(2.19) $\quad \langle |h_1|,\phi_1\rangle > 0$.

For arbitrary $f_1 \in E_1$, $\mathrm{Re}\,\lambda > 0$ we have $\langle |R_1(\lambda)f_1|,\phi_1\rangle \leq$ $\langle R_1(\mathrm{Re}\lambda)|f_1|,\phi_1\rangle = \langle |f_1|,R_1(\mathrm{Re}\lambda)'\phi_1\rangle = (\mathrm{Re}\lambda)^{-1}\langle |f_1|,\phi_1\rangle$. Therefore the ideal $I := \{f_1 \in E_1 : \langle |f_1|,\phi_1\rangle = 0\}$ is invariant under $\{(R_1(\lambda)\}_{\mathrm{Re}\lambda>0}$. Furthermore we have (see (2.17), (2.18)),

$$\begin{aligned}\langle |rR_1(r)|h_1| - |h_1||,\phi_1\rangle &= \langle rR_1(r)|h_1| - |h_1|,\phi_1\rangle \\ &= \langle |h_1|,rR_1(r)'\phi_1 - \phi_1\rangle = 0 \quad \text{for} \quad r > 0\end{aligned}$$

which implies

(2.20) $\quad rR_1(r)|h_1| - |h_1| \in I \quad (r > 0)$.

Denoting by E_2 the quotient space E_1/I and by $\{(R_2(\lambda)\}_{\mathrm{Re}\lambda>0}$ the pseudo-resolvent on E_2 induced by $\{(R_1(\lambda)\}_{\mathrm{Re}\lambda>0}$ in the canonical way, then $h_2 := h_1 + I \neq 0$ (by (2.19)) . Moreover, $\lambda R_2(\lambda+i\beta)h_2 = h_2$ (by (2.16)) and $\lambda R_2(\lambda)|h_2| = |h_2|$ (by (2.20) and Prop.2.6(a)). Now we apply Prop.2.7(b) and obtain

(2.21) $\quad \lambda R_2(\lambda+in\beta)h_2^{[n]} = h_2^{[n]}$ for $\mathrm{Re}\,\lambda > 0$, $n \in \mathbb{Z}$.

In particular, we have $\|R_2(r+in\beta)\| \geq \frac{1}{r}$, thus $\|R(r+in\beta,A)\| = \|R_1(r+in\beta)\| \geq \|R_2(r+in\beta)\| \geq \frac{1}{r}$ for $r > 0$. This finally implies that $in\beta \in \sigma(A)$ for $n \in \mathbb{Z}$.

□

To prove cyclicity of the boundary spectrum in case $s(A)$ is a pole (of arbitrary order) one applies B-III,Lemma 2.8 to reduce the problem to the case of first order poles. Actually, B-III,Lemma 2.8 is true for arbitrary Banach lattices and the proof given in chapter B-III works in the general case as well. For completeness we recall this result.

Proposition 2.11. Let A be the generator of a positive semigroup $\mathcal{T}$ on a Banach lattice E and suppose that the spectral bound $s(A)$ is a pole of the resolvent of order k . Then there is a sequence

(2.22) $\quad I_{-1} := \{0\} \subset I_o \subsetneqq I_1 \subsetneqq \ldots \subsetneqq I_k := E$

of $\mathcal{T}$-invariant closed ideals with the following properties: If A_n is the generator of the semigroup induced by $\mathcal{T}$ on the quotient I_n/I_{n-1} , then we have

(a) $s(A_o) < s(A)$;

(b) If $n \geq 1$ then $s(A_n) = s(A)$ is a first order pole of the resolvent $R(.,A_n)$. The corresponding residue is a strictly positive operator.