Since $(T(t))_{t\geq 0}$ is not positive but Kato's inequality holds, it follows from Theorem 3.8 that there does not exist a strictly positive subeigenvector of A'. In fact, even the following is true.

(3.16) $0 \leq \phi \in D(A')$, $A'\phi \leq \mu\phi$ for some $\mu \in \mathbb{R}$ implies $\phi = 0$.

Proof. Suppose that $0 \leq \phi \in D(A')$ such that $-\phi' = A'\phi \leq \mu\phi$. We can assume that $\mu \geq 0$. Let $\psi(x) = \phi(1-x)$. Then $\psi'(x) = -\phi'(1-x) \leq \mu\phi(1-x) = \mu\psi(x)$. Since $\psi(0) = 0$, we obtain

$\psi(x) = \int_0^x \psi'(y)\, dy \leq \mu \int_0^x \psi(y)\, dy \qquad (x \in [0,1])$.

It follows from Gronwall's lemma that $\psi \leq 0$. Hence $\phi = \psi = 0$.

□

Remark 3.16. Let B be the generator of a strongly continuous semigroup on a real Banach lattice with order continuous norm. Assume that the following two conditions hold.

(K) $\quad \langle(\text{sign } f)Bf, \phi\rangle \leq \langle|f|, B'\phi\rangle \quad (f \in D(B),\ \phi \in D(B')_+)$.

(3.17) $\quad (D(B')_+)^{-\sigma(E',E)} = E'_+$.

Because of (3.17) condition (K) implies that $P_f Bf \leq (\text{sign } f)Bf \leq Bf$ whenever $f \in D(B)_+$.
This is Kato's inequality in the strong form for positive $f \in D(B)$ and is equivalent to $(Bf)^- \in \{f\}^{dd} = \bar{E}_f^-$ $(f \in D(A)_+)$ (recall that E has order continuous norm). By Lemma 1.5 this again is equivalent to

(P) $\quad 0 \leq f \in D(B)$, $\phi \in E'_+$, $\langle f, \phi\rangle = 0$ implies $\langle Bf, \phi\rangle \geq 0$.

It is easy to see that the operator A in the example satisfies conditions (K) and (3.17). Thus the positive minimum principle (P) is not sufficient for the positivity of the semigroup.

In view of the preceding example and remarks one might presume that the existence of a strictly positive set of subeigenvectors of the adjoint of the generator actually implies the positivity of the semigroup. This is not the case.

To give an example consider $E = L^2(\mathbb{R})$ and the operator B given by

$Bf = f^{(3)}$ with domain

$D(B) = \{f \in L^2(\mathbb{R}) : f \in C^2(\mathbb{R})\ ;\ f'' \in AC(\mathbb{R})\ ;\ f, f', f'', f^{(3)} \in L^2(\mathbb{R})\}$