Definition 1.1 (?). The *semigroup dual* of the Banach space E with respect to the strongly continuous semigroup $(T(t))_{t\geq 0}$ is

$$E^* := \{\phi \in E' : \|\cdot\|\text{-}\lim_{t\to 0} T(t)'\phi = \phi\}.$$

(restricted)

The adjoint semigroup on E^* is given by the operators

$$T(t)^* := T(t)'|_{E^*}, \quad t \geq 0.$$

Since $(T(t)^*)_{t\geq 0}$ is strongly continuous on E^* we call its generator $(A^*, D(A^*))$ the *adjoint generator.*

The above definition makes sense since E^* is norm-closed in E' and $(T(t)')$-invariant. The main point is that E^* is still reasonably large. In fact, since $\int_0^t T(s)'\phi\, ds$, understood in the weak sense, is contained in E^* for every $\phi \in E'$, $t \geq 0$ it follows that

$$\sup\{\langle f, \phi\rangle : \phi \in E^*, \|\phi\| \leq 1\} \leq \|f\| \leq M \cdot \sup\{\langle f, \phi\rangle : \phi \in E^*, \|\phi\| \leq 1\}$$

where $M := \limsup_{t\to 0} \|T(t)\|$. In particular, E^* separates E, i.e. E^* is $\sigma(E', E)$-dense in E'. In addition the estimate of $\|\cdot\|$ given above yields muss da ein , sein?

$$\|T(t)^*\| \leq \|T(t)\| \leq M\|T(t)^*\| \quad \text{for all } t \geq 0.$$

In the following proposition we describe the relation between A^* and A'.

Proposition 1.2. *For the adjoint generator A^* of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E the following assertions hold:*

(i) E^* *is the* $\|\cdot\|$*-closure of* $D(A')$, in E'.

(ii) $D(A^*) = \{\phi \in D(A') : A'\phi \in E^*\}$.

(iii) A^* *and* A' *coincide on* $D(A^*)$.

Proof. (i) Take $\phi \in D(A')$ fixed. For every $f \in D(A)$ with $\|f\| \leq 1$ we define a continuously differentiable function

$$t \mapsto \xi_f(t) := \langle T(t)f, \phi\rangle$$

on $[0,1]$ with derivative $\xi_f'(t) = \langle T(t)Af, \phi\rangle = \langle T(t)f, A'\phi\rangle$.

Since $\{\xi_f'(t) : t \in [0,1], f \in D(A), \|f\| \leq 1\}$ is bounded, it follows that the set

$$\{\xi_f : f \in D(A), \|f\| \leq 1\}$$

is equicontinuous at 0, i.e. ? for every $\varepsilon > 0$ there exists $0 < t_0 < 1$ such that

$$|\xi_f(s) - \xi_f(0)| = |\langle f, T(s)'\phi - \phi\rangle| < \varepsilon$$

for every $0 \leq s \leq t_0$ and $f \in D(A)$, $\|f\| \leq 1$. But this implies $\|T(s)'\phi - \phi\| < \varepsilon$ and hence $\phi \in E^*$.

Conversely take $\psi \in E^*$. Then $\frac{1}{t}\int_0^t T(s)'\psi\, ds$, $t > 0$, belongs to $D(A')$ and norm converges to ψ as $t \to 0$, i.e. ψ belongs to the norm closure of $D(A')$.

xi

ξ

(ii) and (iii): Since the weak* topology on E' is weaker than the norm topology, it follows that A' is an extension of A^*. Now take $\phi \in D(A')$ such that $A'\phi \in E^*$. As above define the functions f_f. The assumption on ϕ implies the set of all derivatives

$$\{f'_f : f \in D(A), \|f\| \leq 1\}$$

to be equicontinuous at $t = 0$. This means that for every $\varepsilon > 0$ there exists $0 < t_o < 1$ such that $|f'_f(0) - f'_f(s)| < \varepsilon$ for every $f \in D(A)$, $\|f\| \leq 1$ and $0 < s < t_o$. In particular,

$$\varepsilon > |f'_f(0) - \frac{1}{s}(\xi_f(s) - \xi_f(0))| = | < f, A'\phi - \frac{1}{s}(T(s)'\phi - \phi) > |,$$

hence

$$\varepsilon > \|A'\phi - \frac{1}{s}(T(s)'\phi - \phi)\|$$

for all $0 \leq s \leq t_o$. From this it follows that $\phi \in D(A^*)$.

□

On reflexive Banach spaces we have $A^* = A'$ by the above proposition. In other cases this construction is more interesting.

Example 1.3 (continued). The adjoints of the (left) translation $T(t)$ on $E =$ L^1 $L^1(\mathbb{R})$ are the (right) translations $T(t)'$ on $E' = L^\infty(\mathbb{R})$. The largest subspace of $L^\infty(\mathbb{R})$ on which these translations form a str. cont semigroup ~~which is strongly continuous~~ with respect to the sup-norm, is the space of all bounded uniformly continuous functions on $\mathbb{R}$, i.e. $E^* = C_{bu}(\mathbb{R})$.

Calculating $D(A')$ and $D(A^*)$ respectively, one obtains

$$D(A') = \{f \in L^\infty(\mathbb{R})\colon f \in AC, f' \in L^\infty(\mathbb{R})\},$$
$$D(A^*) = \{f \in L^\infty(\mathbb{R})\colon f \in C^1(\mathbb{R}), f' \in C_{bu}(\mathbb{R})\}.$$

Obviously, the function $x \mapsto |\sin x|$ belongs to $D(A')$, but not to $D(A^*)$.

1.1.1 The Associated Sobolev Semigroups

Since the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ is closed, its domain $D(A)$ becomes a Banach space for the graph norm

$$\|f\|_1 := \|f\| + \|Af\|.$$

We denote this Banach space by E_1 and the continuous injection from E_1 into E by i_1. Since E_1 is invariant under $(T(t))_{t\geq 0}$ - apply Prop.1.6.i – it makes sense to consider the semigroup $(T_1(t))_{t\geq 0}$ of all restrictions $T_1(t) := T(t)|_{E_1}$.
osition
The results of Prop.1.6 imply that $T_1(t) \in \mathcal{L}(E_1)$ and $\|T_1(t)f - f\|_1 \to 0$ as $t \to 0$ for every $f \in E_1$. Thus $(T_1(t))_{t\geq 0}$ is a strongly continuous semigroup on E_1 and has a generator denoted by $(A_1, D(A_1))$.

Using Prop.1.6 again we see that A_1 is the restriction of A to E_1 with maximal domain, i.e., ? $D(A_1) = \{f \in E_1\colon Af \in E_1\} = D(A^2)$ and $A_1 f = Af$ for every $f \in D(A_1)$.

It is now possible to repeat this construction in order to obtain Banach spaces

was meint claude: , e.g., oder ohne , ?

E_n and semigroups $(T_n(t))_{t\geq 0}$ with generators $(A_n, D(A_n))$ which are related as visualized in the following diagram.

$$
\begin{tikzcd}
E \arrow[r, "T(t)"] \arrow[d, "i_1"'] & E \arrow[d, "i_1"] & \\
E_1 \arrow[r, "T_1(t)"] \arrow[ur, "A"] \arrow[d, "i_2"'] & E_1 \arrow[d, "i_2"] & {= D(A)} \\
E_2 \arrow[r, "T_2(t)"] \arrow[ur, "A_1"] & E_2 & {= D(A_1) = D(A^2)} \\
{} \arrow[u, "i_3"] & {} \arrow[u, "i_3"'] &
\end{tikzcd}
$$

zu breit!

For the translation semigroup on $L^p(\mathbb{R})$ (see 2.3) the above construction leads to the usual *Sobolev spaces*. Therefore we ~~might~~ call E_n the *n-th Sobolev space* and $(T_n(t))_{t\geq 0}$ the *n-th Sobolev semigroup* associated to E and $(T(t))_{t\geq 0}$.

Remark 1.4. For $\lambda \in \rho(A)$ the operator $(\lambda - A)$ and the resolvent $R(\lambda, A)$ are isomorphisms from E_1 onto E, resp. from E onto E_1 (show that $\|\cdot\|_1$ and $\|\cdot\|_\lambda$ with $\|\cdot\|_\lambda := \|(\lambda - A)\cdot\|$ are equivalent). In addition, the following diagram commutes.

$$\begin{tikzcd}
E \arrow[r, "T(t)"] \arrow[d, "\lambda - A"'] & E \arrow[d, "R(\lambda, A)"] \\
E_1 \arrow[r, "T_1(t)"] & E_1
\end{tikzcd}$$

Therefore all Sobolev semigroups $(E_n, T_n(t))_{t\geq 0}$, $n \in \mathbb{N}$, are isomorphic.

Remark 1.5. For $\lambda \in \rho(A)$ consider the norm

$$\|f\|_{-1} := \|R(\lambda, A)f\|$$

for every $f \in E$ and define E_{-1} as the completion of E for $\|\cdot\|_{-1}$.

Then $(T(t))_{t\geq 0}$ extends continuously to a strongly continuous semigroup $(T_{-1}(t))_{t\geq 0}$ on E_{-1} and the above diagram can be extended to the negative integers. See Engel-Na for more on these abstract Sobolev spaces

1.2 The ~~F~~ $\mathcal{F}$-Product Semigroup

It is a standard in functional analysis ~~very successful mathematical method~~ to consider a sequence of points in a certain space as a point in a new and larger space. In particular such a method can serve to convert an approximate eigenvector of a linear operator into an eigenvector. Occasionally we will need such a construction and refer to Section V.1 of [Schaefer (1974)] for the details. noch anderes Zitat?

If we try to adapt this construction to strongly continuous semigroups, we encounter the difficulty that the semigroup extended to the larger space will not remain strongly continuous. An idea already used in 3.4 will help to overcome this difficulty.

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space E. Denote by $m(E)$ the Banach space of all bounded E-valued sequences endowed with the norm

$$\|(f_n)_{n\in\mathbb{N}}\| := \sup\{\|f_n\| : n \in \mathbb{N}\}$$

It is clear that every $T(t)$ extends canonically to a bounded linear operator

$$\hat{T}(t)(f_n) := (T(t)f_n)$$

on $m(E)$, but the semigroup $(\hat{T}(t))_{t\geq 0}$ is strongly continuous if and only if T has a bounded generator. Therefore we restrict our attention to the closed, $(\hat{T}(t))$-invariant subspace

$$m^T(E) := \{(f_n) \in m(E) : \lim_{t\to 0} \|T(t)f_n - f_n\| = 0 \text{ uniformly for } n \in \mathbb{N}\}$$

Then the restricted semigroup

$$\hat{T}(t)(f_n) = (T(t)f_n), \quad (f_n) \in m^T(E)$$

is strongly continuous and we denote its generator by $(\hat{A}, D(\hat{A}))$.

The following lemma shows that $\hat{A}$ is obtained canonically from A.

Lemma 1.6. *For the generator $\hat{A}$ of $(\hat{T}(t))_{t\geq 0}$ on $m^T(E)$ one has* the following properties.

(i) $D(\hat{A}) = \{(f_n) \in m^T(E) : f_n \in D(A)$ *and* $(Af_n) \in m^T(E)\}$,

(ii) $\hat{A}(f_n) = (Af_n)$ *for* $(f_n) \in D(\hat{A})$.

For the proof we refer to Lemma 1.4. of [Derndinger (1980)].

Now let F be any filter on $\mathbb{N}$ finer than the Frechet filter (i.e. the filter of sets with finite complement). (In most cases F will be either the Frechet filter or some free ultra filter.)

Then the sub

space of all F-null sequences in $m(E)$

$$c_F(E) := \{(f_n) \in m(E) : F\text{-}\lim \|f_n\| = 0\}$$

is closed in $m(E)$ and invariant under $(\hat{T}(t))_{t\geq 0}$. We call the quotient spaces

$$E_F := m(E)/c_F(E) \quad \text{and} \quad E_F^T := m^T(E)/c_F(E) \cap m^T(E)$$

the *F-product of E* and the *F-product of E with respect to the semigroup T*, respectively.

Thus E_F^T can be considered as a closed linear subspace of E_F. We have $E_F^T = E_F$ if (and only if) T has a bounded generator.

The canonical quotient norm on E_F is given by

$$\|(f_n) + c_F(E)\| = F\text{-}\limsup \|f_n\|.$$

We can apply 3.3 in order to define the *F-product semigroup* $(T_F(t))_{t\geq 0}$ on E_F^T by

$$T_F(t)((f_n) + c_F(E)) := (T(t)f_n) + c_F(E) \cap m^T(E)$$

Thus $T_F(t)$ is the restriction of $T(t)_F$ where $T(t)_F$ denotes the canonical extension of $T(t)$ to the F-product E_F. (Note that $(T(t)_F)_{t\geq 0}$ is not strongly continuous unless T has a bounded generator.)

With the canonical injection $j : f \mapsto (f, f, f, \ldots) + c_F(E)$ from E into E_F^T the operators $T_F(t)$ are extensions of $T(t)$ satisfying $\|T_F(t)\| = \|T(t)\|$. The basic facts about the generator $(A_F, D(A_F))$ of $(T_F(t))_{t\geq 0}$ follow from 3.3 and are collected in the following proposition.

Proposition 1.7. *For the generator* $(A_F, D(A_F))$ *of the F-product semigroup the following holds:*

(i) $D(A_F) = \{(f_n) + c_F(E) : f_n \in D(A); (f_n), (Af_n) \in m^T(E)\}$

(ii) $A_F((f_n) + c_F(E)) = (Af_n) + c_F(E)$

In case A is a bounded operator then $D(A_F) = E_F^T = E_F$ and A_F is the canonical extension of A to E_F.

We will show in A-III,4.5 that the above construction preserves and even improves many spectral properties of the semigroup and its generator.

1.3 The Tensor Product Semigroup

Real- or complex-valued functions of two variables x, y are often limits of functions of the form $\sum_{i=1}^{n} f_i(x)g_i(y)$, which, to some extent, allows one to consider the variables x and y separately.

Since algebraic manipulation with these latter functions is governed by the formal rules of a tensor product, it is customary to identify (for example) the function

$$(x, y) \mapsto f(x)g(y)$$

with the tensor product $f \otimes g$ and to consider limits of linear combinations of such functions as elements of a completed tensor product.

To be more precise, we briefly present the most important examples for this situation.

Examples 1.8. (i) Let (X, Σ, μ) and (Y, Ω, ν) be measure spaces. Identifying for $f_i \in L^p(\mu)$, $g_i \in L^p(\nu)$ the elements $\sum_{i=1}^{n} f_i \otimes g_i$ of the tensor product

$$L^p(\mu) \otimes L^p(\nu)$$

with the (class of $\mu \times \nu$-a.e.-defined) functions

$$(x, y) \mapsto \sum_{i=1}^{n} f_i(x)g_i(y),$$

$L^p(\mu) \otimes L^p(\nu)$ becomes a dense subspace of $L^p(X \times Y, \Sigma \times \Omega, \mu \times \nu)$ for $1 \leq$

$p < \infty$.

(ii) Similarly, let X,Y be compact spaces. Then $C(X) \otimes C(Y)$ becomes a dense subspace of $C(X \times Y)$ by identifying, for $f \in C(X)$ and $g \in C(Y)$, $f \otimes g$ with the function

$$(x, y) \mapsto f(x)g(y) .$$

We do not intend to go ~~into a~~ deeper ~~investigation of~~ into the quite sophisticated problems related to normed tensor products of general Banach spaces, but will rather confine ourselves to the discussion of certain special cases. These will always be related to one of the following standard methods to define a norm on the tensor product of two Banach spaces E, F. Let $u := \sum_{i=1}^{n} f_i \otimes g_i$ be an element of $E \otimes F$. Then

(i) $\|u\|_\pi := \inf\{\sum_{j=1}^{m} \|h_j\|\|k_j\| : u = \sum_{j=1}^{m} h_j \otimes k_j, h_j \in E, k_j \in F\}$ defines the *greatest cross norm* π on $E \otimes F$.

(ii) $\|u\|_\varepsilon := \sup\{\langle u, \phi \otimes \psi\rangle : \phi \in E', \psi \in F', \|\phi\|, \|\psi\| \leq 1\}$ defines the "least cross norm ε" on $E \times F$. Here, $\langle u, \phi \otimes \psi\rangle$ denotes the canonical bilinear form on $(E \otimes F) \times (E' \otimes F')$, i.e. $\langle \sum_{i=1}^{n} f_i \otimes g_i, \phi \otimes \psi\rangle = \sum_{i=1}^{n} \langle f_i, \phi\rangle\langle g_i, \psi\rangle$.

(iii) if E and F are Hilbert spaces, $\|u\|_h = (u|u)_h^{1/2}$, where the scalar product $(\cdot|\cdot)_h$ is defined as in (ii), defines the *Hilbert norm* h on $E \otimes F$.

In the following we write $E \otimes_\alpha F$ for the tensor product of E and F endowed – if applicable – with one of the norms π, ε, h just defined. In each case one has $\|f \otimes g\| = \|f\|\|g\|$ for $f \in E, g \in F$.

By $E \tilde{\otimes}_\alpha F$ we mean the completion of $E \otimes_\alpha F$. Moreover we recall how examples (i) and (ii) above fit into this pattern:

$$L^1(\mu \otimes \nu) = L^1(\mu) \tilde{\otimes}_\pi L^1(\nu), \quad L^2(\mu \otimes \nu) = L^2(\mu) \tilde{\otimes}_h L^2(\nu),$$

$$C(X \otimes Y) = C(X) \tilde{\otimes}_\varepsilon C(Y).$$

Finally we point out that for any $S \in \mathcal{L}(E)$, $T \in \mathcal{L}(F)$, the mapping

$$\sum_{i=1}^{n} f_i \otimes g_i \mapsto \sum_{i=1}^{n} S f_i \otimes T g_i$$

defined on $E \otimes F$ is linear and continuous on $E \otimes_\alpha F$, hence has a continuous extension to $E \tilde{\otimes}_\alpha F$. This operator, as well as its continuous extension, will be denoted by $S \otimes T$ and satisfies $\|S \otimes T\| = \|S\|\|T\|$. The notation $A \otimes B$ will also be used in the obvious way if A and B are not necessarily bounded operators on E and F. We are now ready to consider semigroups induced on tensor product.

Proposition 1.9. *Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be strongly continuous semigroups on Banach spaces E, F, and let A, B be their generators. Then the family $(S(t) \otimes T(t))_{t\geq 0}$ is a strongly continuous semigroup on $E \tilde{\otimes}_\alpha F$. The closure of $A \otimes Id + Id \otimes B$, defined on the core $D(A) \otimes D(B)$, is its generator.*

Proof. It is immediately verified that $(S(t) \otimes T(t))_{t\geq 0}$ is in fact a semigroup of operators on $E \tilde{\otimes}_\alpha F$. The strong continuity need only be verified at $t = 0$ and on elements of the form $u = f \otimes g \in E \otimes F$.

This verification being straightforward, there remains to show that the generator of $(S(t)\otimes T(t))_{t\geq 0}$ is obtained as the closure of $(A\otimes \mathrm{Id}+\mathrm{Id}\otimes B, D(A)\otimes D(B))$.

To this end, let $f\in D(A)$ and $g\in D(B)$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}(T(h)\otimes S(h)(f\otimes g)-f\otimes g) &= \lim_{h\to 0}\frac{1}{h}(T(h)f\otimes (S(h)g-g)+(T(h)f-f)\otimes g)\\ &= (f\otimes Bg)+(Af\otimes g)\end{aligned}$$

Since the elements of the form $f\otimes g$, $f\in D(A)$, $g\in D(B)$, generate the linear subspace $D(A)\otimes D(B)$ of $E\tilde{\otimes}_\alpha F$, this subspace belongs to the domain of the generator. Moreover, $D(A)\otimes D(B)$ is dense in $E\tilde{\otimes}_\alpha F$ and invariant under $(S(t)\otimes T(t))_{t\geq 0}$, hence it is a core of $A\otimes \mathrm{Id}+\mathrm{Id}\otimes B$ by osition Prop.1.9.ii. □

1.4 The Product of Commuting Semigroups

Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be semigroups with generators A and B, respectively on some Banach space E. It is not difficult to see that the following assertions are equivalent.

(a) $S(t)T(t)=S(t)T(t)$ for all $t\geq 0$.

(b) $R(\mu,A)R(\mu,B)=R(\mu,B)R(\mu,A)$ for some $\mu\in\rho(A)\cap\rho(B)$.

(c) $R(\mu,A)R(\mu,B)=R(\mu,B)R(\mu,A)$ for all $\mu\in\rho(A)\cap\rho(B)$.

In that case $U(t) = S(t)T(t)$ $(t\geq 0)$ defines a semigroup $(U(t))_{t\geq 0}$. Using osition Prop.1.9(ii) one easily shows that $D_0 := D(A)\cap D(B)$ is a core for its generator

C and $Cf = Af + Bf$ for all $f \in D_0$.

Notes

For a more complete information on semigroup theory we refer the reader to [Hille-Phillips (1957)], to the ~~recent~~ monographs by [Davies (1980)], [Goldstein (1985a)] and [Pazy (1983)], to the survey article by [Krein-Khazan (1985)] and to the bibliography by [Goldstein (1985b)]. E-N

super !