Corollary. Let $\Phi = \psi \otimes B$ where $0 \leq \psi \in L^{\infty}[-1,0]$ and $0 \leq B \in L(F)$ such that B^n is compact for some $n \in \mathbb{N}$. Then the following holds.

(3.10) $\qquad s(A) \begin{smallmatrix}\leq\\=\\>\end{smallmatrix} \lambda$ if and only if $\psi(\varepsilon_\lambda)\cdot s(B) \begin{smallmatrix}\leq\\=\\>\end{smallmatrix} 1$.

Example 3.10. Let F be a Banach lattice with the Dunford-Pettis property (see Schaefer(1974),Sec.II.9). Take for example $F = C(K)$ or $F = L^1(X,\Sigma,\mu)$. Furthermore define $E = L^1([-1,0],F)$ as usual and let $\{K(s) : s \in [-1,0]\}$ be a family of positive, irreducible, weakly compact operators on F which is bounded.
If we define $\Phi f := \int_{-1}^{0} K(s)f(s)\,ds$ for all $f \in E$, then (RE) has the form

$$(3.11)\qquad \begin{aligned} f(t) &= \int_{-1}^{0} K(s)f(s+t)\,ds\ ,\ t \geq 0\ ,\\ f_0 &= \psi \in E\ .\end{aligned}$$

By Cor.3.2 (3.11) has a unique solution $f \in L^1([-1,\infty),F)$. For Φ_λ we obtain $\Phi_\lambda x = \int_{-1}^{0} e^{\lambda s}K(s)x\,ds,\ x \in F$. In this case we have

$$s(A) \begin{smallmatrix}\leq\\=\\>\end{smallmatrix} \lambda \quad \text{if and only if} \quad s(\Phi_\lambda) \begin{smallmatrix}\leq\\=\\>\end{smallmatrix} 1\ .$$

Proof. By Cor.3.8 it suffices to show that the map $h : \lambda \to s(\Phi_\lambda) = r(\Phi_\lambda)$ is strictly decreasing and continuous.
With the help of [Schaefer (1966),Thm.III.11.4] and [Schaefer (1974), Thm.II.9.9] it is easy to show that $\Phi_\lambda^{\ 2}$ is compact and the continuity of h follows by the above remark. It remains to show that h is strictly decreasing.
Assume $s(\Phi_\lambda) > 0$ for all $\lambda \in \mathbb{R}$. Since $\Phi_\lambda^{\ 2}$ and $\Phi_\mu^{\ 2}$ are compact, $s(\Phi_\lambda)$ and $s(\Phi_\mu)$ are eigenvalues of Φ_λ resp. Φ_μ with corresponding eigenfunctions x_λ resp. x_μ. In the same way $s(\Phi_\lambda)$ and $s(\Phi_\mu)$ are eigenvalues of Φ_λ' resp. Φ_μ' with corresponding eigenfunctions x_λ' resp. x_μ'.
For $0 < x \in F$ and $0 < \mu < \lambda$ we obtain,

$$\Phi_\mu x = \int_{-1}^{0} e^{\mu s}K(s)x\,ds = \int_{-1}^{0} e^{(\mu-\lambda)s}e^{\lambda s}K(s)x\,ds \ > \int_{-1}^{0} e^{\lambda s}K(s)x\,ds \ = \Phi_\lambda x$$

since $K(s)$ are positive and irreducible operators.
Especially, $\Phi_\mu x_\lambda > \Phi_\lambda x_\lambda = r(\Phi_\lambda)x_\lambda$ and by evaluation $\langle\Phi_\mu x_\lambda, x_\mu'\rangle > r(\Phi_\lambda)\langle x_\lambda, x_\mu'\rangle$. Thus $r(\Phi_\mu)\langle x_\lambda, x_\mu'\rangle > r(\Phi_\lambda)\langle x_\lambda, x_\mu'\rangle$. Since the operators Φ_λ are irreducible for each λ (due to the irreducibility of $K(s)$) x_μ' is a strictly positive functional on F. Hence $\langle x_\lambda, x_\mu'\rangle \neq 0$ and thus $r(\Phi_\mu) > r(\Phi_\lambda)$.

□