CMPE 322/327 - Theory of Computation

Week 11: Turing Machines & Decision Problems

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May 9-13, 2022

Outline

1. A Quick Recap
2. Turing Machines
3. Decision Problems
4. Encoding
5. Diagonalization

Definitions

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  7. $F \subseteq Q$: final states

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- configuration: element of $Q \times \Sigma^* \times \Gamma^*$ (current state, remaining input, stack content)
- start configuration on input x: $(s, x, \perp)$
- next configuration relation is binary relation $\xrightarrow[M]{1}$ defined as: $(p, ay, A\beta) \xrightarrow[M]{1} (q, y, \gamma\beta)$ for all $((p, a, A), (q, \gamma)) \in \delta$ with $a \in \Sigma \cup \{\varepsilon\}$ and $y \in \Sigma^*$, $\beta \in \Gamma^*$

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stack:	$\perp$

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stack:	$\perp$		$\perp$		$\perp$		$\perp$		$\perp$		$\perp$		$\perp$	
			[		[		[		[		[		[	
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stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
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state:	1	1	1	1	1	1	1	1	1	2
stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	
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state:	1	1	1	1	1	1	1	1	1	2
stack:	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	
		[	[	[	[	[	[	[		
			[		[	[	[			
						[				

input:	[	[	]	]	]
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stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	
		[	[	[	[	[	[	[		
			[		[	[	[			
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state:	1	1	1	1	1	1	1	1	1	2		1	1	1	1	2
stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$			$\perp$	$\perp$	$\perp$	$\perp$	ε
		[	[	[	[	[	[	[					[	[	[	
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input:	[	[	]	[	[	]	]	]	ε	
state:	1	1	1	1	1	1	1	1	1	2
stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	
		[	[	[	[	[	[	[		
			[		[	[	[			
						[				

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		[	[	[	
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Theorem

CFGs and NPDAs are equivalent:

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4. $A = L_e(M) = L_f(M)$ for some NPDA M

Definition

A deterministic pushdown automaton (DPDA) is an octuple $M = (Q, \Sigma, \Gamma, \delta, \perp, \dashv, s, F)$

1. $\dashv$ is a special symbol not in Σ, called the right endmarker
2. for any $p \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $A \in \Gamma$, the set $\delta \subseteq (Q \times (\Sigma \cup \{\dashv\} \cup \{\varepsilon\}) \times \Gamma) \times (Q \times \Gamma^*)$ contains
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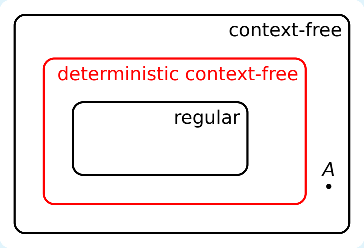

$A = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$

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context-free sets are effectively closed under

- union
- concatenation
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- homomorphic image
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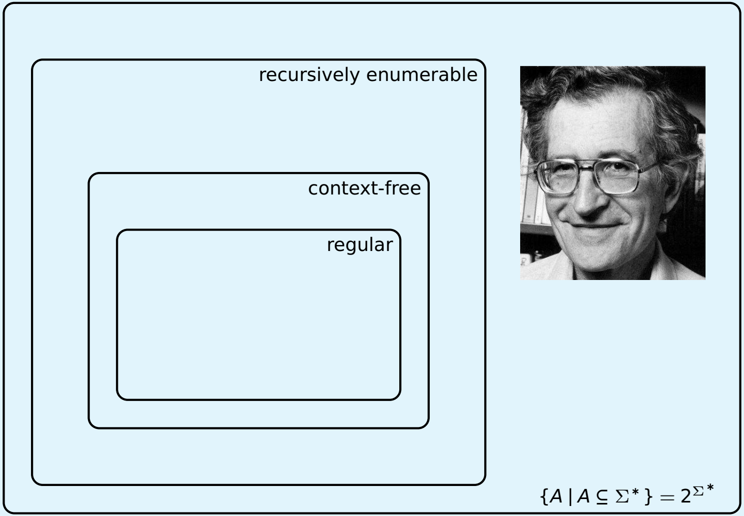
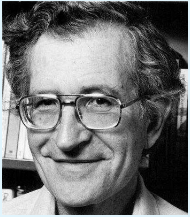
Outline


recursively enumerable
context-free
regular
$\{A \mid A \subseteq \Sigma^*\} = 2^{\Sigma^*}$

recursively enumerable

Turing machines
unrestricted grammars



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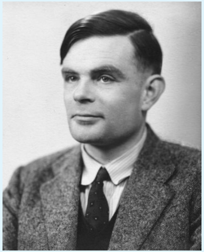

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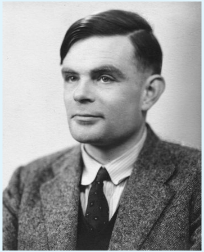

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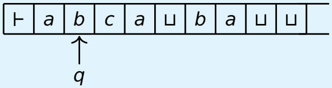

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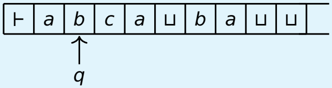

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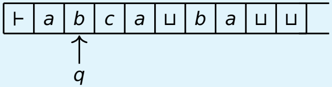

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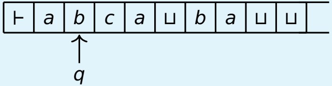

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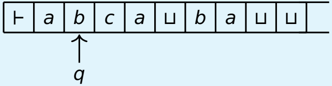

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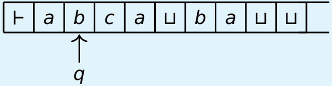

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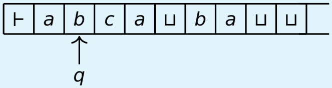

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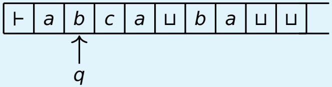

such that

$$\forall a \in \Gamma \;\; \exists b, c \in \Gamma \;\; \exists d, e \in \{L, R\} : \; \delta(t, a) = (t, b, d) \quad \text{and} \quad \delta(r, a) = (r, c, e)$$

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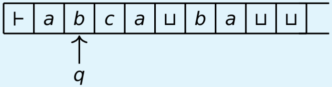

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$$\forall p \in Q \ \ \exists q \in Q: \ \delta(p, \vdash) = (q, \vdash, R)$$

Example

$A = \{a^n b^n c^n \mid n \geqslant 0\} = L(M)$ for TM $M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, 1, t, r)$ with

1. $Q = \{1, 2, 3, 4, 5, t, r\}$
2. $\Sigma = \{a, b, c\}$
3. $\Gamma = \{a, b, c, \vdash, \sqcup, X, x\}$
4.

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, a, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, b, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, a, L)$	$(4, b, L)$	$(4, c, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

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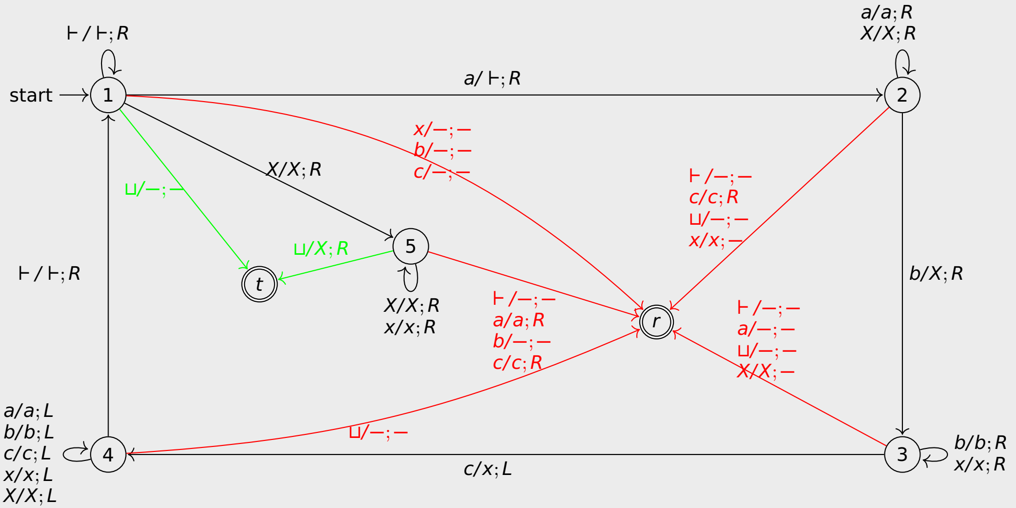
δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, a, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, b, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, a, L)$	$(4, b, L)$	$(4, c, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

Example


start
1
2
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4
5
t
r
⊢ / ⊢; R
a/ ⊢; R
a/a; R
X/X; R
x/—; —
b/—; —
c/—; —
X/X; R
⊔/—; —
⊢ /—; —
c/c; R
⊔/—; —
x/x; —
⊔/X; R
⊢ / ⊢; R
X/X; R
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⊔/—; —
b/b; R
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c/x; L

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,\mathrm{a},R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,\mathrm{b},R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,\mathrm{a},L)$	$(4,\mathrm{b},L)$	$(4,\mathrm{c},L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
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$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega$$

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δ	$\vdash$	a	b	c	$\sqcup$	X	x
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2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, a, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, b, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, a, L)$	$(4, b, L)$	$(4, c, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$a^2b^2c^2$

$$\underset{1}{\vdash}aabbcc\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}abbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}bbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash a\underset{2}{b}bcc\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,\mathrm{a},R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,\mathrm{b},R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,\mathrm{a},L)$	$(4,\mathrm{b},L)$	$(4,\mathrm{c},L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{aabbcc}}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{abbcc}}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{abbcc}}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{3}{\mathrm{a}X\mathrm{bcc}}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \underset{3}{\mathrm{a}X\mathrm{bcc}}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{\mathrm{a}X\mathrm{b}x\mathrm{c}}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{\mathrm{a}X\mathrm{b}x\mathrm{c}}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,\mathrm{a},R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,\mathrm{b},R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,\mathrm{a},L)$	$(4,\mathrm{b},L)$	$(4,\mathrm{c},L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash}\mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \underset{4}{\vdash}\vdash \mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,\mathrm{a},R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,\mathrm{b},R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,\mathrm{a},L)$	$(4,\mathrm{b},L)$	$(4,\mathrm{c},L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \underset{4}{\vdash}\vdash \mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \underset{4}{\vdash}\vdash \mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}x\mathrm{c}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,a,R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,b,R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,a,L)$	$(4,b,L)$	$(4,c,L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$a^2b^2c^2$

$$\underset{1}{\vdash}aabbcc\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{a}abbcc\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}bbcc\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash a\underset{2}{b}bcc\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash aX\underset{3}{b}cc\sqcup^{\omega}$$

$$\xrightarrow[M]{1} \vdash\vdash aXb\underset{3}{c}c\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash aX\underset{4}{b}xc\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash a\underset{4}{X}bxc\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}Xbxc\sqcup^{\omega}$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash} aXbxc\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}Xbxc\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}bxc\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}xc\sqcup^{\omega}$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}c\sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,\mathrm{a},R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,\mathrm{b},R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,\mathrm{a},L)$	$(4,\mathrm{b},L)$	$(4,\mathrm{c},L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \underset{4}{\vdash}\vdash \mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,\mathrm{a},R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,\mathrm{b},R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,\mathrm{a},L)$	$(4,\mathrm{b},L)$	$(4,\mathrm{c},L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash}\mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{1}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash\underset{2}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash} \mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{4}{X}xx\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,\mathrm{a},R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,\mathrm{b},R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,\mathrm{a},L)$	$(4,\mathrm{b},L)$	$(4,\mathrm{c},L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{\mathrm{c}}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash} \mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{4}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash \underset{4}{X}Xxx\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,\mathrm{a},R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,\mathrm{b},R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,\mathrm{a},L)$	$(4,\mathrm{b},L)$	$(4,\mathrm{c},L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash} \mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{4}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash \underset{4}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{\vdash} XXxx\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \underset{4}{\vdash}\vdash \mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\underset{1}{\vdash} aX\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{4}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash \underset{4}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{\vdash} XXxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{1}{X}Xxx\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,a,R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,b,R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,a,L)$	$(4,b,L)$	$(4,c,L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$a^2b^2c^2$

$$\underset{1}{\vdash}aabbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\underset{1}{a}abbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{2}{a}bbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash a\underset{2}{b}bcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash aX\underset{3}{b}cc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash aXb\underset{3}{c}c\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash aX\underset{4}{b}xc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash a\underset{4}{X}bxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}Xbxc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash} aXbxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}Xbxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}bxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}xc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}c\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{4}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash \underset{4}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{\vdash} XXxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{1}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{5}{X}xx\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,a,R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,b,R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,a,L)$	$(4,b,L)$	$(4,c,L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$a^2b^2c^2$

$$\underset{1}{\vdash}aabbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\underset{1}{a}abbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{2}{a}bbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash a\underset{2}{b}bcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash aX\underset{3}{b}cc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash aXb\underset{3}{c}c\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash aX\underset{4}{b}xc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash a\underset{4}{X}bxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{a}Xbxc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash}aXbxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{1}{a}Xbxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash\underset{2}{X}bxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}xc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}c\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{4}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash\underset{4}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{\vdash}XXxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash\underset{1}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{5}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{5}{x}x\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,a,R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,b,R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,a,L)$	$(4,b,L)$	$(4,c,L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$a^2b^2c^2$

$$\underset{1}{\vdash}aabbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\underset{1}{a}abbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{2}{a}bbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash a\underset{2}{b}bcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash aX\underset{3}{b}cc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash aXb\underset{3}{c}c\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash aX\underset{4}{b}xc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash a\underset{4}{X}bxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}Xbxc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash} aXbxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}Xbxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}bxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}xc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}c\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{4}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash \underset{4}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{\vdash} XXxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{1}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{5}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{5}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{5}{x}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(2, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

$\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$

$$\underset{1}{\vdash}\mathrm{aabbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{a}\mathrm{abbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{a}\mathrm{bbcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{2}{b}\mathrm{bcc}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{3}{b}\mathrm{cc}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\mathrm{b}\underset{3}{c}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}X\underset{4}{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{a}\underset{4}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash} \mathrm{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}X\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}\mathrm{b}x\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}x\mathrm{c}\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}\mathrm{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{4}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash \underset{4}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{\vdash} XXxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{1}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{5}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{5}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{5}{x}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXxx\underset{5}{\sqcup}^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,a,R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(2,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,b,R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,a,L)$	$(4,b,L)$	$(4,c,L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$

$a^2b^2c^2 \in L(M)$

$$\underset{1}{\vdash}aabbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\underset{1}{a}abbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{2}{a}bbcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash a\underset{2}{b}bcc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash aX\underset{3}{b}cc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash aXb\underset{3}{c}c\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash aX\underset{4}{b}xc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash a\underset{4}{X}bxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{a}Xbxc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\underset{4}{\vdash} aXbxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{1}{a}Xbxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{2}{X}bxc\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{2}{b}xc\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{3}{x}c\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{3}{c}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{4}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{4}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash \underset{4}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\underset{4}{\vdash} XXxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash \underset{1}{X}Xxx\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash X\underset{5}{X}xx\sqcup^\omega$$

$$\xrightarrow[M]{1} \vdash\vdash\vdash XX\underset{5}{x}x\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXx\underset{5}{x}\sqcup^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXxx\underset{5}{\sqcup}^\omega \xrightarrow[M]{1} \vdash\vdash\vdash XXXxx\underset{t}{\sqcup}^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$

acb

$\underset{1}{\vdash}\mathrm{acb}\sqcup^{\omega}$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, a, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, b, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, a, L)$	$(4, b, L)$	$(4, c, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

acb

$$\underset{1}{\vdash}\text{acb}\sqcup^{\omega} \xrightarrow[M]{1} \underset{1}{\vdash}\,\text{acb}\,\sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathsf{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathsf{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathsf{a}, L)$	$(4, \mathsf{b}, L)$	$(4, \mathsf{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

acb

$$\underset{1}{\vdash}\mathsf{acb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\mathsf{a}}\mathsf{cb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathsf{c}}\mathsf{b}\sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

$\mathrm{acb} \notin L(M)$

$$\underset{1}{\vdash}\mathrm{acb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{a}}\mathrm{cb} \sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{c}}\mathrm{b} \sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \mathrm{c}\underset{r}{\mathrm{b}} \sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1,\vdash,R)$	$(2,\vdash,R)$	$(r,-,-)$	$(r,-,-)$	$(t,-,-)$	$(5,X,R)$	$(r,-,-)$
2	$(r,-,-)$	$(2,a,R)$	$(3,X,R)$	(r,c,R)	$(r,-,-)$	$(2,X,R)$	$(r,-,-)$
3	$(r,-,-)$	$(r,-,-)$	$(3,b,R)$	$(4,x,L)$	$(r,-,-)$	$(r,-,-)$	$(3,x,R)$
4	$(1,\vdash,R)$	$(4,a,L)$	$(4,b,L)$	$(4,c,L)$	$(r,-,-)$	$(4,X,L)$	$(4,x,L)$
5	$(r,-,-)$	(r,a,R)	$(r,-,-)$	(r,c,R)	(t,X,R)	$(5,X,R)$	$(5,x,R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

acb $\notin L(M)$

$$\underset{1}{\vdash}\text{acb}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{\text{acb}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{\text{cb}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \text{cb}\underset{r}{\,} \sqcup^\omega$$

abca

$$\underset{1}{\vdash}\text{abca}\sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathsf{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathsf{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathsf{a}, L)$	$(4, \mathsf{b}, L)$	$(4, \mathsf{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

$\mathsf{acb} \notin L(M)$

$$\underset{1}{\vdash}\mathsf{acb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\mathsf{a}}\mathsf{cb} \sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathsf{c}}\mathsf{b} \sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{r}{\mathsf{c}}\mathsf{b} \sqcup^{\omega}$$

abca

$$\underset{1}{\vdash}\mathsf{abca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\mathsf{a}}\mathsf{bca} \sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

$\mathrm{acb} \notin L(M)$

$$\underset{1}{\vdash}\mathrm{acb}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{acb}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{cb}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \mathrm{cb} \underset{r}{\sqcup}^\omega$$

abca

$$\underset{1}{\vdash}\mathrm{abca}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{abca}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{bca}} \sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \text{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \text{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \text{a}, L)$	$(4, \text{b}, L)$	$(4, \text{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

acb $\notin L(M)$

$$\underset{1}{\vdash}\text{acb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\text{a}}\text{cb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{\text{c}}\text{b}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \text{c}\underset{r}{\text{b}}\sqcup^{\omega}$$

abca

$$\underset{1}{\vdash}\text{abca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\text{a}}\text{bca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{\text{b}}\text{ca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash X\underset{3}{\text{c}}\text{a}\sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, a, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, b, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, a, L)$	$(4, b, L)$	$(4, c, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

acb $\notin L(M)$

$$\underset{1}{\vdash}\mathrm{acb}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{acb}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{cb}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{r}{\mathrm{cb}} \sqcup^\omega$$

abca

$$\underset{1}{\vdash}\mathrm{abca}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{abca}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{bca}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{3}{X\mathrm{ca}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{4}{Xx\mathrm{a}} \sqcup^\omega$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

acb $\notin L(M)$

$$\underset{1}{\vdash}\mathrm{acb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{a}}\mathrm{cb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{c}}\mathrm{b}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{r}{\mathrm{c}}\mathrm{b}\sqcup^{\omega}$$

abca

$$\underset{1}{\vdash}\mathrm{abca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{a}}\mathrm{bca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{b}}\mathrm{ca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash X\underset{3}{\mathrm{c}}\mathrm{a}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{4}{X}x\mathrm{a}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\underset{4}{\vdash} Xx\mathrm{a}\sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, \mathrm{a}, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, \mathrm{b}, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, \mathrm{a}, L)$	$(4, \mathrm{b}, L)$	$(4, \mathrm{c}, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

$\mathrm{acb} \notin L(M)$

$$\underset{1}{\vdash}\mathrm{acb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{a}}\mathrm{cb}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{c}}\mathrm{b}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{r}{\mathrm{c}}\mathrm{b}\sqcup^{\omega}$$

abca

$$\underset{1}{\vdash}\mathrm{abca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{a}}\mathrm{bca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{b}}\mathrm{ca}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash X\underset{3}{\mathrm{c}}\mathrm{a}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\vdash \underset{4}{X}x\mathrm{a}\sqcup^{\omega} \xrightarrow[M]{1} \vdash\underset{4}{\vdash} Xx\mathrm{a}\sqcup^{\omega}$$
$$\xrightarrow[M]{1} \vdash\vdash \underset{1}{X}x\mathrm{a}\sqcup^{\omega}$$

Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, a, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, b, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, a, L)$	$(4, b, L)$	$(4, c, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

acb $\notin L(M)$

$$\underset{1}{\vdash}\mathrm{acb}\sqcup^\omega \xrightarrow[M]{1} \vdash \underset{1}{\mathrm{acb}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{2}{\mathrm{cb}} \sqcup^\omega \xrightarrow[M]{1} \vdash\vdash \underset{r}{\mathrm{cb}} \sqcup^\omega$$

abca

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Example

δ	$\vdash$	a	b	c	$\sqcup$	X	x
1	$(1, \vdash, R)$	$(2, \vdash, R)$	$(r, -, -)$	$(r, -, -)$	$(t, -, -)$	$(5, X, R)$	$(r, -, -)$
2	$(r, -, -)$	$(2, a, R)$	$(3, X, R)$	(r, c, R)	$(r, -, -)$	$(2, X, R)$	$(r, -, -)$
3	$(r, -, -)$	$(r, -, -)$	$(3, b, R)$	$(4, x, L)$	$(r, -, -)$	$(r, -, -)$	$(3, x, R)$
4	$(1, \vdash, R)$	$(4, a, L)$	$(4, b, L)$	$(4, c, L)$	$(r, -, -)$	$(4, X, L)$	$(4, x, L)$
5	$(r, -, -)$	(r, a, R)	$(r, -, -)$	(r, c, R)	(t, X, R)	$(5, X, R)$	$(5, x, R)$
t	$-$	$-$	$-$	$-$	$-$	$-$	$-$
r	$-$	$-$	$-$	$-$	$-$	$-$	$-$

acb $\notin L(M)$

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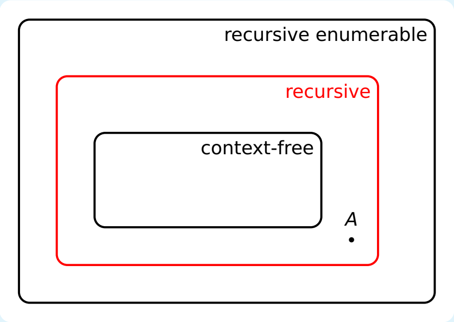

$$A = \{a^n b^n c^n \mid n \geqslant 0\}$$

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if A and $\sim A$ are r.e. then A is recursive

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Proof.

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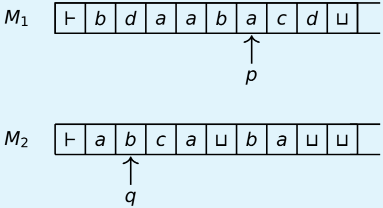

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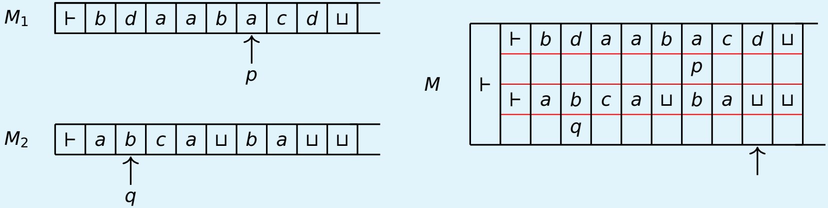

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Outline

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instance: TM M, string x
question: does M halt on x?

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for every set A exactly one of following alternatives holds

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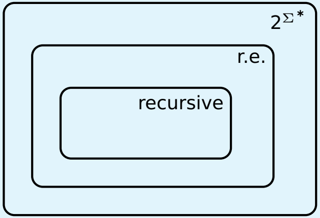

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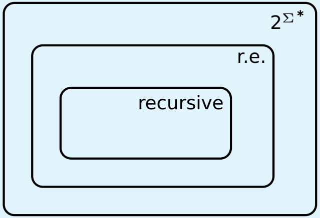

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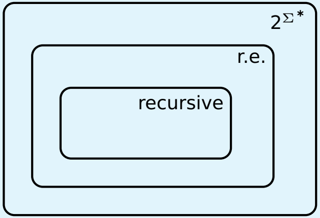

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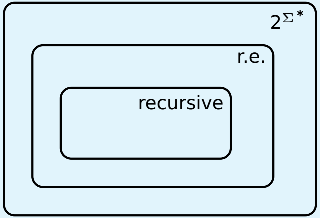

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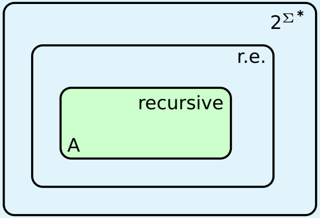

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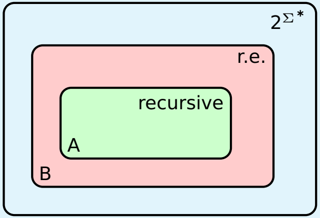

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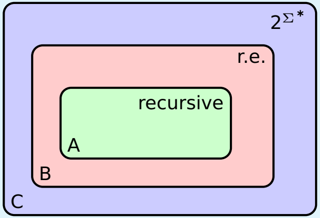

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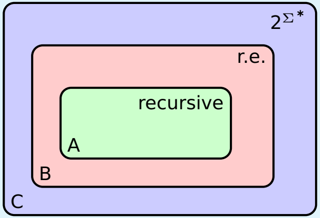

$A \backslash \sim A$	A	B	C
A	1	×	×
B	×	×	3
C	×	3	2

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$\text{enc}(M) = 0^n 1 0^m 1 0^k 1 0^s 1 0^t 1 0^r 1 0^u 1 0^v 1 \quad \cdots \quad 0^p 1 0^a 1 0^q 1 0^b 1 0^c 1 \cdots$

for all $((p, a), (q, b, d)) \in \delta$ with $c = \begin{cases} 0 & \text{if } d = L \\ 1 & \text{if } d = R \end{cases}$

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  for all $((p, a), (q, b, d)) \in \delta$ with $c = \begin{cases} 0 & \text{if } d = L \\ 1 & \text{if } d = R \end{cases}$

- string $x = a_1 a_2 \cdots a_n \in \Sigma^*$
  $\text{enc}(x) = 0^{a_1} 1 0^{a_2} 1 \cdots 0^{a_n} 1$

Example

- $N_{\text{trivial}} = (\{0,1\}, \{0,1\}, \{0,1,2,3\}, 2, 3, \delta, 0, 1, 0)$ with

δ	0	1	2	3
0	$(0,0,R)$	$(0,1,R)$	$(0,2,R)$	$(0,3,R)$
1	$(1,0,R)$	$(1,1,R)$	$(1,2,R)$	$(1,3,R)$

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- $\text{enc}(N_{\text{trivial}}) = 001000010011011$

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  $111101101101011001100101100011000101$

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- $\exists$ computable enumeration of all TMs with input alphabet $\{0,1\}$

$$M_\varepsilon,\ M_0,\ M_1,\ M_{00},\ M_{01},\ M_{10},\ \ldots$$

Definition (Encoding of Membership Problem for Turing Machines)

$\mathrm{MP} = \{\mathrm{enc}(M)\,\#\,\mathrm{enc}(x) \mid x \in L(M)\}$

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Corollary

MP is r.e.

How to Prove Undecidability?

- diagonalization
- reduction

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Outline

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Theorem (Schröder–Bernstein)

$\exists$ bijection $A \to B$ $\quad \Longleftrightarrow \quad$ $\exists$ injective functions $f : A \to B$ and $g : B \to A$

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- Σ^* for finite alphabet Σ is countable

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Proof. (Diagonalization)

- suppose $2^{\Sigma^*} = \{A_0, A_1, A_2, \ldots\}$ is countable

	A_0	A_1	A_2	$\cdots$
x_0				
x_1				
x_2				
$\vdots$				

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- create infinite table

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$$
\begin{array}{c|cccc}
 & A_0 & A_1 & A_2 & \cdots \\
\hline
x_0 & \times & & & \\
x_1 & \checkmark & & & \\
x_2 & \times & & & \\
\vdots & & & &
\end{array}
$$

- $A_0 = \{x_1\}$

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x_0	$\times$	$\checkmark$		
x_1	$\checkmark$	$\times$		
x_2	$\times$	$\checkmark$		
$\vdots$				

- $A_0 = \{x_1\} \quad A_1 = \{x_0, x_2, x_7, x_8, \ldots\}$

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- $A_0 = \{x_1\} \quad A_1 = \{x_0, x_2, x_7, x_8, \dots\} \quad A_2 = \{x_2, x_5\}$

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	A_0	A_1	A_2	$\cdots$
x_0	×	✓	×	
x_1	✓	×	×	
x_2	×	✓	✓	
$\vdots$				

- $A_0 = \{x_1\}$ $\quad A_1 = \{x_0, x_2, x_7, x_8, \dots\}$ $\quad A_2 = \{x_2, x_5\}$
- define $B = \{x_i \mid x_i \notin A_i\} \in 2^{\Sigma^*}$

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	A_0	A_1	A_2	$\cdots$
x_0	✓	✓	×	
x_1	✓	✓	×	
x_2	×	✓	×	
$\vdots$				

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- $B \neq A_i \ \forall i$

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	A_0	A_1	A_2	$\cdots$
x_0	✓	✓	×	
x_1	✓	✓	×	
x_2	×	✓	×	
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Thanks! & Questions?