Positive Semigroups on C*- and W*-Algebras

by

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D-I: Basic Results on Semigroups and Operator Algebras

This is not a systematic introduction into the theory of strongly continuous semigroups on C^* and W^*-algebras. For that we refer to *Bratteli-Robinson (1979), Davies (1976)* and the survey article of *Oseledets (1984)*. We only prepare for the subsequent chapters on spectral theory and asymptotics by fixing the notations and introducing some standard constructions.

1.1 Notations

1. By M we shall denote a C*-algebra with unit $\mathbb{1}$.

$$M^{\mathrm{sa}} := \{x \in M : x^* = x\}$$

is the selfadjoint part of M and

$$M_+ := \{x^* x : x \in M\}$$

the positive cone in M.

If M' is the dual of M, then

$$M'_+ = \{\phi \in M' : \phi(x) \geq 0 \text{ for alle } x \in M_+\}$$

is a weak*-closed generating cone in M' and we call

$$S(M) := \left\{\phi \in M'_+ : \phi(\mathbb{1}) = 1\right\}$$

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the state space of M. For the theory of C*-algebras and related notations we refer to [Pedersen (1979)].

M is called a W*-algebra, if there exists a Banach space M_*, such that its dual space $(M_*)'$ is (isomorphic to) M. We call M_* the predual of M and $\phi \in M_*$ a normal linear functional. It is known that M_* is unique [Sakai (1971), 1.13.3.]. For further properties of M_* and M we refer to [Takesaki (1979), Chapter III].

2. A map $T \in L(M)$ is called positive (in symbols $T \geq 0$) if $T(M_+) \subseteq M_+$ and T is called n-positive ($n \in \mathbb{N}$) if $T \otimes id_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where id_n is the identity map on the C*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive.

We call contraction $T \in L(M)$ a Schwarz map if T satisfies the inequality

$$T(x)\,T(x)^* \leq T\left(xx^*\right) \quad (x \in M).$$

It is well known that every n-positive contraction, $n \geq 2$ and that every positive contraction on a commutative C*-algebra is a Schwarz map [Takesaki (1979), Corollary IV. 3.8.]. As we shall see, the Schwarz inequality is crucial for our investigations.

3. If M is a C*-algebra we assume $T = (T(t))_{t\geq 0}$ to be a strongly continuous semigroup (abbreviated semigroup) while on W*-algebras we consider weak*-semigroups, i.e. the mapping

$$t \to T(t)_* x : \mathbb{R}_+ \to M_*$$

is continuous, M_* the predual of M, and every $T(t)$ w*-continuous. Note that the preadjoint semigroup

$$T_* = \{T(t)_* : T(t) \in t \geq 0\}$$

is weakly, hence strongly continuous on M_* (see e.q., Davies (1980), Prop.1.23). We call T identity preserving if $T(t)\,\mathbb{1} = \mathbb{1}$ and of Schwarz type if every $T(t)$ is a Schwarz map.

For the notations concerning one-parameter semigroups we refer to Part-A. In addition we recommend to compare the results of this section of the book with the corresponding results for commutative C*-algebras, i.e. for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ (see Part-B).

1.2 A Fundamental Inequality for the Resolvent

If $(T(t))_t$ is a strongly continuous semigroup of Schwarz maps on a C*-algebra M (resp. a weak*-semigroup of schwarz type on a W^*-algebra M) with generator A, then the spectral bound $s(A) \leq 0$. Then for $\lambda \in C$, $\mathrm{Re}(\lambda) > 0$, there exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)\, x = \int_0^\infty e^{-\lambda t} T(t)\, x \,\mathrm{d}t \quad (x \in M)$$

where the integral exists in the norm topology.

Theorem 1 *Let $T = (T(t))_{t\geq 0}$ be a semigroup of schwarz type and $T = (S(t))_{t\geq 0}$ a semigroup on a C^*-algebra M with generators A and B, respectively. If*

$$(S(t)\, x)\,(S(t)\, x)^* \leq T(t)\, xx^* \tag{*}$$

for all $x \in M$ and $t \in \mathbb{R}_+$, then

$$(\mu R(\mu, B)\, x)\,(\mu R(\mu, B)\, x)^* \leqslant \mu R(\mu, A)\, xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$. The same result holds if T is a weak-semigroup of schwarz type and S is a weak*-semigroup on a $W^\star$-algebra M such that (*) is fulfilled.*