Wolfgang Arendt, Annette Grabosch, Günther Greiner, Ulrich Groh, Heinrich P. Lotz, Ulrich Moustakas, Rainer Nagel, Frank Neubrander, Ulf Schlotterbeck

One-parameter Semigroups of Positive Operators

Edited by R. Nagel

Lecture Notes in Mathematics

1184

Springer-Verlag
Berlin Heidelberg New York Tokyo

Stand: June 2, 2025

Springer Nature

This Latex version of the book "One-Parameter Semigroups of Positive Operators" is dedicated to the memory of our co-authors, Heinrich P. Lotz and Ulf Schlotterbeck. Their contributions to the first edition remain an inspiration to us all. We miss their presence and remain grateful for the legacy they have left in this work.

Preface

As early as 1948 in the first edition of his fundamental treatise on *Semigroups and Functional Analysis*, E. Hille expressed the need for

> *... developing an adequate theory of transformation semigroups operating in partially ordered spaces* (l.c., Foreword).

In the meantime the theory of one-parameter semigroups of positive linear operators has grown continuously. Motivated by problems in probability theory and partial differential equations W. Feller (1952) and R. S. Phillips (1962) laid the first corner-stones by characterizing the generators of special positive semigroups. In the 60's and 70's the theory of positive operators on ordered Banach spaces was built systematically and is well documented in the monographs of H. H. Schaefer (1974) and A. C. Zaanen (1983). But in this process the original ties with the applications and, in particular, with initial value problems were at times obscured. Only in recent years an adequate and up-to-date theory emerged, largely based on the techniques developed for positive operators and thus recombining the functional analytic theory with the investigation of Cauchy problems having positive solutions to each positive initial value. Even though this development — in particular with respect to applications to concrete evolution equations in transport theory, mathematical biology, and physics — is far from being complete, the present volume is a first attempt to shape the multitude of available results into a coherent theory of one-parameter semigroups of positive linear operators on ordered Banach spaces.

The book is organized as follows. We concentrate our attention on three subjects of semigroup theory: *characterization*, *spectral theory* and *asymptotic behavior*. By *characterization*, we understand the problem of describing special properties of a semigroup, such as positivity, through the generator. By *spectral theory* we mean the investigation of the spectrum of a generator. *Asymptotic behavior* refers to the orbits of the initial values under a given semigroup and phenomena such as stability.

This program (characterization, spectral theory, asymptotic behavior) is worked out on four different types of underlying spaces.

(A) On Banach spaces—Here we present the background for the subsequent discussions related to order.

(B) On spaces $C_0(X)$ (X locally compact), which constitute an important class of ordered Banach spaces and where our results can be presented in a form which makes them accessible also for the non-expert in order-theory.

(C) On Banach lattices, which admit a rich theory and are still sufficiently general as to include many concrete spaces appearing in analysis; e.g., $C_0(X)$, $\mathcal{L}^p(k)$ or l^p.

(D) On non-commutative operator algebras such as C*- or W*-algebras, which are not lattice ordered but still possess an interesting order structure of great importance in mathematical physics.

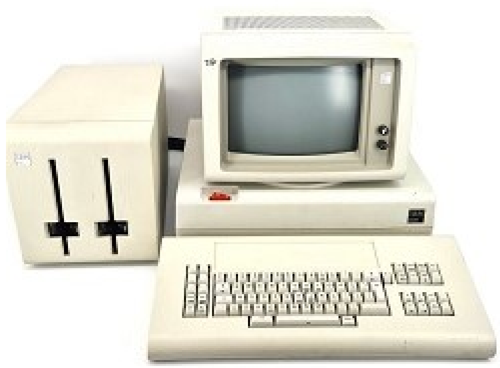
In each of these cases we start with a short collection of basic results and notations, so that the contents of the book may be visualized in the form of a 4×4 matrix in a way which will allow "row readers" (interested in semigroups on certain types of spaces) and "column readers" (interested in certain aspects) to find a path through the book corresponding to their interest.

We display this matrix, together with the names of the authors contributing to the subjects defined through this scheme.

	I Basic Results	II Characterization	III Spectral Theory	IV Asymptotics
A. Banach Spaces	R. Nagel U. Schlotterbeck	W. Arendt H. P. Lotz	G. Greiner R. Nagel	F. Neubrander
B. $C_0(X)$	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas F. Neubrander
C. Banach Lattices	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas R. Nagel F. Neubrander
D. Operator Algebras	U. Groh	U. Groh	U. Groh	U. Groh

This "matrix of contents" has been an indispensable guide line in our discussions on the scope and the spirit of the various contributions. However, we would not have succeeded in completing this manuscript, as a collection of independent contributions (personally accounted for by the authors), under less favorable conditions than we have actually met. For one thing, Rainer Nagel was an unfaltering and undisputed spiritus rector from the very beginning of the project. On the other hand we gratefully acknowledge the influence of Helmut H. Schaefer and his pioneering work on order structures in analysis. It was the team spirit produced by this common mathematical background which, with a little help from our friends, made it possible to overcome most difficulties.

We have prepared the manuscript with the aid of a word processor, but we confess that without the assistance of Klaus Kuhn the pitfalls of such a system would have been greater than its benefits.



The authors

Contents

List of Symbols

The next list describes several symbols that will be later used within the body of the document

(E, φ) see C-I,4

1_C characteristic function of set C

$\bar{f}$ complex conjugate of f

δ_x Dirac measure in x

$\hat{f}, \check{f}$ Fourier (inverse Fourier) transformation

$\mathcal{L}(E)$ Banach space of bounded linear operators on E

$\mathcal{Z}(E)$ center of E

$\omega = \omega(A) = \omega(T)$ growth bound

$\omega(f)$ growth bound of $T(\cdot)f$

$\omega_{ess}(A)$ essential growth bound

$\omega_I(A)$ growth bound of solution of (ACP)

$\varrho(A)$ resolvent set

$\sigma(A)$ spectrum

$\sigma_b(A)$ boundary spectrum

$\sigma_{ess}(A)$ essential spectrum

$\mathrm{Re}f, \mathrm{Im}f$ see C-I,7

$\mathrm{Re}T, \mathrm{Im}T$ see C-I,7

$\vee$ supremum

$\wedge$ infimum

A generator

A' adjoint

$A\sigma(A)$ approximate point spectrum

A^* adjoint generator

$B(H)$ W*-algebra of bounded linear operators on a Hilbert space H

$C(K), C(K,E)$ continuous functions (with values in E)

$C^b(X)$ bounded continuous functions

$C^n, C^{(n)}$ continuous differentiable functions (n-times)

$C_c^\infty(\mathbb{R}^n)$ infinitely differentiable functions with compact support

$C_o(X), C_o(X,E)$ continuous functions vanishing at infinity with values in E

$C_{uu}(X)$ uniformly continuous functions

$d\varrho(f)$ subdifferential of ϱ in f

$dN(f)$ subdifferential of norm in f

$dN^+(f)$ subdifferential of canonical half-norm in f

E' dual Banach space

$E \otimes F$ tensor product

E^* semigroup dual

E_+ positive cone

$E_\mathbb{R}, E_\mathbb{C} = E$ real, complex Banach lattice

E_F $\mathcal{F}$-product of E

E_f see C-I,4

E_F^T $\mathcal{F}$-product of E with respect to semigroup $\mathcal{T}$

E_n n-th Sobolev space

f^+ positive part of f

f^- negative part of f

$f^{[n]}$ see C-II,2.2

I^d, $\{I^d\}_{d=1}^{dd}$ orthogonal band of I (of I^d)

$L^p(\mu)$ p-integrable functions

$M(K)$ regular Borel measures

M^{sa} self-adjoint part

M_*	predual
M_+	positive cone of C*-algebra M
$M_b(X)$	bounded regular Borel measures
M_n	C*-algebra of nxn-matrices
M_p	multiplication operator
$P\sigma(A)$	point spectrum
$P\sigma_b(A)$	boundary point spectrum
$R(\lambda, A)$	resolvent operator
$r(T)$	spectral radius
$R\sigma(A)$	residual spectrum
$r_{ess}(T)$	essential spectral radius
$S(\alpha)$	sector in complex plane
$S(\mathbb{R}^n)$	Schwartz space
$s(A)$	spectral bound
$S(M)$	state space of C*-algebra M
S_f	signum operator with respect to f
$T = (T(t))_{t\geqslant 0}$	(one-parameter) semigroup
$T/$	quotient semigroup
(ACP)	abstract Cauchy problem
(K)	Kato's (equality) inequality
(P')	B-II,1.21
(P)	positive minimum principle
(RCP)	retarded Cauchy problem
(RE)	retarded equation
(T)	translation property
1	function identically 1
AC	absolutely continuous functions
BV	functions of bounded variation
Fix(T)	fixed space of T
Id	identity operator

Im	imaginary part
im	range
K	compact topological space
ker	null-space
Re	real part
sign f	signum of f
span M	linear subspace generated by M
tr	trace
X	locally compact topological space

Part A

One-parameter Semigroups on Banach Spaces

Chapter A-I

Basic Results on Semigroups on Banach Spaces

by
Rainer Nagel and Ulf Schlotterbeck

Since the basic theory of one-parameter semigroups can be found in several excellent books (e.g., Davies [1], Goldstein [4], Pazy [9] or Hille and Phillips [6]), we do not want to give a self-contained introduction to this subject here. It may however be useful to fix our notation, to collect briefly some important definitions and results (Section 1), to present a list of *standard examples* in Section 2 and to discuss standard constructions of new semigroups from a given one in Section 3 on p. 17.

In the entire chapter we denote by E a (real or) complex Banach space and consider one-parameter semigroups of bounded linear operators $T(t)$ on E. By this we understand a subset $\{T(t)\colon t \in \mathbb{R}_+\}$ of $\mathcal{L}(E)$, usually written as $(T(t))_{t\geqslant 0}$, such that

$$\begin{aligned} T(0) &= \mathrm{Id}, \\ T(s+t) &= T(s)\cdot T(t) \text{ for all } s,t \in \mathbb{R}_+. \end{aligned}$$

In more abstract terms this means that the map $t \mapsto T(t)$ is a homomorphism from the additive semigroup $(\mathbb{R}_+, +)$ into the multiplicative semigroup $(\mathcal{L}(E), \cdot)$. Similarly, a one-parameter group $(T(t))_{t\in\mathbb{R}}$ will be a homomorphic image of the group $(\mathbb{R}, +)$ in $(\mathcal{L}(E), \cdot)$.

1 Standard Definitions and Results

We consider a one-parameter semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E and observe that the domain $\mathbb{R}_+$ and range $\mathcal{L}(E)$ of the (semigroup) homomorphism $\tau\colon t \mapsto T(t)$ are topological semigroups for the natural topology on $\mathbb{R}_+$ and any one of the standard operator topologies on $\mathcal{L}(E)$. We single out the strong operator topology on $\mathcal{L}(E)$ and require τ to be continuous.

Definition 1.1 A one-parameter semigroup $(T(t))_{t\geqslant 0}$ is called *strongly continuous* if the map $t \mapsto T(t)$ is continuous for the strong operator topology on $\mathcal{L}(E)$, e.g.,

$$\lim_{t\to t_0} \|T(t)f - T(t_0)f\| = 0$$

for every $f \in E$ and $t, t_0 \geqslant 0$.

Clearly one defines in a similar way *weakly continuous*, resp. *uniformly continuous* (compare A-II, Definition 1.19) semigroups, but since we concentrate on the strongly continuous case we agree on the following terminology.

> If not stated otherwise, a *semigroup* is a strongly continuous one-parameter semigroup of bounded linear operators.

Next we collect a few elementary facts on the continuity and boundedness of one-parameter semigroups.

Remarks 1.2 (i) A one-parameter semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E is strongly continuous if and only if for any $f \in E$ it is true that $T(t)f \to f$ if $t \to 0$.

(ii) For every strongly continuous semigroup there exist constants $M \geqslant 1$, $\omega \in \mathbb{R}$, such that $\|T(t)\| \leqslant M \cdot \mathrm{e}^{\omega t}$ for every $t \geqslant 0$.

(iii) If $T(t)_{t\geqslant 0}$ is a one-parameter semigroup such that $\|T(t)\|$ is bounded for $0 < t \leqslant \delta$ then it is strongly continuous if and only if $\lim_{t\to 0} T(t)f = f$ for every f in a total subset of E.

The exponential estimate from Remark for the growth of $\|T(t)\|$ can be used to define an important characteristic of the semigroup.

Definition 1.3 By the growth bound (or type) of the semigroup $T(t)_{t\geqslant 0}$ we understand the number ω_0,

$$\begin{aligned}\omega_0 &:= \inf\{\omega \in \mathbb{R}\colon \text{there exists } M \in \mathbb{R}_+ \text{ such that } \|T(t)\| \leqslant M \cdot \mathrm{e}^{wt} \text{ for } t \geqslant 0\} \\ &= \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\| = \inf_{t>0} \frac{1}{t} \log \|T(t)\|.\end{aligned}$$

Particularly important are semigroups such that for every $t \geqslant 0$ we have $\|T(t)\| \leqslant M$ (*bounded semigroups*) or $\|T(t)\| \leqslant 1$ (*contraction semigroups*). In both cases we have $\omega_0 \leqslant 0$.

It follows from the subsequent examples and from Definition **??** that ω_0 may be any number $-\infty \leqslant \omega < +\infty$. Moreover the reader should observe that the infimum in Definition **??** need not be attained and that M may be larger than 1 even for bounded semigroups.

Examples 1.4 (i) Take

$$E = \mathbb{C}^2, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T(t) = \mathrm{e}^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then for the ℓ^1-norm on E we obtain $\|T(t)\| = 1+t$, hence $(T(t))_{t\geqslant 0}$ is an unbounded semigroup having growth bound $\omega_0 = 0$.

(ii) Take $E = L^1(\mathbb{R})$ and for $f \in E$, $t \geqslant 0$ define

$$T(t)f(x) := \begin{cases} 2 \cdot f(x+t) & \text{if } x \in [-t, 0] \\ f(x+t) & \text{otherwise.} \end{cases}$$

Each $T(t)$, $t > 0$, satisfies $\|T(t)\| = 2$ as can be seen by taking $f := \mathbb{1}_{[0,t]}$. Therefore $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup which is bounded, hence has $\omega_0 = 0$, but the constant M in (1.2) cannot be chosen to be 1.

The most important object associated to a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is its *generator* which is obtained as the (right) derivative of the map $t \mapsto T(t)$ at $t = 0$. Since for strongly continuous semigroups the functions $t \mapsto T(t)f$, $f \in E$, are continuous but not always differentiable, we have to restrict our attention to those $f \in E$ for which the desired derivative exists. We then obtain the *generator* as a not necessarily everywhere defined operator.

Definition 1.5 To every semigroup $(T(t))_{t\geqslant 0}$ there belongs an operator $(A, D(A))$, called the *generator* and defined on the *domain*

$$\begin{aligned} D(A) &:= \{f \in E \colon \lim_{h\to 0} \frac{T(h)f - f}{h} \text{ exists in } E\} \text{ by} \\ Af &:= \lim_{h\to 0} \frac{T(h)f - f}{h} \qquad (f \in D(A)). \end{aligned}$$

Clearly, $D(A)$ is a linear subspace of E and A is linear from $D(A)$ into E. Only in certain special cases (see Section 2.1) the generator is everywhere defined and therefore bounded (use Proposition 1.9 (ii)) on p. 7). In general, the precise extent of the domain $D(A)$ is essential for the characterization of the generator. But since the domain is canonically associated to the generator of a semigroup, we shall write in most cases A instead of $(A, D(A))$. As a first result we collect some information on the domain of the generator.

Proposition 1.6 *For the generator A of a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E the following assertions hold.*

(i) *If $f \in D(A)$, then $T(t)f \in D(A)$ for every $t \geqslant 0$.*

(ii) *The map $t \mapsto T(t)f$ is differentiable on $\mathbb{R}_+$ if and only if $f \in D(A)$. In that case one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)f = AT(t)f = T(t)Af. \tag{1.1}$$

(iii) *For every $f \in E$ and $t > 0$ the element $\int_0^t T(s)f\,\mathrm{d}s$ belongs to $D(A)$ and one has*

$$A\int_0^t T(s)f\,\mathrm{d}s = T(t)f - f. \tag{1.2}$$

(iv) *If $f \in D(A)$, then*

$$\int_0^t T(s)Af\,\mathrm{d}s = T(t)f - f. \tag{1.3}$$

(v) *The domain $D(A)$ is dense in E.*

The identity (1.1) is of great importance and shows how semigroups are related to certain Cauchy problems. We state this explicitly in the following theorem.

Theorem 1.7 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on the Banach space E. Then the* abstract Cauchy problem

$$\frac{d}{\mathrm{d}t}\xi(t) = A\xi(t), \quad \xi(0) = f_0 \tag{1.4}$$

has a unique solution $\xi\colon \mathbb{R}_+ \to D(A)$ in $C^1(\mathbb{R}_+, E)$ for every $f_0 \in D(A)$. In fact, this solution is given by $\xi(t) := T(t)f_0$.

For more on the relation of semigroups to abstract Cauchy problems we refer to A-II, Section 1. Here we only point out that the above theorem implies that a semigroup is uniquely determined by its generator.

While the generator is bounded only for uniformly continuous semigroups (see Section 2 below), it always enjoys a weaker but useful property.

Definition 1.8 An operator B with domain $D(B)$ on a Banach space E is called *closed* if $D(B)$ endowed with the *graph norm*

$$\|f\|_B := \|f\| + \|Bf\|$$

becomes a Banach space. Equivalently, $(B, D(B))$ is closed if and only if its *graph* $\{(f, Bf)\colon f \in D(B)\}$ is closed in $E \times E$, i.e.,

$$(f_n) \subset D(B),\, f_n \to f \text{ and } Bf_n \to g \text{ implies } f \in D(B) \text{ and } Bf = g.$$

It is clear from this definition that the *closedness* of an operator B depends very much on the size of the domain $D(B)$. For example, a bounded and densely defined operator $(B, D(B))$ is closed if and only if $D(B) = E$.

On the other hand it may happen that $(B, D(B))$ is not closed but has a closed *extension* $(C, D(C))$, i.e., $D(B) \subseteq D(C)$ and $Bf = Cf$ for every $f \in D(B)$. In that case, B is called *closable*, a property which is equivalent to

$$(f_n) \subset D(B), f_n \to 0 \text{ and } Bf_n \to g \text{ implies } g = 0.$$

The smallest closed extension of $(B, D(B))$ will be called the *closure* $\overline{B}$ with domain $D(\overline{B})$. In other words, the graph of $\overline{B}$ is the closure of $\{(f, Bf) : f \in D(B)\}$ in $E \times E$.

Finally we call a subset D_0 of $D(B)$ a *core* for B if D_0 is $\|\cdot\|_B$-dense in $D(B)$. This means that a closed operator is determined (via closure) by its restriction to a core in its domain.

We now collect the fundamental topological properties of semigroup generators, their domains (see also A-II, Corollary 1.34) and their resolvents.

Proposition 1.9 *For the generator A of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ the following hold.*

(i) *The generator A is a closed operator.*

(ii) *If a subspace D_0 of the domain $D(A)$ is dense in E and $(T(t))$-invariant, then it is a core for A.*

(iii) *Define*

$$\begin{aligned} D(A^1) &= D(A) \\ D(A^n) &= \{f \in D(A^{n-1}) : Af \in D(A^{n-1})\} \text{ for } n \geqslant 2, \end{aligned}$$

Then

$$D(A^\infty) := \bigcap_{n\in\mathbb{N}} D(A^n)$$

is dense in E and a core for A.

Example 1.10 Property (iii) above does not hold for general densely defined closed operators. Take $E = C\,[0,1]$, $D(B) = C^1\,[0,1]$ and $Bf = q \cdot f'$ for some nowhere differentiable function $q \in C\,[0,1]$. Then B is closed, but $D(B^2) = \{0\}$.

Proposition 1.11 *For the generator A of a strongly continous semigroup on a Banach space E the following hold. If*

$$\int_0^\infty e^{-\lambda t} T(t) f \, dt$$

exists for every $f \in E$ and some $\lambda \in \mathbb{C}$, then $\lambda \in \varrho(A)$ and

$$R(\lambda, A) f = \int_0^\infty e^{-\lambda t} T(t) f \, dt.$$

In particular,

$$R(\lambda, A)^{n+1} f = \frac{(-1)^n}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n R(\lambda, A) f = \int_0^\infty \mathrm{e}^{-\lambda t} \frac{t^n}{n!} T(t) f \, \mathrm{d}t \tag{1.5}$$

for every $f \in E$, $n \geqslant 0$ *and* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) > \omega_0$.

Remarks 1.12 (i) For continuous Banach space valued functions such as $t \mapsto T(t)f$ we consider the Riemann integral and define

$$\int_0^\infty T(t) f \, \mathrm{d}t \quad \text{as} \quad \lim_{t \to \infty} \int_0^t T(s) f \, \mathrm{d}s.$$

Sometimes such integrals for strongly continuous semigroups are written as $\int_a^b T(t)\,\mathrm{d}t$ but understood in the strong sense.

(ii) Since the generator $(A, D(A))$ determines the semigroup $(T(t))_{t \geqslant 0}$ uniquely, we will speak occasionally of the growth bound of the generator instead of the semigroup, i.e., we write $\omega_0 = \omega_0(A) = \omega_0(\mathcal{T})$ where $\mathcal{T} = (T(t))_{t \geqslant 0}$ denotes the semigroup .

(iii) For one-parameter groups it might seem to be more natural to define the generator as the *derivative* rather than just the *right derivative* at $t = 0$. This yields the same operator as the following result shows.

The strongly continuous semigroup $(T(t))_{t \geqslant 0}$ with generator A can be extended to a strongly continuous one-parameter group $(U(t))_{t \in \mathbb{R}}$ if and only if $-A$ generates a semigroup $(S(t))_{t \geqslant 0}$. In that case $(U(t))_{t \in \mathbb{R}}$ is obtained as

$$U(t) = \begin{cases} T(t) & \text{for } t \geqslant 0, \\ S(-t) & \text{for } t \leqslant 0. \end{cases}$$

We refer to Davies [1, Proposition 1.14] for the details.

2 Standard Examples

In this section we list and discuss briefly the most basic examples of semigroups together with their generators. These semigroups will reappear throughout this book and will be used to illustrate the theory. We start with the class of semigroups mentioned after Definition 1.1 on p. 4.

2.1 Uniformly Continuous Semigroups

It follows from elementary operator theory that for every bounded operator A in $\mathcal{L}(E)$ the sum

$$\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} =: \mathrm{e}^{tA}$$

exists and determines a unique uniformly continuous (semi)group $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$ having A as its generator. Conversely, any uniformly continuous semigroup is of this form.

If the semigroup $(T(t))_{t\geqslant 0}$ is uniformly continuous, then

$$\frac{1}{t}\int_0^t T(s)\,\mathrm{d}s$$

uniformly converges to $T(0) = \mathrm{Id}$ as $t \to 0$. Therefore for some $t' > 0$ the operator

$$\frac{1}{t'}\int_0^{t'} T(s)\,\mathrm{d}s$$

is invertible and every $f \in E$ is of the form

$$f = \frac{1}{t'}\int_0^{t'} T(s)g\,\mathrm{d}s$$

for some $g \in E$. But these elements belong to $D(A)$ by (1.2), hence $D(A) = E$. Since the generator A is closed and everywhere defined, it must be bounded. Remark that bounded operators are always generators of groups, not just semigroups. Moreover, the growth bound ω_0 satisfies $|\omega_0| \leqslant \|A\|$ in this situation.

The above characterization of the generators of uniformly continuous semigroups as the bounded operators shows that these semigroups are—at least in many aspects—rather simple objects.

2.2 Matrix Semigroups

The above considerations especially apply in the situation $E = M_n(\mathbb{C})$. If $n = 2$ and $A \in E$, one can derive an explicit formula for e^{tA}.

Let

$$s = \mathrm{trace}(A), \quad d = \det(A) \quad \text{and} \quad D^2 := \left(\frac{s^2}{4} - d\right).$$

If

$$p_A(\lambda) := \det(\lambda - A) = \lambda^2 - s \cdot \lambda + d,$$

is the characteristic polynomial of A, then $p_A(A) = 0$ by *Cayley-Hamilton* hence

$$(A - s/2 \cdot \mathbb{1})^2 = D^2 \cdot \mathbb{1},$$

where $\mathbb{1}$ denotes the unit matrix.

Now we have to consider two cases.

$D^2 = 0$: Then $(A - s/2 \cdot \mathbb{1})^k = D^k \cdot \mathbb{1} = 0$ for all $(k \geqslant 2)$ and this implies

$$\mathrm{e}^{t\cdot(A-s/2\cdot\mathbb{1})} = \mathbb{1} + t(A - s/2 \cdot \mathbb{1})$$

or

$$\mathrm{e}^{tA} = \mathrm{e}^{t\cdot s/2}\,[(1 - \frac{s}{2}\,t) \cdot \mathbb{1} + t \cdot A].$$

$D^2 \neq 0$: In this case we obtain for every $k \in \mathbb{N}$

$$(A - s/2 \cdot \mathbb{1})^{2k} = D^{2k} \cdot \mathbb{1}$$
$$(A - s/2 \cdot \mathbb{1})^{2k+1} = \frac{1}{D}\,D^{2k+1}(A - s/2 \cdot \mathbb{1}).$$

Thus

$$\begin{aligned}\mathrm{e}^{t(A-s/2\cdot\mathbb{1})} &= \sum_{k=0}^{\infty}\frac{t^k}{k!}(A - s/2 \cdot \mathbb{1})^k = \sum_{\text{k even}}(\cdots) + \sum_{\text{k odd}}(\cdots)\\ &= \cosh(tD)\cdot\mathbb{1} + \frac{1}{D}\sinh(tD)(A - s/2\cdot\mathbb{1})\\ &= \frac{1}{D}\sinh(tD)\cdot A + \Big(\cosh(tD) - \frac{s}{2D}\sinh(tD)\Big)\cdot\mathbb{1}.\end{aligned}$$

by using the power series representation of the hyperbolic functions sinh and cosh.

As summary

$$\mathrm{e}^{tA} = \begin{cases}\mathrm{e}^{t\cdot s/2}\,\left[\frac{1}{D}\sinh(tD)\cdot A + \left(\cosh(tD) - \frac{s}{2D}\sinh(tD)\right)\cdot\mathbb{1}\right], & \text{if } D \neq 0,\\ \mathrm{e}^{t\cdot s/2}\,\left[(1 - \frac{s}{2}\,t)\cdot\mathbb{1} + t\cdot A\right], & \text{if } D = 0.\end{cases}$$

In case A is a real 2×2-matrix and if $D \neq 0$, then $D^2 = s^2/4 - d$ can be positive or negative.

$D^2 > 0$: Since sinh is an odd function, we can choose $D > 0$ in the formula for e^{tA}. Furthermore, sinh and cosh are (by definition) linear combinations of e^{tD} and e^{-tD} simplifying our formula (see below).

$D^2 < 0$: In this case $D = \pm\mathrm{i}|D|$. For $z \in \mathbb{C}$ we have the following identities

$$\sinh(\mathrm{i}z) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) = \mathrm{i}\sin(z) \quad \text{and} \quad \cosh(\mathrm{i}z) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) = \cos(z).$$

Hence sinh is an odd function, $\sinh(\mathrm{i}|D|t) = \mathrm{i}\ \sin(t|D|)$ and $\cosh(\mathrm{i}|D|t) = \cos(|D|t)$

Here are some simple examples.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathrm{e}^{tA} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \Longrightarrow \quad \mathrm{e}^{tA} = \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathrm{e}^{tA} = \begin{pmatrix} \cos(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

2.3 Multiplication Semigroups

Many Banach spaces appearing in applications are Banach spaces of (real or) complex valued functions over a set X. As the most standard examples of these "function spaces", we mention the space $C_0(X)$ of all continuous complex valued functions vanishing at infinity on a locally compact space X, or the spaces $L^p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, of all (equivalence classes of) p-integrable functions on a σ-finite measure space (X, Σ, μ).

On these function spaces $E = C_0(X)$, resp. $E = L^p(X, \Sigma, \mu)$, there is a simple way to define *multiplication operators*.

Take a continuous, resp. measurable function $q\colon X \to \mathbb{C}$ and define

$$M_q f := q \cdot f, \quad \text{i.e.,} \quad M_q f(x) := q(x) \cdot f(x) \quad \text{for } x \in X$$

and for every f in the *maximal* domain $D(M_q) := \{g \in E : q \cdot g \in E\}$.

This natural domain is a dense subspace of $C_0(X)$, resp. $L^p(X, \Sigma, \mu)$, for $1 \leqslant p < \infty$. Moreover, $(M_q, D(M_q))$ is a closed operator. This is easy in case $E = C_0(X)$.

For $E = L^p(X, \Sigma, \mu)$, $1 \leqslant p < \infty$, we consider a sequence $(f_n) \subset E$ such that $\lim_{n\to\infty} f_n = f \in E$ and $\lim_{n\to\infty} q f_n =: g \in E$. Choose a subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f_{n(k)}(x) = f(x)$ and $\lim_{k\to\infty} q(x) f_{n(k)}(x) = g(x)$ for μ-almost every $x \in X$. Then $g = q \cdot f$ and $f \in D(M_q)$, i.e., M_q is closed.

For such multiplication operators many properties can be checked quite directly. For example, the following statements are equivalent.

(a) M_q is bounded.

(b) q is (μ-essentially) bounded.

One has $\|M_q\| = \|q\|_\infty$ in this situation. Observe that on spaces $C(K)$, K compact, there are no densely defined, unbounded multiplication operators.

By defining the multiplication semigroups

$$T(t)f(x) := \exp(t \cdot q(x))f(x), \quad x \in X, f \in E,$$

one obtains the following characterizations.

Proposition *Let M_q be a multiplication operator on $E = C_0(X)$ or $E = L^p(X, \Sigma, \mu)$, $1 \leqslant p < \infty$. Then the properties (a) and (b), resp. (a') and (b'), are equivalent.*

(a) *M_q generates a strongly continuous semigroup.*

(b) $\sup\{\mathrm{Re}(q(x)) \colon x \in X\} < \infty$.

(a') *M_q generates a uniformly continuous semigroup.*

(b') $\sup\{|q(x)| \colon x \in X\} < \infty$.

As a consequence one computes the growth bound of a multiplication semigroup as

$$\omega_0 = \sup\{\mathrm{Re}(q(x)) \colon x \in X\}$$

in the case $E = C_0(X)$ and

$$\omega_0 = \mu\text{-ess-}\sup\{\mathrm{Re}(q(x)) \colon x \in X\}$$

in the case $E = L^p(\mu)$. It is a nice exercise to characterize those multiplication operators which generate strongly continuous groups.

We point out that the above results cover the cases of sequence spaces such as c_0 or ℓ^p, $1 \leqslant p < \infty$. An abstract characterization of generators of multiplication semigroups will be given in C-II, Theorem 5.13.

2.4 Translation (Semi)Groups

Let E to be one of the following function spaces $C_0(\mathbb{R}_+)$, $C_0(\mathbb{R})$, $L^p(\mathbb{R}_+)$ or $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$. Define $T(t)$ to be the (left) translation operator

$$T(t)f(x) := f(x+t)$$

for x, $t \in \mathbb{R}_+$, resp. $x, t \in \mathbb{R}$ and $f \in E$. Then $(T(t))_{t \geqslant 0}$ is a strongly continous semigroup, resp. group of contractions on E and its generator is the first derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ with *maximal* domain. In order to be more precise we have to distinguish the cases $E = C_0$ and $E = L^p$.

The generator of the translation (semi)group on $E = C_0(\mathbb{R}_+)$ is

$$\begin{aligned} Af &:= \frac{\mathrm{d}}{\mathrm{d}x} f = f' \\ D(A) &:= \{f \in E \colon f \text{ differentiable and } f' \in E\}. \end{aligned}$$

Proof For $f \in D(A)$ it follows that for every $x \in \mathbb{R}_{(+)}$

$$\lim_{h \to 0} \frac{T(h)f(x) - f(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}$$

(uniformly in x) and coincides with $Af(x)$. Therefore f is differentiable and $f' \in E$.

On the other hand, take $f \in E$ differentiable such that $f' \in E$. Then

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leqslant \frac{1}{h} \int_x^{x+h} |f'(y) - f'(x)| \, \mathrm{d}y,$$

where the last expression tends to zero uniformly in x as $h \to 0$. Thus $f \in D(A)$ and $f' = Af$. □

The generator of the translation (semi)group on $E = L^p(\mathbb{R}_+)$, $1 \leqslant p < \infty$, is

$$\begin{aligned} Af &:= \frac{\mathrm{d}}{\mathrm{d}x} f = f', \\ D(A) &:= \{f \in E \colon f \text{ absolutely continuous}, f' \in E\}. \end{aligned}$$

Proof Take $f \in D(A)$ such that $\lim_{h \to 0} \frac{1}{h}(T(h)f - f) = g \in E$. Since integration is continuous, we obtain for every $a, b \in \mathbb{R}_{(+)}$ that

$$(*) \quad \frac{1}{h} \int_b^{b+h} f(x) \, \mathrm{d}x - \frac{1}{h} \int_a^{a+h} f(x) \, \mathrm{d}x = \int_a^b \frac{f(x+h) - f(x)}{h} \, \mathrm{d}x$$

converges to $\int_a^b g(x) \, \mathrm{d}x$ as $h \to 0+$. But for almost all a, b the left hand side of $(*)$ converges to $f(b) - f(a)$. By redefining f on a nullset we obtain

$$f(y) = \int_a^y g(x) \, \mathrm{d}x + f(a), \quad y \in \mathbb{R}_{(+)},$$

which is an absolutely continuous function whose derivative is (almost everywhere) equal to g.

On the other hand, let f be absolutely continuous such that $f' \in L^p$. Then

$$\begin{aligned}
\lim_{h\to 0}\int \left|\tfrac{f(x+h)-f(x)}{h} - f'(x)\right|^p \mathrm{d}x &= \lim_{h\to 0}\int \left|\tfrac{1}{h}\int_0^h (f'(x+s)-f'(x))\,\mathrm{d}s\right|^p \mathrm{d}x\\
&= \lim_{h\to 0}\int \left|\int_0^1 (f'(x+uh) - \int f'(x))\,\mathrm{d}u\right|^p \mathrm{d}x\\
&\leqslant \lim_{h\to 0}\int\int_0^1 |f'(x+uh)-f'(x)|^p\,\mathrm{d}u\,\mathrm{d}x\\
&= \int_0^1 \lim_{h\to 0}\int |f'(x+uh)-f'(x)|^p\,\mathrm{d}x\,\mathrm{d}u = 0,
\end{aligned}$$

hence $f \in D(A)$. □

2.5 Rotation Groups

On $E = C(\Gamma)$, resp. $E = L^p(\Gamma, m)$, $1 \leqslant p < \infty$, m Lebesgue measure we have canonical groups defined by rotations of the unit circle Γ with a certain period, i.e., for $0 < \tau \in \mathbb{R}$ the operators

$$R_\tau(t)f(z) := f(\mathrm{e}^{2\pi \mathrm{i}t/\tau}\cdot z), \quad z \in \Gamma$$

yield a group $(R_\tau(t))_{t\in\mathbb{R}}$ having period τ, i.e., $R_\tau(\tau) = \mathrm{Id}$. As in Example 2.4 one shows that its generator has the form

$$\begin{aligned}
D(A) &= \{f \in E\colon f \text{ absolutely continuous}, f' \in E\},\\
Af(z) &= (2\pi\mathrm{i}/\tau)\cdot z\cdot f'(z).
\end{aligned}$$

An isomorphic copy of the group $(R_\tau(t))_{t\in\mathbb{R}}$ is obtained if we consider

$$E = \{f \in C\,[0,1] : f(0) = f(1)\}$$

resp.

$$E = L^p([0,1])$$

and the group of *periodic translations*

$$T(t)f(x) := f(y) \quad \text{for } y \in [0,1]\,, y = x + t \bmod 1$$

with generator

$$D(A) := \{f \in E\colon f \text{ absolutely continuous}, f' \in E, Af := f'.\}$$

2.6 Nilpotent Translation Semigroups

Take $E = L^p([0,\tau], m)$ for $1 \leqslant p < \infty$ and define

$$T(t)f(x) := \begin{cases} f(x+t) & \text{if } x+t \leqslant \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then $(T(t))_{t\geqslant 0}$ is a semigroup satisfying $T(t) = 0$ for $t \geqslant \tau$. Its generator is still the first derivative $A = \frac{d}{\mathrm{d}x}$, but with domain is

$$D(A) = \{f \in E\colon f \text{ absolutely continuous}, f' \in E, f(\tau) = 0\}.$$

In fact, if $f \in D(A)$, then f is absolutely continuous with $f' \in E$. By Proposition 1.6 (i), it follows that $T(t)f$ is absolutely continuous and hence $f(\tau) = 0$.

2.7 One-dimensional Diffusion Semigroup

For the second derivative

$$Bf(x) := \frac{d^2}{dx^2}f(x) = f''(x)$$

we take the domain

$$D(B) := \{f \in C^2\,[0,1] : f'(0) = f'(1) = 0\}$$

in the Banach space $E = C\,[0,1]$. Then $D(B)$ is dense in $C\,[0,1]$, but closed for the graph norm. Obviously, each function

$$e_n(x) := \cos \pi n x, \quad n \in \mathbb{Z},$$

is contained in $D(B)$ and is an eigenfunction of B pertaining to the eigenvalue $\lambda_n := -\pi^2 n^2$. The linear hull span $\{e_n\colon n \in \mathbb{Z}\} =: E_0$ forms a subalgebra of $D(B)$ which by the Stone-Weierstrass theorem is dense in E.

We now use e_n to define bounded linear operators

$$e_n \otimes e_n\colon f \mapsto \left(\int_0^1 f(x)e_n(x)\,\mathrm{d}x\right)e_n = (f|e_n)e_n$$

satisfying $\|e_n \otimes e_n\| \leqslant 1$ and $(e_n \otimes e_n)(e_m \otimes e_m) = \delta_{n,m}(e_n \otimes e_n)$ for $n \in \mathbb{Z}$.

For $t > 0$ we define

$$\begin{aligned} T(t) &:= \sum_{n\in\mathbb{Z}} \exp(-\pi^2 n^2 t)\cdot e_n \otimes e_n \\ &= e_0 \otimes e_0 + 2\sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t)\cdot e_n \otimes e_n, \end{aligned}$$

or

$$T(t)f(x) = \int_0^1 k_t(x,y)f(x)dy$$
$$\text{where } k_t(x,y) = 1 + 2\sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t)\cos\pi nx \cos\pi ny.$$

The Jacobi identity

$$\begin{aligned} w_t(x) &:= 1/(4\pi t)^{\frac{1}{2}} \sum_{m\in\mathbb{Z}} \exp(-(x+2m)^2/4t) \\ &= \frac{1}{2} + \sum_{n\in\mathbb{N}} \exp(-\pi^2 n^2 t)\cos\pi nx \end{aligned}$$

and trigonometric relations show that

$$k_t(x,y) = w_t(x+y) + w_t(x-y)$$

which is a positive function on $[0,1]^2$. Therefore $T(t)$ is a bounded operator on $C\,[0,1]$ with

$$\|T(t)\| = \|T(t)\mathbb{1}\| = \sup_{x\in[0,1]} \int_0^1 k_t(x,y)dy = 1.$$

From the behavior of $T(t)$ on the dense subspace E_0 it follows that $(T(t))_{t\geqslant 0}$ with $T(0) = \mathrm{Id}$ is a strongly continuous semigroup on E and its generator A coincides with B on E_0. Finally, we observe that E_0 is a core for $(A, D(A))$ by Proposition 1.9(ii).

Consequently, $(T(t))_{t\geqslant 0}$ is the semigroup generated by the closure of the second derivative with domain $D(B)$.

2.8 n-dimensional Diffusion Semigroup

On $E = L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, the operators

$$\begin{aligned} T(t)f(x) &:= (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|x-y|^2/4t)f(y)dy \\ &= \mu_t * f(x) \end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$ and $\mu_t(x) := (4\pi t)^{-n/2}\exp(-|x|^2/4t)$ form a strongly continuous semigroup.

In fact the integral exists for every $f \in L^p(\mathbb{R}^n)$ since μ_t is an element of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of all rapidly decreasing smooth functions on $\mathbb{R}^n$.

Moreover,

$$\|T(t)f\|_p \leqslant \|\mu_t\|_1 \|f\|_p = \|f\|_p$$

by Young's inequality (Reed and Simon [11, p.28]). Hence $\|T(t)\| \leqslant 1$ for every $t > 0$.

Next we observe that $\mathcal{S}(\mathbb{R}^n)$ is dense in E and invariant under each $T(t)$. Therefore we can apply the Fourier transformation F which leaves $\mathcal{S}(\mathbb{R}^n)$ invariant and yields

$$F(\mu_t * f) = (2\pi)^{n/2} F(\mu_t) \cdot F(f) = (2\pi)^{n/2} \hat{\mu}_t \cdot \hat{f}$$

where $f \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f} = Ff \in \mathcal{S}(\mathbb{R}^n)$.

In other words, F transforms $(T(t)|_{\mathcal{S}(\mathbb{R}^n)})_{t \geqslant 0}$ into a multiplication semigroup on $\mathcal{S}(\mathbb{R}^n)$ which is pointwise continuous for the usual topology of $\mathcal{S}(\mathbb{R}^n)$. The generator, i.e., the right derivative at 0, of this semigroup is the multiplication operator

$$B\hat{f}(x) := -|x|^2 \hat{f}(x) \quad (x \in \mathbb{R}^n)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Applying the inverse Fourier transformation and observing that the topology of $\mathcal{S}(\mathbb{R}^n)$ is finer than the topology induced from $L^p(\mathbb{R}^n)$, we obtain that $(T(t))_{t \geqslant 0}$ is a semigroup which is strongly continuous (use Remark 1.2 (iii) on p. 4).

Its generator A coincides with

$$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x_1, \ldots, x_n)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is $(T(t))$-invariant, we have determined the generator on a core of its domain (see Proposition 1.9(ii). In particular, the above semigroup *solves* the initial value problem for the *heat equation*

$$\frac{\partial}{\partial t} f(x,t) = \Delta f(x,t), \quad f(x,0) = f_0(x), \quad x \in \mathbb{R}^n.$$

For the analogous discussion of the unitary group on $L^2(\mathbb{R}^n)$ generated by

$$C := \mathrm{i}\Delta$$

we refer to Section IX.7 in Reed and Simon [11].

Analogous examples to 2.7 are valid in $L^p\,[0,1]$, resp. to 2.8 in $C_0(\mathbb{R}^n)$.

3 Standard Constructions

Starting with a semigroup $(T(t))_{t \geqslant 0}$ on a Banach space E it is possible to construct new semigroups on spaces naturally associated with E. Such constructions will be important technical devices in many of the subsequent proofs. Although most of these constructions are rather routine, we present in the sequel a systematic account of them for the convenience of the reader.

We always start with a semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E, and denote its generator by A on the domain $D(A)$.

3.1 Similar Semigroups

There is an easy way how to obtain different (but isomorphic) semigroups out of a given semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E.

Let V be an isomorphism from E onto E. Then $S(t) := VT(t)V^{-1}$, $t \geqslant 0$, defines a strongly continuous semigroup. If A is the generator of $(T(t))_{t\geqslant 0}$ then

$$B := VAV^{-1} \text{ with domain } D(B) := \{f \in E : V^{-1}f \in D(A)\}$$

is the generator of $(S(t))_{t\geqslant 0}$.

3.2 The Rescaled Semigroup

For fixed $\lambda \in \mathbb{C}$ and $\alpha > 0$ the operators

$$S(t) := \exp(\lambda t)T(\alpha t)$$

yield a new semigroup having generator

$$B := \alpha A + \lambda \,\mathrm{Id} \;\text{ with } D(B) = D(A).$$

This *rescaled semigroup* enjoys most of the properties of the original semigroup and the same is true for the corresponding generators. However, by using this procedure certain constants associated with $(T(t))_{t\geqslant 0}$ and A can be normalized. For example, by this rescaling we may in many cases suppose without loss of generality that the growth bound ω_0 is zero.

Another application is the following. For $\lambda \in \mathbb{C}$ and $S(t) := \exp(-\lambda t)T(t)$ the formulas (1.2) and (1.3) yield:

$$\mathrm{e}^{-\lambda t}T(t)f - f = (A-\lambda)\int_0^t \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s \quad \text{or}$$

$$(\mathrm{e}^{\lambda t} - T(t))f = (\lambda - A)\int_0^t \mathrm{e}^{\lambda(t-s)}T(s)f\,\mathrm{d}s \quad \text{for } f \in E,$$

and

$$e^{-\lambda t}T(t)f - f = \int_0^t e^{-\lambda s}T(s)(A-\lambda)f \, ds \quad \text{or}$$

$$(e^{\lambda t} - T(t))f = \int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)f \, ds \quad \text{for } f \in D(A).$$

3.3 The Subspace Semigroup

Assume F to be a closed $(T(t))$-invariant or, equivalently, $R(\lambda, A)$-invariant for $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > \omega_0$, subspace of E. Then the semigroup $(T(t)_|)_{t\geqslant 0}$ of all restrictions $T(t)_| := T(t)_{|F}$ is strongly continuous on F. If $(A, D(A))$ denotes the generator of $(T(t))_{t\geqslant 0}$ it follows from the $(T(t))$-invariance and closedness of F that A maps $D(A) \cap F$ into F. Therefore

$$A_| := A_{|(D(A)\cap F)} \text{ with domain } D(A_|) := D(A) \cap F$$

is the generator of $(T(t)_|)$. Conversely, if F is a closed *linear subspace* of E with $A(D(A)\cap F) \subset F$ such that $A_|$ is a generator on F, then F is $(T(t))$-invariant.

An A-invariant subspace need not necessarily be $(T(t))$-invariant: Take for example the translation group with $T(t)f(x) = f(x+t)$ on $E = C_0(\mathbb{R})$ and the subspace $F := \{f \in E : f(x) = 0 \text{ for } x \leqslant 0\}$.

3.4 The Quotient Semigroup

Let F be a closed $(T(t))$-invariant subspace of E and consider the quotient space $E_/ := {}^E\!/_F$ with quotient map $q : E \to E_/$. The quotient operators

$$T(t)_/ q(f) := q(T(t)f), \quad f \in E,$$

are well defined and form a strongly continuous semigroup $(T(t)_/)_{t\geqslant 0}$ on $E_/$. For the generator $(A_/, D(A_/))$ of $(T(t)_/)_{t\geqslant 0}$ the following holds:

$$D(A_/) = q(D(A)) \quad \text{and} \quad A_/ q(f) = q(Af)$$

for every $f \in D(A)$. Here we use the fact that every $\hat{f} := q(f) \in D(A_/)$ can be written as

$$\hat{f} = \int_0^\infty e^{-\lambda s}\hat{T}(s)_/ \hat{g} \, ds = \int_0^\infty e^{-\lambda s} q(T(s)g) \, ds = q\Big(\int_0^\infty e^{-\lambda s}T(s)g \, ds\Big) = q(h)$$

where $h \in D(A)$ and $\lambda > \omega$ (see Proposition 1.6). In particular we point out that for every $\hat{f} \in D(A/)$ there exist representatives $f \in \hat{f}$ belonging to $D(A)$.

Example We start with the Banach space $E = L^1(\mathbb{R})$ and the translation semigroup $(T(t))_{t\geqslant 0}$ where $T(t)f(x) := f(x+t)$ (see Example 2.4). Then $L^1((-\infty,1])$ can be identified with the closed, $(T(t))$-invariant subspace

$$J := \{f \in E \colon f(x) = 0 \text{ for } 1 < x < \infty\}.$$

There we obtain the subspace semigroup

$$T(t)|_{(-\infty,1]}(x) \cdot f(x+t),$$

where $f \in L^1((-\infty,1])$, $-\infty < x \leqslant 1$ and $t \geqslant 0$.

By 2.4 and 3.2 its generator is

$$A|f := f'$$

for $f \in D(A|) := \{f \in E \colon f \in \text{AC with } f' \in E \text{ and } f(x) = 0 \text{ for } x \geqslant 1\}$.

Next we identify $L^1([0,1])$ with the quotient space $L^1((-\infty,1])/I$ where

$$I := \{f \in L^1((-\infty,1]) \colon f(x) = 0 \text{ for } 0 \leqslant x \leqslant 1\}.$$

Again I is invariant for the restricted semigroup $(T(t)_|)$ and the quotient semigroup $(T(t)|/)$ on $L^1([0,1])$ is the nilpotent translation semigroup as in Example 2.6. In particular it follows that the domain of its generator is

$$D(A_{|/}) = \{f \in L^1([0,1]) \colon f \in \text{AC with } f' \in L^1([0,1]) \text{ and } f(1) = 0\}.$$

3.5 The Adjoint Semigroup

The adjoint operators $(T(t)')_{t\geqslant 0}$ of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E form a semigroup on E' which need, however, not be strongly continuous.

Example Take the translation operators $T(t)f(x) = f(x+t)$ on $E = L^1(\mathbb{R})$ (see Example 2.4) and their adjoints

$$T(t)'f(x) = f(x-t)$$

on $E' = L^\infty(\mathbb{R})$. Then $(T(t)')_{t\in\mathbb{R}}$ is a one-parameter group which is not strongly continuous on $L^\infty(\mathbb{R})$ (take any non-trivial characteristic function).

Since the semigroup $(T(t)')_{t\geqslant 0}$ is obviously *weak*-continuous* in the sense that

$$\lim_{t\to s}\langle f, (T(t)' - T(s)')\varphi\rangle = 0$$

for every $f \in E$, $\varphi \in E'$ and $s, t \geqslant 0$, it is natural to associate $(T(t)')_{t\geqslant 0}$ its a *weak*-generator*

$$A'\varphi := \sigma(E',E)\text{-}\lim \frac{1}{h}(T(h)'\varphi - \varphi) \text{ for every } \varphi \text{ in the domain}$$
$$D(A') := \{\varphi \in E' : \sigma(E',E)\text{-}\lim \frac{1}{h}(T(h)'\varphi - \varphi) \text{ exists}\}.$$

This operator coincides with the *adjoint* of the generator $(A, D(A))$, i.e.,

$$D(A') = \{\varphi \in E' : \text{there exists } \psi \in E' \text{ such that } \langle f, \psi \rangle = \langle Af, \varphi \rangle \text{ for all } f \in D(A)\}$$

and $A'\varphi = \psi$. In particular, A' is a closed and $\sigma(E',E)$-densely defined operator in E'.

It follows that the resolvent $R(\lambda, A')$ of A' is $R(\lambda, A)'$ (Kato [7, Theorem III.5.30]). In particular, the spectra $\sigma(A)$ and $\sigma(A')$ coincide.

However, it is still necessary in some situations to have strong continuity for the adjoint semigroup. In order to achieve this we restrict $T(t)'$ to an appropriate subspace of E'.

Definition (Phillips [10]) The *semigroup dual* of the Banach space E with respect to the strongly continuous semigroup $(T(t))_{t \geqslant 0}$ is

$$E^* := \{\varphi \in E' : \|\cdot\|\text{-}\lim_{t \to 0} T(t)'\varphi = \varphi\}.$$

The adjoint semigroup on E^* is given by the operators

$$T(t)^* := T(t)'|_{E^*}, \quad t \geqslant 0.$$

Since $(T(t)^*)_{t \geqslant 0}$ is strongly continuous on E^* we call its generator $(A^*, D(A^*))$ the *adjoint generator*.

The above definition makes sense since E^* is norm-closed in E' and $(T(t)')$-invariant. The main point is that E^* is still reasonably large. In fact, since $\int_0^t T(s)'\varphi \, \mathrm{d}s$, understood in the weak sense, is contained in E^* for every $\varphi \in E'$ and $t \geqslant 0$, it follows that

$$\sup\{\langle f, \varphi \rangle : \varphi \in E^*, \|\varphi\| \leqslant 1\} \leqslant \|f\| \leqslant M \cdot \sup\{\langle f, \varphi \rangle : \varphi \in E^*, \|\varphi\| \leqslant 1\}$$

where $M := \limsup_{t \to 0} \|T(t)\|$. In particular, E^* separates E, i.e., E^* is $\sigma(E',E)$-dense in E'. In addition, the estimate of $\|\cdot\|$ given above yields

$$\|T(t)^*\| \leqslant \|T(t)\| \leqslant M\|T(t)^*\| \quad \text{for all } t \geqslant 0.$$

In the following proposition we describe the relation between A^* and A'.

Proposition *For the adjoint generator A^* of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$ on E the following assertions hold.*

(i) *E^* is the $\|\cdot\|$-closure of $D(A')$.*

(ii) $D(A^*) = \{\varphi \in D(A') : A'\varphi \in E^*\}$.

(iii) A^* *and* A' *coincide on* $D(A^*)$.

Proof (i) Take $\varphi \in D(A')$ fixed. For every $f \in D(A)$ with $\|f\| \leqslant 1$ we define a continuously differentiable function

$$t \mapsto \xi_f(t) := \langle T(t)f, \varphi\rangle$$

on $[0,1]$ with derivative $\xi'_f(t) = \langle T(t)Af, \varphi\rangle = \langle T(t)f, A'\varphi\rangle$.

Since $\{\xi'_f(t) : t \in [0,1]\,, f \in D(A), \|f\| \leqslant 1\}$ is bounded, it follows that the set

$$\{\xi_f : f \in D(A), \|f\| \leqslant 1\}$$

is equicontinuous at 0, i.e., for every $\varepsilon > 0$ there exists $0 < t_0 < 1$ such that

$$|\xi_f(s) - \xi_f(0)| = |\langle f, T(s)'\varphi - \varphi\rangle| < \varepsilon$$

for every $0 \leqslant s \leqslant t_0$ and $f \in D(A)$, $\|f\| \leqslant 1$. But this implies $\|T(s)'\varphi - \varphi\| < \varepsilon$ and hence $\varphi \in E^*$.

Conversely, take $\psi \in E^*$. Then $\frac{1}{t}\int_0^t T(s)'\psi \,\mathrm{d}s$, $t > 0$, belongs to $D(A')$ and norm converges to ψ as $t \to 0$, i.e., ψ belongs to the norm closure of $D(A')$.

(ii) and (iii): Since the weak* topology on E' is weaker than the norm topology, it follows that A' is an extension of A^*. Now take $\varphi \in D(A')$ such that $A'\varphi \in E^*$. As above define the functions ξ_f. The assumption on φ implies the set of all derivatives

$$\{\xi'_f : f \in D(A), \|f\| \leqslant 1\}$$

to be equicontinuous at $t = 0$. This means that for every $\varepsilon > 0$ there exists $0 < t_o < 1$ such that $|f'_f(0) - f'_f(s)| < \varepsilon$ for every $f \in D(A)$, $\|f\| \leqslant 1$ and $0 < s < t_o$. In particular,

$$\varepsilon > |f'_f(0) - \frac{1}{s}(\xi_f(s) - \xi_f(0))| = | < f, A'\varphi - \frac{1}{s}(T(s)'\varphi - \varphi) > |,$$

hence

$$\varepsilon > \|A'\varphi - \frac{1}{s}(T(s)'\varphi - \varphi)\|$$

for all $0 \leqslant s \leqslant t_o$. From this it follows that $\varphi \in D(A^*)$. □

On reflexive Banach spaces we have $A^* = A'$ by the above proposition. In other cases this construction is more interesting.

Example (continued) The adjoints of the (left) translation $T(t)$ on $E = L^1(\mathbb{R})$ are the (right) translations $T(t)'$ on $E' = L^\infty(\mathbb{R})$. The largest subspace of $L^\infty(\mathbb{R})$ on which these translations form a strongly-continuos semigroup with respect to the sup-norm, is the space of all bounded uniformly continuous functions on $\mathbb{R}$, i.e., $E^* = C_{bu}(\mathbb{R})$.

Calculating $D(A')$ and $D(A^*)$ respectively, one obtains

$$D(A') = \{f \in L^\infty(\mathbb{R})\colon f \in AC, f' \in L^\infty(\mathbb{R})\},$$
$$D(A^*) = \{f \in L^\infty(\mathbb{R})\colon f \in C^1(\mathbb{R}), f' \in C_{bu}(\mathbb{R})\}.$$

Obviously, the function $x \mapsto |\sin x|$ belongs to $D(A')$, but not to $D(A^*)$.

3.6 The Associated Sobolev Semigroups

Since the generator A of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is closed, its domain $D(A)$ becomes a Banach space for the graph norm

$$\|f\|_1 := \|f\| + \|Af\|.$$

We denote this Banach space by E_1 and the continuous injection from E_1 into E by i_1. Since E_1 is invariant under $(T(t))_{t\geqslant 0}$, apply Proposition 1.6 (i), it makes sense to consider the semigroup $(T_1(t))_{t\geqslant 0}$ of all restrictions $T_1(t) := T(t)|_{E_1}$. The results of Proposition 1.6 imply that $T_1(t) \in \mathcal{L}(E_1)$ and $\|T_1(t)f - f\|_1 \to 0$ as $t \to 0$ for every $f \in E_1$. Thus $(T_1(t))_{t\geqslant 0}$ is a strongly continuous semigroup on E_1 and has a generator denoted by $(A_1, D(A_1))$.

Using 1.6 again we see that A_1 is the restriction of A to E_1 with maximal domain, i.e., $D(A_1) = \{f \in E_1\colon Af \in E_1\} = D(A^2)$ and $A_1 f = Af$ for every $f \in D(A_1)$.

It is now possible to repeat this construction in order to obtain Banach spaces E_n and semigroups $(T_n(t))_{t\geqslant 0}$ with generators $(A_n, D(A_n))$ which are related as visualized in the following diagram.

$$\begin{tikzcd}
E \arrow[r, "T(t)"] & E \\
E_1 \arrow[u, "i_1"] \arrow[r, "T_1(t)"] \arrow[ur, "A"] & E_1 = D(A) \arrow[u, "i_1"'] \\
E_2 \arrow[u, "i_2"] \arrow[r, "T_2(t)"] \arrow[ur, "A_1"] & E_2 = D(A_1) = D(A^2) \arrow[u, "i_2"'] \\
\dots \arrow[u, "i_3"] \arrow[ur, "A_1"] & \dots \arrow[u, "i_3"']
\end{tikzcd}$$

For the translation semigroup on $L^p(\mathbb{R})$ (see 2.3) the above construction leads to the usual *Sobolev spaces*. Therefore we might call E_n the *n-th Sobolev space* and $(T_n(t))_{t\geqslant 0}$ the *n-th Sobolev semigroup* associated to E and $(T(t))_{t\geqslant 0}$.

Remark (i) For $\lambda \in \varrho(A)$ the operator $(\lambda - A)$ and the resolvent $R(\lambda, A)$ are isomorphisms from E_1 onto E, resp. from E onto E_1 (show that $\|\cdot\|_1$ and $\|\cdot\|_\lambda$ with $\|\cdot\|_\lambda \coloneqq \|(\lambda - A)\cdot\|$ are equivalent). In addition, the following diagram commutes.

$$\begin{tikzcd}
E \arrow[r, "T(t)"] \arrow[d, "\lambda - A"'] & E \arrow[d, "R(\lambda, A)"] \\
E_1 \arrow[r, "T_1(t)"] & E_1
\end{tikzcd}$$

Therefore all Sobolev semigroups $(E_n, T_n(t))_{t\geqslant 0}$, $n \in \mathbb{N}$, are isomorphic.

(ii) For $\lambda \in \varrho(A)$ consider the norm

$$\|f\|_{-1} \coloneqq \|R(\lambda, A)f\|$$

for every $f \in E$ and define E_{-1} as the completion of E for $\|\cdot\|_{-1}$. Then $(T(t))_{t\geqslant 0}$ extends continuously to a strongly continuous semigroup $(T_{-1}(t))_{t\geqslant 0}$ on E_{-1} and the above diagram can be extended to the negative integers.

3.7 The $\mathcal{F}$-Product Semigroup

It is standard in functional analysis to consider a sequence of points in a certain space as a point in a new and larger space. In particular such a method can serve to convert an approximate eigenvector of a linear operator into an eigenvector. Occasionally we will need such a construction and refer to Section V.1 of Schaefer [12].

If we try to adapt this construction to strongly continuous semigroups we encounter the difficulty that the semigroup extended to the larger space will not remain strongly continuous. An idea already used in 3.4 will help to overcome this difficulty.

Let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a strongly continuous semigroup on the Banach space E. Denote by $m(E)$ the Banach space of all bounded E-valued sequences endowed with the norm

$$\|(f_n)_{n\in\mathbb{N}}\| := \sup\{\|f_n\| : n \in \mathbb{N}\}.$$

It is clear that every $T(t)$ extends canonically to a bounded linear operator

$$\hat{T}(t)(f_n) := (T(t)f_n)$$

on $m(E)$, but the semigroup $(\hat{T}(t))_{t\geqslant 0}$ is strongly continuous if and only if T has a bounded generator. Therefore we restrict our attention to the closed, $(\hat{T}(t))$-invariant subspace

$$m^{\mathcal{T}}(E) := \{(f_n) \in m(E) : \lim_{t\to 0} \|T(t)f_n - f_n\| = 0 \text{ uniformly for } n \in \mathbb{N}\}.$$

Then the restricted semigroup

$$\hat{T}(t)(f_n) = (T(t)f_n), \quad (f_n) \in m^{\mathcal{T}}(E)$$

is strongly continuous and we denote its generator by $(\hat{A}, D(\hat{A}))$.

The following lemma shows that $\hat{A}$ is obtained canonically from A.

Lemma *For the generator $\hat{A}$ of $(\hat{T}(t))_{t\geqslant 0}$ on $m^{\mathcal{T}}(E)$ one has the following properties.*

(i) $D(\hat{A}) = \{(f_n) \in m^{\mathcal{T}}(E) : f_n \in D(A)$ *and* $(Af_n) \in m^{\mathcal{T}}(E)\}$,

(ii) $\hat{A}(f_n) = (Af_n)$ *for* $(f_n) \in D(\hat{A})$.

For the proof we refer to Lemma 1.4. of Derndinger [2].

Now let $\mathcal{F}$ be any filter on $\mathbb{N}$ finer than the Frechét filter (i.e., the filter of sets with finite complement. In most cases $\mathcal{F}$ will be either the Frechét filter or some free ultra filter). The space of all $\mathcal{F}$-null sequences in $m(E)$, i.e.,

$$c_{\mathcal{F}}(E) := \{(f_n) \in m(E) : \mathcal{F}\text{-}\lim \|f_n\| = 0\}$$

is closed in $m(E)$ and invariant under $(\hat{T}(t))_{t\geqslant 0}$. We call the quotient spaces

$$E_{\mathcal{F}} := m(E)/c_{\mathcal{F}}(E) \quad \text{and} \quad E_{\mathcal{F}}^{T} := m^{\mathcal{T}}(E)/(c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E))$$

the *$\mathcal{F}$-product of E* and the *$\mathcal{F}$-product of E with respect to the semigroup T*, respectively.

Thus $E_{\mathcal{F}}^{T}$ can be considered as a closed linear subspace of $E_{\mathcal{F}}$. We have $E_{\mathcal{F}}^{T} = E_{\mathcal{F}}$ if (and only if) T has a bounded generator.

The canonical quotient norm on $E_{\mathcal{F}}$ is given by

$$\|(f_n) + c_{\mathcal{F}}(E)\| = \mathcal{F}\text{-}\limsup \|f_n\|.$$

We can apply Subsection 3.4 in order to define the *$\mathcal{F}$-product semigroup* $(T_{\mathcal{F}}(t))_{t\geqslant 0}$ on $E_{\mathcal{F}}^{T}$ by

$$T_{\mathcal{F}}(t)((f_n) + c_{\mathcal{F}}(E)) := (T(t)f_n) + (c_{\mathcal{F}}(E) \cap m^{\mathcal{T}}(E))$$

Thus $T_{\mathcal{F}}(t)$ is the restriction of $T(t)_F$ where $T(t)_F$ denotes the canonical extension of $T(t)$ to the $\mathcal{F}$-product $E_{\mathcal{F}}$. But note that $(T(t)_F)_{t\geqslant 0}$ is not strongly continuous unless T has a bounded generator.

With the canonical injection $j\colon f \mapsto (f, f, f, \ldots) + c_{\mathcal{F}}(E)$ from E into $E_{\mathcal{F}}^{T}$ the operators $T_{\mathcal{F}}(t)$ are extensions of $T(t)$ satisfying $\|T_{\mathcal{F}}(t)\| = \|T(t)\|$. The basic facts about the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of $(T_{\mathcal{F}}(t))_{t\geqslant 0}$ follow from 3.3 and are collected in the following proposition.

Proposition *For the generator $(A_{\mathcal{F}}, D(A_{\mathcal{F}}))$ of the $\mathcal{F}$-product semigroup the following holds.*

(i) $D(A_{\mathcal{F}}) = \{(f_n) + c_{\mathcal{F}}(E)\colon f_n \in D(A); (f_n), (Af_n) \in m^{\mathcal{T}}(E)\}$,

(ii) $A_{\mathcal{F}}((f_n) + c_{\mathcal{F}}(E)) = (Af_n) + c_{\mathcal{F}}(E)$.

In case A is a bounded operator then $D(A_{\mathcal{F}}) = E_{\mathcal{F}}^{T} = E_{\mathcal{F}}$ and $A_{\mathcal{F}}$ is the canonical extension of A to $E_{\mathcal{F}}$.

We will show in A-III,4.5 that the above construction preserves and even improves many spectral properties of the semigroup and its generator.

3.8 The Tensor Product Semigroup

Real- or complex-valued functions of two variables x, y are often limits of functions of the form $\sum_{i=1}^{n} f_i(x)g_i(y)$ which, to some extent, allows one to consider the variables x and y separately. Since algebraic manipulation with these latter functions is governed by the formal rules of a tensor product, it is customary to identify (for example) the function

$$(x, y) \mapsto f(x)g(y)$$

with the tensor product $f \otimes g$ and to consider limits of linear combinations of such functions as elements of a completed tensor product.

To be more precise, we briefly present the most important examples for this situation.

Examples (i) Let (X, Σ, μ) and (Y, Ω, ν) be measure spaces. If we identify for $f_i \in L^p(\mu)$, $g_i \in L^p(\nu)$ the elements $\sum_{i=1}^n f_i \otimes g_i$ of the tensor product

$$L^p(\mu) \otimes L^p(\nu)$$

with the (class of $\mu \times \nu$-a.e.-defined) functions

$$(x, y) \mapsto \sum_{i=1}^{n} f_i(x) g_i(y),$$

then $L^p(\mu) \otimes L^p(\nu)$ becomes a dense subspace of $L^p(X \times Y, \Sigma \times \Omega, \mu \times \nu)$ for $1 \leqslant p < \infty$.

(ii) Similarly, let X,Y be compact spaces. Then $C(X) \otimes C(Y)$ becomes a dense subspace of $C(X \times Y)$ by identifying, for $f \in C(X)$ and $g \in C(Y)$, $f \otimes g$ with the function

$$(x, y) \mapsto f(x)g(y).$$

We do not intend to go deeper into the quite sophisticated problems related to normed tensor products of general Banach spaces, but will rather confine ourselves to the discussion of certain special cases. These will always be related to one of the following standard methods to define a norm on the tensor product of two Banach spaces E, F.

Let $u := \sum_{i=1}^n f_i \otimes g_i$ be an element of $E \otimes F$. Then

(i) $\|u\|_\pi := \inf\{\sum_{j=1}^m \|h_j\|\|k_j\| \colon u = \sum_{j=1}^m h_j \otimes k_j, h_j \in E, k_j \in F\}$ defines the *greatest cross norm* π on $E \otimes F$.

(ii) $\|u\|_\varepsilon := \sup\{\langle u, \varphi \otimes \psi\rangle \colon \varphi \in E', \psi \in F', \|\varphi\|, \|\psi\| \leqslant 1\}$ defines the *least cross norm* ε on $E \times F$. Here, $\langle u, \varphi \otimes \psi\rangle$ denotes the canonical bilinear form on $(E \otimes F) \times (E' \otimes F')$, i.e., $\langle \sum_{i=1}^n f_i \otimes g_i, \varphi \otimes \psi\rangle = \sum_{i=1}^n \langle f_i, \varphi\rangle\langle g_i, \psi\rangle$.

(iii) if E and F are Hilbert spaces, $\|u\|_h = (u|u)_h^{1/2}$, where the scalar product $(\cdot|\cdot)_h$ is defined as in (ii), defines the *Hilbert norm* h on $E \otimes F$.

In the following we write $E \otimes_\alpha F$ for the tensor product of E and F endowed—if applicable—with one of the norms π, ε, h just defined. In each case one has $\|f \otimes g\| = \|f\|\|g\|$ for $f \in E$, $g \in F$.

By $E \widetilde{\otimes}_\alpha F$ we mean the completion of $E \otimes_\alpha F$. Moreover we recall how examples (i) and (ii) above fit into this pattern

$$L^1(\mu \otimes \nu) = L^1(\mu) \widetilde{\otimes}_\pi L^1(\nu), \quad L^2(\mu \otimes \nu) = L^2(\mu) \widetilde{\otimes}_h L^2(\nu),$$

$$C(X \otimes Y) = C(X)\widetilde{\otimes}_\varepsilon C(Y).$$

Finally, we point out that for any $S \in \mathcal{L}(E), T \in \mathcal{L}(F)$, the mapping

$$\sum_{i=1}^{n} f_i \otimes g_i \mapsto \sum_{i=1}^{n} Sf_i \otimes Tg_i$$

defined on $E \otimes F$ is linear and continuous on $E \otimes_\alpha F$, hence has a continuous extension to $E\widetilde{\otimes}_\alpha F$. This operator, as well as its continuous extension, will be denoted by $S \otimes T$ and satisfies $\|S \otimes T\| = \|S\|\|T\|$. The notation $A \otimes B$ will also be used in the obvious way if A and B are not necessarily bounded operators on E and F. We are now ready to consider semigroups induced on the tensor product.

Proposition *Let $(S(t))_{t\geqslant 0}$ and $(T(t))_{t\geqslant 0}$ be strongly continuous semigroups on Banach spaces E, F, and let A, B be their generators. Then the family $(S(t) \otimes T(t))_{t\geqslant 0}$ is a strongly continuous semigroup on $E\widetilde{\otimes}_\alpha F$. The closure of $A \otimes \mathrm{Id} + \mathrm{Id} \otimes B$, defined on the core $D(A) \otimes D(B)$, is its generator.*

Proof It is immediately verified that $(S(t) \otimes T(t))_{t\geqslant 0}$ is in fact a semigroup of operators on $E\widetilde{\otimes}_\alpha F$. The strong continuity need only be verified at $t = 0$ and on elements of the form $u = f \otimes g \in E \otimes F$.

This verification being straightforward, there remains to show that the generator of $(S(t) \otimes T(t))_{t\geqslant 0}$ is obtained as the closure of

$$(A \otimes \mathrm{Id} + \mathrm{Id} \otimes B, D(A) \otimes D(B)).$$

To this end, let $f \in D(A)$ and $g \in D(B)$. Then

$$\begin{aligned}
&\lim_{h\to 0} \frac{1}{h}(T(h) \otimes S(h)(f \otimes g) - f \otimes g) \\
&= \lim_{h\to 0} \frac{1}{h}(T(h)f \otimes (S(h)g - g) + (T(h)f - f) \otimes g) \\
&= (f \otimes Bg) + (Af \otimes g).
\end{aligned}$$

Since the elements of the form $f \otimes g$, $f \in D(A)$, $g \in D(B)$, generate the linear subspace $D(A) \otimes D(B)$ of $E \otimes_\alpha F$, this subspace belongs to the domain of the generator. Moreover, $D(A) \otimes D(B)$ is dense in $E\widetilde{\otimes}_\alpha F$ and invariant under $(S(t) \otimes T(t))_{t\geqslant 0}$, hence it is a core of $A \otimes \mathrm{Id} + \mathrm{Id} \otimes B$ by Proposition 1.9 (ii). $\square$

3.9 The Product of Commuting Semigroups

Let $(S(t))_{t\geqslant 0}$ and $(T(t))_{t\geqslant 0}$ be semigroups with generators A and B, respectively on some Banach space E. It is not difficult to see that the following assertions are equivalent.

(a) $S(t)T(t) = S(t)T(t)$ for all $t \geqslant 0$.

(b) $R(\mu, A)R(\mu, B) = R(\mu, B)R(\mu, A)$ for some $\mu \in \varrho(A) \cap \varrho(B)$.

(c) $R(\mu, A)R(\mu, B) = R(\mu, B)R(\mu, A)$ for all $\mu \in \varrho(A) \cap \varrho(B)$.

In that case $U(t) = S(t)T(t)$ $(t \geqslant 0)$ defines a semigroup $(U(t))_{t \geqslant 0}$. Using Proposition 1.9 (ii) on p. 7 one easily shows that $D_0 := D(A) \cap D(B)$ is a core for its generator C and $Cf = Af + Bf$ for all $f \in D_0$.

Notes

For more complete information on semigroup theory we refer the reader to Hille and Phillips [6], to the monographs by Davies [1], Goldstein [4] and Pazy [9], to the survey article by Krein and Khazan [8], to the bibliography by Goldstein [5] and to Engel and Nagel [3].

References

[1] E. Davies. *One-parameter Semigroups*. Academic Press, London-New York-San Francisco, 1980.

[2] R. Derndinger. Über das Spektrum positiver Generatoren. *Math. Z.*, 172: 281–293, 1980.

[3] K.-J. Engel and R. Nagel. *A Short Course on Operator Semigroups*. Universitext UTX. Springer, 2006.

[4] J. Goldstein. *Semigroups of Operators and Applications*. Oxford University Press, 1985.

[5] J. Goldstein. A (more-or-less) complete bibliography of semigroups of operators through 1984. Preprints and lecture notes in mathematics, Tulane University, 1985.

[6] E. Hille and R. S. Phillips. *Functional Analysis and Semigroups*, volume 31 of *Colloquium Publications*. American Mathematical Society, Providence, R.I., 1957.

[7] T. Kato. *Perturbation Theory for Linear Operators*. Springer, 1966. 2nd printing: Berlin-Heidelberg-New York: Springer 1976.

[8] S. G. Krein and M. I. Khazan. Differential equations in a Banach space. *J. Soviet Math.*, 30:2154–2239, 1985.

[9] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[10] R. S. Phillips. A note on the abstract Cauchy problem. *Proc. Nat. Acad. Sci. U.S.A.*, 40:244–248, 1954.

[11] M. Reed and B. Simon. *Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness*. Academic Press, New York, 1975.

[12] H. H. Schaefer. *Banach Lattices and Positive Operators*. Springer, New York-Heidelberg-Berlin, 1974.

Chapter A-II

Characterization of Semigroups on Banach Spaces

In this chapter two different problems are treated:

(i) to characterize generators of strongly continuous semigroups;

(ii) to characterize various properties of strongly continuous semigroups in terms of their generators.

In Section 1 the first problem is solved by finding conditions on the Cauchy problem associated with A and also by finding conditions on the resolvent of A. The second problem is treated for a hierarchy of smoothness properties of the semigroup.

Contraction semigroups are considered in Section 2. Here, the first problem has a simple and extremely useful solution: A densely defined operator A is generator of a contraction semigroup if and only if A is dissipative and satisfies a range condition.

Our approach is quite general. We do not only consider contractions with respect to the norm but also with respect to *half-norms*. This will allow us to obtain results on positive contraction semigroups simultaneously by choosing a suitable half-norm (cf. C-II, Section 1).

The last section contains a surprising result: on certain Banach spaces (e.g., L^∞) only bounded operators are generators of strongly continuous semigroups.

1 The Abstract Cauchy Problem, Special Semigroups and Perturbation

by Wolfgang Arendt

Linear differential equations in Banach spaces are intimately connected with the theory of one-parameter semigroups. In fact, given a closed linear operator A with dense domain $D(A)$ the following statement is true (with some reservation regarding a technical detail):
The abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) \quad (t \geqslant 0) \\ u(0) &= f \end{aligned}$$

has a unique solution for every $f \in D(A)$ if and only if A is the generator of a strongly continuous semigroup. This is one characterization of generators which illustrates their important role for applications. The fundamental Hille-Yosida theorem gives a different characterization in terms of the resolvent and yields a powerful tool for actually proving that a given operator is the generator of a semigroup.

Another problem we will treat here is how diverse properties of a semigroup can be described in terms of its generator. This is a reasonable question from the theoretical point of view (since the generator uniquely determines the semigroup).
It is of interest from the practical point of view as well: the generator is the given object, defined by the differential equation. It is useful to dispose of conditions of the generator itself giving information on the solutions (which might not be known explicitely). We discuss smoothness properties such as analyticity, differentiability, norm continuity and compactness of the semigroup.

A frequent method to obtain new generators out of a given one is by perturbation. We will have a brief look at this circle of problems at the end of this section.

The results are explained and illustrated by examples. Proofs are only given when new aspects are presented which are not yet contained in the literature, otherwise we refer to the recent monographs Davies [11], Goldstein [15], Pazy [31].

1.1 The abstract Cauchy problem

Let A be a closed operator on a Banach space E and consider the abstract Cauchy problem

$$\text{(ACP)} \quad \begin{cases} \dot{u}(t) &= Au(t) \quad (t \geqslant 0) \\ u(0) &= f. \end{cases}$$

By a solution of (ACP) for the initial value $f \in D(A)$ we understand a continuously differentiable function $u\colon [0, \infty[\to E$ satisfying $u(0) = f$ and $u(t) \in D(A)$ for all $t \geqslant 0$ such that $u'(t) = Au(t)$ for $t \geqslant 0$.

By A-I, Theorem.1.7 there exists a unique solution of (ACP) for all initial values in the domain $D(A)$ whenever A is the generator of a strongly continuous semigroup. The converse does not hold (see Example 1.4. below). However, for the operator A_1 on the Banach space $E_1 = D(A)$ (see A-I,3.5) with domain $D(A_1) = D(A^2)$ given by $A_1 f = Af$ $(f \in D(A_1))$ the following holds.

Theorem 1.1 *The following assertions are equivalent.*

(a) *For every $f \in D(A)$ there exists a unique solution of (ACP).*

(b) *A_1 is the generator of a strongly continuous semigroup.*

Proof (a) $\implies$ (b): Assume that (a) holds, i.e., for every $f \in D(A)$ there exists a unique solution $u(\cdot, f) \in C^1([0, \infty), E)$ of (ACP). For $f \in E_1$ define

$$T_1(t)f := u(t, f)\,.$$

By the uniqueness of the solutions it follows that $T_1(t)$ is a linear operator on E_1 and $T_1(s+t) = T_1(s)T_1(t)$. Moreover, since $u(\cdot, f) \in C^1$, it follows that $t \mapsto T_1(t)f$ is continuous from $[0, \infty)$ into E_1. We show that $T_1(t)$ is a continuous operator for all $t > 0$.

Let $t > 0$. Consider the mapping $\eta\colon E_1 \to C([0, t], E_1)$ given by

$$\eta(f) = T_1(\cdot)f = u(\cdot, f).$$

We show that η has a closed graph.

In fact, let $f_n \to f$ in E_1 and $\eta(f_n) = u(\cdot, f_n) \to v$ in $C([0, t], E_1)$. Then

$$u(s, f_n) = f_n + \int_0^s Au(r, f_n)\,\mathrm{d}r.$$

Letting $n \to \infty$ we obtain $v(s) = f + \int_0^s Av(r)\,\mathrm{d}r$ for $0 \leqslant s \leqslant t$.

Let

$$\tilde{v}(s) = \begin{cases} T_1(s-t)v(t) & \text{for } s > t, \\ \tilde{v}(s) = v(s) & 0 \leqslant s \leqslant t. \end{cases}$$

Then $\tilde{v}$ is a solution of (ACP). It follows that $\tilde{v}(s) = T_1(s)f$ for all $s \geqslant 0$. Hence $v = \eta(f)$.

We have shown that η has a closed graph and so η is continuous. This implies the continuity of $T_1(t)$. It remains to show that A_1 is the generator of $(T_1(t))_{t \geqslant 0}$.

We first show that for $f \in D(A^2)$ one has

$$AT_1(t)f = T_1(t)Af. \tag{1.1}$$

In fact, let $v(t) = f + \int_0^t u(s, Af)\,\mathrm{d}s$. Then

$$\dot v(t) = u(t, Af) = Af + \int_0^t Au(s, Af)\,\mathrm{d}s = A(f + \int_0^t u(s, Af)\,\mathrm{d}s) = Av(t).$$

Since $v(0) = f$, it follows that $v(t) = u(t, f)$. Hence $Au(t, f) = Av(t) = \dot v(t) = u(t, Af)$. This is (1.1). Now denote by B the generator of $(T_1(t))_{t\geqslant 0}$. For $f \in D(A^2)$, we have

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af$$

and by (1.1),

$$\lim_{t\to 0} A\frac{T_1(t)f - f}{t} = \lim_{t\to 0} \frac{T_1(t)Af - Af}{t} = A^2 f$$

in the norm of E. Hence

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af$$

in the norm of E_1. This shows that $A_1 \subset B$.

In order to show the converse, let $f \in D(B)$. Then

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t}$$

exists in the norm of E. Since

$$\lim_{t\to 0} \frac{T_1(t)f - f}{t} = Af \text{ in the norm of } E,$$

it follows that $Af \in D(A)$, since A is closed. Thus $f \in D(A^2) = D(A_1)$. We have shown that $B = A_1$.

(b) $\implies$ (a): Assume that A_1 is the generator of a strongly continuous semigroup $(T_1(t))_{t\geqslant 0}$ on E_1. Let $f \in D(A)$ and set $u(t) = T_1(t)f$. Then $u \in C([0,\infty), E)$ and $Au(\cdot) \in C([0,\infty), E)$.

Moreover,

$$\int_0^t u(s)\,\mathrm{d}s = \int_0^t T_1(s)f\,\mathrm{d}s \in D(A_1) = D(A^2)$$

and

$$A\int_0^t u(s)\,\mathrm{d}s = u(t) - u(0) = u(t) - f$$

(by A-I, (1.3)). Consequently, $u(t) = f + \int_0^t Au(s)\,\mathrm{d}s$. Hence $u \in C^1([0,\infty), E)$ and $\dot u(t) = Au(t)$. Thus u is a solution of (ACP). We have shown existence.

In order to show uniqueness, assume that u is a solution of (ACP) with initial value 0. We have to show that $u \equiv 0$.

Let $v(t) = \int_0^t u(s)\,\mathrm{d}s$. Then $v(t) \in D(A)$ and $Av(t) = \int_0^t Au(s)\,\mathrm{d}s = \int_0^t \dot{u}(s)\,\mathrm{d}s = u(t) \in D(A)$. Consequently, $v(t) \in D(A^2)$ for all $t \geqslant 0$. Moreover, $\dot{v}(t) = u(t) = Av(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}Av(t) = Au(t) = A_1\dot{v}(t) = A^2v(t)$. Thus $v \in C^1([0,\infty), E_1)$ and $\dot{v}(t) = A_1v(t)$. Since $v(0) = 0$, it follows that $v \equiv 0$. Thus $u \equiv v \equiv 0$. □

If (ACP) has a unique solution for every initial value in $D(A)$, then A is the generator of a strongly continuous semigroup only if some additional assumptions on the solutions (continuous dependence from the initial value) or on A ($\varrho(A) \neq \emptyset$) are made.

Corollary 1.2 *Let A be a closed operator. Consider the following existence and uniqueness condition.*

$$(EU) \qquad \begin{cases} \textit{For every } f \in D(A) \textit{ there exists a unique solution} \\ u(\cdot, f) \in C^1([0,\infty), E) \textit{ of the Cauchy problem associated with } A \\ \textit{having the initial value } u(0, f) = f. \end{cases}$$

The following assertions are equivalent.

(a) *A is the generator of a strongly continuous semigroup.*

(b) *A satisfies (EU) and $\varrho(A) \neq \emptyset$.*

(c) *A satisfies (EU) and for every $\mu \in \mathbb{R}$ there exists $\lambda > \mu$ such that $(\lambda - A)D(A) = E$.*

(d) *A satisfies (EU), has dense domain and for every sequence (f_n) in $D(A)$ satisfying $\lim_{n\to\infty} f_n = 0$ one has $\lim_{n\to\infty} u(t, f_n) = 0$ uniformly in $t \in [0, 1]$.*

Proof It is clear that (a) implies the remaining assertions. So assume that A satisfy (EU). Then by Theorem 1.1, A_1 is a generator. If there exists $\lambda \in \varrho(A)$, then $(\lambda - A)$ is an isomorphism from E_1 onto E and A is similar to A_1 via this isomorphism since $D(A_1) = \{(\lambda - A)^{-1}f : f \in D(A)\}$ and $Af = (\lambda - A)A_1(\lambda - A)^{-1}f$ for all $f \in D(A)$, see A-I, 3.0. Thus A is a generator on E and we have shown that (b) implies (a).

If (c) holds, then there exists $\lambda > s(A_1)$ such that $(\lambda - A)D(A) = E$. We show that $(\lambda - A)$ is injective. Then $\lambda \in \varrho(A)$ since A is closed. Assume that $\lambda f = Af$ for some $f \in D(A)$. Then $f \in D(A^2) = D(A_1)$, and so $f = 0$ since $\lambda \in \varrho(A_1)$. This proves that (c) implies (b).

It remains to show that (d) implies (a). Assertion (d) implies that for all $t \geqslant 0$ there exist bounded operators $T(t) \in \mathcal{L}(E)$ such that $u(t, f) = T(t)f$ if $f \in D(A)$. Moreover, $\sup_{0\leqslant t\leqslant 1} \|T(t)\| < \infty$. It follows that $T(\cdot)f$ is strongly continuous for all $f \in E$ (since it is so for $f \in D(A)$ and $D(A)$ is dense). Let $t > 1$. There exist unique $n \in \mathbb{N}$ and $s \in [0, 1)$ such that $t = n + s$. Let $T(t) := T(1)^n T(s)$. From the uniqueness of the solutions it follows that $T(t)f = u(t, f)$ for all $t \geqslant 0$ as well as $T(t+s)f = T(s)T(t)f$ for all $f \in D(A)$ and $s, t \geqslant 0$. Thus T is a semigroup.

Denote by B its generator. It follows from the definition that $A \subset B$. Moreover, $D(A)$ is invariant under the semigroup. So by A-I, Proposition 1.9 (ii) $D(A)$ is a core of B. Since A is closed this implies that $A = B$. □

Remark 1.3 It is surprising that from condition (b) and (c) in the corollary it follows automatically that $D(A)$ is dense. On the other hand, this condition cannot be omitted in (d). In fact, consider any generator $\tilde{A}$ and its restriction A with domain $D(A) = \{0\}$. Then $\overline{A}$ satisfies the remaining conditions in (d) but is not a generator (if $\dim E > 0$).

Example 1.4 We give a densely defined closed operator A such that there exists a unique solution of (ACP) for all initial values in $D(A)$, but A is not a generator.

Let B be a densely defined unbounded closed operator on a Banach space F. Consider $E = F \oplus F$ and A on E given by

$$A := \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix}$$

with domain $F \times D(B)$.

Then $D(A^2) = \{(f, g) \in F \times D(B) \colon Bg \in F\} = D(A)$ and so $A_1 \in \mathcal{L}(E_1)$. In particular, A_1 is a generator while A is not. For instance condition (b) in Corollary 1.2 does not hold since, for each $\lambda \in \mathbb{C}$

$$(\lambda - A)D(A) = \{(\lambda f - Bg, \lambda g) \colon f \in F, g \in D(B)\} \subset F \times D(B) \neq F \times F = E,$$

so $\varrho(A) = \emptyset$.

As a further illustration, we note that the solution of the corresponding abstract Cauchy problem for the initial value $(f, g) \in F \times D(B)$ is given by

$$u(t) = (f + tBg, g).$$

Since B is unbounded, condition (d) of Corollary 1.2 is clearly violated.

Remark It may happen that a generator A can be extended to a closed operator B. Then one can consider the abstract Cauchy problem ACP(B) associated with B. It also has a solution for every initial value in $D(B)$, but none of the solutions is unique unless $A = B$.

In fact, denote by $(T(t))_{t \geqslant 0}$ the semigroup generated by A. Let $f \in D(B)$. Let $\lambda > \omega(A)$. Then there exists $g \in D(A)$ such that $(\lambda - B)f = (\lambda - A)g$. Let $h = f - g$. Then $h \in \ker(\lambda - B)$. Define u by $u(t) = \mathrm{e}^{\lambda t} h + T(t)g$. Then u is a solution ACP(B) with initial value f. It follows from Corollary 1.2 that there exists a non-trivial solution for the initial value 0.

1.2 One-parameter groups

Generators of one-parameter groups can be characterized similarly by existence and uniqueness of the solutions of the associated Cauchy problem. However, here the assumption of continuous dependence on the initial values can be relaxed (in fact, one has no longer to assume that the continuous dependence is uniform in t).

Theorem 1.6 *Let A be a closed densely defined operator. The following assertions are equivalent.*

(a) *A is generator of a strongly continuous one-parameter group.*

(b) *For every $f \in D(A)$ there exists a unique function $u(\cdot, f) \in C^1(\mathbb{R})$ satisfying $u(t, f) \in D(A)$ for all $t \in \mathbb{R}$ and $u(0, f) = f$ such that $\frac{\mathrm{d}}{\mathrm{d}t}u = Au(t, f)$, and if $f_n \in D(A)$ such that $\lim_{n\to\infty} f_n = 0$, then $\lim_{n\to\infty} u(t, f_n) = 0$ for all $t \in \mathbb{R}$.*

Proof It is clear that (a) implies (b) . If (b) holds then there exist operators $T(t) \in \mathcal{L}(E)$ such that $u(t, f) = T(t)f$ $(t \in \mathbb{R}, f \in D(A))$. It follows from the uniqueness of the solutions that $T(t + s) = T(t)T(s)$ $(s, t \in \mathbb{R})$. Let $f \in E$. There exist $(f_n) \in D(A)$ such that $\lim_{n\to\infty} f_n = f$.

Then $\lim_{n\to\infty} T(t)f_n = T(t)f$ for all $t \in \mathbb{R}$. Since $T(\cdot)f$ is continuous, it follows that $T(\cdot)f$ is measureable. Hence by Hille and Phillips [21, 10.2.1] $\sup_{t\in J} \|T(t)\| < \infty$ for every compact interval $J \subset (0, \infty)$. By the group property this implies that $T(\cdot)$ is norm bounded on bounded subsets of $\mathbb{R}$ and $T(\cdot)f$ is continuous if $f \in D(A)$. Since $D(A)$ is dense, this implies the strong continuity of $(T(t))_{t\in\mathbb{R}}$. □

1.3 The Hille-Yosida Theorem

Given an operator A, frequently it is easier to obtain information about its resolvent than to solve the Cauchy problem. Therefore the following theorem is central in the theory of one-parameter semigroups.

Theorem 1.7 (Hille-Yosida) *Let A be an operator on a Banach space E. The following conditions are equivalent.*

(a) *A is the generator of a strongly continuous semigroup.*

(b) *There exist $w, M \in \mathbb{R}$ such that $(w, \infty) \subset \varrho(A)$ and*

$$\|(\lambda - w)^n R(\lambda - A)^{-n}\| \leqslant M$$

for all $\lambda > w$ and $n \in \mathbb{N}$.

In general it is not easy to give an estimate for the powers of the resolvent in order to apply Theorem 1.7. However, there is an important case when it suffices to consider merely the resolvent.

Corollary 1.8 *For an operator A on a Banach space E the following assertions are equivalent.*

(a) *A is the generator of a strongly continuous contraction semigroup.*

(b) *$(0, \infty) \subset \varrho(A)$ and $\|\lambda R(\lambda, A)\| \leqslant 1$ for all $\lambda > 0$.*

The difficult part in the proof of Theorem 1.7 is to show that (b) implies (a) . One has to construct the semigroup out of the resolvent. We mention two formulas which can be used for the proof.

Proposition 1.9 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$. For $\lambda > 0$ let*

$$A(\lambda) = \lambda^2 R(\lambda, A) - \lambda Id = \lambda A R(\lambda, A).$$

Then

$$T(t)f = \lim_{\lambda \to \infty} \mathrm{e}^{tA(\lambda)} f \tag{1.2}$$

for all $f \in E$ and $t \geqslant 0$.

Yosida's proof consists in showing that the limit in (1.2) exists under the hypothesis (b) of Theorem 1.7 (see Davies [11], Goldstein [16] or Pazy [31]).

The proof of Hille (see Kato [22]) is inspired by the following formula.

Proposition 1.10 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$. Then*

$$T(t)f = \lim_{n \to \infty} (Id - t/nA)^{-n} f = \lim_{n \to \infty} (n/t \cdot R(n/t, A))^n f \tag{1.3}$$

for all $f \in E$ and $t \geqslant 0$.

1.4 Holomorphic semigroups

We now describe a hierarchy of smoothness conditions on the semigroup, starting with the most restrictive class; namely, holomorphic semigroups. The generators of these semigroups can be characterized by a particularly simple condition.

For $\alpha \in (0, \pi]$ we define the sector $S(\alpha)$ in the complex plane by

$$S(\alpha) = \{r\mathrm{e}^{\mathrm{i}\vartheta} : r \geqslant 0, \vartheta \in (-\alpha, \alpha)\}.$$

Definition 1.11 Let $\alpha \in (0, \pi/2]$. A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is called a bounded holomorphic semigroup of angle α if $T(\cdot)$ is the restriction of a holomorphic function

$$T\colon S(\alpha) \to \mathcal{L}(E)$$

satisfying

$$T(z)T(z') = T(z+z') \quad (z, z' \in S(\alpha)), \tag{1.4}$$

$$\text{For each } \alpha_1 \in (0,\alpha) \text{ the set } \{T(z)\colon z \in S(\alpha_1)\} \text{ is uniformly bounded, and} \\ \lim_{n\to\infty} T(z_n)f = f \text{ for every null-sequence } (z_n) \text{ in } S(\alpha_1) \text{ and every } f \in E. \tag{1.5}$$

Remark A function $T\colon S(\alpha) \to L(E)$ is holomorphic with respect to the operator norm if and only if it is strongly holomorphic if and only if it is weakly holomorphic Yosida [40, V.3].

Theorem 1.12 *Let A be a densely defined operator on a Banach space E and $\alpha \in (0, \pi/2]$. Then A is the generator of a bounded holomorphic semigroup of angle α if and only if*

$$S(\alpha + \pi/2) \subset \varrho(A)$$

and for every $\alpha_1 \in (0, \alpha)$ there exists a constant M such that

$$\|R(\lambda, A)\| \leqslant M/|\lambda| \quad (\lambda \in S(\alpha_1 + \pi/2). \tag{1.6}$$

For the proof we refer to Davies [11], for example.

Remark Let A be the generator of a bounded holomorphic semigroup $(T(t))_{t\geqslant 0}$ of angle α, and let $z_0 \in S(\alpha)$. Then z_0A generates a bounded semigroup $(S(t))_{t\geqslant 0}$ given by $S(t) = T(z_0 t)$ $(t \geqslant 0)$ (where again we denote by $T(z)$ the holomorphic extension of $(T(t))_{t\geqslant 0}$ on $S(\alpha)$).

As an application of Theorem 1.12 we prove the following.

Corollary 1.13 *Let A be the generator of a bounded group. Then A^2 generates a bounded holomorphic semigroup of angle $\pi/2$.*

Proof Let $0 < \alpha_1 < \pi/2$ and $\lambda \in S(\alpha_1 + \pi/2)$. There exists $r > 0$ and $\beta \in (-\beta_1, \beta_1)$, where $\beta_1 := (\alpha_1 + \pi/2)/2$ such that $\lambda = r^2 e^{i2\beta}$. Then

$$(\lambda - A^2) = (re^{\mathrm{i}\beta} - A)(re^{\mathrm{i}\beta} + A)\,.$$

It follows that $\lambda \in \varrho(A)$ and

$$R(\lambda, A^2) = R(re^{\mathrm{i}\beta}, A)R(re^{-\mathrm{i}\beta}, -A)\,. \tag{1.7}$$

Since A generates a bounded group, there exists a constant $N \geqslant 0$ such that $\|R(\mu, A)\| \leqslant N/\mathrm{Re}\mu$, $\|R(\mu, -A)\| \leqslant N/\mathrm{Re}\mu$ for all $\mu \in S(\pi/2)$. Consequently, $\|R(\lambda, A^2)\| \leqslant N^2/r^2(\cos\beta)^2 \leqslant 1/r^2[N/\cos\beta]^2 = M/|\lambda|$. ◻

The corollary will be extended below. We first consider an example.

Example (The Laplacian on $E = C_0(\mathbb{R}^n)$ or $L^p(R^n)$ $(1 \leqslant p < \infty)$)

(i) Let $n = 1$. Then $(U(t)f)(x) = f(x+t)$ $(x \in \mathbb{R})$ defines an isometric group on E. Its generator A is given by

$$Af = f'$$

with

$$D(A) = \{f \in C^1(\mathbb{R}) \cap C_0(\mathbb{R})\colon f' \in C_0(\mathbb{R})\} \quad \text{in the case } E = C_0(\mathbb{R})$$

and

$$D(A) = \{f \in E \cap AC(\mathbb{R})\colon f' \in E\} \quad \text{in the case } E = L^p$$

(see A-I, 2.4). Thus A^2 generates a bounded holomorphic semigroup by Corollary 1.13.

(ii) Let $E = C_0(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$. For $i \in \{1, \ldots, n\}$ denote by $(U_i)_{t \geqslant 0}$ the group given by

$$(U_i(t)f)(x) = f(x_1, \ldots, x_{i-1}, x_i + t, \ldots, x_n) \quad (x \in \mathbb{R}^n, t \in \mathbb{R})$$

and by A_i its generator. Since

$$U_i(t)U_j(s) = U_j(s)U_i(t) \quad (s, t \in \mathbb{R}, i, j \in \{1, \ldots, n\}),$$

it follows that the resolvents of A_i commute. So the same is true for the resolvents of A_i^2 (cf. (1.7) and A-I, 3.8).

Denote by $(T_i(t))_{t \geqslant 0}$ the semigroup generated by A_i^2 $(i = 1, \ldots, n)$. Then for $z, z' \in S(\pi/2)$ one has $T_i(z)T_j(z') = T_j(z')T_i(z)$ $(i, j = 1, \ldots, n)$. Consequently, $T(t) \coloneqq T_1(t) \circ \cdots \circ T_n(t)$ $(t \geqslant 0)$ defines a holomorphic semigroup of angle $\pi/2$. According to A-I, 3.8 the domain of its generator A contains $D(A_1^2) \cap \cdots \cap D(A_n^2)$ and, in particular

$$D_0 = \{f \in E \cap C^2(\mathbb{R}^n)\colon D^\alpha f \in E \text{ for every multiindex } \alpha \text{ with } |\alpha| \leqslant 2\} \subset D(A)\,.$$

On D_0 the generator is given by

$$Af = (A_1^2 + \cdots + A_n^2)f = \sum_{i=1}^n \frac{\partial^2}{(\partial x_i)^2} f = \Delta F \quad \text{for all} \quad f \in D_0\,.$$

Let $\alpha \in (0, \pi/2]$. A semigroup $(T(t))_{t \geqslant 0}$ is called *holomorphic of angle* α if it possesses an extension $T\colon S(\alpha) \to L(E)$ for some $\alpha \in (0, \pi/2]$ which satisfies all the requirements of Definition 1.11 except that it is not required to be bounded on any sector $S(\alpha_1)$.

Theorem 1.14 *A densely defined operator A is the generator of a holomorphic semigroup if and only if there exist $M > 0$ and $r \geqslant 0$ such that $\lambda \in \varrho(A)$ and $\|R(\lambda, A)\| \leqslant M/|\lambda|$ whenever* $\mathrm{Re}\lambda > 0$, $|\lambda| \geqslant r$.

Proof It is not difficult to show that A generates a holomorphic semigroup of angle α if and only if for every $\alpha_1 \in (0, \alpha)$ there exists $w \in \mathbb{R}$ such that $A - w$ generates a bounded holomorphic semigroup of angle α_1 (cf. Reed and Simon [35, p.252]). As a consequence one obtains the following. A densely defined operator A generates a holomorphic semigroup of angle $\alpha \in (0, \pi/2]$ if and only if for every $\alpha_1 \in [0, \alpha[$ there exist a constant $M \geqslant 0$ and $r \geqslant 0$ such that

$$S(\alpha_1 + \pi/2) \setminus B(r) \subset \varrho(A) \quad (\text{where} \quad B(r) = \{z \in \mathbb{C} \colon |z| \leqslant r\})$$

and

$$\|R(\lambda, A)\| \leqslant M/|\lambda| \quad \text{for all} \quad \lambda \in S(\alpha_1 + \pi/2) \setminus B(r).$$

This shows that the condition of the theorem is necessary. Conversely, assume that the condition holds. Since $\|R(\lambda, A)\| \to \infty$ when λ approaches $\sigma(A)$ (cf. Lemma 1.21 below), it follows that $\lambda \in \varrho(A)$ and $\|R(\lambda, A)\| \leqslant M/|\lambda|$ if $\mathrm{Re}\lambda = 0$ and $|\lambda| > r$ as well.

Let $c = 1/(2M)$. If $\xi, \eta \in \mathbb{R}$ satisfy $|\xi| \leqslant c|\eta|$, $|\eta| \geqslant r$, then

$$\|\xi R(\mathrm{i}\eta, A)\| \leqslant |\xi| \cdot M/|\eta| \leqslant c \cdot M = 1/2.$$

Hence $R(\xi + \mathrm{i}\eta, A) = \sum_{n=0}^{\infty} (-\xi)^n R(\mathrm{i}\eta, A)^{n+1}$ exists and

$$\begin{aligned}
\|R(\xi + \mathrm{i}\eta, A)\| &\leqslant (|\xi + \mathrm{i}\eta|)^{-1} \cdot |\xi + \mathrm{i}\eta| \cdot \sum_{n=0}^{\infty} |\xi|^n M^{n+1}/|\eta|^{n+1} \\
&\leqslant (|\xi + \mathrm{i}\eta|)^{-1} \cdot 2M(|\xi|^2 + |\eta|^2)^{1/2}/|\eta| \cdot \sum_{n=0}^{\infty} M^n c^n \\
&\leqslant (4M \cdot (c^2 + 1)^{1/2})/|\xi + \mathrm{i}\eta| \\
&\leqslant N/|\xi + \mathrm{i}\eta|\,.
\end{aligned}$$

This together with the assumption implies that there exist $N' \geqslant 0$ and $\alpha \in]0, \pi/2]$ such that $\lambda \in \varrho(A)$ and $\|R(\lambda, A)\| \leqslant N'/|\lambda|$ for all $\lambda \in S(\alpha + \pi/2)$. □

Compared with the Hille-Yosida theorem, Theorem 1.14 gives a very simple criterion for an operator to be the generator of a (holomorphic) semigroup. Merely the resolvent and not its powers have to be estimated. However, the resolvent has to be known in a right half-plane instead of a right half-line.

On the other hand, given a strongly continuous semigroup, merely an estimate on a vertical line implies that the semigroup is holomorphic. More precisely, the following holds.

Corollary *A strongly continuous semigroup with generator A is holomorphic if and only if there exist $w > \omega(A)$ and $M \geqslant 0$ such that one has*

$$\|R(w + \mathrm{i}\eta, A)\| \leqslant \frac{M}{|\eta|} \text{ for all } \quad \eta \in \mathbb{R}.$$

Proof In fact, assume that the condition holds. Since $A - w$ is the generator of a bounded semigroup, one has $\|R(\lambda, A - w)\| \leqslant N/\mathrm{Re}\lambda$ for some $N > 0$ and all $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}\lambda > 0$. Consequently, for every $\alpha \in (0, \pi/2)$,

$$\|R(\lambda, A - w)\| \leqslant (|\lambda|/\mathrm{Re}\lambda)N/|\lambda| \leqslant N(\cos\alpha)^{-1}/|\lambda| for all \lambda \in S(\alpha).$$

The remaining estimate for a sector symmectric to the imaginary axis is given by the proof of Theorem 1.14 and shows that $A - w$ generates a holomorphic semigroup. The reverse implication is clear. □

We now prove the following extension of Corollary 1.13

Theorem 1.15 *Let A be the generator of a strongly continuous group. Then A^2 generates a holomorphic semigroup of angle $\pi/2$.*

Proof There exists $w \geqslant 0$ such that $(\pm A - w)$ generates a bounded semigroup. Consequently, there exists $M \geqslant 0$ such that $\|R(\mu, \pm A - w)\| \leqslant M/\mathrm{Re}\mu$ whenever $\mathrm{Re}\mu > 0$.

Let $\alpha \in (0, \pi/2)$. There exist $r_0 \geqslant 0$ and $\beta \in (0, \pi/2)$ such that

$$S(\alpha + \pi/2) \setminus B(r_0) \subset \{z^2 : z \in S(\beta) + w\}.$$

In fact, the line $\{w + r(\cos\beta + \mathrm{i}\sin\beta) : r \geqslant 0\}$ can be parameterized by

$$z(t) = w + t + \mathrm{i} \cdot t/\varepsilon \quad (t \geqslant 0),$$

where $\varepsilon > 0$ depends on β. Then

$$z(t)^2 = (w + t)^2 - t^2/\varepsilon^2 + \mathrm{i}2t(w + t)/\varepsilon.$$

Thus $\lim_{t\to\infty} \mathrm{Im}z(t)^2/\mathrm{Re}z(t)^2 = 2\varepsilon/(\varepsilon^2 - 1)$. Choose $\beta \in (\pi/4, \pi/2)$ such that $\tan(\alpha + \pi/2) > 2\varepsilon/(\varepsilon^2 - 1)$.

Now let $\lambda \in S(\alpha + \pi/2) \setminus B(r_0)$. Then there exist $\vartheta \in (-\beta, \beta)$ and $r \geqslant 0$ such that $\lambda = (re^{\mathrm{i}\vartheta} + \omega)^2$. Thus $(\lambda - A^2) = (re^{\mathrm{i}\vartheta} + w - A)(re^{\mathrm{i}\vartheta} + w + A)$. Hence $\lambda \in \varrho(A^2)$ and $R(\lambda, A^2) = R(re^{\mathrm{i}\vartheta}, A - w)R(re^{\mathrm{i}\vartheta}, -A - w)$. We conclude that

$$|\lambda| \cdot \|R(\lambda, A^2)\| \leqslant |\lambda| \cdot M^2/(\cos\vartheta)^2 r^2 \leqslant (|\lambda|/r^2) \cdot M^2/(\cos\beta)^2.$$

Thus $|\lambda| \cdot \|R(\lambda, A^2)\|$ is uniformly bounded for $\lambda \in S(\alpha + \pi/2) \setminus B(r_0)$. □

Remark In Theorem 1.15 the assumption that $\pm A$ are generators can be relaxed. In fact, the proof shows the following. If A is a densely defined operator such that $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > 0\} \subset \varrho(\pm A - w)$ and $\|R(\lambda, \pm A - w)\| \leqslant M/\operatorname{Re}\lambda$ for some $M \geqslant 0$, $w \geqslant 0$, then A^2 generates a holomorphic semigroup of angle $\pi/2$.

1.5 Differentiable semigroups

Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator A. Let $t_0 \geqslant 0$ and $f \in E$. Then the function $t \mapsto T(t)f$ is right sided differentiable at t_0 if and only if $T(t_0)f \in D(A)$; and in that case it is differentiable at every $s > t_0$ and the derivative is $AT(s)f$ (this follows from A-I, Proposition 1.6(ii)).

Definition 1.16 A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E is called *eventually differentiable* if there exists $t_0 \geqslant 0$ such that the function $t \mapsto T(t)f$ from (t_0, ∞) into E is differentiable for every $f \in E$. The semigroup is called *differentiable* if t_0 can be chosen 0.

It is not difficult to see that if $(T(t))_{t\geqslant 0}$ is differentiable for $t > t_0$, then for $n \in \mathbb{N}$ it is n-times differentiable at all $s > nt_0$ and $T(s)E \subset D(A^n)$. If $(T(t))_{t\geqslant 0}$ is differentiable, then the function $t \mapsto T(t)f$ from $(0, \infty)$ into E is infinitely often differentiable for every $f \in E$.

Generators of (eventually) differentiable semigroups can be characterized by the spectral behavior of the resolvent, in a similar way as it has been done for holomorphic semigroups in the last section. Whereas the spectrum of the generator of a holomorphic semigroup is included in a sector, the spectrum of the generator of an eventually differentiable semigroup is limited by a function of exponential growth (instead of a line).

Theorem 1.17 *A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is eventually differentiable if and only if its generator A satisfies the following.*
There exist constants $c > 0$, $b > 0$, $M > 0$ such that

$$\Sigma := \{\lambda \in \mathbb{C}\colon c\mathrm{e}^{-b\cdot\operatorname{Re}\lambda} \leqslant |\operatorname{Im}\lambda|\} \subset \varrho(A)$$

and

$$\|R(\lambda, A)\| \leqslant M \cdot |\operatorname{Im}\lambda| \textit{ for all } \lambda \in \Sigma \textit{ satisfying } \operatorname{Re}\lambda \leqslant \omega_0(A).$$

Theorem 1.18 *A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is differentiable if and only if its generator A satisfies the following.*
For all $b > 0$ there exist $c > 0$, $M > 0$ such that

$$\Sigma := \{\lambda \in \mathbb{C}\colon c\mathrm{e}^{-b\cdot\operatorname{Re}\lambda} \leqslant |\operatorname{Im}\lambda|\} \subset \varrho(A)$$

and

$$\|R(\lambda, A)\| \leqslant M \cdot |\operatorname{Im}\lambda| \text{ for all } \lambda \in \Sigma \text{ satisfying } \operatorname{Re}\lambda \leqslant \omega_0(A).$$

For the proofs of these two theorems we refer to Pazy [31, Chapter 3, Theorem 4.7 and 4.8].

1.6 Norm continuous semigroups

Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup and $t' > 0$. If $\lim_{t\to t'} \|T(t) - T(t')\| = 0$, then it follows from the semigroup property, that the function $t \mapsto T(t)$ is norm continuous on the whole half line (t', ∞).

Definition 1.19 A semigroup $(T(t))_{t\geqslant 0}$ is called *eventually norm continuous* if there exists $t' \geqslant 0$ such that the function $t \mapsto T(t)$ from (t', ∞) into $\mathcal{L}(E)$ is norm continuous. The semigroup is called *norm continuous* if t' can be chosen equal to 0.

The spectrum of generators of eventually norm continuous semigroups still is compact in every right half-plane.

Theorem 1.20 *Let A be the generator of an eventually norm continuous semigroup. Then for every $b \in \mathbb{R}$ the set*

$$\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geqslant b\}$$

is bounded.

For the proof of Theorem 1.20 we use the following lemma.

Lemma 1.21 *Let A be an operator and $\lambda \in \varrho(A)$. Then*

$$dist(\lambda, \sigma(A)) = r(R(\lambda, A))^{-1}.$$

Proof One has $\{0\} \cup \sigma(R(\lambda, A)) = \{0\} \cup \{(\lambda - \mu)^{-1} : \mu \in \sigma(A)\}$ (Davies [11, Lemma 2.11]). Hence

$$r(R(\lambda, A)) = \sup\{|\lambda-\mu|^{-1} : \mu \in \sigma(A)\} = (\inf\{|\lambda-\mu| : \mu \in \sigma(A)\})^{-1} = \operatorname{dist}(\lambda, \sigma(A))^{-1}.$$

Proof (Proof of Theorem 1.20) It is enough to show the following. Let $a > \omega(A)$. Then for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $r_0 \geqslant 0$ such that

$$\|R(a + ir, A)^n\|^{1/n} < \varepsilon \quad \text{for all } r \in \mathbb{R} \text{ satisfying } |r| \geqslant r_0.$$

In fact, then we have by the lemma,

$$\operatorname{dist}(a+ir,\sigma(A)) = r(R(a+ir,A))^{-1} \geqslant \varepsilon^{-1} \quad \text{whenever} \quad |r| \geqslant r_0.$$

So let $\varepsilon > 0$. If $\mathrm{Re}\lambda > \omega(A)$, then by A-I, Proposition 1.11,

$$R(\lambda,A)^{n+1} = \frac{1}{n!}\int_0^\infty \mathrm{e}^{-\lambda t} t^n T(t)\,\mathrm{d}t (n \in \mathbb{N}).$$

Let $t' > 0$ such that $t \mapsto T(t)$ is norm continuous on (t',∞). Let $w \in (\omega(A),a)$. There exists $M \geqslant 1$ such that $\|T(t)\| \leqslant M\mathrm{e}^{wt}$ for all $t \geqslant 0$. Let

$$N := M \cdot \int_0^{t'} \mathrm{e}^{-at}\mathrm{e}^{wt}\,\mathrm{d}t.$$

Since $\lim_{n\to\infty} c^n/n! = 0$ for all $c > 0$, there exists $n \in \mathbb{N}$ such that $N \cdot (t')^n/n! < \varepsilon^{n+1}/3$. Choose $T \geqslant t'$ such that $\frac{1}{n!}\int_T^\infty t^n \mathrm{e}^{-at}\|T(t)\|\,\mathrm{d}t < \varepsilon^{n+1}/3$.

Since $(T(t))_{t\geqslant 0}$ is norm continuous for $t \geqslant t'$, it follows from the Riemann-Lebesgue lemma that there exists $r_0 \geqslant 0$ such that $\|\frac{1}{n!}\int_{t'}^T t^n \mathrm{e}^{-irt}\mathrm{e}^{-at}T(t)\,\mathrm{d}t\| < \varepsilon^{n+1}/3$ whenever $|r| \geqslant r_0$.

All together we obtain for $|r| \geqslant r_0$,

$$\begin{aligned}
\|R(a+ir,A)^{n+1}\| &= \frac{1}{n!} \cdot \|\int_0^\infty \mathrm{e}^{-(a+ir)t} t^n T(t)\,\mathrm{d}t\| \\
&\leqslant \frac{1}{n!} \cdot \int_0^{t'} \mathrm{e}^{-at} t^n \|T(t)\|\,\mathrm{d}t \\
&+ \frac{1}{n!} \cdot \|\int_{t'}^T t^n \mathrm{e}^{-irt}\mathrm{e}^{-at} T(t)\,\mathrm{d}t\| \\
&+ \frac{1}{n!} \cdot \int_T^\infty \mathrm{e}^{-at} t^n \|T(t)\|\,\mathrm{d}t \\
&\leqslant \frac{1}{n!} \cdot (t')^n \int_0^{t'} \mathrm{e}^{-at} M \mathrm{e}^{wt}\,\mathrm{d}t + \frac{2}{3} \cdot \varepsilon^{n+1} \\
&\leqslant N \cdot (t')^n/n! + \frac{2}{3} \cdot \varepsilon^{n+1} \\
&\leqslant \varepsilon^{n+1}.
\end{aligned}$$

A complete characterization of eventually norm continuous semigroups in terms of their generator seems not to be known.

Eventually norm continuous semigroups are of particular interest in spectral theory (cf. A-III, Theorem 6.6). Moreover their asymptotic behavior is easy to describe (see A-IV, (1.8)).

Next we describe special norm continuous semigroups.

1.7 Compact semigroups

Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup and $t_0 > 0$. If $T(t_0)$ is compact, then it follows from the semigroup property that $T(t)$ is compact for all $t \geqslant t_0$. Moreover, $t \mapsto T(t)$ is norm continuous at every $t > t_0$.

In fact, since $T(h) \to \mathrm{Id}$ strongly with $h \downarrow 0$, it follows that $\lim_{h\downarrow 0} T(h)f = f$ uniformly on every compact subset K of E. Now let $t \geqslant t_0$. Then $K = T(t)(U)$ is compact (where U denotes the unit ball of E). Hence $\lim\limits_{h\downarrow 0} T(h+t)f = \lim\limits_{h\downarrow 0} T(h)T(t)f$ uniformly for $f \in U$. So the semigroup is right-sided norm continuous on $[t_0, \infty)$ and so norm continuous on (t_0, ∞).

Definition 1.22 A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is called *compact* if $T(t)$ is compact for all $t > 0$. It is called *eventually compact* if there exists $t_0 > 0$ such that $T(t_0)$ is compact (and hence $T(t)$ is compact for all $t \geqslant t_0$).

We want to find a relation between the compactness of the semigroup and the compactness of the resolvent of its generator.

Definition 1.23 Let A be an operator and $\varrho(A) \neq \emptyset$. We say, A has a compact resolvent if $R(\lambda, A)$ is compact for one (and hence all) $\lambda \in \varrho(A)$.

Proposition 1.24 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup and assume that its generator has a compact resolvent. If $t \mapsto T(t)$ is norm continuous at t_0, then $T(t)$ is compact for all $t \geqslant t_0$.*

Proof Considering $(e^{-wt}T(t))_{t\geqslant 0}$ for some $w > 0$ if necessary, we can assume that $\sigma(A) < 0$. Let $S(t) \in \mathcal{L}(E)$ be given by $S(t)f = \int_0^t T(s)f\,\mathrm{d}s$ $(t \geqslant 0)$. Then $AS(t)f = T(t)f - f$ for all $f \in E$, and so $S(t) = R(0, A)(Id - T(t))$ is compact for all $t \geqslant 0$.

Since $t \mapsto T(t)$ is norm continuous for $t \geqslant t_0$, one has $\lim_{h\downarrow 0} \frac{1}{h}(S(t_0+h)-S(t_0)) = T(t_0)$ in the operator norm. Thus $T(t_0)$ is compact as limit of compact operators. □

Theorem 1.25 *A strongly continuous semigroup $(T(t))_{t\geqslant 0}$ is compact if and only if it is norm continuous and its generator A has compact resolvent.*

Proof Assume that $(T(t))_{t\geqslant 0}$ is compact. Then $T(\cdot)$ is norm continuous on $(0, \infty)$, and so

$$\int_0^t e^{-ws}T(s)\,\mathrm{d}s$$

is compact as the norm limit of linear combinations of compact operators, whenever $w > \omega_0(A)$. Since

$$R(w, A) = \lim_{t\to\infty} \int_0^t e^{-ws}T(s)\,\mathrm{d}s$$

in the operator norm, it follows that $R(w, A)$ is compact. This proves one implication. The other follows from Proposition 1.24 . □

Remark 1.26
(i) Generators of eventually compact semigroups do not necessarily have compact resolvent. Consider the nilpotent translation semigroup $(T(t))_{t\geqslant 0}$ on $F := L^1([0,1])$ (see A-I, Example 2.6). Let $E = F\widetilde{\otimes}_\pi F = L^1([0,1]\times[0,1])$ and $S(t) = T(t)\otimes \mathrm{Id}$ $(t \geqslant 0)$. Then $(S(t))_{t\geqslant 0}$ is a strongly continuous semigroup (see A-I, 3.7), with B as its generator. Clearly $(S(t))_{t\geqslant 0}$ is a nilpotent semigroup (so it is eventually compact), but $R(\lambda, B) = R(\lambda, A)\otimes \mathrm{Id}$ is not compact.

(ii) It is obvious that a group $(T(t))_{t\in\mathbb{R}}$ is eventually norm continuous if and only if it is norm continuous at 0, i.e., its generator is bounded.

On the other hand, the generator of the rotation group (A-I, Example 2.5) has a compact resolvent. Hence this condition does not imply any smoothness property of the semigroup.

Positive eventually compact semigroups have remarkable properties in the setting of the Perron-Frobenius theory (see e.g., B-III, Corollary 2.12).

The following scheme indicates the relation between the different classes of semigroups defined so far.

$$
\begin{tikzcd}
\text{holomorphic} \arrow[r] & \text{differentiable} \arrow[r] \arrow[d] & \text{eventually differentiable} \arrow[d] \\
 & \text{norm continuous} \arrow[r] & \text{eventually norm continuous} \\
 & \text{compact} \arrow[r] \arrow[u] & \text{eventually compact} \arrow[u]
\end{tikzcd}
$$

All these classes are different. This is shown by the following examples.

Example 1.27 The nilpotent shift semigroup (A-I, 2.6) is obviously eventually differentiable, eventually compact and eventually norm continuous. But it is not norm continuous and consequently not differentiable or compact.

Example 1.28 We consider multiplication semigroups (see A-I, 2.3). Let $E = C_0(X)$, where X is a locally compact space, or $E = L^p(X, \Sigma, \mu)$, where $1 \leqslant p < \infty$ and (X, Σ, μ) is a σ-finite measure space. Let $m\colon X \to \mathbb{R}$ be continuous [resp., measurable] such that [ess]- $\sup_{x\in X} \mathrm{Re}(m(x)) < \infty$.

Then $Af = m\cdot f$ with domain $D(A) = \{f \in E : m\cdot f \in E\}$ is the generator of the semigroup $(T(t))_{t\geqslant 0}$ given by

$$(T(t)f)(x) = \mathrm{e}^{tm(x)}f(x), \quad (t \geqslant 0, x \in X, f \in E)\,.$$

Observe that $\sigma(A) = \overline{m(X)}$ in case $E = C_0(X)$ and

$\sigma(A) = [\text{ess}]\text{-image}(m) := \{\lambda \in \mathbb{C} : \mu(\{x \in X : |m(x) - \lambda| < \varepsilon\}) \neq 0 \text{ for all } \varepsilon > 0\}$

if $E = L^p$ (see A-II, 2.3). Consequently, $s(A) = \omega(A) = [\text{ess}]\text{-}\sup_{x\in X} \operatorname{Re}(m(x))$.

(i) The semigroup is norm continuous for $t > 0$ if and only if it is eventually norm continuous if and only if $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geqslant b\}$ is bounded for every $b \in \mathbb{R}$. Thus the property proved in Theorem 1.20 characterizes generators of eventually norm continuous semigroups in the case that the semigroup consists of multiplication operators.

Proof Assume that $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geqslant b\}$ is bounded for every $b \in \mathbb{R}$. Let $t' > 0$. We show that the semigroup is norm continuous at t'. Take $\varepsilon > 0$ and $b \in \mathbb{R}$ such that $2\mathrm{e}^{(t'+1)b} < \varepsilon$.

If $\operatorname{Re}(m(x)) \leqslant b$, then

$$|\mathrm{e}^{tm(x)} - \mathrm{e}^{t'm(x)}| \leqslant \mathrm{e}^{t\operatorname{Re}(m(x))} + \mathrm{e}^{t'\operatorname{Re}(m(x))} \leqslant 2\mathrm{e}^{(t'+1)b} < \varepsilon$$

whenever $|t - t'| \leqslant 1$.

By hypothesis, the set $H := \{m(x) : x \in X, \operatorname{Re}(m(x)) \geqslant b\}$ in the case $E = C_0(X)$ and $H := \{m(x) : \operatorname{Re}\lambda \geqslant b$ and for all $\eta > 0, \mu(\{x \in X : |m(x) - \lambda| < \eta\}) \neq 0\}$ in the case $E = L^p$ is a bounded subset of $\mathbb{C}$. Thus $\lim_{t\to t'} |\mathrm{e}^{tz} - \mathrm{e}^{t'z}| = 0$ uniformly for $z \in H$. Hence there exists $\delta \in]0, 1]$ such that

$$[\text{ess}]\text{-}\sup\{|\mathrm{e}^{tm(x)} - \mathrm{e}^{t'm(x)}| : x \in X, \operatorname{Re}(m(x)) > b\} < \varepsilon$$

whenever $|t - t'| < \delta$. Together with the inequality above, we obtain that

$$\|T(t) - T(t')\| = [\text{ess}]\text{-}\sup\{|\mathrm{e}^{tm(x)} - \mathrm{e}^{t'm(x)}| : x \in X\} < \varepsilon$$

whenever $|t - t'| < \delta$. We have shown that the semigroup is norm continuous for $t > 0$ whenever $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geqslant b\}$ is bounded for all $b \in \mathbb{R}$. □

(ii) The semigroup is right-sided differentiable at a point $t > 0$ if and only if there exists $c > 0$ such that $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| > c \cdot \mathrm{e}^{-t\operatorname{Re}\lambda}\} \subset \varrho(A)$.

Proof The semigroup is right-sided differentiable at t if and only if $T(t)E \subset D(A)$ if and only if $\mathrm{e}^{tm} \cdot f \cdot m \in E$ for all $f \in E$ if and only if $\mathrm{e}^{tm} \cdot m$ is [essentially] bounded if and only if $\mathrm{e}^{t\operatorname{Re} m} \cdot \operatorname{Im} m$ is [essentially] bounded if and only if there exists $c > 0$ such that $[\text{ess}]\text{-image}(m) \subset \{\lambda \in \mathbb{C} : \mathrm{e}^{t\operatorname{Re}\lambda}|\operatorname{Im}\lambda| \leqslant c\}$ if and only if there exists $c > 0$ such that $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| > c \cdot \mathrm{e}^{-t\operatorname{Re}\lambda}\} \subset \varrho(A)$. □

(iii) $(T(t))_{t\geqslant 0}$ is a bounded holomorphic semigroup of angle ϑ if and only if $S(\vartheta + \pi/2) \subset \varrho(A)$.

Proof The condition is necessary by Theorem 1.12. Conversely, if $S(\vartheta+\pi/2) \subset \varrho(A)$, then one verifies directly that

$$(T(z)f)(x) = \mathrm{e}^{z\cdot m(x)} f(x) \quad (f \in E, x \in X)$$

defines a family $(T(z))_{z\in S(\vartheta)}$ of bounded operators satisfying conditions (1.4) and (1.5). $\square$

(iv) Choosing $X = \mathbb{N}$ and the counting measure we have $E = c_0$ or ℓ^p. Then A has a compact resolvent if $\lim_{n\to\infty} |m(n)| = \infty$.

In fact, let $\lambda > s(A)$. Then $(R(\lambda, A)f)(n) = (\lambda - m(n))^{-1} f(n)$. Hence $R(\lambda, A)$ is compact if and only if $((\lambda - m(n))^{-1})_{n\in\mathbb{N}} \in c_0$.

The semigroup is compact if and only if it is eventually compact if and only if

$$\lim_{n\to\infty} \mathrm{Re}(m(n)) = -\infty.$$

(v) Now it is easy to give concrete examples. Again let $X = \mathbb{N}$, so that $E = c_0$ or ℓ^p. Let $m(n) = -n + i \cdot \exp(n^2)$. Then the semigroup is compact and (consequently) norm continuous for $t > 0$, but it is not eventually differentiable. Let $m(n) = -n + i\mathrm{e}^{t'n}$. Then the semigroup is differentiable for $t > t'$ but not differentiable at $t \in [0, t')$. If $m(n) = -n + i \cdot n^2$, then the semigroup is differentiable but not holomorphic.

1.8 Perturbation of Generators

A useful way to construct new semigroups out of a given one is by additive perturbation.

Theorem 1.29 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ and let $B \in \mathcal{L}(E)$. Then $A + B$ with domain $D(A + B) = D(A)$ is the generator of a strongly continuous semigroup $(S(t))_{t\geqslant 0}$.*

It is possible to express the new semigroup $(S(t))_{t\geqslant 0}$ by known objects. The product formula

$$S(t)f = \lim_{n\to\infty} (T(t/n)\mathrm{e}^{t/n\cdot B})^n f \tag{1.8}$$

holds for all $t \geqslant 0$ and $f \in E$.

Moreover, $S(t)$ is the solution of the following integral equation

$$S(t)f = T(t)f + \int_0^t T(t-s)BS(s)f \,\mathrm{d}s \quad (t \geqslant 0, f \in E). \tag{1.9}$$

Let $S_0(t) = T(t)$ and

$$S_n(t)f = \int_0^t T(t-s)BS_{n-1}(s)f \,\mathrm{d}s \quad (f \in E) \tag{1.10}$$

for $n \in \mathbb{N}$. Then

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \tag{1.11}$$

where the series converges in the operator norm uniformly on bounded intervals. We refer to Davies [11, III.1], Goldstein [15, I.6] or Pazy [31, Chapter 3] for these results.

Several special properties discussed above are preserved by bounded perturbations.

Theorem 1.30 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator A, and let $B \in \mathcal{L}(E)$.*

(i) *If $(T(t))_{t\geqslant 0}$ is holomorphic or norm continuous or compact, then so is the semigroup $(S(t))_{t\geqslant 0}$ generated by $A + B$.*

(ii) *If A has a compact resolvent then so has $A + B$.*

(iii) *Let $t_0 \geqslant 0$. If $(T(t))_{t\geqslant 0}$ is norm continuous for $t > t_0$ and if B is compact, then $(S(t))_{t\geqslant 0}$ is also norm continuous for $t > t_0$.*

Proof If $(T(t))_{t\geqslant 0}$ is norm continuous for $t > 0$, then $S_n(t)$ in (1.10) is norm continuous at $t > 0$ for every n. Thus $(S(t))_{t\geqslant 0}$ is norm continuous at $t > 0$ by (1.11). There exists $\lambda_0 \in \mathbb{R}$ such that $\|R(\lambda, A)\| \leqslant (2\|B\|)^{-1}$ for $\mathrm{Re}\lambda \geqslant \lambda_0$. Hence $(Id - BR(\lambda, A))^{-1}$ exists for $\mathrm{Re}\lambda \geqslant \lambda_0$. Since

$$(\lambda - (A + B))f = (Id - BR(\lambda, A))(\lambda - A)f \quad \text{ for all } \quad f \in D(A),$$

it follows that $(\lambda - (A + B))^{-1}$ exists and is given by

$$R(\lambda, A + B) = R(\lambda, A)(Id - BR(\lambda, A))^{-1} \tag{1.12}$$

whenever $\mathrm{Re}\lambda \geqslant \lambda_0$. Now if A generates a holomorphic semigroup, there exists $M \geqslant 0$ such that $\|R(\lambda_0 + \mathrm{i}\eta, A)\| \leqslant M/|\eta|$ for all $\eta \in \mathbb{R}$. Consequently,

$$\|R(\lambda_0 + \mathrm{i}\eta, A + B)\| \leqslant \|(Id - BR(\lambda_0 + i\eta, A))^{-1}\| \cdot 2M/|\eta| \leqslant 2M/|\eta|$$

for all $\eta \in \mathbb{R}$. Thus $A + B$ generates a holomorphic semigroup by the corollary of Theorem 1.14. Moreover, it follows from (1.12) that $R(\lambda, A + B)$ is compact whenever $R(\lambda, A)$ is compact. Consequently, by Theorem 1.25 and the assertion proved above, $(S(t))_{t\geqslant 0}$ is compact whenever $(T(t))_{t\geqslant 0}$ is compact.

Finally assume that B is compact and $t_0 \geqslant 0$ such that $(T(t))_{t\geqslant 0}$ is norm continuous for $t > t_0$. Fix $t > t_0$. Denote by U the unit ball of E and fix $s \in (0, t]$. Then

$$\lim_{h\to 0} (T(t + s - h) - T(t - s))f = 0$$

for all $f \in \overline{BS(s)U} =: K$.

Since K is compact it follows that the limit exists uniformly with respect to $f \in K$, i.e., $\lim_{h\to 0} \|(T(t + s - h) - T(t - s))BS(s)\| = 0$. It follows from the dominated convergence theorem that

$$\lim_{h\to 0}\int_0^t \|(T(t+s-h)-T(t-s))BS(s)\|\ \mathrm{d}s = 0. \tag{1.13}$$

Using (1.9) we obtain

$$\begin{aligned}\|S(t+h)-S(t)\| &\leqslant \|T(t+h)-T(t)\| \\ &\qquad + \left\|\int_0^{t+h} T(t+h-s)BS(s)\ \mathrm{d}s - \int_0^t (T(t-s)BS(s)\ \mathrm{d}s\right\| \\ &\leqslant \|T(t+h)-T(t)\| + \int_t^{t+h} \|T(t+h-s)BS(s)\|\,\mathrm{d}s \\ &\qquad + \int_0^t \|(T(t+h-s)-T(t-s))BS(s)\|\ \mathrm{d}s \to 0\end{aligned}$$

when $h\to 0$. □

In C-IV, Example 2.15 a generator A of an eventually differentiable and eventually compact semigroup and a bounded operator B will be given such that the semigroup generated by $A+B$ is not eventually norm continuous.

Using Theorem 1.29 we now prove a perturbation result due to Desch and Schappacher [12]. Instead of assuming that $B\in\mathcal{L}(E)$ we assume that $B\in\mathcal{L}(D(A))$. The short proof given below is due to G. Greiner.

Theorem 1.31 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator A. Assume that $B\colon D(A)\to D(A)$ is linear and continuous for the graph norm on $D(A)$. Then $A+B$ with domain $D(A+B)=D(A)$ is the generator of a strongly continuous semigroup. Moreover, there exists a bounded operator C on E such that $A+B$ is similar to $A+C$.*

Proof We first show that $(Id-BR(\lambda,A))$ is invertible for some $\lambda\in\mathbb{C}$. Choose $\lambda_0\in\varrho(A)$. Then $S:=(\lambda_0-A)BR(\lambda_0,A)\in\mathcal{L}(E)$. Let $\lambda>s(A)$ be so large such that $\|SR(\lambda,A)\|<1$. Then

$$1-(\lambda_0-A)BR(\lambda_0,A)R(\lambda,A)=(1-SR(\lambda,A)$$

is invertible. Consequently, also $(1-BR(\lambda,A))^{-1})$ exists (since

$$\sigma(TR(\lambda_0,A))\setminus\{0\}=\sigma(R(\lambda_0,A)T)\setminus\{0\}$$

where $T=(\lambda_0-A)BR(\lambda,A)$).

Let $C=(A-\lambda)B(A-\lambda)^{-1}\in\mathcal{L}(E)$. Then $A+C$ is the generator of a strongly continuous semigroup by Theorem 1.29 . We show that $A+B$ is similar to $A+C$. In fact, let $U=(1-BR(\lambda,A))$. Then U is an isomorphism on E such that $U(D(A))=D(A)$. Moreover,

$$
\begin{aligned}
U(A+C)U^{-1} &= U(A-\lambda+C)U^{-1}+\lambda \\
&= U[(A-\lambda)-(A-\lambda)BR(\lambda,A)]U^{-1}+\lambda \\
&= U(A-\lambda)[1-BR(\lambda,A)]U^{-1}+\lambda \\
&= U(A-\lambda)+\lambda \\
&= A-\lambda+B+\lambda \\
&= A+B.
\end{aligned}
$$

Corollary 1.32 *Keeping the hypotheses and notations of Theorem 1.31 denote by $(S(t))_{t\geqslant 0}$ the semigroup generated by $A+B$. If $(T(t))_{t\geqslant 0}$ is norm continuous or compact or holomorphic, then $(S(t))_{t\geqslant 0}$ has the corresponding properties. If B is compact as an operator on $D(A)$ endowed with the graph norm and if $(T(t))_{t\geqslant 0}$ is eventually norm continuous, then so is $(S(t))_{t\geqslant 0}$.*

Proof This follows from Theorem 1.30 since $(US(t)U^{-1})_{t\geqslant 0}$ has $A+C$ as generator. □

1.9 Domains of Uniqueness

Given a semigroup $(T(t))_{t\geqslant 0}$, it is frequently difficult to determine the precise domain of its generator A. So it is important to know which (possibly strict) subspaces of $D(A)$ determine the semigroup uniquely. This can be formulated more precisely in the following way.

Let D_0 be a subspace of $D(A)$ and consider the restriction A_0 of A to D_0. Under which condition on D_0 is A the only extension of A_0 which is a generator? One obvious condition is that D_0 is a core. [In fact, in that case, A is the closure of A_0. Since every generator B extending A_0 is closed, it follows that $A \subset B$ and hence $A = B$ since $\varrho(A) \cap \varrho(B) \neq \emptyset$].

We now show that cores are the only domains of uniqueness.

Theorem 1.33 *Let A be the generator of a semigroup and D_0 a subspace of $D(A)$. Consider the restriction A_0 of A to D_0. If D_0 is not a core of A, then there exists an infinite number of extensions of A_0 which are generators.*

Proof If D_0 is not dense in $D(A)$ with respect to the graph norm, then there exists a non-zero linear form φ on $D(A)$ which is continuous for the graph norm such that $\varphi(f) = 0$ for all $f \in D_0$. Let $u \in D(A)$ and $B\colon D(A) \to D(A)$ be given by $Bf = \varphi(f)u$ for all $f \in D(A)$. Then B is continuous for the graph norm. So by Theorem 1.31 the operator $A + B$ with domain $D(A)$ is a generator. Clearly, $A + B \neq A$ if $u \neq 0$ but $Af + Bf = Af$ for all $f \in D_0$. It is obvious that an infinite number of generators can be constructed in that way. □

Corollary 1.34 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator A. Let D_0 be a dense subspace of E. Assume that $D_0 \subset D(A)$ and $T(t)D_0 \subset D_0$ for all $t \geqslant 0$. Then D_0 is a core.*

Proof Let $(S(t))_{t\geqslant 0}$ be a semigroup with generator B such that $B|_{D_0} = A|_{D_0}$, and take $f \in D_0$. Then $u(t) := T(t)f$ satisfies

$$u(0) = f \text{ and } \dot{u}(t) = AT(t)f = BT(t)f = Bu(t) \quad (t \geqslant 0).$$

Since $v(t) = S(t)f$ $(t \geqslant 0)$ also is a solution of the Cauchy problem defined by B with initial value f it follows that $S(t)f = T(t)f$ $(t \geqslant 0)$. Since D_0 is dense in E, it follows that $S(t) = T(t)$ $(t \geqslant 0)$. □

2 Contraction Semigroups and Dissipative Operators

by Wolfgang Arendt

The Hille-Yosida theorem gives a characterization of generators in terms of the resolvent of the operator. However, given an operator A, frequently it is difficult to compute the resolvent (and its powers). So it is desirable to find conditions more immanent on A. This is possible for generators of contraction semigroups.

For later purposes (see B-II and C-II) it will be useful not only to consider semigroups which are contractive with respect to the norm but to consider more general sublinear functionals than the norm as well.

So our setting is the following. By E we denote a real Banach space throughout and $p\colon E \to \mathbb{R}$ is a continuous sublinear function; i.e., p satisfies

$$p(f+g) \leqslant p(f) + p(g) \quad (f, g \in E), \tag{2.1}$$
$$p(\lambda f) = \lambda p(f) \quad (f \in E, \lambda \geqslant 0). \tag{2.2}$$

The continuity of p implies that there exists a constant $c > 0$ such that

$$|p(f)| \leqslant c\|f\| \quad (f \in E). \tag{2.3}$$

Moreover, it follows from (2.1) and (2.2) that

$$p(f) + p(-f) \geqslant p(0) = 0 \quad (f \in E). \tag{2.4}$$

A bounded operator T on E is called *p-contractive* if $p(Tf) \leqslant p(f)$ for all $f \in E$. Similarly, a semigroup $(T(t))_{t\geqslant 0}$ is called *p-contractive* if $T(t)$ is p-contractive for all $t \geqslant 0$. Of course, the most important case in this section is when p is the norm function N given by $N(f) = \|f\|$ $(f \in E)$. An N-contractive operator is just a contraction in the usual sense.

Remark However in Chapter B-II and C-II it will be important to dispose of a variety of sublinear functionals other than N. For example, we will consider N^+ on $C[0,1]$ given by $N^+(f) = \sup_{x\in[0,1]} f(x)$. Then a bounded operator T is N^+-contractive if and only if T is positive and $\|T\| \leqslant 1$.

We first want to solve the following problem. Given the generator A of a semigroup $(T(t))_{t\geqslant 0}$ find a condition on A which is equivalent to $T(t)$ being p-contractive for all $t \geqslant 0$.

The *subdifferential* dp of p at f is defined by

$$\mathrm{d}p(f) = \{\varphi \in E' : \langle g, \varphi\rangle \leqslant p(g) \text{ for all } g \in E, \langle f, \varphi\rangle = p(f)\}. \tag{2.5}$$

It follows from the Hahn-Banach theorem that $\mathrm{d}p(f) \neq \emptyset$ for all $f \in E$.

Definition 2.1 An operator A on E is called *p-dissipative* if for all $f \in D(A)$ there exists $\varphi \in \mathrm{d}p(f)$ such that $\langle Af, \varphi\rangle \leqslant 0$ and A is called *strictly p-dissipative* if for all $f \in D(A)$ the inequality $\langle Af, \varphi\rangle \leqslant 0$ holds for all $\varphi \in \mathrm{d}p(f)$.

For convenience we want to have a distinctive name for the norm function. So we denote by $N\colon E \to \mathbb{R}$ the function given by $N(f) = \|f\|$ throughout. Then (2.5) can be written in the form

$$\mathrm{d}N(f) = \{\varphi \in E' : \|\varphi\| \leqslant 1, \langle f, \varphi\rangle = \|f\|\}. \tag{2.6}$$

A (strictly) N-dissipative operator is simply called (strictly) *dissipative*, which is in accordance with the usual nomenclature.

Examples 2.2

(i) Let $E = C[0,1]$, $f \in E$. Then there exists $x \in [0,1]$ such that $|f(x)| = \|f\|_\infty$. Define $\varphi \in E'$ by $\langle g, \varphi\rangle = (\operatorname{sign} f(x))g(x)$. Then $\varphi \in \mathrm{d}N(f)$. Note that $\mathrm{d}N(f)$ may be an infinite set.

(ii) Let H be a Hilbert space, $f \in H$, $f \neq 0$. Then $\mathrm{d}N(f) = \{\varphi_f\}$ where $\langle g, \varphi_f\rangle = 1/\|f\|\,(g|f)$.

(iii) $A - \|A\|\,\mathrm{Id}$ is strictly dissipative for every bounded operator A.

Proposition 2.3 *Let A be an operator on E. Then A is p-dissipative if and only if*

$$p(f) \leqslant p(f - tAf) \text{ for all } f \in D(A), t > 0. \tag{2.7}$$

If in particular $(w, \infty) \subset \varrho(A)$ for some $w \in \mathbb{R}$, then A is p-dissipative if and only if

$$p(\lambda R(\lambda, A)f) \leqslant p(f) \text{ for all } f \in E, \lambda > w. \tag{2.8}$$

Proof Assume that A is p-dissipative. Let $f \in D(A)$, $t > 0$. There exists $\varphi \in \mathrm{d}p(f)$ such that $\langle Af, \varphi\rangle \leqslant 0$. Hence,

$$p(f) = \langle f, \varphi\rangle = \langle f - tAf + tAf, \varphi\rangle \leqslant \langle f - tAf, \varphi\rangle \leqslant p(f - tAf).$$

So (2.7) holds.

Conversely, let $f \in D(A)$. For every $t > 0$ choose $\varphi_t \in \mathrm{d}p(f - tAf)$. Then $\pm\langle g, \varphi_t\rangle \leqslant p(\pm g) \leqslant c\|g\|$ for all $g \in E$, $t > 0$. Thus the net $(\varphi_t)_{t>0}$ is bounded. Consequently it posseses a $\sigma(E', E)$-limit point φ as $t \to 0$. We show that $\varphi \in \mathrm{d}p(f)$ and $\langle Af, \varphi\rangle \leqslant 0$.

Since $\langle g, \varphi_t\rangle \leqslant p(g)$ for all $t > 0$, it follows that $\langle g, \varphi\rangle \leqslant p(g)$ $(g \in E)$. Moreover, $\langle f, \varphi_t\rangle - t\langle Af, \varphi_t\rangle = p(f - tAf)$ $(t > 0)$. Letting $t \to 0$ yields $\langle f, \varphi\rangle = p(f)$.

We have proved that $\varphi \in \mathrm{d}p(f)$. By hypothesis we have for all $t > 0$,
$p(f) \leqslant p(f - tAf) = \langle f - tAf, \varphi_t\rangle = \langle f, \varphi_t\rangle - t\langle Af, \varphi_t\rangle \leqslant p(f) - t\langle Af, \varphi_t\rangle$.

Consequently $\langle Af, \varphi_t\rangle \leqslant 0$ for all $t > 0$. Thus $\langle Af, \varphi\rangle \leqslant 0$. □

Remark 2.4 The function p is convex. So the one-sided Gateaux-derivatives

$$D_g^+ p(f) = \lim_{t\downarrow 0} 1/t(p(f + tg) - p(f)) \quad \text{and}$$
$$D_g^- p(f) = \lim_{t\uparrow 0} 1/t(p(f + tg) - p(f))$$

exist and satisfy $D_g^- p(f) \leqslant D_g^+ p(f)$ for all $f, g \in E$ (cf. Moreau [30]). Moreover,

$$D_g^+ p(f) = \sup\{\langle g, \varphi\rangle \colon \varphi \in \mathrm{d}p(f)\}, \tag{2.9}$$

$$D_g^- p(f) = \inf\{\langle g, \varphi\rangle \colon \varphi \in \mathrm{d}p(f)\}. \tag{2.10}$$

Thus A is p-dissipative if and only if $D_{Af}^- p(f) \leqslant 0$, and A is strictly p-dissipative if and only if $D_{Af}^+ p(f) \leqslant 0$ for all $f \in D(A)$.

Corollary 2.5 *Let A be a closable operator. If A is p-dissipative, then so is its closure.*

Theorem 2.6 *Let p be a continuous sublinear functional on a real Banach space E. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$. The following assertions are equivalent.*

(a) *$p(T(t)f) \leqslant p(f)$ for all $t \geqslant 0$, $f \in E$.*

(b) *A is strictly p-dissipative.*

(c) *There exists a core D of A such that $A|_D$ is p-dissipative.*

Proof Assume that (a) holds. Let $f \in D(A)$, $\varphi \in \mathrm{d}p(f)$. Then

$$\begin{aligned}\langle Af, \varphi\rangle &= \lim_{t\to 0} 1/t(\langle T(t)f, \varphi\rangle - \langle f, \varphi\rangle) \\ &= \lim_{t\to 0} 1/t(\langle T(t)f, \varphi\rangle - p(f)) \\ &\leqslant \limsup_{t\to 0} 1/t(p(T(t)f) - p(f)) \leqslant 0.\end{aligned}$$

This proves (b) .

It is trivial that (b) implies (c). So let us assume (c) . Then it follows from Corollary 2.5 that A is p-dissipative. Hence, by (2.8), $p(\lambda R(\lambda, A)g) \leqslant p(g)$ for all $g \in E$, $\lambda > \omega_0(A)$. Hence $\lambda R(\lambda, A)$ is p-contractive for $\lambda > \omega_0(A)$. This implies that $T(t)$ is p-contractive by the formula (1.3),

$$T(t) = \lim_{t\to 0}(n/t\, R(n/t, A))^n \text{ (strongly) for } t \geqslant 0.$$

We have shown that for generators, p-dissipativity is equivalent to p-contractivity of the semigroup. Now we will consider a p-dissipative operator A (which is not a generator a priori) and investigate under which additional hypotheses A is the generator of a (necessarily contractive) semigroup. At first we present some consequences of p-dissipativity.

Theorem 2.7 *Let A be a p-dissipative operator. If $D(A)$ is dense, then A is strictly p-dissipative.*

Proof Let $f \in D(A)$, $\varphi \in \mathrm{d}p(f)$. Then for every $t > 0$ and $g \in D(A)$ we have

$$\begin{aligned}\langle Af, \varphi\rangle &= \frac{1}{t}(\langle f + tAf, \varphi\rangle - \langle f, \varphi\rangle) \leqslant \frac{1}{t}(p(f + tAf) - p(f)) \\ &\leqslant \frac{1}{t}(p(f + tg) + tp(Af - g) - p(f)) \\ &\leqslant \frac{1}{t}(p((Id - tA)(f + tg)) + tp(Af - g) - p(f)) \quad \text{(by (2.7))} \\ &\leqslant \frac{1}{t}(p(f) + tp(g - Af) + t^2 p(-Ag) + tp(Af - g) - p(f)) \\ &\leqslant \frac{1}{t}(2tc\|g - Af\| + t^2 c\|Ag\|) \quad \text{(by (2.3))} \\ &= 2c\|g - Af\| + tc\|Ag\|.\end{aligned}$$

Letting $t \to 0$ we obtain $\langle Af, \varphi\rangle \geqslant 2c\|g - Af\|$ for all $g \in D(A)$. Since $D(A)$ is dense in E, this implies that $\langle Af, \varphi\rangle \geqslant 0$. □

We now impose stronger conditions on p. A continuous sublinear function $p : E \to \mathbb{R}$ is called *half-norm* if

$$p(f) + p(-f) > 0 \text{ whenever } f \neq 0; \tag{2.11}$$

and p is called a *strict half-norm* if in addition there exists some constant $d > 0$ such that

$$p(f) + p(-f) \geqslant d\|f\| \text{ for all } f \in E. \tag{2.12}$$

If p is a half-norm, then

$$\|f\|_p = p(f) + p(-f) \quad (f \in E) \tag{2.13}$$

defines a norm on E which is equivalent to the given norm if and only if p is strict.

Remark 2.8 Every half-norm p induces a closed proper cone

$$E_p := \{f \in E : p(-f) \geqslant 0\}$$

on E. Any p-contractive operator T on E leaves the cone E_p invariant (i.e., T is positive for the corresponding ordering).

Conversely, given a closed proper cone E_+ on E, then

$$p(f) := \operatorname{dist}(-f, E_+) = \inf\{\|f + g\| : g \in E_+\}$$

defines a half-norm on E such that $E_+ = E_p$. This half-norm is called the *canonical half-norm* on the ordered Banach space (E, E_+). The canonical half-norm is strict if and only if the cone E_+ is *normal* (this is equivalent to the fact that for every $\varphi \in E'$ there exist positive linear forms φ_1 and φ_2 on E such that $\varphi = \varphi_1 - \varphi_2$ (see Batty and Robinson [3] or Schaefer [37, Chapter V]).

Proposition 2.9 *Let A be a p-dissipative operator where p is a half-norm. If $D(A)$ is dense, then A is closable (and the closure of A is p-dissipative as well (by Corollary 2.5)).*

Proof Let $f_n \in D(A)$, $\lim_{n\to\infty} f_n = 0$, $\lim_{n\to\infty} Af_n = g$. We have to show that $g = 0$. To this end let $h \in D(A)$. Then (2.7) gives

$$p(f_n + th) \leqslant p(f_n + th - tA(f_n + th)) \quad (t > 0).$$

Letting $n \to \infty$ we obtain $p(th) \leqslant p(th - tg - t^2 Ah) \quad (t > 0)$. Hence

$$p(h) \leqslant p((h - g) - tAh) \quad (t > 0)$$

by positive homogeneity. Letting $t \downarrow 0$ finally we obtain $p(h) \leqslant p(h - g)$ for all $h \in D(A)$. Since $D(A)$ is dense by hypothesis, we can approximate g by $h \in D(A)$ and conclude that $p(g) \leqslant p(0) = 0$. Since $\lim_{n\to\infty} A(-f_n) = -g$, we have $p(-g) \leqslant 0$ by symmetry. Hence $p(g) + p(-g) \leqslant 0$ which implies $g = 0$ by (2.11) . □

Lemma 2.10 *Let p be a half-norm and A a p-dissipative operator. Then*

$$\lambda\|f\|_p \leqslant \|(\lambda - A)f\|_p \textit{ for all } f \in D(A), \lambda > 0. \tag{2.14}$$

In particular, $(\lambda - A)$ *is injective for all* $\lambda > 0$. *If* p *is strict and* A *is closed, then* $im(\lambda - A)$ *is closed for all* $\lambda > 0$.

Proof Let $\lambda > 0$, $f \in D(A)$. Then by (2.7) , $\lambda p(\pm f) \leqslant p((\lambda - A)(\pm f))$. Hence

$$\lambda \|f\|_p = \lambda p(f) + \lambda p(-f) \leqslant p((\lambda - A)f) + p(-(\lambda - A)f) = \|(\lambda - A)f\|_p.$$

Thus (2.14) is proved. Now suppose that p is strict. Then $\|\cdot\|_p$ is equivalent to the given norm. Let $\lambda > 0$ and $g \in \overline{\mathrm{Im}(\lambda - A)}$. Then $g = \lim_{n\to\infty}(\lambda - A)f_n$ for some sequence $(f_n)_{n\in\mathbb{N}} \subset D(A)$. It follows from (2.14) that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $f = \lim_{n\to\infty} f_n$. Then

$$\lim_{n\to\infty} Af_n = \lambda \lim_{n\to\infty} f_n - \lim_{n\to\infty}(\lambda - A)f_n = \lambda f - g$$

exists. If A is closed, this implies that $f \in D(A)$ and $Af = \lambda f - g$. Hence $g = (\lambda - A)f \in \mathrm{im}(\lambda - A)$. We have shown that $\mathrm{im}(\lambda - A)$ is closed. □

The following is the main theorem of this section.

Theorem 2.11 *Let* p *be a strict half-norm and* A *an operator on* E. *The following assertions are equivalent.*

(a) *A is the generator of a p-contraction semigroup.*

(b) $D(A)$ *is dense,* A *is p-dissipative and* $im(\lambda - A) = E$ *for some* $\lambda > 0$.

Proof Since p is a strict half-norm, we can assume that $\|f\| = \|f\|_p$ for all $f \in E$. By Theorem 2.6, condition (a) implies (b) .

Now suppose that (b) holds. Then it follows from Lemma 2.10 that $\mu \in \varrho(A)$ and $\|\mu R(\mu, A)\| \leqslant 1$ whenever $\mu > 0$ such that $\mathrm{im}(\mu - A) = E$. So by hypothesis $\lambda \in \varrho(A)$ and $\mathrm{dist}(\lambda, \sigma(A)) \geqslant \|R(\lambda, A)\|^{-1} \geqslant \lambda$. Hence $(0, 2\lambda) \subset \varrho(A)$. Iterating this argument we see that $(0, \infty) \subset \varrho(A)$. It follows from the Hille-Yosida theorem that A generates a contraction semigroup $(T(t))_{t\geqslant 0}$. Finally, from 2.6 it follows that $(T(t))_{t\geqslant 0}$ is p-contractive. □

Of course, the norm function N given by $N(f) = \|f\|$ is a strict half-norm. In the case when $p = N$, Theorem 2.11 is due to Lumer and Phillips [28]. It turns out to be extremely useful in showing that a concrete operator is a generator. Because of its importance we state this special case explicitly below (including the complex case). Before that let us formulate Theorem 2.11 for the case when the operator is merely given on a core.

Corollary 2.12 *Let* p *be a strict half-norm and* A *be a densely defined operator. If* A *is p-dissipative and* $(\lambda - A)$ *has dense range for some* $\lambda > 0$, *then* A *is closable and the closure* $\overline{A}$ *of* A *generates a* p*-contraction semigroup.*

Proof It follows from Proposition 2.9 that A is closable and the closure $\overline{A}$ is p-dissipative. Lemma 2.10 implies that $(\lambda - \overline{A})D(\overline{A}) = E$. So Theorem 2.11 yields the desired conclusion. □

We conclude this section indicating the results for the complex case.

Let E be a complex Banach space and $p\colon E \to \mathbb{R}_+$ be a seminorm on E (i.e., $p(f+g) \leqslant p(f) + p(g)$ and $p(\lambda f) = |\lambda| p(f)$ holds for all $f, g \in E$, $\lambda \in \mathbb{C}$). The subdifferential $\mathrm{d}p(f)$ of p in $f \in E$ is defined by

$$\mathrm{d}p(f) = \{\varphi \in E' \colon \operatorname{Re}\langle g, \varphi\rangle \leqslant p(g) \text{ for all } g \in E \quad \text{and } \langle f, \varphi\rangle = p(f)\}. \tag{2.15}$$

We assume in addition that p is continuous. Then it follows from the Hahn-Banach theorem that $\mathrm{d}p(f) \neq \emptyset$ for any $f \in E$. A linear operator A on E is called *p-dissipative* if for all $f \in D(A)$ there exists $\varphi \in \mathrm{d}p(f)$ such that $\operatorname{Re}\langle Af, \varphi\rangle \leqslant 0$.

The arguments given above show that also in the situation considered here A is p-dissipative if and only if

$$p((1 - tA)f) \geqslant p(f)$$

for all $f \in D(A)$, $t \geqslant 0$.

The results of this section carry over if they are appropriately modified. We explicitly state the most important result for the case when p is the norm. A linear operator A is simply called *dissipative* if it is N-dissipative where $N(f) = \|f\|$ $(f \in E)$.

Theorem 2.13 (Lumer-Phillips) *Let A be a densely defined operator on a complex Banach space E. The following assertions are equivalent.*

(a) *A is closable and the closure of A is the generator of a contraction semigroup.*

(b) *A is dissipative and $(\lambda - A)$ has dense range for some $\lambda > 0$.*

3 Semigroups on L^∞ and H^∞

by Heinrich P. Lotz

In this section we shall prove that on L^∞, on H^∞, and on some other classical Banach spaces every strongly continuous semigroup of operators is uniformly continuous.

Lemma 3.1 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a one-parameter semigroup of operators on a Banach space E. Suppose that $s = \limsup_{t\to 0} \|T(t) - \mathrm{Id}\|$ is finite. If $\lim_{t\to 0} \|(T(t) - \mathrm{Id})^2\| = 0$, then $\mathcal{T}$ is uniformly continuous.*

Proof The identity $2(T(t) - \mathrm{Id}) = T(2t) - \mathrm{Id} - (T(t) - \mathrm{Id})^2$ shows that

$$2\|T(t) - \mathrm{Id}\| - \|(T(t) - \mathrm{Id})^2\| \leqslant \|T(2t) - \mathrm{Id}\|.$$

Hence $2s \leqslant \limsup_{t\downarrow 0} \|T(2t) - \mathrm{Id}\,\|$. Obviously, $\limsup_{t\downarrow 0} \|T(2t) - \mathrm{Id}\,\| = s$ and so, $2s \leqslant s$. Consequently, $s = 0$. □

Remark

(i) If, in Lemma 3.1, $\mathcal{T} = (T(t))_{t\geqslant 0}$ is strongly continuous, in which case $s < \infty$, one can replace $\lim_{t\to 0} \|(T(t) - \mathrm{Id})^2\| = 0$ by the weaker condition

$$\limsup_{t\to 0} r(T(t) - \mathrm{Id}) < 1\,,$$

see Lotz [27, Lemma 2], where r denotes the spectral radius.

(ii) The condition $s < \infty$ in Lemma 3.1 is essential as the following example shows: Let $f\colon \mathbb{R} \to \mathbb{R}$ be non-continuous with $f(s+t) = f(s) + f(t)$ for all $s, t \in \mathbb{R}$ (see Hamel [18]). Then $\{(t, f(t))\colon t \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$. Hence for the semigroups $\mathcal{T} = (T(t))_{t\geqslant 0}$ on $\mathbb{R}^2$ with

$$T(t) = \begin{pmatrix} 1 & f(t) \\ 0 & 1 \end{pmatrix} \qquad \text{for } t \geqslant 0$$

we have $s = \infty$. Therefore $\mathcal{T}$ is not uniformly continuous. However, $(T(t) - \mathrm{Id})^2 = 0$ for all $t \geqslant 0$.

Lemma 3.2 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a one-parameter semigroup of operators on a Banach space E. Then the following assertions are equivalent*

(a) *$\mathcal{T}' = (T(t)')_{t\geqslant 0}$ is a strongly continuous semigroup on the dual E'.*

(b) *$((T(t_n) - \mathrm{Id})x_n)$ converges weakly to zero for every bounded sequence (x_n) in E and every sequence (t_n) in $[0,\infty)$ with $\lim t_n = 0$.*

Moreover, (a) implies

(c) *$\mathcal{T}$ is strongly continuous.*

Proof Let $x' \in E'$ and $t_n \geqslant 0$ be given. Then $\lim \|(T(t_n) - \mathrm{Id})'x'\| = 0$ if and only if $\lim\langle x_n, (T(t_n) - \mathrm{Id})'x'\rangle = 0$ for every bounded sequence (x_n) in E. This implies the equivalence of (a) and (b) . In particular, (a) implies that $((T(t_n) - \mathrm{Id})x)$ converges weakly to zero for every sequence (t_n) in $[0,\infty)$ with $\lim t_n = 0$ and every $x \in E$. Hence $\mathcal{T}$ is strongly continuous by Proposition 1.23 in Davies [11]. □

We recall that a Banach space E is called a *Grothendieck space* if every weak* convergent sequence in E' converges weakly.

Theorem 3.3 *Let E be a Grothendieck space. If $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup in E, then $\mathcal{T}'' = (T(t)'')_{t\geqslant 0}$ is strongly continuous in E''.*

Proof Suppose that (x'_n) is a bounded sequence in E' and that $t_n \geqslant 0$ with $\lim t_n = 0$. Put $V_n := T(t_n) - \mathrm{Id}$. Then $\lim \|V_n x\| = 0$ and therefore $\lim\langle x, V'_n x'_n\rangle = 0$

for every $x \in E$. Hence $(V_n' x_n')$ w*-converges to zero. Since E is a Grothendieck space, $(V_n' x_n')$ converges weakly to zero. Now Lemma 3.2 implies that $(T(t)'')$ is strongly continuous. □

Recall now that a Banach space E is said to have the *Dunford-Pettis property* if $\lim\langle x_n, x_n'\rangle = 0$ whenever (x_n) in E and (x_n') in E' converge weakly to zero.

Theorem 3.4 *Let E be a Banach space with the Dunford-Pettis property and let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a one-parameter semigroup of operators on E. If $\mathcal{T}'' = (T(t)'')_{t\geqslant 0}$ is strongly continuous in E'', then $\mathcal{T}$ is uniformly continuous.*

Proof Suppose that $\mathcal{T}''$ is a strongly continuous semigroup. Then Lemma 3.2 implies that $\mathcal{T}'$ and $\mathcal{T}$ are strongly continuous. Hence by the uniform boundedness principle, $\limsup_{t\to 0} \|(T(t) - Id)\|$ is finite. By Lemma 3.1 it suffices to show that $\lim_{t\to 0} \|(T(t) - Id)^2\| = 0$. Let $t_n \geqslant 0$ with $\lim t_n = 0$ be given. Then there exists a bounded sequence (x_n) in E and a bounded sequence (x_n') in E' such that

$$\|(T(t_n) - Id)^2\| - 1/n \leqslant \langle (T(t_n) - Id)x_n, (T(t_n) - Id)' x_n'\rangle\,.$$

Since $\mathcal{T}'$ and $\mathcal{T}''$ are strongly continuous, Lemma 3.2 implies that $((T(t_n) - Id)x_n)$ and $((T(t_n) - Id)' x_n')$ converge weakly to zero. Since E has the Dunford-Pettis property, $\lim \|(T(t_n) - Id)^2\| = 0$. Consequently, $\lim_{t\to 0} \|(T(t) - Id)^2\| = 0$. □

An immediate consequence of Theorem 3.3 and Theorem 3.4 is the following.

Theorem 3.5 *Let E be a Grothendieck space with the Dunford-Pettis property. Then every strongly continuous semigroup of operators on E is uniformly continuous.*

A compact Hausdorff space is called an *F-space* if the closures of two disjoint open F_σ-sets are disjoint and is called a *Stonean* (res., *σ-Stonean*) space if the closure of every open set (res., open F_σ-set) is open. Every σ-Stonean space is an F-space.

Theorem 3.6 *Every strongly continuous semigroups of operators on one of the following Banach spaces is uniformly continuous.*

(i) *$C(K)$, where K is a compact F-space.*

(ii) *$L^\infty(S, \Sigma, \mu)$ for any measure space (S, Σ, μ).*

(iii) *The Banach space $B(S, \Sigma)$ of all bounded Σ-measurable functions on S if Σ is a σ-algebra of subsets of S.*

(iv) *The Banach space $\mathcal{H}(O)$ of all bounded continuous solutions of*

$$\sum_{1\leqslant i\leqslant n} (\partial^2 f/\partial x_i^2) = 0$$

on an open subset O of $\mathbb{R}^n$.

(v) *The Banach space* $\mathcal{W}(O)$ *of all bounded continuous solutions of*

$$\sum_{1\leqslant i\leqslant n} (\partial^2 f/\partial x_i^2) = (\partial f/\partial x_{n+1})$$

on an open subset O *of* $\mathbb{R}^{n+1}$.

(vi) *The Banach space* $H^\infty(O)$ *of bounded analytic functions on a finitely connected domain* O *of the complex plane.*

Proof By Theorem 3.5 it suffices to show that the spaces listed above are Grothendieck spaces with the Dunford-Pettis property.

(i) If K is compact, then $C(K)$ has the Dunford-Pettis property (cf. Grothendieck [17, Théorème 4]). If K is a compact F-space, then $C(K)$ is a Grothendieck space cf. Seever [38, Theorem 2.5]. The special cases for Stonean and σ-Stonean spaces are due to Grothendieck [17, Théorème 9] and Ando [1], respectively.

(ii) and (iii) It is well known that every σ-order complete AM-space with unit is isometric to a space $C(K)$ where K is a compact σ-Stonean space. Obviously, the spaces under (ii) and (iii) are σ-order complete AM-spaces with unit and therefore, by proof of (i), are Grothendieck spaces with the Dunford-Pettis property.

(iv) and (v) These spaces are order complete vector lattices. This follows from Bauer [4, pp.18-22, Standardbeispiele 1 and 2 p.55]. Since these spaces contain the constant functions on O, they are complete for the supremum-norm. Indeed, if (f_n) is a Cauchy-sequence for this norm, it is easily seen that (f_n) converges in norm to $\inf_n \sup(f_k : n < k)$. Therefore these spaces are σ-order complete AM-spaces with unit and so as before Grothendieck spaces with the Dunford-Pettis property.

(vi) In Bourgain [7] it is shown, that $H^\infty(D)$ is a Grothendieck space, where D is the open unit disc $\{z : |z| < 1\}$, and in Bourgain [8], that this Banach space has the Dunford-Pettis property (see also the summary of Bourgain in Blei and Sidney [6]).

If O is a finitely connected domain and H^∞ does not only contain the constant functions, then $H^\infty(O)$ is isomorphic to a finite direct sum of copies of $H^\infty(D)$. (Note that $H^\infty(D)$ is isomorphic to $\{f \in H^\infty(D) : f(0) = 0\}$ via the map $f \mapsto zf$. Then use Grothendieck [17, p.77 and Proposition 4.4.1]). Hence $H^\infty(O)$ is a Grothendieck space with the Dunford-Pettis property.

Remark (Final) It follows from Theorem 3.6 that on L^∞ the infinitesimal generator of a strongly continuous semigroup is necessarily bounded. It is not obvious that on $L^\infty([0,1])$ there exist closed densely defined unbounded operators.

To see this let A be a closed densely defined unbounded operator from ℓ^2 into $L^\infty([0,1])$ with domain D (such operators can easily be constructed). By the Khintchine inequality, the map $R : (a_n) \mapsto \sum a_n r_n$, where r_n denotes the n^{th} Rademacher function, from ℓ^2 into $L^1([0,1])$ is a topological isomorphism. Hence $T = R'$ maps $L^\infty([0,1])$ onto ℓ^2.

Banach's homomorphism theorem implies that $T^{-1}(D)$ is dense in $L^\infty([0,1])$ and that AT is a closed densely defined unbounded operator on $L^\infty([0,1])$ with domain $T^{-1}(D)$. This solves a problem raised by R. Kaufman.

H. Porta and the author have shown that if a Banach space E has an infinite dimensional separable quotient space and F is an infinite dimensional Banach space, then there always exists a closed densely defined unbounded operator from E into F.

Notes

Section 1: The abstract Cauchy problem is treated systematically in the monographs of Krein [24] and Fattorini [13]. We refer to these books for more details and historical notes. One implication of Theorem 1.1 is proved in Krein [24, Theorem 2.11].

The Hille-Yosida Theorem has been proved independently by Hille [20] and Yosida [39] for contraction semigroups. The extension to arbitrary strongly continuous semigroups is independently due to Feller [14], Miyadera [29] and Phillips [32]. Thus our terminology is slightly incorrect, and some authors refer to the general version as the Hille-Yosida-Phillips theorem which is more correct.

Holomorphic semigroups belong to the standard material of the theory of one-parameter semigroups. Our Theorem 1.14 deviates from the usual presentation since the condition on the resolvent is merely required on a half-plane.

Differentiable semigroups are treated in detail in the book of Pazy [31] who discovered Theorem 1.17 and 1.18 . The spectral property of eventually norm continuous semigroups given in Theorem 1.20 is contained in Hille and Phillips [21, Theorem 16.4.2] with a proof depending on Gelfand theory. For norm continuous semigroups it is contained in Pazy [31] with a simpler proof. The elementary proof we give here is due to G. Greiner.

Theorem 1.29 on the perturbation by bounded operators is due to Phillips [32] who also investigated permanence of smoothness properties by this kind of perturbation. We also refer to Pazy [31, Section 3.1].

The observation that eventually norm continuity is preserved by perturbation by a compact operator (see Theorem 1.30) seems to be new.

The perturbation by continuous operators on the graph of the generator is due to Desch and Schappacher [12]. The short proof we give here is due to G. Greiner and has the advantage to yield the same permanence for smoothness properties as in the classical case, see Corollary 1.32 .

The characterization of a core as "domain of uniqueness" given in Theorem 1.33 seems to be new. In this section we have presented part of the standard theory of one-parameter semigroups including some new aspects. A very elegant brief introduction to one-parameter semigroups is given in the treatise of Kato [22] where one can also find all the results on perturbation theory going beyond the elementary facts we discuss here. A complete information on the general theory can be obtained by consulting the books of Davies [11], Goldstein [15] and Pazy [31]. The monograph of Goldstein [15] contains a variety of examples and applications.

Section 2: Dissipative operators were introduced by Lumer and Phillips [28]. The analogous notion of dispersiveness is due to Phillips [33]. Our approach follows closely Arendt et al. [2] where half-norms were introduced. Related previous results were obtained by Calvert [9], Hasegawa [19], Sato [36], Bénilan and Picard [5] and Picard [34], where the two last consider non-linear semigroups. A further investigation of half-norms can be found in Batty and Robinson [3] who consider ordered Banach spaces other than Banach lattices in great detail. We also refer to the historical notes given there.

Section 3: It had been proved by Kishimoto and Robinson [23] that every generator of a positive semigroup on L^∞ is bounded. That every strongly continuous semigroup on L^∞ is uniformly continuous was first shown by Lotz [25], Lotz [26] and Lotz [27]. The proof of Lemma 3.1 was communicated to the author of this section by T. Coulhon, who independently obtained a particular case of Theorem 3.5 (Coulhon [10]) .

References

[1] T. Ando. Convergent sequences of finitely additive measures. *Pacific J. Math.*, 11:395–404, 1961.

[2] W. Arendt, P. Chernoff, and T. Kato. A generalization of dissipativity and positive semigroups. *J. Operator Theory*, 8:167–180, 1982.

[3] C. Batty and D. Robinson. Positive one-parameter semigroups on ordered spaces. *Acta Appl. Math.*, 2:221–296, 1984.

[4] H. Bauer. *Harmonische Räume und ihre Potentialtheorie*. Springer, Berlin-Heidelberg-New York, 1966.

[5] P. Bénilan and C. Picard. Quelques aspects non lineaires du principe du maximum. In *Séminaire de Théorie du Potentiel*, volume 713 of *Lecture Notes in Mathematics*. Springer, 1979.

[6] R. C. Blei and S. J. Sidney. *Banach Spaces, Harmonic Analysis and Probability Theory: Proceedings of the Special year in analysis*. Lecture notes in mathematics 995. Springer-Verlag, 1983.

[7] J. Bourgain. H^∞ is a Grothendieck space. *Studia Math.*, 75:192–226, 1982.

[8] J. Bourgain. New Banach space properties of the disc algebra and H^∞. *Acta Math.*, 152:1–48, 1984.

[9] B. Calvert. Nonlinear equations of evolution. *Pacific J. Math.*, 39:293–350, 1971.

[10] T. Coulhon. Suites d'opérateurs sur un espace $C(K)$ de Grothendieck. *C. R. Acad. Sci. Paris Sér. I Math.*, 298:13–15, 1984.

[11] E. Davies. *One-parameter Semigroups*. Academic Press, London-New York-San Francisco, 1980.

[12] W. Desch and W. Schappacher. On relatively bounded perturbations of linear C_0-semigroups. *Ann. Scuola Norm. Sup. Pisa*, 11:327–341, 1984.

[13] H. Fattorini. *The Cauchy Problem*. Addison-Wesley, Reading, Massachusetts, 1983.

[14] W. Feller. On the generation of unbounded semigroups of bounded linear operators. *Ann. of Math.*, 58:166–174, 1953.

[15] J. Goldstein. *Semigroups of Operators and Applications*. Oxford University Press, 1985.

[16] J. Goldstein. A (more-or-less) complete bibliography of semigroups of operators through 1984. Preprints and lecture notes in mathematics, Tulane University, 1985.

[17] A. Grothendieck. Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$. *Canad. J. Math.*, 5:129–173, 1953.

[18] G. Hamel. Eine basis aller zahlen und die unstetigen lösungen der funktionalgleichung: $f(x+y) = f(x) + f(y)$. *Math. Ann.*, 60:459–462, 1905.

[19] M. Hasegawa. On contraction semigroups and (di)-operators. *J. Math. Soc. Japan*, 18:290–302, 1966.

[20] E. Hille. *Functional Analysis and Semigroups*, volume 31 of *Colloquium Publications*. American Mathematical Society, Providence, R.I., 1948.

[21] E. Hille and R. S. Phillips. *Functional Analysis and Semigroups*, volume 31 of *Colloquium Publications*. American Mathematical Society, Providence, R.I., 1957.

[22] T. Kato. *Perturbation Theory for Linear Operators*. Springer, 1966. 2nd printing: Berlin-Heidelberg-New York: Springer 1976.

[23] A. Kishimoto and D. W. Robinson. Subordinate semigroups and order properties. *J. Austral. Math. Soc. Ser. A*, 31:59–76, 1981.

[24] S. G. Krein. *Linear Differential Equations in Banach Spaces*, volume 29 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, R.I., 1971.

[25] H. P. Lotz. Uniform convergence of operators on L^∞. Semesterbericht Funktionalanalysis, 1982.

[26] H. P. Lotz. Tauberian theorems for operators on L^∞ and similar spaces. In K. D. Bierstedt and E. Fuchssteiner, editors, *Functional Analysis, Surveys and Recent Results III*. North Holland, Amsterdam, 1984.

[27] H. P. Lotz. Uniform convergence of operators on L^∞ and similar spaces. *Math. Z.*, 190:207–220, 1985.

[28] G. Lumer and R. S. Phillips. Dissipative operators in a Banach space. *Pacific J. Math.*, 11:679–698, 1961.

[29] I. Miyadera. Generation of a strongly continuous semigroup of operators. *Tôhoku Math. J.*, 4:109–114, 1952.

[30] J. J. Moreau. Fonctionelles convexes. Séminaire sur les équations aux dérivées partielles, Collège de France, 1966. 1966–67.

[31] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[32] R. S. Phillips. A note on the abstract Cauchy problem. *Proc. Nat. Acad. Sci. U.S.A.*, 40:244–248, 1954.

[33] R. S. Phillips. Semigroups of positive contraction operators. *Czechoslovak Math. J.*, 12:294–313, 1962.

[34] C. Picard. Opérateurs φ-accretifs et génération des semi-groupes non linéaires. *C. R. Acad. Sci. Paris Sér. I Math.*, 275:639–641, 1972.

[35] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV. Analysis of Operators*. Academic Press, New York, 1978.

[36] K. I. Sato. On the generators of non-negative contraction semigroups in Banach lattices. *J. Math. Soc. Japan*, 20:423–436, 1968.

[37] H. H. Schaefer. *Topological Vector Spaces*. Springer, 1966. 4th printing: New York-Heidelberg-Berlin: Springer 1980.

[38] G. L. Seever. Measures on F-spaces. *Trans. Amer. Math. Soc.*, 133:267–280, 1968.

[39] K. Yosida. On the differentiability and representation of one-parameter semigroups of linear operators. *J. Math. Soc. Japan*, 1:15–21, 1948.

[40] K. Yosida. *Functional Analysis*. Springer, 1965.

Chapter A-III
Spectral Theory

by
Günther Greiner and Rainer Nagel

1 Introduction

In this chapter, we start a systematic analysis of the spectrum of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ on a complex Banach space E. By the spectrum of the semigroup we understand the spectrum $\sigma(A)$ of the generator A of $\mathcal{T}$. In particular, we are interested in the precise relations between $\sigma(A)$ and $\sigma(T(t))$. The heuristic formula

$$T(t) = \mathrm{e}^{tA}$$

serves as a leitmotiv and suggests relations of the form

$$\sigma(T(t)) = \mathrm{e}^{t\sigma(A)} = \{\mathrm{e}^{t\lambda} \colon \lambda \in \sigma(A)\},$$

called *spectral mapping theorem*. These—or similar—relations will be of great use in Chapter IV and enable us to determine the asymptotic behavior of the semigroup $\mathcal{T}$ by the spectrum of its generator.

As motivation and also as a preliminary step, we concentrate here on the *spectral radius*

$$r(T(t)) := \sup\{|\lambda| : \lambda \in \sigma(T(t))\}, \quad t \geqslant 0, \tag{1.1}$$

and show how it is related to the *spectral bound*

$$s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \tag{1.2}$$

of the generator A and to the *growth bound*

$$\omega_0 := \inf\{\omega \in \mathbb{R} : \|T(t)\| \leqslant M_\omega \cdot \mathrm{e}^{\omega t} \text{ for all } t \geqslant 0 \text{ and suitable } M_\omega\} \tag{1.3}$$

of the semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$. (Recall that sometimes we write $\omega_0(\mathcal{T})$ or $\omega_0(A)$ instead of ω_0). The Examples 1.3 and 1.4 below illustrate the main difficulties to be encountered.

Proposition 1.1 *Let ω_0 be the growth bound of the strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$. Then*

$$r(T(t)) = \mathrm{e}^{\omega_0 t} \tag{1.4}$$

for every $t \geqslant 0$.

Proof From A-I, (1.1) we know that

$$\omega_0(\mathcal{T}) = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|.$$

Since the spectral radius of $T(t)$ is given as

$$r(T(t)) = \lim_{n\to\infty} \|T(nt)\|^{1/n},$$

we obtain for $t > 0$

$$r(T(t)) = \lim_{n\to\infty} \exp\left(\frac{t}{nt} \log \|T(nt)\|\right) = \mathrm{e}^{\omega_0 t}.$$

It was shown in A-I, Proposition 1.11 that the spectral bound $s(A)$ is always dominated by the growth bound ω_0 and therefore $\mathrm{e}^{s(A)t} \leqslant r(T(t))$. If the above mentioned spectral mapping theorem holds—as is the case for bounded generators (e.g., see Theorem VII.3.11 of Dunford and Schwartz [5])—we obtain the equality

$$\mathrm{e}^{s(A)t} = r(T(t)) = \mathrm{e}^{\omega_0(\mathcal{T})t},$$

hence $s(A) = \omega_0(\mathcal{T})$. Therefore, the following corollary is a consequence of the definitions of $s(A)$ and $\omega_0(\mathcal{T})$.

Corollary 1.2 *Consider the semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ generated by some bounded linear operator $A \in \mathcal{L}(E)$. If* $\operatorname{Re}\lambda < 0$ *for each $\lambda \in \sigma(A)$, then* $\lim_{t\to\infty} \|T(t)\| = 0$.

Through this corollary we have re-established a famous result of Liapunov which assures that the solutions of the linear Cauchy problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in \mathbb{C}^n \quad \text{and} \quad A = (a_{ij})_{n\times n}$$

are *stable*, i.e., they converge to zero as $t \to \infty$ if (and only if) the real parts of all eigenvalues of the matrix A are smaller than zero.

For unbounded generators the situation is much more difficult and $s(A)$ may differ drastically from $\omega_0(\mathcal{T})$.

Example 1.3 (Banach function space, Greiner et al. [11]) Consider the Banach space E of all complex valued continuous functions on $\mathbb{R}_+$ which vanish at infinity and are integrable for $\mathrm{e}^x\,\mathrm{d}x$, i.e.,

$$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathrm{e}^x\,\mathrm{d}x)$$

endowed with the norm

$$\|f\| := \|f\|_\infty + \|f\|_1 = \sup\{|f(x)| : x \in \mathbb{R}_+\} + \int_0^\infty |f(x)|\mathrm{e}^x\,\mathrm{d}x.$$

The translation semigroup

$$T(t)f(x) := f(x+t)$$

is strongly continuous on E and one shows as in A-I, 2.4 that its generator is given by

$$Af = f', \quad D(A) = \{f \in E : f \in C^1(\mathbb{R}_+), f' \in E\}.$$

First we observe that $\|T(t)\| = 1$ for every $t \geqslant 0$, hence $\omega_0(\mathcal{T}) = 0$. Moreover it is clear that λ is an eigenvalue of A as soon as $\operatorname{Re}\lambda < -1$ (in fact: the function

$$x \mapsto e_\lambda(x) := \mathrm{e}^{\lambda x}$$

belongs to $D(A)$ and is an eigenvector of A), hence $s(A) \geqslant -1$. For $f \in E$ and $\operatorname{Re}\lambda > -1$

$$\|\cdot\|_1\text{-}\lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s$$

exists since $\|T(s)f\|_1 \leqslant \mathrm{e}^{-s}\|f\|_1$ for $s \geqslant 0$, and

$$\|\cdot\|_\infty\text{-}\lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s$$

exists since $\int_0^\infty \mathrm{e}^x|f(x)|\,\mathrm{d}x < \infty$. Therefore $\int_0^\infty \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s$ exists in E for every $f \in E$, $\operatorname{Re}\lambda > -1$.

As we observed in A-I, Proposition 1.11, this implies $\lambda \in \varrho(A)$. Therefore $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a semigroup having $s(A) = -1$, but $\omega_0(\mathcal{T}) = 0$.

Example 1.4 (Hilbert space, Zabczyk [27]) For every $n \in \mathbb{N}$ consider the n-dimensional Hilbert space $H_n := \mathbb{C}^n$ and operators $A_n \in \mathcal{L}(H_n)$ defined by the matrices

$$A_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdot & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & 0 \end{pmatrix}_{n\times n}.$$

These matrices are nilpotent and therefore $\sigma(A_n) = \{0\}$. The elements

$$x_n := n^{-1/2}(1, \dots, 1) \in H_n$$

satisfy the following properties.

(i) $\|x_n\| = 1$ for every $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \|A_n x_n - x_n\| = 0$,

(iii) $\lim_{n\to\infty} \|\exp(tA_n)x_n - \mathrm{e}^t x_n\| = 0.$

Consider now the Hilbert space

$$H := \bigoplus_{n\in\mathbb{N}} H_n \text{ and the operator } A := (A_n + 2\pi\mathrm{i}n)_{n\in\mathbb{N}}$$

with maximal domain in H.

Analogously we define a semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ by

$$T(t) := (\mathrm{e}^{2\pi\mathrm{i}nt} \exp(tA_n))_{n\in\mathbb{N}}.$$

Since $\|\exp(tA_n)\| \leqslant \mathrm{e}^t$ for every $n \in \mathbb{N}$, $t \geqslant 0$, and since $t \mapsto T(t)x$ is continuous on each component E_n, it follows that $\mathcal{T}$ is strongly continuous. Its generator is the operator A as defined above.

For $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda > 0$, we have $\lim_{n\to\infty} \|R(\lambda - 2\pi\mathrm{i}n, A_n)\| = 0$, hence

$$(R(\lambda, A_n + 2\pi\mathrm{i}n))_{n\in\mathbb{N}} = (R(\lambda - 2\pi\mathrm{i}n, A_n))_{n\in\mathbb{N}}$$

is a bounded operator on H representing the resolvent $R(\lambda, A)$. Therefore we obtain $s(A) \leqslant 0$. On the other hand, each $2\pi\mathrm{i}n$ is an eigenvalue of A, hence $s(A) = 0$.

Take now $x_n \in H_n$ as above and consider the sequence $(x_n)_{n\in\mathbb{N}}$. From (iii) it follows that for $t > 0$ the number e^t is an approximate eigenvalue of $T(t)$ with approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ (see Definition 2.1 below). Therefore

$$\mathrm{e}^t \leqslant r(T(t)) \leqslant \|T(t)\|$$

and hence $\omega_0(\mathcal{T}) \geqslant 1$. On the other hand, it is easy to see that $\|T(t)\| = \mathrm{e}^t$, hence $\omega_0(\mathcal{T}) = 1$.

Finally, if we take $S(t) := \mathrm{e}^{-t/2}T(t)$, we obtain a semigroup $\mathcal{S}$ having spectral bound $-\frac{1}{2}$ but satisfying $\lim_{t\to\infty} \|S(t)\| = \infty$ in contrast with Corollary 1.2.

These examples show that neither the conclusion of Corollary 1.2, i.e., "$s(A) < 0$ implies stability", nor the "spectral mapping theorem"

$$\sigma(T(t)) = \exp(t \cdot \sigma(A))$$

is valid for arbitrary strongly continuous semigroups. A careful analysis of the general situation will be given in Section 6 below, but we will first develop systematically the necessary spectral theoretic tools for unbounded operators.

2 The Fine Structure of the Spectrum

As usual, to a closed linear operator A with dense domain $D(A)$ in a Banach space E, we associate its spectrum $\sigma(A)$, its resolvent set $\varrho(A)$ and its resolvent

$$\lambda \mapsto R(\lambda, A) := (\lambda - A)^{-1}$$

which is a holomorphic map from $\varrho(A)$ into $\mathcal{L}(E)$. In contrast to the finite dimensional situation, where a linear operator fails to be surjective if and only if it fails to be injective, we now have to distinguish different cases of *non-invertibility* of $\lambda - A$. This distinction gives rise to a subdivision of $\sigma(A)$ into different subsets. We point out that these subsets need not be disjoint. Our definitions are justified by the fact that for each of the following subsets of $\sigma(A)$ there exist canonical constructions converting the corresponding spectral values into eigenvalues (see Proposition 2.2.(ii) and Proposition 4.4 below).

Definition 2.1 For a closed, densely defined, linear operator A with domain $D(A)$ in the Banach space E denote by the

(i) *point spectrum* $P\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not injective.

(ii) *approximate point spectrum* $A\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not injective or $(A - \lambda)D(A)$ is not closed in E .

(iii) *residual spectrum* $R\sigma(A)$ the set of all $\lambda \in \mathbb{C}$ such that $(A - \lambda)D(A)$ is not dense in E.

From these definitions it follows that $\lambda \in P\sigma(A)$ if and only if there exists a non-zero *eigenvector* $f \in D(A)$ such that $Af = \lambda f$, i.e., λ is an *eigenvalue*. It follows from the *Open Mapping Theorem* that $\lambda \in A\sigma(A)$ if and only if λ is an *approximate eigenvalue*, i.e., there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset D(A)$, called an *approximate eigenvector*, such that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|Af_n - \lambda f_n\| = 0$.

Clearly we have $P\sigma(A) \subset A\sigma(A)$ and $\sigma(A) = A\sigma(A) \cup R\sigma(A)$ where the union need not be disjoint.

The following proposition is a first indication that the subdivision we made yields nice properties.

Proposition 2.2 *For a closed, densely defined, linear operator* $(A, D(A))$ *in a Banach space* E *the following holds.*

(i) *The topological boundary* $\partial\sigma(A)$ *of* $\sigma(A)$ *is contained in* $A\sigma(A)$.

(ii) $R\sigma(A) = P\sigma(A')$ *for the adjoint operator* A' *on* E'.

Proof (i) Take $\lambda_0 \in \partial\sigma(A)$ and $\lambda_n \in \varrho(A)$ such that $\lambda_n \to \lambda_0$. Since

$$\|R(\lambda_n, A)\| \geqslant r(R(\lambda_n, A)) = (\operatorname{dist}(x, \sigma(A)))^{-1}$$

(see Proposition 2.5.(ii)), by the uniform boundedness principle we find $f \in E$ such that

$$\lim_{n\to\infty} \|R(\lambda_n, A)f\| = \infty.$$

Define $g_n \in D(A)$ by

$$g_n := \|R(\lambda_n, A)f\|^{-1} R(\lambda_n, A)f$$

and use the identity

$$(\lambda_0 - A)g_n = (\lambda_0 - \lambda_n)g_n + (\lambda_n - A)g_n$$

to show that $(g_n)_{n\in\mathbb{N}}$ is an approximate eigenvector corresponding to λ_0.

(ii) This is a simple consequence of the Hahn-Banach theorem. □

In order to illuminate the above definitions we now return to the Standard Examples introduced in Section 2 of A-I and discuss the fine structure of the spectrum of these strongly continuous semigroups, i.e., of their generators and their semigroup operators.

Example 2.3 (The Spectrum of Multiplication Semigroups) Take $E = C_0(X)$ for some locally compact space X and take a continuous function $q\colon X \mapsto \mathbb{C}$ whose real part is bounded above. As observed in A-I,2.3 the multiplication operator

$$M_q : f \mapsto q \cdot f$$

with maximal domain $D(M_q)$ generates the multiplication semigroup

$$T(t)f := \mathrm{e}^{tq} \cdot f\,,\ f \in E.$$

Since M_q is bounded if and only if q is bounded, we conclude that M_q is invertible (with bounded inverse $M_{1/q}$) if and only if

$$0 \notin \overline{\{q(x) : x \in X\}}.$$

Therefore we obtain

$$\sigma(M_q) = \overline{q(X)} = \overline{\{q(x) : x \in X\}},$$

and

$$\sigma(T(t)) = \overline{\{\exp(tq(x)) : x \in X\}}.$$

In particular the following *weak spectral mapping theorem* is valid

$$\sigma(T(t)) = \overline{\exp(t\sigma(M_q))}.$$

In addition, we observe that to each spectral value of A (resp. of $T(t)$) there exists an approximate eigenvector and hence

$$\sigma(A) = A\sigma(A) \text{ and } \sigma(T(t)) = A\sigma(T(t)).$$

Since each Dirac functional is an eigenvector for the adjoint multiplication operator, we obtain

$$q(X) \subset R\sigma(M_q) \text{ and } \mathrm{e}^{tq(X)} \subset R\sigma(T(t)).$$

The eigenvalues of M_q can be characterized as follows.

> $\lambda \in P\sigma(M_q)$ if and only if the set $\{x \in X\colon q(x) = \lambda\}$ has non empty interior (analogously for $P\sigma(T(t))$).

For example, it follows that $P\sigma(M_q) = \emptyset$ for $E = C_0(\mathbb{R}_+)$ and $q(x) = -x$, $x \in \mathbb{R}_+$.

On $E = L^p(X, \Sigma, \mu)$ analogous results are valid, but their exact formulation—using the notion *essential range*, see Goldstein [8]—is left to the reader.

Example 2.4 (The Spectrum of Translation Semigroups) We consider the translation semigroup

$$T(t)f(x) := f(x+t)$$

on $E = C_0(\mathbb{R}_+)$ (or $L^p(\mathbb{R}_+)$, see A-I,2.4). Its generator A is the first derivative and for every $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda < 0$, the function $\varepsilon_\lambda\colon x \mapsto \mathrm{e}^{\lambda x}$ belongs to $D(A)$ and satisfies

$$A\varepsilon_\lambda = \lambda\varepsilon_\lambda,$$

hence $\lambda \in P\sigma(A)$.

Since $\mathcal{T} = (T(t))_{t \geqslant 0}$ is a contraction semigroup it follows that

$$\sigma(A) = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant 0\} \text{ and } \mathrm{i}\mathbb{R} \subset A\sigma(A)$$

(use Proposition 2.2.(i)) or show directly that $f_n(x) = \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{-x/n}$ defines an approximate eigenvector for $\mathrm{i}\alpha$, $\alpha \in \mathbb{R}$). Using the same functions one obtains

$$P\sigma(T(t)) = \{\mathrm{e}^{\lambda t}\colon \operatorname{Re}\lambda < 0\} = \{z \in \mathbb{C}\colon |z| < 1\},$$
$$\sigma(T(t)) = \{z \in \mathbb{C}\colon |z| \leqslant 1\} \text{ for every } t > 0.$$

In the case of the translation group on $E = C_0(\mathbb{R})$ one has $\sigma(A) \subset \mathrm{i}\mathbb{R}$. As above one obtains approximate eigenvectors for every $\alpha \in \mathbb{R}$ from $f_n(x) = \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{-|x|/n}$, hence

$$\sigma(A) = A\sigma(A) = \mathrm{i}\mathbb{R}.$$

The generator A of the nilpotent translation semigroup A-I,2.6 has empty spectrum by A-I, Proposition 1.11. The resolvent is given by

$$R(\lambda, A)f(x) = \mathrm{e}^{\lambda x} \int_x^\infty \mathrm{e}^{-\lambda s} f(s)\, \mathrm{d}s \quad (f \in L^p([0, \tau]), \lambda \in \mathbb{C}).$$

Finally, the generator of the periodic translation group from A-I,2.5 on

$$E = \{f \in C[0, 1] : f(0) = f(1)\}$$

has point spectrum

$$P\sigma(A) = 2\pi\mathrm{i}\mathbb{Z}$$

with eigenfunctions $\varepsilon_n(x) := \exp(2\pi\mathrm{i}nx)$. In Section 5 we show that $\sigma(A) = 2\pi\mathrm{i}\mathbb{Z}$.

We now return to the general theory and recall from Corollary 1.2 that it is very useful (e.g., for stability theory) to be able to convert spectral values of the generator A into spectral values of the semigroup operator $T(t)$ and vice versa. As shown in Examples 1.3 and 1.4 this is not possible in general. Therefore we tackle first a much easier *spectral mapping theorem*: the relation between $\sigma(A)$ and $\sigma(R(\lambda_0))$, where $R(\lambda_0) := R(\lambda_0, A)$ for some $\lambda_0 \in \varrho(A)$.

Proposition 2.5 *Let $(A, D(A))$ be a densely defined closed linear operator with non-empty resolvent set $\varrho(A)$. For each $\lambda_0 \in \varrho(A)$ the following assertions hold.*

(i) $\sigma(R(\lambda_0))\backslash\{0\} = (\lambda_0 - \sigma(A))^{-1}$, *in particular,* $r(R(\lambda_0)) = (\mathrm{dist}(\lambda_0, \sigma(A)))^{-1}$.

(ii) *Analogous statements hold for the point-, approximate point-, residual spectra of A and $R(\lambda_0, A)$.*

(iii) *The point α is isolated in $\sigma(A)$ if and only if $(\lambda_0 - \alpha)^{-1}$ is isolated in $\sigma(R(\lambda_0))$. In that case the residues (resp. the pole orders) in α and in $(\lambda_0 - \alpha)^{-1}$ coincide.*

Proof (i) is well known. It can be found for example in Dunford and Schwartz [5, VII.9.2].

(ii) We show that $\alpha \in A\sigma(A)$ if $(\lambda_0 - \alpha)^{-1} \in A\sigma(R(\lambda_0))$ and leave the proof of the remaining statements to the reader.

Take $(f_n)_{n\in\mathbb{N}} \subset E$ such that $\|f_n\| = 1$, $\|(\lambda_0 - \alpha)^{-1} f_n - R(\lambda_0, A)f_n\| \to 0$ and $\|R(\lambda_0, A)f_n\| \geqslant \frac{1}{2}|\lambda_0 - \alpha|^{-1}$. Define

$$g_n := \|R(\lambda_0, A)f_n\|^{-1} R(\lambda_0, A)f_n \in D(A)$$

and deduce from

$$\begin{aligned}(\alpha - A)g_n &= \|R(\lambda_0, A)f_n\|^{-1} \cdot [(\lambda_0 - A) - (\lambda_0 - \alpha)]R(\lambda_0, A)f_n \\ &= \|R(\lambda_0, A)f_n\|^{-1} \cdot (\lambda_0 - \alpha)[(\lambda_0 - \alpha)^{-1} - R(\lambda_0, A)]f_n\end{aligned}$$

that (g_n) is an approximate eigenvector of A to the eigenvalue α.

(iii) First we recall the wellknown *resolvent equation*. For any $z, \lambda_0 \in \varrho(A)$ we have $R(\lambda_0, A) - R(z, A) = -(\lambda_0 - z)R(\lambda_0, A)R(z, A))$. From this it follows that

$$(\lambda_0 - z)^2 \cdot R(z, A) = \Big((\lambda_0 - z)^{-1} - R(\lambda_0, A)\Big)^{-1} - (\lambda_0 - z).$$

If we now take a circle Γ with center α and sufficiently small radius, then the residue P of $R(\cdot, A)$ at α is

$$\begin{aligned} P &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma R(z, A)\, \mathrm{d}z = \\ &= \frac{1}{2\pi \mathrm{i}} \left[\int_\Gamma (\lambda_0 - z)^{-2} R((\lambda_0 - z)^{-1}\, R(\lambda_0, A))\, \mathrm{d}z - \int_\Gamma (\lambda_0 - z)^{-1}\, \mathrm{d}z \right]. \end{aligned}$$

If λ_0 lies in the exterior of Γ, the second integral is zero. The substitution

$$\tilde{z} := (\lambda_0 - \tilde{z})^{-1}$$

yields a path $\tilde{\Gamma}$ around $(\lambda_0 - \alpha)^{-1}$ and we obtain

$$P = \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\Gamma}} R(\tilde{z}, R(\lambda_0, A))\, \mathrm{d}\tilde{z}$$

which is the residue of $R(\cdot, R(\lambda_0, A))$ at $(\lambda_0 - \alpha)^{-1}$. The final assertion on the pole order follows from the identities

$$V_{-n} = ((\lambda_0 - \alpha)^{-1} R(\lambda_0, A))^{n-1} U_{-n} \quad (n \in \mathbb{N}),$$

where U_n, resp. V_n stand for the n-th coefficient in the Laurent series of $R(\cdot, A)$, resp. $R(\cdot, R(\lambda_0, A))$ at α, resp. $(\lambda_0 - \alpha)^{-1}$. This has already been proved for $n = 1$ and follows for $n > 1$ by induction using the relations

$$U_{-n-1} = (A - \alpha) U_{-n} \quad \text{and} \quad V_{-n-1} = \Big(R(\lambda_0, A) - (\lambda_0 - \alpha)^{-1}\Big) V_{-n}.$$

3 Spectral Decomposition

In the next two sections we develop some important techniques for our further investigation of semigroups and their generators. Even though these methods are well known (compare, e.g., Section VII.3 of Dunford and Schwartz [5]) or rather technical, it is useful to present them in a coherent way.

Our interest in this section is the following: Let E be a Banach space and $\mathcal{T} = (T(t))_{t \geqslant 0}$ a strongly continuous semigroup with generator A. Suppose that the spectrum $\sigma(A)$ splits into the disjoint union of two closed subsets σ_1 and σ_2. Does there exist a corresponding decomposition of the space E and the semigroup $\mathcal{T}$?

In the following definition, we explain what we understand by “corresponding decomposition”.

Definition 3.1 Assume that $\sigma(A)$ is the disjoint union

$$\sigma(A) = \sigma_1 \cup \sigma_2$$

of two non-empty closed subsets σ_1, σ_2. A decomposition

$$E = E_1 \oplus E_2$$

of E into the direct sum of two non-trivial closed $\mathcal{T}$-invariant subspaces is called a *spectral decomposition* corresponding to $\sigma_1 \cup \sigma_2$ if the spectrum $\sigma(A_i)$ of the generator A_i of $\mathcal{T}_i := (T(t)_{|E_i})_{t \geqslant 0}$ coincides with σ_i for $i = 1, 2$.

For a better understanding of the above definition we recall that to every direct sum decomposition $E = E_1 \oplus E_2$ there corresponds a continuous projection $P \in \mathcal{L}(E)$ such that $PE = E_1$ and $P^{-1}(0) = E_2$. Moreover, the subspaces E_1, E_2 are $\mathcal{T}$-invariant if and only if P commutes with the semigroup $\mathcal{T}$, i.e., $T(t)P = PT(t)$ for every $t \geqslant 0$. In this case it follows that the domain $D(A)$ of the generator A splits analogously and $D(A) \cap E_i$ is the domain $D(A_i)$ of the generator A_i of the restricted semigroup $\mathcal{T}_i$, $i = 1, 2$. We write

$$A = A_1 \oplus A_2.$$

and say that A *commutes with* P and call P a *spectral projection*. In terms of the generator A this means that for $f \in D(A)$ we have $Pf \in D(A)$ and $APf = PAf$.

The existence of such projections reduces the semigroup $\mathcal{T}$ into two (possibly simpler) semigroups $\mathcal{T}_1$, $\mathcal{T}_2$ such that

$$\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \quad \text{and} \quad \sigma(T(t)) = \sigma(T_1(t)) \cup \sigma(T_2(t)).$$

For example, in some cases (see Theorem 3.3 below) it can be shown that one of the reduced semigroups has additional properties.

In order to achieve such decompositions we will assume that $\sigma(A)$ decomposes into sets σ_1 and σ_2 and will then try to find a corresponding spectral projection. Unfortunately such spectral decompositions do not exist in general.

Example 3.2 Take the rotation semigroup from A-I,2.4 on the Banach space $L^p(\Gamma)$, $1 \leqslant p < \infty$, $\tau = 2\pi$. It was stated in Example 2.4 and will be proved in Section 5 that its generator A has spectrum

$$\sigma(A) = P\sigma(A) = \mathrm{i}\mathbb{Z}$$

where $\varepsilon_k(z) := z^k$ spans the eigenspace corresponding to $\mathrm{i}k$, $k \in \mathbb{Z}$.

Now, $\sigma(A)$ is the disjoint union of $\sigma_1 := \{0, \mathrm{i}, 2\mathrm{i}, \ldots\}$ and $\sigma_2 := \{-\mathrm{i}, -2\mathrm{i}, \ldots\}$. By a result of M. Riesz there is no projection $P \in \mathcal{L}(L^1(\Gamma))$ satisfying $P\varepsilon_k = \varepsilon_k$

for $k \geqslant 0$, $P\varepsilon_k = 0$ for $k < 0$, hence there is no spectral decomposition of $L^1(\Gamma)$ corresponding to σ_1, σ_2 (Lindenstrauss and Tzafriri [19, p.165]).

On the other hand, for $L^p(\Gamma)$, $1 < p < \infty$, such a spectral projection exists (l.c., 2.c.15). As long as $p \neq 2$ we can always decompose $\sigma(A)$ into suitable subsets admitting no spectral decomposition (l.c., remark before 2.c.15). Clearly, for $p = 2$ such spectral decompositions always exist.

In the above example both subsets σ_1, σ_2 of $\sigma(A)$ are unbounded. But as soon as one of these sets is bounded a corresponding spectral decomposition can always be found.

Theorem 3.3 *Let $\mathcal{T}$ be a strongly continuous semigroup on a Banach space E and assume that the spectrum $\sigma(A)$ of the generator A can be decomposed into the disjoint union of two non-empty closed subsets σ_1 and σ_2.*

If σ_1 is compact, then there exists a unique corresponding spectral decomposition $E = E_1 \oplus E_2$ such that the restricted semigroup $\mathcal{T}_1$ has a bounded generator.

Proof We assume the reader to be familiar with the spectral decomposition theorem for bounded operators (see, e.g., Dunford and Schwartz [5, p.572]) and apply the spectral mapping theorem for the resolvent (Proposition 2.5.(i)) in order to decompose $R(\lambda, A)$ instead of A.

For $\lambda_0 > \omega_0(\mathcal{T})$ it follows from Proposition 2.5 that $\sigma(R(\lambda_0, A)) \setminus \{0\} = (\lambda_0 - \sigma(A))^{-1}$. From $\sigma(A) = \sigma_1 \cup \sigma_2$ we obtain a decomposition of $\sigma(R(\lambda_0, A)) \setminus \{0\}$ into

$$\tau_1 := (\lambda_0 \setminus \sigma_1)^{-1}, \quad \tau_2 := (\lambda_0 \setminus \sigma_2)^{-1}.$$

Since σ_1 is compact, the set τ_1 is compact and does not contain 0. Only in the case that σ_2 is unbounded, the point 0 will be an accumulation point of τ_2. Therefore $\sigma(R(\lambda_0, A)) \cup \{0\}$ is the disjoint union of the closed sets τ_1 and $\tau_2 \cup \{0\}$.

Take now P to be the spectral projection of $R(\lambda_0, A)$ corresponding to this decomposition. Then P commutes with $R(\lambda_0, A)$ (by definition), with $R(\lambda, A)$ for every $\lambda > \omega_0(\mathcal{T})$ (use the series representation of the resolvent), with $T(t)$ for each $t \geqslant 0$ (use A-II, Proposition 1.10) and therefore with the generator A (in the sense explained above). In particular, we obtain

$$R(\lambda_0, A)P = R(\lambda_0, A_1), \quad R(\lambda_0, A)(Id - P) = R(\lambda_0, A_2)$$

for the generator A_1 of $T_1 = (T(t)P)_{t\geqslant 0}$ and A_2 of $T_2 = (T(t)(Id - P))_{t\geqslant 0}$. Applying the Spectral Mapping Theorem 2.5 we conclude

$$\sigma(A_1) = \sigma_1 \text{ and } \sigma(A_2) = \sigma_2,$$

i.e., P is a spectral projection corresponding to σ_1, σ_2. Finally, the above spectral decomposition of $R(\lambda_0, A)$ is unique and satisfies $0 \notin \sigma(R(\lambda_0, A_1))$. Therefore $R(\lambda_0, A_1)^{-1} = (\lambda_0 - A_1)$ is bounded. $\square$

Example If we do not require $\mathcal{T}_1$ to be uniformly continuous, the above spectral decomposition need not be unique, as can be seen from the following example.

Consider a decomposition $E = E_1 \oplus E_2$ and add a direct summand E_3 with a strongly continuous semigroup T_3 whose generator A_3 has empty spectrum (e.g., A-I,Example 2.6). Then still $\sigma(A) = \sigma_1 \cup \sigma_2$, but $E_1 \oplus (E_2 \oplus E_3)$ and $(E_1 \oplus E_3) \oplus E_2$ are two different spectral decompositions corresponding to σ_1, σ_2.

The importance of the above theorem stems from the fact that $\mathcal{T}_1$ has a bounded generator and therefore is easy to deal with. In particular the asymptotic behavior of $\mathcal{T}_1$ can be deduced from the location of σ_1.

Corollary 3.4 *Assume that $\sigma(A)$ splits into non-empty closed sets σ_1, σ_2 where σ_1 is compact and consider the corresponding spectral decomposition $E = E_1 \oplus E_2$ for which $\mathcal{T}_1$ is uniformly continuous.*

For all constants $\nu, \omega \in \mathbb{R}$ satisfying

$$\nu < \inf\{\operatorname{Re}\lambda : \lambda \in \sigma_1\} \leqslant \sup\{\operatorname{Re}\lambda : \lambda \in \sigma_1\} < \omega$$

there exist $m \leqslant 1$, $M \geqslant 1$ such that

$$m \cdot \mathrm{e}^{\nu t}\|f\| \leqslant \|T_1(t)f\| \leqslant M \cdot \mathrm{e}^{\omega t}\|f\|$$

for every $f \in E_1$, $t \geqslant 0$.

Proof Since the generator A_1 of $\mathcal{T}_1$ is bounded, we have $T_1(t) = \exp(tA_1)$ and $\sigma(T_1(t)) = \exp(t\sigma(A_1))$. Therefore by the remark following Proposition 1.1, the spectral bound $s(A_1)$ coincides with the growth bound $\omega_0(T_1)$ and we have the upper estimate. The lower estimate is obtained by applying the same reasoning to $-A_1$ which generates the semigroup $(T_1(t)^{-1})_{t\geqslant 0}$ on E_1. □

It is clear from Examples 1.3 and 1.4 on page 73 that no norm estimates for $(T_2(t))_{t\geqslant 0}$ can be obtained from the location of σ_2. Only by adding appropriate hypotheses we will achieve spectral decompositions admitting norm estimates on both components (see Theorem 6.6 below).

Another way of obtaining such norm estimates is by constructing spectral decompositions starting from a semigroup operator $T(t_0)$ (instead of A, and $R(\lambda, A)$ resp., as in Theorem 3.3).

Corollary 3.5 *If $\sigma(T(t_0)) = \tau_1 \cup \tau_2$ for two non-empty, closed, disjoint sets τ_1, τ_2 and if P is the spectral projection corresponding to $T(t_0)$ and τ_1, τ_2, then $\sigma(A)$ splits into closed subsets σ_1, σ_2 and P is the corresponding spectral projection for $\mathcal{T}$ and σ_1, σ_2.*

Proof The spectral projection P of $T(t_0)$ is obtained by integrating $R(\lambda, T(t_0))$ (see, e.g., Dunford and Schwartz [5, Section VII.3]). Since every $T(t)$, $t \geqslant 0$, commutes with $T(t_0)$, it must commute with $R(\lambda, T(t_0))$, hence with P. The statement on the decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$ follows from the Spectral Inclusion Theorem 6.2 below. □

This decomposition can be applied to the study of the asymptotic behavior of $\mathcal{T}$. In the situation of Corollary 3.5 assume

$$\sup\{|\lambda|: \lambda \in \tau_2\} < \alpha < \inf\{|\lambda|: \lambda \in \tau_1\}$$

for some $\alpha > 0$. If we set $\beta := (\log \alpha)/t_0$ and use Pazy [21, Chap.I, Theorem 6.5], we obtain $\omega_0(\mathcal{T}_2) < \beta$ and $\omega_0(\mathcal{T}_1^{-1}) < \beta$ by Proposition 1.1. Therefore we have constants m, M with $m \le 1 \le M$ such that

$$\|T(t)f\| \leqslant M \cdot \mathrm{e}^{\beta t}\|f\| \quad \text{for } f \in E_2,$$
$$\|T(t)f\| \geqslant m \cdot \mathrm{e}^{-\beta t}\|f\| \quad \text{for } f \in E_1.$$

As nice as they might look, results of this type are unsatisfactory. We need information on the semigroup in order to estimate its asymptotic behavior. In Chapter IV we will try to obtain such results by exploiting information about the generator only.

Example 3.6 (Isolated singularities and poles)

In case that λ_0 is an isolated point of $\sigma(A)$ the holomorphic function $\lambda \mapsto R(\lambda, A)$ can be expanded as a Laurent series

$$R(\lambda, A) = \sum_{n=-\infty}^{+\infty} U_n(\lambda - \lambda_0)^n \text{ for } 0 < |\lambda - \lambda_0| < \delta \text{ and some } \delta > 0.$$

The coefficients U_n are bounded linear operators given by

$$U_n = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (z - \lambda_0)^{-(n+1)} R(z, A)\, \mathrm{d}z,\ n \in \mathbb{Z}, \tag{3.1}$$

where $\Gamma = \{z \in \mathbb{C}: |z - \lambda_0| = \delta/2\}$. The coefficient U_{-1} is the spectral projection corresponding to the spectral set $\{\lambda_0\}$ (see Definition 3.1). It is called the *residue* of $R(\cdot, A)$ at λ_0 and will be denoted by P. From (3.1) one deduces

$$U_{-(n+1)} = (A - \lambda_0)^n \circ P \quad \text{and} \quad U_{-(n+1)} \circ U_{-(m+1)} = U_{-(n+m+1)} \text{ for } n, m \geqslant 0.$$

If there exists $k > 0$ such that $U_{-k} \neq 0$ while $U_{-n} = 0$ for all $n > k$, the point λ_0 is called a *pole of* $R(\cdot, A)$ *of order* k. In view of (3.2) this is true if $U_{-k} \neq 0$ and $U_{-(k+1)} = 0$. In this case one can retrieve U_{-k} as

$$U_{-k} = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^k R(\lambda, A). \tag{3.2}$$

The dimension of PE (i.e., the dimension of the spectral subspace corresponding to $\{\lambda_0\}$) is called the *algebraic multiplicity* m_a of λ_0, while the *geometric multi-*

plicity is $m_g := \dim\ker(\lambda_0 - A)$. In case $m_a = 1$, we call λ_0 an *algebraically simple pole*.

If k is the pole order ($k = \infty$ in case of an essential singularity), we have

$$\max\{m_g, k\} \leqslant m_a \leqslant k \cdot m_g \tag{3.3}$$

where $\infty \cdot 0 = \infty$.

These inequalities yield the following implications.

(i) $m_a < \infty$ if and only if λ_0 is a pole with $m_g < \infty$,

(ii) if λ_0 is a pole with order k, then $\lambda_0 \in P\sigma(A)$ and $PE = \ker(\lambda_0 - A)^k$.

If A has compact resolvent, then every point of $\sigma(A)$ is a pole of finite algebraic multiplicity. This is a consequence of Proposition 2.5.(iii) and the well-known Riesz-Schauder Theory for compact operators (see Dunford and Schwartz [5, VII.4.5]).

Example 3.7 ((The essential spectrum)) For an operator $T \in \mathcal{L}(E)$ the *Fredholm domain* $\varrho_F(T)$ is

$$\begin{aligned} \varrho_F(T) &:= \{\lambda \in \mathbb{C} \colon \lambda - T \text{ is a Fredholm operator}\} \\ &= \{\lambda \in \mathbb{C} \colon \ker(\lambda - T) \text{ and } E/\mathrm{im}(\lambda - T) \text{ are finite dimensional}\}. \end{aligned} \tag{3.4}$$

An equivalent characterization of $\varrho_F(T)$ is obtained through the *Calkin algebra* $\mathcal{L}(E)/\mathcal{K}(E)$, where $\mathcal{K}(E)$ stands for the closed ideal of all compact operators. In fact, $\varrho_F(T)$ coincides with the resolvent set of the canonical image of T in the Calkin algebra. The complement of $\varrho_F(T)$ is called *essential spectrum* of T and denoted by $\sigma_{\mathrm{ess}}(T)$. The corresponding spectral radius, called *essential spectral radius*, satisfies

$$r_{\mathrm{ess}}(T) := \sup\{|\lambda| \colon \lambda \in \sigma_{\mathrm{ess}}(T)\} = \lim_{n\to\infty} \|T^n\|_{\mathrm{ess}}^{1/n}, \tag{3.5}$$

where

$$\|T\|_{\mathrm{ess}} = \mathrm{dist}(T, \mathcal{K}(E)) := \inf\{\|T - K\| \colon K \in \mathcal{K}(E)\}$$

is the norm of T in $\mathcal{L}(E)/\mathcal{K}(E)$.

For every compact operator K we have $\|T - K\|_{\mathrm{ess}} = \|T\|_{\mathrm{ess}}$, hence

$$r_{\mathrm{ess}}(T - K) = r_{\mathrm{ess}}(T). \tag{3.6}$$

A detailed analysis of $\varrho_F(T)$ can be found in Section IV.5.6 of Kato [16]. In particular we recall that the poles of $R(\cdot, T)$ with finite algebraic multiplicity belong to $\varrho_F(T)$. Conversely, an element of the unbounded component of $\varrho_F(T)$ either belongs to $\varrho(T)$ or is a pole of finite algebraic multiplicity.

Thus $r_{\mathrm{ess}}(T)$ can be characterized as

$r_{\text{ess}}(T)$ is the smallest $r \in \mathbb{R}_+$ such that every $\lambda \in \sigma(T), |\lambda| > r$ is a pole of finite algebraic multiplicity. (3.7)

Now, if $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a strongly continuous semigroup, then VIII.1, Lemma 4 of Dunford and Schwartz [5] applied to the function $t \mapsto \log \|T(t)\|_{\text{ess}}$ ensures that

$$\omega_{\text{ess}}(\mathcal{T}) := \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|_{\text{ess}} = \inf\{\frac{1}{t} \log \|T(t)\|_{\text{ess}} : t > 0\} \tag{3.8}$$

is well defined (possibly $-\infty$). By the definition of $\omega_{\text{ess}}(\mathcal{T})$ and (3.5) we have

$$r_{\text{ess}}(T(t)) = \exp(t\omega_{\text{ess}}(\mathcal{T})), \quad t \geqslant 0. \tag{3.9}$$

Obviously, $\omega_{\text{ess}} \leqslant \omega_0$ and equality occurs if and only if $r_{\text{ess}}(T(t)) = r(T(t))$ for $t \geqslant 0$.

If $\omega_{\text{ess}} < \omega_0$, there exists an eigenvalue λ of $T(t)$ satisfying $|\lambda| = r(T(t))$, hence by Theorem 6.3 below there exists $\lambda_1 \in P\sigma(A)$ such that $\operatorname{Re}\lambda_1 = \omega_0$. Thus $\omega_{\text{ess}} < \omega_0$ implies $s(A) = \omega_0(\mathcal{T})$, i.e., we have

$$\omega_0(\mathcal{T}) = \max\{\omega_{\text{ess}}(\mathcal{T}), s(A)\}. \tag{3.10}$$

As a final observation we point out that

$$\omega_{\text{ess}}(\mathcal{T}) = \omega_{\text{ess}}(\mathcal{S}), \tag{3.11}$$

whenever $\mathcal{T}$ is generated by A and $\mathcal{S}$ is generated by $A + K$ for some compact operator K (see Proposition 2.8 and Proposition 2.9 of B-IV).

4 The Spectrum of Induced Semigroups

In the previous section we tried to decompose a semigroup into the direct sum of two, hopefully simpler objects. Here we present other methods to reduce the complexity of a semigroup and its generator. Forming subspace or quotient semigroups as in A-I,3.2, A-I,3.3 are such methods. But also the constructions of new semigroups on canonically associated spaces such as the dual space, see A-I,3.4, or the $\mathcal{F}$-product, see A-I,3.6, might be helpful. We review these constructions under the spectral theoretical point of view and collect a number of technical properties for later use.

We start by studying the spectrum of subspace and quotient semigroups. To that purpose assume that the strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ leaves invariant some closed subspace N of the Banach space E. There are canonically induced semigroups $\mathcal{T}_|$ on N, resp. $\mathcal{T}_/$ on E/N and their generators $A_|$, resp. $A_/$ are canonically obtained from the generator A of $\mathcal{T}$ (see A-I, Section 3). The following example shows that the spectra of A, $A_|$ and $A_/$ may differ quite drastically.

Example 4.1 As in the example in A-I,3.3 we consider the translation semigroup on $E = L^1(\mathbb{R})$ and the invariant subspace

$$N := \{f \in E\colon f(x) = 0 \text{ for } x \geqslant 1\}.$$

Then $\sigma(A) = \mathrm{i}\mathbb{R}$, but $\sigma(A_|) = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant 0\}$. Next we take the translation invariant subspace

$$M := \{f \in N\colon f(x) = 0 \text{ for } 0 \leqslant x \leqslant 1\}$$

and obtain $\sigma(A_{|/}) = \emptyset$ for the generator $A_{|/}$ of the quotient semigroup $\mathcal{T}_{|/}$ (use the fact that $\mathcal{T}_{|/}$ is nilpotent).

In the next proposition we collect the information on $\sigma(A)$ which in general can be obtained from the "subspace spectrum" $\sigma(A_|)$ and the "quotient spectrum" $\sigma(A_/)$.

Proposition 4.2 *Using the standard notations the following inclusions hold.*

(i) $\varrho(A) \subset [\varrho(A_|) \cap \varrho(A_/)] \cup [\sigma(A_|) \cap \sigma(A_/)]$,

(ii) $[\varrho(A_|) \cap \varrho(A_/)] \subset \varrho(A)$,

(iii) $\varrho_+(A) \subset [\varrho(A_|) \cap \varrho(A_/)]$,

where $\varrho_+(A)$ denotes the connected component of $\varrho(A)$ which is unbounded to the right.

Proof (i) Assume $\lambda \in \varrho(A)$, i.e., $(\lambda - A)$ is a bijection from $D(A)$ onto E. Since N is T-invariant, we have $D(A_|) = D(A) \cap N$ and $(\lambda - A)D(A_|) \subset N$. If $(\lambda - A)D(A_|) = N$, then $R(\lambda, A)N = D(A_|)$ and the induced operators $R(\lambda, A)_|$, resp. $R(\lambda, A)_/$ are the inverses of $(\lambda - A_|)$, resp. $(\lambda - A_/)$. If $(\lambda - A)D(A_|) \neq N$, then $\lambda \in \sigma(A_|)$.

In addition there exists $f \in D(A)\backslash N$ such that $g := (\lambda - A)f \in N$. Hence for $\hat{f} := f + N$, $\hat{g} := g + N \in E_/$ it follows that $(\lambda - A_/)\hat{f} = \hat{g} = 0$, i.e., $\lambda \in \sigma(A_/)$

(ii) Take $\lambda \in \varrho(A_|) \cap \varrho(A_/)$. Then $(\lambda - A)$ is injective since $(\lambda - A)f = 0$ implies $(\lambda - A_/)\hat{f} = 0$, hence $\hat{f} = 0$, i.e., $f \in N$ and therefore $f = 0$.

In addition, $(\lambda - A)$ is surjective: For $g \in E$ there exists $\hat{f} \in E_/$ such that $(\lambda - A_/)\hat{f} = \hat{g}$, i.e., there exists $h \in N$ such that $(\lambda - A)f - g = h = (\lambda - A)k$ for some $k \in D(A_|)$. Therefore we obtain $(\lambda - A)(f - k) = g$.

(iii) The integral representation of the resolvent for $\lambda > \omega_0(\mathcal{T})$ (see A-I, Proposition 1.11) shows that $R(\lambda, A)N \subset N$. By the power series expansion for holomorphic functions this extends to all $\lambda \in \varrho_+(A)$. Therefore the restriction $R(\lambda, A)_|$ coincides with the resolvent $R(\lambda, A_|)$. On the other hand $R(\lambda, A)_/$ is well defined on $E_/$ and satisfies

$$R(\lambda, A)_/(f + N) = R(\lambda, A)f + N$$

(use again the integral representation). This proves that $R(\lambda, A)_/ = R(\lambda, A_/)$. □

Corollary 4.3 *Under the above assumptions take a point* μ *in the closure of* $\varrho_+(A)$. *Then*

(i) $\mu \in \sigma(A)$ *if and only if* $\mu \in \sigma(A_|)$ *or* $\mu \in \sigma(A_/)$.

(ii) μ *is a pole of* $R(\cdot, A)$ *if and only if* μ *is a pole of* $R(\cdot, A_|)$ *and of* $R(\cdot, A_/)$.

In that case,

$$\max\{k_|, k_/\} \leqslant k \leqslant k_| + k_/$$

for the respective pole orders. Note that hereby pole orders 0 *are allowed.*

Proof (i) This follows from Proposition 4.2 (ii) and (iii).

(ii) By the previous assertion we may assume that for some $\delta > 0$ the pointed disc

$$\{\lambda \in \mathbb{C} \colon 0 < |\lambda - \mu| < \delta\}$$

is contained in $\varrho(A) \cap \varrho(A_|) \cap \varrho(A_/)$.

Call U_n the coefficients of the Laurent expansion of $R(\cdot, A)$. Since N is $R(\lambda, A)$-invariant for $\lambda \in \varrho_+(A)$, the same holds for each U_n. With the obvious notations we have $R(\lambda, A) = \sum_n U_n(\lambda - \mu)^n$, $R(\lambda, A)_| = \sum U_{n|}(\lambda - \mu)^n$ and $R(\lambda, A)_/ = \sum U_{n/}(\lambda - \mu)^n$ which shows $\max\{k_|, k_/\} \leqslant k$.

If $R(\cdot, A)_|$ has a pole in μ of order ℓ, then $U_{-(\ell+1)|} = 0$, i.e., $U_{-(\ell+1)} N = \{0\}$. Similarly, it follows that $U_{-(m+1)} E \subset N$ if $R(\cdot, A)_/$ has a pole in μ of order m. Therefore $U_{-(\ell+1)} \circ U_{-(m+1)} = 0$.

The relations (3.2) imply $U_{-(m+\ell+1)} = 0$, hence the pole order of $R(\cdot, A)$ is dominated by $\ell + m$. □

4.1 Spectrum of the adjoint semigroup

We recall from A-I,3.4 that to every strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ there corresponds a strongly continuous adjoint semigroup $\mathcal{T}^* = (T(t)^*)_{t \geqslant 0}$ on the semigroup dual

$$E^* = \{\varphi \in E' \colon \lim_{t \to \infty} \|T(t)'\varphi - \varphi\| = 0\}.$$

Its generator A^* is the maximal restriction of the adjoint A' to E^*. For these operators the spectra coincide, or more precisely.

(i) $\sigma(T(t)) = \sigma(T(t)') = \sigma(T(t)^*)$,
$R\sigma(T(t)) = P\sigma(T(t)') = P\sigma(T(t)^*)$,

(ii) $\sigma(A) = \sigma(A') = \sigma(A^*)$, $R\sigma(A) = P\sigma(A') = P\sigma(A^*)$,

(iii) $s(A) = s(A^*)$, $\omega_0(A) = \omega_0(A^*)$.

Proof The left part of these equalities is either well known or has been stated in 2.2(ii). The first statement of (iii) follows from (ii), while the second is an immediate consequence of the estimate $\|T(t)^*\| \leqslant \|T(t)\| \leqslant M \cdot \|T(t)^*\|$ given in A-I,3.4.

As a sample for the remaining assertions we show that $0 \notin \sigma(A)$ if and only if $0 \notin \sigma(A^*)$: If A and therefore A' is invertible, it follows from A-I,3.4 that A^* is a bijection from $D(A^*)$ onto E^*.

Conversely assume that A^* is invertible. Then A' must be injective by the Proposition in A-I,3.4. Moreover $A'(D(A'))$ contains $A^*(D(A^*)) = E^*$ and is $\sigma(E', E)$-dense in E'. By standard duality arguments it follows that A is injective with dense image. Next we show that $A(D(A))$ is closed: For $f \in D(A)$ choose $\varphi \in D(A')$ such that $\|\varphi\| \leqslant 1$ and $|\langle f, \varphi\rangle| \geqslant \frac{1}{2}\|f\|$. Then

$$\begin{aligned}\|(A^*)^{-1}\|\|Af\| \geqslant \|(A^*)^{-1}\||\langle Af, \varphi\rangle| &\geqslant |\langle Af, (A^*)^{-1}\varphi\rangle| \\ = |\langle f, \varphi\rangle| &\geqslant \frac{1}{2}\|f\|,\end{aligned}$$

hence

$$\|Af\| \geqslant \frac{1}{2}\|(A^*)^{-1}\|^{-1}\|f\|,$$

and $A(D(A))$ is closed since A is closed. □

4.2 Spectrum of the $\mathcal{F}$-product semigroup

As stated in A-I,3.6 the $\mathcal{F}$-product semigroup $\mathcal{T}_{\mathcal{F}} = (T_{\mathcal{F}}(t))_{t\geqslant 0}$ on $E_{\mathcal{F}}^{\mathcal{T}}$ of a strongly continuous semigroup $\mathcal{T}$ on E serves to convert sequences in E into points in $E_{\mathcal{F}}^{\mathcal{T}}$. In particular it can be used to convert approximate eigenvectors of the generator A into eigenvectors of $A_{\mathcal{F}}$.

Proposition 4.4 *Let A be the generator of a strongly continuous semigroup. Then the generator $A_{\mathcal{F}}$ of the $\mathcal{F}$-product semigroup satisfies.*

(i) $A\sigma(A) = A\sigma(A_{\mathcal{F}}) = P\sigma(A_{\mathcal{F}})$,

(ii) $\sigma(A) = \sigma(A_{\mathcal{F}})$.

Remark 4.5 In case A is bounded, then the canonical extension $A_{\mathcal{F}}$ is a generator and $E_{\mathcal{F}}^{\mathcal{T}} = E_{\mathcal{F}}$ (cf. A-I,3.6). Thus the proposition applies to bounded linear operators and their canonical extensions to the $\mathcal{F}$-product $E_{\mathcal{F}}$.

Proof (Proof of the proposition) (i) The inclusion $P\sigma(A_{\mathcal{F}}) \subset A\sigma(A_{\mathcal{F}})$ holds trivially.

We show that $A\sigma(A_{\mathcal{F}}) \subset A\sigma(A)$: Take $\lambda \in A\sigma(A_{\mathcal{F}})$ and an associated approximate eigenvector $(\hat{f}^m)_{n\in\mathbb{N}}$, i.e., $\hat{f}^m \in D(A_{\mathcal{F}})$, $\|\hat{f}^m\| = 1$ and $(\lambda - A_{\mathcal{F}})\hat{f}^m \to 0$ as $m \to \infty$.

By the considerations in A-I,3.6 we can represent each $\hat{f}^m$ as a normalized sequence $(f_n^m)_{n\in\mathbb{N}}$ in $D(A)$ such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|(\lambda - A)f_n^m\| = 0.$$

Therefore we can find a sequence $g_k = f_k^{m(k)}$ satisfying

$$\lim_{k\to\infty} \|(\lambda - A)g_k\| = 0,$$

i.e., $\lambda \in A\sigma(A)$.

Finally we show $A\sigma(A) \subset P\sigma(A_{\mathcal{F}})$: For $\lambda \in A\sigma(A)$ take a corresponding approximate eigenvector (f_n). By A-I,(3.2) we have

$$\begin{aligned}\|T(t)f_n - f_n\| &\leqslant \|T(t)f_n - e^{\lambda t} f_n\| + |e^{\lambda t} - 1| \\ &= \left\| \int_0^t e^{\lambda(t-s)} T(s)(\lambda - A)f_n \, ds \right\| + |e^{\lambda t} - 1|\end{aligned}$$

which converges to zero uniformly in n as $t \to 0$, i.e., $(f_n) \in m^{\mathcal{T}}(E)$. From the characterization of $D(A_{\mathcal{F}})$ given in A-I,3.6 it follows that

$$\hat{f} := (f_n) + c_F(E) \in D(A_{\mathcal{F}}) \quad \text{and} \quad A_{\mathcal{F}}\hat{f} = \lambda\hat{f}$$

i.e., $\lambda \in P\sigma(A_{\mathcal{F}})$.

(ii) The inclusion $A\sigma(A) \subset \sigma(A_{\mathcal{F}})$ follows from (i).

Now we show $R\sigma(A) \subset R\sigma(A_{\mathcal{F}})$. For $\lambda \in R\sigma(A)$ choose $f \in E$ such that $\|(\lambda - A)g - f\| \geqslant 1$ for every $g \in D(A)$. Then $\|(\lambda - A_{\mathcal{F}})g - \hat{f}\| \geqslant 1$ for every $\hat{g} \in D(A_{\mathcal{F}})$ and $\hat{f} = (f, f, \dots) + c_F(E)$. Therefore $\lambda \in R\sigma(A_{\mathcal{F}})$.

We now show $\varrho(A) \subset \varrho(A_{\mathcal{F}})$: Assume $\lambda \in \varrho(A)$. By (i) $(\lambda - A_{\mathcal{F}})$ has to be injective. Choose $\hat{f} = (f_1, f_2, \dots) + c_{\mathcal{F}}(E)$ such that $(f_n) \in m^{\mathcal{T}}(E)$. Then $(R(\lambda, A)f_n) \in m^{\mathcal{T}}(E)$ and $(\lambda - A_{\mathcal{F}})((R(\lambda, A)f_n) + c_{\mathcal{F}}(E)) = (f_n) + c_{\mathcal{F}}(E)$, i.e., $(\lambda - A_{\mathcal{F}})$ is surjective and $\lambda \in \varrho(A_{\mathcal{F}})$. □

Applying this proposition to a single operator $T(t)$, we obtain

$$A\sigma(T(t)) = P\sigma(T(t)_{\mathcal{F}}).$$

Note that in general $A\sigma(T(t)) \neq P\sigma(T_{\mathcal{F}}(t))$ (see the Examples 1.3 and 1.4 in combination with Theorem 6.3).

5 The Spectrum of Periodic Semigroups

In this section we determine the spectrum of a particularly simple class of strongly continuous semigroups and thereby achieve a rather complete description of the semigroup itself. Besides being nice and simple these semigroups gain their importance as building blocks for the general theory.

Definition 5.1 A strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ on a Banach space E is called *periodic* if $T(t_0) = \mathrm{Id}$ for some $t_0 > 0$.

The *period* τ of $\mathcal{T}$ is obtained as

$$\tau := \inf\{t_0 > 0\colon T(t_0) = \mathrm{Id}\}.$$

We immediately observe that periodic semigroups are groups with inverses $T(t)^{-1} = T(n\tau - t)$ for $0 \leqslant t \leqslant n\tau$, τ the period of $\mathcal{T}$. Moreover, they are bounded, hence the growth bound is zero and $\sigma(A) \subset \mathrm{i}\mathbb{R}$.

Lemma 5.2 *Let T be a strongly continuous semigroup with period $\tau > 0$ and generator A. Then*

$$\sigma(A) \subset 2\pi\mathrm{i}/\tau \cdot \mathbb{Z}$$

and

$$R(\mu, A) = (1 - \mathrm{e}^{-\mu\tau})^{-1} \int_0^\tau \mathrm{e}^{-\mu s} T(s)\ \mathrm{d}s \tag{5.1}$$

for $\mu \notin 2\pi\mathrm{i}/\tau \cdot \mathbb{Z}$.

Proof From the basic identities A-I,(3.1) and A-I,(3.2) for $t = \tau$, it follows that $(\mu - A)$ has a left and right inverse if $\mu \neq 2\pi\mathrm{i}n/\tau$, $n \in \mathbb{Z}$, and that the inverse is given by the above expression. □

The representation of $R(\mu, A)$ given in A-I, Proposition 1.11 shows that the resolvent of the generator of a periodic semigroup is a meromorphic function having only poles of order one and the residues

$$P_n := \lim_{\mu\to\mu_n} (\mu - \mu_n) R(\mu, A) \quad \text{in} \quad \mu_n := 2\pi\mathrm{i}n/\tau, \quad n \in \mathbb{Z},$$

are

$$P_n = \frac{1}{\tau} \int_0^\tau \exp(-\mu_n s) T(s)\,\mathrm{d}s. \tag{5.2}$$

Moreover, it follows that the spectrum of A consists of eigenvalues only and each P_n is the spectral projection belonging to μ_n (see 3.6). Another way of looking at P_n is given by saying that P_n is the n-th Fourier coefficient of the τ-periodic function $s \mapsto T(s)$. From this it follows that no non-zero $\varphi \in E'$ vanishes on all $P_n E$ simultaneously. By the Hahn-Banach theorem we conclude that span $\cup_{n\in\mathbb{Z}}$ $P_n E$ is dense in E.

Since $P_n E \subset D(A)$, we obtain from A-I,(3.1) that

$$AP_n f = \mu_n P_n f \tag{5.3}$$

for every $f \in E$, $n \in \mathbb{Z}$. This and A-I,(3.2) imply

$$T(t)P_n f = \exp(\mu_n t) \cdot P_n f \tag{5.4}$$

for every $t \geqslant 0$. Therefore μ_n is an eigenvalue of A and $\exp(\mu_n t)$ is an eigenvalue of $T(t)$ if and only if $P_n \neq 0$. In that case, $P_n E$ is the corresponding eigenspace and we have the following lemma.

Lemma 5.3 *For a τ-periodic semigroup $\mathcal{T}$ we take $\mu_n := 2\pi i n/\tau$, $n \in \mathbb{Z}$, and consider*

$$P_n := \frac{1}{\tau}\int_0^\tau \exp(-\mu_n s)T(s)\,\mathrm{d}s.$$

Then the following assertions are equivalent.

(a) $P_n \neq 0$,

(b) $\mu_n \in P\sigma(A)$,

(c) $\exp(\mu_n t) \in P\sigma(T(t))$ *for every* $t > 0$.

The action of A, resp. $T(t)$ in the subspaces $P_n E$, $n \in \mathbb{Z}$, is determined by (5.3) and (5.4) resp. Moreover,

$$P_m P_n f = \frac{1}{\tau}\int_0^\tau \exp(-\mu_m s)T(s)P_n f\,\mathrm{d}s = \frac{1}{\tau}\int_0^\tau \exp((\mu_n - \mu_m)s)P_n f\,\mathrm{d}s = 0$$

for $n \neq m$, the subspaces $P_n E$ are "orthogonal". Since their union is total in E, one expects to be able to extend the representations (5.3) and (5.4) of A and $T(t)$. This is possible if

$$\sum_{-\infty}^{+\infty} P_n = \mathrm{Id},$$

where the series should be summable for the strong operator topology.

Unfortunately this is false in general since the family of projections

$$Q_H := \sum_{n \in H} P_n,$$

where H runs through all finite subsets of $\mathbb{Z}$, may be unbounded (see the example below). Nevertheless the following is true.

Theorem 5.4 *Let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a τ-periodic semigroup on a Banach space E with generator A and associated spectral projections*

$$P_n := \frac{1}{\tau}\int_0^\tau \exp(-\mu_n s)T(s)\,\mathrm{d}s, \quad \mu_n := 2\pi\mathrm{i}n/\tau, \quad n \in \mathbb{Z}.$$

For every $f \in D(A)$ one has $f = \sum_{-\infty}^{+\infty} P_n f$ and therefore

(i) $T(t)f = \sum_{-\infty}^{+\infty} \exp(\mu_n t)P_n f$ *if $f \in D(A)$,*

(ii) $Af = \sum_{-\infty}^{+\infty} \mu_n P_n f$ *if $f \in D(A^2)$.*

Proof It suffices to prove the first statement. Then (i) and (ii) follow by (5.3) and (5.4).

We assume $\tau = 2\pi$ and show first that $\sum_{-\infty}^{+\infty} P_n f$ is summable for $f \in D(A)$: For $g := Af$ we obtain $P_n g = P_n Af = AP_n f = inP_n f$. Take H to be a finite subset of $\mathbb{Z} \setminus \{0\}$ and $\varphi \in E'$. Then

$$\left|\sum_{n\in H}\langle P_n f, \varphi\rangle\right| = \left|\sum_{n\in H}\frac{1}{in}\langle P_n g, \varphi\rangle\right| \leqslant \left(\sum_{n\in H}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n\in H}|\langle P_n g, \varphi\rangle|^2\right)^{1/2}.$$

From Bessel's inequality we obtain for the second factor

$$\sum_{n\in H}|\langle P_n g, \varphi\rangle|^2 \leqslant \frac{1}{2\pi}\cdot\int_0^{2\pi}|\langle T(s)g, \varphi\rangle|^2\,\mathrm{d}s \leqslant \|\varphi\|^2\cdot\frac{1}{2\pi}\cdot\int_0^{2\pi}\|T(s)g\|^2\,\mathrm{d}s.$$

With the constant $c := \left(\frac{1}{2\pi}\cdot\int_0^{2\pi}\|T(s)g\|^2\,\mathrm{d}s\right)^{1/2}$ we obtain

$$\left\|\sum_{n\in H}P_n f\right\| \leqslant c\left(\sum_{n\in H}n^{-2}\right)^{1/2}$$

for every finite subset H of $\mathbb{Z}$, i.e., $\sum_{-\infty}^{+\infty} P_n f$ is summable.

Next we set $h := \sum_{-\infty}^{+\infty} P_n f$ and observe that for every $\varphi' \in E'$ the Fourier coefficients of the continuous, τ-periodic functions $s \mapsto \langle T(s)h, \varphi\rangle$ and $s \mapsto \langle T(s)f, \varphi\rangle$ coincide. Therefore these functions are identical for $s \geqslant 0$ and in particular for $s = 0$, i.e., $\langle h, \varphi\rangle = \langle f, \varphi\rangle$. By the Hahn-Banach Theorem we obtain $f = h$. □

The above theorem contains rather precise information on periodic semigroups. In particular, it characterizes periodic semigroups by the fact that $\sigma(A)$ is contained in $\mathrm{i}\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$ and the eigenfunctions of A form a total subset of E.

For a periodic semigroup with bounded generator only a finite number of spectral projections P_n are distinct from 0 and we have the following characterization.

Corollary 5.5 *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a semigroup with bounded generator on some Banach space E.*

This semigroup has period τ/k for some $k \in \mathbb{N}$ if and only if there exist finitely many pairwise orthogonal projections P_n, $-m \leqslant n \leqslant m$, $P_{-m} \neq 0$ or $P_m \neq 0$, such that

(i) $\sum_{-m}^{+m} P_n = \mathrm{Id}$,

(ii) $T(t) = \sum_{-m}^{+m} \exp(2\pi \mathrm{i} nt/\tau) P_n$,

(iii) $A = \sum_{-m}^{+m} (2\pi \mathrm{i} n/\tau) P_n$.

Example 5.6 From A-I,2.5 we recall briefly the rotation group

$$R_\tau(t) f(z) := f(\exp(2\pi \mathrm{i} nt/\tau) \cdot z)$$

on $E = C(\Gamma)$, resp. $E = L^p(\Gamma, m)$ for $1 \leqslant p < \infty$. The spectrum of the generator $Af(z) = (2\pi \mathrm{i}/\tau) z \cdot f'(z)$ is $\sigma(A) = (2\pi \mathrm{i}/\tau) \cdot \mathbb{Z}$. The eigenfunctions $\varepsilon_n(z) := z^n$ yield the projections

$$P_n = (1/2\pi \mathrm{i}) \cdot \varepsilon_{-(n+1)} \otimes \varepsilon_n, \text{ i.e.,}$$

$$P_n f(z) = (1/2\pi \mathrm{i}) \cdot (\int_\Gamma f(w) w^{-(n+1)}\, dw) \cdot z^n.$$

It is left as an exercise to compute the norms of $Q_m := \sum_{-m}^{+m} P_n$ in $L^p(\Gamma, m)$ for various p and then check the assertions of Theorem 5.4.

Clearly, this proves some classical convergence theorems for Fourier series (compare Davies [3, Chap.8.1]).

6 Spectral Mapping Theorems

We now return to the question posed in the introduction to this chapter: In which form and under which conditions is it true that the spectrum $\sigma(T(t))$ of the semigroup operators is obtained—via the exponential map—from the spectrum $\sigma(A)$ of the generator, or briefly

Do we have $\sigma(T(t)) = \exp(t\sigma(A))$ or at least $\sigma(T(t)) = \overline{\exp(t\sigma(A))}$?

This and similar statements will be called *spectral mapping theorems* for the semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ and its generator A. In addition, we saw in Proposition 1.1 that the validity of such a spectral mapping theorem implies

$$s(A) = \omega_0(A)$$

for the spectral- and growth bounds and therefore guarantees that the location of the spectrum of A determines the asymptotic behavior of $\mathcal{T}$. As we have seen in Examples 1.3 and 1.4 the last statement does not hold in general. We therefore present a detailed analysis, where and why it fails and what additional assumptions are needed for its validity. Before doing so, we have another look at the examples.

Example 6.1 (The counterexamples revisited)

(i) Take the nilpotent translation semigroup from A-I,2.6. Then $\sigma(A) = \emptyset$ and $\sigma(T(t)) = 0$ for every $t > 0$. By this trivial example and since $\mathrm{e}^z \neq 0$ for every $z \in \mathbb{C}$, it is natural to read the spectral mapping theorem modulo the addition of $\{0\}$, i.e.,

$$\sigma(T(t)) \setminus \{0\} = \exp(t\sigma(A)) \text{ for } t \geqslant 0.$$

(ii) The spectrum of the generator A of the τ-periodic rotation group $(R_\tau(t))_{t\geqslant 0}$ on $C(\Gamma)$ is $\sigma(A) = 2\pi\mathrm{i}/\tau \cdot \mathbb{Z}$ and $\exp(2\pi\mathrm{i}nt/\tau)$, $n \in \mathbb{Z}$, is an eigenvalue of $R_\tau(t)$ for every $t \geqslant 0$ (see Example 5.6. If t/τ is irrational, these eigenvalues form a dense subset of Γ. Since the spectrum is closed, we obtain $\sigma(T(t)) = \Gamma$ for these t. Therefore in this example the spectral mapping theorem is valid only in the following "weak" form

$$\sigma(T(t)) = \overline{\exp(t\sigma(A))}, \quad t \geqslant 0.$$

(iii) By Example 1.3 there exists a semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ with generator A such that $s(A) = -1$ and $\omega_0(\mathcal{T}) = 0$. This implies that for preassigned real numbers $\alpha < \beta$ there exists a semigroup $\mathcal{S} = (S(t))_{t\geqslant 0}$ with generator B such that $s(B) = \alpha$ and $\omega_0(\mathcal{S}) = \beta$. Indeed, take $S(t) = \mathrm{e}^{\beta t}T((\beta - \alpha)t)$ and observe that $B = (\beta-\alpha)A + \beta\,\mathrm{Id}$. In that case $\exp(t\sigma(B))$ is contained in the circle about 0 with radius $\mathrm{e}^{\alpha t}$ while $\sigma(S(t))$ has spectral values satisfying $|\lambda| = r(S(t)) = \mathrm{e}^{\beta t} > \mathrm{e}^{\alpha t}$.

(iv) The Example 1.3 can be strengthend in order to yield a semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ with generator A such that $\sigma(A) = \emptyset$, but $\|T(t)\| = r(T(t)) = 1$ for $t \geqslant 0$, i.e., $s(A) = -\infty$, $\omega_0 = 0$ and $s(A) < \omega_0$. Take the translation semigroup on the Banach space

$$E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathrm{e}^{x^2}\,\mathrm{d}x)$$

with $\|f\| := \sup\{|f(x)| : x \in \mathbb{R}_+\} + \int_0^\infty |f(x)|\mathrm{e}^{x^2}\,\mathrm{d}x$ (see Greiner et al. [11]).

(v) Another modification of Example 1.3 yields a group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ satisfying $s(A) < \omega_0$. Therefore the spectral mapping theorem does not hold in the setting of groups (see Wolff (1981)).

The next few theorems form the core of this chapter. We show that only one part of the spectrum and one inclusion is responsible for the failure of the spectral mapping theorem. The usefulness of this detailed analysis will become clear in the subsequent chapters on stability and asymptotics.

Proposition 6.2 (Spectral Inclusion Theorem)

Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$ on some Banach space E. Then

$$\exp(t\sigma(A)) \subset \sigma(T(t)) \text{ for } t \geqslant 0.$$

More precisely we have the following inclusions.

$$\exp(t \cdot P\sigma(A)) \subset P\sigma(T(t)), \tag{6.1}$$
$$\exp(t \cdot A\sigma(A)) \subset A\sigma(T(t)), \tag{6.2}$$
$$\exp(t \cdot R\sigma(A)) \subset R\sigma(T(t)). \tag{6.3}$$

Proof Since $\mathrm{e}^{\lambda t} - T(t) = (\lambda - A) \int_0^t \mathrm{e}^{\lambda(t-s)} T(s)\, \mathrm{d}s$ (see A-I,(3.1)), it follows that $(\mathrm{e}^{\lambda t} - T(t))$ is not bijective if $(\lambda - A)$ fails to be bijective proving the main inclusion.

The inclusion (6.1) becomes evident from the following proof of (6.2). Take $\lambda \in A\sigma(A)$ and an associated approximate eigenvector $(f_n) \subset D(A)$. Then

$$g_n := \mathrm{e}^{\lambda t} f_n - T(t) f_n = \int_0^t \mathrm{e}^{\lambda(t-s)} T(s)(\lambda - A) f_n \, \mathrm{d}s$$

converges to zero as $n \to \infty$. Consequently, $\mathrm{e}^{\lambda t} \in A\sigma(T(t))$ and, in fact, the same approximate eigenvector (f_n) does the job for all $t \geqslant 0$.

For the proof of (6.3) we take $\lambda \in R\sigma(A)$ and observe that $(\mathrm{e}^{\lambda t} - T(t)) f = (\lambda - A)(\int_0^t \mathrm{e}^{\lambda(t-s)} T(s) f \, \mathrm{d}s) \in (\lambda - A) D(A)$ for every $f \in E$. □

As we know from the Examples 6.1, the converse inclusions do not hold in general, i.e., not every spectral value of a semigroup operator $T(t)$ comes—via the exponential map—from a spectral value of the generator. But at least this is true for some important parts of the spectrum.

Theorem 6.3 (Spectral Mapping Theorem for Point and Residual Spectrum) *Let A be the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$. Then*

$$\exp(t \cdot P\sigma(A)) = P\sigma(T(t)) \setminus \{0\}, \tag{6.4}$$

$$\exp(t \cdot R\sigma(A)) = R\sigma(T(t)) \setminus \{0\} \text{ for } t \geqslant 0. \tag{6.5}$$

Proof For the proof of (6.4), take $t_0 > 0$ and $0 \neq \lambda \in P\sigma(T(t_0))$.

After rescaling the semigroup $\mathcal{T} = (T(t))_{t \geqslant 0}$ to the semigroup

$$(\exp(-t \cdot \log \lambda / t_0) T(t))_{t \geqslant 0},$$

we may assume $\lambda = 1$. Then the closed, $\mathcal{T}$-invariant subspace

$$G := \{g \in E : T(t_0) g = g\}$$

is non trivial. The restricted semigroup $T_|$ is periodic with period $\tau \leqslant t_0$ and the spectrum of its generator $A_|$ contains at least one eigenvalue $\mu = 2\pi \mathrm{i} n / t_0$ for some

$n \in \mathbb{Z}$ (see Lemma 5.3). Since every eigenvalue of $A_|$ is an eigenvalue of A, we obtain that $1 \in \exp(t_0 \cdot P\sigma(A))$. The converse inclusion has been proved in (6.1).

In fact, even more can be said: Let $g \in G$ be an eigenvector of $T(t_0)$ corresponding to the eigenvalue $\lambda = 1$. For each $n \in \mathbb{Z}$ define

$$g_n := P_n g = 1/t_0 \cdot \int_0^{t_0} \exp(-2\pi i n s/t_0) T(s) g \, \mathrm{d}s \in G$$

as in Section 5. If $g_n \neq 0$, then g_n is an eigenvector of $A_|$, hence of A with eigenvalue $2\pi i n/t_0$ as soon as g_n is distinct from zero. Since $D(A_|)$ is dense in G it follows from Theorem 5.4 that this holds for at least one $n \in \mathbb{Z}$. And from the proof of (6.1) we know that this g_n is in fact an eigenvector for each $T(t)$, $t \geqslant 0$.

Since $R\sigma(A) = P\sigma(A^*)$ and $R\sigma(T(t)) = P\sigma(T(t)^*)$ (see Proposition 4.4) the assertion (6.5) follows from (6.4) . □

Note that the proof is essentially an application of the structure theorem for periodic semigroups as given in Theorem 5.4. The information gained there can be reformulated into statements on the eigenspaces of A and $T(t)$.

Corollary 6.4 *For the eigenspaces of the generator A, resp. of the semigroup operators $T(t)$, $t > 0$, the following holds for $\mu \in \mathbb{C}$.*

(i) $\ker(\mu - A) = \bigcap_{s \geqslant 0} \ker(\mathrm{e}^{\mu s} - T(s))$,

(ii) $\ker(\mathrm{e}^{\mu t} - T(t)) = \overline{\mathrm{span}_{n\in\mathbb{Z}}\{\ker(\mu + 2\pi i n/t - A)\}}$.

We note that an analogous statements is valid for $\ker(\mu - A')$ and $\ker(\mathrm{e}^{\mu t} - T(t)')$ if we take in (ii) the $\sigma(E', E)$-closure.

Without proof (see Greiner [9, Proposition 1.10]) we add another corollary showing that poles of the resolvent of $T(t)$ correspond necessarily to poles of the resolvent of the generator. Again the converse is not true as shown by Example 5.6.

Corollary 6.5 *Assume that $\mathrm{e}^{\mu t}$ is a pole of order k of $R(\cdot, T(t))$ with residue P and Q as the k-th coefficient of the Laurent series. Then*

(i) *$\mu + 2\pi i n/t$ is a pole of $R(\cdot, A)$ of order $\leqslant k$ for every $n \in \mathbb{Z}$,*

(ii) *the residues P_n in $\mu + 2\pi i n/t$ yield $PE = \overline{\mathrm{span}_{n\in\mathbb{Z}}\{P_n E\}}$,*

(iii) *the k-th coefficient of the Laurent series of $R(\cdot, A)$ at $\mu + 2\pi i n/t$ is*

$$Q_n = (t \cdot \mathrm{e}^{\mu t})^{1-k} \cdot Q \circ (1/t) \int_0^t \mathrm{e}^{-(\mu + 2\pi i n/t)s} T(s) \, \mathrm{d}s.$$

From Proposition 6.2 and Theorem 6.3 it follows that the approximate point spectrum is the trouble maker in the sense that not every approximate eigenvalue of $T(t)$ corresponds to an approximate eigenvalue of the generator A. Since nothing more can be said in general, we now look for additional hypotheses on the semigroup implying the spectral mapping theorem.

As a simple example we assume $T(t_0)$ to be compact for some $t_0 > 0$. Then $\sigma(T(t)) \setminus \{0\} = P\sigma(T(t)) \setminus \{0\}$ for $t \geqslant t_0$ and the spectral mapping theorem is valid by (6.4). A different class of semigroups verifying the spectral mapping theorem is given by the uniformly continuous semigroups (compare Corollary 1.2).

Both cases, and many more, are included in the following result.

Theorem 6.6 (Spectral Mapping Theorem for Eventually Norm Continuous Semigroups) *Let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be an eventually norm continuous semigroup with generator A. Then the spectral mapping theorem is valid, i.e.,*

$$\sigma(T(t)) \setminus \{0\} = \mathrm{e}^{t\cdot\sigma(A)} \text{ for every } t \geqslant 0. \tag{6.6}$$

Proof By the previous considerations it suffices to show that $A\sigma(T(t)) \setminus \{0\} \subset \mathrm{e}^{t\cdot\sigma(A)}$ for $t > 0$. This will be done by converting approximate eigenvectors into eigenvectors in the semigroup $\mathcal{F}$-product (see subsection 4.2). The assertion then follows from (6.4) and Proposition 4.4.(ii).

Assume $t \mapsto T(t)$ to be norm continuous for $t \geqslant t_0$. Moreover it suffices to consider $1 \in A\sigma(T(t_1))$ for some $t_1 > 0$, i.e., we have a normalized sequence $(f_n)_{n\in\mathbb{N}} \subset E$ such that

$$\lim_{n\to\infty} \|T(t_1)f_n - f_n\| = 0.$$

Choose $k \in \mathbb{N}$ such that $kt_1 > t_0$ and define $g_n := T(kt_1)f_n$. Then

$$\lim_{n\to\infty} \|g_n\| = \lim_{n\to\infty} \|T(t_1)^k f_n\| = \lim_{n\to\infty} \|f_n\| = 1$$

and

$$\lim_{n\to\infty} \|T(t_1)g_n - g_n\| = 0,$$

i.e., $(g_n)_{n\in\mathbb{N}}$ yields an approximate eigenvector of $T(t_1)$ with approximate eigenvalue 1. But the semigroup $\mathcal{T}$ is uniformly continuous on sets of the form $T(t_0)V$, V bounded in E. In particular, it is uniformly continuous on the sequence $(g_n)_{n\in\mathbb{N}}$, which therefore defines an element g in the semigroup $\mathcal{F}$-product $E_{\mathcal{F}}$.

Obviously, g is an eigenvector of $T_{\mathcal{F}}(t_1)$ with eigenvalue 1 and by (6.4) we obtain an eigenvalue $2\pi\mathrm{i}n/t_1$ of $A_{\mathcal{F}}$ for some $n \in \mathbb{Z}$. The coincidence of $\sigma(A)$ and $\sigma(A_{\mathcal{F}})$ proves the assertion. □

We point out that the above spectral mapping theorem implies the coincidence of spectral bound and growth bound for eventually norm continuous semigroups, hence we have generalized the Liapunov Stability Theorem (see 1.2) to a much larger class of semigroups. As mentioned before, this will be of great use in many applications. Therefore we state explicitly the spectral mapping theorem for several important classes of semigroups all of which are eventually norm continuous (cf. the diagram preceding A-II, Example 1.27).

Corollary 6.7 *The spectral mapping theorem 6.6 holds for each of the following classes of strongly continuous semigroups.*

(i) *eventually compact semigroups,*

(ii) *eventually differentiable semigroups,*

(iii) *holomorphic semigroups,*

(iv) *uniformly continuous semigroups.*

Another application of the above spectral mapping theorem can be made to tensor product semigroups (see A-I,3.7). Let $\mathcal{T}_1 = (T_1(t))_{t\geqslant 0}$, $\mathcal{T}_2 = (T_2(t))_{t\geqslant 0}$ be strongly continuous semigroups on Banach spaces E_1, E_2 with generator A_1, A_2. The tensor product semigroup $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$ on some (appropriate) tensor product $E := E_1 \otimes E_2$ has the generator $A = A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$, but in general the spectrum of A is not determined by the spectra of A_1, A_2. But with an additional hypothesis the following can be proved.

Corollary 6.8 *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are eventually norm continuous, then*

$$\sigma(A) = \sigma(A_1) + \sigma(A_2),$$

where A is the generator of the tensor product semigroup

$$\mathcal{T}_1 \otimes \mathcal{T}_2 = (T_1(t) \otimes T_2(t))_{t\geqslant 0}.$$

Proof Clearly, the tensor product semigroup is eventually norm continuous and hence the Spectral Mapping Theorem 6.6 is valid for all three semigroups $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}$. Moreover the spectrum of the tensor product of bounded operators is the product of the spectra (Reed and Simon [23, XIII.9]). Therefore

$$\sigma(T_1(t) \otimes T_2(t)) = \sigma(T_1(t)) \cdot \sigma(T_2(t)), \quad t \geqslant 0.$$

Consequently we have the following identity for every $t \geqslant 0$

$$\begin{aligned} \mathrm{e}^{t\cdot\sigma(A)} &= \sigma(T_1(t) \otimes T_2(t)) \setminus \{0\} \\ &= \sigma(T_1(t)) \cdot \sigma(T_2(t)) \setminus \{0\} \\ &= \mathrm{e}^{t\cdot\sigma(A_1)} \cdot \mathrm{e}^{t\cdot\sigma(A_2)} \\ &= \mathrm{e}^{t(\sigma(A_1)+\sigma(A_2))}. \end{aligned}$$

From this identity we want to deduce $\sigma(A) = \sigma(A_1) + \sigma(A_2)$.

"$\subseteq$" Take $\xi \in \sigma(A)$. Then for every $t > 0$ there exist $\mu_t \in \sigma(A_1)$, $\lambda_t \in \sigma(A_2)$ and $n_t \in \mathbb{Z}$ such that $\xi = \mu_t + \lambda_t + 2\pi \mathrm{i} n_t/t$.

Since the real parts of μ_t, λ_t are bounded above, they lie in some interval $[a, b]$. But $\sigma(A_k) \cap ([a, b] + \mathrm{i}\mathbb{R})$ is compact for $k = 1, 2$ since A_k is the generator of an eventually norm continuous semigroup (see A-II, Theorem 1.20). By taking t sufficiently small, we conclude that $n_{t'} = 0$ for some $t' > 0$, i.e., $\xi = \mu_{t'} + \lambda_{t'}$.

"$\supseteq$" Choose $\mu \in \sigma(A_1)$, $\lambda \in \sigma(A_2)$. For every $t > 0$ there exist $\eta_t \in \sigma(A)$, $m_t \in \mathbb{Z}$ such that $\mu + \lambda = \eta_t + 2\pi \mathrm{i} m_t/t$. Since $\operatorname{Re}\mu + \operatorname{Re}\lambda = \operatorname{Re}\eta_t$ and $\{\mathrm{i}\eta_t : t > 0\}$ is bounded, $\mathcal{T} = (T_1(t) \otimes T_2(t))_{t\geqslant 0}$ being eventually norm continuous, it follows that $m_{t'} = 0$ for some $t' > 0$. □

7 Weak Spectral Mapping Theorems

In the previous section we showed under which hypotheses a spectral mapping theorem of the form

$$\sigma(T(t)) \setminus \{0\} = \mathrm{e}^{t\cdot\sigma(A)}, \quad t \geqslant 0, \tag{7.1}$$

is valid for the generator A of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$.

Among the various examples showing that (7.1) does not hold in general we recall the following. Take the Banach space $E = c_0$, the multiplication operator $A(x_n)_{n\in\mathbb{N}} = (\mathrm{i}nx_n)_{n\in\mathbb{N}}$ with maximal domain and the corresponding semigroup $T(t)(x_n)_{n\in\mathbb{N}} = (\mathrm{e}^{\mathrm{i}nt}x_n)_{n\in\mathbb{N}}$. Then $\sigma(A) = \{in\colon n \in \mathbb{N}\}$ and the spectral mapping theorem is valid only in the following weak form

$$\sigma(T(t)) = \overline{\exp(t\cdot\sigma(A))}, \quad t \geqslant 0. \tag{7.2}$$

In this section we prove similar weak spectral mapping theorems. We start with a generalization of the above example.

Consider the Banach space $E = C_0(X, \mathbb{C}^n)$ of all continuous $\mathbb{C}^n$-valued functions vanishing at infinity on some locally compact space X. In analogy to A-I,2.3 we associate to every continuous function $q\colon X \to M(n)$, where $M(n)$ denotes the space of all complex $n \times n$-matrices, a "multiplication operator" $M_q\colon f \to q\cdot f$ such that $(q\cdot f)(x) = q(x)\cdot f(x)$, $x \in X$, on the maximal domain $D(M_q) = \{f \in E\colon q\cdot f \in E\}$. If $\|\mathrm{e}^{tq(x)}\|$ is uniformly bounded for $0 \leqslant t \leqslant 1$ and $x \in X$, it follows that M_q generates the multiplication semigroup

$$(T(t)f)(x) = \mathrm{e}^{tq(x)}\cdot f(x), \quad f \in E, \quad x \in X, \quad t \geqslant 0.$$

Since M_q has a bounded inverse if and only if $q(x)^{-1}$ exists and is uniformly bounded for $x \in X$, it follows that the eigenvalues of each matrix $q(x)$ are always contained in $\sigma(M_q)$. In fact, much more can be said, in case the function is bounded.

Lemma 7.1 *The spectrum of the matrix valued multiplication operator M_q, where $q\colon X \to M(n)$ is bounded, is given by $\sigma(M_q) = \overline{\bigcup_{x\in X} \sigma(q(x))}$.*

Proof It remains to show that $0 \notin \overline{\bigcup_{x\in X} \sigma(q(x))}$ implies $0 \notin \sigma(M_q)$. Since $\det q(x)$ is the product of n eigenvalues (according to their multiplicity) of $q(x)$, the hypothesis implies that $d := \inf\{|\det q(x)|\colon x \in X\} > 0$. By Formula 4.12 in Chapter I of Kato [16] we obtain

$$\|q(x)^{-1}\| \leqslant \gamma \cdot \|q(x)\|^{n-1} \cdot |\det q(x)|^{-1} \leqslant \gamma/d \cdot \|M_q\|^{n-1}$$

for every $x \in X$ and a constant γ depending only on the norm chosen on $\mathbb{C}^n$. Therefore, $x \mapsto q(x)^{-1}$ defines a bounded continuous function on X which obviously yields the inverse of M_q, i.e., $0 \notin \sigma(M_q)$. □

Theorem 7.2 *Let $A = M_q$ be a matrix multiplication operator on $C_0(X, \mathbb{C}^n)$ generating a strongly continuous semigroup $(T(t))_{t\geqslant 0}$,*

$$T(t) := M_{\mathrm{e}^{tq}} \quad \textit{for } t \geqslant 0.$$

Then the Weak Spectral Mapping Theorem 7.2 holds true, i.e.,

$$\sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))}.$$

Proof By the Spectral Inclusion Proposition 6.2 we always have $\exp(t\sigma(A)) \subset \sigma(T(t))$. Since $T(t)$ is a matrix multiplication operator with a bounded function, we obtain from Lemma 7.1

$$\sigma(T(t)) = \overline{\bigcup_{x\in X} \sigma(\exp(tq(x)))} = \overline{\bigcup_{x\in X} \exp(t\sigma(q(x)))} \subset \overline{\exp(t\sigma(A))}$$

which proves the assertion. □

Corollary 7.3 *The growth bound $\omega_0(A)$ and the spectral bound $s(A)$ coincide for matrix multiplication semigroups.*

The above results remain valid for other Banach spaces of $\mathbb{C}^n$-valued functions such as $L^p(X, \mathbb{C}^n)$, $1 \leqslant p < \infty$.

The example given at the beginning of this section can be generalized in a different way. In fact, $A(x_n) := (\mathrm{i}nx_n)$ on $E = c_0$ generates a bounded group, and we will show that this property too ensures that the Weak Spectral Mapping Theorem 7.2 holds. Without any boundedness assumption on $(T(t))_{t\in\mathbb{R}}$ this result cannot be true (see Hille and Phillips [14, Sec.23.16] or Wolff [25]).

Theorem 7.4 *Let $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ be a strongly continuous group on some Banach space E such that $\|T(t)\| \leqslant p(t)$ for some polynomial p and all $t \in \mathbb{R}$. Then the Weak Spectral Mapping Theorem 7.2 holds, i.e.,*

$$\sigma(T(t)) = \overline{\exp(t\cdot\sigma(A))} \text{ for all } t \in \mathbb{R}.$$

From the proof we isolate a series of lemmas assuming the hypothesis made in Theorem 7.4. Moreover we recall from Fourier analysis that the Fourier transformation $\varphi \mapsto \hat{\varphi}$,

$$\hat{\varphi}(\alpha) := \int_{-\infty}^{\infty} \varphi(x)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}x$$

and its inverse $\Psi \mapsto \check{\Psi}$,

$$\check{\Psi}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\alpha)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}\alpha$$

are topological isomorphisms of the Schwartz space $\mathcal{S}(= \mathcal{S}(\mathbb{R}))$. Since the subspace $\mathcal{D}$ of all functions having compact support is dense in $\mathcal{S}$, it follows that $\{\varphi \in \mathcal{S} : \check{\varphi} \in \mathcal{D}\}$ is also dense in $\mathcal{S}$.

Lemma 7.5 *For every function $\varphi \in \mathcal{S}$ we obtain an operator $T(\varphi) \in \mathcal{L}(E)$ by*

$$T(\varphi)f := \int_{-\infty}^{\infty} \varphi(s)T(s)f\,\mathrm{d}s, \quad f \in E.$$

This operator can be represented as

$$T(\varphi)f = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} \check{\varphi}(\alpha)[R(\varepsilon-\mathrm{i}\alpha, A)f - R(-\varepsilon-\mathrm{i}\alpha, A)f]d\alpha, \quad f \in E. \tag{7.3}$$

Proof That $T(\varphi)$ is well-defined follows from the polynomial boundedness of $(T(t))_{t\in\mathbb{R}}$. In fact, $\varphi \to T(\varphi)$ is continuous from $\mathcal{S}$ into $(\mathcal{L}(E), \|\cdot\|)$. We obtain

$$\begin{aligned} T(\varphi)f &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \varphi(s)\mathrm{e}^{-\varepsilon|s|}T(s)f\,\mathrm{d}s \\ &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\check{\varphi}(\alpha)\mathrm{e}^{\mathrm{i}\alpha s}\mathrm{e}^{-\varepsilon|s|}T(s)f\,d\alpha\,\mathrm{d}s \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\check{\varphi}(\alpha)\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\alpha s}\mathrm{e}^{-\varepsilon|s|}T(s)f\,\mathrm{d}s d\alpha \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\check{\varphi}(\alpha)[R(\varepsilon-\mathrm{i}\alpha, A)f - R(-\varepsilon-\mathrm{i}\alpha, A)f]d\alpha. \end{aligned}$$

Here we used that polynomially bounded groups have growth bound 0, hence $\omega_0(A) = \omega_0(-A) = 0$. Therefore the integral representation of $R(\varepsilon - \mathrm{i}\alpha, A)$ (cf. A-I, Proposition 1.11) exists for $\varepsilon \neq 0$. ◻

Lemma 7.6 *If $E \neq \{0\}$, then $\sigma(A) \neq \emptyset$.*

Proof If $\sigma(A) = \emptyset$, then (7.3) implies $T(\varphi) = 0$ whenever $\check{\varphi}$ has compact support. Since these functions form a dense subspace of $\mathcal{S}$, we conclude that $T(\varphi) = 0$ for all $\varphi \in \mathcal{S}$. Choosing an approximate identity $(\psi_n)_{n \in \mathbb{N}} \subset \mathcal{D}$, we obtain

$$f = T(0)f = \lim_{n \to \infty} T(\psi_n) f = 0$$

for every $f \in E$. □

Proof (Proof of Theorem 7.4 (1st part)) By the Spectral Inclusion (see Proposition 6.2), we have to show that every spectral value of $T(t)$ can be approximated by exponentials of spectral values of A. In view of the rescaling procedure it suffices to prove this when $-1 \in \varrho(T(\pi))$, provided that the following condition is satisfied.

$$\text{There exists } \varepsilon > 0 \text{ such that } \bigcup_{k \in \mathbb{Z}} \mathrm{i}\,[2k + 1 - 2\varepsilon, 2k + 1 + 2\varepsilon] \subset \varrho(A). \tag{7.4}$$

Assume now that (7.4) holds. Then each of the sets

$$\sigma_k := \{\mathrm{i}\alpha \in \sigma(A) \colon \alpha \in [2k - 1, 2k + 1]\}$$

is a spectral set of A with corresponding spectral projection P_k. If we choose $\varphi_0 \in \mathcal{D}$ such that

$$\begin{aligned} \operatorname{supp}(\varphi_0) &\subseteq [-1 + \varepsilon, 1 - \varepsilon], \\ \varphi_0(x) &= 1 \quad \text{for } x \in [-1 + 2\varepsilon, 1 - 2\varepsilon], \end{aligned}$$

it follows from (7.3) and the integral representation of P_k (cf. (3.1)) that $P_0 = T(\check{\varphi}_0)$.

More generally, since

$$\left(\mathrm{e}^{\mathrm{i}2k} \check{\varphi}_0\right)\hat{}\,(\alpha) = \varphi_0(\alpha - 2k),$$

the assertions (7.3) and (7.4) imply

$$P_k = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2ks} \check{\varphi}_0(s) T(s)\, \mathrm{d}s \text{ for } k \in \mathbb{Z}. \tag{7.5}$$

At this point we isolate another lemma.

Lemma 7.7 span $\bigcup_{k \in \mathbb{Z}} P_k E$ *is dense in E.*

Proof The closure of span $\bigcup_{k \in \mathbb{Z}} P_k E$ is a $\mathcal{T}$-invariant subspace G of E. Consider the quotient group $(T(t)_/)_{t \in \mathbb{R}}$ induced on E/G. The spectrum of its generator $A_/$ is contained in $\sigma(A)$ by Proposition 4.2.(ii). Moreover, the spectral projection corresponding to $\sigma(A_/) \cap \sigma_k$ is the quotient operator $P_{k/}$. Obviously $P_{k/} = 0$, hence

$\sigma(A_/) \cap \sigma_k = \emptyset$ for every $k \in \mathbb{Z}$ and $\sigma(A_/) = \emptyset$. By Lemma 7.6 this implies $E/G = \{0\}$, i.e., $G = E$. □

Proof (Proof of Theorem 7.4 (2^nd^ part)) We return to the situation of the first part of the proof. Using (7.5), the spectral projection P_k can be transformed into

$$\begin{aligned} P_k &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2ks} \check{\varphi}_0(s) T(s)\, \mathrm{d}s \\ &= \sum_{m\in\mathbb{Z}} \int_{(m-1/2)\pi}^{(m+1/2)\pi} \mathrm{e}^{\mathrm{i}2ks} \check{\varphi}_0(s) T(s)\, \mathrm{d}s \\ &= \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}2ks} \sum_{m\in\mathbb{Z}} \check{\varphi}_0(s+m\pi) T(s+m\pi)\, \mathrm{d}s, \end{aligned}$$

i.e., $P_k f$ is the k-th Fourier coefficient of the π-periodic, continuous function $\xi_f : s \mapsto \sum_{m\in\mathbb{Z}} \check{\varphi}_0(s+m\pi)T(s+m\pi)f$, $f \in E$. Since the projections P_k are mutually orthogonal, i.e., $P_k P_m = 0$ for $k \neq m$, it follows that $g = \sum_{n\in\mathbb{Z}} P_n g$ for every $g \in \operatorname{span} \bigcup_{k\in\mathbb{Z}} P_k E$. In particular, the Fourier coefficients of the function ξ_g are absolutely summable, hence the Fourier series of ξ_g converges to ξ.

For $s = 0$ we obtain

$$g = \sum_{n\in\mathbb{Z}} P_n g \cdot \mathrm{e}^{-\mathrm{i}n0} = \sum_{m\in\mathbb{Z}} \check{\varphi}_0(0+m\pi) T(0+m\pi) g \qquad \left(g \in \operatorname{span} \bigcup_{k\in\mathbb{Z}} P_k E \right).$$

Since $\operatorname{span} \bigcup_{k\in\mathbb{Z}} P_k E$ is dense (Lemma 7.7), we conclude that

$$\sum_{m\in\mathbb{Z}} \varphi_0(m\pi) T(m\pi) = \mathrm{Id}\,. \tag{7.6}$$

As the final step we construct the inverse operator of $\mathrm{Id} + T(\pi)$ showing that $-1 \in \varrho(T(\pi))$. We define $\psi_0(\alpha) := \varphi_0(\alpha) \cdot (1 + \mathrm{e}^{\mathrm{i}\pi\alpha})^{-1}$, $\alpha \in \mathbb{R}$. Then we have $\psi_0 \in \mathcal{S}$ and $\psi_0 \cdot (1 + \mathrm{e}^{\mathrm{i}\pi\cdot}) = \varphi_0$, hence $\check{\psi}_0(x) + \check{\psi}_0(x+\pi) = \check{\varphi}_0(x)$ for all $x \in \mathbb{R}$. Then (7.6) implies

$$\begin{aligned} \mathrm{Id} &= \sum_{m\in\mathbb{Z}} \check{\varphi}_0(m\pi) T(m\pi) \\ &= \sum_{m\in\mathbb{Z}} (\check{\psi}_0(m\pi) + \check{\psi}_0((m+1)\pi)) T(m\pi) \\ &= \left[\sum_{m\in\mathbb{Z}} \check{\psi}_0(m\pi) T(m\pi) \right] (\mathrm{Id} + T(\pi)). \end{aligned}$$

□

In the rest of this section we discuss the behavior of the single spectral values λ of $T(t)$, $t > 0$. The goal is a characterization of $\sigma(T(t))$ involving only properties of the generator. By the rescaling procedure A-I,3.1 we may assume $\lambda = 1$ and $t = 2\pi$.

From the Spectral Inclusion Theorem (Proposition 6.2 on page 94) we know that $1 \in \varrho(T(2\pi))$ implies $\mathrm{i}\mathbb{Z} \subset \varrho(A)$. As seen in many examples the converse does not hold and we are now looking for additional conditions. Henceforth we assume $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ and define for $k \in \mathbb{Z}$

$$Q_k := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ks} T(s)\,\mathrm{d}s = \frac{1}{2\pi}(1 - T(2\pi))R(\mathrm{i}k, A) \tag{7.7}$$

(cf. Formula A-I, (3.1)).

Our approach is based on Fejér's Theorem (for Banach space valued functions). Let us recall this result. Suppose $\xi\colon [0, 2\pi] \to E$ is a continuous function and let $\xi_k := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ks}\xi(s)\,\mathrm{d}s$ be its k-th Fourier coefficient. Then the Fourier series is Césaro summable to ξ in every point $t \in (0, 2\pi)$. Moreover one has

$$\frac{1}{2}(\xi(0) + \xi(2\pi)) = C_1\text{-}\sum_{k\in\mathbb{Z}} \xi_n := \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1}\Big(\sum_{k=-n}^{n} \xi_k\Big). \tag{7.8}$$

This result enables us to prove the following proposition.

Proposition 7.8 *Let $(T(t))_{t\geqslant 0}$ be a semigroup on a Banach space E and denote its generator by A. Then the following conditions are equivalent.*

(a) $1 \in \varrho(T(2\pi))$,

(b) $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ *and the series* $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)f$ *is Césaro-summable for every* $f \in E$,

(c) $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ *and the series* $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)Q_k f$ *is Césaro-summable for every* $f \in E$.

Proof (a) $\Rightarrow$ (b) The Spectral Inclusion Theorem 6.2 implies $\mathrm{i}\mathbb{Z} \subset \varrho(A)$. By (7.7) we have $R(\mathrm{i}k, A) = 2\pi \cdot (1 - T(2\pi))^{-1}Q_k$. Since $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable (towards $\frac{1}{2}(f + T(2\pi)f)$) (see (7.8), it follows that $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)f$ is Césaro-summable as well.

(b) $\Leftrightarrow$ (c) If we use A-I,(3.1) and integrate by parts, we obtain

$$\begin{aligned} R(\mathrm{i}k, A)Q_k f &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ks} T(s)R(\mathrm{i}k, A)f\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left[R(\mathrm{i}k, A)f - \int_0^s \mathrm{e}^{-\mathrm{i}kt}T(t)f\,\mathrm{d}t\right]\mathrm{d}s \\ &= R(\mathrm{i}k, A)f - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt}(2\pi - t)T(t)f\,\mathrm{d}t. \end{aligned}$$

Fejér's theorem ensures that

$$\sum_{k\in\mathbb{Z}}(1/2\pi)\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt}(2\pi - t)T(t)f\,\mathrm{d}t$$

is Césaro summable. Hence $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)Q_k f$ is Césaro-summable if and only if $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)f$ is.

(b) $\Rightarrow$ (a) We have $Q_k = \frac{1}{2\pi}(1 - T(2\pi))R(\mathrm{i}k, A)$. Furthermore we know by (7.7) and (7.8) that $\sum_{k\in\mathbb{Z}} Q_k f$ is Césaro-summable towards $\frac{1}{2}(f + T(2\pi)f)$. If we define $S\colon E \to E$ by $Sf := \frac{1}{2}f + \frac{1}{2\pi}\cdot C_1\text{-}\sum_k R(\mathrm{i}k, A)f$, then we have

$$\begin{aligned}(1 - T(2\pi))Sf &= \frac{1}{2}(f - T(2\pi)f) + \frac{1}{2\pi}\cdot C_1\text{-}\sum_k (1 - T(2\pi))R(\mathrm{i}k, A)f\\ &= \frac{1}{2}(f - T(2\pi)f) + \frac{1}{2}(f + T(2\pi)f) = f.\end{aligned}$$

Since S commutes with $T(2\pi)$, it follows that S is the inverse of $(1 - T(2\pi))$ thus $1 \in \varrho(T(2\pi))$. □

Based on the equivalence of (a) and (b), one can state a characterization of the spectrum of $T(t)$ in terms of the generator and its resolvent only. However, in general it is difficult to verify the summability condition stated in (b).

In Hilbert spaces the boundedness of the resolvents will suffice (see Theorem 7.10 below).

Lemma 7.9 *Let $(T(t))_{t\geqslant 0}$ be a semigroup on some Hilbert space H and assume $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ for the generator A. Then we have*

(i) *$(Q_k f)_{k\in\mathbb{Z}} \subset \ell^2(H)$ for every $f \in H$, and*

(ii) *if $\sup_{k\in\mathbb{Z}} \|R(\mathrm{i}k, A)\| < \infty$, then $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A)f_k$ is summable whenever $(f_k)_{k\in\mathbb{Z}} \in \ell^2(H)$.*

Proof (i) We consider the Hilbert space $L^2([0, 2\pi], H)$ and obtain

$$\begin{aligned}0 &\leqslant \left\| T(\cdot)f - \sum_{k=-n}^{n} Q_k f\cdot \mathrm{e}^{\mathrm{i}k\cdot}\right\|^2\\ &= \int_0^{2\pi} \|T(s)f\|^2\,\mathrm{d}s - \int_0^{2\pi}\sum_{k=-n}^{n}(T(s)f|\mathrm{e}^{\mathrm{i}ks}Q_k f)\,\mathrm{d}s -\\ &\quad \int_0^{2\pi}\sum_{k=-n}^{n}(\mathrm{e}^{\mathrm{i}ks}Q_k f|T(s)f)\,\mathrm{d}s + \int_0^{2\pi}\left(\sum_{k=-n}^{n}\mathrm{e}^{\mathrm{i}ks}Q_k f|\sum_{\ell=-n}^{n}\mathrm{e}^{i\ell s}Q_\ell f\right)\mathrm{d}s\\ &= \int_0^{2\pi}\|T(s)f\|^2\,\mathrm{d}s - 2\pi\sum_{k=-n}^{n}\|Q_k f\|^2 \text{ (use (7.5)) }.\end{aligned}$$

It follows that $\sum_{k\in\mathbb{Z}} \|Q_k f\|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} \|T(s)f\|^2\,\mathrm{d}s < \infty$.

(ii) Fix $\lambda > 0$ sufficiently large and set

$$g_k := (1 + \lambda R(\mathrm{i}k, A)) f_k,\ k \in \mathbb{Z}.$$

Using the resolvent equation and then (A-I,(3.1)), we obtain

$$R(\mathrm{i}k, A) f_k = R(\lambda + \mathrm{i}k, A) g_k = [1 - \mathrm{e}^{-2\pi\lambda} T(2\pi)]^{-1} \int_0^{2\pi} \mathrm{e}^{-\lambda s}\mathrm{e}^{-\mathrm{i}ks} T(s) g_k\,\mathrm{d}s.$$

This yields for every finite subset N of $\mathbb{Z}$ that

$$\Big\|\sum_{k\in N} R(\mathrm{i}k, A) f_k\Big\| \leqslant \|(1 - \mathrm{e}^{-2\pi\lambda} T(2\pi))^{-1}\| \cdot \int_0^{2\pi} \|T(s)\| \Big\|\sum_{k\in N} \mathrm{e}^{-\mathrm{i}ks} g_k\Big\|\,\mathrm{d}s$$

$$\leqslant \|(1 - \mathrm{e}^{-2\pi\lambda} T(2\pi))^{-1}\| \cdot \left(\int_0^{2\pi} \|T(s)\|^2\,\mathrm{d}s\right)^{1/2} \cdot \left(\int_0^{2\pi} \Big\|\sum_{k\in N} \mathrm{e}^{-\mathrm{i}ks} g_k\Big\|^2\,\mathrm{d}x\right)^{1/2}$$

$$= c\Big(\sum_{k\in N} \|g_k\|^2\Big)^{1/2} \leqslant c(1 + \lambda M)\left(\sum_{k\in N} \|f_k\|^2\right)^{1/2}.$$

Here $c := \|(1-\mathrm{e}^{-2\pi\lambda} T(2\pi))^{-1}\| \cdot \left(\int_0^{2\pi} \|T(s)\|^2\,\mathrm{d}s\right)^{1/2}$ and $M := \sup_{k\in\mathbb{Z}} \|R(\mathrm{i}k, A)\|$ do the job. □

Theorem 7.10 *Let A be the generator of a semigroup $(T(t))_{t\geqslant 0}$ on some Hilbert space H. Then the following form of the spectral mapping theorem is valid.*

> $\mathrm{e}^{\lambda t} \in \sigma(T(t)) \setminus \{0\}$ *if and only if either* $\mu_k := \lambda + 2\pi \mathrm{i}k/t \in \sigma(A)$ *for some* $k \in \mathbb{Z}$ *or if* $(\|R(\mu_k, A)\|)_{k\in\mathbb{Z}}$ *is unbounded.*

Proof If $\mathrm{e}^{\lambda t} \notin \sigma(T(t))$, it follows from the Spectral Inclusion Theorem 6.2 that $\mu_k \notin \sigma(A)$ for every $k \in \mathbb{Z}$ and from Formula (3.1) in A-I, that $\|R(\mu_k, A)\|$ is bounded. For the converse inclusion it suffices to assume $t = 2\pi$ and $\lambda = 0$ (use the rescaling procedure A-I,3.1). Assuming that $\mathrm{i}\mathbb{Z} \subset \varrho(A)$ and $\|R(\mathrm{i}k, A)\|$ is bounded, then $\sum_{k\in\mathbb{Z}} R(\mathrm{i}k, A) Q_k f$ is summable by Lemma 7.9. Since every summable series is Cesàro-summable, condition (c) of Proposition 7.8 is satisfied and we conclude $1 \in \varrho(T(2\pi))$. □

As an immediate consequence we obtain an interesting characterization of the growth bound ω_0 of semigroups on Hilbert spaces.

Corollary 7.11 *The growth bound of a semigroup $(T(t))_{t\geqslant 0}$ on a Hilbert space H satisfies*

$$\omega_0 = \inf\{\lambda \in \mathbb{R} \colon \lambda + \mathrm{i}\mathbb{R} \subset \varrho(A) \text{ and } \|R(\lambda + \mathrm{i}\mu, A)\| \text{ is bounded for } \mu \in \mathbb{R}\}. \quad (7.9)$$

The Example 1.3 above in combination with C-III,Corollary 1.3 shows that (7.9) is not valid in arbitrary Banach spaces.

Notes

Section 1: It was already known to Hille and Phillips [14] that for strongly continuous semigroups $(T(t))_{t\geqslant 0}$ with generator A the Spectral Mapping Theorem "$\sigma(T(t)) = \exp(t\sigma(A))$" may be violated in various ways [l.c.,Sec.23.16]. The simple Examples 1.3 and 1.4 are due to Wolff (see Greiner et al. [11]) and Zabczyk [27]. A more sophisticated example of a positive group with "$s(A) < \omega_0(A)$" is given in Wolff [25].

Section 2: In Definition 2.1 we define the residual spectrum of A in such a way that it coincides with the point spectrum of the adjoint A' (see Proposition 2.2.(ii)). It therefore differs slightly from the one used, e.g., by Schaefer [24]. The spectral mapping theorem for the resolvent (Theorem 2.5) is well known and can, e.g., be deduced from Lemma 9.2 and Theorem 3.11 of Chap.VII in Dunford and Schwartz [5].

Section 3: The general theory of spectral decompositions can be found in Kato [16], Chap.III,§ 6.4]. Applications to isolated singularities like 3.6 are discussed extensively in [l.c., Chap. III, §6.5] and Yosida [26, Chap.VIII, Sec.8]. There are many ways to introduce an "essential spectrum" (see the footnote on page 243 of Kato [16]). However each one yields the same "essential spectral radius".

Section 4: These results are taken from Derndinger [4] and Greiner [9].

Section 5: Periodic semigroups are studied explicitly in Bart [1], but most of the results of this section seem to be well known.

Section 6: The *Spectral Inclusion Theorem* 6.2 and the *Spectral Mapping Theorem* 6.6 for eventually norm continuous semigroups date back to Hille and Phillips [14]. Davies [3] gives an elegant proof using Banach algebra techniques. Tensor products of operators and their spectral theory have been studied by Ichinose and others (see Chap. XIII.9 of Reed and Simon [23]). The spectral mapping theorem in Corollary 6.8 generalizes Theorem XIII.35 of Reed and Simon [23] (see also Herbst [12]).

Section 7: Matrix valued multiplication semigroups appear as solution, via Fourier transformation, of systems of partial differential equations. Kreiss initiated a systematic investigation (see, e.g., Kreiss [17], Kreiss [18], Miller and Strang [20]) and the Weak Spectral Mapping Theorem 7.2 must have been known to him. The direct proof of the Weak Spectral Mapping Theorem 7.4 for polynomially bounded groups seems to be new. The result can also be deduced from the theory of spectral subspaces of group representations (see, e.g., Combes and Delaroche [2]), since the Arveson spectrum of a strongly continuous one-parameter group can be identified with the spectrum of the generator (see Evans [6]). The final part of this section is taken from Greiner [10] and yields a new approach to Gearhart's characterization of the spectrum of semigroups on Hilbert spaces (Gearhart [7]). Different proofs can be found in Herbst [13], Howland [15] and Prüss [22].

References

[1] H. Bart. Periodic strongly continuous semigroups. *Ann. Mat. Pura Appl.*, 115:311–318, 1977.

[2] F. Combes and C. Delaroche. Representations des groupes localement compacts et applications aux algèbres d'opérateurs. *Astérisque*, 55, 1978. Séminaire d'Orleans.

[3] E. Davies. *One-parameter Semigroups*. Academic Press, London-New York-San Francisco, 1980.

[4] R. Derndinger. Über das Spektrum positiver Generatoren. *Math. Z.*, 172: 281–293, 1980.

[5] N. Dunford and J. Schwartz. *Linear Operators, Part I: General Theory*. Wiley, New York, 1958.

[6] D. Evans. On the spectrum of a one-parameter strongly continuous representation. *Math. Scand.*, 39:80–82, 1976.

[7] L. Gearhart. Spectral theory for contraction semigroups on Hilbert spaces. *Trans. Amer. Math. Soc.*, 236:385–394, 1978.

[8] J. Goldstein. *Semigroups of Operators and Applications*. Oxford University Press, 1985.

[9] G. Greiner. Zur Perron-Frobenius Theorie stark stetiger Halbgruppen. *Math. Z.*, 177:401–423, 1981.

[10] G. Greiner. Some applications of Fejer's theorem to one-parameter semigroups. Semesterbericht Funktionalanalysis, Universität Tübingen, 1985.

[11] G. Greiner, J. Voigt, and M. Wolff. On the spectral bound of the generator of semigroups of positive operators. *J. Operator Theory*, pages 245–256, 1981.

[12] I. W. Herbst. Contraction semigroups and the spectrum of $A_1 \otimes I + I \otimes A_2$. *J. Operator Theory*, 7:61–78, 1982.

[13] I. W. Herbst. The spectrum of Hilbert space semigroups. *J. Operator Theory*, 10:87–94, 1983.

[14] E. Hille and R. S. Phillips. *Functional Analysis and Semigroups*, volume 31 of *Colloquium Publications*. American Mathematical Society, Providence, R.I., 1957.

[15] J. S. Howland. On a theorem of Gearhart. *Integral Equations and Operator Theory*, 7:138–142, 1984.

[16] T. Kato. *Perturbation Theory for Linear Operators*. Springer, 1966. 2nd printing: Berlin-Heidelberg-New York: Springer 1976.

[17] H. O. Kreiss. Über sachgemässe Cauchyprobleme für Systeme von linearen partiellen Differentialgleichungen. Technical Report 127, Roy. Inst. Technol., Stockholm, 1958.

[18] H. O. Kreiss. Über Matrizen die beschränkte Halbgruppen erzeugen. *Math. Scand.*, 7:71–80, 1959.

[19] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces II, Function Spaces*. Springer, Berlin-Heidelberg-New York, 1979.

[20] J. Miller and G. Strang. Matrix theorems for partial differential and difference equations. *Math. Scand.*, 18:113–133, 1966.

[21] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[22] J. Prüss. On the spectrum of C_0-semigroups. *Trans. Amer. Math. Soc.*, 284: 847–857, 1984.

[23] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV. Analysis of Operators*. Academic Press, New York, 1978.

[24] H. H. Schaefer. *Banach Lattices and Positive Operators*. Springer, New York-Heidelberg-Berlin, 1974.

[25] M. P. H. Wolff. A remark on the spectral bound of the generator of a semigroup of positive operators with application to stability theory. In P. L. Butzer, B. Sz.-Nagy, and E. Görlich, editors, *Functional Analysis and Approximation*, pages 39–50. Birkhäuser, Basel-Boston-Stuttgart, 1981.

[26] K. Yosida. *Functional Analysis*. Springer, 1965.

[27] J. Zabczyk. A note on C_0-semigroups. *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astronom. Phys.*, 23:895–898, 1975.

Chapter A-IV

Asymptotics of Semigroups on Banach Spaces

by

Frank Neubrander

In this chapter, we study the asymptotic behavior of the solutions of the inhomogeneous initial value problem

$$\dot{u}(t) = Au(t) + F(t) \quad \text{for} \quad t \geqslant 0 \quad \text{and} \quad u(0) = f, \tag{*}$$

where A is the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E, and $F(\cdot)$ is a function from $\mathbb{R}_+$ into E.

In Section 1, we investigate whether – and at what rate – a solution $T(\cdot)f$ of the homogeneous problem converges to the zero solution as $t \to \infty$. In Section 2, we consider the long-term behavior of the solutions of (*) for different classes of forcing terms F.

1 Stability: Homogeneous Case

Let $(T(t))_{t\geqslant 0}$ be a semigroup on E with generator A. An initial value $f \in D(A)$ is said to be *stable* if the solution $t \mapsto T(t)f$ of

$$\dot{u}(t) = Au(t), \quad u(0) = f \tag{ACP}$$

converges to zero as $t \to \infty$. The semigroup is called stable if every solution converges to zero, i.e., if every initial value $f \in D(A)$ is stable.

If the space E is finite-dimensional, then the stability of the semigroup implies exponential decay. More precisely, the following statements are equivalent.

(a) $\|T(t)f\| \to 0$ for every $f \in \mathbb{C}^n$,

(b) $\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}$ for some $M \geqslant 1$ and $\omega > 0$.

Moreover, these statements hold if and only if

(c) $s(A) = \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\} < 0$,

see A-III, Corollary 1.2.

As discussed in Chapter A-III, the situation becomes significantly more intricate in the infinite-dimensional setting. For unbounded generators, we must distinguish between *strong* and *generalized (mild)* solutions of $\dot{u}(t) = Au(t)$, as well as between various notions of stability. If A is the generator of a strongly continuous semigroup on a Banach space E and $f \in D(A)$, then $T(\cdot)f$ is the unique solution or, equivalently, the strong solution of (ACP) with initial value f; see A-II, Corollary 1.2 For an arbitrary $f \in E$, the function $T(\cdot)f$ is referred to as a generalized or mild solution of (ACP). Next, we introduce several constants that characterize the growth of solutions of (ACP).

Definition 1.1 (1^{st} part) Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. Then we define the following constants.

(i) $\omega(f) := \inf\{\omega\colon \|T(t)f\| \leqslant M\mathrm{e}^{\omega t}$ for some M and every $t \geqslant 0\}$ is called the *exponential growth bound* of $T(\cdot)f$,

(ii) $\omega_1(A) := \sup\{\omega(f)\colon f \in D(A)\}$ is called the *exponential growth bound for the solutions of the Cauchy problem* $\dot{u}(t) = Au(t)$,

(iii) $\omega_0(A) := \sup\{\omega(f)\colon f \in E\}$ is called the (exponential) *growth bound for the mild solutions of the Cauchy problem* $\dot{u}(t) = Au(t)$.

Note that, by the Principle of Uniform Boundedness,

$$\sup\{\omega(f)\colon f \in E\} = \inf\{\omega\colon \|T(t)\| \leqslant M\mathrm{e}^{\omega t} \text{ for some } M \text{ and every } t \geqslant 0\}.$$

Hence, $\omega_0(A)$ coincides with the growth bound of the semigroup $(T(t))_{t\geqslant 0}$, as defined in A-I, I.3. Using the constants defined above, we obtain the following stability concepts.

Definition 1.1 (2^{nd} part) The semigroup is called

(iv) *uniformly exponentially stable* if $\omega_0(A) < 0$,

(v) *exponentially stable* if $\omega_1(A) < 0$,

(vi) *uniformly stable* if $\|T(t)f\| \to 0$ as $t \to \infty$ for every $f \in E$,

(vii) *stable* if $\|T(t)f\| \to 0$ as $t \to \infty$ for every $f \in D(A)$.

The interrelation between these stability concepts is given by

$$\begin{tikzcd} & \text{(v)} \arrow[dr] & \\ \text{(iv)} \arrow[ur] \arrow[dr] & & \text{(vii)} \\ & \text{(vi)} \arrow[ur] & \end{tikzcd}$$

If A is a bounded operator, that is, if $D(A) = E$, then

$$\text{(iv)} \Leftrightarrow \text{(v) and (vi)} \Leftrightarrow \text{(vii)}.$$

However, if A is unbounded, all stability notions may differ, as illustrated in the following examples.

Example 1.2 (i) Let $E = c_0$. Then

$$A\colon (x_n)_{n\in\mathbb{N}} \mapsto (-1/n \cdot x_n)_{n\in\mathbb{N}}$$

generates the semigroup

$$T(t)(x_n)_{n\in\mathbb{N}} = (\mathrm{e}^{-t/n} x_n)_{n\in\mathbb{N}}.$$

It is easy to see that $\|T(t)\| = 1$ and that $\|T(t)f\| \to 0$ for every $f \in c_0$.

Moreover, since A is a bounded operator, $D(A) = E$. This provides an example of a (uniformly) stable but not exponentially stable semigroup.

The translation semigroups generated by the first derivative on $C_0(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$ for $1 < p < \infty$ offer further examples of (uniformly) stable but not exponentially stable semigroups.

Moreover, as shown in A-II, Example 1.14, the Laplacian Δ on $C_0(\mathbb{R}^n)$ generates a bounded holomorphic semigroup given by

$$T(t)f(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-|x-y|^2/4t} f(y)\,\mathrm{d}y\,.$$

This semigroup is not exponentially stable because $0 \in \sigma(\Delta)$ ($\mathrm{im}(\Delta) \neq C_0(\mathbb{R}^n)$); see Corollary 1.5 below. To see that the semigroup is (uniformly) stable, observe that for every fixed $x \in \mathbb{R}^n$, the kernel $k_t(x,y) := (4\pi t)^{-n/2}\mathrm{e}^{-|x-y|^2/4t}$ defines a probability density with $\int_{\mathbb{R}^n} k_t(x,y)\,\mathrm{d}y = 1$. Hence, $\|T(t)\| \le 1$; in fact, $\|T(t)\| = 1$ since k_t also forms an approximate identity. Let $f \in C_0(\mathbb{R}^n)$ and all $\epsilon > 0$. Then there exists a compactly supported $g \in C_0(\mathbb{R}^n)$ such that $\|f - g\| \le \epsilon$. Therefore,

$$\|T(t)f\| \le \|T(t)\|\|f-g\| + \|T(t)g\| \le \epsilon + (4\pi t)^{-n/2} \int_{\mathbb{R}^n} |g(y)|\,\mathrm{d}y\,,$$

which implies $\|T(t)f\| \to 0$ as $t \to \infty$ for all $f \in C_0(\mathbb{R}^n)$; see also B-III, Example 1.7. This shows that the Laplacian on $C_0(\mathbb{R}^n)$ (and also on $L^p(\mathbb{R}^n)$ for

$1 < p < \infty$, see Example 1.15 below) generates a uniformly stable but not exponentially stable semigroup.

(ii) Note that the condition

$$0 \leqslant \omega_0(A) = \inf\{\omega\colon \|T(t)\| \leqslant M\mathrm{e}^{\omega t} \text{ for all } t \geqslant 0\}$$

does not exclude the possibility that the semigroup is exponentially stable.

To see this, consider $E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathrm{e}^x\,\mathrm{d}x)$. Then, as shown in A-III, Example 1.3, the translation semigroup satisfies $\|T(t)\| = 1$, and hence $\omega_0(A) = 0$. For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -1$ and every $f \in E$, the resolvent of the generator is given as $R(\lambda, A)f = \int_0^\infty \mathrm{e}^{\lambda t} T(t) f \,\mathrm{d}t$. From the equation A-I, (3.2), it follows that

$$T(t)f = \mathrm{e}^{\lambda t}\left(f - \int_0^t \mathrm{e}^{-\lambda s} T(s)(\lambda - A) f \,\mathrm{d}s\right),$$

and from the existence of the limit

$$\lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s} T(s)(\lambda - A) f \,\mathrm{d}s,$$

it follows that $\|T(t)f\| \leqslant M\mathrm{e}^{\lambda t}$ for every $f \in D(A)$ and some constant M depending on f. This yields $\omega_1(A) \leq -1 < 0 = \omega_0(A)$. Thus, we have a semigroup that is exponentially stable, but not uniformly exponentially stable.

(iii) Rescaling this semigroup (see A-I, 3.1), we obtain a semigroup with $\omega_1(A) = -1/2$ and $\omega_0(A) = 1/2$. Therefore, there exist exponentially stable (and hence, stable) semigroups that are not bounded, and hence, not uniformly stable. This example illustrates that there may be a significant difference between the long-term behavior of the semigroup $(T(t))_{t\geqslant 0}$ (i.e., the set of all mild solutions) and the long-term behavior of the strong solutions $\{T(\cdot)f : f \in D(A)\}$ of (ACP).

In what follows, we characterize the exponential growth bounds $\omega(f)$, $\omega_1(A)$, and $\omega_0(A)$ in terms of certain abscissas of simple or absolute convergence of the Laplace transform of $T(\cdot)f$. These characterizations will serve as one of the tools for establishing that, for certain semigroups beyond the class of eventually norm-continuous semigroups (see A-III, Theorem 6.6), the growth bounds $\omega_0(A)$ and/or $\omega_1(A)$ coincide with the spectral bound $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$. For results of this type, see, for example, B-IV, C-IV, and D-IV.

We begin by observing that $s(A)$ can be interpreted as the abscissa of holomorphy of the Laplace transform $\lambda \mapsto \int_0^\infty \mathrm{e}^{-\lambda t} T(t)\,\mathrm{d}t$ of the semigroup $(T(t))_{t\geqslant 0}$.

Furthermore, we recall that the Laplace transform exists for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \operatorname{Re}\mu$, provided it exists for some $\mu \in \mathbb{C}$. This follows from

$$\int_0^t \mathrm{e}^{-\lambda s} f(s)\,\mathrm{d}s = \mathrm{e}^{-(\lambda-\mu)t} \int_0^t \mathrm{e}^{\mu s} f(s)\,\mathrm{d}s + (\lambda-\mu)\int_0^t \mathrm{e}^{-(\lambda-\mu)s}\int_0^s \mathrm{e}^{\mu r} f(r)\,\mathrm{d}r\,\mathrm{d}s\,. \tag{1.1}$$

Note that even boundedness of

$$\left\{\int_0^t \mathrm{e}^{-\mu s} f(s)\,\mathrm{d}s \colon t > 0\right\}$$

implies the existence of the Laplace transform for $\operatorname{Re}\lambda > \operatorname{Re}\mu$. Therefore, the subset of $\mathbb{C}$ for which the Laplace transform exists is always a half-plane $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > \gamma\} \cup H$, where H is a subset of the line $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda = \gamma\}$.

In the following theorem, we show that the bound of the half-plane for which the Laplace transform of $T(\cdot)f$ ($f \in E$) exists absolutely, and the bound of the half-plane for which the Laplace transform of $T(\cdot)Af$ ($f \in D(A)$) exists strongly, coincide with the growth bound $\omega(f) = \inf\{\omega\colon \|T(t)f\| \leqslant M\mathrm{e}^{\omega t}$ for all $t \geqslant 0\}$.

Theorem 1.3 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. Then, for every $f \in E$,*

$$\omega(f) = \limsup_{t\to\infty} \frac{1}{t}\log\|T(t)f\|, \tag{1.2}$$

and

(i) $\omega(f) = \inf\{\operatorname{Re}\lambda\colon \int_0^\infty \|\mathrm{e}^{-\lambda t}T(t)f\|\,\mathrm{d}t$ *exists*$\}$.

If $\ker(A) = \{0\}$*, then for every $f \in D(A)$ we have*

(ii) $\omega(f) = \inf\{\operatorname{Re}\lambda\colon \int_0^\infty \mathrm{e}^{-\lambda t}T(t)Af\,\mathrm{d}t$ *exists*$\}$.

Proof The proof of (1.2) is omitted (see Hille and Phillips [11, p.306]. To prove (i) and (ii), we need the following lemma.

Lemma *Let $F \in C(\mathbb{R}_+, \mathbb{R}_+)$ be such that $\int_0^\infty F(r)\,\mathrm{d}r$ exists. If there exist positive constants m and n such that F satisfies the local growth condition $F(t+s) \leqslant m\cdot F(s)$ for all $s \geqslant 0$ and $t \in [0,n]$, then* $\lim_{s\to\infty} F(s) = 0$. □

Proof (Proof of Lemma) Let $\varepsilon > 0$. Since $\int_0^\infty F(r)\,\mathrm{d}r < \infty$, there exists $a > 0$ such that

$$A(a) := \int_a^\infty F(r)\,\mathrm{d}r < \frac{n}{m}\varepsilon.$$

Now fix any $s > a + n$. We claim that there exists $r \in [s-n, s]$ such that $F(r) \leqslant \frac{1}{n}A(a)$. Indeed, suppose that $F(r) > \frac{1}{n}A(a)$ for all $r \in [s-n, s]$. Then

$$A(a) = \int_{s-n}^s \frac{1}{n}A(a)\,\mathrm{d}r < \int_{s-n}^s F(r)\,\mathrm{d}r \leq \int_a^\infty F(r)\,\mathrm{d}r = A(a),$$

which is a contradiction. Hence, such an $r \in [s-n, s]$ must exist. Finally, if $s > a+n$ and $r \in [s-n, s]$, then $0 \leq s-r \leq n$ and, therefore, $F(s) = F(s-r+r) \leqslant m \cdot F(r) \leqslant m \cdot \frac{A(a)}{n} < \varepsilon$. This shows that $\lim_{s\to\infty} F(s) = 0$. ☐

To prove part (i) of Theorem 1.3, define $b := \inf\{\operatorname{Re}\lambda\colon \int_0^\infty \|^{-\lambda t}T(t)f\|\,\mathrm{d}t \text{ exists}\}$. A straightforward application of the lemma shows that $\omega(f) \leqslant b$. The definition of $\omega(f)$ yields the reverse inequality.

It remains to prove part (ii) of Theorem 1.3. Assume that $\ker(A) = \{0\}$ and let $f \in D(A)$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega(f)$. From the equation

$$\int_0^t \mathrm{e}^{-\lambda s}T(s)Af\,\mathrm{d}s = \mathrm{e}^{-\lambda t}T(t)f - f + \lambda\int_0^t \mathrm{e}^{-\lambda s}T(s)f\,\mathrm{d}s$$

it follows that $\int_0^\infty \mathrm{e}^{-\lambda t}T(t)Af\,\mathrm{d}t$ exists. Therefore,

$$c := \inf\{\operatorname{Re}\lambda\colon \int_0^\infty \mathrm{e}^{-\lambda t}T(t)Af\,\mathrm{d}t \text{ exists}\} \leqslant \omega(f).$$

Next, we show that $c < 0$ implies $c = \omega(f)$. For $c < 0$, it follows from (*) that $\int_0^\infty T(s)Af\,\mathrm{d}s$ exists. Since $\int_0^r T(s)Af\,\mathrm{d}s = T(r)f - f$, we see that $g := \lim_{r\to\infty} T(r)f$ exists. But for every $t \geqslant 0$, $T(t)g = g$ which implies that $g \in \ker(A) = \{0\}$ or $g = 0$. Hence, $\int_0^\infty T(s)Af\,\mathrm{d}s = -f$. Now, choosing $r < 0$, $b < r < 0$, and integrating by parts, we obtain

$$\begin{aligned}
-T(t)f &= \lim_{u\to\infty}\int_t^u \mathrm{e}^{rs}\mathrm{e}^{-rs}T(s)Af\,\mathrm{d}s \\
&= \lim_{u\to\infty}\Big(\mathrm{e}^{ru}\int_0^u \mathrm{e}^{-rs}T(s)Af\,\mathrm{d}s - \mathrm{e}^{rt}\int_0^t \mathrm{e}^{-rs}T(s)Af\,\mathrm{d}s \\
&\qquad\qquad - r\int_t^u \mathrm{e}^{rs}\int_0^s \mathrm{e}^{-rv}T(v)Af\,\mathrm{d}v\,\mathrm{d}s\Big) \\
&= -\mathrm{e}^{rt}\int_0^t \mathrm{e}^{-rs}T(s)Af\,\mathrm{d}s - r\int_t^\infty \mathrm{e}^{rs}\int_0^s \mathrm{e}^{-rv}T(v)Af\,\mathrm{d}v\,\mathrm{d}s.
\end{aligned}$$

From $\left\|\int_0^t \mathrm{e}^{-rs}T(s)Af\,\mathrm{d}s\right\| \leqslant M$ for some M and every $t \geqslant 0$ we conclude that $\|T(t)f\| \leqslant \tilde{M}\mathrm{e}^{rt}$ for all $t \geqslant 0$ and some constant $\tilde{M}$. Hence, $\omega(f) \leqslant r$ for every $c < r < 0$, i.e., $\omega(f) \leqslant c$.

If $c \geqslant 0$ and $w > c$, then $\left\|\int_0^t \mathrm{e}^{-ws}T(s)Af\,\mathrm{d}s\right\| \leqslant M$ for all $t \geqslant 0$. By

$$\begin{aligned}
T(t)f - f &= \int_0^t \mathrm{e}^{ws}\mathrm{e}^{-ws}T(s)Af\,\mathrm{d}s \\
&= \mathrm{e}^{wt}\int_0^t \mathrm{e}^{-ws}T(s)Af\,\mathrm{d}s - w\int_0^t \mathrm{e}^{ws}\int_0^s \mathrm{e}^{-wr}T(r)Af\,\mathrm{d}r\,\mathrm{d}s,
\end{aligned}$$

we obtain $\|T(t)f - f\| \leqslant M\mathrm{e}^{wt} + M(\mathrm{e}^{wt} - 1) \leq 2M\mathrm{e}^{wt}$. Hence, $\omega(f) \leq w$ for every $w > c$, that is, $\omega(f) \leqslant c$. □

Finally, from (1.2) and the Uniform Boundedness Principle, it follows that the growth bound

$$\omega_1(A) = \sup\{\omega(f)\colon f \in D(A)\}$$

satisfies

$$\begin{aligned}\omega_1(A) = {} & \inf\{\omega\colon \text{for every } f \in D(A) \text{ there exists a constant } M \text{ such that} \\ & \qquad \|T(t)f\| \leqslant M\mathrm{e}^{\omega t} \text{ for every } t \geqslant 0\} \qquad (1.3)\\ = {} & \limsup_{t\to\infty} \frac{1}{t} \log \|T(t)R(\lambda, A)\| \quad (\lambda \in \varrho(A)).\end{aligned}$$

The following theorem plays a central role in the stability theory of positive semigroups. We show that the constant $\omega_1(A)$ coincides both with the abscissa of simple convergence of the Laplace transform of the semigroup and with the abscissa of absolute convergence of the Laplace transform of the strong solutions of (ACP).

Theorem 1.4 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. Then*

$$\begin{aligned}\omega_1(A) &= \inf\{\operatorname{Re}\lambda\colon \textstyle\int_0^\infty \mathrm{e}^{-\lambda t}T(t)f \,\mathrm{d}t \textit{ exists as an improper Riemann} \qquad (1.4)\\ &\qquad\qquad \textit{integral for every } f \in E\}\\ &= \inf\{\operatorname{Re}\lambda\colon \textstyle\int_0^\infty \|\mathrm{e}^{-\lambda t}T(t)f\| \,\mathrm{d}t \textit{ exists for every } f \in D(A)\}.\end{aligned}$$

Remark (i) One can show that the abscissas of uniform, strong, and weak convergence of the Laplace transform coincide (see C-III, Theorem I.2, last part of the proof). Therefore, by Theorem 1.4,

$$\begin{aligned}\omega_1(A) &= \inf\left\{\operatorname{Re}\lambda\colon \text{weak-}\lim_{t\to\infty} \textstyle\int_0^t \mathrm{e}^{-\lambda s}T(s)\,\mathrm{d}s \text{ exists}\right\} \qquad (1.5)\\ &= \inf\left\{\operatorname{Re}\lambda\colon \text{uniform-}\lim_{t\to\infty} \textstyle\int_0^t \mathrm{e}^{-\lambda s}T(s)\,\mathrm{d}s \text{ exists}\right\}.\end{aligned}$$

(ii) In Equations (1.4) and (1.5), the term "$\operatorname{Re}\lambda$" may be replaced by "$\lambda \in \mathbb{R}$" (use (*)).

Proof (Proof of Theorem 1.4) The equality

$$\omega_1(A) = \inf\left\{\operatorname{Re}\lambda\colon \int_0^\infty \|\mathrm{e}^{-\lambda t}T(t)f\| \,\mathrm{d}t \text{ exists for all } f \in D(A)\right\}$$

follows from the definition of $\omega_1(A)$ and the lemma in the proof of Theorem 1.3. We aim to prove that

$$\omega_1(A) = \inf\left\{\operatorname{Re}\lambda\colon \int_0^\infty \mathrm{e}^{-\lambda s} T(s) f \,\mathrm{d}s \text{ exists for every } f \in E\right\} =: b.$$

The identity $T(t)f = \mathrm{e}^{\lambda t}\left\{f - \int_0^t \mathrm{e}^{-\lambda s} T(s)(\lambda - A) f \,\mathrm{d}s\right\}$ yields

$$\omega_1(A) \leqslant \inf\left\{\operatorname{Re}\lambda\colon \int_0^\infty \mathrm{e}^{-\lambda t} T(t) f \,\mathrm{d}t \text{ exists for every } f \in \operatorname{im}(\lambda - A)\right\}.$$

Therefore, $\omega_1(A) \leqslant b$. Take $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_1(A)$. Then $\int_0^\infty \mathrm{e}^{-\lambda t} T(t) f \,\mathrm{d}t$ exists for every $f \in D(A)$. Define $g := \int_0^\infty \mathrm{e}^{-\lambda t} T(t) f \,\mathrm{d}t$. Then $g \in D(A)$ and $\int_0^n \mathrm{e}^{-\lambda t} T(t) f \,\mathrm{d}t = \sum_{k=0}^{n-1} \mathrm{e}^{-\lambda k} T(k) g$. Since $\operatorname{Re}\lambda > \omega_1(A)$, the sum converges for every $g \in D(A)$. Therefore, the integral converges as $n \to \infty$ for every $f \in E$. For every $t \in \mathbb{R}_+$, define a bounded operator T_t by $f \mapsto \int_0^t \mathrm{e}^{-\lambda s} T(s) f \,\mathrm{d}s$. As seen above, $T_n f$ converges as $n \to \infty$ for every $f \in E$. It follows from the Uniform Boundedness Principle that $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded.

For every $t \in \mathbb{R}_+$, there exist $n \in \mathbb{N}$ and $t' \in [0,1)$ such that $T_t = T_{t'} + \mathrm{e}^{-\lambda t'} T(t') T_n$. Since the operator families on the right side are uniformly bounded, the same is true for $(T_t f)_{t\geqslant 0}$. Since $(T_t f)_{t\geqslant 0}$ converges for every $f \in D(A)$, it follows that $(T_t f)_{t\geqslant 0}$ converges for every $f \in E$. Thus, $b \leqslant \omega_1(A)$. □

The inequality

$$\omega_0(A) \geqslant \inf\left\{\operatorname{Re}\lambda\colon \int_0^\infty \|\mathrm{e}^{-\lambda t} T(t) f\| \,\mathrm{d}t \text{ exists for every } f \in E\right\}$$

combined with the lemma of Theorem 1.3 implies that the growth bound $\omega_0(A)$ coincides with the abscissa of absolute convergence of the Laplace transform of $(T(t))_{t\geqslant 0}$, i.e.,

$$\omega_0(A) = \inf\left\{\operatorname{Re}\lambda\colon \int_0^\infty \|\mathrm{e}^{-\lambda t} T(t) f\| \,\mathrm{d}t \text{ exists for every } f \in E\right\}. \tag{1.6}$$

As seen in A-I, Proposition 1.11, if $\int_0^\infty \mathrm{e}^{-\lambda t} T(t) f \,\mathrm{d}t$ exists for every $f \in E$, then $\lambda \in \varrho(A)$ and $R(\lambda, A) f = \int_0^\infty \mathrm{e}^{-\lambda t} T(t) f \,\mathrm{d}t$. This and Theorem 1.4 yield the following corollary.

Corollary 1.5 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. Then*

$$s(A) \leqslant \omega_1(A) \leqslant \omega_0(A).$$

Example 1.2(c) shows that uniform exponential stability is not equivalent to $\sigma(A) \subset \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \leqslant q < 0\}$. The following example shows that even strong solutions can be unstable when $s(A) < 0$. We construct a semigroup where $s(A) < \omega_1(A) < \omega_0(A)$.

Example 1.6 As in A-III, Example 1.4, consider the semigroup $(T(t))_{t\geqslant 0}$ on the Hilbert space $E = \{(x^1, x^2, ...) : x^n \in \mathbb{C}^n : \sum_{j=1}^{\infty} \|x^j\|^2 < \infty\}$, defined by

$$T(t) := (\mathrm{e}^{2\pi int} \exp(tA_n))_{n\in\mathbb{N}},$$

where

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix}_{n\times n}.$$

This semigroup satisfies $\|T(t)\| = e^t$ for all $t \geq 0$. Hence, $\omega_0(A) = 1$, while the generator $A = (2\pi in + A_n)_{n\in\mathbb{N}}$ has spectral bound $s(A) = 0$. We first show that $\omega_1(A) = \omega_0(A)$; this will later be used to construct a semigroup for which $s(A) < \omega_1(A) < \omega_0(A)$. Let $e_n = n^{-1/2}(1, ..., 1) \in \mathbb{C}^n$.

Then, for fixed n,

$$\begin{aligned}
&\|\exp(tA_n)e_n\|^2 = \\
&\quad = \frac{1}{n}\left\|\left(1 + t + \cdots + \frac{t^{n-1}}{(n-1)!}, 1 + t + \cdots + \frac{t^{n-2}}{(n-2)!}, \dots, 1 + t, 1\right)\right\|^2 \\
&\quad = \frac{1}{n}\sum_{r=0}^{n-1}\sum_{j,s=0}^{r}\frac{1}{j!s!}t^{j+s} = \frac{1}{n}\sum_{r=0}^{n-1}\left(\sum_{j=0}^{r}\frac{1}{j!}t^j\right)^2 = \frac{1}{n}\sum_{r=0}^{n-1}\sum_{j,s=0}^{r}\frac{1}{j!s!}t^{j+s} \\
&\quad = \frac{1}{n}\sum_{r=0}^{n-1}\sum_{i=0}^{2r}t^i\sum_{j+s=i}\frac{1}{j!s!} = \frac{1}{n}\sum_{r=0}^{n-1}\sum_{i=0}^{2r}\frac{(2t)^i}{i!} \geqslant \frac{1}{n^2}\sum_{i=0}^{n-1}\frac{(2t)^i}{i!}.
\end{aligned}$$

For $0 < q < 1$, define $x_q \in E$ as $x_q := (qe_1, 2q^2e_2, ..., nq^ne_n, ...)$. Then $x_q \in D(A)$ and

$$\begin{aligned}
\|T(t)x_q\|^2 &= \sum_{n=1}^{\infty} q^{2n}\|\exp(tA)e_n\|^2 \geqslant \sum_{n=1}^{\infty} n^2q^{2n}\left(\frac{1}{n^2}\sum_{i=0}^{n-1}\frac{1}{i!}(2t)^i\right) \\
&= \sum_{i=0}^{\infty}\sum_{n=i+1}^{\infty}\left(q^{2n}\frac{1}{i!}(2t)^i\right) = \sum_{i=0}^{\infty} q^{2i+2}(1-q^2)^{-1}\frac{1}{i!}(2t)^i \\
&= \frac{q^2}{1-q^2}\sum_{i=0}^{\infty}\frac{1}{i!}(2tq^2)^i = \frac{q^2}{1-q^2}\mathrm{e}^{2tq^2}.
\end{aligned}$$

It follows that $\omega(x_q) \geqslant q^2$. Thus,

$$1 = \sup\{\omega(x_q) \colon 0 < q < 1\} \leqslant \omega_1(A) \leqslant \omega_0(A) = 1.$$

Rescaling the semigroup (i.e., looking at $e^{-3/2 \cdot t} T(t)$), we obtain a semigroup generator A on the Hilbert space E with $-3/2 = s(A)$ and $\omega_1(A) = \omega_0(A) = -1/2$. On the other hand, Example 1.2(c) produces a semigroup in a Banach space F with generator B such that $-1 = s(B) = \omega_1(B)$ while $\omega_0(B) = 0$. Now the operator $C := A \oplus B$ on the Banach space $E \oplus F$ is a semigroup generator for which

$$s(C) = \max\{s(A), s(B)\} = -1, \quad \omega_1(C) = \max\{\omega_1(A), \omega_1(B)\} = -1/2$$
$$\text{and} \quad \omega_0(C) = \max\{\omega_0(A), \omega_0(B)\} = 0.$$

Remark 1.7 For eventually norm continuous semigroups—particularly compact, differentiable, holomorphic, or nilpotent ones— the spectral mapping theorem

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \tag{1.7}$$

holds. Consequently,

$$s(A) = \omega_1(A) = \omega_0(A) \tag{1.8}$$

is valid (Corollary 1.5 and A-III, Corollary 6.7). Hence, if A is the generator of an eventually norm-continuous semigroup, the exponential growth bounds of the strong and mild solutions of the abstract Cauchy problem $\dot{u}(t) = Au(t), u(0) = x$ are determined by the spectral bound $s(A) = \sup\{\operatorname{Re}\lambda \colon \lambda \in \sigma(A)\}$.

Remark 1.8 In general, the growth bound $\omega_0(A)$ can be obtained using the Hille-Yosida theorem (see A-II, Theorem 1.7) as

$$\omega_0(A) = \inf\{w \colon \|R(\lambda, A)^n\| \leqslant M(\operatorname{Re}\lambda - w)^{-n} \text{ for some } M \text{ and every } n \in \mathbb{N} \text{ and } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda > w\}. \tag{1.9}$$

Due to the difficulty of estimating all powers of the resolvent, this characterization is of limited practical use. However, if A is the generator of a semigroup on a Hilbert space H, then it is shown in A-III, Corollary 7.11 that

$$\omega_0(A) = \inf\{w \colon \|R(\lambda, A)\| \leqslant M_w \quad \text{for } \operatorname{Re}\lambda > w\}. \tag{1.10}$$

Unfortunately, the identity (1.10) does not hold on arbitrary Banach spaces. However, as we will see in Section 1 of C-IV, the identity

$$s(A) = \omega_1(A) = \inf\{w \colon \|R(\lambda, A)\| \leqslant M_w \quad \text{for } \operatorname{Re}\lambda > w\} \tag{1.11}$$

holds for every positive semigroup on a Banach lattice. Consequently, for positive semigroups with $s(A) = \omega_1(A) < \omega_0(A)$ (see Example 1.2 (ii)), the identity (1.10) is not applicable. Nevertheless, we can establish the following theorem.

Theorem 1.9 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. Suppose there exist constants $a \geqslant 0$ and $q \geqslant s(A)$, and that there exist $C > 0$ and $n \in \mathbb{N}$ such that*

$$\|R(\lambda, A)\| \leqslant C|\lambda|^{n-2}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > q$ and $|\operatorname{Im}\lambda| > a$. Then

$$\sup\{\omega(f), f \in D(A^n) \leqslant q\}.$$

Proof The hypothesis $\|R(\lambda, A)\| \leqslant C|\lambda|^{n-2}$ is invariant under rescaling. That is, the resolvent $R(\lambda, -b + A)$ of the generator $-b + A$ of the rescaled semigroup $\mathrm{e}^{-bt}T(t)$ satisfies $\|R(\lambda, -b + A)\| \leqslant \tilde{C}|\lambda|^{n-2}$ for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > q - b$ and $|\operatorname{Im}\lambda| > a + 2b$, for a suitable constant $\tilde{C}$. Therefore, we may assume without loss of generality that $b := \max(\omega_0(A), q) < 0$. Let $\omega_0(A) < p < 0$, and set $p' := \max\{p, q\} < 0$. Then, for every $f \in D(A)$, the inversion formula for the Laplace transform yields

$$T(t)f = \frac{1}{2\pi i}\int_{p'-i\infty}^{p'+i\infty} \mathrm{e}^{\lambda t} R(\lambda, A)f\, d\lambda. \tag{1.12}$$

(For a proof of the vector-valued inversion formula, see Widder [24, p.66]; also refer to the notes in this section.) Using the resolvent equation, we obtain

$$R(\lambda, A)^n R(0, A) = \sum_{k=1}^{n}(-1)^{k+1}\lambda^{-k}R(0, A)^{n+1-k} + (-1)^n\lambda^{-n}R(\lambda, A).$$

Since $\frac{1}{2\pi i}\int_{p'-i\infty}^{p'+i\infty} \mathrm{e}^{\lambda t}\,\lambda^{-k}\,d\lambda = 0$ for $k \geqslant 1$, $p' < 0$ and $t > 0$, it follows that

$$T(t)R(0, A)^n f = (-1)^n\frac{1}{2\pi i}\int_{p'-i\infty}^{p'+i\infty} \mathrm{e}^{\lambda t}\lambda^{-n}R(\lambda, A)f\, d\lambda \tag{1.13}$$

for every $f \in E$ and $t > 0$.

If $q < p'$, then by Cauchy's Integral Theorem and the growth bound $\|R(\lambda, A)\| \leqslant C|\lambda|^{n-2}$, we can shift the path of integration to $\operatorname{Re}\lambda = q$, yielding

$$T(t)R(0, A)^n f = (-1)^n\frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} \mathrm{e}^{\lambda t}\lambda^{-n}R(\lambda, A)f\, d\lambda.$$

Thus, we estimate

$$\|T(t)R(0, A)^n f\| \leqslant c'\mathrm{e}^{qt}\|f\|\int_{-\infty}^{\infty}(q^2 + s^2)^{-1}\,\mathrm{d}s = M\mathrm{e}^{qt}\|f\|.$$

Equivalently,

$$\|T(t)f\| \leqslant M e^{qt} \|A^n f\| \text{for } f \in D(A^n).$$

In view of the characterizations given in Section 1 of A-II, the semigroups considered in the theorem above are holomorphic if $n = 1$. In this case, one may apply (1.7) to obtain the stronger statement (1.8).

Rather than imposing conditions on the resolvent of A, we now adopt a different perspective and characterize the property "$\omega_0(A) < 0$" directly in terms of the semigroup $(T(t))_{t\geqslant 0}$.

Proposition 1.10 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. Then the following statements are equivalent.*

(a) $\omega_0(A) < 0$.

(b) $\lim_{t\to\infty} \|T(t)\| = 0$.

(c) $\|T(t')\| < 1$ *for some* $t' > 0$.

Proof The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are immediate; To prove $(c) \Rightarrow (a)$, note that $\omega_0(A) = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|$ (see A-I,(1.1)). Assume $\|T(t')\| < 1$ for some $t' > 0$. For $t = nt' + s$ with $s \in [0, t']$, we have $\|T(t)\| \leq \|T(t')\|^n \|T(s)\|$, so that

$$\frac{\log \|T(t)\|}{t} \leqslant \frac{n \log \|T(t')\|}{nt' + s} + \frac{\log \|T(s)\|}{nt' + s}.$$

Since $\|T(s)\|$ is bounded on compact intervals, the second term tends to zero as $n \to \infty$. The first term tends to $\frac{\log \|T(t')\|}{t'} < 0$. Thus, $\omega_0(A) < 0$, which proves $(c) \Rightarrow (a)$. □

Other less obvious characterizations of the property "$\omega_0(A) < 0$" are provided in the following theorem. The equivalence of (a) and (c) is known as *Datko's Theorem*.

Theorem 1.11 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. Then the following statements are equivalent.*

(a) $\omega_0(A) < 0$.

(b) $s(A) < 0$ *and there is* $t_0 > 0$ *such that* $|\lambda| < 1$ *for every* $\lambda \in A\sigma(T(t_0))$.

(c) *For some (equivalently, every)* $p \geqslant 1$ *the integral* $\int_0^\infty \|T(t)f\|^p \, dt$ *exists for every* $f \in E$.

Proof The implication "(a) ⇒ (b)" follows from the fact that $r(T(t)) = e^{\omega_0(A)t} < 1$ and $s(A) \leqslant \omega_0(A) < 0$. For the point and residual spectrum, the spectral mapping theorem is valid (see A-III, Theorem 6.3). Since the approximate point spectrum is closed, the additional assumption in (b) implies that $|\lambda| \leqslant r < 1$ for all $\lambda \in A\sigma(T(t_0))$. Consequently,

$$\exp(\omega_0(A) \cdot t_0) = r(T(t_0)) \leqslant \max\{\exp(t_0 \cdot s(A)), r\} < 1,$$

which implies $\omega_0(A) < 0$. This proves "(b) ⇒ (a)". For a proof of the equivalence of (a) and (c), we refer to Datko [4] or Pazy [17, Theorem 4.4.1]. □

By rescaling a given semigroup $(T(t))_{t\geqslant 0}$, one obtains the following corollary from (1.1) and statement (c) of the preceding theorem.

Corollary 1.12 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space E. Then the set of $\lambda \in \mathbb{C}$ for which*

$$\int_0^\infty \|\mathrm{e}^{-\lambda t}T(t)f\| \,\mathrm{d}t$$

exists for every $f \in E$ is an open right half-plane.

In the next theorem, we present two necessary conditions for the stability of the semigroup $(T(t))_{t\geqslant 0}$ in terms of its generator A.

We will see in Chapter C-IV that for positive semigroups a condition similar to statement (ii) below is even sufficient for stability.

It is important to note that stable semigroups need not be uniformly bounded (see Example 1.2(c)) and that the equality $s(A) = \omega_0(A) = 0$ does not imply boundedness or even stability of the semigroup (see also A-I, Example 1.4.(i)).

Theorem 1.13 *Let A be the generator of a stable semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. Then the following assertions hold.*

(i) $s(A) \leqslant 0$ *and* $\operatorname{Re}\lambda < 0$ *for every* $\lambda \in P\sigma(A) \cup R\sigma(A)$.

(ii) $\lim_{\lambda\downarrow 0} \lambda R(\lambda, A)f$ *exists for every* $f \in D(A)$.

Proof (i) If $(T(t))_{t\geqslant 0}$ is stable, then $\|T(t)f\| \leqslant M_f$ for every $f \in D(A)$. Therefore, $s(A) \leqslant \omega_1(A) \leqslant 0$.

Assume, by contradiction, that there exists $\lambda \in P\sigma(A)$ with $\operatorname{Re}\lambda = 0$. Then, by A-III, Corollary 6.4, there exists $g \neq 0$ such that $T(t)g = \mathrm{e}^{\lambda t}g$ for all $t \geqslant 0$. Since $|\mathrm{e}^{\lambda t}| = 1$, this contradicts the stability of the semigroup.

Now assume there exists $\lambda \in R\sigma(A) = P\sigma(A')$ with $\operatorname{Re}\lambda = 0$. Then there exists $0 \neq \varphi \in E'$ with $T(t)^*\varphi = \exp(\lambda t)\varphi$ for all $t \geqslant 0$. Choose $f \in D(A)$ such that $\langle f, \varphi\rangle \neq 0$. Then $|\langle T(t)f, \varphi\rangle| = |\langle f, \varphi\rangle| > 0$ for every $t \geqslant 0$, which again contradicts the stability of the semigroup.

(ii) From the stability of the semigroup and the identity $\int_0^t T(s)Af \,\mathrm{d}s = T(t)f - f$, it follows that $\int_0^\infty T(s)Af \,\mathrm{d}s$ exists for every $f \in D(A)$. Since $\omega_1(A) \leqslant 0$, we may apply Theorem 1.4 to write $R(\lambda, A)Af = \int_0^\infty \mathrm{e}^{-\lambda s}T(s)Af \,\mathrm{d}s$ for every $\lambda > 0$. By a classical theorem of Laplace transform theory, (for a proof of the vector-valued version one may follow Widder [25, p.196]), we conclude that $\lim_{\lambda\to 0+} R(\lambda, A)Af$ exists and is equal to $\int_0^\infty T(s)Af \,\mathrm{d}s$. The identity $R(\lambda, A)Af = \lambda R(\lambda, A)f - f$ yields the existence of $\lim_{\lambda\to 0+} \lambda R(\lambda, A)f$ for every $f \in D(A)$. □

Bounded holomorphic semigroups (see A-II, Definition 1.11) satisfy the estimate $\|AT(t)\| \leqslant \frac{m}{t}$ Goldstein [9, p.33], and hence $T(t)f \to 0$ as $t \to \infty$ for every $f \in \operatorname{im}(A)$. If $\operatorname{im}(A)$ is dense (i.e., $0 \notin R\sigma(A)$), then we obtain uniform stability and the following corollary.

Corollary 1.14 *Let A be the generator of a bounded holomorphic semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. Then the following statements are equivalent.*

(a) $0 \notin P\sigma(A) \cup R\sigma(A)$.

(b) $(T(t))_{t\geqslant 0}$ *is uniformly stable.*

Example 1.15 The Laplacian Δ generates a bounded holomorphic semigroups on $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$ (see the example proceeding Corollary 1.13 in Chapter A-II). All solutions of the equation $\Delta f = 0$ are either constant or unbounded, so $0 \notin P\sigma(\Delta)$. If $1 < p < \infty$, then the adjoint of the Laplacian on $L^p(\mathbb{R}^n)$ is the Laplacian on $L^q(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, $0 \notin P\sigma(\Delta) \cup R\sigma(\Delta)$, and by Corollary 1.14, the Laplacian generates a uniformly stable semigroup on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. However, since $\operatorname{im}\Delta \neq L^p(\mathbb{R}^n)$, Corollary 1.5 implies that the semigroup is not exponentially stable.

As seen in Theorem 1.4, exponential stability can be characterized by the condition that the abscissa of convergence of the Laplace transform of $(T(t))_{t\geqslant 0}$ is less than zero. This should be compared to the following result, which characterizes (uniform) stability in terms of integrability and the kernel of the generator.

Theorem 1.16 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. The following assertions are equivalent.*

(a) $(T(t))_{t\geqslant 0}$ *is stable.*

(b) $\ker(A) = \{0\}$ *and* $\int_0^\infty T(t)f \,\mathrm{d}t$ *exists for all* $f \in \operatorname{im}(A)$.

Furthermore the following statements are equivalent,

(a') $(T(t))_{t\geqslant 0}$ *is stable and bounded.*

(b') $(T(t))_{t\geqslant 0}$ *is uniformly stable.*

(c') $(T(t))_{t\geqslant 0}$ *is bounded and there is a dense subspace D such that* $\int_0^\infty T(t)f \,\mathrm{d}t$ *exists for every $f \in D$.*

Proof If $(T(t))_{t\geqslant 0}$ is stable, then by Theorem 1.13, $\ker(A) = \{0\}$, and

$$\int_0^t T(s)Af \,\mathrm{d}s = T(t)f - f \to -f \text{ as } t \to \infty\,.$$

Hence, $(a) \Rightarrow (b)$.

Conversely, suppose $\int_0^t T(s)Af \,\mathrm{d}s$ converges as $t \to \infty$. Then, by the identity above, the limit $g := \lim_{t\to\infty} T(t)f$ exists. Since $\ker(A) = \{0\}$, it follows that $g = 0$, so $T(t)f \to 0$. Thus, $(b) \Rightarrow (a)$.

The implication $(a') \Rightarrow (b')$ is immediate.

If $T(t)f \to 0$ for every $f \in E$, then $\|T(t)\| \leqslant M$, and $0 \notin R\sigma(A)$ by Theorem 1.13. Therefore, $D := \operatorname{im}(A)$ is dense in E, and for every $f \in D$, the integral $\int_0^\infty T(t)f \, dt$ exists. This proves $(b') \Rightarrow (c')$.

It remains to prove $(c') \Rightarrow (a')$. Define

$$G := \{h \in E : h = \int_0^\infty T(t)g \, dt \text{ for some } g \in D\}.$$

We claim that G is dense in E. Indeed, for any $g \in D$ and $s > 0$, $g - T(s)g \in D$. Define $h_s = \frac{1}{s}\int_0^\infty T(t)(g - T(s)g) \, dt = \frac{1}{s}\int_0^s T(t)g \, dt$. Then $h_s \in G$, and $h_s \to g$ as $s \to 0$. Therefore, $D \subset \overline{G}$ or $\overline{G} = E$ since D is dense in E. Now, let $h \in G$, so $h = \int_0^\infty T(t)g \, dt$ for some $g \in D$. Then $T(t)h = T(t)\int_0^\infty T(s)g \, ds = \int_t^\infty T(s)g \, ds \to 0$ as $t \to \infty$. Since $\|T(t)\| \leqslant M$, it follows that $T(t)f \to 0$ for every $f \in E$. □

Remark 1.17 (i) If A is the generator of a stable semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E, then by the previous theorem,

$$\operatorname{im}(A) \subset \{f \in E : \int_0^\infty T(t)f \, dt \text{ exists}\} =: H.$$

If $g \in H$, then $\int_0^\infty T(t)g \, dt \in D(A)$ and $A\int_0^\infty T(t)g \, dt = -g$. Thus, $g \in \operatorname{im}(A)$, and the dense subspace $\operatorname{im}(A)$ is given by

$$\operatorname{im}(A) = \{f \in E : \int_0^\infty T(t)f \, dt \text{ exists}\} \tag{1.14}$$

whenever A generates a stable semigroup $(T(t))_{t\geqslant 0}$.

(ii) If $\omega(f) < 0$ for every $f \in D(A)$, then $(T(t))$ is stable, but might not be exponentially stable if $0 = \omega_1(A) = \sup\{\omega(f) : f \in D(A)\}$. In this case, one can show–via a proof similar to that of Theorem 1.4–that the spectrum $\sigma(A)$ must be contained in the open left half-plane, i.e., $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$.

(iii) If one defines a semigroup $(T(t))_{t\geqslant 0}$ to be weakly stable if $\langle T(t)f, \varphi\rangle \to 0$ as $t \to \infty$ for all $f \in D(A)$ and $\varphi \in E'$ or as weakly uniformly stable if the above holds for all $f \in E$ and $\varphi \in E'$, then Theorems 1.13 and 1.16 can be reformulated in a weak form (i.e., replacing stable by weakly stable and *lim* by *weak-lim*). The proofs require only obvious modifications. If A has a compact resolvent, or if A generates a bounded holomorphic semigroup, then weak stability implies stability. In general, this implication fails; e.g., the translation semigroup on $L^2(\mathbb{R})$ is weakly uniformly stable but not stable (see also B-IV, Example 1.2).

2 Stability: Inhomogeneous Case

Using the results of the previous section, we now investigate the long-term behavior of solutions to the inhomogeneous initial value problem

$$\dot{u}(t) = Au(t) + F(t) \quad , \quad u(0) = f, \tag{2.1}$$

where A generates a strongly continuous semigroup on a Banach space E and $F(\cdot)$ is a locally integrable function from $\mathbb{R}_+$ into E, referred to as the *forcing term*. A function $u(\cdot)$ is called a *(strong) solution* of (2.1) if $u(\cdot)\colon \mathbb{R}_+ \to D(A)$, $u(\cdot) \in C^1(\mathbb{R}_+, E)$, and (2.1) holds for all $t \geqslant 0$. The assumption that A generates a semigroup $(T(t))_{t\geqslant 0}$ guarantees the uniqueness of the solution to (2.1). If $u(\cdot)$ is a solution, then for fixed $t > 0$, the function $v(s) := T(t-s)u(s)$, $0 \leqslant s \leqslant t$, is differentiable and $v'(s) = T(t-s)F(s)$. Since $F(\cdot)$ is locally integrable, it follows that

$$\int_0^t T(t-s)F(s)\,\mathrm{d}s = v(t) - v(0) = u(t) - T(t)f.$$

Thus, the solution $u(t)$ of (2.1) is given by

$$u(t) = T(t)f + \int_0^t T(t-s)F(s)\,\mathrm{d}s\,. \tag{2.2}$$

Example Let $(T(t))_{t\geqslant 0}$ be a semigroup that is not eventually differentiable. Then there exists $g \in E$ such that $t \mapsto T(t)g$ is not differentiable on $(0,\infty)$. Consider the initial value problem $\dot{u}(t) = Au(t) + T(t)g$, $u(0) = 0$. This problem has no (strong) solution $u(\cdot)$, because otherwise we would have

$$u(t) = \int_0^t T(t-s)T(s)g\,\mathrm{d}s = tT(t)g,$$

which would imply that $t \to T(t)g$ is differentiable on $\mathbb{R}_+$, a contradiction.

Whenever the expression (2.2) is well-defined, we refer to it as a *generalized* (or *mild*) solution of (2.1). If $F(\cdot)$ is continuous and $f \in D(A)$, then the generalized solution of (2.1) is a strong solution if and only if the function $v(t) := \int_0^t T(t-s)F(s)\,\mathrm{d}s$ is differentiable (see Pazy [17, Chapter 4,2.4]. There are several sufficient conditions on the generator A, the forcing term $F(\cdot)$, or the space E under which every mild solution is a strong solution of (2.1) (see Travis [21] or Pazy [17, Section 4.2]).

Our aim in this section is to study the asymptotic behavior of the solutions of (2.1) as $t \to \infty$. To that end, we consider absolutely integrable or periodic forcing terms $F(\cdot)$, and assume that the semigroup is uniformly stable.

Similar results for integrable and convergent forcing terms $F(\cdot)$ can be obtained under the assumption of uniform stability (see Pazy [17, p.119] or Neubrander

[16]). However, if the semigroup is positive, these results remain valid even under the weaker assumption of stability (see Section C-IV). Recall from Theorem 1.13(i) that for stable semigroups, the range $\operatorname{im}(A)$ is dense in E.

Theorem 2.1 *Let A be the generator of a uniformly stable semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. If there exists $g \in \operatorname{im}(A)$ such that $\int_0^\infty \|F(s)-g\|\,\mathrm{d}s < \infty$, then every generalized solution $u(\cdot)$ of (2.1) converges as $t \to \infty$, and $\lim_{t\to\infty} u(t) = -h$, where $h \in D(A)$ satisfies $Ah = g$.*

Proof If $u(\cdot)$ is a generalized solution of (2.1), then by (2.2), we have

$$u(t) = T(t)f + \int_0^t T(s)g\,\mathrm{d}s + \int_0^t T(t-s)(F(s)-g)\,\mathrm{d}s.$$

By uniform stability and the identity $\int_0^t T(s)Ah\,\mathrm{d}s = T(t)h - h$ (see A-I, Proposition 1.6), the first term converges to zero and the second term converges to $-h$. It remains to show that the third term also converges to zero. Let $\varepsilon > 0$ and define $G(s) := F(s) - g$. Then for any $r > 0$,

$$\begin{aligned}\left\|\int_0^t T(t-s)G(s)\,\mathrm{d}s\right\| &\leqslant \left\|\int_0^r T(t-r+r-s)G(s)\,\mathrm{d}s\right\| + \left\|\int_r^t T(t-s)G(s)\,\mathrm{d}s\right\| \\ &\leqslant \left\|T(t-r)\int_0^r T(r-s)G(s)\,\mathrm{d}s\right\| + M\int_r^\infty \|G(s)\|\,\mathrm{d}s\,,\end{aligned}$$

where $\|T(t)\| \leq M$ for all $t \geq 0$. Since the semigroup is uniformly stable, we obtain $T(t-r)\int_0^r T(r-s)G(s)\,\mathrm{d}s \to 0$ as $t \to \infty$ for every $r \geqslant 0$. Therefore, $\|\int_0^t T(t-s)G(s)\,\mathrm{d}s\| \leqslant \varepsilon$ for all sufficiently large t. □

In the following theorem, we show that if A generates a uniformly stable semigroup, the forcing term $F(\cdot)$ is p-periodic and $\int_0^p T(p-s)F(s)\,\mathrm{d}s \in \operatorname{im}(\mathrm{Id}-T(p))$, then (2.1) admits a unique p-periodic mild solution that is *asymptotically stable*; i.e., for every generalized solution $v(\cdot)$ of (2.1),

$$\lim_{t\to\infty} \|v(t) - u(t)\| = 0.$$

(Notice that, by Theorem 1.13 and A-III, Lemma 5.3, $\overline{\operatorname{im}(\mathrm{Id}-T(p))} = E$.)

Lemma 2.2 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E, and let $F(\cdot)$ be a p-periodic, locally integrable function with $p > 0$. Then the following statements are equivalent.*

(a) *$\dot u(t) = Au(t) + F(t)$ admits a (unique) generalized p-periodic solution.*

(b) *There exists a (unique) $f \in E$ such that $(\mathrm{Id}-T(p))f = \int_0^p T(p-s)F(s)\,\mathrm{d}s$.*

Proof (a) ⇒ (b): Let $f := u(0)$ be the initial value for which (2.1) admits a p-periodic mild solution. Then for every $t \geq 0$,

$$\begin{aligned} u(t) &= u(t+p) \\ &= T(t)T(p)f + \int_0^p T(t+p-s)F(s)\,\mathrm{d}s + \int_p^{t+p} T(t+p-s)F(s)\,\mathrm{d}s \\ &= T(t)\left[T(p)f + \int_0^p T(p-s)F(s)\,\mathrm{d}s\right] + \int_0^t T(t-s)F(s)\,\mathrm{d}s. \end{aligned}$$

Since $u(t) = T(t)f + \int_0^t T(t-s)F(s)\,\mathrm{d}s$, it follows that

$$T(t)f = T(t)\left[T(p)f + \int_0^p T(p-s)F(s)\,\mathrm{d}s\right].$$

This implies $f = T(p)f + \int_0^p T(p-s)F(s)\,\mathrm{d}s$. If $u(\cdot)$ is a unique periodic solution with $u(0) = f$, then f is the unique element in E for which this identity holds.

(b) $\Rightarrow$ (a): Define $u(\cdot)$ as in (2.2). Then

$$u(t+p) = T(t)\left[T(p)f + \int_0^p T(p-s)F(s)\,\mathrm{d}s\right] + \int_0^t T(t-s)F(s)\,\mathrm{d}s = u(t).$$

If f is unique, then, by the above considerations, the solution is also unique. □

Remark 2.3 Let A be the generator of a strongly continuous semigroup for which the spectral mapping theorem holds (see A-III, Section 6), and let F be a p-periodic forcing term. If $\frac{2\pi i n}{p} \in \varrho(A)$ for every $n \in \mathbb{Z}$, then, by Lemma 2.2, (2.1) admits a unique p-periodic solution with initial value $(\mathrm{Id} - T(p))^{-1}\left(\int_0^p T(p-s)F(s)\,\mathrm{d}s\right)$.

As a consequence of Theorem 1.13 and A-III, Corollary 6.4, a uniformly stable semigroup admits at most one $f \in E$ satisfying

$$(\mathrm{Id} - T(p))f = \int_0^p T(p-s)F(s)\,\mathrm{d}s\,.$$

This fact, together with Lemma 2.2, is used to prove the following theorem.

Theorem 2.4 *Let A be the generator of a uniformly stable semigroup $(T(t))_{t\geqslant 0}$ and let $F(\cdot)$ be a p-periodic, locally integrable function such that*

$$(\mathrm{Id} - T(p))f = \textstyle\int_0^p T(p-s)F(s)\,\mathrm{d}s \ \textit{ for some } \ f \in E\,.$$

Then the unique p-periodic generalized solution

$$u(t) = T(t)f + \int_0^t T(t-s)F(s)\,\mathrm{d}s$$

is asymptotically stable.

Example 2.5 Let E be the Banach space $C_0(\mathbb{R}_+)$ of continuous functions vanishing at infinity. Define $A = \frac{d}{dx}$ with domain $D(A) = \{f \in E : f' \in C^1 \text{ and } f' \in E\}$ is the generator of the uniformly stable translation semigroup $T(t)f(x) := f(t+x)$. Applying (1.14), we obtain $\operatorname{im}(A) = \{f\colon \int_0^\infty f(x)\, \mathrm{d}x \text{ exists}\}$ is dense in $C_0(\mathbb{R}_+)$. Let $r \in \operatorname{im}(A)$ and let $F(\cdot)$ be a p-periodic, real-valued function. We apply Theorem 2.4 to the initial value problem

$$\frac{d}{dt}u(t,x) = \frac{d}{dx}u(t,x) + r(x)F(x+t), \quad u(0,\cdot) \in D(A). \tag{*}$$

We rewrite (*) as

$$\dot{v}(t) = Av(t) + G(t), \tag{**}$$

where $v(t) = u(t,\cdot)$ and $G\colon \mathbb{R}_+ \to E$ is defined by $G(t)(x) = r(x)F(x+t)$. Then G is p-periodic with values in E and $h_0 := \int_0^p T(p-t)G(t)\, \mathrm{d}t$ is the function $x \mapsto \left[\int_0^p T(p-t)G(t)\, \mathrm{d}t\right](x) = F(x)\int_x^{x+p} r(s)\, \mathrm{d}s$. Now define $f := \sum_{k=0}^\infty T(kp)h_0$, which corresponds to the function $x \mapsto F(x)\int_x^\infty r(s)\, \mathrm{d}s$. It is then clear that $(\mathrm{Id} - T(p))f = h_0$. Therefore, (**) admits a unique p-periodic generalized solution by Theorem 2.4), even though $\mathrm{i}\mathbb{R} \subset \sigma(A)$ (cf. Remark 2.3).

The unique p-periodic generalized solution $u(t,\cdot)$ is given explicitly by

$$u(t,x) = F(x+t)\int_{x+t}^\infty r(s)\, \mathrm{d}s + F(x+t)\int_x^{x+t} r(s)\, \mathrm{d}s = F(x+t)\int_x^\infty r(s)\, \mathrm{d}s.$$

Finally, for every solution $v(t,\cdot)$ of (*), Theorem 2.4 implies that

$$\sup\left\{\left|v(t,x) - F(x+t)\int_x^\infty r(s)\, \mathrm{d}s\right| : x \in \mathbb{R}_+\right\} \to 0 \quad \text{as} \quad t \to 0.$$

Notes

Section 1: The exponential growth bounds $\omega(f)$ and $\omega_0(A)$ as well as the characterizations (1.2), (1.6) and Theorem 1.3 (i) can be found in Hille and Phillips [11]. Growth bounds similar to $\omega_1(A)$ were first considered in D'Jacenko [6] and in Zabczyk [28, Proposition 2]. Example 1.2(2) is taken from Wolff [26]; other *counterexamples* can be found in Hille and Phillips [11], Foias [8], Triggiani [22], Zabczyk [27] and Greiner et al. [10]. Statements (1.2), (1.6) and Theorem 1.3 (i) are semigroup versions of results in classical Laplace transform theory, see Hille and Phillips [11] and Widder [24]. Theorem 1.3 (ii) is a semigroup version of Theorem 1.2.7 and 1.2.8 in Doetsch [7]. The lemma in the proof of Theorem 1.3 is taken from Mil'stein [13]. Theorem 1.4 and Corollary 1.5 can be found in Neubrander [15]. Example 1.6 follows Remark 2 in Zabczyk [27]. Statement (1.8) is sometimes called the *spectrum determined growth assumption*, see, for example, Triggiani [23]. Theorem 1.9 is due to Slemrod [20]. The proof presented here is based on the following sharper version of the inversion formula for the Laplace transform, which improves on the one given in Hille and Phillips [11, p.349]. Using Widder [24, p.66] or Doetsch [7, p.212] one can establish the following theorem (see Neubrander [14].

Theorem 2.6 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space E. For every $f \in D(A)$ and $p > \omega_1(A)$ we have*

$$T(t)f = \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} \mathrm{e}^{\mu t} R(\mu, A) f \, d\mu.$$

The equivalence of the statements (1.12), (1.13) and $\omega_0(A) < 0$ were observed by many authors, see for example, Balakrishnan [1, p.178], or Benchimol [2]. Theorem 1.11 is due to Datko [3]; for a proof see Pazy [17, p.116]. Theorems 1.13 and 1.16 can be found in Neubrander [16] and Corollary 1.14 is due to Komatsu [12]. An example of an unstable semigroup generator A with $\mathrm{Re}\,\mu < 0$ for all $\mu \in \sigma(A)$ is given in Datko [5].

Section 2: For a discussion of well-posedness of inhomogeneous Cauchy problems, we refer to Goldstein [9, p.83], and Pazy [17, p.105]. Further results on the asymptotic behavior of the solutions of the inhomogeneous problem can be found in Rao and Hengartner [19], Zaidman [29], Pazy [17], and Neubrander [16]. Results similar to Lemma 2.2 and Theorem 2.4 are due to Prüss [18]. For a discussion of the asymptotic behavior of the solutions of $\dot{u}(t) = A(t)u(t) + F(t)$ see Datko [4] and Pazy [17, p.172].

References

[1] A. Balakrishnan. *Applied Functional Analysis. Application of Mathematics. 3.* Springer, Berlin-Heidelberg-New York, 1976.

[2] C. Benchimol. Feedback stabilizability in Hilbert spaces. *Appl. Math. Optim.*, 4:223–248, 1978.

[3] R. Datko. Extending a theorem of A.M. Liapunov to Hilbert space. *J. Math. Anal. Appl.*, 32:610–616, 1970.

[4] R. Datko. Uniform asymptotic stability of evolutionary processes in a Banach space. *SIAM J. Math. Anal.*, 3:428–445, 1972.

[5] R. Datko. An example of an unstable neutral differential equation. *Internat. J. Control*, 38:263–267, 1983.

[6] S. V. D'Jacenko. Semigroups of almost negative type and their applications. *Soviet Math. Dokl.*, 17:1189–1193, 1976.

[7] G. Doetsch. *Handbuch der Laplace Transformation, Band I.* Birkhäuser, Basel, 1950.

[8] C. Foias. Sur une question de M. Reghis. *An. Univ. Timişoara Ser. Ştiinţe Mat.*, 11:111–114, 1973.

[9] J. Goldstein. *Semigroups of Operators and Applications*. Oxford University Press, 1985.

[10] G. Greiner, J. Voigt, and M. Wolff. On the spectral bound of the generator of semigroups of positive operators. *J. Operator Theory*, 5:245–256, 1981.

[11] E. Hille and R. S. Phillips. *Functional Analysis and Semigroups*, volume 31 of *Colloquium Publications*. American Mathematical Society, Providence, R.I., 1957.

[12] H. Komatsu. Fractional powers of operators III, Negative powers. *J. Math. Soc. Japan*, 21:205–228, 1969.

[13] G. N. Mil'stein. Exponential stability of positive semigroups in a linear topological space I,II. *Soviet Math.*, 19:35–42, 51–61, 1975.

[14] F. Neubrander. *Well-posedness of higher order abstract Cauchy problems*. PhD thesis, Universität Tübingen, 1984.

[15] F. Neubrander. Laplace transform and asymptotic behavior of strongly continuous semigroups. *Houston J. Math.*, 12:549–561, 1985.

[16] F. Neubrander. Asymptotic behavior of solutions of inhomogeneous abstract Cauchy problems. In *Proceedings of the Conference on Physical Mathematics and Nonlinear Partial Differential Equations, Morgantown 1983*, pages 157–173. Marcel Dekker, 1985.

[17] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[18] J. Prüss. On the spectrum of C_0-semigroups. *Trans. Amer. Math. Soc.*, 284: 847–857, 1984.

[19] A. S. Rao and W. Hengartner. On the existence of a unique almost periodic solution of an abstract differential equation. *J. London Math. Soc.*, 8:577–581, 1974.

[20] M. Slemrod. Asymptotic behavior of C_0-semigroups as determined by the spectrum of the generator. *Indiana Univ. Math. J.*, 25:783–892, 1976.

[21] C. Travis. Differentiability of weak solutions to an abstract inhomogeneous differential equation. *Proc. Amer. Math. Soc.*, 82:425–430, 1981.

[22] R. Triggiani. Pathological asymptotic behavior of systems in Banach spaces. *J. Math. Anal. Appl.*, 49:411–429, 1975.

[23] R. Triggiani. On the stabilizability problem in Banach spaces. *J. Math. Anal. Appl.*, 52:383–403, 1975.

[24] D. V. Widder. *The Laplace Transform*. Princeton University Press, Princeton, N.J., 1946.

[25] D. V. Widder. *An Introduction to Transform Theory*. Academic Press, New York, 1971.

[26] M. P. H. Wolff. A remark on the spectral bound of the generator of a semigroup of positive operators with application to stability theory. In P. L. Butzer, B. Sz.-Nagy, and E. Görlich, editors, *Functional Analysis and Approximation*, pages 39–50. Birkhäuser, Basel-Boston-Stuttgart, 1981.

[27] J. Zabczyk. A note on C_0-semigroups. *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astronom. Phys.*, 23:895–898, 1975.

[28] J. Zabczyk. Stabilization of boundary control systems. In *International Symposium on Systems Optimization and Analysis 1978*, Lecture Notes in

Control Theory and Information. Springer, Berlin-Heidelberg-New York, 1979.

[29] S. D. Zaidman. *Abstract Differential Equations*. Pitman, London, 1979.

Part B
Positive Semigroups on Spaces $C_0(X)$

Chapter B-I

Basic Results of Spaces $C_0(X)$

by

Rainer Nagel and Ulf Schlotterbeck

This part of the book is devoted to one-parameter semigroups of operators on spaces of continuous functions of type $C_0(X)$. Such spaces are Banach lattices of a very special kind. We treat this case separately since we feel that an intermingling with the abstract Banach lattice situation considered in Part C would have made it difficult to appreciate the easy accessibility and the pilot function of methods and results available here. In this chapter we fix the notation we are going to use and collect some basic facts about the spaces we are considering.

If X is a locally compact topological space, then $C_0(X)$ denotes the space of all continuous complex-valued functions on X which vanish at infinity, endowed with the supremum-norm. If X is compact, then any continuous function on X "vanishes at infinity" and $C_0(X)$ is the space of all continuous functions on X. We often write $C(X)$ instead of $C_0(X)$ in this situation.

We sometimes study real-valued functions and write the corresponding real spaces as $C_0(X,\mathbb{R})$ and $C(X,\mathbb{R})$, and the notations $C_0(X,\mathbb{C})$ and $C(X,\mathbb{C})$ are used if there might be confusion between both cases.

1 Algebraic and Order-Structure: Ideals and Quotients

Any space $C_0(X)$ is a commutative C*-algebra under the sup-norm and the pointwise multiplication, and by the *Gelfand Representation Theorem* any commutative C*-algebra can, on the other hand, be canonically represented as an algebra $C_0(X)$ on a suitable locally compact space X. The algebraic notions of ideal, quotient, homomorphism are well known and need not be explained further.

Another natural and important structure of $C_0(X)$ is the *pointwise* ordering under which $C_0(X,\mathbb{R})$ is a (real) Banach lattice and $C_0(X,\mathbb{C})$ a complex Banach lattice in the sense explained in Chapter C-I. Concerning the order structure of $C_0(X)$ we use the following notations. For a function f in $C_0(X,\mathbb{R})$

(i) A function f is called *positive,* $f \geqslant 0$, if $f(t) \geqslant 0$ for all $t \in X$,

(ii) We write $f > 0$ if f is positive but does not vanish identically,

(iii) We call f *strictly positive,* $f \gg 0$, if $f(t) > 0$ for all $t \in X$.

The notion of an order ideal explained in Chapter C-I applies of course to the Banach lattices $C_0(X)$ and might give rise to confusion with the corresponding algebraic notion.

However, since we are mainly considering closed ideals and since a closed linear subspace I of $C_0(X)$ is a lattice ideal if and only if I is an algebraic ideal, we may and will simply speak of closed ideals, without specifying whether we think of the algebraic or the order theoretic meaning of this word.

An important and frequently used characterization of these objects is the following: A subspace I of $C_0(X)$ is a closed ideal if and only if there exists a closed subset A of X such that a function f belongs to I if and only if f vanishes on A. The set A is of course uniquely determined by I and is called the *support* of I. If $I = I_A$ is a closed ideal with support A, then I_A is naturally isomorphic to $C_0(X\backslash A)$ and the quotient $C_0(X)/I$ (under the natural quotient structure) is again a Banach algebra and a Banach lattice that can be identified canonically (via the map $f + I \to f_{|A}$) with $C_0(A)$.

2 Linear Forms and Duality

The *Riesz Representation Theorem* asserts that the dual of $C_0(X)$ can be identified in a natural way with the space of bounded regular Borel measures on X. While there is no natural algebra structure on this dual, the dual ordering (see Chapter C-I) makes $C_0(X)'$ into a Banach lattice. We will occasionally make use of the order structure of $C_0(X)'$, but since at least its measure theoretic interpretation is supposed to be well-known, it may suffice here to refer to Chapter C-I, Sections 3 and 7, for a more detailed discussion. We recall only some basic notations here.

If μ is a linear form on $C_0(X,\mathbb{R})$, then

(i) $\mu \geqslant 0$ means $\mu(f) \geqslant 0$ for all $f \geqslant 0$; μ is then called *positive*,

(ii) $\mu > 0$ means that μ is positive but does not vanish identically,

(iii) $\mu \gg 0$ means that $\mu(f) > 0$ for any $f > 0$; μ is then called *strictly positive*.

If μ is a linear form on $C_0(X,\mathbb{C})$, then μ can be written uniquely as

$$\mu = \mu_1 + \mathrm{i}\mu_2$$

where μ_1 and μ_2 map $C_0(X, \mathbb{R})$ into $\mathbb{R}$ (decomposition into *real* and *imaginary parts*).

We call μ *positive* (*strictly positive*) and use the above notations if $\mu_2 = 0$ and μ_1 is positive (strictly positive). We point out that strictly positive linear forms need not exist on $C_0(X)$, but if X is separable, then a strictly positive linear form is obtained by summing a suitable sequence of point measures.

The coincidence of the notions of a closed algebraic and a closed lattice ideal in $C_0(X)$ has of course its effect on the algebraic and the lattice theoretic notions of a homomorphism. The case of a homomorphism into another space $C_0(Y)$ will be discussed below. As for homomorphisms into the scalar field, we have essentially coincidence between the algebraic and the order theoretic meaning of this word, more exactly:

> *A linear form $\mu \neq 0$ on $C_0(X)$ is a lattice homomorphism if and only if μ is, up to normalization, an algebra homomorphism* (algebra homomorphisms $\neq 0$ must necessarily have norm 1).

Since the algebra homomorphisms $\neq 0$ on $C_0(X)$ are known to be the point measures (denoted by δ_t) on X and since on the other hand μ is a lattice homomorphism if and only if

$$|\mu(f)| = \mu(|f|) \quad \text{for all } f,$$

it follows that this latter condition on μ is equivalent to $\mu = \alpha\delta_t$ for a suitable t in X and a positive real number α.

This can be summarized by saying that X can be canonically identified, via the map $t \to \delta_t$, with the subset of the dual $C_0(X)'$ consisting of the non-zero algebra homomorphisms, which is also the set of all normalized lattice homomorphisms. This identification is a topological isomorphisms with respect to the weak*-topology of $C_0(X)'$.

3 Linear Operators

A linear mapping T from $C_0(X, \mathbb{R})$ into $C_0(Y, \mathbb{R})$ is called

(i) *positive* (notation: $T \geqslant 0$) if Tf is a positive whenever f is positive,

(ii) *lattice homomorphism* if $|Tf| = T|f|$ for all f,

(iii) *Markov-operator* if X and Y are compact and T is a positive operator mapping $\mathbb{1}_X$ to $\mathbb{1}_Y$.

In the case of complex scalars, T can be decomposed into real and imaginary parts. We call T positive in this situation if the imaginary part of T is $= 0$ and the real part is positive. The terms *Markov operator* and *lattice homomorphism* are defined as above. Note that a complex lattice homomorphism is necessarily positive, and that the *complexification* of a real lattice homomorphism is a complex lattice homomorphism. Positive Operators are always continuous.

Note that the adjoint of a Markov operator T maps positive normalized measures into positive normalized measures while the adjoint of an algebra homomorphism (lattice homomorphism) maps point measures into (multiples of) point measures. Therefore the adjoint of a Markov lattice homomorphism as well as the adjoint of an algebra homomorphism induces a continuous map φ from Y (viewed as a subset of the weak dual $C(Y)'$) into X (viewed as a subset of $C(X)'$).

This mapping φ determines T in a natural and unique way, so that the following are equivalent assertions on a linear mapping T from a space $C(X)$ into a space $C(Y)$.

(a) T is a Markov lattice homomorphism.

(b) T is a Markov algebra homomorphism.

(c) There exists a continuous map φ from Y into X such that $Tf = f \circ \varphi$ for all $f \in C(X)$.

If T is, in addition, bijective, then the mapping φ in (c) is a homeomorphism from Y onto X. This characterization of homomorphisms carries over mutatis mutandis to situations where the conditions on X, Y or T are less restrictive. For later reference we explicitly state the following.

(i) Let K be compact. Then $T \in \mathcal{L}(C(K))$ is a lattice homomorphism if and only if there is a mapping φ from K into K and a function $h \in C(K)$ such that $Tf(s) = h(s)f(\varphi(s))$ holds for all $s \in K$. The mapping φ is continuous in every point t with $h(t) \neq 0$.

(ii) Let X be locally compact and $T \in \mathcal{L}(C_0(X))$. Then T is a lattice isomorphism if and only if there is a homeomorphism φ from X onto X and a bounded continuous function h on X such that $h(s) \geqslant \delta > 0$ for all s and $Tf(s) = h(s)f(\varphi(s))$ $(s \in X)$. Moreover, T is an algebraic $*$-isomorphism if and only if T is a lattice isomorphism and the function h above is $\equiv \mathbb{1}$.

Notes

For the representation theory of commutative C*-algebras we refer to Takesaki [2]. This and the other mentioned properties like algebraic ideals, their connections with closed sets, the representation of lattice or algebraic homomorpism etc. we refer to Semadeni [1].

References

[1] Z. Semadeni. *Banach Spaces of Continuous Functions*. Polish Scientific Publishers, Warszawa, 1971.

[2] M. Takesaki. *Theory of Operator Algebras I*. Springer, New York-Heidelberg-Berlin, 1979.

Part C

Positive Semigroups on Banach Lattices

Chapter C-I

Basic Results on Banach Lattices and Positive Operators

by

Rainer Nagel und Ulf Schlotterbeck

This introductory chapter is intended to give a brief exposition of those results on Banach lattices and ordered Banach spaces which are indispensable for an understanding of the subsequent chapters. We do not give proofs of the results, since these can easily be found in the literature (e.g., in Schaefer [1]). We rather want to give the reader, who is unfamiliar with the results or the terminology used in this book, the necessary information for an intelligent reading of the main discussions. Since relatively few general results on ordered Banach spaces are needed, we will primarily talk about Banach lattices. The scalar field will be $\mathbb{R}$ except for the last three sections, where complex Banach lattices will be discussed.

The notion of a Banach lattice was devised to obtain a common abstract setting within which one could talk about phenomena related to positivity. This has previously been studied in various types of spaces of real-valued functions, such as the spaces $C(K)$ of continuous functions on a compact topological space K, the Lebesgue spaces $L^1(\mu)$ or more generally the spaces $L^p(\mu)$ constructed over a measure space (X, Σ, μ) for $1 \leqslant p \leqslant \infty$. Thus it is a good idea to think of such spaces first in order to get a feeling for the concrete meaning of the abstract notions we introduce. Later we will see that the connections between *abstract* Banach lattices and the *concrete* function lattices $C(K)$ and $L^1(\mu)$ are closer than one might think at first. We will use without further explanation the terms *order relation* (ordering), *ordered set*, *majorant*, *minorant*, *supremum*, *infimum*.

By definition, a Banach lattice is a Banach space $(E, \|\cdot\|)$ which is endowed with an order relation, usually written $\leqslant$, such that $(E, \leqslant)$ is a lattice and

(LO1) $f \leqslant g$ implies $f + h \leqslant g + h$ for all f, g, h in E,

(LO2) $f \geqslant 0$ implies $\lambda f \geqslant 0$ for all f in E and $\lambda \geqslant 0$, i.e., the ordering is compatible with the Banach space structure and the linear structure of E. We elaborate this in more detail now.

Any (real) vector space with an ordering satisfying (LO_1) and (LO_2) is called an *ordered vector space*. The property expressed in (LO_1) is sometimes called *translation invariance* and implies that the ordering of an ordered vector space E is completely determined by the positive part $E_+ = \{f \in E : f \geqslant 0\}$ of E. In fact, one has $f \leqslant g$ if and only if $g - f \in E_+$. (LO_1) together with (LO_2) furthermore imply that the positive part of E is a convex set and a cone with vertex 0 (often called the *positive cone* of E). It is easily verified that conversely any proper convex cone C with vertex 0 in E is the positive part of E with respect to a uniquely determined compatible ordering.

An ordered vector space E is called a *vector lattice* if any two elements f, g in E have a *supremum*, which is denoted by $\sup(f,g)$ or by $f \vee g$, and an *infimum*, denoted by $\inf(f,g)$ or by $f \wedge g$. It is obvious that the existence of, e.g., the supremum of any two elements in an ordered vector space implies the existence of the supremum of any finite number of elements in E and, since $f \leqslant g$ is equivalent to $-g \leqslant -f$ this automatically implies the existence of finite infima.

This gives rise to the following definitions.

$|f| := \sup(f, -f)$ is called the *absolute value* of f,
$f^+ := \sup(f, 0)$ is called the *positive part* of f,
$f^- := \sup(-f, 0)$ is called the *negative part* of f.

Note that the negative part of f is positive. We call two elements f and g of a vector lattice *orthogonal* or *lattice disjoint* and write $f \perp g$ if $\inf(|f|, |g|) = 0$.

However, suprema (infima) of infinite majorized subsets need not exist in a vector lattice. If they do, then the vector lattice is called *order complete* (*countably order complete* or *σ-order complete* if suprema of countable majorized subsets exist). At any rate, the binary relations *sup* and *inf* in a vector lattice automatically satisfy the (infinite) distributive laws

$$\inf(\sup_\alpha f_\alpha, h) = \sup_\alpha(\inf(f_\alpha, h)),$$
$$\sup(\inf_\alpha f_\alpha, h) = \inf_\alpha(\sup(f_\alpha, h)),$$

whenever one side exists.

Apart from this, the above definitions allow us to formulate the axiom of compatibility between norm and order requested in a Banach lattice in the following short way.

$$\text{(LN)} \quad |f| \leqslant |g| \text{ implies } \|f\| \leqslant \|g\|. \tag{0.1}$$

A norm on a vector lattice is called a *lattice norm* if it satisfies (LN). With these notations we can now give the definition of a Banach lattice as follows.

A Banach lattice *is a Banach space E endowed with an ordering $\leqslant$ such that $(E, \leqslant)$ is a vector lattice and the norm on E is a lattice norm. By a* normed vector lattice *we understand a vector lattice endowed with a lattice norm.*

There is a number of elementary, but very important formulas valid in any vector lattice, such as

$$f = f^+ - f^-, \quad |f + g| \leqslant |f| + |g|,$$
$$|f| = f^+ + f^-, \quad f + g = \sup(f, g) + \inf(f, g)$$

(see, e.g., Schaefer [1, Chap. II, §1]).

Let us note in passing the following consequences.

(i) The lattice operations $(f, g) \mapsto \sup(f, g)$ and $(f, g) \mapsto \inf(f, g)$ and the mappings $f \mapsto f^+$, $f \mapsto f^-$, $f \mapsto |f|$ are uniformly continuous.

(ii) The positive cone is closed.

(iii) *Order intervals*, i.e., sets of the form

$$[f, g] = \{h \in E \colon f \leqslant h \leqslant g\}$$

are closed and bounded.

Instead of dwelling upon a detailed discussion of the above equalities and inequalities let us rather formulate the following principle, which allows us to verify any of them and to invent, prove or disprove new ones whenever necessary.

Any general formula relating a finite number of *variables* to each other by means of lattice operations and/or linear operations is valid in any Banach lattice as soon as it is valid in the real number system.

In fact, we see below that any Banach lattice E is, as a vector lattice, *locally* of type $C(X)$, more exactly: Given any finite number $x_1, \ldots, x_n$ of elements in E, there is a compact topological space X and a vector sublattice J of E which is isomorphic to $C(X)$ and contains $x_1, \ldots, x_n$ (see Section. 4). The above principle is an easy consequence of the following: In a space $C(X)$ it is clear that a formula of the type considered above need only be verified pointwise, i.e., in $\mathbb{R}$.

The reader may now be prepared to follow a concise presentation of the most basic facts on Banach lattices.

1 Sublattices, Ideals, Bands

The notion of a *vector sublattice* of a vector lattice E is self-explanatory, but it should be pointed out that a vector subspace F of E which is a vector lattice for the ordering induced by E need not be a vector sublattice of E (formation of suprema may differ in E and in F), and that a vector sublattice need not contain (or may lead to different) infinite suprema and infima. The following are necessary and sufficient conditions on a vector subspace G of E to be a vector sublattice.

(a) $|h| \in G$ for all $h \in G$,

(b) $h^+ \in G$ for all $h \in G$,

(c) $h^- \in G$ for all $h \in G$.

A subset B of a vector lattice is called *solid* if $f \in B$, $|g| \leqslant |f|$ implies $g \in B$. Thus a norm on a vector lattice is a lattice norm if and only if its unit ball is solid. A solid linear subspace is called an *ideal*. Ideals are automatically vector sublattices since $|\sup(f,g)| \leqslant |f| + |g|$. On the other hand, a vector sublattice F is an ideal in E if $g \in F$ and $0 \leqslant f \leqslant g$ imply $f \in F$. A *band* in a vector lattice E is an ideal which contains arbitrary suprema, or more exactly:

> *B is a band in E if B is an ideal in E and* $\sup M$ *is contained in B whenever M is contained in B and has a supremum in E.*

Since the notions of sublattice, ideal, band are invariant under the formation of arbitrary intersections, there exists, for any subset B of E, a uniquely determined smallest sublattice (ideal, band) of E containing B, i.e., the *sublattice* (*ideal*, *band*) *generated by B*.

If we denote by B^d the set

$$\{h \in E : \ \inf(|h|, |f|) = 0 \text{ for all } f \in B\},$$

then B^d is a band for any subset B of E, and $(B^d)^d = B^{dd}$ is a band containing B. If E is a normed vector lattice (more generally, if E is archimedean ordered, then B^{dd} is the band generated by B (see, e.g., Schaefer [1]).

If two ideals I, J of a vector lattice E have trivial intersection $\{0\}$, then I and J are *lattice disjoint*, i.e., $I \subset J^d$. Thus if E is the direct sum of two ideals I, J, then one has automatically $I = J^d$ and $J = I^d$, hence $I = I^{dd}$ and $J = J^{dd}$ must be bands in this situation. In general, an ideal I need not have a complementary ideal J even if $I = I^{dd}$ is a band. This amounts to the same as saying that even if $I = I^{dd}$ (which is always true if I is a band in a normed vector lattice), one need not necessarily have $E = I + I^d$.

An ideal I is called a *projection band* if it does have a complementary ideal, and in this case the projection of E onto I with kernel I^d is called the *band projection* belonging to I. An example of a band which is not a projection band is furnished by the subspace of $C([0,1])$ consisting of the functions vanishing on $[0, 1/2]$.

The *Riesz Decomposition Theorem* asserts that in an order complete vector lattice every band is a projection band. As a consequence, if E is order complete and B is an arbitrary subset of E, then E is the direct sum of the complementary bands B^d and B^{dd}.

This theorem, which is quite easy to prove, is widely used in analysis and gives an abstract background to, e.g., the decomposition of a measure into atomic and diffuse parts (the atomic measures being those contained in the band generated by the point measures, the diffuse measures those disjoint to the latter). Or, more specifically, to the well-known decomposition of a measure on $[a, b]$ into an atomic part and a diffuse part, which latter can in turn be decomposed into the sum of a measure which is *absolutely continuous* (which means, contained in the band generated by Lebesgue measure) and a *singular measure* (which means, a diffuse measure disjoint to Lebesgue measure).

A band in a normed vector lattice is necessarily closed. By contrast, an ideal need not be closed, but the closure of an ideal is again an ideal. The situation, where every closed ideal is a band, will be briefly discussed in Section 5.

2 Order Units, Weak Order Units, Quasi-Interior Points

An element u in the positive cone of a vector lattice E is called an *order unit* if the ideal generated by u is all of E. If the band generated by u is all of E (which is equivalent to $\{u\}^d = 0$ whenever E is archimedean, hence in particular if E is a normed vector lattice), then u is called a *weak order unit* of E. If E is a Banach lattice, then any order unit in E is an interior point of the positive cone E_+, and conversely any interior point of E_+ must be an order unit of E. Every space $C(K)$ has order units (namely, the strictly positive functions), and conversely by the Kakutani-Krein Representation Theorem (see Section 4), every Banach lattice with an order unit is isomorphic to a space $C(K)$.

If an element u in the positive cone of a Banach lattice E has the property that the closed ideal generated by u is all of E, then u is called a *quasi-interior point* of E_+. Quasi-interior points of the positive cone exist, e.g., in any separable Banach lattice. If $E = C(K)$, then the quasi-interior points and the interior points of E_+ coincide, while the weak order units of E are the (positive) functions vanishing on a nowhere dense subset of K. If E is a space $L^p(\mu)$ with σ-finite μ and $1 \leqslant p < \infty$, then the weak order units and the quasi-interior points of E_+ coincide with the functions strictly positive μ-a.e., while E_+ does not contain any interior point, except in finite dimensions.

3 Linear Forms and Duality

A linear functional φ on a vector lattice E is called

(i) *order-bounded* if φ is bounded on order intervals of E,

(ii) *positive* if $\varphi(f) \geqslant 0$ for all $f \geqslant 0$,

(iii) *strictly positive* if $\varphi(f) > 0$ for all $f > 0$.

Any positive linear functional is order bounded, and the positive functionals form a proper convex cone with vertex 0 in the linear space $E^{\#}$ of all order bounded functionals, thus defining a natural ordering given by

$$\varphi \leqslant \psi \text{ if and only if } \varphi(f) \leqslant \psi(f) \text{ for all } f \in E_+$$

under which $E^{\#}$ is an order complete vector lattice. In particular, positive part, negative part and absolute value exist for any order bounded functional on E, the absolute value of $\varphi \in E^{\#}$ being given by

$$|\varphi|(f) = \sup\{\varphi(h) \colon |h| \leqslant f \text{ for } f \in E_+\}.$$

As a consequence, one has $|\varphi(f)| \leqslant |\varphi|(|f|)$ for all f in E whenever φ is order bounded, and $|\varphi(f)| \leqslant \varphi(|f|)$ if and only if φ is positive. An order bounded linear functional φ is called *order-continuous* (*σ-order-continuous*) if both positive and negative part of φ have the property that they transform any decreasing net (any decreasing sequence) with infimum 0 into a net (sequence) converging to 0 in $\mathbb{R}$. The order-continuous (σ-order-continuous) functionals form a band in $E^{\#}$.

In general, a vector lattice E need not admit any non-zero order-bounded linear functional. However, if E is a normed lattice, then any continuous functional is order-bounded, and if E is a Banach lattice, then one has coincidence between $E^{\#}$ and E'. Still, order-continuous functionals $\neq 0$ need not exist on a Banach lattice. Situations where every order-bounded functional is order-continuous will be briefly discussed in Section 5.

If E is a Banach lattice, then the dual norm on $E' = E^{\#}$ is a lattice norm, hence E' is an order-complete Banach lattice under the natural norm and order. The evaluation map from E into the second dual E'' is a lattice homomorphism (for the definition see Section 6) into the band of order-continuous functionals on E'. In particular, every dual Banach lattice E admits sufficiently many order-continuous functionals to separate the points of E.

4 AM- and AL-Spaces

If the norm on a Banach lattice E satisfies

$$\text{(M)} \quad \|\sup(f, g)\| = \sup(\|f\|, \|g\|) \text{ for } f, g \in E_+,$$

then E is called an abstract M-space or an *AM-space*. If, in addition, the unit ball of E contains a largest element u, then u must be an order unit of E and E is then called an *(AM)-space with unit*. Condition (M) in E implies that in the dual of E one has

$$\text{(L)} \quad \|f+g\| = \|f\| + \|g\| \text{ for } f,g \in E'_+.$$

Any Banach lattice satisfying (L) is called an abstract L-space or an *AL-space*. Thus the dual of an AM-space is an AL-space.

It is quite easy to verify that, on the other hand, the dual of an AL-space is an AM-space with unit, the unit being the uniquely determined linear functional that coincides with the norm on the positive cone. Putting this together, one gets that the second dual of an AM-space E is an AM-space with unit. If E already has a unit u, then u is also the unit of E'', so that the ideal of E'' generated by E is all of E''. By contrast, if E is an AL-space, then E is an ideal (even a band) in E''. Infinite-dimensional AL- or AM-spaces are never reflexive.

The importance of AL- and AM-spaces in the general theory of Banach lattices is due to the fact that these spaces have very special concrete representations as function lattices and that, on the other hand, any general Banach lattice E is in a very intimate way connected to certain families of AL- and AM-spaces canonically associated with E. Let us first discuss the natural representations of AM- and AL-spaces.

If E is an AM-space with unit u, then the set K of lattice homomorphisms from E into $\mathbb{R}$ taking the value 1 on u is a non-empty, $\sigma(E',E)$-compact subset of E' and the natural evaluation map from E into $\mathbb{R}^K$ maps E isometrically onto the continuous real-valued functions on K (cf. Section 6). This is the *Kakutani-Krein Representation Theorem*, which is an order-theoretic counterpart to the Gelfand Representation Theorem in the theory of commutative C*-algebras. If E is an AM-space without unit, then the second dual of E has a unit and thus gives a representation of E as a closed sublattice of a space $C(K)$.

If E is an AL-space, then the representation of the dual of E as a space $C(K)$ leads to an interpretation of the elements of the bidual of E as Radon measures on K. If E_+ has a quasi-interior point h, then in this interpretation E consists exactly of the measures absolutely continuous with respect to (the measure corresponding to) h, thus by the *Radon-Nikodym-Theorem*, $E = L^1(K,h)$. In general, a similar argument leads to a representation of E as a space $L^1(X,\mu)$ constructed over a locally compact space X.

If E is an arbitrary Banach lattice and $f \in E_+$, then the ideal I generated by f in E (which is the union of the positive multiples of the interval $[-f,f]$) can be made into an AM-space with unit f by endowing it with the gauge function p_f of $[-f,f]$. We denote (I,p_f) by E_f. On the other hand, if f' is a positive linear functional on E, then the mapping $q_{f'}: f \mapsto \langle |f|, f'\rangle$ is a semi-norm on E. The kernel J of $q_{f'}$ is an ideal in E, and the completion of E/J with respect to the norm canonically derived from $q_{f'}$ becomes an AL-space which we denote by (E,x'). A good illustration for these constructions is the following.

If $E = C(K)$ and if μ is a positive linear form (Radon measure) on E, then (E, μ) is just $L^1(K, \mu)$; if $E = L^p(\mu)$ ($1 \leqslant p < \infty$, μ finite), then $E_{1_X} = L^\infty(\mu)$.

5 Special Connections Between Norm and Order

If an increasing net $(x_\alpha)_{\alpha \in A}$ in a normed vector lattice is convergent, then its limit must be the supremum as a consequence of the closedness of the positive cone. On the other hand, if $\{x_\alpha : \alpha \in A\}$ has a supremum, the net $(x_\alpha)_{\alpha \in A}$ need not converge. A Banach lattice is said to have *order-continuous norm* (*σ-order-continuous norm*) if any increasing net (sequence) which has a supremum is automatically convergent. This is of course equivalent to saying that any decreasing net (sequence) with an infimum is convergent. Since infimum and limit must coincide, the order continuity (σ-order continuity) of the norm in a Banach lattice is also equivalent to the property that any decreasing net (sequence) with infimum 0 converges to 0.

A Banach lattice with order-continuous norm must be order complete, but σ-order-continuity of the norm need not imply order completeness. At any rate, one has the following characterization.

A Banach lattice E has order-continuous norm if and only if any one of the following equivalent assertions holds.

(a) E is σ-order complete and has σ-order-continuous norm.

(b) Every order interval in E is weakly compact.

(c) E is (under evaluation) an ideal in E''.

(d) Every continuous linear form on E is order continuous.

(e) Every closed ideal in E is a projection band.

An even more stringent condition than order-continuity of the norm is that any increasing norm-bounded net be convergent. This condition is satisfied if and only if any one of the following equivalent assertions holds.

(a) E is (under evaluation) a band in E''.

(b) E is weakly sequentially complete.

(c) Every order-continuous linear form on E' belongs to E.

(d) No closed sublattice of E is isomorphic to c_0.

The most important examples of non-reflexive Banach lattices with this property are the AL-spaces.

6 Positive Operators, Lattice Homomorphisms

A linear mapping T from an ordered Banach space E into an ordered Banach space F is called *positive* (notation: $T \geqslant 0$) if $Tf \in F_+$ for all $f \in E_+$; T is called *strictly positive* if $T \geqslant 0$ and $\{f \in E: T|f| = 0\} = \{0\}$. The set of all positive linear mappings is a convex cone in the space $\mathcal{L}(E, F)$ of all linear mappings from E into F defining the *natural ordering* of $\mathcal{L}(E, F)$.

The linear subspace of $\mathcal{L}(E, F)$ generated by the positive maps (i.e. the space of linear maps that can be written as differences of positive maps) is denoted by $\mathcal{L}^r(E; F)$ and its elements are called *regular* mappings. If E and F are Banach lattices, then any regular mapping from E into F is continuous, but $\mathcal{L}^r(E; F)$ is in general a proper subspace of the space $\mathcal{L}(E, F)$ of all continuous linear mappings. One has coincidence of $\mathcal{L}^r(E; F)$ and $\mathcal{L}(E, F)$, e.g., when $E = F$ is an order complete AM-space with unit or an AL-space. At any rate, if F is order complete, then $\mathcal{L}^r(E; F)$ under the natural ordering is an order-complete vector lattice, and a Banach lattice under the norm

$$T \mapsto \|T\|_r = \||T|\|,$$

the right hand side denoting the operator norm of the absolute value of T. The absolute value of $T \in \mathcal{L}^r(E; F)$, if it exists, is given by

$$|T|(f) := \sup\{Th: |h| \leqslant f, \ f \in E_+\}.$$

Thus, T is positive if and only if $|Tf| \leqslant T|f|$ holds for any $f \in E$.

An operator $T \in \mathcal{L}(E, F)$ is called a *lattice homomorphism* if $|Tf| = T|f|$ holds for all $f \in E$. Lattice homomorphisms are alternatively characterized by any one of the following conditions holding for all f and $g \in E$.

(i) $(Tf)^+ = T(f^+)$,

(ii) $(Tf)^- = T(f^-)$,

(iii) $T(f \vee g) = Tf \vee Tg$,

(iv) $T(f \wedge g) = Tf \wedge Tg$,

(v) $T(f^+) \wedge T(f^-) = 0$.

The kernel of a lattice homomorphism is an ideal. If T is bijective, then T is a lattice homomorphism if and only if T and T^{-1} are positive.

7 Complex Banach Lattices

Although the notion of a Banach lattice is intrinsically related to the real number system, it is possible and often desirable to extend discussions to complexifications of Banach lattices in such a way that the order-related terms introduced in the real situation essentially retain their meaning. Thus we define a *complex Banach lattice* E to be the complexification of a real Banach lattice $E_{\mathbb{R}}$ in the sense that

$$E = E_{\mathbb{R}} \oplus \mathrm{i}E_{\mathbb{R}}$$

which means more exactly $E = E_{\mathbb{R}} \times E_{\mathbb{R}}$ with scalar multiplication

$$(\alpha + \mathrm{i}\beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y).$$

The space $E_{\mathbb{R}}$ will sometimes be called the *underlying real Banach lattice* or the *real part* of E. The classical complex Banach spaces $C(X)$, $L^p(\mu)$ are complex Banach lattices in this sense, the underlying real Banach lattices being the corresponding (real) subspaces of real-valued functions. We want to extend the formation of absolute values, which is a priori defined only in the real part of E, in such a way that in the classical situation $E = C(X)$ or $E = L^p(\mu)$ the usual absolute value of a function is obtained. This is in fact possible by putting, for $h = f + ig$ in E,

$$|h| = \sup\{\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\vartheta} h\right) : 0 \leqslant \vartheta \leqslant 2\pi\}.$$

The only problem with this definition being the existence of the right hand side without the assumption of order-completeness on $E_{\mathbb{R}}$.

But for this we just have to observe that the set

$$M = \{\mathrm{Re}(\mathrm{e}^{\mathrm{i}\vartheta} h) \colon 0 \leqslant \vartheta \leqslant 2\pi\}$$

is contained and order bounded in the ideal generated in $E_{\mathbb{R}}$ by $|f| + |g|$, which in turn is by the Kakutani-Krein Representation Theorem isomorphic to a space $C_{\mathbb{R}}(X)$ under the pointwise ordering. Now the pointwise supremum of M in $\mathbb{R}^X$ is readily seen to be a continuous function (namely, the modulus of the complex valued continuous function corresponding to $f + \mathrm{i}g$), so that M has a supremum in $C_{\mathbb{R}}(X) = (E_{\mathbb{R}})_{|f|+|g|}$.

Since the mapping $f \mapsto |f|$ now has a meaning in E, the definition of an ideal can be extended formally unchanged to the complex situation. We observe that $|f + \mathrm{i}g| = |f - \mathrm{i}g| \leqslant |f| + |g|$ implies that any ideal J in a complex Banach lattice is conjugation invariant and itself the complexification of the ideal $J \cap E_{\mathbb{R}}$ of the real part of E.

Suffice it now to say that the meaning of most of the terms introduced for real Banach lattices can be extended to the complex situation. Use the complex modulus or else, if the formation of suprema or infima is involved, relate them to real parts. For example $f \in E$ is called *positive* if $f = |f|$ which means that f is a positive element of $E_{\mathbb{R}}$, E is called order complete if $E_{\mathbb{R}}$ is order complete, and an ideal J is called a band if the real part of J is a band.

We refer to Chapter II, Section 11 of Schaefer [1] for a detailed discussion of this and restrict ourselves to a short discussion of linear mappings.

Let E and F be complex Banach lattices with real parts $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$. Then a linear mapping T from E into F is determined by its restriction T_0 to $E_{\mathbb{R}}$, and T_0 can be written in the form $T_0 = T_1 + iT_2$ with real-linear mappings T_j from $E_{\mathbb{R}}$ into $F_{\mathbb{R}}$. Thus $L(E, F)$ is the complexification of the real linear space $L(E_{\mathbb{R}}, F_{\mathbb{R}})$. With the above notation, T is called *real* if T_2 is $= 0$, *positive* if T is real and T_1 is positive, and a *lattice homomorphism* if T is real and T_1 is a lattice homomorphism. Lattice homomorphisms are characterized by the equality $|Th| = T|h|$ as in the real case.

8 The Signum Operator

We discuss in some detail how a mapping of the form

$$g \mapsto (\operatorname{sign} f)g$$

which has an obvious meaning, depending on f, in spaces $C(K)$, can be defined in an abstract complex Banach lattice. We prove the following.

Let E be a complex Banach lattice and let $f \in E$. If either E is order-complete or $|f|$ is a quasi-interior point in E_+, then there exists a unique linear mapping S_f, called the *signum operator* with respect to f, with the following properties.

(i) $S_f \bar{f} = |f|$, where $\bar{f} = \mathrm{Re}(f) - \mathrm{i} \cdot \mathrm{Im}(f)$,

(ii) $|S_f g| \leqslant |g|$ for every g in E,

(iii) $S_f g = 0$ for every g in E orthogonal to f.

In fact, if $E = C(K)$ and if $|f|$ is a quasi-interior point in E, then $|f|$ is a strictly positive function and multiplication with the function $\operatorname{sign} f = f \cdot |f|^{-1}$ has the desired properties. Uniqueness follows from Zaanen [2, Chap. 20]. We reduce the general situation to the case just considered in the following way.

(1) If $|f|$ is quasi-interior to E_+, then $E_{|f|}$ is a dense subspace of E isomorphic to a space $C(K)$, and with the above arguments one gets a uniquely determined operator S_0 on $E_{|f|}$ with the desired properties. Since (ii) implies the continuity of S_0 for the norm induced by E, S_0 can be extended to E.

(2) If f is arbitrary, then, as above, one obtains an operator S_0 on $E_{|f|}$ with (i) and (ii). If E is order complete, an extension S_f of S_0 to E satisfying (i)–(iii) is possible as soon as S_0 can be extended to the band $\{x\}^{dd}$ of E.

- On the complementary band $\{x\}^d$ one has necessarily the values $= 0$ for S_f.
- The extension to $\{x\}^{dd}$ is obtained as follows: If S_0 is positive (which means $f \geqslant 0$), then $S_f h = \sup\{S_f g \colon g \in [0, h] \cap E_{|f|}$ for $h \geqslant 0\}$ will do.

In general, the problem can be reduced to this situation by decomposing S_0 into a sum of the form $S_0 = (S_1 - S_2) + i(S_3 - S_4)$ with positive operators S_j. Such a decomposition of S_0 exists since the order completeness of E implies the order completeness of $E_{|f|} = C(K)$ and since every continuous linear operator on a space $C(K)$ is necessarily order-bounded.

9 The Center of $\mathcal{L}(E)$

We give a short description of a special, but important class of operators.

Let E be a (complex) Banach lattice. For $T \in \mathcal{L}(E)$ the following conditions are equivalent.

(a) $f \perp g$ implies $Tf \perp g \quad (f, g \in E)$,

(b) $\pm T \leqslant \|T\|\mathrm{Id}$,

(c) $TJ \subseteq J$ for every ideal J in E.

If E is countably order complete, then this is equivalent to

(d) $TJ \subseteq J$ for every projection band J in E.

The last assertion also means that T commutes with every band projection.

The set of all $T \in \mathcal{L}(E)$ satisfying the above conditions is called the *center* of $\mathcal{L}(E)$ and denoted $\mathcal{Z}(E)$. Under its natural ordering, the operator norm and the composition product is $\mathcal{Z}(E)$ isomorphic to a Banach lattice algebra $C(K)$ with K compact.

The following examples may illustrate the situation and explain why the term *multiplication operator* is often used for operators in $\mathcal{Z}(E)$.

(i) If $E = L^p(X, \Sigma, \mu)$ $(1 \leqslant p \leqslant \infty)$ with σ-finite μ, then $Z(E)$ is isomorphic to $L^\infty(\mu)$ via the natural identification of a function $f \in L^\infty(\mu)$ with the multiplication operator $g \mapsto f \cdot g$ on E.

(ii) If X is locally compact, $E = C_0(X)$, then similarly $\mathcal{Z}(E) \cong C^b(X)$ via the identification of $f \in C^b(X)$ with the mapping $g \mapsto f \cdot g$ $(g \in C_0(X))$.

Notes

For the theory of Banach lattices and positive operators we refer to Schaefer [1].

References

[1] H. H. Schaefer. *Banach Lattices and Positive Operators*. Springer, New York-Heidelberg-Berlin, 1974.

[2] A. C. Zaanen. *Riesz Spaces II*. North Holland, Groningen, 1983.

Chapter C-II

Characterization of Positive Semigroups on Banach Lattices

by

Wolfgang Arendt

In this chapter our first goal is to find conditions on a generator A of a semigroup $(T(t))_{t\geqslant 0}$ which are equivalent to the positivity of the semigroup. After the preparations in A-II, Section 2 this is easy if in addition we ask that the semigroup be contractive: $T(t)$ is a positive contraction for all $t \geqslant 0$ if and only if A is dispersive (Section 1). For arbitrary (not necessarily contractive) semigroups a condition on the generator had been found in the case when $E = C(K)$ (K compact), namely the positive minimum principle (P) (see B-II, Theorem 1.6). One may easily reformulate this condition in arbitrary Banach lattices and show its necessity. However, only in special cases (for example if A is bounded (see Section 1)) the positive minimum principle is sufficient for the positivity of the semigroup. In fact, on $L^2(\mathbb{R})$ there exists a non-positive semigroup whose generator satisfies (P) (Section 3).

Looking for another condition we consider the Laplacian Δ as a prototype. Defined on a suitable domain, Δ generates a positive semigroup on $L^p(\mathbb{R}^n)$. Kato proved the following distributional inequality for the Laplacian:

$$\mathrm{Re}(\operatorname{sign} \bar{f})\Delta f \leqslant \Delta|f|$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\Delta f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In Section 3 we will show that an abstract version of Kato's inequality for a generator A together with an additional condition is equivalent to the positivity of the semigroup generated by A.

Domination of one semigroup by another can be characterized by an analoguous condition for the generators (Section 4). The results will be applied to Schrödinger operators on $L^p(\mathbb{R}^n)$.

Finally, in Section 5 we show that $(T(t))_{t\geqslant 0}$ is a lattice semigroup (i.e., $|T(t)f| = T(t)|f|$ for all $t \geqslant 0$, $f \in E$) if and only if A satisfies Kato's equality. This parallels the case when $E = C_0(X)$, but if E has order continuous norm the strong form of Kato's equality can be considered (in particular, $f \in D(A)$ implies $|f| \in D(A)$ if A is the generator of such a semigroup).

1 Positive Contraction Semigroups and Bounded Generators

In this section we first characterize generators of positive contraction semigroups on a real Banach lattice E. For that we use the results developed in A-II, Section 2 for the canonical half-norm $N^+ : E \to \mathbb{R}$ given by

$$N^+(f) = \|f^+\| \quad (f \in E). \tag{1.1}$$

Remark It is easy to see that $N^+(f) = \inf\{\|f + g\| : g \in E_+\} = \mathrm{dist}(-f, E_+)$ (cf. A-II, Remark 2.8).

It is obvious that N^+ is a strict half-norm (see A-II, (2.12)).

The subdifferential of N^+ is given by

$$\partial N^+(f) = \{\varphi \in E'_+ : \|\varphi\| \leqslant 1, \langle f, \varphi\rangle = \|f^+\|\} \tag{1.2}$$

(this follows from the definition, see A-II, (2.5)).

Examples 1.1 (i) Let $E = C_0(X)$ (X locally compact). Let $f \in E$. There exists $x \in X$ such that $f(x) = \|f^+\|_\infty$. Then $\delta_x \in \partial N^+(f)$.

(ii) Let $E = L^p(X, \Sigma, \mu)$, where (X, Σ, μ) is a σ-finite measure space and $1 < p < \infty$. Let $f \in E$ satisfy $f^+ \neq 0$. Let

$$\varphi(x) = \begin{cases} c \cdot f(x)^{p-1} & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leqslant 0 \end{cases}$$

where $c > 0$ is such that $\int f(x)\varphi(x)\,\mathrm{d}x = \|f^+\|$. Then $\mathrm{d}N^+(f) = \{\varphi\}$.

(iii) Let $E = L^1(X, \Sigma, \mu)$, where (X, Σ, μ) is a σ-finite measure space, and $f \in E$. Let $\varphi \in L^\infty(X, \Sigma, \mu)_+$. Then $\varphi \in \partial N^+(f)$ if and only if

$$\begin{aligned} \varphi(x) &= 1 && \text{if } f(x) > 0, \\ 0 \leqslant \varphi(x) &\leqslant 1 && \text{if } f(x) = 0 \text{ and} \\ \varphi(x) &= 0 && \text{if } f(x) < 0. \end{aligned}$$

An operator A on E is called [strictly] dispersive if A is [strictly] N^+-dissipative; that is, for every $f \in D(A)$ one has $\langle Af, \varphi\rangle \leqslant 0$ for some [resp., all] $\varphi \in \mathrm{d}N^+(f)$ (see A-II, Section 2). Generators of positive contraction semigroups are characterized by the following theorem which is due to Phillips [26]

Theorem 1.2 *Let A be a densely defined operator on a real Banach lattice E. The following assertions are equivalent.*

(a) *A is the generator of a positive contraction semigroup.*

(b) *A is dispersive and $(\lambda - A)$ is surjective for some $\lambda > 0$.*

Frequently an operator is known explicitly only on a core. In that case one can use the following result.

Corollary 1.3 *Let A be a densely defined dispersive operator on a real Banach lattice E. If $(\lambda - A)D(A)$ is dense in E for some $\lambda > 0$, then A is closable and the closure $\overline{A}$ of A is the generator of a positive contraction semigroup.*

Theorem 1.2 and Corollary 1.3 immediately follow from A-II, Theorem 2.11 and A-II, Corollary 2.12 if one observes the following.

Lemma 1.4 *A bounded linear operator T on a Banach lattice E is a positive contraction if and only if $\|(Tf)^+\| \leqslant \|f^+\|$ for all $f \in E$ (i.e., if T is N^+-contractive).*

Proof (Proof of the lemma) If T is a positive contraction, then $0 \leqslant (Tf)^+ \leqslant Tf^+$ and so

$$N^+(Tf) = \|(Tf)^+\| \leqslant \|Tf^+\| \leqslant \|f^+\| = N^+(f)$$

for all $f \in E$.

Conversely, assume that T is an N^+-contraction and let $f \geqslant 0$. Then $\|(Tf)^-\| = N^+(T(-f)) \leqslant N^+(-f) = \|f^-\| = 0$. Hence $(Tf)^- = 0$; i.e., $Tf \geqslant 0$. We have proved that T is positive. In particular, $|Tf| \leqslant T|f|$ for all $f \in E$. Hence $\|Tf\| = \|\,|Tf|\,\| \leqslant \|T|f|\,\| \leqslant N^+(T|f|) \leqslant N^+(|f|) = \|f\|$ for all $f \in E$. So T is a contraction. $\square$

Examples 1.5 (i) Consider the second derivative with Dirichlet boundary condition on $E = C_0(0,1)$; i.e., we let $Af = f''$ with domain $D(A) = \{f \in C^2[0,1] : f(0) = f(1) = f''(0) = f''(1) = 0\}$. A is dispersive. In fact, let $f \in D(A)$. Then there exists $x \in (0,1)$ such that $f(x) = \sup_{y \in [0,1]} f(y) = \|f^+\|_\infty$. Thus $\delta_x \in \mathrm{d}N^+(f)$. But $\langle Af, \delta_x \rangle = f''(x) \leqslant 0$ since f has a maximum in x. Let $g \in E$. Define

$$f_0(x) = 1/2[\mathrm{e}^x \int_0^1 \mathrm{e}^{-y} g(y)\,\mathrm{d}y - \mathrm{e}^{-x} \int_0^1 \mathrm{e}^{y} g(y)\,\mathrm{d}y].$$

Then $f_0 \in C^2[0,1]$ and $f_0 - f_0'' = g$. There exist $a, b \in \mathbb{R}$ such that $f(x) = f_0(x) + a\mathrm{e}^x + b\mathrm{e}^{-x}$ defines a function $f \in C^2[0,1]$ satisfying $f(0) = f(1) = 0$. Since $f - f'' = f_0 - f_0'' = g$ this implies that $f \in D(A)$ and $f - Af = g$. We have shown that $(Id - A)$ is surjective. It follows from Theorem 1.2 that A is the generator of a positive contraction semigroup.

(ii) Let $E = L^p[0,1]$ $(1 \leqslant p < \infty)$ and A be given by $Af = f''$ on $D(A) = \{f \in E \colon f \in C^1[0,1], f' \in AC[0,1], f'' \in L^p[0,1], f(0) = f(1) = 0\}$. Then A is the generator of a positive contraction semigroup.

Proof A is dispersive. In fact, let $f \in D(A)$. Since the set $M = \{x \in (0,1) \colon f(x) > 0\}$ is open, there exists a countable set of disjoint intervals (a_n, b_n) such that $M = \cup_{n \in \mathbb{N}} (a_n, b_n)$.

First case: $p > 1$. Let $\varphi \in \partial N^+(f)$. Then there exists $c \geqslant 0$ such that $\varphi(x) = c f(x)^{p-1}$ for all $x \in M$ and and $\varphi(x) = 0$ if $f(x) \leqslant 0$ (see Example 1.1, (ii)). Thus integration by parts yields

$$\begin{aligned}\langle Af, \varphi\rangle &= \sum_n \int_{a_n}^{b_n} f''(x)\varphi(x)\ \mathrm{d}x \\ &= -c \sum_n \int_{a_n}^{b_n} f'(x) f'(x)(p-1) f(x)^{p-2}\, dx \\ &\leqslant 0.\end{aligned}$$

Second case: $p = 1$. Let $\varphi(x) = 1$ for $x \in M$ and $\varphi(x) = 0$ for $x \notin M$. Then $\varphi \in \mathrm{d}N^+(f)$ and

$$\langle Af, \varphi\rangle = \sum_n \int_{a_n}^{b_n} f''(x)\ \mathrm{d}x = \sum_n (f'(b_n) - f'(a_n)) \leqslant 0$$

since $f'(b_n) \leqslant 0$ and $f'(a_n) \geqslant 0$ for all $n \in \mathbb{N}$.

We have shown that A is dispersive. As in (i) one shows that $(Id - A)$ is surjective. Now the claim follows from Theorem 1.2 . □

(iii) Consider $E = C_0(\mathbb{R}^n)$. Let $D(A) = \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space of all infinitely differentiable rapidly decreasing functions) and $Af = \Delta f$ $(f \in D(A))$. Then A is closable and the closure of A generates a positive contraction semigroup on E.

Proof A is dispersive. In fact, let $f \in D(A)$. If $f^+ = 0$, then $\varphi := 0 \in \mathrm{d}N^+(f)$. So assume that $f^+ \neq 0$. Then there exists $x \in \mathbb{R}^n$ such that $f(x) = \|f\|_\infty = \sup\{f(y) \colon y \in \mathbb{R}^n\}$. Thus $\delta_x \in \mathrm{d}N^+(f)$. Since f has a maximum in x it follows that $\langle Af, \delta_x\rangle = (\Delta f)(x) = \mathrm{tr}(\partial^2 f / \partial x_i \partial x_j)(x) \leqslant 0$. Moreover,

$$(Id - \Delta) \text{ is an isomorphism from } \mathcal{S}(\mathbb{R}^n) \text{ onto } \mathcal{S}(\mathbb{R}^n). \tag{1.3}$$

In fact, the Fourier transform $f \mapsto \hat{f}$ is a bijection from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. But $[(Id - \Delta)f]\hat{} = M\hat{f}$ where $(Mg)(y) = (1 + \sum_{i=1}^n y_i^2) g(y)$ $(g \in \mathcal{S}(\mathbb{R}^n))$. It follows from (1.3) that $(Id - A)D(A)$ is dense in E. So the claim follows from Corollary 1.3. □

Remark In addition one can show that the closure $\overline{A}$ of A is given by $\overline{A} f = \Delta f$ with domain $D(\overline{A}) = \{f \in E \colon \Delta f \in E\}$ where for $f \in C_0(\mathbb{R}^n)$ the expression

Δf is understood in the sense of distributions. Moreover, the space $C_c^\infty(\mathbb{R}^n)$ (of all infinitely differentiable functions with compact support) is a core of $\overline{A}$ (cf. (iv)).

(iv) Let $E = L^p(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$ and A be given by $Af = \Delta f$ with domain $D(A) = \{f \in L^p(\mathbb{R}^n) \colon \Delta f \in L^p(\mathbb{R}^n)\}$ where for $f \in L^p(\mathbb{R}^n)$ the expression Δf is understood in the sense of distributions. Then A is the generator of a positive contraction semigroup. Moreover, the space $C_c^\infty(\mathbb{R}^n)$ is a core of A.

Proof It is easy to see that A is closed. Let A_0 denote the restriction of A to $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$. Then $A_0 f = \Delta f$ in the classical sense for all $f \in \mathcal{S}$. One can show in an analogous way as in (ii) that A_0 is dispersive. Moreover, it follows from (1.3) that $(Id - A_0)D(A_0)$ is dense. Hence by Corollary 1.3 the closure $\overline{A_0}$ of A_0 is the generator of a positive contraction semigroup.

By construction one has $\overline{A_0} \subset A$. We prove that $\overline{A_0} = A$. For that it is enough to show that

$$(Id - A) \text{ is injective.} \tag{1.4}$$

In fact, since the restriction $(Id - \overline{A_0})$ of $(Id - A)$ is bijective from $D(\overline{A_0})$ onto E it follows from (1.4) that $D(\overline{A_0}) = D(A)$. So let us show (1.4) . Assume that there is $f \in E$ such that $f - \Delta f = 0$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\langle \varphi - \Delta\varphi, f\rangle = 0. \tag{1.5}$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}$ for the topology of $\mathcal{S}$, it follows from (1.3) that $(Id - \Delta)C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}$. Hence (1.5) implies that $\langle \varphi, f\rangle = 0$ for all $\varphi \in S$. Consequently, $f = 0$. □

Remark Using the Fourier transform one can show that the semigroups in example (iii) and (iv) are given by

$$(T(t)f)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-(x-y)^2/4t) f(y)\, \mathrm{d}y \tag{1.6}$$

$(f \in E)$, where $z^2 := \sum_{i=1}^n z_i^2$ $(z \in \mathbb{R}^n)$.

(v) The following example is the analog of (i) for higher dimension. Let $\Omega \subset \mathbb{R}^n$ be a bounded open and connected set and $E = C_0(\Omega)$. We assume that the Dirichlet problem

$$\begin{array}{lll} u(x) - \Delta u(x) = 0 & & (x \in \Omega) \\ u(x) \qquad\quad\ = b(x) & & (x \in \partial\Omega) \end{array} \tag{1.7}$$

has a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for every $b \in C(\partial\Omega)$. For example, this is the case if the boundary $\partial\Omega$ is C^2 (see Gilbarg and Trudinger [12, Section 2.8 and Theorem. 6.13]). Let A be given by $Af = \Delta f$ on $D(A) = \{f \in C^2(\Omega) \cap C_0(\Omega) \colon \Delta f \in C_0(\Omega)\}$. Then A is closable and the closure of A is the generator of a positive contraction semigroup.

Proof $D(A)$ is clearly dense in E. Moreover, one can show as in (iii) that A is dispersive. It remains to prove that $(Id - A)D(A)$ is dense in E. The space $C_c^\infty(\Omega)$ of all infinitely differentiabel functions on Ω with compact support contained in Ω is dense in $E \cap C_c^\infty(\Omega)$. We show that there exists $f \in D(A)$ satisfying $(Id - A)f = g$. Let $\bar{g}\colon \mathbb{R}^n \to \mathbb{R}$ be given by $\bar{g}(x) = g(x)$ if $x \in \Omega$ and 0 if $x \notin \Omega$. Then $\bar{g} \in \mathcal{S}(\mathbb{R}^n)$. By (1.3) there exists $\bar{f} \in \mathcal{S}(\mathbb{R}^n)$ such that $\bar{f} - \Delta\bar{f} = \bar{g}$. Consider the function $b \in C(\partial\Omega)$ given by $b(x) = \bar{f}(x)$ for all $x \in \partial\Omega$. Then by our hypothesis there exists $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfying (1.7). Let $f(x) = \bar{f}(x) - u(x)$ $(x \in \Omega)$. Then $f \in C^2(\Omega) \cap C(\Omega)$ and $(f - \Delta f)(x) = g(x)$ $(x \in \Omega)$. Thus $\Delta f = f - g$ vanishes on $\partial\Omega$. Hence $f \in D(A)$ and $f - Af = g$. □

(vi) Let $\Omega \subset \mathbb{R}^n$ be as in (v) and $E = L^p(\Omega)$. Define $Af = \Delta f$ on $D(A) = \{f \in C^2(\Omega) \cap C_0(\Omega)\colon \Delta f \in C_0(\Omega)\}$. Then A is closable and the closure of A is the generator of a positive contraction semigroup on E.

Proof $D(A)$ is dense and one can show in an analoguous manner as in (iv) that A is dispersive. We know from (iv) that $C_c^\infty(\Omega) \subset (Id - A)D(A)$. Thus $(Id - A)D(A)$ is dense in E and the claim follows from Corollary 1.3. □

We now turn to the problem to characterize generators of arbitrary (not necessarily contractive) positive semigroups. Of course, as in B-II, Section 1 one sees that a semigroup $(T(t))_{t \geqslant 0}$ is positive if and only if $R(\lambda, A) \geqslant 0$ for all $\lambda > \omega(A)$ where A denotes the generator of $(T(t))_{t \geqslant 0}$. We are looking for an intrinsic condition on A.

The positive minimum principle which is characteristic for generators of strongly continuous semigroups on $C(K)$ (see B-II, Theorem 1.6) can be reformulated on an arbitrary Banach lattice E.

Definition 1.6 An operator A on E satisfies the *positive minimum principle* if for all $f \in D(A)_+$, $\varphi \in E'_+$,

$$\langle f, \varphi \rangle = 0 \quad \text{implies} \quad \langle Af, \varphi \rangle \geqslant 0. \tag{P}$$

Remark It is easy to see that this definition coincides with that given in B-II, Section 1 in the case when $E = C(K)$ (K compact). [In fact, suppose that for all $f \in D(A)_+$ and $x \in K$, $f(x) = 0$ implies $(Af)(x) \geqslant 0$. Let $g \in D(A)_+$, $\mu \in M(K)_+$ such that $\langle g, \mu \rangle = 0$. Then $g(x) = 0$ for all $x \in \operatorname{supp} \mu$. Thus by hypothesis, $(Ag)(x) \geqslant 0$ for all $x \in \operatorname{supp} \mu$. Consequently $\langle Ag, \mu \rangle \geqslant 0$. This proves one direction. The other is obvious by considering point measures.]

Proposition 1.7 *The generator of a strongly continuous positive semigroup satisfies the positive minimum principle* (P).

Proof Let $(T(t))_{t \geqslant 0}$ be a strongly continuous positive semigroup with generator A and $0 \leqslant f \in D(A)$, $\varphi \in E'_+$ such that $\langle f, \varphi \rangle = 0$. Then $\langle Af, \varphi \rangle = \lim_{t \to 0} 1/t \langle T(t)f - f, \varphi \rangle = \lim_{t \to 0} 1/t \langle T(t)f, \varphi \rangle \geqslant 0$. □

We will see that the positive minimum principle is not sufficient for the positivity of the semigroup, in general (Remark 3.16). However, the following special case is of interest.

Theorem 1.8 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach lattice E. Assume that*

(i) *there exists $w \in \mathbb{R}$ such that $\|T(t)\| \leqslant \mathrm{e}^{wt}$ for all $t \geqslant 0$;*

(ii) *there exists a core D_0 of A such that $f \in D_0$ implies $|f| \in D_0$.*

If the restriction of A to D_0 satisfies the positive minimum principle, then the semigroup is positive.

Remark Elementary examples show that neither a) nor b) hold for generators of positive semigroups, in general.

The proof of Theorem 1.8 is based on the following proposition.

Proposition 1.9 *Let A be a densely defined dissipative operator which possesses a core D_0 such that $f \in D_0$ implies $|f| \in D_0$. If the restriction of A to D_0 satisfies the positive minimum principle (P), then A is dispersive.*

Proof (Proof of Proposition 1.9) By A-II, Proposition 2.9, it is enough to show that $A_0 := A|_{D_0}$ is dispersive. Let $f \in D_0$ and $\varphi \in \mathrm{d}N^+(f)$. Then $\varphi \in E'_+$, $\|\varphi\| \leqslant 1$ and $\langle f, \varphi\rangle = \|f^+\|$. Hence, $\langle f^-, \varphi\rangle = \langle f^-, \varphi\rangle + \langle f, \varphi\rangle - \|f^+\| = \langle f^+, \varphi\rangle - \|f^+\| \leqslant 0$. Thus $\langle f^-, \varphi\rangle = 0$. Consequently, $\langle f^+, \varphi\rangle = \langle f, \varphi\rangle = \|f^+\|$; and so $\varphi \in \mathrm{d}N(f^+)$. Since A is dissipative it follows that $\langle Af^+, \varphi\rangle \leqslant 0$. Moreover, since A satisfies (P) we have $\langle Af^-, \varphi\rangle \geqslant 0$. So we finally obtain, $\langle Af, \varphi\rangle = \langle Af^+, \varphi\rangle - \langle Af^-, \varphi\rangle \leqslant 0$.□

Proof (Proof of Theorem 1.8) The operator $A - w$ satisfies (P) as well. So it follows from Proposition 1.9 that $A - w$ is dispersive. Consequently, the semigroup $(\mathrm{e}^{-wt}T(t))_{t\geqslant 0}$, which is generated by $A - w$, is positive. Thus $(T(t))_{t\geqslant 0}$ is positive as well. □

Next we give a reformulation of the positive minimum principle. For $0 < u \in E_+$ we denote by E_u the principal ideal generated by u. If $g \in E_+$, then $g \in \overline{E_u}$ if and only if $\lim_{n\to\infty} \|u - nu \wedge g\| = 0$.

Lemma 1.10 *An operator A on E satisfies (P) if and only if*

$$(Au)^- \in E_u \text{ for all } u \in D(A)_+ := D(A) \cap E_+. \tag{1.8}$$

Proof Let $u \in D(A)_+$, $g = Au$. Assume that $(Au)^- \in \overline{E}_u$. Then, if $0 \leqslant \varphi \in E'_+$ such that $\langle u, \varphi\rangle = 0$ one has $\langle f, \varphi\rangle = 0$ for all $f \in \overline{E}_u$, hence $\langle (Au)^-, \varphi\rangle = 0$ and consequently $\langle Au, \varphi\rangle \geqslant 0$. This proves one direction. To prove the other assume that $g^- \notin \overline{E}_u$. Then there exists $\varphi \in (E_u)^\circ$ such that $\langle g^-, \varphi\rangle \neq 0$. Define $\psi_0(f) = \sup \varphi([0, f] \cap E_{(g^-)})$ for $f \in E_+$. Then ψ_0 is positive homogeneous on

E_+. Thus the linear extension of ψ_0 defines a positive linear form ψ on E. We have $\langle g^-, \psi\rangle = \langle g^-, \varphi\rangle > 0$ and $\langle g^+, \psi\rangle = 0$. Thus $\langle Au, \psi\rangle = -\langle g^-, \psi\rangle < 0$. But $\langle u, \psi\rangle \leqslant \langle u, \varphi\rangle = 0$. Thus (P) does not hold. □

Bounded generators of positive semigroups can now be characterized as follows.

Theorem 1.11 *Let A be a bounded operator on a Banach lattice E. The following assertions are equivalent:*

(a) $e^{tA} \geqslant 0$ $(t \geqslant 0)$.

(b) $f \in E_+$, $\varphi \in E'_+$, $\langle f, \varphi\rangle = 0$ *implies* $\langle Af, \varphi\rangle \geqslant 0$.

(c) $(Af)^- \in \overline{E_f}$ *for all* $f \in D(A)_+$.

(d) $A + \|A\| \cdot \mathrm{Id} \geqslant 0$.

Proof It follows by Proposition 1.7 that (a) implies (b) . Since $\|e^{tA}\| \leqslant e^{t\|A\|}$ $(t \geqslant 0)$, (b) implies (a) by Theorem 1.8 . The equivalence of (b) and (c) is established by Lemma 1.10. If (d) holds, then $e^{t(A+\|A\|)} \geqslant 0$ $(t \geqslant 0)$. Thus $e^{tA} = e^{-t\|A\|}e^{t(A+\|A\|)} \geqslant 0$, $(t \geqslant 0)$. We have shown that (a), (b) and (c) are equivalent and (d) implies (a) It remains to show that (a) implies (d) . Since assertions (a) and (d) are satisfied for A if and only if they are satisfied for A', we can assume that E is order complete (considering A' instead of A if necessary). Assume that (a) holds. Then by what we have proved above (c) holds as well. In particular

$$(Au)^- \in \{u\}^{dd} \text{ for all } u \in E_+. \tag{1.9}$$

Let $\lambda \geqslant 0$ and $f \in E_+$ such that $g = (A + \lambda)f \ngeqslant 0$. We have to show that $\lambda \leqslant \|A\|$. Denote by P the band projection onto the band generated by g^-. Then $PAf + \lambda Pf = Pg = g^- < 0$. Since by (1.9), $[A(\mathrm{Id} - P)f]^- \in (\mathrm{Id} - P)E$, it follows $0 > \lambda Pf + PAf = \lambda Pf + PAPf + PA(\mathrm{Id} - P)f \geqslant \lambda Pf + PAPf + P(A(\mathrm{Id} - P)f)^+ \geqslant \lambda Pf + PAPf$. Hence $0 \leqslant \lambda Pf < -PAPf$. This implies that $Pf \neq 0$ and $\lambda\|Pf\| \leqslant \|PAPf\| \leqslant \|A\| \cdot \|Pf\|$. Consequently $\lambda \leqslant \|A\|$. □

Remark 1.12 It follows from the proof of Theorem 1.11 that on a σ-order complete Banach lattice condition (1.9) is equivalent to the positivity of the semigroup $(e^{tA})_{t\geqslant 0}$.

Examples 1.13 Let $E = \ell^p$ $(1 \leqslant p \leqslant \infty)$ or $E = c_0$.

(i) An operator $A \in \mathcal{L}(E))$ can be canonically represented by a matrix (a_{ij}). It follows from Theorem 1.11 that $e^{tA} \geqslant 0$ for all $t \geqslant 0$ if and only if $a_{ij} \geqslant 0$ whenever $i \neq j$.

(ii) Let A be the generator of a strongly continuous contraction semigroup $(T(t))_{t\geqslant 0}$ on E. Suppose that the space c_{00} of all sequences which vanish off a finite set is a core of A. Let $(a_{nm})_{m\in\mathbb{N}} = (Ae_n)$ where $e_n = (\delta_{nm})_{m\in\mathbb{N}}$ denotes the n^{th} unit vector. Then it follows from Theorem 1.8 that the semigroup is positive if and only if $a_{nm} \geqslant 0$ whenever $n \neq m$.

2 Kato's Inequality

A strongly continuous semigroup on $C(K)$ (K compact) or a norm continuous semigroup on an arbitrary Banach lattice is positive if and only if its generator A satisfies the positive minimum principle (P) from Definition 1.6. However, we will see that in general (P) is not sufficient for the positivity of the semigroup. One reason seems to be that (P) involves merely positive elements in $D(A)$ but $D(A)_+$ can be small if the semigroup is not positive (cf. Remark 3.16). Our aim in this section is to find a different condition on the generator which is necessary for the positivity of the semigroup.

We recall from Chapter C-I, Section 8 definition and properties of the signum operator.

Proposition 2.1 *Let E be a σ-order complete (real or complex) Banach lattice. For every $f \in E$ there exists a unique linear operator (sign f) on E which satisfies*

$$|(\operatorname{sign} f)g| \leqslant |g| \quad (g \in E) \tag{2.1}$$

$$(\operatorname{sign} f)g = 0 \quad \textit{if} \quad \inf\{|f|, |g|\} = 0 \tag{2.2}$$

$$(\operatorname{sign} \bar{f})f = |f| \quad \textit{(where } \bar{f} := \operatorname{Re} f - i \operatorname{Im} f\textit{)}. \tag{2.3}$$

The operator $(\widehat{\operatorname{sign}} f)$ (which is non-linear in general) is defined by

$$(\widehat{\operatorname{sign}} f)g = (\operatorname{sign} f)g + (\operatorname{Id} - P_{|f|})|g| \tag{2.4}$$

where for $h \in E_+$ we denote by P_h the band projection onto the band $\{h\}^{dd}$ generated by h.

If E is a real σ-order complete Banach lattice, then

$$\operatorname{sign} f = P_{(f^+)} - P_{(f^-)}. \tag{2.5}$$

Example 2.2 Let $f \in E := L^p(X, \Sigma, \mu)$ (real or complex) where (X, Σ, μ) is a σ-finite measure space and $1 \leqslant p \leqslant \infty$. Define

$$m(x) = \begin{cases} f(x)/|f(x)| & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0. \end{cases}$$

Then sign f is the multiplication operator defined by m; i.e.,

$$(\operatorname{sign} f)g = m \cdot g \quad (g \in E),$$

Moreover,

$$(\widehat{\operatorname{sign}} f)g = m \cdot g + 1_{[f(x)=0]}|g| \quad (g \in E).$$

The operator $\widehat{\text{sign}} f$ is related to the Gateaux-derivative (B-II, Definition 3.2) of the modulus. We explain this by an example.

Example 2.3 Let E be the real or complex space $L^p(X, \Sigma, \mu)$ where (X, Σ, μ) is a σ-finite measure space and $1 \leqslant p < \infty$. Let $f, g \in E$ and $x \in X$. Then by B-II, Lemma 2.4

$$\lim_{t \downarrow 0} 1/t\,(|f(x) + tg(x)| - |f(x)|) = \begin{cases} \operatorname{Re}(\operatorname{sign} \overline{f(x)})g(x) & \text{if } f(x) \neq 0 \\ |g(x)| & \text{if } f(x) = 0. \end{cases}$$

If $\Theta\colon E \to E_+$ denotes the modulus function given by $\Theta(h) = |h|$, then it follows from the dominated convergence theorem that Θ is right-sided Gateaux-differentiable and

$$D_g\Theta(f) = \operatorname{Re}(\widehat{\text{sign}} \bar{f})g. \tag{2.6}$$

Later we will see that 2.6 holds in every Banach lattice with order continuous norm. Now let A be the generator of a strongly continuous positive semigroup $(T(t))_{t \geqslant 0}$. The positivity of the semigroup is equivalent to

$$|T(t)f| \leqslant T(t)|f| \quad (t \geqslant 0\,, f \in E). \tag{2.7}$$

In order to deduce from 2.7 a property for the generator A it is natural trying to differentiate 2.7 at $t = 0$. Let us assume for a moment that $E = L^p(X, \Sigma, \mu)$ (as in Example 2.3). Let $f \in D(A)$ and $0 \leqslant \varphi \in D(A')$. Then by (2.7) ,

$$\langle |T(t)f|, \varphi \rangle \leqslant \langle T(t)|f|, \varphi \rangle \quad (t \geqslant 0) \tag{2.8}$$

where the equality holds for $t = 0$. Hence the inequality remains valid if we differentiate at 0 on both sides of (2.8) . Since $\varphi \in D(A')$ we obtain $\mathrm{d}/\mathrm{d}t|_{t=0}\langle T(t)|f|, \varphi \rangle = \langle |f|, A'\varphi \rangle$ on the right side. By (2.6) and the chain rule B-II, Proposition 2.3 one obtains $\mathrm{d}/\mathrm{d}t|_{t=0}|T(t)f| = \operatorname{Re}((\widehat{\text{sign}} \bar{f})Af)$ on the left side. Since $\operatorname{Re}((\operatorname{sign} \bar{f})Af) \leqslant \operatorname{Re}((\widehat{\text{sign}} \bar{f})Af)$, this finally gives

$$\operatorname{Re}\langle (\operatorname{sign} \bar{f})Af, \varphi \rangle \leqslant \langle |f|, A'\varphi \rangle \quad (f \in D(A)\,, 0 \leqslant \varphi \in D(A')). \tag{K}$$

We refer to this as *Kato's inequality*, since it represents an abstract version of the classical inequality proved by Kato for the Laplacian (see Example 2.5).

We will see in the next section that, together with an additional condition, this inequality is characteristic for the positivity of the semigroup.

By a different proof, we now show that Kato's inequality holds for generators of positive semigroups in general.

Theorem 2.4 *The generator A of a strongly continuous positive semigroup $(T(t))_{t \geqslant 0}$ on a σ-order complete (real or complex) Banach lattice E satisfies Kato's inequality; i.e.,*

$$\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af,\varphi\rangle \leqslant \langle|f|,A'\varphi\rangle \quad (f\in D(A), 0\leqslant\varphi\in D(A')).$$

Proof Let $f\in D(A)$, $0\leqslant\varphi\in D(A')$. Then

$$\begin{aligned}
\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af,\varphi\rangle &= \lim_{t\to 0} 1/t\,\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})(T(t)f-f),\varphi\rangle \\
&= \lim_{t\to 0} 1/t\,\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})T(t)f-|f|,\varphi\rangle \\
&\leqslant \lim_{t\to 0} 1/t\langle|T(t)f|-|f|,\varphi\rangle \\
&\leqslant \lim_{t\to 0} 1/t\langle T(t)|f|-|f|,\varphi\rangle \\
&= \lim_{t\to 0}\langle|f|,1/t(T(t)'\varphi-\varphi)\rangle \\
&= \langle|f|,A'\varphi\rangle.
\end{aligned}$$

Let $D(A')_+ = E'_+\cap D(A')$. Consider the condition

$$\overline{D(A')_+}^{\sigma(E',E)} = E'_+ \tag{2.9}$$

(which is satisfied if the semigroup is positive). If (K) and (2.9) hold, then Kato's inequality holds in the strong form as well, whenever it makes sense; i.e.,

$$\mathrm{Re}((\mathrm{sign}\,\bar{f})Af) \leqslant A|f| \quad (\text{whenever } f,|f|\in D(A)). \tag{2.10}$$

Example 2.5 Kato's inequality in its classical form says the following (see Kato (1973) or [Reed-Simon (1975); X.27]). Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that the distributional Laplacian satisfies $\Delta f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the inequality

$$\mathrm{Re}((\mathrm{sign}\,\bar{f})\Delta f) \leqslant \Delta|f|$$

holds in the sense of distributions; i.e., $\langle\varphi,\mathrm{Re}((\mathrm{sign}\,\bar{f})\Delta f)\rangle \leqslant \langle\varphi,\Delta|f|\rangle$ (= $\langle\Delta\varphi,|f|\rangle$) holds for all $0\leqslant\varphi\in C_c^\infty(\mathbb{R}^n)$. Note that the closure of Δ defined on the domain $C_c^\infty(\mathbb{R}^n)$ generates a strongly continuous positive semigroup on $L^p(\mathbb{R}^n)$ $(1\leqslant p<\infty)$ (see Example (iv) and Example 4.7).

We want to discuss the relation between the classical (distributional) inequality and our version given in Theorem 2.4 . Let

$$\mathcal{A} = \sum_{|\alpha|\leqslant m} a_\alpha D^\alpha$$

be a differential operator, where $a_\alpha\in C_c^\infty(\mathbb{R}^n)$. Here we let

$$D^\alpha = (\partial/\partial x_1)^{\alpha_1}\ldots(\partial/\partial x_n)^{\alpha_n}$$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, ($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) of order $|\alpha| := \alpha_1 + \ldots + \alpha_n$. We say that $\mathcal{A}$ satisfies Kato's inequality in the sense of distributions if

$$\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})\mathcal{A}f, \varphi\rangle \leqslant \langle|f|, \mathcal{A}^*\varphi\rangle \tag{K_d}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$, $0 \leqslant \varphi \in C_c^\infty(\mathbb{R}^n)$, where $\mathcal{A}^*$ denotes the formal adjoint of $\mathcal{A}$.

Let now A be the generator of a positive semigroup $(T(t))_{t\geqslant 0}$ on $E := L^p(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$ or $C_0(\mathbb{R}^n)$. Assume that there exists a core D_0 of A such that $C_c^\infty \subset D_0$ and $Af = \mathcal{A}f$ in the sense of distributions for all $f \in D_0$. Then $\mathcal{A}$ satisfies Kato's inequality in the sense of distributions.

In fact, let $0 \leqslant \varphi \in C_c^\infty(\mathbb{R}^n)$. Then $\langle Af, \varphi\rangle = \langle\mathcal{A}f, \varphi\rangle = \langle f, \mathcal{A}^*\varphi\rangle$ for all $f \in D_0$. Since D_0 is a core of A, this implies that $\varphi \in D(A')$ and $A'\varphi = \mathcal{A}^*\varphi$. Thus (K) gives $\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})\mathcal{A}f, \varphi\rangle = \mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af), \varphi\rangle \leqslant \langle|f|, A'\varphi\rangle = \langle|f|, \mathcal{A}^*\varphi\rangle = \langle\mathcal{A}|f|, \varphi\rangle$ for all $f \in C_c^\infty(\mathbb{R}^n)$, $0 \leqslant \varphi \in C_c^\infty(\mathbb{R}^n)$. This is Kato's inequality in the distributional sense.

Remark It has been proved by Miyajima and Okazawa [21]] that

(K_d) implies that $m \leqslant 2$ and that the principal part $\mathcal{A}_0 = \sum_{|\alpha|=2} a_\alpha D^\alpha$ of $\mathcal{A}$ is elliptic; i.e., if we write the operator $\mathcal{A}_0$ in the form $\mathcal{A}_0 = \sum_{i,j=1}^2 b_{ij}\partial^2/\partial x_i \partial x_j$, then the matrix $(b_{ij}(x))$ is positive semidefinite for all $x \in \mathbb{R}^n$.

Finally we formulate Theorem 2.4 for the space $E := C_0(X)$ (X locally compact) (which is not σ-order complete unless X is σ-Stonian). Recall, for $f \in C_0(X)$ sign f is defined as a Borel function and for any bounded Borel function g on X and any $\varphi \in M(X) = C_0(X)'$, we let $\langle g, \varphi\rangle = \int g(x)d\varphi(x)$ (see B-II, Section 2).

Theorem 2.6 *Let X be a locally compact space and A be the generator of a strongly continuous positive semigroup on $C_0(X)$. Then*

$$\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af, \varphi\rangle \leqslant \langle|f|, A'\varphi\rangle \quad (f \in D(A), \varphi \in D(A')_+). \tag{K}$$

The proof of Theorem 2.4 can be taken over literally. Also the analogue of the proof given for L^p-spaces (preceding Theorem 2.4) is valid if one uses B-II, Lemma 2.6.

3 A Characterization of Generators of Positive Semigroups

In this section we confine ourselves to real Banach lattices. This does not mean a restriction since every positive semigroup on a complex Banach lattice leaves the real part of the space invariant.

Remark 3.1 Let $(S(t))_{t\geqslant 0}$ be a semigroup on a complex Banach lattice E with generator A. Then $S(t)E_{\mathbb{R}} \subset E_{\mathbb{R}}$ for all $t \geqslant 0$ if and only if

$$f \in D(A) \implies \bar{f} \in D(A) \text{ and } A\bar{f} = (Af)^{-}. \tag{3.1}$$

In that case the generator $A_{\mathbb{R}}$ of the restriction semigroup on $E_{\mathbb{R}}$ is given by $A_{\mathbb{R}}f = Af$ on $D(A_{\mathbb{R}}) = D(A) \cap E_{\mathbb{R}}$.

We will see below that for generators of a strongly continuous semigroup Kato's inequality alone is not sufficient to ensure the positivity of the semigroup. So we introduce another condition.

Definition 3.2 A subset M' of E' is called *strictly positive* if for every $f \in E_+$, $\langle f, \varphi\rangle = 0$ for all $\varphi \in M'$ implies $f = 0$. Accordingly, an element φ of E'_+ is called *strictly positive* if the set $\{\varphi\}$ is strictly positive.

Example 3.3 Let $E = L^p(X, \mu)$ $(1 \leqslant p < \infty)$, where (X, μ) is a σ-finite measure space. Then $\varphi \in E' = L^q(X, \mu)$ (where $1/p + 1/q = 1$) is strictly positive if and only if $\varphi(x) > 0$ μ-a.e. Note that strictly positive elements of E' always exist in this case.

Definition 3.4 Let B be an operator on a Banach lattice F and let $u \in F$. Then u is called a *positive subeigenvector* of B if

(i) $0 < u \in D(B)$ and

(ii) $Bu \leqslant \lambda u$ for some $\lambda \in \mathbb{R}$.

Proposition 3.5 *Let $(T(t))_{t\geqslant 0}$ be a positive semigroup on a real Banach lattice with generator A. Then there exists a strictly positive set M' of subeigenvectors of A' (the adjoint of the generator A). Moreover, if there exist strictly positive linear forms on E, then there exists a strictly positive subeigenvector of A'.*

Proof Let $\lambda > 0$ be such that $R(\lambda, A) = (\lambda - A)^{-1}$ exists and such that $R(\lambda, A) \geqslant 0$. Let $N' \subset E'_+$ be strictly positive. Then $M' := \{R(\lambda, A)'\psi : \psi \in N'\} \subset D(A')_+$. We show that M' is strictly positive. Indeed, let $f \in E_+$ such that $\langle f, \varphi\rangle = 0$ for all $\varphi \in M'$. Then $\langle R(\lambda, A)f, \psi\rangle = 0$ for all $\psi \in N'$. Hence $R(\lambda, A)f = 0$ since N' is strictly positive. Consequently, $f = 0$. The set M' consists of subeigenvectors of A'. In fact, let $\psi \in N'$, $\varphi = R(\lambda, A)'\psi$. Then $A'\varphi = \lambda\varphi - \psi \leqslant \lambda\varphi$. □

The fact that $\varphi \in D(A')_+$ is a subeigenvector can be expressed by the semigroup (if it is positive).

Proposition 3.6 *Assume that A is the generator of a positive semigroup $(T(t))_{t\geqslant 0}$ on a real Banach lattice E. Let $\varphi \in D(A')_+$ and $\lambda \in \mathbb{R}$. Then*

$$A'\varphi \leqslant \lambda\varphi \textit{ if and only if } T(t)'\varphi \leqslant e^{\lambda t}\varphi \ (t \geqslant 0).$$

Proof If $T(t)'\varphi \leqslant \mathrm{e}^{\lambda t}\varphi$ for all $t \geqslant 0$, then

$$A'\varphi = \sigma(E', E)\text{-}\lim_{t\to 0} 1/t \cdot (T(t)'\varphi - \varphi) \leqslant \lim_{t\to 0} 1/t \cdot (\mathrm{e}^{\lambda t}\varphi - \varphi) = \lambda\varphi.$$

For the converse let $f \in E_+$. Then

$$\begin{aligned}\langle f, T(t)'\varphi\rangle &= \langle f, \varphi\rangle + \int_0^t \langle f, T(s)'A'\varphi\rangle \, \mathrm{d}s \\ &\leqslant \langle f, \varphi\rangle + \lambda \int_0^t \langle f, T(s)'\varphi\rangle \, \mathrm{d}s.\end{aligned}$$

It follows from Gronwall's lemma that $\langle f, T(t)'\varphi\rangle \leqslant \mathrm{e}^{\lambda t}\langle f, \varphi\rangle$. □

Remark 3.7 (i) Using Proposition 3.6 it is immediately clear that $(T(t))_{t\geqslant 0}$ is irreducible if and only if every positive subeigenvector of A' is strictly positive (cf. C-III, Definition 3.1).
(ii) In the proof of the "only if" - part of Proposition 3.6 we needed the positivity of the semigroup in order to be able to apply Gronwall's lemma. However, if instead of assuming that the semigroup is positive we suppose that A satisfies Kato's inequality and $A'\varphi \leqslant \lambda\varphi$ for some strictly positive $\varphi \in D(A')$ then we will show that $T(t)'\varphi \leqslant \mathrm{e}^{\lambda t}\varphi$ and that the semigroup is positive (see Corollary 3.9).

The following is our characterization.

Theorem 3.8 *Let $(T(t))_{t\geqslant 0}$ be a semigroup on a σ-order complete real Banach lattice E. The semigroup is positive if and only if its generator A satisfies the following condition. There exists a core D_0 of A and a strictly positive set M' of subeigenvectors of A' such that*

$$\langle(\operatorname{sign} f)Af, \varphi\rangle \leqslant \langle|f|, A'\varphi\rangle \quad \textit{for all } f \in D_0\,,\ \varphi \in M'. \tag{K}$$

Corollary 3.9 *Assume in addition that E' contains a strictly positive functional. Then the semigroup is positive if and only if there exists a core D_0 of A and a strictly positive subeigenvector φ of A' such that*

$$\langle(\operatorname{sign} f)Af, \varphi\rangle \leqslant \langle|f|, A'\varphi\rangle \quad \textit{for all } f \in D_0. \tag{K}$$

From the proof of Theorem 3.8 we isolate the following

Proposition 3.10 *Let B be a densely defined operator on E and D_0 be a core of B. Suppose that $\varphi \in D(B')_+$ is such that $B'\varphi \leqslant 0$. Denote by p the sublinear functional given by $p(f) = \langle f^+, \varphi\rangle$. If*

$$\langle(\operatorname{sign} f)Bf, \varphi\rangle \leqslant \langle|f|, B'\varphi\rangle \quad (f \in D_0), \tag{K}$$

then B is p-dissipative.

Proof Let $f \in D_0$. Set $P_+ := P_{f^+}$, $P_- := P_{f^-}$ and let $P := \mathrm{Id} - P_+ - P_-$, $Q = P_+ + 1/2\,P$ and $\psi = Q'\varphi$. We show that

$$\psi \in \mathrm{d}p(f). \tag{3.2}$$

Let $g \in E$. Since $0 \leqslant Q \leqslant \mathrm{Id}$ we have $\langle g, \psi\rangle = \langle Qg, \varphi\rangle \leqslant \langle Qg^+, \varphi\rangle \leqslant \langle g^+, \varphi\rangle = p(g)$. Moreover, $\langle f, \psi\rangle = \langle Qf, \varphi\rangle = \langle P_+ f + 1/2\,Pf, \varphi\rangle = \langle f^+, \varphi\rangle = p(f^+)$. So (3.2) follows by the definition of $\mathrm{d}p(f)$. We show that

$$\langle Bf, \psi\rangle \leqslant 0. \tag{3.3}$$

One has trivially

$$\langle (P_+ + P_- + P)Bf, \varphi\rangle = \langle f, B'\varphi\rangle. \tag{3.4}$$

Addition of (3.4) and (K) gives $\langle (2P_+ + P)Bf, \varphi\rangle \leqslant \langle 2f^+, B'\varphi\rangle \leqslant 0$. Hence $\langle Bf, \psi\rangle = \langle QBf, \varphi\rangle \leqslant 0$. Thus we have proved that $B|D_0$ is p-dissipative. Hence B is p-dissipative as well (by A-II, Corollary 2.5). □

Proof (Proof of Theorem 3.8) Proposition 3.5 and Theorem 2.4 yield one implication. In order to show the other assume that the condition in Theorem 3.8 is satisfied. We have to show that $T(t) \geqslant 0$ for all $t \geqslant 0$. Let $\varphi \in M'$. Consider the half-norm $p(f) = \langle f^+, \varphi\rangle$ and the operator $B = A - \lambda$, where $\lambda \in \mathbb{R}$ is such that $A'\varphi \leqslant \lambda\varphi$. Then B satisfies $B'\varphi \leqslant 0$ and K as well. So it follows from Proposition 3.10 that B is p-dissipative. Since B generates the semigroup $(\mathrm{e}^{-\lambda t}T(t))_{t\geqslant 0}$ we obtain fromA-II, Theorem 2.6 that $p(\mathrm{e}^{-\lambda t}T(t)f) \leqslant p(f)$ $(f \in E\,, t \geqslant 0)$. Hence,

$$\langle (T(t)f)^+, \varphi\rangle \leqslant \mathrm{e}^{\lambda t}\langle f^+, \varphi\rangle \quad (f \in E, t \geqslant 0). \tag{3.5}$$

Now let $t > 0$ and $f \leqslant 0$; then $f^+ = 0$. It follows from (3.5) that $\langle (T(t)f)^+, \varphi\rangle \leqslant 0$. Since $\varphi \in M'$ is arbitrary and M' is strictly positive, it follows that $(T(t)f)^+ = 0$; i.e., $T(t)f \leqslant 0$. This implies that $T(t) \geqslant 0$. □

Remark 3.11

(i) The proof of Theorem 3.8 shows the following. If A is the generator of a positive semigroup and E' contains strictly positive linear forms, then there exist a continuous half-norm p on E and $w \in \mathbb{R}$ such that $A - w$ is p-dissipative. We stress that p cannot be replaced by the norm (or by N^+), since in general none of the semigroups $(\mathrm{e}^{-wt}T(t))_{t\geqslant 0}$ $(w \in \mathbb{R})$ is contractive for the norm (cf. Derndinger [10] and Batty and Davies [5]).

(ii) Using Proposition 3.10 one can show with the help of the proof of A-II, Proposition 2.9 that a densely defined operator is closable whenever there exists a strictly positive set M' of subeigenvectors of A' such that (K) holds for all $f \in D(A)$ and $\varphi \in M'$.

Remark 3.12 In Theorem 3.8 and Corollary 3.9 one can replace inequality (K) by the inequality

$$\langle P_{(f^+)} Af, \varphi \rangle \leqslant \langle f^+, A'\varphi \rangle, \tag{3.6}$$

(with the notation of Proposition 3.10).

Indeed, (3.6) for $-f$ yields $\langle -P_{(f^-)} Af, \varphi \rangle \leqslant \langle f^-, A'\varphi \rangle$. Adding up both inequalities one obtains $\langle (\operatorname{sign} f) Af, \varphi \rangle \leqslant \langle |f|, A'\varphi \rangle$. On the other hand, if A generates a positive semigroup, one sees by the obvious alterations in the proof of Theorem 2.4 that (3.6) holds for all $f \in D(A)$ and $\varphi \in D(A')_+$.

Next we formulate the result for the space $C_0(X)$, where X is a locally compact space (concerning the notation cf. Theorem 2.6 and Section 2 of B-II).

Theorem 3.13 *Let A be the generator of a semigroup on $C_0(X)$. The semigroup is positive if and only if there exists a core D_0 of A and a strictly positive set M' of subeigenvectors of A' such that*

$$\langle (\operatorname{sign} f) Af, \varphi \rangle \leqslant \langle |f|, A'\varphi \rangle \quad \textit{for all } f \in D_0, \varphi \in M'. \tag{K}$$

This theorem can be proved in the same way as Theorem 3.8 .

Remark If X is separable, then there exist strictly positive measures on $C_0(X)$. In that case the analogue of Corollary 3.9 holds as well.

Now we want to discuss the results obtained so far.

As a first example we consider the first derivative with boundary conditions on

$$E = L^p[0,1] \quad (1 \leqslant p < \infty).$$

By $\mathrm{AC}[0,1]$ we denote the space of all absolutely continuous functions on $[0,1]$. Let $A_{\max}$ be given by

$$\begin{aligned} D(A_{\max}) &= \{f \in \mathrm{AC}[0,1] : f' \in L^p[0,1]\} \\ A_{\max} f &= f' \quad (f \in D(A_{\max})). \end{aligned}$$

The following lemma is easy to prove.

Lemma 3.14 *Let $f \in AC[0,1]$. Then $|f| \in AC[0,1]$ and $|f|' = (\operatorname{sign} f) \cdot f'$ (a.e.).*

As a consequence of the lemma, $D(A_{\max})$ is a sublattice of E and

$$(\operatorname{sign} f) A_{\max} f = A_{\max} |f| \quad (f \in D(A_{\max})). \tag{3.7}$$

For $\lambda > 0$ one has

$$\ker(\lambda - A_{\max}) = \mathbb{R} \cdot e_\lambda \quad \text{where } e_\lambda(x) = \mathrm{e}^{\lambda x}. \tag{3.8}$$

Hence $A_{\max}$ is not a generator. We impose the following boundary conditions. Let $d \in \mathbb{R}$. Consider the restriction A_d of $A_{\max}$ to the domain

$$D(A_d) = \{f \in D(A_{\max}) \colon f(1) = \mathrm{d}f(0)\}.$$

Then A_d is the generator of the semigroup $(T_d(t))_{t\geqslant 0}$ given by

$$T_d(t)f(x) = d^n \cdot f(x+t-n) \quad \text{if } x+t \in [n, n+1) \quad (n \in \mathbb{N}). \tag{3.9}$$

This is not difficult to prove. Actually (3.9) defines a group if $d \neq 0$ and if we allow $t \in \mathbb{R}$, $n \in \mathbb{Z}$. For $d = 0$ one obtains the nilpotent shift semigroup on E. It follows from (3.9) that the semigroup $(T_d(t))_{t\geqslant 0}$ is positive if and only if $d \geqslant 0$.

Let us fix $d < 0$. Let $A = A_d$ and $T(t) = T_d(t)$ for $t \geqslant 0$. Then $(T(t))_{t\geqslant 0}$ is a semigroup which is *not positive*. Nevertheless its generator A satisfies Kato's inequality. Even the equality is valid; i.e.,

$$\langle (\mathrm{sign}\, f)Af, \varphi\rangle = \langle |f|, A'\varphi\rangle \quad \text{for all } f \in D(A), 0 \leqslant \varphi \in D(A'). \tag{3.10}$$

Proof It is not difficult to see that

$$\begin{aligned} D(A') &= \{\varphi \in \mathrm{AC}[0,1] \colon \varphi' \in L^q[0,1], \varphi(0) = d\varphi(1)\} \\ A'\varphi &= -\varphi' \quad \text{for all } \varphi \in D(A'). \end{aligned} \tag{3.11}$$

where $1/p + 1/q = 1$. Let $\varphi \in D(A')_+$. Since $d < 0$, it follows that $\varphi(0) = \varphi(1) = 0$. Hence for $f \in D(A)$,

$$\begin{aligned} \langle (\mathrm{sign}\, f)Af, \varphi\rangle &= \langle (\mathrm{sign}\, f)f', \varphi\rangle = \langle |f|', \varphi\rangle \\ &= \int_0^1 |f|'(x)\, \varphi(x)\ dx \\ &= |f(1)|\varphi(1) - |f(0)|\varphi(0) - \int_0^1 |f(x)|\, \varphi'(x)\ \mathrm{d}x \\ &= |f(1)|\varphi(1) - |f(0)|\varphi(0) + \langle |f|, A'\varphi\rangle \\ &= \langle |f|, A'\varphi\rangle \end{aligned}$$

Remark 3.15 The equality (3.10) does not hold for all $\varphi \in D(A')$. In fact, this would imply that $|f| \in D(A)$ and $(\mathrm{sign}\, f)Af = A|f|$ for all $f \in D(A)$. Thus by Corollary 5.8 below the semigroup would be positive. The reason why in this example the equality holds will be explained from a more general point of view in Section 5 (see Remark 5.12).

Relation (3.10) shows that A also satisfies Kato's inequality formally in the strong sense. In order to formulate this more precisely, observe that it follows from (3.8) that $D(A_{\max}) = D(A) + \mathbb{R} \cdot e_\lambda$ (for any fixed $0 < \lambda \in \varrho(A)$). Thus the extension $A_{\max}$ of A satisfies the following.

$$A_{\max} \quad \text{is closed.} \tag{3.12}$$
$$D(A_{\max}) \quad \text{is a sublattice of } E. \tag{3.13}$$
$$D(A) \quad \text{has codimension one in } D(A_{\max}) \tag{3.14}$$
$$(\operatorname{sign} f)Af = A_{\max}|f| \quad \text{for all } f \in D(A). \tag{3.15}$$

It is also remarkable that there exists a dense sublattice

$$D_0 := \{f \in D(A) : f(0) = f(1) = 0\}$$

of E which is included in $D(A)$. But D_0 is not a core of A (this would imply the positivity of the semigroup by Theorem 1.8 if $|d| \leqslant 1$). Since $(T(t))_{t\geqslant 0}$ is not positive but Kato's inequality holds, it follows from Theorem 3.8 that there does not exist a strictly positive subeigenvector of A'. In fact, even the following is true.

$$0 \leqslant \varphi \in D(A'),\ A'\varphi \leqslant \mu\varphi \text{ for some } \mu \in \mathbb{R} \text{ implies } \varphi = 0. \tag{3.16}$$

Proof Suppose that $0 \leqslant \varphi \in D(A')$ such that $-\varphi' = A'\varphi \leqslant \mu\varphi$. We can assume that $\mu \geqslant 0$. Let $\psi(x) = \varphi(1-x)$. Then $\psi'(x) = -\varphi'(1-x) \leqslant \mu\varphi(1-x) = \mu\psi(x)$. Since $\psi(0) = 0$, we obtain

$$\psi(x) = \int_0^x \psi'(y)\ \mathrm{d}y \leqslant \mu \int_0^x \psi(y)\ \mathrm{d}y \quad (x \in [0,1]).$$

It follows from Gronwall's lemma that $\psi \leqslant 0$. Hence $\varphi = \psi = 0$. □

Remark 3.16 Let B be the generator of a strongly continuous semigroup on a real Banach lattice with order continuous norm. Assume that the following two conditions hold.

$$\langle(\operatorname{sign} f)Bf, \varphi\rangle \leqslant \langle|f|, B'\varphi\rangle \quad (f \in D(B),\ \varphi \in D(B')_+). \tag{K}$$
$$(D(B')_+)^{-\sigma(E',E)} = E'_+. \tag{3.17}$$

Because of (3.17) condition (K) implies that $P_f Bf \leqslant (\operatorname{sign} f)Bf \leqslant Bf$ whenever $f \in D(B)_+$.

This is Kato's inequality in the strong form for positive $f \in D(B)$ and is equivalent to $(Bf)^- \in \{f\}^{dd} = \overline{E_f}$ $(f \in D(A)_+)$ (recall that E has order continuous norm). By Lemma 1.10 this again is equivalent to

$$0 \leqslant f \in D(B),\ \varphi \in E'_+,\ \langle f, \varphi\rangle = 0 \text{ implies } \langle Bf, \varphi\rangle \geqslant 0. \tag{P}$$

It is easy to see that the operator A in the example satisfies conditions (K) and (3.17). Thus the positive minimum principle (P) is not sufficient for the positivity of the semigroup.

In view of the preceding example and remarks one might presume that the existence of a strictly positive set of subeigenvectors of the adjoint of the generator actually implies the positivity of the semigroup. This is not the case.

To give an example consider $E = L^2(\mathbb{R})$ and the operator B given by $Bf = f^{(3)}$ with domain

$$D(B) = \{f \in L^2(\mathbb{R})\colon f \in C^2(\mathbb{R}); f'' \in \mathrm{AC}(\mathbb{R}); f, f', f'', f^{(3)} \in L^2(\mathbb{R})\}$$

Then B is the generator of a unitary group $(U(t))_{t\in\mathbb{R}}$. In particular, B is skew-adjoint, i.e. $B' = -B$. Moreover, we claim that

$$B' \text{ has a strictly positive subeigenvector } \varphi. \tag{3.18}$$

Proof Let $\lambda > 0$ and $\varphi \in C^3(\mathbb{R})$ such that $\varphi(x) = \mathrm{e}^{-|x|}$ for $|x| \geqslant 1$, $\varphi(x) > 0$ for all $x \in \mathbb{R}$, $\varphi(0) = 1$ and $\varphi'(0) = \varphi''(0) = 0$. Then $\varphi \in D(B')$. Moreover, $-\varphi^{(3)}(x) \leqslant \varphi(x)$ for $|x| \geqslant 1$. Hence there exists $\mu \geqslant 1$ such that

$$B'\varphi = -\varphi^{(3)} \leqslant \mu\varphi.$$

□

But the semigroup $(U(t))_{t\geqslant 0}$ is not positive. In fact, we show that there exists $f \in D(B)$ such that

$$\langle(\operatorname{sign} f)Bf, \varphi\rangle > \langle|f|, B'\varphi\rangle. \tag{3.19}$$

Proof Let $f \in D(B)$ be such that $f(x) = \mathrm{e}^{-x}\sin x$ in a neighborhood of 0, while $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$. Then

$$\langle(\operatorname{sign} f)Bf, \varphi\rangle = -\int_{-\infty}^{0} f^{(3)}(x)\varphi(x)\,\mathrm{d}x + \int_{0}^{\infty} f^{(3)}(x)\varphi(x)\,\mathrm{d}x.$$

Hence,

$$\begin{aligned}
\langle|f|, B'\varphi\rangle &= \int_{-\infty}^{0}(-f(x))(-\varphi^{(3)}(x))\,\mathrm{d}x + \int_{0}^{\infty} f(x)(-\varphi^{(3)}(x))\,\mathrm{d}x \\
&= -\int_{-\infty}^{0} f^{(3)}(x)\varphi(x)\,\mathrm{d}x + \int_{0}^{\infty} f^{(3)}(x)\varphi(x)\,\mathrm{d}x \\
&\quad + [f''\varphi]_{-\infty}^{0} - [f''\varphi]_{0}^{\infty} \quad (\text{since } \varphi''(0) = \varphi'(0) = 0) \\
&= \langle(\operatorname{sign} f)Bf, \varphi\rangle + 2f''(0)\varphi(0) \\
&< \langle(\operatorname{sign} f)Bf, \varphi\rangle \quad (\text{since } f''(0)\varphi(0) = f''(0) = -2).
\end{aligned}$$

□

We now show that B satisfies Kato's inequality for positive elements, though; i.e.,

$$P_f Bf \leqslant Bf \quad \text{for all } f \in D(B)_+. \tag{3.20}$$

In fact, more is true. B is *local*, i.e.

$$f \perp g \quad \text{implies} \quad Bf \perp g \quad \text{for all } f \in D(B), g \in L^2(\mathbb{R}). \tag{3.21}$$

Proof Let A be the generator of the translation group which, in particular, is a lattice semigroup (see Section 5). We obtain from Proposition 5.4 below that A is local. Hence $B = A^3$ is local as well. □

This example shows that even if there exists a strictly positive subeigenvector of the adjoint of the generator, Kato's inequality for positive elements alone does not suffice for the positivity of the semigroup. Note also that (because of the order continuous norm) Kato's inequality holds for positive elements if and only if the positive minimum principle is satisfied (see Remark 3.16).

4 Domination of Semigroups

Frequently it is useful to be able to compare two semigroups on a Banach lattice with respect to the ordering (for example, in order to decide whether a semigroup is stable (see Chapter A-IV and Example 4.14).

In this section we assume that E is a σ-order complete complex Banach lattice. Let $(T(t))_{t\geqslant 0}$ be a positive semigroup with generator A and $(S(t))_{t\geqslant 0}$ a semigroup with generator B. We say, $(T(t))_{t\geqslant 0}$ *dominates* $(S(t))_{t\geqslant 0}$ if

$$|S(t)f| \leqslant T(t)|f| \quad \text{for all } f \in E, t > 0. \tag{4.1}$$

We first observe that domination of the semigroup is equivalent to domination of the resolvents.

Proposition 4.1 *The semigroup* $(T(t))_{t\geqslant 0}$ *dominates* $(S(t))_{t\geqslant 0}$ *if and only if*

$$|R(\lambda, B)f| \leqslant R(\lambda, A)|f| \quad (f \in E) \quad \textit{for large real } \lambda. \tag{4.2}$$

Proof (4.2) follows from (4.1) since the resolvent is given by the Laplace transform of the semigroup. Conversely, if (4.2) holds, then

$$\begin{aligned} |S(t)f| &= \lim_{n\to\infty} |((n/t)R(n/t, B))^n f| \\ &\leqslant \lim_{n\to\infty} ((n/t)R(n/t, A))^n |f| \\ &= T(t)|f| \quad (t \geqslant 0, f \in E). \end{aligned}$$

One can describe domination by an inequality for the generators in a manner analoguous to the characterization of positive semigroups in Section 1; however, no positive subeigenvectors are needed here.

Theorem 4.2 *Let $(T(t))_{t\geqslant 0}$ be a positive semigroup with generator A and $(S(t))_{t\geqslant 0}$ a semigroup with generator B. The following assertions are equivalent.*

(a) $|S(t)f| \leqslant T(t)|f|$ *for all* $f \in E$, $t \geqslant 0$.

(b) $\mathrm{Re}\langle(\mathrm{sign}\, f)Bf, \varphi\rangle \leqslant \langle |f|, A'\varphi\rangle$ *for all* $f \in D(B)$, $\varphi \in D(A')_+$.

Proof (a) implies (b) . Let $f \in D(B)$, $\varphi \in D(A')_+$. Then

$$\begin{aligned}\mathrm{Re}\langle(\mathrm{sign}\,\bar f)Bf, \varphi\rangle &= \mathrm{Re}\langle(\mathrm{sign}\,\bar f)\lim_{t\to 0} 1/t(S(t)f - f), \varphi\rangle \\ &= \langle \lim_{t\to 0} 1/t(\mathrm{Re}\langle(\mathrm{sign}\,\bar f)S(t)f\rangle - |f|), \varphi\rangle \\ &\leqslant \lim_{t\to 0}\langle 1/t(|S(t)f| - |f|), \varphi\rangle \\ &\leqslant \lim_{t\to 0}\langle 1/t(T(t)|f| - |f|), \varphi\rangle = \langle |f|, A'\varphi\rangle.\end{aligned}$$

(b) implies (a) . Let $\lambda > \max\{\omega(A), \omega(B)\}$ and $g \in E$. We show that

$$|R(\lambda, B)g| \leqslant R(\lambda, A)|g|. \tag{4.3}$$

Let $\psi \in E'_+$. Then $\varphi := R(\lambda, A)'\psi \in D(A')_+$. Setting $f := R(\lambda, B)g \in D(B)$ we obtain by (b)

$$\begin{aligned}\langle |R(\lambda, B)g|, \psi\rangle = \langle |f|, (\lambda - A')\varphi\rangle &\leqslant \langle \lambda|f|, \varphi\rangle - \mathrm{Re}\langle(\mathrm{sign}\,\bar f)Bf, \varphi\rangle \\ &= \mathrm{Re}\langle(\mathrm{sign}\,\bar f)(\lambda f - Bf), \varphi\rangle \\ &= \mathrm{Re}\langle(\mathrm{sign}\,\bar f)g, \varphi\rangle \\ &\leqslant \langle |g|, \varphi\rangle = \langle R(\lambda, A)|g|, \psi\rangle\end{aligned}$$

Since $\psi \in E'_+$ is arbitrary (4.3) follows. ◻

In order to deduce that (b) implies (a) in Theorem 4.2 , it is not necessary to assume that B is a generator. Merely a range condition is sufficient. The precise formulation is the following.

Theorem 4.3 *Let $(T(t))_{t\geqslant 0}$ be a positive semigroup with generator A. Let B be a densely defined operator such that*

$$\mathrm{Re}\langle(\mathrm{sign}\,\bar f)Bf, \varphi\rangle \leqslant \langle |f|, A'\varphi\rangle \tag{4.4}$$

for all $f \in D(B)$, $\varphi \in D(A')_+$. Then B is closable. Moreover, if $(\lambda - B)D(B)$ is dense in E for some $\lambda > \max\{0, s(A)\}$, then $\overline{B}$ (the closure of B) generates a semigroup which is dominated by $(T(t))_{t\geqslant 0}$.

Proof 1. We show that B is closable. Let $u_n \in D(B)$ satisfy $u_n \to 0$ and $Bu_n \to v$ ($n \to \infty$). We have to show that $v = 0$. Considering $A - \mu$ and $B - \mu$ for some $\mu > s(A)$ instead of A and B we may assume that $s(A) < 0$. Then there exists a strictly positive set $M' \subset E'$ such that

$$\varphi \in D(A') \text{ and } A'\varphi \leqslant 0 \text{ for all } \varphi \in M' \tag{4.5}$$

(see the proof of Proposition 3.5).

Let $\varphi \in M'$ and p be the seminorm given by $p(f) = \langle |f|, \varphi\rangle$. We show that B is p-dissipative (see end of A-II, Section 2). Let $f \in D(B)$, $\psi = (\operatorname{sign} \bar{f})'\varphi$. Then it is easy to see that $\psi \in \mathrm{d}p(f)$. Moreover, by (4.4) and (4.5) one obtains that $\operatorname{Re}\langle Bf, \psi\rangle = \operatorname{Re}\langle(\operatorname{sign} \bar{f})Bf, \varphi\rangle \leqslant \langle |f|, A'\varphi\rangle \leqslant 0$. Thus B is p-dissipative. By the proof of A-II, Proposition 2.9 one sees that $p(v) = 0$; i.e., $\langle |v|, \varphi\rangle = 0$. Since $\varphi \in M'$ was arbitrary we conclude that $v = 0$.

2. Let $\lambda > \lambda_0 := \max\{s(A), 0\}$. We show that for $f \in D(B)$,

$$g = (\lambda - B)f \text{ implies } |f| \leqslant R(\lambda, A)|g|. \tag{4.6}$$

Let $\psi \in E'_+$. We have to show that $\langle |f|, \psi\rangle \leqslant \langle R(\lambda, A)|g|, \psi\rangle$. Let $\varphi = R(\lambda, A)'\psi \in D(A')_+$. Then by (4.4)

$$\begin{aligned}
\langle |f|, \psi\rangle &= \langle |f|, (\lambda - A')\varphi\rangle \\
&= \operatorname{Re}\langle(\operatorname{sign} \bar{f})(\lambda f), \varphi\rangle - \langle |f|, A'\varphi\rangle \\
&\leqslant \operatorname{Re}\langle(\operatorname{sign} \bar{f})(\lambda - B)f, \varphi\rangle = \operatorname{Re}\langle(\operatorname{sign} \bar{f})g, \varphi\rangle \\
&\leqslant \langle |g|, \varphi\rangle = \langle R(\lambda, A)|g|, \psi\rangle.
\end{aligned}$$

It follows from (4.6) that for $\lambda > \lambda_0$ and $f \in D(\overline{B})$

$$g = (\lambda - \overline{B})f \text{ implies } |f| \leqslant R(\lambda, A)|g|. \tag{4.7}$$

In particular, $(\lambda - \overline{B})$ is injective for $\lambda > \lambda_0$. Moreover,

$$\begin{aligned}
&|R(\lambda, \overline{B})g| \leqslant R(\lambda, A)|g| \quad \text{for all } g \in E \\
&\text{whenever} \lambda_0 < \lambda \in \varrho(\overline{B}).
\end{aligned} \tag{4.8}$$

Assume now that $\mu > \lambda_0$ such that $(\mu - B)D(B)$ is dense in E. Then $(\mu - \overline{B})D(\overline{B}) = E$. (Indeed, let $h \in E$. There exists $f_n \in D(B)$ such that $g_n := (\mu - B)f_n \to h$ $(n \to \infty)$. By (4.6) it follows that $|f_n - f_m| \leqslant R(\lambda, A)|g_n - g_m|$. Thus (f_n) is a Cauchy sequence. Let $f = \lim_{n\to\infty} f_n$. Then $f \in D(\overline{B})$ and $(\mu - \overline{B})f = h$.) Thus $\mu \in \varrho(\overline{B})$.

It follows from the hypothesis that there exists $\lambda_1 \in \varrho(\overline{B})$ such that $\lambda_0 < \lambda_1$. Since $R(\lambda, A) \leqslant R(\lambda_1, A)$ (by B-II, Lemma 1.9) , it follows from (4.8) that $\|R(\lambda, \overline{B})\| \leqslant \|R(\lambda, A)\| \leqslant \|R(\lambda_1, A)\| := c$; hence
$\operatorname{dist}(\lambda, \sigma(\overline{B})) = r(R(\lambda, \overline{B}))^{-1} \geqslant \|R(\lambda, \overline{B})\|^{-1} \geqslant 1/c$ for all $\lambda \in \varrho(\overline{B}) \cap [\lambda_1, \infty)$. This implies that $[\lambda_1, \infty) \subset \varrho(\overline{B})$. Moreover, it follows from (4.8) that

$$|R(\lambda, \overline{B})^n f| \leqslant R(\lambda, A)^n |f| \quad (f \in E, n \in \mathbb{N}, \lambda_1 < \lambda). \tag{4.9}$$

Let $w > \omega(A)$, λ_1. Then it follows from (4.9) that

$\|(\lambda - w)^n R(\lambda, \overline{B})^n\| \leqslant \|(\lambda - w)^n R(\lambda, A)^n\|$ for all $\lambda > w$, $n \in \mathbb{N}$. So by the Hille-Yosida theorem, $\overline{B}$ is the generator of a semigroup $(S(t))_{t \geqslant 0}$. Finally, the domination of $(S(t))_{t \geqslant 0}$ by $(T(t))_{t \geqslant 0}$ follows from (4.8) and Proposition 4.1 . ◻

Example 4.4 (i) Let E be a σ-order complete complex Banach lattice and $(T(t))_{t \geqslant 0}$ be a positive semigroup with generator A. Let $M \in \mathcal{Z}(E)$ (the center of E (see C-I, Section 9). For example, if $E = L^p(X, \mu)$ (where (X, μ) is a σ-finite measure space and $1 \leqslant p \leqslant \infty$) then M is the multiplication operator defined by a function in $L^\infty(X, \mu)$.

Let $B = A + M$. Then B generates a semigroup $(S(t))_{t \geqslant 0}$. Assume that $\mathrm{Re}M \leqslant 0$. Let $f \in D(B)$ and $\varphi \in D(A')_+$. Then

$$\begin{aligned}
\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf, \varphi\rangle &= \mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af, \varphi\rangle + \mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Mf, \varphi\rangle \\
&= \mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af, \varphi\rangle + \mathrm{Re}\langle M|f|, \varphi\rangle \\
&\leqslant \langle |f|, A'\varphi\rangle.
\end{aligned}$$

Thus, by Theorem 4.2 , $S(t))_{t \geqslant 0}$ is dominated by $(T(t))_{t \geqslant 0}$.

(ii) Let E be an order complete complex Banach lattice and B be a regular bounded operator on E. Then B can be written as $B = B_0 + M$ where $M \in \mathcal{Z}(E)$ and $B_0 \in \mathcal{L}(E)^r$ such that $\inf\{|B_0|, \mathrm{Id}\} = 0$. Let $A = |B_0| + \mathrm{Re}M$. Then the semigroup $(\mathrm{e}^{tB})_{t \geqslant 0}$ is dominated by $(\mathrm{e}^{tA})_{t \geqslant 0}$.

In fact, let $f \in E$. Then
$\mathrm{Re}[(\mathrm{sign}\,\bar{f})Bf] = \mathrm{Re}[(\mathrm{sign}\,\bar{f})B_0 f] + \mathrm{Re}M \cdot |f| \leqslant |B_0||f| + \mathrm{Re}M \cdot |f| = A|f|$.
This implies condition (b) in Theorem 4.2.

Domination and positivity are characterized simultaneously as follows.

Proposition 4.5 *Let E be a σ-order complete real Banach lattice. Let $(T(t))_{t \geqslant 0}$ be a positive semigroup with generator A and let $(S(t))_{t \geqslant 0}$ be a semigroup with generator B. The following are equivalent.*

(a) $0 \leqslant S(t) \leqslant T(t)$ *for all* $t \geqslant 0$.

(b) $\langle P_{(f^+)}Bf, \varphi\rangle \leqslant \langle f^+, A'\varphi\rangle$ *for all* $f \in D(B)$, $\varphi \in D(A')_+$.

(c) $\langle P_{(f^+)}Bf, \varphi\rangle \leqslant \langle f^+, A'\varphi\rangle$ *for all* $f \in D_0$, $\varphi \in D(A')_+$, *where* D_0 *is a core of* B.

Remark 4.6 Condition (b) implies (4.4) (cf. Remark 3.12).

Proof (Proof of Proposition 4.5) One proves as in Theorem 4.2 that (a) implies (b) . It is trivial that (b) implies (c). Assume that (c) holds. Let $\lambda > \lambda_0 = \max\{s(A), s(B), 0\}$. In a similar way as (4.6) one shows that for all $f \in D_0$

$$\lambda f - Bf = g \text{ implies } f^+ \leqslant R(\lambda, A)g^+. \tag{4.10}$$

Since D_0 is a core of B it follows that (4.10) also holds for all $f \in D(B)$. This implies that $(R(\lambda, B)g)^+ \leqslant R(\lambda, A)g^+$ for all $g \in E$, $\lambda > \lambda_0$. Consequently, $0 \leqslant R(\lambda, B) \leqslant R(\lambda, A)$ for all $\lambda > \lambda_0$. Hence (a) holds. □

In the following example we apply Theorem 4.3 to Schrödinger operators. Here the range condition is proved by an elegant argument due to Kato [15] with the help of Kato's classical inequality.

Example 4.7 (Schrödinger operators on L^p). Let $E = L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, and $V \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\mathrm{Re}V \geqslant 0$. Define B on E by $Bf = \Delta f - Vf$ with domain $D(B) = C_c^\infty(\mathbb{R}^n)$. Then B is closable and $\overline{B}$ is the generator of a semigroup $(S(t))_{t\geqslant 0}$ which is dominated by the diffusion semigroup (Example 1.5(iv) and A-I, 2.8). If $V \geqslant 0$, then $(S(t))_{t\geqslant 0}$ is positive.

Proof Denote by A the generator of the diffusion semigroup. Then $C_c^\infty := C_c^\infty(\mathbb{R}^n)$ is a core of A and $Af = \Delta f$ for $f \in C_c^\infty$ (see Example 1.5(iv)). Let $0 \leqslant \varphi \in D(A')$. Then

$$\begin{aligned}\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf, \varphi\rangle &= \mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af, \varphi\rangle - \langle(\mathrm{Re}V)|f|, \varphi\rangle \leqslant \mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af, \varphi\rangle \\ &\leqslant \langle|f|, A'\varphi\rangle \quad \text{for all } f \in C_c^\infty\end{aligned}$$

by Theorem 2.4 . Thus (4.4) holds. We show that $(\lambda - B)$ has dense range for $\lambda > 0$. If not, then there exists $0 \neq \varphi \in E' = L^q(\mathbb{R}^n)$ such that $\langle(\lambda - \Delta + V)f, \varphi\rangle = 0$ for all $f \in C_c^\infty$; i.e., $(\lambda - \Delta + V)\varphi = 0$ in the sense of distributions. By Kato's classical inequality (see Example 2.5) this implies that

$$(\lambda - \Delta + \mathrm{Re}V)|\varphi| \leqslant \lambda|\varphi| - \mathrm{Re}[(\mathrm{sign}\,\overline{\varphi})(\lambda\varphi - \Delta\varphi + V\varphi)] = 0$$

(here we use that $\Delta\varphi = \lambda\varphi + V\varphi \in L^1_{\mathrm{loc}}$). Hence $(\lambda - \Delta)|\varphi| \leqslant -(\mathrm{Re}V)|\varphi| \leqslant 0$. Since $(\lambda - \Delta)^{-1}$ is a positive linear mapping from $\mathcal{S}(\mathbb{R}^n)'$ onto $\mathcal{S}(\mathbb{R}^n)'$, this implies that $\varphi = 0$. It follows from Theorem 2.4 that $\overline{B}$ is the generator of a semigroup $(S(t))_{t\geqslant 0}$ which is dominated by the semigroup generated by A.

If $V = \mathrm{Re}V \geqslant 0$, we may consider the real space $L^p(\mathbb{R}^n)$. Then for every $f \in C_c^\infty$, $0 \leqslant \varphi \in D(A')$ we have

$$\begin{aligned}\langle P_{(f^+)}Bf, \varphi\rangle &= \langle P_{(f^+)}Af, \varphi\rangle - \langle Vf^+, \varphi\rangle \\ &\leqslant \langle P_{(f^+)}Af, \varphi\rangle \\ &\leqslant \langle f^+, A'\varphi\rangle\end{aligned}$$

by (3.6). It follows from Proposition 4.5 that $(S(t))_{t\geqslant 0}$ is positive. □

Finally, if it is known that the semigroup $(S(t))_{t\geqslant 0}$ is positive, domination can be characterized as follows.

Proposition 4.8 *Let E be a real Banach lattice, $(T(t))_{t\geqslant 0}$ a positive semigroup with generator A and $(S(t))_{t\geqslant 0}$ a positive semigroup with generator B. Consider the following conditions.*

(a) $S(t) \leqslant T(t) \quad (t \geqslant 0)$.

(b) $\langle Bf, \varphi\rangle \leqslant \langle f, A'\varphi\rangle \quad$ *for all* $f \in D(B)_+, \varphi \in D(A')_+$.

(c) $Bf \leqslant Af \quad$ *for* $0 \leqslant f \in D(A) \cap D(B)$.

Then (a) and (b) are equivalent and imply (c) . Moreover, if $D(A) \subset D(B)$ or $D(B) \subset D(A)$, then (c) implies (a) .

Proof Assume that (a) holds. Then for $f \in D(B)_+$, $\varphi \in D(A')_+$,

$$\begin{aligned}\langle Bf, \varphi\rangle &= \lim_{t\to 0} 1/t\langle S(t)f - f, \varphi\rangle \leqslant \lim_{t\to 0} 1/t\langle T(t)f - f, \varphi\rangle \\ &= \langle f, A'\varphi\rangle.\end{aligned}$$

So (b) holds. (c) is proved similarly.

Now assume (b) . Let $\lambda > \max\{s(A), s(B)\}$. Let $g \in E_+$, $\psi \in E'_+$. Then

$$\begin{aligned}&\langle R(\lambda, B)g - R(\lambda, A)g, \psi\rangle \\ &\qquad = \langle R(\lambda, A)g, \lambda R(\lambda, B)'\psi - \psi\rangle - \langle \lambda R(\lambda, A)g - g, R(\lambda, B)'\psi\rangle \\ &\qquad = \langle f, B'\varphi\rangle - \langle Af, \varphi\rangle \leqslant 0,\end{aligned}$$

where $f = R(\lambda, A)g \in D(A)_+$ and $\varphi = R(\lambda, B)'\psi \in D(B')_+$. Hence $R(\lambda, B) \leqslant R(\lambda, A)$ and (a) follows.

Finally, we prove that (c) implies (a) if $D(B) \subset D(A)$, say. Let $\lambda > \max\{s(A), s(B)\}$. Then $(A - B)R(\lambda, B)$ is a positive operator.
Hence $R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(A - B)R(\lambda, B) \geqslant 0$. This implies (a) . □

The preceding results can be applied to the perturbation by multiplication operators. Let (X, μ) be a σ-finite measure space and $E = L^p(X, \mu)$ $(1 \leqslant p < \infty)$. Consider a positive semigroup $(T(t))_{t\geqslant 0}$ with generator A. Let $m\colon X \to \mathbb{R}$ be a measurable function such that $m(x) \leqslant 0$ for all $x \in X$. Let $D(m) = \{f \in E\colon f\cdot m \in E\}$. Define the operator B with domain $D(B) = D(A) \cap D(m)$ by $Bf = Af + m \cdot f$ $(f \in D(B))$.

Theorem 4.9 *If there exists a quasi-interior subeigenvector u of A such that $u \in D(m)$, then B is closable and the closure $\overline{B}$ of B is the generator of a positive semigroup $(S(t))_{t\geqslant 0}$ which is dominated by $(T(t))_{t\geqslant 0}$.*

For the proof of the theorem we need the following lemma.

Lemma 4.10 *Let A and B be generators of positive semigroups $(T(t))_{t\geqslant 0}$ and $(S(t))_{t\geqslant 0}$, respectively. If $(T(t))_{t\geqslant 0}$ dominates $(S(t))_{t\geqslant 0}$, then $s(B) \leqslant s(A)$.*

Proof (Proof of Lemma 4.10) Let $\lambda > s(A)$. Then for all $\mu > \max\{\lambda, s(B)\}$ one has $0 \leqslant R(\mu, A) \leqslant R(\lambda, A)$ (by B-II, Lemma 1.9), and so $\|R(\mu, B)\| \leqslant \|R(\mu, A)\| \leqslant \|R(\lambda, A)\|$. Thus $\operatorname{dist}(\mu, \sigma(B)) \geqslant \|R(\mu, B)\|^{-1} \geqslant \|R(\lambda, A)\|^{-1}$. This implies that $[\lambda, \infty) \subset \varrho(B)$. Hence $s(B) \leqslant \lambda$. $\square$

Proof (Proof of Theorem 4.9) There exists $\mu > 0$ such that $Au \leqslant \mu u$. Let $\lambda > \max\{s(A), \mu\}$. Then $\lambda R(\lambda, A)u = \lambda R(\lambda, A)u + u \leqslant \mu R(\lambda, A)u + u$. Hence $R(\lambda, A)u \leqslant c \cdot u$ where $c > 0$. It follows that $R(\lambda, A)E_u \subset E_u \cap D(A) \subset D(B)$. Hence $D(B)$ is dense. Let $f \in D(B)$, $\varphi \in D(A')_+$ and set $P_+ := P_{f^+}$, $P_- := P_{f^-}$. Then

$$\langle P_+ Bf, \varphi\rangle \leqslant \langle f^+, A'\varphi\rangle. \tag{4.11}$$

In fact,

$$\begin{aligned}
\langle P_+ Bf, \varphi\rangle &= \langle P_+ Af, \varphi\rangle + \langle P_+ m \cdot f, \varphi\rangle \\
&= \langle P_+ Af, \varphi\rangle + \langle m \cdot f^+, \varphi\rangle \\
&\leqslant \langle P_+ Af, \varphi\rangle \\
&\leqslant \langle f^+, A'\varphi\rangle \quad \text{(by (3.6))}.
\end{aligned}$$

But (4.11) implies (4.4). So it follows from Theorem 4.3 that B is closable. Moreover, if we can show that $(\lambda - \overline{B})D(\overline{B})$ is dense in E, it follows that $\overline{B}$ is the generator of a semigroup $(S(t))_{t\geqslant 0}$. In that case (4.11) implies that $(S(t))_{t\geqslant 0}$ is dominated by $(T(t))_{t\geqslant 0}$ (by Proposition 4.5).

Now we show that $(\lambda - \overline{B})D(\overline{B})$ is dense in E. Let $m_n = \sup\{m, -n1_X\}$ $(n \in \mathbb{N})$ and $B_n = A + m_n$. Then B_n is the generator of a positive semigroup and it follows from Proposition 4.8 that $0 \leqslant R(\lambda, B_{n+1}) \leqslant R(\lambda, B_n) \leqslant R(\lambda, A)$ for all $n \in \mathbb{N}$, $\lambda > s(A)$. (Note that $s(B_n) \leqslant s(A)$ by Lemma 4.10). Let $0 \leqslant f \in E_u$ and $g_n = R(\lambda, B_n)f$. Then $g = \inf_{n\in\mathbb{N}} g_n = \lim_{n\to\infty} g_n$ exists. Moreover $g_n \in D(B)$ and $\lim_{n\to\infty}(\lambda - B)g_n = f + \lim_{n\to\infty}(B_n - B)g_n = f$, since $|(B_n - B)g_n| \leqslant (m_n - m)|g_n| = (m_n - m)|R(\lambda, B_n)f| \leqslant (m_n - m)R(\lambda, A)|f| \leqslant c'(m_n - m)u$ for some positive constant c'. But $\lim_{n\to\infty}(m_n - m)u = 0$ since $u \in D(m)$. Thus $g \in D(\overline{B})$ and $(\lambda - \overline{B})g = f$. We have shown that $E_u \subset (\lambda - \overline{B})D(\overline{B})$. Hence $(\lambda - \overline{B})D(\overline{B})$ is dense in E. $\square$

Example 4.11 If in the situation explained before Theorem 4.9 $D(A) \subset L^\infty(X, \mu)$ and $m \in L^p(X, \mu)$, then the hypotheses of Theorem 4.9 are satisfied.

Now we want to indicate how the results of this section look like for $C_0(X)$. In fact, most of them carry over with a different interpretation of *"sign"* but the same proofs. We want to state the analogs of Theorem 4.2 and Theorem 4.3 explicitly but omit the proof. Here we use the notation of B-II, Section 2.

Theorem 4.12 *Let $E = C_0(X)$ where X is locally compact. Let $(T(t))_{t\geqslant 0}$ be a strongly continuous positive semigroup with generator A and $(S(t))_{t\geqslant 0}$ a semigroup with generator B. The following assertions are equivalent.*

(a) $|S(t)f| \leqslant T(t)|f|$ *for all* $f \in E$, $t > 0$.

(b) $\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf, \varphi\rangle \leqslant \langle |f|, A'\varphi\rangle$ *for all* $f \in D(B)$, $\varphi \in D(A')_+$.

Recall that by definition $\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf, \varphi\rangle = \int [(\mathrm{sign}\,\overline{f(x)}) \cdot (Bf)(x)]\ \mathrm{d}\varphi(x)$ where $\mathrm{sign}\, f(x) = f(x)/|f(x)|$ if $f(x) \neq 0$ and $\mathrm{sign}\, 0 = 0$.

Theorem 4.13 *Let* $E = C_0(X)$ *(X locally compact) and let* $(T(t))_{t\geqslant 0}$ *be a positive semigroup on E with generator A. Let B be a densely defined operator such that*

$$\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf, \varphi\rangle \leqslant \langle |f|, A'\varphi\rangle \quad \textit{for all} \quad f \in D(B),\ \varphi \in D(A')_+. \tag{4.12}$$

Then B is closable. Moreover, if $(\lambda - B)D(B)$ *is dense in E for some* $\lambda > \max\{0, s(A)\}$, *then* $\overline{B}$ *(the closure of B) generates a semigroup which is dominated by* $(T(t))_{t\geqslant 0}$.

Example 4.14 Let $E := C([-1,0],\mathbb{C})$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$, $\mu \in M[-1,0]_+$ and $\nu \in M[-1,0]$ such that $\mu(\{0\}) = \nu(\{0\}) = 0$.

Then the operator A given by

$$Af = f' \text{ on } D(A) = \{f \in C^1([-1,0],\mathbb{C})\colon f'(0) = \alpha f(0) + \langle f, \mu\rangle\}$$

generates a strongly continuous positive semigroup $(T(t))_{t\geqslant 0}$ (see B-II, Example 1.22).

Consider the operator B given by

$$Bf = f' \text{ with domain } D(B) = \{f \in C^1([-1,0],\mathbb{C})\colon f'(0) = \beta f(0) + \langle f, \nu\rangle\}.$$

We claim that

$$\begin{gathered}B \text{ is the generator of a strongly continuous semigroup}\\ (S(t))_{t\geqslant 0}. \text{ Moreover, } (S(t))_{t\geqslant 0} \text{ is dominated by } (T(t))_{t\geqslant 0}\\ \text{if and only if } \mathrm{Re}\beta \leqslant \alpha \text{ and } |\nu| \leqslant \mu.\end{gathered} \tag{4.13}$$

Proof (Proof of equation (4.13)***)*** We first assume that $\alpha := \mathrm{Re}\beta$ and $\mu = |\nu|$. We show that (4.12) is satisfied. Consider the operator $A_{\max}$ on $C[-1,0]$ given by $A_{\max} f = f'$ with domain $D(A_{\max}) = C^1[-1,0]$. We know by B-II, Example 2.12 that
$\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Af, \varphi\rangle \leqslant \mathrm{Re}\langle(\widehat{\mathrm{sign}\bar{f}})(Af), \varphi\rangle = \langle |f|, (A_{\max})'\varphi\rangle$ for all $f \in D(A_{\max})$, $0 \leqslant \varphi \in D((A_{\max})')$. In particular

$$\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf, \varphi\rangle \leqslant \langle |f|, A'\varphi\rangle \tag{4.14}$$

holds for all $f \in D(B)$, $0 \leqslant \varphi \in D((A_{\max})')$. It is not difficult to see that $D(A') = D((A_{\max})') + \mathbb{C}\delta_0$, and since $D((A_{\max})') = BV[-1,0]$ (see B-II, Example 2.12) this is an order direct sum. Thus, in view of (4.14), it remains to show that

$$\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf,\delta_0\rangle \leqslant \langle|f|,A'\delta_0\rangle \tag{4.15}$$

for all $f \in D(B)$. By the definition of A, $\delta_0 \in D(A')$ and $A'\delta_0 = \alpha\delta_0 + \mu$. Hence for $f \in D(B)$,

$$\begin{aligned}
\mathrm{Re}\langle(\mathrm{sign}\,\bar{f})Bf,\delta_0\rangle &= \mathrm{Re}[(\mathrm{sign}\,\bar{f})f'](0)\\
&= \mathrm{Re}[(\mathrm{sign}\,\overline{f(0)})\cdot(\beta f(0) + \langle f,\nu\rangle)]\\
&\leqslant \mathrm{Re}\beta\,|f(0)| + |\langle f,\nu\rangle|\\
&\leqslant \alpha|f(0)| + \langle|f|,\mu\rangle = \langle|f|,A'\delta_0\rangle.
\end{aligned}$$

Thus (4.15) and so also (4.12) are proved.

As in the proof in Example B-II, 1.22 one shows that $\lambda - B$ is surjective for large real λ. Hence by Theorem 4.13, B is the generator of a strongly continuous semigroup $(S(t))_{t\geqslant0}$ which is dominated by $(T(t))_{t\geqslant0}$. This proves the first assertion of (4.13) and the sufficiency of the second.

Now we assume that the semigroup $(S(t))_{t\geqslant0}$ is dominated by $(T(t))_{t\geqslant0}$. We have to show that $\mathrm{Re}\beta \leqslant \alpha$ and $|\nu| \leqslant \mu$. Since $\delta_0 \in D(A') \cap D(B')$ we have for all $f \in C[-1,0]_+$ satisfying $f(0) = 0$,

$$\begin{aligned}
|\langle f,\nu\rangle| = |\langle f,B'\delta_0\rangle| &= \lim_{t\to0+} 1/t\,|\langle S(t)f - f,\delta_0\rangle|\\
&= \lim_{t\to0+} 1/t\,|(S(t)f)(0)|\\
&\leqslant \lim_{t\to0+} 1/t\,((T(t)|f|)(0)\\
&= \lim_{t\to0+} \langle|f|,1/t(T(t)'\delta_0 - \delta_0)\rangle\\
&= \langle|f|,A'\delta_0\rangle = \langle|f|,\mu\rangle.
\end{aligned}$$

Since $\mu(\{0\}) = \nu\{0\}) = 0$, this implies that $|\nu| \leqslant \mu$.

Moreover, for arbitrary $f \in C[-1,0]_+$ we have

$$\begin{aligned}
\langle f,\mathrm{Re}\beta\,\delta_0 + \mathrm{Re}\nu\rangle &= \lim_{t\to0+} 1/t\;\mathrm{Re}\langle(S(t)f - f),\delta_0\rangle\\
&\leqslant \lim_{t\to0+} 1/t\;\mathrm{Re}\langle(T(t)f - f),\delta_0\rangle\\
&= \langle f,A'\delta_0\rangle = \langle f,\alpha\delta_0 + \mu\rangle.
\end{aligned}$$

Consequently, $(\mathrm{Re}\beta)\delta_0 + \mathrm{Re}\nu \leqslant \alpha\delta_0 + \mu$. This implies $\mathrm{Re}\beta \leqslant \alpha$ since $\mu(\{0\}) = \nu\{0\}) = 0$. ◻

Remark It is of interest to find a condition on B which implies that the semigroup $(S(t))_{t\geqslant0}$ is stable (see A-IV, Section 1). Using the positivity of $(T(t))_{t\geqslant0}$ it is shown in B-IV, Example 3.9, that $(T(t))_{t\geqslant0}$ is stable if and only if $\|\mu\| + \alpha < 0$. Since a semigroup which is dominated by a stable semigroup is itself stable we obtain from (4.13) that $(S(t))_{t\geqslant0}$ is stable if $\|\nu\| + \mathrm{Re}\beta < 0$.

We conclude this section discussing the following question. Let $(S(t))_{t\geqslant 0}$ be a semigroup which is dominated by some positive semigroup. Does there exist a smallest semigroup $(T(t))_{t\geqslant 0}$ which dominates $(S(t))_{t\geqslant 0}$? More precisely, we look for a positive semigroup $(T(t))_{t\geqslant 0}$ dominating $(S(t))_{t\geqslant 0}$ such that $(T(t))_{t\geqslant 0}$ is dominated by any other positive semigroup which dominates $(S(t))_{t\geqslant 0}$. If such a minimal dominating semigroup exists, it is unique and we call it the *modulus semigroup* of $(S(t))_{t\geqslant 0}$.

Example 4.15 (the modulus semigroup associated with $\Delta - V$). Let E be the complex space $L^p(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$ and $V \in L^p_{\text{loc}}(\mathbb{R}^n)$ satisfying $\text{Re}V \geqslant 0$. Denote by B the closure of $\Delta - V$ on C_c^∞ (cf. Example 4.7) . The modulus semigroup of the semigroup $(S(t))_{t\geqslant 0}$ generated by B exists and its generator A is given by $Af = \Delta f - (\text{Re}V)f$ for all $f \in C_c^\infty$ (and C_c^∞ is a core of A, see Example 4.7).

Proof The operator A defined above generates a positive semigroup (see Example 4.7). For $f \in C_c^\infty$, $\varphi \in D(A')_+$ one has

$$\begin{aligned}\text{Re}\langle(\text{sign}\,\bar f)Bf,\varphi\rangle &= \text{Re}\langle(\text{sign}\,\bar f)(\Delta f - Vf),\varphi\rangle\\ &= \text{Re}\langle(\text{sign}\,\bar f)\Delta f,\varphi\rangle - \langle(\text{Re}V)|f|,\varphi\rangle \quad = \text{Re}\langle(\text{sign}\,\bar f)Af,\varphi\rangle\\ &\leqslant \langle|f|,A'\varphi\rangle\end{aligned}$$

by Theorem 2.4 . Since C_c^∞ is a core of B, it follows from Theorem 4.3 that the semigroup generated by A dominates $(S(t))_{t\geqslant 0}$. Let C be the generator of a semigroup $(U(t))_{t\geqslant 0}$ dominating $(S(t))_{t\geqslant 0}$. Then

$$\begin{aligned}\text{Re}\langle(\text{sign}\,\bar f)Af,\varphi\rangle &= \text{Re}\langle(\text{sign}\,\bar f)\Delta f,\varphi\rangle - \langle(\text{Re}V)|f|,\varphi\rangle\\ &= \text{Re}\langle(\text{sign}\,\bar f)Bf,\varphi\rangle\\ &\leqslant \langle|f|,C'\varphi\rangle\end{aligned}$$

for all $f \in C_c^\infty$, $\varphi \in D(C')_+$ by Theorem 4.2 . It follows from Theorem 4.3 that $(U(t))_{t\geqslant 0}$ dominates the semigroup generated by A. ◻

Example 4.16 Let A_0 be the generator of a positive semigroup on an order complete Banach lattice E and $M \in \mathcal{Z}(E)$. The semigroup generated by $A_0 + M$ possesses a modulus semigroup. Its generator is $A_0 + \text{Re}M$. (This can be proved as the assertion in Example 4.15 .)

If a semigroup has a bounded regular generator, then it possesses a modulus semigroup. Its generator is bounded too (see C-I, Section 6 for the notion of regular operators).

Theorem 4.17 *Let B be a regular, bounded operator on an order complete complex Banach lattice E. The semigroup $(e^{tB})_{t\geqslant 0}$ possesses a modulus semigroup. Its generator is $A = |B_0| + \text{Re}M$, where $B = B_0 + M$ is the unique decomposition of B in $\mathcal{L}^r(E)$ satisfying $M \in \mathcal{Z}(E)$), $B_0 \in \mathcal{Z}(E)^d$.*

For the proof we need the following result which is of independent interest.

Lemma 4.18 *Let A be the generator of a positive semigroup on a Banach lattice E. If $Af \geqslant 0$ for all $f \in D(A)_+$, then A is bounded.*

Proof (Proof of Lemma 4.18) There exists $M \geqslant 1$ such that $\|R(\lambda, A)\| \leqslant M/\lambda$ for all $\lambda \geqslant \omega(A) + 1$. Fix $\mu \geqslant \omega(A) + 1$. Then $AR(\mu, A)Af = \mu R(\mu, A)Af - Af = \mu^2 R(\mu, A)f - \mu f - Af$; hence $0 \leqslant Af \leqslant \mu^2 R(\mu, A)f$ whenever $f \in D(A)_+$. Thus $\|Af\| \leqslant c\|f\|$ for all $f \in D(A)_+$ (where $c := \mu^2\|R(\mu, A)\|$). Consequently,

$$\|(\lambda R(\lambda, A) - \mathrm{Id})f\| = \|AR(\lambda, A)f\| \leqslant c\|R(\lambda, A)f\| \leqslant (Mc/\lambda)\|f\|$$

for all $f \in E_+$ and all $\lambda \geqslant \omega(A) + 1$. Hence

$$\|(\lambda R(\lambda, A) - \mathrm{Id})g\| \leqslant Mc/\lambda(\|g^+\| + \|g^-\|) \leqslant (2Mc/\lambda)\|g\| \quad \text{for all} \quad g \in E.$$

Thus $\lambda R(\lambda, A)$ is invertible if λ is large enough and $D(A) = \mathrm{im}(\lambda R(\lambda, A)) = E$. □

Proof (Proof of Theorem 4.17) Let $A = |B_0| + \mathrm{Re}M$. It has been shown in Example 4.4(ii) that $(\mathrm{e}^{tA})_{t\geqslant 0}$ dominates $(\mathrm{e}^{tB})_{t\geqslant 0}$. Let $(U(t))_{t\geqslant 0}$ be a positive semigroup dominating $(\mathrm{e}^{tB})_{t\geqslant 0}$ and C its generator. We first show that C is bounded. Let $f \in D(C)_+$. Then $\mathrm{Re}(Bf) = \lim_{t\to 0} 1/t(\mathrm{Re}(\mathrm{e}^{tB}f) - f) \leqslant \lim_{t\to 0} 1/t(U(t)f - f) = Cf$. Hence $(C + |B|)f \geqslant (C - \mathrm{Re}B)f \geqslant 0$ for all $f \in D(C)_+$. By Lemma 4.18 this implies that $C + |B|$ is bounded. Hence C is bounded as well. Since $C + \|C\| \cdot \mathrm{Id} \geqslant 0$ by Theorem 1.11, C is regular. Let $C = C_0 + N$ where $C_0 \in \mathcal{Z}(E)^d$ and $N \in \mathcal{Z}(E)$. Since $C \geqslant \mathrm{Re}B$ by what we just proved, it follows that $N \geqslant \mathrm{Re}M$. Let $f \in E_+$, $\varphi \in E'_+$ satisfy $\langle f, \varphi\rangle = 0$. Then for all $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\langle \mathrm{Re}(\mathrm{e}^{i\alpha}B)f, \varphi\rangle &= \lim_{t\to 0} 1/t\langle \mathrm{Re}(\mathrm{e}^{i\alpha}\mathrm{e}^{tB})f, \varphi\rangle \\
&\leqslant \lim_{t\to 0} 1/t\langle \mathrm{e}^{tC}f, \varphi\rangle \\
&= \langle Cf, \varphi\rangle.
\end{aligned}$$

Thus $C - \mathrm{Re}(\mathrm{e}^{i\alpha}B)$ satisfies the positive minimum principle (P) for all $\alpha \in \mathbb{R}$. It follows from Theorem 1.11 that $C - \mathrm{Re}(\mathrm{e}^{i\alpha}B) + (\|C\| + \|B\|)\,\mathrm{Id} \geqslant 0$ for all $\alpha \in \mathbb{R}$. Applying the band projection onto $\mathcal{Z}(E)^d$ on both sides of this inequality one obtains that

$$|B_0| = \sup_{\alpha\in\mathbb{R}} \mathrm{Re}(\mathrm{e}^{i\alpha}B) \leqslant C_0$$

(since $|T| = \sup_{\alpha\in\mathbb{R}} \mathrm{Re}(\mathrm{e}^{i\alpha}T)$ for all $T \in \mathcal{L}^r(E)$, see C-I, Section 7). We have proved that $\mathrm{Re}M \leqslant N$ and $|B_0| \leqslant C_0$. This implies that

$$\mathrm{Re}((\mathrm{sign}\,\bar{f})Bf) = \mathrm{Re}((\mathrm{sign}\,\bar{f})B_0 f) + (\mathrm{Re}M)|f| \leqslant C_0|f| + N|f| = C|f|$$

for all $f \in E$. It follows from Theorem 4.2 that $(\mathrm{e}^{tB})_{t\geqslant 0}$ is dominated by $(\mathrm{e}^{tC})_{t\geqslant 0}$. □

Remark The proof of Theorem 4.17 shows that any semigroup dominating a semigroup whose generator is bounded and regular has a bounded generator as well.

Example 4.19 Let $E = \ell^p$ $(1 \leqslant p < \infty)$ or c_0 and $B \in \mathcal{L}^r(E)$ be given by the matrix (b_{ij}). The generator A of the modulus semigroup of $(\mathrm{e}^{tB})_{t \geqslant 0}$ is given by the matrix (a_{ij}) where $a_{ij} = |b_{ij}|$ when $i \neq j$ and $a_{ii} = \mathrm{Re} b_{ii}$.

A related question is under which condition a semigroup $(S(t))_{t \geqslant 0}$ is dominated by some positive semigroup. Of course, a necessary condition is that every $S(t)$ is a regular operator. On an AL-space this condition is automatically satisfied. But Kipnis [16] gives an example of a strongly continuous semigroup on ℓ^1 which is not dominated. On the other hand, it has been independently shown by Kipnis [16] and Kubokawa [18] that every contraction semigroup on an L^1-space possesses a modulus semigroup (which is contractive as well).

5 Semigroups of Disjointness Preserving Operators

In this section we consider a special case of domination. Recall from C-I, Section 6 that a linear operator S on E is called *lattice homomorphism* if

$$|Sf| = S|f| \quad \text{for all } f \in E. \tag{5.1}$$

An operator $S \in \mathcal{L}(E)$ is called *disjointness preserving* if

$$f \perp g \quad \text{implies} \quad Sf \perp Sg \quad \text{for all } f, g \in E. \tag{5.2}$$

Note that an operator S is a lattice homomorphism if and only if S is positive and disjointness preserving.

In the following we will consider *disjointness preserving semigroups* (by this we mean semigroups of disjointness preserving operators) and *lattice semigroups* (i.e., semigroups of lattice homomorphisms). For example, the semigroup $(T_d(t))_{t \geqslant 0}$ defined in Section 3 after Theorem 3.13 is disjointness preserving for all $d \in \mathbb{R}$ and a lattice semigroup if $d \geqslant 0$.

Proposition 5.1 *A bounded operator S on a complex Banach lattice E is disjointness preserving if and only if there exists a linear operator $|S|$ on E such that*

$$|Sf| = |S||f| \quad (f \in E). \tag{5.3}$$

In that case the operator $|S|$ is uniquely determined by (5.3)*. $|S|$ is a lattice homomorphism and the* modulus *of S (i.e., one has $|S| \leqslant T$ for all $T \in \mathcal{L}(E)$ such that $|Sf| \leqslant T|f|$ $(f \in E)$).*

For the proof of the proposition we refer to Arendt [2] and de Pagter [9].

Proposition 5.2 *Let $(S(t))_{t\geqslant 0}$ be a disjointness preserving semigroup. Let $T(t) = |S(t)|$ $(t \geqslant 0)$. Then $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup.*

Proof Let $0 \leqslant s, t$ and $f \in E_+$. Then by (5.1), $T(s)T(t)f = T(s)|S(t)f| = |S(s)S(t)f| = |S(s+t)f| = T(s+t)f$. Since span $E_+ = E$, it follows that $(T(t))_{t\geqslant 0}$ is a semigroup. Moreover, for $f \in E_+$, $\lim_{t\to 0} T(t)f = \lim_{t\to 0} |S(t)f| = |f| = f$. This implies that $(T(t))_{t\geqslant 0}$ is strongly continuous. □

Example 5.3 Let $d \in \mathbb{C}$ and $S(t) = T_d(t)$ be given by 3.8 . Then $|T_d(t)| = T_{|d|}(t)$ $(t \geqslant 0)$.

Proposition 5.4 *Let B be the generator of a disjointness preserving semigroup $(S(t))_{t\geqslant 0}$ on a Banach lattice E. Then B is* local*; i.e.*

$$f \perp g \quad \textit{implies} \quad Bf \perp g \quad \textit{for all } f \in D(B), g \in E. \tag{5.4}$$

Proof Let $f \in D(B)$ and $g \in E$ such that $\inf\{|f|, |g|\} = 0$. Then

$$\begin{aligned} |1/t(S(t)f - f)| \wedge |g| &\leqslant |1/tS(t)f| \wedge |g| + 1/t|f| \wedge |g| \\ &= 1/t|S(t)f| \wedge |g| \\ &\leqslant 1/t|S(t)f| \wedge |S(t)g - g| + (1/t|S(t)f|) \wedge |S(t)g| \\ &= 1/t|S(t)f| \wedge |S(t)g - g| \\ &\leqslant |S(t)g - g|. \end{aligned}$$

Letting $t \to \infty$ one obtains $|Bf| \wedge |g| = 0$. □

We now describe the relation between the generator of a disjointness preserving semigroup and the generator of the modulus semigroup.

Theorem 5.5 *Assume that E is a complex Banach lattice with order continuous norm. Let $(S(t))_{t\geqslant 0}$ be a semigroup with generator B. The following assertions are equivalent.*

(a) *$(S(t))_{t\geqslant 0}$ is disjointness preserving.*

(b) *There exists a semigroup $(T(t))_{t\geqslant 0}$ with generator A such that*

$$f \in D(B) \textit{ implies } |f| \in D(A) \textit{ and } \operatorname{Re}((\widehat{\operatorname{sign}\bar{f}})Bf) = A|f|. \tag{5.5}$$

Moreover, if these equivalent conditions are satisfied, then $T(t) = |S(t)|$ for all $t \geqslant 0$.

Remark

i) By B-II, Lemma 2.9 the relation (5.5) is equivalent to

$$\langle \mathrm{Re}((\mathrm{sign}\,\bar{f})Bf), \varphi\rangle = \langle |f|, A'\varphi\rangle (f \in D(B), \varphi \in D(A')).$$

ii) It is remarkable that, in contrast with the situation considered in Theorem 3.8, here condition (b) implies the positivity of $(T(t))_{t\geqslant 0}$ without further assumptions.

The basic idea of the proof of Theorem 5.5 is to differentiate the equation $|S(t)f| = T(t)|f|$ (where $T(t) = |S(t)|$, cf. (5.3)). For that we need that the modulus function is differentiable. If $E = L^p(X, \Sigma, \mu)$ $(1 \leqslant p < \infty)$ this had been proved in Section 2 Example 2.3. We extend this result to Banach lattices with order continuous norm.

Proposition 5.6 *Let E be a real or complex Banach lattice with order continuous norm. Then the modulus function $\Theta\colon E \to E$ (given by $\Theta(h) = |h|$) is right-sided Gateaux differerentiable and*

$$D_g\Theta(f) = \mathrm{Re}((\widehat{\mathrm{sign}\,\bar{f}})g) \quad (f, g \in E). \tag{5.6}$$

Proof Let $f, g \in E$. Define $k\colon \mathbb{R} \to E$ by $k(t) = |f + tg| - |f|$. Then $k(0) = 0$ and k is convex (i.e., $k(\lambda s + (1-\lambda)t) \leqslant \lambda k(s) + (1-\lambda)k(t)$ for all $s, t \in \mathbb{R}, \lambda \in [0, 1]$). We show that

$$k(s)/s \leqslant k(t)/t \tag{5.7}$$

whenever $s < t$, $s, t \neq 0$.

First case: $s < t < 0$. Choose $\lambda = t/s \in (0, 1)$. Then $t = (1-\lambda)0+\lambda s$. Consequently, $k(t) \leqslant (1-\lambda)k(0) + \lambda k(s) = t/s\, k(s)$.

Second case: $s < 0 < t$. Let $0 < \lambda := t/(t-s) < 1$. Then $0 = \lambda s + (1-\lambda)t$. Hence $0 = k(0) \leqslant \lambda k(s) + (1-\lambda)k(t) = t/(t-s)\, k(s) - s/(t-s)\, k(t)$, which implies (5.7).

Third case: $0 < s < t$. Let $\lambda = s/t \in (0, 1)$. Then $s = (1-\lambda)0 + \lambda t$. Consequently, $k(s) \leqslant (1-\lambda)k(0) + \lambda k(t) = s/t\, k(t)$, which implies (5.7).

It follows from (5.7) that the net $(k(t)/t)_{t>0}$ is decreasing and bounded below (by $-k(-1)$, for instance). Since E has order continuous norm, it follows that $D_g\Theta(f) = \lim_{t\to 0+} k(t)/t$ exists.

It remains to show that $D_g\Theta(f) = \mathrm{Re}(\widehat{\mathrm{sign}\,\bar{f}})g$. First of all denote by P the band projection onto $\{f\}^{dd}$. Then it follows from the definition of $D_g\Theta(f)$ that $D_g\Theta(f) = PD_g\Theta(f) + (\mathrm{Id} - P)D_g\Theta(f) = D_{Pg}\Theta(f) + |(\mathrm{Id} - P)g|$. Thus it remains to show that

$$D_h\Theta(f) = \mathrm{Re}((\mathrm{sign}\,\bar{f})h) \quad \text{whenever} \quad h \in \{f\}^{dd}. \tag{5.8}$$

According to the Kakutani-Krein theorem there exists a compact space K such that $E_{|f|}$ can be identified with $C(K)$. Then by B-II, Lemma 2.4

$$\lim_{t\to 0+} 1/t(|f+th|-|f|)(x) = \mathrm{Re}(\mathrm{sign}(\overline{f(x)})h(x)) \quad (x \in K). \tag{5.9}$$

Let $\varphi \in E'_+$. Then φ restricted to $E_{|f|}$ can be identified with a regular Borel measure μ on $C(K)$. So it follows from (5.9) and the dominated convergence theorem that

$$\begin{aligned}\langle D_h\Theta(f), \varphi\rangle &= \lim_{t\to 0+} 1/t\langle(|f+th|-|f|), \varphi\rangle \\ &= \int_K \mathrm{Re}(\mathrm{sign}(\bar{f}(x)))h(x))\ \mathrm{d}\mu(x) \\ &= \langle \mathrm{Re}((\mathrm{sign}\,\bar{f})h), \varphi\rangle\end{aligned}$$

(the last identity holds since by the definition of sign $\bar{f} \in \mathcal{L}(E)$, we have $(\mathrm{sign}\,\bar{f})h \in E_{|f|} = C(K)$ whenever $h \in C(K)$ and $((\mathrm{sign}\,\bar{f})h)(x) = (\mathrm{sign}\,\overline{f(x)})h(x)$ (see C-I, Section 8)).
Consequently, $D_h\Theta(f) = \mathrm{Re}(\mathrm{sign}\,\bar{f})h$ whenever $h \in E_{|f|}$. Since $D_h\Theta(f)$ is continuous in h (in fact, $|D_h\Theta(f) - D_k\Theta(f)| \leqslant |h-k|$ for all $h, k \in E$) and $E_{|f|}$ is dense in $\{f\}^{dd}$, it follows that (5.8) holds for all $h \in \{f\}^{dd}$. □

Remark 5.7 a) By the same argument as given in the proof one sees that Θ is left-sided Gateaux differentiable and

$$D_g^-\Theta(f) = \mathrm{Re}((\mathrm{sign}\,\overline{f})g) - P_f^d|g|$$

for all $f, g \in E$, where $D_g^-\Theta(f) = \lim_{t\to 0} 1/t(\Theta(f+tg) - \Theta(f))$ and P_f^d denotes the band projection onto $\{f\}^d$. In particular,

$$D_g^+\Theta(f) = D_g^-\Theta(f) \quad \text{whenever} \quad g \in \{f\}^{dd}. \tag{5.10}$$

b) The proof of Proposition 5.6 shows that every convex function $\Theta\colon E \to \mathbb{R}_+$ (where E is a Banach lattice with order continuous norm) is right- (and left-) sided Gateaux differentiable [(cf. Arendt (1982)]).

Proof (Proof of Theorem 5.5) Assume that (a) holds. Let $f \in D(B)$. Then $S(t)f$ is differentiable at t. By the chain rule B-II, Proposition 2.3, $T(t)|f| = |S(t)f|$ is also differentiable and $\mathrm{d}/\mathrm{d}t|_{t=0}T(t)|f| = \mathrm{d}/\mathrm{d}t|_{t=0}|S(t)f| = \mathrm{Re}(\widehat{\mathrm{sign}\,\bar{f}})Bf$ (by Proposition 5.6 . Hence $|f| \in D(A)$ and $A|f| = \mathrm{Re}(\widehat{\mathrm{sign}\,\bar{f}})Bf$.

Conversely, assume that (b) holds. Let $s > 0, f \in E$. We show that $|S(s)f| = T(s)|f|$. This implies that $S(s)$ is disjointness preserving and $|S(s)| = T(s)$ (by Proposition 5.1 . Since $D(B)$ is dense we can assume that $f \in D(B)$. Let $\xi(t) = T(s-t)|S(t)f|$ $(t \in [0, s])$. Since by assumption $|S(t)f| \in D(A)$ one obtains

$$\begin{aligned} \mathrm{d}/\mathrm{d}t\, \xi(t) &= -AT(s-t)|S(t)f| + T(s-t)\,\mathrm{d}/\mathrm{d}r|_{r=t}|S(r)f| \\ &= -AT(s-t)|S(t)f| + T(s-t)(\mathrm{Re}(\widehat{\mathrm{sign}}\overline{S(t)f})BS(t)f) \\ &\quad \text{(by Proposition 5.6 and the chain rule B-II, Proposition 2.3)} \\ &= 0 \quad \text{by the assumption (b)}. \end{aligned}$$

Hence $\xi(0) = \xi(s)$; i.e., $|S(s)f| = T(s)|f|$. □

The case when $S(t) = T(t)$ ($t \geqslant 0$ is of special interest: it yields a characterization of generators of lattice semigroups.

Recall that if a semigroup $(T(t))_{t\geqslant 0}$ is positive, i.e., if

$$|T(t)f| \leqslant T(t)|f| \quad (f \in E), \tag{5.11}$$

then its generator A satisfies Kato's inequality. We now obtain from Theorem 5.5 the semigroup consists of lattice homomorphisms (i.e., the equality holds in (5.11)) if and only if A satisfies Kato's equality. The precise statement is the following.

Corollary 5.8 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach lattice E with order continuous norm. The following assertions are equivalent.*

(a) $(T(t))_{t\geqslant 0}$ *is a lattice semigroup.*

(b) $f \in D(A)$ *implies* $|f| \in D(A)$ *and* $\mathrm{Re}((\widehat{\mathrm{sign}}\,\bar{f})Af) = A|f|$

(c) $f \in D(A)$ *implies* $|f|, \bar{f} \in D(A)$ *and* $\mathrm{Re}((\mathrm{sign}\,\bar{f})Af) = A|f|$ *(Kato's equality).*

Proof The equivalence of (a) and (b) follows directly from Theorem 5.5 . If (a) holds, then A is local by Proposition 5.4 . Thus $(\mathrm{sign}\,\bar{f})Af = (\widehat{\mathrm{sign}}\,\bar{f})Af$ for all $f \in D(A)$ and so (c) holds since (b) is valid.

Assume now that (c) holds. Then Kato's equality implies that $Af \in \{f\}^{dd}$ whenever $f \in D(A)_+$. Since $D(A)$ is a sublattice of E by hypothesis, this implies that A is local, Thus (b) follows from (c) . □

In the case when E is real this result can be reformulated.

Corollary 5.9 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a real Banach lattice E with order continuous norm. The following assertions are equivalent.*

(a) $(T(t))_{t\geqslant 0}$ *is a lattice semigroup.*

(b) $D(A)$ *is a sublattice and A is local.*

Proof Assume that (b) holds. Let $f \in D(A)$ and $P_+ := P_{f^+}$ and $P_- := P_{f^-}$. Then $(P_+)Af^- = (P_-)Af^+ = 0$ since A is local. Hence $(\operatorname{sign} f)Af = (P_+ - P_-)Af = (P_+ - P_-)(Af^+ + Af^-) = (P_+)Af^+ + (P_-)Af^- = Af^+ + Af^- = A|f|$ □

Thus Kato's equality holds and it follows from Corollary 5.8 that $(T(t))_{t\geqslant 0}$ is a lattice semigroup. The other implication follows directly from Corollary 5.8 .

Example 5.10 Let $E = L^p(X,\mu)$ (where (X,μ) is a σ-finite measure space and $1 \leqslant p < \infty$) and let A_0 be the generator of a semigroup of lattice homomorphisms. Let $h \in L^\infty$ and $B = A_0 + h$ (i.e., B is given by $Bf = A_0 f + h \cdot f$ for $f \in D(B) = D(A_0)$). Let $A = A_0 + \operatorname{Re} h$. Since A_0 generates a semigroup of lattice homomorphisms, we have $|f| \in D(A_0)$ whenever $f \in D(A_0)$ and $\operatorname{Re}((\operatorname{sign} f)A_0 f) = A_0|f|$. Hence $\operatorname{Re}((\widehat{\operatorname{sign} \bar{f}})Bf) = \operatorname{Re}((\widehat{\operatorname{sign} \bar{f}})A_0 f) + (\operatorname{Re} h)\cdot|f|) = A_0|f| + (\operatorname{Re} h)\cdot |f| = A|f|$ for all $f \in D(B)$. Thus it follows from Theorem 5.5 that B generates a disjointness preserving semigroup whose modulus semigroup is generated by A.

Next we describe when a disjointness preserving semigroup is positive.

Proposition 5.11 *Let E be a complex Banach lattice with order continuous norm and B be the generator of a disjointness preserving semigroup $(S(t))_{t\geqslant 0}$. The semigroup is positive if and only if B is real and $\operatorname{span} D(B)_+ = D(B)$.*

Proof The conditions are clearly necessary. In order to prove sufficiency, we can assume that E is real. Denote by A the generator of $(T(t))_{t\geqslant 0}$, where $T(t) = |S(t)|$. Let $f \in D(B)_+$. Since B is local we have $Bf = P_f Bf = (\operatorname{sign} f)Bf = A|f| = Af$. By assumption, $\operatorname{span} D(B)_+ = D(B)$. Thus it follows that $B \subset A$. This implies that $B = A$ since $\varrho(B) \cap \varrho(A) \neq \emptyset$. □

Remark 5.12 If B is the generator of a disjointness preserving semigroup $(S(t))_{t\geqslant 0}$ on a real Banach lattice E with order continuous norm then Kato's inequality holds in the reverse sense; i.e.,

$$\langle (\operatorname{sign} f)Bf, \varphi\rangle \geqslant \langle |f|, B'\varphi\rangle \text{ for all } f \in D(B), \varphi \in D(B')_+.$$

(cf. (3.9) for a concrete example). In fact, let $T(t) = |S(t)|$ and denote by A the generator of $(T(t))_{t\geqslant 0}$. Let $f \in D(B)$, $\varphi \in D(B')_+$. Then

$$\begin{aligned}
\langle (\operatorname{sign} f)Bf, \varphi\rangle &= \langle A|f|, \varphi\rangle \\
&= \lim_{t\to 0}(1/t)\langle T(t)|f| - |f|, \varphi\rangle \\
&\geqslant \lim_{t\to 0} 1/t\langle S(t)|f| - |f|, \varphi\rangle \\
&= \langle |f|, B'\varphi\rangle.
\end{aligned}$$

Finally, we come back to Corollary 5.9 . If in condition (b) we demand that $D(A)$ is not only a sublattice but an ideal of E we obtain a characterization of multiplication semigroups.

Here we call a semigroup $(T(t))_{t\geqslant 0}$ *multiplication semigroup* if $T(t)$ is a multiplication operator (i.e., an element of the center) for every $t > 0$.

Theorem 5.13 *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a σ-order complete real or complex Banach lattice E. The following assertions are equivalent.*

(a) *$(T(t))_{t\geqslant 0}$ is a multiplication semigroup.*

(b) *There exists $\lambda \in \varrho(A)$ such that $R(\lambda, A)$ is a multiplication operator.*

(c) *$R(\lambda, A)$ is a multiplication operator for all $\lambda \in \varrho(A)$*

(d) *A is local and $D(A)$ is an ideal in E*

(e) *If $f \in D(A)$ then $Pf \in D(A)$ for every band projection P on E and $APf = PAf$.*

Proof Assume that (a) holds and let $\lambda \in \varrho(A)$. Since $R(\lambda, A)$ is the Laplace transform of the semigroup, it follows that $R(\lambda, A)$ is local since $T(t)$ is local for all $t \geqslant 0$. This implies $R(\lambda, A) \in \mathcal{Z}(E)$ (see C-I, Section 9).

We show that (b) implies (e) . Assume that $\lambda \in \varrho(A)$ such that $R(\lambda, A)$ is a multiplication operator. Let P be a band projection. Then $PR(\lambda, A) = R(\lambda, A)P$. Let $f \in D(A)$, $g := (\lambda - A)f$. Then $Pf = PR(\lambda, A)g = R(\lambda, A)Pg$. Hence $Pf \in D(A)$ and $(\lambda - A)Pf = Pg$. Thus $APf = \lambda Pf - Pg = P(\lambda f - g) = PAf$.

We show that (e) implies (c) . Let $\lambda \in \varrho(A)$ and P be a band projection. We have to show that $PR(\lambda, A) = R(\lambda, A)P$. Let $g \in E$, $f := R(\lambda, A)g$. Then $Pf \in D(A)$ and $APf = PAf$. Hence $PR(\lambda, A)g = Pf = R(\lambda, A)P(\lambda - A)f = R(\lambda, A)Pg$. It follows from C-I, Section 9 that $R(\lambda, A) \in \mathcal{Z}(E)$.

(c) implies (a) since $T(t) = \lim_{n\to\infty}[n/t\, R(n/t, A)]^n$ strongly for all $t > 0$.

It remains to show the equivalence of (d) and (e) . Assume that (d) holds, let $f \in D(A)$ and P be a band projection. Then $Pf \in D(A)$ and $(Id - P)f \in D(A)$ by the assumption. Since A is local we have

$$APf = PAPf + (Id - P)APf = PAPf = PAPf + PA(Id - P)f = PAf.$$

Conversely, assume (e) . Let $f \in D(A)$ and $|g| \leqslant |f|$. Then there exists a band projection P such that $Pf = g$. Hence $g \in D(A)$. We have shown that $D(A)$ is an ideal. Assume that $\inf\{|h|, |f|\} = 0$. Denote by P the band projection onto $\{|h|\}^{dd}$. Then $PAf = APf = A0 = 0$. Thus $Af \in \{|h|\}^d$. We have proved that A is local. □

Corollary 5.14 *A multiplication semigroup $(T(t))_{t\geqslant 0}$ on a complex Banach lattice E with order continuous norm is positive if and only if its generator A is real; i.e., $f \in D(A)$ implies $\bar{f} \in D(A)$ and $A\bar{f} = \overline{(Af)}$.*

Proof The condition is equivalent to $T(t)E_{\mathbb{R}} \subset E_{\mathbb{R}}$ $(t \geqslant 0)$ (cf. Remark 3.1), so it is clearly necessary. Conversely, if A is real, then denote by $(T_{\mathbb{R}}(t))_{t\geqslant 0}$ the restriction semigroup on $E_{\mathbb{R}}$ and by $A_{\mathbb{R}}$ its generator. Then $A_{\mathbb{R}}$ is local (since A is local) and $D(A_{\mathbb{R}})$ is a sublattice of $E_{\mathbb{R}}$. Thus $(T_{\mathbb{R}}(t))_{t\geqslant 0}$ is a lattice semigroup (and so positive) by Corollary 5.9 . □

The class of bounded operators which generate a lattice semigroup is very restricted.

Proposition 5.15 *Let E be a real or complex Banach lattice and $A \in \mathcal{L}(E)$. The following assertions are equivalent.*

(a) $A \in \mathcal{Z}(E)$

(b) e^{tA} *is disjointness preserving for all* $t \geqslant 0$

(c) $\mathrm{e}^{tA} \in \mathcal{Z}(E)$ *for all* $t \in \mathbb{R}$.

Moreover, if $A \in \mathcal{Z}(E)$ is real, then $\mathrm{e}^{tA} \geqslant 0$ for all $t \in \mathbb{R}$.

Proof Since $\mathcal{Z}(E)$ is a closed subalgebra of $\mathcal{L}(E)$ (see C-I, Section 9) , it is clear that (a) implies (c) . Assertion (b) follows trivially from (c) . If (b) holds, then A is local by Proposition 5.4 . Hence $A \in \mathcal{Z}(E)$

The last assertion follows from the fact that $\mathcal{Z}(E)$ is isomorphic to a space $C(K)$ as a Banach lattice and a Banach algebra. □

Proposition 5.16 *Let E be a complex Banach lattice. Every strongly continuous group $(T(t))_{t\in\mathbb{R}}$ of real operators contained in $\mathcal{Z}(E)$ has a bounded generator.*

Proof Let $(T(t))_{t\geqslant 0}$ be a strongly continuous multiplication semigroup. There exist $\omega \in \mathbb{R}$, $M \geqslant 1$ such that $\|T(t)\| \leqslant M\mathrm{e}^{\omega|t|}$ $(t \geqslant 0)$. Then $\|f\|_1 := \sup_{t\geqslant 0} \|\mathrm{e}^{-\omega t}T(t)f\|$ defines an equivalent lattice norm on E for which $\|T(t)\|_1 \leqslant \mathrm{e}^{\omega t}$ $(t \geqslant 0)$. Since $\mathcal{Z}(E)$ is isometrically isomorphic to a space $C(K)$ (as a Banach lattice), for an operator $S \in \mathcal{Z}(E)$ one has $\|S\| = \inf\{c > 0\colon |S| \leqslant c \cdot Id\}$. Hence the operator norm of S is independent of which lattice norm equivalent to the given one is considered on E. Consequently, $\|T(t)\| = \|T(t)\|_1 \leqslant \mathrm{e}^{\omega t}$ $(t \geqslant 0)$.

If $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group contained in $\mathcal{Z}(E)$ then it follows that $\|T(t)\| \leqslant \mathrm{e}^{\omega|t|}$ $(t \in \mathbb{R})$ for some $\omega \geqslant 0$. If in addition the operators $T(t)$ are real one obtains from the above expression for the operator norm that

$$\mathrm{e}^{-\omega t} \cdot \mathrm{Id} \leqslant T(t) \leqslant \mathrm{e}^{\omega t} \cdot \mathrm{Id} \quad (t \geqslant 0).$$

Consequently, $\lim_{t\to 0} \|T(t) - Id\| = 0$. □

The assumption that the group consists of real operators is essential in Proposition 5.16 . In fact, many differential operators on $L^2(\mathbb{R}^n)$ generate a strongly continuous group which (via Fourier transformation) is similar to a multiplication group. A concrete example is the Laplacian (A-I, Example 2.8).

On the other hand, if $E = C(K)$ (K compact), then every strongly continous multiplication semigroup $(T(t))_{t\geqslant 0}$ has a bounded generator.

[In fact, let $m_t := T(t)1$ $(t \neq 0)$. Then $\lim_{t\downarrow 0} \|T(t) - \mathrm{Id}\| = \lim_{t\downarrow 0} \|m_t - 1\|_\infty = 0$.]

Lemma 5.17 *Let E be a real Banach lattice with order continuous norm. Let $A \in \mathcal{L}(E)$. Assume that there exists a dense sublattice D of E such that for all $f \in D$, $g \in E$, $f \perp g$ implies $Af \perp g$. Then $A \in \mathcal{Z}(E)$.*

Proof Let $0 \leqslant f \in D$, $\phi \in E^*$ such that $\langle f, \varphi\rangle = 0$. Since $Af \in (f)^{dd}$ by assumption, it follows that $\langle Af, \varphi\rangle = 0$. Thus $A|_D$ and $-A|_D$ satisfy (P). It follows from Theorem 1.8 that $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$ is a positive group. Thus $A \in \mathcal{Z}(E)$ by Proposition 5.15
. □

Let A be the generator of a positive semigroup and $B \in \mathcal{L}(E)$. The semigroup generated by $A+B$ is positive whenever $(\mathrm{e}^{tB})_{t\geqslant 0}$ is positive (this follows from 1.8). However this condition is not necessary. [For example, let $A \in \mathcal{L}(E)$ such that $(\mathrm{e}^{tA})_{t\geqslant 0}$ is positive and let $B = -A$. Then $A+B$ generates a positive semigroup, but $(\mathrm{e}^{tB})_{t\geqslant 0}$ is positive only if $A \in \mathcal{Z}(E)$] The situation is different when A generates a lattice semigroup.

Theorem 5.18 *Let E be a real Banach lattice with order continuous norm and A be the generator of a lattice semigroup. Let $B \in \mathcal{L}(E)$. The semigroup generated by $A + B$ is positive if and only if $(\mathrm{e}^{tB})_{t\geqslant 0}$ is positive. The semigroup generated by $A + B$ is a lattice semigroup if and only if $B \in \mathcal{Z}(E)$.*

Proof Assume that $A + B$ generates a positive semigroup. Let $f \in D(A)_+$, $\varphi \in E^*_+$ such that $\langle f, \varphi\rangle = 0$. Since A is local, it follows that $\langle Af, \varphi\rangle = 0$.
But $\langle (A + B)f, \varphi\rangle \geqslant 0$ by Proposition 1.7 . Hence $\langle Bf, \varphi\rangle \geqslant 0$. We have shown that $B|_{D(A)}$ satisfies the positive minimum principle (Definition 1.6). Since $D(A)$ is a sublattice of E (by Corollary 5.9 , it follows from Theorem 1.8 that $(\mathrm{e}^{tB})_{t\geqslant 0}$ is positive.

By Corollary 5.9 the operator $A + B$ generates a lattice semigroup if and only if $A + B$ is local. Since A is local, this is equivalent to $B|_{D(A)}$ being local. By Lemma 5.17 this is true if and only if $B \in \mathcal{Z}(E)$. □

Notes

Section 1: The notion of dispersiveness is due to Phillips [26] who uses a semi-scalar product instead of the subdifferential of the canonical half-norm. Our approach follows Arendt et al. [4]. Bounded generators of positive semigroups on a special class of ordered Banach spaces (which includes Banach lattices and C*-algebras) were characterized by the positive minimum principle by Evans and Hanche-Olsen [11]. The equivalence of (a) and (d) in Theorem 1.11 is due to Nagel and Uhlig [22]. Theorem 1.8 has been obtained independently by Arendt (1984a) and van Casteren (1984).

Section 2: The classical distributional Kato's inequality for the Laplacian is due to Kato [14]. It is a most elegant tool to prove essential selfadjointness of Schrödinger operators with domain $C_0^\infty(\mathbb{R}^n)$ (cf. Example 4.7).

The relation between Kato's inequality and positivity of $(e^{t\Delta})_{t\geqslant 0}$ was pointed out by Simon [30]. A criterion for a form negative operator on a space L^2 to generate a positive semigroup is given by Beurling and Deny [7], see also Reed and Simon [27, Vol. IV, Section XIII.12]. It was a conjecture of Nagel that some abstract version of Kato's inequality characterizes the positivity of the semigroup (cf., Nagel and Uhlig [22]). The necessity of Kato's inequality in the form given in Theorem 2.4 was first proved in Arendt [1, Remark 3.10] with a different proof. The proof we give here appeared in Arendt [3]. Miyajima and Okazawa [21] use this inequality to show that a differential operator on which generates a positive semigroup is necessarily of order 2 and has an elliptic principal part. This result is generalized to the spaces $L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$ suitable, by Miyajima [20].

Section 3: In this section we closely follow Arendt [3]. Theorem 3.8 , in a similar form but with different proof, has been obtained independently by Schep [29].

Section 4: The characterization of domination by Kato's inequality on a Hilbert space is due to Simon [30]. Further contributions are due to Hess et al. [13] and Kishimoto and Robinson [17]. Theorem 4.3 is due to [Arendt (1984b)]. The result on Schrödinger operators on $L^p(\mathbb{R}^n)$ stated in Example 4.7 is due to Kato [15]. The case $p = 2$ was proved in Kato [14], where the classical Kato's inequality was established. Extensive information on Schrödinger semigroups on $L^p(\mathbb{R}^n)$ is given in Simon [31]. Other recent results on the L^p-theory of Schrödinger operators are obtained by Davies [8], Okazawa [24] and Voigt [32].

The existence of the modulus semigroup of semigroups with bounded, regular generator (Theorem 4.17) is due to Derndinger [10] (in the real case).

Proposition 5.15 had been proved in Schaefer et al. [28] by a completely different method.

Section 5: The characterization of generators of lattice semigroups on a Banach lattice with order continuous norm (Corollary 5.8) is due to Nagel and Uhlig [22]. An extension of this result to arbitrary Banach lattices is given by Arendt [1] from which the proof of Proposition 5.6 is taken as well.

Local closed operators having an ideal as domain (i.e., operators satisfying condition (d) of Theorem 5.13 are investigated in detail by Nakano [23] who calls them dilatators. Peetre [25] characterizes differential operators by locality (see also Luxemburg [19]). In the context of C*-algebras local operators are investigated by **?**] and Batty and Robinson [6].

References

[1] W. Arendt. Kato's equality and spectral decomposition for positive C_0-groups. *Manuscripta Math.*, 40:277–298, 1982.

[2] W. Arendt. Spectral properties of Lamperti operators. *Indiana Univ. Math. J.*, 32:199–215, 1983.

[3] W. Arendt. Generators of positive semigroups. In F. Kappel and W. Schappacher, editors, *Infinite-dimensional Systems, Retahof 1983*, volume 1076 of *Lecture Notes in Mathematics*, pages 1–15. Springer, Berlin-Heidelberg-New York, 1984.

[4] W. Arendt, P. Chernoff, and T. Kato. A generalization of dissipativity and positive semigroups. *J. Operator Theory*, 8:167–180, 1982.

[5] C. Batty and E. Davies. Positive semigroups and resolvents. *J. Operator Theory*, 10:357–363, 1983.

[6] C. Batty and D. Robinson. The characterization of differential operators by locality: abstract derivations. *Ergodic Theory Dynam. Systems*, 5:171–183, 1985.

[7] A. Beurling and J. Deny. Espaces de Dirichlet I: Le cas élémentaire. *Acta Math.*, 99:203–224, 1958.

[8] E. Davies. Spectral properties of some second order elliptic operators on L^p-spaces. In R. Nagel, U. Schlotterbeck, and M. Wolff, editors, *Aspects of Positivity in Functional Analysis*. North Holland, Amsterdam, 1986.

[9] B. de Pagter. A note on disjointness preserving operators. *Proc. Amer. Math. Soc.*, 90:543–550, 1984.

[10] R. Derndinger. Betragshalbgruppen normstetiger Operatorhalbgruppen. *Arch. Math.*, 42:371–375, 1984.

[11] D. Evans and H. Hanche-Olsen. The generators of positive semigroups. *J. Funct. Anal.*, 32:207–212, 1979.

[12] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin, 1977.

[13] H. Hess, R. Schrader, and D. A. Uhlenbrock. Domination of semigroups and generalization of Kato's inequality. *Duke Math. J.*, 44:893–904, 1977.

[14] T. Kato. Schrödinger operators with singular potentials. *Israel J. Math.*, 13: 135–148, 1973.

[15] T. Kato. L^p-theory of Schrödinger operators with a singular potential. In R. Nagel, U. Schlotterbeck, and M. P. H. Wolff, editors, *Aspects of Positivity in Functional Analysis*. North Holland, Amsterdam, 1986.

[16] C. Kipnis. Majoration des semi-groupes de contractions de L^1 et applications. *Ann. Inst. H. Poincaré Sect. B*, 10:369–384, 1974.

[17] A. Kishimoto and D. W. Robinson. Positivity and monotonicity properties of C_0-semigroups, II. *Comm. Math. Phys.*, 75:85–101, 1980.

[18] Y. Kubokawa. Ergodic theorems for contraction semigroups. *J. Math. Soc. Japan*, 27:184–193, 1975.

[19] W. A. J. Luxemburg. *Some Aspects of the Theory of Riesz Spaces*, volume 4 of *The University of Arkansas Lecture Notes in Mathematics*. Fayetteville, 1979.

[20] S. Miyajima. Generators of positive C_0-semigroups. In R. Nagel, U. Schlotterbeck, and M. P. H. Wolff, editors, *Aspects of Positivity in Functional Analysis*. North Holland, Amsterdam, 1986.

[21] S. Miyajima and N. Okazawa. Generators of positive C_0-semigroups on Banach lattices. Preprint, 1984.

[22] R. Nagel and H. Uhlig. An abstract Kato inequality for generators of positive semigroups on Banach lattices. *J. Operator Theory*, 6:113–123, 1981.

[23] H. Nakano. *Modern Spectral Theory*, volume II of *Tokyo Mathematical Book Series*. Maruzen Co., Tokyo, 1950.

[24] N. Okazawa. An L^p-theory for Schrödinger operators with nonnegative potentials. *J. Math. Soc. Japan*, 36:675–688, 1984.

[25] J. Peetre. Une charactérisation abstraite des opérateurs différentiels. *Math. Scand.*, 7:211–218, 1959. Rectification à l'article précédent. Math. Scand. 8 (1960), 116–120.

[26] R. S. Phillips. Semigroups of positive contraction operators. *Czechoslovak Math. J.*, 12:294–313, 1962.

[27] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV. Analysis of Operators*. Academic Press, New York, 1978.

[28] H. H. Schaefer, M. P. H. Wolff, and W. Arendt. On lattice isomorphisms with positive real spectrum and groups of positive operators. *Math. Z.*, 164:115–123, 1978.

[29] A. R. Schep. Weak Kato-inequalities and positive semigroups. *Math. Z.*, 190: 305–314, 1985.

[30] B. Simon. An abstract Kato's inequality for generators of positivity preserving semigroups. *Indiana Univ. Math. J.*, 26:1067–1073, 1977.

[31] B. Simon. Schrödinger semigroups. *Bull. Amer. Math. Soc.*, 7:447–526, 1982.

[32] J. Voigt. Positivity in time dependent linear transport theory. *Acta Appl. Math.*, 2:311–331, 1984.

Part D

Positive Semigroups on C*- and W*-Algebras

Chapter D-I

Basic Results on Semigroups and Operator Algebras

by

Ulrich Groh

This is not a systematic introduction to the theory of strongly continuous semigroups on C*- and W*-algebras. We only prepare for the following chapters on spectral and asymptotic theory by fixing the notations and introducing some standard constructions. For results on strongly continuous semigroups on Banach spaces, we refer to Chapter A-I.

1 Notations

1. Let M denote a C*-algebra with unit $\mathbb{1}$, where $M^{sa} := \{x \in M : x^* = x\}$ is the self-adjoint part of M and $M_+ := \{x^*x : x \in M\}$ is the positive cone in M. If M' is the dual of M, then $M'_+ := \{\varphi \in M' : \varphi(x) \geqslant 0, x \in M_+\}$ is a weak*-closed generating cone in M' and $S(M) := \{\varphi \in M'_+ : \varphi(\mathbb{1}) = 1\}$ is called the state space of M. For the theory of C*-algebras and related notions see Pedersen [5].

2. We say that M is a W*-algebra if there exists a Banach space M_* such that its dual $(M_*)'$ is (isomorphic to) M. We call M_* the *predual* of M and $\varphi \in M_*$ a *normal linear functional*. It is known that M_* is unique. For this and other properties of M_*, see Takesaki [6, Chapter III].

3. A map $T \in \mathcal{L}(M)$ is called *positive* (in symbols $T \geqslant 0$) if $T(M_+) \subseteq M_+$. It is called *n-positive* ($n \in \mathbb{N}$) if $T \otimes \mathrm{Id}_n$ is positive from $M \otimes M_n$ in $M \otimes M_n$, where Id_n is the identity map on the C*-algebra M_n of all $n \times n$-matrices. Obviously, every n-positive map is positive.

We call a contraction $T \in \mathcal{L}(M)$ a *Schwarz map* if T satisfies the so called *Schwarz-inequality*

$$T(x)T(x)^* \leqslant T(xx^*)$$

for all $x \in M$. It is well known that every n-positive contraction, for $n \geqslant 2$ and every positive contraction on a commutative C*-algebra is a Schwarz map. (Takesaki [6, Chapter IV]) As we shall see, the Schwarz inequality is crucial for our investigations.

4. If M is a C*-algebra, we assume that $\mathcal{T} = (T(t))_{t \geqslant 0}$ is a strongly continuous semigroup (abbreviated as semigroup), while for W*-algebras we consider weak*-semigroups, i.e. the mapping $(t \mapsto T(t)x)$ is continuous from $\mathbb{R}_+$ into $(M, \sigma(M, M_*))$, where M_* is the predual of M, and every $T(t) \in \mathcal{T}$ is $\sigma(M, M_*)$-continuous. Note that the preadjoint semigroup

$$\mathcal{T}_* = \{T(t)_* : T(t) \in \mathcal{T}\}$$

is weakly, hence strongly continuous on M_*. (Chapter A-I, Proposition 1.2)

5. We call the semigroup $\mathcal{T}$ *identity preserving* if $T(t)\mathbb{1} = \mathbb{1}$ and of *Schwarz type* if every $T(t)$ is a Schwarz map.

For the notations concerning one-parameter semigroups we refer to Part A. In addition, we recommend to compare the results of this section with the corresponding results for commutative C*-algebras, i.e., for $C_0(X)$, $C(K)$ and $L^\infty(\mu)$ in Part B.

2 A Fundamental Inequality for the Resolvent

If $\mathcal{T} = (T(t))_{t \geqslant 0}$ is a strongly continuous semigroup of Schwarz maps on a C*-algebra M (resp. a weak*-semigroup of Schwarz type on a W*-algebra M) with generator A, then the spectral bound satisfies $s(A) \leqslant 0$. The resolvent $R(\lambda, A)$ exists for $\mathrm{Re}(\lambda) > 0$ and is positive for $\lambda \in \mathbb{R}_+$. There exists a representation for the resolvent $R(\lambda, A)$ given by the formula

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t \quad (x \in M)$$

where the integral exists in the norm topology.

The next theorem relates the domination of two semigroups to an inequality for the corresponding resolvent operators. This inequality will be needed later and can be used to characterize semigroups of Schwarz type on C*-algebras.

Theorem 2.1 *Let $\mathcal{T} = (T(t))_{t \geqslant 0}$ be a semigroup of Schwarz type with generator A and $\mathcal{S} = (S(t))_{t \geqslant 0}$ a semigroup with generator B on a* C*-*algebra M. If*

$$(S(t)x)(S(t)x)^* \leqslant T(t)(xx^*) \tag{*}$$

for all $x \in M$ and $t \in \mathbb{R}_+$. Then

$$(\mu R(\mu, B)x)\,(\mu R(\mu, B)x)^* \leqslant \mu R(\mu, A)xx^*$$

for all $x \in M$ and $\mu \in \mathbb{R}_+$. The same result holds if $\mathcal{T}$ is a weak-semigroup of Schwarz type and $\mathcal{S}$ is a weak*-semigroup on a* W^**-algebra M such that $(*)$ is fulfilled.*

Proof From the assumption $(*)$ it follows that

$$\begin{aligned} 0 \leqslant (S(r)x - S(t)x)\,(S(r)x - S(t)x)^* & \\ &= (S(r)x)(S(r)x)^* - (S(r)x)(S(t)x)^* \\ &\quad - (S(t)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \\ &\leqslant T(r)xx^* + T(t)xx^* - (S(r)x)(S(t)x)^* - (S(t)x)(S(r)x)^* \end{aligned}$$

for every $r, t \in \mathbb{R}_+$ and therefore

$$(S(r)x)(S(r)x)^* + (S(t)x)(S(t)x)^* \leqslant T(r)xx^* + T(t)xx^*.$$

Obviously, $\|S(t)\| \leqslant 1$ for all $t \in \mathbb{R}_+$. Then for all $\mu \in \mathbb{R}_+$ and $x \in M$

$$\begin{aligned} (R(\mu, B)x)\,(R(\mu, B)x)^* &= \left(\int_0^\infty e^{-\mu r} S(r)x\,\mathrm{d}r\right)\left(\int_0^\infty e^{-\mu t} S(t)x\,\mathrm{d}t\right)^* \\ &= \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}((S(r)x)(S(t)x)^* + (S(t)x)(S(r)x)^*\,\mathrm{d}r\,\mathrm{d}t\right) \\ &\leqslant \frac{1}{2}\left(\int_0^\infty \int_0^\infty e^{-\mu(r+t)}(T(r)xx^* + T(t)xx^*)\,\mathrm{d}r\,\mathrm{d}t\right) \\ &= \left(\int_0^\infty e^{-\mu s} ds\right)\left(\int_0^\infty e^{-\mu t} T(t)xx^*\,\mathrm{d}t\right) = \mu^{-1} R(\mu, A)xx^* \end{aligned}$$

where the handling of the integral is justified by Bourbaki [1, Chap. V, §8, n° 4, Proposition 9]. The claim is obtained by multiplying both sides by μ^2. ◻

Corollary 2.2 *Let $\mathcal{T}$ be a semigroup of Schwarz maps (resp. weak*-semigroup of Schwarz maps). Then for all $\lambda \in \mathbb{C}$ with* $\mathrm{Re}\,(\lambda) > 0$ *we have*

$$(R(\lambda, A)x)\,(R(\lambda, A)x)^* \leqslant \mathrm{Re}(\lambda)^{-1} R(\mathrm{Re}(\lambda), A)xx^*\,,\ x \in M.$$

In particular for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $x \in M$

$$(\mu R(\mu + \mathrm{i}\alpha, A)x)(\mu R(\mu + \mathrm{i}\alpha, A)x)^* \leqslant \mu R(\mu, A)(xx^*).$$

Proof Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Then the semigroup

$$S := \left(e^{-\mathrm{i}(\lambda)t} T(t)\right)_{t \geqslant 0}$$

fulfills the assumption of Thm. 2.1 and $B := A - \mathrm{i}\lambda$ is the generator of S. Consequently $R(\lambda, A) = R(\mathrm{Re}\lambda, B)$ and the corollary follows from Thm. 2.1. □

Remark 2.3 Since

$$T(t)x = \lim_n \left(\frac{n}{t} R\left(\frac{n}{t}, A\right)\right)^n x, \quad x \in M,$$

it follows from above, that $\mathcal{T}$ is a semigroup of Schwarz-type, if and only if $\mu R(\mu, A)$ is a Schwarz-operator for every $\mu \in \mathbb{R}_+$.

As in Section C-III the following notion will be an important tool for the spectral theory of semigroups on C*- and W*-algebras.

Definition 2.4 Let E be a Banach space and let D be a non-empty open subset of $\mathbb{C}$. A family $\mathcal{R}\colon D \mapsto L(E)$ is called a *pseudo-resolvent* on D with values in E if

$$R(\lambda) - R(\mu) = -(\lambda - \mu) R(\lambda) R(\mu) \qquad \text{(Resolvent Equation)}$$

for all λ, μ in D and $R \in \mathcal{R}$.

If $\mathcal{R}$ is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon \mathrm{Re}(\lambda) > 0\}$ with values in a C*- or W*-algebra, then $\mathcal{R}$ is called of Schwarz type if

$$(R(\lambda)x)(R(\lambda)x)^* \leqslant (\mathrm{Re}\lambda)^{-1} R(\mathrm{Re}\lambda) x x^*$$

and *identity preserving* if $\lambda R(\lambda)\mathbb{1} = \mathbb{1}$ for all $\lambda \in D$ and $R \in \mathcal{R}$. For examples and properties of a pseudo-resolvent, see C-III, 2.5.

We state what will be used without further reference.

(i) If $\alpha \in \mathbb{C}$ and $x \in E$ such that $(\alpha - \lambda)R(\lambda)x = x$ for some $\lambda \in D$, then $(\alpha - \mu)R(\mu)x = x$ for all $\mu \in D$ (use the *resolvent equation*).

(ii) If F is a closed subspace of E such that $R(\lambda)F \subseteq F$ for some $\lambda \in D$, then $R(\mu)F \subseteq F$ for all μ in a neighborhood of λ. This follows from the fact that for all $\mu \in D$ near λ the pseudo-resolvent in μ is given by

$$R(\mu) = \sum_n (\lambda - \mu)^n R(\lambda)^{n+1}.$$

Definition 2.5 We call a semigroup $\mathcal{T}$ on the *predual* M_* of a W*-algebra M *identity preserving and of Schwarz type* if its adjoint weak*-semigroup has these properties. Similarly, a pseudo-resolvent $\mathcal{R}$ on $D = \{\lambda \in \mathbb{C}\colon \mathrm{Re}(\lambda) > 0\}$ with values in M_* is said to be identity preserving and of Schwarz type if $\mathcal{R}'$ has these properties.

For a semigroup of contractions on a Banach space we have

$$\begin{aligned}\mathrm{Fix}\,(T) &= \bigcap_{t\geqslant 0} \ker(\mathrm{Id} - T(t))\\ &= \ker(\mathrm{Id} - \lambda R(\lambda, A)) = \mathrm{Fix}\,((\lambda R(\lambda, A))\end{aligned}$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. Therefore a semigroup of contractions on M is identity preserving, if and only if the pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon \mathrm{Re}(\lambda) > 0\}$ given by

$$R(\lambda) := R(\lambda, A)_{|D}$$

is identity preserving. By Corollary 2.2 an analogous statement holds for *Schwarz type*.

3 Induction and Reduction

1. If E is a Banach space and $\mathcal{S} \subseteq \mathcal{L}(E)$ is a semigroup of bounded operators, then a closed subspace F is called $\mathcal{S}$-invariant, if $SF \subseteq F$ for all $S \in \mathcal{S}$. We call the semigroup $\mathcal{S}_{|F} := \{S_{|F}\colon S \in \mathcal{S}\}$ the reduced semigroup. Note that for a one-parameter semigroup $\mathcal{T}$ (resp., pseudo-resolvent $\mathcal{R}$) the reduced semigroup is again strongly continuous (resp. $\mathcal{R}_{|F}$ is again a pseudo-resolvent). (Compare A-I, 3.2).

2. Let M be a W*-algebra, $p \in M$ a projection and $S \in \mathcal{L}(M)$ such that

$$S(p^{\perp}M) \subseteq p^{\perp}M \quad \text{and} \quad S(Mp^{\perp}) \subseteq Mp^{\perp},$$

where $p^{\perp} := \mathbb{1} - p$. Since for all $x \in M$

$$p[S(x) - S(pxp)] = p[S(p^{\perp}xp) + S(xp^{\perp})]p = 0,$$

we obtain $p(Sx)p = p(S(pxp))p$. Therefore, the map

$$S_p := (x \mapsto p(Sx)p)\colon pMp \to pMp$$

is well defined and we call S_p the *induced map*. If S is an identity preserving Schwarz map, then it is easy to see that S_p is again a Schwarz map such that $S_p(p) = p$.

3. If $\mathcal{T} = (T(t))_{t\geqslant 0}$ is a weak*-semigroup on M which is of Schwarz type and if $T(t)(p^{\perp}) \leqslant p^{\perp}$ for all $t \in \mathbb{R}_+$, then T leaves $p^{\perp}M$ and $Mp^{\perp}$ invariant. One can verify that the induced semigroup $T_p = (T(t)p)_{t\geqslant 0}$ is again a weak*-semigroup.

If $\mathcal{R}$ is an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \mathrm{Re}(\lambda) > 0\}$ with values in M such that $R(\mu)p^{\perp} \leqslant p^{\perp}$ for some $\mu \in \mathbb{R}_+$, then $p^{\perp}M$ and $Mp^{\perp}$ are $\mathcal{R}$-invariant. It follows directly that the induced pseudo-resolvent $\mathcal{R}_p$ has both the Schwarz type property and is identity preservation.

4. Let φ be a positive normal linear functional on a W*-algebra M such that $T_*\varphi = \varphi$ for some identity preserving Schwarz map T on M with preadjoint $T_* \in L(M_*)$. Then $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$ where $s(\varphi)$ is the support projection of φ.

Let

$$L_\varphi := \{x \in M\colon \varphi(xx^*) = 0\} \quad \text{and} \quad M_\varphi := L_\varphi \cap L_\varphi^*.$$

Since φ is T_*-invariant and T is a Schwarz map, the subspaces L_φ and M_φ are T-invariant. From $M_\varphi = s(\varphi)^\perp M s(\varphi)^\perp$ and $T(s(\varphi)^\perp) \leqslant 1$ it follows that $T(s(\varphi)^\perp) \leqslant s(\varphi)^\perp$.

Let $T_{s(\varphi)}$ be the induced map on $M_{s(\varphi)}$ and define

$$s(\varphi)M_*s(\varphi) := \{\psi \in M_*\colon \psi = s(\varphi)\psi s(\varphi)\}$$

where $\langle s(\varphi)\psi s(\varphi), x\rangle := \langle \psi, s(\varphi)xs(\varphi)\rangle$ $(x \in M)$. For any $\psi \in s(\varphi)M_s(\varphi)$ and all $x \in M$, the following equalities holds

$$\begin{aligned}(T_*\psi)(x) = \psi(Tx) &= \langle \psi, s(\varphi)(Tx)s(\varphi)\rangle \\ &= \langle \psi, s(\varphi)(T(s(\varphi)xs(\varphi)))s(\varphi)\rangle = \langle T_*\psi, s(\varphi)xs(\varphi)\rangle,\end{aligned}$$

hence $T_*\psi \in s(\varphi)M_*s(\varphi)$. Since the dual of $s(\varphi)M_*s(\varphi)$ is $M_{s(\varphi)}$, it follows that the adjoint of the reduced map $T_{*|}$ is identity preserving and of Schwarz type.

For example, if $\mathcal{T}$ is an identity preserving semigroup of Schwarz type on M_* such that $\varphi \in \mathrm{Fix}(T)$, then the semigroup $T_{|(s(\varphi)M_*s(\varphi))}$ is again identity preserving and of Schwarz type. Furthermore, if $\mathcal{R}$ is a pseudo-resolvent on

$$D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$$

with values in M_* which is identity preserving and of Schwarz type such that $R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$, then $\mathcal{R}_{|s(\varphi)M_*s(\varphi)}$ has the same properties.

Notes

We refer to Bratteli and Robinson [2], Davies [3] and the survey article of Oseledets [4].

References

[1] N. Bourbaki. *Éléments des Mathématiques, Intégration, Chapitre 5: Intégration des Mesures*. Hermann, Paris, 1955.

[2] O. Bratteli and D. Robinson. *Operator Algebras and Quantum Statistical Mechanics I*. Springer, New York-Heidelberg-Berlin, 1979. Volume II published 1981.

[3] E. Davies. *Quantum Theory of Open Systems*. Academic Press, London-New York-San Francisco, 1976.

[4] V. I. Oseledets. Completely positive linear maps, non Hamiltonian evolution and quantum stochastic processes. *J. Soviet Math.*, 25:1529–1557, 1984.

[5] G. K. Pedersen. *C*-Algebras and their Automorphism Groups*. Academic Press, London, New York, San Francisco, 1979.

[6] M. Takesaki. *Theory of Operator Algebras I*. Springer, New York-Heidelberg-Berlin, 1979.

Chapter D-II

Characterization of Positive Semigroups on W*-Algebras

by
Ulrich Groh

Since the positive cone of a C*-algebra has non-empty interior, many results of Chapter B-II can be applied verbatim to the characterization of the generator of positive semigroups on C*-algebras. On the other hand, a concrete and detailed representation of such generators has been found only in the uniformly continuous case (see Lindblad [4]). A third area of active research has been the following: Which maps on C*-algebras (in particular, which derivations) commuting with certain automorphism groups are automatically generators of strongly continuous positive semigroups. For more information we refer to the survey article of Evans [3].

1 Semigroups on Properly Infinite W*-Algebras

The aim of this section is to show that strongly continuous semigroups of Schwarz maps on properly infinite W*-algebras are already uniformly continuous. In particular, our theorem is applicable to such semigroups on $B(H)$.

It is worthwhile to remark that the result of Lotz [5] on the uniform continuity of every strongly continuous semigroup on L^∞ (see A-II, Sec.3) does not extend to arbitrary W*-algebras.

Example 1.1 Take $M = \mathcal{B}(H)$, H infinite dimensional, and choose a projection $p \in M$ such that Mp is topologically isomorphic to H. Therefore $M = H \oplus M_0$, where $M_0 = \ker(x \mapsto xp)$. Next, take a strongly, but not uniformly continuous semigroup $\mathcal{T}$ on H and consider the strongly continuous semigroup $\mathcal{T} \oplus \mathrm{Id}$ on M.

For results on the classification theory of W*-algebras needed in our approach we refer to Sakai [7, 2.2] and Takesaki [10, V.1].

Theorem 1.2 *Every strongly continuous one-parameter semigroup of Schwarz type on a properly infinite* W**-algebra* M *is uniformly continuous.*

Proof Let $\mathcal{T} = (T(t)_{t\geqslant 0})$ be strongly continuous on M and suppose $\mathcal{T}$ not to be uniformly continuous. Then there exists a sequence (T_n) in $\mathcal{T}$ and $\varepsilon > 0$ such that $\|T_n - \mathrm{Id}\| \geqslant \varepsilon$, but $T_n \longrightarrow \mathrm{Id}$ in the strong operator topology. We claim that for every sequence (p_k) of mutually orthogonal projections and all bounded sequences (x_k) in M

$$\lim_n \|(T_n - \mathrm{Id})(p_k x_k p_k)\| = 0$$

uniformly in $k \in \mathbb{N}$. This follows from the *Lemma of Phillips* (Schaefer [9]) and the fact that the sequence $(p_k x_k p_k)$ is summable in the $s^*(M, M_*)$-topology (compare Elliot [2], Lemma 2.).

Let (p_k) be a sequence of mutually orthogonal projections in M such that every p_k is equivalent to $\mathbb{1}$ via some $u_k \in M$ [7, 2.2]. Without loss of generality we may assume $\|(T_n - \mathrm{Id})(u_n)\| \leqslant n^{-1}$ since the semigroup T is strongly continuous. Thus we obtained the following.

(i) $\lim_n \|(T_n - \mathrm{Id})(p_k x_k p_k)\| = 0$ uniformly in $k \in \mathbb{N}$ for every bounded sequence (x_k) in M.

(ii) Every projection p_k is equivalent to 1 via some $u_k \in M$.

(iii) $\|(T_n - \mathrm{Id})u_n\| \leqslant n^{-1}$ for all $n \in \mathbb{N}$.

For the following construction see A-I,3.6 and D-II,Sec.2. Take

(i) $\widehat{M}$ be an ultrapower of M,

(ii) let $p := \widehat{(p_k)} \in \widehat{M}$,

(iii) let $T := \widehat{(T_n)} \in L(\widehat{M})$

(iv) and let $u := \widehat{(u_k)} \in \widehat{M}$.

Then T is identity preserving and of Schwarz type on $\widehat{M}$.

Since $u^*u = p$ and $uu^* = \mathbb{1}$, it follows $pu^* = u^*$ and $(uu^*)x(uu^*) = x$ for all $x \in \widehat{M}$. Finally, $T(pxp) = pxp$ for all $x \in \widehat{M}$ which follows from (i), and $T(u^*) = T(pu^*) = pu^* = u^*$ and $T(u) = u$, which follows from (iii). Using the Schwarz, inequality we obtain

$$T(uu^*) = T(\mathbb{1}) \leqslant \mathbb{1} = uu^* = T(u)T(u)^*.$$

From D-III, Lemma 1.1., we conclude $T(ux) = uT(x)$ and $T(xu^*) = T(x)u^*$ for all $x \in \widehat{M}$. Hence

$$\begin{aligned} T(x) &= T(uu^*xuu^*) = uT(u^*xu)u^* = uT(pu^*xup)u^* \\ &= upu^*xupu^* = uu^*xuu^* = x \end{aligned}$$

for all $x \in \widehat{M}$. From this we obtain that for every bounded sequence (x_k) in M

$$\lim_k \|T_k x_k - x_k\| = 0$$

for some subsequence of the T_k's and of the x_k's. This conflicts with our assumption at the beginning, hence the theorem is proved. □

Notes

Let M be a W*-algebra and H be an infinite-dimensional Hilbert-space. Then the W*-tensor product $N := M\overline{\otimes}\mathcal{B}(H)$ is a properly infinite W*-algebra (Sakai [7, Thm. 2.6.6]). Let $\mathcal{S}$ be the semigroup

$$S(t) = T(t) \otimes \mathrm{Id}_H \quad (t \geqslant 0).$$

Then $S(t)$ is a Schwarz-map on N and contractive (Takesaki [10, Prop. IV.5.13.]), hence the smigoup $\mathcal{S}$ is equicontinuous in $\mathcal{L}(N)$.

Let $x \in M$ and $\xi \in H$. Since the norm on N is a cross-norm, we obtain

$$\lim_{t\to 0} \|(S(t) - \mathrm{Id})x \otimes \xi\| = \lim_{t\to 0} \|(S(t) - \mathrm{Id})x\|\|\xi\| = 0.$$

From Schaefer [8, III.4.5] it follows that $\mathcal{S}$ is strongly-continuous, hence norm-continuous on N from which we conclude, that $\mathcal{T}$ is norm-continuous on M.

Remark 1.3 If M is a finite W*-algebra of Type I, then M is a Grothendieck space and has the Dunford-Pettis property. Hence we can apply the results of Lotz [5]. However, has W*-algebra have the Dunford-Pettis property iff it is finite and of Type I (Chu and Iochum [1]). But is known that every W*-algebra is a Grothendick space (Pfitzner [6].

References

[1] C. H. Chu and B. Iochum. The dunford-pettis property in C*-algebras. *Studia Math.*, 97:59–64, 1990. URL https://api.semanticscholar.org/CorpusID:117828608.

[2] G. Elliot. Convergent sequences of finitely additive measures. *J. Funct. Anal.*, 11:204–206, 1972.

[3] D. Evans. Quantum dynamical semigroups, symmetry groups, and locality. *Acta Appl. Math.*, 2:333–352, 1984.

[4] G. Lindblad. On the generators of quantum dynamical semigroups. *Comm. Math. Phys.*, 48:119–130, 1976.

[5] H. P. Lotz. Uniform convergence of operators on L^∞ and similar spaces. *Math. Z.*, 190:207–220, 1985.

[6] H. Pfitzner. Weak compactness in the dual of a C*-algebra is determined commutatively. *Math. Ann.*, 298:349–371, 1994.

[7] S. Sakai. *C*-Algebras and W*-Algebras*. Springer, Berlin-Heidelberg-New York, 1971.

[8] H. H. Schaefer. *Topological Vector Spaces*. Springer, 1966. 4th printing: New York-Heidelberg-Berlin: Springer 1980.

[9] H. H. Schaefer. *Banach Lattices and Positive Operators*. Springer, New York-Heidelberg-Berlin, 1974.

[10] M. Takesaki. *Theory of Operator Algebras I*. Springer, New York-Heidelberg-Berlin, 1979.

Chapter D-III

Spectral Theory of Positive Semigroups on W*-Algebras and their Preduals

by

Ulrich Groh

Motivated by the classical results of Perron and Frobenius one expects the following spectral properties for the generator A of a positive semigroup on a C*-algebra.

> The spectral bound $s(A) := \sup\{\operatorname{Re}\lambda \colon \lambda \in \sigma(A)\}$ belongs to the spectrum $\sigma(A)$ and the boundary spectrum $\sigma_b(A) := \sigma(A) \cap \{s(A) + \mathrm{i}\mathbb{R}\}$ possesses a certain symmetric structure, called cyclicity.

Results of this type have been proved in Chapter B-III for positive semigroups on commutative C*-algebras, however in the non-commutative case the situation is more complicated. While "$s(A) \in \sigma(A)$" still holds (see Greiner et al. [6] or the notes of this chapter), the cyclicity of the boundary spectrum $\sigma_b(A)$ is true only under additional assumptions on the semigroup (e.g., irreducibility, see Section 1 below).

For technical reasons we consider mostly strongly continuous semigroups on the predual of a W*-algebra M or its adjoint semigroup which is a weak*-continuous semigroup on M.

1 Spectral Theory for Positive Semigroups on Preduals

The aim of this section is to develop a Perron-Frobenius theory for identity preserving semigroups of Schwarz type on W*-algebras. However we will show in the example preceding Theorem 1.11 on page 226 below that the boundary spectrum is no longer cyclic. The appropriate hypothesis on the semigroup implying the desired results seems to be the concept of *irreducibility*.

Let us first recall some facts on normal linear functionals. If φ is a normal linear functional on a W*-algebra M, then there exists a partial isometry $u \in M$ and a positive linear functional $|\varphi| \in M_*$ such that

$$\varphi(x) = |\varphi|(xu) =: (u|\varphi|)(x) \quad (x \in M),$$
$$u^*u = s(|\varphi|),$$

where $s(|\varphi|)$ denotes the support projection of $|\varphi|$ in M. We refer to this as the *polar decomposition* of φ. In addition, $|\varphi|$ is *uniquely determined* by the following two conditions.

$$\left.\begin{aligned} \|\varphi\| &= \||\varphi|\| \\ |\varphi(x)|^2 &\leqslant |\varphi|(xx^*) \quad (x \in M) \end{aligned}\right\} (*)$$

For the polar decomposition of the adjoint φ^*, where $\varphi^*(x) = \overline{\varphi(x^*)}$, we obtain

$$\varphi^* = u^*|\varphi^*|, \quad |\varphi^*| = u|\varphi|u^* \quad \text{and} \quad uu^* = s(|\varphi^*|).$$

It is easy to see that $u^* \in s(|\varphi|)M$ (Takesaki [15, Theorem III.4.2 & Proposition III.4.6]).

If Ψ is a subset of the state space of a C*-algebra M, then Ψ is called *faithful* if $0 \leqslant x \in M$ and $\psi(x) = 0$ for all $\psi \in \Psi$ implies $x = 0$. Moreover Ψ is called *subinvariant* for a positive map $T \in \mathcal{L}(M)$ (resp.., positive semigroup $\mathcal{T}$) if $T'\psi \leqslant \psi$ for all $\psi \in \Psi$ (resp. $T(t)'\psi \leqslant \psi$ for all $T(t) \in \mathcal{T}$ and $\psi \in \Psi$). Recall that for every positive map $T \in \mathcal{L}(M)$ there exists a state φ on M such that $T'\varphi = r(T)\varphi$, where $r(T)$ denotes the spectral radius of T (Groh [7, Theorem 2.1]).

Let us start our investigation with two lemmata where $\operatorname{Fix}(T)$ is the fixed space of T, i.e., the set $\{x \in M : Tx = x\}$.

Lemma 1.1 *Suppose M to be a* C**-algebra and $T \in \mathcal{L}(M)$ an identity preserving Schwarz map.*

(i) *Let $b\colon M \times M \to M$ be a sesquilinear map such that $b(z,z) \geqslant 0$ for all $z \in M$. Then $b(x,x) = 0$ for some $x \in M$ if and only if $b(x,y) = 0$ and $b(y,x) = 0$ for all $y \in M$.*

(ii) *If there exists a faithful family Ψ of subinvariant states for T on M, then $\operatorname{Fix}(T)$ is a* C**-subalgebra of M and $T(xy) = xT(y)$ for all $x \in \operatorname{Fix}(T)$ and $y \in M$.*

Proof (i) Take $0 \leqslant \psi \in M^*$ and consider $f := \psi \circ b$. Then f is a positive semidefinite sesquilinear form on M with values in $\mathbb{C}$. From the Cauchy-Schwarz inequality it follows that $f(x,x) = 0$ for some $x \in M$ if and only if $f(x,y) = 0$ and $f(y,x) = 0$ for all $y \in M$. Since the positive cone M^*_+ is generating, assertion (i) is proved.

(ii) Since T is positive, it follows that $T(x)^* = T(x^*)$ for all $x \in M$. Hence $\operatorname{Fix}(T)$ is a self adjoint subspace of M, i.e., invariant under the involution on M. For every

$x, y \in M$ define
$$b(x, y) := T(xy^*) - T(x)T(y)^*.$$

Then b satisfies the assumptions of (i).

If $x \in \text{Fix}\,(T)$, then
$$0 \leqslant xx^* = (Tx)(Tx)^* \leqslant T(xx^*),$$

hence
$$0 \leqslant \psi(T(xx^*) - xx^*) = 0 \quad \text{for all } \psi \in \Psi.$$

But this implies $T(xx^*) = T(x)T(x)^* = xx^*$ and consequently, $b(x, x) = 0$. Hence $T(xy^*) = xT(y)^*$ for all $y \in M$ and (ii) is proved. □

Lemma 1.2 *Let M be a* W**-algebra, T an identity preserving Schwarz map on M and $S \in \mathcal{L}(M)$ such that $S(x)(Sx)^* \leqslant T(xx^*)$ for every $x \in M$.*

(i) *If $v \in M$ such that $S(v^*) = v^*$ and $T(v^*v) = v^*v$, then $T(xv) = S(x)v$ for all $x \in M$.*

(ii) *Suppose there exists $\varphi \in M_*$ with polar decomposition $\varphi = u|\varphi|$ such that $S_*\varphi = \varphi$ and $T_*|\varphi| = |\varphi|$. If the closed subspace $s(|\varphi|)M$ is T-invariant, then $Su^* = u^*$ and $T(u^*u) = u^*u$.*

Proof (i) Define a positive semidefinite sesquilinear map $b : M \times M \mapsto M$ by
$$b(x, y) := T(xy^*) - S(x)S(y)^* \quad (x, y \in M).$$

Since $b(v^*, v^*) = 0$ we obtain $b(x, v^*) = 0$ for all $x \in M$, hence $T(xv) = S(x)v$. (Lemma 1.1 (i))

(ii) Since $s(|\varphi|)M$ is a closed right ideal, the closed face $F := s(|\varphi|)(M_+)s(|\varphi|)$ determines $s(|\varphi|)M$ uniquely, i.e.,
$$s(|\varphi|)M = \{x \in M : xx^* \in F\}$$

(Pedersen [13, Theorem 1.5.2]). Since T is a Schwarz map and $s(|\varphi|)M$ is T-invariant, it follows $TF \subseteq F$. On the other hand, if $x \in s(|\varphi|)M$, then $xx^* \in F$. Consequently,
$$0 \leqslant S(x)S(x)^* \leqslant T(xx^*) \in F,$$

whence $S(x) \in s(|\varphi|)M$.

Next we show $T(u^*u) = u^*u$ and $Su^* = u^* \in s(|\varphi|)M$. First of all
$$\begin{aligned} 0 &\leqslant (Su^* - u^*)(Su^* - u^*)^* \\ &\leqslant T(u^*u) - u^*S(u^*)^* - (Su^*)u + u^*u. \end{aligned}$$

Since $S_*\varphi = \varphi$, $T_*|\varphi| = |\varphi|$ and $\varphi = u|\varphi|$ it follows

$$\begin{aligned}0 &\leqslant |\varphi|((Su^* - u^*)(Su^* - u^*)^*)\\ &\leqslant 2|\varphi|(u^*u) - |\varphi|(S(u^*)u)^* - |\varphi|(S(u^*)u)\\ &= 2|\varphi|(uu^*) - \varphi(u^*)^* - \varphi(u^*)\\ &= 2(|\varphi|(u^*u) - |\varphi|(u^*u)) = 0.\end{aligned}$$

But $(Su^* - u^*)(Su^* - u^*) \in F$ and $|\varphi|$ is faithful on F. Hence we obtain $Su^* = u^*$. Consequently,

$$0 \leqslant u^*u = (Su^*)(Su^*)^* \leqslant T(u^*u)$$

and $T(u^*u) = u^*u$ by the faithfulness and T-invariance of $|\varphi|$. □

Remark 1.3 Take S and T as in Lemma 1.2 (ii). If V_{u^*} (resp. V_u) is the map $(x \mapsto xu^*)$ (resp. $(x \mapsto xu)$) on M, then V_{u^*} is a continuous bijection from $Ms(|\varphi|)$ onto $Ms(|\varphi^*|)$ with inverse V_u (because $V_u \circ V_{u^*} = \mathrm{Id}_{Ms(|\varphi|)}$ and $V_{u^*} \circ V_u = \mathrm{Id}_{Ms(|\varphi^*|)}$). Let $x \in M$. From $T(xu) = S(x)u$ we obtain $T(xu)u^* = S(x)uu^*$. In particular, if $Ms(|\varphi^*|)$ is S-invariant, then

$$(V_{u^*} \circ T \circ V_u)(x) = T(xu)u^* = S(x)$$

for every $x \in Ms(|\varphi^*|)$. Let $T_|$ (resp. $S_|$) be the restriction of T to $Ms(|\varphi|)$ (resp. of S to $Ms(|\varphi^*|)$). Then the following diagram is commutative:

$$\begin{array}{ccc} Ms(|\varphi|) & \xrightarrow{T_|} & Ms(|\varphi|) \\ {\scriptstyle V_u}\big\downarrow & & \big\downarrow{\scriptstyle V_{u^*}} \\ Ms(|\varphi^*|) & \xrightarrow[S_|]{} & Ms(|\varphi^*|) \end{array}$$

In particular, $\sigma(S_|) = \sigma(T_|)$. Therefore we may deduce spectral properties of $S_|$ from $T_|$ and vice versa. More concrete applications of Lemma 1.2 will follow.

We now investigate the fixed space $\mathrm{Fix}\,(\mathcal{R}) := \mathrm{Fix}\,(\lambda R(\lambda))$, $\lambda \in D$, of a pseudo-resolvent $\mathcal{R}$ with values in the predual of a W*-algebra M.

Proposition 1.4 *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}\,\lambda > 0\}$ with values in the predual M_* of a* W*-*algebra M and suppose $\mathcal{R}$ to be identity preserving and of Schwarz type.*

(i) *If $\alpha \in \mathbb{R}$ and $\psi \in M_*$ such that $(\gamma - \mathrm{i}\alpha)R(\gamma)\psi = \psi$ for some $\gamma \in D$, then $\lambda R(\lambda)|\psi| = |\psi|$ and $\lambda R(\lambda)|\psi^*| = |\psi^*|$ for all $\lambda \in D$.*

(ii) $\mathrm{Fix}\,(\mathcal{R})$ *is invariant under the involution in M_*. If $\psi \in$* $\mathrm{Fix}\,(\mathcal{R})$ *is self adjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of* $\mathrm{Fix}\,(\mathcal{R})$.

Proof If $(\gamma - \mathrm{i}\alpha)R(\gamma)\psi = \psi$ then $(\lambda - \mathrm{i}\alpha)R(\lambda)\psi = \psi$ for all $\lambda \in D$. In particular, $\mu R(\mu + \mathrm{i}\alpha)\psi = \psi$ $(\mu \in \mathbb{R}_+)$. For all $x \in M$ we obtain

$$\begin{aligned}|\psi(x)|^2 &= |<\mu R(\mu+\mathrm{i}\alpha)'x,\psi>|^2 \leqslant\\ &\leqslant \|\psi\| < (\mu R(\mu+\mathrm{i}\alpha)'x)(\mu R(\mu+\mathrm{i}\alpha)'x)^*,\psi>\leqslant\\ &\leqslant \|\psi\| < \mu R(\mu)'(xx^*),|\psi|>\end{aligned}$$

(D-I, Corollary 2.2). Since

$$\begin{aligned}\|\psi\| &= \||\psi|\| = |\psi|(1) =\\ &=< \mu R(\mu)'1,|\psi|>= \|\mu R(\mu)|\psi|\|,\end{aligned}$$

we obtain $\mu R(\mu)|\psi| = |\psi|$ by the uniqueness theorem (*) above for the absolut value— therefore $|\psi| \in \operatorname{Fix}(\mathcal{R})$. Since

$$0 \leqslant (\mu R(\mu)'x)(\mu R(\mu)'x)^* \leqslant \mu R(\mu)'xx^*,$$

the map $R(\mu)$ is positive. Consequently $(\mu+\mathrm{i}\alpha)R(\mu)\psi^* = \psi^*$ from which $|\psi^*| \in \operatorname{Fix}(\mathcal{R})$ follows. If $\varphi \in \operatorname{Fix}(\mathcal{R})$ is selfadjoint with Jordan decomposition $\varphi = \varphi^+ - \varphi^-$, then $|\varphi| = \varphi^+ + \varphi^-$ (Takesaki [15, Theorem III.4.2.]). From this we obtain that φ^+ and φ^- are in $\operatorname{Fix}(\mathcal{R})$. □

Corollary 1.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on M_* with generator A and suppose $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.*

(i) *If $\alpha \in \mathbb{R}$ and $\psi \in \ker(\mathrm{i}\alpha - A)$, then $|\psi|$ and $|\psi^*|$ are elements of* $\operatorname{Fix}(\mathcal{T}) = \ker(A)$.

(ii) $\operatorname{Fix}(\mathcal{T})$ *is invariant under the involution of M_*. If $\psi \in \operatorname{Fix}(\mathcal{T})$ is selfadjoint, then the positive part ψ^+ and the negative part ψ^- of ψ are elements of* $\operatorname{Fix}(\mathcal{T})$.

The proof follows immediately from Proposition 1.4 and the fact that $\ker(A) = \operatorname{Fix}(\lambda R(\lambda, A))$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.

If $\mathcal{T}$ is the semigroup of translations on $L^1(\mathbb{R})$ and A' the generator of the adjoint weak*-semigroup, then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$, while $P\sigma(A') \cap \mathrm{i}\mathbb{R} = \mathrm{i}\mathbb{R}$. For that reason we cannot expect a simple connection between these two sets. But as we shall see below, if a semigroup on the predual of a W*-algebra has sufficiently many invariant states, then the point spectra contained in $\mathrm{i}\mathbb{R}$ of A and A' are identical.

Helpful for these investigations will be the next lemma.

Lemma 1.6 *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda > 0\}$ with values in a Banach space E such that $\|R(\mu+\mathrm{i}\alpha)\| \leqslant 1$ for all $(\mu,\alpha) \in \mathbb{R}_+ \times \mathbb{R}$. Then*

$$\dim \operatorname{Fix}(\lambda R(\lambda+\mathrm{i}\alpha)) \leqslant \dim \operatorname{Fix}(\lambda R(\lambda+\mathrm{i}\alpha)')$$

for all $\lambda \in D$.

Proof Let $(\mu,\alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $S := \mu R(\mu+\mathrm{i}\alpha)$. Since S is a contraction, its adjoint S' maps the dual unit ball E_1' into itself.

Let $\mathfrak{U}$ be a free ultrafilter on $[1, \infty[$ which converges to 1. Since E_1' is $\sigma(E', E)$-compact,

$$\psi_0 := \lim_{\mathfrak{U}} (\lambda - 1) R(\lambda, S)' \psi$$

exists for each $\psi \in E_1'$. Since S' is $\sigma(E', E)$-continuous and since $S' R(\lambda, S)' = \lambda R(\lambda, S') - \mathrm{Id}$ we conclude $\psi_0 \in \operatorname{Fix}(S')$.

Take now $0 \neq x_0 \in \operatorname{Fix}(S)$ and choose $\psi \in E_1'$ such that $\psi(x_0)$ is different from zero. From the considerations above it follows

$$\psi_0(x_0) = \lim_{\mathfrak{U}} (\lambda - 1) \psi(R(\lambda, S) x_0) = \psi(x_0) \neq 0$$

hence $0 \neq \psi_0 \in \operatorname{Fix}(S)$. Therefore $\operatorname{Fix}(S')$ separates the points of $\operatorname{Fix}(S)$.

From this it follows that

$$\dim \operatorname{Fix}(S) \leqslant \dim \operatorname{Fix}(S') .$$

Since $\mathcal{R}$ and $\mathcal{R}'$ are pseudo-resolvents, the assertion is proved. □

Corollary 1.7 *Let $\mathcal{T}$ be a semigroup of contractions on a Banach space E with generator A. Then*

$$\dim \ker(\mathrm{i}\alpha - A) \leqslant \dim \ker(\mathrm{i}\alpha - A')$$

for all $\alpha \in \mathbb{R}$.

This follows from Lemma 1.6 on page 223 because $\operatorname{Fix}(\lambda R(\lambda + \mathrm{i}\alpha)) = \ker(\mathrm{i}\alpha - A)$.

Proposition 1.8 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra and suppose that there exists a faithful family Ψ of $\mathcal{T}$-invariant states. Then for all $\alpha \in \mathbb{R}$ we have*

$$\dim \ker(\mathrm{i}\alpha - A) = \dim \ker(\mathrm{i}\alpha - A')$$

and

$$P\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A') \cap \mathrm{i}\mathbb{R}.$$

Proof The inequality $\dim \ker(\mathrm{i}\alpha - A) \leqslant \dim \ker(\mathrm{i}\alpha - A')$ follows from Corollary 1.7.

Let $D = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ and $\mathcal{R}$ the pseudo-resolvent induced by $R(\lambda, A)$ on D. Then $\mathcal{R}$ is identity preserving and of Schwarz type. Take $\mathrm{i}\alpha \in P\sigma(A)$ ($\alpha \in \mathbb{R}$) and choose $0 < \mu \in \mathbb{R}$.

If $\psi_\alpha \in M_*$ is of norm one with polar decomposition $\psi_\alpha = u_\alpha |\psi_\alpha|$ such that $\psi_\alpha = (\mu - \mathrm{i}\alpha) R(\mu) \psi_\alpha$ then $\mu R(\mu) |\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4 (i) on page 222).

Since
$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant 1 - s(|\psi_\alpha|),$$
we obtain $\mu R(\mu)' s(|\psi_\alpha|) = s(|\psi_\alpha|)$ by the faithfulness of Ψ. Hence the maps $S := (\mu - \mathrm{i}\alpha)R(\mu)'$ and $T := \mu R(\mu)'$ fulfill the assumptions of Lemma 1.2 (ii) on page 221. Therefore $Su_\alpha^* = u_\alpha^*$ or $(\mu - \mathrm{i}\alpha)R(\mu)' u_\alpha^* = u_\alpha^*$ which implies $u_\alpha^* \in D(A')$ and $A' u_\alpha^* = \mathrm{i}\alpha u_\alpha^*$.

If $\mathrm{i}\alpha \in P\sigma(A')$, $\alpha \in \mathbb{R}$, choose $0 \neq v_\alpha$ such that
$$v_\alpha = (\mu - \mathrm{i}\alpha)R(\mu)' v_\alpha \quad (\mu \in \mathbb{R}_+)$$
and $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$.

Since
$$0 \leqslant v_\alpha v_\alpha^* = ((\mu - \mathrm{i}\alpha)R(\mu)' v_\alpha)((\mu - \mathrm{i}\alpha)R(\mu)' v_\alpha)^* \leqslant \mu R(\mu)'(v_\alpha v_\alpha^*),$$
we obtain $\mu R(\mu)'(v_\alpha v_\alpha^*) = v_\alpha v_\alpha^*$ because Ψ is faithful.

Hence from Lemma 1.2 (i) on page 221 it follows that
$$\mu R(\mu)'(x v_\alpha^*) = ((\mu - \mathrm{i}\alpha)R(\mu)' x) v_\alpha^*$$
for all $x \in M$.

Let ψ_α be the normal linear functional $(x \mapsto \psi(x v_\alpha^*))$ on M and note that $\psi_\alpha(v_\alpha) \neq 0$. Then
$$\begin{aligned}\langle x, (\mu - \mathrm{i}\alpha)R(\mu)\psi_\alpha\rangle &= \langle ((\mu - \mathrm{i}\alpha)R(\mu)' x) v_\alpha^*, \psi\rangle \\ &= \langle \mu R(\mu)'(x v_\alpha^*), \psi\rangle = \psi(x v_\alpha^*) = \psi_\alpha(x)\end{aligned}$$
for all $x \in M$. Consequently $\mathrm{i}\alpha \in P\sigma(A)$ and
$$\dim \ker((\mathrm{i}\alpha - A')) \leqslant \dim \ker((\mathrm{i}\alpha - A))$$
which proves the assertion. □

Remark 1.9 From the above proof we obtain the following: If $0 \neq \psi_\alpha \in \ker(\mathrm{i}\alpha - A)$ for $\alpha \in \mathbb{R}$ with polar decomposition $\psi_\alpha = u_\alpha |\psi_\alpha|$ $(\alpha \in \mathbb{R})$, then $A' u_\alpha = \mathrm{i}\alpha u_\alpha$.

Conversely, if $0 \neq v_\alpha \in \ker(\mathrm{i}\alpha - A')$, then there exists $\psi \in \Psi$ such that $\psi(v_\alpha v_\alpha^*) \neq 0$ and the normal linear form
$$\psi_\alpha := (x \mapsto \psi(x v_\alpha^*))$$
is an eigenvector of A pertaining to the eigenvalue $\mathrm{i}\alpha$.

If $\mathcal{T}$ is a C_0-semigroup of Markov operators on a commutative C*-algebra with generator A, it has been shown in B-III, that the boundary spectrum $\sigma(A) \cap \mathrm{i}\mathbb{R}$

of its generator is additively cyclic. This is no longer true in the non commutative case.

Example 1.10 For $0 \neq \lambda \in \mathrm{i}\mathbb{R}$ and $t \in \mathbb{R}$ let

$$u_t := \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \in M_2(\mathbb{C}).$$

The semigroup of *-automorphisms $(x \mapsto u_t x u_t^*)$ on $M_2(\mathbb{C})$ is identity preserving and of Schwarz type, but the spectrum of its generator is $\{0, \lambda, \lambda^*\}$ hence is not additively cyclic.

It turns out that, in order to obtain a non commutative analogue of the Perron-Frobenius theorems, one has to consider semigroups which are irreducible.

Recall that a semigroup $\mathcal{S}$ of positive operators on an ordered Banach space (E, E_+) is called *irreducible* if no closed face of E_+, different from $\{0\}$ and E_+, is invariant under $\mathcal{S}$. In the context of W*-algebras M we call a semigroup $\mathcal{S}$ of positive maps on M *weak*-irreducible* if no $\sigma(M, M_*)$-closed face of M_+ is $\mathcal{S}$-invariant.

Since the norm closed faces of M_* and the $\sigma(M, M_*)$-closed faces of M are related by formation of polars with respect to the dual system $\langle M, M_* \rangle$ (see Pedersen [13, Theorem 3.6.11 and Theorem 3.10.7.]) a semigroup $\mathcal{S}$ is (norm) irreducible on M_* if and only if its adjoint semigroup is weak*-irreducible.

Theorem 1.11 *Let $\mathcal{T}$ be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra and suppose $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.*

(i) *The fixed space of $\mathcal{T}$ is one dimensional and spanned by a faithful normal state.*

(ii) *$P\sigma(A) \cap \mathrm{i}\mathbb{R}$ is an additive subgroup of* $\mathrm{i}\mathbb{R}$,

$$\sigma(A) = \sigma(A) + (P\sigma(A) \cap \mathrm{i}\mathbb{R})$$

and every eigenvalue in $\mathrm{i}\mathbb{R}$ *is simple.*

(iii) *The fixed space of the adjoint weak*-semigroup $\mathcal{T}'$ is one-dimensional.*

(iv) *$P\sigma(A') \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R}$ for the generator A' of the adjoint semigroup, and every $\gamma \in P\sigma(A') \cap \mathrm{i}\mathbb{R}$ is simple.*

Proof Since $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$, there exists $\psi \in \operatorname{Fix}(\mathcal{T})_+$ of norm one (Corollary 1.5). If $F := \{x \in M_+ : \psi(x) = 0\}$, then F is a $\sigma(M, M_*)$-closed, $\mathcal{T}'$-invariant face in M, hence $F = \{0\}$. Therefore every $0 \neq \psi \in \operatorname{Fix}(\mathcal{T})_+$ is faithful.

Let $\psi_1, \psi_2 \in \operatorname{Fix}(\mathcal{T})_+$ be states such that $f := \psi_1 - \psi_2$ is different from zero. If $f = f^+ - f^-$ is the Jordan decomposition of f, then f^+ and f^- are elements of

$\operatorname{Fix}(\mathcal{T})$, whence faithful. Since the support projections of these two normal linear functionals are orthogonal, we obtain $f^+ = 0$ or $f^- = 0$ which implies $\psi_1 \leqslant \psi_2$ or $\psi_2 \leqslant \psi_1$. Consequently $\psi_2 = \psi_1$.

Since $\operatorname{Fix}(\mathcal{T})$ is positively generated (Corollary 1.5 on page 223), $\operatorname{Fix}(\mathcal{T}) = \{\lambda\varphi\colon \lambda \in \mathbb{C}\} =: \mathbb{C}.\varphi$ for some faithful normal state φ.

Let $\mu \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ such that $\mathrm{i}\alpha \in P\sigma(A)$. If $\psi_\alpha = u_\alpha|\psi_\alpha|$ is a normalized eigenvector of A pertaining to $\mathrm{i}\alpha$, then $\varphi = |\psi_\alpha| = |\psi_\alpha^*|$ (Corollary 1.5 and the above considerations). Hence $u_\alpha u_\alpha^* = u_\alpha^* u_\alpha = s(\varphi) = 1$.

Since
$$(\mu - \mathrm{i}\alpha)R(\mu, A)\psi_\alpha = \psi_\alpha$$
and
$$\mu R(\mu, A)|\psi_\alpha| = |\psi_\alpha|,$$
we obtain by Lemma 1.2 (ii) on page 221 that
$$\mu R(\mu, A) = V_\alpha \circ \mu R(\mu + \mathrm{i}\alpha, A) \circ V_\alpha^{-1} \quad (1)$$
where V_α is the map $(x \mapsto x u_\alpha)$ on M.

Similarly, for $i\beta \in P\sigma(A)$ we find V_β such that $1 = u_\beta u_\beta^* = u_\beta u_\beta^*$ and
$$\mu R(\mu, A) = V_\beta \circ \mu R(\mu + i\beta, A) \circ V_\beta^{-1}. \quad (2)$$

Hence
$$\mu R(\mu, A) = V_{\alpha\beta} \circ \mu R(\mu + i(\alpha + \beta), A) \circ V_{\alpha\beta}^{-1} \quad (3)$$
where $V_{\alpha\beta} := V_\alpha \circ V_\beta$.

Since u_α is unitary in M, it follows from (1) that $\mathrm{i}\alpha$ is an eigenvalue which is simple because $\operatorname{Fix}(T) = \operatorname{Fix}(\mu R(\mu, A))$ is one dimensional.

From (3) it follows that $i(\alpha + \beta) \in P\sigma(A)$ since $0 \in P\sigma(A)$ and $V_{\alpha\beta}$ is bijective. From the identity (1) we conclude that $\sigma(R(\mu, A)) = \sigma(R(\mu + \mathrm{i}\alpha))$, which proves
$$\sigma(A) + (P\sigma(A) \cap \mathrm{i}\mathbb{R}) \subseteq \sigma(A).$$

The other inclusion is trivial since $0 \in P\sigma(A)$. □

Remarks 1.12 (i) Let φ be the normal state on M such that $\operatorname{Fix}(T) = \mathbb{C}.\varphi$ and let $H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$. From the proof of Theorem 1.10 it follows that there exists a family $\{u_\eta\colon \eta \in H\}$ of unitaries in M such that $A' u_\eta = -\eta u_\eta$ and $A(u_\eta\varphi) = \eta(u_\eta\varphi)$ for all $\eta \in H$.

(ii) If the group H is generated by a single element, i.e., $H = \mathrm{i}\gamma\mathbb{Z}$ for some $\gamma \in \mathbb{R}$, then $\{u_\gamma^k\colon k \in \mathbb{Z}\}$ is a complete family of eigenvectors pertaining to the eigenvalues in H, where $u_\gamma \in M$ is unitary such that $A' u_\gamma = \mathrm{i}\gamma u_\gamma$.

Proposition 1.13 *Suppose that $\mathcal{T}$ and M satisfy the assumptions of Theorem 1.10, and let N_* be the closed linear subspace of M_* generated by the eigenvectors of A pertaining to the eigenvalues in* $\mathrm{i}\mathbb{R}$*. Denote by $\mathcal{T}_0$ the restriction of $\mathcal{T}$ to N_*. Then*

(i) $G := (\mathcal{T}_0)^- \subseteq L_s(N_*)$ *is a compact, Abelian group in the strong operator topology.*

(ii) $\mathrm{Id}_{|N_*} \in \overline{\{T_0(t) \colon t > s\}} \subseteq L_s(N_*)$ *for all* $0 < s \in \mathbb{R}$.

Proof For $\eta \in H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$ let

$$U(\eta) := \{\psi \in D(A) : A\psi = \eta\psi\}$$

and $U = \{U(\eta) \colon \eta \in H\}$. Then $(U)^- = N_*$.

For each $\psi \in U$ there exists $\eta \in H$ such that

$$\{T_0(t)\psi \colon t \in \mathbb{R}_+\} = \{e^{-\eta t}\psi \colon t \in \mathbb{R}_+\}.$$

Consequently this set is relatively compact in $L_s(N_*)$. From [Schaefer (1966),III.4.5] we obtain that G is compact in the strong operator topology.

Next choose $\psi_1, \ldots, \psi_n \in U$, $0 < s \in \mathbb{R}$ and $\delta > 0$. Since $T_0(t)\psi_i = e^{\eta_i t}\psi_i$ $(1 \leqslant i \leqslant n)$ for some $\eta_i \in H$, it follows from a theorem of Kronecker (see, Jacobs [11, Satz 6.1., p.77]) that there exists $s < t$ such that

$$|(1, 1, \ldots, 1) - (e^{\eta_1 t}, e^{\eta_2 t}, \ldots, e^{\eta_n t})| < \delta,$$

hence

$$\sup\{\|\psi_i - T_0(t)\psi_i\| \colon 1 \leqslant i \leqslant n\} < \delta$$

or $\mathrm{Id}_{|N_*} \in \overline{\{T_0(t) \colon t > s\}} \subseteq L_s(N_*)$.

Finally we prove the group property of G. Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{R}$ such that $\lim_{\mathfrak{U}} T_0(t) = \mathrm{Id}$ in the strong operator topology. For positive $s \in \mathbb{R}$ let $S := \lim_{\mathfrak{U}} T(t-s)$. Then $ST_0(s) = T_0(s)S = \mathrm{Id}$, hence $T_0(s)^{-1}$ exists in G for all $s \in \mathbb{R}_+$. From this it follows that G is a group. □

Remark 1.14 (i) Let $\kappa : \mathbb{R} \to G$ be given by

$$\kappa(t) = \begin{cases} T_0(t) & \text{if } 0 \leqslant t, \\ T_0(t)^{-1} & \text{if } t \leqslant 0. \end{cases}$$

Then κ is a continuous homomorphism with dense range, i.e., (G, κ) is solenoidal (see Hewitt and Ross [10]).

(ii) The compact group G and the discret group $P\sigma(A) \cap \mathrm{i}\mathbb{R}$ are dual as locally compact Abelian groups.

(iii) Let (G, κ) be a solenoidal compact group and let $N_* = L^1(G)$. Then the induced lattice semigroup $T = (\kappa(t))_{t \geqslant 0}$ fulfils the assertions of Theorem 1.10. For example, if G is the dual of $\mathbb{R}_d$, then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\mathbb{R}$. Since the fixed space of $\kappa(t)$ is given by

$$\mathrm{Fix}\,(\kappa(t)) = \overline{\left(\bigcup_{k\in\mathbb{Z}} \ker(\frac{2\pi ik}{t} - A)\right)},$$

however no $T(t) \in \mathcal{T}$ is irreducible.

(iv) If $\mathcal{T}$ is the irreducible semigroup of Schwarz type on the predual of $B(H)$ given in Evans [3], then $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$.

2 Spectral Properties of Uniformly Ergodic Semigoups

The aim of this section is the study of spectral properties of semigroups which are uniformly ergodic, identity preserving and of Schwarz type. For the basic theory of uniformly ergodic semigroups on Banach spaces we refer to Dunford and Schwartz [2].

Our first result yields an estimate for the dimension of the eigenspaces pertaining to eigenvalues of a pseudo-resolvent.

Proposition 2.1 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \mathrm{Re}\,\lambda > 0\}$ with values in the predual of a* W**-algebra M. If $Fix\lambda R(\lambda)$ is finite dimensional for some $\lambda \in D$, then*

$$\dim\,\mathrm{Fix}\,((\gamma - \mathrm{i}\alpha)\,R(\gamma)) \leqslant \dim\,\mathrm{Fix}\,(\lambda R(\lambda))$$

for all $\gamma \in D$ and $\alpha \in \mathbb{R}$.

Proof By D-IV, Remark 3.2.c, we may assume without loss of generality that there exists a faithful family of $\mathcal{R}$-invariant normal states on M. In particular the fixed space N of the adjoint pseudo-resolvent $\mathbb{R}\mathcal{R}'$ is a W*-subalgebra of M with $\mathbb{1} \in N$ (by Lemma 1.1 (ii)). Since N is finite dimensional, there exist a natural number n and a set $P := \{p_1, .., p_n\}$ of minimal, mutually orthogonal projections in N such that $\sum_{k=1}^{n} p_k = \mathbb{1}$. These projections are also mutually orthogonal in M with sum $\mathbb{1}$.

Let R_j be the $\sigma(M, M_*)$-closed right ideal $p_j M$ and L_j the closed left invariant subspace $M_* p_j$ for $(1 \leqslant j \leqslant n)$. Since the map $\mu R(\mu)'$, $\mu \in \mathbb{R}_+$ is an identity preserving Schwarz map, we obtain from Lemma 1.1.b that for all $x \in N$ and $y \in M$,

$$\mu R(\mu)'(xy) = x(\mu\mathcal{R}'(\mu)y).$$

In particular, R_j, resp. L_j are invariant under $\mathcal{R}'$, respectively, $\mathcal{R}$. Furthermore, if $\psi \in L_j$ with polar decomposition $\psi = u|\psi|$, then $u^*u \leqslant s(|\psi|) \leqslant p_j$. Consequently, $|\psi| \in L_j$.

Let now $\alpha \in \mathbb{R}$ and suppose that there exists $\psi_\alpha \in L_j$ of norm 1, $\psi_\alpha = u_\alpha|\psi_\alpha|$, such that

$$\psi_\alpha \in \text{Fix}\,((\lambda - \mathrm{i}\alpha)R(\lambda))\,, \lambda \in D.$$

Since $\lambda R(\lambda)|\psi_\alpha| = |\psi_\alpha|$ (Proposition 1.4 on page 222), we obtain

$$\mu R(\mu)'(1 - s(|\psi_\alpha|)) \leqslant (1 - s(|\psi_\alpha|), \mu \in \mathbb{R}_+.$$

From the existence of a faithful family of $\mathcal{R}$-invariant normal states and since $\mathcal{R}'$ is identity preserving, it follows that

$$\mu R(\mu)' s(|\psi_\alpha|) = s(|\psi_\alpha|).$$

Thus $s(|\psi_\alpha|) \leqslant p_j$ and even $s(|\psi_\alpha|) = p_j$ by the minimality property of p_j.

On the other hand, $\psi_\alpha^* \in \text{Fix}\,((\lambda + \mathrm{i}\alpha)R(\lambda))$. As above we obtain

$$\mu R(\mu)' s(|\psi_\alpha^*|) = s(|\psi_\alpha^*|).$$

Consequently, the closed left ideals $Ms(|\psi_\alpha^*|)$ and $Ms(|\psi_\alpha|)$ are $\mathcal{R}'$-invariant.

Next fix $\mu \in \mathbb{R}_+$, let $S := (\mu - \mathrm{i}\alpha)R(\mu)'$ and $T = \mu R(\mu)'$. Then

$$(Sx)(Sx)^* \leqslant T(xx^*),\ S_*(\psi_\alpha^*) = \psi_\alpha^*,\ T_*(|\psi_\alpha^*|) = |\psi_\alpha^*|,$$

and T is an identity preserving Schwarz map. Since $s(|\psi_\alpha^*|)M$ is T-invariant, the assumptions of Lemma 1.2 on page 221 are fulfilled and we obtain for every $x \in M$

$$S(x)u_\alpha^* = T(xu_\alpha^*).$$

The closed left ideal Mp_j is S-invariant, therefore it follows

$$S(x) = T(xu_\alpha^*)u_\alpha\,,\ x \in Mp_j$$

(see Remark 1.3 on page 222). Since u_α does not depend on $\mu \in \mathbb{R}_+$, we obtain for all $\mu \in \mathbb{R}_+$

$$\mu R(\mu + \mathrm{i}\alpha)'x = \mu R(\mu)'(xu_\alpha^*)u_\alpha.$$

Consequently, the holomorphic functions

$$(\mu \mapsto \mu R(\mu)'(xu_\alpha)u_\alpha^*) \quad \text{and} \quad (\mu \mapsto \mu R(\mu + \mathrm{i}\alpha)'x)$$

coincide on $\mathbb{R}_+$ from which we conclude

$$\lambda R(\lambda + \mathrm{i}\alpha)'x = \lambda R(\lambda)'(xu_\alpha^*)u_\alpha$$

for every $\lambda \in D$ and all $x \in Mp_j$.

Since the map $(y \mapsto yu_\alpha)$ is a continuous bijection from $M(u_\alpha u_\alpha^*)$ onto Mp_j with inverse $(y \mapsto yu_\alpha^*)$, we can deduce that

$$\begin{aligned}\dim \operatorname{Fix}\big((\lambda-\mathrm{i}\alpha)R(\lambda)'|Mp_j\big) &= \dim \operatorname{Fix}\big(\lambda R(\lambda)')|M(u_\alpha u_\alpha^*\big)\\ &\leqslant \dim \operatorname{Fix}(\mathcal{R}') .\end{aligned}$$

Since $\bigoplus_{j=1}^n Mp_j = M$ and $\bigoplus_{j=1}^n L_j = M_*$, we obtain

$$\begin{aligned}\dim \operatorname{Fix}((\lambda-\mathrm{i}\alpha)R(\lambda)')) &= \dim \operatorname{Fix}(\lambda R(\lambda)') ,\\ &= \dim \operatorname{Fix}(\lambda R(\lambda)) ,\end{aligned}$$

and the assertion follows from Lemma 1.6 on page 223. □

Before going on let us recall the basic facts of the *ultrapower* $\hat{E}$ of a Banach space E with respect to some free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ (compare A-I,3.6). If $\ell^\infty(E)$ is the Banach space of all bounded functions on $\mathbb{N}$ with values in E, then

$$c_{\mathfrak{U}}(E) := \{(x_n) \in \ell^\infty(E) \colon \lim_{\mathfrak{U}} \|x_n\| = 0\}$$

is a closed subspace of $\ell^\infty(E)$ and equal to the kernel of the seminorm

$$\|(x_n)\| := \lim_{\mathfrak{U}} \|x_n\| ,\ (x_n) \in \ell^\infty(E).$$

By the *ultrapower* $\hat{E}$ we understand the quotient space $\ell^\infty(E)/c_{\mathfrak{U}}(E)$ with norm

$$\|\hat{x}\| = \lim_{\mathfrak{U}} \|x_n\| ,\ (x_n) \in \hat{x} \in \hat{E}.$$

Moreover, for a bounded linear operator $T \in L(E)$, we denote by $\hat{T}$ the well defined operator $\hat{T}\hat{x} := (Tx_n) + c_{\mathfrak{U}}(E)$, $(x_n) \in \hat{x}$.

It is clear by virtue of $(x \mapsto (x, x, ..) + c_{\mathfrak{U}}(E))$ that each $x \in E$ defines an element $\hat{x} \in \hat{E}$. This isometric embedding as well as the operator map $(T \mapsto \hat{T})$ are called canonical. In particular, if $\mathcal{R} \colon (D \to L(E))$ is a pseudo-resolvent, then

$$\hat{\mathcal{R}} := (\lambda \mapsto R(\lambda)^\wedge) : D \to L(\hat{E})$$

is a pseudo-resolvent, too. Recall that the approximative point spectrum $A\sigma(T)$ is equal to the point spectrum $P\sigma(\hat{T})$ (see, e.g., Schaefer [14, Chapter V, §1]).

This construction gives us the possibility to characterize uniformly ergodic semigroups with finite dimensional fixed space.

Lemma 2.2 *Let $\mathcal{R}$ be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda > 0\}$ such that $\|R(\mu + \mathrm{i}\alpha)\| \leqslant 1$ for all $(\mu, \alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and suppose*

$$0 < \dim \operatorname{Fix}\big((\lambda-\mathrm{i}\alpha)\hat{R}(\lambda)\big) < \infty \quad \textit{for some} \quad \lambda \in D ,\ \alpha \in \mathbb{R}.$$

For the canonical extension $\hat{R}$ on some ultrapower $\hat{E}$, the following assertions hold.

(i) $(\lambda - \mathrm{i}\alpha)^{-1}$ *is a pole of the resolvent* $R(., R(\lambda))$ *for all* $\lambda \in D$.

(ii) $\dim \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) = \dim \operatorname{Fix}\big((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\big)$ *for all* $\lambda \in D$.

(iii) $\mathrm{i}\alpha$ *is a pole of the pseudo-resolvent* $\mathcal{R}$ *and the residue of* $\mathcal{R}$ *and* $R(., R(\lambda))$ *in* $\mathrm{i}\alpha$ *respectively* $(\lambda - \mathrm{i}\alpha)^{-1}$ *are identical.*

Proof Take a normalized sequence (x_n) in E with

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)x_n - x_n\| = 0.$$

The existence of such a sequence follows from the fact that the fixed space of $(\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)$ is non trivial. Suppose (x_n) is not relatively compact. Then we may assume that there exists $\delta > 0$ such that

$$\|x_n - x_m\| > \delta \quad \text{for} \quad n \neq m.$$

Take $k \in \mathbb{N}$ and let $\hat{x}_k$ be the image of (x_{n+k}) in $\hat{E}$. Since

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)x_{n+k} - x_{n+k}\| = 0,$$

the so defined $\hat{x}_k$'s belong to $\operatorname{Fix}\big((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\big)$. Since this space is finite dimensional there exist $j < \ell$, such that

$$\|\hat{x}_j - \hat{x}_\ell\| \leqslant \frac{\delta}{2}.$$

From the definition of the norm in $\hat{E}$ it follows that there are natural numbers $n < m$ such that

$$\|x_n - x_m\| \leqslant \frac{\delta}{2},$$

leading to a contradiction.

Therefore every approximate eigenvector of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ pertaining to α is relatively compact. In particular, it has a convergent subsequence from which it follows that the fixed space of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ is non trivial.

Obviously

$$\dim \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda)) \leqslant \dim \operatorname{Fix}\big((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\big).$$

If the last inequality is strict, then there exists $\gamma > 0$ and a normalized $\hat{x} \in \operatorname{Fix}\big((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\big)$ such that

$$\gamma \leqslant \|\hat{y} - \hat{x}\|$$

for all $y \in \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$.

Take a normalized sequence $(x_n) \in \hat{x}$. Then (x_n) has a convergent subsequence, whence we may assume that $\lim_n x_n = z$ exists in E. Thus $0 \neq z \in \operatorname{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$. From this we obtain the contradiction

$$0 \leqslant \gamma \leqslant \|\hat{z} - \hat{x}\| = \lim \|z - x_n\| = 0$$

Consequently,

$$\dim \text{Fix}\left((\lambda - \mathrm{i}\alpha)R(\lambda)\right) = \dim \text{Fix}\left(\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right).$$

Let $\{x_1, ..., x_n\}$ be a base of $\text{Fix}\left((\lambda - \mathrm{i}\alpha)R(\lambda)\right)$ and choose $\{\varphi_1, ..., \varphi_n\}$ in $\text{Fix}\left((\lambda - \mathrm{i}\alpha)R(\lambda)'\right)$ such that $\varphi_k(x_j) = \delta_{k,j}$ (Lemma 1.6. Then

$$E = \text{Fix}\left((\lambda - \mathrm{i}\alpha)R(\lambda)\right) \oplus \bigcap_{j=1}^{n} \ker(\varphi_j),$$

where both subspaces on the right are $(\lambda - \mathrm{i}\alpha)R(\lambda)$-invariant and 1 is a pole of $(\lambda - \mathrm{i}\alpha)R(\lambda)|_{\text{Fix}(()(\lambda - \mathrm{i}\alpha)R(\lambda))}$ by the finite dimensionality of $\text{Fix}\left((\lambda - \mathrm{i}\alpha)R(\lambda)\right)$.

Suppose 1 belongs to the spectrum of S where S is the restriction of $(\lambda - \mathrm{i}\alpha)R(\lambda)$ to $\bigcap_{j=1}^{n} \ker \varphi_j$. Then there exists a normalized sequence (y_n) in $\bigcap_{j=1}^{n} \ker(\varphi_j)$ such that

$$\lim_n \|(\lambda - \mathrm{i}\alpha)R(\lambda)y_n - y_n\| = 0.$$

Therefore (y_n) has an accumulation point different from zero contained in

$$\text{Fix}\left((\lambda - \mathrm{i}\alpha)R(\lambda)\right) \cap (\bigcap_{j=1}^{n} \ker \varphi_j).$$

This contradiction implies that 1 does not belong to the spectrum of S.

Since $\text{Fix}\left((\lambda - \mathrm{i}\alpha)R(\lambda)\right)$ is finite dimensional, it follows from general spectral theory that $(\lambda - \mathrm{i}\alpha)^{-1}$ is a pole of $R(., R(\lambda))$ for every λ. Thus (i) and (ii) are proved and assertion (iii) follows from the resolvent equality as in the proof of Greiner [4, Proposition 1.2]. □

Proposition 2.3 *Let $\mathcal{T}$ be a semigroup of contractions on a Banach space E with generator A. Then the following assertions are equivalent.*

(a) *Each* $\mathrm{i}\alpha$*,* $\alpha \in \mathbb{R}$*, is a pole of the resolvent* $R(., A)$ *such that the corresponding residue has finite rank.*

(b) $\dim \text{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda, A)\right) < \infty$ *for some (hence all)* $\lambda \in \mathbb{C}$*,* $\operatorname{Re}\lambda > 0$ *and the canonical extensions* $\hat{R}(\lambda, A)$ *of* $R(\lambda, A)$ *to some ultrapower.*

Proof Let P_α be the residue of the resolvent $R(., A)$ in $\mathrm{i}\alpha$. Then $P_\alpha = \lim_{\lambda \to \mathrm{i}\alpha}(\lambda - \mathrm{i}\alpha)R(\lambda, A)$ in the operator norm of $L(E)$. Since the canonical map $(T \mapsto \hat{T})$ is isometric and since $\hat{E}$ is an ultrapower, we obtain

$$\hat{P}_\alpha = \lim_{\lambda \to \mathrm{i}\alpha} (\lambda - \mathrm{i}\alpha)\hat{R}(\lambda, A)$$

in $L(\hat{E})$ and $\text{rank}\,(P_\alpha) = \text{rank}\,(\hat{P}_\alpha)$. Because of

$$\hat{P}_\alpha(\hat{E}) = \text{Fix}\left((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\right)$$

one part of the corollary is proved. The other follows from Lemma 2.2 on page 231. □

Remarks 2.4 (i) By the results in Lin [12] a semigroup of contractions on a Banach space is uniformly ergodic if and only if 0 is a pole of the generator with order $\leqslant 1$. The residue of the resolvent in 0 and the assocciated ergodic projection are identical.

(ii) Let M be a W*-algebra with predual M_*, $\mathfrak{U}$ a free ultrafilter on $\mathbb{N}$ and $\widehat{M}$ (resp. $(M_*)^\wedge$) the ultrapower of M (resp. M_*) with respect to $\mathfrak{U}$. Then it is easy to see that $c_{\mathfrak{U}}(M)$ is a two sided ideal in $\ell^\infty(M)$ hence $\widehat{M}$ is a C*-algebra, but in general not a W*-algebra. Note that the unit of $\widehat{M}$ is the canonical image of 1. For $\hat{x} \in \hat{M}$ and $\hat{\varphi} \in (M_*)^\wedge$ let $J : (M_*)^\wedge \to \widehat{M}'$ be defined by

$$\langle x, J(\hat{\varphi})\rangle := \lim_{\mathfrak{U}} \varphi_n(x_n)\,,\ (x_n) \in \hat{x}\,,\ (\varphi_n) \in \hat{\varphi}.$$

Then J is well defined and an isometric embedding. It turns out that $J((M_*)^\wedge)$ is a translation invariant subspace of $\widehat{M'}$. Hence there exists a central projection $z \in \widehat{M''}$ such that $J((M_*)^\wedge) = \widehat{M''}z$ (Groh [9, Proposition 2.2]).

Below we identify $(M_*)^\wedge$ via J with this translation invariant subspace. From the construction the following is obvious: If T is an identity preserving Schwarz map with preadjoint $T_* \in L(M_*)$, then $\widehat{T}$ is an identity preserving Schwarz map on $\widehat{M}$ such that $(T_*)^\wedge = \widehat{T}'|(M_*)^\wedge$.

Theorem 2.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type with generator A on the predual of a* W*-*algebra M. If $\mathcal{T}$ is uniformly ergodic with finite dimensional fixed space, then every $\gamma \in \sigma(A) \cap \mathrm{i}\mathbb{R}$ is a pole of the resolvent $R(.,A)$ and* $\dim \ker(\gamma - A) \leqslant \dim \mathrm{Fix}\,(T)$.

Proof Let $D = \{\lambda \in \mathbb{C}\colon\ \mathrm{Re}\,\lambda > 0\}$ and $\mathcal{R}$ the M_*-valued pseudo-resolvent of Schwarz type induced by $R(.,A)$ on D. Then

$$P = \lim_{\mu\downarrow 0} \mu R(\mu)$$

exists in the uniform operator topology. Since $P(E) = \mathrm{Fix}\,(T)$, we obtain $\hat{P}(\hat{E}) = \mathrm{Fix}\,(\hat{T})$ and $\dim \mathrm{Fix}\,(T) = \dim \mathrm{Fix}\,(\hat{T}) < \infty$, where $\hat{P}$ is the canonical extension of P onto $(M_*)^\wedge$. Since $\hat{P} = \lim_{\mu\downarrow 0} \mu R(\mu)^\wedge$ it follows that

$$\dim \mathrm{Fix}\,\big((\lambda - \mathrm{i}\alpha)\hat{R}(\lambda)\big) \leqslant \dim \mathrm{Fix}\,\big(\hat{T}\big) < \infty$$

for all $\alpha \in \mathbb{R}$ (Proposition 2.1 on page 229). Therefore the assertion follows from Lemma 2.2 on page 231. □

The consequences of this result for the asymptotic behavior of one-parameter semigroups will be discussed in D-IV, Section 4.

Notes

Section 1: The Perron-Frobenius theory for a single positive operator on a non-commutative operator algebra is worked out in Albeverio and Hoegh-Krohn [1] and Groh [7]. The limitations of the theory (in the continuous as in the discrete case) are explained by the example following Remark 1.9 on page 225 (see also Groh [8].Therefore we concentrate on irreducible semigroups. Our main result Theorem 1.11 on page 226 extends B-III, Thm.3.6 to the non-commutative setting.

Section 2: Theorem 2.5 on page 234 has its roots in the Niiro-Sawashima Theorem for a single irreducible positive operator on a Banach lattice (see Schaefer [14, V.5.4]). The analogous semigroup result on Banach lattices is due to Greiner [5]. The ultrapower technique in our proof is developed in Groh [9].

References

[1] S. Albeverio and R. Hoegh-Krohn. Frobenius theory for positive maps on von Neumann algebras. *Comm. Math. Phys.*, 64:83–94, 1978.

[2] N. Dunford and J. Schwartz. *Linear Operators, Part I: General Theory*. Wiley, New York, 1958.

[3] D. Evans. Irreducible quantum dynamical semigroups. *Comm. Math. Phys.*, 54:293–297, 1977.

[4] G. Greiner. Zur Perron-Frobenius Theorie stark stetiger Halbgruppen. *Math. Z.*, 177:401–423, 1981.

[5] G. Greiner. Spektrum und Asymptotik stark stetiger Halbgruppen positiver Operatoren. *Sitzungsber. Heidelb. Akad. Wiss. Math.-Nat. Kl.*, pages 55–80, 1982.

[6] G. Greiner, J. Voigt, and M. Wolff. On the spectral bound of the generator of semigroups of positive operators. *J. Operator Theory*, pages 245–256, 1981.

[7] U. Groh. The peripheral point spectrum of Schwarz operators on C^*-algebras. *Math. Z.*, 176:311–318, 1981.

[8] U. Groh. Some observations on the spectra of positive operators on finite dimensional C^*-algebras. *Linear Algebra Appl.*, 42:213–222, 1982.

[9] U. Groh. Uniform ergodic theorems for identity preserving Schwarz maps on W*-algebras. *J. Operator Theory*, 11:395–404, 1984.

[10] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis I*. Springer, Berlin-Heidelberg-New York, 1963.

[11] K. Jacobs. Gleichverteilung mod 1. In *Selecta Mathematica*, volume IV, pages 57–93. Springer, Berlin-Heidelberg-New York, 1972.

[12] M. Lin. On the uniform ergodic theorem II. *Proc. Amer. Math. Soc.*, 46: 217–225, 1974.

[13] G. K. Pedersen. *C*-Algebras and their Automorphism Groups*. Academic Press, London, New York, San Francisco, 1979.

[14] H. H. Schaefer. *Banach Lattices and Positive Operators*. Springer, New York-Heidelberg-Berlin, 1974.

[15] M. Takesaki. *Theory of Operator Algebras I*. Springer, New York-Heidelberg-Berlin, 1979.

Chapter D-IV

Asymptotics of Positive Semigroups on C*- and W*-Algebras

by
Ulrich Groh

1 Stability of Positive Semigroups

As explained in A-III, Section 1, it is possible to deduce uniform exponential stability of strongly continuous semigroups from the location of the spectrum of its generator if the spectral bound $s(A)$ and the growth bound ω_0 coincide. In this section we prove $s(A) = \omega_0$ for positive semigroups on C*-algebras and preduals of W*-algebras. A more general discussion of the "$s(A) = \omega_0$" problem can be found in Greiner et al. [7]. For the results of this section the existence of a unit is essential.

Theorem 1.1 *Let M be a* C*-*algebra with unit and* $\mathcal{T} = (T(t))_{t\geqslant 0}$ *a positive semigroup on M. Then*

$$-\infty < s(A) = \omega_0 \in \sigma(A).$$

Proof For every $t \geqslant 0$ there exists φ_t in the state space $S(M)$ of M such that

$$T(t)'\varphi_t = r(T(t))\varphi_t = \exp(\omega_0 t)\varphi_t$$

(see, e.g., Groh [8, 2.1]).

Let $n \in \mathbb{N}$ and

$$E_n := \{\varphi \in S(M) : T(2^{-n})\varphi = \exp(\omega_0 2^{-n})\varphi\}.$$

Then $\emptyset \neq E_{n+1} \subseteq E_n$, $(n \in \mathbb{N})$. Since $S(M)$ is $\sigma(M, M')$-compact, there exists $\varphi \in \bigcap_{n\in\mathbb{N}} E_n$. Then $T(t)'\varphi = \exp(\omega_0 t)\varphi$ follows for all $0 \leqslant t$ because the adjoint semigroup $(T(t)')_{t\geqslant 0}$ is a weak*-semigroup on M'.

Suppose $-\infty = \omega_0$. Then for $t > 0$ either $r(T(t)) = 0$ (A-III,Prop.1.1) or $T(t)'\varphi = 0$, in particular $\varphi(T(t)\mathbb{1}) = 0$. From this we obtain the contradiction $\varphi(\mathbb{1}) = 0$. Hence $-\infty < \omega_0$ and $\exp(\omega_0 t) \in \varrho(T(t)')$ for every $t \in \mathbb{R}_+$. Thus $\omega_0 \in \sigma(A)$ or $\omega_0 = s(A)$. □

Remark 1.2 (i) If we consider the nilpotent translation semigroup on the C*-algebra $C_0(\,[0,1[\,)$, then $\sigma(A) = \emptyset$ and $\omega_0 = -\infty$. This shows that the existence of a unit is essential.

(ii) The equality $s(A) = \omega_0$ still holds for positive semigroups on commutative C*-algebras without unit (see B-IV, Rem.1.2.b).

Theorem 1.3 *Let M be a* W**-algebra with predual M_* and let $(T(t))_{t\geqslant 0}$ be a positive semigroup on M_*. Then $s(A) = \omega_0$.*

Proof For all $\lambda > s(A)$ and $\varphi \in M_*$

$$R(\lambda, A)\varphi = \int_0^\infty e^{-\lambda t} T(s)\varphi ds$$

which follows as in C-III, Section 1 or Greiner et al. [7, Theorem 3]. Since $\|\varphi\| = \varphi(\mathbb{1})$ for every $\varphi \in M_*^+$ and since the norm is additive on the positive cone of M_*, the integral

$$\int_0^\infty e^{\lambda t} \|T(s)\varphi\| ds$$

exists for all $\varphi \in M_*$ and all $\lambda > s(A)$. From this the assumption follows by A-IV,Thm.1.11. □

Corollary 1.4 *Let M be a* C**-algebra and $(T(t))_{t\geqslant 0}$ a positive semigroup on M'. Then $s(A) = \omega_0$ holds.*

This follows from the fact that the bidual of a C*-algebra is a W*-algebra (see Takesaki [23, Theorem III.2.4.]).

Remark 1.5 A simple modification of A-III, Example 1.4 (take c_0 instead of ℓ^2) shows that Theorem 1.3 is no longer true for non-positive semigroups (for details see Groh and Neubrander [12, Beispiel 2.5]).

While the growth bound ω_0 characterizes uniform exponential stability of the semigroup there are other (and weaker) stability concepts (cf. A-IV, Section 1).

Definition 1.6 Let E be a Banach space and $(T(t))_{t\geqslant 0}$ a semigroup on E. We call the semigroup

(i) *uniformly exponentially stable* if $\|T(t)\| \leqslant Me^{-\omega t}$ for some $\omega, M > 0$ and all $t \geqslant 0$.

(ii) *uniformly stable* if $\lim_{t\to\infty} T(t) = 0$ in the strong operator topology.

(iii) *weakly stable* if $\lim_{t\to\infty} T(t) = 0$ in the weak operator topology.

Surprisingly all these properties coincide for positive semigroups on C*-algebras with unit.

Theorem 1.7 *Let M be a C*-algebra with unit and $(T(t))_{t\geqslant 0}$ a positive semigroup on M. Then the following assertion are equivalent.*

(a) $s(A) < 0$.

(b) *The semigroup $(T(t))_{t\geqslant 0}$ is uniformly exponentially stable.*

(c) *The semigroup $(T(t))_{t\geqslant 0}$ is uniformly stable.*

(d) *The semigroup $(T(t))_{t\geqslant 0}$ is weakly stable.*

Proof Since $s(A) = \omega_0$ by Theorem 1.3, it suffices to show that (d) implies (a). For $t > 0$ there exists $\varphi \in S(M)$ such that

$$T(t)'\varphi = r(T(t))\varphi.$$

Then for $x \in M$

$$\varphi(T(t)^n x) = (r(T(t)))^n \varphi(x) \to 0$$

as $n \to \infty$. Therefore $r(T(t)) < 1$ or $\omega_0 < 0$. Since $s(A) \leqslant \omega_0$ the assertion follows. □

Remark 1.8 Consider the translation semigroup $(T(t))_{t\geqslant 0}$ on $C_0(\mathbb{R}_+)$. Then $\|T(t)\| = 1$, hence $s(A) = 1$, but $(T(t))_{t\geqslant 0}$ is strongly stable. The same holds for the translation semigroup on $L^1(\mathbb{R}_+)$. Thus Theorem 1.7 is not true for semigroups on C*-algebras without unit or on preduals of W*-algebras. For the discussion of the commutative situation we refer to B-IV, Section 1.

2 Stability of Implemented Semigroups

Let H be a Hilbert space, $\mathcal{U} = (U(t))_{t\geqslant 0}$ a strongly continuous semigroup on H with generator B and $M \subseteq \mathcal{B}(H)$ a W*-algebra, where $\mathcal{B}(H)$ is the W*-algebra of all bounded linear operators on H. Suppose $\mathcal{U}(t)^* M U(t) \subseteq M$. Then one can define a weak*-continuous semigroup $\mathcal{T}$ on M by

$$T(t)x := U(t)^* x U(t) \quad (t \in \mathbb{R}_+, x \in M).$$

We call $\mathcal{T}$ an *implemented semigroup*. Every map $T(t) \in \mathcal{T}$ of an implemented semigroup is weak*-continuous and n-positive for every $n \in \mathbb{N}$.

Remarks 2.1 (i) Because of

$$\|T(t)\| = \|T(t)\mathbb{1}\| = \|U(t)^* U(t)\| = \|U(t)\|^2$$

it follows that $\omega_0(\mathcal{T}) = 2\,\omega_0(\mathcal{U})$.

(ii) If $\mathcal{T}$ is an implemented semigroup, then the preadjoint semigroup is strongly continuous on M_*. Therefore $s(A) = \omega_0$ for $\mathcal{T}$ by Theorem 1.3.

(iii) Since $\mathcal{U}$ is a strongly continuous semigroup on H, the same is true for the adjoint semigroup $\mathcal{U}^* = \{U(t)^* \colon U(t) \in \mathcal{U}\}$ and its generator is given by B^*. In analogy to Bratteli and Robinson [3, 3.2.55] the following assertions for $x \in M$ are equivalent.

(a) $x \in D(A)$, A the generator of $\mathcal{T}$.

(b) For $\xi \in D(B)$ it follows $x\xi \in D(B^*)$ and the linear mapping

$$(\xi \mapsto x(B\xi) + B^*(x\xi)) : D(B) \to H \tag{*}$$

has a continuous extension to H.

Then for A is given as the continuous extension of (*), i.e., $Ax = xB + B^*x$ for $x \in D(A)$

In the next theorem we give some equivalent conditions for the uniform exponential stability of an implemented semigroup. As we shall see, the operator equality

$$yB + B^*y = -x \quad (x, y \in M_+)$$

is necessary and sufficient, which is in complete analogy to the classical Liapunov stability result.

Theorem 2.2 *Let M be a* W**-algebra on a Hilbert space H and let $\mathcal{T} = (T(t))_{t\geqslant 0}$ be a weak*-semigroup on M with generator A implemented by the semigroup $\mathcal{U}$ on H with generator B. Then the following assertions are equivalent.*

(a) $\omega_0(\mathcal{T}) = s(A) < 0$.

(b) *The semigroup $(U(t))_{t\geqslant 0}$ is uniformly exponentially stable.*

(c) *There exists $0 \leqslant x \in D(A)$ such that $Ax = -\mathbb{1}$.*

(d) *There exists $0 \leqslant x \in D(A)$ such that $x(D(B)) \subseteq D(B^*)$ and $xB + B^*x = -\mathbb{1}$.*

(e) *For every $0 \leqslant x \in D(A)$ there exists $0 \leqslant y \in D(A)$ such that $Ay = -x$.*

(f) *For every $0 \leqslant x \in D(A)$ there exists $0 \leqslant y \in D(A)$ such that $y(D(B)) \subseteq D(B^*)$ and $yB + B^*y = -x$.*

(g) *$\int_0^\infty \|U(s)\xi\|^2 ds$ exists for all $\xi \in H$.*

(h) *$\int_0^\infty |(T(s)x)\xi|\zeta) ds$ exists for all $\xi, \zeta \in H$ and all $x \in M$.*

Proof The equivalence of (a) and (b) follows from Remark 2.1 (i), whereas (c) and (d)), resp. (e) and (f) are equivalent by the Remark 2.1 (iii)

(a) $\implies$ (c): Since $s(A) < 0$ the resolvent $R(0, A)$ exists and is a positive map on M. Therefore $R(0, A)1 \in D(A)_+$ or $Ax = -\mathbb{1}$ for some $x \in D(A)_+$.

(c) $\implies$ (e): Let $x \in D(A)_+$ such that $Ax = -\mathbb{1}$. Then

$$T(t)x - x = \int_0^t T(s)Ax\,\mathrm{d}s = -\int_0^t T(s)\mathbb{1}\,\mathrm{d}s \quad (t \geqslant 0),$$

hence

$$0 \leqslant \int_0^t T(s)\mathbb{1}\,\mathrm{d}s \leqslant x \quad (t \in \mathbb{R}_+).$$

Since the family $(\int_0^t T(s)\mathbb{1}\,\mathrm{d}s)_{t\geqslant 0}$ is increasing and bounded,

$$\lim_{t\to\infty} \int_0^t T(s)\mathbb{1}\,\mathrm{d}s$$

exists in the weak operator topology on $\mathcal{B}(H)$.

Since on bounded sets of M, the weak operator topology is equivalent to the $\sigma(M, M_*)$-topology, for every $\varphi \in M_*$ the integral $\int_0^\infty \varphi(T(s)\mathbb{1})\,\mathrm{d}s$ exists (Sakai [19, 1.15.2.]). Take $x \in M_+$ and $\varphi \in M_*^+$. Then $x \leqslant \|x\|\mathbb{1}$ and therefore

$$\varphi(T(s)x) \leqslant \|x\|\varphi(T(s)\mathbb{1}) \quad (s \in \mathbb{R}_+).$$

Hence $\int_0^\infty \varphi(T(s)x)ds$ exists. Since the positive cones of M and M_* are generating, $\int_0^\infty \varphi(T(s)x)ds$ exists for every $x \in M$ and $\varphi \in M_*$. Therefore $R(0, A)$ exists and is positive which proves (e).

(c) $\implies$ (g): From the last paragraph we obtain that for all $\xi \in H$

$$\int_0^\infty \|U(s)\|^2 ds = \int_0^\infty (T(s)1\xi|\xi)ds$$

exists.

(g) $\implies$ (h): It follows from the polarization identity that the integral

$$\int_0^\infty (U(s)\xi|U(s)\zeta)ds$$

exists for all $\xi, \zeta \in H$. Using Takesaki [23, Theorem III.4.2 and Theorem II.2.6], we conclude as in the implication from (c) to (e) that for all $\xi, \zeta \in H$ the integral

$$\int_0^\infty ((T(s)x)\xi|\zeta)ds \quad (x \in M)$$

is finite.

(g) $\implies$ (a): Since the vector states are dense in the predual of M and since the preadjoint semigroup of $\mathcal{T}$ is strongly continuous, it is easy to see that the integral

$$\int_0^\infty \varphi(T(s)x)ds$$

exists for all $x \in M$ and $\varphi \in M_*$ (Takesaki [23, Theorem II.2.6]). Therefore, the resolvent $R(0, A)$ exists and is positive, hence $s(A) < 0$. □

3 Convergence of Positive Semigroups

In this section the asymptotic behavior of positive semigroups $(T(t))_{t\geqslant 0}$ on W*-algebras will be described in more detail. Essentially we distinguish three cases.

(i) The Cesàro means $\frac{1}{s}\int_0^s T(t)dt$ converge strongly to a projection P onto the fixed space of $(T(t))_{t\geqslant 0}$ (see Proposition 3.3 & 3.4).

(ii) The maps $T(t)$ converge strongly to P (see Proposition 3.7, 3.8 &. 3.9).

(iii) The maps $T(t)$ behave asymptotically as a periodic group (Theorem 3.11).

Much of the following is based on the theory of weakly compact operator semigroups. Therefore the following compactness criterium is quite useful.

Proposition 3.1 *Let M be a* W*-*algebra, $\mathcal{T}$ a bounded semigroup of positive maps on M_* and suppose that there exists a faithful family Φ of $\mathcal{T}$-subinvariant states in M_*. Then $\mathcal{T}$ is relatively compact in the weak operator topology of $\mathcal{L}(M_*)$. In particular, $\mathcal{T}$ is strongly ergodic, i.e.,*

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s T(t)x\,\mathrm{d}t$$

exists for every x in M and yields a projection onto Fix $(\mathcal{T})$.

Proof Since the positive cone of M_* is generating, it is enough to show that for every $0 \leqslant \varphi \in M_*$ the orbit $\{T(t)\varphi \colon t \in \mathbb{R}_+\}$ is relatively weak compact. For this we use Takesaki [23, Theorem III.5.4.(iii)].

Let $(p_n)_{n\in\mathbb{N}}$ be a decreasing sequence of projections in M such that $\inf_n p_n = 0$. Then $\lim_n \varphi(p_n) = 0$ for every $\varphi \in \Phi$. Since

$$(T(t)p_n)^2 \leqslant T(t)p_n, \quad t \in \mathbb{R}_+,$$

we obtain by a classical *inequality of Kadison* that

$$0 \leqslant \varphi((T(t)p_n)^2) \leqslant \varphi(T(t)p_n) \leqslant \varphi(p_n),$$

hence $\lim_n \varphi(T(t)p_n) = 0$ uniformly in $t \in \mathbb{R}_+$. Since the family Φ is faithful on M, it follows from Takesaki [23, Proposition III.5.3] that $(T(t)p_n)$ converges to zero in the $s(M, M_*)$-topology uniformly in $t \in \mathbb{R}_+$. Since this topology is finer than the weak*-topology on M, we obtain the relative compactness of $\mathcal{T}$ which implies the strong ergodicity. □

Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a W*-algebra M. We call

$$p_r := \sup\{s(|\varphi|) : \varphi \in \text{ Fix }(T)\}$$

the recurrent projection associated with $\mathcal{T}$. For a motivation of this definition compare, e.g., Davies [4, Section 6.3].

Since $T(t)|\varphi| = |\varphi|$ for all $\varphi \in$ Fix (T) (D-III, Cor. 1.5), we obtain $T(t)'p_r \geqslant p_r$ (see D-I,Sec.3.(c)). Let $\mathcal{T}^{(r)}$ be the reduced semigroup on $p_r M_* p_r$ with generator $A^{(r)}$. Then $\mathcal{T}^{(r)}$ is identity preserving and of Schwarz type. Similarly, if $\mathcal{R}$ is a pseudo-resolvent on $D = \{\lambda \in \mathbb{C}\colon \text{ Re}(\lambda) > 0\}$ with values in M_* such that $\mathcal{R}$ is identity preserving and of Schwarz type, then the recurrent projection associated with $\mathcal{R}$ is defined using Fix $(\mathcal{R})$.

Remark 3.2 (i) Let $\varphi \in M_*$ and $\alpha \in \mathbb{R}$ such that $(\mu - \mathrm{i}\alpha)R(\mu)\varphi = \varphi$ for some $\mu \in \mathbb{R}_+$. Since $s(|\varphi|)$ and $s(|\varphi^*|)$ are majorized by p_r (D-III,Prop.1.4), it follows that φ and φ^* are in $p_r M_* p_r$.

(ii) From (i) and the observation that the family $\{|\varphi|\colon \varphi \in \text{Fix}(\mathcal{T})\}$ is faithful on $p_r M p_r$ and consists of $\mathcal{T}^{(r)}$-invariant elements, it follows that

- $P\sigma(A) \cap \mathrm{i}\mathbb{R} = P_\sigma(A^{(r)}) \cap \mathrm{i}\mathbb{R}$.
- $\ker((\mathrm{i}\alpha - A)) \subset p_r M_* p_r$ for all $\alpha \in \mathbb{R}$.
- The semigroup $\mathcal{T}^{(r)}$ is relatively compact in the weak operator topology and therefore strongly ergodic.

(iii) Similarly, let $\mathcal{R}$ be an identity preserving pseudo-resolvent with values in M_* on $D = \{\lambda \in \mathbb{C}\colon \text{ Re}(\lambda) > 0\}$ which is of Schwarz type. It follows as in (b) that $\text{Fix}((\lambda - \mathrm{i}\alpha)R(\lambda))$ is contained in $p_r M_* p_r$ for all $\lambda \in D$ and $\alpha \in \mathbb{R}$, where p_r is the associated recurrent projection.

We now give a characterization of strong ergodicity of semigroups which are identity preserving and of Schwarz type. For this we need that the Cesàro means

$$C(s)x = \frac{1}{s}\int_0^s T(t)x\,dt \quad (x \in M, 0 \leqslant s \in \mathbb{R})$$

are Schwarz maps. We omit the simple calculation (compare D-I,Thm.2.1).

Proposition 3.3 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W*-*algebra M. Then the following assertions are equivalent.*

(a) *$\mathcal{T}$ is strongly ergodic on M_*.*

(b) $\sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)'p_r = 1$.

(c) $s^*(M, M_*)\text{-}\lim_{s\to\infty} C(s)' p_r = 1$.

Proof Suppose that (a) holds. Since $\mathrm{Fix}\,(T)$ separates $\mathrm{Fix}(T')$ (see Krengel [14, Chap.2,Thm.1.4]), the fixed space of $\mathcal{T}'$ is non trivial, hence $p_r \neq 0$. Let $0 \leqslant \psi \in M_*$, then $\psi_0 := \lim_{s\to\infty} C(s)\psi \in \mathrm{Fix}\,(T)$ and $s(\psi_0) \leqslant p_r$. Therefore

$$\begin{aligned}\lim_{s\to\infty} \psi(C(s)' p_r) &= \lim_{s\to\infty} (C(s)\psi)(p_r) = \psi_0(p_r) \\ &= \psi_0(1) = \lim_{s\to\infty} (C(s)\psi)(1) = \psi(1)\end{aligned}$$

which proves (b).

Suppose that (b) is satisfied. Since $C(s)' p_r \leqslant 1$ for all $s \in \mathbb{R}_+$, we obtain (c). (Use that for $(x_\alpha) \in M_+$ we have $\lim_\alpha x_\alpha = 0$ in the weak*-topology if and only if $\lim_\alpha x_\alpha = 0$ in the $s^*(M, M_*)$-topology.)

Suppose that (c) holds. Since each $C(s)'$ is an identity preserving Schwarz map, we obtain for all $x \in M$

$$\begin{aligned}(C(s)'((1-p_r)x))(C(s)'((1-p_r)x)^*) &\leqslant C(s)'((1-p_r)xx^*(1-p_r)) \\ &\leqslant \|x\|^2 C(s)'(1-p_r),\end{aligned}$$

hence

$$s^*(M, M_*)\text{-}\lim_{s\to\infty} C(s)'((1-p_r)x) = 0.$$

In particular, we obtain for all $x \in \mathrm{Fix}(\mathcal{T}')$ that $x = \sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)' x = \sigma(M, M_*)\text{-}\lim_{s\to\infty} C(s)'(p_r x)$.

Especially for $0 \neq x \in \mathrm{Fix}\,(\mathcal{T})$ we obtain $p_r x p_r \neq 0$. Since the W*-algebra $p_r M p_r$ is the dual of $p_r M_* p_r$ and since $\mathcal{T}^{(r)}$ is strongly ergodic, it follows that the fixed space of $\mathcal{T}$ separates the points of $\mathrm{Fix}\,(\mathcal{T}')$. Thus $\mathcal{T}$ is strongly ergodic (Krengel [14, Chap. 2, Thm. 1.4]). ◻

It follows from the result above that the semigroup in Evans [5] cannot be strongly ergodic on $\mathcal{B}(H)_*$ since the associated recurrent projection is zero. But for irreducible semigroups we have the following result.

Proposition 3.4 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W**-algebra M. Then the following assertions are equivalent.*

(a) *$\mathcal{T}$ is irreducible and $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$.*

(b) *$\mathcal{T}$ is relatively compact in the weak operator topology and the fixed space of $\mathcal{T}$ is generated by a faithful state.*

(c) *$\mathcal{T}$ is strongly ergodic and the fixed space of $\mathcal{T}$ is generated by a faithful state.*

(d) *The fixed space of $\mathcal{T}$ is generated by a faithful state.*

Proof Suppose (a) is satisfied. Since $\text{Fix}(\mathcal{T}) \neq \{0\}$, there exists a faithful normal state φ on M such that $\text{Fix}(\mathcal{T}) = \mathbb{C}\,\varphi$ (D-III, Thm.1.10.). Therefore $\mathcal{T}$ is relatively compact in the weak operator topology by Proposition 3.1., whence (b) holds and the implications from (b) to (c) and (c) to (d) are obvious.

Suppose that (d) holds. Let φ be a faithful normal state on M such that $\text{Fix}(T) = \varphi\mathbb{C}$. By Proposition 3.1 the semigroup $\mathcal{T}$ is strongly ergodic. Therefore the fixed space of $\mathcal{T}$ separates the points of $\text{Fix}(T')$. Consequently $\text{Fix}(T') = \mathbb{C}1$. Thus the ergodic projection associated with $\mathcal{T}$ is given by $P = 1 \otimes \varphi$, i.e., $P\psi = \psi(1)\varphi$ for all $\psi \in M_*$. Let F be a closed $\mathcal{T}$-invariant face of M_*^+. If $0 \neq \psi \in F$ then

$$\lim_{s\to\infty} C(s)\psi = \psi(1)\varphi \in F.$$

Hence $\varphi \in F$ and therefore $F = M_*^+$ by the faithfulness of φ which proves (a). □

The next theorem is an extension of D-III, Thm.1.10 and shows the usefulness of the theory of semitopological semigroups. Assume $\mathcal{T} \subseteq \mathcal{L}(M_*)$ to be relatively compact in the weak operator topology. Since $\mathcal{T}$ is commutative its closure $\mathcal{S} = (\mathcal{T})^- \subseteq L_w(M_*)$ contains a unique minimal ideal $\mathcal{K}$, called the kernel of $\mathcal{S}$, which is a compact Abelian group (**?**], Junghenn [13] & Krengel [14, § 2.4]). The identity Q of $\mathcal{K}$ is a projection onto the closed linear span of all eigenvectors of A pertaining to the eigenvalues in $\mathrm{i}\mathbb{R}$.

Moreover, the dual group of $\mathcal{K}$ can be identified with the subgroup of $\mathrm{i}\mathbb{R}$ generated by $P\sigma(A) \cap \mathrm{i}\mathbb{R}$. We call Q the semigroup projection associated with $\mathcal{T}$. On the other hand, $\mathcal{T}$ is always strongly ergodic with projection P onto $\text{Fix}(\mathcal{T})$. Obviously, the relation

$$0 \leqslant P \leqslant Q \leqslant \mathrm{Id}$$

holds, where the order relation is defined by the inclusion of the range spaces.

There are two extreme cases. First, $Q = \mathrm{Id}$ and $\operatorname{rank}(P)$. This corresponds to the Halmos-von Neumann Theorem in commutative ergodic theory and is discussed, at least for irreducible semigroups, in Olesen et al. [18].

Second, $\mathrm{Id} > Q = P$, in particular $\operatorname{rank}(P) = 1$. This latter case will be investigated in detail for $M = \mathcal{B}(H)$, the W*-algebra of all bounded linear operators on a Hilbert space H. But we first need some preparations.

Theorem 3.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W**-algebra M and suppose there exists a faithful family of $\mathcal{T}$-invariant states on M. Let N be the $\sigma(M, M_*)$-closed linear span of all eigenvectors of A' pertaining to the eigenvalues in* $\mathrm{i}\mathbb{R}$. *If Q is the semigroup projection associated with $\mathcal{T}$, then the following holds.*

(i) *The adjoint of Q is a faithful normal conditional expectation from M onto the* W**-subalgebra N.*

(ii) *The restriction of T' to N can be embedded into a $\sigma(M, M_*)$-continuous, one-parameter group of *-automorphisms.*

(iii) *If, in addition, $\mathcal{T}$ is irreducible and if φ is the normal state generating the fixed space of $\mathcal{T}$, then $\varphi|_N$ is a faithful normal trace.*

Proof Consider $H := P\sigma(A) \cap \mathrm{i}\mathbb{R}$ which is not empty by assumptions. From Proposition 3.1 it follows that $\mathcal{T}$ is relatively compact in the weak operator topology. Let K be the semigroup kernel of $\overline{\mathcal{T}}w \subset L(M_*)$ and Q the unit of K. Recall that $Q\psi_n = \psi_n$ for all $\psi_n \in M_*$ such that $A\psi_n = n\psi_n$ $(n \in H)$. Let $\mathcal{E}$ be the family of all eigenvectors of A' pertaining to the eigenvalues in H.

Then $\mathcal{E}$ is closed with respect to the multiplication in M and the formation of adjoints. Thus N is a W*-subalgebra of M, Sakai [19, Corollary 1.7.9.], and $\mathcal{T}_0(t)' := T(t)'_{|N}$ is multiplicative (for this see D-III, Lemma 1.1).

Since $Q \in \overline{\mathcal{T}}w \subseteq L_w(M_*)$, there exists an ultrafilter $\mathfrak{U}$ on $\mathbb{R}_+$ such that

$$\lim_{\mathfrak{U}} \langle T(t)\psi, x\rangle = \langle Q\psi, x\rangle$$

for all $x \in M$ and $\psi \in M_*$. If $n \in H$ and $\psi_n \in M_*$ such that $A\psi_n = n\psi_n$, then for all $x \in M$ we obtain

$$\langle \psi_n, x\rangle = \langle Q\psi_n, x\rangle = \lim_{\mathfrak{U}} \langle T(t)\psi_n, x\rangle = (\lim_{\mathfrak{U}} e^{nt})\langle \psi_n, x\rangle,$$

hence $\lim_{\mathfrak{U}} e^{nt} = 1$. From this it follows that for all $\psi \in M_*$ we have

$$\langle \psi, Q'(u_n)\rangle = \lim_{\mathfrak{U}} \langle \psi, T(t)'u_n\rangle = (\lim_{\mathfrak{U}} e^{nt})\langle \psi, u_n\rangle = \langle \psi, u_n\rangle.$$

Hence $N \subseteq Q'(M)$.

For γ in the dual group of K and $x \in M$ we define x_γ by

$$\psi(x_\gamma) := \int_K \langle S\psi, x\rangle\langle S, \gamma\rangle^* \,\mathrm{d}m(S) \quad (\psi \in M_*^+).$$

Then $x_\gamma \in M$ and $T(t)'x_\gamma = \langle QT(t), \gamma\rangle x_\gamma$. Therefore $x_\gamma \in N$. Thus the inclusion $Q'M \subseteq N$ is proved if we can show that $Q'M$ belongs to the $\sigma(M, M_*)$-closed linear span of $\{x_\gamma : \gamma \in K, x \in M\}$. For this it is enough to show that every linear form $\psi \in M_*$ such that $\psi(x_\gamma) = 0$ for all $\gamma \in K$ satisfies $\psi(Qx) = 0$ for all $x \in M$. But if $\psi(x_\gamma) = 0$, then

$$\int_K \langle S\psi, x\rangle\langle S, \gamma\rangle^* \,\mathrm{d}m(S) = 0, \gamma \in K.$$

Since the map $(S \mapsto \psi(Sx))$ is continuous on K and since the elements of K form a complete orthonormal basis in $L^2(K, \mathrm{d}m)$, we obtain $\psi(Sx) = 0$ for all $S \in K$, in particular $\psi(Qx) = 0$ as desired.

Since the range of Q' is a W*-subalgebra of M it follows from Takesaki [23, Theorem III.3.4] that Q' is a completely positive, normal conditional expectation. This Q' is faithful, i.e., $\ker(Q') \cap M_+ = \{0\}$ since $Q\varphi = \varphi$ for the faithful linear form φ.

Let φ be the faithful normal state generating $\mathrm{Fix}\,(T)$ and let $\mathcal{U}$ be a family of unitary eigenvectors of A' pertaining to the eigenvalues in H (see D-III, Remark 1.11). If $u_1, u_2 \in U$, then

$$\varphi(u_1 u_2^*) = \varphi(T_0(t)'(u_1 u_2^*)) = e^{(n_1 - n_2)t} \varphi(u_1 u_2^*).$$

Therefore

$$\varphi(u_1 u_2^*) = \begin{cases} 0 & \text{if } n_1 \neq n_2, \\ 1 & \text{if } n_1 = n_2. \end{cases}$$

Hence $\varphi(u_1 u_2^*) = \varphi(u_2^* u_1)$ from which it follows that $\tau := \varphi|_N$ is a faithful normal trace. □

Remarks 3.6 (i) Since $QM_* = N_*$ and $Q'M = N$, where N_* is as in D-III, Proposition 1.12, it follows from general duality theory that $(N_*)' = N$.

(ii) If $\psi \in N_*$, then $|\psi| \in N_*$. To see this, note that $Q\psi = \psi$ and Q is an identity preserving Schwarz map. Then the assertion follows from D-III, Proposition 1.4.

(iii) If $\psi \in N_*$, then $|T_0(t)\psi| = T_0(t)|\psi|$ for all $t \in \mathbb{R}$. This follows immediately from the fact that $\mathcal{T}_0(t)'$ is a *-automorphismus on N.

(iv) Let us add a few words concerning the structure of N: If $\mathcal{T}$ is irreducible and K is the semigroup kernel of $\mathcal{T}^- \subseteq L_w(M_*)$, then $(S \mapsto S') : K \to L((N, \sigma(N, N_*)))$ is a representation of the compact, Abelian group K as group of *-automorphism such that the fixed space is one dimensional. Therefore we are able to apply the results of Olesen et al. [18]. There are three possibilities for N.

1. $N = L^\infty(K, \mathrm{d}m)$ and $\mathcal{T}|_N$ is the translation group on N.
2. $N \cong R$ where $\mathcal{R}$ is the (unique) hyperfinite factor of type II_1. In that case (the image of) K is approximately inner on $\mathcal{R}$ [l.c., Theorem 5.8].
3. There exists a closed subgroup G of K such that

   $$N = L^\infty(K/G, \mathrm{d}m) \otimes R$$

   where R is as in (ii) and dm the normalized Haar measure on K/G [l.c., Theorem 5.15].

So far we have studied weak*-semigroups on general W*-algebras. We apply now these results to weak*-semigroup on $\mathcal{B}(H)$. To do this we call a triple $(M, \varphi, \mathcal{T})$ a W*-dynamical system if M is a W*-algebra, $\mathcal{T}$ a weak*-semigroup of identity preserving Schwarz maps on M and φ a faithful family of $\mathcal{T}$-invariant normal states. We call $(M, \varphi, \mathcal{T})$ irreducible, if the preadjoint semigroup is irreducible (alternatively, if the fixed space of $\mathcal{T}$ is one dimensional).

Proposition 3.7 *Let $(\mathcal{B}(H), \varphi, \mathcal{T})$ be a* W*-*dynamical system on the* W*-*algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H. Then the following assertions are equivalent:*

(a) $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$,

(b) $\lim_{s\to\infty} T(s)_* = P_*$ *in the strong operator topology on* $\mathcal{L}(\mathcal{B}(H)_*)$.

Proof Obviously (b) implies (a). Suppose that (a) is fulfilled. Then the ergodic projection P_* of the preadjoint semigroup is equal to the associated semigroup projection. Consequently there exists an ultrafilter $\mathfrak{U}$ on $\mathbb{R}_+$ such that $\lim_{\mathfrak{U}} T(t) = P$ in the weak operator topology. We claim that the convergence holds even in the strong operator topology. Taking this for granted it follows, since for every $t \in \mathbb{R}_+$ $T(t)$ is a contraction, that

$$\lim_{t\to\infty} \|T(t)_*\varphi\| = 0$$

for all $\varphi \in \ker(P_*)$. Since $T(t)_*\psi = \psi$ for every $\psi \in \mathrm{im}(P_*)$ and

$$\mathcal{B}(H)_* = \mathrm{im}(P_*) \oplus \ker(P_*)$$

the assertion is proved.

It remains to show that $\lim_{\mathfrak{U}} T(t)_* = P_*$ in the strong operator topology. Choose $0 \leqslant \varphi \in \mathcal{B}(H)_*$, $\|\varphi\| \leqslant 1$ and let $\varphi_t := T(t)_*\varphi$ $(t > 0)$. $\varphi_0 := P_*\varphi$ and let $\{p_i : i \in A\}$ be an increasing net of projections of finite rank in $\mathcal{B}(H)$ with strong limit 1. Since the set $K := \{\varphi_t : t \geqslant 0\}$ is relatively compact in the $\sigma(\mathcal{B}(H)_*, \mathcal{B}(H))$-topology, there exists for every $\delta > 0$ an index $i_0 \in A$ such that

$$\|(1 - p_i)\psi(1 - p_i)\| \leqslant \delta$$

for every $\psi \in K$ and $i \geqslant i_0$ (Takesaki [23, Theorem III.5.4.(vi)]). In particular

$$|\psi(1 - p_i)| \leqslant \delta, \quad \psi \in K,\ i(0) \leqslant i.$$

Let $p := p_{i(0)}$. Then for all x in the unit ball of M it follows that

$$\begin{aligned} &|(\varphi_t - \varphi_0)(x)| \leqslant \\ &\quad |(\varphi_t - \varphi_0)(pxp)| + |(\varphi_t - \varphi_0)((1-p)xp)| \\ &+ |(\varphi_t - \varphi_0)(x(1-p))| \leqslant\leqslant |(\varphi_t - \varphi_0)(pxp)| + 4\sqrt{\delta}. \end{aligned}$$

Since the W*-algebra $p\mathcal{B}(H)p$ is finite dimensional, there exists $U \in \mathfrak{U}$ such that

$$\|(\varphi_t - \varphi_0)|_{p\mathcal{B}(H)p}\| \leqslant \delta.$$

for all $t \in U$. Consequently

$$\|(\varphi_t - \varphi_0)\| \leqslant (\delta + 4\sqrt{\delta})$$

for all $t \in U$. Therefore $\lim_{\mathfrak{U}} T(t)_* \varphi = P_* \varphi$ in the strong operator topology. Since the positive cone of $\mathcal{B}(H)_*$ is generating, the assertion is proved. $\square$

We show next, that for irreducible W*-dynamical systems on $\mathcal{B}(H)$ the above properties always hold.

Theorem 3.8 *Let $(\mathcal{B}(H), \varphi, \mathcal{T})$ be an irreducible W*-dynamical system. Then*

$$P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}.$$

Proof Let N be the W*-subalgebra of $M = \mathcal{B}(H)$ generated by the eigenvectors of A pertaining to the eigenvalues on $\mathrm{i}\mathbb{R}$ and let Q be the faithful normal conditional expectation from M onto N (Proposition 3.7). Since M is atomic, N is atomic (Størmer [22]). N is finite since there exists a finite, faithful normal trace on N. In particular the center of N is isomorphic to ℓ^∞.

Let $\mathcal{S}$ be the restriction of $\mathcal{T}$ to the center. Then $\mathcal{S}$ is a weak*-semigroup such that every $S(t) \in \mathcal{S}$ is $\sigma(\ell^\infty, \ell^1)$-continuous and a *-automorphism. From this it follows that $S(t)$ is induced by some continuous flow $\kappa_t : \mathbb{N} \to \mathbb{N}$. Indeed, if $\delta_n((\xi_m)) = \xi_n$ $(n \in \mathbb{N}, (\xi_m) \in \ell^\infty)$, then $\delta_n \circ S(t)$ is a normal scalar valued *-homomorphism hence of the form δ_m for some $m = \kappa_t(n)$. But the function $t \mapsto \kappa_t$ is continuous from $\mathbb{R}$ into $\mathbb{N}$, whence constant. Hence $S(t) = \mathrm{Id}$. But the semigroup S is weak*-irreducible on the center. Consequently, the center is one dimensional. Using [Takesaki, Theorem V.1.27] we obtain $N = B(H_n)$ where H_n is a finite dimensional Hilbert space. But if $0 \neq \mathrm{i}\alpha \in P\sigma(A) \cap \mathrm{i}\mathbb{R}$ then $\mathrm{i}\alpha\mathbb{Z} \subset P\sigma(A)$ by D-III,Thm.1.10, whence N must be infinite dimensional. Therefore $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$ as desired. $\square$

Corollary 3.9 *If $(\mathcal{B}(H), \varphi, T)$ is an irreducible W*-dynamical system, then*

$$\lim_{s \to \infty} T(s) = 1 \otimes \varphi$$

in the strong operator topology on $L(\mathcal{B}(H)_*)$, where φ is the unique normal state generating the fixed space of T_*.

We are now going to discuss the asymptotic behavior of positive semigroups whose generator has boundary point spectrum different from 0. The standard example is the following. If Γ is the unit circle, dm the normalized Haar measure on Γ and $0 < \tau \in \mathbb{R}$, then we define the maps $T_\tau(t)$, $t \in \mathbb{R}_+$, on $L^1(\Gamma, m)$ by

$$(T_\tau(t)f)(\xi) = f(\xi \exp(\frac{2\pi i}{\tau} t)) \quad (f \in L^1(\Gamma, \mathrm{d}m), \xi \in \Gamma).$$

Then $\mathcal{T} := (T_\tau(t))_{t \geqslant 0}$ forms a strongly continuous one parameter semigroup which is identity preserving and of Schwarz type. Since $\mathcal{T}$ is periodic of period τ, it follows that 0 is a pole of the resolvent of its generator B with residuum $P = 1 \otimes 1$

and $\{\frac{2\pi \mathrm{i}}{\tau} \cdot k : k \in \mathbb{Z}\} = \sigma(B)$. Thus $\mathcal{T}$ is irreducible and uniformly ergodic on $L^1(\Gamma, \mathrm{d}m)$ (see A-II, Section 5).

Now let $\mathcal{T}$ be a semigroup on a predual M_* of a von Neumann-algebra M. It is called *partially periodic*, if there exists a projection $Q \in L(M_*)$ reducing T such that $Q(M_*) \cong L^1(\Gamma, \mathrm{d}m)$ and $T_{|\mathrm{im}(Q)}$ is conjugate to a periodic semigroup on $L^1(\Gamma, \mathrm{d}m)$.

In the main result we present a non commutative version of Nagel [17] showing that certain dynamical systems are partially periodic semigroups.

Proposition 3.10 *Let $\mathcal{T}$ be an irreducible, identity preserving semigroup of Schwarz type with generator A on the predual of a* W**-algebra M.*

If $\mathcal{T}$ is uniformly ergodic, then $\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$. If additionally $\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \{0\}$, there exists a strictly positive projection Q on M_* which is identity preserving and completely positive such that

(i) *Q reduces $\mathcal{T}$ and $Q(M_*) \cong L^1(\Gamma)$, Γ being the one dimensional torus.*

(ii) *The restriction T_0 of $\mathcal{T}$ to im(Q) is irreducible and conjugate to a rotation semigroup of period $\tau = \frac{2\pi}{\alpha}$ on Γ.*

(iii) *The spectral bound $s(A_{|\ \ker(Q)}$ is strictly smaller than* 0.

Proof By D-III, Thm.1.11 and D-III, Thm.2.5 it follows that

$$\sigma(A) \cap \mathrm{i}\mathbb{R} = P\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\alpha\mathbb{Z}$$

for some $\alpha \in \mathbb{R}$. Suppose $\alpha \neq 0$. Since $\sigma(A) + \mathrm{i}\alpha\mathbb{Z} = \sigma(A)$ and since every $n \in \mathrm{i}\alpha\mathbb{Z}$ is isolated, it follows that there exists $\delta > 0$ such that

$$\sigma(A) \setminus \mathrm{i}\alpha\mathbb{Z} \subseteq \{\lambda \in \mathbb{C}:\ \mathrm{Re}(\lambda) \leqslant \delta\}.$$

Let $\{u_\alpha^k : k \in \mathbb{Z}\}$ be a family of unitary eigenvectors of A' pertaining to the eigenvalues in $\mathrm{i}\mathbb{R}$. Then $Q'(M)$ is a commutative W*-algebra. For $\tau := \frac{2\pi}{\alpha}$, we obtain $T(\tau)u_\alpha^k = u_\alpha^k$, hence $T|_{\mathrm{im}(Q)}$ is periodic. From the Halmos-von Neumann theorem (see Schaefer [21, Thm. III.7.11]) it follows that $T|_{\mathrm{im}(Q)}$ is conjugate to the rotation semigroup of period τ on $L^1(\Gamma, m)$. □

Using this proposition we obtain the following theorem.

Theorem 3.11 *Let $T = (T(t))_{t \geqslant 0}$ be a uniformly ergodic, identity preserving semigroup of Schwarz type on the predual of a* W**-algebra M and suppose*

$$\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \{0\}.$$

Then there exists a partially periodic, identity preserving semigroup $S = (S(t))_{t \geqslant 0}$ of Schwarz type on M_* such that

$$\lim_{t\to\infty}(T(t) - S(t)) = 0$$

in the strong operator topology.

Proof Let φ be the normal state on M generating the fixed space of $\mathcal{T}$. Let $\mathcal{S} = (S(t))_{t\geqslant 0}$ where $S(t) := T(t)\circ Q$ and Q is as in 2.6. Obviously, $\mathcal{S}$ is partially periodic and $\varphi \in \operatorname{Fix}(\mathcal{S})$. Let H_φ be the GNS-Hilbert space pertaining to φ. Since φ is fixed under $\mathcal{T}$, $\mathcal{S}$ and Q, these objects have a canonical extension to H_φ (in the following denoted by the same symbols). If $H_0 := \ker(Q) \subseteq H_\varphi$, then it is easy to see that H_0 is invariant under the extension to H_φ and for the multiplication maps we defined in D-III, Remark 1.3.

Consequently, using the results in Groh and Kümmerer [11], it follows that there exists $c \in \mathbb{R}$ such that for all γ near 0 and all $\beta \in \mathbb{R}$:

$$\|R(\gamma + \mathrm{i}\beta A_0)\| \leqslant c, \tag{*}$$

where $A_0 := A_{|\ker(Q)}$ (the norm taken in $L(H_\varphi)$). Using the result in A-III,Cor.7.11 it follows that

$$\lim_{t\to\infty}\|T(t)|_{H_0}\| = 0.$$

Since the $s(M, M_*)$-topology on the unit ball of M is nothing else than the restriction of the norm topology on H_φ, we obtain

$$s(M, M_*)\text{-}\lim_{t\to\infty}(T(t)' - S(t)')(x) = 0$$

uniformly on M_1. From this the assertion follows. □

4 Uniform Ergodic Theorems

As we have seen, uniformly ergodic semigroups have strong spectral properties. In this section we study sufficient conditions which imply uniform ergodicity thereby generalizing results of Groh [9]. We first need some preparations.

Lemma 4.1 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$ with values in the predual of a* W**-algebra M. If the fixed space of $\mathcal{R}$ is infinite dimensional, then there exists a sequence of states in* $\operatorname{Fix}(\mathcal{R})$ *such that the corresponding support projections are mutually orthogonal in M.*

Proof Let $\Phi = \{\varphi \in \operatorname{Fix}(\mathcal{R}) : \varphi \text{ state on } M\}$ and let $p = \sup\{s(\varphi) : \varphi \in \Phi\}$. Since $\lambda R(\lambda)\varphi = \varphi$ for all $\varphi \in \Phi$ and $\lambda \in D$, it follows $\mu R(\mu)(\mathbb{1} - s(\varphi)) = (\mathbb{1} - s(\varphi))$. Hence $\mu R(\mu)(\mathbb{1} - p) = (\mathbb{1} - p)$ for all $\mu \in \mathbb{R}_+$.

Let $\mathcal{R}_1$ be the induced pseudo-resolvent on pM_*p (D-I, Section 3.(c)). Then the family Φ is faithful on M_p and contained in the fixed space of $\mathcal{R}_1$. The adjoint

$\mu R_1(\mu)'$ is an identity preserving Schwarz map. Consequently it follows from D-III, Lemma 1.1.(b) and, the $\sigma(M_p, (M_p)_*)$-continuity of $\mu R_1(\mu)'$ that $\operatorname{Fix}\left(R_1'\right)$ is a W*-subalgebra of M_p and by D-III, Lemma 1.5, $\dim \operatorname{Fix}(\mathcal{R}) \leqslant \dim \operatorname{Fix}\left(R_1'\right)$.

If $\operatorname{Fix}(\mathcal{R})$ is infinite dimensional, let (p_n) be a sequence of mutually orthogonal projections in $\operatorname{Fix}(R_1') \subseteq M_p$ and choose a sequence (φ_n) in Φ such that $\varphi_n(p_n) \neq 0$. For $n \in \mathbb{N}$ let ψ_n be the normal state

$$\psi_n(x) = \varphi_n(p_n)^{-1}\varphi_n(p_n x p_n)$$

on M. Because of $s(\psi_n) \leqslant p_n \leqslant p$, the support projections of the ψ_n's are mutually orthogonal in M. For $\mu \in \mathbb{R}_+$ and $x \in M$ we obtain

$$\begin{aligned}
\langle x, \mu R(\mu)\psi_n\rangle &= \varphi_n(p_n)^{-1}\langle \mu p_n (R(\mu)'x)p_n, \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\langle \mu p_n p(R(\mu)p'x)p_n, \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\langle \mu p_n (pR_1(\mu)'xp)p_n, \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\langle \mu (p_n R_1(\mu)'xp_n), \varphi_n\rangle = \\
&= \varphi_n(p_n)^{-1}\varphi_n(x) = \psi_n(x).
\end{aligned}$$

Therefore $\psi_n \in \operatorname{Fix}(\mathcal{R})$ for all $n \in \mathbb{N}$. □

Remark 4.2 (i) If $\dim \operatorname{Fix}(\mathcal{R}) \geqslant 2$ then the Jordan decomposition of self adjoint linear functionals implies that at least two states in $\operatorname{Fix}(\mathcal{R})$ have orthogonal support (compare D-III, Theorem 1.10.(a)).

(ii) If $\mathcal{R}$ is a pseudo-resolvent with values in a W*-algebra such that $\operatorname{Fix}(\mathcal{R}')$ is contained in M_*, then by D-III, Lemma 1.2, there exists a sequence of normal states in $\operatorname{Fix}(\mathcal{R}')$ with orthogonal supports in M.

Lemma 4.3 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent of Schwarz type on $D = \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > 0\}$ with values in the predual of a* W**-algebra M. If the fixed space of the canonical extension $\widehat{\mathcal{R}}$ of $\mathcal{R}$ to some ultrapower of M_* is infinite dimensional, then there exists a sequence (z_n) in M_1^+ and a sequence of states (φ_n) in M_* such that*

(i) $\lim_n z_n = 0$ *in the $s^*(M, M_*)$-topology,*

(ii) $\lim_n \|(Id - \lambda R(\lambda))\varphi_n\| = 0$ *for all $\lambda \in D$,*

(iii) $\varphi_n(z_n) \geqslant \frac{1}{2}$ *for all $n \in \mathbb{N}$.*

Proof Let $(M_*)^\wedge$ be the ultrapower of M_* with respect to some free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$. Since $(M_*)^\wedge$ is the predual of a W*-subalgebra of $\widehat{M}$ (see D-III, Remark 2.4.(b)), there exists a sequence of states $(\hat{\psi}_n)$ in $\operatorname{Fix}\left(\widehat{\mathcal{R}}\right)$ such that the corresponding support projections are mutually orthogonal in $\widehat{M}$ (Lemma 4.1). For every $n \in \mathbb{N}$ let $(\psi_{n,k})$ be a representing sequence of states,

$$\varphi := \sum_{n,k} 2^{-(n+k+1)} \psi_{n,k}$$

and

$$p := \sup\{s(\psi_{n,k}) : n, k = 1, \ldots\}$$

in M. Then φ is a normal state on M which is faithful on the W*-algebra M_p. Since

$$1 = \langle \psi_{n,k}, s(\psi_{n,k})\rangle = \psi_{n,k}(p) \quad (n, k \in \mathbb{N}),$$

it follows $\hat{\psi}_n(\hat{p}) = 1$ where $\hat{p}$ is the canonical image of p in $\widehat{M}$. But this implies $s(\hat{\psi}_n) \leqslant \hat{p}$ in $\widehat{M}$. Since $\widehat{M}_1^+$ is $\sigma(\widehat{M}, \widehat{M}')$-dense in $(\widehat{M}'')_1^+$ (Kaplansky's density theorem Sakai [19, 1.9.1] with Sakai [19, 1.8.9 and 1.8.12]), there exists for all $n \in \mathbb{N}$ a net $(z_{n,\gamma})$ in $\widehat{M}_1^+$ such that

$$\sigma(\widehat{M}'', \widehat{M}')\text{-}\lim_{\gamma} \hat{z}_{n,\gamma} = s(\hat{\psi}_n).$$

From Sakai [19, 1.7.8] and the above considerations, we obtain that the net $(p\hat{z}_{n,\gamma}\hat{p})$ converges to $s(\hat{\psi}_n)$ in the $\sigma(\widehat{M}'', \widehat{M}')$-topology. Therefore we may assume $\hat{z}_{n,\gamma} \in (\widehat{M}'_p)_1^+$.

In the following we denote by $\hat{\varphi}$ the canonical image of φ in $(M_*)^\wedge$.

Since the projections $s(\hat{\psi}_n)$ are mutually orthogonal, there exists a real sequence (r_n), $0 < r_n < 1$, $\lim_n r_n = 0$ and $\hat{\varphi}(s(\hat{\psi}_n)) \leqslant \frac{1}{2} r_n$. For all $n \in \mathbb{N}$ choose $\hat{z}_n \in (\widehat{M}'_p)_1^+$ such that

$$|\langle \hat{\varphi}, s(\hat{\psi}_n) - \hat{z}_n\rangle| \leqslant \frac{1}{2} r_n,$$
$$|\langle \hat{\psi}_n, s(\hat{\psi}_n) - \hat{z}_n\rangle| \leqslant \frac{1}{2} r_n.$$

Hence $\hat{\varphi}(\hat{z}_n) \leqslant r_n$ and $\hat{\psi}_n(\hat{z}_n) \geqslant \frac{1}{2}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ let $(z_{n,k}) \in \hat{z}_n$ be a representing sequence in $(M_p)_1^+ = p(M_1^+)p$ (note that $M_{\hat{p}} = \widehat{M_p}$) and fix $\mu \in \mathbb{R}_+$. Since $\mu R(\mu)'\hat{\psi}_n = \hat{\psi}_n$, $\hat{\varphi}(\hat{z}_n) \leqslant r_n$ and $\hat{\psi}_n(\hat{z}_n) \geqslant \frac{1}{2}$, there exists for all $n \in \mathbb{N}$ an element $U_n \in \mathfrak{U}$ such that for all $k \in U_n$ and we obtain

(i′) $\varphi(z_{n,k}) \leqslant r_n$,

(ii′) $\|(Id - \mu R(\mu))\psi_{n,k}\| \leqslant r_n$,

(iii′) $\psi_{n,k}(z_{n,k}) \geqslant \frac{1}{2}$.

Inductively we find a sequence (z_n) in $(M_p)_1^+$ and a sequence of states (φ_n) in M_* such that for all $n \in \mathbb{N}$

(i″) $\lim_n \varphi_n(z_n) = 0$,

(ii″) $\lim_n \|(Id - \mu R(\mu))\varphi_n\| = 0$,

(iii″) $\varphi_n(z_n) \geqslant \frac{1}{2}$.

But φ is faithful on M_p. Therefore condition (ii) implies that $\lim_n z_n = 0$ in the $s^*(M_p, (M_p)_*)$-topology (Takesaki [23, Proposition III.5.4]). Since

$$s^*(M_p, (M_p)_*) = s^*(M, M_*)|_{M_p},$$

(i) follows immediately from (ii). Using the resolvent equation for $\mathcal{R}$ it is easy to see that (ii) implies

$$\lim_n \|(Id - \lambda R(\lambda))\varphi_n\| = 0$$

for all $\lambda \in D$ and the proof is complete. □

Without further comments, we will use following facts in this section.

(1) A sequence (φ_n) in M'_+ converges in the $\sigma(M', M)$-topology if and only if it converges in $\sigma(M', M'')$-topology (Akemann et al. [1]).

(2) We can decompose $\varphi \in M'_+$ into its normal and singular part $\varphi = \varphi^{(n)} + \varphi^{(s)}$, $0 \leqslant \varphi^{(n)} \in M_*$, $0 \leqslant \varphi^{(s)} \in M_*^{\perp}$ and $\|\varphi\| = \|\varphi^{(n)}\| + \|\varphi^{(s)}\|$ (Takesaki [23, Theorem III.2.14]).

(3) If (φ_k) is a sequence in M_* convergeing to zero in the $\sigma(M_*, M)$-topology and if (x_n) is a sequence in M converging to zero in the $s^*(M, M_*)$-topology, then $\lim_n \varphi_k(x_n) = 0$ uniformly in $k \in \mathbb{N}$ (Takesaki [23, Lemma III.5.5]).

Theorem 4.4 *Let $\mathcal{R}$ be an identity preserving pseudo-resolvent on*

$$D = \{\lambda \in \mathbb{C} \colon \ \mathrm{Re}(\lambda) > 0\}$$

with values in a W**-algebra M which is of Schwarz type and let $\mathcal{R}'$ br its adjoint pseudo-resolvent. Any one of the following conditions implies* dim Fix $\left(\widehat{\mathcal{R}}\right) < \infty$ *in some ultrapower of M.*

(i) *The fixed space of $\mathcal{R}'$ is finite dimensional.*

(ii) $\lim_{\mu \to 0} \mu R(\mu) = P$ *exists in the strong operator topology and rank$(P) < \infty$.*

(iii) *The fixed space of $\mathcal{R}'$ is contained in M_*.*

(iv) *Every map $\mu R(\mu)$, $\mu \in \mathbb{R}_+$ is irreducible on M.*

Proof Suppose that the dimension of the fixed space of $\widehat{\mathcal{R}'}$ in some ultrapower $\widehat{(M‘)}$ of M' is infinite dimensional. Since $\widehat{(M‘)}$ is the predual of the W*-algebra $\widehat{M}$ and $\mathcal{R}'$ is identity preserving (since $R'\mathbb{1} = R\mathbb{1} = \mathbb{1}$) and of Schwarz type (because $\mu R''(\mu) = (\mu R(\mu))''$ is a Schwarz map for all $\mu \in \mathbb{R}_+$), we may apply Lemma 4.3.

Suppose that the fixed space of the canonical extension of $\mathcal{R}'$ to some ultrapower of M' is infinite dimensional. Thus we may choose a sequence of states (φ_k) in M' and a sequence (z_k) in $(M'')_1$, $0 \leqslant z_k$, satisfying (i)–(ii) of Lemma 4.3. Remark (3) above implies that no subsequence of (φ_k) can converge in the $\sigma(M', M'')$-topology.

(i) If φ is a $\sigma(M', M)$-accumulation point of (φ_k), then $\varphi \in \mathrm{Fix}\,(\mathcal{R}')$. Since $\mathrm{Fix}\,(\mathcal{R}')$ is finite dimensional, the set of accumulation points of the sequence (φ_k) is metrizable in the $\sigma(M', M)$-topology. Hence there exists a sequence $(k(n))$ of natural numbers such that $\sigma(M', M)\text{-}\lim_n \varphi_{k(n)} = \varphi$. Consequently, by Remark (1) above , $\varphi = \sigma(M', M'')\text{-}\lim_n \varphi_{k(n)}$. But this leads to a contradiction proving(i).

(ii) Since $\dim \mathrm{Fix}\,(\mathcal{R}) = \dim \mathrm{Fix}\,(\mathcal{R}') = \mathrm{rank}\,(P) < \infty$, (ii) follows from (i).

(iii) Suppose that the fixed space of R' is infinite dimensional. Since $\mathrm{Fix}\,(\mathcal{R}') \subseteq M_*$, there exists a sequence of states (ψ_n) in $\mathrm{Fix}\,(\mathcal{R}')$ with mutually orthogonal support projections in M (Lemma 4.1). Since every $\sigma(M', M)$-accumulation point of the ψ_n's belongs to $\mathrm{Fix}\,(\mathcal{R}')$, hence is normal, the sequence (ψ_n) is relatively $\sigma(M_*, M)$-compact.

By Eberlein's theorem, we may assume that this sequence is weakly convergent (Schaefer [20]). By the orthogonality of the $s(\psi_n)$'s this sequence converges to zero in the $s^*(M, M_*)$-topology, hence $\lim_n \psi_k(s(\psi_n)) = 0$ uniformly in $k \in \mathbb{N}$, a contradiction. Consequently $\dim \mathrm{Fix}\,(\mathcal{R}) < \infty$ and (i) is proved.

(iv) We prove $\dim \mathrm{Fix}\,(\mathcal{R}') = 1$ and apply (i) once again and need the following observation: If ψ is a faithful state on M, then the normal part is faithful too. Indeed, if $0 \neq x \in M$ such that $\psi^{(n)}(x) = 0$, choose a projection $0 \neq p \in M$ such that $\psi^{(n)}(p) = \psi^{(s)}(p) = 0$ (use Takesaki [23, Theorem III.3.8]). Hence $\psi(p) = 0$ which conflicts with the faithfulness of ψ.

If $2 \leqslant \dim \mathrm{Fix}\,(\mathcal{R}')$ there are states ψ_1 and ψ_2 in $\mathrm{Fix}\,(\mathcal{R}')$ such that the corresponding support projections are orthogonal in M'' (Remark 4.2). Since every $\mathcal{R}'$-invariant state ψ is faithful on M, $\psi_i^{(n)} \neq 0$ (otherwise the norm closed face $\{\psi(x) = 0 : x \in M_+\}$ would be non trivial and $\mu R(\mu)$-invariant). The support projections of the $\psi_i^{(n)}$'s in M'' are orthogonal (since $\psi_1^{(n)} \leqslant \psi_i$) and different from zero. Let (z_γ) be a net in M_1^+ such that

$$\sigma(M'', M')\text{-}\lim_{\gamma} z_\gamma = s(\psi_1^{(n)}).$$

Then $\lim_\gamma \psi_1^{(n)}(z_\gamma) = 1$ but $\lim_\gamma \psi_2^{(n)}(z_\gamma) = 0$. Let z be a $\sigma(M, M_*)$-accumulation point of (z_γ) in M_+. Since every $\psi_i^{(n)}$ is normal, $\psi_1^{(n)}(z) = 1$ but $\psi_2^{(n)}(z) = 0$. The first condition implies $z \neq 0$ while the second shows that $\psi_2^{(n)}$ cannot be faithful. This is a contradiction and it implies $\dim \mathrm{Fix}\,(\mathcal{R}') = 1$, hence (iv). □

The next corollary is an easy application of Theorem 4.4 and of D-III, Proposition 2.3.

Corollary 4.5 *Let $\mathcal{T}$ be an identity preserving semigroup of Schwarz type on the predual of a* W*-algebra M. Then the following assertions are equivalent.*

(a) *$\mathcal{T}$ is uniformly ergodic with finite dimensional fixed space.*

(b) *The adjoint weak*-semigroup is strongly ergodic with finite dimensional fixed space.*

(c) *Every $\mathcal{T}''$-invariant state is normal.*

Proof If (a) is fulfilled, then the semigroup $\mathcal{T}$ is strongly ergodic on M_*. Since

$$\dim \operatorname{Fix}(\mathcal{T}) = \dim \operatorname{Fix}(\mathcal{T}') < \infty,$$

there exist normal states $\varphi_1, \dots, \varphi_n$ in $\operatorname{Fix}(\mathcal{T})$ and $x_1, \dots, x_k$ in $\operatorname{Fix}(\mathcal{T}')$ such that $\varphi_n(x_m) = \delta_{n,m}$ $(1 \leqslant n, m \leqslant k)$. Then

$$P = \sum_{i=1}^{k} \varphi_i \otimes x_i$$

is the associated ergodic projection. If $(C(s))_{s>0}$ is the family of Cesàro means of $\mathcal{T}$, then

$$\lim_{s\to\infty} C(s)''(\psi) = \sum_{i=1}^{k} \varphi_i(\psi) x_i \in M_*$$

for every $\psi \in M'$. Hence $\operatorname{Fix}(\mathcal{T}'') \subseteq M_*$ which implies (c).

If (c) is fulfilled, then $\operatorname{Fix}(\mathcal{T}') = \operatorname{Fix}(\mathcal{T}'')$. Therefore the fixed space of $\mathcal{T}'$ separates the points of $\operatorname{Fix}(\mathcal{T}'')$, hence $\mathcal{T}'$ is strongly ergodic on M (Krengel [14, Chap.2, Thm.1.4]).

If (b) holds, then

$$P = \lim_{\mu\to 0} \mu R(\mu, A')$$

exists in the strong operator topology with A' is the generator of $\mathcal{T}'$. Therefore $\dim \operatorname{Fix}\left(\widehat{\mu R(\mu)}\right) < \infty$ in some ultrapower of M (Theorem 4.4). It follows from D-III, Proposition 2.3 that 0 is a pole of the resolvent of $R(\cdot, A)$. Therefore $\mathcal{T}$ is uniformly ergodic. □

Notes

Section 1: The stability concepts appearing in Theorem 1.7 coincide not only for positive semigroups on C*-algebras but on any order unit Banach space. We refer to Batty and Robinson [2] for this more general setting and to B-IV, Section 1 for the analogous results on $C_0(X)$.

Section 2: Theorem 2.2 generalizes the Liapunov stability theorem from the matrix algebra $B(\mathbb{C}^n)$ to arbitrary W*-algebras. For the algebra $\mathcal{B}(H)$ it is due to Mil'stein [16] and in the general form to Groh and Neubrander [12].

Section 3: From the many papers dealing more or less explicitly with the asymptotic behavior of semigroups on operator algebras we quote Frigerio and Verri [6] and Watanabe [24]. The background for our ergodic theorems (Proposition 3.3 & 3.4) can be found best in Krengel [14]. The "automatic" convergence theorem for an irreducible W*-dynamical system on $\mathcal{B}(H)$ stated in Corollary 3.9 is the continuous version of a result in Groh [10]. Finally, the characterization of convergence towards a periodic semigroup through spectral properties of the generator—Theorem 3.11—is due to Nagel [17] in the commutative case, i.e., in $L^1(\mu)$ (see also C-IV, Thm.2.14).

Section 4: Again we refer to Krengel [14] for the (uniform) ergodic theory for a single operator or a one-parameter semigroup on a Banach space. The characterization given in Corollary 4.5 for positive semigroups on W*-algebras is based on a sophisticated use of ultrapower techniques and has its discrete forerunners in Lotz [15] and Groh [9].

References

[1] C. Akemann, P. Dodds, and J. Gamlen. Weak compactness in the dual space of a W*-algebra. *J. Funct. Anal.*, 16:446–450, 1972.

[2] C. Batty and D. Robinson. Positive one-parameter semigroups on ordered spaces. *Acta Appl. Math.*, 2:221–296, 1984.

[3] O. Bratteli and D. Robinson. *Operator Algebras and Quantum Statistical Mechanics I*. Springer, New York-Heidelberg-Berlin, 1979. Volume II published 1981.

[4] E. Davies. *Quantum Theory of Open Systems*. Academic Press, London-New York-San Francisco, 1976.

[5] D. Evans. Irreducible quantum dynamical semigroups. *Comm. Math. Phys.*, 54:293–297, 1977.

[6] A. Frigerio and M. Verri. Long-time asymptotic properties of dynamical semigroups on W^*-algebras. *Math. Z.*, 180:275–286, 1982.

[7] G. Greiner, J. Voigt, and M. Wolff. On the spectral bound of the generator of semigroups of positive operators. *J. Operator Theory*, pages 245–256, 1981.

[8] U. Groh. The peripheral point spectrum of Schwarz operators on C^*-algebras. *Math. Z.*, 176:311–318, 1981.

[9] U. Groh. Uniform ergodic theorems for identity preserving Schwarz maps on W*-algebras. *J. Operator Theory*, 11:395–404, 1984.

[10] U. Groh. Spectrum and asymptotic behaviour of completely positive maps on $B(H)$. *Math. Japon.*, 29:395–402, 1984.

[11] U. Groh and B. Kümmerer. Bibounded operators w*-algebras. *Math. Scand.*, 50:269 – 285, 1982. doi: 10.7146/math.scand.a-11959.

[12] U. Groh and F. Neubrander. Stabilität starkstetiger, positiver operatorhalbgruppen auf c*-algebren. *Math. Ann.*, 256:509 – 516, 1981.

[13] H. D. Junghenn. *Almost periodic compactifications and applications to one parameter semigroups*. PhD thesis, The George Washington University, 1971.

[14] U. Krengel. *Ergodic Theorems*. de Gruyter, Berlin, New York, 1985.

[15] H. P. Lotz. Uniform ergodic theorems for Markov operators on $C(X)$. *Math. Z.*, 178:145–156, 1981.

[16] G. N. Mil'stein. Exponential stability of positive semigroups in a linear topological space I,II. *Soviet Math.*, 19:35–42, 51–61, 1975.

[17] R. Nagel. What can positivity do for stability? In K. D. Bierstedt and B. Fuchssteiner, editors, *Functional Analysis, Surveys and Recent Results III*. North Holland, Amsterdam, 1984.

[18] D. Olesen, G. K. Pedersen, and M. Takesaki. Ergodic actions of compact Abelian groups. *J. Operator Theory*, 3:237–269, 1980.

[19] S. Sakai. *C*-Algebras and W*-Algebras*. Springer, Berlin-Heidelberg-New York, 1971.

[20] H. H. Schaefer. *Topological Vector Spaces*. Springer, 1966. 4th printing: New York-Heidelberg-Berlin: Springer 1980.

[21] H. H. Schaefer. *Banach Lattices and Positive Operators*. Springer, New York-Heidelberg-Berlin, 1974.

[22] E. Størmer. On projection maps on von Neumann algebras. *Math. Scand.*, 50: 42–50, 1972.

[23] M. Takesaki. *Theory of Operator Algebras I*. Springer, New York-Heidelberg-Berlin, 1979.

[24] S. Watanabe. Asymptotic behaviour and eigenvalues of dynamical semigroups on operator algebras. *J. Math. Anal. Appl.*, 86:411–424, 1982.

Index