Applying the triangle inequality to $T(t)f = e^{rt}\big(Pf + (e^{-rt}T(t)f-Pf)\big)$ and using (2.2) one easily deduces (2.3).

□

Let us point out the following consequence of Corollary 2.2 :

For every positive, non-zero initial value f the solution $T(.)f$ of the abstract Cauchy problem $\dot{u} = Au$ decreases or increases exponentially in norm according to the sign of $r = s(A)$. If $s(A) = 0$ then $T(.)f$ tends to an equilibrium state which is unique up to a constant and non-zero whenever the initial value is positive and non-zero.

In order to apply Thm.2.1 and its corollary to concrete problems one needs conditions which ensure that the semigroup is eventually compact. We discuss this problem for the spaces $C(K)$, K compact, in more detail. The crucial tool is the following characterization of weakly compact subsets in the dual space $M(K) = C(K)'$ due to Grothendieck (1953).

<u>Proposition</u> 2.3. Let K be a compact space. For a subset $M \subset M(K) = C(K)'$ the following assertions are equivalent:

(i) M is relatively compact for the weak topology $\sigma(M(K),M(K)')$;

(ii) for each weak null sequence (f_n) in $C(K)$, $\lim_{n\to\infty}\langle f_n,\nu\rangle = 0$ uniformly for $\nu \in M$;

(iii) for each sequence (U_n) of disjoint open subsets of K, $\lim_{n\to\infty}\nu(U_n) = 0$ uniformly for ν in M.

For a proof of this result see e.g. II.9.8 in Schaefer (1974). We use this proposition in order to describe weakly compact operators on spaces $C(K)$. As usual we identify in the natural way the bounded Borel functions on K with a subspace $B(K)$ of $M(K)' = C(K)''$; in general, $C(K) \subsetneqq B(K) \subsetneqq C(K)''$.

<u>Proposition</u> 2.4. Let K be a compact space, G be a Banach space and let $R : C(K) \to G$ be a bounded linear operator.

(a) The following assertions are equivalent :

(i) R is weakly compact;

(ii) for every bounded Borel function g on K we have $R''g \in G$;

(iii) for every Borel set $C \subset K$ we have $R''(\chi_C) \in G$.