As a consequence one computes the growth bound of a multiplication semigroup as follows:

$$\omega = \sup\{\operatorname{Re} q(x) : x \in X\} \qquad \text{in the case} \quad E = C_o(X) ,$$
$$\omega = \mu\text{-ess-sup}\{\operatorname{Re} q(x) : x \in X\} \qquad \text{in the case} \quad E = L^p(\mu) .$$

It is a nice exercise to characterize those multiplication operators which generate strongly continuous groups.
We point out that the above results cover the cases of sequence spaces such as c_o or l^p , $1 \leq p < \infty$. An abstract characterization of generators of multiplication semigroups will be given in C-II,Thm.5.13.

2.4. Translation (Semi)Groups

Let E to be one of the following function spaces $C_o(\mathbb{R}_+)$, $C_o(\mathbb{R})$ or $L^p(\mathbb{R}_+)$, $L^p(\mathbb{R})$ for $1 \leq p < \infty$. Define $T(t)$ to be the (left) translation operator

$$T(t)f(x) := f(x+t)$$

for $x , t \in \mathbb{R}_+$, resp. $x , t \in \mathbb{R}$ and $f \in E$. Then $(T(t))_{t\geq 0}$ is a strongly continous semigroup, resp. group of contractions on E and its generator is the first derivative $\frac{d}{dx}$ with 'maximal' domain. In order to be more precise we have to distinguish the cases $E = C_o$ and $E = L^p$:

(i) The generator of the translation (semi)group on $E = C_o(\mathbb{R}_+)$ is

$$Af := \frac{d}{dx}f = f' ,$$
$$D(A) := \{f \in E : f \text{ differentiable and } f' \in E\} .$$

Proof. For $f \in D(A)$ it follows that for every $x \in \mathbb{R}_{(+)}$

$$\lim_{h\to 0} \frac{T(h)f(x) - f(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists}$$

(uniformly in x) and coincides with $Af(x)$. Therefore f is differentiable and $f' \in E$.
On the other hand, take $f \in E$ differentiable such that $f' \in E$. Then

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leq \frac{1}{h} \int_x^{x+h} |f'(y) - f'(x)| dy ,$$

where the last expression tends to zero uniformly in x as $h \to 0$. Thus $f \in D(A)$ and $f' = Af$.

□