$t \to h_t(x) f(\phi(t,x)) = (T(t)f)(x)$ is continuous on $\mathbb{R}$. It follows from the dominated convergence theorem that $T(\cdot)f$ is weakly continuous. Hence $(T(t))_{t\in\mathbb{R}}$ is strongly continuous (see e.g., [Davies (1980), Prop. 1.23]).

□

The group defined by (3.12) is positive whenever $(h_t)_{t\in\mathbb{R}} \subset C^b(X)_+$. We now show that every positive group on $C_o(X)$ is of the form (3.12).

<u>Proposition</u> 3.9. Let $(T(t))_{t\in\mathbb{R}}$ be a strongly continuous group of positive operators on $C_o(X)$. Then there exist a continuous flow on X and a continuous cocycle $(h_t)_{t\in\mathbb{R}}$ of ϕ such that (3.12) holds.

<u>Proof</u>. Since $T(t)$ and $T(t)^{-1} = T(-t)$ are positive operators, $T(t)$ actually is a lattice isomorphism. Then there exist a homeomorphism ϕ_t on X and $h_t \in C^b(X)_+$ such that $T(t)f = h_t \cdot f \circ \phi_t$ for all $f \in C_o(X)$ $(t \in \mathbb{R})$. The group property of $(T(t))_{t\in\mathbb{R}}$ then implies that $\phi(t,x) := \phi_t(x)$ defines a flow on X and that $(h_t)_{t\in\mathbb{R}}$ is a cocycle of ϕ. It remains to show that ϕ and $(h_t)_{t\in\mathbb{R}}$ are continuous.
At first we consider the flow. Since we have $\phi_{t+s} = \phi_t \circ \phi_s$ and every ϕ_t is a homeomorphism on X, it is enough to establish continuity of ϕ at points $(0,x_o) \in \mathbb{R} \times X$. Given a compact neighbourhood V of $x_o = \phi(0,x_o)$, there exists a continuous function $f : X \to [0,1]$ satisfying $f(x_o) = 1$ and $\operatorname{supp} f \subset V$. There exists $t_o > 0$ such that $\|T(t)f - f\| < \frac{1}{2}$ for $|t| \le t_o$. Let $W := \{x \in X: |f(x)| > \frac{1}{2}\}$; then for $|t| \le t_o$ and $x \in W$ we have $|h_t(x) \cdot f(\phi(t,x)) - f(x)| < \frac{1}{2}$ and $|f(x)| > \frac{1}{2}$; hence $f(\phi(t,x)) > 0$. This implies that $\phi(t,x) \in V$ whenever $|t| \le t_o$ and $x \in W$.
To prove continuity of the cocycle we first remark that by strong continuity of $(T(t))_{t\in\mathbb{R}}$ the mapping $(t,x) \to (T(t)f)(x)$ is continuous on $\mathbb{R} \times X$ for every fixed $f \in C_o(X)$. Given compact subsets $A \subset \mathbb{R}$, $B \subset X$, the set $C := \phi(A\times B)$ is compact; hence there exists $f \in C_o(X)$ such that $f|_C = 1$. For $(t,x) \in A \times B$ we have $h_t(x) = (T(t)f)(x)$. Thus $(t,x) \to h_t(x)$ is continuous on $A \times B$.

□

<u>Corollary</u> 3.10. Let ϕ be a separately continuous flow. Then ϕ is continuous. If $(h_t)_{t\in\mathbb{R}}$ is a cocycle of ϕ such that $t \to h_t(x)$ is continuous for every $x \in X$, then $(h_t)_{t\in\mathbb{R}}$ is continuous.