Proof. Let $(T(t))_{t\geq 0}$ be a strongly continuous multiplication semigroup. There exist $w \in \mathbb{R}$, $M \geq 1$ such that $\|T(t)\| \leq Me^{w|t|}$ $(t \geq 0)$. Then $\|f\|_1 := \sup_{t\geq 0}\|e^{-wt}T(t)|f|\|$ defines an equivalent lattice norm on E for which $\|T(t)\|_1 \leq e^{wt}$ $(t \geq 0)$. Since $Z(E)$ is isometrically isomorphic to a space $C(K)$ (as a Banach lattice), for an operator $S \in Z(E)$ one has $\|S\| = \inf\{c > 0 : |S| \leq c\cdot Id\}$. Hence the operator norm of S is independent of which lattice norm equivalent to the given one is considered on E. Consequently, $\|T(t)\| = \|T(t)\|_1 \leq e^{wt}$ $(t \geq 0)$.

If $(T(t))_{t\geq 0}$ is a strongly continuous group contained in $Z(E)$, then it follows that $\|T(t)\| \leq e^{w|t|}$ $(t \in \mathbb{R})$ for some $w \geq 0$. If in addition the operators $T(t)$ are real one obtains from the above expression for the operator norm that

$$e^{-wt}\cdot Id \leq T(t) \leq e^{wt}\cdot Id \qquad (t \geq 0).$$

Consequently, $\lim_{t\downarrow 0}\|T(t) - Id\| = 0$.

□

The assumption that the group consists of real operators is essential in Proposition 5.16. In fact, many differential operators on $L^2(\mathbb{R}^n)$ generate a strongly continuous group which (via Fourier transformation) is similar to a multiplication group. A concrete example is the Laplacian (A-I,Example 2.8).
On the other hand, if $E = C(K)$ (K compact), then every strongly continous multiplication semigroup $(T(t))_{t\geq 0}$ has a bounded generator.
[In fact, let $m_t = T(t)1$ $(t \geq 0)$. Then $\lim_{t\downarrow 0}\|T(t) - Id\| = \lim_{t\downarrow 0}\|m_t - 1\|_\infty = 0$.]

Lemma 5.17. Let E be a real Banach lattice with order continuous norm. Let $A \in L(E)$. Assume that there exists a dense sublattice D of E such that for all $f \in D$, $g \in E$, $f \perp g$ implies $Af \perp g$. Then $A \in Z(E)$.

Proof. Let $0 \leq f \in D$, $\phi \in E'_+$ such that $\langle f,\phi\rangle = 0$. Since $Af \in \{f\}^{dd}$ by assumption, it follows that $\langle Af,\phi\rangle = 0$. Thus $A_{|D}$ and $-A_{|D}$ satisfy (P). It follows from Thm.1.8 that $(e^{tA})_{t\in\mathbb{R}}$ is a positive group. Thus $A \in Z(E)$ by Prop.5.15.

□

Let A be the generator of a positive semigroup and $B \in L(E)$. The semigroup generated by $A + B$ is positive whenever $(e^{tB})_{t\geq 0}$ is positive (this follows from (1.8)). However this condition is not