Lemma 1.9. Let B be an operator on $C(K)$ (more generally, on a Banach lattice). If $\mu_1, \mu_2 \in \rho(B) \cap \mathbb{R}$ such that $0 \leq R(\mu_1,B)$, $0 \leq R(\mu_2,B)$ and $\mu_1 < \mu_2$, then $[\mu_1,\mu_2] \subset \rho(B)$ and

$$0 \leq R(\mu_2,B) \leq R(\mu,B) \leq R(\mu_1,B) \qquad \text{for all } \mu \in [\mu_1,\mu_2] .$$

Proof. Let $M := \{\mu \in \rho(B) \cap [\mu_1,\mu_2] : [\mu,\mu_2] \subset \rho(B)$ and $R(\lambda,B) \geq 0$ for all $\lambda \in [\mu,\mu_2]\}$.
a) The set M is open. In fact, let $\mu \in M$. Then for small $h > 0$ one has $R(\mu-h,B) = \sum_{n=0}^{\infty} h^n R(\mu,B)^{n+1} \geq 0$.
b) M is closed. In fact, by the resolvent equation one has for $\mu \in M$, $R(\mu_1,B) - R(\mu,B) = (\mu - \mu_1)R(\mu_1,B)R(\mu,B) \geq 0$, hence $R(\mu,B) \leq R(\mu_1,B)$. Consequently, $\operatorname{dist}(\mu,\sigma(B)) \geq 1/\|R(\mu,B)\| \geq 1/\|R(\mu_1,B)\|$ for all $\mu \in M$. This implies that M is closed.
The assertions a) and b) imply that $M = [\mu_1,\mu_2]$.

□

Remark. a) The lemma shows in particular that the resolvent of the generator A of a positive semigroup is decreasing on $(s(A),\infty)$.
b) There exists a linear operator B on $\mathbb{R}^n$ such that $R(\mu,B) \geq 0$ on some interval $[\mu_1,\mu_2] \subset \rho(B) \cap \mathbb{R}$ but $(e^{tB})_{t\geq 0}$ is not positive (see Greiner-Voigt-Wolff (1981)).

Remark. Theorem 1.8 does not hold in $C_o(X)$, in general. In fact, the operator A on $C_o(0,1]$ given by $Af(x) = f'(x) + \alpha/x\, f(x)$ $(x \in (0,1])$ with domain $D(A) = \{f \in C^1[0,1] : f'(0) = f(0) = 0\}$ where $\alpha \in (0,1)$ satisfies the following: $\rho(A) = \mathbb{C}$, $R(\lambda,A) \geq 0$ for all $\lambda \in \mathbb{R}$. But A is not the generator of a semigroup (even if more general classes than C_o-semigroups are admitted). See Arendt (1985b) for this example and a general theory of resolvent positive operators. Another example is given by Batty-Davies (1983).

Next we investigate consequences of the positive minimum principle for a densely defined operator which is not a priori assumed to be a generator. For that we will make use of the theory of half-norms developed in A-II,Sec.2.

For $0 \ll u \in C(K)$ let

$$(1.4) \qquad p_u(f) = \inf\{\lambda \in \mathbb{R}_+ : f \leq \lambda u\} = \sup_{x\in K} f^+(x)/u(x) .$$