Proof. The first equivalence follows easily from Thm.3.7 . The additional statement is a consequence of the strict monotonicity of h.

□

Remarks. 1. We note that in Prop.3.6 and Thm.3.7 it actually suffices that some power of Φ_λ is compact.

2. The equivalence (3.9) reduces the problem of determining $s(A)$ to the determination of the spectral bounds of the operators Φ_λ on the "smaller" Banach space F.
In particular, $s(A) < 0$ if and only if $s(\Phi_0) < 1$.

3. We call the identity "$s(\Phi_\lambda) = 1$" a generalized characteristic equation (see also the remark following B-IV,Thm.3.7). The usual characteristic equation (see for example [Hale (1977),p.168ff] and [Heijmans (1984), Sec.5]) is an equation determining all eigenvalues of the generator A. In fact, if F is finite dimensional the characterization of the spectral values λ of A in Prop.3.6.(c) reduces to solving the complex equation $\det(Id - \Phi_\lambda) = 0$. Obviously, there is no analogous identity characterizing $\sigma(A)$ for infinite dimensional F. However, in order to determine the long term behavior of the solutions of (RE) it is often enough to know the spectral bound $s(A)$. Under the assumptions of Cor.3.8 (in particular if Φ is positive) Formula (3.9) gives a tool to reduce this problem to the determination of the real solution of $s(\Phi_\lambda) = 1$.

Example 3.9. We give an example of a large class of operators Φ satisfying the above assumptions.
For $\psi \in (L^1[-1,0])' = L^\infty[-1,0]$ and $B \in L(F)$ we denote by $\Phi := \psi\otimes B$ the operator defined by $\Phi(h\otimes x) = \psi(h)\cdot Bx$ for $h \in L^1[-1,0]$, $x \in F$. Note that $E = L^1([-1,0],F)$ is isomorphic to $L^1[-1,0] \tilde{\otimes}_\pi F$ (see [Schaefer (1966), Chap.III,6.5]). The operator Φ is bounded from E into F. We assume that ψ and B, hence Φ are positive.

Then the following holds and is stated without proof.

Lemma. (a) If B is compact, then Φ is compact. If B is surjective, then $\Phi(D(A_0)) = F$.
(b) $\sigma(\Phi_\lambda) = \psi(\varepsilon_\lambda)\cdot\sigma(B)$ for each $\lambda \in \mathbb{C}$. Hence the map $\mu \to s(\Phi_\mu)$ is continuous and strictly decreasing on $\mathbb{R}$.

For this type of "retarding functionals" Φ we obtain a simple characterization of the spectral bound.