unique solution for every $g \in E$ if and only if $1 \in \rho(\Phi_\lambda)$. According to the proof of 3.5(b) this is equivalent to $\lambda \in \rho(A)$.
If Φ_λ is compact, then $\sigma(\Phi_\lambda)\setminus\{0\} \subset P\sigma(\Phi_\lambda)$. Thus the assertion follows from (a) and (b).

□

The previous results will now be used to characterize the spectral bound of A and hence the stability of the solutions of (RE).

<u>Theorem</u> 3.7. Let $A := \frac{d}{dx}$, $D(A) := \{f \in AC([-1,0],F) : f' \in L^1([-1,0],F)$ and $f(0) = \Phi(f)\}$ be the generator of the solution semigroup on $E := L^1([-1,0],F)$ corresponding to (RE). If F is a Banach lattice and $0 \leq \Phi \in \mathcal{L}(E,F)$, then the following assertions hold for $\lambda \in \mathbb{R}$.

(a) If $s(\Phi_\lambda) < 1$, then $s(A) < \lambda$.

(b) Let $\Phi(D(A_o)) = F$ or let Φ_λ be compact for all $\lambda \in \mathbb{R}$. In addition, suppose that the map $\mu \to s(\Phi_\mu)$ is strictly decreasing at $\mu = s(A)$. If $s(\Phi_\lambda) = 1$, then $s(A) = \lambda$.

(c) Let Φ_λ be compact for all $\lambda \in \mathbb{R}$ or let $\Phi(D(A_o)) = F$ and suppose that $\mu \to s(\Phi_\mu)$ is continuous from the right. If $s(\Phi_\lambda) > 1$, then $s(A) > \lambda$.

<u>Proof</u>. (a) Suppose $r := s(A) \geq \lambda$. The positivity of $(T(t))_{t\geq 0}$ implies $r \in \sigma(A)$ (see C-III,Thm.1.1.(a)) and by Prop.3.6 (a) this implies $1 \in \sigma(\Phi_r)$ so that $s(\Phi_r) \geq 1$. Since $\lambda \leq r$ this yields $s(\Phi_\lambda) \geq s(\Phi_r) \geq 1$.
(b) Let $s(\Phi_\lambda) = 1$. Since $1 \in \sigma(\Phi_\lambda)$ (see C-III,Thm.1.1(a)) $\lambda \in \sigma(A)$ by Prop.3.6(c) whence $s(A) \geq \lambda$. If $r := s(A)$ we deduce as in the proof of (a) that $s(\Phi_r) \geq 1$. Now $r > \lambda$ would imply $s(\Phi_\lambda) > s(\Phi_r) \geq 1$ (by the strict monotonicity of $\mu \to s(\Phi_\mu)$), a contradiction. Hence we conclude $s(A) = r = \lambda$.
(c) The hypotheses and Lemma 3.4 imply that the map $\mu \to s(\Phi_\mu)$ is continuous. Let $s(\Phi_\lambda) > 1$. Since $s(\Phi_\mu) \leq \|\Phi_\mu\| \leq \|\Phi\|\cdot\|\varepsilon_\mu\|$ we see that $s(\Phi_\mu)$ tends to zero as $\mu \to \infty$. Therefore there must exist $\mu' > \lambda$ such that $s(\Phi_{\mu'}) = 1$. Now Prop.3.6.(c) implies $\mu' \in \sigma(A)$ whence $s(A) \geq \mu' > \lambda$.

□

<u>Corollary</u> 3.8. Under the hypotheses of Thm.3.7, suppose that the mapping $h : \mu \to s(\Phi_\mu)$ is continuous from the right and strictly decreasing. Then the following equivalence holds.

(3.9) $\quad s(A) \substack{<\\=\\>} \lambda$ if and only if $s(\Phi_\lambda) \substack{<\\=\\>} 1$.

In particular, $\lambda = s(A)$ is the only real solution of $s(\Phi_\lambda) = 1$.