Proof. We can assume that $s(A) = 0$ and we will denote the negative coefficients of the Laurent series of $R(.,A)$ at 0 by Q_n. Thus the following relations hold (see A-III,3.6),

$$Q_n = \frac{1}{2\pi i}\cdot\int_\gamma z^{n-1}R(z,A)\,dz \quad (n \in \mathbb{N}) \;;$$

(2.17) $$Q_n \neq 0 \text{ if } n \leq k \text{ and } Q_n = 0 \text{ for } n > k \;;$$

$$Q_n = A^{n-1}\,Q_1 \quad (n \in \mathbb{N}) \;; \quad Q_k = \lim_{z\to 0} z^k\cdot R(z,A) \;;$$

We define I_n as follows $(n = 0,1,\ldots,k-1)$:

$$I_n := \{f \in E : Q_{n+1}|f| = Q_{n+2}|f| = \ldots = Q_k|f| = 0\} .$$

At first we restrict our attention to I_{k-1}.
Since $R(\lambda,A)$ is positive if $\lambda > 0$ (Cor.1.3), it follows from (2.17) that Q_k is a positive bounded operator, hence $I_{k-1} = \{f \in E : Q_k|f| = 0\}$ is a closed ideal. Since Q_k commutes with $R(\lambda,A)$ (see (2.17)), it follows that I_{k-1} is a $\mathcal{T}$-invariant ideal. By A-III,Cor.4.3 the generators ${}^A|I_{k-1}$ and A_k induced by A on I_{k-1} and E/I_{k-1} respectively have a pole at 0. The coefficients of the Laurent series are the operators induced by Q_n on E/I_{k-1} and I_{k-1} respectively.
Suppose that the pole order of $R(.,A_k)$ is greater than 1, say m. Then $Q_{m/} = \lim_{z\to 0} z^m R(z,A_k)$ is a positive non-zero operator, hence we find for every $x \in E_+$ an element $y \in I_{k-1}$ such that $Q_m x + y \geq 0$. Then we have

$0 \leq Q_k|Q_m x + y| = Q_k Q_m x + Q_k y = Q_{k+m-1}x + Q_k y = 0 + Q_k y \leq Q_k|y| = 0$
hence $Q_m x = (Q_m x + y) - y \in I_{k-1}$ $(x \in E_+)$. It follows that $Q_{m/} = 0$ which is a contradiction.
So far we know that the resolvent of A_k has a pole of order ≤ 1. Moreover, since $Q_k|I_{k-1} = 0$, the resolvent of ${}^A|I_{k-1}$ has a pole of order $\leq k-1$. From A-III,Cor.4.3 it follows that the pole order of A_k and ${}^A|I_{k-1}$ is precisely 1 and $k-1$, respectively. The residue $Q_{1/I_{k-1}} = \lim_{z\to 0} zR(z,A_k)$ is positive since $R(z,A_k) \geq 0$ for $z > 0$ (Cor.1.3). To prove that it is strictly positive we assume $Q_{1/I_{k-1}}(|x + I_{k-1}|) = 0$ which means $Q_1|x| \in I_{k-1}$ hence $Q_k|x| = A^{k-1}Q_1|x| = 0$, that is, $x \in I_{k-1}$ or $x + I_{k-1} = 0$.
Applying what we have proved so far to I_{k-1} and ${}^A|I_{k-1}$ we obtain I_{k-2}, A_{k-1}, and so on. After k steps $(n=1)$ we conclude that I_o is $\mathcal{T}$-invariant and that the order of the pole of $R(.,{}^A|I_o)$ is 0,