

Extended Notes for B-II

July 2, 2025

4 Positive semigroups generated by elliptic operators on spaces of continuous functions

Important examples of semigroups on $C_0(\Omega)$ or $C(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open and bounded, are generated by elliptic differential operators. In the following we put together a series of results starting with the Laplacian subject to Dirichlet and to Robin boundary conditions and ending with the Dirichlet-to-Neumann operator on $C(\partial\Omega)$. Each time we obtain a positive irreducible semigroup. We consider $\mathbb{K} = \mathbb{R}$ throughout this section.

4.1 The Laplacian

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. We say that Ω is *Dirichlet-regular* if for every $g \in C(\partial\Omega)$ there exists a (unique) function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\Delta u = 0 \quad \text{and} \quad (1)$$

$$u|_{\partial\Omega} = g. \quad (2)$$

This means that the Dirichlet problem is well-posed. This property is very well understood and precise characterizations in terms of barriers or of capacity are known. If Ω has Lipschitz boundary, then Ω is Dirichlet regular. In dimension $d = 2$ it suffices that Ω is simply connected.

We refer to Arendt-Urban [14], Section 6.9 or Gilbarg-Trudinger [21], Section 2.8 for further information on the Dirichlet Problem.

The Dirichlet Laplacian Δ_0 on $C_0(\Omega)$ is defined by

$$\Delta_0 u := \Delta u \quad (3)$$

$$D(\Delta_0) := \{u \in C_0(\Omega) : \Delta u \in C_0(\Omega)\}. \quad (4)$$

Here Δu is to be understood in the sense of distributions.

Theorem 4.1. *The following are equivalent.*

- (a) Ω is Dirichlet regular;
- (b) Δ_0 generates a positive semigroup \mathcal{T} on $C_0(\Omega)$.

In that case the semigroup \mathcal{T} is holomorphic of angle $\pi/2$. Moreover $T(t)$ is compact for all $t > 0$. If Ω is connected, then the semigroup is irreducible. Moreover,

$$\|T(t)\| \leq M e^{-\varepsilon t} \quad (t \geq 0) \quad (5)$$

for some $\varepsilon > 0$, $M \geq 1$.

This result is due to Arendt-Bénilan [7] besides irreducibility on which we comment later. In Example C-II.1.5 (e), the generation result was obtained if Ω has C^2 -boundary.

The implication (a) \Rightarrow (b) of Theorem 4.1 is proved below in order to show how the Dirichlet problem comes into play and leads to a result with minimal regularity assumptions on the boundary of Ω .

We use the following abstract generation result which is of independent interest. By C-II, Theorem 1.2 a densely defined operator A generates a contractive positive semigroup if and only if A is dispersive and $(\lambda - A)$ is surjective for some $\lambda > 0$. We now describe the case $\lambda = 0$.

Theorem 4.2. *Let A be a densely defined operator on a real or complex Banach lattice E . The following are equivalent.*

- (a) *A generates a positive, contractive semigroup and $\mathfrak{s}(A) \leq 0$.*
- (b) *A is dispersive and surjective.*

In particular, (b) implies that A is closed.

Dispersive operators are defined before C-II, Theorem 1.2. A densely defined operator A on $C_0(\Omega)$ is dispersive iff for $u \in D(A)$, $x_0 \in \overline{\Omega}$:

$$u(x_0) = \sup_{x \in \overline{\Omega}} u(x) > 0 \text{ implies } (Au)(x_0) \leq 0.$$

Proof. (Theorem 4.2.) (b) \Rightarrow (a)

Consider the equivalent norm

$$\|u\|_1 := \|u^+\| + \|u^-\|$$

on E . Since A is dispersive it is dissipative with respect to this new norm as is easy to see. Now Theorem 4.5 of Arendt, Chalendar and Moletsare [1] implies that A generates a contraction semigroup \mathcal{T} and A is invertible. Since A is dispersive, it follows from C-II, Theorem 1.2 that \mathcal{T} is positive and contractive (with respect to the original norm). Since $R(\lambda, A) \geq 0$ for $\lambda > 0$, it follows that $-A^{-1} \geq 0$. Now C-I, Theorem 1.1 (vi) implies that $s(A) \leq 0$.

(a) \Rightarrow (b) is obvious from C-II, Theorem 1.2. □

Proof. (Theorem 4.1.) (a) \Rightarrow (b)

The operator Δ_0 is dispersive by the maximum principle. If Ω is Dirichlet regular, then Δ_0 is surjective. In fact, let $f \in C_0(\Omega)$. Extend f by 0 to \mathbb{R}^n and let $w = \Gamma * f$, where Γ is the fundamental solution of Laplace's equation, see Gilbarg and Trudinger [21], (2.12). Then $w \in C(\mathbb{R}^n)$ and $\Delta w = f$ in the sense of distributions. Let $g = w|_{\partial\Omega}$ and let $v \in C^2(\overline{\Omega}) \cap C(\overline{\Omega})$ be the solution of the Dirichlet problem; i.e. $v|_{\partial\Omega} = g$ and $\Delta v = 0$ in Ω . Then $u := w - v \in D(\Delta_0)$ and $\Delta u = f$.

We have shown that Δ_0 satisfies condition (b) of Theorem 4.2. Thus Δ_0 generates a positive, contractive C_0 -semigroup $(T(t))_{t \geq 0}$ on $C_0(\Omega)$ and $s(\Delta_0) \leq 0$. Since by C-IV Theorem 1.1 (iv) $\mathfrak{s}(\Delta_0) = \omega_0(\Delta_0)$, it is exponentially stable.

We refer to Arendt and Bénilan [7] for the proof of (b) \Rightarrow (a). □

We want to add two further comments on the Dirichlet Laplacian Δ_0 on $C_0(\Omega)$. The first concerns its domain

$$D(\Delta_0) = \{u \in C_0(\Omega) : \Delta u \in C_0(\Omega)\}.$$

This distributional domain is not contained in $C^2(\Omega)$ for any open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, see Arendt-Urban [14, Theorem 6.60].

Our second comment concerns the proof of holomorphy. It can be given via Gaussian estimates (see the Extended Notes for C-II). In our context, a short proof based on Kato's inequality of C-II, Section 2 is more appealing (see Arendt-Batty [6]).

Finally, we comment on irreducibility. On $C_0(\Omega)$ it is a strong property. By C-III, Theorem 3.2 (ii) it means that for $0 \leq f \in C_0(\Omega)$, $f \neq 0$,

$$(T(t)f)(x) > 0 \text{ for all } x \in \Omega, t > 0.$$

On $L^2(\Omega)$ irreducibility is much weaker (meaning that $(T(t)f)(x) > 0$ x -a.e.), but easy to prove (see the Extended Notes to C-I). In the paper Arendt, ter Elst, Glück [3] an argument based on Banach lattice technique shows how irreducibility on $L^2(\Omega)$ can be carried over to $C_0(\Omega)$ or even to $C(\overline{\Omega})$ in the case of Robin boundary conditions which we consider now.

By $H^1(\Omega) := \{u \in L^2(\Omega) : \partial_j u \in L^2(\Omega) \text{ for } j = 1, \dots, n\}$ we denote the first Sobolev space.

We assume that Ω has Lipschitz boundary. Then there exists a unique bounded operator $\text{tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ such that $\text{tr}(u) = u|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega})$. It is called the *trace operator*.

Here the space $L^2(\partial\Omega)$ is defined with respect to the surface measure (i.e. the $(d-1)$ -dimensional Hausdorff measure) on $\partial\Omega$.

The normal derivative $\partial_\nu u$ of u is defined as follows. Let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Let $h \in L^2(\partial\Omega)$. We say that h is the (outer) normal derivative of u and write $\partial_\nu u = h$ if

$$\int_{\Omega} \Delta u v + \int_{\Omega} \nabla u \nabla v = \int_{\partial\Omega} h v$$

for all $v \in C^1(\overline{\Omega})$.

If $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$ we say $\partial_\nu u \in L^2(\partial\Omega)$ if there exists $h \in L^2(\partial\Omega)$ such that $\partial_\nu u = h$.

Remark. Since Ω has Lipschitz boundary the outer normal $\nu(z)$ exists for almost all $z \in \partial\Omega$ and $\nu \in L^\infty(\partial\Omega)$. But we do not use this outer normal and rather define $\partial_\nu u$ weakly by the validity of Green's formula.

Let $\beta \in L^\infty(\partial\Omega)$. We define the Laplacian Δ^β with Robin boundary conditions as follows:

$$D(\Delta^\beta) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \partial_\nu u + \beta \text{tr}(u) = 0\} \quad (6)$$

$$\Delta^\beta u := \Delta u. \quad (7)$$

We call Δ^β briefly the Robin-Laplacian. Note that for $\beta = 0$, we obtain Neumann boundary conditions, and $\Delta^0 =: \Delta^N$ is the Neumann Laplacian.

The following result is valid.

Theorem 4.3 (4.3). Assume that $\Omega \subset \mathbb{R}^d$ is bounded, open, connected with Lipschitz boundary, and let $\beta \in L^\infty(\partial\Omega)$. Then Δ^β generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on $C(\overline{\Omega})$. Moreover, $T(t)$ is compact for all $t > 0$.

The generation property on $C(\overline{\Omega})$ is due to Nittka [24]. A major point is to show that the resolvent of the corresponding operator on $L^2(\Omega)$ leaves $C(\overline{\Omega})$ invariant. Given $f \in C(\overline{\Omega})$, $u \in H^1(\Omega)$ such that $u - \Delta u = f$, $\partial_\nu u + \beta u|_{\partial\Omega} = 0$.

Theorem 4.4 (4.8). *Assume (4.2) and (4.4). Then $\Delta - V$ generates a positive, irreducible semigroup on $C(\partial\Omega)$. If $V \geq 0$, then the semigroup is contractive.*

If Ω is of class C^∞ similar results have been obtained by Escher [20] and Engel [19]. Under the very general conditions here, Theorem 4.8 is due to Arendt and ter Elst [13]. There it is shown that N_V is resolvent-

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one has to show that $u \in C(\overline{\Omega})$. Nittka extends u to an open set $\tilde{\Omega}$ containing $\overline{\Omega}$ by reflecting u along the graph. Then u becomes the solution of an elliptic problem on $\tilde{\Omega}$. Continuity on $\tilde{\Omega}$, and hence on $\overline{\Omega}$, follows from the De Giorgi-Nash Theorem.

Irreducibility is due to Arendt, ter Elst and Glück [3], Theorem 4.5.

Since the semigroup is holomorphic, by C-II, Theorem 3.2 (ii), it implies that

$$\inf_{x \in \overline{\Omega}} (T(t)f)(x) > 0 \quad (4.1)$$

for all $t > 0$ and $0 \leq f \in C(\overline{\Omega})$, $f \neq 0$.

Denote by $\mathfrak{s}(\Delta^\beta)$ the spectral bound of Δ^β . By C-III, Theorem 3.8 (iv), $\mathfrak{s}(\Delta^\beta)$ is the unique eigenvalue with a positive eigenfunction $u_0 \geq 0$, $u_0 \neq 0$. It follows from (4.1) that u_0 is strictly positive; i.e.,

$$\inf_{x \in \overline{\Omega}} u_0(x) > 0,$$

a remarkable property, which has important applications to semi-linear problems, see Arendt-Daners [?].

The spectral bound $\mathfrak{s}(\Delta^\beta)$ determines the asymptotic behavior of the semigroup \mathcal{T} . In fact, the following follows from B-III Proposition 3.5.

Corollary 4.5 (4.4). *There exist a strictly positive Borel measure μ on $\overline{\Omega}$, $M \geq 0$ and $\varepsilon > 0$ such that $\langle \mu, u_0 \rangle = 1$ and*

$$\|T(t) - e^{\mathfrak{s}(\Delta^\beta)t}P\| \leq Me^{-\varepsilon t}$$

for all $t \geq 0$, where $P \in \mathcal{L}(C(\overline{\Omega}))$ is given by

$$Pf = \langle \mu, f \rangle u_0.$$

The theorem says that the rescaled semigroup $(e^{-\mathfrak{s}(\Delta^\beta)t}T(t))_{t \geq 0}$ converges in the operator norm to the rank-1-projection P exponentially fast.

5 Elliptic operators in divergence form

The preceding results extend to elliptic operators in divergence form with bounded measurable coefficients. Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

Let $a_{k,\ell}, b_k, c_k, c_0 \in L^\infty(\Omega)$, $k, \ell = 1, \dots, n$ such that for some $\alpha > 0$

$$\sum_{k,\ell=1}^n a_{k,\ell}(x) \xi_k \xi_\ell \geq \alpha |\xi|^2$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$, where $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$.

Let $H_{\text{loc}}^1(\Omega) := \{u \in L_{\text{loc}}^2(\Omega) : \partial_k u \in L_{\text{loc}}^2(\Omega), k = 1, \dots, n\}$.

Define $\mathcal{A} : H_{\text{loc}}^1(\Omega) \rightarrow C_c^\infty(\Omega)'$ by

$$\langle \mathcal{A}u, v \rangle = \sum_{k,\ell=1}^d \partial_k(a_{k\ell} \partial_\ell u) + \sum_{k=1}^d \partial_k(b_k u) + \sum_{k=1}^d c_k \partial_k u + c_0 u.$$

We define A_0 as the part of \mathcal{A} in $C_0(\Omega)$; i.e.

$$\mathcal{D}(A_0) := \{u \in C_0(\Omega) \cap H_0^1(\Omega) : \mathcal{A}u \in C_0(\Omega)\}$$

$$A_0 u := \mathcal{A}u.$$

Then Theorem 4.1 holds with Δ_0 replaced by A_0 . It is remarkable that Dirichlet regularity of Ω is the right regularity condition at the boundary, a discovery due to Stampacchia. We refer to Arendt and B enilan [7], Section 4 for a proof of the following result.

Theorem 5.1 (4.5). *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded, open, connected, Dirichlet regular set. Then A_0 generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on $C_0(\Omega)$. Moreover, $T(t)$ is compact for all $t > 0$.*

Remark 5.2. *The proof of holomorphy depends on Gaussian estimates, which in [7] were merely known if the $b_k \in W^{1,\infty}(\Omega)$. Later, it was shown by Davies [15] that they always hold.*

Also the results for Robin boundary conditions Theorem 4.3 and 4.4 can be extended to elliptic operators in divergence form on $C(\overline{\Omega})$; see Theorem 4.5 in Arendt, ter Elst, Gl uck [3]. It uses results of Nittka [24].

6 Elliptic operators in non-divergence forms

The techniques for elliptic operators in non-divergence form are quite different than those used in the divergence-case form. But the results are similar.

Let $\Omega \subset \mathbb{R}^n$ be open and connected. We assume that Ω satisfies the uniform exterior cone condition. This means the following. There exists a finite, right circular cone V such that for each $x \in \partial\Omega$ there exists a cone V_x which is congruent to V such that $V_x \cap \overline{\Omega} = \{x\}$.

Let $a_{k\ell} = a_{\ell k} \in C(\overline{\Omega})$, $b_k \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$, $c \leq 0$ such that

$$\sum_{k,\ell=1}^n a_{k\ell}(x) \xi_k \xi_\ell \geq \mu |\xi|^2$$

for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$ and some $\mu > 0$.

For $u \in W_{\text{loc}}^{2,n}(\Omega)$ we define

$$\mathcal{A}u = \sum_{k,\ell=1}^n \partial_k a_{k\ell} \partial_\ell u + \sum_{k=1}^n b_k \partial_k u + cu.$$

Thus $\mathcal{A} : W_{\text{loc}}^{2,n}(\Omega) \rightarrow L_{\text{loc}}^n(\Omega)$ is linear. Here

$$W_{\text{loc}}^{2,n}(\Omega) := \{u \in L_{\text{loc}}^n(\Omega) : \partial_k u \in L_{\text{loc}}^n(\Omega), \partial_k \partial_\ell u \in L_{\text{loc}}^n(\Omega) \text{ for all } k, \ell = 1, \dots, n\}.$$

We consider the operator A on $C_0(\Omega)$ defined by

$$\mathcal{D}(A) := \{u \in C_0(\Omega) \cap W_{\text{loc}}^{2,n}(\Omega) : \mathcal{A}u \in C_0(\Omega)\}$$

$$Au := \mathcal{A}u.$$

Then the following holds.

Theorem 6.1 (4.6). *The operator A generates a positive, irreducible, contractive holomorphic semigroup $(T(t))_{t \geq 0}$ on $C_0(\Omega)$. Moreover*

$$\|T(t)\| \leq Me^{-\varepsilon t} \quad (t \geq 0)$$

for some $\varepsilon > 0$, $M \geq 1$. The resolvent of A is compact.

This result is proved by Arendt and Schätzle [8], Proposition 4.7. The monograph of Lunardi [23] is devoted to the study of holomorphic semigroups generated by elliptic operators in non-divergence form and Theorem 4.6 is a generalization of results found therein.

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of Lunardi's results which were obtained under the more restrictive assumption of a C^2 -boundary. Positivity and irreducibility are a consequence of the Alexandrov maximum principle. For C^2 -boundary also results on $L^p(\Omega)$ spaces are obtained by Denk, Hieber and Prüss [18] whose main interest lies in proving maximal regularity and establishing a bounded H^∞ -calculus.

However, in the situation of Theorem 4.6, without assuming merely the uniform exterior cone condition on Ω , it seems not to be known whether the semigroup extends to a strongly continuous semigroup on $L^p(\Omega)$ for some $p \in [1, \infty)$.

Theorem 4.6 is extended by Arendt and Schätzle [9] to unbounded open sets which satisfy the locally uniform exterior cone condition. However, in the case of unbounded Ω the semigroup converges merely strongly to 0 (and not exponentially fast).

7 The Dirichlet-to-Neumann operator on $C(\partial\Omega)$

Let Ω be a bounded, open, connected subset of \mathbb{R}^n with Lipschitz boundary and let $V \in L^\infty(\Omega)$. We consider the Dirichlet-to-Neumann operator with respect to $\Delta - V$ on the space $C(\partial\Omega)$. For that we first establish well-posedness of the Dirichlet Problem.

We assume throughout this subsection that

$$u \in C_0(\Omega), \Delta u - Vu = 0 \text{ implies } u = 0. \tag{4.2}$$

This is exactly the condition that the solutions of the Dirichlet problem with respect to $\Delta - V$ formulated in Proposition 4.7 are unique. An equivalent condition is

$$u \in H_0^1(\Omega), \Delta u - Vu = 0 \text{ implies } u = 0; \quad (4.3)$$

(which means that $0 \notin \sigma(\Delta_0 - V)$ where Δ_0 is the Dirichlet Laplacian on $L^2(\Omega)$).

Proposition 7.1 (4.7). *Assume (4.2). Let $g \in C(\partial\Omega)$. Then there exists a unique $u_g \in C(\overline{\Omega})$ such that*

$$(\Delta - V)u_g = 0 \quad \text{and} \quad u_g|_{\partial\Omega} = g.$$

Thus, u_g is a harmonic function with respect to $\Delta - V$ which has to be understood in the sense of distributions; i.e.,

$$\int_{\Omega} u_g \Delta \varphi - \int_{\Omega} V u_g \varphi = 0$$

for all $\varphi \in C_c^\infty(\Omega)$.

For a simple proof of Proposition 4.7 we refer to [11].

Next we define the Dirichlet-to-Neumann operator N_V with respect to $\Delta - V$ on $C(\partial\Omega)$ as follows.

$$\mathcal{D}(N_V) := \{g \in C(\partial\Omega) : u_g \in H^1(\Omega), \text{ and } \partial_\nu u_g \in C(\partial\Omega)\} \quad (8)$$

$$N_V g := -\partial_\nu u_g. \quad (9)$$

Recall that $\partial_\nu u_g \in C(\partial\Omega)$ means that there exists $h \in C(\partial\Omega)$ such that

$$\int_{\Omega} \Delta u_g \varphi + \int_{\Omega} \nabla u_g \nabla \varphi = \int_{\partial\Omega} h \varphi$$

for all $\varphi \in C^1(\overline{\Omega})$. Then we put $\partial_\nu u_g := h$.

We will need the hypothesis that $-\Delta_0 + V$ is form-positive i.e.

$$\int_{\Omega} (|\nabla u|^2 + V|u|^2) \geq 0 \quad (4.4)$$

for all $u \in H_0^1(\Omega)$.

Theorem 7.2 (4.8). *Assume (4.2) and (4.4). Then $\Delta - V$ generates a positive, irreducible semigroup on $C(\partial\Omega)$. If $V \geq 0$, then the semigroup is contractive.*

If Ω is of class C^∞ similar results have been obtained by Escher [20] and Engel [19]. Under the very general conditions here, Theorem 4.8 is due to Arendt and ter Elst [13]. There it is shown that N_V is resolvent-positive and that the domain is dense (which is the main difficulty). Then by B-I, Theorem 1.8 N_V generates a positive semigroup. Irreducibility is surprising. In fact, even though Ω is supposed to be connected, $\partial\Omega$ might not be connected (consider a ring for example). The fact that the semigroup is irreducible shows that the operator N_V is non-local in quite a dramatic way.

A first result on irreducibility (on $L^2(\partial\Omega)$) was obtained by Arendt and Mazzeo [4].

It is not known so far whether the semigroup generated by N_V is holomorphic if $\partial\Omega$ has Lipschitz boundary. If the boundary is of class $C^{n+\alpha}$ with $\alpha > 0$, then it is holomorphic of angle $\pi/2$. This is due to ter Elst and Ouhabaz [25].

The operator N_V is also called voltage-to-current map and has physical meaning. One version of the famous Calderón-Problem is the question whether for $V_1, V_2 \in L^\infty(\Omega)$, such that $0 \notin \sigma(\Delta_{V_1}) \cup \sigma(\Delta_{V_2})$,

$$N_{V_1} = N_{V_2} \text{ implies } V_1 = V_2.$$

This is true under the only assumption that Ω has Lipschitz boundary; see Theorem 1.1 by Krupchyk and Uhlmann [22].

Finally we mention that N_V may generate a positive semigroup even if (4.4) is violated. This and other surprising phenomena were discovered by Daners [16], and led

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to the new theory of eventually positive semigroups; see e.g. [17].

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