D_o of A such that $C_c^\infty \subset D_o$ and $Af = \mathcal{A}f$ in the sense of distributions for all $f \in D_o$. Then $\mathcal{A}$ satisfies Kato's inequality in the sense of distributions.

In fact, let $0 \le \phi \in C_c^\infty(\mathbb{R}^n)$. Then $\langle Af,\phi\rangle = \langle \mathcal{A}f,\phi\rangle = \langle f,\mathcal{A}^*\phi\rangle$ for all $f \in D_o$. Since D_o is a core of A, this implies that $\phi \in D(A')$ and $A'\phi = \mathcal{A}^*\phi$. Thus (K) gives $\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})\mathcal{A}f,\phi\rangle = \mathrm{Re}\langle((\mathrm{sign}\,\bar{f})Af),\phi\rangle \le \langle|f|,A'\phi\rangle = \langle|f|,\mathcal{A}^*\phi\rangle = \langle\mathcal{A}|f|,\phi\rangle$ for all $f \in C_c^\infty(\mathbb{R}^n)$, $0 \le \phi \in C_c^\infty(\mathbb{R}^n)$. This is Kato's inequality in the distributional sense.

Remark. It has been proved by Miyajima and Okasawa (1984) that (K_d) implies that $m \le 2$ and that the principal part $\mathcal{A}_o = \sum_{|\alpha|=2} a_\alpha D^\alpha$ of $\mathcal{A}$ is elliptic; i.e., if we write the operator $\mathcal{A}_o$ in the form $\mathcal{A}_o = \sum_{i,j=1}^{2} b_{ij}\,\partial^2/\partial x_i\partial x_j$, then the matrix $(b_{ij}(x))$ is positive semidefinite for all $x \in \mathbb{R}^n$.

Finally we formulate Theorem 2.4 for the space $E := C_o(X)$ (X locally compact) (which is not σ-order complete unless X is σ-Stonian). Recall, for $f \in C_o(X)$, sign f is defined as a Borel function and for any bounded Borel function g on X and any $\phi \in M(X) = C_o(X)'$ we let $\langle g,\phi\rangle = \int g(x)\,d\phi(x)$ (see B-II,Sec.2).

Theorem 2.6. Let X be a locally compact space and A be the generator of a strongly continuous positive semigroup on $C_o(X)$. Then

(K) $\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af,\phi\rangle \le \langle|f|,A'\phi\rangle \qquad (f \in D(A)\,,\ \phi \in D(A')_+)$.

The proof of Theorem 2.4 can be taken over literally. Also the analogue of the proof given for L^p-spaces (preceding Theorem 2.4) is valid if one uses B-II,Lemma 2.6.

3. A CHARACTERIZATION OF GENERATORS OF POSITIVE SEMIGROUPS

In this section we confine ourselves to real Banach lattices. This does not mean a restriction since every positive semigroup on a complex Banach lattice leaves the real part of the space invariant.