$= g - \int_0^t T(s)f\,ds$, hence $0 \leq \int_0^t T(s)f\,ds \leq g$ for every $t \geq 0$. For $\alpha > 0$ we have $\int_0^t e^{-\alpha s}T(s)f\,ds \leq \int_0^t T(s)f\,ds \leq g$. Using C-III,Thm.1.2 we conclude that $R(\alpha,A)f \leq g$ for $\alpha > \max\{0,s(A)\}$. By the Uniform Boundedness Principle we know that $\{R(\alpha,A) : \alpha > \max\{0,s(A)\}\}$ is uniformly bounded. Since $\omega_1(A) = s(A) \in \sigma(A)$ (see Thm.1.3) it follows that $\omega_1(A) < 0$.

□

Next we show that weak uniform stability implies uniform stability provided that E is weakly sequentially complete (see C-I,Sec.5) and $(\mathrm{im}\,A)_+ := A(D(A)) \cap E_+$ is a total subset of E . The left translations on $L^2(\mathbb{R}_+)$ are stable. Hence, by A-IV,Rem.1.17(a), $\mathrm{im}\,A = \{f \in L^2(\mathbb{R}_+) : \int_0^\infty f(x)\,dx \text{ exists}\}$ and we see that $(\mathrm{im}\,A)_+$ is a total subset of $L^2(\mathbb{R}_+)$. On the other hand, $(\mathrm{im}\,A)_+ = \{0\}$ for the generator of the non stable, but weakly stable semigroup of left translations on $L^2(\mathbb{R})$.

<u>Proposition</u> 1.7. Let A be the generator of a positive semigroup $(T(t))_{t\geq 0}$ on a weakly sequentially complete Banach lattice E , such that $(\mathrm{im}\,A)_+$ is total in E . Then $(T(t))_{t\geq 0}$ is uniformly stable if and only if it is weakly uniformly stable.

<u>Proof</u>. If $(T(t))_{t\geq 0}$ is weakly uniformly stable, then $(T(t))$ is bounded by the Uniform Boundedness Principle. Using the weak version of A-IV,Thm.1.14 , $\int_0^\infty \langle T(t)f,\phi\rangle\,dt$ exists for every $f \in (\mathrm{im}\,A)_+$ and $\phi \in E'_+$. It follows that the net $(\int_0^r T(t)f\,dt\,)_{r\geq 0}$ is weakly Cauchy. Hence $\sigma(E',E)\text{-}\lim_{r\to\infty} \int_0^r T(t)f\,dt$ exists for every $f \in (\mathrm{im}\,A)_+$. Since the net is monotone one obtains convergence in norm by Dini's Theorem [Schaefer (1974), II.Thm.5.9]. Now uniform stability follows from A-IV,Thm.1.16.

□

In A-IV,Thm.1.13 we have seen that a generator A of a stable semigroup satisfies necessarily $s(A) \leq 0$, $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in P\sigma(A) \cup R\sigma(A)$ and, by $\lambda R(\lambda,A)f = R(\lambda,A)Af + f$, that $\lim_{\lambda\to 0+} R(\lambda,A)g$ exists for all $g \in \mathrm{Im}\,A$. For positive semigroups similar properties are even sufficient for stability.

<u>Lemma</u> 1.8. Let A be the generator of a positive semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E with $s(A) \leq 0$. Given $f \in E_+$ then $\lim_{\lambda\to 0+} R(\lambda,A)f$ exists if and only if $\lim_{t\to\infty} \int_0^t T(s)f\,ds$ exists.