Definition 1.6. Let E be a Banach space and $(T(t))_{t\geq 0}$ a semigroup on E. We call the semigroup

1. uniformly exponentially stable, if $\|T(t)\| \leq Me^{-wt}$ for some w, $M > 0$ and all $t \geq 0$.

2. uniformly stable, if $\lim_{t\to\infty} T(t) = 0$ in the strong operator topology.

3. weakly stable, if $\lim_{t\to\infty} T(t) = 0$ in the weak operator topology.

Surprisingly all these properties coincide for positive semigroups on C*-algebras with unit.

Theorem 1.7. Let M be a C*-algebra with unit and $(T(t))_{t\geq 0}$ a positive semigroup on M. Then the following assertion are equivalent.

1. $s(A) < 0$.

2. The semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable.

3. The semigroup $(T(t))_{t\geq 0}$ is uniformly stable.

4. The semigroup $(T(t))_{t\geq 0}$ is weakly stable.

Proof. Since '$s(A) = \omega$' by Theorem 1.3, it suffices to show that 4. implies 1.. For $t > 0$ there exists $\phi \in \mathcal{S}(M)$ such that

$$T(t)'\phi = r(T(t))\phi .$$

Then for $x \in M$

$$\phi(T(t)^n x) = (r(T(t)))^n \phi(x) \to 0$$

as $n \to \infty$. Therefore $r(T(t)) < 1$ or $\omega < 0$. Since $s(A) \leq \omega$ the assertion follows.

□