If there exists $k > 0$ such that $U_{-k} \neq 0$ while $U_{-n} = 0$ for all $n > k$ the point λ_o is called a <u>pole of</u> $\underline{R(\cdot,A)}$ <u>of order</u> $\underline{k}$. In view of (3.2) this is true if $U_{-k} \neq 0$ and $U_{-(k+1)} = 0$. In this case one can retrieve U_{-k} as

(3.3) $$U_{-k} = \lim_{\lambda\to\lambda_o} (\lambda - \lambda_o)^k R(\lambda,A) .$$

The dimension of PE (i.e., the dimension of the spectral subspace corresponding to $\{\lambda_o\}$) is called <u>algebraic multiplicity</u> m_a of λ_o, while the <u>geometric multiplicity</u> is $m_g := \dim \ker(\lambda_o - A)$. In case $m_a = 1$ we call λ_o an <u>algebraically simple pole</u>.
If k is the pole order ($k = \infty$ in case of an essential singularity) we have

(3.4) $$\max\{m_g,k\} \leq m_a \leq k\cdot m_g ,$$

where $\infty\cdot 0 = \infty$. These inequalities yield the following implications:

- $m_a < \infty$ if and only if λ_o is a pole with $m_g < \infty$,
- if λ_o is a pole with order k, then $\lambda_o \in P\sigma(A)$ and $PE = \ker(\lambda_o - A)^k$.

If A has compact resolvent then every point of $\sigma(A)$ is a pole of finite algebraic multiplicity. This is a consequence of Prop.2.5(iii) and the well-known Riesz-Schauder Theory for compact operators (see [Dunford-Schwartz (1958),VII.4.5]).

3.7. <u>The essential spectrum</u>.

For $T \in L(E)$ the <u>Fredholm domain</u> $\rho_F(T)$ is

(3.5) $$\begin{aligned} \rho_F(T) &:= \{\lambda \in \mathbb{C} : \lambda - T \text{ is a Fredholm operator}\} \\ &= \{\lambda \in \mathbb{C} : \ker(\lambda - T) \text{ and } E/\mathrm{im}(\lambda - T) \\ &\qquad\qquad \text{are finite dimensional}\} . \end{aligned}$$

An equivalent characterization of $\rho_F(T)$ is obtained through the Calkin algebra $L(E)/K(E)$, where $K(E)$ stands for the closed ideal of all compact operators. In fact, $\rho_F(T)$ coincides with the resolvent set of the canonical image of T in the Calkin algebra. The complement of $\rho_F(T)$ is called <u>essential spectrum</u> of T and denoted by $\sigma_{ess}(T)$. The corresponding spectral radius, called <u>essential spectral radius</u>, satisfies

(3.6) $$r_{ess}(T) := \sup\{|\lambda| : \lambda \in \sigma_{ess}(T)\} = \lim_{n\to\infty} \|T^n\|_{ess}^{1/n} ,$$

where $\|T\|_{ess} = \mathrm{dist}(T,K(E)) := \inf\{\|T - K\| : K\in K(E)\}$ is the norm of T in $L(E)/K(E)$.
For every compact operator K we have $\|T - K\|_{ess} = \|T\|_{ess}$, hence

(3.7) $$r_{ess}(T - K) = r_{ess}(T) .$$