<u>Corollary</u> 2.8. Let $(T(t))_{t\geq 0}$ be a lattice semigroup on $C_o(X)$ with generator $A$. If $f \in D(A)$ and $f(x) \neq 0$ for all $x \in X$, then

$$|f| \in D(A) \quad \text{and} \quad \mathrm{Re}[(\mathrm{sign}\,\bar{f})(Af)] = A|f| .$$

<u>Proof</u>. If $f \in D(A)$ and $f(x) \neq 0$ for all $x \in X$, then $(\mathrm{si\hat{g}n}\,\bar{f})(Af) = (\mathrm{sign}\,\bar{f})(Af) \in C_o(X)$. Hence by (2.9), $\langle \mathrm{Re}[(\mathrm{sign}\,\bar{f})(Af)],\phi\rangle = \langle |f|,A'\phi\rangle$ for all $\phi \in D(A')$. So the assertion follows from the following lemma.

□

<u>Lemma</u> 2.9. Let $A$ be a densely defined closed operator on a (real or complex) Banach space $G$. Let $f, g \in G$ such that $\langle f,\phi\rangle = \langle g,A'\phi\rangle$ for all $\phi \in D(A')$. Then $g \in D(A)$ and $Ag = f$.

<u>Proof</u>. Denote by $G(A) := \{(h,Ah) : h \in D(A)\} \subset G\times G$ the graph of $A$. Assume that $(g,f) \notin G(A)$. Since $G(A)$ is closed, it follows from the Hahn-Banach theorem that there exists $(\psi_1,\psi_2) \in G'\times G'$ such that $\langle h,\psi_1\rangle + \langle Ah,\psi_2\rangle = 0$ for all $h \in D(A)$ and $\langle g,\psi_1\rangle + \langle f,\psi_2\rangle \neq 0$. By the definition of $A'$ this implies that $\psi_2 \in D(A')$ and $A'\psi_2 = -\psi_1$. Hence $\langle f,\psi_2\rangle \neq -\langle g,\psi_1\rangle = \langle g,A'\psi_2\rangle$. So the condition in the lemma is violated.

□

Next we prove a converse of Corollary 2.8.

<u>Theorem</u> 2.10. Let $A$ be the generator of a real semigroup $(T(t))_{t\geq 0}$ on $C(K,\mathbb{C})$, where $K$ is compact. Then $(T(t))_{t\geq 0}$ is a lattice semigroup if and only if
$f \in D(A)$, $f(x) \neq 0$ for all $x \in K$ implies $|f| \in D(A)$ and
$A|f| = \mathrm{Re}((\mathrm{sign}\,\bar{f})Af)$.

<u>Remark</u>. Although we assume that $(T(t))_{t\geq 0}$ is a real semigroup (i.e., $T(t)C(K,\mathbb{R}) \subset C(K,\mathbb{R})$ for all $t \geq 0$), it is important for the proof that we consider the space of all complex-valued continuous functions on $K$. In fact, if $K$ is connected, the condition in the theorem is always trivially satisfied for all $f \in C(K,\mathbb{R})$.

<u>Proof</u>. It follows from Cor. 2.8 that the condition is necessary. So assume that the condition is satisfied. Since $(T(t))_{t\geq 0}$ is real, the restriction $T_{\mathbb{R}}(t)$ of $T(t)$ to $C(K,\mathbb{R})$ $(t \geq 0)$ defines a strongly continuous semigroup. Its generator $A_{\mathbb{R}}$ is a restriction of $A$. Since $D(A_{\mathbb{R}})$ is dense in $C(K,\mathbb{R})$, there exists a strictly positive