Examples 1.2. (a) The left-translation semigroup on $C_o(\mathbb{R}_+)$ or the semigroup generated by the Laplacian on $C_o(\mathbb{R}^n)$, see B-III,Ex.1.7, are uniformly stable but not exponentially stable.

(b) The left translations $T(t)f(x) = f(x+t)$ on $C_o(\mathbb{R})$ form a group of isometries. Hence $(T(t))_{t\geq 0}$ is not stable. However, $(T(t))_{t\geq 0}$ is weakly stable. Indeed, identifying $C_o(\mathbb{R})'$ with the space of all bounded Borel measures on $\mathbb{R}$, for $f \in C_o(\mathbb{R})$, $\mu \in C_o(\mathbb{R})'$ we have

$$\langle T(t)f,\mu\rangle = \int (T(t)f)(x)\, d\mu(x)$$

Obviously, $T(t)f$ tends pointwise to 0 as $t \to \infty$ and is dominated by the μ-integrable function $\|f\|_\infty \cdot 1$. Thus Lebesgue's Dominated Convergence Theorem implies $\lim \langle T(t)f,\mu\rangle = 0$.

(c) Finally we give an example of a positive semigroup on $C_o(X)$ which is not weakly stable but satisfies $\mathrm{Re}(P\sigma(A) \cup R\sigma(A)) < 0$. (Compare with A-IV,Cor.1.14).

Consider in the space $\mathbb{C}\setminus\{0\}$ a flow ϕ having the following properties:

- The orbits starting at z with $|z| \neq 1$ spiral towards the unit circle Γ ;
- 1 is a fixed point and $\Gamma\setminus\{1\}$ is a homoclinic orbit (i.e. $\lim_{t\to+\infty} \phi(t,z) = \lim_{t\to-\infty} \phi(t,z) = 1$ for every $z \in \Gamma$).

A concrete example of this type is the flow governed by the following differential equations for the polar coordinates (i.e. $z = r\cdot e^{i\omega}$)

$$\dot{r} = 1 - r$$

$$\dot{\omega} = 1 + (r^2 - 2r\cdot\cos\omega)$$

The locally compact set $X := \{z \in \mathbb{C} : 0 < |z| < 2 , z \neq 1\}$ is invariant under the flow ϕ and we consider on the space $C_o(X)$ the semigroup $(T(t))_{t\geq 0}$ associated with ϕ (i.e. $T(t)f = f\circ\phi_t$, $f \in C_o(X)$). We claim that

(i) $(T(t))_{t\geq 0}$ is not weakly uniformly stable ;

(ii) $P\sigma(A)\cap i\mathbb{R} = \emptyset$;

(iii) $R\sigma(A)\cap i\mathbb{R} = \emptyset$.

Proof of (i): Given $z \in X$, $|z| \neq 1$, there exist sequences (t_n) , (s_n) both tending to ∞ such that $\phi(t_n,z) \to 1$ and $\phi(s_n,z) \to -1$. Hence for $f \in C_o(X)$ we have

$\langle T(t_n)f,\delta_z\rangle = f(\phi(t_n,z)) \to 0$,

$\langle T(s_n)f,\delta_z\rangle = f(\phi(s_n,z)) \to f(-1)$.

Thus $\lim_{t\to\infty} \langle T(t)f,\delta_z\rangle$ does not exist for every $f \in C_o(X)$.

Proof of (ii): Assume that $T(t)f = e^{i\alpha t}f$ for every $t \geq 0$ and some $\alpha \in \mathbb{R}$ (cf. A-III,Cor.6.4). Given $z \in X$, there exists a sequence (t_n) such that $\phi(t_n,z) \to 1$, hence