Unfortunately, (1.1) does not hold for positive semigroups in general. In A-IV, Example 1.2(2), we have seen that for the generator A of the (positive) translation semigroup on the Banach lattice $C_o(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x dx)$ the strict inequalitiy $\omega_1(A) < \omega(A)$ is valid. For positive semigroups on certain nice Banach lattices (1.1) is true. One of these nice Banach lattices is $C_o(X)$. This will be proved in Theorem 1.4.

For compact X, (1.1) was already proved in B-II, Cor.1.14 and B-III, Thm.1.6 respectively. Actually much more is true and for positive semigroups on $C(K)$, K compact, all stability concepts mentioned in chapter A-IV are mutually equivalent.

Theorem 1.1. Let A be the generator of a positive semigroup $(T(t))_{t\geq 0}$ on $C(K)$, K compact. Then

$$(1.2) \qquad s(A) = \inf\{\lambda \in \mathbb{R} : Af \leq \lambda f \text{ for some } 0 \ll f \in D(A)\}$$

Moreover, $s(A) = \omega(A) \in R\sigma(A) = P\sigma(A')$ and the following statements are mutually equivalent:

(i) $s(A) < 0$,

(ii) $(T(t))_{t\geq 0}$ is uniformly exponentially stable,

(iii) $(T(t))_{t\geq 0}$ is weakly stable; i.e. $\langle T(t)f,\mu\rangle \to 0$ as $t \to \infty$ for every $f \in D(A)$ and every $\mu \in C(X)'$.

Proof. (1.2) follows directly from A-III,4.4 and the results from B-II and B-III mentioned above. It remains to show the implication (iii) → (i).

If $\langle T(t)f,\mu\rangle \to 0$ for every $\mu \in C(K)'$, then, by the Uniform Boundedness Principle, $\|T(t)f\| \leq M_f$ for every $f \in D(A)$. Using $s(A) \leq \sup\{\omega(f) : f \in D(A)\} = \omega_1(A)$ (A-IV, Thm.1.4) we obtain that $s(A) \leq 0$. Suppose $0 = s(A)$. From B-III, Thm.1.6 it follows that $s(A) \in P\sigma(A')$, hence there is $0 < \mu \in C(K)'$ such that $T(t)'\mu = \mu$ for $t \geq 0$. Since $D(A)$ is dense, there exists $f \in D(A)$ such that $\langle f,\mu\rangle \neq 0$. Then $|\langle T(t)f,\mu\rangle| = |\langle f,\mu\rangle| > 0$ which contradicts the weak stability. Therefore $s(A) < 0$.

□

For spaces $C_o(X)$, X locally compact, the different stability concepts are no longer equivalent.