Let ϕ be the faithful normal state generating $\mathrm{Fix}(\mathcal{T})$ and let U be a family of unitary eigenvectors of A' pertaining to the eigenvalues in H (see D-III, Remark 1.11). If $u_1, u_2 \in U$ then

$$\phi(u_1u_2^*) = \phi(T_o(t)'(u_1u_2^*)) = e^{(\eta_1-\eta_2)t}\,\phi(u_1u_2^*) .$$

Therefore

$$\phi(u_1u_2^*) = \begin{cases} 0 & \text{if } \eta_1 \neq \eta_2 , \\ 1 & \text{if } \eta_1 = \eta_2 . \end{cases}$$

Hence $\phi(u_1u_2^*) = \phi(u_2^*u_1)$ from which it follows that $\tau := \phi_{|N}$ is a faithful normal trace.

□

Remarks 3.6. (a) Since $QM_* = N_*$ and $Q'M = N$, where N_* is as in D-III, Proposition 1.12, it follows from general duality theory that $(N_*)' = N$.

(b) If $\psi \in N_*$ then $|\psi| \in N_*$. To see this note that $Q\psi = \psi$ and Q is an identity preserving Schwarz map. Then the assertion follows from D-III, Proposition 1.4 .

(c) If $\psi \in N_*$, then $|T_o(t)\psi| = T_o(t)|\psi|$ for all $t \in \mathbb{R}$. This follows immediately from the fact that $T_o(t)'$ is a $*$-automorphismus on N.

(d) Let us add a few words concerning the structure of N: If $\mathcal{T}$ is irreducible and $\mathcal{K}$ is the semigroup kernel of $\mathcal{T}^- \subseteq L_w(M_*)$, then

$$(S \to S') : \mathcal{K} \to L((N, \sigma(N, N_*))$$

is a representation of the compact, Abelian group $\mathcal{K}$ as group of $*$-automorphism such that the fixed space is one dimensional. Therefore we are able to apply the results of [Olesen-Pedersen-Takesaki (1980)]. There are three possibilities for N :

(i) $N \cong L^\infty(\mathcal{K}, dm)$ and $\mathcal{T}_{|N}$ is the translation group on N.

(ii) $N \cong R$ where R is the (unique) hyperfinite factor of typ II_1. In that case (the image of) $\mathcal{K}$ is approximately inner on R [l.c., Theorem 5.8].