Theorem 3.6. Suppose $\mathcal{T} = (T(t))$ is an irreducible semigroup with generator A and spectral bound $s(A) = 0$. Assume that there exists a positive linear form $\Psi \neq 0$ such that $A'\Psi = 0$. (This is automatically satisfied whenever X is compact (see Thm.1.6).)
If $P\sigma(A) \cap i\mathbb{R}$ is non-empty, then the following assertions are true:

(a) $P\sigma(A) \cap i\mathbb{R}$ is a (additive) subgroup of $i\mathbb{R}$.

(b) The eigenspaces corresponding to $\lambda \in P\sigma(A) \cap i\mathbb{R}$ are one-dimensional.

(c) If $Ah = i\alpha h$ $(h \neq 0,\ \alpha \in \mathbb{R})$, then h has no zeros in X. In case $\alpha = 0$ then $h(x)/|h(x)|$ is constant; otherwise, $\{h(x)/|h(x)| : x \in X\}$ is a dense subset of Γ.

(d) If $Ah = i\alpha h$ $(h \neq 0,\ \alpha \in \mathbb{R})$, then

$$(3.5) \qquad S_h(D(A)) = D(A) \quad \text{and} \quad S_h^{-1} \circ A \circ S_h = (A + i\alpha).$$

In particular, spectrum and point spectrum of A are invariant under translations by $i\alpha$.

(e) 0 is the only eigenvalue admitting a positive eigenfunction.

Proof. By Prop.3.5(c) the invariant linear form Ψ is strictly positive and it satisfies $T(t)'\Psi = \Psi$ $(t \geq 0)$.

(d) Supposing $Ah = i\alpha h$ $(h \neq 0,\ \alpha \in \mathbb{R})$ then $A|h| = 0$ by (2.14) and (2.15). By Prop.3.5(b) $|h|$ is strictly positive, thus Thm.2.4(b) implies (3.5).

(b) Assertion (d) implies that S_h maps $\ker(i\alpha + A)$ onto $\ker A$ whenever $i\alpha \in P\sigma(A) \cap i\mathbb{R}$. Moreover, we have seen in the proof of (d) that $\ker A \neq \{0\}$ hence it is one-dimensional by Prop.3.5(d).

(a) Assume that $Ah = i\alpha h$, $Ag = i\beta g$ $(\alpha, \beta \in \mathbb{R},\ h \neq 0,\ g \neq 0)$. By (3.5) we have $S_{\bar g} A S_g = A + i\beta$ and $S_h A S_{\bar h} = A - i\alpha$, therefore

$$(3.6) \qquad A + i(\beta - \alpha) = S_h(A + i\beta)S_{\bar h} = S_h S_{\bar g} A S_g S_{\bar h}.$$

It follows that $\ker[A + i(\beta - \alpha)] = S_h S_{\bar g}(\ker A) \neq \{0\}$, hence $i(\beta - \alpha) \in P\sigma(A)$.

(e) If $Af = \lambda f$ where $f > 0$, then

$$(3.7) \qquad \lambda \cdot \langle f, \Psi\rangle = \langle Af, \Psi\rangle = \langle f, A'\Psi\rangle = 0.$$

Since Ψ is strictly positive we have $\langle f, \Psi\rangle > 0$ hence $\lambda = 0$.

(c) We already know that $Ah = i\alpha h$ implies that $A|h| = 0$. It follows from Prop.3.5(b) that h is strictly positive; i.e., h has no zeros in X. By Prop.3.5(d) $\ker A$ is one-dimensional hence every