3.6). Another way of looking at P_n is given by saying that P_n is the n-th Fourier coefficient of the τ-periodic function $s \to T(s)$. From this it follows that no non-zero $\phi \in E'$ vanishes on all P_nE simultaneously. By the Hahn-Banach theorem we conclude that spann $\cup_{n\in\mathbb{Z}} P_nE$ is dense in E .
Since $P_nE \subset D(A)$ we obtain from A-I,(3.1) that

$$AP_nf = \mu_nP_nf \tag{5.3}$$

for every $f \in E$, $n \in \mathbb{Z}$. This and A-I,(3.2) imply

$$T(t)P_nf = \exp(\mu_nt)\cdot P_nf \tag{5.4}$$

for every $t \geq 0$. Therefore μ_n is an eigenvalue of A and $\exp(\mu_nt)$ is an eigenvalue of $T(t)$ if and only if $P_n \neq 0$. In that case, P_nE is the corresponding eigenspace and we have the following lemma.

<u>Lemma</u> 5.3. For a τ-periodic semigroup $\mathcal{T}$ we take $\mu_n := 2\pi in/\tau$, $n \in \mathbb{Z}$ and consider

$$P_n := \tau^{-1}\cdot\int_0^\tau \exp(-\mu_ns)T(s)\,ds .$$

Then the following assertions are equivalent:

(a) $P_n \neq 0$

(b) $\mu_n \in P\sigma(A)$

(c) $\exp(\mu_nt) \in P\sigma(T(t))$ for every $t > 0$.

The action of A , resp. $T(t)$ on the subspaces P_nE , $n \in \mathbb{Z}$, is determined by (5.3), resp. (5.4). Moreover,

$$\begin{aligned} P_mP_nf &= \tau^{-1}\cdot\int_0^\tau \exp(-\mu_ms)T(s)P_nf\,ds = \\ &= \tau^{-1}\cdot\int_0^\tau \exp((\mu_n-\mu_m)s)P_nf\,ds = 0 \end{aligned}$$

for $n \neq m$, i.e. the subspaces P_nE are "orthogonal". Since their union is total in E one expects to be able to extend the representations (5.3) and (5.4) of A and $T(t)$. This is possible if

$$\sum_{-\infty}^{+\infty} P_n = Id ,$$

where the series should be summable for the strong operator topology. Unfortunately this is false in general since the family of projections

$$Q_H := \sum_{n\in H} P_n ,$$

where H runs through all finite subsets of $\mathbb{Z}$, may be unbounded (see the example below). Nevertheless the following is true.