Since $u(t) = (T(t)g)(0) \in D(B)$ for $t \geq 0$ the above calculation shows that $u$ is right-sided differentiable at $0$ and differentiable for $t > 0$; hence

$$\dot{u}(t) = Bu(t) + \Phi(T(t)g).$$

By the translation property (T) we have $T(t)g = u_t$, indeed

$$u_t(s) = u(t+s) = \begin{cases} g(t+s) & \text{if } t+s \leq 0 \\ T(t+s)g(0) & \text{if } t+s > 0 \end{cases} = T(t)g(s).$$

Therefore $\dot{u}(t) = Bu(t) + \Phi(u_t)$, i.e. $u$ solves (RCP).

In order to show uniqueness of the solution we take $w$ to be a solution of (RCP) satisfying $w_o = 0$. Let $x(t) := w_t$, $t \geq 0$. It is easy to see that $x(t) \in C^1([-1,0],F)$; moreover, since $\dot{w}_t(0) = \dot{w}(t) = Bw(t) + \Phi(w_t)$ we obtain $x(t) \in D(A)$. By the definition of $A$ we have $Ax(t) = \dot{w}_t$. On the other hand, $x(\cdot) \in C^1([0,\infty),E)$ and

$$(\dot{x}(t))(s) = \lim_{h\to 0} 1/h\cdot(w_{t+h}(s) - w_t(s))$$
$$= \lim_{h\to 0} 1/h\cdot(w_t(h+s) - w_t(s)) = \dot{w}_t(s), \text{ whence } \dot{x}(t) = \dot{w}_t .$$

Therefore we obtain $\dot{x}(t) = Ax(t)$. As $x(0) = w_o = 0$ it follows by the well-posedness of the abstract Cauchy problem corresponding to $A$ that $x(t) = 0$ for each $t \geq 0$. This proves $w \equiv 0$.

□

<u>Remarks</u> 1. By similar arguments the following can be proved. If $u$ is a solution of (RCP) such that $u_o \in D(A)$, then $x$ given by $x(t) := u_t$ is a solution of the abstract Cauchy problem associated with the operator $A$ defined in (3.1). In this sense, (RCP) and the semigroup generated by $A$ correspond to each other.

2. If additionally to the assumptions of Cor.3.2 $B \in L(F)$ then $u$ is a solution of (RCP) for every $g \in E$. [Indeed, a careful inspection shows that the proof of Cor.3.2 can be generalized to this situation, since $u(t) = (T(t)g)(0) \in F = D(B)$ for all $g \in E$ and $t \geq 0$.]

3. For general $g \in E$ the retarded Cauchy problem (RCP) may not have a solution. Indeed, if $u$ is a solution of (RCP) then the following is valid for $0 \leq s \leq t$:

$$\frac{d}{ds}S(t-s)u(s) = -BS(t-s)u(s) + S(t-s)\dot{u}(s)$$
$$= -BS(t-s)u(s) + S(t-s)Bu(s) + S(t-s)\Phi(u_s) = S(t-s)\Phi(u_s).$$

Hence

$$u(t) - S(t)u(0) = \int_0^t S(t-s)\Phi(u_s)\, ds.$$

Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup which is not differentiable (for examples see A-II,1.28). Define $g \in E$ by $g(s) := \tilde{g}$ for all $s \in [-1,0]$ where $\tilde{g} \in F$ is chosen such that