For convenience we want to have a distinctive name for the norm function. So we denote by $N : E \to \mathbb{R}$ the function given by $N(f) = \|f\|$ throughout. Then (2.5) can be written in the form

$$(2.6) \qquad dN(f) = \{\phi \in E' : \|\phi\| \leq 1 \ , \ \langle f,\phi\rangle = \|f\|\} \ .$$

A [strictly] N -dissipative operator is simply called <u>[strictly] dissipative</u> (which is in accordance with the usual nomenclature).

<u>Example</u> 2.2. a) Let $E = C[0,1]$, $f \in E$.
Then there exists $x \in [0,1]$ such that $|f(x)| = \|f\|_\infty$. Define $\phi \in E'$ by $\langle g,\phi\rangle = (\text{sign } f(x))g(x)$. Then $\phi \in dN(f)$. Note that $dN(f)$ may be an infinite set.
b) Let H be a Hilbert space, $f \in H$, $f \neq 0$. Then $dN(f) = \{\phi_f\}$ where $\langle g,\phi_f\rangle = 1/\|f\|(g|f)$.
c) $A - \|A\|\text{Id}$ is strictly dissipative for every bounded operator A .

<u>Proposition</u> 2.3. Let A be an operator on E.
Then A is p-dissipative if and only if

$$(2.7) \qquad p(f) \leq p(f - tAf) \qquad \text{for all } f \in D(A) \ , \ t > 0 \ .$$

If in particular $(w,\infty) \subset \rho(A)$ for some $w \in \mathbb{R}$, then A is p-dissipative if and only if

$$(2.8) \qquad p(\lambda R(\lambda,A)f) \leq p(f) \qquad \text{for all } f \in E \ , \ \lambda > w \ .$$

<u>Proof</u>. Assume that A is p -dissipative. Let $f \in D(A)$, $t > 0$. There exists $\phi \in dp(f)$ such that $\langle Af,\phi\rangle \leq 0$. Hence, $p(f) = \langle f,\phi\rangle = \langle f - tAf + tAf , \phi\rangle \leq \langle f - tAf,\phi\rangle \leq p(f - tAf)$. So (2.7) holds.
Converse, let $f \in D(A)$. For every $t > 0$ choose $\phi_t \in dp(f - tAf)$. Then $\pm\langle g,\phi_t\rangle \leq p(\pm g) \leq c\|g\|$ for all $g \in E$, $t > 0$. Thus the net $(\phi_t)_{t>0}$ is bounded. Consequently it posseses a $\sigma(E',E)$ - limit point ϕ as $t \to 0$. We show that $\phi \in dp(f)$ and $\langle Af,\phi\rangle \leq 0$.
Since $\langle g,\phi_t\rangle \leq p(g)$ for all $t > 0$ it follows that $\langle g,\phi\rangle \leq p(g)$ $(g \in E)$. Moreover, $\langle f,\phi_t\rangle - t\langle Af,\phi_t\rangle = p(f - tAf)$ $(t > 0)$. Letting $t \to 0$ yields $\langle f,\phi\rangle = p(f)$.
We have proved that $\phi \in dp(f)$. By hypothesis we have for all $t > 0$, $p(f) \leq p(f-tAf) = \langle f-tAf,\phi_t\rangle = \langle f,\phi_t\rangle - t\langle Af,\phi_t\rangle \leq p(f) - t\langle Af,\phi_t\rangle$.
Consequently $\langle Af,\phi_t\rangle \leq 0$ for all $t > 0$. Thus $\langle Af,\phi\rangle \leq 0$.

□