ding to $T(t_o)$ and τ_1, τ_2, then $\sigma(A)$ splits into closed subsets σ_1, σ_2 and P is the corresponding spectral projection for $\mathcal{T}$ and σ_1, σ_2.

Proof. The spectral projection P of $T(t_o)$ is obtained by integrating $R(\lambda,T(t_o))$ (see e.g. [Dunford-Schwartz (1958), Section VII.3]). Since every $T(t)$, $t \geqq 0$, commutes with $T(t_o)$ it must commute with $R(\lambda,T(t_o))$, hence with P. The statement on the decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$ follows from the Spectral Inclusion Theorem 6.2 below.

□

This decomposition can be applied as follows to the study of the asymptotic behavior of $\mathcal{T}$: In the situation of Cor.3.5 assume

$$\sup\{|\lambda| : \lambda \in \tau_2\} < \alpha < \inf\{|\lambda| : \lambda \in \tau_1\}.$$

If we set $\beta := (\log\alpha)/t_o$ and use [Pazy(1984),Chap.I,Thm.6.5] we obtain $\omega(\mathcal{T}_2) < \beta$ and $\omega(\mathcal{T}_1^{-1}) < \beta$ by Prop.1.1. Therefore we have constants m, $M \geqq 1$ such that

$$\|T(t)f\| \leqq M\cdot e^{\beta t}\|f\| \quad \text{for } f \in E_2,$$
$$\|T(t)f\| \geqq m\cdot e^{\beta t}\|f\| \quad \text{for } f \in E_1.$$

As nice as they might look results of this type are unsatisfactory : we need information on the semigroup in order to estimate its asymptotic behavior. In Chapter IV we will try to obtain such results by exploiting information about the generator only.

3.6 Isolated singularities and poles.

In case that λ_o is an isolated point of $\sigma(A)$ the holomorphic function $\lambda \mapsto R(\lambda,A)$ can be expanded as a Laurent series $R(\lambda,A) = \sum_{n=-\infty}^{+\infty} U_n(\lambda - \lambda_o)^n$ for $0 < |\lambda - \lambda_o| < \delta$ and some $\delta > 0$. The coefficients U_n are bounded linear operators given by

$$(3.1)\quad U_n = \frac{1}{2\pi i}\int_\Gamma (z-\lambda_o)^{-(n+1)}R(z,A)\,dz \quad, n \in \mathbb{Z},$$

where $\Gamma = \{z \in \mathbb{C} : |z - \lambda_o| = \delta/2\}$.

The coefficient U_{-1} is the spectral projection corresponding to the spectral set $\{\lambda_o\}$ (see Def.3.1), it is called the residue of $R(\cdot,A)$ at λ_o, and will be denoted by P. From (3.1) one deduces

$$(3.2)\quad U_{-(n+1)} = (A-\lambda_o)^n\circ P \quad \text{and}$$
$$U_{-(n+1)}\circ U_{-(m+1)} = U_{-(n+m+1)} \quad \text{for } n, m \geqq 0.$$