tions under the spectral theoretical point of view and collect a number of technical properties for later use.

We start by studying the spectrum of subspace and quotient semigroups. To that purpose assume that the strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ leaves invariant some closed subspace N of the Banach space E. There are canonically induced semigroups $\mathcal{T}_|$ on N, resp. $\mathcal{T}_/$ on $E/_N$ and their generators $A_|$, resp. $A_/$ are canonically obtained from the generator A of $\mathcal{T}$ (see A-I, Section 3). The following example shows that the spectra of A, $A_|$ and $A_/$ may differ quite drastically.

Example 4.1. As in the example in A-I,3.3 we consider the translation semigroup on $E = L^1(\mathbb{R})$ and the invariant subspace $N := \{f \in E : f(x) = 0 \text{ for } x \geq 1\}$. Then $\sigma(A) = i\mathbb{R}$ but $\sigma(A_|) = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq 0\}$. Next we take the translation invariant subspace $M := \{f \in N : f(x) = 0 \text{ for } 0 \leq x \leq 1\}$ and obtain $\sigma(A_{|/}) = \emptyset$ for the generator $A_{|/}$ of the quotient semigroup $\mathcal{T}_{|/}$ (use the fact that $\mathcal{T}_{|/}$ is nilpotent).

In the next proposition we collect the information on $\sigma(A)$ which in general can be obtained from the 'subspace spectrum' $\sigma(A_|)$ and the 'quotient spectrum' $\sigma(A_/)$.

Proposition 4.2. Using the standard notations the following inclusions hold:

$$\rho_+(A) \underset{\text{(iii)}}{\subset} [\rho(A_|)\cap\rho(A_/)] \underset{\text{(ii)}}{\subset} \rho(A) \underset{\text{(i)}}{\subset} [\rho(A_|)\cap\rho(A_/)] \cup [\sigma(A_|)\cap\sigma(A_/)],$$

where $\rho_+(A)$ denotes the connected component of $\rho(A)$ which is unbounded to the right.

Proof. (i) Assume $\lambda \in \rho(A)$, i.e. $(\lambda-A)$ is a bijection from $D(A)$ onto E. Since N is $\mathcal{T}$-invariant we have $D(A_|) = D(A) \cap N$ and $(\lambda-A)D(A_|) \subset N$. If $(\lambda-A)D(A_|) = N$ then $R(\lambda,A)N = D(A_|)$ and the induced operators $R(\lambda,A)_|$, resp. $R(\lambda,A)_/$ are the inverses of $(\lambda-A_|)$, resp. $(\lambda-A_/)$. If $(\lambda-A)D(A_|) \neq N$ then $\lambda \in \sigma(A_|)$. In addition there exists $f \in D(A)\setminus N$ such that $g := (\lambda-A)f \in N$. Hence for $\hat{f} := f+N$, $\hat{g} := g+N \in E_/$ it follows that $(\lambda-A_/)\hat{f} = \hat{g} = 0$, i.e. $\lambda \in \sigma(A_/)$.

(ii) Take $\lambda \in \rho(A_|) \cap \rho(A_/)$. Then $(\lambda-A)$ is injective: $(\lambda-A)f = 0$ implies $(\lambda-A_/)\hat{f} = 0$, hence $\hat{f} = 0$, i.e. $f \in N$ and therefore $f = 0$.