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One-parameter Semigroups of Positive Operators

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Part A
One-parameter Semigroups on Banach Spaces

Chapter A-I
Basic Results on Semigroups on Banach Spaces

Since the basic theory of one-parameter semigroups can be found in several excellent books (e.g. Davies (1980), Goldstein (1985a), Pazy (1983) or Hille and Phillips (1957)), we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, to collect briefly some important definitions and results (Section 1), to present a list of *standard examples* in Section 2 and to discuss standard constructions of new semigroups from a given one in Section 3 on p. 15.

In the entire chapter we denote by E a (real or) complex Banach space and consider one-parameter semigroups of bounded linear operators $T(t)$ on E. By this we understand a subset $\{T(t)\colon t \in \mathbb{R}_+\}$ of $\mathcal{L}(E)$, usually written as $(T(t))_{t\geq 0}$, such that

$$T(0) = \mathrm{Id},$$
$$T(s+t) = T(s)\cdot T(t) \text{ for all } s,t \in \mathbb{R}_+.$$

In more abstract terms this means that the map $t \mapsto T(t)$ is a homomorphism from the additive semigroup $(\mathbb{R}_+, +)$ into the multiplicative semigroup $(\mathcal{L}(E), \cdot)$. Similarly, a one-parameter group $(T(t))_{t\in\mathbb{R}}$ will be a homomorphic image of the group $(\mathbb{R}, +)$ in $(\mathcal{L}(E), \cdot)$.

1 Standard Definitions and Results

We consider a one-parameter semigroup $(T(t))_{t\geq 0}$ on a Banach space E and observe that the domain $\mathbb{R}_+$ and the range $\mathcal{L}(E)$ of the (semigroup) homomorphism $\tau\colon t \mapsto T(t)$ are topological semigroups for the natural topology on $\mathbb{R}_+$ and any one of the standard operator topologies on $\mathcal{L}(E)$. We single out the strong operator topology on $\mathcal{L}(E)$ and require τ to be continuous.

Definition 1.1 A one-parameter semigroup $(T(t))_{t\geq 0}$ is called *strongly continuous* if the map $t \mapsto T(t)$ is continuous for the strong operator topology on $\mathcal{L}(E)$, e.g.,

$$\lim_{t \to t_0} \|T(t)f - T(t_0)f\| = 0$$

for every $f \in E$ and $t, t_0 \geq 0$.

Clearly one defines in a similar way *weakly continuous*, resp. *uniformly continuous* (compare A-II, Def. 1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology.

> If not stated otherwise, a *semigroup* is a strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

Remarks 1.2 *(i) A one-parameter semigroup $(T(t))_{t \geq 0}$ on a Banach space E is strongly continuous if and only if for any $f \in E$ it is true that $T(t)f \to f$ if $t \to 0$.*

(ii) For every strongly continuous semigroup there exist constants $M \geq 1$, $w \in \mathbb{R}$ such that $\|T(t)\| \leq M \cdot e^{wt}$ for every $t \geq 0$.

(iii) If $(T(t))_{t \geq 0}$ is a one-parameter semigroup such that $\|T(t)\|$ is bounded for $0 < t \leq \delta$ then it is strongly continuous if and only if $\lim_{t \to 0} T(t)f = f$ for every f in a total subset of E.

The exponential estimate from Remark 1.2 (ii) for the growth of $\|T(t)\|$ can be used to define an important characteristic of the semigroup.

Definition 1.3 By the growth bound (or type) of the semigroup $(T(t))_{t \geq 0}$ we understand the number

$$\begin{aligned} \omega_0 &:= \inf\{w \in \mathbb{R} \colon \text{There exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leq Me^{wt} \text{ for } t \geq 0\} \\ &= \lim_{t \to \infty} \frac{1}{t} \log \|T(t)\| = \inf_{t > 0} \frac{1}{t} \log \|T(t)\|. \end{aligned}$$

Particularly important are semigroups such that for every $t \geq 0$ we have $\|T(t)\| \leq M$ (*bounded semigroups*) or $\|T(t)\| \leq 1$ (*contraction semigroups*). In both cases we have $\omega_0 \leq 0$.

It follows from the subsequent examples and from Def. 1.3 that ω_0 may be any number $-\infty \leq \omega < +\infty$. Moreover the reader should observe that the infimum in Def. 1.3 need not be attained and that M may be larger than 1 even for bounded semigroups.

Examples 1.4 *(i) Take $E = \mathbb{C}^2$,*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then for the ℓ^1-norm on E we obtain $\|T(t)\| = 1 + t$, hence $(T(t))_{t \geq 0}$ is an unbounded semigroup having growth bound $\omega_0 = 0$.

(ii) Take $E = L^1(\mathbb{R})$ *and for* $f \in E$, $t \geq 0$ *define*

$$T(t)f(x) := \begin{cases} 2 \cdot f(x+t) & \text{if } x \in [-t, 0] \\ f(x+t) & \text{otherwise.} \end{cases}$$

Each $T(t)$, $t > 0$, *satisfies* $\|T(t)\| = 2$ *as can be seen by taking* $f := \chi_{[0,t]}$. *Therefore* $(T(t))_{t \geq 0}$ *is a strongly continuous semigroup which is bounded, hence has* $\omega_0 = 0$, *but the constant* M *in (1.3) cannot be chosen to be* 1.

The most important object associated to a strongly continuous semigroup $(T(t))_{t \geq 0}$ is its *generator* which is obtained as the (right)derivative of the map $t \mapsto T(t)$ at $t = 0$. Since for strongly continuous semigroups the functions $t \mapsto T(t)f$, $f \in E$, are continuous but not always differentiable, we have to restrict our attention to those $f \in E$ for which the desired derivative exists. We then obtain the *generator* as a not necessarily everywhere defined operator.

Definition 1.5 To every semigroup $(T(t))_{t \geq 0}$ there belongs an operator $(A, D(A))$, called the *generator* and defined on the *domain*

$$D(A) := \{f \in E : \lim_{h \to 0} \frac{T(h)f - f}{h} \text{ exists in } E\}$$

by

$$Af := \lim_{h \to 0} \frac{T(h)f - f}{h} \quad (f \in D(A)).$$

Clearly, $D(A)$ is a linear subspace of E and A is linear from $D(A)$ into E. Only in certain special cases (see 2.1) the generator is everywhere defined and therefore bounded (use Prop. 1.9 (ii)) on p. 7). In general, the precise extent of the domain $D(A)$ is essential for the characterization of the generator. But since the domain is canonically associated to the generator of a semigroup, we shall write in most cases A instead of $(A, D(A))$.

As a first result we collect some information on the domain of the generator.

Proposition 1.6 *For the generator* A *of a semigroup* $(T(t))_{t \geq 0}$ *on a Banach space* E *the following assertions hold.*

(i) If $f \in D(A)$, *then* $T(t)f \in D(A)$ *for every* $t \geq 0$.

(ii) The map $t \mapsto T(t)f$ *is differentiable on* $\mathbb{R}_+$ *if and only if* $f \in D(A)$. *In that case one has*

$$\frac{d}{dt}T(t)f = AT(t)f = T(t)Af. \qquad \text{(A-I.1)}$$

(iii) For every $f \in E$ *and* $t > 0$ *the element* $\int_0^t T(s)f\,ds$ *belongs to* $D(A)$ *and one has*

$$A \int_0^t T(s)f\,ds = T(t)f - f. \qquad \text{(A-I.2)}$$

(iv) If $f \in D(A)$, then

$$\int_0^t T(s)Af\,ds = T(t)f - f. \tag{A-I.3}$$

(v) The domain $D(A)$ is dense in E.

The identity (A-I.1) is of great importance and shows how semigroups are related to certain Cauchy problems. We state this explicitly in the following theorem.

Theorem 1.7 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Banach space E. Then the* abstract Cauchy problem

$$\frac{d}{dt}\xi(t) = A\xi(t), \quad \xi(0) = f_0 \tag{A-I.4}$$

has a unique solution $\xi\colon \mathbb{R}_+ \to D(A)$ in $C^1(\mathbb{R}_+, E)$ for every $f_0 \in D(A)$. In fact, this solution is given by $\xi(t) := T(t)f_0$.

For more on the relation of semigroups to abstract Cauchy problems we refer to A-II,Section 1. Here we only point out that the above theorem implies that a semigroup is uniquely determined by its generator.

While the generator is bounded only for uniformly continuous semigroups (see Sec. 2 below), it always enjoys a weaker but useful property.

Definition 1.8 An operator B with domain $D(B)$ on a Banach space E is called *closed* if $D(B)$ endowed with the *graph norm*

$$\|f\|_B := \|f\| + \|Bf\|$$

becomes a Banach space. Equivalently, $(B, D(B))$ is closed if and only if its *graph* $\{(f, Bf)\colon f \in D(B)\}$ is closed in $E \times E$, i.e.,

$$f_n \in D(B), f_n \to f \text{ and } Bf_n \to g \text{ implies } f \in D(B) \text{ and } Bf = g.$$

It is clear from this definition that the *closedness* of an operator B depends very much on the size of the domain $D(B)$. For example, a bounded and densely defined operator $(B, D(B))$ is closed if and only if $D(B) = E$.

On the other hand it may happen that $(B, D(B))$ is not closed but has a closed *extension* $(C, D(C))$, i.e. $D(B) \subseteq D(C)$ and $Bf = Cf$ for every $f \in D(B)$. In that case, B is called *closable*, a property which is equivalent to

$$f_n \in D(B), f_n \to 0 \text{ and } Bf_n \to g \text{ implies } g = 0.$$

The smallest closed extension of $(B, D(B))$ will be called the *closure* $\overline{B}$ with domain $D(\overline{B})$. In other words, the graph of $\overline{B}$ is the closure of $\{(f, Bf)\colon f \in D(B)\}$ in $E \times E$.

Finally we call a subset D_0 of $D(B)$ a *core* for B if D_0 is $\|\cdot\|_B$-dense in $D(B)$. This means that a closed operator is determined (via closure) by its restriction to a core in its domain.

We now collect the fundamental topological properties of semigroup generators, their domains (see also A-II,Cor.1.34) and their resolvents.

Proposition 1.9 *For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ the following hold.*

(i) The generator A is a closed operator.
(ii) If a subspace D_0 of the domain $D(A)$ is dense in E and $(T(t))$-invariant, then it is a core for A.
(iii) Define $D(A^n) := \{f \in D(A^{n-1})\colon Af \in D(A^{n-1})\}$, $D(A^1) = D(A)$. Then $D(A^\infty) := \bigcap_{n\in\mathbb{N}} D(A^n)$ is dense in E and a core for A.

Example 1.10 Property (iii) above does not hold for general densely defined closed operators. Take $E = C[0,1]$, $D(B) = C^1[0,1]$ and $Bf = q \cdot f'$ for some nowhere differentiable function $q \in C[0,1]$. Then B is closed, but $D(B^2) = \{0\}$.

Proposition 1.11 *For the generator A of a strongly continous semigroup on a Banach space E the following hold.*

If $\int_0^\infty e^{-\lambda t}T(t)f\,dt$ exists for every $f \in E$ and some $\lambda \in \mathbb{C}$, then $\lambda \in \rho(A)$ and $R(\lambda, A)f = \int_0^\infty e^{-\lambda t}T(t)f\,dt$. In particular,

$$R(\lambda, A)^{n+1}f = \frac{(-1)^n}{n!}\left(\frac{d}{d\lambda}\right)^n R(\lambda, A)f = \int_0^\infty e^{-\lambda t}\frac{t^n}{n!}T(t)f\,dt \qquad \text{(A-I.5)}$$

for every $f \in E$, $n \geq 0$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$.

Remarks 1.12 *(i) For continuous Banach space valued functions such as $t \mapsto T(t)f$ we consider the Riemann integral and define*

$$\int_0^\infty T(t)f\,\mathrm{d}t \quad as \quad \lim_{t\to\infty}\int_0^t T(s)f\,\mathrm{d}s.$$

Sometimes such integrals for strongly continuous semigroups are written as $\int_a^b T(t)\,dt$ but understood in the strong sense.

(ii) Since the generator $(A, D(A))$ determines the semigroup uniquely, we will speak occasionally of the growth bound of the generator instead of the semigroup, i.e. we write $\omega_0 = \omega_0(A) = \omega_0((T(t))_{t\geq 0}$.

(iii) For one-parameter groups it might seem to be more natural to define the generator as the derivative *rather than just the* right derivative *at $t = 0$. This yields the same operator as the following result shows.*
The strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator A can be extended to a strongly continuous one-parameter group $(U(t))_{t\in\mathbb{R}}$ if and only if $-A$ generates a semigroup $(S(t))_{t\geq 0}$.
In that case $(U(t))_{t\in\mathbb{R}}$ is obtained as

$$U(t) = \begin{cases} T(t) & \text{for } t \geq 0, \\ S(-t) & \text{for } t \leq 0. \end{cases}$$

We refer to Davies (1980, Prop.1.14) for the details.

2 Standard Examples

In this section we list and discuss briefly the most basic examples of semigroups together with their generators. These semigroups will reappear throughout this book and will be used to illustrate the theory. We start with the class of semigroups mentioned after Definition 1.1 on p. 3.

2.1 Uniformly Continuous Semigroups

It follows from elementary operator theory that for every bounded operator A in $\mathcal{L}(E)$ the sum

$$\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} =: e^{tA}$$

exists and determines a unique uniformly continuous (semi)group $(e^{tA})_{t\in\mathbb{R}}$ having A as its generator.

Conversely, any uniformly continuous semigroup is of this form.

If the semigroup $(T(t))_{t\geq 0}$ is uniformly continuous, then $\frac{1}{t}\int_0^t T(s)\,ds$ uniformly converges to $T(0) = \mathrm{Id}$ as $t \to 0$. Therefore for some $t' > 0$ the operator $\frac{1}{t'}\int_0^{t'} T(s)\,ds$ is invertible and every $f \in E$ is of the form $f = \frac{1}{t'}\int_0^{t'} T(s)g\,ds$ for some $g \in E$. But these elements belong to $D(A)$ by (1.3), hence $D(A) = E$. Since the generator A is closed and everywhere defined, it must be bounded.

Remark that bounded operators are always generators of groups, not just semigroups. Moreover, the growth bound ω satisfies $|\omega| \leq \|A\|$ in this situation.

The above characterization of the generators of uniformly continuous semigroups as the bounded operators shows that these semigroups are—at least in many aspects—rather simple objects.

2.2 Matrix Semigroups

The above considerations expecially apply in the situation $E = \mathbb{C}^n$. If $n = 2$ and $A = (a_{ij})_{2\times 2}$ the following explicit formulas for e^{tA} might be of interest.

Set (i) $s := \text{trace } A$, (ii) $d := \det A$ (iii) and $D := (s^2 - 4d)^{1/2}$. Then if $D \neq 0$

$$e^{tA} = e^{ts/2} \cdot [D^{-1} 2 \sinh(tD/2) \cdot A + (\cosh(tD/2) - sD^{-1}\sinh(tD/2)) \cdot \mathrm{Id}]$$

and if $D = 0$

$$e^{ts/2} \cdot [tA + (1 - ts/2) \cdot \mathrm{Id}].$$

2.3 Multiplication Semigroups

Many Banach spaces appearing in applications are Banach spaces of (real or) complex valued functions over a set X. As the most standard examples of these "function spaces", we mention the space $C_0(X)$ of all continuous complex valued functions vanishing at infinity on a locally compact space X, or the spaces $L^p(X, \Sigma, \mu)$, $1 \le p \le \infty$, of all (equivalence classes of) p-integrable functions on a σ-finite measure space (X, Σ, μ).

On these function spaces $E = C_0(X)$, resp. $E = L^p(X, \Sigma, \mu)$, there is a simple way to define *multiplication operators*.

Take a continuous, resp. measurable function $q\colon X \to \mathbb{C}$ and define

$$M_q f := q \cdot f, \quad \text{i.e.} \quad M_q f(x) := q(x) \cdot f(x) \quad \text{for } x \in X$$

and for every f in the *maximal* domain $D(M_q) := \{g \in E\colon q \cdot g \in E\}$.

This natural domain is a dense subspace of $C_0(X)$, resp. $L^p(X, \Sigma, \mu)$, for $1 \le p < \infty$. Moreover, $(M_q, D(M_q))$ is a closed operator. This is easy in case $E = C_0(X)$.

For $E = L^p(X, \Sigma, \mu)$, $1 \le p < \infty$, we consider a sequence $(f_n) \subset E$ such that $\lim_{n\to\infty} f_n = f \in E$ and $\lim_{n\to\infty} q f_n =: g \in E$. Choose a subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f_{n(k)}(x) = f(x)$ and $\lim_{k\to\infty} q(x) f_{n(k)}(x) = g(x)$ for μ-almost every $x \in X$. Then $g = q \cdot f$ and $f \in D(M_q)$, i.e. M_q is closed.

For such multiplication operators many properties can be checked quite directly. For example, the following statements are equivalent.

(a) M_q is bounded.
(b) q is (μ-essentially) bounded.

One has $\|M_q\| = \|q\|_\infty$ in this situation.

Observe that on spaces $C(K)$, K compact, there are no densely defined, unbounded multiplication operators.

By defining the multiplication semigroups

$$T(t)f(x) := \exp(t \cdot q(x))f(x), \quad x \in X, f \in E,$$

one obtains the following characterizations.

Proposition 2.13 *Let M_q be a multiplication operator on $E = C_0(X)$ or $E = L^p(X, \Sigma, \mu)$, $1 \le p < \infty$. Then the properties (a) and (b), resp. (a') and (b'), are equivalent.*

(a) M_q generates a strongly continuous semigroup.
(b) $\sup\{\Re q(x) : x \in X\} < \infty$.
(a') M_q generates a uniformly continuous semigroup.
(b') $\sup\{|q(x)| : x \in X\} < \infty$.

As a consequence one computes the growth bound of a multiplication semigroup as

$$\omega_0 = \sup\{\Re q(x)\colon x \in X\}$$

in the case $E = C_0(X)$ and

$$\omega_0 = \mu\text{-ess-}\sup\{\Re q(x) \colon x \in X\}$$

in the case $E = L^p(\mu)$. It is a nice exercise to characterize those multiplication operators which generate strongly continuous groups.

We point out that the above results cover the cases of sequence spaces such as c_0 or ℓ^p, $1 \le p < \infty$. An abstract characterization of generators of multiplication semigroups will be given in C-II,Thm.5.13.

2.4 Translation (Semi)Groups

Let E to be one of the following function spaces $C_0(\mathbb{R}_+)$, $C_0(\mathbb{R})$ or $L^p(\mathbb{R}_+)$, $L^p(\mathbb{R})$ for $1 \le p < \infty$. Define $T(t)$ to be the (left) translation operator

$$T(t)f(x) := f(x+t)$$

for x ,, $t \in \mathbb{R}_+$, resp. $x, t \in \mathbb{R}$ and $f \in E$. Then $(T(t))_{t\ge0}$ is a strongly continous semigroup, resp. group of contractions on E and its generator is the first derivative $\frac{d}{dx}$ with *maximal* domain. In order to be more precise we have to distinguish the cases $E = C_0$ and $E = L^p$.

The generator of the translation (semi)group on $E = C_0(\mathbb{R}_+)$ is

$$\begin{aligned} Af &:= \frac{d}{\mathrm{d}x} f = f' \\ D(A) &:= \{f \in E \colon f \text{ differentiable and } f' \in E\}. \end{aligned}$$

Proof For $f \in D(A)$ it follows that for every $x \in \mathbb{R}_{(+)}$

$$\lim_{h\to0} \frac{T(h)f(x) - f(x)}{h} = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

(uniformly in x) and coincides with $Af(x)$. Therefore f is differentiable and $f' \in E$.

On the other hand, take $f \in E$ differentiable such that $f' \in E$. Then

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \le \frac{1}{h} \int_x^{x+h} |f'(y) - f'(x)| \ \mathrm{d}y,$$

where the last expression tends to zero uniformly in x as $h \to 0$. Thus $f \in D(A)$ and $f' = Af$. □

The generator of the translation (semi)group on $E = L^p(\mathbb{R}_+)$, $1 \le p < \infty$, is

$$\begin{aligned} Af &:= \frac{d}{dx} f = f', \\ D(A) &:= \{f \in E \colon f \text{ absolutely continuous}, f' \in E\}. \end{aligned}$$

Proof Take $f \in D(A)$ such that $\lim_{h \to 0} \frac{1}{h}(T(h)f - f) = g \in E$. Since integration is continuous, we obtain for every $a, b \in \mathbb{R}_{(+)}$ that

$$(*) \quad \frac{1}{h} \int_{b+h}^{b} f(x)\, dx - \frac{1}{h} \int_{a+h}^{a} f(x)\, dx = \int_{a}^{b} \frac{f(x+h) - f(x)}{h}\, dx$$

converges to $\int_a^b g(x)\ \mathrm{d}x$ as $h \to 0+$. But for almost all a, b the left hand side of $(*)$ converges to $f(b) - f(a)$. By redefining f on a nullset we obtain

$$f(y) = \int_a^y g(x)\, dx + f(a), \quad y \in \mathbb{R}_{(+)},$$

which is an absolutely continuous function whose derivative is (almost everywhere) equal to g.

On the other hand, let f be absolutely continuous such that $f' \in L^p$. Then

$$\begin{aligned} \lim_{h \to 0} \int \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right|^p dx &= \lim_{h \to 0} \int \left| \frac{1}{h} \int_0^h (f'(x+s) - f'(x))\, ds \right|^p dx \\ &= \lim_{h \to 0} \int \left| \int_0^1 (f'(x+uh) - f'(x))\, du \right|^p dx \\ &\le \lim_{h \to 0} \int \int_0^1 |f'(x+uh) - f'(x)|^p\, du\, dx \\ &= \int_0^1 \lim_{h \to 0} \int |f'(x+uh) - f'(x)|^p\, dx\, du = 0, \end{aligned}$$

hence $f \in D(A)$. □

2.5 Rotation Groups

On $E = C(\Gamma)$, resp. $E = L^p(\Gamma, m)$, $1 \le p < \infty$, m Lebesgue measure we have canonical groups defined by rotations of the unit circle Γ with a certain period, i.e. for $0 < \tau \in \mathbb{R}$ the operators

$$R_\tau(t) f(z) := f(e^{2\pi i t/\tau} \cdot z), \quad z \in \Gamma$$

yield a group $(R_\tau(t))_{t \in \mathbb{R}}$ having period τ, i.e. $R_\tau(\tau) = \mathrm{Id}$. As in Example 2.4 one shows that its generator has the form

$$D(A) = \{f \in E \colon f \text{ absolutely continuous}, f' \in E\}, Af(z) \quad = (2\pi i/\tau) \cdot z \cdot f'(z).$$

An isomorphic copy of the group $(R_\tau(t))_{t\in\mathbb{R}}$ is obtained if we consider $E = \{f \in C[0,1]\colon f(0) = f(1)\}$, resp. $E = L^p([0,1])$ and the group of 'periodic translations'

$$T(t)f(x) := f(y) \quad \text{for } y \in [0,1], y = x + t \bmod 1$$

with generator

$$D(A) := \{f \in E\colon f \text{ absolutely continuous}, f' \in E\},$$
$$Af := f'.$$

2.6 Nilpotent Translation Semigroups

Take $E = L^p([0,\tau], m)$ for $1 \le p < \infty$ and define

$$T(t)f(x) := \begin{cases} f(x+t) & \text{if } x + t \le \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then $(T(t))_{t\ge 0}$ is a semigroup satisfying $T(t) = 0$ for $t \ge \tau$. Its generator is still the first derivative $A = \frac{d}{\mathrm{d}x}$, but with domain is

$$D(A) = \{f \in E\colon f \text{ absolutely continuous}, f' \in E, f(\tau) = 0\}.$$

In fact, if $f \in D(A)$, then f is absolutely continuous with $f' \in E$. By Prop. 1.6 (i) on p. 5 it follows that $T(t)f$ is absolutely continuous and hence $f(\tau) = 0$.

2.7 One-dimensional Diffusion Semigroup

For the second derivative

$$Bf(x) := \frac{d^2}{dx^2} f(x) = f''(x)$$

we take the domain

$$D(B) := \{f \in C^2[0,1]\colon f'(0) = f'(1) = 0\}$$

in the Banach space $E = C[0,1]$. Then $D(B)$ is dense in $C[0,1]$, but closed for the graph norm. Obviously, each function

$$e_n(x) := \cos \pi n x, \quad n \in \mathbb{Z},$$

is contained in $D(B)$ and is an eigenfunction of B pertaining to the eigenvalue $\lambda_n := -\pi^2 n^2$. The linear hull span $\{e_n : n \in \mathbb{Z}\} =: E_0$ forms a subalgebra of $D(B)$ which by the Stone-Weierstrass theorem is dense in E.

We now use e_n to define bounded linear operators

$$e_n \otimes e_n : f \mapsto (\int_0^1 f(x) e_n(x)\ \mathrm{d}x) e_n = (f|e_n) e_n$$

satisfying $\|e_n \otimes e_n\| \leq 1$ and $(e_n \otimes e_n)(e_m \otimes e_m) = \delta_{n,m}(e_n \otimes e_n)$ for $n \in \mathbb{Z}$.

For $t > 0$ we define

$$\begin{aligned} T(t) &:= \sum_{n \in \mathbb{Z}} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n \\ &= e_0 \otimes e_0 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cdot e_n \otimes e_n, \end{aligned}$$

or

$$T(t) f(x) = \int_0^1 k_t(x, y) f(x) dy$$

$$\text{where } k_t(x, y) = 1 + 2 \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t) \cos \pi n x \cos \pi n y.$$

The Jacobi identity

$$\begin{aligned} w_t(x) &:= 1/(4\pi t)^{\frac{1}{2}} \sum_{m \in \mathbb{Z}} \exp(-(x + 2m)^2/4t) \\ &= \frac{1}{2} + \sum_{n \in \mathbb{N}} \exp(-\pi^2 n^2 t) \cos \pi n x \end{aligned}$$

and trigonometric relations show that

$$k_t(x, y) = w_t(x + y) + w_t(x - y)$$

which is a positive function on $[0, 1]^2$. Therefore $T(t)$ is a bounded operator on $C[0, 1]$ with

$$\|T(t)\| = \|T(t)1\| = \sup_{x \in [0,1]} \int_0^1 k_t(x, y) dy = 1.$$

From the behavior of $T(t)$ on the dense subspace E_0 it follows that $(T(t))_{t \geq 0}$ with $T(0) = \mathrm{Id}$ is a strongly continuous semigroup on E and its generator A coincides with B on E_0. Finally, we observe that E_0 is a core for $(A, D(A))$ by Prop.1.9(ii).

Consequently, $(T(t))_{t \geq 0}$ is the semigroup generated by the closure of the second derivative with domain $D(B)$.

2.8 n-dimensional Diffusion Semigroup

On $E = L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, the operators

$$\begin{aligned} T(t)f(x) &:= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|x-y|^2/4t) f(y)dy \\ &= \mu_t * f(x) \end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$ and $\mu_t(x) := (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ form a strongly continuous semigroup:

In fact the integral exists for every $f \in L^p(\mathbb{R}^n)$ since μ_t is an element of the Schwartz space $S(\mathbb{R}^n)$ of all rapidly decreasing smooth functions on $\mathbb{R}^n$.

Moreover,

$$\|T(t)f\|_p \le \|\mu_t\|_1 \|f\|_p = \|f\|_p$$

by Young's inequality, (Reed and Simon, 1975, p.28), hence $\|T(t)\| \le 1$ for every $t > 0$. Next we observe that $S(\mathbb{R}^n)$ is dense in E and invariant under each $T(t)$. Therefore we can apply the Fourier transformation F which leaves $S(\mathbb{R}^n)$ invariant and yields

$$F(\mu_t * f) = (2\pi)^{n/2} F(\mu_t) \cdot F(f) = (2\pi)^{n/2} \hat{\mu}_t \cdot \hat{f}$$

where $f \in S(\mathbb{R}^n)$, $\hat{f} = Ff \in S(\mathbb{R}^n)$.

In other words, F transforms $(T(t)|_{S(\mathbb{R}^n)})_{t \ge 0}$ into a multiplication semigroup on $S(\mathbb{R}^n)$ which is pointwise continuous for the usual topology of $S(\mathbb{R}^n)$. The generator, i.e., the right derivative at 0, of this semigroup is the multiplication operator

$$B\hat{f}(x) := -|x|^2 \hat{f}(x) \quad (x \in \mathbb{R}^n)$$

for every $f \in S(\mathbb{R}^n)$.

Applying the inverse Fourier transformation and observing that the topology of $S(\mathbb{R}^n)$ is finer than the topology induced from $L^p(\mathbb{R}^n)$, we obtain that $(T(t))_{t \ge 0}$ is a semigroup which is strongly continuous (use Rem. 1.2 (iii) on p. 4).

Its generator A coincides with

$$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x_1, \ldots, x_n)$$

for every $f \in S(\mathbb{R}^n)$.

Since $S(\mathbb{R}^n)$ is $(T(t))$-invariant, we have determined the generator on a core of its domain (see Prop.1.9.ii).

In particular, the above semigroup *solves* the initial value problem for the *heat equation*

$$\frac{\partial}{\partial t} f(x,t) = \Delta f(x,t), \quad f(x,0) = f_0(x), \quad x \in \mathbb{R}^n.$$

For the analogous discussion of the unitary group on $L^2(\mathbb{R}^n)$ generated by

$$C := i\Delta$$

we refer to Section IX.7 in Reed and Simon (1975).

Analogous examples to 2.7 are valid in $L^p[0, 1]$, resp. to 2.8 in $C_0(\mathbb{R}^n)$.

3 Standard Constructions

Starting with a semigroup $(T(t))_{t\geq 0}$ on a Banach space E it is possible to construct new semigroups on spaces naturally associated with E. Such constructions will be important technical devices in many of the subsequent proofs. Although most of these constructions are rather routine, we present in the sequel a systematic account of them for the convenience of the reader.

We always start with a semigroup $(T(t))_{t\geq 0}$ on a Banach space E, and denote its generator by A on the domain $D(A)$.

3.1 Similar Semigroups

There is an easy way how to obtain different (but isomorphic) semigroups out of a given semigroup $(T(t))_{t\geq 0}$ on a Banach space E.

Let V be an isomorphism from E onto E. Then $S(t) := VT(t)V^{-1}$, $t \geq 0$, defines a strongly continuous semigroup. If A is the generator of $(T(t))_{t\geq 0}$ then

$$B := VAV^{-1} \text{ with domain } D(B) := \{f \in E : V^{-1}f \in D(A)\}$$

is the generator of $(S(t))_{t\geq 0}$.

3.2 The Rescaled Semigroup

For fixed $\lambda \in \mathbb{C}$ and $\alpha > 0$ the operators

$$S(t) := \exp(\lambda t)T(\alpha t)$$

yield a new semigroup having generator

$$B := \alpha A + \lambda \mathrm{Id} \text{ with } D(B) = D(A).$$

This *rescaled semigroup* enjoys most of the properties of the original semigroup and the same is true for the corresponding generators. However, by using this procedure certain constants associated with $(T(t))_{t\geq 0}$ and A can be normalized. For example, by this rescaling we may in many cases suppose without loss of generality that the growth bound ω_0 is zero.

Another application is the following. For $\lambda \in \mathbb{C}$ and $S(t) := \exp(-\lambda t)T(t)$ the formulas *(1.3)* and *(1.4)* yield:

$$e^{-\lambda t}T(t)f - f = (\lambda - A)\int_0^t e^{-\lambda s}T(s)f\,ds$$

or

$$(e^{\lambda t} - T(t))f = (\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)f\,ds \quad \text{for } f \in E, \tag{3.1}$$

and

$$e^{-\lambda t}T(t)f - f = \int_0^t e^{-\lambda s}T(s)(\lambda - A)f\,ds$$

or

$$(e^{\lambda t} - T(t))f = \int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)f\,ds \quad \text{for } f \in D(A). \tag{3.2}$$

3.3 The Subspace Semigroup

Assume F to be a closed $(T(t))$-invariant or, equivalently, $R(\lambda, A)$-invariant for $\lambda \in \mathbb{C}$, $\Re\lambda > \omega_0$, subspace of E. Then the semigroup $(T(t)_|)_{t\geq 0}$ of all restrictions $T(t)_| := T(t)_{|F}$ is strongly continuous on F. If $(A, D(A))$ denotes the generator of $(T(t))_{t\geq 0}$ it follows from the $(T(t))$-invariance and closedness of F that A maps $D(A) \cap F$ into F. Therefore

$$A_| := A_{|(D(A)\cap F)} \text{ with domain } D(A_|) := D(A) \cap F$$

is the generator of $(T(t)_|)$. Conversely, if F is a closed *linear subspace* of E with $A(D(A) \cap F) \subset F$ such that $A_|$ is a generator on F, then F is $(T(t))$-invariant.

An A-invariant subspace need not necessarily be $(T(t))$-invariant: Take for example the translation group with $T(t)f(x) = f(x + t)$ on $E = C_0(\mathbb{R})$ and $F := \{f \in E : f(x) = 0 \text{ for } x \leq 0\}$.

3.4 The Quotient Semigroup

Let F be a closed $(T(t))$-invariant subspace of E and consider the quotient space $E_/ := E/F$ with quotient map $q : E \to E_/$. The quotient operators

$$T(t)_/ q(f) := q(T(t)f), \quad f \in E,$$

are well defined and form a strongly continuous semigroup $(T(t)_/)_{t\geq 0}$ on $E_/$. For the generator $(A_/, D(A_/))$ of $(T(t)_/)_{t\geq 0}$ the following holds:

$$D(A_/) = q(D(A)) \quad \text{and} \quad A_/ q(f) = q(Af)$$

for every $f \in D(A)$. Here we use the fact that every $\hat{f} := q(f) \in D(A_/)$ can be written as

$$\hat{f} = \int_0^\infty e^{-\lambda s}\hat{T}(s)_/\hat{g}\, ds = \int_0^\infty e^{-\lambda s} q(T(s)g)\, ds = q(\int_0^\infty e^{-\lambda s}T(s)g\, ds) = q(h)$$

where $h \in D(A)$ and $\lambda > \omega$ (see Prop. **??**). In particular we point out that for every $\hat{f} \in D(A/)$ there exist representatives $f \in \hat{f}$ belonging to $D(A)$.

Example 3.14 We start with the Banach space $E = L^1(\mathbb{R})$ and the translation semigroup $(T(t))_{t\geq 0}$ where $T(t)f(x) := f(x+t)$ (see Example 2.4). Then $L^1((-\infty, 1])$ can be identified with the closed, $(T(t))$-invariant subspace

$$J := \{f \in E\colon f(x) = 0 \text{ for } 1 < x < \infty\}.$$

There we obtain the subspace semigroup

$$T(t)|_{(-\infty,1]}(x) \cdot f(x+t),$$

where $f \in L^1((-\infty, 1])$, $-\infty < x \leq 1$ and $t \geq 0$.

By 2.4 and 3.2 its generator is

$$A|f := f'$$

for $f \in D(A|) := \{f \in E\colon f \in \text{AC with } f' \in E \text{ and } f(x) = 0 \text{ for } x \geq 1\}$.

Next we identify $L^1([0, 1])$ with the quotient space $L^1((-\infty, 1])/I$ where

$$I := \{f \in L^1((-\infty, 1])\colon f(x) = 0 \text{ for } 0 \leq x \leq 1\}.$$

Again I is invariant for the restricted semigroup $(T(t)_|)$ and the quotient semigroup $(T(t)|/)$ on $L^1([0, 1])$ is the nilpotent translation semigroup as in Example 2.6. In particular it follows that the domain of its generator is

$$D(A_{|/}) = \{f \in L^1([0, 1])\colon f \in \text{AC with } f' \in L^1([0, 1]) \text{ and } f(1) = 0\}.$$

3.5 The Adjoint Semigroup

The adjoint operators $(T(t)')_{t\geq 0}$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E form a semigroup on E' which need, however, not be strongly continuous.

Example 3.15 Take the translation operators $T(t)f(x) = f(x+t)$ on $E = L^1(\mathbb{R})$ (see Example 2.4) and their adjoints

$$T(t)'f(x) = f(x-t)$$

on $E' = L^\infty(\mathbb{R})$. Then $(T(t)')_{t\in\mathbb{R}}$ is a one-parameter group which is not strongly continuous on $L^\infty(\mathbb{R})$ (take any non-trivial characteristic function).

Since the semigroup $(T(t)')_{t\geq 0}$ is obviously *weak*-continuous* in the sense that $\lim_{t\to s}\langle f, (T(t)' - T(s)')\phi\rangle = 0$ for every $f \in E$, $\phi \in E'$ and $s, t \geq 0$, it is natural to associate $(T(t)')_{t\geq 0}$ its a *weak*-generator*

$$A'\phi := \sigma(E', E)\text{-}\lim \frac{1}{h}(T(h)'\phi - \phi) \text{ for every } \phi \text{ in the domain}$$
$$D(A') := \{\phi \in E' : \sigma(E', E)\text{-}\lim \frac{1}{h}(T(h)'\phi - \phi) \text{ exists}\}.$$

This operator coincides with the *adjoint* of the generator $(A, D(A))$, i.e.

$$D(A') = \{\phi \in E' : \text{there exists } \psi \in E' \text{ such that } \langle f, \psi\rangle = \langle Af, \phi\rangle \text{ for all } f \in D(A)\}$$

and $A'\phi = \psi$. In particular, A' is a closed and $\sigma(E', E)$-densely defined operator in E'.

It follows from (Kato, 1966, Thm.III.5.30) that the resolvent $R(\lambda, A')$ of A' is $R(\lambda, A)'$. In particular, the spectra $\sigma(A)$ and $\sigma(A')$ coincide.

However, it is still necessary in some situations to have strong continuity for the adjoint semigroup. In order to achieve this we restrict $T(t)'$ to an appropriate subspace of E'.

Definition 3.16 (Phillips (1954)) The *semigroup dual* of the Banach space E with respect to the strongly continuous semigroup $(T(t))_{t\geq 0}$ is

$$E^* := \{\phi \in E' : \|\cdot\|\text{-}\lim_{t\to 0} T(t)'\phi = \phi\}.$$

The adjoint semigroup on E^* is given by the operators

$$T(t)^* := T(t)'|_{E^*}, \quad t \geq 0.$$

Since $(T(t)^*)_{t\geq 0}$ is strongly continuous on E^* we call its generator $(A^*, D(A^*))$ the *adjoint generator*.

The above definition makes sense since E^* is norm-closed in E' and $(T(t)')$-invariant. The main point is that E^* is still reasonably large. In fact, since $\int_0^t T(s)'\phi \, \mathrm{d}s$, understood in the weak sense, is contained in E^* for every $\phi \in E'$ and $t \geq 0$, it follows that

$$\sup\{\langle f, \phi\rangle : \phi \in E^*, \|\phi\| \leq 1\} \leq \|f\| \leq M \cdot \sup\{\langle f, \phi\rangle : \phi \in E^*, \|\phi\| \leq 1\}$$

where $M := \limsup_{t\to 0} \|T(t)\|$. In particular, E^* separates E, i.e. E^* is $\sigma(E', E)$-dense in E'. In addition the estimate of $\|\cdot\|$ given above yields

$$\|T(t)^*\| \le \|T(t)\| \le M\|T(t)^*\| \quad \text{for all } t \ge 0.$$

In the following proposition we describe the relation between A^* and A'.

Proposition 3.17 *For the adjoint generator A^* of a strongly continuous semigroup $(T(t))_{t\ge 0}$ on E the following assertions hold.*

(i) E^* is the $\|\cdot\|$-closure of $D(A')$.
(ii) $D(A^*) = \{\phi \in D(A') \colon A'\phi \in E^*\}$.
(iii) A^* and A' coincide on $D(A^*)$.

Proof (i) Take $\phi \in D(A')$ fixed. For every $f \in D(A)$ with $\|f\| \le 1$ we define a continuously differentiable function

$$t \mapsto \xi_f(t) := \langle T(t)f, \phi\rangle$$

on $[0, 1]$ with derivative $\xi'_f(t) = \langle T(t)Af, \phi\rangle = \langle T(t)f, A'\phi\rangle$.

Since $\{\xi'_f(t) \colon t \in [0, 1], f \in D(A), \|f\| \le 1\}$ is bounded, it follows that the set

$$\{\xi_f \colon f \in D(A), \|f\| \le 1\}$$

is equicontinuous at 0, i.e., for every $\varepsilon > 0$ there exists $0 < t_0 < 1$ such that

$$|\xi_f(s) - \xi_f(0)| = |\langle f, T(s)'\phi - \phi\rangle| < \varepsilon$$

for every $0 \le s \le t_0$ and $f \in D(A)$, $\|f\| \le 1$. But this implies $\|T(s)'\phi - \phi\| < \varepsilon$ and hence $\phi \in E^*$.

Conversely, take $\psi \in E^*$. Then $\frac{1}{t}\int_0^t T(s)'\psi\, ds$, $t > 0$, belongs to $D(A')$ and norm converges to ψ as $t \to 0$, i.e. ψ belongs to the norm closure of $D(A')$.

(ii) and (iii): Since the weak* topology on E' is weaker than the norm topology, it follows that A' is an extension of A^*. Now take $\phi \in D(A')$ such that $A'\phi \in E^*$. As above define the functions ξ_f. The assumption on ϕ implies the set of all derivatives

$$\{\xi'_f \colon f \in D(A), \|f\| \le 1\}$$

to be equicontinuous at $t = 0$. This means that for every $\varepsilon > 0$ there exists $0 < t_o < 1$ such that $|f'_f(0) - f'_f(s)| < \varepsilon$ for every $f \in D(A)$, $\|f\| \le 1$ and $0 < s < t_o$. In particular,

$$\varepsilon > |f'_f(0) - \frac{1}{s}(\xi_f(s) - \xi_f(0))| = | < f, A'\phi - \frac{1}{s}(T(s)'\phi - \phi) > |,$$

hence

$$\varepsilon > \|A'\phi - \frac{1}{s}(T(s)'\phi - \phi)\|$$

for all $0 \le s \le t_o$. From this it follows that $\phi \in D(A^*)$. $\square$

On reflexive Banach spaces we have $A^* = A'$ by the above proposition. In other cases this construction is more interesting.

Example 3.18 (continued) The adjoints of the (left) translation $T(t)$ on $E = L^1(\mathbb{R})$ are the (right) translations $T(t)'$ on $E' = L^\infty(\mathbb{R})$. The largest subspace of $L^\infty(\mathbb{R})$ on which these translations form a strongly-continuos semigroup with respect to the sup-norm, is the space of all bounded uniformly continuous functions on $\mathbb{R}$, i.e. $E^* = C_{bu}(\mathbb{R})$.

Calculating $D(A')$ and $D(A^*)$ respectively, one obtains

$$D(A') = \{f \in L^\infty(\mathbb{R}) \colon f \in AC, f' \in L^\infty(\mathbb{R})\},$$
$$D(A^*) = \{f \in L^\infty(\mathbb{R}) \colon f \in C^1(\mathbb{R}), f' \in C_{bu}(\mathbb{R})\}.$$

Obviously, the function $x \mapsto |\sin x|$ belongs to $D(A')$, but not to $D(A^*)$.

3.6 The Associated Sobolev Semigroups

Since the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ is closed, its domain $D(A)$ becomes a Banach space for the graph norm

$$\|f\|_1 := \|f\| + \|Af\|.$$

We denote this Banach space by E_1 and the continuous injection from E_1 into E by i_1. Since E_1 is invariant under $(T(t))_{t\geq 0}$, apply Prop. 1.6 (i), it makes sense to consider the semigroup $(T_1(t))_{t\geq 0}$ of all restrictions $T_1(t) := T(t)|_{E_1}$. The results of Prop. 1.6 imply that $T_1(t) \in \mathrm{L}E_1$ and $\|T_1(t)f - f\|_1 \to 0$ as $t \to 0$ for every $f \in E_1$. Thus $(T_1(t))_{t\geq 0}$ is a strongly continuous semigroup on E_1 and has a generator denoted by $(A_1, D(A_1))$.

Using 1.6 again we see that A_1 is the restriction of A to E_1 with maximal domain, i.e. $D(A_1) = \{f \in E_1 \colon Af \in E_1\} = D(A^2)$ and $A_1 f = Af$ for every $f \in D(A_1)$.

It is now possible to repeat this construction in order to obtain Banach spaces E_n and semigroups $(T_n(t))_{t\geq 0}$ with generators $(A_n, D(A_n))$ which are related as visualized in the following diagram.

$$
\begin{tikzcd}
E \arrow[rr, "T(t)"] \arrow[d, "i_1"'] & & E \arrow[d, "i_1"] \\
E_1 \arrow[rr, "T_1(t)"] \arrow[urr, "A"] \arrow[d, "i_2"'] & & E_1 = D(A) \arrow[d, "i_2"] \\
E_2 \arrow[rr, "T_2(t)"] \arrow[urr, "A_1"] & & E_2 = D(A_1) = D(A^2)
\end{tikzcd}
$$

For the translation semigroup on $L^p(\mathbb{R})$ (see 2.3) the above construction leads to the usual *Sobolev spaces*. Therefore we might call E_n the *n-th Sobolev space* and $(T_n(t))_{t\geq 0}$ the *n-th Sobolev semigroup* associated to E and $(T(t))_{t\geq 0}$.

Remark 3.19 For $\lambda \in \rho(A)$ the operator $(\lambda - A)$ and the resolvent $R(\lambda, A)$ are isomorphisms from E_1 onto E, resp. from E onto E_1 (show that $\|\cdot\|_1$ and $\|\cdot\|_\lambda$ with $\|\cdot\|_\lambda := \|(\lambda - A)\cdot\|$ are equivalent). In addition, the following diagram commutes.

$$
\begin{tikzcd}
E \arrow[r, "T(t)"] \arrow[d, "\lambda - A"'] & E \arrow[d, "{R(\lambda, A)}"] \\
E_1 \arrow[r, "T_1(t)"] & E_1
\end{tikzcd}
$$

Therefore all Sobolev semigroups $(E_n, T_n(t))_{t\geq 0}$, $n \in \mathbb{N}$, are isomorphic.

Remark 3.20 For $\lambda \in \rho(A)$ consider the norm

$$\|f\|_{-1} := \|R(\lambda, A) f\|$$

for every $f \in E$ and define E_{-1} as the completion of E for $\|\cdot\|_{-1}$.

Then $(T(t))_{t\geq 0}$ extends continuously to a strongly continuous semigroup $(T_{-1}(t))_{t\geq 0}$ on E_{-1} and the above diagram can be extended to the negative integers.

4 The $\mathcal{F}$-Product Semigroup

It is standard in functional analysis to consider a sequence of points in a certain space as a point in a new and larger space. In particular such a method can serve to convert an approximate eigenvector of a linear operator into an eigenvector. Occasionally we will need such a construction and refer to Section V.1 of Schaefer (1974).

If we try to adapt this construction to strongly continuous semigroups we encounter the difficulty that the semigroup extended to the larger space will not remain strongly continuous. An idea already used in 3.4 will help to overcome this difficulty.

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space E. Denote by $m(E)$ the Banach space of all bounded E-valued sequences endowed with the norm

$$\|(f_n)_{n\in\mathbb{N}}\| := \sup\{\|f_n\| : n \in \mathbb{N}\}.$$

It is clear that every $T(t)$ extends canonically to a bounded linear operator

$$\hat{T}(t)(f_n) := (T(t)f_n)$$

on $m(E)$, but the semigroup $(\hat{T}(t))_{t\geq 0}$ is strongly continuous if and only if T has a bounded generator. Therefore we restrict our attention to the closed, $(\hat{T}(t))$-invariant subspace

$$m^{\mathcal{T}}(E) := \{(f_n) \in m(E) : \lim_{t\to 0} \|T(t)f_n - f_n\| = 0 \text{ uniformly for } n \in \mathbb{N}\}.$$

Then the restricted semigroup

$$\hat{T}(t)(f_n) = (T(t)f_n), \quad (f_n) \in m^{\mathcal{T}}(E)$$

is strongly continuous and we denote its generator by $(\hat{A}, D(\hat{A}))$.

The following lemma shows that $\hat{A}$ is obtained canonically from A.

Lemma 4.21 *For the generator $\hat{A}$ of $(\hat{T}(t))_{t\geq 0}$ on $m^{\mathcal{T}}(E)$ one has the following properties.*

(i) $D(\hat{A}) = \{(f_n) \in m^{\mathcal{T}}(E) : f_n \in D(A)$ *and* $(Af_n) \in m^{\mathcal{T}}(E)\}$,
(ii) $\hat{A}(f_n) = (Af_n)$ *for* $(f_n) \in D(\hat{A})$.

For the proof we refer to Lemma 1.4. of [Derndinger (1980)].

Now let $\mathcal{F}$ be any filter on $\mathbb{N}$ finer than the Frechét filter (i.e. the filter of sets with finite complement. In most cases F will be either the Frechét filter or some free ultra filter.) The space of all $\mathcal{F}$-null sequences in $m(E)$, i.e.

$$c_{\mathcal{F}}(E) := \{(f_n) \in m(E) : \mathcal{F}\text{-}\lim \|f_n\| = 0\}$$

is closed in $m(E)$ and invariant under $(\hat{T}(t))_{t\geq 0}$. We call the quotient spaces

$$E_{\mathcal{F}} := m(E)/c_{\mathcal{F}}(E) \quad \text{and} \quad E_{\mathcal{F}}^{T} := m^{\mathcal{T}}(E)/c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E)$$

the *$\mathcal{F}$-product of E* and the *$\mathcal{F}$-product of E with respect to the semigroup T*, respectively.

Thus $E_{\mathcal{F}}^{T}$ can be considered as a closed linear subspace of $E_{\mathcal{F}}$. We have $E_{\mathcal{F}}^{T} = E_{\mathcal{F}}$ if (and only if) T has a bounded generator.

The canonical quotient norm on $E_{\mathcal{F}}$ is given by

$$\|(f_n) + c_{\mathcal{F}}(E)\| = \mathcal{F}\text{-}\limsup \|f_n\|.$$

We can apply Subsec. 3.4 in order to define the *$\mathcal{F}$-product semigroup* $(T_{\mathcal{F}}(t))_{t\geq 0}$ on $E_{\mathcal{F}}^{T}$ by

$$T_{\mathcal{F}}(t)((f_n) + c_{\mathcal{F}}(E)) := (T(t)f_n) + c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E)$$

Thus $T_{\mathcal{F}}(t)$ is the restriction of $T(t)_F$ where $T(t)_F$ denotes the canonical extension of $T(t)$ to the $\mathcal{F}$-product $E_{\mathcal{F}}$. (Note that $(T(t)_F)_{t\geq 0}$ is not strongly continuous unless T has a bounded generator.)

With the canonical injection $j\colon f \mapsto (f, f, f, \ldots) + c_{\mathcal{F}}(E)$ from E into $E_{\mathcal{F}}^{T}$ the operators $T_{\mathcal{F}}(t)$ are extensions of $T(t)$ satisfying $\|T_{\mathcal{F}}(t)\| = \|T(t)\|$. The basic facts about the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of $(T_{\mathcal{F}}(t))_{t\geq 0}$ follow from 3.3 and are collected in the following proposition.

Proposition 4.22 *For the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of the $\mathcal{F}$-product semigroup the following holds.*

(i) $D(A_{\mathcal{F}}) = \{(f_n) + c_{\mathcal{F}}(E)\colon f_n \in D(A); (f_n), (Af_n) \in m^{\mathcal{T}}(E)\}$,
(ii) $A_{\mathcal{F}}((f_n) + c_{\mathcal{F}}(E)) = (Af_n) + c_{\mathcal{F}}(E)$.

In case A is a bounded operator then $D(A_{\mathcal{F}}) = E_{\mathcal{F}}^{T} = E_{\mathcal{F}}$ and $A_{\mathcal{F}}$ is the canonical extension of A to $E_{\mathcal{F}}$.

We will show in A-III,4.5 that the above construction preserves and even improves many spectral properties of the semigroup and its generator.

5 The Tensor Product Semigroup

Real- or complex-valued functions of two variables x, y are often limits of functions of the form $\sum_{i=1}^{n} f_i(x)g_i(y)$ which, to some extent, allows one to consider the variables x and y separately. Since algebraic manipulation with these latter functions is governed by the formal rules of a tensor product, it is customary to identify (for example) the function

$$(x, y) \mapsto f(x)g(y)$$

with the tensor product $f \otimes g$ and to consider limits of linear combinations of such functions as elements of a completed tensor product.

To be more precise, we briefly present the most important examples for this situation.

Examples 5.23 *(i) Let (X, Σ, μ) and (Y, Ω, ν) be measure spaces. If we identify for $f_i \in L^p(\mu)$, $g_i \in L^p(\nu)$ the elements $\sum_{i=1}^{n} f_i \otimes g_i$ of the tensor product*

$$L^p(\mu) \otimes L^p(\nu)$$

with the (class of $\mu \times \nu$-a.e.-defined) functions

$$(x, y) \mapsto \sum_{i=1}^{n} f_i(x) g_i(y),$$

then $L^p(\mu) \otimes L^p(\nu)$ becomes a dense subspace of $L^p(X \times Y, \Sigma \times \Omega, \mu \times \nu)$ for $1 \le p < \infty$.

(ii) Similarly, let X,Y be compact spaces. Then $C(X) \otimes C(Y)$ becomes a dense subspace of $C(X \times Y)$ by identifying, for $f \in C(X)$ and $g \in C(Y)$, $f \otimes g$ with the function

$$(x, y) \mapsto f(x)g(y).$$

We do not intend to go deeper into the quite sophisticated problems related to normed tensor products of general Banach spaces, but will rather confine ourselves to the discussion of certain special cases. These will always be related to one of the following standard methods to define a norm on the tensor product of two Banach spaces E, F.

Let $u := \sum_{i=1}^{n} f_i \otimes g_i$ be an element of $E \otimes F$. Then

(i) $\|u\|_\pi := \inf\{\sum_{j=1}^{m} \|h_j\|\|k_j\| \colon u = \sum_{j=1}^{m} h_j \otimes k_j, h_j \in E, k_j \in F\}$ defines the *greatest cross norm* π on $E \otimes F$.

(ii) $\|u\|_\varepsilon := \sup\{\langle u, \phi \otimes \psi\rangle \colon \phi \in E', \psi \in F', \|\phi\|, \|\psi\| \le 1\}$ defines the *least cross norm* ε on $E \times F$. Here, $\langle u, \phi \otimes \psi\rangle$ denotes the canonical bilinear form on $(E \otimes F) \times (E' \otimes F')$, i.e., $\langle \sum_{i=1}^{n} f_i \otimes g_i, \phi \otimes \psi\rangle = \sum_{i=1}^{n} \langle f_i, \phi\rangle\langle g_i, \psi\rangle$.

(iii) if E and F are Hilbert spaces, $\|u\|_h = (u|u)_h^{1/2}$, where the scalar product $(\cdot|\cdot)_h$ is defined as in (ii), defines the *Hilbert norm* h on $E \otimes F$.

In the following we write $E \otimes_\alpha F$ for the tensor product of E and F endowed—if applicable—with one of the norms π, ε, h just defined. In each case one has $\|f \otimes g\| = \|f\|\|g\|$ for $f \in E$, $g \in F$.

By $E \widetilde{\otimes}_\alpha F$ we mean the completion of $E \otimes_\alpha F$. Moreover we recall how examples (i) and (ii) above fit into this pattern

$$L^1(\mu \otimes \nu) = L^1(\mu) \widetilde{\otimes}_\pi L^1(\nu), \quad L^2(\mu \otimes \nu) = L^2(\mu) \widetilde{\otimes}_h L^2(\nu),$$

$$C(X \otimes Y) = C(X) \widetilde{\otimes}_\varepsilon C(Y).$$

Finally, we point out that for any $S \in \mathcal{L}(E)$, $T \in \mathrm{L}F$, the mapping

$$\sum_{i=1}^{n} f_i \otimes g_i \mapsto \sum_{i=1}^{n} S f_i \otimes T g_i$$

defined on $E \otimes F$ is linear and continuous on $E \otimes_\alpha F$, hence has a continuous extension to $E \widetilde{\otimes}_\alpha F$. This operator, as well as its continuous extension, will be denoted by $S \otimes T$ and satisfies $\|S \otimes T\| = \|S\|\|T\|$. The notation $A \otimes B$ will also be used in the obvious way if A and B are not necessarily bounded operators on E and F. We are now ready to consider semigroups induced on the tensor product.

Proposition 5.24 *Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be strongly continuous semigroups on Banach spaces E, F, and let A, B be their generators. Then the family $(S(t)\otimes T(t))_{t\geq 0}$ is a strongly continuous semigroup on $E\widetilde{\otimes}_\alpha F$. The closure of $A\otimes \mathrm{Id}+\mathrm{Id}\otimes B$, defined on the core $D(A)\otimes D(B)$, is its generator.*

Proof It is immediately verified that $(S(t)\otimes T(t))_{t\geq 0}$ is in fact a semigroup of operators on $E\widetilde{\otimes}_\alpha F$. The strong continuity need only be verified at $t=0$ and on elements of the form $u=f\otimes g\in E\otimes F$.

This verification being straightforward, there remains to show that the generator of $(S(t)\otimes T(t))_{t\geq 0}$ is obtained as the closure of

$$(A\otimes \mathrm{Id}+\mathrm{Id}\otimes B, D(A)\otimes D(B)).$$

To this end, let $f\in D(A)$ and $g\in D(B)$. Then

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{h}(T(h)\otimes S(h)(f\otimes g)-f\otimes g)\\&=\lim_{h\to 0}\frac{1}{h}(T(h)f\otimes (S(h)g-g)+(T(h)f-f)\otimes g)\\&=(f\otimes Bg)+(Af\otimes g).\end{aligned}$$

Since the elements of the form $f\otimes g$, $f\in D(A)$, $g\in D(B)$, generate the linear subspace $D(A)\otimes D(B)$ of $E\otimes_\alpha F$, this subspace belongs to the domain of the generator. Moreover, $D(A)\otimes D(B)$ is dense in $E\widetilde{\otimes}_\alpha F$ and invariant under $(S(t)\otimes T(t))_{t\geq 0}$, hence it is a core of $A\otimes \mathrm{Id}+\mathrm{Id}\otimes B$ by Prop. 1.9 (ii). □

6 The Product of Commuting Semigroups

Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be semigroups with generators A and B, respectively on some Banach space E. It is not difficult to see that the following assertions are equivalent.

(a) $S(t)T(t)=S(t)T(t)$ for all $t\geq 0$.
(b) $R(\mu,A)R(\mu,B)=R(\mu,B)R(\mu,A)$ for some $\mu\in\rho(A)\cap\rho(B)$.
(c) $R(\mu,A)R(\mu,B)=R(\mu,B)R(\mu,A)$ for all $\mu\in\rho(A)\cap\rho(B)$.

In that case $U(t)=S(t)T(t)$ $(t\geq 0)$ defines a semigroup $(U(t))_{t\geq 0}$. Using Prop. 1.9 (ii) on p. 7 one easily shows that $D_0:=D(A)\cap D(B)$ is a core for its generator C and $Cf=Af+Bf$ for all $f\in D_0$.

Notes

For more complete information on semigroup theory we refer the reader to Hille and Phillips (1957), to the monographs by Davies (1980), Goldstein (1985a) and Pazy (1983), to the survey article by Krein and Khazan (1985), to the bibliography by Goldstein (1985b) and to Engel and Nagel (2000).

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Part B
Positive Semigroups on Spaces $C_0(X)$

Chapter B-I
Basic Results on $C_0(X)$

This part of the book is devoted to one-parameter semigroups of operators on spaces of continuous functions of type $C_0(X)$. Such spaces are Banach lattices of a very special kind. We treat this case separately since we feel that an intermingling with the abstract Banach lattice situation considered in Part C would have made it difficult to appreciate the easy accessibility and the pilot function of methods and results available here. In this chapter we want to fix the notation we are going to use and to collect some basic facts about the spaces we are considering.

If X is a locally compact topological space, then $C_0(X)$ denotes the space of all continuous complex-valued functions on X which vanish at infinity, endowed with the supremum-norm. If X is compact, then any continuous function on X "vanishes at infinity" and $C_0(X)$ is the space of all continuous functions on X. We often write $C(X)$ instead of $C_0(X)$ in this situation.

We sometimes study real-valued functions and write the corresponding real spaces as $C_0(X,\mathbb{R})$ and $C(X,\mathbb{R})$, and the notations $C_0(X,\mathbb{C})$ and $C(X,\mathbb{C})$ are used if there might be confusion between both cases.

1 Algebraic and Order-Structure: Ideals and Quotients

Any space $C_0(X)$ is a commutative C*-algebra under the sup-norm and the pointwise multiplication, and by the *Gelfand Representation Theorem* any commutative C*-algebra can, on the other hand, be canonically represented as an algebra $C_0(X)$ on a suitable locally compact space X. The algebraic notions of ideal, quotient, homomorphism are well known and need not be explained further.

Another natural and important structure of $C_0(X)$ is the *pointwise* ordering, under which $C_0(X,\mathbb{R})$ is a (real) Banach lattice and $C_0(X,\mathbb{C})$ a complex Banach lattice in the sense explained in Chapter C-I.

Concerning the order structure of $C_0(X)$ we use the following notations. For a function f in $C_0(X, \mathbb{R})$

A function f is called *positive*, $f \geq 0$, if $f(t) \geq 0$ for all $t \in X$,
We write $f > 0$ if f is positive but does not vanish identically,
We call f *strictly positive* if $f(t) > 0$ for all $t \in X$.

The notion of an order ideal explained in Chapter C-I applies of course to the Banach lattices $C_0(X)$ and might give rise to confusion with the corresponding algebraic notion.

However, since we are mainly considering closed ideals and since a closed linear subspace I of $C_0(X)$ is a lattice ideal if and only if I is an algebraic ideal, we may and will simply speak of closed ideals without specifying whether we think of the algebraic or the order theoretic meaning of this word.

An important and frequently used characterization of these objects is the following: A subspace I of $C_0(X)$ is a closed ideal if and only if there exists a closed subset A of X such that a function f belongs to I if and only if f vanishes on A. The set A is of course uniquely determined by I and is called the *support* of I. If $I = I_A$ is a closed ideal with support A, then I_A is naturally isomorphic to $C_0(X \backslash A)$ and the quotient $C_0(X)/I$ (under the natural quotient structure) is again a Banach algebra and a Banach lattice that can be identified canonically (via the map $f + I \to f_{|A}$) with $C_0(A)$.

2 Linear Forms and Duality

The *Riesz Representation Theorem* asserts that the dual of $C_0(X)$ can be identified in a natural way with the space of bounded regular Borel measures on X. While there is no natural algebra structure on this dual, the dual ordering (see Chapter C-I) makes $C_0(X)'$ into a Banach lattice. We will occasionally make use of the order structure of $C_0(X)'$, but since at least its measure theoretic interpretation is supposed to be well-known, it may suffice here to refer to Chapter C-I, Sections 3 and 7, for a more detailed discussion and to recall only some basic notations here.

If μ is a linear form on $C_0(X, \mathbb{R})$, then

$\mu \geq 0$ means $\mu(f) \geq 0$ for all $f \geq 0$; μ is then called *positive*,
$\mu > 0$ means that μ is positive but does not vanish identically,
$\mu \gg 0$ means that $\mu(f) > 0$ for any $f > 0$; μ is then called *strictly positive*.

If μ is a linear form on $C_0(X, \mathbb{C})$, then μ can be written uniquely as $\mu = \mu_1 + i\mu_2$ where μ_1 and μ_2 map $C_0(X, \mathbb{R})$ into $\mathbb{R}$ (decomposition into *real* and *imaginary parts*). We call μ positive (strictly positive) and use the above notations if $\mu_2 = 0$ and μ_1 is positive (strictly positive). We point out that strictly positive linear forms need not exist on $C_0(X)$, but if X is separable, then a strictly positive linear form is obtained by summing a suitable sequence of point measures.

The coincidence of the notions of a closed algebraic and a closed lattice ideal in $C_0(X)$ has of course its effect on the algebraic and the lattice theoretic notions of a homomorphism. The case of a homomorphism into another space $C_0(Y)$ will be discussed below.

As for homomorphisms into the scalar field, we have essentially coincidence between the algebraic and the order theoretic meaning of this word, more exactly: A linear form $\mu \neq 0$ on $C_0(X)$ is a lattice homomorphism if and only if μ is, up to normalization, an algebra homomorphism (algebra homomorphisms $\neq 0$ must necessarily have norm 1). Since the algebra homomorphisms $\neq 0$ on $C_0(X)$ are known to be the point measures (denoted by δ_t) on X and since on the other hand μ is a lattice homomorphism if and only if $|\mu(f)|$ equals $\mu(|f|)$ for all f, it follows that this latter condition on μ is equivalent to $\mu = \alpha\delta_t$ for a suitable t in X and a positive real number α.

This can be summarized by saying that X can be canonically identified, via the map $t \longrightarrow \delta_t$, with the subset of the dual $C_0(X)'$ consisting of the non-zero algebra homomorphisms, which is also the set of all normalized lattice homomorphisms. This identification is a topological isomorphisms with respect to the weak*-topology of $C_0(X)'$.

3 Linear Operators

A linear mapping T from $C_0(X, \mathbb{R})$ into $C_0(Y, \mathbb{R})$ is called

positive (notation: $T \geq 0$) if Tf is a positive whenever f is positive,
lattice homomorphism if $|Tf| = T|f|$ all f,
Markov-operator if X and Y are compact and T is a positive operator mapping 1_X to 1_Y.

In the case of complex scalars, T can be decomposed into real and imaginary parts. We call T positive in this situation if the imaginary part of T is $= 0$ and the real part is positive. The terms *Markov operator* and *lattice homomorphism* are defined as above. Note that a complex lattice homomorphism is necessarily positive, and that the *complexification* of a real lattice homomorphism is a complex lattice homomorphism. Positive Operators are always continuous.

Note that the adjoint of a Markov operator T maps positive normalized measures into positive normalized measures while the adjoint of an algebra homomorphism (lattice homomorphism) maps point measures into (multiples of) point measures. Therefore the adjoint of a Markov lattice homomorphism as well as the adjoint of an algebra homomorphism induces a continuous map ϕ from Y (viewed as a subset of the weak dual $C(Y)'$) into X (viewed as a subset of $C(X)'$).

This mapping ϕ determines T in a natural and unique way, so that the following are equivalent assertions on a linear mapping T from a space $C(X)$ into a space $C(Y)$.

(a) T is a Markov lattice homomorphism.
(b) T is a Markov algebra homomorphism.

(c) There exists a continuous map ϕ from Y into X such that $Tf = f \circ \phi$ for all $f \in C(X)$.

If T is, in addition, bijective, then the mapping ϕ in (c) is a homeomorphism from Y onto X. This characterization of homomorphisms carries over mutatis mutandis to situations where the conditions on X, Y or T are less restrictive. For later reference we explicitly state the following.

(i) Let K be compact. Then $T \in \mathrm{L}C(K)$ is a lattice homomorphism if and only if there is a mapping ϕ from K into K and a function $h \in C(K)$ such that $Tf(s) = h(s)f(\phi(s))$ holds for all $s \in K$. The mapping ϕ is continuous in every point t with $h(t) \neq 0$.

(ii) Let X be locally compact and $T \in \mathrm{L}C_0(X)$. Then T is a lattice isomorphism if and only if there is a homeomorphism ϕ from X onto X and a bounded continuous function h on X such that $h(s) \geq \delta > 0$ for all s and $Tf(s) = h(s)f(\phi(s))$ $(s \in X)$. Moreover, T is an algebraic $*$-isomorphism if and only if T is a lattice isomorphism and the function h above is $\equiv 1$.

Notes

For the representation theory of commutative C*-algebras we refer to Takesaki (1979). This and the other mentioned properties like algebraic ideals, their connections with closed sets, the representation of latice or algebraic homomorpism etc. we refer to the excellent book Semadeni (1971).

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Part C
Positive Semigroups on Banach Lattices

Chapter C-I
Basic Results on Banach Lattices and Positive Operators

This introductory chapter is intended to give a brief exposition of those results on Banach lattices and ordered Banach spaces which are indispensable for an understanding of the subsequent chapters. We do not give proofs of the results we are going to present, since these can easily be found in the literature (e.g., in Schaefer (1974)). We rather want to give the reader, who is unfamiliar with these results or with the terminology used in this book, the necessary information for an intelligent reading of the main discussions. Since relatively few general results on ordered Banach spaces are needed, we will primarily talk about Banach lattices. The scalar field will be $\mathbb{R}$ except for the last three sections, where complex Banach lattices will be discussed.

The notion of a Banach lattice was devised to obtain a common abstract setting within which one could talk about phenomena related to positivity that had previously been studied in various types of spaces of real-valued functions, such as the spaces $C(K)$ of continuous functions on a compact topological space K, the Lebesgue spaces $L^1(\mu)$ or more generally the spaces $L^p(\mu)$ constructed over a measure space (X, Σ, μ) for $1 \leq p \leq \infty$. Thus it is a good idea to think of such spaces first in order to get a feeling for the concrete meaning of the abstract notions we are going to introduce. Later we will see that the connections between abstract Banach lattices and the *concrete* function lattices $C(K)$ and $L^1(\mu)$ are closer than one might think at first. We will use without further explanation the terms order relation (ordering), *ordered set*, *majorant*, *minorant*, *supremum*, *infimum*.

By definition, a Banach lattice is a Banach space $(E, \|\cdot\|)$ which is endowed with an order relation, usually written $\leq$, such that $(E, \leq)$ is a lattice and the ordering is compatible with the Banach space structure of E. We are going to elaborate this in more detail now. The axioms of compatibility between the linear structure of E and the order are the following

$$f \leq g \text{ implies } f + h \leq g + h \text{ for all } f, g, h \text{ in } E \tag{LO1}$$

$$f \geq 0 \text{ implies } \lambda f \geq 0 \text{ for all } f \text{ in } E \text{ and } \lambda \geq 0. \tag{LO2}$$

Any (real) vector space with an ordering satisfying (LO_1) and (LO_1) is called an *ordered vector space*. The property expressed in (LO_1) is sometimes called *translation invariance* and implies that the ordering of an ordered vector space E is completely determined by the positive part $E_+ = \{f \in E : f \geq 0\}$ of E. In fact, one has $f \leq g$ if and only if $g - f \in E_+$. (LO_1) together with (LO_2) furthermore imply that the positive part of E is a convex set and a cone with vertex 0 (often called the *positive cone* of E). It is easily verified that conversely any proper convex cone C with vertex 0 in E is the positive part of E with respect to a uniquely determined compatible ordering.

An ordered vector space E is called a *vector lattice* if any two elements f, g in E have a supremum, which is denoted by $\sup(f, g)$ or by $f \vee g$, and an infimum, denoted by $\inf(f, g)$ or by $f \wedge g$. It is obvious that the existence of, e.g., the supremum of any two elements in an ordered vector space implies the existence of the supremum of any finite number of elements in E and, since $f \leq g$ is equivalent to $-g \leq -f$ this automatically implies the existence of finite infima. However, suprema (infima) of infinite majorized subsets need not exist in a vector lattice. If they do, then the vector lattice is called *order complete* (*countably order complete* or *σ-order complete* if suprema of countable majorized subsets exist). At any rate, the binary relations *sup* and *inf* in a vector lattice automatically satisfy the (infinite) distributive laws

$$\inf(\sup_\alpha f_\alpha, h) = \sup_\alpha(\inf(f_\alpha, h)),$$
$$\sup(\inf_\alpha f_\alpha, h) = \inf_\alpha(\sup(f_\alpha, h)),$$

whenever one side exists. This gives rise to the following definitions.

$$\sup(f, -f) = |f| \text{ is called the } \textit{absolute value} \text{ of } f,$$
$$\sup(f, 0) = f^+ \text{ is called the } \textit{positive part} \text{ of } f,$$
$$\sup(-f, 0) = f^- \text{ is called the } \textit{negative part} \text{ of } f.$$

Note that the negative part of f is positive.

We call two elements f, g of a vector lattice *orthogonal* or *lattice disjoint* and write $f \perp g$ if $\inf(|f|, |g|) = 0$.

Apart from this, the above definitions allow us to formulate the axiom of compatibility between norm and order requested in a Banach lattice in the following short way:

$$|f| \leq |g| \text{ implies } \|f\| \leq \|g\|. \tag{LN}$$

A norm on a vector lattice is called a *lattice norm*, if it satisfies (LN). Whese notations we can now give the definition of a Banach lattice as follows.

> *A* Banach lattice *is a Banach space E endowed with an ordering $\leq$ such that $(E, \leq)$ is a vector lattice and the norm on E is a lattice norm. By a* normed vector lattice *we understand a vector lattice endowed with a lattice norm.*

There is a number of elementary, but very important formulas valid in any vector lattice, such as

$$f = f^+ - f^- \qquad |f+g| \le |f| + |g|$$
$$|f| = f^+ + f^- \qquad f + g = \sup(f,g) + \inf(f,g)$$

(see, e.g., [Schaefer (1974)]).

Let us note in passing the following consequences.

(i) The lattice operations $(f,g) \mapsto \sup(f,g)$ and $(f,g) \mapsto \inf(f,g)$ and the mappings $f \mapsto f^+$, $f \mapsto f^-$, $f \mapsto |f|$ are uniformly continuous.
(ii) The positive cone is closed.
(iii) *Order intervals*, i.e., sets of the form

$$[f,g] = \{h \in E : f \le h \le g\}$$

are closed and bounded.

Instead of dwelling upon a detailed discussion of the above equalities and inequalities let us rather formulate the following principle, which allows us to verify any of them and to invent, prove or disprove new ones whenever necessary.

> *Any general formula relating a finite number of* variables *to each other by means of lattice operations and/or linear operations is valid in any Banach lattice as soon as it is valid in the real number system.*

In fact, we are going to see below that any Banach lattice E is, as a vector lattice, *locally* of type $C(X)$, more exactly: Given any finite number $x_1, \ldots, x_n$ of elements in E, there is a compact topological space X and a vector sublattice J of E which is isomorphic to $C(X)$ and contains $x_1, \ldots, x_n$ (see Section. 4). The above principle is an easy consequence of the following: In a space $C(X)$ it is clear that a formula of the type considered need only be verified pointwise, i.e. in $\mathbb{R}$.

The reader may now be prepared to follow a concise presentation of the most basic facts on Banach lattices.

1 Sublattices, Ideals, Bands

The notion of a *vector sublattice* of a vector lattice E is self-explanatory, but it should be pointed out that a vector subspace F of E which is a vector lattice for the ordering induced by E need not be a vector sublattice of E (formation of suprema may differ in E and in F), and that a vector sublattice need not contain (or may lead to different) infinite suprema and infima. The following are necessary and sufficient conditions on a vector subspace G of E to be a vector sublattice.

(a) $|h| \in G$ for all $h \in G$,
(b) $h^+ \in G$ for all $h \in G$,
(c) $h^- \in G$ for all $h \in G$.

A subset B of a vector lattice is called *solid* if $f \in B$, $|g| \le |f|$ implies $g \in B$. Thus a norm on a vector lattice is a lattice norm if and only if its unit ball is solid. A solid linear subspace is called an *ideal*. Ideals are automatically vector sublattices since $|\sup(f,g)| \le |f| + |g|$. On the other hand, a vector sublattice F is an ideal in E if $g \in F$ and $0 \le f \le g$ imply $f \in F$. A *band* in a vector lattice E is an ideal which contains arbitrary suprema, or more exactly:

B is a band in E if B is an ideal in E and $\sup M$ *is contained in B whenever M is contained in B and has a supremum in E.*

Since the notions of sublattice, ideal, band are invariant under the formation of arbitrary intersections there exists, for any subset B of E, a uniquely determined smallest sublattice (ideal, band) of E containing B: the *sublattice* (*ideal*, *band*) *generated by B*.

If we denote by B^d the set $\{h \in E\colon \inf(|h|,|f|) = 0 \text{ for all } f \in B\}$, then B^d is a band for any subset B of E, and $(B^d)^d = B^{dd}$ is a band containing B. If E is a normed vector lattice (more generally, if E is archimedean ordered, see e.g., Schaefer (1974)), then B^{dd} is the band generated by B.

If two ideals I, J of a vector lattice E have trivial intersection $\{0\}$, then I and J are *lattice disjoint*, i.e. $I \subset J^d$. Thus if E is the direct sum of two ideals I, J, then one has automatically $I = J^d$ and $J = I^d$, hence $I = I^{dd}$ and $J = J^{dd}$ must be bands in this situation. In general, an ideal I need not have a complementary ideal J even if $I = I^{dd}$ is a band. This amounts to the same as saying that even if $I = I^{dd}$ (which is always true if I is a band in a normed vector lattice) one need not necessarily have $E = I + I^d$.

An ideal I is called a *projection band* if it does have a complementary ideal, and in this case the projection of E onto I with kernel I^d is called the *band projection* belonging to I. An example of a band which is not a projection band is furnished by the subspace of $C([0,1])$ consisting of the functions vanishing on $[0, 1/2]$.

The *Riesz Decomposition Theorem* asserts that in an order complete vector lattice every band is a projection band. As a consequence, if E is order complete and B is an arbitrary subset of E, then E is the direct sum of the complementary bands B^d and B^{dd}.

This theorem, which is quite easy to prove, is widely used in analysis and gives an abstract background to, e.g., the decomposition of a measure into atomic and diffuse parts (the atomic measures being those contained in the band generated by the point measures, the diffuse measures those disjoint to the latter). Or, more specifically, to the well-known decomposition of a measure on $[a,b]$ into an atomic part and a diffuse part, which latter can in turn be decomposed into the sum of a measure which is *absolutely continuous* (which means, contained in the band generated by Lebesgue measure) and a *singular measure* (which means, a diffuse measure disjoint to Lebesgue measure).

A band in a normed vector lattice is necessarily closed. By contrast, an ideal need not be closed, but the closure of an ideal is again an ideal. The situation, where every closed ideal is a band, will be briefly discussed in Section 5.

2 Order Units, Weak Order Units, Quasi-Interior Points

An element u in the positive cone of a vector lattice E is called an *order unit* if the ideal generated by u is all of E. If the band generated by u is all of E (which is equivalent to $\{u\}^d = 0$ whenever E is archimedean, hence in particular if E is a normed vector lattice), then u is called a *weak order unit* of E. If E is a Banach lattice, then any order unit in E is an interior point of the positive cone E_+, and conversely any interior point of E_+ must be an order unit of E. Every space $C(K)$ has order units (namely, the strictly positive functions), and conversely by the Kakutani-Krein Representation Theorem (see Section 4), every Banach lattice with an order unit is isomorphic to a space $C(K)$.

If an element u in the positive cone of a Banach lattice E has the property that the closed ideal generated by u is all of E, then u is called a *quasi-interior point* of E_+. Quasi-interior points of the positive cone exist, e.g., in any separable Banach lattice. If $E = C(K)$, then the quasi-interior points and the interior points of E_+ coincide, while the weak order units of E are the (positive) functions vanishing on a nowhere dense subset of K. If E is a space $L^p(\mu)$ with σ-finite μ and $1 \le p < \infty$, then the weak order units and the quasi-interior points of E_+ coincide with the functions strictly positive μ-a.e., while E_+ does not contain any interior point.

3 Linear Forms and Duality

A linear functional ϕ on a vector lattice E is called

order-bounded, if ϕ is bounded on order intervals of E,
positive, if $\phi(f) \ge 0$ for all $f \ge 0$,
strictly positive, if $\phi(f) > 0$ for all $f > 0$.

Any positive linear functional is order bounded, and the positive functionals form a proper convex cone with vertex 0 in the linear space $E^\#$ of all order bounded functionals, thus defining a natural ordering (given by $\phi \le \psi$ if and only if $\phi(f) \le \psi(f)$ for all $f \in E_+$ under which $E^\#$ is an order complete vector lattice. In particular, positive part, negative part and absolute value exist for any order bounded functional on E, the absolute value of $\phi \in E^\#$ being given by

$$|\phi|(f) = \sup\{\phi(h) \colon |h| \le f \text{ for } f \in E_+ \ .$$

As a consequence, one has $|\phi(f)| \le |\phi|(|f|)$ for all f in E whenever ϕ is order bounded, and $|\phi(f)| \le \phi(|f|)$ if and only if ϕ is positive. An order bounded linear functional ϕ is called *order-continuous* (*σ-order-continuous*) if both positive and negative part of ϕ have the property that they transform any decreasing net (any decreasing sequence) with infimum 0 into a net (sequence) converging to 0 in $\mathbb{R}$. The order-continuous (σ-order-continuous) functionals form a band in $E^\#$.

In general, a vector lattice E need not admit any non-zero order-bounded linear functional. However, if E is a normed lattice, then any continuous functional is order-bounded, and if E is a Banach lattice, then one has coincidence between $E^\#$ and E'. Still, order-continuous functionals $\neq 0$ need not exist on a Banach lattice. Situations where every order-bounded functional is order-continuous will be briefly discussed in Section 5.

If E is a Banach lattice, then the dual norm on $E' = E^\#$ is a lattice norm, hence E' is an order-complete Banach lattice under the natural norm and order. The evaluation map from E into the second dual E'' is a lattice homomorphism (for the definition see Section 6) into the band of order-continuous functionals on E'. In particular, every dual Banach lattice E admits sufficiently many order-continuous functionals to separate the points of E.

4 AM- and AL-Spaces

If the norm on a Banach lattice E satisfies

$$\|\sup(f,g)\| = \sup(\|f\|,\|g\|) \text{ for } f,g \in E_+, \tag{M}$$

then E is called an abstract M-space or an *AM-space*. If, in addition, the unit ball of E contains a largest element u, then u must be an order unit of E and E is then called an *(AM)-space with unit*. Condition (M) in E implies that in the dual of E one has

$$\|f+g\| = \|f\| + \|g\| \text{ for } f,g \in E_+. \tag{L}$$

Any Banach lattice satisfying (L) is called an abstract L-space or an *AL-space*. Thus the dual of an AM-space is an AL-space. It is quite easy to verify, that on the other hand, the dual of an AL-space is an AM-space with unit, the unit being the uniquely determined linear functional that coincides with the norm on the positive cone. Putting this together, one gets that the second dual of an AM-space E is an AM-space with unit. If E already has a unit u, then u is also the unit of E'', so that the ideal of E'' generated by E is all of E''. By contrast, if E is an AL-space, then E is an ideal (even a band) in E''. Infinite-dimensional AL- or AM-spaces are never reflexive.

The importance of AL- and AM-spaces in the general theory of Banach lattices is due to the fact that these spaces have very special concrete representations as function lattices and that, on the other hand, any general Banach lattice E is in a very intimate way connected to certain families of AL- and AM-spaces canonically associated with E. Let us first discuss the natural representations of AM- and AL-spaces.

If E is an AM-space with unit u, then the set K of lattice homomorphisms from E into $\mathbb{R}$ taking the value 1 on u is a non-empty, compact subset of the weak dual of E and the natural evaluation map from E into $\mathbb{R}^K$ maps E isometrically onto the continuous real-valued functions on K (cf. Section 6). This is the *Kakutani-Krein Representation Theorem*, which is an order-theoretic counterpart to the Gelfand

Representation Theorem in the theory of commutative C*-algebras. If E is an AM-space without unit, then the second dual of E has a unit and thus gives a representation of E as a closed sublattice of a space $C(K)$.

If E is an AL-space, then the representation of the dual of E as a space $C(K)$ leads to an interpretation of the elements of E' as Radon measures on K. If E_+ has a quasi-interior point h, then in this interpretation E consists exactly of the measures absolutely continuous with respect to (the measure corresponding to) h, thus by the *Radon-Nikodym-Theorem*, $E = L^1(K, h)$. In general, a similar argument leads to a representation of E as a space $L^1(X, \mu)$ constructed over a locally compact space X.

If E is an arbitrary Banach lattice and $f \in E_+$, then the ideal I generated by f in E (which is the union of the positive multiples of the interval $[-f, f]$) can be made into an AM-space with unit f by endowing it with the gauge function p_f of $[-f, f]$. We denote (I, p_f) by E_f. On the other hand, if f' is a positive linear functional on E, then the mapping $q_{f'}: f \mapsto \langle |f|, f' \rangle$ is a semi-norm on E. The kernel J of $q_{f'}$ is an ideal in E, and the completion of E/J with respect to the norm canonically derived from $q_{f'}$ becomes an AL-space which we denote by (E, x'). A good illustration for these constructions is the following.

If $E = C(K)$ and if μ is a positive linear form (Radon measure) on E, then (E, μ) is just $L^1(K, \mu)$; if $E = L^p(\mu)$ $(1 \le p < \infty$, μ finite), then $E_{1_X} = L^\infty(\mu)$.

5 Special Connections Between Norm and Order

If an increasing net $(x_\alpha)_{\alpha \in A}$ in a normed vector lattice is convergent, then its limit must be the supremum as a consequence of the closedness of the positive cone. On the other hand, if $\{x_\alpha : \alpha \in A\}$ has a supremum, the net $(x_\alpha)_{\alpha \in A}$ need not converge. A Banach lattice is said to have *order-continuous norm* (*σ-order-continuous norm*) if any increasing net (sequence) which has a supremum is automatically convergent. This is of course equivalent to saying that any decreasing net (sequence) with an infimum is convergent, and since infimum and limit must coincide, the order continuity (σ-order continuity) of the norm in a Banach lattice is also equivalent to the propperty that any decreasing net (sequence) with infimum 0 converges to 0.

A Banach lattice with order-continuous norm must be order complete, but σ-order-continuity of the norm need not imply order completeness. At any rate, one has the following characterization.

A Banach lattice E has order-continuous norm if and only if any one of the following equivalent assertions holds.

(a) E is σ-order complete and has σ-order-continuous norm.
(b) Every order interval in E is weakly compact.
(c) E is (under evaluation) an ideal in E''.
(d) Every continuous linear form on E is order continuous.
(e) Every closed ideal in E is a projection band.

An even more stringent condition than order-continuity of the norm is that any increasing norm-bounded net be convergent. This condition is satisfied if and only if any one of the following equivalent assertions holds.

(a) E is (under evaluation) a band in E''.
(b) E is weakly sequentially complete.
(c) Every order-continuous linear form on E' belongs to E.
(d) No closed sublattice of E is isomorphic to c_0.

The most important examples of non-reflexive Banach lattices with this property are the AL-spaces.

6 Positive Operators, Lattice Homomorphisms

A linear mapping T from an ordered Banach space E into an ordered Banach space F is called *positive* (notation: $T \geq 0$) if $Tf \in F_+$ for all $f \in E_+$; T is called *strictly positive* if $T \geq 0$ and $\{f \in E\colon T|f| = 0\} = \{0\}$. The set of all positive linear mappings is a convex cone in the space $\mathrm{L}E, F$ of all linear mappings from E into F defining the *natural ordering* of $\mathrm{L}E, F$. The linear subspace of $\mathrm{L}E, F$ generated by the positive maps (i.e. the space of linear maps that can be written as differences of positive maps) is denoted by $\mathcal{L}^r(E; F)$ and its elements are called *regular* mappings. If E and F are Banach lattices, then any regular mapping from E into F is continuous, but $\mathcal{L}^r(E; F)$ is in general a proper subspace of the space $\mathrm{L}E, F$ of all continuous linear mappings. One has coincidence of $\mathcal{L}^r(E; F)$ and $\mathrm{L}E, F$, e.g., when $E = F$ is an order complete AM-space with unit or an AL-space. At any rate, if F is order complete, then $\mathcal{L}^r(E; F)$ under the natural ordering is an order-complete vector lattice, and a Banach lattice under the norm

$$T \mapsto \|T\|_r = \||T|\|,$$

the right hand side denoting the operator norm of the absolute value of T. The absolute value of $T \in \mathcal{L}^r(E; F)$, if it exists, is given by

$$|T|(f) := \sup\{Th\colon |h| \leq f,\ f \in E_+.\}$$

Thus T is positive if and only if $|Tf| \leq T|f|$ holds for any f in E.

An operator $T \in \mathrm{L}E, F$ is called a *lattice homomorphism* if $|Tf| = T|f|$ holds for all $f \in E$. Lattice homomorphisms are alternatively characterized by any one of the following conditions holding for all f, and $g \in E$.

(i) $(Tf)^+ = T(f^+)$,
(ii) $(Tf)^- = T(f^-)$,
(iii) $T(f \vee g) = Tf \vee Tg$,
(iv) $T(f \wedge g) = Tf \wedge Tg$,
(v) $T(f^+) \wedge T(f^-) = 0$.

The kernel of a lattice homomorphism is an ideal. If T is bijective, then T is a lattice homomorphism if and only if T and T^{-1} are positive.

7 Complex Banach Lattices

Although the notion of a Banach lattice is intrinsically related to the real number system, it is possible and often desirable to extend discussions to complexifications of Banach lattices in such a way that the order-related terms introduced in the real situation essentially retain their meaning. Thus we define a *complex Banach lattice* E to be the complexification of a real Banach lattice $E_{\mathbb{R}}$ in the sense that

$$E = E_{\mathbb{R}} \oplus \mathrm{i}\, E_{\mathbb{R}}$$

which means more exactly $E = E_{\mathbb{R}} \times E_{\mathbb{R}}$ with scalar multiplication

$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y).$$

The space $E_{\mathbb{R}}$ will sometimes be called the *underlying real Banach lattice* or the *real part* of E. The classical complex Banach spaces $C(X)$, $L^p(\mu)$ are complex Banach lattices in this sense, the underlying real Banach lattices being the corresponding (real) subspaces of real-valued functions. We want to extend the formation of absolute values, which is a priori defined only in the real part of E, in such a way that in the classical situation $E = C(X)$ or $E = L^p(\mu)$ the usual absolute value of a function is obtained. This is in fact possible by putting, for $h = f + ig$ in E,

$$|h| = \sup\{\Re(e^{i\theta} h) \colon 0 \le \theta \le 2\pi\},$$

the only problem with this definition being the existence of the right hand side without the assumption of order-completeness on $E_{\mathbb{R}}$, where the set in brackets is an order bounded subset of $E_{\mathbb{R}}$. But for this we just have to observe that the set $M = \{\Re(e^{i\theta} h) \colon 0 \le \theta \le 2\pi\}$ is contained and order bounded in the ideal generated in $E_{\mathbb{R}}$ by $|f| + |g|$, which in turn is by the Kakutani-Krein Representation Theorem isomorphic to a space $C_{\mathbb{R}}(X)$ under the pointwise ordering. Now the pointwise supremum of M in $\mathbb{R}^X$ is readily seen to be a continuous function (namely, the modulus of the complex valued continuous function corresponding to $f + ig$), so that M has a supremum in $C_{\mathbb{R}}(X) = (E_{\mathbb{R}})_{|f|+|g|}$.

Since the mapping $f \mapsto |f|$ now has a meaning in E, the definition of an ideal can be extended formally unchanged to the complex situation. We observe that $|f + ig| = |f - ig| \le |f| + |g|$ implies that any ideal J in a complex Banach lattice is conjugation invariant and itself the complexification of the ideal $J \cap E_{\mathbb{R}}$ of the real part of E.

Suffice it now to say that the meaning of most of the terms introduced for real Banach lattices can be extended to the complex situation under retention (mutatis mutandis) of the corresponding results valid in the real case by either using the complex modulus or else, if the formation of suprema or infima is involved, by relating them to real parts. For example $f \in E$ is called *positive* if $f = |f|$ which means that f is a positive element of $E_{\mathbb{R}}$, E is called order complete if $E_{\mathbb{R}}$ is order complete, and an ideal J is called a band if the real part of J is a band. We refer to Chapter II, Section 11 of Schaefer (1974) for a detailed discussion of this and restrict ourselves to a short discussion of linear mappings.

Let E and F be complex Banach lattices with real parts $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$. Then a linear mapping T from E into F is determined by its restriction T_0 to $E_{\mathbb{R}}$, and T_0 can be written in the form $T_0 = T_1 + iT_2$ with real-linear mappings T_j from $E_{\mathbb{R}}$ into $F_{\mathbb{R}}$. Thus $L(E, F)$ is the complexification of the real linear space $L(E_{\mathbb{R}}, F_{\mathbb{R}})$. With the above notation, T is called *real* if T_2 is $= 0$, *positive* if T is real and T_1 is positive, and a *lattice homomorphism* if T is real and T_1 is a lattice homomorphism. Lattice homomorphisms are characterized by the equality $|Th| = T|h|$ as in the real case.

8 The Signum Operator

We discuss in some detail how a mapping of the form

$$g \mapsto (\operatorname{sign} f)g$$

which has an obvious meaning, depending on f, in spaces $C(K)$, can be defined in an abstract complex Banach lattice. We prove the following

Let E be a complex Banach lattice and let $f \in E$. If either E is order-complete or $|f|$ is a quasi-interior point in E_+, then there exists a unique linear mapping S_f, called the *signum operator* with respect to f, with the following properties.

(i) $S_f \bar{f} = |f|$, where $\bar{f} = \Re f - \mathrm{i} \cdot \Im f$,
(ii) $|S_f g| \le |g|$ for every g in E,
(iii) $S_f g = 0$ for every g in E orthogonal to f.

In fact, if $E = C(K)$ and if $|f|$ is a quasi-interior point in E, then $|f|$ is a strictly positive function and multiplication with the function sign $f = f \cdot |f|^{-1}$ has the desired properties. Uniqueness follows from Zaanen (1983, Chap. 20). We reduce the general situation to the case just considered in the following way.

— If $|f|$ is quasi-interior to E_+, then $E_{|f|}$ is a dense subspace of E isomorphic to a space $C(K)$, and with the above arguments one gets a uniquely determined operator S_0 on $E_{|f|}$ with the desired properties. Since (ii) implies the continuity of S_0 for the norm induced by E, S_0 can be extended to E.
— If f is arbitrary, then, as above, one gets an operator S_0 on $E_{|f|}$ with (i) and (ii). If E is order complete, an extension S_f of S_0 to E satisfying (i)–(iii) is possible as soon as S_0 can be extended to the band $\{x\}^{dd}$ of E.

  - On the complementary band $\{x\}^d$ one has necessarily the values = 0 for S_f.
  - The extension to $\{x\}^{dd}$ is obtained as follows: If S_0 is positive (which means $f \ge 0$) then

$$S_f h = \sup\{S_f g\colon g \in [0,h] \cap E_{|f|} \text{ for } h \geq 0\}$$

will do.

In general, the problem can be reduced to this situation by decomposing S_0 into a sum of the form $S_0 = (S_1 - S_2) + i(S_3 - S_4)$ with positive operators S_j. Such a decomposition of S_0 exists since the order completeness of E implies the order completeness of $E_{|f|} = C(K)$ and since every continuous linear operator on a space $C(K)$ is necessarily order-bounded.

9 The Center of $\mathcal{L}(E)$

We give a short description of a special, but important class of operators.

Let E be a (complex) Banach lattice. For $T \in \mathcal{L}(E)$ the following conditions are equivalent.

(a) $f \perp g$ implies $Tf \perp g \quad (f, g \in E)$,
(b) $\pm T \leq \|T\| \mathrm{Id}$,
(c) $TJ \subseteq J$ for every ideal J in E.

If E is countably order complete, then this is equivalent to:

(d) $TJ \subseteq J$ for every projection band J in E

The last assertion also means that T commutes with every band projection.

The set of all $T \in \mathcal{L}(E)$ satisfying the above conditions is called the *center* of $\mathcal{L}(E)$ and denoted $\mathcal{Z}(E)$. Under its natural ordering, the operator norm and the composition product is $\mathcal{Z}(E)$ isomorphic to a Banach lattice algebra $C(K)$ with K compact. The following examples may illustrate the situation and explain why the term *multiplication operator* is often used for operators in $\mathcal{Z}(E)$.

(i) If $E = L^p(X, \Sigma, \mu)$ $(1 \leq p \leq \infty)$ with σ-finite μ, then $Z(E)$ is isomorphic to $L^\infty(\mu)$ via the natural identification of a function $f \in L^\infty(\mu)$ with the multiplication operator $g \mapsto f \cdot g$ on E.
(ii) If X is locally compact, $E = C_0(X)$, then similarly $\mathcal{Z}(E) \cong C^b(X)$ via the identification of $f \in C^b(X)$ with the mapping $g \mapsto f \cdot g$ $(g \in C_0(X))$.

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