Chapter 1

Characterization of Semigroups on Banach Spaces

In this chapter two different problems are treated:

(i) to characterize generators of strongly continuous semigroups;

(ii) to characterize various properties of strongly continuous semigroups in terms of their generators.

In Section 1 the first problem is solved by finding conditions on the Cauchy problem associated with A and also by finding conditions on the resolvent of A. The second problem is treated for a hierarchy of smoothness properties of the semigroup.

Contraction semigroups are considered in Section 2. Here, the first problem has a simple and extremely useful solution: A densely defined operator A is generator of a contraction semigroup if and only if A is dissipative and satisfies a range condition.

Our approach is quite general. We do not only consider contractions with respect to the norm but also with respect to "half-norms". This will allow us to obtain results on positive contraction semigroups simultaneously by choosing a suitable half-norm (cf. C-II,Sec.1).

The last section contains a surprising result: on certain Banach spaces (e.g., L^∞) only bounded operators are generators of strongly continuous semigroups.

1.1 The Abstract Cauchy Problem, Special Semigroups and Perturbation

Linear differential equations in Banach spaces are intimately connected with the theory of one-parameter semigroups. In fact, given a closed linear operator A with dense domain $D(A)$ the following statement is true (with some reservation regarding a technical detail): The abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) \quad (t \geq 0) \\ u(0) &= f \end{aligned}$$

has a unique solution for every $f \in D(A)$ if and only if A is the generator of a strongly continuous semigroup.

This is one characterization of generators which illustrates their important role for applications. The fundamental Hille-Yosida theorem gives a different characterization in terms of the resolvent and yields a powerful tool for actually proving that a given operator is the generator of a semigroup.

Another problem we will treat here is how diverse properties of a semigroup can be described in terms of its generator. This is a reasonable question from the theoretical point of view (since the generator uniquely determines the semigroup). It is of interest from the practical point of view as well: the generator is the given object, defined by the differential equation. It is useful to dispose of conditions of the generator itself giving information on the solutions (which might not be known explicitly). We discuss smoothness properties such as analyticity, differentiability, norm continuity and compactness of the semigroup.

A frequent method to obtain new generators out of a given one is by perturbation. We will have a brief look at this circle of problems at the end of this section.

The results are explained and illustrated by examples. Proofs are only given when new aspects are presented which are not yet contained in the literature, otherwise we refer to the recent monographs Davies (1980), Goldstein (1985a), Pazy (1983).

1.1.1 The Abstract Cauchy Problem

Let A be a closed operator on a Banach space E and consider the abstract Cauchy problem

$$\text{(ACP)} \quad \begin{cases} \dot{u}(t) &= Au(t) \quad (t \geq 0) \\ u(0) &= f. \end{cases}$$

By a solution of (ACP) for the initial value $f \in D(A)$ we understand a continuously differentiable function $u : [0, \infty) \to E$ satisfying $u(0) = f$ and $u(t) \in D(A)$ for all $t \geq 0$ such that $\dot{u}(t) = Au(t)$ for $t \geq 0$.

By A-I,Thm.1.7 there exists a unique solution of (ACP) for all initial values in the domain $D(A)$ whenever A is the generator of a strongly continuous semigroup. The converse does not hold (see Example 1.4. below). However, for the operator A_1 on the Banach space $E_1 = D(A)$ (see A-I,3.5) with domain $D(A_1) = D(A^2)$ given by $A_1 f = Af$ $(f \in D(A_1))$ the following holds.

Theorem 1.1. *The following assertions are equivalent.*

(a) For every $f \in D(A)$ there exists a unique solution of (ACP).

(b) A_1 is the generator of a strongly continuous semigroup.

Proof. (i) implies (ii). Assume that (i) holds; i.e., for every $f \in D(A)$ there exists a unique solution $u(\cdot, f) \in C^1([0,\infty), E)$ of (ACP). For $f \in E_1$ define $T_1(t)f := u(t,f)$ $(t \geq 0)$. By the uniqueness of the solutions it follows that $T_1(t)$ is a linear operator on E_1 and $T_1(s+t) = T_1(s)T_1(t)$. Moreover, since $u(\cdot, f) \in C^1$, it follows that $t \mapsto T_1(t)f$ is continuous from $[0,\infty)$ into E_1. We show that $T_1(t)$ is a continuous operator for all $t > 0$.

Let $t > 0$. Consider the mapping $\eta : E_1 \to C([0,t], E_1)$ given by $\eta(f) = T_1(\cdot)f = u(\cdot, f)$. We show that η has a closed graph. In fact, let $f_n \to f$ in E_1 and $\eta(f_n) = u(\cdot, f_n) \to v$ in $C([0,t], E_1)$. Then $u(s, f_n) = f_n + \int_0^s Au(r, f_n)dr$. Letting $n \to \infty$ we obtain $v(s) = f + \int_0^s Av(r)dr$ for $0 \leq s \leq t$. Let $\tilde{v}(s) = T_1(s-t)v(t)$ for $s > t$, and $\tilde{v}(s) = v(s)$ for $0 \leq s \leq t$.

Then $\tilde{v}$ is a solution of (ACP). It follows that $\tilde{v}(s) = T_1(s)f$ for all $s \geq 0$. Hence $v = \eta(f)$. We have shown that η has a closed graph and so η is continuous. This implies the continuity of $T_1(t)$. It remains to show that A_1 is the generator of $(T_1(t))_{t\geq 0}$.

We first show that for $f \in D(A^2)$ one has

$$AT_1(t)f = T_1(t)Af. \tag{1.1}$$

In fact, let $v(t) = f + \int_0^t u(s, Af)ds$. Then $\dot{v}(t) = u(t, Af) = Af + \int_0^t Au(s, Af)ds = A(f + \int_0^t u(s, Af)ds) = Av(t)$. Since $v(0) = f$, it follows that $v(t) = u(t,f)$. Hence $Au(t,f) = Av(t) = \dot{v}(t) = u(t, Af)$. This is (1.1).

Now denote by B the generator of $(T_1(t))_{t\geq 0}$. For $f \in D(A^2)$ we have

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af$$

and by (1.1),

$$\lim_{t\to 0} A\frac{T_1(t)f - f}{t} = \lim_{t\to 0} \frac{T_1(t)Af - Af}{t} = A^2 f \text{ in the norm of } E.$$

Hence $\lim_{t\to 0} \frac{T_1(t)f-f}{t} = Af$ in the norm of E_1.

This shows that $A_1 \subset B$. In order to show the converse, let $f \in D(B)$. Then $\lim_{t\to 0} A\frac{T_1(t)f-f}{t}$ exists in the norm of E. Since $\lim_{t\to 0} \frac{T_1(t)f-f}{t} = Af$ in the norm of E, it follows that $Af \in D(A)$, since A is closed. Thus $f \in D(A^2) = D(A_1)$. We have shown that $B = A_1$.

(ii) implies (i). Assume that A_1 is the generator of a strongly continuous semigroup $(T_1(t))_{t\geq 0}$ on E_1. Let $f \in D(A)$ and set $u(t) = T_1(t)f$. Then $u \in C([0,\infty), E)$ and $Au(\cdot) \in C([0,\infty), E)$. Moreover, $\int_0^t u(s)ds = \int_0^t T_1(s)f ds \in D(A_1) = D(A^2)$ and $A\int_0^t u(s)ds = u(t) - u(0) = u(t) - f$ (by A-I, (1.3)). Consequently, $u(t) = f + A\int_0^t u(s)ds = f + \int_0^t Au(s)ds$. Hence $u \in C^1([0,\infty), E)$ and $\dot{u}(t) = Au(t)$. Thus u is a solution of (ACP). We have shown existence.

Then $\tilde{v}$ is a solution of (ACP). It follows that $\tilde{v}(s) = T_1(s)f$ for all $s \geq 0$. Hence $v = \eta(f)$. We have shown that η has a closed graph and so η is continuous. This implies the continuity of $T_1(t)$. It remains to show that A_1 is the generator of $(T_1(t))_{t\geq 0}$.

We first show that for $f \in D(A^2)$ one has

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Now denote by B the generator of $(T_1(t))_{t\geq 0}$. For $f \in D(A^2)$ we have

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af$$

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Hence $\lim_{t\to 0} \frac{T_1(t)f-f}{t} = Af$ in the norm of E_1.

This shows that $A_1 \subset B$. In order to show the converse, let $f \in D(B)$. Then $\lim_{t\to 0} A\frac{T_1(t)f-f}{t}$ exists in the norm of E. Since $\lim_{t\to 0} \frac{T_1(t)f-f}{t} = Af$ in the norm of E, it follows that $Af \in D(A)$, since A is closed. Thus $f \in D(A^2) = D(A_1)$. We have shown that $B = A_1$.

(ii) implies (i). Assume that A_1 is the generator of a strongly continuous semigroup $(T_1(t))_{t\geq 0}$ on E_1. Let $f \in D(A)$ and set $u(t) = T_1(t)f$. Then $u \in C([0,\infty), E)$ and $Au(\cdot) \in C([0,\infty), E)$. Moreover, $\int_0^t u(s)ds = \int_0^t T_1(s)fds \in D(A_1) = D(A^2)$ and $A\int_0^t u(s)ds = u(t) - u(0) = u(t) - f$ (by A-I, (1.3)). Consequently, $u(t) = f + A\int_0^t u(s)ds = f + \int_0^t Au(s)ds$. Hence $u \in C^1([0,\infty), E)$ and $\dot{u}(t) = Au(t)$. Thus u is a solution of (ACP). We have shown existence.

In order to show uniqueness, assume that u is a solution of (ACP) with initial value 0. We have to show that $u \equiv 0$. Let $v(t) = \int_0^t u(s)ds$. Then $v(t) \in D(A)$ and $Av(t) = \int_0^t Au(s)ds = \int_0^t \dot{u}(s)ds = u(t) \in D(A)$. Consequently, $v(t) \in D(A^2)$ for all $t \geq 0$. Moreover, $\dot{v}(t) = u(t) = Av(t)$ and $\frac{d}{dt}Av(t) = Au(t) = A\dot{v}(t) = A^2 v(t)$. Thus $v \in C^1([0,\infty), E_1)$ and $\dot{v}(t) = A_1 v(t)$. Since $v(0) = 0$, it follows that $v \equiv 0$. Thus $u \equiv v \equiv 0$. □

If (ACP) has a unique solution for every initial value in $D(A)$, then A is the generator of a strongly continuous semigroup only if some additional assumptions on the solutions (continuous dependence from the initial value) or on A ($\rho(A) \neq \emptyset$) are made.

Corollary 1.2. *Let A be a closed operator. Consider the following existence and uniqueness condition.*

(EU) For every $f \in D(A)$ there exists a unique solution $u(\cdot, f) \in C^1([0,\infty), E)$ of the Cauchy problem associated with A having the initial value $u(0, f) = f$.

The following assertions are equivalent.

(a) A is the generator of a strongly continuous semigroup.

(b) A satisfies (EU) and $\rho(A) \neq \emptyset$.

(c) A satisfies (EU) and for every $\mu \in \mathbb{R}$ there exists $\lambda > \mu$ such that $(\lambda - A)D(A) = E$.

(d) *A satisfies (EU), has dense domain and for every sequence (f_n) in $D(A)$ satisfying $\lim_{n\to\infty} f_n = 0$ one has $\lim_{n\to\infty} u(t, f_n) = 0$ uniformly in $t \in [0, 1]$.*

Proof. It is clear that (i) implies the remaining assertions. So assume that A satisfy (EU). Then by Theorem 1.1., A_1 is a generator. If there exists $\lambda \in \rho(A)$, then $(\lambda - A)$ is an isomorphism from E_1 onto E and A is similar to A_1 via this isomorphism [since $D(A_1) = \{(\lambda - A)^{-1}f : f \in D(A)\}$ and $Af = (\lambda - A)A_1(\lambda - A)^{-1}f$ for all $f \in D(A)$, see A-I,3.0]. Thus A is a generator on E and we have shown that (ii) implies (i).

If (iii) holds, then there exists $\lambda > s(A_1)$ such that $(\lambda - A)D(A) = E$. We show that $(\lambda - A)$ is injective. Then $\lambda \in \rho(A)$ since A is closed. Assume that $Af = \lambda f$ for some $f \in D(A)$. Then $f \in D(A^2) = D(A_1)$, and so $f = 0$ since $\lambda \in \rho(A_1)$. This proves that (iii) implies (ii).

It remains to show that (iv) implies (i). Assertion (iv) implies that for all $t \geq 0$ there exist bounded operators $T(t) \in \mathcal{L}(E)$ such that $u(t, f) = T(t)f$ if $f \in D(A)$. Moreover, $\sup_{0\leq t\leq 1} \|T(t)\| < \infty$. It follows that $T(\cdot)f$ is strongly continuous for all $f \in E$ (since it is so for $f \in D(A)$ and $D(A)$ is dense). Let $t > 1$. There exist unique $n \in \mathbb{N}$ and $s \in [0, 1)$ such that $t = n + s$. Let $T(t) := T(1)^n T(s)$. From the uniqueness of the solutions it follows that $T(t)f = u(t, f)$ for all $t \geq 0$ as well as $T(t+s)f = T(s)T(t)f$ for all $f \in D(A)$ and $s, t \geq 0$. Thus $(T(t))_{t\geq 0}$ is a semigroup. Denote by B its generator. It follows from the definition that $A \subset B$. Moreover, $D(A)$ is invariant under the semigroup. So by A-I,Prop.1.9.(ii) $D(A)$ is a core of B. Since A is closed this implies that $A = B$. □

Remark 1.3. It is surprising that from condition (ii) and (iii) in the corollary it follows automatically that $D(A)$ is dense. On the other hand this condition cannot be omitted in (iv). In fact, consider any generator $\tilde{A}$ and its restriction A with domain $D(A) = \{0\}$. Then A satisfies the remaining conditions in (iv) but is not a generator (if $\dim E > 0$).

Example 1.4. We give a densely defined closed operator A, such that there exists a unique solution of (ACP) for all initial values in $D(A)$, but A is not a generator. Let B be a densely defined unbounded closed operator on a Banach space F. Consider $E = F \oplus F$ and A on E given by

$$A = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

with domain $F \times D(B)$.

Then $D(A^2) = \{(f, g) \in F \times D(B) : Bg \in F\} = D(A)$ and so $A_1 \in \mathcal{L}(E_1)$. In particular, A_1 is a generator. But A is not. For instance condition (ii) in Corollary 1.2. does not hold, since for each $\lambda \in \mathbb{C}$,

$$(\lambda - A)D(A) = \{(\lambda f - Bg, \lambda g) : f \in F, g \in D(B)\} \subset F \times D(B) \neq F \times F = E.$$

So $\rho(A) = \emptyset$.

As a further illustration, we note that the solution of the corresponding abstract Cauchy problem for the initial value $(f, g) \in F \times D(B)$ is given by $u(t) = (f + tBg, g)$. Since B is unbounded, condition (iv) of Corollary 1.2. is clearly violated.