If the last inequality is strict, then there exists $\gamma > 0$ and a normalized $\hat{x} \in \mathrm{Fix}((\lambda - i\alpha)\hat{R}(\lambda))$ such that

$$\gamma \leq \|\hat{y} - \hat{x}\|$$

for all $y \in \mathrm{Fix}((\lambda - i\alpha)R(\lambda))$. Take a normalized sequence $(x_n) \in \hat{x}$. Then (x_n) has a convergent subsequence whence we may assume that $\lim_n x_n = z$ exists in E . Thus $0 \neq z \in \mathrm{Fix}((\lambda - i\alpha)R(\lambda))$. From this we obtain the contradiction

$$\gamma \leq \|\hat{z} - \hat{x}\| = \lim \|z - x_n\| = 0 .$$

Consequently

$$\dim \mathrm{Fix}((\lambda - i\alpha)R(\lambda)) = \dim \mathrm{Fix}((\lambda - i\alpha)\hat{R}(\lambda)) .$$

Let $\{x_1,..,x_n\}$ be a base of $\mathrm{Fix}((\lambda - i\alpha)R(\lambda))$ and choose $\{\phi_1,..,\phi_n\}$ in $\mathrm{Fix}((\lambda - i\alpha)R(\lambda)')$ such that $\phi_k(x_j) = \delta_{k,j}$ (Lemma 1.6). Then

$$E = \mathrm{Fix}((\lambda - i\alpha)R(\lambda)) \quad \oplus \quad (\cap_{j=1}^{n} \ker\phi_j) ,$$

where both subspaces on the right are $(\lambda - i\alpha)R(\lambda)$-invariant and 1 is a pole of $(\lambda-i\alpha)R(\lambda)_{|\mathrm{Fix}((\lambda-i\alpha)R(\lambda))}$ by the finite dimensionality of $\mathrm{Fix}((\lambda-i\alpha)R(\lambda))$. Suppose 1 belongs to the spectrum of S where S is the restriction of $(\lambda-i\alpha)R(\lambda)$ to $\cap_{j=1}^{n} \ker\phi_j$. Then there exists a normalized sequence (y_n) in $\cap_{j=1}^{n} \ker\phi_j$ such that

$$\lim_n \|(\lambda - i\alpha)R(\lambda)y_n - y_n\| = 0 .$$

Therefore (y_n) has an accumulation point different from zero in

$$\mathrm{Fix}((\lambda - i\alpha)R(\lambda)) \cap (\cap_{j=1}^{n} \ker\phi_j) .$$

This contradiction implies that 1 does not belong to the spectrum of S . Since $\mathrm{Fix}((\lambda - i\alpha)R(\lambda))$ is finite dimensional, it follows from general spectral theory that $(\lambda - i\alpha)^{-1}$ is a pole of $R(.,R(\lambda))$ for every λ . Thus (a) and (b) are proved. Assertion (c) follows from the resolvent equality as in the proof of [Greiner (1981), Proposition 1.2].

$\square$