(a) For every p, $1 \le p < \infty$ the following assertions are equivalent.

(i) The operators $T(t)$ defined by $T(t)f := f\circ\phi_t$ for $f \in L^p(\mu)$, $t \ge 0$, are well-defined as bounded linear operator on $L^p(\mu)$ and $(T(t))_{t\ge0}$ is a strongly continuous semigroup.

(ii) There exist constants $t_o > 0$, $M > 0$ such that $\mu(\phi_t^{-1}(C)) \le M\cdot\mu(C)$ for every open (compact) set $C \subset X$ and every $t \le t_o$.

(b) In case the conditions (i) and (ii) are fulfilled then $(T(t))_{t\ge0}$ is a semigroup of lattice homomorphisms on $L^p(\mu)$ and $C_c(X)\cap D(A)$ is a core of the generator.

Proof. (a) Since μ is assumed to be regular, the inequality stated in (ii) holds true for all Borel sets provided it is true for all open sets (compact sets, respectively).

(i)$\to$(ii): Assume that $(T(t))$ is a strongly continuous semigroup on $L^p(\mu)$, $1 \le p < \infty$. For $t_o > 0$ we define $M := (\sup\{\|T(t)\|: 0\le t\le t_o\})^{1/p}$. Given a Borel set $C \subset X$ we write $C(t) := \phi_t^{-1}(C)$.

Then we have $T(t)\chi_C = \chi_{C(t)}$, hence

$\mu(\phi_t^{-1}(C)) = \|\chi_{C(t)}\|_p^p = \|T(t)\chi_C\|_p^p \le M\cdot\|\chi_C\|_p^p = M\cdot\mu(C)$.

(ii)$\to$(i): Since the inequality in (ii) holds for all Borel sets, $\phi_t^{-1}(C)$ is a μ-null set whenever C is a μ-null set. Thus given Borel functions f, g such that $f = g$ μ-a.e. then $f\circ\phi_t = g\circ\phi_t$ μ-a.e.. Moreover, for $0 \le f \in L^p(\mu)$, there exists an increasing sequence (h_n) of simple functions converging pointwise to f. Then $(h_n\circ\phi_t)$ is a monotone sequence converging pointwise to $f\circ\phi_t$. Using the fact that $\chi_C\circ\phi_t = \chi_{C(t)}$, $C(t)$ as above, and the assumption $\mu(C(t)) \le M\cdot\mu(C)$ it is easy to see that $\|h_n\circ\phi_t\|_p^p \le M\cdot\|h_n\|_p^p \le M\cdot\|f\|_p^p$. Thus by the Monotone Convergence Theorem we have $f\circ\phi_t \in L^p(\mu)$ and $\|f\circ\phi_t\|_p \le M^{1/p}\|f\|_p$. It follows that $T(t)$ is a bounded linear operator on $L^p(\mu)$ and $\|T(t)\| \le M^{1/p}$ for $0 \le t \le t_o$. Since ϕ is semiflow we have $T(0) = Id$ and $T(t+s) = T(s)T(t)$ $(0 \le s,t < \infty)$. It remains to prove strong continuity. Since ϕ is continuous and (4.12) holds, we know that $T(t)(C_c(X)) \subset C_c(X)$ and that $T(t)f$ tends to f uniformly as $t \to 0$ provided that $f \in C_c(X)$. It follows that $\lim_{t\to0}\|T(t)f - f\|_p = 0$ for $f \in C_c(X)$. Since $C_c(X)$ is dense in $L^p(\mu)$ and $\|T(t)\| \le M^{1/p}$ for $0 \le t \le t_o$, the semigroup is strongly continuous.

(b) Obviously every operator $T(t)$ defined in assertion (i) of (a) is a lattice homomorphism. Above we pointed out that $C_c(X)$ is