Since $\|\exp(tA_n)\| \leq e^t$ for every $n \in \mathbb{N}$, $t \geq 0$, and since $t \to T(t)x$ is continuous on each component E_n it follows that $\mathcal{T}$ is strongly continuous. Its generator is the operator A as defined above.

For $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda > 0$, we have $\lim_{n\to\infty} \|R(\lambda-2\pi in, A_n)\| = 0$, hence

$$(R(\lambda, A_n+2\pi in))_{n\in\mathbb{N}} = (R(\lambda-2\pi in, A_n))_{n\in\mathbb{N}}$$

is a bounded operator on E representing the resolvent $R(\lambda,A)$. Therefore we obtain $s(A) \leq 0$. On the other hand, each $2\pi in$ is an eigenvalue of A, hence $s(A) = 0$.

Take now $x_n \in E_n$ as above and consider the sequence $(x_n)_{n\in\mathbb{N}}$. From (iii) it follows that for $t > 0$ the number e^t is an approximate eigenvalue of $T(t)$ with approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ (see Def.2.1 below). Therefore $e^t \leq r(T(t)) \leq \|T(t)\|$ and hence $\omega(\mathcal{T}) \geq 1$. On the other hand, it is easy to see that $\|T(t)\| = e^t$, hence $\omega(\mathcal{T}) = 1$.

Finally if we take $S(t) := e^{-t/2}\cdot T(t)$ we obtain a semigroup having spectral bound $-\frac{1}{2}$ but such that $\lim_{t\to\infty} \|S(t)\| = \infty$ in contrast with Cor. 1.2 .

These examples show that neither the conclusion of Cor.1.2 , i.e. '$s(A) < 0$ implies stability', nor the 'spectral mapping theorem'

$$\sigma(T(t)) = \exp(t\cdot\sigma(A))$$

is valid for arbitrary strongly continuous semigroups. A careful analysis of the general situation will be given in Section 6 below, but we will first develop systematically the necessary spectral theoretic tools for unbounded operators.

2. THE FINE STRUCTURE OF THE SPECTRUM

As usual, with a closed linear operator A with dense domain $D(A)$ in a Banach space E, we associate its spectrum $\sigma(A)$, its resolvent set $\rho(A)$ and its resolvent

$$\lambda \to R(\lambda,A) := (\lambda - A)^{-1}$$

which is a holomorphic map from $\rho(A)$ into $\mathcal{L}(E)$. In contrast to the finite dimensional situation, where a linear operator fails to be surjective if and only if it fails to be injective, we now have to distinguish different cases of 'non-invertibility' of $\lambda - A$. This distinction gives rise to a subdivision of $\sigma(A)$ into different subsets. We point out that these subsets need not be disjoint, but our defini-