Proposition 3.4. Take the operators $A$ , $B$ and $\Phi$ as above. For every $\lambda \in \mathbb{C}$ the following equivalence holds:

(3.4) $\lambda \in \sigma(A)$ if and only if $\lambda \in \sigma(B + \Phi_\lambda)$.

Proof. By definition, $\lambda \in \rho(A)$ if and only if for every $g \in E$ there exists a unique $f \in D(A)$ such that $\lambda f - f' = g$. This equality is satisfied if and only if there exists $x \in F$ such that

$$f(t) = \int_t^0 e^{\lambda(t-s)} g(s)\, ds + e^{\lambda t}\cdot x \quad \text{for } -1 \le t \le 0 .$$

On the other hand $f \in D(A)$ if and only if $x \in D(B)$ and $\lambda x - g(0) = Bx + \Phi H_\lambda g + \Phi_\lambda x$ where $H_\lambda g(t) := \int_t^0 e^{\lambda(t-s)} g(s)\, ds$ .
Thus $\lambda \in \rho(A)$ if and only if for every $g \in E$ there exists a unique $x \in D(B)$ such that $(\lambda - B - \Phi_\lambda)x = g(0) + \Phi H_\lambda g$. Notice that the map $x \to x + \Phi H_\lambda(\varepsilon_\mu \otimes x)$ $(x \in F)$ is surjective on $F$ if $\mu$ is chosen so large that $\|\Phi H_\lambda(\varepsilon_\mu \otimes x)\| \le 1/2\cdot\|x\|$ for all $x \in F$. Hence the map $g \to g(0) + \Phi H_\lambda g$ is surjective from $E$ onto $F$ and this shows that $\lambda \in \rho(A)$ if and only if $\lambda - B - \Phi_\lambda$ is invertible.

□

An immediate consequence of the proof is the following corollary.

Corollary. With the notations of the above proposition and $A_o$ as in the proof of Thm.3.1 we have:

(a) $R(\lambda,A)g = \varepsilon_\lambda \otimes R(\lambda,B+\Phi_\lambda)(g(0)+\Phi H_\lambda g) + H_\lambda g$ for $\lambda \in \rho(A)$ , $g \in E$.

(b) $R(\lambda,A_o)g = \varepsilon_\lambda \otimes R(\lambda,B)g(0) + H_\lambda g$ for $\lambda \in \rho(A_o)$ , $g \in E$.

We now turn to the aspect of positivity in (RCP) and its impact on the asymptotic behavior of the solution semigroup $(T(t))_{t\ge 0}$ .
To this end we let $F$ be a Banach lattice which makes $E = C([-1,0],F)$ into a Banach lattice as well. Furthermore, let $(S(t))_{t\ge0}$ be a positive semigroup with generator $B$ and let $\Phi \in L(E,F)$ be a positive operator. As before we restrict our attention to the case that $B - w$ generates a positive contraction semigroup for some $w \in \mathbb{R}$. Indeed, if $B$ generates a bounded positive semigroup $(S(t))_{t\ge0}$ on $F$, then $\|x\| := \sup_{t\ge0}\|S(t)|x|\|$ for $x \in F$ defines an equivalent lattice norm on $F$, for which $(S(t))_{t\ge0}$ is contractive.

Proposition 3.5. If $\Phi \in L(E,F)$ is a positive operator and if $B$ generates a positive semigroup on $F$, then the semigroup $(T(t))_{t\ge0}$ on $E$ generated by $Af := f'$ with domain $D(A) := \{f \in C^1 : f(0) \in D(B),\ f'(0) = Bf(0) + \Phi f\}$ is positive.