Let $r \in (\mu, s(B+\Phi_\mu)]$ and assume $r \in \rho(B+\Phi_\mu)$. By the definition of $\mu$ we have $r \in \rho(B+\Phi_{\mu+\varepsilon})$ for all $\varepsilon > 0$. By C-III,Thm.1.1 $R(r,B+\Phi_{\mu+\varepsilon}) \geq 0$ and by the assumption $R(r,B+\Phi_\mu) \geq 0$ as well. Now C-III,Thm.1.1 implies $r > s(B+\Phi_\mu)$ which yields a contradiction to the choice of $r$. Thus $r \in \sigma(B+\Phi_\mu)$ for every $r \in (\mu, s(B+\Phi_\mu)]$ and hence $\mu \in \sigma(B+\Phi_\mu)$. Consequently $s(A) \geq \mu$.
Finally we assume $s(A) > \mu$. The definition of $\mu$ yields $s(A) > s(B+\Phi_{s(A)})$. Hence $s(A) \in \rho(B+\Phi_{s(A)})$ and thus $s(A) \in \rho(A)$ by Prop. 3.4 . This yields a contradiction, since $A$ generates a positive semigroup, hence $s(A) = \mu$.

□

An immediate consequence of the preceding lemma is the following stability criterion.

Corollary 3.8. Let $\Phi \in \mathcal{L}(E,F)$ be positive and let $B$ be the generator of a positive semigroup. The following assertions are equivalent:

(i) The semigroup generated by $A$ is exponentially stable in $E$.
(ii) The semigroup generated by $B + \Phi_o$ is exponentially stable in $F$.

Proof. We can assume that there exists $\lambda \in \mathbb{R}$ with $\sigma(B+\Phi_\lambda) \neq \emptyset$. The implication "(i)→(ii)" follows immediately from Thm. 3.7.(a). It remains to show "(ii)→(i)". Let $s(B+\Phi_o) < 0$. By the lemma and since $\lambda \to s(B+\Phi_\lambda)$ is non-increasing we have $s(A) = \mu = \sup\{\lambda : s(B+\Phi_\lambda) > \lambda\} < 0$. Thus the semigroup generated by $A$ is exponentially stable.

Remark. In the situation of Thm.3.7(c) we have the stronger result that $s(A)$ and $s(B + \Phi_o)$ have the same sign.

Example 3.9 (see also C-II,Ex.4.14). Take $E = C([-1,0],\mathbb{C})$, $\alpha \in \mathbb{C}$ and $\mu \in M[-1,0]_+$ such that $\mu(\{0\}) = 0$. Then the operator $A$ given by $Af = f'$ on $D(A) = \{f \in C^1([-1,0],\mathbb{C}) : f'(0) = \alpha f(0) + \langle f,\mu\rangle\}$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$. In fact, this follows from Thm.3.1 if we put $F = \mathbb{C}$ , $\Phi = \mu$ and $B$ the multiplication by $\alpha$. Moreover $\Phi_o$ is the multiplication by $\langle\varepsilon_o,\Phi\rangle = \|\Phi\|$ (notice $\Phi\geq 0$) and $s(B + \Phi_o) = \alpha + \|\Phi\|$. Since $\omega(A) = s(A)$ by B-IV,(1.1) we obtain from Cor.3.8 that $A$ generates a uniformly exponentially stable semigroup if and only if $\alpha + \|\Phi\| < 0$.