Theorem 3.3. Let $E$ be a Grothendieck space. If $\mathcal{T} = (T(t))_{t\geq 0}$ is a strongly continuous semigroup in $E$, then $\mathcal{T}'' = (T(t)'')_{t\geq 0}$ is strongly continuous in $E''$.

Proof. Suppose that $(x_n')$ is a bounded sequence in $E'$ and that $t_n \geq 0$ with $\lim t_n = 0$. Put $V_n = T(t_n) - Id$. Then $\lim \|V_n x\| = 0$ and therefore $\lim \langle x, V_n' x_n' \rangle = 0$ for every $x \in E$. Hence $(V_n' x_n')$ w*-converges to zero. Since $E$ is a Grothendieck space $(V_n' x_n')$ converges weakly to zero. Now Lemma 3.2 implies that $(T(t)'')$ is strongly continuous.

□

Recall now that a Banach space $E$ is said to have the Dunford-Pettis property if $\lim \langle x_n, x_n' \rangle = 0$ whenever $(x_n)$ in $E$ and $(x_n')$ in $E'$ converge weakly to zero.

Theorem 3.4. Let $E$ be a Banach space with the Dunford-Pettis property and let $\mathcal{T} = (T(t))_{t\geq 0}$ be a one-parameter semigroup of operators on $E$. If $\mathcal{T}'' = (T(t)'')_{t\geq 0}$ is strongly continuous in $E''$, then $\mathcal{T}$ is uniformly continuous.

Proof. Suppose that $\mathcal{T}''$ is a strongly continuous semigroup. Then Lemma 3.2 implies that $\mathcal{T}'$ and $\mathcal{T}$ are strongly continuous. Hence by the uniform boundedness principle, $\lim\sup_{t\to 0} \|(T(t) - Id)\|$ is finite. By Lemma 3.1 it suffices to show that $\lim_{t\to 0} \|(T(t) - Id)^2\| = 0$. Let $t_n \geq 0$ with $\lim t_n = 0$ be given. Then there exists a bounded sequence $(x_n)$ in $E$ and a bounded sequence $(x_n')$ in $E'$ such that $\|(T(t_n) - Id)^2\| = \langle (T(t_n)-Id)x_n, (T(t_n)-Id)'x_n' \rangle$.
Since $\mathcal{T}'$ and $\mathcal{T}''$ are strongly continuous, Lemma 3.2 implies that $((T(t_n) - Id)x_n)$ and $((T(t_n) - Id)'x_n')$ converge weakly to zero. Since $E$ has the Dunford-Pettis property, $\lim \|(T(t_n) - Id)^2\| = 0$. Consequently, $\lim_{t\to 0} \|(T(t) - Id)^2\| = 0$.

□

An immediate consequence of Theorem 3.3 and Thoerem 3.4 is the following.

Theorem 3.5. Let $E$ be a Grothendieck space with the Dunford-Pettis property. Then every strongly continuous semigroup of operators on $E$ is uniformly continuous.