Lemma 4.6. Let A be the generator of a positive group on a Banach lattice E which has order continuous norm. Given $\mu \in \rho(A) \cap \mathbb{R}$ then every $g \in E_{\mathbb{R}}$ is representable as sum of two elements g_1 and g_2 such that

(a) $g \geq 0$ if and only if both g_1 and g_2 are positive;

(b) $R(\mu,A)g_1 = (R(\mu,A)g)^+$;

(c) $R(\mu,A)g_2 = -(R(\mu,A)g)^-$.

We need another lemma. It can be formulated for arbitrary positive semigroups on Banach lattices.

Lemma 4.7. Let $(T(t))_{t \geq 0}$ be a positive semigroup on a Banach lattice E with generator A . Given $\mu \in \rho(A) \cap \mathbb{R}$ and $h \in E_+$ then the following assertions are equivalent:

(i) $R(\mu,A)h \geq 0$;

(ii) $\{\int_0^t e^{-\mu s} T(s)h \, ds : t \in \mathbb{R}_+\}$ is bounded in E .

Proof. (i)→(ii): We have
$\int_0^t e^{-\mu s} T(s)h \, ds = (Id - e^{-\mu t} T(t))R(\mu,A)h$ (see A-I,(3.2)).
Since $R(\mu,A)h \geq 0$ and $T(t)$ is a positive operator we obtain
$\int_0^t e^{-\mu s} T(s)h \, ds = R(\mu,A)h - e^{-\mu t} T(t)R(\mu,A)h \leq R(\mu,A)h$ which implies assertion (ii) .
(ii)→(i): The assumption implies that $\int_0^\infty e^{-\nu s} T(s)h \, ds :=$ $\lim_{t\to\infty} \int_0^t e^{-\nu s} T(s)h \, ds$ exists for $\nu > \mu$. Using that A is a closed operator it follows that $(\nu - A)(\int_0^\infty e^{-\nu s} T(s)h \, ds) = h$. For ν sufficiently close to μ such that $\nu \in \rho(A) \cap \mathbb{R}$ we have $R(\nu,A)h =$ $\int_0^\infty e^{-\nu s} T(s)h \, ds \geq 0$. By continuity we conclude $R(\mu,A)h \geq 0$.

□

By now we are prepared to prove the spectral decomposition for positive groups. Before we formulate the theorem we recall the following consequence of Thm.4.2 : For any $\mu \in \rho(A) \cap \mathbb{R}$ the line $\mu + i\mathbb{R}$ is a subset of the resolvent set and divides $\sigma(A)$ into disjoint sets. Both sets will be unbounded in general.

Theorem 4.8. Let $(T(t))_{t \in \mathbb{R}}$ be strongly continuous group of positive operators on a Banach lattice E with order continuous norm.
If A is the generator and $\mu \in \rho(A) \cap \mathbb{R}$ then
$I_\mu := \{f \in E : R(\mu,A)|f| \geq 0\}$ and $J_\mu := \{f \in E : R(\mu,A)|f| \leq 0\}$
are $(T(t))_{t \in \mathbb{R}}$-invariant projection bands, the direct sum of them