Extended Notes for B-II.

Section 4 Positive semigroups generated by elliptic operators on spaces of continuous functions

Important examples of semigroups on $C_0(\Omega)$ or $C(\bar{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open, bounded, are generated by elliptic differential operators. In the following we put together a series of results starting with the Laplacian subject to Dirichlet and to Robin boundary conditions and ending with the Dirichlet-to-Neumann operator on $C(\partial\Omega)$. Each time we obtain a positive irreducible semigroup. We consider $\mathbb{K} = \mathbb{R}$ throughout this section.

The Laplacian

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. We say that Ω is <u>Dirichlet-regular</u> if for every $g \in C(\partial\Omega)$ there exists a (unique) function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\Delta u = 0 \quad \text{and} \quad u_{|\partial\Omega} = g.$$

This means that the Dirichlet problem is well-posed.

This property is very well understood and precise characterizations in terms of barriers or of capacity are known.

If Ω has Lipschitz boundary, then Ω is Dirichlet regular.

In dimension $d=2$ it suffices that Ω is simply connected.

We refer to Arendt–Urban [AU24, Section 6.9] or Gilbarg–Trudinger [GT83, Section 2.8] for further information on the Dirichlet Problem.

The Dirichlet Laplacian Δ_0 on $C_0(\Omega)$ is defined by

$$\Delta_0 u := \Delta u$$

$$D(\Delta_0) := \{ u \in C_0(\Omega) : \Delta u \in C_0(\Omega) \}.$$

Here Δu is to be understood in the sense of distributions.

<u>Theorem</u> 4.1 TFAE

(a) Ω is Dirichlet regular;

(b) Δ_0 generates a positive semigroup T on $C_0(\Omega)$.

In that case the semigroup T is holomorphic of angle $\pi/2$. Moreover

$T(t)$ is compact for all $t > 0$ and

$\omega_0(\Delta_0) < 0$. If Ω is connected, then the semigroup is irreducible. Moreover, $\|T(t)\| \leq Me^{-\varepsilon t}$ $(t \geq 0$ for some $\varepsilon > 0$, $M \geq 1$.

This result is due to Arendt - Bénilan [ArBe99] besides irreducibility on which we comment later.

In Example C-II 1.5 e) the generation result was obtained if Ω has C^2-boundary.

The implication (a) $\Rightarrow$ (b) of Theorem 4.1 is proved below in order to show how the Dirichlet problem comes into play and leads to a result with minimal regularity assumptions on the boundary of Ω.

We use the following abstract generation result which is of independent interest.
By C-II, Theorem 1.2 a densely defined operator A generates a contractive positive semigroup iff A is dispersive and $(\lambda - A)$ is surjective for some $\lambda > 0$.
We now describe the case $\lambda = 0$.

<u>Theorem</u> 4.2. Let A be a densely defined operator on a real or complex Banach lattice E.
T.F.A.E.

(a) A generates a positive, contractive semigroup and $s(A) < 0$.

(b) A is dispersive and surjective.

In particular, (b) implies that A is closed.
Dispersive operators are defined before

C-II, Theorem 1.2. A densely defined operator A on $C_0(\Omega)$ is dispersive iff for $u \in D(A)$, $x_0 \in \Omega$

$u(x_0) = \sup_{x\in\Omega} u(x) > 0$ implies $(Au)(x_0) \leq 0$.

Proof of Theorem 4.2. (b) $\Rightarrow$ (a)

Consider the equivalent norm $|||u||| := \|u^+\| + \|u^-\|$ on E. Since A is dispersive it is dissipative with respect to this new norm as is easy to see. Now Theorem 4.5 of Arendt, Chalendar and Moletsane [ACM 24] implies that A generates a contraction semigroup T and A is invertible. Since A is dispersive, it follows from C-II, Theorem 1.2 that T is positive and contractive (with respect to the original norm). Since $R(\lambda, A) \geq 0$ for $\lambda > 0$, it follows that $-A^{-1} \geq 0$. Now C-III, Theorem 1.1 (ii) implies that $s(A) < 0$.

(a) $\Rightarrow$ (b) is obvious from C-II, Theorem 1.2. $\square$

Proof of Theorem 4.1. (a) $\Rightarrow$ (b)

The operator Δ_0 is dispersive by the maximum principle. If Ω is Dirichlet regular, then Δ_0 is surjective. In fact, let $f \in C_0(\Omega)$. Extend f by 0 to $\mathbb{R}^n$ and let $w = \Gamma * f$, where Γ is the fundamental solution of Laplace's equation, see Gilbarg and Trudinger [GT83, (2.12)]. Then $w \in C(\mathbb{R}^n)$ and $\Delta w = f$ in the sense of distributions. Let $g = w_{|\partial\Omega}$ and let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be the solution of the

Dirichlet problem; i.e. $v_{|\partial\Omega} = g$ and $\Delta v = 0$ in Ω. Then $u := w - v \in D(\Delta_0)$ and $\Delta u = f$.

We have shown that Δ_0 satisfies condition (b) of Theorem 4.2. Thus Δ_0 generates a positive, contractive C_0-semigroup $(T(t))_{t \geq 0}$ on $C_0(\Omega)$ and $s(\Delta_0) < 0$.

Since by C-IV Theorem 1.1 (i) $s(\Delta_0) = \omega_0(\Delta_0)$, it is exponentially stable.

We refer to Arendt and Bénilan [ArBe 99] for the proof of (b) $\Rightarrow$ (a).

$\square$

We want to add two further comments on the Dirichlet Laplacian Δ_0 on $C_0(\Omega)$.
The first concerns its domain

$$D(\Delta_0) = \{ u \in C_0(\Omega) : \Delta u \in C_0(\Omega) \}$$

This distributional domain is not contained in $C^2(\Omega)$ for any open set $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, see Arendt-Urban [AU24, Theorem 6.60].

Our second comment concerns the proof of holomorphy. It can be given via Gaussian estimates (see the Extended Notes for C-II)
In our context, a short proof based on Kato's inequality of C-II, Section 2 is

more appealing (see Arendt - Batty [ArBa 92]).

Finally, we comment on irreducibility. On $C_0(\Omega)$ it is a strong property. By C-III, Theorem 3.2 (ii) it means that for $0 \le f \in C_0(\Omega)$, $f \neq 0$,

$$(T(t)f)(x) > 0 \quad \text{for all } x \in \Omega,\ t > 0.$$

On $L^2(\Omega)$ irreducibility is much weaker (meaning that $(T(t)f)(x) > 0$ x-a.e.), but easy to prove (see the Extended Notes to C-II).

In the paper Arendt, ter Elst, Glück [AEG 20] an argument based on Banach lattice techniques shows how irreducibility on $L^2(\Omega)$ can be carried over to $C_0(\Omega)$ or even to $C(\overline{\Omega})$ in the case of Robin boundary conditions, which we consider now.

By $H^1(\Omega) := \{u \in L^2(\Omega) : \partial_j u \in L^2(\Omega)$ for $j = 1, \dots, n\}$

we denote the first Sobolev space.

We assume that Ω
has Lipschitz boundary. Then
there exists a unique bounded

operator $\quad tr : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$

such that $\quad tr\, u = u_{|\partial\Omega}$ for all

$u \in C^1(\overline{\Omega})$. It is called the

<u>trace operator</u>.

Here the space $L^2(\partial\Omega)$
is defined with
respect to the surface measure
(i.e. the $(d-1)$-dimensional
Hausdorff measure) on $\partial\Omega$.

The <u>normal derivative</u> $\partial_\nu u$ of u is defined as follows.

Let $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Let $h \in L^2(\partial\Omega)$. We say that h is the <u>(outer) normal derivative</u> of u and write $\partial_\nu u = h$ if

$$\int_\Omega \Delta u \, v + \int_\Omega \nabla u \nabla v = \int_{\partial\Omega} h v$$

for all $v \in C^1(\bar{\Omega})$.

If $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$ we say $\partial_\nu u \in L^2(\partial\Omega)$ if there exist $h \in L^2(\partial\Omega)$ such that $\partial_\nu u = h$.

<u>Remark</u>: Since Ω has Lipschitz boundary the outer normal $\nu(z)$ exists for almost all $z \in \partial\Omega$ and $\nu \in L^\infty(\partial\Omega)$. But we do not use this outer normal and rather define $\partial_\nu u$ weakly by the validity of

Green's formula.

Let $\beta \in L^\infty(\Omega)$. We define the Laplacian Δ^β with Robin boundary conditions as follows

$$D(\Delta^\beta) := \{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \\ \partial_\nu u + \beta \operatorname{tr}(u) = 0 \}$$

$$\Delta^\beta u := \Delta u .$$

We call Δ^β briefly the <u>Robin-Laplacian</u>. Note that for $\beta = 0$, we obtain <u>Neumann boundary conditions</u>, and $\Delta^N := \Delta^0$ is the <u>Neumann Laplacian</u>.

The following result is valid.

<u>Theorem</u> 4.3. Assume that $\Omega \subset \mathbb{R}^d$ is bounded, open, connected with Lipschitz boundary and let $\beta \in L^\infty(\partial\Omega)$. Then Δ^β generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on $C(\bar{\Omega})$. Moreover, $T(t)$ is compact for all $t > 0$.

The generation property on $C(\bar{\Omega})$ is due to Nittka [Ni11]. A major point is to show that the resolvent of the corresponding operator on $L^2(\Omega)$ leaves $C(\bar{\Omega})$ invariant. Given $f \in C(\bar{\Omega})$, $u \in H^1(\Omega)$ such that

$$u - \Delta u = f \ , \ \partial_\nu u + \beta u_{|\partial\Omega} = 0.$$

Theorem 4.8. Assume (4.2) and (4.4). Then $\Delta - V$ generates a positive, irreducible semigroup on $C(\partial\Omega)$. If $V \geq 0$, then the semigroup is contractive.

If Ω is of class C^∞ similar results have been obtained by Escher [Es 94] and Engel [En 03]. Under the very general conditions here, Theorem 4.8 is due to Arendt and ter Elst [AtE 20]. There it is shown that N_V is resolvent-