follow Widder (1971),p.196) we conclude that $\lim_{\lambda\to 0+} R(\lambda,A)Af$ exists and is equal to $\int_0^\infty T(s)Af\,ds$. The identity $R(\lambda,A)Af = \lambda R(\lambda,A)f - f$ yields the existence of $\lim_{\lambda\to 0+} \lambda R(\lambda,A)f$ for every $f \in D(A)$.

□

Bounded holomorphic semigroups (see A-II,Def.1.11) satisfy $\|AT(t)\| \leq m\cdot t^{-1}$ [Goldstein (1985a),p.33], hence $T(t)f \to 0$ as $t \to \infty$ whenever $f \in \operatorname{im} A$. If $\operatorname{im} A$ is dense (i.e., $0 \notin R\sigma(A)$) we obtain uniform stability and the following corollary.

Corollary 1.14. Let A be the generator of a bounded holomorphic semigroup $(T(t))_{t\geq 0}$ on a Banach space E . Then the following statements are equivalent.

(a) $0 \notin P\sigma(A) \cup R\sigma(A)$.

(b) $(T(t))_{t\geq 0}$ is uniformly stable.

Example 1.15 The Laplacian Δ generates a bounded holomorphic semigroups on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ (see the example proceeding Cor.1.13 of Chap. A-II). All solutions of $\Delta f = 0$ are either constant or unbounded, therefore $0 \notin P\sigma(\Delta)$. If $1 < p < \infty$, then the adjoint of the Laplacian on $L^p(\mathbb{R}^n)$ is the Laplacian on $L^q(\mathbb{R}^n)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore $0 \notin R\sigma(\Delta) \cup P\sigma(\Delta')$ and we obtain by Cor.1.14 that Δ generates uniformly stable semigroups on the space $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ which are, by $\operatorname{im}\Delta \neq L^p(\mathbb{R}^n)$ and Cor.1.5, not exponentially stable.

As seen in Thm.1.4, exponential stability can be defined by saying that the abscissa of convergence of the Laplace transform of $(T(t))_{t\geq 0}$ is less than zero. This should be compared to the assertion of our final theorem.

Theorem 1.16. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E . The following assertions are equivalent:

(a) $(T(t))_{t\geq 0}$ is stable.

(b) $\ker A = \{0\}$ and $\int_0^\infty T(t)f\,dt$ exists for all $f \in \operatorname{im} A$.

Furthermore the following statements are equivalent: