Therefore

$$\lim_{s\to\infty} \psi(C(s)'p_r) = \lim_{s\to\infty} (C(s)\psi)(p_r) =$$

$$= \psi_o(p_r) = \psi_o(1) = \lim_{s\to\infty} (C(s)\psi)(1) = \psi(1) \ ,$$

which proves (b).

Suppose that (b) is satisfied. Since $C(s)'p_r \leqq 1$ for all $s\in\mathbb{R}_+$ we obtain (c). (Use that for $(x_\alpha)\in M_+$ we have $\lim_\alpha x_\alpha = 0$ in the weak*-topology if and only if $\lim_\alpha x_\alpha = 0$ in the $s^*(M,M_*)$-topology.)

Suppose that (c) holds. Since each $C(s)'$ is an identity preserving Schwarz map we obtain for all $x\in M$:

$$(C(s)'((1-p_r)x))(C(s)'((1-p_r)x)^*) \leqq$$

$$\leqq C(s)'((1-p_r)xx^*(1-p_r)) \leqq$$

$$\leqq \|x\|^2 \ C(s)'(1-p_r) \ ,$$

hence

$$s^*(M,M_*)\text{-}\lim_{s\to\infty} C(s)'((1-p_r)x) = 0 \ .$$

In particular we obtain for all $x\in\mathrm{Fix}(\mathcal{T}')$ that

$$x = \sigma(M,M_*)\text{-}\lim_{s\to\infty} C(s)'x = \sigma(M,M_*)\text{-}\lim_{s\to\infty} C(s)'(p_r x) \ .$$

Especially for $0 \neq x \in \mathrm{Fix}(\mathcal{T})$ we obtain $p_r x p_r \neq 0$. Since the W*-algebra $p_r M p_r$ is the dual of $p_r M_* p_r$ and since $\mathcal{T}^{(r)}$ is strongly ergodic, it follows that the fixed space of $\mathcal{T}$ separates the points of $\mathrm{Fix}(\mathcal{T}')$. Thus $\mathcal{T}$ is strongly ergodic ([Krengel (1985), Chap. 2, Thm. 1.4]).

□

It follows from the result above that the semigroup in [Evans (1977)] cannot be strongly ergodic on $B(H)_*$ since the associated recurrent projection is zero. But for irreducible semigroups we have the following result.