Applying the proposition to a single operator $T(t)$ we obtain $A\sigma(T(t)) = P\sigma(T(t)_F)$. Note that in general $A\sigma(T(t)) \neq P\sigma(T_F(t))$ (see the Examples 1.3 and 1.4 in combination with Theorem 6.3).

5. THE SPECTRUM OF PERIODIC SEMIGROUPS

In this section we determine the spectrum of a particularly simple class of strongly continuous semigroups and thereby achieve a rather complete description of the semigroup itself. Besides being nice and simple these semigroups gain their importance as building blocks for the general theory.

Definition 5.1. A strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on a Banach space E is called periodic if $T(t_o) = Id$ for some $t_o > 0$. The period τ of $\mathcal{T}$ is obtained as $\tau := \inf\{t_o > 0 : T(t_o) = Id\}$.

We immediately observe that periodic semigroups are groups with inverses $T(t)^{-1} = T(n\tau - t)$ for $0 \leq t \leq n\tau$, τ the period of $\mathcal{T}$. Moreover, they are bounded, hence the growth bound is zero and $\sigma(A) \subset i\mathbb{R}$.

Lemma 5.2. Let $\mathcal{T}$ be a strongly continuous semigroup with period $\tau > 0$ and generator A . Then

$$\sigma(A) \subset 2\pi i/\tau \cdot \mathbb{Z} \text{ , and}$$

$$R(\mu,A) = (1-e^{-\mu\tau})^{-1} \int_0^\tau e^{-\mu s} T(s)\, ds \tag{5.1}$$

for $\mu \notin 2\pi i/\tau \cdot \mathbb{Z}$.

Proof. From the basic identities A-I,(3.1) and A-I,(3.2) for $t = \tau$, it follows that $(\mu - A)$ has a left and right inverse if $\mu \neq 2\pi i n/\tau$, $n \in \mathbb{Z}$, and that the inverse is given by the above expression.

□

The representation of $R(\mu,A)$ given in A-I,Prop.1.11 shows that the resolvent of the generator of a periodic semigroup is a meromorphic function having only poles of order one and the residues

$P_n := \lim_{\mu\to\mu_n} (\mu-\mu_n)R(\mu,A)$ in $\mu_n := 2\pi i n/\tau$, $n \in \mathbb{Z}$, are

$$P_n = \tau^{-1}\int_0^\tau \exp(-\mu_n s)T(s)\, ds \text{ .} \tag{5.2}$$

Moreover, it follows that the spectrum of A consists of eigenvalues only and each P_n is the spectral projection belonging to μ_n (see