This example shows that even if there exists a strictly positive subeigenvector of the adjoint of the generator, Kato's inequality for positive elements alone does not suffice for the positivity of the semigroup. Note also that (because of the order continuous norm) Kato's inequality holds for positive elements if and only if the positive minimum principle is satisfied (see Remark 3.14).

4. DOMINATION OF SEMIGROUPS

Frequently it is useful to be able to compare two semigroups on a Banach lattice with respect to the ordering (for example, in order to decide whether a semigroup is stable (see Chapter A-IV and Example 4.14).
In this section we assume that E is a σ-order complete complex Banach lattice. Let $(T(t))_{t\geq 0}$ be a positive semigroup with generator A and $(S(t))_{t\geq 0}$ a semigroup with generator B. We say, $(T(t))_{t\geq 0}$ dominates $(S(t))_{t\geq 0}$ if

(4.1) $$|S(t)f| \leq T(t)|f| \quad \text{for all } f \in E,\ t > 0 .$$

We first observe that domination of the semigroup is equivalent to domination of the resolvents.

Proposition 4.1. The semigroup $(T(t))_{t\geq 0}$ dominates $(S(t))_{t\geq 0}$ if and only if

(4.2) $$|R(\lambda,B)f| \leq R(\lambda,A)|f| \quad (f \in E) \quad \text{for large real } \lambda .$$

Proof. (4.2) follows from (4.1) since the resolvent is given by the Laplace transform of the semigroup. Conversely, if (4.2) holds, then

$$\begin{aligned} |S(t)f| &= \lim_{n\to\infty} |((n/t)R(n/t,B))^n f| \\ &\leq \lim_{n\to\infty} ((n/t)R(n/t,A))^n |f| \\ &= T(t)|f| \qquad (t \geq 0,\ f \in E). \end{aligned}$$

□