Proof. If $(\gamma - i\alpha)R(\gamma)\psi = \psi$ then $(\lambda - i\alpha)R(\lambda)\psi = \psi$ for all $\lambda \in D$. In particular, $\mu R(\mu + i\alpha)\psi = \psi$ $(\mu \in \mathbb{R}_+)$. For all $x \in M$ we obtain

$$|\psi(x)|^2 = |\langle \mu R(\mu+i\alpha)'x,\psi\rangle|^2 \leq$$

$$\leq \|\psi\| \langle (\mu R(\mu+i\alpha x)'x)(\mu R(\mu+i\alpha x)'x)^*,\psi\rangle \leq$$

$$\leq \|\psi\| \langle \mu R(\mu)'(xx^*),|\psi|\rangle$$

(D-I, Corollary 2.2). Since

$$\|\psi\| = \|\ |\psi|\ \| = |\psi|(1) =$$

$$= \langle \mu R(\mu)'1,|\psi|\rangle = \|\ \mu R(\mu)|\psi|\ \| ,$$

we obtain $\mu R(\mu)|\psi| = |\psi|$ by the uniqueness theorem (*) mentioned at the beginning. Therefore $|\psi| \in \mathrm{Fix}(R)$. Since

$$0 \leq (\mu R(\mu)'x)(\mu R(\mu)'x)^* \leq \mu R(\mu)'xx^* ,$$

the map $R(\mu)$ is positive. Consequently $(\mu+i\alpha)R(\mu)\psi^* = \psi^*$ from which $|\psi^*| \in \mathrm{Fix}(R)$ follows.
If $\phi \in \mathrm{Fix}(R)$ is selfadjoint with Jordan decomposition $\phi = \phi^+ - \phi^-$, then $|\phi| = \phi^+ + \phi^-$ [Takesaki (1979), Theorem III.4.2.]. From this we obtain that ϕ^+ and ϕ^- are in $\mathrm{Fix}(R)$.

□

Corollary 1.5. Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on M_* with generator A and suppose $P\sigma(A) \cap i\mathbb{R} \neq \emptyset$.

(a) If $\alpha \in \mathbb{R}$ and $\psi \in \ker(i\alpha - A)$, then $|\psi|$ and $|\psi^*|$ are elements of $\mathrm{Fix}(\mathcal{T}) = \ker(A)$.

(b) $\mathrm{Fix}(\mathcal{T})$ is invariant under the involution of M_*. If $\psi \in \mathrm{Fix}(\mathcal{T})$ is self adjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of $\mathrm{Fix}(\mathcal{T})$.

The proof follows immediately from D-I, Corollary 2.2 and the fact that $\ker(A) = \mathrm{Fix}(\lambda R(\lambda,A))$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$.

If $\mathcal{T}$ is the semigroup of translations on $L^1(\mathbb{R})$ and A' the gene-