quotient semigroup $(T(t)_{|/})$ on $L^1([0,1])$ is the nilpotent translation semigroup as in ~~Example~~ 2.6 . In particular it follows that the domain of the generator is
$D(A_{|/}) = \{f \in L^1([0,1]) : f \in AC \text{ with } f' \in L^1([0,1]) \text{ and } f(1) = 0\}$.

3.4. The Adjoint Semigroup

The adjoint operators $(T(t)')_{t\geq 0}$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E form a semigroup on E' which need, however, not be strongly continuous.

Example. Take the translation operators $T(t)f(x) = f(x+t)$ on $E = L^1(\mathbb{R})$ (see Example 2.4) and their adjoints

$$T(t)'f(x) = f(x-t)$$

on $E' = L^\infty(\mathbb{R})$. Then $(T(t)')_{t\in\mathbb{R}}$ is a one-parameter group which is not strongly continuous on $L^\infty(\mathbb{R})$ (take any non-trivial characteristic function).

Since the semigroup $(T(t)')_{t\geq 0}$ is obviously weak*-continuous in the sense that $\lim_{t\to s} \langle f,(T(t)'-T(s)')\phi\rangle = 0$ for every $f \in E$, $\phi \in E'$ and s , $t \geq 0$, it is natural to associate $(T(t)')_{t\geq 0}$ its a weak*-generator

$$A'\phi := \sigma(E',E)\text{-}\lim \tfrac{1}{h}(T(h)'\phi-\phi) \quad \text{for every } \phi \text{ in the domain}$$

$$D(A') := \{\phi \in E' : \sigma(E',E)\text{-}\lim \tfrac{1}{h}(T(h)'\phi-\phi) \text{ exists}\}.$$

This operator coincides with the adjoint of the generator $(A,D(A))$, i.e.

$$D(A') = \{\phi \in E' : \text{there exists } \psi \in E' \text{ such that } \langle f,\psi\rangle = \langle Af,\phi\rangle \text{ for all } f \in D(A)\}$$

and $A'\phi = \Psi$.

In particular, A' is a closed and $\sigma(E',E)$-densely defined operator in E' .

It follows from Thm.III.5.30 in Kato (1966) that the resolvent $R(\lambda,A')$ of A' is $R(\lambda,A)'$. In particular, the spectra $\sigma(A)$ and $\sigma(A')$ coincide. But it is still necessary in some situations to have strong continuity for the adjoint semigroup. In order to achieve this we restrict $T(t)'$ to an appropriate subspace of E' .

Definition (Phillips, 1955). The semigroup dual of the Banach space E with respect to the strongly continuous semigroup $(T(t))_{t\geq 0}$ is

$$E^* := \{\phi \in E' : \|\cdot\|\text{-}\lim_{t\to 0} T(t)'\phi = \phi\}.$$