Detailed Mathematical Derivations

Given - Hazard (failure rate): $z(t) = h(t) = \dfrac{0.4}{0.2t+1}$ - Cumulative hazard: $Z(t) = H(t) = \int_0^t z(u)\,du$

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1) Cumulative hazard Z(t)

Substitution: $w = 0.2u + 1,\ dw = 0.2\,du,\ du = 5\,dw$, limits $u : 0 \to t \Rightarrow w : 1 \to 0.2t + 1$:

$$Z(t) = \int_0^t \frac{0.4}{0.2u+1}\,du = \int_1^{0.2t+1} \frac{0.4}{w} \cdot 5\,dw = 2\int_1^{0.2t+1} \frac{1}{w}\,dw = 2\ln(0.2t+1).$$

Result:

$$\boxed{Z(t) = 2\ln(0.2t+1)}$$

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2) Reliability R(t)

Using $R(t) = \exp(-Z(t))$:

$$R(t) = \exp\bigl(-2\ln(0.2t+1)\bigr) = (0.2t+1)^{-2} = \frac{1}{(0.2t+1)^2}.$$

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3) PDF and CDF

$$f(t) = z(t)\,R(t) = \frac{0.4}{0.2t+1} \cdot \frac{1}{(0.2t+1)^2} = \frac{0.4}{(0.2t+1)^3}, \qquad F(t) = 1 - R(t) = 1 - \frac{1}{(0.2t+1)^2}.$$

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4) Mean Time to Failure (MTTF)

Substitution $u = 0.2t + 1,\ du = 0.2\,dt,\ dt = 5\,du$, limits $t : 0 \to \infty \Rightarrow u : 1 \to \infty$:

$$\mathrm{MTTF} = \int_0^\infty R(t)\,dt = \int_0^\infty \frac{1}{(0.2t+1)^2}\,dt = 5\int_1^\infty u^{-2}\,du = 5\left[-\frac{1}{u}\right]_1^\infty = 5.$$

Result:

$$\boxed{\mathrm{MTTF} = 5}$$

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5) Second moment and variance

Direct form with $u = 0.2t + 1,\ t = 5(u - 1),\ dt = 5\,du$:

$$E[T^2] = \int_0^\infty t^2 f(t)\,dt = \int_0^\infty t^2 \frac{0.4}{(0.2t+1)^3}\,dt = 50\int_1^\infty \frac{(u-1)^2}{u^3}\,du = 50\int_1^\infty \left(\frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3}\right) du.$$

Since $\int_1^\infty \frac{1}{u}\,du = \infty$, we obtain

$$\boxed{E[T^2] = \infty \ \Rightarrow\ \mathrm{Var}(T) = \infty,\ \sigma = \infty}$$

(Equivalent via parts: $E[T^2] = -\int_0^\infty t^2\,dR(t) = [-t^2R(t)]_0^\infty + 2\int_0^\infty tR(t)\,dt$, and $t^2R(t) \not\to 0$.)

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6) Median and MRLT

Median from $R(t_{0.5}) = 0.5$:

$$\frac{1}{(0.2t_{0.5}+1)^2} = 0.5 \Rightarrow (0.2t_{0.5}+1)^2 = 2 \Rightarrow 0.2t_{0.5}+1 = \sqrt{2} \Rightarrow t_{0.5} = 5(\sqrt{2}-1) \approx 2.0711.$$

Mean residual life:

$$m(t) = \frac{\int_t^\infty R(u)\,du}{R(t)} = \frac{\frac{5}{0.2t+1}}{\frac{1}{(0.2t+1)^2}} = 5(0.2t+1) = t + 5.$$

Mode

PDF: $f(t) = 0.4(0.2t+1)^{-3},\ t \ge 0.$

Derivative:

$$f'(t) = 0.4(-3)(0.2t+1)^{-4} \cdot 0.2 = -0.24(0.2t+1)^{-4} < 0 \quad (t \ge 0).$$

Hence $f(t)$ is strictly decreasing on $[0,\infty)$, maximum at boundary:

$$\boxed{\text{mode} = 0, \quad f(0) = 0.4.}$$

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7) Hazard type

$$z(t) = \frac{0.4}{0.2t+1} = 0.4(0.2t+1)^{-1}, \quad z'(t) = 0.4(-1)(0.2t+1)^{-2} \cdot 0.2 = -\frac{0.08}{(0.2t+1)^2} < 0,$$

so decreasing hazard rate (DHR).

Summary

- $Z(t) = 2\ln(0.2t + 1)$
- $R(t) = 1/(0.2t + 1)^2,\ f(t) = 0.4/(0.2t + 1)^3,\ F(t) = 1 - 1/(0.2t + 1)^2$
- $\mathrm{MTTF} = 5$
- $E[T^2] = \infty \Rightarrow \mathrm{Var}(T) = \infty,\ \sigma = \infty,\ \mathrm{CV} = \infty$
- $t_{0.5} = 5(\sqrt{2} - 1) \approx 2.0711$, $m(t) = t + 5$, mode $= 0$, DHR