$\lim_{t\to\infty} T(t)$ - if it exists in some operator topology - is always a projection P onto the fixed space of $(T(t))_{t\geq 0}$ which coincides with the kernel of A. In case $P = 0$ we have stability which was discussed in Sec.1 . In this section we mainly consider the case $s(A) = 0 \in P\sigma(A)$ and show that the symmetric structure of the boundary spectrum of the generator of a positive semigroup yields interesting results.

We begin our discussion by considering quasi-compact semigroups. Using the general results presented in Sec.2 of B-IV and the spectral theoretical result of Chapter C-III we obtain the following.

Theorem 2.1. Let $(T(t))_{t\geq 0}$ be a positive semigroup on a Banach lattice E which is bounded, quasi-compact and has spectral bound zero. Then there exists a positive projection P of finite rank and suitable constants $\delta > 0$, $M \geq 1$ such that

(2.1) $\|T(t) - P\| \leq M\cdot e^{-\delta t}$ for all $t \geq 0$.

Proof. By Thm.2.9 of B-IV the set $\{\lambda \in \sigma(A) : \operatorname{Re} \lambda = 0\}$ is finite and by Thm.2.10 of C-III imaginary additively cyclic. Thus it contains only the value $s(A) = 0$. Then by B-IV,(2.5) we have

$$T(t) = \sum_{j=0}^{k-1} \frac{1}{j!}\cdot t^j A^j \circ P + R(t) \quad (t\geq 0)$$

where P is the residue of $R(.,A)$ at 0 , k is the pole order and $\|R(t)\| \leq M\cdot e^{-\delta t}$ for suitable constants $\delta > 0$, $M \geq 1$. Since we assumed that $(T(t))_{t\geq 0}$ is bounded, the pole order k has to be 1 .

□

Before discussing a concrete example we formulate some remarks related to Theorem 2.1.

Remarks 2.2. (a) If one has a positive semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ satisfying $\omega_{ess}(\mathcal{T}) < \omega(\mathcal{T})$ then the rescaled semigroup with $\tilde{T}(t) := \exp(-\omega(\mathcal{T}))T(t)$ is quasi-compact and has spectral bound zero. In order to apply Theorem 2.1 we still need the boundedness of $(\tilde{T}(t))_{t\geq 0}$ (see the following remarks).

(b) Without assuming boundedness of the semigroup one can conclude that $T(t) - \sum_{j=0}^{k-1} \frac{1}{j!}\cdot t^j A^j \circ P$ tends to zero exponentially.

(c) In the proof of Theorem 2.1 we saw that a quasi-compact semigroup of positive operators having spectral bound zero is bounded if and only if the pole order at zero is one. This is automatically true