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One-parameter Semigroups of Positive Operators

Edited by R. Nagel

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This Latex version of the book "One-Parameter Semigroups of Positive Operators" is dedicated to the memory of our co-authors, Heinrich P. Lotz, Ulrich Moustakas, and Ulf Schlotterbeck. Their contributions to the first edition remain an inspiration to us all. We miss their presence and remain grateful for the legacy they have left in this work.

Preface

As early as 1948 in the first edition of his fundamental treatise on Semigroups and Functional Analysis, E. Hille expressed the need for "developing an adequate theory of transformation semigroups operating in partially ordered spaces" (l.c., Foreword). In the meantime the theory of one-parameter semigroups of positive linear operators has grown continuously. Motivated by problems in probability theory and partial differential equations W. Feller (1952) and R. S. Phillips (1962) laid the first cornerstones by characterizing the generators of special positive semigroups. In the 60's and 70's the theory of positive operators on ordered Banach spaces was built systematically and is well documented in the monographs of H. H. Schaefer (1974) and A. C. Zaanen (1983). But in this process the original ties with the applications and, in particular, with initial value problems were at times obscured. Only in recent years an adequate and up-to-date theory emerged, largely based on the techniques developed for positive operators and thus recombining the functional analytic theory with the investigation of Cauchy problems having positive solutions to each positive initial value. Even though this development — in particular with respect to applications to concrete evolution equations in transport theory, mathematical biology, and physics — is far from being complete, the present volume is a first attempt to shape the multitude of available results into a coherent theory of one-parameter semigroups of positive linear operators on ordered Banach spaces.

The book is organized as follows. We concentrate our attention on three subjects of semigroup theory: *characterization*, *spectral theory* and *asymptotic behavior*. By *characterization*, we understand the problem of describing special properties of a semigroup, such as positivity, through the generator. By *spectral theory* we mean the investigation of the spectrum of a generator. *Asymptotic behavior* refers to the orbits of the initial values under a given semigroup and phenomena such as stability.

This program (characterization, spectral theory, asymptotic behavior) is worked out on four different types of underlying spaces:

(A) On Banach spaces — Here we present the background for the subsequent discussions related to order.

(B) On spaces $C_0(X)$ (X locally compact), which constitute an important class of ordered Banach spaces and where our results can be presented in a form which makes them accessible also for the non-expert in order-theory.

(C) On Banach lattices, which admit a rich theory and are still sufficiently general as to include many concrete spaces appearing in analysis; e.g., $C_0(X)$, $\mathcal{L}^p(k)$ or l^p.

(D) On non-commutative operator algebras such as C^*- or W^*-algebras, which are not lattice ordered but still possess an interesting order structure of great importance in mathematical physics.

In each of these cases we start with a short collection of basic results and notations, so that the contents of the book may be visualized in the form of a 4×4 matrix in a way which will allow "row readers" (interested in semigroups on certain types of spaces) and "column readers" (interested in certain aspects) to find a path through the book corresponding to their interest.

We display this matrix, together with the names of the authors contributing to the subjects defined through this scheme:

	I Basic Results	II Characterization	III Spectral Theory	IV Asymptotics
A. Banach Spaces	R. Nagel U. Schlotterbeck	W. Arendt H. P. Lotz	G. Greiner R. Nagel	F. Neubrander
B. $C_0(X)$	R. Nagel U. Schlotterbeck	W. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas F. Neubrander
C. Banach Lattices	R. Nagel U. Schlotterbeck	G. Arendt	G. Greiner	A. Grabosch G. Greiner U. Moustakas R. Nagel F. Neubrander
D. Operator Algebras	U. Groh	U. Groh	U. Groh	U. Groh

This "matrix of contents" has been an indispensable guide line in our discussions on the scope and the spirit of the various contributions. However, we would not have succeeded in completing this manuscript, as a collection of independent contributions (personally accounted for by the authors), under less favorable conditions than we have actually met. For one thing, Rainer Nagel was an unfaltering and undisputed spiritus rector from the very beginning of the project. On the other hand we gratefully acknowledge the influence of Helmut H. Schaefer and his pioneering work on order structures in analysis. It was the team spirit produced by this common mathematical background which, with a little help from our friends, made it possible to overcome most difficulties.

We have prepared the manuscript with the aid of a word processor, but we confess that without the assistance of Klaus Kuhn the pitfalls of such a system would have been greater than its benefits.

The authors

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F. Neubrander

Part B
Positive Semigroups on Spaces $C_0(X)$

Chapter B-I
Basic Results on $C_0(X)$
R. Nagel & U. Schlotterbeck

This part of the book is devoted to a study of one-parameter semigroups of operators on spaces of continuous functions of type $C_o(X)$. Spaces which are Banach lattices of a very special kind. We treat this case separately since we feel that an intermingling with the abstract Banach lattice situation considered in Part C would have made it difficult to appreciate the easy accessibility and the pilot function of methods and results available here. In this chapter we want to fix the notation we are going to use and to collect some basic facts about the spaces we are considering.

If X is a locally compact topological space, then $C_o(X)$ denotes the space of all continuous complex-valued functions on X which vanish at infinity, endowed with the supremum-norm. If X is compact, then any continuous function on X "vanishes at infinity" and $C_o(X)$ is the space of all continuous functions on X. We often write $C(X)$ instead of $C_o(X)$ in this situation.

We sometimes study real-valued functions and write the corresponding real spaces as $C_o(X,\mathbb{R})$ and $C(X,\mathbb{R})$, and the notations $C_o(X,\mathbb{C})$ and $C(X,\mathbb{C})$ are used if there is the possibility of confusion between both cases.

We refer to the book of Semadeni [2] or [1] for the details.

1 Algebraic and Order-Structure: Ideals and Quotients

Any space $C_o(X)$ is a commutative C^*-algebra under the sup-norm and the pointwise multiplication, and by the Gelfand Representation Theorem any commutative C^*-algebra can, on the other hand, be canonically represented as an algebra $C_o(X)$ on a suitable locally compact space X. The algebraic notions of ideal, quotient, homomorphism are well known and need not be explained further.

Another natural and important structure of $C_o(X)$ is the pointwise ordering, under which $C_o(X,\mathbb{R})$ is a (real) Banach lattice and $C_o(X,\mathbb{C})$ a complex Banach lattice in the sense explained in Chapter C-I.

Concerning the order structure of $C_o(X)$ we use the following notations: For a function f in $C_o(X,\mathbb{R})$

$f \geqslant 0$ means $f(t) \geqslant 0$ for all $t \in X$ (f is then called positive),

$f > 0$ means that f is positive but does not vanish identically,

$f \gg 0$ means that $f(t) > 0$ for all t in X (f is then called strictly positive).

The notion of an order ideal explained in Chapter C-I applies of course to the Banach lattices $C_o(X)$ and might give rise to confusion with the corresponding algebraic notion.

However, since we are mainly considering closed ideals and since a closed linear subspace I of $C_o(X)$ is a lattice ideal if and only if I is an algebraic ideal, we may and will simply speak of closed ideals without specifying whether we think of the algebraic or the order theoretic meaning of this word.

An important and frequently used characterization of these objects is the following: A subspace I of $C_o(X)$ is a closed ideal if and only if there exists a closed subset A of X such that a function f belongs to I if and only if f vanishes on A. A is of course uniquely determined by I and is called the support of I. If $I = I_A$ is a closed ideal with support A then I_A is naturally isomorphic to $C_o(X \backslash A)$ and the quotient $C_o(X)/I$ (under the natural quotient structure) again a Banach algebra and a Banach lattice that can be identified canonically (via the map $f + I \to f|_A$) with $C_o(A)$.

2 Linear Forms and Duality

The Riesz Representation Theorem asserts that the dual of $C_o(X)$ can be identified in a natural way with the space of bounded regular Borel measures on X. While there is no natural algebra structure on this dual, the dual ordering (see C-I) makes $C_o(X)'$ into a Banach lattice. We will occasionally make use of the order structure of $C_o(X)'$ but since at least its measure theoretic interpretation is supposed to be well-known, it may suffice here to refer to Chapter C-I, Sections 3 and 7, for a more detailed discussion and to recall some basic notations here: If μ is a linear form on $C_o(X, \mathbb{R})$ then

$\mu \geqslant 0$ means $\mu(f) \geqslant 0$ for all $f \geqslant 0$; μ is then called positive,

$\mu > 0$ means that $\mu \geqslant 0$, but μ does not vanish identically,

$\mu \gg 0$ means that $\mu(f) > 0$ for any $f > 0$; μ is then called strictly positive.

If μ is a linear form on $C_o(X, \mathbb{C})$, then μ can be written uniquely as $\mu = \mu_1 + i\mu_2$ where μ_1 and μ_2 map $C_o(X, \mathbb{R})$ into $\mathbb{R}$ (decomposition into real and imaginary parts). We call μ positive (strictly positive) and use the above notations if $\mu_2 = 0$ and μ_1 is positive (strictly positive). We point out that strictly positive linear forms need not exist on $C_o(X)$, but if X is separable then a strictly positive linear form is obtained by summing a suitable sequence of point measures.

The coincidence of the notions of a closed algebraic and a closed lattice ideal in $C_o(X)$ has of course its effect on the algebraic and the lattice theoretic notions of a homomorphism. The case of a homomorphism into another space $C_o(Y)$ will be discussed below. As for homomorphisms into the scalar field, we have essentially coincidence between the algebraic and the order theoretic meaning of this word, more exactly: A linear form $\mu \neq 0$ on $C_o(X)$ is a lattice homomorphism if and only if μ is, up to normalization, an algebra homomorphism (algebra homomorphisms $\neq 0$ must necessarily have norm 1). Since the algebra homomorphisms $\neq 0$ on $C_o(X)$ are known to be the point measures (denoted by δ_t) on X and since on the other hand μ is a lattice homomorphism if and only if $|\mu(f)|$ equals $\mu(|f|)$ for all f, it follows that this latter condition on μ is equivalent to $\mu = \alpha\delta_t$ for a suitable t in X and a positive real number α.

This can be summarized by saying that X can be canonically identified, via the map $t \longrightarrow \delta_t$, with the subset of the dual $C_o(X)'$ consisting of the non-zero algebra homomorphisms, which is also the set of all normalized lattice homomorphisms. This identification is a topological isomorphism with respect to the weak*-topology of $C_o(X)'$.

3 Linear Operators

A linear mapping T from $C_o(X, \mathbb{R})$ into $C_o(Y, \mathbb{R})$ is called:

positive (notation: $T \geqslant 0$), if Tf is a positive function whenever f is positive,

a lattice homomorphism if $|Tf| = T|f|$ for all f,

a Markov-operator if X and Y are compact and T is a positive operator mapping 1_X to 1_Y.

In the case of complex scalars T can be decomposed into real and imaginary parts. We call T positive in this situation if the imaginary part of T is $= 0$ and the real part is positive. The terms "Markov operator" and "lattice homomorphism" are defined formally in the same way as above. Note that a complex lattice homomorphism is necessarily positive, and that the complexification of a real lattice homomorphism is a complex lattice homomorphism. Positive Operators are always continuous.

Since the adjoint of a Markov operator T maps positive normalized measures into positive normalized measures while the adjoint of an algebra homomorphism (lattice homomorphism) maps point measures into (multiples of) point measures, the adjoint of a Markov lattice homomorphism as well as the adjoint of an algebra homomorphism induces a continuous map φ from Y (viewed as a subset of the weak dual $C(Y)'$) into X (viewed as a subset of $C(X)'$).

This mapping φ determines T in a natural and unique way, so that the following are equivalent assertions on a linear mapping T from a space $C(X)$ into a space $C(Y)$:

(a) T is a Markov lattice homomorphism.

(b) T is a Markov algebra homomorphism.

(c) There exists a continuous map φ from Y into X such that $Tf = f \circ \varphi$ for all $f \in C(X)$

If T is in addition bijective, then the mapping φ in (c) is a homeomorphism from Y onto X. This characterization of homomorphisms carries over mutatis mutandis to situations where the conditions on X, Y or T are less restrictive. For later reference we explicitly state:

(i) Let K be compact. Then $T \in L(C(K))$ is a lattice homomorphism if and only if there is a mapping φ from K into K and a function $h \in C(K)$ such that $Tf(s) = h(s)f(\varphi(s))$ holds for all $s \in K$. φ is continuous in every point t with $h(t) \neq 0$.

(ii) Let X be locally compact, $T \in L(C_o(X))$. T is a lattice isomorphism if and only if there is a homeomorphism φ from X onto X and a bounded continuous function h on X such that $h(s) \geqslant \delta > 0$ for all s and $Tf(s) = h(s)f(\varphi(s))$ $(s \in X)$. T is an algebraic $*$-isomorphism if and only if T is a lattice isomorphism and the function h above is $\equiv 1$.

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Chapter B-II
Characterization of Positive Semigroups on $C_0(X)$

W. Arendt

Chapter B-III

Spectral Theory of Positive Semigroups on $C_0(X)$

G. Greiner

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Asymptotics of Positive Semigroups on $C_0(X)$

F. Neubrander & G. Greiner & A. Grabosch/Ulrich Moustakas

Part C
Positive Semigroups on Banach Lattices

Chapter C-I

Basic Results on Banach Lattices and Positive Operators

R. Nagel & U. Schlotterbeck

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Characterization of Positive Semigroups on Banach Lattices and Positive Operators
W. Arendt

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Asymptotics of Positive Semigroups on Banach Lattices

G. Greiner/F. Neubrander & G. Greiner & A. Grabosch/Ulrich Moustakas

Part D
Positive Semigroups on $C^\star$- and $W^\star$-Algebras

Chapter D-I

Basic Results on Semigroups and Operator Algebras

U. Groh

Chapter D-II
Characterization of Positive Semigroups on W*-Algebras
U. Groh

Since the positive cone of a C*-algebra has non-empty interior many results of Chapter B-II can be applied verbatim to the characterization of the generator of positive semigroups on C*-algebras. On the other hand a concrete and detailed representation of such generators has been found only in the uniformly continuous case (see Lindblad (1976)). A third area of active research has been the following: Which maps on C*-algebras (in particular, which derivations) commuting with certain automorphism groups are automatically generators of strongly continuous positive semigroups. For more information we refer to the survey article of [1].

1 Semigroups on Properly Infinite W*-Algebras

The aim of this section is to show that strongly continuous semigroups of Schwarz maps on properly infinite W*-algebras are already uniformly continuous. In particular, our theorem is applicable to such semigroups on $B(H)$.

It is worthwhile to remark, that the result of [2] on the uniform continuity of every strongly continuous semigroup on L^∞ (see A-II, Sec.3) does not extend to arbitrary W*-algebras.

Example 1.1 Take $M = B(H)$, H infinite dimensional, and choose a projection $p \in M$ such that Mp is topologically isomorphic to H. Therefore $M = H \oplus M_0$, where $M_0 = \ker(x \mapsto xp)$. Next take a strongly, but not uniformly continuous, semigroup S on H and consider the strongly continuous semigroup $S \oplus \mathrm{Id}$ on M.

For results from the classification theory of W*-algebras needed in our approach we refer to [3, 2.2] and [4, V.1].

Theorem 1.2 *Every strongly continuous one-parameter semigroup of Schwarz type on a properly infinite W*-algebra M is uniformly continuous.*

Proof Let $T = (T(t)_{t \geqslant 0})$ be strongly continuous on M and suppose T not to be uniformly continuous. Then there exists a sequence $(T_n) \subset T$ and $\epsilon > 0$ such that

$\|T_n - \mathrm{Id}\,\| \geqslant \epsilon$ but $T_n \longrightarrow \mathrm{Id}$ in the strong operator topology. We claim that for every sequence (P_k) of mutually orthogonal projections and all bounded sequences (x_k) in M

$$\lim_n \|(T_n - \mathrm{Id})(P_k x_k P_k)\| = 0$$

uniformly in $k \in \mathbb{N}$. This follows from an application of the *Lemma of Phillips* and the fact that the sequence $(P_k x_k P_k)$ is summable in the $s^*(M, M_*)$-topology (compare Elliot (1972)).

Let (P_k) be a sequence of mutually orthogonal projections in M such that every P_k is equivalent to 1 via some $u_k \in M$ [3, 2.2]. Without loss of generality we may assume $\|(T_n - \mathrm{Id})(u_n)\| \leqslant n^{-1}$ since the semigroup T is strongly continuous. Thus we obtained the following:

(i) $\lim_n \|(T_n - \mathrm{Id})(P_k x_k P_k)\| = 0$ uniformly in $k \in \mathbb{N}$ for every bounded sequence (x_k) in M.

(ii) Every projection P_k is equivalent to 1 via some $u_k \in M$.

(iii) $\|(T_n - \mathrm{Id})u_n\| \leqslant n^{-1}$ for all $n \in \mathbb{N}$.

For the following construction see A-I,3.6 and D-II,Sec.2. Let

(i) $\widehat{M}$ be an ultrapower of M,

(ii) let $p := \widehat{(P_k)} \in \widehat{M}$,

(iii) $T := \widehat{(T_n)} \in L(\widehat{M})$

(iv) and $u := \widehat{(u_k)} \in \widehat{M}$.

Then T is identity preserving and of Schwarz type on $\widehat{M}$. Since $u^*u = p$ and $uu^* = 1$ it follows $pu^* = u^*$ and $(uu^*)x(uu^*) = x$ for all $x \in \widehat{M}$. Finally, $T(pxp) = pxp$ for all $x \in \widehat{M}$, which follows from (i), and $T(u^*) = T(pu^*) = pu^* = u^*$ and $T(u) = u$, which follows from (iii). Using the Schwarz inequality we obtain

$$T(uu^*) = T(1) \leqslant 1 = uu^* = T(u)T(u)^*.$$

Using D-III, Lemma 1.1. we conclude $T(ux) = uT(x)$ and $T(xu^*) = T(x)u^*$ for all $x \in \widehat{M}$. Hence

$$\begin{aligned} T(x) &= T(uu^*xuu^*) = uT(u^*xu)u^* = uT(pu^*xup)u^* \\ &= upu^*xupu^* = uu^*xuu^* = x \end{aligned}$$

for all $x \in \widehat{M}$. From this we obtain that for every bounded sequence (x_k) in M

$$\lim_m \|T_m x_m - x_m\| = 0$$

for some subsequence of the T_n's and of the x_k's. This conflicts with our assumption at the beginning, hence the theorem is proved. □

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Chapter D-III

Spectral Theory of Positive Semigroups on W*-Algebras and their Preduals

U. Groh

Chapter D-IV

Asymptotics of Positive Semigroups on $C^\star$- and $W^\star$-Algebras

U. Groh