It remains to show that (iv) implies (i). Assertion (iv) implies that for all $t \geq 0$ there exist bounded operators $T(t) \in L(E)$ such that $u(t,f) = T(t)f$ if $f \in D(A)$. Moreover, $\sup_{0 \leq t \leq 1} \|T(t)\| < \infty$. It follows that $T(\cdot)f$ is strongly continuous for all $f \in E$ (since it is so for $f \in D(A)$ and $D(A)$ is dense). Let $t > 1$. There exist unique $n \in \mathbb{N}$ and $s \in [0,1)$ such that $t = n + s$. Let $T(t) := T(1)^n T(s)$. From the uniqueness of the solutions it follows that $T(t)f = u(t,f)$ for all $t \geq 0$ as well as $T(t+s)f = T(s)T(t)f$ for all $f \in D(A)$ and $s,t \geq 0$. Thus $(T(t))_{t \geq 0}$ is a semigroup.
Denote by B its generator. It follows from the definition that $A \subset B$. Moreover, $D(A)$ is invariant under the semigroup. So by A-I,Prop.1.9.(ii) $D(A)$ is a core of B. Since A is closed this implies that $A = B$.

□

Remark 1.3. It is surprising that from condition (ii) and (iii) in the corollary it follows automatically that $D(A)$ is dense.
On the other hand this condition cannot be omitted in (iv). In fact, consider any generator $\tilde{A}$ and its restriction A with domain $D(A) = \{0\}$. Then A satisfies the remaining conditions in (iv) but is not a generator (if $\dim E > 0$).

Example 1.4. We give a densely defined closed operator A, such that there exists a unique solution of (ACP) for all initial values in $D(A)$, but A is not a generator.
Let B be a densely defined unbounded closed operator on a Banach space F. Consider $E = F \oplus F$ and A on E given by

$$A = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

with domain $F \times D(B)$.
Then $D(A^2) = \{(f,g) \in F \times D(B) : Bg \in F\} = D(A)$ and so $A_1 \in L(E_1)$. In particular, A_1 is a generator. But A is not. For instance condition (ii) in Corollary 1.2. does not hold, since for each $\lambda \in \mathbb{C}$,

$$(\lambda - A)D(A) = \{(\lambda f - Bg, \lambda g) : f \in F,\ g \in D(B)\}$$
$$\subset F \times D(B) \neq F \times F = E.$$

So $\rho(A) = \emptyset$.

As a further illustration, we note that the solution of the corresponding abstract Cauchy problem for the initial value $(f,g) \in F \times D(B)$ is given by $u(t) = (f + tBg, g)$. Since B is unbounded, condition (iv) of Corollary 1.2. is clearly violated.