estimated. However, the resolvent has to be known in the right half-plane instead of a right half-line.

On the other hand, given a strongly continuous semigroup, merely an estimate on a vertical line implies that the semigroup is holomorphic. More precisely, the following holds.

Corollary. A strongly continuous semigroup with generator A is holomorphic if and only if there exist $w > \omega(A)$ and $M \geq 0$ such that one has $\|R(w+i\eta,A)\| \leq M/|\eta|$ for all $\eta \in \mathbb{R}$.

Proof. In fact, assume that the condition holds. Since $A-w$ is the generator of a bounded semigroup one has $\|R(\lambda,A-w)\| \leq N/\mathrm{Re}\lambda$ for some $N > 0$ and all $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}\lambda > 0$. Consequently, for every $\alpha \in (0,\pi/2)$, $\|R(\lambda,A-w)\| \leq (|\lambda|/\mathrm{Re}\lambda)\cdot N/|\lambda| \leq N(\cos\alpha)^{-1}/|\lambda|$ for all $\lambda \in S(\alpha)$. The remaining estimate for a sector around the imaginary axis is given by the proof of Thm.1.14 and shows that $A-w$ generates a holomorphic semigroup. The reverse implication is clear.

□

We now prove the following extension of Cor.1.13.

Theorem 1.15. Let A be the generator of a strongly continuous group. Then A^2 generates a holomorphic semigroup of angle $\pi/2$.

Proof. There exists $w \geq 0$ such that $(\pm A - w)$ generates a bounded semigroup. Consequently, there exists $M \geq 0$ such that
$\|R(\mu,\pm A - w)\| \leq M/\mathrm{Re}\,\mu$ whenever $\mathrm{Re}\,\mu > 0$.
Let $\alpha \in (0,\pi/2)$. There exist $r_o \geq 0$ and $\beta \in (0,\pi/2)$ such that $S(\alpha+\pi/2)\backslash B(r_o) \subset \{z^2 : z \in S(\beta) + w\}$.
[In fact, the line $\{w + r(\cos\beta + i\sin\beta) : r \geq 0\}$ can be parameterized by $z(t) = w + t + i\cdot t/\varepsilon$ $(t \geq 0)$ (where $\varepsilon > 0$ depends on β). Then $z(t)^2 = (w+t)^2 - t^2/\varepsilon^2 + i2t(w+t)/\varepsilon$.
Thus $\lim_{t\to\infty} \mathrm{Im}z(t)^2/\mathrm{Re}z(t)^2 = 2\varepsilon/(\varepsilon^2-1)$. Choose $\beta \in (\pi/4,\pi/2)$ such that $\tan(\alpha + \pi/2) > 2\varepsilon/(\varepsilon^2-1)$.]
Now let $\lambda \in S(\alpha+\pi/2)\backslash B(r_o)$. Then there exist $\theta \in (-\beta,\beta)$ and $r \geq 0$ such that $\lambda = (re^{i\theta}+w)^2$. Thus $(\lambda-A^2) = (re^{i\theta}+w-A)(re^{i\theta}+w-A)$. Hence $\lambda \in \rho(A^2)$ and $R(\lambda,A^2) = R(re^{i\theta},A-w)R(re^{i\theta},-A-w)$. We conclude that $|\lambda|\cdot\|R(\lambda,A^2)\| \leq |\lambda|\cdot M^2/(\cos\theta)^2 r^2 \leq (|\lambda|/r^2)\cdot M^2/(\cos\beta)^2$. Thus $|\lambda|\cdot R(\lambda,A^2)$ is uniformly bounded for $\lambda \in S(\alpha+\pi/2)\backslash B(r_o)$.

□