4. UNIFORM ERGODIC THEOREMS

As we have seen, uniformly ergodic semigroups have nice spectral properties. In this section we study sufficient conditions which imply uniform ergodicity thereby generalizing results the results of Groh (1984b). We first need some preparations.

Lemma 4.1. Let R be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in the predual of a W*-algebra M. If the fixed space of R is infinite dimensional, then there exists a sequence of states in Fix(R) such that the corresponding support projections are mutually orthogonal in M.

Proof. Let $\Phi = \{\phi \in \mathrm{Fix}(R) : \phi \text{ state on } M\}$ and let $p = \sup\{s(\phi) : \phi \in \Phi\}$. Since $\lambda R(\lambda)\phi = \phi$ for all $\phi \in \Phi$ and $\lambda \in D$ it follows $\mu R(\mu)(1-s(\phi)) \leq (1-s(\phi))$. Hence $\mu R(\mu)(1-p) \leq (1-p)$ for all $\mu \in \mathbb{R}_+$. Let $R_|$ be the induced pseudo-resolvent on pM_*p (D-I, Section 3.(c)). Then the family Φ is faithful on M_p and contained in the fixed space of $R_|$. The adjoint $\mu R_|(\mu)'$ is an identity preserving Schwarz map. Consequently it follows from D-III, Lemma 1.1.(b) and the $\sigma(M_p,(M_p)_*)$-continuity of $\mu R_|(\mu)'$, that $\mathrm{Fix}(R_|')$ is a W*-subalgebra of M_p and by D-III, Lemma 1.5 it follows that $\dim \mathrm{Fix}(R) \leq \dim \mathrm{Fix}(R_|')$. If Fix(R) is infinite dimensional, let (p_n) be a sequence of mutually orthogonal projections in $\mathrm{Fix}(R_|') \subseteq M_p$ and choose a sequence (ϕ_n) in Φ such that $\phi_n(p_n) \neq 0$. For $n \in \mathbb{N}$ let ψ_n be the normal state

$$\psi_n(x) = \phi_n(p_n)^{-1}\phi_n(p_n x p_n)$$

on M. Because of $s(\psi_n) \leq p_n \leq p$, the support projections of the ψ_n's are mutually orthogonal in M. For $\mu \in \mathbb{R}_+$ and $x \in M$ we obtain:

$$\langle x, \mu R(\mu)\psi_n\rangle = \phi_n(p_n)^{-1}\langle \mu p_n(R(\mu)'x)p_n, \phi_n\rangle =$$

$$= \phi_n(p_n)^{-1}\langle \mu p_n p(R(\mu)p'x)p_n, \phi_n\rangle =$$

$$= \phi_n(p_n)^{-1}\langle \mu p_n(pR_|(\mu)'xp)p_n, \phi_n\rangle =$$

$$= \phi_n(p_n)^{-1}\langle \mu(p_n R_|(\mu)'xp_n), \phi_n\rangle =$$

$$= \phi_n(p_n)^{-1}\phi_n(x) = \psi_n(x) .$$

Therefore $\psi_n \in \mathrm{Fix}(R)$ for all $n \in \mathbb{N}$.

□