Two sufficient conditions for a resolvent to grow slowly are stated in the following proposition. Its simple proof is omitted.

Proposition 2.9. Suppose $(T(t))_{t\geq 0}$ is a positive semigroup with generator A. Each of the following conditions guarantees that the resolvent grows slowly.

(a) $(T(t))_{t\geq 0}$ is bounded and $s(A) = 0$;
(b) $s(A)$ is a first order pole of the resolvent.

In case $s(A)$ is a pole of order greater than 1, the resolvent does not grow slowly. We will treat this case in Cor.2.12.

Theorem 2.10. The boundary spectrum of a positive semigroup with slowly growing resolvent is cyclic.

Proof. Without loss of generality we can assume that $s(A) = 0$. Given $i\beta \in \sigma(A)$ $(\beta \in \mathbb{R})$, then $i\beta \in A\sigma(A)$ (A-III,Prop.2.2) and $(\lambda-i\beta)^{-1} \in A\sigma(R(\lambda,A))$ (A-III,Prop.2.5). We consider an $\mathcal{F}$-product $E_{\mathcal{F}}$ of E and for convenience write E_1 instead of $E_{\mathcal{F}}$. The canonical extensions of $R(\lambda,A)$ to E_1 form a positive pseudo-resolvent $\{(R_1(\lambda)\}_{Re\lambda>0}$ on E_1. By Prop.2.6(a) and A-III,4.5 there exists $h_1 \in E_1$, $h_1 \neq 0$ such that

$$(2.16)\quad \lambda R_1(\lambda+i\beta)h_1 = h_1 \quad \text{for } \operatorname{Re}\lambda > 0 .$$

By (2.13) we have

$$(2.17)\quad |h_1| = |rR_1(r+i\beta)h_1| \leq rR_1(r)|h_1| \quad (r > 0).$$

Next we choose any $\phi \in E_1'$ such that $\langle h_1,\phi\rangle \neq 0$. Since $\|R_1(\lambda)'\| = \|R_1(\lambda)\| = \|R(\lambda,A)\|$, the assumption of slow growth implies that $\{\lambda R_1(\lambda)'|\phi| : \lambda > 0\}$ is bounded in E_1', hence $\sigma(E_1',E_1)$-relatively compact by Alaoglu's Theorem. Thus there exist
$\phi_1 \in \bigcap_{\varepsilon>0}\overline{\{rR_1(r)'|\phi| : 0<r<\varepsilon\}}^{\sigma}$.
Using the resolvent equation (2.8) we get for $r > 0$, $\varepsilon > 0$:
$(1 - rR_1(r)')\varepsilon R_1(\varepsilon)'|\phi| = \varepsilon(r-\varepsilon)^{-1}\big(rR_1(r)'|\phi| - \varepsilon R_1(\varepsilon)'|\phi|\big)$.
Since the right hand side tends to 0 as $\varepsilon \to 0$, we have
$(1 - rR_1(r)')\phi_1 = 0$ or

$$(2.18)\quad \lambda R_1(\lambda)'\phi_1 = \phi_1 \quad (\operatorname{Re}\lambda > 0) .$$

Moreover, from $0 < |\langle h_1,\phi\rangle| \leq \langle |h_1|,|\phi|\rangle \leq \langle rR_1(r)|h_1|,|\phi|\rangle = \langle |h_1|,rR_1(r)'|\phi|\rangle$ it follows that