$$
\begin{aligned}
-T(t)f &= \lim_{u\to\infty} \int_t^u e^{rs}e^{-rs}\, T(s)Af\, ds \\
&= \lim_{u\to\infty} \Big(e^{ru}\int_0^u e^{-rs}\, T(s)Af\, ds - e^{rt}\int_0^t e^{-rs}\, T(s)Af\, ds \\
&\qquad - r\int_t^u e^{rs}\int_0^s e^{-rv}\, T(v)Af\, dv\, ds\Big) \\
&= -e^{rt}\int_0^t e^{-rs}\, T(s)Af\, ds - r\int_t^\infty e^{rs}\int_0^s e^{-rv}\, T(v)Af\, dv\, ds\ .
\end{aligned}
$$

From $\|\int_0^t e^{-rs}\, T(s)Af\, ds\| \leq M$ for some M and every $t \geq 0$ we conclude that $\|T(t)f\| \leq \tilde{M}e^{rt}$ for all $t \geq 0$ and some constant $\tilde{M}$. Hence $\omega(f) \leq r$ for every $b < r < 0$, i.e., $\omega(f) \leq b$.
If $b \geq 0$ and $w > b$, then $\|\int_0^t e^{-ws}\, T(s)Af\, ds\| \leq M$ for all $t \geq 0$.
By

$$
\begin{aligned}
T(t)f - f &= \int_0^t e^{ws}e^{-ws}\, T(s)Af\, ds \\
&= e^{wt}\int_0^t e^{-ws}\, T(s)Af\, ds - w\int_0^t e^{ws}\int_0^s e^{-wr}\, T(r)Af\, dr\, ds
\end{aligned}
$$

we obtain $\|T(t)f - f\| \leq Me^{wt} + M(e^{wt} - 1) \leq 2Me^{wt}$. Hence $\omega(f) \leq w$ for every $w > b$, i.e., $\omega(f) \leq b$.

□

From (1.2) and the Uniform Boundedness Principle it follows that the growth bound $\omega_1(A) = \sup\{\omega(f) : f \in D(A)\}$ satisfies

$$
\begin{aligned}
(1.3)\qquad \omega_1(A) &= \inf\{w : \text{for every } f \in D(A) \text{ there exists a constant} \\
&\qquad M \text{ such that } \|T(t)f\| \leq Me^{wt} \text{ for every } t \geq 0\} \\
&= \limsup_{t\to\infty} 1/t\cdot\log\|T(t)R(\lambda,A)\| \qquad (\lambda \in \rho(A))\ .
\end{aligned}
$$

The subsequent theorem will be of particular importance in the stability theory of positive semigroups. We show that the constant $\omega_1(A)$ coincides with the abscissa of simple convergence of the Laplace transform of the semigroup and with the abscissa of absolute convergence of the Laplace transform of the strong solutions of (ACP).

Theorem 1.4. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E . Then

$$
\begin{aligned}
(1.4)\qquad \omega_1(A) &= \inf\{\operatorname{Re}\lambda : \int_0^\infty e^{-\lambda t}\, T(t)f\, dt \text{ exists as an improper} \\
&\qquad \text{Riemann integral for every } f \in E\} \\
&= \inf\{\operatorname{Re}\lambda : \int_0^\infty \|e^{-\lambda t}\, T(t)f\|dt \text{ exists for every} \\
&\qquad f \in D(A)\}\ .
\end{aligned}
$$

Remarks. (a) One can show that the abscissas of uniform, strong and weak convergence of the Laplace transform coincide (see C-III,Thm.1.2, last part of the proof). Therefore, by Thm.1.4,