Let $F = C[0,1]$, $E = C([-1,0]\times[0,1])$ and let $\tilde{B}$ be defined by $\tilde{B}h = h''$ with domain $D(\tilde{B}) := \{h \in C^2[0,1] : h'(0) = h'(1) = 0\}$. Denote by M_b and M_d the respective multiplication operators for $0 \leq b,d \in F$. Then (3.11) takes the abstract form

$$\dot{u}(t) = \tilde{B}u(t) - M_d u(t) + M_b u(t-1)$$

$$u_o = \psi \in E .$$

It is well-known that $\tilde{B}$ generates a positive contraction semigroup and has compact resolvent (see A-I,2.7). The same is true for the operator $B := \tilde{B} - M_d$ (see A-II, Thm.1.29 and Thm.1.30). Thus by the above results the solution semigroup of (3.11) is positive and its asymptotic behavior can be investigated by the "undelayed" equation

$$\dot{u}(t) = (\tilde{B} + M_h)u(t) \ , \text{ where } \ h := b - d .$$

Let $h(x) < 0$ for all $x \in [0,1]$.
Then $s(\tilde{B} + M_h) \leq \max\{h(x) : x \in [0,1]\} < 0$. Hence the solutions of (3.11) are uniformly exponentially stable.

Interpretation. The solution u of (3.11) can be interpreted as the density of a population, distributed over an "area" $[0,1]$.
The operator $\frac{\partial^2}{\partial^2 x}$ describes the internal migration of the population and the functions b and d are the "place specific" birth- resp. death rate of the population members. The time delay 1 stands for the gestation period. The stability condition $h(x) < 0$ for all $x \in [0,1]$ means that the death rate has to majorize the birth rate in each spatial point to lead to extinction of the population, no matter whether the equation with or without delay is considered.

Example 3.12. An interesting example from cell biology is given by Gyllenberg-Heijmans (1985). They investigate a balance equation for the size distribution of a cell population which is structured by size. To point out the main ideas we will restrict the complex situation to the simple case of linear cell growth and refer to the original paper for details and the more general case.
Let $0 < r < 1$ and let $\alpha = r$ be the minimal cell size. Furthermore let $F = L^1([\alpha,1])$ and $E = C([-r,0],F)$. The retarded differential equation of interest is the following.

$$\begin{aligned} &\frac{d}{dt}u(t) = Bu(t) + Lu(t-r) \\ &\qquad u = \Psi \in E. \end{aligned} \tag{3.12}$$