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Answer 1

a)

Assume that $x \in (A \cap B)$, then it means $x \in A$, $x \in B$
$(x \in A \rightarrow x \in (A \cup \overline{B})) \quad \wedge \quad (x \in B \rightarrow x \in (\overline{A} \cup B))$
$(x \in (A \cup \overline{B}) \wedge x \in (\overline{A} \cup B)) \rightarrow x \in ((A \cup \overline{B}) \cap (\overline{A} \cup B))$

This means that if I take a x from the set $A \cap B$, this x is also going to be in $(A \cup \overline{B}) \cap (\overline{A} \cup B)$. This fact proves that $A \cap B$ is a subset of $(A \cup \overline{B}) \cap (\overline{A} \cup B)$.

b)

Assume that $x \in (\overline{A} \cap \overline{B})$, then it means $x \in \overline{A}$, $x \in \overline{B}$
$(x \in \overline{B} \rightarrow x \in (A \cup \overline{B})) \quad \wedge \quad (x \in \overline{A} \rightarrow x \in (\overline{A} \cup B))$
$(x \in (A \cup \overline{B}) \wedge x \in (\overline{A} \cup B)) \rightarrow x \in ((A \cup \overline{B}) \cap (\overline{A} \cup B))$

This means that if I take a x from the set $\overline{A} \cap \overline{B}$, this x is also going to be in $(A \cup \overline{B}) \cap (\overline{A} \cup B)$. This fact proves that $\overline{A} \cap \overline{B}$ is a subset of $(A \cup \overline{B}) \cap (\overline{A} \cup B)$.

Answer 2

$f : X \rightarrow YxZ$, $A \subseteq Y$, $B \subseteq Y$, $C \subseteq Z$

$i)$ $A \ x \ C \subseteq Y \ x \ Z \qquad since \ A \subseteq Y, C \subseteq Z$
$f^{-1}(AxC) \subseteq X$
$B \ x \ C \subseteq Y \ x \ Z \qquad since \ B \subseteq Y, C \subseteq Z$
$f^{-1}(BxC) \subseteq X$
$f^{-1}(AxC) \cap f^{-1}(BxC) \subseteq X$

$ii)$ $(A \subseteq Y) \wedge (B \subseteq Y) \rightarrow (A \cap B) \subseteq Y, \ \ (A \cap B)xC \subseteq YxZ \qquad since \ C \subseteq Z$
$f^{-1}((A \cap B)xC) \subseteq X$

$iii)$ Showing that each side is a subset of the other side is enough, to prove the given equality.

$Case\ 1:$

$$f^{-1}((A\cap B)xC) \subseteq f^{-1}(AxC)\cap f^{-1}(BxC)$$
$$x \in f^{-1}((A\cap B)xC) \subseteq X$$

$$\begin{aligned} Assume\ that\ f(x) = y \rightarrow\ & y \in (A\cap B)\ x\ C \\ & y \in (\ (AxC)\cap(BxC)\ ) \\ & (y \in (AxC))\ \wedge\ (y \in (BxC)) \\ & (\ f^{-1}(y) \in f^{-1}(AxC)\ )\ \wedge\ (\ f^{-1}(y) \in f^{-1}(BxC)\ ) \\ & (\ x \in f^{-1}(AxC)\ )\ \wedge\ (\ x \in f^{-1}(BxC)\ ) \\ & x \in (\ f^{-1}(AxC)\ \cap\ f^{-1}(BxC)\ ) \end{aligned} \tag{1}$$

$Case\ 2:$

$$f^{-1}(AxC)\cap f^{-1}(BxC) \subseteq f^{-1}((A\cap B)xC)$$
$$x \in (\ f^{-1}(AxC)\cap f^{-1}(BxC)\ ) \subseteq X$$
$$(x \in f^{-1}(AxC)) \wedge (x \in f^{-1}(BxC))$$

$$\begin{aligned} Assume\ that\ f(x) = y \rightarrow\ & y \in A\ x\ C \\ & y \in B\ x\ C \\ & y \in (\ (AxC)\cap(BxC)\ ) \\ & y \in\ (A\cap B)\ x\ C \\ & f^{-1}(y) \in\ f^{-1}((A\cap B)xC) \\ & x \in\ f^{-1}((A\cap B)xC) \end{aligned} \tag{2}$$

$These\ two\ cases\ prove\ the\ given\ equality\ f^{-1}((A\cap B)xC) = f^{-1}(AxC)\cap f^{-1}(BxC).$

Answer 3

a)

Since $x^2 + 5$ cannot be smaller than 0 this function's domain is R. However, for $x = 2\ and\ x = -2\ f(2) = f(-2) = ln(9)$ this function is not one-to-one. $f(x) = ln(x^2+5)$ has its lowest value at $x = 0$ and it is $ln(5)$ so it means that this function is not onto. This function is neither one-to-one, nor it is not onto.

b)

Since this function is in the form of $e^{g(x)}$, it cannot be negative. So this function is not onto. On the other hand, power of x is an odd number, for every distinct x x^7 will be different, so ase$^{x^7}$. This means that $f(x) = e^{e^{x^7}}$ is one-to-one. This function is one-to-one but it is not onto.

Answer 4

a)

Let $A = \{a_1,\ a_2,\ a_3,\ ...,\ a_i,\ ...\}$ and $B = \{b_1,\ b_2,\ b_3,\ ...,\ b_i,\ ...\}$.

$$\begin{aligned} A\ x\ B\ = \{&(a_1,\ b_1),\ (a_1,\ b_2),\ (a_1,\ b_3),\ ...,\ (a_1,\ b_i),\ ... \\ &(a_2,\ b_1),\ (a_2,\ b_2),\ (a_2,\ b_3),\ ...,\ (a_2,\ b_i),\ ... \\ &(a_3,\ b_1),\ (a_3,\ b_2),\ (a_3,\ b_3),\ ...,\ (a_3,\ b_i),\ ... \\ &. \\ &. \\ &(a_i,\ b_1),\ (a_i,\ b_2),\ (a_i,\ b_3),\ ...,\ (a_i,\ b_i),\ ... \\ &. \\ &. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \} \end{aligned} \tag{3}$$

Using the "Zigzag Method", we can find a one-to-one correspondence for $A\ x\ B$ as following,

$A\,x\,B\ = \{(a_1, b_1),\ (a_1, b_2),\ (a_2, b_1),\ (a_3, b_1),\ (a_2, b_2),\ (a_1, b_3),\ (a_1, b_4),\ (a_2, b_3),\ (a_3, b_2),\ (a_4, b_1),\ ...\}$

b)

Using the Part-C of this question. Assume that B is countable, then we can easily say that A is countable too since subset of of a countable set is also countable (Part-C). However, this contradicts with the given premise that A is uncountable so B is uncountable.

c)

Consider $B = \{b_1,\ b_2,\ b_3,\ b_4,\ ...\}$ since A is a subset of B we can easily say that $B = A \cup \{b'_1,\ b'_2,\ b'_3,\ ...\}$ since left side of equality is countable and $\{b'_1,\ b'_2,\ b'_3,\ ...\}$ part is countable A must be countable as well so that we can find a one-to-one correspondence. So A is countable.

For $B = \{b'_1,\ a_1,\ b'_2,\ a_2,\ b'_3,\ a_3,\ ...\}$ (where $A = \{a_1,\ a_2,\ a_3,\ ...\}$), we can right the correspondence as $b_1 = b'_1$, $b_2 = a_1$, $b_3 = b'_2$, $b_4 = a_2$, $b_5 = b'_3$, $b_6 = a_3$.

Answer 5

If $f_1(x)$ is $O(f_2(x))$, then we can say that

$$|f_1(x)| \leq c.|f_2(x)|$$

a)

$ln|f_1(x)| \leq ln|c.f_2(x)|$

$ln|f_1(x)| \leq ln|c|\ +\ ln|f_2(x)| \leq ln|c|\ .\ ln|f_2(x)| \qquad$ (for which $\frac{ln|f_2(x)|}{ln|f_2(x)|-1} \leq ln|c|$)

$ln|f_1(x)| \leq c'\ .\ ln|f_2(x)|$

So it can be written as $ln|f_1(x)|$ is $O(ln|f_2(x)|)$

b)

$3^{f_1(x)} \leq 3^{c.f_2(x)}$
$3^{f_1(x)} \leq 3^{c.f_2(x)} \leq 3^{f_2(x)} \,.\, 3^{c.f_2(x)}$
$3^{c.f_1(x)} \leq c' \,.\, 3^{f_2(x)}$ (for which $3^{c.f_2(x)} \leq c'$)
So it can be written as $3^{f_1(x)}$ is $O(3^{f_2(x)})$

Answer 6

a)

$i) \quad x \;=\; y$

$$\begin{aligned} (3^x-1)mod(3^x-1) \;&=\; 3^{(x\ mod\ x)}-1 \\ 0 = 1 - 1 \end{aligned} \tag{4}$$

$ii) \quad x \;<\; y$

$$\begin{aligned} (3^x-1)mod(3^y-1) \;&=\; 3^x-1 \\ &(3^y-1) \mid (3^x-1)-(3^x-1) \\ &(3^y-1) \mid 0 \end{aligned} \tag{5}$$

$iii) \quad y \;<\; x$
$x\ mod\ y \equiv\ m\ (m \in Z^+,\ m < y),\ yk = m - x\ (k \in Z^+)$

$$\begin{aligned} &(3^y-1) \mid 3^m - 3^x \\ &(3^y-1) \mid 3^x(3^{m-x}-1) \\ &(3^y-1) \mid 3^x(3^{yk}-1) \end{aligned} \tag{6}$$

This is true since $(3^y-1) \mid (3^{yk}-1)$ (for every $k \in Z^+$)

In conclusion, for every possible x and y combination where $x, y \in Z^+$ given statement is true.

b)

$gcd(a, 0) = 0$
$gcd(a, b) = gcd(b, a\ mod\ b)$
$gcd(123, 277) = gcd(277, 123) = gcd(123, 31) = gcd(31, 30) = gcd(30, 1) = gcd(1, 0) = 1$