Then the following holds true:

(2.7) $Ah^{[n]} = (\alpha+in\beta)h^{[n]}$ for all $n \in \mathbb{Z}$.

In case $|h|$ is a quasi-interior point of E_+ , then $S_h D(A) = D(A)$ and $A + i\beta = S_h^{-1} A S_h$.

Proof. Without loss of generality we may assume that $\alpha = 0$. Then the assumption (2.6) implies that $T(t)|h| = |h|$ and $T(t)h = e^{i\beta t}h$ for $t \geq 0$ (see A-III,Cor.6.4). In particular, the principal ideal $E_{|h|}$ is invariant under every operator $T(t)$. By the Kakutani-Krein Theorem (C-I,Sec.4) we can identify $E_{|h|}$ with a space $C(K)$, K compact. Then the restrictions $\tilde{T}(t) := T(t)|E_{|h|}$ are positive operators on $C(K)$ satisfying $\tilde{T}(t)|\tilde{h}| = |\tilde{h}|$ and $\tilde{T}(t)\tilde{h} = e^{i\beta t}\tilde{h}$. From B-III,Thm.2.4(a) we conclude $\tilde{T}(t)\tilde{h}^{[n]} = e^{i\beta t}\tilde{h}^{[n]}$ for all $t \geq 0$, $n \in \mathbb{Z}$. Translating this back to $T(t)$ and E this means precisely $T(t)h^{[n]} = e^{in\beta}h^{[n]}$ $(n \in \mathbb{Z})$, hence $h^{[n]} \in D(A)$ and $Ah^{[n]} = in\beta h^{[n]}$.

Moreover, by B-III,Thm.2.4(a) we have $e^{i\beta t}\tilde{T}(t) = S_{\tilde{h}}^{-1}\tilde{T}(t)S_{\tilde{h}}$. If $|h|$ is a quasi-interior point this relation extends by continuity from the dense subspace $E_{|h|}$ to the whole space E , i.e., we have $e^{i\beta t}T(t) = S_h^{-1}T(t)S_h$ for all $t \geq 0$.

□

In the proof above we could not apply assertion (b) of B-III,Thm.2.4 because the semigroup $(\tilde{T}(t))$ on $E_{|h|} \cong C(K)$ need not be strongly continuous with respect to the sup-norm.

As a first application of Thm.2.2 we prove a cyclicity result for the point spectrum of contraction semigroups on a class of Banach lattices which includes the L^p-spaces.

Corollary 2.3. Suppose E is a Banach lattice such that the norm is strictly monotone on E_+ (i.e., $0 \leq f < g \Rightarrow \|f\| < \|g\|$).

If $(T(t))$ is a positive contraction semigroup with $s(A) = 0$, then $P\sigma_b(A) = P\sigma(A) \cap i\mathbb{R}$ is imaginary additively cyclic.

Proof. Suppose that $Ah = i\beta h$ $(\beta \in \mathbb{R}$, $h \in E)$. Then we have $T(t)h = e^{i\beta t}h$ $(t \geq 0)$ and $|h| = |T(t)h| \leq T(t)|h|$ since $T(t)$ is positive. Moreover, $\|h\| \leq \|T(t)|h|\| \leq \|h\|$ since $\|T(t)\| \leq 1$. The assumption on the norm of E implies that $T(t)|h| = |h|$ for all $t \geq 0$, equivalently $A|h| = 0$. Now we can apply Thm.2.2 in order to obtain the desired result.

□