Green's Formula and Robin-Laplacian

Mathematical Notes (from Handwriting)

1 Green's Formula

Let $\beta \in L^\infty(\partial\Omega)$. We define the Laplacian Δ^β with Robin boundary conditions as follows. Let

$$\mathrm{D}(\Delta^\beta) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \tag{1}$$
$$\partial_\nu u + \beta \,\mathrm{tr}(u) = 0\} \tag{2}$$
$$\Delta^\beta u := \Delta u. \tag{3}$$

We call Δ^β briefly the **Robin-Laplacian**. Note that for $\beta = 0$, we obtain **Neumann boundary conditions**, and $\Delta^N := \Delta^0$ is the **Neumann Laplacian**.

The following result is valid.

Theorem 1.1 (4.3)**.** *Assume that $\Omega \subset \mathbb{R}^d$ is bounded, open, connected with Lipschitz boundary, and let $\beta \in L^\infty(\partial\Omega)$. Then Δ^β generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on $C(\overline{\Omega})$. Moreover, $T(t)$ is compact for all $t > 0$.*

Irreducibility has strong consequences. One has $\sigma(\Delta^\beta) = \sigma_p(\Delta^\beta) \subset \mathbb{R}$. Denote by $\mathrm{s}(\Delta^\beta)$ the spectral bound of Δ^β. Then $\mathrm{s}(\Delta^\beta)$ is the largest eigenvalue of Δ^β. It is the unique eigenvalue with a positive eigenfunction $0 < u_0 \in \mathrm{D}(\Delta^\beta)$. The eigenfunction u_0 is strictly positive; i.e. there exists $\delta > 0$ such that $u_0(x) \geq \delta > 0$ for all $x \in \overline{\Omega}$.

The spectral bound $\mathrm{s}(\Delta^\beta)$ determines the asymptotic behavior of the semigroup $\mathcal{T}$. In fact, the following follows from B-II Proposition 3.5.

Corollary 1.2 (4.4)**.** *There exist a strictly positive Borel measure μ on $\overline{\Omega}$, $M \geq 0$ and $\varepsilon > 0$ such that $\langle \mu, u_0 \rangle = 1$ and*

$$\|T(t) - e^{\mathrm{s}(\Delta^\beta)t} P\| \leq M e^{-\varepsilon t} \tag{4}$$

for all $t \geq 0$, where $P \in \mathcal{L}(C(\overline{\Omega}))$ is given by

$$Pf = \langle \mu, f \rangle u_0. \tag{5}$$

The theorem says that the semigroup converges in the operator norm to the rank-1-projection P exponentially fast.

2 Elliptic Operators in Divergence Form

The preceding results extend to elliptic operators in divergence form for bounded measurable coefficients.

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $a_{k,\ell}, b_k, c_k, c_0 \in L^\infty(\Omega)$, $k, \ell = 1, \ldots, d$ such that for some $\alpha > 0$

$$\sum_{k,\ell=1}^{d} a_{k,\ell}(x)\xi_k\xi_\ell \geq \alpha|\xi|^2 \tag{6}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^d$, where $|\xi|^2 = \xi_1^2 + \ldots + \xi_d^2$.

Let $H^1_{loc}(\Omega) := \{v \in L^2_{loc}(\Omega) : D_k v \in L^2_{loc}(\Omega), k = 1, \ldots, d\}$.

Define $\mathcal{A} : H^1_{loc}(\Omega) \to C_0'(\Omega)$ by

$$\langle \mathcal{A}u, v \rangle = \sum_{k,\ell=1}^{d} \int_\Omega a_{k,\ell}(x) D_\ell u D_k v \, dx + \sum_{k=1}^{d} \int_\Omega b_k(x) D_k u v \, dx \tag{7}$$

$$+ \sum_{k=1}^{d} \int_\Omega c_k(x) u D_k v \, dx + \int_\Omega c_0(x) u v \, dx. \tag{8}$$

We define A_0 as the part of $\mathcal{A}$ in $C_0(\Omega)$; i.e.

$$\mathrm{D}(A_0) := \{u \in C_0(\Omega) \cap H^1_0(\Omega) : \mathcal{A}u \in C_0(\Omega)\} \tag{9}$$

$$A_0 u := \mathcal{A}u. \tag{10}$$

Then Theorem 4.1 holds with Δ_0 replaced by A_0. It is remarkable that Dirichlet regularity of Ω is the right boundary condition again. This is due to fundamental results of Stampacchia and co-authors. We refer to Arendt and Bénilan 1999 for a proof of the following result.

Theorem 2.1 (4.4)**.** *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded, open, connected Dirichlet regular set. Then A_0 generates a positive, irreducible, holomorphic semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on $C_0(\Omega)$. Moreover, $T(t)$ is compact for all $t > 0$.*

Also the results for Robin boundary conditions Theorems 4.3 and 4.4 can be extended for elliptic operators in divergence form on $C_0(\Omega)$; see Arendt and Bénilan 1999 for a proof of the following result.

3 Elliptic Operators in Non-Divergence Form on $C_0(\Omega)$

To Do

The Dirichlet-to-Neumann operator on $C(\partial\Omega)$ – for this case irreducibility is very surprising.

References for Notes to B-II 2025

W. Arendt, A.F.M. ter Elst, J. Glück: Strict positivity for the principal eigenfunction of elliptic operators with various boundary conditions. Adv. Nonlinear Stud. 2020; 20(3): 633–650

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