As an immediate consequence we obtain an interesting characterization of the growth bound ω of semigroups on Hilbert spaces.

Corollary 7.11. The growth bound of a semigroup $(T(t))_{t\geq 0}$ on a Hilbert space H satisfies

$$\omega = \inf \{ \lambda \in \mathbb{R} : \lambda + i\mathbb{R} \subset \rho(A) \text{ and } \|R(\lambda+i\mu,A)\| \text{ is bounded for } \mu \in \mathbb{R} \} . \tag{7.9}$$

The Example 1.3 above in combination with C-III,Cor.1.3 shows that (7.9) is not valid in arbitrary Banach spaces.

NOTES.

Section 1. It was already known to Hille-Phillips (1957) that for strongly continuous semigroups $(T(t))_{t\geq 0}$ with generator A the spectral mapping theorem "$\sigma(T(t)) = \exp(t\sigma(A))$" may be violated in various ways [l.c.,Sec.23.16]. The simple Examples 1.3 and 1.4 are due to Wolff (see Greiner-Voigt-Wolff (1981)) and Zabczyk (1975). A more sophisticated example of a positive group with "$s(A) < \omega(A)$" is given in Wolff (1981).

Section 2. In Definition 2.1 we define the residual spectrum of A in such a way that it coincides with the point spectrum of the adjoint A' (see Prop. 2.2.(ii)). It therefore differs slightly from the one used, e.g., by Schaefer (1974). The spectral mapping theorem for the resolvent (Thm.2.5) is well known and can, e.g., be deduced from Lemma 9.2 and Thm.3.11 of Chap.VII in Dunford-Schwartz (1958).

Section 3. The general theory of spectral decompositions can be found in [Kato (1966), Chap.III,§ 6.4]. Applications to isolated singularities like 3.6 are discussed extensively in [l.c., Chap.III, §6.5] and [Yosida (1965), Chap.VIII, Sec.8]. There are many ways to introduce an "essential spectrum" (see the footnote on page 243 of Kato (1966)). However each one yields the same "essential spectral radius".

Section 4. These results are taken from Derndinger (1980) and Greiner (1981).

Section 5. Periodic semigroups are studied explicitely in Bart (1977) but most of the results of this section seem to be well known.

Section 6. The Spectral Inclusion Theorem 6.2 and the Spectral Mapping Theorem 6.6 for eventually norm continuous semigroups date back to Hille-Phillips (1957). Davies (1980) gives an elegant proof using Banach algebra techniques. Tensor products of operators and their spectral theory have been studied by Ichinose and others (see Chap. XIII.9 of Reed-Simon (1978)). The spectral mapping theorem in Corollary 6.8 generalizes Thm.XIII.35 of Reed-Simon (1978) (see also Herbst (1982)).

Section 7. Matrix valued multiplication semigroups appear as solution, via Fourier transformation, of systems of partial differential equations. Kreiss initiated a systematic investigation (see, e.g., Kreiss (1958), Kreiss (1959), Miller-Strang