denotes a compact space and $C(K)$ the space of all real-valued continuous functions on K. It will be essential that K is compact for all what follows since it will be needed that the positive cone of $C(K)$ has interior points.

We reformulate condition (ii) of Theorem 1.3 for unbounded operators.

Definition 1.5. An (unbounded) operator A on $C(K)$ is said to satisfy the positive minimum principle if

$$\text{(P)} \qquad \begin{array}{l} \text{for every } 0 \leq f \in D(A) \text{ and } x \in K, \\ f(x) = 0 \text{ implies } (Af)(x) \geq 0 . \end{array}$$

Our next theorem shows that the positive minimum principle characterizes the positivity of the semigroup; and in fact, the proof is very elementary. Using more involved arguments we will later prove a much stronger result (Theorem 1.13).

Theorem 1.6. Let A be the generator of a strongly continuous semigroup on $C(K)$. Then the semigroup is positive if and only if the generator A satisfies the positive minimum principle (P).

Proof. The necessity of the condition is proved as "(i) implies (ii)" in Theorem 1.3. Assume that (P) holds. We claim that $R(\lambda,A) \geq 0$ for sufficiently large real λ. (This implies the positivity of the semigroup by Prop. 1.1). Let $s := \inf\{\lambda \in \mathbb{R} : [\lambda,\infty) \subset \rho(A)\}$. Then $s \leq \omega(A) < \infty$. Let $0 \ll u \in C(K)$. Then $\lambda_o := \inf\{\lambda > s : R(\mu,A)u \gg 0$ for all $\mu \in (\lambda,\infty)\} < \infty$ since $\lim_{\mu\to\infty} \mu R(\mu,A)u = u$.
We claim that $\lambda_o = s$.
In fact, if this is not true, then $[\lambda_o,\infty) \subset \rho(A)$ and $R(\lambda_o,A)u \geq 0$ but $R(\lambda_o,A)u$ is not strictly positive. Consequently there exists $x \in K$ such that $(R(\lambda_o,A)u)(x) = 0$. Then (P) implies that $A(R(\lambda_o,A)u)(x) \geq 0$. Hence, $0 < u(x) = \lambda_o(R(\lambda_o,A)u)(x) - A(R(\lambda_o,A)u)(x) \leq 0$, a contradiction. We have shown that $R(\lambda,A)u \gg 0$ for all $u \gg 0$ and $\lambda > s$. Since $\{u \in C(K) : u \gg 0\}$ is dense in $C(K)_+$, it follows that $R(\lambda,A) \geq 0$ for all $\lambda > s$.

□

Remark 1.7. The proof of Theorem 1.6 shows that for the generator A of a positive semigroup on $C(K)$, $R(\lambda,A)u \gg 0$ whenever $0 \ll u \in C(K)$ and $[\lambda,\infty) \subset \rho(A)$. In particular, $R(\lambda,A) \geq 0$ whenever $[\lambda,\infty) \subset \rho(A)$.