Remark 2.4. The function p is convex. So the one-sided Gateaux-derivatives

$D_g^+ p(f) = \lim_{t\downarrow 0} 1/t\ (p(f+tg) - p(f))$ and

$D_g^- p(f) = \lim_{t\uparrow 0} 1/t\ (p(f+tg) - p(f))$

exist and satisfy $D_g^- p(f) \le D_g^+ p(f)$ for all $f, g \in E$ (cf. Moreau (1966)). Moreover,

$$D_g^+ p(f) = \sup \{\langle g,\phi\rangle : \phi \in dp(f)\}, \tag{2.9}$$

$$D_g^- p(f) = \inf \{\langle g,\phi\rangle : \phi \in dp(f)\}. \tag{2.10}$$

Thus A is p-dissipative if and only if $D_{Af}^- p(f) \le 0$, and A is strictly p-dissipative if and only if $D_{Af}^+ p(f) \le 0$ for all $f \in D(A)$.

Corollary 2.5. Let A be a closable operator. If A is p-dissipative, then so is its closure.

Theorem 2.6. Let p be a continuous sublinear functional on a real Banach space E. Let A be the generator of a strongly continuous semigroup $(T(t))_{t\ge 0}$. The following assertions are equivalent.

(i) $p(T(t)f) \le p(f)$ for all $t \ge 0$, $f \in E$.

(ii) A is strictly p-dissipative.

(iii) There exists a core D of A such that $A_{|D}$ is p-dissipative.

Proof. Assume that (i) holds. Let $f \in D(A)$, $\phi \in dp(f)$. Then

$$\begin{aligned}\langle Af,\phi\rangle &= \lim_{t\to 0} 1/t(\langle T(t)f,\phi\rangle - \langle f,\phi\rangle)\\ &= \lim_{t\to 0} 1/t(\langle T(t)f,\phi\rangle - p(f))\\ &\le \limsup_{t\to 0} 1/t(p(T(t)f) - p(f)) \le 0 .\end{aligned}$$

This proves (ii).

It is trivial that (ii) implies (iii). So let us assume (iii). Then it follows from Cor. 2.5 that A is p-dissipative. Hence by (2.8) $p(\lambda R(\lambda,A)g) \le p(g)$ for all $g \in E$, $\lambda > \omega(A)$. Hence $\lambda R(\lambda,A)$ is p-contractive for $\lambda > \omega(A)$. This implies that $T(t)$ is p-contractive by the formula (1.3)

$T(t) = \lim_{t\to 0} (n/tR(n/t,A))^n$ (strongly) for $t \ge 0$.

□

We have shown that for generators, p-dissipativity is equivalent to p-contractivity of the semigroup. Now we will consider a p-dissipative operator A (which is not a generator a priori) and investigate under which additional hypotheses A is the generator of a