The other part follows from this since $\omega(A) = \omega(m^{-1}\cdot m\cdot A) \leq \|m^{-1}\|_\infty \, \omega(m\cdot A)$.

□

In the following lemma a condition (P') is introduced which is dual to the positive minimum principle.

Lemma 1.21. Let A be the generator of a strongly continuous positive semigroup on $C(K)$. Then for $f \in C(K)_+$, $0 \leq \mu \in D(A')$

(P') $\langle f,\mu\rangle = 0$ implies $\langle f,A'\mu\rangle \geq 0$.

Proof. $\langle f,A'\mu\rangle = \lim_{t\to 0} \frac{1}{t} \langle T(t)f - f, \mu\rangle = \lim_{t\to 0} \frac{1}{t} \langle T(t)f, \mu\rangle \geq 0$.

□

Example 1.22. Let $K = [-1,0]$. Let $\alpha \in \mathbb{R}$ and μ be a measure on $[-1,0]$ such that $\mu(\{0\}) = 0$. Define the operator A on $C[-1,0]$ by $Af = f'$ with domain $D(A) = \{f \in C^1[-1,0] : f'(0) = \alpha f(0) + \langle f,\mu\rangle\}$.

Claim: A is the generator of a positive semigroup if and only if $\mu \geq 0$.

Proof of the claim. Assume that A generates a positive semigroup. By the definition of A one has $\delta_o \in D(A')$ and $A'\delta_o = \alpha\delta_o + \mu$. So it follows from (P') that $\langle f,\mu\rangle = \langle f,A'\delta_o\rangle \geq 0$ for all $f \in C[-1,0]_+$ such that $f(0) = 0$. By Lemma 1.2 this implies that $\mu \geq 0$.

In order to show the converse assume that $\mu \geq 0$.

a) We show that A is densely defined. Consider the normed space $F = C^1[-1,0]$ with the supremum norm. Then $\psi : F \to \mathbb{R}$ given by $\psi(f) = f'(0) - \alpha f(0) - \langle f,\mu\rangle$ is a discontinuous linear form on F. Consequently $D(A) = \ker \psi$ is dense in F. Since F is dense in $C[-1,0]$, $D(A)$ is dense in $C[-1,0]$ as well.

b) A satisfies (P) (see Def. 1.5). In fact, let $f \in D(A)_+$ and $x \in [-1,0]$ such that $f(x) = 0$. It is clear that $Af(x) = f'(x) \geq 0$ if $x < 0$. But if $f(0) = 0$, then $Af(0) = f'(0) = \langle f,\mu\rangle \geq 0$ since $f \in D(A)$.

c) We show that $(\lambda - A)$ is bijective for $\lambda > \alpha + \|\mu\|$. Let $g \in C[-1,0]$. The solutions of the equation $\lambda f - f' = g$ ($f \in C[-1,0]$) are given by $f(x) = e^{\lambda x}[\int_x^0 e^{-\lambda y} g(y)\, dy + c]$ where $c \in \mathbb{R}$. Moreover, $f \in D(A)$ if and only if

$$(2.8) \quad c(\lambda - \alpha - \int_{-1}^0 e^{\lambda x}\, d\mu(x) = g(0) + \int_{-1}^0 e^{\lambda x} \int_x^0 e^{-\lambda y} g(y)\, dy d\mu(x).$$