Proof. From A-I,(1.1) we know that

$$\omega(\mathcal{T}) = \lim_{t\to\infty} \tfrac{1}{t}\log\|T(t)\| \ .$$

Since the spectral radius of $T(t)$ is given as

$$r(T(t)) = \lim_{n\to\infty} \|T(nt)\|^{1/n}$$

we obtain for $t > 0$

$$\begin{aligned} r(T(t)) &= \lim_{n\to\infty} \exp(\ t(nt)^{-1} \log \|T(nt)\|\ ) \\ &= e^{\omega t} \ . \end{aligned}$$

□

It was shown in A-I,Prop.1.11 that the spectral bound $s(A)$ is always dominated by the growth bound ω and therefore $e^{s(A)t} \leq r(T(t))$. If the above mentioned spectral mapping theorem holds - as is the case for bounded generators (e.g., see Thm. VII.3.11 of Dunford-Schwartz (1958)) we obtain the equality

$$e^{s(A)t} = r(T(t)) = e^{\omega t}$$

hence $s(A) = \omega$. Therefore the following corollary is a consequence of the definitions of $s(A)$ and ω .

Corollary 1.2. Consider the semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ generated by some bounded linear operator $A \in \mathcal{L}(E)$. If $\mathrm{Re}\lambda < 0$ for each $\lambda \in \sigma(A)$ then $\lim_{t\to\infty}\|T(t)\| = 0$.

Through this corollary we have re-established a famous result of Liapunov which assures that the solutions of the linear Cauchy problem

$$\dot{x}(t) = Ax(t) \ , \ x(0) = x_o \in \mathbb{C}^n \quad \text{and} \quad A = (a_{ij})_{n\times n}$$

are 'stable' ; i.e., they converge to zero as $t \to \infty$, if the real parts of all eigenvalues of the matrix A are smaller than zero.
For unbounded generators the situation is much more difficult and $s(A)$ may differ drastically from ω .

Example 1.3. (Banach function space, Greiner-Voigt-Wolff (1981)) Consider the Banach space E of all complex valued continuous functions on $\mathbb{R}_+$ which vanish at infinity and are integrable for $e^x dx$, i.e.

$$E := C_o(\mathbb{R}_+) \cap L^1(\mathbb{R}_+,\ e^x dx)$$

endowed with the norm

$$\|f\| := \|f\|_\infty + \|f\|_1 = \sup\{\ |f(x)|\ :\ x \in \mathbb{R}_+\ \} + \int_0^\infty |f(x)|e^x\, dx \ .$$