$$\leq \lim_{t \to 0} 1/t \langle T(t)|f| - |f|, \phi\rangle$$

$$= \lim_{t \to 0} \langle |f|, 1/t(T(t)'\phi - \phi)\rangle$$

$$= \langle |f|, A'\phi\rangle.$$

□

Let $D(A')_+ = E'_+ \cap D(A')$. Consider the condition

$$\overline{D(A')_+}^{\sigma(E',E)} = E'_+ \tag{2.9}$$

(which is satisfied if the semigroup is positive). If (K) and (2.9) hold, then Kato's inequality holds in the strong form as well, whenever it makes sense; i.e.,

$$\operatorname{Re}((\operatorname{sign} \bar{f})Af) \leq A|f| \qquad (\text{whenever } f, |f| \in D(A)). \tag{2.10}$$

<u>Example</u> 2.5. Kato's inequality in its classical form says the following (see Kato (1973) or [Reed-Simon (1975); X.27]).
Let $f \in L^1_{loc}(\mathbb{R}^n)$ be such that the distributional Laplacian satisfies $\Delta f \in L^1_{loc}(\mathbb{R}^n)$. Then the inequality

$$\operatorname{Re}((\operatorname{sign} \bar{f})\Delta f) \leq \Delta|f|$$

holds in the sense of distributions; i.e.,
$\langle\phi, \operatorname{Re}((\operatorname{sign} \bar{f})\Delta f)\rangle \leq \langle\phi, \Delta|f|\rangle$ $(= \langle\Delta\phi, |f|\rangle)$ holds for all $0 \leq \phi \in C_c^\infty(\mathbb{R}^n)$. Note that the closure of Δ defined on $C_c^\infty(\mathbb{R}^n)$ generates a strongly continuous positive semigroup on $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ (see Example 1.5.d and Example 4.7)).

We want to discuss the relation between the classical (distributional) inequality and our version given in Theorem 2.4.

Let $$A = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

be a differential operator, where $a_\alpha \in C_c^\infty(\mathbb{R}^n)$. Here we let $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$ for all multi-indices $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$ $(\mathbb{N}_0 := \mathbb{N}\cup\{0\})$ of order $|\alpha| := \alpha_1 + \dots + \alpha_n$.
We say that A satisfies Kato's inequality in the sense of distributions if

$$\operatorname{Re}\langle(\operatorname{sign} \bar{f})\, Af, \phi\rangle \leq \langle |f|, A^*\phi\rangle \tag{K_d}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \phi \in C_c^\infty(\mathbb{R}^n)$, where A^* denotes the formal adjoint of A.
Let now A be the generator of a positive semigroup $(T(t))_{t\geq 0}$ on $E := L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ or $C_o(\mathbb{R}^n)$. Assume that there exists a core