b) The semigroup is right-sided differentiable in a fixed point $t > 0$ if and only if there exists $c > 0$ such that $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| > c\cdot e^{-t\mathrm{Re}\lambda}\} \subset \rho(A)$.

Proof. The semigroup is right-sided differentiable in t if and only if $T(t)E \subset D(A)$ if and only if $e^{tm}\cdot f\cdot m \in E$ for all $f \in E$ if and only if $e^{tm}\cdot m$ is [essentially] bounded if and only if $e^{t\mathrm{Re}\, m}\cdot \mathrm{Im}\, m$ is [essentially] bounded if and only if there exists $c > 0$ such that [ess]-image(m) $\subset \{\lambda \in \mathbb{C} : e^{t\mathrm{Re}\lambda}|\mathrm{Im}\lambda| \leq c\}$ if and only if there exists $c > 0$ such that $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| > c\cdot e^{-t\mathrm{Re}\lambda}\} \subset \rho(A)$.

□

c) $(T(t))_{t\geq 0}$ is a bounded holomorphic semigroup of angle θ if and only if $S(\theta+\pi/2) \subset \rho(A)$.

Proof. The condition is necessary by Theorem 1.12.
Conversely, if $S(\theta+\pi/2) \subset \rho(A)$, then one verifies directly that $(T(z)f)(x) = e^{z\cdot m(x)}f(x)$ $(f\in E,\ x\in X)$ defines a family $(T(z))_{z\in S(\theta)}$ of bounded operators satisfying conditions (1.4) and (1.5).

□

d) Choosing $X = \mathbb{N}$ and μ the counting measure we have $E = c_o$ or ℓ^p $(1\leq p<\infty)$. Then A has a compact resolvent if $\lim_{n\to\infty} |m(n)| = \infty$.
[In fact, let $\lambda > s(A)$. Then $(R(\lambda,A)f)(n) = (\lambda-m(n))^{-1}f(n)$. Hence $R(\lambda,A)$ is compact if and only if $((\lambda-(m(n))^{-1})_{n\in\mathbb{N}} \in c_o$.]
The semigroup is compact if and only if it is eventually compact if and only if $\lim_{n\to\infty} \mathrm{Re}(m(n)) = -\infty$.

e) Now it is easy to give concrete examples. Again let $X = \mathbb{N}$, so that $E = c_o$ or ℓ^p $(1\leq p<\infty)$. Let $m(n) = -n + i\cdot\exp(n^2)$. Then the semigroup is compact and (consequently) norm continuous for $t > 0$, but it is not eventually differentiable. Let $m(n) = -n + ie^{t'n}$. Then the semigroup is differentiable for $t > t'$ but not differentiable in $t \in [0,t')$. If $m(n) = -n + i\cdot n^2$, then the semigroup is differentiable but not holomorphic.

Perturbation of Generators

A useful way to construct new semigroups out of a given one is by additive perturbation.