

# Chapter 1

## Basic Results on $C_0(X)$

This part of the book is devoted to one-parameter semigroups of operators on spaces of continuous functions of type  $C_0(X)$ . Such spaces are Banach lattices of a very special kind. We treat this case separately since we believe that combining it with the abstract Banach lattice situation considered in Part C would make it difficult to appreciate the straightforward accessibility and the guiding function of methods and results available here. In this chapter, we want to establish the notation we will use and collect some basic facts about the spaces under consideration.

If  $X$  is a locally compact topological space, then  $C_0(X)$  denotes the space of all continuous complex-valued functions on  $X$  that vanish at infinity, endowed with the supremum-norm. If  $X$  is compact, then any continuous function on  $X$  “vanishes at infinity”, and  $C_0(X)$  is the space of all continuous functions on  $X$ . We often write  $C(X)$  instead of  $C_0(X)$  in this situation.

We sometimes study real-valued functions and write the corresponding real spaces as  $C_0(X, \mathbb{R})$  and  $C(X, \mathbb{R})$ . The notations  $C_0(X, \mathbb{C})$  and  $C(X, \mathbb{C})$  are used if there might be confusion between both cases.

### 1 Algebraic and Order-Structure: Ideals and Quotients

Any space  $C_0(X)$  is a commutative  $C^*$ -algebra under the sup-norm and point-wise multiplication, and by the Gelfand Representation Theorem, any commutative  $C^*$ -algebra can be canonically represented as an algebra  $C_0(X)$  on a suitable

locally compact space  $X$ . The algebraic notions of ideal, quotient, and homomorphism are well-known and need no further explanation.

Another natural and important structure of  $C_0(X)$  is the *pointwise* ordering, under which  $C_0(X, \mathbb{R})$  is a (real) Banach lattice and  $C_0(X, \mathbb{C})$  is a complex Banach lattice as explained in Chapter C-I.

For the order structure of  $C_0(X)$ , we use the following notations. For a function  $f$  in  $C_0(X, \mathbb{R})$ :

A function  $f$  is called *positive*,  $f \geq 0$ , if  $f(t) \geq 0$  for all  $t \in X$ ,

We write  $f > 0$  if  $f$  is positive but does not vanish identically,

We call  $f$  *strictly positive* if  $f(t) > 0$  for all  $t \in X$ .

The notion of an order ideal explained in Chapter C-I naturally applies to the Banach lattices  $C_0(X)$  and might lead to confusion with the corresponding algebraic notion.

However, since we primarily consider closed ideals and since a closed linear subspace  $I$  of  $C_0(X)$  is a lattice ideal if and only if  $I$  is an algebraic ideal, we may simply speak of closed ideals without specifying whether we refer to the algebraic or the order theoretic meaning of this term.

An important and frequently used characterization of these objects is the following: A subspace  $I$  of  $C_0(X)$  is a closed ideal if and only if there exists a closed subset  $A$  of  $X$  such that a function  $f$  belongs to  $I$  if and only if  $f$  vanishes on  $A$ . The set  $A$  is uniquely determined by  $I$  and is called the support of  $I$ . If  $I = I_A$  is a closed ideal with support  $A$ , then  $I_A$  is naturally isomorphic to  $C_0(X \setminus A)$ , and the quotient  $C_0(X)/I$  (under the natural quotient structure) is again a Banach algebra and a Banach lattice that can be canonically identified (via the map  $f + I \rightarrow f|_A$ ) with  $C_0(A)$ .

## 2 Linear Forms and Duality

The *Riesz Representation Theorem* states that the dual of  $C_0(X)$  can be identified naturally with the space of bounded regular Borel measures on  $X$ . Although there is no natural algebra structure on this dual, the dual ordering (see Chapter C-I) makes  $C_0(X)'$  into a Banach lattice. We occasionally make use of the order structure of  $C_0(X)'$ , but since at least its measure theoretic interpretation is presumed to be well-known, we refer to Chapter C-I, Sections 3 and 7, for a more detailed discussion and recall only some basic notations here.

If  $\mu$  is a linear form on  $C_0(X, \mathbb{R})$ , then:

- $\mu \geq 0$  means  $\mu(f) \geq 0$  for all  $f \geq 0$ ;  $\mu$  is then called *positive*,
- $\mu > 0$  means that  $\mu$  is positive but does not vanish identically,
- $\mu \gg 0$  means that  $\mu(f) > 0$  for any  $f > 0$ ;  $\mu$  is then called *strictly positive*.

If  $\mu$  is a linear form on  $C_0(X, \mathbb{C})$ , then  $\mu$  can be written uniquely as  $\mu = \mu_1 + i\mu_2$  where  $\mu_1$  and  $\mu_2$  map  $C_0(X, \mathbb{R})$  into  $\mathbb{R}$  (decomposition into *real* and *imaginary parts*). We call  $\mu$  positive (strictly positive) and use the above notations if  $\mu_2 = 0$  and  $\mu_1$  is positive (strictly positive). We note that strictly positive linear forms need not exist on  $C_0(X)$ , but if  $X$  is separable, then a strictly positive linear form is obtained by summing a suitable sequence of point measures.

The coincidence of the notions of a closed algebraic and a closed lattice ideal in  $C_0(X)$  naturally affects the algebraic and lattice theoretic notions of a homomorphism. The case of a homomorphism into another space  $C_0(Y)$  will be discussed below. Regarding homomorphisms into the scalar field, we have essentially a coincidence between the algebraic and order theoretic meanings of this term. More precisely: A linear form  $\mu \neq 0$  on  $C_0(X)$  is a lattice homomorphism if and only if  $\mu$  is an algebra homomorphism up to normalization (algebra homomorphisms  $\neq 0$  must necessarily have norm 1). Since the non-zero algebra homomorphisms on  $C_0(X)$  are known to be the point measures (denoted by  $\delta_t$ ) on  $X$ , and since  $\mu$  is a lattice homomorphism if and only if  $|\mu(f)|$  equals  $\mu(|f|)$  for all  $f$ , it follows that this latter condition on  $\mu$  is equivalent to  $\mu = \alpha\delta_t$  for a suitable  $t$  in  $X$  and a positive real number  $\alpha$ .

This can be summarized as follows:  $X$  can be canonically identified, via the map  $t \rightarrow \delta_t$ , with the subset of the dual  $C_0(X)'$  consisting of the non-zero algebra homomorphisms, which is also the set of all normalized lattice homomorphisms. This identification is a topological isomorphism with respect to the weak\*-topology of  $C_0(X)'$ .

### 3 Linear Operators

A linear mapping  $T$  from  $C_0(X, \mathbb{R})$  into  $C_0(Y, \mathbb{R})$  is called:

- positive* (notation:  $T \geq 0$ ) if  $Tf$  is positive whenever  $f$  is positive,
- lattice homomorphism* if  $|Tf| = T|f|$  for all  $f$ ,

*Markov operator* if  $X$  and  $Y$  are compact and  $T$  is a positive operator mapping  $1_X$  to  $1_Y$ .

In the case of complex scalars,  $T$  can be decomposed into real and imaginary parts. We call  $T$  positive in this situation if the imaginary part of  $T$  is 0 and the real part is positive. The terms *Markov operator* and *lattice homomorphism* are defined as above. Note that a complex lattice homomorphism is necessarily positive, and that the *complexification* of a real lattice homomorphism is a complex lattice homomorphism. Positive operators are always continuous.

Note that the adjoint of a Markov operator  $T$  maps positive normalized measures into positive normalized measures, while the adjoint of an algebra homomorphism (lattice homomorphism) maps point measures into (multiples of) point measures. Therefore, the adjoint of a Markov lattice homomorphism as well as the adjoint of an algebra homomorphism induces a continuous map  $\phi$  from  $Y$  (viewed as a subset of the weak dual  $C(Y)'$ ) into  $X$  (viewed as a subset of  $C(X)'$ ).

This mapping  $\phi$  determines  $T$  in a natural and unique way, so that the following are equivalent assertions on a linear mapping  $T$  from a space  $C(X)$  into a space  $C(Y)$ :

- (a)  $T$  is a Markov lattice homomorphism.
- (b)  $T$  is a Markov algebra homomorphism.
- (c) There exists a continuous map  $\phi$  from  $Y$  into  $X$  such that  $Tf = f \circ \phi$  for all  $f \in C(X)$ .

If  $T$  is, in addition, bijective, then the mapping  $\phi$  in (c) is a homeomorphism from  $Y$  onto  $X$ . This characterization of homomorphisms carries over mutatis mutandis to situations where the conditions on  $X$ ,  $Y$ , or  $T$  are less restrictive. For later reference, we explicitly state the following:

- (i) Let  $K$  be compact. Then  $T \in \mathcal{L}(C(K))$  is a lattice homomorphism if and only if there is a mapping  $\phi$  from  $K$  into  $K$  and a function  $h \in C(K)$  such that  $Tf(s) = h(s)f(\phi(s))$  holds for all  $s \in K$ . The mapping  $\phi$  is continuous at every point  $t$  where  $h(t) \neq 0$ .
- (ii) Let  $X$  be locally compact and  $T \in \mathcal{L}(C_0(X))$ . Then  $T$  is a lattice isomorphism if and only if there is a homeomorphism  $\phi$  from  $X$  onto  $X$  and a bounded continuous function  $h$  on  $X$  such that  $h(s) \geq \delta > 0$  for all  $s$  and  $Tf(s) = h(s)f(\phi(s))$  ( $s \in X$ ). Moreover,  $T$  is an algebraic  $*$ -isomorphism if and only if  $T$  is a lattice isomorphism and the function  $h$  above is identically 1.

# Bibliography

H. H. Schaefer. *Banach Lattices and Positive Operators*. Springer, New York-Heidelberg-Berlin, 1974.