Chapter 1

Spectral Theory

1.1 Introduction

In this chapter we start a systematic analysis of the spectrum of a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on a complex Banach space E. By the spectrum of the semigroup we understand the spectrum $\sigma(A)$ of the generator A of T. In particular we are interested in precise relations between $\sigma(A)$ and $\sigma(T(t))$. The heuristic formula

$$\text{“}T(t) = e^{tA}\text{”}$$

serves as a leitmotiv and suggests relations of the form

$$\text{“}\sigma(T(t)) = e^{t\sigma(A)} = \{e^{t\lambda} : \lambda \in \sigma(A)\}\text{”},$$

called “spectral mapping theorem”. These - or similar - relations will be of great use in Chapter IV and enable us to determine the asymptotic behavior of the semigroup T by the spectrum of the generator.

As a motivation as well as a preliminary step we concentrate here on the spectral radius

$$r(T(t)) := \sup\{|\lambda| : \lambda \in \sigma(T(t))\}, \quad t \geq 0$$

and show how it is related to the spectral bound

$$s(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\}$$

of the generator A and to the growth bound

$$\omega := \inf\{\omega \in \mathbb{R} : \|T(t)\| \leq M_\omega \cdot e^{\omega t} \text{ for all } t \geq 0 \text{ and suitable } M_\omega\}$$

of the semigroup $T = (T(t))_{t\geq 0}$. (Recall that we sometimes write $\omega(T)$ or $\omega(A)$ instead of ω). The Examples 1.3 and 1.4 below illustrate the main difficulties to be encountered.

Proposition 1.1. *Let ω be the growth bound of the strongly continuous semigroup $T = (T(t))_{t\geq 0}$. Then*

$$r(T(t)) = e^{\omega t}$$

for every $t \geq 0$.