Then the following assertions hold:

(a) $(\lambda - i\alpha)^{-1}$ is a pole of the resolvent $R(.,R(\lambda))$ for all $\lambda \in D$.

(b) $\dim \mathrm{Fix}((\lambda-i\alpha)R(\lambda)) = \dim \mathrm{Fix}((\lambda-i\alpha)\hat{R}(\lambda))$ for all $\lambda \in D$.

(c) $i\alpha$ is a pole of the pseudo-resolvent R and the residue of R and $R(.,R(\lambda))$ in $i\alpha$ respectively $(\lambda - i\alpha)^{-1}$ are identical.

Proof. Take a normalized sequence (x_n) in E with

$$\lim_n \|(\lambda - i\alpha)R(\lambda)x_n - x_n\| = 0 .$$

The existence of such a sequence follows from the fact that the fixed space of $(\lambda-i\alpha)\hat{R}(\lambda)$ is non trivial. Suppose (x_n) is not relatively compact. Then we may assume that there exists $\delta>0$ such that

$$\|x_n - x_m\| > \delta \quad \text{for} \quad n \neq m .$$

Take $k \in \mathbb{N}$ and let $\hat{x}_k$ be the image of (x_{n+k}) in $\hat{E}$. Since

$$\lim_n \|(\lambda - i\alpha)R(\lambda)x_{n+k} - x_{n+k}\| = 0 ,$$

the so defined $\hat{x}_k$'s belong to $\mathrm{Fix}((\lambda - i\alpha)\hat{R}(\lambda))$. Since this space is finite dimensional there exist $j < \ell$ such that

$$\|\hat{x}_j - \hat{x}_\ell\| \leq \frac{\delta}{2} .$$

From the definition of the norm in $\hat{E}$ it follows that there are natural numbers $n < m$ such that

$$\|x_n - x_m\| \leq \frac{\delta}{2}$$

which leads to a contradiction. Therefore every approximate eigenvector of $(\lambda - i\alpha)R(\lambda)$ pertaining to 1 is relatively compact. In particular it has a convergent subsequence from which it follows that the fixed space of $(\lambda - i\alpha)R(\lambda)$ is non trivial.

Obviously

$$\dim \mathrm{Fix}((\lambda - i\alpha)R(\lambda)) \leq \dim \mathrm{Fix}((\lambda - i\alpha)\hat{R}(\lambda)) .$$