<u>Theorem</u> 4.9. If there exists a quasi-interior subeigenvector u of A such that $u \in D(m)$, then B is closable and the closure $\bar{B}$ of B is the generator of a positive semigroup $(S(t))_{t\geq 0}$ which is dominated by $(T(t))_{t\geq 0}$.

For the proof of the theorem we need the following lemma.

<u>Lemma</u> 4.10. Let A and B be generators of positive semigroups $(T(t))_{t\geq 0}$ and $S(t)_{t\geq 0}$, respectively. If $(T(t))_{t\geq 0}$ dominates $(S(t))_{t\geq 0}$, then $s(B) \leq s(A)$.

<u>Proof of Lemma 4.10</u>. Let $\lambda > s(A)$. Then for all $\mu > \max\{\lambda, s(B)\}$ one has $0 \leq R(\mu,A) \leq R(\lambda,A)$ (by B-II,Lemma 1.9), and so $\|R(\mu,B)\| \leq \|R(\mu,A)\| \leq \|R(\lambda,A)\|$. Thus $\operatorname{dist}(\mu,\sigma(B)) \geq \|R(\mu,B)\|^{-1} \geq \|R(\lambda,A)\|^{-1}$. This implies that $[\lambda,\infty) \subset \rho(B)$. Hence $s(B) \leq \lambda$.

□

<u>Proof of Theorem 4.9</u>. There exists $\mu > 0$ such that $Au \leq \mu u$.
Let $\lambda > \max\{s(A),\mu\}$. Then $\lambda R(\lambda,A)u = AR(\lambda,A)u+u \leq \mu R(\lambda,A)u + u$. Hence $R(\lambda,A)u \leq c\cdot u$ where $c > 0$. It follows that $R(\lambda,A)E_u \subset E_u \cap D(A) \subset D(B)$. Hence $D(B)$ is dense.
Let $f \in D(B)$, $\phi \in D(A')_+$ and set $P_+ := P_{f^+}$, $P_- := P_{f^-}$. Then

$$(4.11) \qquad \langle P_+ Bf,\phi\rangle \leq \langle f^+,A'\phi\rangle .$$

In fact,
$$\begin{aligned}\langle P_+ Bf, \phi\rangle &= \langle P_+ Af, \phi\rangle + \langle P_+ m\cdot f, \phi\rangle\\ &= \langle P_+ Af, \phi\rangle + \langle m\cdot f^+, \phi\rangle\\ &\leq \langle P_+ Af, \phi\rangle\\ &\leq \langle f^+, A'\phi\rangle \qquad \text{(by (3.6))}.\end{aligned}$$

But (4.11) implies (4.4). So it follows from Theorem 4.3 that B is closable. Moreover, if we can show that $(\lambda - \bar{B})D(\bar{B})$ is dense in E, it follows that $\bar{B}$ is the generator of a semigroup $(S(t))_{t\geq 0}$. In that case (4.11) implies that $(S(t))_{t\geq 0}$ is dominated by $(T(t))_{t\geq 0}$ (by Proposition 4.5).
Now we show that $(\lambda - \bar{B})D(\bar{B})$ is dense in E.
Let $m_n = \sup\{m, -n1_X\}$ $(n \in \mathbb{N})$ and $B_n = A + m_n$. Then B_n is the generator of a positive semigroup and it follows from Proposition 4.8 that $0 \leq R(\lambda,B_{n+1}) \leq R(\lambda,B_n) \leq R(\lambda,A)$ for all $n \in \mathbb{N}$, $\lambda > s(A)$. (Note that $s(B_n) \leq s(A)$ by Lemma 4.10). Let $0 \leq f \in E_u$ and $g_n = R(\lambda,B_n)f$. Then $g = \inf_{n\in\mathbb{N}} g_n = \lim_{n\to\infty} g_n$ exists. Moreover $g_n \in D(B)$ and $\lim_{n\to\infty} (\lambda - B)g_n = f + \lim_{n\to\infty} (B_n - B)g_n = f$, since $|(B_n - B)g_n| \leq (m_n - m)|g_n| = (m_n - m)|R(\lambda,B_n)f| \leq (m_n - m)R(\lambda,A)|f| \leq c'(m_n - m)u$ for some positive constant c'.