Chapter 4

Asymptotics of Positive Semigroups on C^*- and W^*-Algebras

1 Stability of Positive Semigroups

As explained in A-III, Section 1, it is possible to deduce uniform exponential stability of strongly continuous semigroups from the location of the spectrum of its generator if the spectral bound $s(A)$ and the growth bound ω_0 coincide. In this section we prove $s(A) = \omega_0$ for positive semigroups on C*-algebras and preduals of W*-algebras. A more general discussion of the "$s(A) = \omega_0$" problem can be found in Greiner et al. (1981). For the results of this section the existence of a unit is essential.

Theorem 1.1. *Let M be a* C*-*algebra with unit and $\mathcal{T} = (T(t))_{t\geqslant 0}$ a positive semigroup on M. Then*

$$-\infty < s(A) = \omega_0 \in \sigma(A).$$

Proof. For every $t \geqslant 0$ there exists φ_t in the state space $S(M)$ of M such that

$$T(t)'\varphi_t = r(T(t))\varphi_t = \exp(\omega_0 t)\varphi_t$$

(see, e.g., Groh (1981, 2.1)).

Let $n \in \mathbb{N}$ and

$$E_n := \{\varphi \in S(M) \colon T(2^{-n})\varphi = \exp(\omega_0 2^{-n})\varphi\}.$$

Then $\emptyset \neq E_{n+1} \subseteq E_n$, $(n \in \mathbb{N})$. Since $S(M)$ is $\sigma(M, M')$-compact, there exists $\varphi \in \bigcap_{n\in\mathbb{N}} E_n$. Then $T(t)'\varphi = \exp(\omega_0 t)\varphi$ follows for all $0 \leqslant t$ because the adjoint semigroup $(T(t)')_{t\geqslant 0}$ is a weak*-semigroup on M'.

Suppose $-\infty = \omega_0$. Then for $t > 0$ either $r(T(t)) = 0$ (A-III,Prop.1.1) or $T(t)'\varphi = 0$, in particular $\varphi(T(t)\mathbb{1}) = 0$. From this we obtain the contradiction $\varphi(\mathbb{1}) = 0$. Hence $-\infty < \omega_0$ and $\exp(\omega_0 t) \in \varrho(T(t)')$ for every $t \in \mathbb{R}_+$. Thus $\omega_0 \in \sigma(A)$ or $\omega_0 = s(A)$. $\square$

Remark 1.2. (i) If we consider the nilpotent translation semigroup on the C*-algebra $C_0([0,1[)$, then $\sigma(A) = \emptyset$ and $\omega_0 = -\infty$. This shows that the existence of a unit is essential.

(ii) The equality $s(A) = \omega_0$ still holds for positive semigroups on commutative C*-algebras without unit (see B-IV, Rem.1.2.b).

Theorem 1.3. *Let M be a* W**-algebra with predual M_* and let $(T(t))_{t\geqslant 0}$ be a positive semigroup on M_*. Then $s(A) = \omega_0$.*

Proof. For all $\lambda > s(A)$ and $\varphi \in M_*$

$$R(\lambda, A)\varphi = \int_0^\infty e^{-\lambda t} T(s)\varphi ds$$

which follows as in C-III, Section 1 or Greiner et al. (1981, Theorem 3). Since $\|\varphi\| = \varphi(\mathbb{1})$ for every $\varphi \in M_*^+$ and since the norm is additive on the positive cone of M_*, the integral

$$\int_0^\infty e^{\lambda t}\|T(s)\varphi\| ds$$

exists for all $\varphi \in M_*$ and all $\lambda > s(A)$. From this the assumption follows by A-IV,Thm.1.11. □

Corollary 1.4. *Let M be a* C**-algebra and $(T(t))_{t\geqslant 0}$ a positive semigroup on M'. Then $s(A) = \omega_0$ holds.*

This follows from the fact that the bidual of a C*-algebra is a W*-algebra (see Takesaki (1979, Theorem III.2.4.)).

Remark 1.5. A simple modification of A-III, Example 1.4 (take c_0 instead of ℓ^2) shows that Theorem 1.3 is no longer true for non-positive semigroups (for details see Groh and Neubrander (1981, Beispiel 2.5)).

While the growth bound ω_0 characterizes uniform exponential stability of the semigroup there are other (and weaker) stability concepts (cf. A-IV, Section 1).

Definition 1.6. Let E be a Banach space and $(T(t))_{t\geqslant 0}$ a semigroup on E. We call the semigroup

(i) *uniformly exponentially stable* if $\|T(t)\| \leqslant Me^{-\omega t}$ for some ω, $M > 0$ and all $t \geqslant 0$.

(ii) *uniformly stable* if $\lim_{t\to\infty} T(t) = 0$ in the strong operator topology.

(iii) *weakly stable* if $\lim_{t\to\infty} T(t) = 0$ in the weak operator topology.

Surprisingly all these properties coincide for positive semigroups on C*-algebras with unit.

Theorem 1.7. *Let M be a C*-algebra with unit and $(T(t))_{t\geqslant 0}$ a positive semigroup on M. Then the following assertion are equivalent.*

(a) $s(A) < 0$.

(b) *The semigroup $(T(t))_{t\geqslant 0}$ is uniformly exponentially stable.*

(c) *The semigroup $(T(t))_{t\geqslant 0}$ is uniformly stable.*

(d) *The semigroup $(T(t))_{t\geqslant 0}$ is weakly stable.*

Proof. Since $s(A) = \omega_0$ by Theorem 1.3, it suffices to show that (d) implies (a). For $t > 0$ there exists $\varphi \in S(M)$ such that

$$T(t)'\varphi = r(T(t))\varphi.$$

Then for $x \in M$

$$\varphi(T(t)^n x) = (r(T(t)))^n \varphi(x) \to 0$$

as $n \to \infty$. Therefore $r(T(t)) < 1$ or $\omega_0 < 0$. Since $s(A) \leqslant \omega_0$ the assertion follows. ☐

Remark 1.8. Consider the translation semigroup $(T(t))_{t\geqslant 0}$ on $C_0(\mathbb{R}_+)$. Then $\|T(t)\| = 1$, hence $s(A) = 1$, but $(T(t))_{t\geqslant 0}$ is strongly stable. The same holds for the translation semigroup on $L^1(\mathbb{R}_+)$. Thus Theorem 1.7 is not true for semigroups on C*-algebras without unit or on preduals of W*-algebras. For the discussion of the commutative situation we refer to B-IV, Section 1.

2 Stability of Implemented Semigroups

Let H be a Hilbert space, $\mathcal{U} = (U(t))_{t\geqslant 0}$ a strongly continuous semigroup on H with generator B and $M \subseteq \mathcal{B}(H)$ a W*-algebra, where $\mathcal{B}(H)$ is the W*-algebra of all bounded linear operators on H. Suppose $\mathcal{U}(t)^* M U(t) \subseteq M$. Then one can define a weak*-continuous semigroup $\mathcal{T}$ on M by

$$T(t)x := U(t)^* x U(t) \quad (t \in \mathbb{R}_+, x \in M).$$

We call $\mathcal{T}$ an *implemented semigroup*. Every map $T(t) \in \mathcal{T}$ of an implemented semigroup is weak*-continuous and n-positive for every $n \in \mathbb{N}$.

Remark 2.1. (i) Because of

$$\|T(t)\| = \|T(t)\mathbb{1}\| = \|U(t)^* U(t)\| = \|U(t)\|^2$$

it follows that $\omega_0(\mathcal{T}) = 2\,\omega_0(\mathcal{U})$.

(ii) If $\mathcal{T}$ is an implemented semigroup, then the preadjoint semigroup is strongly continuous on M_*. Therefore $s(A) = \omega_0$ for $\mathcal{T}$ by Theorem 1.3.

(iii) Since $\mathcal{U}$ is a strongly continuous semigroup on H, the same is true for the adjoint semigroup $\mathcal{U}^* = \{U(t)^* \colon U(t) \in \mathcal{U}\}$ and its generator is given by B^*. In analogy to Bratteli and Robinson (1979, 3.2.55) the following assertions for $x \in M$ are equivalent.

(a) $x \in D(A)$, A the generator of $\mathcal{T}$.

(b) For $\xi \in D(B)$ it follows $x\xi \in D(B^*)$ and the linear mapping

$$(\xi \mapsto x(B\xi) + B^*(x\xi)) : D(B) \to H \tag{*}$$

has a continuous extension to H.

Then for A is given as the continuous extension of (*), i.e., $Ax = xB + B^*x$ for $x \in D(A)$

In the next theorem we give some equivalent conditions for the uniform exponential stability of an implemented semigroup. As we shall see, the operator equality

$$yB + B^*y = -x \quad (x, y \in M_+)$$

is necessary and sufficient, which is in complete analogy to the classical Liapunov stability result.

Theorem 2.2. *Let M be a W^*-algebra on a Hilbert space H and let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a weak*-semigroup on M with generator A implemented by the semigroup $\mathcal{U}$ on H with generator B. Then the following assertions are equivalent.*

(a) $\omega_0(\mathcal{T}) = s(A) < 0$.

(b) The semigroup $(U(t))_{t \geqslant 0}$ is uniformly exponentially stable.

(c) There exists $0 \leqslant x \in D(A)$ such that $Ax = -\mathbb{1}$.

(d) There exists $0 \leqslant x \in D(A)$ such that $x(D(B)) \subseteq D(B^*)$ and $xB + B^*x = -\mathbb{1}$.

(e) For every $0 \leqslant x \in D(A)$ there exists $0 \leqslant y \in D(A)$ such that $Ay = -x$.

(f) For every $0 \leqslant x \in D(A)$ there exists $0 \leqslant y \in D(A)$ such that $y(D(B)) \subseteq D(B^*)$ and $yB + B^*y = -x$.

(g) $\int_0^\infty \|U(s)\xi\|^2 ds$ exists for all $\xi \in H$.

(h) $\int_0^\infty |(T(s)x)\xi|\zeta) ds$ exists for all $\xi, \zeta \in H$ and all $x \in M$.

Proof. The equivalence of (a) and (b) follows from Remark 2.1 (i), whereas (c) and (d)), resp. (e) and (f) are equivalent by the Remark 2.1 (iii)

(a) $\implies$ (c): Since $s(A) < 0$ the resolvent $R(0, A)$ exists and is a positive map on M. Therefore $R(0, A)1 \in D(A)_+$ or $Ax = -\mathbb{1}$ for some $x \in D(A)_+$.

(c) $\implies$ (e): Let $x \in D(A)_+$ such that $Ax = -\mathbb{1}$. Then

$$T(t)x - x = \int_0^t T(s)Ax \,\mathrm{d}s = -\int_0^t T(s)\mathbb{1} \,\mathrm{d}s \quad (t \geqslant 0),$$

hence

$$0 \leqslant \int_0^t T(s)\mathbb{1} \,\mathrm{d}s \leqslant x \quad (t \in \mathbb{R}_+).$$

Since the family $(\int_0^t T(s)\mathbb{1} \,\mathrm{d}s)_{t \geqslant 0}$ is increasing and bounded,

$$\lim_{t \to \infty} \int_0^t T(s)\mathbb{1} \,\mathrm{d}s$$

exists in the weak operator topology on $\mathcal{B}(H)$.

Since on bounded sets of M, the weak operator topology is equivalent to the $\sigma(M, M_*)$-topology, for every $\varphi \in M_*$ the integral $\int_0^\infty \varphi(T(s)\mathbb{1})\,\mathrm{d}s$ exists (Sakai (1971, 1.15.2.)). Take $x \in M_+$ and $\varphi \in M_*^+$. Then $x \leqslant \|x\|\mathbb{1}$ and therefore

$$\varphi(T(s)x) \leqslant \|x\|\varphi(T(s)\mathbb{1}) \quad (s \in \mathbb{R}_+).$$

Hence $\int_0^\infty \varphi(T(s)x)ds$ exists. Since the positive cones of M and M_* are generating, $\int_0^\infty \varphi(T(s)x)ds$ exists for every $x \in M$ and $\varphi \in M_*$. Therefore $R(0, A)$ exists and is positive which proves (e).

(c) $\implies$ (g): From the last paragraph we obtain that for all $\xi \in H$

$$\int_0^\infty \|U(s)\|^2 ds = \int_0^\infty (T(s)1\xi|\xi)ds$$

exists.

(g) $\implies$ (h): It follows from the polarization identity that the integral

$$\int_0^\infty (U(s)\xi|U(s)\zeta)ds$$

exists for all $\xi, \zeta \in H$. Using Takesaki (1979, Theorem III.4.2 and Theorem II.2.6), we conclude as in the implication from (c) to (e) that for all $\xi, \zeta \in H$ the integral

$$\int_0^\infty ((T(s)x)\xi|\zeta)ds \quad (x \in M)$$

is finite.

(g) $\implies$ (a): Since the vector states are dense in the predual of M and since the preadjoint semigroup of $\mathcal{T}$ is strongly continuous, it is easy to see that the integral

$$\int_0^\infty \varphi(T(s)x)ds$$

exists for all $x \in M$ and $\varphi \in M_*$ (Takesaki (1979, Theorem II.2.6)). Therefore, the resolvent $R(0, A)$ exists and is positive, hence $s(A) < 0$. □

3 Convergence of Positive Semigroups

In this section the asymptotic behavior of positive semigroups $(T(t))_{t \geqslant 0}$ on W*-algebras will be described in more detail. Essentially we distinguish three cases.

(i) The Cesàro means $\frac{1}{s}\int_0^s T(t)dt$ converge strongly to a projection P onto the fixed space of $(T(t))_{t \geqslant 0}$ (see Proposition 3.3 & 3.4).

(ii) The maps $T(t)$ converge strongly to P (see Proposition 3.7, **??** &. **??**).

(iii) The maps $T(t)$ behave asymptotically as a periodic group (Theorem 3.11).

Much of the following is based on the theory of weakly compact operator semigroups. Therefore the following compactness criterium is quite useful.

Proposition 3.1. *Let M be a W^*-algebra, $\mathcal{T}$ a bounded semigroup of positive maps on M_* and suppose that there exists a faithful family Φ of $\mathcal{T}$-subinvariant states in M_*. Then $\mathcal{T}$ is relatively compact in the weak operator topology of $\mathcal{L}(M_*)$. In particular, $\mathcal{T}$ is strongly ergodic, i.e.,*

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s T(t)x \,\mathrm{d}t$$

exists for every x in M and yields a projection onto $\mathrm{Fix}\,(\mathcal{T})$.

Proof. Since the positive cone of M_* is generating, it is enough to show that for every $0 \leqslant \varphi \in M_*$ the orbit $\{T(t)\varphi \colon t \in \mathbb{R}_+\}$ is relatively weak compact. For this we use Takesaki (1979, Theorem III.5.4.(iii)).

Let $(p_n)_{n\in\mathbb{N}}$ be a decreasing sequence of projections in M such that $\inf_n p_n = 0$. Then $\lim_n \varphi(p_n) = 0$ for every $\varphi \in \Phi$. Since

$$(T(t)p_n)^2 \leqslant T(t)p_n, \quad t \in \mathbb{R}_+,$$

we obtain by a classical *inequality of Kadison* that

$$0 \leqslant \varphi((T(t)p_n)^2) \leqslant \varphi(T(t)p_n) \leqslant \varphi(p_n),$$

hence $\lim_n \varphi(T(t)p_n) = 0$ uniformly in $t \in \mathbb{R}_+$. Since the family Φ is faithful on M, it follows from Takesaki (1979, Proposition III.5.3) that $(T(t)p_n)$ converges to zero in the $s(M, M_*)$-topology uniformly in $t \in \mathbb{R}_+$. Since this topology is finer than the weak*-topology on M, we obtain the relative compactness of $\mathcal{T}$ which implies the strong ergodicity. ☐

Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a W^*-algebra M. We call

$$p_r \coloneqq \sup\{s(|\varphi|) \colon \varphi \in \mathrm{Fix}\,(T)\}$$

the recurrent projection associated with $\mathcal{T}$. For a motivation of this definition compare, e.g., Davies (1976, Section 6.3).

Since $T(t)|\varphi| = |\varphi|$ for all $\varphi \in \mathrm{Fix}\,(T)$ (D-III, Cor. 1.5), we obtain $T(t)'p_r \geqslant p_r$ (see D-I,Sec.3.(c)). Let $\mathcal{T}^{(r)}$ be the reduced semigroup on $p_r M_* p_r$ with generator $A^{(r)}$. Then $\mathcal{T}^{(r)}$ is identity preserving and of Schwarz type. Similarly, if $\mathcal{R}$ is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \mathrm{Re}(\lambda) > 0\}$ with values in M_* such that $\mathcal{R}$ is identity preserving and of Schwarz type, then the recurrent projection associated with $\mathcal{R}$ is defined using $\mathrm{Fix}\,(\mathcal{R})$.

Remark 3.2. (i) Let $\varphi \in M_*$ and $\alpha \in \mathbb{R}$ such that $(\mu - \mathrm{i}\alpha)R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$. Since $s(|\varphi|)$ and $s(|\varphi^*|)$ are majorized by p_r (D-III,Prop.1.4), it follows that φ and φ^* are in $p_r M_* p_r$.

(ii) From (i) and the observation that the family $\{|\varphi| \colon \varphi \in \mathrm{Fix}(\mathcal{T})\}$ is faithful on $p_r M p_r$ and consists of $\mathcal{T}^{(r)}$-invariant elements, it follows that

- $P\sigma(A) \cap \mathrm{i}\mathbb{R} = P_\sigma(A^{(r)}) \cap \mathrm{i}\mathbb{R}$.
- $\mathrm{Ker}((\mathrm{i}\alpha - A)) \subset p_r M_* p_r$ for all $\alpha \in \mathbb{R}$.

– The semigroup $\mathcal{T}^{(r)}$ is relatively compact in the weak operator topology and therefore strongly ergodic.

(iii) Similarly, let $\mathcal{R}$ be an identity preserving pseudo-resolvent with values in M_* on $D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}(\lambda) > 0\}$ which is of Schwarz type. It follows as in (b) that Fix$((\lambda - \mathrm{i}\alpha)R(\lambda))$ is contained in $p_r M_* p_r$ for all $\lambda \in D$ and $\alpha \in \mathbb{R}$, where p_r is the associated recurrent projection.

We now give a characterization of strong ergodicity of semigroups which are identity preserving and of Schwarz type. For this we need that the Cesàro means

$$C(s)x = \frac{1}{s}\int_0^s T(t)x dt \quad (x \in M, 0 \leqslant s \in \mathbb{R})$$

are Schwarz maps. We omit the simple calculation (compare D-I,Thm.2.1).

Proposition 3.3. *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W^**-algebra M. Then the following assertions are equivalent.*

(a) $\mathcal{T}$ *is strongly ergodic on* M_*.

(b) $\sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)'p_r = 1.$

(c) $s^*(M, M_*)\text{-}\lim_{s\to\infty} C(s)'p_r = 1.$

Proof. Suppose that (a) holds. Since $\mathrm{Fix}\,(T)$ separates Fix(T') (see Krengel (1985, Chap.2,Thm.1.4)), the fixed space of $\mathcal{T}'$ is non trivial, hence $p_r \neq 0$. Let $0 \leqslant \psi \in M_*$, then $\psi_0 := \lim_{s\to\infty} C(s)\psi \in\ \mathrm{Fix}\,(T)$ and $s(\psi_0) \leqslant p_r$. Therefore

$$\begin{aligned}\lim_{s\to\infty} \psi(C(s)'p_r) &= \lim_{s\to\infty} (C(s)\psi)(p_r) = \psi_0(p_r) \\ &= \psi_0(1) = \lim_{s\to\infty} (C(s)\psi)(1) = \psi(1)\end{aligned}$$

which proves (b).

Suppose that (b) is satisfied. Since $C(s)'p_r \leqslant 1$ for all $s \in \mathbb{R}_+$, we obtain (c). (Use that for $(x_\alpha) \in M_+$ we have $\lim_\alpha x_\alpha = 0$ in the weak*-topology if and only if $\lim_\alpha x_\alpha = 0$ in the $s^*(M, M_*)$-topology.)

Suppose that (c) holds. Since each $C(s)'$ is an identity preserving Schwarz map, we obtain for all $x \in M$

$$\begin{aligned}(C(s)'((1-p_r)x))(C(s)'((1-p_r)x)^*) &\leqslant C(s)'((1-p_r)xx^*(1-p_r)) \\ &\leqslant \|x\|^2 C(s)'(1-p_r),\end{aligned}$$

hence

$$s^*(M, M_*)\text{-}\lim_{s\to\infty} C(s)'((1-p_r)x) = 0.$$

In particular, we obtain for all $x \in$ Fix$(\mathcal{T}')$ that $x = \sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)'x = \sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)'(p_r x)$.

Especially for $0 \neq x \in\ \mathrm{Fix}\,(\mathcal{T})$ we obtain $p_r x p_r \neq 0$. Since the W^*-algebra $p_r M p_r$ is the dual of $p_r M_* p_r$ and since $\mathcal{T}^{(r)}$ is strongly ergodic, it follows that the fixed space of $\mathcal{T}$ separates the points of $\mathrm{Fix}\,(\mathcal{T}')$. Thus $\mathcal{T}$ is strongly ergodic (Krengel (1985, Chap. 2, Thm. 1.4)). □

It follows from the result above that the semigroup in Evans (1977) cannot be strongly ergodic on $\mathcal{B}(H)_*$ since the associated recurrent projection is zero. But for irreducible semigroups we have the following result.

Proposition 3.4. *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a W^*-algebra M. Then the following assertions are equivalent.*

(a) $\mathcal{T}$ is irreducible and $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.

(b) $\mathcal{T}$ is relatively compact in the weak operator topology and the fixed space of $\mathcal{T}$ is generated by a faithful state.

(c) $\mathcal{T}$ is strongly ergodic and the fixed space of $\mathcal{T}$ is generated by a faithful state.

(d) The fixed space of $\mathcal{T}$ is generated by a faithful state.

Proof. Suppose (a) is satisfied. Since $\mathrm{Fix}\,(\mathcal{T}) \neq \{0\}$, there exists a faithful normal state φ on M such that $\mathrm{Fix}\,(\mathcal{T}) = \mathbb{C}\,\varphi$ (D-III, Thm.1.10.). Therefore $\mathcal{T}$ is relatively compact in the weak operator topology by Proposition 3.1., whence (b) holds and the implications from (b) to (c) and (c) to (d) are obvious.

Suppose that (d) holds. Let φ be a faithful normal state on M such that $\mathrm{Fix}\,(T) = \varphi\mathbb{C}$. By Proposition 3.1 the semigroup $\mathcal{T}$ is strongly ergodic. Therefore the fixed space of $\mathcal{T}$ separates the points of $\mathrm{Fix}(T')$. Consequently $\mathrm{Fix}(T') = \mathbb{C}1$. Thus the ergodic projection associated with $\mathcal{T}$ is given by $P = 1 \otimes \varphi$, i.e., $P\psi = \psi(1)\varphi$ for all $\psi \in M_*$. Let F be a closed $\mathcal{T}$-invariant face of M_*^+. If $0 \neq \psi \in F$ then

$$\lim_{s \to \infty} C(s)\psi = \psi(1)\varphi \in F.$$

Hence $\varphi \in F$ and therefore $F = M_*^+$ by the faithfulness of φ which proves (a). $\square$

The next theorem is an extension of D-III, Thm.1.10 and shows the usefulness of the theory of semitopological semigroups. Assume $\mathcal{T} \subseteq \mathcal{L}(M_*)$ to be relatively compact in the weak operator topology. Since $\mathcal{T}$ is commutative its closure $\mathcal{S} = (\mathcal{T})^- \subseteq L_w(M_*)$ contains a unique minimal ideal $\mathcal{K}$, called the kernel of $\mathcal{S}$, which is a compact Abelian group (DeLeeuw and Glicksberg (1961), Junghenn (1971) & Krengel (1985, § 2.4)). The identity Q of $\mathcal{K}$ is a projection onto the closed linear span of all eigenvectors of A pertaining to the eigenvalues in $\mathrm{i}\mathbb{R}$.

Moreover, the dual group of $\mathcal{K}$ can be identified with the subgroup of $\mathrm{i}\mathbb{R}$ generated by $P\sigma(A) \cap \mathrm{i}\mathbb{R}$. We call Q the semigroup projection associated with $\mathcal{T}$. On the other hand, $\mathcal{T}$ is always strongly ergodic with projection P onto $\mathrm{Fix}\,(\mathcal{T})$. Obviously, the relation

$$0 \leqslant P \leqslant Q \leqslant \mathrm{Id}$$

holds, where the order relation is defined by the inclusion of the range spaces.

There are two extreme cases. First, $Q = \mathrm{Id}$ and $\mathrm{rank}(P)$. This corresponds to the Halmos-von Neumann Theorem in commutative ergodic theory and is discussed, at least for irreducible semigroups, in Olesen et al. (1980).

Second, $\mathrm{Id} > Q = P$, in particular $\mathrm{rank}\,(P) = 1$. This latter case will be investigated in detail for $M = \mathcal{B}(H)$, the W^*-algebra of all bounded linear operators on a Hilbert space H. But we first need some preparations.

Theorem 3.5. *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a W^*-algebra M and suppose there exists a faithful family of $\mathcal{T}$-invariant states on M. Let N be the $\sigma(M, M_*)$-closed linear span of all eigenvectors of A' pertaining to the eigenvalues in $\mathrm{i}\mathbb{R}$. If Q is the semigroup projection associated with $\mathcal{T}$, then the following holds.*

(i) The adjoint of Q is a faithful normal conditional expectation from M onto the W^*-subalgebra N.

*(ii) The restriction of T' to N can be embedded into a $\sigma(M, M_*)$-continuous, one-parameter group of *-automorphisms.*

(iii) If, in addition, $\mathcal{T}$ is irreducible and if φ is the normal state generating the fixed space of $\mathcal{T}$, then $\varphi|_N$ is a faithful normal trace.

Proof. Consider $H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$ which is not empty by assumptions. From Proposition 3.1 it follows that $\mathcal{T}$ is relatively compact in the weak operator topology. Let K be the semigroup kernel of $\overline{\mathcal{T}}w \subset L(M_*)$ and Q the unit of K. Recall that $Q\psi_n = \psi_n$ for all $\psi_n \in M_*$ such that $A\psi_n = n\psi_n$ $(n \in H)$. Let $\mathcal{E}$ be the family of all eigenvectors of A' pertaining to the eigenvalues in H.

Then $\mathcal{E}$ is closed with respect to the multiplication in M and the formation of adjoints. Thus N is a W^*-subalgebra of M, Sakai (1971, Corollary 1.7.9.), and $\mathcal{T}_0(t)' := T(t)'_{|N}$ is multiplicative (for this see D-III, Lemma 1.1).

Since $Q \in \overline{\mathcal{T}}w \subseteq L_w(M_*)$, there exists an ultrafilter $\mathfrak{U}$ on $\mathbb{R}_+$ such that

$$\lim_{\mathfrak{U}} \langle T(t)\psi, x\rangle = \langle Q\psi, x\rangle$$

for all $x \in M$ and $\psi \in M_*$. If $n \in H$ and $\psi_n \in M_*$ such that $A\psi_n = n\psi_n$, then for all $x \in M$ we obtain

$$\langle \psi_n, x\rangle = \langle Q\psi_n, x\rangle = \lim_{\mathfrak{U}} \langle T(t)\psi_n, x\rangle = (\lim_{\mathfrak{U}} e^{nt})\langle \psi_n, x\rangle,$$

hence $\lim_{\mathfrak{U}} e^{nt} = 1$. From this it follows that for all $\psi \in M_*$ we have

$$\langle \psi, Q'(u_n)\rangle = \lim_{\mathfrak{U}} \langle \psi, T(t)'u_n\rangle = (\lim_{\mathfrak{U}} e^{nt})\langle \psi, u_n\rangle = \langle \psi, u_n\rangle.$$

Hence $N \subseteq Q'(M)$.

For γ in the dual group of K and $x \in M$ we define x_γ by

$$\psi(x_\gamma) := \int_K \langle S\psi, x\rangle \langle S, \gamma\rangle^* \,\mathrm{d}m(S) \quad (\psi \in M_*^+).$$

Then $x_\gamma \in M$ and $T(t)'x_\gamma = \langle QT(t), \gamma\rangle x_\gamma$. Therefore $x_\gamma \in N$. Thus the inclusion $Q'M \subseteq N$ is proved if we can show that $Q'M$ belongs to the $\sigma(M, M_*)$-closed linear span of $\{x_\gamma \colon \gamma \in K, x \in M\}$. For this it is enough to show that every linear form $\psi \in M_*$ such that $\psi(x_\gamma) = 0$ for all $\gamma \in K$ satisfies $\psi(Qx) = 0$ for all $x \in M$. But if $\psi(x_\gamma) = 0$, then

$$\int_K \langle S\psi, x\rangle \langle S, \gamma\rangle^* \,\mathrm{d}m(S) = 0, \gamma \in K.$$

Since the map $(S \mapsto \psi(Sx))$ is continuous on K and since the elements of K form a complete orthonormal basis in $L^2(K, \mathrm{d}m)$, we obtain $\psi(Sx) = 0$ for all $S \in K$, in particular $\psi(Qx) = 0$ as desired.

Since the range of Q' is a W^*-subalgebra of M it follows from Takesaki (1979, Theorem III.3.4) that Q' is a completely positive, normal conditional expectation. This Q' is faithful, i.e., $\mathrm{Ker}(Q') \cap M_+ = \{0\}$ since $Q\varphi = \varphi$ for the faithful linear form φ.

Let φ be the faithful normal state generating $\mathrm{Fix}\,(T)$ and let $\mathcal{U}$ be a family of unitary eigenvectors of A' pertaining to the eigenvalues in H (see D-III, Remark 1.11). If $u_1, u_2 \in U$, then

$$\varphi(u_1 u_2^*) = \varphi(T_0(t)'(u_1 u_2^*)) = e^{(n_1 - n_2)t} \varphi(u_1 u_2^*).$$

Therefore

$$\varphi(u_1 u_2^*) = \begin{cases} 0 & \text{if } n_1 \neq n_2, \\ 1 & \text{if } n_1 = n_2. \end{cases}$$

Hence $\varphi(u_1 u_2^*) = \varphi(u_2^* u_1)$ from which it follows that $\tau := \varphi|_N$ is a faithful normal trace. □

Remark 3.6. (i) Since $QM_* = N_*$ and $Q'M = N$, where N_* is as in D-III, Proposition 1.12, it follows from general duality theory that $(N_*)' = N$.

(ii) If $\psi \in N_*$, then $|\psi| \in N_*$. To see this, note that $Q\psi = \psi$ and Q is an identity preserving Schwarz map. Then the assertion follows from D-III, Proposition 1.4.

(iii) If $\psi \in N_*$, then $|T_0(t)\psi| = T_0(t)|\psi|$ for all $t \in \mathbb{R}$. This follows immediately from the fact that $\mathcal{T}_0(t)'$ is a *-automorphismus on N.

(iv) Let us add a few words concerning the structure of N: If $\mathcal{T}$ is irreducible and K is the semigroup kernel of $\mathcal{T}^- \subseteq L_w(M_*)$, then $(S \mapsto S') : K \to L((N, \sigma(N, N_*)))$ is a representation of the compact, Abelian group K as group of *-automorphism such that the fixed space is one dimensional. Therefore we are able to apply the results of Olesen et al. (1980). There are three possibilities for N.

1. $N = L^\infty(K, \mathrm{d}m)$ and $\mathcal{T}|_N$ is the translation group on N.
2. $N \cong R$ where $\mathcal{R}$ is the (unique) hyperfinite factor of type II_1. In that case (the image of) K is approximately inner on $\mathcal{R}$ [l.c., Theorem 5.8].
3. There exists a closed subgroup G of K such that

$$N = L^\infty(K/G, \mathrm{d}m) \otimes R$$

   where R is as in (ii) and $\mathrm{d}m$ the normalized Haar measure on K/G [l.c., Theorem 5.15].

So far we have studied weak*-semigroups on general W*-algebras. We apply now these results to weak*-semigroup on $\mathcal{B}(H)$. To do this we call a triple $(M, \varphi, \mathcal{T})$ a W^*-dynamical system if M is a W^*-algebra, $\mathcal{T}$ a weak*-semigroup of identity preserving Schwarz maps on M and φ a faithful family of $\mathcal{T}$-invariant normal states. We call $(M, \varphi, \mathcal{T})$ irreducible, if the preadjoint semigroup is irreducible (alternatively, if the fixed space of $\mathcal{T}$ is one dimensional).

Proposition 3.7. *Let $(\mathcal{B}(H), \varphi, \mathcal{T})$ be a W^*-dynamical system on the W^*-algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H. Then the following assertions are equivalent:*

(a) $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$,

(b) $\lim_{s\to\infty} T(s)_* = P_*$ *in the strong operator topology on* $\mathcal{L}(\mathcal{B}(H)_*)$.

Proof. Obviously (b) implies (a). Suppose that (a) is fulfilled. Then the ergodic projection P_* of the preadjoint semigroup is equal to the associated semigroup projection. Consequently there exists an ultrafilter $\mathfrak{U}$ on $\mathbb{R}_+$ such that $\lim_{\mathfrak{U}} T(t) = P$ in the weak operator topology. We claim that the convergence holds even in the strong operator topology. Taking this for granted it follows, since for every $t \in \mathbb{R}_+$ $T(t)$ is a contraction, that

$$\lim_{t\to\infty} \|T(t)_*\varphi\| = 0$$

for all $\varphi \in \operatorname{Ker}(()P_*)$. Since $T(t)_*\psi = \psi$ for every $\psi \in \mathrm{im}(P_*)$ and

$$\mathcal{B}(H)_* = \mathrm{im}(P_*) \oplus \operatorname{Ker}(()P_*)$$

the assertion is proved.

It remains to show that $\lim_{\mathfrak{U}} T(t)_* = P_*$ in the strong operator topology. Choose $0 \leqslant \varphi \in \mathcal{B}(H)_*$, $\|\varphi\| \leqslant 1$ and let $\varphi_t := T(t)_*\varphi$ $(t > 0)$. $\varphi_0 := P_*\varphi$ and let $\{p_i : i \in A\}$ be an increasing net of projections of finite rank in $\mathcal{B}(H)$ with strong limit 1. Since the set $K := \{\varphi_t : t \geqslant 0\}$ is relatively compact in the $\sigma(\mathcal{B}(H)_*, \mathcal{B}(H))$-topology, there exists for every $\delta > 0$ an index $i_0 \in A$ such that

$$\|(1 - p_i)\psi(1 - p_i)\| \leqslant \delta$$

for every $\psi \in K$ and $i \geqslant i_0$ (Takesaki (1979, Theorem III.5.4.(vi))). In particular

$$|\psi(1 - p_i)| \leqslant \delta, \quad \psi \in K,\ i(0) \leqslant i.$$

Let $p := p_{i(0)}$. Then for all x in the unit ball of M it follows that

$$\begin{gathered} |(\varphi_t - \varphi_0)(x)| \leqslant \\ |(\varphi_t - \varphi_0)(pxp)| + |(\varphi_t - \varphi_0)((1-p)xp)| \\ +|(\varphi_t - \varphi_0)(x(1-p))| \leqslant\leqslant |(\varphi_t - \varphi_0)(pxp)| + 4\sqrt{\delta}. \end{gathered}$$

Since the W^*-algebra $p\mathcal{B}(H)p$ is finite dimensional, there exists $U \in \mathfrak{U}$ such that

$$\|(\varphi_t - \varphi_0)|_{p\mathcal{B}(H)p}\| \leqslant \delta.$$

for all $t \in U$. Consequently

$$\|(\varphi_t - \varphi_0)\| \leqslant (\delta + 4\sqrt{\delta})$$

for all $t \in U$. Therefore $\lim_{\mathfrak{U}} T(t)_*\varphi = P_*\varphi$ in the strong operator topology. Since the positive cone of $\mathcal{B}(H)_*$ is generating, the assertion is proved. ☐

We show next, that for irreducible W^*-dynamical systems on $\mathcal{B}(H)$ the above properties always hold.

Theorem 3.8. *Let $(\mathcal{B}(H), \varphi, \mathcal{T})$ be an irreducible W^*-dynamical system. Then*

$$P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}.$$

Proof. Let N be the W^*-subalgebra of $M = \mathcal{B}(H)$ generated by the eigenvectors of A pertaining to the eigenvalues on $\mathrm{i}\mathbb{R}$ and let Q be the faithful normal conditional expectation from M onto N (Proposition 3.7). Since M is atomic, N is atomic (Størmer (1972)). N is finite since there exists a finite, faithful normal trace on N. In particular the center of N is isomorphic to ℓ^∞.

Let $\mathcal{S}$ be the restriction of $\mathcal{T}$ to the center. Then $\mathcal{S}$ is a weak*-semigroup such that every $S(t) \in \mathcal{S}$ is $\sigma(\ell^\infty, \ell^1)$-continuous and a *-automorphism. From this it follows that $S(t)$ is induced by some continuous flow $\kappa_t : \mathbb{N} \to \mathbb{N}$. Indeed, if $\delta_n((\xi_m)) = \xi_n$ $(n \in \mathbb{N}, (\xi_m) \in \ell^\infty)$, then $\delta_n \circ S(t)$ is a normal scalar valued *-homomorphism hence of the form δ_m for some $m = \kappa_t(n)$. But the function $t \mapsto \kappa_t$ is continuous from $\mathbb{R}$ into $\mathbb{N}$, whence constant. Hence $S(t) = \mathrm{Id}$. But the semigroup S is weak*-irreducible on the center. Consequently, the center is one dimensional. Using [Takesaki, Theorem V.1.27] we obtain $N = B(H_n)$ where H_n is a finite dimensional Hilbert space. But if $0 \neq \mathrm{i}\alpha \in P\sigma(A) \cap \mathrm{i}\mathbb{R}$ then $\mathrm{i}\alpha\mathbb{Z} \subset P\sigma(A)$ by D-III,Thm.1.10, whence N must be infinite dimensional. Therefore $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$ as desired. □

Corollary 3.9. *If $(\mathcal{B}(H), \varphi, T)$ is an irreducible W^*-dynamical system, then*

$$\lim_{s \to \infty} T(s) = 1 \otimes \varphi$$

in the strong operator topology on $L(\mathcal{B}(H)_*)$, where φ is the unique normal state generating the fixed space of T_*.

We are now going to discuss the asymptotic behavior of positive semigroups whose generator has boundary point spectrum different from 0. The standard example is the following. If Γ is the unit circle, $\mathrm{d}m$ the normalized Haar measure on Γ and $0 < \tau \in \mathbb{R}$, then we define the maps $T_\tau(t)$, $t \in \mathbb{R}_+$, on $L^1(\Gamma, m)$ by

$$(T_\tau(t)f)(\xi) = f(\xi \exp(\frac{2\pi i}{\tau}t)) \quad (f \in L^1(\Gamma, \mathrm{d}m), \xi \in \Gamma).$$

Then $\mathcal{T} := (T_\tau(t))_{t \geqslant 0}$ forms a strongly continuous one parameter semigroup which is identity preserving and of Schwarz type. Since $\mathcal{T}$ is periodic of period τ, it follows that 0 is a pole of the resolvent of its generator B with residuum $P = 1 \otimes 1$ and $\{\frac{2\pi \mathrm{i}}{\tau} \cdot k : k \in \mathbb{Z}\} = \sigma(B)$. Thus $\mathcal{T}$ is irreducible and uniformly ergodic on $L^1(\Gamma, \mathrm{d}m)$ (see A-II, Section 5).

Now let $\mathcal{T}$ be a semigroup on a predual M_* of a von Neumann-algebra M. It is called *partially periodic*, if there exists a projection $Q \in L(M_*)$ reducing T such that $Q(M_*) \cong L^1(\Gamma, \mathrm{d}m)$ and $T_{|\mathrm{im}(Q)}$ is conjugate to a periodic semigroup on $L^1(\Gamma, \mathrm{d}m)$.

In the main result we present a non commutative version of Nagel (1984) showing that certain dynamical systems are partially periodic semigroups.

Proposition 3.10. *Let $\mathcal{T}$ be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a W^*-algebra M.*

If $\mathcal{T}$ is uniformly ergodic, then $\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$. If additionally $\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \{0\}$, there exists a strictly positive projection Q on M_* which is identity preserving and completely positive such that

- *(i) Q reduces $\mathcal{T}$ and $Q(M_*) \cong L^1(\Gamma)$, Γ being the one dimensional torus.*
- *(ii) The restriction T_0 of $\mathcal{T}$ to im(Q) is irreducible and conjugate to a rotation semigroup of period $\tau = \frac{2\pi}{\alpha}$ on Γ.*
- *(iii) The spectral bound $s(A_{|\ \mathrm{Ker}(()Q)})$ is strictly smaller than 0.*

Proof. By D-III, Thm.1.11 and D-III, Thm.2.5 it follows that

$$\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z}$$

for some $\alpha \in \mathbb{R}$. Suppose $\alpha \neq 0$. Since $\sigma(A) + \mathrm{i}\alpha\mathbb{Z} = \sigma(A)$ and since every $n \in \mathrm{i}\alpha\mathbb{Z}$ is isolated, it follows that there exists $\delta > 0$ such that

$$\sigma(A) \setminus \mathrm{i}\alpha\mathbb{Z} \subseteq \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}(\lambda) \leqslant \delta\}.$$

Let $\{u_\alpha^k : k \in \mathbb{Z}\}$ be a family of unitary eigenvectors of A' pertaining to the eigenvalues in $\mathrm{i}\mathbb{R}$. Then $Q'(M)$ is a commutative W^*-algebra. For $\tau := \frac{2\pi}{\alpha}$, we obtain $T(\tau)u_\alpha^k = u_\alpha^k$, hence $T|_{\mathrm{im}(Q)}$ is periodic. From the Halmos-von Neumann theorem (see Schaefer (1974, Thm. III.7.11)) it follows that $T|_{\mathrm{im}(Q)}$ is conjugate to the rotation semigroup of period τ on $L^1(\Gamma, m)$. ☐

Using this proposition we obtain the following theorem.

Theorem 3.11. *Let $T = (T(t))_{t\geqslant 0}$ be a uniformly ergodic, identity preserving semigroup of Schwarz type on the predual of a W^*-algebra M and suppose*

$$\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \{0\}.$$

Then there exists a partially periodic, identity preserving semigroup $S = (S(t))_{t\geqslant 0}$ of Schwarz type on M_* such that

$$\lim_{t\to\infty} (T(t) - S(t)) = 0$$

in the strong operator topology.

Proof. Let φ be the normal state on M generating the fixed space of $\mathcal{T}$. Let $\mathcal{S} = (S(t))_{t\geqslant 0}$ where $S(t) := T(t) \circ Q$ and Q is as in 2.6. Obviously, $\mathcal{S}$ is partially periodic and $\varphi \in \mathrm{Fix}\,(\mathcal{S})$. Let H_φ be the GNS-Hilbert space pertaining to φ. Since φ is fixed under $\mathcal{T}$, $\mathcal{S}$ and Q, these objects have a canonical extension to H_φ (in the following denoted by the same symbols). If $H_0 := \mathrm{Ker}(()Q) \subseteq H_\varphi$, then it is easy to see that H_0 is invariant under the extension to H_φ and for the multiplication maps we defined in D-III, Remark 1.3.

Consequently, using the results in Groh and Kümmerer (1982), it follows that there exists $c \in \mathbb{R}$ such that for all γ near 0 and all $\beta \in \mathbb{R}$:

$$\|R(\gamma + \mathrm{i}\beta A_0)\| \leqslant c, \tag{*}$$

where $A_0 := A_{|\ \mathrm{Ker}(()Q)}$ (the norm taken in $L(H_\varphi)$). Using the result in A-III,Cor.7.11 it follows that

$$\lim_{t\to\infty} \|T(t)|_{H_0}\| = 0.$$

Since the $s(M, M_*)$-topology on the unit ball of M is nothing else than the restriction of the norm topology on H_φ, we obtain

$$s(M, M_*)\text{-}\lim_{t\to\infty} (T(t)' - S(t)')(x) = 0$$

uniformly on M_1. From this the assertion follows. □

4 Uniform Ergodic Theorems

As we have seen, uniformly ergodic semigroups have strong spectral properties. In this section we study sufficient conditions which imply uniform ergodicity thereby generalizing results of Groh (1984a). We first need some preparations.

Lemma 4.1. *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C} \colon \mathrm{Re}(\lambda) > 0\}$ with values in the predual of a W^*-algebra M. If the fixed space of $\mathcal{R}$ is infinite dimensional, then there exists a sequence of states in $\mathrm{Fix}\,(\mathcal{R})$ such that the corresponding support projections are mutually orthogonal in M.*

Proof. Let $\Phi = \{\varphi \in \mathrm{Fix}\,(\mathcal{R}) : \varphi \text{ state on } M\}$ and let $p = \sup\{s(\varphi) : \varphi \in \Phi\}$. Since $\lambda R(\lambda)\varphi = \varphi$ for all $\varphi \in \Phi$ and $\lambda \in D$, it follows $\mu R(\mu)(\mathbb{1} - s(\varphi)) = (\mathbb{1} - s(\varphi))$. Hence $\mu R(\mu)(\mathbb{1} - p) = (\mathbb{1} - p)$ for all $\mu \in \mathbb{R}_+$.

Let $\mathcal{R}_1$ be the induced pseudo-resolvent on pM_*p (D-I, Section 3.(c)). Then the family Φ is faithful on M_p and contained in the fixed space of $\mathcal{R}_1$. The adjoint $\mu R_1(\mu)'$ is an identity preserving Schwarz map. Consequently it follows from D-III, Lemma 1.1.(b) and, the $\sigma(M_p, (M_p)_*)$-continuity of $\mu R_1(\mu)'$ that $\mathrm{Fix}\,(R_1')$ is a W^*-subalgebra of M_p and by D-III, Lemma 1.5, $\dim\, \mathrm{Fix}\,(\mathcal{R}) \leqslant \dim\, \mathrm{Fix}\,(R_1')$.

If $\mathrm{Fix}\,(\mathcal{R})$ is infinite dimensional, let (p_n) be a sequence of mutually orthogonal projections in $\mathrm{Fix}(R_1') \subseteq M_p$ and choose a sequence (φ_n) in Φ such that $\varphi_n(p_n) \neq 0$. For $n \in \mathbb{N}$ let ψ_n be the normal state

$$\psi_n(x) = \varphi_n(p_n)^{-1}\varphi_n(p_n x p_n)$$

on M. Because of $s(\psi_n) \leqslant p_n \leqslant p$, the support projections of the ψ_n's are mutually orthogonal in M. For $\mu \in \mathbb{R}_+$ and $x \in M$ we obtain

$$\begin{aligned}
\langle x, \mu R(\mu)\psi_n\rangle &= \varphi_n(p_n)^{-1}\langle \mu p_n(R(\mu)'x)p_n, \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\langle \mu p_n p(R(\mu)p'x)p_n, \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\langle \mu p_n(pR_1(\mu)'xp)p_n, \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\langle \mu(p_n R_1(\mu)'xp_n), \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\varphi_n(x) = \psi_n(x).
\end{aligned}$$

Therefore $\psi_n \in \operatorname{Fix}(\mathcal{R})$ for all $n \in \mathbb{N}$. □

Remark 4.2. (i) If $\dim \operatorname{Fix}(\mathcal{R}) \geqslant 2$ then the Jordan decomposition of self adjoint linear functionals implies that at least two states in $\operatorname{Fix}(\mathcal{R})$ have orthogonal support (compare D-III, Theorem 1.10.(a)).

(ii) If $\mathcal{R}$ is a pseudo-resolvent with values in a W^*-algebra such that $\operatorname{Fix}(\mathcal{R}')$ is contained in M_*, then by D-III, Lemma 1.2, there exists a sequence of normal states in $\operatorname{Fix}(\mathcal{R}')$ with orthogonal supports in M.

Lemma 4.3. *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$ with values in the predual of a W^*-algebra M. If the fixed space of the canonical extension $\widehat{\mathcal{R}}$ of $\mathcal{R}$ to some ultrapower of M_* is infinite dimensional, then there exists a sequence (z_n) in M_1^+ and a sequence of states (φ_n) in M_* such that*

(i) $\lim_n z_n = 0$ *in the $s^*(M, M_*)$-topology,*

(ii) $\lim_n \|(Id - \lambda R(\lambda))\varphi_n\| = 0$ *for all $\lambda \in D$,*

(iii) $\varphi_n(z_n) \geqslant \frac{1}{2}$ *for all $n \in \mathbb{N}$.*

Proof. Let $(M_*)^\wedge$ be the ultrapower of M_* with respect to some free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$. Since $(M_*)^\wedge$ is the predual of a W^*-subalgebra of $\widehat{M}$ (see D-III, Remark 2.4.(b)), there exists a sequence of states $(\hat{\psi}_n)$ in $\operatorname{Fix}\left(\widehat{\mathcal{R}}\right)$ such that the corresponding support projections are mutually orthogonal in $\widehat{M}$ (Lemma 4.1). For every $n \in \mathbb{N}$ let $(\psi_{n,k})$ be a representing sequence of states,

$$\varphi := \sum_{n,k} 2^{-(n+k+1)} \psi_{n,k}$$

and

$$p := \sup\{s(\psi_{n,k}) : n, k = 1, \ldots\}$$

in M. Then φ is a normal state on M which is faithful on the W^*-algebra M_p. Since

$$1 = \langle \psi_{n,k}, s(\psi_{n,k}) \rangle = \psi_{n,k}(p) \quad (n, k \in \mathbb{N}),$$

it follows $\hat{\psi}_n(\hat{p}) = 1$ where $\hat{p}$ is the canonical image of p in $\widehat{M}$. But this implies $s(\hat{\psi}_n) \leqslant \hat{p}$ in $\widehat{M}$. Since $\widehat{M}_1^+$ is $\sigma(\widehat{M}, \widehat{M}')$-dense in $(\widehat{M}'')_1^+$ (Kaplansky's density theorem Sakai (1971, 1.9.1) with Sakai (1971, 1.8.9 and 1.8.12)), there exists for all $n \in \mathbb{N}$ a net $(z_{n,\gamma})$ in $\widehat{M}_1^+$ such that

$$\sigma(\widehat{M}'', \widehat{M}')\text{-}\lim_{\gamma} \hat{z}_{n,\gamma} = s(\hat{\psi}_n).$$

From Sakai (1971, 1.7.8) and the above considerations, we obtain that the net $(p\hat{z}_{n,\gamma}\hat{p})$ converges to $s(\hat{\psi}_n)$ in the $\sigma(\widehat{M}'', \widehat{M}')$-topology. Therefore we may assume $\hat{z}_{n,\gamma} \in (\widehat{M}_p')_1^+$.

In the following we denote by $\hat{\varphi}$ the canonical image of φ in $(M_*)^\wedge$.

Since the projections $s(\hat{\psi}_n)$ are mutually orthogonal, there exists a real sequence (r_n), $0 < r_n < 1$, $\lim_n r_n = 0$ and $\hat{\varphi}(s(\hat{\psi}_n)) \leqslant \frac{1}{2} r_n$. For all $n \in \mathbb{N}$ choose $\hat{z}_n \in (\widehat{M'_p})_1^+$ such that

$$|\langle \hat{\varphi}, s(\hat{\psi}_n) - \hat{z}_n \rangle| \leqslant \frac{1}{2} r_n,$$
$$|\langle \hat{\psi}_n, s(\hat{\psi}_n) - \hat{z}_n \rangle| \leqslant \frac{1}{2} r_n.$$

Hence $\hat{\varphi}(\hat{z}_n) \leqslant r_n$ and $\hat{\psi}_n(\hat{z}_n) \geqslant \frac{1}{2}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ let $(z_{n,k}) \in \hat{z}_n$ be a representing sequence in $(M_p)_1^+ = p(M_1^+)p$ (note that $M_{\hat{p}} = \widehat{M_p}$) and fix $\mu \in \mathbb{R}_+$. Since $\mu R(\mu)' \hat{\psi}_n = \hat{\psi}_n$, $\hat{\varphi}(\hat{z}_n) \leqslant r_n$ and $\hat{\psi}_n(\hat{z}_n) \geqslant \frac{1}{2}$, there exists for all $n \in \mathbb{N}$ an element $U_n \in \mathfrak{U}$ such that for all $k \in U_n$ and we obtain

(i′) $\varphi(z_{n,k}) \leqslant r_n$,

(ii′) $\|(Id - \mu R(\mu))\psi_{n,k}\| \leqslant r_n$,

(iii′) $\psi_{n,k}(z_{n,k}) \geqslant \frac{1}{2}$.

Inductively we find a sequence (z_n) in $(M_p)_1^+$ and a sequence of states (φ_n) in M_* such that for all $n \in \mathbb{N}$

(i″) $\lim_n \varphi_n(z_n) = 0$,

(ii″) $\lim_n \|(Id - \mu R(\mu))\varphi_n\| = 0$,

(iii″) $\varphi_n(z_n) \geqslant \frac{1}{2}$.

But φ is faithful on M_p. Therefore condition (ii″) implies that $\lim_n z_n = 0$ in the $s^*(M_p, (M_p)_*)$-topology (Takesaki (1979, Proposition III.5.4)). Since

$$s^*(M_p, (M_p)_*) = s^*(M, M_*)|_{M_p},$$

(i) follows immediately from (ii″). Using the resolvent equation for $\mathcal{R}$ it is easy to see that (ii″) implies

$$\lim_n \|(Id - \lambda R(\lambda))\varphi_n\| = 0$$

for all $\lambda \in D$ and the proof is complete. □

Without further comments, we will use following facts in this section.

(1) A sequence (φ_n) in M'_+ converges in the $\sigma(M', M)$-topology if and only if it converges in $\sigma(M', M'')$-topology (Akemann et al. (1972)).

(2) We can decompose $\varphi \in M'_+$ into its normal and singular part $\varphi = \varphi^{(n)} + \varphi^{(s)}$, $0 \leqslant \varphi^{(n)} \in M_*$, $0 \leqslant \varphi^{(s)} \in M_*^\perp$ and $\|\varphi\| = \|\varphi^{(n)}\| + \|\varphi^{(s)}\|$ (Takesaki (1979, Theorem III.2.14)).

(3) If (φ_k) is a sequence in M_* convergeing to zero in the $\sigma(M_*, M)$-topology and if (x_n) is a sequence in M converging to zero in the $s^*(M, M_*)$-topology, then $\lim_n \varphi_k(x_n) = 0$ uniformly in $k \in \mathbb{N}$ (Takesaki (1979, Lemma III.5.5)).

Theorem 4.4. *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent on*

$$D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$$

with values in a W^*-algebra M which is of Schwarz type and let $\mathcal{R}'$ br its adjoint pseudo-resolvent. Any one of the following conditions implies $\dim\operatorname{Fix}\left(\widehat{\mathcal{R}}\right) < \infty$ in some ultrapower of M.

(i) The fixed space of $\mathcal{R}'$ is finite dimensional.

(ii) $\lim_{\mu\to 0}\mu R(\mu) = P$ exists in the strong operator topology and rank$(P) < \infty$.

(iii) The fixed space of $\mathcal{R}'$ is contained in M_*.

(iv) Every map $\mu R(\mu)$, $\mu \in \mathbb{R}_+$ is irreducible on M.

Proof. Suppose that the dimension of the fixed space of $\widehat{\mathcal{R}'}$ in some ultrapower $\widehat{(M`)}$ of $M'$ is infinite dimensional. Since $\widehat{(M`)}$ is the predual of the W^*-algebra $\widehat{M}$ and $\mathcal{R}'$ is identity preserving (since $R'\mathbb{1} = R\mathbb{1} = \mathbb{1}$) and of Schwarz type (because $\mu R''(\mu) = (\mu R(\mu))''$ is a Schwarz map for all $\mu \in \mathbb{R}_+$), we may apply Lemma 4.3.

Suppose that the fixed space of the canonical extension of $\mathcal{R}'$ to some ultrapower of M' is infinite dimensional. Thus we may choose a sequence of states (φ_k) in M' and a sequence (z_k) in $(M'')_1$, $0 \leqslant z_k$, satisfying (i)–(ii) of Lemma 4.3. Remark (3) above implies that no subsequence of (φ_k) can converge in the $\sigma(M', M'')$-topology.

(i) If φ is a $\sigma(M', M)$-accumulation point of (φ_k), then $\varphi \in \operatorname{Fix}(\mathcal{R}')$. Since $\operatorname{Fix}(\mathcal{R}')$ is finite dimensional, the set of accumulation points of the sequence (φ_k) is metrizable in the $\sigma(M', M)$-topology. Hence there exists a sequence $(k(n))$ of natural numbers such that $\sigma(M', M)\text{-}\lim_n \varphi_{k(n)} = \varphi$. Consequently, by Remark (1) above , $\varphi = \sigma(M', M'')\text{-}\lim_n \varphi_{k(n)}$. But this leads to a contradiction proving(i).

(ii) Since $\dim\operatorname{Fix}(\mathcal{R}) = \dim\operatorname{Fix}(\mathcal{R}') = \operatorname{rank}(P) < \infty$, (ii) follows from (i).

(iii) Suppose that the fixed space of R' is infinite dimensional. Since $\operatorname{Fix}(\mathcal{R}') \subseteq M_*$, there exists a sequence of states (ψ_n) in $\operatorname{Fix}(\mathcal{R}')$ with mutually orthogonal support projections in M (Lemma 4.1). Since every $\sigma(M', M)$-accumulation point of the ψ_n's belongs to $\operatorname{Fix}(\mathcal{R}')$, hence is normal, the sequence (ψ_n) is relatively $\sigma(M_*, M)$-compact.

By Eberlein's theorem, we may assume that this sequence is weakly convergent (Schaefer (1966)). By the orthogonality of the $s(\psi_n)$'s this sequence converges to zero in the $s^*(M, M_*)$-topology, hence $\lim_n \psi_k(s(\psi_n)) = 0$ uniformly in $k \in \mathbb{N}$, a contradiction. Consequently $\dim\operatorname{Fix}(\mathcal{R}) < \infty$ and (i) is proved.

(iv) We prove $\dim\operatorname{Fix}(\mathcal{R}') = 1$ and apply (i) once again and need the following observation: If ψ is a faithful state on M, then the normal part is faithful too. Indeed, if $0 \neq x \in M$ such that $\psi^{(n)}(x) = 0$, choose a projection $0 \neq p \in M$ such that $\psi^{(n)}(p) = \psi^{(s)}(p) = 0$ (use Takesaki (1979, Theorem III.3.8)). Hence $\psi(p) = 0$ which conflicts with the faithfulness of ψ.

If $2 \leqslant \dim\operatorname{Fix}(\mathcal{R}')$ there are states ψ_1 and ψ_2 in $\operatorname{Fix}(\mathcal{R}')$ such that the corresponding support projections are orthogonal in M'' (Remark 4.2). Since every

$\mathcal{R}'$-invariant state ψ is faithful on M, $\psi_i^{(n)} \neq 0$ (otherwise the norm closed face $\{\psi(x) = 0 : x \in M_+\}$ would be non trivial and $\mu R(\mu)$-invariant). The support projections of the $\psi_i^{(n)}$'s in M'' are orthogonal (since $\psi_1^{(n)} \leqslant \psi_i$) and different from zero. Let (z_γ) be a net in M_1^+ such that

$$\sigma(M'', M')\text{-}\lim_{\gamma} z_\gamma = s(\psi_1^{(n)}).$$

Then $\lim_\gamma \psi_1^{(n)}(z_\gamma) = 1$ but $\lim_\gamma \psi_2^{(n)}(z_\gamma) = 0$. Let z be a $\sigma(M, M_*)$-accumulation point of (z_γ) in M_+. Since every $\psi_i^{(n)}$ is normal, $\psi_1^{(n)}(z) = 1$ but $\psi_2^{(n)}(z) = 0$. The first condition implies $z \neq 0$ while the second shows that $\psi_2^{(n)}$ cannot be faithful. This is a contradiction and it implies $\dim\ \mathrm{Fix}\,(\mathcal{R}') = 1$, hence (iv).

□

The next corollary is an easy application of Theorem 4.4 and of D-III, Proposition 2.3.

Corollary 4.5. *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a W^*-algebra M. Then the following assertions are equivalent.*

- *(a) $\mathcal{T}$ is uniformly ergodic with finite dimensional fixed space.*
- *(b) The adjoint weak*-semigroup is strongly ergodic with finite dimensional fixed space.*
- *(c) Every $\mathcal{T}''$-invariant state is normal.*

Proof. If (a) is fulfilled, then the semigroup $\mathcal{T}$ is strongly ergodic on M_*. Since

$$\dim\ \mathrm{Fix}\,(\mathcal{T}) = \dim\ \mathrm{Fix}\,(\mathcal{T}') < \infty,$$

there exist normal states $\varphi_1, \ldots, \varphi_n$ in $\mathrm{Fix}\,(\mathcal{T})$ and $x_1, \ldots, x_k$ in Fix$(\mathcal{T}')$ such that $\varphi_n(x_m) = \delta_{n,m}$ $(1 \leqslant n, m \leqslant k)$. Then

$$P = \sum_{i=1}^{k} \varphi_i \otimes x_i$$

is the associated ergodic projection. If $(C(s))_{s>0}$ is the family of Cesàro means of $\mathcal{T}$, then

$$\lim_{s\to\infty} C(s)''(\psi) = \sum_{i=1}^{k} \varphi_i(\psi) x_i \in M_*$$

for every $\psi \in M'$. Hence $\mathrm{Fix}\,(\mathcal{T}'') \subseteq M_*$ which implies (c).

If (c) is fulfilled, then $\mathrm{Fix}\,(\mathcal{T}') = \mathrm{Fix}\,(\mathcal{T}'')$. Therefore the fixed space of $\mathcal{T}'$ separates the points of $\mathrm{Fix}\,(\mathcal{T}'')$, hence $\mathcal{T}'$ is strongly ergodic on M (Krengel (1985, Chap.2, Thm.1.4)).

If (b) holds, then

$$P = \lim_{\mu\to 0} \mu R(\mu, A')$$

exists in the strong operator topology with A' is the generator of $\mathcal{T}'$. Therefore $\dim\ \mathrm{Fix}\left(\widehat{\mu R(\mu)}\right) < \infty$ in some ultrapower of M (Theorem 4.4). It follows from D-III, Proposition 2.3 that 0 is a pole of the resolvent of $R(\cdot, A)$. Therefore $\mathcal{T}$ is uniformly ergodic. □

Notes

Section 1: The stability concepts appearing in Theorem 1.7 coincide not only for positive semigroups on C^*-algebras but on any order unit Banach space. We refer to Batty and Robinson (1984) for this more general setting and to B-IV, Section 1 for the analogous results on $C_0(X)$.

Section 2: Theorem 2.2 generalizes the Liapunov stability theorem from the matrix algebra $B(\mathbb{C}^n)$ to arbitrary W^*-algebras. For the algebra $\mathcal{B}(H)$ it is due to Mil'stein (1975) and in the general form to Groh and Neubrander (1981).

Section 3: From the many papers dealing more or less explicitly with the asymptotic behavior of semigroups on operator algebras we quote Frigerio and Verri (1982) and Watanabe (1982). The background for our ergodic theorems (Proposition 3.3 & 3.4) can be found best in Krengel (1985). The "automatic" convergence theorem for an irreducible W^*-dynamical system on $\mathcal{B}(H)$ stated in Corollary 3.9 is the continuous version of a result in Groh (1984b). Finally, the characterization of convergence towards a periodic semigroup through spectral properties of the generator—Theorem 3.11—is due to Nagel (1984) in the commutative case, i.e., in $L^1(\mu)$ (see also C-IV, Thm.2.14).

Section 4: Again we refer to Krengel (1985) for the (uniform) ergodic theory for a single operator or a one-parameter semigroup on a Banach space. The characterization given in Corollary 4.5 for positive semigroups on W^*-algebras is based on a sophisticated use of ultrapower techniques and has its discrete forerunners in Lotz (1981) and Groh (1984a).

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