$$\begin{aligned} &\leq \lim_{t\to 0+} 1/t\ ((T(t)|f|)(0) \\ &= \lim_{t\to 0+} <|f|, 1/t(T(z)'\delta_o - \delta_o)> \\ &= <|f|, A'\delta_o> = <|f|, \mu> . \end{aligned}$$

Since $\mu(\{0\}) = \nu\{0\}) = 0$, this implies that $|\nu| \leq \mu$.

Moreover, for arbitrary $f \in C[-1,0]_+$ we have

$<f, \mathrm{Re}\beta\ \delta_o + \mathrm{Re}\nu> = \lim_{t\to 0+} 1/t\ \mathrm{Re}<(S(t)f - f), \delta_o>$

$\leq \lim_{t\to 0+} 1/t\ \mathrm{Re}<(T(t)f - f), \delta_o> = <f, A'\delta_o> = <f, \alpha\delta_o + \mu>$.

Consequently, $(\mathrm{Re}\beta)\delta_o + \mathrm{Re}\nu \leq \alpha\delta_o + \mu$. This implies $\mathrm{Re}\beta \leq \alpha$ since $\mu(\{0\}) = \nu\{0\}) = 0$.

□

We conclude this section discussing the following question. Let $(S(t))_{t\geq 0}$ be a semigroup which is dominated by some positive semigroup. Does there exist a smallest semigroup $(T(t))_{t\geq 0}$ which dominates $(S(t))_{t\geq 0}$? More precisely, we look for a positive semigroup $(T(t))_{t\geq 0}$ dominating $(S(t))_{t\geq 0}$ such that $(T(t))_{t\geq 0}$ is dominated by any other positive semigroup which dominates $(S(t))_{t\geq 0}$. If such a minimal dominating semigroup exists, it is unique and we call it the <u>modulus semigroup</u> of $(S(t))_{t\geq 0}$.

<u>Example</u> 4.15 (the modulus semigroup associated with $\Delta - V$).
Let E be the complex space $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ and $V \in L^p_{loc}(\mathbb{R}^n)$ satisfying $\mathrm{Re}V \geq 0$. Denote by B the closure of $\Delta - V$ on C_c^∞ (cf. Example 4.7). The modulus semigroup of the semigroup $(S(t))_{t\geq 0}$ generated by B exists and its generator A is given by $Af = \Delta f - (\mathrm{Re}V)f$ for all $f \in C_c^\infty$ (and C_c^∞ is a core of A , see Example 4.7).

<u>Proof</u>. The operator A defined above generates a positive semigroup (see Example 4.7). For $f \in C_c^\infty$, $\phi \in D(A')_+$ one has
$\mathrm{Re}<(\mathrm{sign}\ \bar{f})Bf, \phi> = \mathrm{Re}<(\mathrm{sign}\ \bar{f})(\Delta f - Vf), \phi> =$
$\mathrm{Re}<(\mathrm{sign}\ \bar{f})\Delta f, \phi> - <(\mathrm{Re}V)|f|, \phi> = \mathrm{Re}<(\mathrm{sign}\ \bar{f})Af, \phi> \leq <|f|, A'\phi>$ by Thm.2.4. Since C_c^∞ is a core of B , it follows from Thm.4.3 that the semigroup generated by A dominates $(S(t))_{t\geq 0}$. Let C be the generator of a semigroup $(U(t))_{t\geq 0}$ dominating $(S(t))_{t\geq 0}$. Then
$\mathrm{Re}<(\mathrm{sign}\ \bar{f})Af, \phi> = \mathrm{Re}<(\mathrm{sign}\ \bar{f})\Delta f, \phi> - <(\mathrm{Re}V)|f|, \phi> = \mathrm{Re}<(\mathrm{sign}\ \bar{f})Bf, \phi>$
$\leq <|f|, C'\phi>$ for all $f \in C_c^\infty$, $\phi \in D(C')_+$ by Thm.4.2. It follows from Thm.4.3 that $(U(t))_{t\geq 0}$ dominates the semigroup generated by A .

□