and Thm.6.3 of A-III) we have

(1.5) $\sigma(T(1)) \cap \Gamma = \{1\}$ and $1 \notin R\sigma(T(1))$.

From (1.4) it follows that $\lim_{n\to\infty} \|T(n) - T(n+1)\| = 0$ and therefore $\lim_{t\to\infty} \|T(t) - T(t+1)\| = 0$. Thus given $g \in \operatorname{im}(Id - T(1))$ then $g = f - T(1)f$ for some $f \in E$ hence $\|T(t)g\| = \|(T(t) - T(t+1))f\| \leq \|(T(t) - T(t+1))\| \cdot \|f\| \to 0$. The second assertion of (1.5) ensures that $\operatorname{im}(Id - T(1))$ is dense in E . Since the semigroup is bounded we have $\lim_{t\to\infty} \|T(t)f\| = 0$ for every $f \in \overline{\operatorname{im}(Id - T(1))} = E$, i.e., $(T(t))$ is uniformly stable.

(i)$\Rightarrow$(iii) is always true and follows from A-IV,Thm.1.13 .

(iii)$\Rightarrow$(ii): The adjoint semigroup $(T(t)')_{t\geq 0}$ is eventually norm-continuous and bounded and we have $R\sigma(A') = P\sigma(A'') = P\sigma(A)$. Thus the implication "(ii)$\Rightarrow$(i)" can be applied and we obtain that $(T(t)')_{t\geq 0}$ is stable. Then A-IV,Thm.1.13 yields $0 \notin P\sigma(A') = R\sigma(A)$.

□

As an application of Thm.1.5 we consider the Laplacian as generator on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, (see A-I,2.8). For $p = 1$ the constant functions are eigenvectors of the adjoint operator, hence $0 \in R\sigma(\Delta)$. Thus the semigroup is not stable on $L^1(\mathbb{R}^n)$. On the other hand, for $1 \leq p < \infty$ there does not exist a non-zero function $h \in L^p(\mathbb{R}^n)$ with $\Delta h = 0$. Hence Δ generates a stable semigroup on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.
(That $\ker \Delta = \{0\}$ can be deduced from the following two facts:

- since the semigroup consists of contractions and since the norm is strictly monotone on E_+ it follows that $\ker \Delta$ is a sublattice. Thus irreducibility of the semigroup (see A-I,2.8 and C-III,Ex.3.4(a)) implies that $\dim \ker \Delta \leq 1$;
- The semigroup commutes with the translations on $\mathbb{R}^n$, hence $\ker \Delta$ is invariant under translations.)

In the next results we give conditions on the range of the generator which ensure stability. We begin with a generalization of Cor.1.4(b) .

Propositon 1.6. Let A be the generator of a positive semigroup on a (real or complex) Banach lattice, $D(A)_- := -(D(A) \cap E_+)$.
Then $\omega_1(A) < 0$ if and only if $E_+ \subset \operatorname{im} A(D(A)_-)$.

Proof. If $\omega_1(A) < 0$ then $s(A) < 0$ (A-IV,Cor.1.5), hence $A^{-1} = -R(0,A) \leq 0$ by C-III,Thm.1.1 .
If $E_+ \subset \operatorname{im} A(D(A)_-)$, then, for every $f \in E_+$, there exists $g \in D(A)_+$ such that $Ag = -f$. We have $0 \leq T(t)g = g + \int_0^t T(s)Ag\, ds$