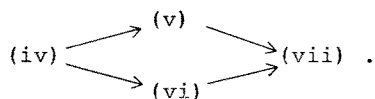


The interrelation between these stability concepts is given by



If A is a bounded operator, i.e., if $D(A) = E$, then $(iv) \Leftrightarrow (v)$ and $(vi) \Leftrightarrow (vii)$. If A is unbounded then the stability notions may differ as we will see in the following examples.

Examples 1.2. (a) Let $E = C_0$. Then $A : (x_n)_{n \in \mathbb{N}} \rightarrow (-1/n \cdot x_n)_{n \in \mathbb{N}}$ generates the semigroup $T(t)(x_n) = (e^{-t/n} x_n)_{n \in \mathbb{N}}$. It is easy to see that $\|T(t)\| = 1$ and $\|T(t)f\| \rightarrow 0$ for every $f \in C_0$. Moreover, A is a bounded operator, hence $D(A) = E$. This gives an example for a (uniformly) stable but not exponentially stable semigroup. The translation semigroups generated by the first derivative on $C_0(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$ for $1 < p < \infty$ give further examples for (uniformly) stable but not exponentially stable semigroups. Moreover, as seen in A-II, Ex.1.14, the Laplacian Δ on $C_0(\mathbb{R}^n)$ generates a bounded holomorphic semigroup given by

$$T(t)f(x) = (4\pi t)^{-n/2} \cdot \int_{\mathbb{R}^n} e^{-(x-y)^2/4t} f(y) dy$$

which cannot be exponentially stable because $0 \in \sigma(\Delta)$ ($\text{im} \Delta \neq C_0(\mathbb{R}^n)$), see Cor.1.5 below. By a straightforward $(2-\varepsilon)$ -argument using $(4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-y^2/4t) dy = 1$ one can easily show that $\|T(t)f\| \rightarrow 0$ for all $f \in C_0(\mathbb{R}^n)$ (see also B-III, Ex.1.7).

Therefore, the Laplacian on $C_0(\mathbb{R}^n)$ (and also on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, see Ex.1.15 below) generates a (uniformly) stable but not exponentially stable semigroup.

(b) Note that the condition $0 \leq \omega(A) = \inf\{\omega : \|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0\}$ does not exclude that the semigroup $(T(t))_{t \geq 0}$ is exponentially stable. In fact, as shown in A-III, 1.3 the translation semigroup $(T(t))_{t \geq 0}$ on $E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x dx)$ satisfies $\|T(t)\| = 1$, hence $\omega(A) = 0$, and for every $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > -1$ the resolvent of the generator is given as $R(\lambda, A)f = \int_0^\infty e^{\lambda t} T(t)f dt$ for every $f \in E$. From the equation A-I, 3.2

$$T(t)f = e^{\lambda t} \left(f - \int_0^t e^{-\lambda s} T(s)(\lambda - A)f ds \right)$$

and the existence of $\lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} T(s)(\lambda - A)f ds$ it follows that $\|T(t)f\| \leq M \cdot e^{\lambda t}$ for every $f \in D(A)$ and some constant M depending