<u>Theorem</u> 1.25. A strongly continuous semigroup $(T(t))_{t\geq 0}$ is compact if and only if it is norm continuous and its generator A has compact resolvent.

<u>Proof</u>. Assume that $(T(t))_{t\geq 0}$ is compact. Then $T(\cdot)$ is norm continuous on $(0,\infty)$, and so $\int_0^t e^{-ws}T(s)ds$ is compact as the norm limit of linear combinations of compact operators, where $w > \omega(A)$. Since $R(w,A) = \lim_{t\to\infty} \int_0^t e^{-ws}T(s)ds$ in the operator norm, it follows that $R(w,A)$ is compact. This proves one implication. The other follows from Proposition 1.24.

□

<u>Remark</u> 1.26.a) Generators of eventually compact semigroups do not necessarily have compact resolvent. Consider the nilpotent translation semigroup $(T(t))_{t\geq 0}$ on $F := L^1[0,1]$ (see A-I,Ex.2.6). Let $E = F \otimes_\pi F = L^1([0,1] \times [0,1])$ and $S(t) = T(t) \otimes \mathrm{Id}$ $(t\geq 0)$. Then $(S(t))_{t\geq 0}$ is a strongly continuous semigroup (see A-I,3.7). Denote by B its generator. $(S(t))_{t\geq 0}$ is a nilpotent semigroup (so it is eventually compact), but $R(\lambda,B) = R(\lambda,A) \otimes \mathrm{Id}$ is not compact.
b) It is obvious that a group $(T(t))_{t\in\mathbb{R}}$ is eventually norm continuous if and only if it is norm continuous in 0; i.e., its generator is bounded.
On the other hand, the generator of the rotation group (A-I,Ex.2.5) has a compact resolvent. Hence this condition does not imply any smoothness property of the semigroup.

Positive eventually compact semigroups have remarkable properties in the setting of the Perron-Frobenius theory (see e.g., B-III,Cor.2.12).

The following scheme indicates the relation between the different classes of semigroups defined so far.

$$\begin{array}{ccccc}
\text{holomorphic} & \to & \text{differentiable} & \to & \text{eventually differentiable} \\
 & & \downarrow & & \downarrow \\
 & & \text{norm continuous} & \to & \text{eventually norm continuous} \\
 & & \uparrow & & \uparrow \\
 & & \text{compact} & \to & \text{eventually compact}
\end{array}$$

All these classes are different. This is shown by the following examples.