rator of the adjoint weak*-semigroup, then $P\sigma(A) \cap i\mathbb{R} = \emptyset$, while $P\sigma(A') \cap i\mathbb{R} = i\mathbb{R}$. For that reason we cannot expect a simple connection between these two sets. But as we shall see below, if a semigroup on the predual of a W*-algebra has sufficiently many invariant states, then the point spectra of A and A' contained in $i\mathbb{R}$ are identical. Helpful for these investigations will be the next lemma.

Lemma 1.6. Let R be a pseudo-resolvent on $D = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ with values in a Banach space E such that $\|\mu R(\mu + i\alpha)\| \leq 1$ for all $(\mu,\alpha) \in \mathbb{R}_+ \times \mathbb{R}$. Then

$$\dim \mathrm{Fix}(\lambda R(\lambda + i\alpha)) \leq \dim \mathrm{Fix}(\lambda R(\lambda + i\alpha)')$$

for all $\lambda \in D$.

Proof. Let $(\mu,\alpha) \in \mathbb{R}_+ \times \mathbb{R}$ and $S := \mu R(\mu + i\alpha)$. Since S is a contraction, its adjoint S' maps the dual unit ball E'_1 into itself. Let $\mathcal{U}$ be a free ultrafilter on $[1,\infty)$ which converges to 1. Since E'_1 is $\sigma(E',E)$-compact,

$$\psi_o := \lim_{\mathcal{U}} (\lambda - 1) R(\lambda, S)'\psi$$

exists for all $\psi \in E'_1$. Since S' is $\sigma(E',E)$-continuous and since $S'R(\lambda,S)' = \lambda R(\lambda,S') - \mathrm{Id}$ we conclude $\psi_o \in \mathrm{Fix}(S')$.

Take now $0 \neq x_o \in \mathrm{Fix}(S)$ and choose $\psi \in E'_1$ such that $\psi(x_o)$ is different from zero. From the considerations above it follows

$$\psi_o(x_o) = \lim_{\mathcal{U}} (\lambda - 1)\ \psi(R(\lambda,S)x_o) = \psi(x_o) \neq 0 ,$$

hence $0 \neq \psi_o \in \mathrm{Fix}(S)$. Therefore $\mathrm{Fix}(S')$ separates the points of $\mathrm{Fix}(S)$. From this it follows that

$$\dim \mathrm{Fix}(S) \leq \dim \mathrm{Fix}(S') .$$

Since R and R' are pseudo-resolvents, the assertion is proved.

□

Corollary 1.7. Let $\mathcal{T}$ be a semigroup of contractions on a Banach space E with generator A. Then

$$\dim \ker(i\alpha - A) \leq \dim \ker(i\alpha - A')$$

for all $\alpha \in \mathbb{R}$.