Üstel Dağılım

Reliability

$$R(t) = Pr(X > t) = \int_t^\infty \frac{1}{\theta} e^{-x/\theta} dx = [-e^{-x/\theta}]_t^\infty = e^{-t/\theta}$$

Arıza Hızı (Hazard)

$$z(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\theta} e^{-t/\theta}}{e^{-t/\theta}} = \frac{1}{\theta} \quad \text{(sabit)}$$

Ortalama Arızaya Kadar Geçen Süre (MTTF)

$$E[X] = \int_0^\infty x \frac{1}{\theta} e^{-x/\theta} dx$$

Parçalı integral ile:

$u = x \; dv = \frac{1}{\theta} e^{-x/\theta} dx \implies v = -e^{-x/\theta}$

$$E[X] = [-xe^{-x/\theta}]_0^\infty + \int_0^\infty e^{-x/\theta} dx = 0 + \theta = \theta$$

İkinci Moment ve Varyans

Değişken değişimi: $y = x/\theta$, $x = \theta y$, $dx = \theta dy$

$$E[X^2] = \int_0^\infty x^2 \frac{1}{\theta} e^{-x/\theta} dx = \theta^2 \int_0^\infty y^2 e^{-y} dy = \theta^2 \cdot 2! = 2\theta^2$$

$$Var(X) = E[X^2] - (E[X])^2 = 2\theta^2 - \theta^2 = \theta^2$$

Mod

$$f'(x) = -\frac{1}{\theta^2} e^{-x/\theta} < 0 \quad (x > 0) \implies f \text{ tekdüze azalır} \implies t_{mod} = 0$$

Medyan

$$\int_0^{t_{med}} \frac{1}{\theta} e^{-x/\theta} dx = [-e^{-x/\theta}]0^{tmed} = 1 - e^{-t_{med}/\theta} = \frac{1}{2}$$

$$e^{-t_{med}/\theta} = \frac{1}{2} \implies t_{med} = \theta \ln 2$$

Ortalama Kalan Ömür (MRL)

$$MRL(x) = \frac{1}{R(x)} \int_x^\infty R(t)dt = \frac{1}{e^{-x/\theta}} \int_x^\infty e^{-t/\theta}dt$$

$$= e^{x/\theta} \left[-\theta e^{-t/\theta}\right]_x^\infty = e^{x/\theta}(0 + \theta e^{-x/\theta}) = \theta$$

Özet

$R(t) = e^{-t/\theta}, \quad z(t) = \frac{1}{\theta}, \quad MTTF = \theta, \quad Var(X) = \theta^2, \quad t_{mod} = 0, \quad t_{med} = \theta \ln 2, \quad MRL(x) = \theta$

Gamma Dağılımı (şekil α, ölçek β)

Olasılık yoğunluğu (pdf):

$$f(x) = \frac{1}{\Gamma(\alpha)\,\beta^\alpha}\, x^{\alpha-1}e^{-x/\beta}, \qquad x \geq 0,\ \alpha > 0,\ \beta > 0.$$

Sağkalım (Reliability) $R(t)$

Tanım:

$$R(t) = P(X > t) = \int_t^\infty f(x)\,dx.$$

Pdf'yi yerine koyup değişken dönüşümü $y = x/\beta$ ($x = \beta y$, $dx = \beta\,dy$) ile:

$$R(t) = \frac{1}{\Gamma(\alpha)\,\beta^\alpha} \int_t^\infty x^{\alpha-1}e^{-x/\beta}\,dx = \frac{1}{\Gamma(\alpha)} \int_{t/\beta}^\infty y^{\alpha-1}e^{-y}\,dy = \frac{\Gamma(\alpha,\, t/\beta)}{\Gamma(\alpha)}.$$

Burada $\Gamma(\alpha, x) = \int_x^\infty y^{\alpha-1}e^{-y}\,dy$ **üst (upper) eksik gamma** fonksiyonudur.

Arıza hızı (Hazard) $z(t)$

$$z(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\Gamma(\alpha)\,\beta^\alpha}\,t^{\alpha-1}e^{-t/\beta}}{\frac{\Gamma(\alpha,\,t/\beta)}{\Gamma(\alpha)}} = \frac{t^{\alpha-1}e^{-t/\beta}}{\beta^\alpha\,\Gamma(\alpha,\,t/\beta)}.$$

MTTF $= \mathbb{E}[X]$

$$\mathbb{E}[X] = \int_0^\infty x\,f(x)\,dx = \frac{1}{\Gamma(\alpha)\,\beta^\alpha} \int_0^\infty x^\alpha e^{-x/\beta}\,dx$$

$y = x/\beta \Rightarrow x = \beta y$, $dx = \beta dy$:

$$\mathbb{E}[X] = \frac{1}{\Gamma(\alpha)\,\beta^\alpha} \int_0^\infty (\beta y)^\alpha e^{-y}\,\beta\,dy = \frac{\beta^{\alpha+1}}{\Gamma(\alpha)\,\beta^\alpha} \int_0^\infty y^\alpha e^{-y}\,dy = \beta\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\,\beta.$$

İkinci moment ve Varyans

$$\mathbb{E}[X^2] = \int_0^\infty x^2 f(x)\,dx = \frac{1}{\Gamma(\alpha)\,\beta^\alpha}\int_0^\infty x^{\alpha+1}e^{-x/\beta}\,dx \overset{y=x/\beta}{=} \frac{1}{\Gamma(\alpha)}\beta^2\int_0^\infty y^{\alpha+1}e^{-y}\,dy = \beta^2\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \beta^2\alpha(\alpha{+}1)$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \beta^2\alpha(\alpha+1) - (\alpha\beta)^2 = \alpha\,\beta^2.$$

Mod $t_{\mathbf{mod}}$

$$\frac{d}{dx}\log f(x) = \frac{d}{dx}[(\alpha-1)\ln x - x/\beta + C] = \frac{\alpha-1}{x} - \frac{1}{\beta}.$$

Sıfıra eşitleyince $x = (\alpha-1)\beta$. Dolayısıyla

$$t_{\mathrm{mod}} = \begin{cases} (\alpha-1)\beta, & \alpha > 1, \\ 0, & \alpha \le 1 \text{ (tepe 0'da).} \end{cases}$$

Medyan $t_{\mathbf{med}}$

Medyan, $F(t_{\mathrm{med}}) = \frac{1}{2}$ koşulundan gelir:

$$F(t) = \int_0^t f(x)\,dx = \frac{1}{\Gamma(\alpha)}\int_0^{t/\beta} y^{\alpha-1}e^{-y}\,dy = \frac{\gamma(\alpha,\,t/\beta)}{\Gamma(\alpha)}.$$

Burada $\gamma(\alpha,x) = \int_0^x y^{\alpha-1}e^{-y}\,dy$ **alt (lower) eksik gamma**. Kapalı form **genelde yoktur**; t_{med} şu denklemin çözümüdür:

$$\frac{\gamma(\alpha,\,t_{\mathrm{med}}/\beta)}{\Gamma(\alpha)} = \tfrac{1}{2} \quad\iff\quad t_{\mathrm{med}} = \beta\cdot\gamma^{-1}\Big(\alpha,\ \tfrac{1}{2}\,\Gamma(\alpha)\Big).$$

(Özel durum: $\alpha = 1$ üstel; $t_{\mathrm{med}} = \beta\ln 2$.)

Ortalama kalan ömür (MRL) MRL(x)

Tanım:

$$\mathrm{MRL}(x) = \frac{\int_x^\infty R(t)\,dt}{R(x)} = \mathbb{E}[X - x \mid X > x].$$

Önce payı hesaplayalım:

$$\int_x^\infty R(t)\,dt = \int_x^\infty \Big(\int_t^\infty f(u)\,du\Big)dt = \int_x^\infty (u{-}x)f(u)\,du = \underbrace{\int_x^\infty uf(u)\,du}_{A} - x\,R(x).$$

A için:

$$A = \frac{1}{\Gamma(\alpha)\,\beta^\alpha}\int_x^\infty u^\alpha e^{-u/\beta}\,du = \frac{\beta}{\Gamma(\alpha)}\int_{x/\beta}^\infty y^\alpha e^{-y}\,dy = \frac{\beta\,\Gamma(\alpha+1,\,x/\beta)}{\Gamma(\alpha)}.$$

Dolayısıyla $R(x) = \dfrac{\Gamma(\alpha,\, x/\beta)}{\Gamma(\alpha)}$ ile birlikte

$$\mathrm{MRL}(x) = \frac{A - xR(x)}{R(x)} = \beta\,\frac{\Gamma(\alpha+1,\, x/\beta)}{\Gamma(\alpha,\, x/\beta)} - x.$$

Weibull Dağılımı (şekil k, ölçek θ) — $z(t)$ notasyonu ile

Olasılık yoğunluğu (pdf):

$$f(x) = \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1} \exp\!\left[-\left(\frac{x}{\theta}\right)^{k}\right], \qquad x \ge 0,\ k > 0,\ \theta > 0.$$

Sağkalım (Reliability) $R(t)$:

$$R(t) = Pr(X > t) = \int_t^\infty f(x)\,dx = \int_t^\infty \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1} \exp\!\left[-\left(\frac{x}{\theta}\right)^{k}\right] dx.$$

Değişken dönüşümü $y = (x/\theta)^k \Rightarrow dy = (k/\theta)(x/\theta)^{k-1}dx$ ile

$$R(t) = \int_{(t/\theta)^k}^\infty e^{-y}\,dy = \Big[-e^{-y}\Big]_{(t/\theta)^k}^\infty = e^{-(t/\theta)^k}.$$

Arıza hızı (Hazard) $z(t)$:

$$z(t) = \frac{f(t)}{R(t)} = \frac{\frac{k}{\theta}\left(\frac{t}{\theta}\right)^{k-1} e^{-(t/\theta)^k}}{e^{-(t/\theta)^k}} = \frac{k}{\theta}\left(\frac{t}{\theta}\right)^{k-1}.$$

MTTF ($= \mathbb{E}[X]$) — integral ile

$$\mathbb{E}[X] = \int_0^\infty R(t)\,dt = \int_0^\infty \exp\!\left[-\left(\frac{t}{\theta}\right)^{k}\right] dt.$$

$y = (t/\theta)^k \Rightarrow dt = \frac{\theta}{k} y^{\frac{1}{k}-1} dy$:

$$\mathbb{E}[X] = \frac{\theta}{k}\int_0^\infty y^{\frac{1}{k}-1} e^{-y}\,dy = \frac{\theta}{k}\,\Gamma\!\left(\frac{1}{k}\right) = \theta\,\Gamma\!\left(1 + \frac{1}{k}\right),$$

çünkü $\Gamma(1+u) = u\,\Gamma(u)$.

İkinci moment ve Varyans

$$\mathbb{E}[X^2] = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty x^2 \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1} e^{-(x/\theta)^k}\,dx.$$

$y = (x/\theta)^k \Rightarrow x = \theta y^{1/k},\ dx = \frac{\theta}{k} y^{\frac{1}{k}-1} dy$:

$$\mathbb{E}[X^2] = \int_0^\infty (\theta y^{1/k})^2 \frac{k}{\theta} y^{\frac{k-1}{k}} e^{-y}\, \frac{\theta}{k} y^{\frac{1}{k}-1} dy = \theta^2 \int_0^\infty y^{\frac{2}{k}} e^{-y}\,dy = \theta^2\,\Gamma\Big(1+\frac{2}{k}\Big).$$

Dolayısıyla

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \theta^2\Big[\Gamma\Big(1+\frac{2}{k}\Big) - \Gamma\Big(1+\frac{1}{k}\Big)^2\Big].$$

Mod $t_{\mathbf{mod}}$

$$\frac{d}{dt}\log f(t) = \frac{d}{dt}\Big[\log k - \log\theta + (k-1)\log t - (k-1)\log\theta - \Big(\frac{t}{\theta}\Big)^k\Big] = \frac{k-1}{t} - \frac{k}{\theta}\left(\frac{t}{\theta}\right)^{k-1}.$$

Sıfıra eşitleyince $(k-1)/t = (k/\theta)(t/\theta)^{k-1} \Rightarrow (t/\theta)^k = (k-1)/k$.

$$t_{\text{mod}} = \begin{cases} \theta\left(\dfrac{k-1}{k}\right)^{1/k}, & k > 1, \\ 0, & k \le 1 \text{ (tepe 0'da)}. \end{cases}$$

Medyan $t_{\mathbf{med}}$ — integralden

$$\int_0^{t_{\text{med}}} f(x)\,dx = \frac{1}{2} \quad\Longleftrightarrow\quad 1 - e^{-(t_{\text{med}}/\theta)^k} = \frac{1}{2} \ \Rightarrow\ t_{\text{med}} = \theta(\ln 2)^{1/k}.$$

Ortalama kalan ömür (MRL) MRL(x)

Tanım (integralle):

$$\mathrm{MRL}(x) = \frac{\int_x^\infty R(t)\,dt}{R(x)} = \frac{\int_x^\infty e^{-(t/\theta)^k}\,dt}{e^{-(x/\theta)^k}}.$$

$y = (t/\theta)^k \Rightarrow dt = \frac{\theta}{k} y^{\frac{1}{k}-1} dy$ ve alt sınır $a = (x/\theta)^k$:

$$\int_x^\infty e^{-(t/\theta)^k} dt = \frac{\theta}{k}\int_a^\infty y^{\frac{1}{k}-1} e^{-y}\,dy = \frac{\theta}{k}\,\Gamma\Big(\frac{1}{k},\,a\Big),$$

burada $\Gamma(s,a) = \int_a^\infty y^{s-1} e^{-y} dy$ **üst eksik gamma**. Dolayısıyla

$$\mathrm{MRL}(x) = \frac{\frac{\theta}{k}\Gamma(1/k,\,a)}{e^{-a}} = \frac{\theta}{k} e^a \Gamma\Big(\frac{1}{k},\,a\Big) = \theta\, e^a \Gamma\Big(1+\frac{1}{k},\,a\Big) - x,$$

son eşitlik $\Gamma(s+1,a) = s\,\Gamma(s,a) + a^s e^{-a}$ özdeşliğinden gelir.

Parametrizasyon notu

- Weibull'un iki parametresi vardır: şekil k ve ölçek θ (bazı kaynaklarda λ veya η).
- Üstel dağılımda olduğu gibi "ölçek θ" yerine hız (rate) $r = 1/\theta$ yazılabilir, fakat Weibull'da bu daha az kullanılır. Bu durumda

$$f(x) = k\,r\,(rx)^{k-1}\,e^{-(rx)^k}, \quad R(t) = e^{-(rt)^k}, \quad z(t) = k\,r\,(rt)^{k-1}.$$

- Özel durum: $k = 1$ olursa Weibull Exp(θ)'ye iner; $z(t) = 1/\theta$ sabit olur.

Özet

- $R(t) = e^{-(t/\theta)^k}$, $z(t) = \dfrac{k}{\theta}\left(\dfrac{t}{\theta}\right)^{k-1}$.
- $\mathrm{MTTF} = \theta\,\Gamma\left(1 + \dfrac{1}{k}\right)$, $\mathrm{Var}(X) = \theta^2\left[\Gamma\left(1 + \dfrac{2}{k}\right) - \Gamma\left(1 + \dfrac{1}{k}\right)^2\right]$.
- $t_{\mathrm{mod}} = \theta\left(\dfrac{k-1}{k}\right)^{1/k}$ ($k > 1$; yoksa 0).
- $t_{\mathrm{med}} = \theta(\ln 2)^{1/k}$.
- $\mathrm{MRL}(x) = \dfrac{\theta}{k}e^{(x/\theta)^k}\Gamma\left(\dfrac{1}{k},\,(x/\theta)^k\right) \;=\; \theta\,e^{(x/\theta)^k}\Gamma\left(1 + \dfrac{1}{k},\,(x/\theta)^k\right) - x$.
