$\{s(\Phi_\lambda)\}$ is the boundary spectrum $\sigma_b(\Phi_\lambda)$ (see C-III, Cor.2.12) of Φ_λ. Moreover, $s(\Phi_\lambda)$ is a pole of the resolvent with residue of finite rank. Such spectral sets vary continuously under smooth perturbations of Φ_λ (see [Dunford-Schwartz (1958),VII.6, Thm.9]), thus $\lambda \to s(\Phi_\lambda)$ is continuous.

□

For the operators A_o and A as defined in the beginning of this section we obtain an explicit representation of their resolvents.

Lemma 3.5. For the resolvents of the operators A_o, resp. A, on E the following statements hold.

(a) For every $\lambda \in \mathbb{C}$ we have $\lambda \in \rho(A_o)$ and

$$R(\lambda,A_o)g(t) = \int_t^0 e^{\lambda(t-s)}\, g(s)\, ds\ ,\quad g \in E. \tag{3.7}$$

(b) For $\lambda \in \mathbb{C}$ satisfying $1 \in \rho(\Phi_\lambda)$ we have $\lambda \in \rho(A)$ and

$$R(\lambda,A)g = R(\lambda,A_o)g + \varepsilon_\lambda \otimes R(1,\Phi_\lambda)\Phi R(\lambda,A_o)g\ ,\quad g \in E. \tag{3.8}$$

Proof. (a) $\rho(A_o) = \mathbb{C}$ follows directly from $(T_o(t))_{t\geq 0}$ being nilpotent (see A-III,Prop.1.1). For $g \in E$ we obtain $R(\lambda,A_o)g = f$ where f is a solution of $\lambda f - f' = g$.
Thus $R(\lambda,A_o)g(t) = \int_t^0 e^{\lambda(t-s)}\, g(s)ds + e^{\lambda t}\cdot x$ for some $x \in F$. The condition $f \in D(A_o)$ now implies $x = 0$ and Formula (3.7).
(b) The assertion $\lambda \in \rho(A)$ means that for every $g \in E$ the equation $\lambda f - f' = g$ has exactly one solution f in $D(A)$. As in case (a) we have $f(t) = \int_t^0 e^{\lambda(t-s)}\, g(s)ds + e^{\lambda t}\cdot x$ for some $x \in F$ and hence $R(\lambda,A)g = f = R(\lambda,A_o)g + \varepsilon_\lambda\otimes x$. The condition $R(\lambda,A)g \in D(A)$ implies $x - \Phi_\lambda(x) = \Phi R(\lambda,A_o)g$. Hence $x = R(1,\Phi_\lambda)\Phi R(\lambda,A_o)g$ if $1 \in \rho(\Phi_\lambda)$ and thus (3.8) follows.

□

Proposition 3.6. For each $\lambda \in \mathbb{C}$ the following implications hold.

(a) If $\lambda \in \sigma(A)$, then $1 \in \sigma(\Phi_\lambda)$.
(b) If $1 \in P\sigma(\Phi_\lambda)$, then $\lambda \in P\sigma(A)$.
If, in addition, $\Phi(D(A_o)) = F$ or if Φ_λ is compact for all $\lambda \in \mathbb{C}$, then the following equivalence holds:
(c) $\lambda \in \sigma(A)$ if and only if $1 \in \sigma(\Phi_\lambda)$.

Proof. (a) This implication follows immediately from Lemma 3.5(b).
(b) If $x \neq 0$ satisfies $x - \Phi_\lambda(x) = 0$, then $f := \varepsilon_\lambda\otimes x \in D(A)$ and $\lambda f - f' = 0$.
(c) If $\Phi(D(A_o)) = F$, then the equation $x - \Phi_\lambda x = \Phi R(\lambda,A_o)g$ has a