A S Y M P T O T I C S   O F   P O S I T I V E

S E M I G R O U P S   O N   C*- A N D   W⁺- A L G E B R A S

1. STABILITY OF POSITIVE SEMIGROUPS

As explained in A-III, Section 1 , it is possible to deduce uniform exponential stability of strongly continuous semigroups from the location of the spectrum of its generator if the spectral bound $s(A)$ and the growth bound ω coincide. In this section we prove '$s(A) = \omega$' for positive semigroups on C*-algebras and preduals of W*-algebras. A more general discussion of the "$s(A) = \omega$" problem can be found in [Greiner-Voigt-Wolff (1981)]. For the results of this section the existence of a unit is essential.

Theorem 1.1. Let M be a C*-algebra with unit and $T = (T(t))_{t \geq 0}$ a positive semigroup on M . Then

$$-\infty < s(A) = \omega \in \sigma(A) .$$

Proof. For every $t \geq 0$ there exists ϕ_t in the state space $S(M)$ of M such that

$$T(t)'\phi_t = r(T(t))\phi_t = \exp(\omega t)\phi_t$$

(see, e.g., [Groh (1981), 2.1]). Let $n \in \mathbb{N}$ and

$$E_n := \{\phi \in S(M) : T(2^{-n})'\phi = \exp(\omega 2^{-n})\phi \} .$$

Then $\emptyset \neq E_{n+1} \subseteq E_n$ $(n \in \mathbb{N})$. Since $S(M)$ is $\sigma(M,M')$-compact there exists $\phi \in \cap_{n \in \mathbb{N}} E_n$ and $T(t)'\phi = \exp(\omega t)\phi$ follows because the adjoint semigroup $(T(t)')_{t \geq 0}$ is a weak*-semigroup on M' . Suppose $-\infty = \omega$. Then for $t > 0$ $r(T(t)) = 0$ (A-III,Prop.1.1) or $T(t)'\phi = 0$, in particular $\phi(T(t)1) = 0$. From this we obtain the contra-

diction $\phi(1) = 0$. Hence $-\infty < \omega$ and $\exp(\omega t) \in P\sigma(T(t)')$ for every $t \in \mathbb{R}_+$. Thus $\omega \in \sigma(A)$ or $\omega = s(A)$.

□

Remark 1.2.   (a)   If we consider the nilpotent translation semigroup on the C*-algebra $C_0([0,1))$ then $\sigma(A) = \emptyset$ and $\omega = -\infty$. This shows that the existence of a unit is essential.

(b)   '$s(A) = \omega$' still holds for positive semigroups on commutative C*-algebras without unit (see B-IV, Rem.1.2.b).

Theorem 1.3.   Let M be a W*-algebra with predual M_* and let $(T(t))_{t \geq 0}$ be a positive semigroup on M_* . Then $s(A) = \omega$.

Proof. For all $\lambda > s(A)$ and $\phi \in M_*$

$$R(\lambda, A)\phi = \int_0^\infty e^{-\lambda t} T(s)\phi \, ds$$

which follows as in C-III, Section 1 or [Greiner-Voigt-Wolff (1981), Theorem 3]. Since $\|\phi\| = \phi(1)$ for every $\phi \in M_*^+$ and since the norm is additive on the positive cone of M_* the integral

$$\int_0^\infty e^{\lambda t} \|T(s)\phi\| \, ds$$

exists for all $\phi \in M_*$ and all $\lambda > s(A)$. From this the assumption follows by A-IV,Thm.1.11.

□

Corollary 1.4.   Let M be a C*-algebra and $(T(t))_{t \geq 0}$ a positive semigroup on M' . Then $s(A) = \omega$ holds.

This follows from the fact that the bidual of a C*-algebra is a W*-algebra (see [Takesaki (1979), Theorem III.2.4.]).

Remark 1.5. A simple modification of A-III, Example 1.4 (take c_0 instead of ℓ^2) shows that Theorem 1.3 is no longer true for non-positive semigroups (for details see [Groh-Neubrander (1981), Beispiel 2.5]).

While the growth bound ω characterizes uniform exponential stability of the semigroup there are other (and weaker) stability concepts (cf. A-IV, Section 1).

<u>Definition</u> 1.6. Let E be a Banach space and $(T(t))_{t \geq 0}$ a semigroup on E . We call the semigroup

1.   <u>uniformly exponentially stable</u>, if $\|T(t)\| \leq Me^{-wt}$ for some w, M > 0 and all t ≥ 0 .

2.   <u>uniformly stable</u>, if $\lim_{t \to \infty} T(t) = 0$ in the strong operator topology.

3.   <u>weakly stable</u>, if $\lim_{t \to \infty} T(t) = 0$ in the weak operator topology.

Surprisingly all these properties coincide for positive semigroups on C*-algebras with unit.

<u>Theorem</u> 1.7. Let M be a C*-algebra with unit and $(T(t))_{t \geq 0}$ a positive semigroup on M . Then the following assertion are equivalent.

1.   s(A) < 0 .

2.   The semigroup $(T(t))_{t \geq 0}$ is uniformly exponentially stable.

3.   The semigroup $(T(t))_{t \geq 0}$ is uniformly stable.

4.   The semigroup $(T(t))_{t \geq 0}$ is weakly stable.

<u>Proof</u>. Since 's(A) = ω' by Theorem 1.3, it suffices to show that 4. implies 1. . For t > 0 there exists $\phi \in S(M)$ such that

$$T(t)'\phi = r(T(t))\phi .$$

Then for x ∈ M

$$\phi(T(t)^n x) = (r(T(t)))^n \phi(x) \to 0$$

as n → ∞ . Therefore r(T(t)) < 1 or ω < 0 . Since s(A) ≤ ω the assertion follows.

□

Remark 1.8. If we consider the translation semigroup $(T(t))_{t \geq 0}$ on $C_0(\mathbb{R}_+)$, then $\|T(t)\| = 1$, hence $s(A) = 1$, but $(T(t))_{t \geq 0}$ is uniformly stable. The same holds for the translation semigroup on $L^1(\mathbb{R}_+)$. Thus Theorem 1.7 is not true for semigroups on C^*-algebras with unit or on preduals of W^*-algebras. For the discussion of the commutative situation we refer to B-IV, Section 1.

2. STABILITY OF IMPLEMENTED SEMIGROUPS

Let H be a Hilbert space, $\mathcal{U} = (U(t))_{t \geq 0}$ a strongly continuous semigroup on H with generator B and $M \subseteq B(H)$ be a W^*-algebra, where B(H) is the W^*-algebra of all bounded linear operators on H . Suppose $U(t)MU(t)^* \subseteq M$. Then one can define a weak*-continuous semigroup $T = (T(t))_{t \geq 0}$ on M by $T(t)x := U(t)xU(t)^*$ $(t \in \mathbb{R}_+, x \in M)$. We call T an __implemented semigroup__. Every map T(t) of an implemented semigroup is weak*-continuous and n-positive for every $n \in \mathbb{N}$.

Remarks 2.1. (a) Because of

$$\|T(t)\| = \|T(t)1\| = \|U(t)U(t)^*\| = \|U(t)\|^2$$

it follows that $\omega(T) = 2\omega(\mathcal{U})$.

(b) If $(T(t))_{t \geq 0}$ is an implemented semigroup, then the preadjoint semigroup is strongly continuous on M_* . Therefore $s(A) = \omega$ for $(T(t))_{t \geq 0}$ by Theorem 1.3.

(c) Since $(U(t))_{t \geq 0}$ is a (strongly continuous) semigroup the same is true for the adjoint semigroup $(U(t)^*)_{t \geq 0}$ and its generator is given by B^* . In analogy to [Bratteli-Robinson (1979), 3.2.55] the following assertions for $x \in M$ are equivalent:

(i) $x \in D(A)$.

(ii) For $\xi \in D(B)$ it follows $x\xi \in D(B^*)$ and the linear mapping

(*) $(\xi \to x(B\xi)+B^*(x\xi)): D(B) \to H$

has a continuous extension to H .

Then   Ax   is given as the continuous extension of   (*) . We shortly
write $Ax = xB + B^*x$.

In the next theorem we give some equivalent conditions for the uni-
form exponential stability of an implemented semigroup . As we shall
see, the operator equality

$$yB + B^*y = -x   (x,y \in M_+)$$

is necessary and sufficient, which is in complete analogy to the
classical Liapunov stability result.

Theorem 2.2.   Let   M   be a W*-algebra on a Hilbert space   H   and let
$T = (T(t))_{t \geq 0}$ be a weak*-semigroup on   M   with generator   A   imple-
mented by the semigroup $(U(t))_{t \geq 0}$ on   H   with generator   B . Then
the following assertions are equivalent.

(a) $\omega(T) = s(A) < 0$.

(b)   The semigroup $(U(t))_{t \geq 0}$ is uniformly exponentially stable.

(c)   There exists $0 \leq x \in D(A)$ such that $Ax = -1$.

(d)   There exists $0 \leq x \in D(A)$ such that $x(D(B)) \subseteq D(B^*)$ and
      $xB+B^*x = -1$.

(e)   For every $0 \leq x \in D(A)$ there   exists $0 \leq y \in D(A)$ such that $Ay = -x$.

(f)   For every $0 \leq x \in D(A)$ there exists $0 \leq y \in D(A)$ such that
      $y(D(B)) \subseteq D(B^*)$ and $yB+B^*y = -x$.

(g) $\int_0^\infty \|U(s)\xi\|^2 ds$ exists for all $\xi \in H$.

(h) $\int_0^\infty ((T(s)x)\xi | \zeta) ds$ exists for all $\xi, \zeta \in H$ and all $x \in M$.

Proof. The equivalence of (a) and (b) follows from Remark 2.1.(a)
whereas (c) and (d), resp., (e) and (f) are equivalent by the Remark
2.1.(c).

(a) $\to$ (c):   Since $s(A) < 0$ the resolvent $R(0,A)$ exists and is a
positive map on   M . Therefore $R(0,A)1 \in D(A)_+$ or $Ax = -1$ for some
$x \in D(A)_+$.

(c) → (e): Let $x \in D(A)_+$ such that $Ax = -1$. Then

$$T(t)x - x = \int_o^t T(s)Axds = -\int_o^t T(s)1ds \quad (t \geq 0) \ ,$$

hence

$$0 \leq \int_o^t T(s)1ds \leq x \quad (t \in R_+) \ .$$

Since the family $(\int_o^t T(s)1ds)_{t \geq 0}$ is increasing and bounded,

$$\lim_{t \to \infty} \int_o^t T(s)1ds$$

exists in the weak operator topology on $B(H)$. Since on bounded sets of M the weak operator topology is equivalent to the $\sigma(M,M_*)$-topology, [Sakai (1971), 1.15.2.], for every $\phi \in M_*$ the integral $\int_o^\infty \phi(T(s)1)ds$ exists. Take $x \in M_+$ and $\phi \in M_*^+$. Then $x \leq \|x\|1$ and therefore

$$\phi(T(s)x) \leq \|x\|\phi(T(s)1) \quad (s \in R_+).$$

Hence $\int_o^\infty \phi(T(s)x)ds$ exists. Since the positive cones of M and M_* are generating, $\int_o^\infty \phi(T(s)x)ds$ exists for every $x \in M$ and $\phi \in M_*$. Therefore $R(0,A)$ exists and is positive which proves (e).

(c) → (g) From the last paragraph we obtain that for all $\xi \in H$

$$\int_o^\infty \|U(s)\|^2 ds = \int_o^\infty (T(s)1\xi|\xi)ds$$

exists.

(g) → (h): It follows from the polarization identity that the integral

$$\int_o^\infty (U(s)\xi|U(s)\zeta)ds$$

exists for all $\xi,\zeta \in H$. Using [Takesaki (1979), Theorem III.4.2 and Theorem II.2.6] we conclude as in the implication from (c) to (e) that for all $\xi,\zeta \in H$ the integral

$$\int_o^\infty (((T(s)x)\xi|\zeta)ds \quad (x \in M)$$

is finite.

(g) → (a): Since the vector states are dense in the predual of M
([Takesaki (1979), Theorem II.2.6]) and since the preadjoint semi-
group of T is strongly continuous, it is easy to see that the inte-
gral

$$\int_0^\infty \phi(T(s)x)\,ds$$

exists for all x∈M and φ∈M$_*$. Therefore, the resolvent R(0,A)
exists and is positive, hence s(A) < 0 .

<div align="right">□</div>

3. CONVERGENCE OF POSITIVE SEMIGROUPS

In this section the asymptotic behavior of positive semigroups
$(T(t))_{t\geq0}$ will be described in more detail. Essentially we distin-
guish three cases:

1. The Cesàro means $\frac{1}{s}\int_0^s T(t)dt$ converge strongly to a projection
P onto the fixed space of $(T(t))_{t\geq0}$ (see Proposition 3.3 and 3.4)

2. The maps T(t) converge strongly to P (see Proposition 3.7,
3.8 and 3.9).

3. The maps T(t) behave asymptotically as a periodic group (Theo-
rem 3.11).

Much of the following is based on the theory of weakly compact oper-
ator semigroups. Therefore the following compactness criterium is
quite useful.

Proposition 3.1. Let M be a W*-algebra, T a bounded semigroup
of positive maps on M$_*$ and suppose that there exists a faithful
family Φ of T-subinvariant states in M$_*$. Then T is relatively
compact in the weak operator topology of $L(M_*)$. In particular, T
is strongly ergodic, i.e. $\lim_{s\to\infty} \frac{1}{s}\int_0^s T(t)x\,dt$ exists for every x
in M and yields a projection onto Fix(T) .

Proof. Since the positive cone of M$_*$ is generating, it is enough to
show that for every 0≤ψ∈M$_*$ the orbit {T(t)ψ : t∈R$_+$} is relatively
weak compact. For this we use [Takesaki(1979),Theorem III.5.4.(iii)].

Let $(p_n)_{n \in \mathbb{N}}$ be a decreasing sequence of projections in M such that $\inf_n p_n = 0$. Then $\lim_n \phi(p_n) = 0$ for every $\phi \in \Phi$. Since

$$(T(t)p_n)^2 \leq T(t)p_n \ , \ t \in \mathbb{R}_+ \ ,$$

we obtain by a classical inequality of Kadison that

$$0 \leq \phi((T(t)p_n)^2) \leq \phi(T(t)p_n) \leq \phi(p_n) \ ,$$

hence $\lim_n \phi(T(t)p_n) = 0$ uniformly in $t \in \mathbb{R}_+$. Since the family Φ is faithful on M, it follows from [Takesaki (1979), Proposition III.5.3] that $(T(t)p_n)$ converges to zero in the $s(M,M_*)$-topology uniformly in $t \in \mathbb{R}_+$. Since this topology is finer than the weak*-topology on M we obtain the relative compactness of T which implies the strong ergodicity.

□

Let T be an identity preserving semigroup of Schwarz type on the predual of a W*-algebra M. We call

$$p_r := \sup\{s(|\phi|) \ : \ \phi \in \text{Fix}(T)\}$$

the __recurrent projection__ associated with T. For a motivation of this definition compare, e.g., [Davies (1976), Section 6.3].

Since $T(t)|\phi| = |\phi|$ for all $\phi \in \text{Fix}(T)$ (D-III, Cor. 1.5) we obtain $T(t)'p_r \geq p_r$ (see D-I, Sec.3.(c)). Let $T^{(r)}$ be the reduced semigroup on $p_r M_* p_r$ with generator $A^{(r)}$. Then $T^{(r)}$ is identity preserving and of Schwarz type. Similarly, if R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0\}$ with values in M_* such that R is identity preserving and of Schwarz type, then the recurrent projecton associated with R is defined using $\text{Fix}(R)$.

__Remark 3.2.__ (a) Let $\phi \in M_*$ and $\alpha \in \mathbb{R}$ such that

$$(\mu - i\alpha)R(\mu)\phi = \phi \quad \text{for some} \quad \mu \in \mathbb{R}_+ \ .$$

Since $s(|\phi|)$ and $s(|\phi^*|)$ are majorized by p_r (D-III,Prop.1.4) it follows that ϕ and ϕ^* are in $p_r M_* p_r$.

(b) From (a) and the observation that the family $\{|\phi| : \phi \in \text{Fix}(T)\}$ is

faithful on $p_r M p_r$ and consists of $T^{(r)}$-invariant elements, it follows that:

(i) $P\sigma(A) \cap i\mathbb{R} = P\sigma(A^{(r)}) \cap i\mathbb{R}$.

(ii) $\ker(i\alpha - A) \subseteq p_r M_* p_r$ for all $\alpha \in \mathbb{R}$.

(iii) The semigroup $T^{(r)}$ is relatively compact in
       the weak operator topology and therefore strongly ergodic.

(c) Similarly, let R be an identity preserving pseudo-resolvent with values in M_* on $D = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0\}$ which is of Schwarz type. It follows as in (b) that $\text{Fix}((\lambda - i\alpha)R(\lambda))$ is contained in $p_r M_* p_r$ for all $\lambda \in D$ and $\alpha \in \mathbb{R}$, where p_r is the associated recurrent projection.

We now give a characterization of strong ergodicity of semigroups which are identity preserving and of Schwarz type. For this we need that the Cesàro means $C(s)$, where

$$C(s)x = \frac{1}{s} \int_o^s T(t)x \, dt \qquad (x \in M , \ 0 < s \in \mathbb{R})$$

are Schwarz maps. We omit the simple calculation (compare D-I,Thm.2.1).

Proposition 3.3. Let T be an identity preserving semigroup of Schwarz type on the predual of a W*-algbra M . Then the following assertions are equivalent:

(a) T is strongly ergodic on M_* .

(b) $\sigma(M,M_*)-\lim_{s \to \infty} C(s)'p_r = 1$.

(c) $s^*(M,M_*)-\lim_{s \to \infty} C(s)'p_r = 1$.

Proof.   Suppose that (a) holds. Since $\text{Fix}(T)$ separates $\text{Fix}(T')$ (see [Krengel (1985), Chap.2,Thm.1.4]), the fixed space of T' is non trivial, hence $p_r \neq 0$. Let $0 \leq \psi \in M_*$, then

$$\psi_o := \lim_{s \to \infty} C(s)\psi \in \text{Fix}(T)$$

and $s(\psi_o) \leq p_r$.

Therefore

$$\lim_{s \to \infty} \psi(C(s)'p_r) = \lim_{s \to \infty} (C(s)\psi)(p_r) =$$

$$= \psi_0(p_r) = \psi_0(1) = \lim_{s \to \infty} (C(s)\psi)(1) = \psi(1) \ ,$$

which proves (b).

Suppose that (b) is satisfied. Since $C(s)'p_r \leq 1$ for all $s \in \mathbb{R}_+$ we obtain (c). (Use that for $(x_\alpha) \in M_+$ we have $\lim_\alpha x_\alpha = 0$ in the weak*-topology if and only if $\lim_\alpha x_\alpha = 0$ in the $s^*(M,M_*)$-topology.)

Suppose that (c) holds. Since each $C(s)'$ is an identity preserving Schwarz map we obtain for all $x \in M$:

$$(C(s)'((1-p_r)x))(C(s)'((1-p_r)x)^*) \leq$$

$$\leq C(s)'((1-p_r)xx^*(1-p_r)) \leq$$

$$\leq \|x\|^2 C(s)'(1-p_r) \ ,$$

hence

$$s^*(M,M_*)-\lim_{s \to \infty} C(s)'((1-p_r)x) = 0 \ .$$

In particular we obtain for all $x \in \text{Fix}(T')$ that

$$x = \sigma(M,M_*)-\lim_{s \to \infty} C(s)'x = \sigma(M,M_*)-\lim_{s \to \infty} C(s)'(p_r x) \ .$$

Especially for $0 \neq x \in \text{Fix}(T)$ we obtain $p_r x p_r \neq 0$. Since the W*-algebra $p_r M p_r$ is the dual of $p_r M_* p_r$ and since $T^{(r)}$ is strongly ergodic, it follows that the fixed space of T separates the points of $\text{Fix}(T')$. Thus T is strongly ergodic ([Krengel (1985), Chap. 2, Thm. 1.4]).

<div align="right">□</div>

It follows from the result above that the semigroup in [Evans (1977)] cannot be strongly ergodic on $B(H)_*$ since the associated recurrent projection is zero. But for irreducible semigroups we have the following result.

<u>Proposition</u> 3.4. Let T be an identity preserving semigroup of Schwarz type on the predual of a W*-algebra M . Then the following assertions are equivalent.

(a) T is irreducible and $P\sigma(A) \cap i\mathbb{R} \neq \emptyset$.

(b) T is relatively compact in the weak operator topology and the fixed space of T is generated by a faithful state.

(c) T is strongly ergodic and the fixed space of T is generated by a faithul state.

(d)   The fixed space of T is generated by a faithful state.

<u>Proof</u>. Suppose (a) is satisfied. Since $\text{Fix}(T) \neq \{0\}$ there exists a faithful normal state ϕ on M such that $\text{Fix}(T) = \phi\mathbb{C}$ (D-III, Thm.1.10.). Therefore T is relatively compact in the weak operator topology by Proposition 3.1., whence (b) holds.

The implications from (b) to (c) and (c) to (d) are trivial.

Suppose that (d) holds. Let ϕ be a faithful normal state on M such that $\text{Fix}(T) = \phi\mathbb{C}$. By Proposition 3.1 the semigroup T is strongly ergodic. Therefore the fixed space of T separates the points of $\text{Fix}(T')$. Consequently $\text{Fix}(T') = \mathbb{C}1$. Thus the ergodic projection associated with T is given by $P = 1 \otimes \phi$, i.e. $P\psi = \psi(1)\phi$ for all $\psi \in M_*$. Let F be a closed T-invariant face of M_*^+ . If $0 \neq \psi \in F$ then

$$\lim_{s\to\infty} C(s)\psi = \psi(1)\phi \in F .$$

Hence $\phi \in F$ and therefore $F = M_*^+$ by the faithfulness of ϕ which proves (a).

□

The next theorem is an extension of D-III,Thm.1.10 and shows the usefulness of the theory of semitopological semigroups. Assume $T \subseteq L(M_*)$ to be relatively compact in the weak operator topology. Since T is commutative its closure $S = (T)^- \subseteq L_w(M_*)$ contains a unique minimal ideal K , called the kernel of S , which is a compact Abelian group ([DeLeeuw-Glicksberg (1961); Junghenn (1971); Krengel (1985), § 2.4]. The identity Q of K is a projection onto

the closed linear span of all eigenvalues of A pertaining to the
eigenvalues in i$\mathbb{R}$. Moreover, the dual group of K can be identi-
fied with the subgroup of i$\mathbb{R}$ generated by $P_\sigma(A) \cap$ i$\mathbb{R}$. We call Q
the semigroup projection associated with T . On the other hand, T
is always strongly ergodic with projection P onto Fix(T) . Obvi-
ously, the relation

$$0 \leqslant P \precsim Q \leqslant \text{Id}$$

holds, where the order relation is defined by the inclusion of the
range spaces.

There are two extreme cases: First $Q = \text{Id}$ and $\text{rank}(P) = 1$. This
corresponds to the Halmos-von Neumann Theorem in commutative ergodic
theory and is discussed, at least for irreducible semigroups, in
[Olesen-Pedersen-Takesaki (1980)]. Second, $\text{Id} > Q = P$, in parti-
cular $\text{rank}(P) = 1$. This latter case will be investigated in detail
for $M = B(H)$, the W*-algebra of all bounded linear operators on a
Hilbert space H . But we first need some preparations.

Theorem 3.5. Let T be an identity preserving semigroup of Schwarz
type on the predual of a W*-algbra M and suppose there exists a
faithful family of T-invariant states on M . Let N be the
$\sigma(M, M_*)$-closed linear span of all eigenvectors of A' pertaining to
the eigenvalues in i$\mathbb{R}$. If Q is the semigroup projection asso-
ciated with T the following holds:

(a) The adjoint of Q is a faithful normal conditional expectation
from M onto the W*-subalgebra N .

(b) The restriction of T' to N can be embedded into a $\sigma(M, M_*)$-
continuous, one-parameter group of *-automorphisms.

(c) If, in addition, T is irreducible and if ϕ is the normal
state generating the fixed space of T , then $\phi_{|N}$ is a faithful
normal trace.

Proof.   Consider $H := P_\sigma(A) \cap$ i$\mathbb{R}$ which is not empty by assump-
tions. From Proposition 3.1 it follows that T is relatively compact
in the weak operator topology. Let K be the semigroup kernel of
$\overline{T} \subseteq L_w(M_*)$ and Q the unit of K . Recall that $Q\psi_\eta = \psi_\eta$ for all
$\psi_\eta \in M_*$ such that $A\psi_\eta = \eta\psi_\eta$ ($\eta \in H$). Let U be the family of all

eigenvectors of A' pertaining to the eigenvalues in H . Then U is closed with respect to the multiplication in M and the formation of adjoints. Thus N is a W*-subalgebra of M [Sakai (1971), Corollary 1.7.9.] and $T_0(t)' := T(t)'|_N$ is multiplicative (for this see D-III, Lemma 1.1).

Since $Q \in T^- \subseteq L(_w(M_*)$ there exists an ultrafilter U on $\mathbb{R}_+$ such that $\lim_U <T(t)\psi,x> = <Q\psi,x>$ for all $x \in M$ and $\psi \in M_*$. If $\eta \in H$ and $\psi_\eta \in M_*$ such that $A\psi_\eta = \eta\psi_\eta$, then for all $x \in M$:

$$<\psi_\eta,x> = <Q\psi_\eta,x> = \lim_U <T(t)\psi_\eta,x> = (\lim_U e^{\eta t})<\psi_\eta,x> ,$$

hence $\lim_U e^{\eta t} = 1$. From this it follows that for all $\psi \in M_*$ we have

$$<\psi,Q'(u_\eta)> = \lim_U <\psi,T(t)'u_\eta> =$$

$$= (\lim_U e^{\eta t})<\psi,u_\eta> = <\psi,u_\eta> .$$

Hence $N \subseteq Q'(M)$.

For γ in the dual group of K and $x \in M$ we define x_γ by

$$\psi(x_\gamma) := \int_K <S\psi,x><S,\gamma>^* dm(S) \qquad (\psi \in M_*^+) .$$

Then $x_\gamma \in M$ and $T(t)'x_\gamma = <QT(t),\gamma>x_\gamma$. Therefore $x_\gamma \in N$. Thus the inclusion $Q'M \subseteq N$ is proved if we can show that $Q'M$ belongs to the $\sigma(M,M_*)$-closed linear span of $\{x_\gamma : \gamma \in K , x \in M\}$. For this it is enough to show that every linear form $\psi \in M_*$ such that $\psi(x_\gamma) = 0$ for all $\gamma \in K$ satisfies $\psi(Qx) = 0$ for all $x \in M$. But if $\psi(x_\gamma) = 0$ then

$$\int_K <S\psi,x><S,\gamma>^* dm(S) = 0 , \quad \gamma \in K .$$

Since the map $(S \to \psi(Sx))$ is continuous on K and since the elements of K form a complete orthonormal basis in $L^2(K,dm)$, we obtain $\psi(Sx) = 0$ for all $S \in K$, in particular $\psi(Qx) = 0$ as desired.

Since the range of Q' is a W*-subalgebra of M it follows from [Takesaki (1979), Theorem III.3.4] that Q' is a completely positive, normal conditional expectation. Q' is faithful, i.e. $\ker(Q') \cap M_+ = \{0\}$ since $Q\phi = \phi$ for the faithful linear form ϕ .

Let ϕ be the faithful normal state generating Fix(T) and let U be a family of unitary eigenvectors of A' pertaining to the eigenvalues in H (see D-III, Remark 1.11). If u_1, $u_2 \in U$ then

$$\phi(u_1 u_2{}^*) = \phi(T_0(t)'(u_1 u_2{}^*)) = e^{(\eta_1 - \eta_2)t} \phi(u_1 u_2{}^*) .$$

Therefore

$$\phi(u_1 u_2{}^*) = \begin{cases} 0 & \text{if } \eta_1 \neq \eta_2 , \\[2mm] 1 & \text{if } \eta_1 = \eta_2 . \end{cases}$$

Hence $\phi(u_1 u_2{}^*) = \phi(u_2{}^* u_1)$ from which it follows that $\tau := \phi|_N$ is a faithful normal trace.

$\square$

Remarks 3.6. (a) Since $QM_* = N_*$ and $Q'M = N$, where N_* is as in D-III, Proposition 1.12, it follows from general duality theory that $(N_*)' = N$.

(b) If $\psi \in N_*$ then $|\psi| \in N_*$. To see this note that $Q\psi = \psi$ and Q is an identity preserving Schwarz map. Then the assertion follows from D-III, Proposition 1.4 .

(c) If $\psi \in N_*$, then $|T_0(t)\psi| = T_0(t)|\psi|$ for all $t \in \mathbb{R}$. This follows immediately from the fact that $T_0(t)'$ is a *-automorphism on N .

(d) Let us add a few words concerning the structure of N: If T is irreducible and K is the semigroup kernel of $T^- \subseteq L_w(M_*)$, then

$$(S \to S') : K \to L((N, \sigma(N, N_*))$$

is a representation of the compact, Abelian group K as group of *-automorphism such that the fixed space is one dimensional. Therefore we are able to apply the results of [Olesen-Pedersen-Takesaki (1980)]. There are three possibilities for N :

(i) $N \cong L^\infty(K, dm)$ and $T|_N$ is the translation group on N .

(ii) $N \cong R$ where R is the (unique) hyperfinite factor of typ II_1. In that case (the image of) K is approximately inner on R [l.c., Theorem 5.8].

(iii)   There exists a closed subgroup G of K such that

$$N \cong L^{\infty}(^K/_G, dm_/) \otimes R \ ,$$

where R is as in (ii) and $dm_/$ the normalized Haar measure on $^K/_G$ [l.c., Theorem 5.15].

So far we have studied weak*-semigroups on general W*-algebras. We now want to apply the results of this section to weak*-semigroup on B(H) . This is of interest in view of the results in [Davies (1976)]. To do this we call a triple (M, Φ, T) a W*-<u>dynamical system</u> if M is a W*-algebra, a weak*-semigroup of identity preserving Schwarz maps on M and Φ a faithful family of T-invariant normal states. We call (M, Φ, T) <u>irreducible</u>, if the preadjoint semigroup is irreducible (alternatively, if the fixed space of T is one dimensional).

<u>Proposition</u> 3.7. Let $(B(H), \Phi, T)$ be a W*-dynamical system on the W*-algebra B(H) of all bounded linear operators on a Hilbert space H . Then the following assertions are equivalent:

(a) $P\sigma(A) \cap i\mathbb{R} = \{0\}$.

(b) $\lim_{s \to \infty} T(s)_* = P_*$ in the strong operator topology on $L(B(H)_*)$.

<u>Proof</u>. Obviously (b) implies (a). Suppose that (a) is fulfilled. Then the ergodic projection P_* of the preadjoint semigroup is equal to the associated semigroup projection. Consequently there exists an ultrafilter U on $\mathbb{R}_+$ such that $\lim_U T(t) = P$ in the weak operator topology. We claim that the convergence holds even in the strong operator topology. Taking this for granted it follows, since for every $t \in \mathbb{R}_+$ T(t) is a contraction, that

$$\lim_{t \to \infty} \|T(t)_* \phi\| = 0$$

for all $\phi \in \ker(P_*)$. Since $T(t)_* \psi = \psi$ for every $\psi \in im(P_*)$ and because

$$B(H)_* = im(P_*) \oplus \ker(P_*)$$

the assertion is proved.

It remains to show that $\lim_U T(t)_* = P_*$ in the strong operator topology. Choose $0 \leq \phi \in B(H)_*$, $\|\phi\| \leq 1$, let $\phi_t := T(t)_* \phi$ (t>0) ,

$\phi_o := P_* \phi$ and let $\{p_i : i \in \Lambda\}$ be an increasing net of projections of finite rank in $B(H)$ with strong limit 1. Since the set $K := \{\phi_t : t \geq 0\}$ is relatively compact in the $\sigma(B(H)_*, B(H))$-topology, there exists for every $\delta > 0$ an index $i_o \in \Lambda$ such that

$$\| (1 - p_i) \psi (1 - p_i) \| \leq \delta$$

for every $\psi \in K$ and $i \geq i_o$ [Takesaki (1979), Theorem III.5.4.(vi)]. In particular

$$| \psi (1 - p_i) | \leq \delta \quad , \quad \psi \in K, \ i(o) \leq i \ .$$

Let $p := p(i(o))$. Then for all x in the unit ball of M it follows that

$$| (\phi_t - \phi_o)(x) | \leq$$

$$\leq | (\phi_t - \phi_o)(pxp) | + | (\phi_t - \phi_o)((1-p)xp) | +$$

$$+ | (\phi_t - \phi_o)(x(1-p)) | \leq$$

$$\leq | (\phi_t - \phi_o)(pxp) | + 4\sqrt{\delta} \ .$$

Since the W*-algebra $pB(H)p$ is finite dimensional, there exists $U \in \mathcal{U}$ such that

$$\| (\phi_t - \phi_o)|_{pB(H)p} \| \leq \delta$$

for all $t \in U$. Consequently

$$\| (\phi_t - \phi_o) \| \leq (\delta + 4\sqrt{\delta})$$

for all $t \in U$. Therefore $\lim_{\mathcal{U}} T(t)_* \phi = P_* \phi$ in the strong operator topology. Since the positive cone of $B(H)_*$ is generating, the assertion is proved.

$\square$

For irreducible W*-dynamical systems on $B(H)$ the above properties always hold.

Theorem 3.8. Let $(B(H), \Phi, T)$ be an irreducible W*-dynamical system. Then

$$P\sigma(A) \cap i\mathbb{R} = \{0\} \ .$$

Proof. Let N be the W*-subalgebra of M = B(H) generated by the
eigenvectors of A pertaining to the eigenvalues on $i\mathbb{R}$ and let Q
be the faithful normal conditional expectation from M onto N
(Theorem 3.7.). Since M is atomic, N is atomic [Størmer (1972)].
N is finite since there exists a finite, faithful normal trace on
N . In particular the center of N is isomorphic to ℓ^{∞} . Let S be
the restriction of T to the center. Then S is a weak*-semigroup
such that every $S(t) \in S$ is $\sigma(\ell^{\infty}, \ell^{1})$-continuous and a *-automorphism.
From this it follows that S(t) is induced by some continuous flow
κ_t: N → N . Indeed, if $\delta_n((\xi_m)) = \xi_n$ $(n \in \mathbb{N}$, $(\xi_m) \in \ell^{\infty})$, then $\delta_n \circ S(t)$
is a normal scalar valued *-homomorphism hence of the form δ_m for
some $m = \kappa_t(n)$. But the function $(t \to \kappa_t)$ is continuous from $\mathbb{R}$
into N , whence constant. Hence S(t) = Id . But the semigroup S is
weak*-irreducible on the center. Consequently the center is one di-
mensional. Using [Takesaki, Theorem V.1.27] we obtain $N = B(H_n)$
where H_n is a finite dimensional Hilbert space. But if $0 \neq i\alpha$
$\in P\sigma(A) \cap i\mathbb{R}$ then $i\alpha\mathbb{Z} \subseteq P\sigma(A)$ by D-III,Thm.1.10, whence N must be
infinite dimensional. Therefore $P\sigma(A) \cap i\mathbb{R} = \{0\}$ as desired.

□

An immediate and interesting consequence of Theorem 3.8 and Proposi-
tion 3.7 is the following.

Corollary 3.9. If $(B(H), \phi, T)$ is an irreducible W*-dynamical system,
then

$$\lim_{s \to \infty} T(s) = 1 \otimes \phi$$

in the strong operator topology on $L(B(H)_*)$, where ϕ is the
unique normal state generating the fixed space of T_* .

We are now going to discuss the asymptotic behavior of positive semi-
groups whose generator has boundary point spectrum different from
0 . The standard example is the following:

If Γ is the unit circle, m the normalized Haar measure on Γ and
$0 < \tau \in \mathbb{R}$, then we define the maps $R_\tau(t)$, $t \in \mathbb{R}_+$, on $L^1(\Gamma, m)$ by

$$(R_\tau(t)f)(\xi) = f(\xi \exp(\frac{2\pi i}{\tau} t))   (f \in L^1(\Gamma, m), \xi \in \Gamma) .$$

Then $R := (R_\tau(t))_{t \geq 0}$ forms a strongly continuous one parameter
semigroup which is identity preserving and of Schwarz type. Since R

is periodic of period τ it follows that 0 is a pole of the resolvent of its generator B with residuum $P = 1 \otimes 1$ and $\{\frac{2\tau i}{\tau} k : k \in \mathbb{Z}\}$ $= \sigma(B)$. Thus R is irreducible and uniformly ergodic on $L^1(\Gamma, m)$ (see A-II, Section 5).

Now let T be a semigroup on M_* . It is called <u>partially periodic</u>, if there exists a projection $Q \in L(M_*)$ reducing T such that $Q(M_*)$ $\cong L^1(\Gamma, m)$ and $T_{|im(Q)}$ is conjugate to a periodic semigroup on $L^1(\Gamma, m)$. In the main result we present a non commutative version of [Nagel (1984)] showing that certain dynamical systems converge to partially periodic semigroups.

<u>Proposition</u> 3.10. Let T be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a W*-algebra M . If T is uniformly ergodic, then $\sigma(A) \cap i\mathbb{R} =$ $P\sigma(A) \cap i\mathbb{R} = i\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$. If additionally $\sigma(A) \cap i\mathbb{R} \neq \{0\}$, there exists a strictly positive projection Q on M_* which is identity preserving and completely positive such that:

(a) Q reduces T and $Q(M_*) \cong L^1(\Gamma)$, Γ being the one dimensional torus.

(b) The restriction T_0 of T to $im(Q)$ is irreducible and conjugate to a rotation semigroup of period $\tau = \frac{2\pi}{\alpha}$ on Γ .

(c) The spectral bound $s(A_{|ker(Q)})$ is strictly smaller than 0 .

<u>Proof.</u> By D-III,Thm.1.11 and D-III,Thm.2.5 it follows that

$$\sigma(A) \cap i\mathbb{R} = P\sigma(A) \cap i\mathbb{R} = i\alpha\mathbb{Z}$$

for some $\alpha \in \mathbb{R}$. Suppose $\alpha \neq 0$. Since $\sigma(A) + i\alpha\mathbb{Z} = \sigma(A)$ and since every $\eta \in i\alpha\mathbb{Z}$ is isolated, it follows that there exists $\delta > 0$ such that

$$\sigma(A) \setminus i\alpha\mathbb{Z} \subseteq \{\lambda \in \mathbb{C} : Re(\lambda) \leq \delta\} .$$

Let $\{u_\alpha^k : k \in \mathbb{Z}\}$ be a family of unitary eigenvectors of A' pertaining to the eigenvalues in $i\mathbb{R}$. Then $Q'(M)$ is a commutative W*-algebra. Let $\tau := \frac{2\pi}{\alpha}$. Then $T(\tau)'u_\alpha^k = u_\alpha^k$, hence $T_{|im(Q)}$ is periodic. From the Halmos-von Neumann theorem (see [Schaefer (1974),

Thm. III.7.11]) it follows that $T_1|_{im(Q)}$ is conjugate to the rotation semigroup of period τ on $L^1(\Gamma,m)$.

□

Using this proposition we obtain

Theorem 3.11. Let $T = (T(t))_{t \geq 0}$ be a uniformly ergodic, identity preserving semigroup of Schwarz type on the predual of a W*-algebra M and suppose $\sigma(A) \cap i\mathbb{R} \neq \{0\}$. Then there exists a partially periodic, identity preserving semigroup $S = (S(t))_{t \geq 0}$ of Schwarz type on M_* such that

$$\lim_{t \to \infty} (T(t) - S(t)) = 0$$

in the strong operator topology.

Proof. Let ϕ be the normal state on M generating the fixed space of T . Let $S = (S(t))_{t \geq 0}$ where $S(t) := T(t) \circ Q$ and Q is as in 2.6. Obviously, S is partially periodic and $\phi \in Fix(S)$. Let H_ϕ be the GNS-Hilbert space pertaining to ϕ . Since ϕ is fixed under T , S and Q these objects have a canonical extension to H_ϕ (in the following denoted by the same symbols). If $H_O := ker(Q) \subseteq H_\phi$ then it is easy to see that H_O is invariant under the extension to H_ϕ of the multiplication maps we defined in D-III, Remark 1.3. Consequently, using the results in Groh-Kümmerer (1982) it follows that there exists $c \in \mathbb{R}$ such that for all γ near 0 and all $\beta \in \mathbb{R}$:

$$\| R(\gamma + i\beta, A_O) \| \leq c \qquad (*)$$

where $A_O := A|_{ker(Q)}$ (the norm taken in $L(H_\phi)$). Using the result in A-III,Cor.7.11 it follows that

$$\lim_{t \to \infty} \|T(t)|H_O\| = 0 .$$

Since the $s(M,M_*)$-topology on the unit ball of M is nothing else than the restriction of the norm topology on H_ϕ , we obtain

$$s(M,M_*)-\lim_{t \to \infty} (T(t)' - S(t)')(x) = 0$$

uniformly on M_1 . From this the assertion follows.

□

4. UNIFORM ERGODIC THEOREMS

As we have seen, uniformly ergodic semigroups have nice spectral properties. In this section we study sufficient conditions which imply uniform ergodicity thereby generalizing results the results of Groh (1984b). We first need some preparations.

Lemma 4.1. Let R be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0\}$ with values in the predual of a W*-algebra M. If the fixed space of R is infinite dimensional, then there exists a sequence of states in $\text{Fix}(R)$ such that the corresponding support projections are mutually orthogonal in M.

Proof. Let $\Phi = \{\phi \in \text{Fix}(R) : \phi \text{ state on } M\}$ and let $p = \sup\{s(\phi) : \phi \in \Phi\}$. Since $\lambda R(\lambda)\phi = \phi$ for all $\phi \in \Phi$ and $\lambda \in D$ it follows $\mu R(\mu)(1-s(\phi)) \leq (1-s(\phi))$. Hence $\mu R(\mu)(1-p) \leq (1-p)$ for all $\mu \in \mathbb{R}_+$. Let $R_|$ be the induced pseudo-resolvent on pM_*p (D-I, Section 3.(c)). Then the family Φ is faithful on M_p and contained in the fixed space of $R_|$. The adjoint $\mu R_|(\mu)'$ is an identity preserving Schwarz map. Consequently it follows from D-III, Lemma 1.1.(b) and the $\sigma(M_p, (M_p)_*)$-continuity of $\mu R_|(\mu)'$, that $\text{Fix}(R_|')$ is a W*-subalgebra of M_p and by D-III, Lemma 1.5 it follows that $\dim \text{Fix}(R) \leq \dim \text{Fix}(R_|')$. If $\text{Fix}(R)$ is infinite dimensional, let (p_n) be a sequence of mutually orthogonal projections in $\text{Fix}(R_|') \subseteq M_p$ and choose a sequence $\{\phi_n\}$ in Φ such that $\phi_n(p_n) \neq 0$. For $n \in \mathbb{N}$ let ψ_n be the normal state

$$\psi_n(x) = \phi_n(p_n)^{-1}\phi_n(p_n x p_n)$$

on M. Because of $s(\psi_n) \leq p_n \leq p$, the support projections of the ψ_n's are mutually orthogonal in M. For $\mu \in \mathbb{R}_+$ and $x \in M$ we obtain:

$$\langle x, \mu R(\mu)\psi_n \rangle = \phi_n(p_n)^{-1}\langle \mu p_n(R(\mu)'x)p_n, \phi_n \rangle =$$

$$= \phi_n(p_n)^{-1}\langle \mu p_n p(R(\mu)p'x)p_n, \phi_n \rangle =$$

$$= \phi_n(p_n)^{-1}\langle \mu p_n(pR_|(\mu)'xp)p_n, \phi_n \rangle =$$

$$= \phi_n(p_n)^{-1}\langle \mu(p_n R_|(\mu)'xp_n), \phi_n \rangle =$$

$$= \phi_n(p_n)^{-1}\phi_n(x) = \psi_n(x).$$

Therefore $\psi_n \in \text{Fix}(R)$ for all $n \in \mathbb{N}$. □

<u>Remark</u> 4.2. (a) If dim Fix(R) $\geq$ 2 then it follows from the Jordan decomposition of self adjoint linear functionals, that there are at least two states in Fix(R) which have orthogonal support (compare the proof of D-III, Theorem 1.10.(a)).

(b) If R is a pseudo-resolvent with values in a W*-algebra such that Fix(R') is contained in M_* , then it follows from the proof of D-III, Lemma 1.2 that there exists a sequence of normal states in Fix(R') whith orthogonal supports in M .

<u>Lemma</u> 4.3.   Let R be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0\}$ with values in the predual of a W*-algebra M . If the fixed space of the canonical extension $\hat{R}$ of R to some ultrapower of M_* is infinite dimensional, then there exists a sequence (z_n) in M_1^+ and a sequence of states (ϕ_n) in M_* such that:

(a) $\lim_n z_n = 0$ in the s*(M,M_*)-topology.

(b) $\lim_n \|(\text{Id} - \lambda R(\lambda))\phi_n\| = 0$ for all $\lambda \in D$.

(c) $\phi_n(z_n) \geq \dfrac{1}{2}$ for all $n \in \mathbb{N}$.

<u>Proof</u>. Let $(M_*)^\wedge$ be the ultrapower of M_* with respect to some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Since $(M_*)^\wedge$ is the predual of a W*-subalgebra of $\hat{M}''$ (see D-III, Remark 2.4.(b)), there exists a sequence of states $(\hat{\psi}_n)$ in Fix($\hat{R}$) such that the corresponding support projections are mutually orthogonal in $\hat{M}''$ (Lemma 4.1). For every $n \in \mathbb{N}$ let $(\psi_{n,k}) \in \hat{\psi}_n$ be a representing sequence of states , let

$$\phi := \sum_{n,k} 2^{-(n+k+1)} \psi_{n,k}$$

and let

$$p := \sup\{s(\psi_{n,k}) : n,k=1,..\}$$

in M . Then ϕ is a normal state on M which is faithful on the W*-algebra M_p . Since

$$1 = \langle \psi_{n,k}, s(\psi_{n,k}) \rangle = \psi_{n,k}(p) \quad (n,k \in \mathbb{N})$$

it follows $\hat{\psi}_n(\hat{p}) = 1$ where $\hat{p}$ is the canonical image of p in $\hat{M}$.

But this implies $s(\hat{\psi}_n) \leq \hat{p}$ in $\hat{M}''$. Since $\hat{M}_1^+$ is $\sigma(\hat{M}'',\hat{M}')$-dense in $(\hat{M}'')_1^+$ (Kaplanskys density theorem [Sakai (1971), 1.9.1] in combination with [Sakai (1971), 1.8.9 and 1.8.12]), there exists for all $n \in \mathbb{N}$ a net $(\hat{z}_{n,\gamma})$ in $\hat{M}_1^+$ such that

$$\sigma(\hat{M}'',\hat{M}')-\lim_{\gamma} \hat{z}_{n,\gamma} = s(\hat{\psi}_n) \ .$$

From [Sakai (1971), 1.7.8] and the considerations above we obtain that the net $(\hat{p}\hat{z}_{n,\gamma}\hat{p})$ converges to $s(\hat{\psi}_n)$ in the $\sigma(\hat{M}'',\hat{M}')$-topology. Therefore we may assume $\hat{z}_{n,\gamma} \in (\hat{M_p})_1^+$. In the following we denote by $\hat{\phi}$ the canonical image of ϕ in $(M_*)\hat{\ }$.

Since the projections $s(\hat{\psi}_n)$ are mutually orthogonal, there exists a real sequence (r_n), $0 < r_n < 1$, $\lim_n r_n = 0$ and $\hat{\phi}(s(\hat{\psi}_n)) \leq \frac{1}{2} r_n$. For all $n \in \mathbb{N}$ choose $\hat{z}_n \in (\hat{M_p})_1^+$ such that

$$|<\hat{\phi},s(\hat{\psi}_n) - \hat{z}_n>| \leq \frac{1}{2} r_n \ ,$$

$$|<\hat{\psi}_n,s(\hat{\psi}_n) - \hat{z}_n>| \leq \frac{1}{2} r_n \ .$$

Hence $\hat{\phi}(\hat{z}_n) \leq r_n$ and $\hat{\psi}_n(\hat{z}_n) \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ let $(z_{n,k}) \in \hat{z}_n$ be a representing sequence in $(M_p)_1^+ = p(M_1^+)p$ (note that $\hat{M_p} = (M_p)\hat{\ }$) and fix $\mu \in \mathbb{R}_+$. Since $\mu R(\mu)\hat{\ }\hat{\psi}_n = \hat{\psi}_n$, $\hat{\phi}(\hat{z}_n) \leq r_n$ and $\hat{\psi}_n(\hat{z}_n) \geq \frac{1}{2}$ there exists for all $n \in \mathbb{N}$ an element $U_n \in \mathcal{U}$ such that for all $k \in U_n$:

(i)' $\phi(z_{n,k}) \leq r_n$,

(ii)' $\|(Id - \mu R(\mu))\psi_{n,k}\| \leq r_n$,

(iii)' $\psi_{n,k}(z_{n,k}) \geq \frac{1}{2}$.

Inductively we find a sequence (z_n) in $(M_p)_1^+$ and a sequence of states (ϕ_n) in M_* such that for all $n \in \mathbb{N}$:

(i)'' $\lim_n \phi_n(z_n) = 0$,

(ii)'' $\lim_n \|(Id - \mu R(\mu))\phi_n\| = 0$,

(iii)'' $\phi_n(z_n) \geq \frac{1}{2}$.

Since ϕ is faithful on M_p , condition (i)'' implies that $\lim_n z_n = 0$ in the $s^*(M_p,(M_p)_*)$-topology [Takesaki(1979), Proposition III.5.4].

Since $s^*(M_p, (M_p)_*) = s^*(M,M_*)|M_p$ (a) follows immediately from (i)''. Using the resolvent equation for R it is easy to see that (ii)'' implies

$$\lim_n \|(Id - \lambda R(\lambda))\phi_n\| = 0$$

for all $\lambda \in D$ and the proof is complete.

$\square$

Without further comments we will make use of the following facts in the rest of this section :

(1) A sequence (ϕ_n) in M'_+ converges in the $\sigma(M',M)$-topology if and only if it converges in $\sigma(M',M'')$-topology [Akeman-Dodds-Gamlen (1972)].

(2) We can decompose $\phi \in M'_+$ into its normal and singular part $\phi = \phi^{(n)} + \phi^{(s)}$, $0 \le \phi^{(n)} \in M_*$, $0 \le \phi^{(s)} \in M_*^\perp$ and $\|\phi\| = \|\phi^{(n)}\| + \|\phi^{(s)}\|$ [Takesaki (1979), Theorem III.2.14].

(3) If (ϕ_n) is a sequence in M_* which converges to zero in the $\sigma(M_*,M)$-topology and if (x_n) is a sequence in M which converges to zero in the $s^*(M,M_*)$-topology, then $\lim_n \phi_k(x_n) = 0$ uniformly in $k \in \mathbb{N}$ [Takesaki (1979), Lemma III.5.5].

Theorem 4.4. Let R be an identity preserving pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$ with values in a W*-algebra M which is of Schwarz type and let R' its adjoint pseudo-resolvent. Any one of the following conditions implies $\dim Fix(\hat{R}) < \infty$ in some ultrapower of M .

(a) The fixed space of R' is finite dimensional.

(b) $\lim_{\mu \downarrow o} \mu R(\mu) = P$ exists in the strong operator topology and rank(P) $< \infty$.

(c) The fixed space of R' is contained in M_* .

(d) Every map $\mu R(\mu)$, $\mu \in \mathbb{R}_+$, is irreducible on M .

Proof. Suppose that the dimension of the fixed space of $(R'')^\hat{}$ in some ultrapower $(M')^\hat{}$ of M' is infinite dimensional. Since $(M')^\hat{}$

is the predual of the W*-algebra $\hat{M}''$ and R' is identity preserving (since $R''1 = R1 = 1$) and of Schwarz type (because $\mu R''(\mu) = (\mu R(\mu))''$ is a Schwarz map for all $\mu \in R_+$) we may apply Lemma 4.3.

Suppose that the fixed space of the canonical extension of R' to some ultrapower of M' is infinite dimensional. Thus we may choose a sequence of states (ϕ_k) in M' and a sequence (z_k) in $(M'')_1^+$ satisfying (a) - (b) of Lemma 4.3. Remark (3) above implies that no subsequence of (ϕ_k) can converge in the $\sigma(M',M'')$-topology.

(a) If ϕ is a $\sigma(M',M)$-accumulation point of (ϕ_k) , then $\phi \in \mathrm{Fix}(R')$. Since $\mathrm{Fix}(R')$ is finite dimensional, the set of accumulation points of the sequence (ϕ_k) is metrizable in the $\sigma(M',M)$-topology. Hence there exists a sequence $(k(n))$ of natural numbers, such that $\sigma(M',M)\text{-}\lim_n \phi_{k(n)} = \phi$. Consequently, $\phi = \sigma(M',M'')\text{-}\lim_n \phi_{k(n)}$ by Remark (1) above. But this leads to a contradiction , which proves (a).

(b) Since $\dim \mathrm{Fix}(R) = \dim \mathrm{Fix}(R') = \mathrm{rank}(P) < \infty$, (b) follows from (a).

(c) Suppose that the fixed space of R' is infinite dimensional. Since $\mathrm{Fix}(R') \subseteq M_*$ there exists a sequence of states (ψ_n) in $\mathrm{Fix}(R')$ with mutually orthogonal support projections in M (Lemma 4.1). Since every $\sigma(M',M)$-accumulation point of the ψ_n's belongs to $\mathrm{Fix}(R')$, hence is normal, the sequence (ψ_n) is relatively $\sigma(M_*,M)$-compact. By Eberleins theorem, we may assume that this sequence is weakly convergent. By the orthogonality of the $s(\psi_n)$'s this sequence converges to zero in the $s^*(M,M_*)$-topology, hence $\lim_n \psi_k(s(\psi_n)) = 0$ uniformly in $k \in \mathbb{N}$, a contradiction. Consequently $\dim \mathrm{Fix}(R) < \infty$ and (c) is proved.

(d) We prove $\dim \mathrm{Fix}(R') = 1$ and apply (a) once again. Useful for this is the following observation : If ψ is a faithful state on M then the normal part is faithful too. Indeed, if $0 \neq x \in M$ such that $\psi^{(n)}(x) = 0$ choose a projection $0 \neq p \in M$ such that $\psi^{(n)}(p) = \psi^{(s)}(p) = 0$ (use [Takesaki (1979), Theorem III.3.8]), hence $\psi(p) = 0$ which conflicts with the faithfulness of ψ .

If $2 \leq \dim \mathrm{Fix}(R')$ there are states ψ_1 and ψ_2 in $\mathrm{Fix}(R')$ such that the corresponding support projections are orthogonal in M'' (Remark 4.2). Since every R'-invariant state ψ is faithful on M , $\psi_i^{(n)} \neq 0$ (otherwise the norm closed face $\{\psi(x) = 0 : x \in M_+\}$ would

be non trivial and $\mu R(\mu)$-invariant). The support projections of the $\psi_i^{(n)}$'s in M'' are orthogonal (since $\psi_i^{(n)} \leq \psi_i$) and different from zero. Let (z_γ) be a net in M_1^+ such that

$$\sigma(M'',M')-\lim_\gamma z_\gamma = s(\psi_1^{(n)}) \quad .$$

Then $\lim_\gamma \psi_1^{(n)}(z_\gamma) = 1$ whereas $\lim_\gamma \psi_2^{(n)}(z_\gamma) = 0$. Let z be a $\sigma(M,M_*)$-accumulation point of (z_γ) in M_+ . Since every $\psi_i^{(n)}$ is normal, $\psi_1^{(n)}(z) = 1$ whereas $\psi_2^{(n)}(z) = 0$. The first condition implies $z \neq 0$ whereas the second shows that $\psi_2^{(n)}$ cannot be faithful. Since this is a contradiction, it follows dim $Fix(R') = 1$ which proves (d).

$\square$

The next corollary is an easy application of Theorem 4.4 and of D-III, Proposition 2.3.

Corollary 4.5. Let T be an identity preserving semigroup of Schwarz type on the predual of a W*-algebra M . Then the following assertions are equivalent:

(a) T is uniformly ergodic with finite dimensional fixed space.

(b)   The adjoint weak*-semigroup is strongly ergodic with finite dimensional fixed space.

(c)   Every T''-invariant state is normal.

Proof. If (a) is fulfilled then the semigroup T is strongly ergodic on M_* . Since

$$\dim Fix(T) = \dim Fix(T') < \infty$$

there exist normal states $\phi_1,..,\phi_n$ in $Fix(T)$ and $x_1,..,x_k$ in $Fix(T')$ such that $\phi_n(x_m) = \delta_{n,m}$ $(1 \leq n,m \leq k)$ and

$$P = \sum_{i=1}^k \phi_i \otimes x_i$$

is the associated ergodic projection. If $(C(s))_{s>0}$ is the family of Césaro means of T , then

$$\lim_{s \to \infty} C(s)''(\psi) = \sum_{i=1}^{k} \phi_i(\psi) x_i \in M_*$$

for every $\psi \in M'$. Hence $\mathrm{Fix}(T'') \subseteq M_*$ which proves (c).

If (c) is fulfilled then $\mathrm{Fix}(T) = \mathrm{Fix}(T'')$. Therefore the fixed space of T' separates the points of $\mathrm{Fix}(T'')$ which shows that T' is strongly ergodic on M ([Krengel (1985), Chap.2, Thm.1.4]). If (b) holds then

$$P = \lim_{\mu \downarrow o} \mu R(\mu, A')$$

exists in the strong operator topology, where A' is the generator of T' . Therefore $\dim \mathrm{Fix}(\mu R(\mu)^\wedge) < \infty$ in some ultrapower of M (Theorem 4.4.(b)). It follows from D-III, Proposition 2.3 that 0 is a pole of the resolvent of $R(.,A)$. Therefore T is uniformly ergodic.

□

NOTES.

Section 1. The stability concepts appearing in Theorem 1.7 coincide not only for positive semigroups on C*-algebras but on any order unit Banach space. We refer to Batty-Robinson (1984) for this more general setting and to B-IV, Section 1 for the analogous results on $C_o(X)$.

Section 2. Theorem 2.2 generalizes the Liapunov stability theorem from the matrix algebra $B(\mathbb{C}^n)$ to arbitrary W*-algebras. For the algebra $B(H)$ it is due to Mil'stein (1975) and in the general form to Groh-Neubrander (1981).

Section 3. From the many papers dealing more or less explicitely with the asymptotic behavior of semigroups on operator algebras we quote Frigerio-Verri (1982) and Watanabe (1982). The background for our ergodic theorems (Prop.3.3 and Prop. 3.4) can be found best in Krengel (1985). The "automatic" convergence theorem for an irreducible W*-dynamical system on $B(H)$ stated in Corollary 3.9 is the continuous version of a result in Groh (1984c). Finally, the characterization of convergence towards a periodic semigroup through spectral properties of the generator (Thm. 3.11) is due to Nagel (1984) in the commutative, i.e. $L^2(\mu)$, case (see also C-IV, Thm.2.14).

Section 4. Again we refer to Krengel (1985) for the (uniform) ergodic theory for a single operator or a one-parameter semigroup on a Banach space. The characterization given in Corollary 4.5 for positive semigroups on W*-algebras is based on a sophisticated use of ultrapower techniques and has its discrete forerunners in Lotz (1981) and Groh (1984b).