3. IRREDUCIBLE SEMIGROUPS

In the case of matrices it is well known that considerably stronger results are available if one considers positive matrices which are irreducible. The concept of irreducibility can be extended to our setting and in many cases one can check easily whether a given semigroup has this property (see Ex.3.4). We will show that irreducible semigroups have many interesting properties. For example, the spectrum $\sigma(A)$ is always non-empty, positive eigenfunctions are strictly positive and if $s(A)$ is a pole, it is algebraically (and geometrically) simple (see Prop.3.5). Moreover, in certain cases irreducibility ensures that $\sigma_b(A)$ and $P\sigma_b(A)$ are not only cyclic subsets but 'subgroups' (see Thm.3.5 and Thm.3.11 for details).
We start the discussion with several, mutually equivalent, definitions of irreducibility.

Definition 3.1. A positive semigroup $\mathcal{T} = (T(t))$ on $E = C_o(X)$, X locally compact, with generator A is called irreducible if one of the following mutually equivalent conditions is satisfied:

(i) There is no $\mathcal{T}$-invariant closed ideal except $\{0\}$ and E.

(ii) Given $0 < f \in E$, $0 < \phi \in E'$, then $\langle T(t_o)f,\phi\rangle > 0$ for some $t_o \geq 0$.

(iii) For every $f \in E_+$ we have $\cup_{t\geq 0}\{x \in X : (T(t)f)(x) > 0\} = X$.

(iv) For some (every) $\lambda > s(A)$ there exists no closed ideal which is invariant under $R(\lambda,A)$ except $\{0\}$ and E.

(v) For some (every) $\lambda > s(A)$ we have: $R(\lambda,A)f$ is strictly positive whenever $f > 0$.

(vi) $\cup_{t\geq 0}\operatorname{supp} T(t)'\delta_x$ is dense in X for every $x \in X$.

That these six conditions are actually equivalent can be seen as follows:
(i) $\rightarrow$ (ii): Suppose there are $0 < f \in E$, $0 < \phi \in E'$ such that $\langle T(t)f,\phi\rangle = 0$ for every $t \geq 0$. Then the ideal I generated by $\{T(t)f : t \geq 0\}$ satisfies $\{0\} \subsetneqq I \subset \{g \in E : \phi(|g|) = 0\} \subsetneqq E$. Obviously I is $\mathcal{T}$-invariant.

(ii) $\rightarrow$ (iii): Given $0 < f \in E$, $x \in X$. By (ii) there exists t_o such that $(T(t_o)f)(x) = \langle T(t_o)f,\delta_x\rangle > 0$.

(iii) $\rightarrow$ (vi): Suppose that $\cup_{t\geq 0}\operatorname{supp} T(t)'\delta_y$ is not dense for some $y \in X$. Then there exists $f_o \in E$, $f_o > 0$ such that