

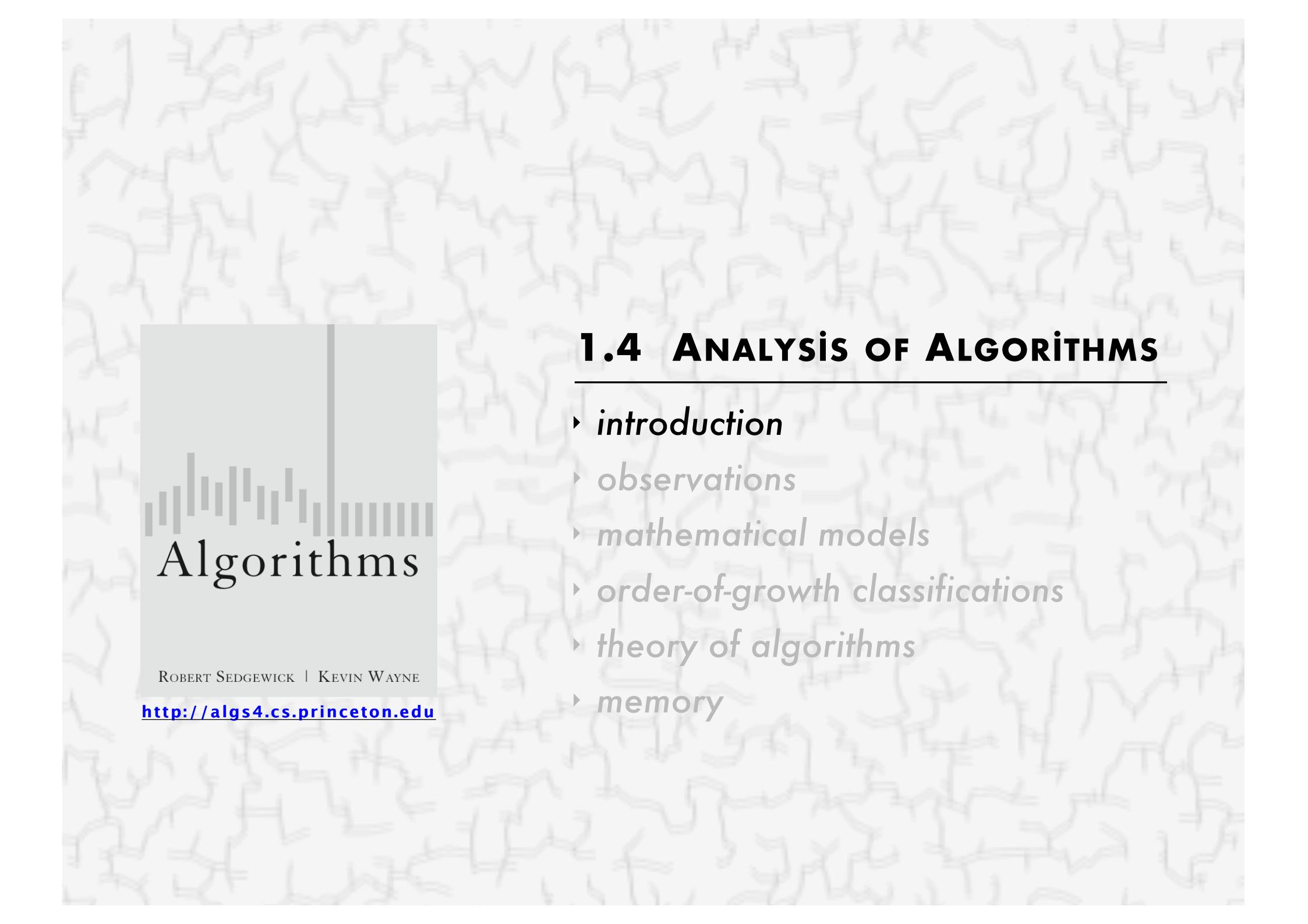


<http://algs4.cs.princeton.edu>

## 1.4 ANALYSIS OF ALGORITHMS

---

- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *theory of algorithms*
- *memory*



# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 1.4 ANALYSIS OF ALGORITHMS

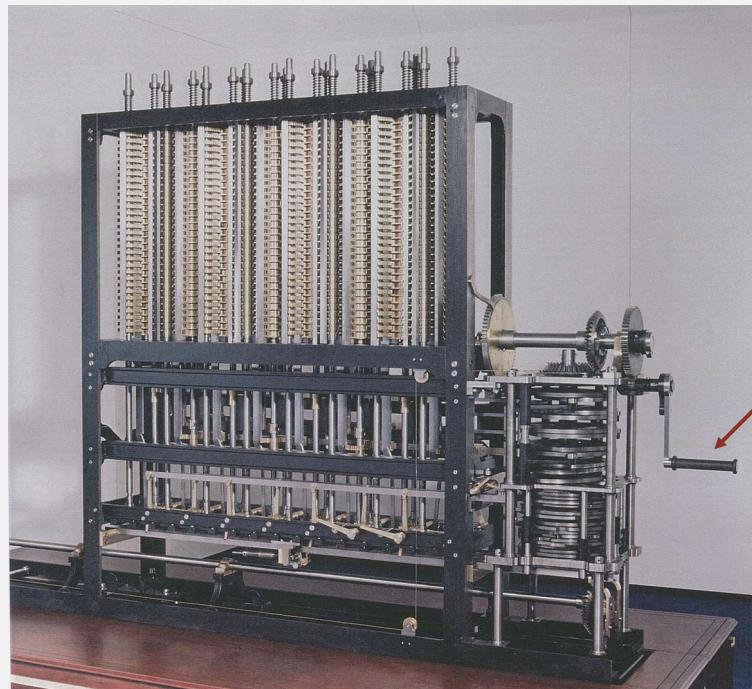
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# Running time

---

*“As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—By what course of calculation can these results be arrived at by the machine in the shortest time? ” — Charles Babbage (1864)*



how many times do you  
have to turn the crank?

Analytic Engine

## Cast of characters

---



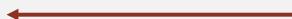
**Programmer** needs to develop  
a working solution.



**Client** wants to solve  
problem efficiently.



**Theoretician** wants  
to understand.



**Student** might play  
any or all of these  
roles someday.



# Reasons to analyze algorithms

---

Predict performance.

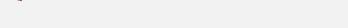


Compare algorithms.



this course

Provide guarantees.



Understand theoretical basis.



theory of algorithms

**Primary practical reason:** avoid performance bugs.



**client gets poor performance because programmer  
did not understand performance characteristics**



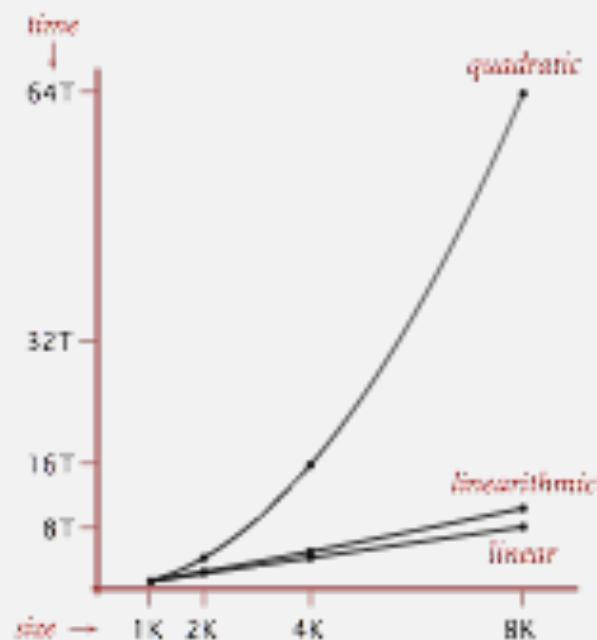
## Some algorithmic successes

### Discrete Fourier transform.

- Break down waveform of  $N$  samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ....
- Brute force:  $N^2$  steps.
- FFT algorithm:  $N \log N$  steps, enables new technology.



Friedrich Gauss  
1805



# Some algorithmic successes

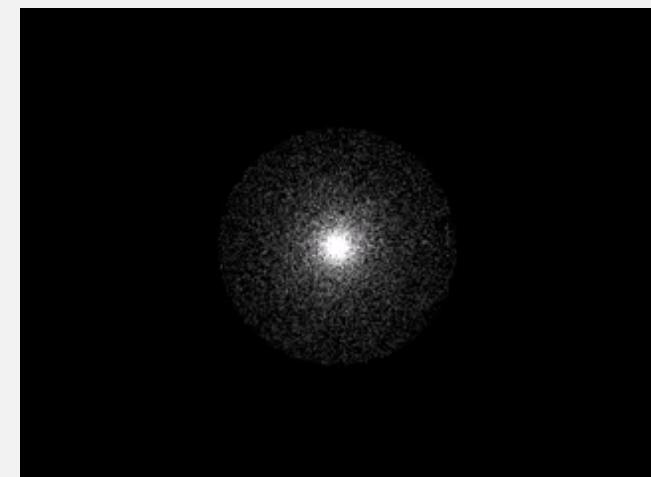
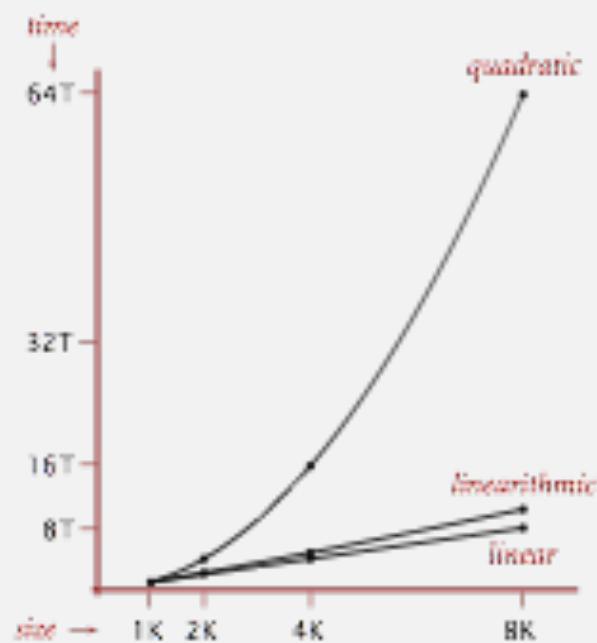
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## N-body simulation.

- Simulate gravitational interactions among  $N$  bodies.
- Brute force:  $N^2$  steps.
- Barnes-Hut algorithm:  $N \log N$  steps, enables new research.



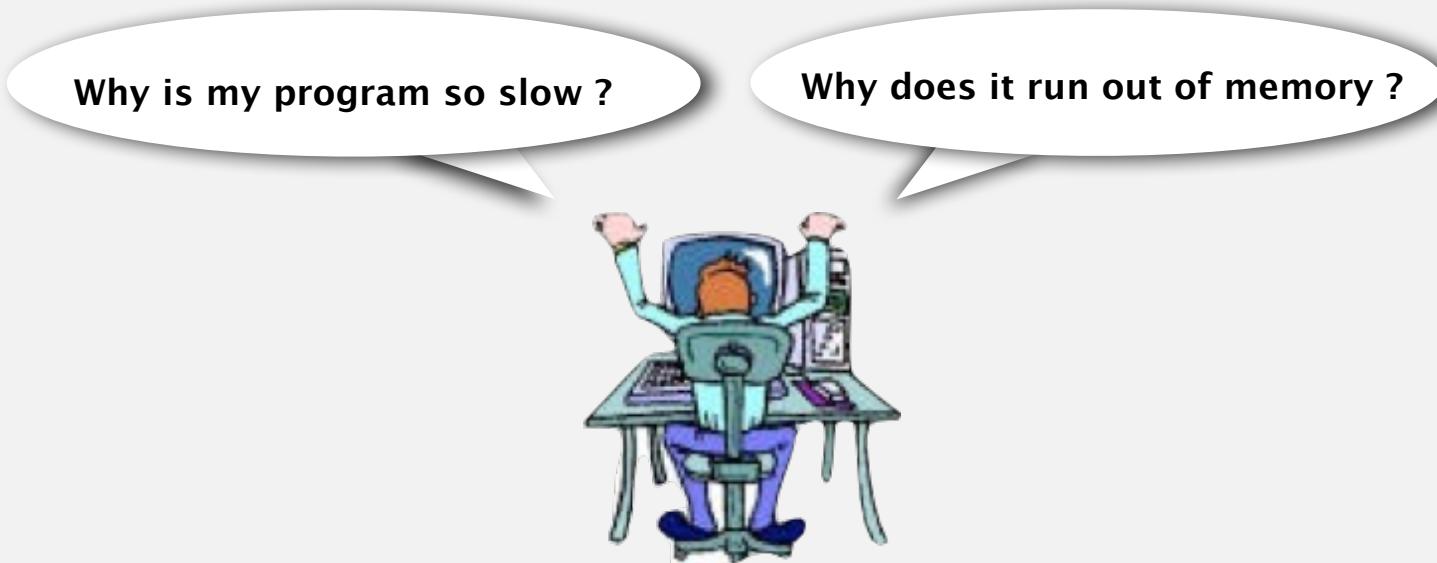
Andrew Appel  
PU '81



## The challenge

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Q. Will my program be able to solve a large practical input?



Insight. [Knuth 1970s] Use **scientific method** to understand performance.

# Scientific method applied to analysis of algorithms

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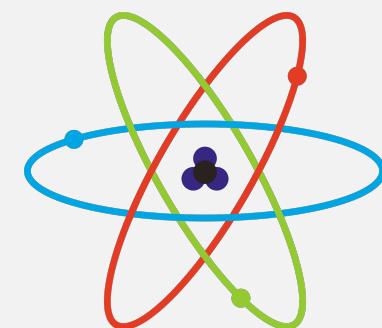
A framework for predicting performance and comparing algorithms.

## Scientific method.

- **Observe** some feature of the natural world.
- **Hypothesize** a model that is consistent with the observations.
- **Predict** events using the hypothesis.
- **Verify** the predictions by making further observations.
- **Validate** by repeating until the hypothesis and observations agree.

## Principles.

- Experiments must be **reproducible**.
- Hypotheses must be **falsifiable**.



Feature of the natural world. Computer itself.



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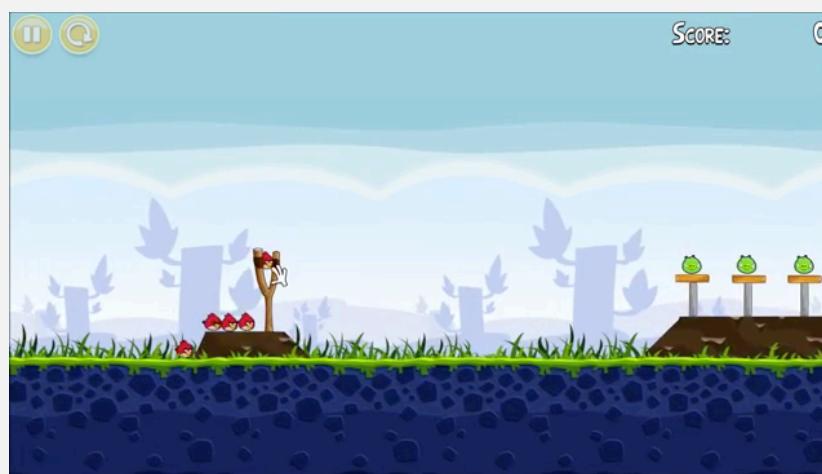
## Example: 3-SUM

---

3-SUM. Given  $N$  distinct integers, how many triples sum to exactly zero?

```
% more 8ints.txt
8
30 -40 -20 -10 40 0 10 5
```

```
% java ThreeSum 8ints.txt
4
```



	a[i]	a[j]	a[k]	sum
1	30	-40	10	0
2	30	-20	-10	0
3	-40	40	0	0
4	-10	0	10	0

Context. Deeply related to problems in computational geometry.

## 3-SUM: brute-force algorithm

---

```
public class ThreeSum
{
    public static int count(int[] a)
    {
        int N = a.length;
        int count = 0;
        for (int i = 0; i < N; i++)
            for (int j = i+1; j < N; j++)
                for (int k = j+1; k < N; k++)
                    if (a[i] + a[j] + a[k] == 0)
                        count++;
        return count;
    }

    public static void main(String[] args)
    {
        In in = new In(args[0]);
        int[] a = in.readAllInts();
        StdOut.println(count(a));
    }
}
```

← check each triple  
← for simplicity, ignore integer overflow

# Measuring the running time

---

Q. How to time a program?

A. Manual.



% java ThreeSum 1Kints.txt



70

% java ThreeSum 2Kints.txt



528

% java ThreeSum 4Kints.txt



4039

# Measuring the running time

---

Q. How to time a program?

A. Automatic.

```
public class Stopwatch (part of stdlib.jar)
```

```
Stopwatch()
```

*create a new stopwatch*

```
double elapsedTime()
```

*time since creation (in seconds)*

```
public static void main(String[] args)
{
    In in = new In(args[0]);
    int[] a = in.readAllInts();
    Stopwatch stopwatch = new Stopwatch();
    StdOut.println(ThreeSum.count(a));
    double time = stopwatch.elapsedTime();
    StdOut.println("elapsed time " + time);
}
```

# Measuring the running time

---

Q. How to time a program?

A. Automatic.

```
public class Stopwatch
```

```
    Stopwatch()
```

*create a new stopwatch*

```
    double elapsedTime()
```

*time since creation (in seconds)*

```
public class Stopwatch
```

```
{
```

```
    private final long start = System.currentTimeMillis();
```

```
    public double elapsedTime()
```

```
{
```

```
    long now = System.currentTimeMillis();
```

```
    return (now - start) / 1000.0;
```

```
}
```

**implementation (part of stdlib.jar)**

```
}
```

## Empirical analysis

---

Run the program for various input sizes and measure running time.

% 

## Empirical analysis

---

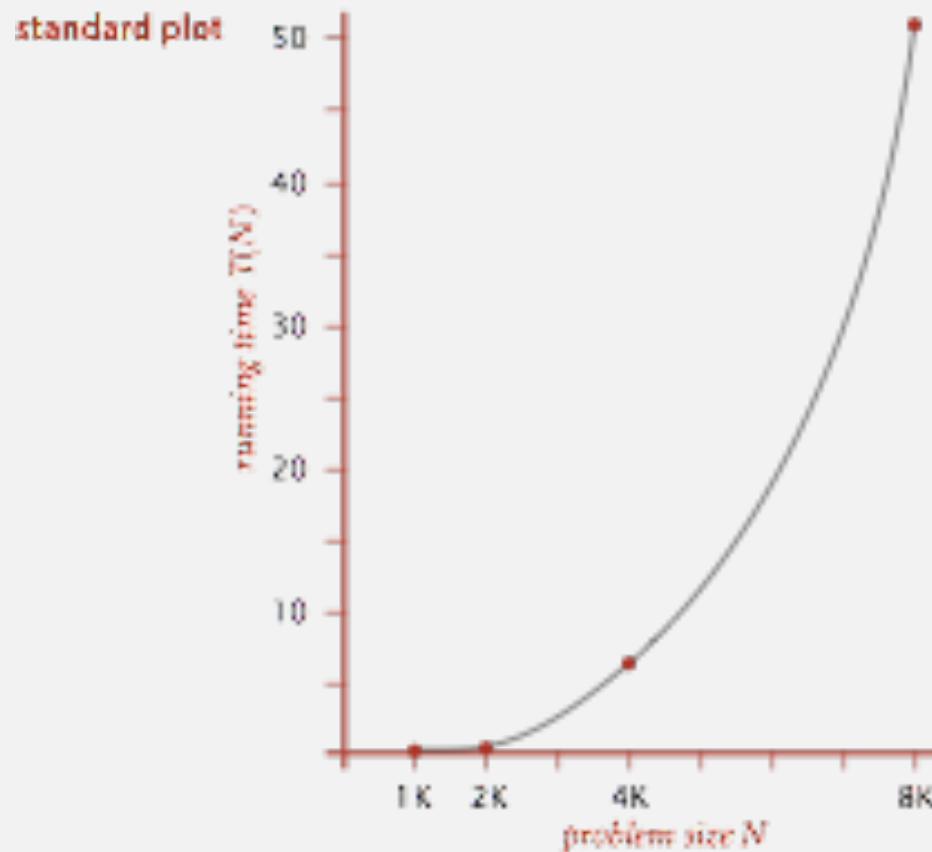
Run the program for various input sizes and measure running time.

N	time (seconds) <sup>†</sup>
250	0
500	0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

## Data analysis

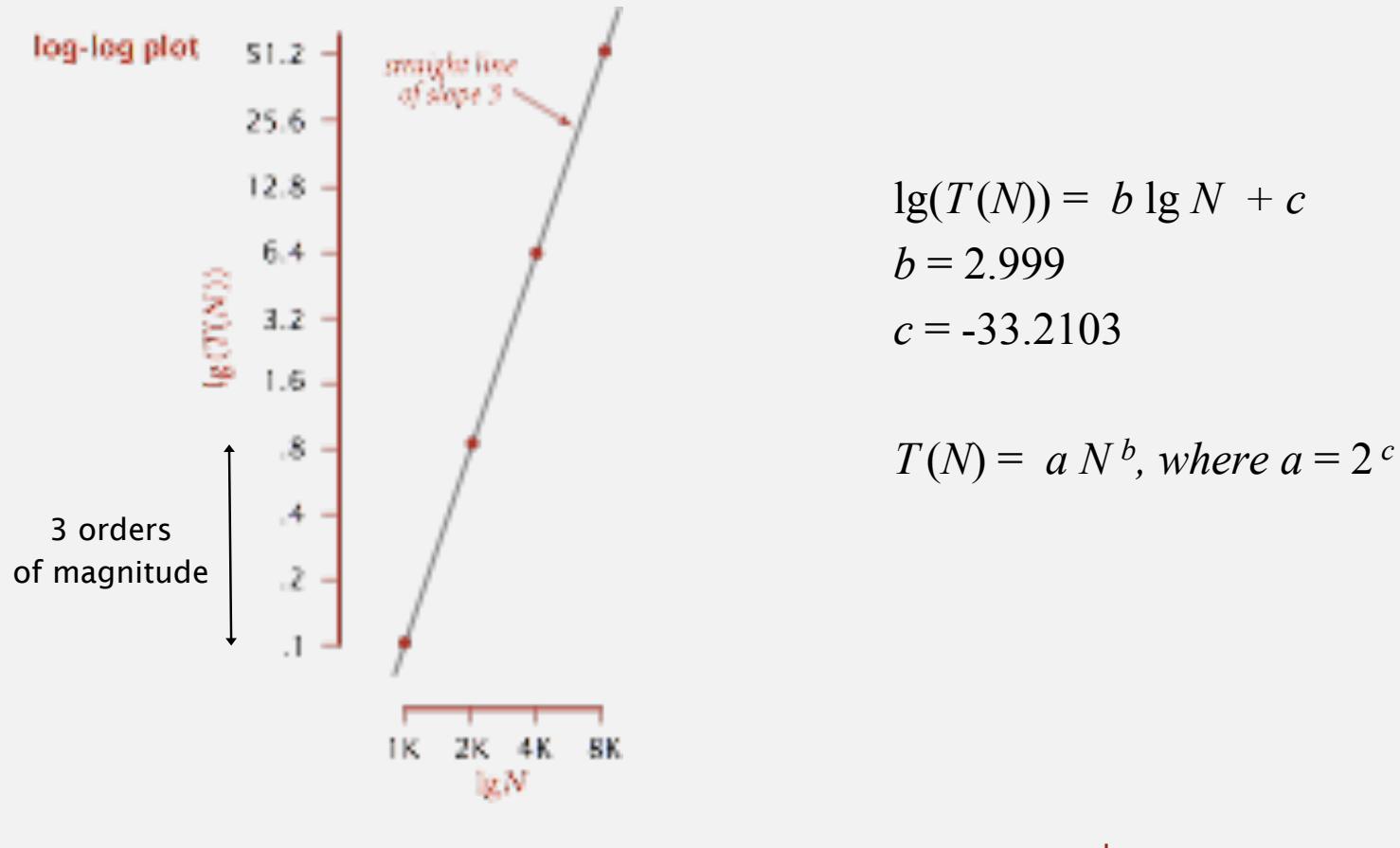
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Standard plot. Plot running time  $T(N)$  vs. input size  $N$ .



# Data analysis

Log-log plot. Plot running time  $T(N)$  vs. input size  $N$  using log-log scale.



Regression. Fit straight line through data points:  $a N^b$ .

Hypothesis. The running time is about  $1.006 \cdot 10^{-10} \cdot N^{2.999}$  seconds.

## Prediction and validation

---

Hypothesis. The running time is about  $1.006 \cdot 10^{-10} \cdot N^{2.999}$  seconds.



"order of growth" of running  
time is about  $N^3$  [stay tuned]

### Predictions.

- 51.0 seconds for  $N=8,000$ .
- 408.1 seconds for  $N=16,000$ .

### Observations.

N	time (seconds) <sup>†</sup>
8,000	51.1
8,000	51
8,000	51.1
16,000	410.8

validates hypothesis!

# Doubling hypothesis

**Doubling hypothesis.** Quick way to estimate  $b$  in a power-law relationship.

Run program, doubling the size of the input.

N	time (seconds) <sup>†</sup>	ratio	lg ratio
250	0		-
500	0	4.8	2.3
1,000	0.1	6.9	2.8
2,000	0.8	7.7	2.9
4,000	6.4	8	3
8,000	51.1	8	3

$$\begin{aligned}\frac{T(2N)}{T(N)} &= \frac{a(2N)^b}{aN^b} \\ &= 2^b\end{aligned}$$

←  $\lg (6.4 / 0.8) = 3.0$



seems to converge to a constant  $b \approx 3$

**Hypothesis.** Running time is about  $a N^b$  with  $b = \lg$  ratio.

**Caveat.** Cannot identify logarithmic factors with doubling hypothesis.

## Doubling hypothesis

---

**Doubling hypothesis.** Quick way to estimate  $b$  in a power-law relationship.

**Q.** How to estimate  $a$  (assuming we know  $b$ ) ?

**A.** Run the program (for a sufficient large value of  $N$ ) and solve for  $a$ .

N	time (seconds) †
8,000	51.1
8,000	51
8,000	51.1

$$51.1 = a \cdot 8000^3$$

$$\Rightarrow a = 0.998 \cdot 10^{-10}$$

**Hypothesis.** Running time is about  $0.998 \cdot 10^{-10} \cdot N^3$  seconds.

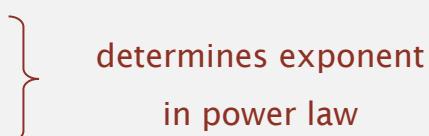


almost identical hypothesis  
to one obtained via linear regression

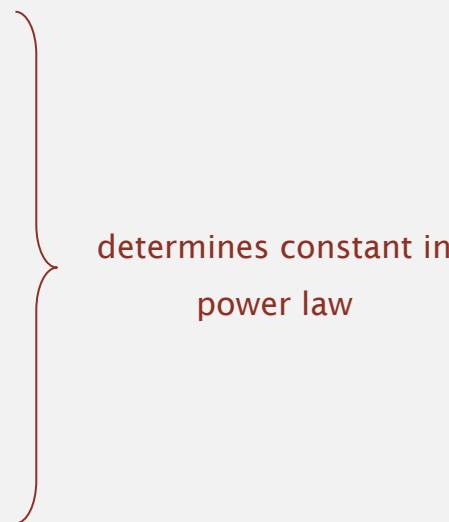
# Experimental algorithmics

---

## System independent effects.

- Algorithm.
  - Input data.
- 
- determines exponent  
in power law

## System dependent effects.

- Hardware: CPU, memory, cache, ...
  - Software: compiler, interpreter, garbage collector, ...
  - System: operating system, network, other apps, ...
- 
- determines constant in  
power law

**Bad news.** Difficult to get precise measurements.

**Good news.** Much easier and cheaper than other sciences.



e.g., can run huge number of experiments

## War story (from COS 126)

---

Q. How long does this program take as a function of  $N$ ?

```
String s = StdIn.readString();
int N = s.length();
...
for (int i = 0; i < N; i++)
    for (int j = 0; j < N; j++)
        distance[i][j] = ...
...
```

N	time
1,000	0.11
2,000	0.35
4,000	1.6
8,000	6.5

Jenny  $\sim c_1 N^2$  seconds

N	time
250	0.5
500	1.1
1,000	1.9
2,000	3.9

Kenny  $\sim c_2 N$  seconds



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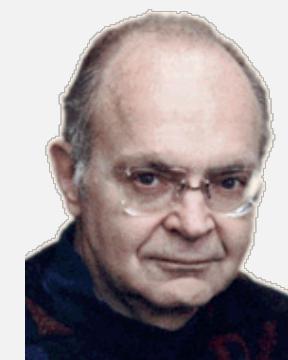
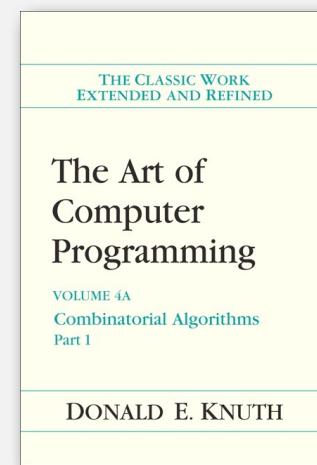
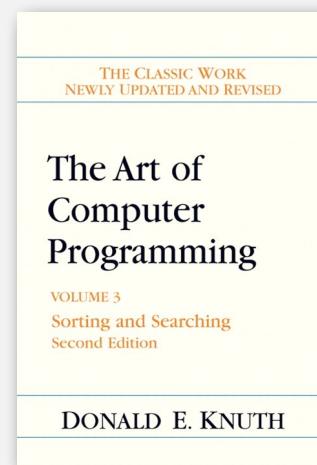
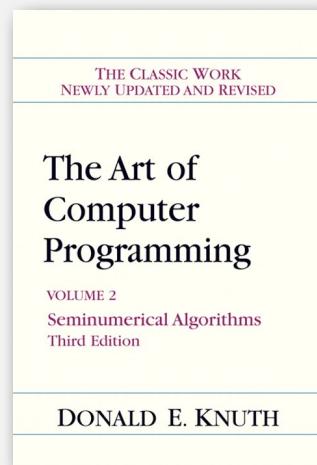
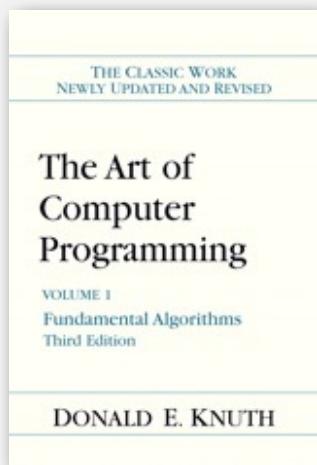
- *introduction*
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# Mathematical models for running time

---

Total running time: sum of cost · frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.



**Donald Knuth**  
1974 Turing Award

In principle, accurate mathematical models are available.

# Cost of basic operations

---

Challenge. How to estimate constants.

operation	example	nanoseconds †
integer add	$a + b$	2.1
integer multiply	$a * b$	2.4
integer divide	$a / b$	5.4
floating-point add	$a + b$	4.6
floating-point multiply	$a * b$	4.2
floating-point divide	$a / b$	13.5
sine	<code>Math.sin(theta)</code>	91.3
arctangent	<code>Math.atan2(y, x)</code>	129
...	...	...

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

## Cost of basic operations

---

Observation. Most primitive operations take constant time.

operation	example	nanoseconds †
variable declaration	<code>int a</code>	$c_1$
assignment statement	<code>a = b</code>	$c_2$
integer compare	<code>a &lt; b</code>	$c_3$
array element access	<code>a[i]</code>	$c_4$
array length	<code>a.length</code>	$c_5$
1D array allocation	<code>new int[N]</code>	$c_6 N$
2D array allocation	<code>new int[N][N]</code>	$c_7 N^2$

Caveat. Non-primitive operations often take more than constant time.

novice mistake: abusive string concatenation

## Example: 1-SUM

---

Q. How many instructions as a function of input size  $N$ ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    if (a[i] == 0)  
        count++;
```

N array accesses

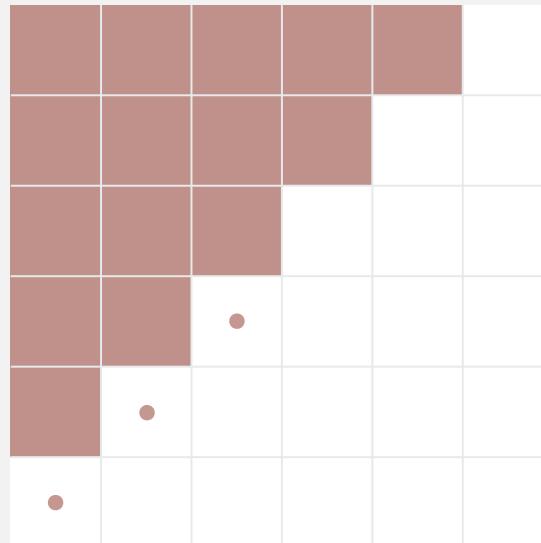
operation	frequency
variable declaration	2
assignment statement	2
less than compare	$N + 1$
equal to compare	$N$
array access	$N$
increment	$N$ to $2N$

## Example: 2-SUM

Q. How many instructions as a function of input size  $N$ ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    for (int j = i+1; j < N; j++)  
        if (a[i] + a[j] == 0)  
            count++;
```

Pf. [ n even]



$$\begin{aligned}0 + 1 + 2 + \dots + (N - 1) &= \frac{1}{2}N(N - 1) \\&= \binom{N}{2}\end{aligned}$$

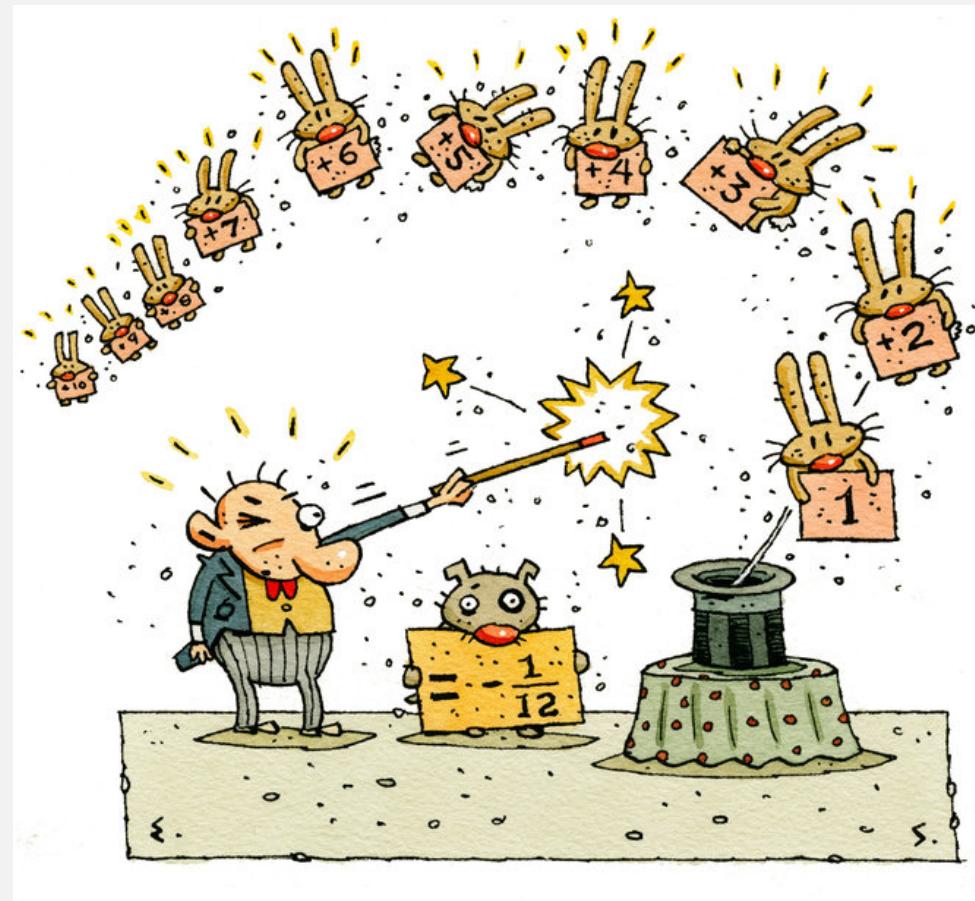
$$0 + 1 + 2 + \dots + (N - 1) = \frac{1}{2}N^2 - \frac{1}{2}N$$

half of square      half of  
diagonal

## String theory infinite sum

---

$$1 + 2 + 3 + 4 + \dots = -\frac{1}{12}$$



<http://www.nytimes.com/2014/02/04/science/in-the-end-it-all-adds-up-to.html>

## Example: 2-SUM

Q. How many instructions as a function of input size  $N$ ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    for (int j = i+1; j < N; j++)  
        if (a[i] + a[j] == 0)  
            count++;
```

$$\begin{aligned}0 + 1 + 2 + \dots + (N - 1) &= \frac{1}{2}N(N - 1) \\&= \binom{N}{2}\end{aligned}$$

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$\frac{1}{2}(N + 1)(N + 2)$
equal to compare	$\frac{1}{2}N(N - 1)$
array access	$N(N - 1)$
increment	$\frac{1}{2}N(N - 1)$ to $N(N - 1)$



tedious to count exactly

# Simplifying the calculations

---

*“ It is convenient to have a measure of the amount of work involved in a computing process, even though it be a very crude one. We may count up the number of times that various elementary operations are applied in the whole process and then given them various weights. We might, for instance, count the number of additions, subtractions, multiplications, divisions, recording of numbers, and extractions of figures from tables. In the case of computing with matrices most of the work consists of multiplications and writing down numbers, and we shall therefore only attempt to count the number of multiplications and recordings. ” — Alan Turing*

## ROUNDING-OFF ERRORS IN MATRIX PROCESSES

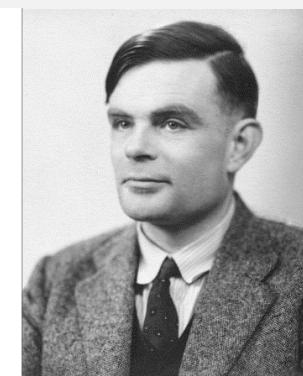
By A. M. TURING

(National Physical Laboratory, Teddington, Middlesex)

[Received 4 November 1947]

### SUMMARY

A number of methods of solving sets of linear equations and inverting matrices are discussed. The theory of the rounding-off errors involved is investigated for some of the methods. In all cases examined, including the well-known 'Gauss elimination process', it is found that the errors are normally quite moderate: no exponential build-up need occur.



## Simplification 1: cost model

Cost model. Use some basic operation as a proxy for running time.

```
int count = 0;  
for (int i = 0; i < N; i++)  
    for (int j = i+1; j < N; j++)  
        if (a[i] + a[j] == 0)  
            count++;
```

$$\begin{aligned}0 + 1 + 2 + \dots + (N - 1) &= \frac{1}{2} N(N - 1) \\&= \binom{N}{2}\end{aligned}$$

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$\frac{1}{2} (N + 1)(N + 2)$
equal to compare	$\frac{1}{2} N(N - 1)$
<b>array access</b>	$N(N - 1)$
increment	$\frac{1}{2} N(N - 1)$ to $N(N - 1)$

cost model = array accesses

(we assume compiler/JVM do not  
optimize any array accesses away!)

## Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size  $N$ .
- Ignore lower order terms.
  - when  $N$  is large, terms are negligible
  - when  $N$  is small, we don't care

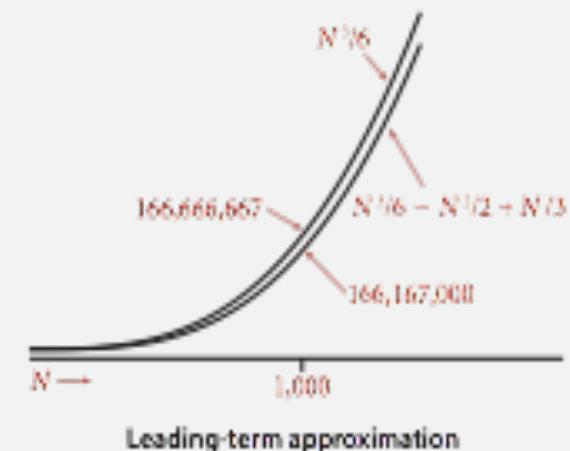
**Ex 1.**  $\frac{1}{6}N^3 + 20N + 16 \sim \frac{1}{6}N^3$

**Ex 2.**  $\frac{1}{6}N^3 + 100N^{4/3} + 56 \sim \frac{1}{6}N^3$

**Ex 3.**  $\frac{1}{6}N^3 - \frac{1}{2}N^2 + \frac{1}{3}N \sim \frac{1}{6}N^3$

discard lower-order terms

(e.g.,  $N = 1000$ : 166.67 million vs. 166.17 million)



**Technical definition.**  $f(N) \sim g(N)$  means  $\lim_{N \rightarrow \infty} \frac{f(N)}{g(N)} = 1$

## Simplification 2: tilde notation

---

- Estimate running time (or memory) as a function of input size  $N$ .
- Ignore lower order terms.
  - when  $N$  is large, terms are negligible
  - when  $N$  is small, we don't care

operation	frequency	tilde notation
variable declaration	$N + 2$	$\sim N$
assignment statement	$N + 2$	$\sim N$
less than compare	$\frac{1}{2} (N + 1) (N + 2)$	$\sim \frac{1}{2} N^2$
equal to compare	$\frac{1}{2} N(N - 1)$	$\sim \frac{1}{2} N^2$
array access	$N(N - 1)$	$\sim N^2$
increment	$\frac{1}{2} N(N - 1)$ to $N(N - 1)$	$\sim \frac{1}{2} N^2$ to $\sim N^2$

## Example: 2-SUM

---

Q. Approximately how many array accesses as a function of input size  $N$ ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    for (int j = i+1; j < N; j++)  
        if (a[i] + a[j] == 0)  
            count++;
```

"inner loop"

$$\begin{aligned}0 + 1 + 2 + \dots + (N - 1) &= \frac{1}{2}N(N - 1) \\&= \binom{N}{2}\end{aligned}$$

A.  $\sim N^2$  array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

## Example: 3-SUM

Q. Approximately how many array accesses as a function of input size  $N$ ?

```
int count = 0;  
for (int i = 0; i < N; i++)  
    for (int j = i+1; j < N; j++)  
        for (int k = j+1; k < N; k++)  
            if (a[i] + a[j] + a[k] == 0)  
                count++;
```

"inner loop"

A.  $\sim \frac{1}{2} N^3$  array accesses.

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!} \\ \sim \frac{1}{6}N^3$$

Bottom line. Use cost model and tilde notation to simplify counts.

## Diversion: estimating a discrete sum

---

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

Ex 1.  $1 + 2 + \dots + N$ .

$$\sum_{i=1}^N i \sim \int_{x=1}^N x dx \sim \frac{1}{2} N^2$$

Ex 2.  $1^k + 2^k + \dots + N^k$ .

$$\sum_{i=1}^N i^k \sim \int_{x=1}^N x^k dx \sim \frac{1}{k+1} N^{k+1}$$

Ex 3.  $1 + 1/2 + 1/3 + \dots + 1/N$ .

$$\sum_{i=1}^N \frac{1}{i} \sim \int_{x=1}^N \frac{1}{x} dx = \ln N$$

Ex 4. 3-sum triple loop.

$$\sum_{i=1}^N \sum_{j=i}^N \sum_{k=j}^N 1 \sim \int_{x=1}^N \int_{y=x}^N \int_{z=y}^N dz dy dx \sim \frac{1}{6} N^3$$

## Estimating a discrete sum

---

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

Ex 4.  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 2$$

$$\int_{x=0}^{\infty} \left(\frac{1}{2}\right)^x dx = \frac{1}{\ln 2} \approx 1.4427$$

Caveat. Integral trick doesn't always work!

# Estimating a discrete sum

Q. How to estimate a discrete sum?

A3. Use Maple or Wolfram Alpha.

The screenshot shows the WolframAlpha PRO interface. The search bar contains the input: "sum(sum(sum(1, k=j+1..N), j = i+1..N), i = 1..N)". Below the input are several small icons. To the right are "Examples" and "Random" buttons. The main result area is titled "Sum:" and displays the mathematical expression: 
$$\sum_{i=1}^N \left( \sum_{j=i+1}^N \left( \sum_{k=j+1}^N 1 \right) \right) = \frac{1}{6} N(N^2 - 3N + 2)$$
.

wolframalpha.com

```
[wayne:nobel.princeton.edu] > maple15
    |\^/|      Maple 15 (X86 64 LINUX)
--||\|  |||_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2011
\ MAPLE / All rights reserved. Maple is a trademark of
<____ _> Waterloo Maple Inc.
           |      Type ? for help.
> factor(sum(sum(sum(1, k=j+1..N), j = i+1..N), i = 1..N));
```

$$\frac{N(N - 1)(N - 2)}{6}$$

# Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.



$$T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$$

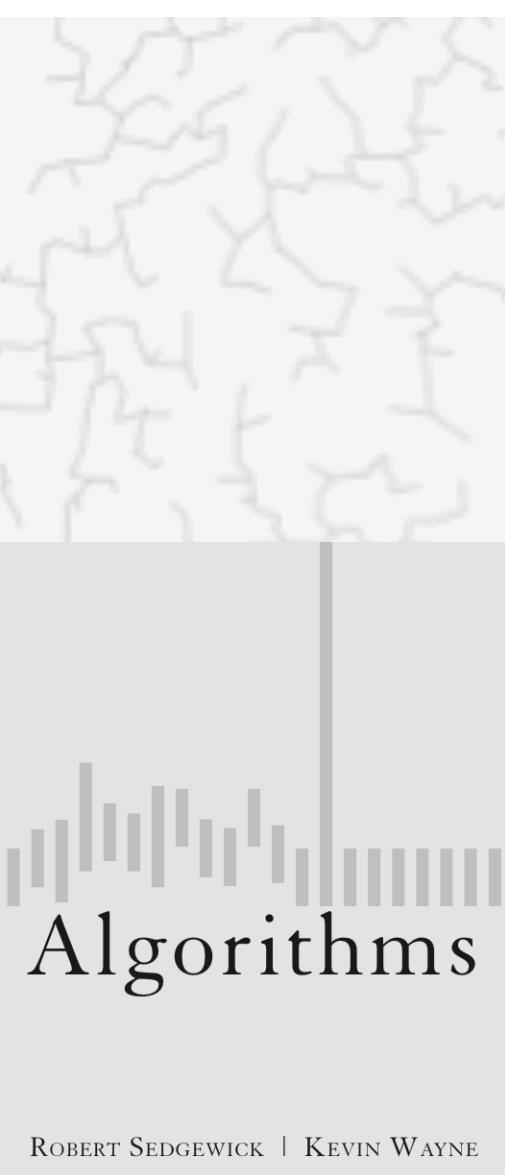
costs (depend on machine, compiler)

$A$  = array access  
 $B$  = integer add  
 $C$  = integer compare  
 $D$  = increment  
 $E$  = variable assignment

frequencies  
(depend on algorithm, input)

The diagram shows the formula  $T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$ . Above the formula, the text "costs (depend on machine, compiler)" is written in red. Below the formula, the meanings of the variables A through E are listed: A is array access, B is integer add, C is integer compare, D is increment, and E is variable assignment. To the right of the formula, four red arrows point from the terms  $c_1 A$ ,  $c_2 B$ ,  $c_3 C$ , and  $c_4 D$  to the word "frequencies" and the parenthetical note "(depend on algorithm, input)".

Bottom line. We use approximate models in this course:  $T(N) \sim c N^3$ .



# Algorithms

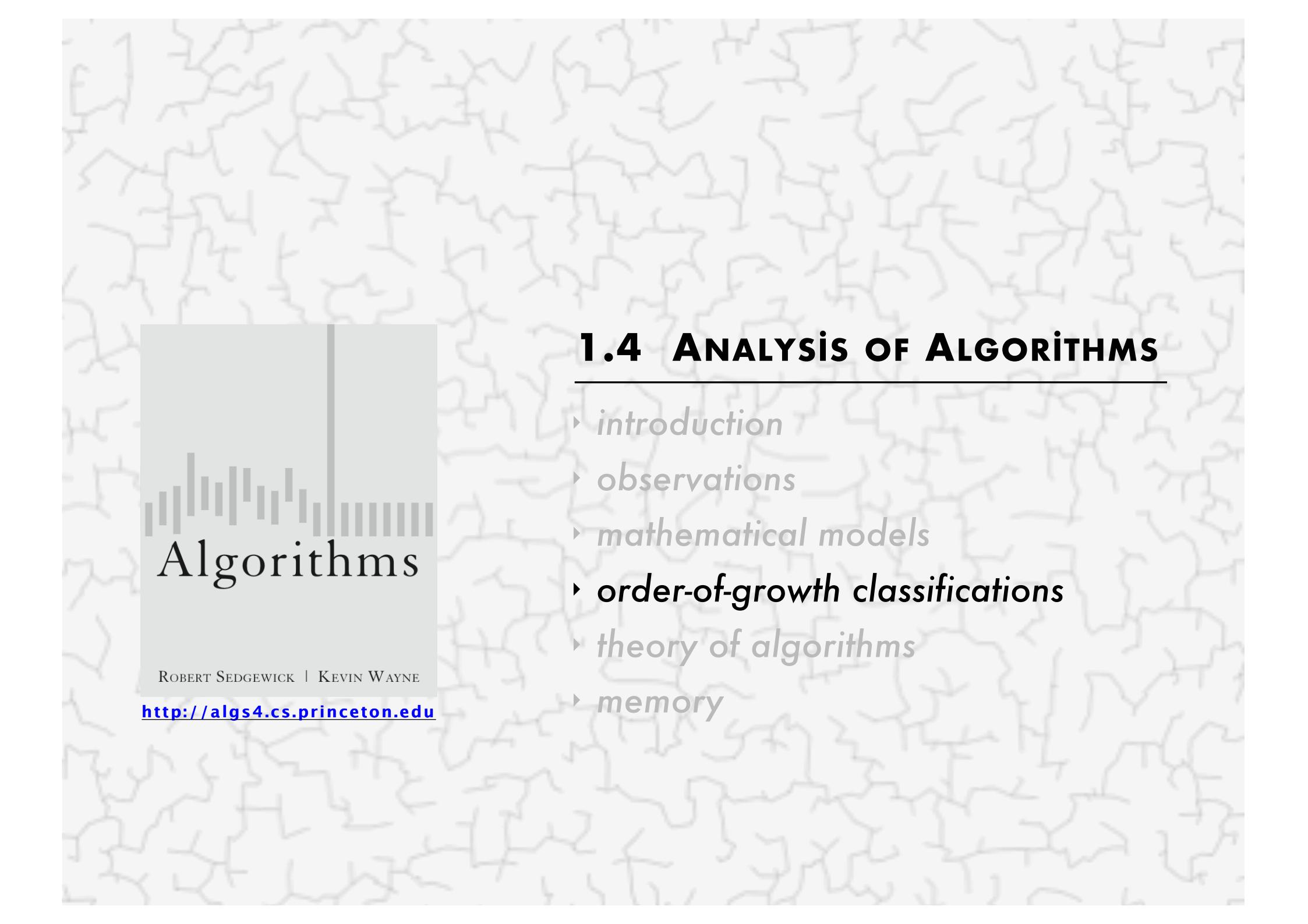
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## 1.4 ANALYSIS OF ALGORITHMS

---

- *introduction*
- *observations*
- ***mathematical models***
- *order-of-growth classifications*
- *theory of algorithms*
- *memory*



# Algorithms

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## 1.4 ANALYSIS OF ALGORITHMS

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- *memory*

## Common order-of-growth classifications

---

**Definition.** If  $f(N) \sim c g(N)$  for some constant  $c > 0$ , then the **order of growth** of  $f(N)$  is  $g(N)$ .

- Ignores leading coefficient.
- Ignores lower-order terms.

**Ex.** The order of growth of the **running time** of this code is  $N^3$ .

```
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        for (int k = j+1; k < N; k++)
            if (a[i] + a[j] + a[k] == 0)
                count++;
```

**Typical usage.** With running times.

where leading coefficient  
depends on machine, compiler, JVM, ...

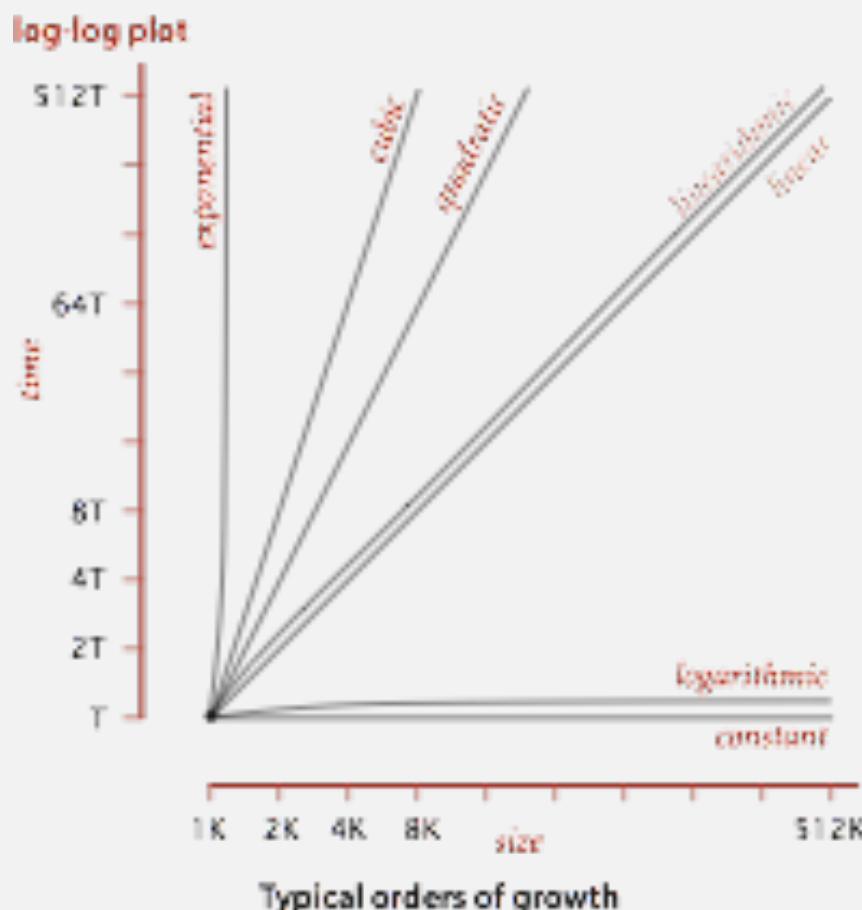
# Common order-of-growth classifications

---

Good news. The set of functions

$1$ ,  $\log N$ ,  $N$ ,  $N \log N$ ,  $N^2$ ,  $N^3$ , and  $2^N$

suffices to describe the order of growth of most common algorithms.



# Common order-of-growth classifications

---

order of growth	name	typical code framework	description	example	$T(2N) / T(N)$
1	<b>constant</b>	$a = b + c;$	statement	add two numbers	1
$\log N$	<b>logarithmic</b>	<pre>while (N &gt; 1) {   N = N / 2; ... }</pre>	divide in half	binary search	$\sim 1$
$N$	<b>linear</b>	<pre>for (int i = 0; i &lt; N; i++) { ... }</pre>	loop	find the maximum	2
$N \log N$	<b>linearithmic</b>	[see mergesort lecture]	divide and conquer	mergesort	$\sim 2$
$N^2$	<b>quadratic</b>	<pre>for (int i = 0; i &lt; N; i++)     for (int j = 0; j &lt; N; j++)         { ... }</pre>	double loop	check all pairs	4
$N^3$	<b>cubic</b>	<pre>for (int i = 0; i &lt; N; i++)     for (int j = 0; j &lt; N; j++)         for (int k = 0; k &lt; N; k++)             { ... }</pre>	triple loop	check all triples	8
$2^N$	<b>exponential</b>	[see combinatorial search lecture]	exhaustive search	check all subsets	$T(N)$

# Practical implications of order-of-growth

growth rate	problem size solvable in minutes			
	1970s	1980s	1990s	2000s
1	any	any	any	any
$\log N$	any	any	any	any
$N$	millions	tens of millions	hundreds of millions	billions
$N \log N$	hundreds of thousands	millions	millions	hundreds of millions
$N^2$	hundreds	thousand	thousands	tens of thousands
$N^3$	hundred	hundreds	thousand	thousands
$2^N$	20	20s	20s	30

**Bottom line.** Need linear or linearithmic alg to keep pace with Moore's law.

# Practical implications of order-of-growth

---

growth rate	problem size solvable in minutes				time to process millions of inputs			
	1970s	1980s	1990s	2000s	1970s	1980s	1990s	2000s
1	any	any	any	any	instant	instant	instant	instant
$\log N$	any	any	any	any	instant	instant	instant	instant
N	millions	tens of millions	hundreds of millions	billions	minutes	seconds	second	instant
$N \log N$	hundreds of thousands	millions	millions	hundreds of millions	hour	minutes	tens of seconds	seconds
$N^2$	hundreds	thousand	thousands	tens of thousands	decades	years	months	weeks
$N^3$	hundred	hundreds	thousand	thousands	never	never	never	millennia

# Practical implications of order-of-growth

---

growth rate	name	description	effect on a program that runs for a few seconds	
			time for 100x more data	size for 100x faster computer
1	constant	independent of input size	-	-
$\log N$	logarithmic	nearly independent of input size	-	-
N	linear	optimal for N inputs	a few minutes	100x
$N \log N$	linearithmic	nearly optimal for N inputs	a few minutes	100x
$N^2$	quadratic	not practical for large problems	several hours	10x
$N^3$	cubic	not practical for medium problems	several weeks	4–5x
$2^N$	exponential	useful only for tiny problems	forever	1x

## Binary search demo

---

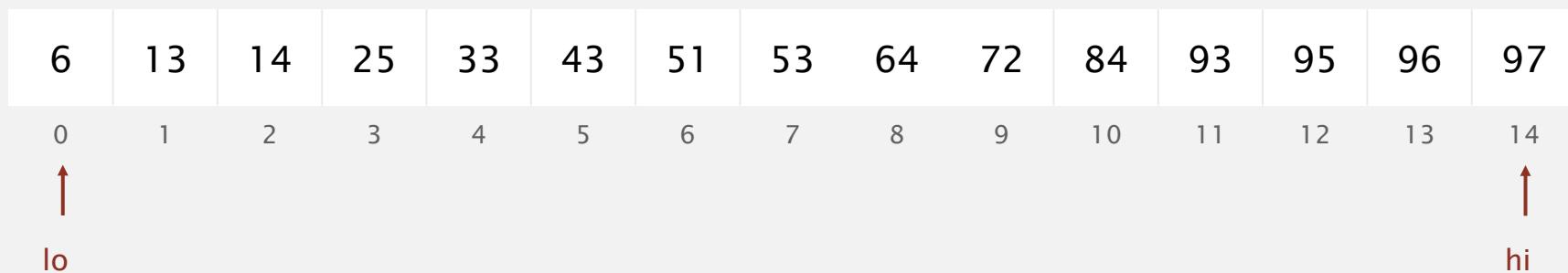
**Goal.** Given a sorted array and a key, find index of the key in the array?

**Binary search.** Compare key against middle entry.

- Too small, go left.
- Too big, go right.
- Equal, found.



**successful search for 33**



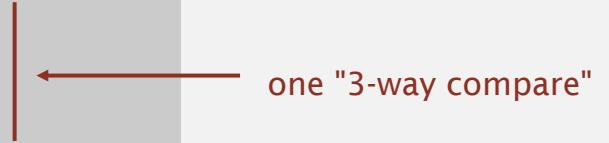
# Binary search: Java implementation

---

## Trivial to implement?

- First binary search published in 1946.
- First bug-free one in 1962.
- Bug in Java's `Arrays.binarySearch()` discovered in 2006.

```
public static int binarySearch(int[] a, int key)
{
    int lo = 0, hi = a.length-1;
    while (lo <= hi)
    {
        int mid = lo + (hi - lo) / 2;
        if      (key < a[mid]) hi = mid - 1;
        else if (key > a[mid]) lo = mid + 1;
        else return mid;
    }
    return -1;
}
```



one "3-way compare"

**Invariant.** If key appears in the array a[], then  $a[lo] \leq \text{key} \leq a[hi]$ .

## Binary search: mathematical analysis

---

**Proposition.** Binary search uses at most  $1 + \lg N$  key compares to search in a sorted array of size  $N$ .

**Def.**  $T(N) = \# \text{ key compares to binary search a sorted subarray of size } \leq N$ .

**Binary search recurrence.**  $T(N) \leq T(N/2) + 1$  for  $N > 1$ , with  $T(1) = 1$ .

$\uparrow$        $\uparrow$   
left or right half      possible to implement with one  
(floored division)      2-way compare (instead of 3-way)

**Pf sketch.** [assume  $N$  is a power of 2]

$$\begin{aligned} T(N) &\leq T(N/2) + 1 && [\text{given}] \\ &\leq T(N/4) + 1 + 1 && [\text{apply recurrence to first term}] \\ &\leq T(N/8) + 1 + 1 + 1 && [\text{apply recurrence to first term}] \\ &\vdots \\ &\leq T(N/N) + 1 + 1 + \dots + 1 && [\text{stop applying, } T(1) = 1] \\ &= 1 + \lg N \end{aligned}$$

## Binary search: mathematical analysis

---

**Proposition.** Binary search uses at most  $1 + \lg N$  key compares to search in a sorted array of size  $N$ .

**Def.**  $T(N) \equiv \# \text{ key compares to binary search a sorted subarray of size } \leq N$ .

**Binary search recurrence.**  $T(N) \leq T(\lfloor N/2 \rfloor) + 1$  for  $N > 1$ , with  $T(0) = 0$ .

For simplicity, we prove when  $N = 2^n - 1$  for some  $n$ , so  $\lfloor N/2 \rfloor = 2^{n-1} - 1$ .

$$\begin{aligned} T(2^n - 1) &\leq T(2^{n-1} - 1) + 1 \\ &\leq T(2^{n-2} - 1) + 1 + 1 \\ &\leq T(2^{n-3} - 1) + 1 + 1 + 1 \\ &\quad \dots \\ &\leq T(2^0 - 1) + 1 + 1 + \dots + 1 \\ &= n \end{aligned}$$

given

apply recurrence to first term

apply recurrence to first term

stop applying,  $T(0) = 1$

# An $N^2 \log N$ algorithm for 3-SUM

## Algorithm.

- Step 1: Sort the  $N$  (distinct) numbers.
- Step 2: For each pair of numbers  $a[i]$  and  $a[j]$ , binary search for  $-(a[i] + a[j])$ .

## input

30 -40 -20 -10 40 0 10 5

## sort

-40 -20 -10 0 5 10 30 40

## Analysis.

Order of growth is  $N^2 \log N$ .

- Step 1:  $N^2$  with insertion sort.
- Step 2:  $N^2 \log N$  with binary search.

## binary search

(-40, -20) 60

(-40, -10) 50

(-40, 0) 40

(-40, 5) 35

(-40, 10) 30

⋮ ⋮

(-20, -10) 30

⋮ ⋮

(-10, 0) 10

⋮ ⋮

(10, 30) -40

(10, 40) -50

(30, 40) -70

only count if  
 $a[i] < a[j] < a[k]$   
to avoid  
double counting

## Remark.

Can achieve  $N^2$  by modifying  
binary search step.

## Comparing programs

---

**Hypothesis.** The sorting-based  $N^2 \log N$  algorithm for 3-SUM is significantly faster in practice than the brute-force  $N^3$  algorithm.

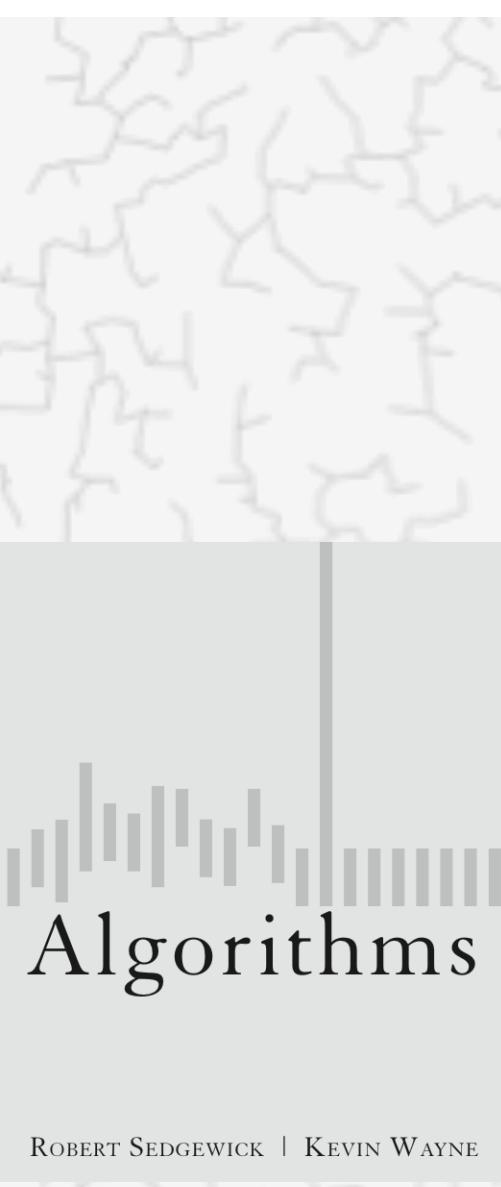
N	time (seconds)
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1

**ThreeSum.java**

N	time (seconds)
1,000	0.14
2,000	0.18
4,000	0.34
8,000	0.96
16,000	3.67
32,000	14.88
64,000	59.16

**ThreeSumDeluxe.java**

**Guiding principle.** Typically, better order of growth  $\Rightarrow$  faster in practice.



# Algorithms

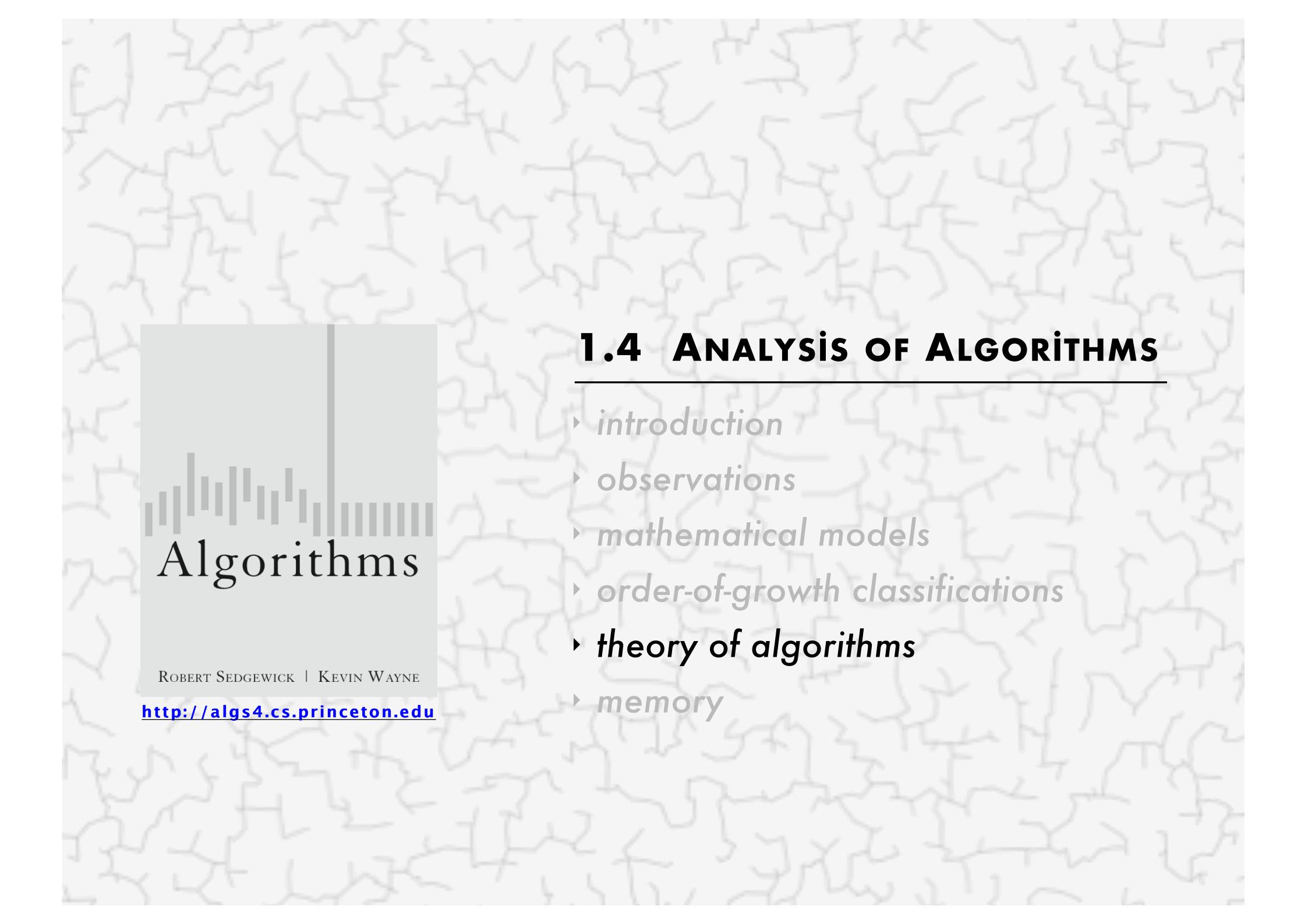
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## 1.4 ANALYSIS OF ALGORITHMS

---

- *introduction*
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- ***order-of-growth classifications***
- *theory of algorithms*
- *memory*



# Algorithms

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# Types of analyses

---

**Best case.** Lower bound on cost.

- Determined by “easiest” input.
- Provides a goal for all inputs.

**Worst case.** Upper bound on cost.

- Determined by “most difficult” input.
- Provides a guarantee for all inputs.

**Average case.** Expected cost for random input.

- Need a model for “random” input.
- Provides a way to predict performance.

this course

**Ex 1.** Array accesses for brute-force 3-SUM.

Best:  $\sim \frac{1}{2} N^3$

Average:  $\sim \frac{1}{2} N^3$

Worst:  $\sim \frac{1}{2} N^3$

**Ex 2.** Comparisons for binary search.

Best:  $\sim 1$

Average:  $\sim \lg N$

Worst:  $\sim \lg N$

## Types of analyses

---

**Best case.** Lower bound on cost.

**Worst case.** Upper bound on cost.

**Average case.** “Expected” cost.

**Actual data might not match input model?**

- Need to understand input to effectively process it.
- Approach 1: design for the worst case.
- Approach 2: randomize, depend on probabilistic guarantee.

# Theory of algorithms

---

## Goals.

- Establish “difficulty” of a problem.
- Develop “optimal” algorithms.

## Approach.

- Suppress details in analysis: analyze “to within a constant factor.”
- Eliminate variability in input model: focus on the worst case.

**Upper bound.** Performance guarantee of algorithm for any input.

**Lower bound.** Proof that no algorithm can do better.

**Optimal algorithm.** Lower bound = upper bound (to within a constant factor).

# Commonly-used notations in the theory of algorithms

---

notation	provides	example	shorthand for	used to
<b>Big Theta</b>	asymptotic order of growth	$\Theta(N^2)$	$\frac{1}{2} N^2$ $10 N^2$ $5 N^2 + 22 N \log N + 3N$ ⋮	classify algorithms
<b>Big Oh</b>	$\Theta(N^2)$ and smaller	$O(N^2)$	$10 N^2$ $100 N$ $22 N \log N + 3 N$ ⋮	develop upper bounds
<b>Big Omega</b>	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2} N^2$ $N^5$ $N^3 + 22 N \log N + 3 N$ ⋮	develop lower bounds

# Theory of algorithms: example 1

---

## Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 1-SUM = “*Is there a 0 in the array?*”

## Upper bound.

A specific algorithm.

- Ex. Brute-force algorithm for 1-SUM: Look at every array entry.
- Running time of the optimal algorithm for 1-SUM is  $O(N)$ .

## Lower bound.

Proof that no algorithm can do better.

- Ex. Have to examine all  $N$  entries (any unexamined one might be 0).
- Running time of the optimal algorithm for 1-SUM is  $\Omega(N)$ .

## Optimal algorithm.

- Lower bound equals upper bound (to within a constant factor).
- Ex. Brute-force algorithm for 1-SUM is optimal: its running time is  $\Theta(N)$ .

## Theory of algorithms: example 2

---

### Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 3-SUM.

### Upper bound. A specific algorithm.

- Ex. Brute-force algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is  $O(N^3)$ .

# Theory of algorithms: example 2

---

## Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 3-SUM.

## Upper bound. A specific algorithm.

- Ex. Improved algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is  $O(N^2 \log N)$ .

## Lower bound. Proof that no algorithm can do better.

- Ex. Have to examine all  $N$  entries to solve 3-SUM.
- Running time of the optimal algorithm for solving 3-SUM is  $\Omega(N)$ .

## Open problems.

- Optimal algorithm for 3-SUM?
- Subquadratic algorithm for 3-SUM?
- Quadratic lower bound for 3-SUM?

# Algorithm design approach

---

## Start.

- Develop an algorithm.
- Prove a lower bound.

## Gap?

- Lower the upper bound (discover a new algorithm).
- Raise the lower bound (more difficult).

## Golden Age of Algorithm Design.

- 1970s-.
- Steadily decreasing upper bounds for many important problems.
- Many known optimal algorithms.

## Caveats.

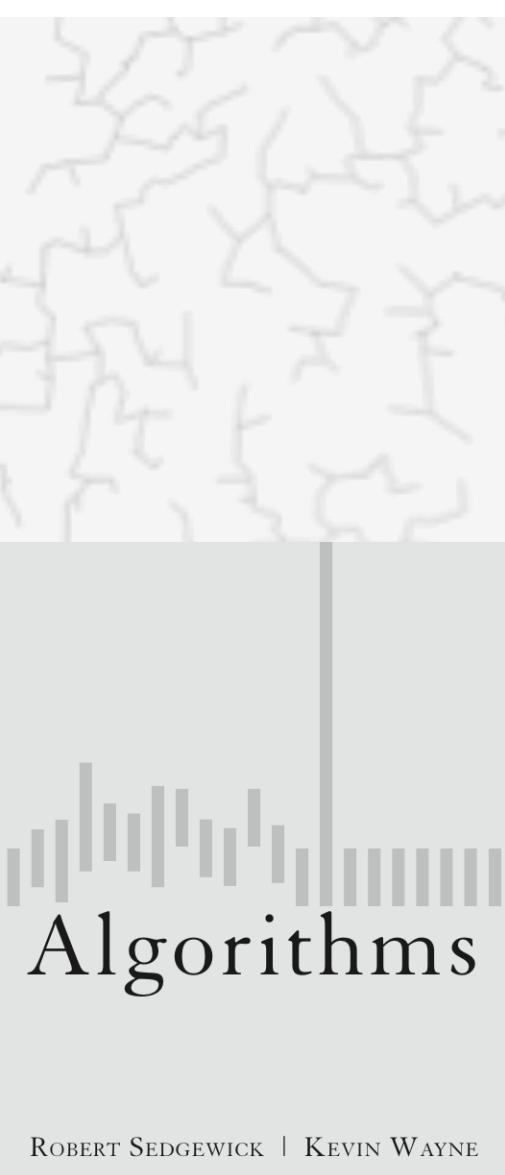
- Overly pessimistic to focus on worst case?
- Need better than “to within a constant factor” to predict performance.

# Commonly-used notations in the theory of algorithms

notation	provides	example	shorthand for	used to
Tilde	leading term	$\sim 10 N^2$	$10 N^2$ $10 N^2 + 22 N \log N$ $10 N^2 + 2 N + 37$	provide approximate model
Big Theta	asymptotic order of growth	$\Theta(N^2)$	$\frac{1}{2} N^2$ $10 N^2$ $5 N^2 + 22 N \log N + 3N$	classify algorithms
Big Oh	$\Theta(N^2)$ and smaller	$O(N^2)$	$10 N^2$ $100 N$ $22 N \log N + 3 N$	develop upper bounds
Big Omega	$\Theta(N^2)$ and larger	$\Omega(N^2)$	$\frac{1}{2} N^2$ $N^5$ $N^3 + 22 N \log N + 3 N$	develop lower bounds

Common mistake. Interpreting big-Oh as an approximate model.

This course. Focus on approximate models: use Tilde-notation



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## 1.4 ANALYSIS OF ALGORITHMS

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## Basics

---

Bit. 0 or 1.

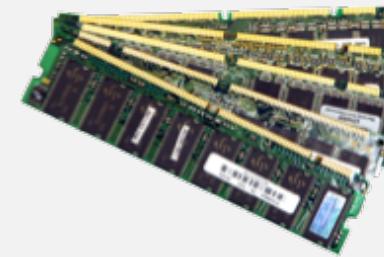
NIST  
↓

most computer scientists  
↓

Byte. 8 bits.

Megabyte (MB). 1 million or  $2^{20}$  bytes.

Gigabyte (GB). 1 billion or  $2^{30}$  bytes.



64-bit machine. We assume a 64-bit machine with 8-byte pointers.

- Can address more memory.
- Pointers use more space.



some JVMs "compress" ordinary object  
pointers to 4 bytes to avoid this cost



# Typical memory usage for primitive types and arrays

---

type	bytes
boolean	1
byte	1
char	2
int	4
float	4
long	8
double	8

primitive types

type	bytes
char[]	$2N + 24$
int[]	$4N + 24$
double[]	$8N + 24$

one-dimensional arrays

type	bytes
char[][]	$\sim 2MN$
int[][]	$\sim 4MN$
double[][]	$\sim 8MN$

two-dimensional arrays

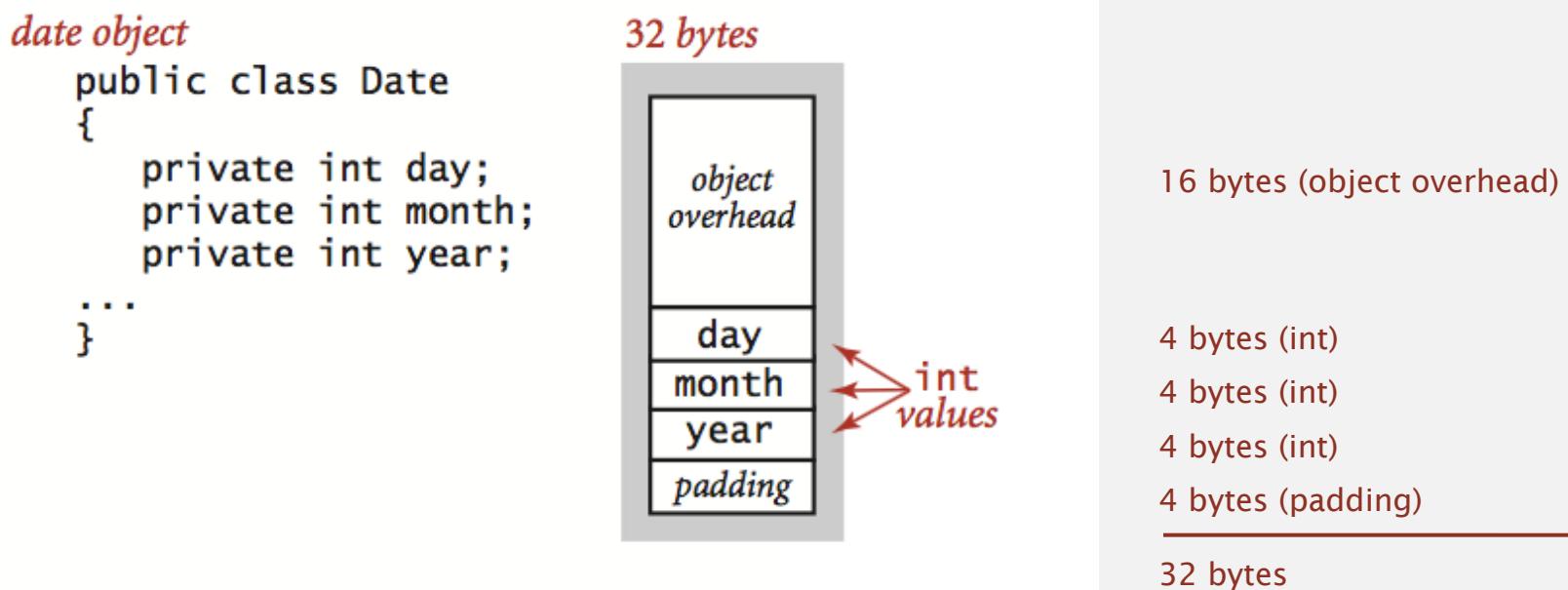
# Typical memory usage for objects in Java

Object overhead. 16 bytes.

Reference. 8 bytes.

Padding. Each object uses a multiple of 8 bytes.

Ex 1. A Date object uses 32 bytes of memory.



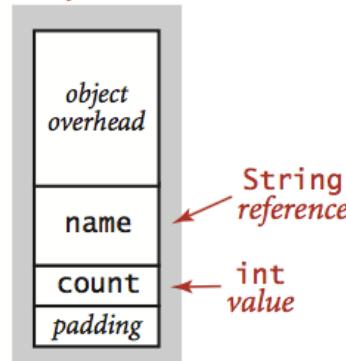
# Typical memory usage for objects in Java

---

*counter object*

```
public class Counter
{
    private String name;
    private int count;
    ...
}
```

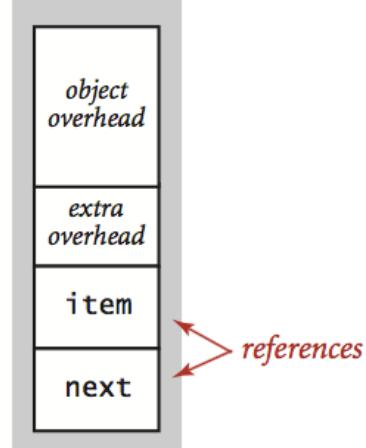
*32 bytes*



*node object (inner class)*

```
public class Node
{
    private Item item;
    private Node next;
    ...
}
```

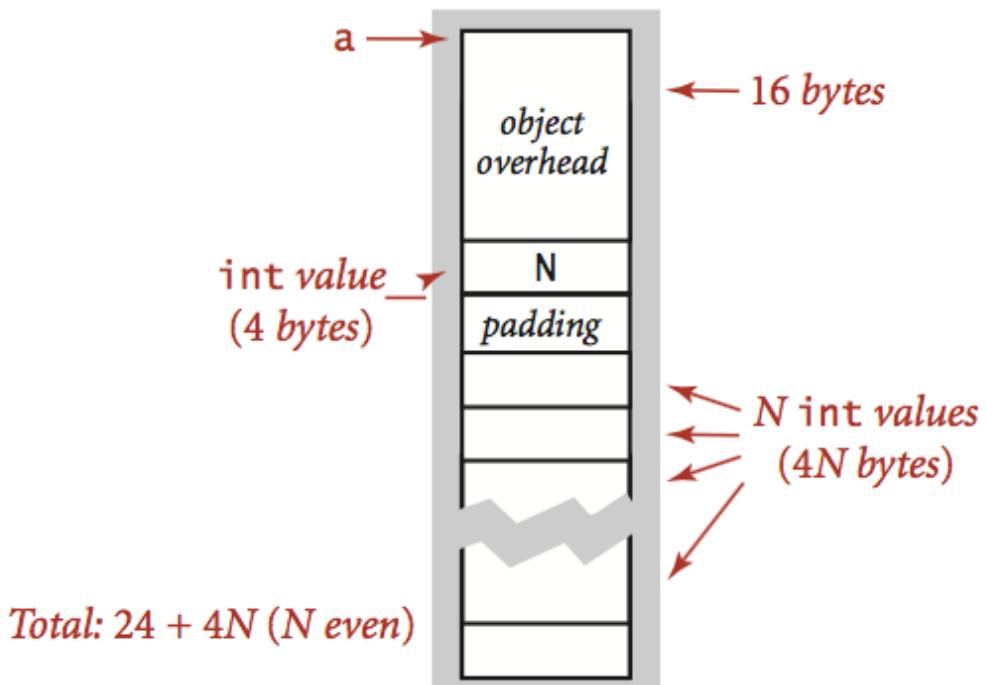
*40 bytes*



# Typical memory usage for objects in Java

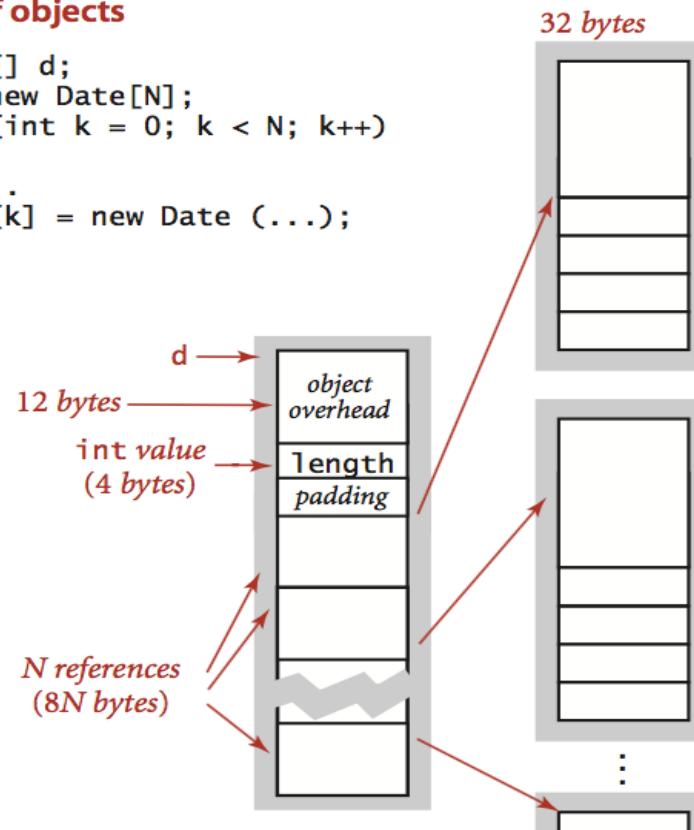
## array of int values

```
int[] a = new int[N];
```



## array of objects

```
Date[] d;  
d = new Date[N];  
for (int k = 0; k < N; k++)  
{  
    ...  
    a[k] = new Date (...);  
}
```



$$\text{Total: } 24 + 8N + N \times 32 = 24 + 40N$$

## summary

type	bytes
<code>int[]</code>	$\sim 4N$
<code>double[]</code>	$\sim 8N$
<code>Date[]</code>	$\sim 40N$
<code>double[][]</code>	$\sim 8NM$

## Typical memory usage for objects in Java

---

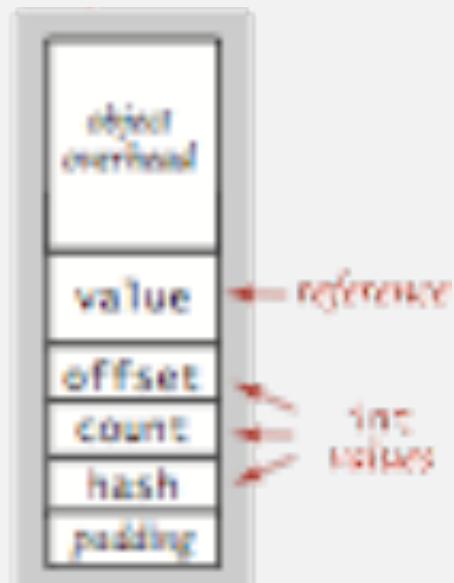
Object overhead. 16 bytes.

Reference. 8 bytes.

Padding. Each object uses a multiple of 8 bytes.

Ex 2. A virgin String of length  $N$  uses  $\sim 2N$  bytes of memory.

```
public class String
{
    private char[] value;
    private int offset;
    private int count;
    private int hash;
    ...
}
```



16 bytes (object overhead)

8 bytes (reference to array)  
2 $N$  + 24 bytes (char[] array)

4 bytes (int)

4 bytes (int)

4 bytes (int)

4 bytes (padding)

---

2 $N$  + 64 bytes

## Typical memory usage summary

---

Total memory usage for a data type value:

- Primitive type: 4 bytes for int, 8 bytes for double, ...
  - Object reference: 8 bytes.
  - Array: 24 bytes + memory for each array entry.
  - Object: 16 bytes + memory for each instance variable.
  - Padding: round up to multiple of 8 bytes.
- + 8 extra bytes per inner class object  
(for reference to enclosing class)

Shallow memory usage: Don't count referenced objects.

Deep memory usage: If array entry or instance variable is a reference, count memory (recursively) for referenced object.

# Memory profiler

Classmexer library. Measure memory usage by querying JVM.

<http://www.javamex.com/classmexer>

```
import com.javamex.classmexer.MemoryUtil;

public class Memory {
    public static void main(String[] args) {
        Date date = new Date(12, 31, 1999);
        StdOut.println(MemoryUtil.memoryUsageOf(date));
        String s = "Hello, World";
        StdOut.println(MemoryUtil.memoryUsageOf(s));           ← shallow
        StdOut.println(MemoryUtil.deepMemoryUsageOf(s));       ← deep
    }
}
```

```
% javac -cp .:classmexer.jar Memory.java
% java -cp .:classmexer.jar -javaagent:classmexer.jar Memory
32
40           ← don't count char[]
88           ← 2N + 64

```

use **-XX:-UseCompressedOops**  
on OS X to match our model

## Example

---

Q. How much memory does WeightedQuickUnionUF use as a function of  $N$ ?  
Use tilde notation to simplify your answer.

```
public class WeightedQuickUnionUF
{
    private int[] id;
    private int[] sz;
    private int count;

    public WeightedQuickUnionUF(int N)
    {
        id = new int[N];
        sz = new int[N];
        for (int i = 0; i < N; i++) id[i] = i;
        for (int i = 0; i < N; i++) sz[i] = 1;
    }
    ...
}
```

The diagram illustrates the memory layout of the `WeightedQuickUnionUF` class. A vertical grey bar represents the object's footprint. Red arrows point from the right side to specific parts of the code, indicating their memory requirements:

- A top arrow points to the class header and opening brace, labeled "16 bytes (object overhead)".
- An arrow points to the declaration of the `id` array, labeled "8 + (4N + 24) bytes each (reference + int[] array)".
- An arrow points to the declaration of the `sz` array, labeled "4 bytes (int)".
- An arrow points to the declaration of the `count` variable, labeled "4 bytes (padding)".
- A final arrow at the bottom points to the total memory usage, labeled "8N + 88 bytes".

A.  $8N + 88 \sim 8N$  bytes.



# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 1.4 ANALYSIS OF ALGORITHMS

---

- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *theory of algorithms*
- ***memory***

## Turning the crank: summary

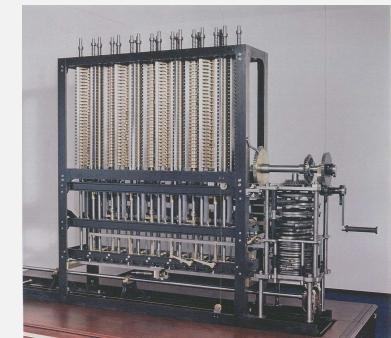
---

### Empirical analysis.

- Execute program to perform experiments.
- Assume power law and formulate a hypothesis for running time.
- Model enables us to **make predictions**.

### Mathematical analysis.

- Analyze algorithm to count frequency of operations.
- Use tilde notation to simplify analysis.
- Model enables us to **explain behavior**.



### Scientific method.

- Mathematical model is independent of a particular system; applies to machines not yet built.
- Empirical analysis is necessary to validate mathematical models and to make predictions.



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