quad (\mu \in \mathbb{R}_+)$$

and $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$.

Since

$$0 \leqslant v_\alpha v_\alpha^* = ((\mu - \mathrm{i}\alpha)R(\mu)'v_\alpha)((\mu - \mathrm{i}\alpha)R(\mu)'v_\alpha)^* \leqslant \mu R(\mu)'(v_\alpha v_\alpha^*)$$

we obtain $\mu R(\mu)'(v_\alpha v_\alpha^*) = v_\alpha v_\alpha^*$ because Ψ is faithful.

Hence from Lemma 1.2 (i) on page 3 it follows

$$\mu R(\mu)'(x v_\alpha^*) = ((\mu - \mathrm{i}\alpha)R(\mu)'x)v_\alpha^*$$

for all $x \in M$.

Let ψ_α be the normal linear functional $(x \mapsto \psi(x v_\alpha^*))$ on M and note that $\psi_\alpha(v_\alpha) \neq 0$. Then

$$\begin{aligned}
\langle x, (\mu - \mathrm{i}\alpha)R(\mu)\psi_\alpha \rangle &= \langle ((\mu - \mathrm{i}\alpha)R(\mu)'x)v_\alpha^*, \psi \rangle \\
&= \langle \mu R(\mu)'(x v_\alpha^*), \psi \rangle = \psi(x v_\alpha^*) = \psi_\alpha(x)
\end{aligned}$$

for all $x \in M$. Consequently $i\alpha \in P\sigma(A)$ and

$$\dim \operatorname{Kern}((i\alpha - A')) \leqslant \dim \operatorname{Kern}((i\alpha - A))$$

which proves the assertion. □

Remark 1.9. From the above proof we obtain the following: If $0 \neq \psi_\alpha \in \operatorname{Kern}(i\alpha - A)$ with polar decomposition $\psi_\alpha = u_\alpha |\psi_\alpha|$ ($\alpha \in \mathbb{R}$) then $A'u_\alpha = i\alpha u_\alpha$.

Conversely, if $0 \neq v_\alpha \in \operatorname{Kern}(i\alpha - A')$, then there exists $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$ and the normal linear form

$$\psi_\alpha := (x \mapsto \psi(x v_\alpha^*))$$

is an eigenvector of A pertaining to the eigenvalue $i\alpha$.

If T is a C_0-semigroup of Markov operators on a commutative C*-algebra with generator A, it has been shown in B-III, that the boundary spectrum $\sigma(A) \cap i\mathbb{R}$ of its generator is additively cyclic. This is no longer true in the non commutative case.

Example 1.10. For $0 \neq \lambda \in i\mathbb{R}$ and $t \in \mathbb{R}$ let

$$u_t := \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \in M_2(\mathbb{C})$$

The semigroup of *-automorphisms $(x \mapsto u_t x u_t^*)$ on $M_2(\mathbb{C})$ is identity preserving and of Schwarz type but the spectrum of its generator is $\{0, \lambda, \lambda^*\}$ hence is not additively cyclic.

It turns out that, in order to obtain a non commutative analogue of the Perron-Frobenius theorems, one has to consider semigroups which are irreducible.

Recall that a semigroup $\mathcal{S}$ of positive operators on an ordered Banach space (E, E_+) is called *irreducible* if no closed face of E_+, different from $\{0\}$ and E_+, is invariant under $\mathcal{S}$. Here a face F in E is a subcone of E_+ such that the conditions $0 \leqslant x \leqslant y$, $x \in E, y \in F$ imply $x \in F$ (compare Definitions 3.1 in B-III and C-III). In the context of W*-algebras M we call a semigroup $\mathcal{S}$ of positive maps on M *weak*-irreducible*, if no $\sigma(M, M_*)$-closed face of M_+ is $\mathcal{S}$-invariant.

Since the norm closed faces of M_* and the $\sigma(M, M_*)$-closed faces of M are related by formation of polars with respect to the dual system $\langle M, M_* \rangle$ (see Pedersen (1979, Theorem 3.6.11 and Theorem 3.10.7.)) a semigroup $\mathcal{S}$ is (norm) irreducible on M_* if and only if its adjoint semigroup is weak*-irreducible.

Theorem 1.11. *Let $\mathcal{T}$ be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-algebra *and suppose $P\sigma(A) \cap i\mathbb{R} \neq \emptyset$.*

(i) *The fixed space of $\mathcal{T}$ is one dimensional and spanned by a faithful normal state.*

(ii) *$P\sigma(A) \cap i\mathbb{R}$ is an additive subgroup of $i\mathbb{R}$,*

$$\sigma(A) = \sigma(A) + (P\sigma(A) \cap i\mathbb{R})$$

and every eigenvalue in $i\mathbb{R}$ is simple.

(iii) *The fixed space of the adjoint weak*-semigroup T' is one-dimensional.*

(iv) $P\sigma(A') \cap i\mathbb{R} = P\sigma(A) \cap i\mathbb{R}$ *for the generator A' of the adjoint semigroup, and every $\gamma \in P\sigma(A') \cap i\mathbb{R}$ is simple.*

Proof. Since $P\sigma(A) \cap i\mathbb{R} \neq \emptyset$ there exists $\psi \in \mathrm{Fix}(\mathcal{T})_+$ of norm one (Corollary 1.5). If $F := \{x \in M_+ : \psi(x) = 0\}$ then F is a $\sigma(M, M_*)$-closed, $\mathcal{T}'$-invariant face in M, hence $F = \{0\}$. Therefore every $0 \neq \psi \in \mathrm{Fix}(T)_+$ is faithful.

Let $\psi_1, \psi_2 \in \mathrm{Fix}(\mathcal{T})_+$ be states such that $f := \psi_1 - \psi_2$ is different from zero. If $f = f^+ - f^-$ is the Jordan decomposition of f, then f^+ and f^- are elements of $\mathrm{Fix}(\mathcal{T})$, whence faithful. Since the support projections of these two normal linear functionals are orthogonal, we obtain $f^+ = 0$ or $f^- = 0$ which implies $\psi_1 \leqslant \psi_2$ or $\psi_2 \leqslant \psi_1$. Consequently $\psi_2 = \psi_1$.

Since $\mathrm{Fix}(\mathcal{T})$ is positively generated (Corollary 1.5 on page 5), $\mathrm{Fix}(\mathcal{T}) = \mathbb{C}.\varphi$ for some faithful normal state φ.

Let $\mu \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ such that $i\alpha \in P\sigma(A)$. If $\psi_\alpha = u_\alpha |\psi_\alpha|$ is a normalized eigenvector of A pertaining to $i\alpha$, then $\varphi = |\psi_\alpha| = |\psi_\alpha^*|$ (Corollary 1.5 and the above considerations). Hence $u_\alpha u_\alpha^* = u_\alpha^* u_\alpha = s(\varphi) = 1$.

Since
$$(\mu - i\alpha)R(\mu, A)\psi_\alpha = \psi_\alpha$$

and
$$\mu R(\mu, A)|\psi_\alpha| = |\psi_\alpha|$$

we obtain by Lemma 1.2 (ii) on page 3 that

$$\mu R(\mu, A) = V_\alpha \circ \mu R(\mu + i\alpha, A) \circ V_\alpha^{-1} \quad (1)$$

where V_α is the map $(x \mapsto x u_\alpha)$ on M.

Similarly for $i\beta \in P\sigma(A)$, we find V_β such that $1 = u_\beta u_\beta^* = u_\beta u_\beta^*$ and

$$\mu R(\mu, A) = V_\beta \circ \mu R(\mu + i\beta, A) \circ V_\beta^{-1} \quad (2)$$

Hence
$$\mu R(\mu, A) = V_{\alpha\beta} \circ \mu R(\mu + i(\alpha + \beta), A) \circ V_{\alpha\beta}^{-1} \quad (3)$$

where $V_{\alpha\beta} := V_\alpha \circ V_\beta$.

Since u_α is unitary in M, it follows from (1) that $i\alpha$ is an eigenvalue which is simple because $\mathrm{Fix}(T) = \mathrm{Fix}(\mu R(\mu, A))$ is one dimensional.

From (3) it follows that $i(\alpha + \beta) \in P\sigma(A)$ since $0 \in P\sigma(A)$ and $V_{\alpha\beta}$ is bijective. From the identity (1) we conclude that $\sigma(R(\mu, A)) = \sigma(R(\mu + i\alpha))$, which proves

$$\sigma(A) + (P\sigma(A) \cap i\mathbb{R}) \subseteq \sigma(A)$$

The other inclusion is trivial since $0 \in P\sigma(A)$. $\qquad\square$

Remarks 1.12.   (i) Let φ be the normal state on M such that $\mathrm{Fix}(T) = \mathbb{C}\varphi$ and let $H := P\sigma(A) \cap i\mathbb{R}$. From the proof of Theorem 1.10 it follows that there exists a family $\{u_\eta : \eta \in H\}$ of unitaries in M such that $A' u_\eta = -\eta u_\eta$ and $A(u_\eta \varphi) = \eta(u_\eta \varphi)$ for all $\eta \in H$.

(ii) If the group H is generated by a single element, i.e., $H = i\gamma\mathbb{Z}$ for some $\gamma \in \mathbb{R}$ then the family $\{u_\gamma^k \colon k \in \mathbb{Z}\}$ is a complete family of eigenvectors pertaining to the eigenvalues in H, where $u_\gamma \in M$ is unitary such that $A'u_\gamma = i\gamma u_\gamma$.

Proposition 1.13. *Suppose that $\mathcal{T}$ and M satisfy the assumptions of Theorem 1.10, and let N_* be the closed linear subspace of M_* generated by the eigenvectors of A pertaining to the eigenvalues in $i\mathbb{R}$. Denote by $\mathcal{T}_0$ the restriction of $\mathcal{T}$ to N_*. Then*

(i) $G := (T_0)^- \subseteq L_s(N_*)$ is a compact, Abelian group.

(ii) $\mathrm{Id}\,|N_* \in \{T_0(t)\colon t > s\}^- \subseteq L_s(N_*)$ for all $0 < s \in \mathbb{R}$.

Proof. For $\eta \in H := P\sigma(A) \cap i\mathbb{R}$ let

$$U(\eta) := \{\psi \in D(A) : A\psi = \eta\psi\}$$

and $U = \{U(\eta)\colon \eta \in H\}$. Then $(U)^- = N_*$.

For each $\psi \in U$ there exists $\eta \in H$ such that

$$\{T_0(t)\psi\colon t \in \mathbb{R}_+\} = \{e^{-\eta t}\psi\colon t \in \mathbb{R}_+\}$$

Consequently this set is relatively compact in $L_s(N_*)$. From [Schaefer (1966),III.4.5] we obtain that G is compact.

Next choose $\psi_1, ..., \psi_n \in U$, $0 < s \in \mathbb{R}$ and $\delta > 0$. Since $T_0(t)\psi_i = e^{\eta_i t}\psi_i$ $(1 \leqslant i \leqslant n)$ for some $\eta_i \in H$, it follows from a theorem of Kronecker (see, Jacobs (1972, Satz 6.1., p.77)) that there exists $s < t$ such that

$$|(1, 1, ..., 1) - (e^{\eta_1 t}, e^{\eta_2 t}, ..., e^{\eta_n t})| < \delta$$

hence

$$\sup\{\|\psi_i - T_0(t)\psi_i\|\colon 1 \leqslant i \leqslant n\} < \delta$$

or $\mathrm{Id}\,|N_* \in \{T_0(t)\colon t > s\}^- \subseteq L_s(N_*)$.

Finally we prove the group property of G. Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{R}$ such that $\lim_\mathfrak{U} T_0(t) = \mathrm{Id}$ in the strong operator topology. For positive $s \in \mathbb{R}$ let $S := \lim_\mathfrak{U} T(t-s)$. Then $ST_0(s) = T_0(s)S = \mathrm{Id}$, hence $T_0(s)^{-1}$ exists in G for all $s \in \mathbb{R}_+$. From this it follows that G is a group. $\qquad\square$

Remark 1.14. (i) Let $\kappa : \mathbb{R} \to G$ be given by

$$\kappa(t) = \begin{cases} T_0(t) & \text{if } 0 \leqslant t \\ T_0(t)^{-1} & \text{if } t \leqslant 0 \end{cases}$$

Then κ is a continuous homomorphism with dense range, i.e., (G, κ) is solenoidal (see Hewitt and Ross (1963)).

(ii) The compact group G and the discret group $P\sigma(A) \cap i\mathbb{R}$ are dual in the sense of locally compact Abelian groups.

(iii) Let (G, κ) be a solenoidal compact group and let $N_* = L^1(G)$. Then the induced lattice semigroup $T = (\kappa(t))_{t \geqslant 0}$ fulfils the assertions of Theorem 1.10. For example, if G is the dual of $\mathbb{R}_d$, then $P\sigma(A) \cap i\mathbb{R} = i\mathbb{R}$. Since the fixed space of $\kappa(t)$ is given by

$$\mathrm{Fix}(\kappa(t)) = \left(\bigcup_{k \in \mathbb{Z}} \mathrm{Kern}(\frac{2\pi i k}{t} - A) \right)^-$$

no $T(t) \in \mathcal{T}$ is irreducible.

(iv) If $\mathcal{T}$ is the irreducible semigroup of Schwarz type on the predual of $B(H)$ given in Evans (1977) then $P\sigma(A) \cap i\mathbb{R} = \emptyset$.

2 Spectral Properties of Uniformly Ergodic Semigoups

The aim of this section is the study of spectral properties of semigroups which are uniformly ergodic, identity preserving and of Schwarz type. For the basic theory of uniformly ergodic semigroups on Banach spaces we refer to Dunford and Schwartz (1958).

Our first result yields an estimate for the dimension of the eigenspaces pertaining to eigenvalues of a pseudo-resolvent.

Proposition 2.1. *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} \colon \Re \lambda > 0\}$ with values in the predual of a W*-algebra M. If $Fix \lambda R(\lambda)$ is finite dimensional for some $\lambda \in D$, then*

$$\dim \mathrm{Fix}((\gamma - i\alpha)\, R(\gamma)) \leqslant \dim \mathrm{Fix}(\lambda R(\lambda))$$

for all $\gamma \in D$ and $\alpha \in \mathbb{R}$.

Proof. By D-IV, Remark 3.2.c we may assume without loss of generality that there exists a faithful family of $\mathcal{R}$-invariant normal states on M. In particular the fixed space N of the adjoint pseudo-resolvent $\mathbb{R}\mathcal{R}'$ is a W*-subalgebra of M with $\mathbb{1} \in N$ (by Lemma 1.1 (ii)). Since N is finite dimensional there exist a natural number n and a set $P := \{p_1, .., p_n\}$ of minimal, mutually orthogonal projections in N such that $\sum_{k=1}^n p_k = \mathbb{1}$. These projections are also mutually orthogonal in M with sum $\mathbb{1}$.

Let R_j be the $\sigma(M, M_*)$-closed right ideal $p_j M$ and L_j the closed left invariant subspace $M_* p_j$ for $(1 \leqslant j \leqslant n)$. Sine the map $\mu R(\mu)'$, $\mu \in \mathbb{R}_+$ is an identity preserving Schwarz map, we obtain from Lemma 1.1.b that for all $x \in N$ and $y \in M$,

$$\mu R(\mu)'(xy) = x(\mu \mathcal{R}'(\mu)y).$$

In particular, R_j, resp. L_j are invariant under $\mathbb{R}\mathcal{R}'$, respectively, $\mathcal{R}$. Furthermore, if $\psi \in L_j$ with polar decomposition $\psi = u|\psi|$, then $u^* u \leqslant s(|\psi|) \leqslant p_j$. Consequently, $|\psi| \in L_j$.

Let now $\alpha \in \mathbb{R}$ and suppose that there exists $\psi_\alpha \in L_j$ of norm 1, $\psi_\alpha = u_\alpha |\psi_\alpha|$, such that

$$\psi_\alpha \in \mathrm{Fix}((\lambda - i\alpha)R(\lambda)), \lambda \in D.$$

Since $\lambda R(\lambda)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4 on page 4), we obtain

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant (1 - s(|\psi_\alpha|)), \mu \in \mathbb{R}_+.$$

From the existence of a faithful family of $\mathcal{R}$-invariant normal states and since $\mathcal{R}'$ is identity preserving it follows that

$$\mu R(\mu)'s(|\psi_\alpha|) = s(|\psi_\alpha|)$$

Thus $s(|\psi_\alpha|) \leqslant p_j$ and even $s(|\psi_\alpha|) = p_j$ by the minimality property of p_j.

On the other hand, $\psi_\alpha^* \in \text{Fix}((\lambda + i\alpha)R(\lambda))$. As above we obtain

$$\mu R(\mu)'s(|\psi_\alpha^*|) = s(|\psi_\alpha^*|)$$

Consequently, the closed left ideals $Ms(|\psi_\alpha^*|)$ and $Ms(|\psi_\alpha|)$ are $\mathcal{R}'$-invariant.

Next fix $\mu \in \mathbb{R}_+$, let $S := (\mu - i\alpha)R(\mu)'$ and $T = \mu R(\mu)'$. Then $(Sx)(Sx)^* \leqslant T(xx^*)$, $S_*(\psi_\alpha^*) = \psi_\alpha^*$, $T_*(|\psi_\alpha^*|) = |\psi_\alpha^*|$, and T is an identity preserving Schwarz map. Since $s(|\psi_\alpha^*|)M$ is T-invariant, the assumptions of Lemma 1.2 on page 3 are fulfilled and we obtain for every $x \in M$

$$S(x)u_\alpha^* = T(xu_\alpha^*).$$

The closed left ideal Mp_j is S-invariant, therefor it follows

$$S(x) = T(xu_\alpha^*)u_\alpha \,,\, x \in Mp_j$$

(see Remark 1.3 on page 4). Since u_α does not depend on $\mu \in \mathbb{R}_+$ we obtain for all $\mu \in \mathbb{R}_+$

$$\mu R(\mu + i\alpha)'x = \mu R(\mu)'(xu_\alpha^*)u_\alpha$$

Consequently, the holomorphic functions $(\mu \mapsto \mu R(\mu)'(xu_\alpha)u_\alpha^*)$ and $(\mu \mapsto \mu R(\mu + i\alpha)'x)$ coincide on $\mathbb{R}_+$ from which we conclude

$$\lambda R(\lambda + i\alpha)'x = \lambda R(\lambda)'(xu_\alpha^*)u_\alpha$$

for every $\lambda \in D$ and all $x \in Mp_j$.

Since the map $(y \mapsto yu_\alpha)$ is a continuous bijection from $M(u_\alpha u_\alpha^*)$ onto Mp_j and its inverse is the map $(y \mapsto yu_\alpha^*)$, we can deduce that

$$\dim \text{Fix}((\lambda - i\alpha)R(\lambda)'|Mp_j) = \dim \text{Fix}(\lambda R(\lambda)')|M(u_\alpha u_\alpha^*)$$
$$\leqslant \dim \text{Fix}(\mathcal{R}')$$

Since $\bigoplus_{j=1}^n Mp_j = M$ and $\bigoplus_{j=1}^n L_j = M_*$ we obtain

$$\dim \text{Fix}((\lambda - i\alpha)R(\lambda)')) = \dim \text{Fix}(\lambda R(\lambda)')$$
$$= \dim \text{Fix}(\lambda R(\lambda))$$

and the assertion follows from Lemma 1.6 on page 5. $\qquad\square$

Before going on let us recall the basic facts of the *ultrapower* $\hat{E}$ of a Banach space E with respect to some free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ (compare A-I,3.6). If $\ell^\infty(E)$ is the Banach space of all bounded functions on $\mathbb{N}$ with values in E, then

$$c_\mathfrak{U}(E) := \{(x_n) \in \ell^\infty(E) \colon \lim_\mathfrak{U} \|x_n\| = 0\}$$

is a closed subspace of $\ell^\infty(E)$ and equal to the kernel of the seminorm

$$\|(x_n)\| := \lim_\mathfrak{U} \|x_n\|, \, (x_n) \in \ell^\infty(E)$$

By the *ultrapower* $\hat{E}$ we understand the quotient space $\ell^\infty(E)/c_\mathfrak{U}(E)$ with norm

$$\|\hat{x}\| = \lim_\mathfrak{U} \|x_n\|, \, (x_n) \in \hat{x} \in \hat{E}$$

Moreover, for a bounded linear operator $T \in L(E)$, we denote by $\hat{T}$ the well defined operator $\hat{T}\hat{x} := (Tx_n) + c_\mathfrak{U}(E), (x_n) \in \hat{x}$.

It is clear by virtue of $(x \mapsto (x, x, ..) + c_\mathfrak{U}(E))$ that each $x \in E$ defines an element $\hat{x} \in \hat{E}$. This isometric embedding as well as the operator map $(T \mapsto \hat{T})$ are called canonical. In particular, if $\mathcal{R} \colon (D \to L(E))$ is a pseudo-resolvent, then

$$\hat{\mathcal{R}} := (\lambda \mapsto R(\lambda)^\wedge) : D \to L(\hat{E})$$

is a pseudo-resolvent, too. Recall that the approximative point spectrum $A\sigma(T)$ is equal to the point spectrum $P\sigma(\hat{T})$ (see, e.g., Schaefer (1974, Chapter V, §1)).

This construction gives us the possibility to characterize uniformly ergodic semi-groups with finite dimensional fixed space.

Lemma 2.2. *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \Re\lambda > 0\}$ such that $\|R(\mu + \mathrm{i}\alpha)\| \leqslant 1$ for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and suppose*

$$0 < \dim \mathrm{Fix}((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)) < \infty \quad \text{for some} \quad \lambda \in D\,, \alpha \in \mathbb{R}$$

and the canonical extension $\hat{R}$ on some ultrapower $\hat{E}$. Then the following assertions hold:

(i) *$(\lambda - \mathrm{i}\alpha)^{-1}$ is a pole of the resolvent $R(., R(\lambda))$ for all $\lambda \in D$.*

(ii) *$\dim \mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) = \dim \mathrm{Fix}((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda))$ for all $\lambda \in D$.*

(iii) *$\mathrm{i}\alpha$ is a pole of the pseudo-resolvent $\mathcal{R}$ and the residue of $\mathcal{R}$ and $R(., R(\lambda))$ in $\mathrm{i}\alpha$ respectively $(\lambda - \mathrm{i}\alpha)^{-1}$ are identical.*

Proof. Take a normalized sequence (x_n) in E with

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)x_n - x_n\| = 0$$

The existence of such a sequence follows from the fact that the fixed space of $(\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)$ is non trivial. Suppose (x_n) is not relatively compact. Then we may assume that there exists $\delta > 0$ such that

$$\|x_n - x_m\| > \delta \quad \text{for} \quad n \neq m$$

Take $k \in \mathbb{N}$ and let $\hat{x}_k$ be the image of (x_{n+k}) in $\hat{E}$. Since

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)x_{n+k} - x_{n+k}\| = 0$$

the so defined $\hat{x}_k$'s belong to $\mathrm{Fix}((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda))$. Since this space is finite dimensional there exist $j < \ell$ such that

$$\|\hat{x}_j - \hat{x}_\ell\| \leqslant \frac{\delta}{2}$$

From the definition of the norm in $\hat{E}$ it follows that there are natural numbers $n < m$ such that

$$\|x_n - x_m\| \leqslant \frac{\delta}{2}$$

which leads to a contradiction.

Therefore every approximate eigenvector of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ pertaining to α is relatively compact. In particular it has a convergent subsequence from which it follows that the fixed space of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ is non trivial.

Obviously

$$\dim \mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) \leqslant \dim \mathrm{Fix}((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda))$$

If the last inequality is strict, then there exists $\gamma > 0$ and a normalized $\hat{x} \in \mathrm{Fix}((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda))$ such that

$$\gamma \leqslant \|\hat{y} - \hat{x}\|$$

for all $y \in \mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$.

Take a normalized sequence $(x_n) \in \hat{x}$. Then (x_n) has a convergent subsequence whence we may assume that $\lim_n x_n = z$ exists in E. Thus $0 \neq z \in \mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$. From this we obtain the contradiction

$$\gamma \leqslant \|\hat{z} - \hat{x}\| = \lim \|z - x_n\| = 0$$

Consequently

$$\dim \mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) = \dim \mathrm{Fix}(\lambda - \mathrm{i}\alpha)\hat{R}(\lambda))$$

Let $\{x_1, ..., x_n\}$ be a base of $\mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$ and choose $\{\varphi_1, ..., \varphi_n\}$ in $\mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)')$ such that $\varphi_k(x_j) = \delta_{k,j}$ (Lemma 1.6). Then

$$E = \mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) \oplus \left(\bigcap_{j=1}^{n} \ker \varphi_j\right)$$

where both subspaces on the right are $(\lambda - \mathrm{i}\alpha)R(\lambda)$-invariant and 1 is a pole of $(\lambda - \mathrm{i}\alpha)R(\lambda)|_{\mathrm{Fix}(()(\lambda - \mathrm{i}\alpha)R(\lambda))}$ by the finite dimensionality of $\mathrm{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$.

Suppose 1 belongs to the spectrum of S where S is the restriction of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ to $\bigcap_{j=1}^{n} \ker \varphi_j$. Then there exists a normalized sequence (y_n) in $\bigcap_{j=1}^{n} \ker \varphi_j$ such that

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)y_n - y_n\| = 0$$

Therefore (y_n) has an accumulation point different from zero in

$$\text{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) \cap (\bigcap_{j=1}^{n} \ker \varphi_j)$$

This contradiction implies that 1 does not belong to the spectrum of S.

Since $\text{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$ is finite dimensional, it follows from general spectral theory that $(\lambda - \mathrm{i}\alpha)^{-1}$ is a pole of $R(., R(\lambda))$ for every λ.

Thus (i) and (ii) are proved and assertion (iii) follows from the resolvent equality as in the proof of Greiner (1981, Proposition 1.2).   □

Proposition 2.3. *Let $\mathcal{T}$ be a semigroup of contractions on a Banach space E with generator A. Then the following assertions are equivalent.*

(a) *Each $\mathrm{i}\alpha$, $\alpha \in \mathbb{R}$, is a pole of the resolvent $R(., A)$ such that the corresponding residue has finite rank.*

(b) $\dim \text{Fix}((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda, A)) < \infty$ *for some (hence all) $\lambda \in \mathbb{C}$, $\Re\lambda > 0$ and the canonical extensions $\hat{R}(\lambda, A)$ of $R(\lambda, A)$ to some ultrapower.*

Proof. Let P_α be the residue of the resolvent $R(., A)$ in $\mathrm{i}\alpha$. Then $P_\alpha = \lim_{\lambda \to \mathrm{i}\alpha}(\lambda - \mathrm{i}\alpha)R(\lambda, A)$ in the operator norm of $L(E)$. Since the canonical map $(T \mapsto \hat{T})$ is isometric and since $\hat{E}$ is an ultrapower, we obtain

$$\hat{P}_\alpha = \lim_{\lambda \to \mathrm{i}\alpha}(\lambda - \mathrm{i}\alpha)\hat{R}(\lambda, A)$$

in $L(\hat{E})$ and $\text{rank}\,(P_\alpha) = \text{rank}\,(\hat{P}_\alpha)$. Because of

$$\hat{P}_\alpha(\hat{E}) = \text{Fix}((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda))$$

one part of the corollary is proved. The other follows from Lemma 2.2 on page 12.   □

Remarks 2.4.   (i) By the results in Lin (1974) a semigroup of contractions on a Banach space is uniformly ergodic if and only if 0 is a pole of the generator with order $\leqslant 1$. The residue of the resolvent in 0 and the assocciated ergodic projection are identical.

(ii) Let M be a W*-algebra with predual M_*, $\mathfrak{U}$ a free ultrafilter on $\mathbb{N}$ and $\widehat{M}$ (resp. $(M_*)^\wedge$) the ultrapower of M (resp. M_*) with respect to $\mathfrak{U}$. Then it is easy to see that $c_{\mathfrak{U}}(M)$ is a two sided ideal in $\ell^\infty(M)$ hence $\widehat{M}$ is a C*-algebra, but in general not a W*-algebra. Note that the unit of $\widehat{M}$ is the canonical image of 1. For $\hat{x} \in \hat{M}$ and $\hat{\varphi} \in (M_*)^\wedge$ let $J : (M_*)^\wedge \to \widehat{M}'$ be defined by

$$\langle x, J(\hat{\varphi})\rangle := \lim_{\mathfrak{U}} \varphi_n(x_n)\,, \ (x_n) \in \hat{x}\,, \ (\varphi_n) \in \hat{\varphi}$$

J is well defined and is an isometric embedding. It turns out that $J((M_*)^\wedge)$ is a translation invariant subspace of $\widehat{M}'$. Hence there exists a central projection $z \in \widehat{M}''$ such that $J((M_*)^\wedge) = \widehat{M}''z$ (Groh (1984, Proposition 2.2)).

Below we identify $(M_*)^\wedge$ via J with this translation invariant subspace. From the construction the following is obvious: If T is an identity preserving Schwarz map with preadjoint $T_* \in L(M_*)$, then $\widehat{T}$ is an identity preserving Schwarz map on $\widehat{M}$ such that $(T_*)^\wedge = \widehat{T'}|(M_*)^\wedge$.

Theorem 2.5. *Let T be an identity preserving semigroup of Schwarz type with generator A on the predual of a* W**-algebra M. If T is uniformly ergodic with finite dimensional fixed space, then every $\gamma \in \sigma(A) \cap i\mathbb{R}$ is a pole of the resolvent $R(., A)$ and* dim Kern$(\gamma - A) \leqslant$ dim Fix(T).

Proof. Let $D = \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$ and $\mathcal{R}$ the M_*-valued pseudo-resolvent of Schwarz type induced by $R(., A)$ on D. Then

$$P = \lim_{\mu \downarrow 0} \mu R(\mu)$$

exists in the uniform operator topology and rank $(P) = $ dim Fix$(T) < \infty$. From this we obtain rank $(P) = $ rank $(\hat{P}) < \infty$ where $\hat{P}$ is the canonical extension of P onto $(M_*)^\wedge$. Since $\hat{P} = \lim_{\mu \downarrow 0} \mu R(\mu)^\wedge$ it follows that

$$\text{dim Fix}((\lambda - i\alpha)\hat{R}(\lambda)) \leqslant \text{rank}\,(\hat{P}) < \infty$$

for all $\alpha \in \mathbb{R}$ (Proposition 2.1 on page 10). Therefore the assertion follows from Lemma–2.2 on page 12. $\qquad\square$

The consequences of this result for the asymptotic behavior of one-parameter semigroups will be discussed in D-IV, Section 4.

Notes

Section 1: The Perron-Frobenius theory for a single positive operator on a non-commutative operator algebra is worked out in Albeverio and Hoegh-Krohn (1978) and Groh (1981). The limitations of the theory (in the continuous as in the discrete case) are explained by the example following Remark 1.9 on page 7 (see also Groh (1982).Therefore we concentrate on irreducible semigroups. Our main result Theorem 1.11 on page 7 extends B-III, Thm.3.6 to the non-commutative setting.

Section 2: Theorem 2.5 on page 15 has its roots in the Niiro-Sawashima Theorem for a single irreducible positive operator on a Banach lattice (see Schaefer (1974, V.5.4)). The analogous semigroup result on Banach lattices is due to Greiner (1982). The ultrapower technique in our proof is developed in Groh (1984).

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