Chapter 1

Basic Results on Semigroups and Operator Algebras

This is not a systematic introduction into the theory of strongly continuous semigroups on C*- and W*-algebras. For that we refer to [2], [3] and the survey article of [5]. We only prepare for the subsequent chapters on spectral theory and asymptotics by fixing the notations and introducing some standard constructions.

1.1 Notations

1. By M we shall denote a C*-algebra with unit 1, with

$$M^{sa} := \{x \in M : x^* = x\}$$

the self-adjoint part of M and

$$M_+ := \{x^*x : x \in M\}$$

is the positive cone in M.

If M' is the dual of M, then

$$M'_+ := \{\psi \in M' : \psi(x) \geqslant 0, x \in M_+\}$$

is a weak*-closed generating cone in M' and

$$S(M) := \{\psi \in M'_+ : \psi(1) = 1\}$$

is called the state space of M.

For the theory of C*-algebras and related notions we refer to [6].

2. M is called a W*-algebra, if there exists a Banach space M_*, such that its dual $(M_*)'$ is (isomorphic to) M. We call M_* the predual of M and $\psi \in M_*$ a normal linear functional. It is known that M_* is unique [7, 1.13.3].

For further properties of M_* we refer to [8, Chapter III].

3. A map $T \in L(M)$ is called *positive* (in symbols $T \geqslant 0$) if $T(M_+) \subseteq M_+$. $T \in L(M)$ is called n*-positive* ($n \in \mathbb{N}$) if $T \otimes \mathrm{Id}_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where Id_n is the identity map on the C*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive.

We call a contraction $T \in L(M)$ a Schwarz map if T satisfies the inequality

$$T(x)T(x)^* \leqslant T(xx^*)\,,\; x \in M.$$

It is well known that every n-positive contraction, $n \geqslant 2$ and that every positive contraction on a commutative C*-algebra is a Schwarz map ([8, Corollary IV. 3.8.]). As we shall see, the Schwarz inequality is crucial for our investigations.

4. If M is a C*-algebra we assume $\mathcal{T} = (T(t))_{t\geqslant 0}$ to be a strongly continuous semigroup (abbreviated semigroup) while on W*-algebras we consider weak*-semigroups, i.e. the mapping $(t \mapsto T(t)x)$ is continuous from $\mathbb{R}_+$ into $(M, \sigma(M, M_*))$, M_* the predual of M, and every $T(t) \in T$ is $\sigma(M, M_*)$-continuous. Note that the preadjoint semigroup

$$T_* = \{T(t)_* : T(t) \in T\}$$

is weakly, hence strongly continuous on M_* (see e.g., [4, Prop. 1.23]). We call $\mathcal{T}$ identity preserving if $T(t)1 = 1$ and of Schwarz type if every $T(t)$ is a Schwarz map.

For the notations concerning one-parameter semigroups we refer to Part A. In addition we recommend to compare the results of this section of the book with the corresponding results for commutative C*-algebras, i.e. for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ (see Part B).

1.2 A Fundamental Inequality for the Resolvent

If $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup of Schwarz maps on a C*-algebra M (resp. a weak*-semigroup of Schwarz type on a W*-algebra M) with generator A, then the spectral bound $s(A) \leqslant 0$. Then for $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$, there exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)x \;\mathrm{d}t\,,\; x \in M$$

where the integral exists in the norm topology.

In [2] it is shown that $\mathcal{T}$ is a semigroup of Schwarz type if and only if $\mu R(\mu, A)$ is a Schwarz map for every $\mu \in \mathbb{R}_+$. Here we relate the domination of two semigroups to an inequality for the corresponding resolvent operator. This inequality will be needed later.

Theorem 1.1. *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a semigroup of Schwarz type and $\mathcal{S} = (S(t))_{t\geqslant 0}$ a semigroup on a C*-algebra M with generators A and B, respectively. If*

$$(*) \qquad (S(t)x)(S(t)x)^* \leqslant T(t)xx^*$$

for all $x \in M$ and $t \in \mathbb{R}_+$, then

$$(\mu R(\mu, B)x)(\mu R(\mu, B)x)^* \leqslant \mu R(\mu, A)xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$.

The same result holds if $\mathcal{T}$ is a weak-semigroup of Schwarz type and $\mathcal{S}$ is a weak*-semigroup on a W*-algebra M such that $(*)$ is fulfilled.*

Proof. From the assumption $(*)$ it follows that

$$\begin{aligned} 0 &\leqslant (S(r)x - S(t)x)(S(r)x - S(t)x)^* = \\ &= (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \\ &\leqslant T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* \end{aligned}$$

for every $r, t \in \mathbb{R}_+$. Hence

$$(S(r)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \leqslant T(r)xx^* + T(t)xx^*.$$

Obviously, $\|S(t)\| \leqslant 1$ for all $t \in \mathbb{R}_+$. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$:

$$\begin{aligned} (R(\mu, B)x)(R(\mu, B)x)^* &= (\int_0^\infty e^{-\mu r} S(r)x\, dr)(\int_0^\infty e^{-\mu t} S(t)x \ \mathrm{d}t)^* \\ &= (\int_0^\infty \int_0^\infty e^{-\mu(r+t)} (S(r)x)(S(t)x)^*\, dr \ \mathrm{d}t) \\ &\leqslant \int_0^\infty \int_0^\infty e^{-\mu(r+t)} (T(r)xx^* + T(t)xx^*)/2\, dr \ \mathrm{d}t \\ &= \int_0^\infty e^{-\mu r} T(r)xx^*\, dr \int_0^\infty e^{-\mu t} \ \mathrm{d}t \\ &= \mu^{-1} \int_0^\infty e^{-\mu r} T(r)xx^*\, dr = R(\mu, A)xx^*. \end{aligned}$$

Here we used the inequality derived above in the first step. The second step follows from $S(t)$ being a contraction semigroup and the third step is achieved by integration. ☐

Remark 1.2. The assumption that T is a semigroup of Schwarz type cannot be weakened in general to T being a positive contraction semigroup. This is shown by examples in [4] where $S(t)x$ is given by $e^{tB}x$ for a skew-adjoint generator B and $T(t)x \equiv x$.

Corollary 1.3. *Let $T = (T(t))_{t\geqslant 0}$ be a semigroup of Schwarz type on a C*-algebra M with generator A. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$:*

$$(\mu R(\mu, A)x)(\mu R(\mu, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof. Just set $S = T$ in Theorem 1.1.

$$= \frac{1}{2}(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}((S(r)x)(S(t)x)^*$$

$$+ (S(t)x)(S(r)x)^*)\, dr\, dt$$

$$\leqslant \frac{1}{2}(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*)\, dr\, dt$$

$$= (\int_0^\infty e^{-\mu s}\, ds)(\int_0^\infty e^{-\mu t} T(t)xx^*\, dt) = \mu^{-1} R(\mu, A)xx^*$$

where the handling of the integral is justified by [1, §8, n° 4, Proposition 9]. □

Corollary 1.4. *Let T be a semigroup of Schwarz maps (resp., weak*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with Re$(\lambda) > 0$:*

$$(R(\lambda, A)x)(R(\lambda, A)x)^* \leqslant (Re\lambda)^{-1} R(Re\lambda, A)xx^*\ ,\ x \in M.$$

In particular for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$, $x \in M$:

$$(\mu R(\mu + i\alpha, A)x)(\mu R(\mu + i\alpha, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof. Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Then the semigroup

$$S := (e^{-i\mathrm{Im}(\lambda)t} T(t))_{t \geqslant 0}$$

fulfils the assumption of Thm **??** and $B := A - i\lambda$ is the generator of S. Consequently $R(\lambda, A) = R(\mathrm{Re}\lambda, B)$ and the corollary follows from Theorem **??**. □ □

As in Section C-III the following notion will be an important tool for the spectral theory of semigroups.

Definition 1.5. Let E be a Banach space and $\emptyset \neq D$ an open subset of $\mathbb{C}$. A family $R : D \to L(E)$ is called a pseudo-resolvent on D with values in E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu) R(\lambda) R(\mu)$$

for all λ and μ in D.

If R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in a C*- or W*-algebra, then R is called of Schwarz type if

$$(R(\lambda)x)(R(\lambda)x)^* \leqslant (\mathrm{Re}\lambda)^{-1} R(\mathrm{Re}\lambda)xx^*$$

for all $\lambda \in D$ and $x \in M$. R is called identity preserving if $\lambda R(\lambda)1 = 1$ for all $\lambda \in D$.

For examples and properties of a pseudo-resolvent see C-III, 2.5. We state what will be used without further reference.

(i) If $\alpha \in \mathbb{C}$ and $x \in E$ such that $(\alpha - \lambda)R(\lambda)x = x$ for some $\lambda \in D$, then $(\alpha - \mu)R(\mu)x = x$ for all $\mu \in D$ (use the "resolvent equation").

(ii) If F is a closed subspace of E such that $R(\lambda)F \subseteq F$ for some $\lambda \in D$, then $R(\mu)F \subseteq F$ for all μ in a neighbourhood of λ. This follows from the fact that for all $\mu \in D$ near λ the pseudo-resolvent in μ is given by

$$R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}.$$

Definition 1.6. We call a semigroup T on the predual M_* of a W*-algebra M identity preserving and of Schwarz type, if its adjoint weak*-semigroup has these properties. Likewise, a pseudo-resolvent R on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in M_* is called identity preserving and of Schwarz type, if R' has these properties.

Since for a semigroup of contractions on a Banach space

$$\mathrm{Fix}(T) = \bigcap_{t \geqslant 0} \ker(\mathrm{Id} - T(t)) =$$

$$= \ker(\mathrm{Id} - \lambda R(\lambda, A)) = \mathrm{Fix}(\lambda R(\lambda, A))$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, it follows that a semigroup of contractions on M is identity preserving if and only if the (pseudo)-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ given by

$$R(\lambda) := R(\lambda, A)|_D$$

is identity preserving. By Corollary **??** an analogous statement holds for "Schwarz type".

1.3 Induction and Reduction

1. If E is a Banach space and $S \subseteq L(E)$ a semigroup of bounded operators, then a closed subspace F is called S-invariant, if $SF \subseteq F$ for all $S \in S$. We call the semigroup $S|_F := \{S|_F : S \in S\}$ the reduced semigroup. Note that for a one-parameter semigroup T (resp., pseudo-resolvent R) the reduced semigroup is again strongly continuous (resp. $R|_F$ is again a pseudo-resolvent) (compare the construction in A-I,3.2).

2. Let M be a W*-algebra, $p \in M$ a projection and $S \in L(M)$ such that $S(p^\perp M) \subseteq p^\perp M$ and $S(Mp^\perp) \subseteq Mp^\perp$, where $p^\perp := 1 - p$. Since for all $x \in M$:

$$p[S(x) - S(pxp)] = p[S(p^\perp xp) + S(xp^\perp)]p = 0,$$

we obtain $p(Sx)p = p(S(pxp))p$. Therefore the map

$$S_p := (x \mapsto p(Sx)p) : pMp \to pMp$$

is well defined. We call S_p the induced map. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$.

If $T = (T(t))_{t \geqslant 0}$ is a weak*-semigroup on M which is of Schwarz type and if $T(t)(p^\perp) \leqslant p^\perp$ for all $t \in \mathbb{R}_+$, then T leaves $p^\perp M$ and $Mp^\perp$ invariant. It is easy to see that the induced semigroup $T_p = (T(t)_p)_{t \geqslant 0}$ is again a weak*-semigroup.

If R is an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in M such that $R(\mu)p^\perp \leqslant p^\perp$ for some $\mu \in \mathbb{R}_+$, then $p^\perp M$ and $Mp^\perp$ are R-invariant. Again, the induced pseudo-resolvent R_p is of Schwarz type and identity preserving.

3. Let φ be a positive normal linear functional on a W*-algebra M such that $T_*\varphi = \varphi$ for some identity preserving Schwarz map T on M with preadjoint $T_* \in L(M_*)$. Then $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$ where $s(\varphi)$ is the support projection of φ.

To see this let $L_\varphi := \{x \in M : \varphi(xx^*) = 0\}$ and $M_\varphi := L_\varphi \cap L_\varphi^*$. Since φ is T_*-invariant, and T is a Schwarz map, the subspaces L_φ and M_φ are T-invariant. From $M_\varphi = s(\varphi)^\perp M s(\varphi)^\perp$ and $T(s(\varphi)^\perp) \leqslant 1$ it follows that $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$.

Let $T_{s(\varphi)}$ be the induced map on $M_{s(\varphi)}$. If

$$s(\varphi)M_*s(\varphi) := \{\psi \in M_* : \psi = s(\varphi)\psi s(\varphi)\}$$

where $\langle s(\varphi)\psi s(\varphi), x\rangle := \langle \psi, s(\varphi)xs(\varphi)\rangle$ ($x \in M$), and if $\psi \in s(\varphi)M_*s(\varphi)$, then for all $x \in M$:

$$(T_*\psi)(x) = \psi(Tx) = \langle \psi, s(\varphi)(Tx)s(\varphi)\rangle =$$
$$= \langle \psi, s(\varphi)(T(s(\varphi)xs(\varphi)))s(\varphi)\rangle = \langle T_*\psi, s(\varphi)xs(\varphi)\rangle,$$

hence $T_*\psi \in s(\varphi)M_*s(\varphi)$. Since the dual of $s(\varphi)M_*s(\varphi)$ is $M_{s(\varphi)}$, it follows that the adjoint of the reduced map $T_*|$ is identity preserving and of Schwarz type.

For example, if T is an identity preserving semigroup of Schwarz type on M_* such that $\varphi \in \mathrm{Fix}(T)$, then the semigroup $T|(s(\varphi)M_*s(\varphi))$ is again identity preserving and of Schwarz type. Furthermore, if R is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in M_* which is identity preserving and of Schwarz type such that $R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$, then $R|s(\varphi)M_*s(\varphi)$ has the same properties.

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