# CMPE 322/327 - Theory of Computation Week 11: Turing Machines & Decision Problems

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# Outline

A Ouick Recap

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- 1 A Quick Recap
- 2 Turing Machine
- B Decision Problems
- 4 Encoding
- Diagonalization

A Ouick Recap

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• NPDA is septuple  $M = (Q, \Sigma, \Gamma, \delta, s, \bot, F)$  with

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  - $\Sigma$ : input alphabet

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  - $\Sigma$ : input alphabet  $\Gamma$ : stack alphabet

Diagonalization

### Definitions

- NPDA is septuple  $M = (Q, \Sigma, \Gamma, \delta, s, \bot, F)$  with
  - Q: finite set of states
  - $\Sigma$ : input alphabet
  - Γ: stack alphabet
  - $\delta$ : finite subset of  $(Q \times (Σ \cup {ε}) \times Γ) \times (Q \times Γ^*)$

A Ouick Recap

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  - $s \in O$ : start state

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  - $s \in Q$ : start state
  - $\bigcirc$  ⊥ ∈ Γ: initial stack symbol
  - $F \subseteq Q$ : final states

### Example

A Ouick Recap

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 $A = \{x \in \{[,]\}^* \mid x \text{ is balanced}\}\$  is accepted by NPDA  $M = (Q, \Sigma, \Gamma, \delta, 1, \bot, F)$  with

- ①  $Q = \{1, 2\}$
- $\Sigma = \{[,]\}$
- $\Gamma = \{\bot, [\}$
- **4**  $F = \{2\}$
- s = 1
- $\delta = \{((1, [, \bot), (1, [\bot)), ((1, ], [), (1, \varepsilon)), ((1, [, [), (1, [[)), ((1, \varepsilon, \bot), (2, \varepsilon))\})\}$

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  - $\delta$ : finite subset of  $(Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma) \times (Q \times \Gamma^*)$
  - $s \in O$ : start state
  - $\bot \in \Gamma$ : initial stack symbol
  - $F \subseteq O$ : final states
- configuration: element of  $Q \times \Sigma^* \times \Gamma^*$

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  - $s \in O$ : start state
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  - $F \subseteq O$ : final states
- configuration: element of  $Q \times \Sigma^* \times \Gamma^*$  (current state, remaining input, stack content)

A Ouick Recap

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  - $s \in Q$ : start state

  - $\mathcal{F} \subseteq Q$ : final states
- configuration: element of  $Q \times \Sigma^* \times \Gamma^*$  (current state, remaining input, stack content)
- start configuration on input x:  $(s, x, \bot)$

A Ouick Recap

- NPDA is septuple  $M = (Q, \Sigma, \Gamma, \delta, s, \bot, F)$  with
  - O: finite set of states

  - Σ: input alphabet
     Γ: stack alphabet
     δ: finite subset of (
     s ∈ Q: start state  $\delta$ : finite subset of  $(Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma) \times (Q \times \Gamma^*)$

  - $\bigcirc$  ⊥ ∈ Γ: initial stack symbol
  - $F \subseteq O$ : final states
- configuration: element of  $Q \times \Sigma^* \times \Gamma^*$  (current state, remaining input, stack content)
- start configuration on input x:  $(s, x, \bot)$
- next configuration relation is binary relation  $\frac{1}{M}$  defined as:  $(p, ay, A\beta) \frac{1}{M} (q, y, \gamma\beta)$ for all  $((p, a, A), (q, \gamma)) \in \delta$  with  $a \in \Sigma \cup \{\epsilon\}$  and  $y \in \Sigma^*, \beta \in \Gamma^*$

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 $A = \{x \in \{[,]\}^* \mid x \text{ is balanced}\}\$  is accepted by NPDA  $M = (Q, \Sigma, \Gamma, \delta, 1, \bot, F)$  with

- $Q = \{1, 2\}$
- $\Sigma = \{[,]\}$
- $\Gamma = \{\bot, [\}$
- **4**  $F = \{2\}$
- s = 1
- $\delta = \{((1, [, \bot), (1, [\bot)), ((1, ], [), (1, \varepsilon)), ((1, [, [), (1, [[)), ((1, \varepsilon, \bot), (2, \varepsilon))\}$

input:

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input:

state: stack:

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input:

state: stack:

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```
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```

Decision Problems

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- s = 1
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input: state: 1 1 stack:  $\perp \perp$  Decision Problems

A Ouick Recap

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                                                 input:
                                                 state:
                                                               1 1
                                                 stack:
                                                               \perp \perp
```

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```
A = \{x \in \{[,]\}^* \mid x \text{ is balanced}\}\ is accepted by NPDA M = (Q, \Sigma, \Gamma, \delta, 1, \bot, F) with
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       F = \{2\}
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                                                 input:
                                                 state:
                                                                1 1 1
                                                 stack:
                                                                \bot \bot \bot
```

Decision Problems

### Example

A Ouick Recap

```
A = \{x \in \{[,]\}^* \mid x \text{ is balanced}\} is accepted by NPDA M = (Q, \Sigma, \Gamma, \delta, 1, \bot, F) with Q = \{1, 2\}
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```
input:
state:
             1111
stack:
             \bot \bot \bot \bot \bot
```

```
A = \{x \in \{[,]\}^* \mid x \text{ is balanced}\}\ is accepted by NPDA M = (Q, \Sigma, \Gamma, \delta, 1, \bot, F) with
  Q = \{1, 2\}
        \Sigma = \{[,]\}
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  6
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                                                 input:
                                                 state:
                                                                1111
                                                 stack:
                                                                \bot \bot \bot \bot \bot
```

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                                                  input:
```

11111  $\bot$   $\bot$   $\bot$   $\bot$   $\bot$   $\bot$ 

state:

stack:

Decision Problems

```
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                                                 input:
                                                 state:
                                                                11111
                                                                \bot \bot \bot \bot \bot \bot
                                                 stack:
```

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```
input:
state:
             111111
             \bot \bot \bot \bot \bot \bot \bot
stack:
```

Decision Problems

A Ouick Recap

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```
input:
state:
             111111
             \bot \bot \bot \bot \bot \bot \bot
stack:
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```
input:
state:
             1111111
stack:
             \bot \bot \bot \bot \bot \bot \bot \bot
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```
input:
state:
             1111111
stack:
             \bot \bot \bot \bot \bot \bot \bot \bot
```

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```
input:
state:
             11111111
stack:
             \bot \bot \bot \bot \bot \bot \bot \bot \bot
```

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```
A = \{x \in \{[,]\}^* \mid x \text{ is balanced}\}\ is accepted by NPDA M = (Q, \Sigma, \Gamma, \delta, 1, \bot, F) with
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Decision Problems

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```
input:
state:
             11111111
stack:
             \bot \bot \bot \bot \bot \bot \bot \bot \bot
```

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```
input:
state:
       111111111
stack:
```

Decision Problems

A Ouick Recap

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stack:
```

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```
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```

Decision Problems

### Example

A Ouick Recap

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#### Example

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```
input:
state:
          111111111
                                        11111
stack:
                                        \bot \bot \bot \bot \bot \bot
```

Decision Problems

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```
input:
state:
           111111111
                                            11117
stack:
                                            \bot \bot \bot \bot \bot \varepsilon
```

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```
input:
state:
           111111111
                                            11112
stack:
                                            \bot \bot \bot \bot \bot \varepsilon
```

Decision Problems

• 
$$\frac{n}{M} = (\frac{1}{M})^n \quad \forall n \ge 0$$

• 
$$\frac{n}{M} = (\frac{1}{M})^n \quad \forall n \ge 0$$

• 
$$\frac{*}{M} = \bigcup_{n \geq 0} \frac{n}{M}$$

NPDA  $M = (Q, \Sigma, \Gamma, \delta, s, \bot, F)$ 

• 
$$\frac{n}{M} = (\frac{1}{M})^n \quad \forall n \ge 0$$

• 
$$\frac{*}{M} = \bigcup_{n \geq 0} \frac{n}{M}$$

•  $x \in \Sigma^*$  is accepted by final state if  $(s, x, \bot) \xrightarrow{*}_{M} (q, \varepsilon, \alpha)$  with  $q \in F$ 

• 
$$\frac{n}{M} = (\frac{1}{M})^n \quad \forall n \ge 0$$

• 
$$\frac{*}{M} = \bigcup_{n \geq 0} \frac{n}{N}$$

- $x \in \Sigma^*$  is accepted by final state if  $(s, x, \bot) \xrightarrow{*}_{M} (q, \varepsilon, \alpha)$  with  $q \in F$
- $L_f(M) = \{x \in \Sigma^* \mid x \text{ is accepted by final state}\}$

• 
$$\frac{n}{M} = (\frac{1}{M})^n \quad \forall n \ge 0$$

• 
$$\frac{*}{M} = \bigcup_{n \geq 0} \frac{n}{M}$$

- $x \in \Sigma^*$  is accepted by final state if  $(s, x, \bot) \xrightarrow{*} (q, \varepsilon, \alpha)$  with  $q \in F$
- $L_f(M) = \{x \in \Sigma^* \mid x \text{ is accepted by final state}\}$
- $x \in \Sigma^*$  is accepted by empty stack if  $(s, x, \bot) \xrightarrow{*}_{M} (q, \varepsilon, \varepsilon)$

• 
$$\frac{n}{M} = (\frac{1}{M})^n \quad \forall n \ge 0$$

• 
$$\frac{*}{M} = \bigcup_{n \geq 0} \frac{n}{M}$$

- $x \in \Sigma^*$  is accepted by final state if  $(s, x, \bot) \xrightarrow{*} (q, \varepsilon, \alpha)$  with  $q \in F$
- $L_f(M) = \{x \in \Sigma^* \mid x \text{ is accepted by final state}\}$
- $x \in \Sigma^*$  is accepted by empty stack if  $(s, x, \bot) \xrightarrow{*}_{M} (q, \varepsilon, \varepsilon)$
- $L_e(M) = \{x \in \Sigma^* \mid x \text{ is accepted by empty stack}\}$

• 
$$\frac{n}{M} = (\frac{1}{M})^n \quad \forall n \ge 0$$

• 
$$\frac{*}{M} = \bigcup_{n \geq 0} \frac{n}{M}$$

- $x \in \Sigma^*$  is accepted by final state if  $(s, x, \bot) \xrightarrow{*} (q, \varepsilon, \alpha)$  with  $q \in F$
- $L_f(M) = \{x \in \Sigma^* \mid x \text{ is accepted by final state}\}$
- $x \in \Sigma^*$  is accepted by empty stack if  $(s, x, \bot) \xrightarrow{*}_{M} (q, \varepsilon, \varepsilon)$
- $L_e(M) = \{x \in \Sigma^* \mid x \text{ is accepted by empty stack}\}$

#### Theoren

A Ouick Recap

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CFGs and NPDAs are equivalent:

- 1 A = L(G) for some CFG  $G \iff$
- $A = L_f(M)$  for some NPDA M

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A Ouick Recap

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CFGs and NPDAs are equivalent:

- A = L(G) for some CFG  $G \iff$
- $A = L_f(M)$  for some NPDA  $M \iff$
- $A = L_e(M)$  for some NPDA M

#### Theoren

A Ouick Recap

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CFGs and NPDAs are equivalent:

- 2  $A = L_f(M)$  for some NPDA  $M \iff$
- $A = L_e(M)$  for some NPDA  $M \iff$
- 4  $A = L_e(M) = L_f(M)$  for some NPDA M

A deterministic pushdown automaton (DPDA) is an octuple  $M = (Q, \Sigma, \Gamma, \delta, \bot, \dashv, s, F)$ 

- $\bullet$  is a special symbol not in  $\Sigma$ , called the right endmarker
- 2 for any  $p \in Q$ ,  $a \in \Sigma \cup \{\epsilon\}$ ,  $A \in \Gamma$ , the set  $\delta \subseteq (Q \times (\Sigma \cup \{\exists\} \cup \{\epsilon\}) \times \Gamma) \times (Q \times \Gamma^*)$  contains
  - at most one element of the form  $((p, a, A), (q, \beta))$
  - exactly one transition of the form  $((p, a, A), (q, \beta))$  or  $((p, \varepsilon, A), (q, \beta))$

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#### Remark

DPDAs are strictly less powerful than NPDAs

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### Remark

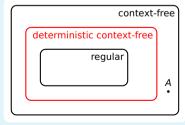
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## Remark

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$$A = \{a^i b^j c^k \mid i = i \text{ or } i = k\}$$

#### Theoren

A Ouick Recap

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context-free sets are effectively closed under

- union
- concatenation
- asterate
- · homomorphic image
- homomorphic preimage

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A Ouick Recap

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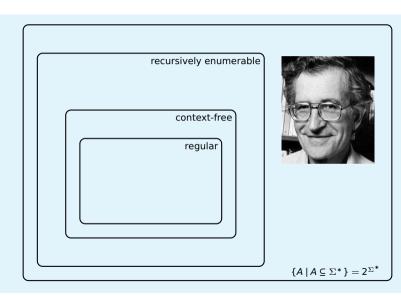
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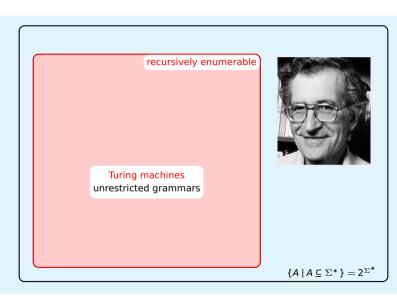
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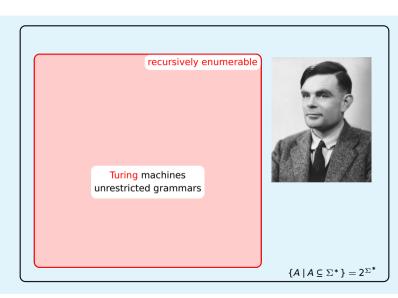
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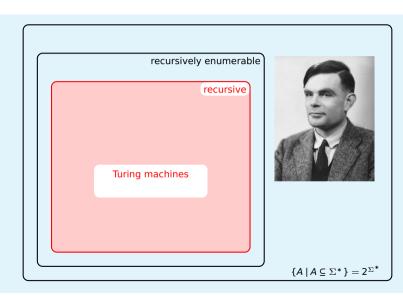
# Outline

- 1 A Quick Recap
- 2 Turing Machines
- 3 Decision Problem
- 4 Encoding
- Diagonalization









A Ouick Recap

A Ouick Recap

Turing machine (TM) is 9-tuple  $M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)$  with

Q: finite set of states

A Ouick Recap

- Q: finite set of states
- $\Sigma$ : input alphabet

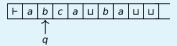
A Ouick Recap

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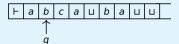
 $\Sigma$ : input alphabet

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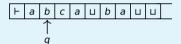
A Ouick Recap

- Q: finite set of states
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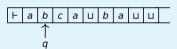


A Ouick Recap

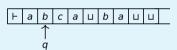
- Q: finite set of states
- $\bigcirc$   $\Sigma$ : input alphabet
- **6**  $\Gamma \supseteq \Sigma$ : tape alphabet
- Φ ⊢ ∈ Γ − Σ: left endmarker
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- Q: finite set of states
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- $δ: Q × Γ → Q × Γ × {L, R}: (partial) transition function$

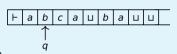


- Q: finite set of states
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- $\delta: Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ : (partial) transition function
- $s \in O$ : start state



A Ouick Recap

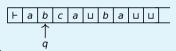
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Turing machine (TM) is 9-tuple  $M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)$  with

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- $\emptyset$   $s \in Q$ : start state
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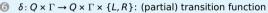


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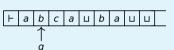
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- $s \in O$ : start state
- $t \in Q$ : accept state
- $r \in O \{t\}$ : reject state



$$\forall a \in \Gamma \ \exists b, c \in \Gamma \ \exists d, e \in \{L, R\} \colon \delta(t, a) = (t, b, d) \ \text{and} \ \delta(r, a) = (r, c, e)$$

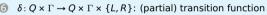


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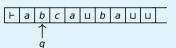
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such that

$$\forall a \in \Gamma \exists b, c \in \Gamma \exists d, e \in \{L, R\} : \delta(t, a) = (t, b, d) \text{ and } \delta(r, a) = (r, c, e)$$
  
 $\forall p \in Q \exists q \in Q : \delta(p, F) = (q, F, R)$ 



Encodina

A Ouick Recap

 $A = \{a^n b^n c^n \mid n \ge 0\} = L(M) \text{ for TM } M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, 1, t, r) \text{ with }$ 

- $\Sigma = \{a, b, c\}$
- $\Gamma = \{a, b, c, \vdash, \sqcup, X, x\}$

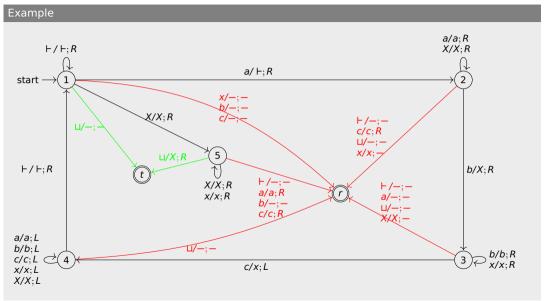
4	δ	-	а	b	С	П	X	X
	1	( <b>1</b> , ⊢, <i>R</i> )	( <b>2</b> , <b>⊢</b> , <i>R</i> )	( <i>r</i> , -, -)	( <b>r</b> , -, -)	(t, -, -)	(5, X, R)	( <i>r</i> , -, -
	2	$(1, \vdash, R)$ $(r, -, -)$ $(r, -, -)$	(2, a, R)	(3, X, R)	(r, c, R)	(r, -, -)	(2, X, R)	( <i>r</i> , –, –
	3	( <i>r</i> , -, -)	(r, -, -)	(3, b, R)	(4, x, L)	(r, -, -)	(r, -, -)	(3, x, R)
	4	( <b>1</b> , ⊢, <i>R</i> )	(4, a, L)	(4, b, L)	(4, c, L)	(r, -, -)	(4, X, L)	(4, x, L)
	5	$(1,\vdash,R)$ $(r,-,-)$	(r, a, R)	(r, -, -)	(r, c, R)	(t, X, R)	(5, X, R)	(5, x, R)

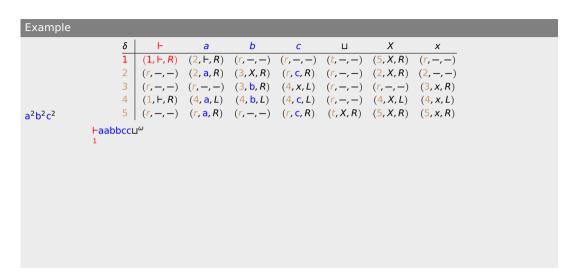
A Ouick Recap

 $A = \{a^n b^n c^n \mid n \ge 0\} = L(M) \text{ for TM } M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, 1, t, r) \text{ with }$ 

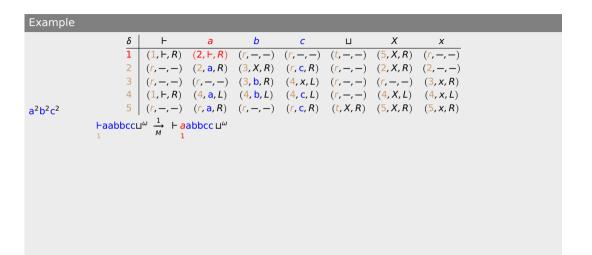
- $\Sigma = \{a, b, c\}$
- $\Gamma = \{a, b, c, \vdash, \sqcup, X, x\}$

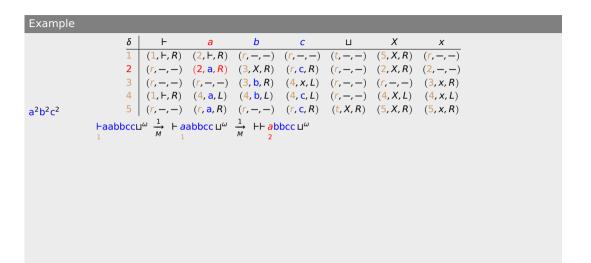
4	δ	⊢	a	b	С	П	X	X
	1	( <b>1</b> , ⊢, <i>R</i> )	( <b>2</b> , <b>⊢</b> , <b>R</b> )	( <i>r</i> , -, -)	( <b>r</b> , -, -)	(t, -, -)	(5, X, R)	( <i>r</i> , –, –
	2	(r, -, -)	(2, a, R)	(3, X, R)	(r, c, R)	(r, -, -)	(2, X, R)	( <i>r</i> , -, -
	3	(r, -, -)	(r, -, -)	(3, b, R)	(4, x, L)	(r, -, -)	(r, -, -)	(3, x, R)
	4	$(1,\vdash,R)$	(4, a, L)	(4, b, L)	(4, c, L)	(r, -, -)	(4, X, L)	(4, x, L)
	5	(r, -, -)	(r, a, R)	(r, -, -)	(r, c, R)	(t, X, R)	(5, X, R)	(5, x, R)
	t	_	_	_	_	_	_	_
	r	_	_	_	_	_	_	_





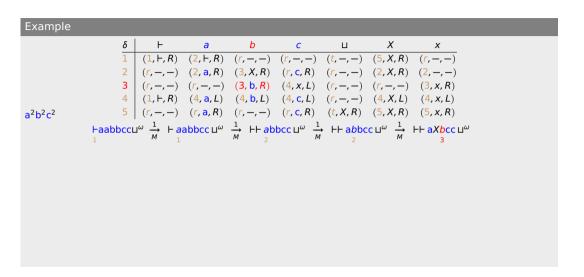
Diagonalization

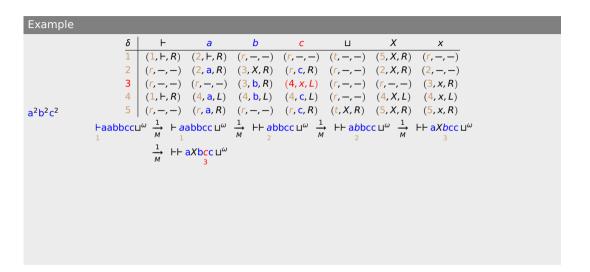


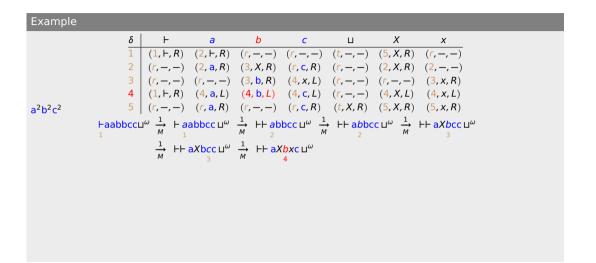


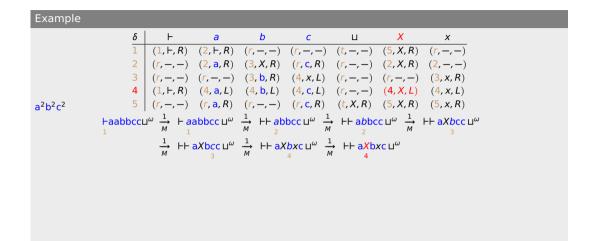
Diagonalization

Example								
δ	⊢ (1, ⊢, R)	а	b	C	П	X	X	
1	( <b>1</b> , ⊢, <i>R</i> )	( <b>2</b> , <b>⊢</b> , <i>R</i> )	( <u>r</u> , -, -)	( <u>r</u> , -, -)	(t, -, -)	(5, X, R)	( <u>r</u> , -, -)	
2	( <i>r</i> , -, -) ( <i>r</i> , -, -)	(2, a, R)	(3, X, R)	(r, c, R)	(r, -, -)	(2, X, R)	(2, -, -)	
3	(r, -, -)	(r, -, -)	(3, b, R)	(4, x, L)	(r, -, -)	(r, -, -)	(3, x, R)	
4	$(1,\vdash,R)$ (r,-,-)	(4, a, L)	(4, b, L)	$(4, \mathbf{c}, \mathbf{L})$	(r, -, -)	(4, X, L)	(4, x, L)	
							(5, x, R)	
FaabbccL 1	$1^{\omega} \xrightarrow{1}_{M} F$ aa	ibbcc ⊔ <sup>ω</sup>	$\frac{1}{M}$ H ab	bcc ⊔ <sup>ω</sup> 1/M	FF abbc	c u <sup>ω</sup>		

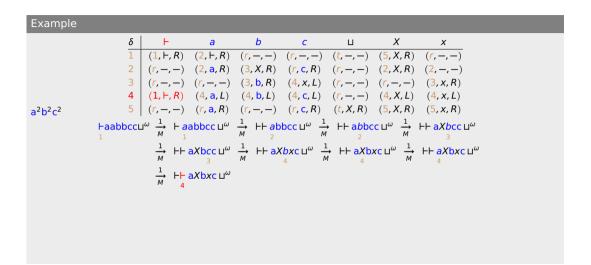


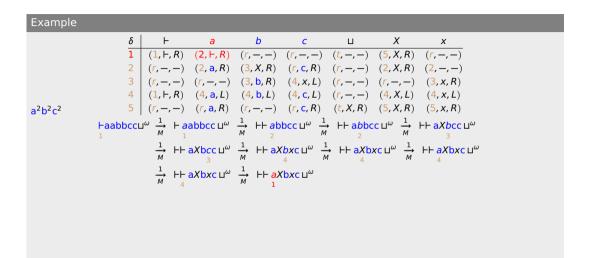


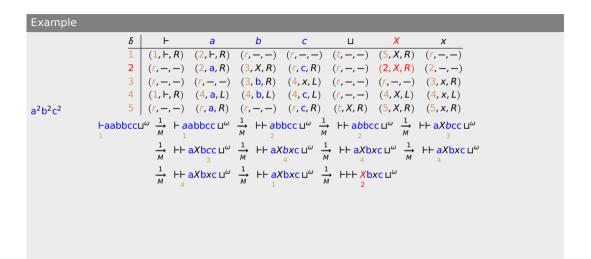




$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	Example									
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		δ	F	a	b	C	Ш	X	X	
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		1	( <b>1</b> , <b>⊢</b> , <i>R</i> )	$(2,\vdash,R)$	( <i>r</i> , -, -)	(r, -, -)	(t, -, -)	(5, X, R)	( <i>r</i> , -, -)	
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		2	(r, -, -)	(2, a, R)	(3, X, R)	(r, c, R)	(r, -, -)	(2, X, R)	(2, -, -)	
$\vdash_{1}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash_{1}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash_{2}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash_{1}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M}$		3	(r, -, -)	( <i>r</i> , -, -)	(3, b, R)	(4, x, L)	( <i>r</i> , -, -)	(r, -, -)	(3, x, R)	
$\vdash_{1}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash_{1}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash_{2}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash_{1}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M}$ habbee $\sqcup^{\omega} \xrightarrow{1}_{M}$		4	$(1, \vdash, R)$	(4, a, L)	(4, b, L)	$(4, \mathbf{c}, \mathbf{L})$	(r, -, -)	(4, X, L)	(4, x, L)	
	$a^2b^2c^2$	5	(r, -, -)	(r, a, R)	(r, -, -)	(r, c, R)	(t, X, R)	(5, X, R)	(5, x, R)	
$\frac{1}{2}$ ++ aXbcc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$		⊢aabbccL	$1^{\omega} \xrightarrow{1 \atop M} \vdash a_1$	abbcc ⊔ <sup>ω</sup>	$\frac{1}{M}$ $\vdash \vdash ab$	$bcc \sqcup^{\omega} \stackrel{1}{\xrightarrow{M}}$	→ ⊢⊢ a <i>b</i> bc	$C \sqcup^{\omega} \xrightarrow{1}_{M}$	⊢⊢ <b>a</b> X <b>b</b> cc ⊔ <sup>ω</sup>	
M 3 M 4 M 4 M 4			$\xrightarrow{1}_{M}$ $\vdash$	aXbcc ⊔ <sup>ω</sup> 3	$\xrightarrow{1}_{M}$ $\vdash\vdash$ $aX$	$\frac{1}{4}$ $\frac{1}{M}$	→ ⊢⊢ aXb	$X \subset \coprod^{\omega} \xrightarrow{1}_{M}$	⊢⊢ aXbxc ⊔ <sup>ω</sup>	







Example		
	δ	
	2 $(r,-,-)$ (2, a, R) (3, X, R) $(r,c,R)$ $(r,-,-)$ (2, X, R) (2, -, -)	
	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	
	4 $(1, \vdash, R)$ $(4, a, L)$ $(4, b, L)$ $(4, c, L)$ $(r, -, -)$ $(4, X, L)$ $(4, x, L)$	
$a^2b^2c^2$		
	$\frac{1}{M}$ $\frac{1}$	
	$\frac{1}{M}$ + H aXbcc $\coprod^{\omega}$ $\frac{1}{M}$ + H aXbxc $\coprod^{\omega}$ $\frac{1}{M}$ + H aXbxc $\coprod^{\omega}$ $\frac{1}{M}$ + H aXbxc $\coprod^{\omega}$	
	$\frac{1}{M} \stackrel{\text{H-}}{=} \frac{1}{M} \frac{1}{M} \stackrel{\text{H-}}{=} \frac{1}{M} \stackrel{\text{H-}}{=} \frac{1}{M} \frac{1}{M} \stackrel{\text{H-}}{=} \frac{1}{M} \stackrel{\text{H-}}$	

Example								
	δ	⊢	a	b	C	П	X	X
	1	( <b>1</b> , ⊢, <i>R</i> )	$(2,\vdash,R)$	( <b>r</b> , -, -)	(r, -, -)	(t, -, -)	(5, X, R)	(r,-,-)
	2	(r, -, -)	(2, a, R)	(3, X, R)	(r, c, R)	(r, -, -)	(2, X, R)	(2, -, -)
	3	(r, -, -)	(r, -, -)	(3, b, R)	(4, x, L)	(r, -, -)	(r, -, -)	(3, x, R)
		$(1,\vdash,R)$						
$a^2b^2c^2$	5	( <i>r</i> , -, -)	(r, a, R)	(r, -, -)	(r, c, R)	(t, X, R)	(5, X, R)	(5, x, R)
	FaabbccL	$J^{\omega} \xrightarrow{1}_{M} \vdash \underset{1}{a}$	abbcc ⊔ <sup>ω</sup>	$\frac{1}{M}$ ++ $\frac{ab}{2}$	$bcc \sqcup^{\omega} \stackrel{1}{\xrightarrow{M}}$	→ FF a <i>b</i> bc	$C \sqcup^{\omega} \xrightarrow{1}_{M}$	⊢⊢ <b>a</b> X <i>b</i> cc ⊔ <sup>ω</sup> ₃
		$\frac{1}{M}$ $\vdash$	aXbcc ⊔ <sup>ω</sup>	$\frac{1}{M}$ $\vdash \vdash aX$	$\int_{4}^{b} x c \sqcup^{\omega} \frac{1}{M}$	→ FF aXb	$X \subset \coprod^{\omega} \xrightarrow{\frac{1}{M}}$	⊢⊢ aXbxc ⊔ <sup>ω</sup>
		$\xrightarrow{\frac{1}{M}}$ $\vdash$	aXbxc ⊔ <sup>ω</sup>	$\frac{1}{M}$ $\vdash \vdash aX$	$(bxc \sqcup^{\omega} \frac{1}{N})$	+ +++ Xb. 2	$X \subset \coprod^{\omega} \xrightarrow{1}_{M}$	⊢⊢ <i>Χb</i> xc ⊔ <sup>α</sup>
		$\frac{1}{M}$ $\vdash$	- XXxc ⊔ <sup>ω</sup>					

# Example (r, -, -) (2, a, R) (3, X, R) (r, c, R) (r, -, -) (2, X, R) (2, -, -) 3 (r,-,-) (r,-,-) (3,b,R) (4,x,L) (r,-,-) (r,-,-) (3,x,R)4 $(1, \vdash, R)$ $(4, \mathsf{a}, L)$ $(4, \mathsf{b}, L)$ $(4, \mathsf{c}, L)$ (r, -, -) (4, X, L) (4, x, L) $5 \mid (r, -, -) \mid (r, a, R) \mid (r, -, -) \mid (r, c, R) \mid (t, X, R) \mid (5, X, R) \mid (5, X, R)$ $a^2b^2c^2$ $\vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ abbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ abbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ a $\vdash$ abbcc $\sqcup^{\omega} \vdash$ $\frac{1}{M}$ H aXbcc $\coprod^{\omega}$ $\frac{1}{M}$ H aXbxc $\coprod^{\omega}$ $\frac{1}{M}$ H aXbxc $\coprod^{\omega}$ H aXbxc $\coprod^{\omega}$ $\frac{1}{M}$ H- $\frac{1$ $\frac{1}{M}$ $\vdash\vdash\vdash XXXC \sqcup^{\omega} \xrightarrow{1} \vdash\vdash\vdash XXXC \sqcup^{\omega}$

Example								
	δ	⊢	a	b	C	П	X	X
	1	( <b>1</b> , ⊢, <i>R</i> )	$(2,\vdash,R)$	( <b>r</b> , -, -)	( <b>r</b> , -, -)	(t, -, -)	(5, X, R)	( <i>r</i> , -, -)
	2	( <i>r</i> , -, -)	(2, a, R)	(3, X, R)	(r, c, R)	(r, -, -)	(2, X, R)	(2, -, -)
	3			(3, b, R)				
	4	$(1,\vdash,R)$	(4, a, L)	(4, b, L)	(4, c, L)	(r, -, -)	(4, X, L)	(4, x, L)
$a^2b^2c^2$	5	(r, -, -)	(r, a, R)	(r, -, -)	(r, c, R)	(t, X, R)	(5, X, R)	(5, x, R)
	⊢aabbccL 1	$J^{\omega} \xrightarrow{1}_{M} \vdash \underset{1}{a}$	abbcc ⊔ <sup>ω</sup>	$\frac{1}{M}$ $\vdash\vdash ab$	$bcc \sqcup^{\omega} \stackrel{1}{\xrightarrow{M}}$	→ FF a <i>b</i> bo	$C \sqcup^{\omega} \xrightarrow{1}_{M}$	⊢⊢ <b>a</b> X <b>b</b> cc ⊔ <sup>ω</sup>
		$\frac{1}{M}$ $\vdash$	aXbcc ⊔ <sup>ω</sup>	$\frac{1}{M}$ $\vdash\vdash a\lambda$	$\frac{1}{4}$	→	$X \subset \coprod^{\omega} \xrightarrow{1}_{M}$	⊢⊢ aXbxc ⊔ <sup>ω</sup>
		$\xrightarrow{1}_{M}$ $\vdash$	aXbxc ⊔ <sup>ω</sup>	$\frac{1}{M}$ $\vdash$ $\vdash$ $\frac{a}{M}$	$(bxc \sqcup^{\omega} \frac{1}{N})$	+ +++ Xb 2	$X \subset \coprod^{\omega} \xrightarrow{1}_{M}$	⊢⊢⊢ Xbxc ⊔ <sup>ω</sup>
		$\xrightarrow{1}_{M}$ $\vdash$ $\vdash$	- XXxc ⊔ <sup>ω</sup> ³	$\frac{1}{M}$ +++ $\lambda$	< X x c ⊔ <sup>ω</sup> - 1/Λ	$\stackrel{\Gamma}{\rightarrow}$ $\vdash\vdash\vdash XX$	(χχ⊔ <sup>ω</sup> 4	

# Example (r, -, -) (2, a, R) (3, X, R) (r, c, R) (r, -, -) (2, X, R) (2, -, -) $3 \mid (r, -, -) \mid (r, -, -) \mid (3, b, R) \mid (4, x, L) \mid (r, -, -) \mid (r, -, -) \mid (3, x, R)$ 4 $(1, \vdash, R)$ $(4, \mathbf{a}, L)$ $(4, \mathbf{b}, L)$ $(4, \mathbf{c}, L)$ (r, -, -) (4, X, L) (4, x, L)[r, -, -) [r, a, R] [r, -, -) [r, c, R] [t, X, R] [t, X, R] [t, X, R] $a^2b^2c^2$ $\vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ abbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ aXbcc $\sqcup^{\omega} \vdash$ $\frac{1}{M}$ ++ aXbxc $\cup^{\omega}$ $\frac{1}{M}$ ++ aXbxc $\cup^{\omega}$ $\frac{1}{M}$ ++ aXbxc $\cup^{\omega}$ $\frac{1}{M}$ H- $\frac{1$ $\frac{1}{M}$ HH XXXCU $^{\omega}$ $\frac{1}{M}$ HH XXXCU $^{\omega}$ $\frac{1}{M}$ HH XXXXU $^{\omega}$ $\frac{1}{M}$ HH XXXXU $^{\omega}$

Example								
	δ   Η	· a	b	С	П	X	Х	
	1 (1, F	$(2, \vdash, R)$	(r, -, -)	(r, -, -)	(t, -, -)	(5, X, R)	(r, -, -)	
		$(2, \mathbf{a}, R)$						
		(r,-,-)						
		$(4, \mathbf{a}, L)$						
$a^2b^2c^2$		(r, a, R)						
	$\vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1}_{M}$	⊢ aabbcc ⊔ <sup>ω</sup>	$\frac{1}{M}$ $\vdash$ $\vdash$ $\frac{ab}{2}$	$bcc \sqcup^{\omega} \xrightarrow{1}_{M}$	⊢⊢ abbo	$C \sqcup^{\omega} \xrightarrow{1}_{M}$	⊢⊢ <b>a</b> X <b>b</b> cc ⊔ <sup>ω</sup> ³	
	$\frac{1}{M}$	⊢⊢ aXbcc ⊔ <sup>ω</sup>	$\frac{1}{M}$ $\vdash$ $\vdash$ $aX$	$bxc \sqcup^{\omega} \xrightarrow{1 \atop M}$	+	$KC \sqcup^{\omega} \xrightarrow{1}_{M}$	⊢⊢ aXbxc ⊔ <sup>ω</sup>	
	$\frac{1}{M}$	⊢⊢ aXbxc ⊔ <sup>ω</sup>	$\frac{1}{M}$ $\vdash \vdash aX$	$bxc \sqcup^{\omega} \xrightarrow{1}_{M}$	► ⊢⊢ Xb.	$X \subset \coprod^{\omega} \xrightarrow{1}_{M}$	⊢⊢⊢ Xbxc ⊔ <sup>a</sup>	)
	$\xrightarrow{1}_{M}$	⊢⊢⊢ XXxc ⊔ <sup>ω</sup> ³	$\frac{1}{M}$ $\vdash\vdash\vdash \lambda$	$(X \times C \sqcup^{\omega} \stackrel{1}{\underset{M}{\longrightarrow}}$	→ +++ XX	$\begin{array}{c} XX \sqcup^{\omega} & \xrightarrow{1} \\ 4 \end{array}$	⊢⊢⊢ <i>XXxx</i> ⊔ 4	ω
	$\xrightarrow{\frac{1}{M}}$	⊢⊢⊢ XXxx ⊔ <sup>ω</sup> 4						

Example					
ě	5	b c	Ц	X	X
1	$ (1,\vdash,R)  (2,\vdash,R) $	(r,-,-) $(r,-,-)$	(t,-,-)	(5, X, R)	( <i>r</i> , -, -)
2	(r,-,-) (2, a, R)	(3, X, R) $(r, c, c)$	(r,-,-)	(2, X, R)	(2, -, -)
3					
	$\{ (1,\vdash,R)  (4,a,L)$				
$a^2b^2c^2$	(r,-,-) $(r,a,R)$	(r,-,-) $(r,c,$	(t, X, R)	(5, X, R)	(5, x, R)
Haabbo 1	$CCL^{\omega} \xrightarrow{1}_{M} \vdash \underset{1}{aabbcc} L^{\omega}$	$\stackrel{1}{\longrightarrow}$ ${\mapsto}$ ${\mapsto}$ $\stackrel{abbcc}{\longrightarrow}$	$\frac{1}{M}$ ++ abbo	$\Box \subset \sqcup^{\omega} \xrightarrow{1}_{M}$	⊢⊢ <b>a</b> X <i>b</i> cc ⊔ <sup>ω</sup>
	$\frac{1}{M}$ $\vdash$	$\stackrel{1}{\xrightarrow{M}} \vdash AXbxc \sqcup$	$\stackrel{\omega}{\longrightarrow} \stackrel{1}{\longrightarrow} \vdash \vdash \stackrel{aXb}{\overset{a}{\longrightarrow}}$	$X \subset \coprod^{\omega} \xrightarrow{1}_{M}$	⊢⊢ $_{4}^{aXbxc}$ ⊔ $_{4}^{\omega}$
	$\xrightarrow{\frac{1}{M}}$ $\vdash$	$\stackrel{1}{\longrightarrow} \vdash \vdash \underset{1}{aXbxc} \sqcup$	$\omega \xrightarrow{1}_{M} \vdash \vdash \vdash X_{b}$	$XC \sqcup^{\omega} \xrightarrow{1}_{M}$	$\vdash\vdash\vdash X_{bxc} \sqcup^{\omega}$
	$\xrightarrow{\frac{1}{M}}$ $\vdash \vdash \vdash XXx \subset \sqcup^{G}$	$\stackrel{\cup}{\longrightarrow} \stackrel{1}{\longrightarrow} \vdash \vdash \vdash XXX_{\stackrel{\bullet}{C}} \bot$	$1^{\omega} \xrightarrow{1}_{M} \vdash \vdash \vdash XX$	$(xx \sqcup^{\omega} \xrightarrow{1}_{M}$	⊢⊢⊢ <i>XXxx</i> ⊔ <sup>a</sup>
	$\xrightarrow{1}_{M}$ $\vdash\vdash\vdash XXxx \sqcup^{c}$	$\stackrel{\omega}{\longrightarrow} \stackrel{1}{\longrightarrow} \stackrel{\vdash}{\vdash} XXxx L$	$J^\omega$		

# Example $1 (1, \vdash, R) (2, \vdash, R) (r, -, -) (r, -, -) (t, -, -) (5, X, R) (r, -, -)$ (r, -, -) (2, a, R) (3, X, R) (r, c, R) (r, -, -) (2, X, R) (2, -, -) $3 \mid (r, -, -) \mid (r, -, -) \mid (3, b, R) \mid (4, x, L) \mid (r, -, -) \mid (r, -, -) \mid (3, x, R)$ 4 $(1, \vdash, R)$ $(4, \mathbf{a}, L)$ $(4, \mathbf{b}, L)$ $(4, \mathbf{c}, L)$ (r, -, -) (4, X, L) (4, x, L)[r, -, -) [r, a, R] [r, -, -) [r, c, R] [t, X, R] [t, X, R] [t, X, R] $a^2b^2c^2$ $\vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ abbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ abbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ aXbcc $\sqcup^{\omega}$ $\frac{1}{M}$ ++ aXbcc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ $\frac{1}{M}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ bxbxc $\coprod^{\omega}$ +++ Xbxc $\coprod^{\omega}$ +++ Xbxc $\coprod^{\omega}$ $\frac{1}{M}$ +++ XXxcu $^{\omega}$ $\frac{1}{M}$ +++ XXxcu $^{\omega}$ $\frac{1}{M}$ +++ XXxxu $^{\omega}$ $\frac{1}{M}$ HHH XXxx $\coprod^{\omega}$ $\frac{1}{M}$ HHH XXxx $\coprod^{\omega}$ $\frac{1}{M}$ HHH XXxx $\coprod^{\omega}$

# Example 1 $(1, \vdash, R)$ $(2, \vdash, R)$ (r, -, -) (r, -, -) (t, -, -) (5, X, R) (r, -, -)(r,-,-) (2, a, R) (3, X, R) (r, c, R) (r, -, -) (2, X, R) (2, -, -) $3 \mid (r, -, -) \mid (r, -, -) \mid (3, b, R) \mid (4, x, L) \mid (r, -, -) \mid (r, -, -) \mid (3, x, R)$ 4 $(1, \vdash, R)$ $(4, \mathbf{a}, L)$ $(4, \mathbf{b}, L)$ $(4, \mathbf{c}, L)$ (r, -, -) (4, X, L) (4, x, L) $5 \mid (r, -, -) \mid (r, a, R) \mid (r, -, -) \mid (r, c, R) \mid (t, X, R) \mid (5, X, R) \mid (5, X, R)$ $a^2b^2c^2$ $\vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ abbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ abbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ aXbcc $\sqcup^{\omega}$ $\frac{1}{M}$ ++ aXbcc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ $\xrightarrow{1}$ H- aXbxc $\sqcup^{\omega}$ $\xrightarrow{1}$ H- aXbxc $\sqcup^{\omega}$ $\xrightarrow{1}$ H- Xbxc $\sqcup^{\omega}$ $\xrightarrow{1}$ H- Xbxc $\sqcup^{\omega}$ $\frac{1}{M}$ +++ XXxcu $^{\omega}$ $\frac{1}{M}$ +++ XXxcu $^{\omega}$ $\frac{1}{M}$ +++ XXxxu $^{\omega}$ $\frac{1}{M}$ HHH XXXX $\sqcup^{\omega}$ $\frac{1}{M}$ HHH XXXX $\sqcup^{\omega}$ $\frac{1}{M}$ HHH XXXX $\sqcup^{\omega}$

Example		
	<u>δ</u>	
	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	
	$3 \mid (r, -, -) \mid (r, -, -) \mid (3, b, R) \mid (4, x, L) \mid (r, -, -) \mid (r, -, -) \mid (3, x, R)$	
	4 $(1, \vdash, R)$ $(4, a, L)$ $(4, b, L)$ $(4, c, L)$ $(r, -, -)$ $(4, X, L)$ $(4, x, L)$	
$a^2b^2c^2$		
	$\frac{1}{M}$ $+$	
	$\frac{1}{M} + H = aXbcc \sqcup^{\omega} + \frac{1}{M} + H = aXbxc \sqcup^{\omega} + \frac{1}{M} + H = aXbxc \sqcup^{\omega} + \frac{1}{M} + H = aXbxc \sqcup^{\omega}$	
	$\frac{1}{M}$ ++ $\frac{1}{M}$ aXbxc $\coprod^{\omega}$ $\frac{1}{M}$ ++ $\frac{1}{M}$ bxc $\coprod^{\omega}$ +++ $\frac{1}{M}$ bxc $\coprod^{\omega}$ +++ $\frac{1}{M}$ bxc $\coprod^{\omega}$	
	$\frac{1}{M}$ +++ XXxc $^{\square}$ $\frac{1}{M}$ +++ XXxc $^{\square}$ $\frac{1}{M}$ +++ XXxx $^{\square}$ $\frac{1}{M}$ +++ XXxx $^{\square}$	
	$\frac{1}{M} \hspace{0.2cm} \vdash \vdash \vdash XX \times X \sqcup^{\omega} \hspace{0.2cm} \frac{1}{M} \hspace{0.2cm} \vdash \vdash \vdash XX \times X \sqcup^{\omega} \hspace{0.2cm} \frac{1}{M} \hspace{0.2cm} \vdash \vdash \vdash XX \times X \sqcup^{\omega} \hspace{0.2cm} \frac{1}{M} \hspace{0.2cm} \vdash \vdash \vdash XX \times X \sqcup^{\omega}$	
	$\frac{1}{M}$ $\vdash \vdash \vdash XX \times X \sqcup^{\omega}$	

Example	
	$\delta$ $\vdash$ a b c $\sqcup$ X X
	1 $(1, \vdash, R)$ $(2, \vdash, R)$ $(r, -, -)$ $(r, -, -)$ $(t, -, -)$ $(5, X, R)$ $(r, -, -)$
	3 $(r,-,-)$ $(r,-,-)$ $(3,b,R)$ $(4,x,L)$ $(r,-,-)$ $(r,-,-)$ $(3,x,R)$
	4 $(1, \vdash, R)$ $(4, a, L)$ $(4, b, L)$ $(4, c, L)$ $(r, -, -)$ $(4, X, L)$ $(4, x, L)$
$a^2b^2c^2$	5 $(r, -, -)$ $(r, a, R)$ $(r, -, -)$ $(r, c, R)$ $(t, X, R)$ $(5, X, R)$
⊢aal 1	$\frac{1}{M}$ $+$
	$\frac{1}{M} \vdash \vdash aXbxc \sqcup^{\omega} \xrightarrow{1}_{M} \vdash \vdash aXbxc \sqcup^{\omega} \xrightarrow{1}_{M} \vdash \vdash aXbxc \sqcup^{\omega} \xrightarrow{1}_{M} \vdash \vdash aXbxc \sqcup^{\omega}$
	$\frac{1}{M} + \frac{1}{4} \frac{AXbxc}{4} \perp \frac{1}{M} + \frac{1}{4} \frac{AXbxc}{4} \perp \frac{1}{M} + \frac{1}{4} \frac{Xbxc}{4} \perp \frac{1}{M} + \frac{1}{4} \frac{Xbxc}{4$
	$\frac{1}{M} \text{ FFF } XXXC \sqcup^{\omega} \xrightarrow{1}_{M} \text{ FFF } XXXC \sqcup^{\omega} \xrightarrow{1}_{M} \text{ FFF } XXXX \sqcup^{\omega} \xrightarrow{1}_{M} \text{ FFF } XXXX \sqcup^{\omega}$
	$\frac{1}{M} \vdash \vdash XXxx \sqcup^{\omega} \xrightarrow{\frac{1}{M}} \vdash \vdash XXxx \sqcup^{\omega} \xrightarrow{\frac{1}{M}} \vdash \vdash XXxx \sqcup^{\omega} \xrightarrow{\frac{1}{M}} \vdash \vdash XXxx \sqcup^{\omega}$
	$\xrightarrow{\frac{1}{M}} \begin{array}{ccc} \vdash \vdash XXxX \sqcup^{\omega} & \xrightarrow{\frac{1}{M}} & \vdash \vdash \vdash XXxX \sqcup^{\omega} \\ & & 5 \end{array}$

Example	
	$\delta$ $\vdash$ $a$ $b$ $c$ $\sqcup$ $X$ $x$
	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$
	$3 \mid (r, -, -) \mid (r, -, -) \mid (3, b, R) \mid (4, x, L) \mid (r, -, -) \mid (r, -, -) \mid (3, x, R)$
	4 $(1, \vdash, R)$ $(4, a, L)$ $(4, b, L)$ $(4, c, L)$ $(r, -, -)$ $(4, X, L)$ $(4, x, L)$
$a^2b^2c^2$	
	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
	$\frac{1}{M} \vdash \vdash aXbxc \sqcup^{\omega} \xrightarrow{\frac{1}{M}} \vdash \vdash aXbxc \sqcup^{\omega} \xrightarrow{\frac{1}{M}} \vdash \vdash aXbxc \sqcup^{\omega} \xrightarrow{\frac{1}{M}} \vdash \vdash aXbxc \sqcup^{\omega}$
	$\frac{1}{M} \underset{4}{\vdash} + \frac{1}{A}Xbxc \sqcup^{\omega} \xrightarrow{\frac{1}{M}} \underset{1}{\vdash} + \frac{1}{A}Xbxc \sqcup^{\omega} \xrightarrow{\frac{1}{M}} \underset{2}{\vdash} + \vdash + \underset{2}{\downarrow} Xbxc \sqcup^{\omega}$
	$\frac{1}{M}  \vdash \vdash \vdash XXXC \sqcup^{\omega}  \frac{1}{M}  \vdash \vdash \vdash XXXC \sqcup^{\omega}  \frac{1}{M}  \vdash \vdash \vdash XXXX \sqcup^{\omega}  \frac{1}{M}  \vdash \vdash \vdash XXXX \sqcup^{\omega}$
	$\frac{1}{M} \begin{array}{cccccccccccccccccccccccccccccccccccc$
	$\begin{array}{cccccccccccccccccccccccccccccccccccc$

## Example 1 $(1, \vdash, R)$ $(2, \vdash, R)$ (r, -, -) (r, -, -) (t, -, -) (5, X, R) (r, -, -)(r, -, -) (2, a, R) (3, X, R) (r, c, R) (r, -, -) (2, X, R) (2, -, -)(r,-,-) (r,-,-) (3,b,R) (4,x,L) (r,-,-) (r,-,-) (3,x,R)4 (1, $\vdash$ , R) (4, a, L) (4, b, L) (4, c, L) (r, -, -) (4, X, L) (4, X, L) | (r, -, -) (r, a, R) (r, -, -) (r, c, R) (t, X, R) (5, X, R) $a^2b^2c^2 \in L(M)$ $\vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ aabbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ abbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ abbcc $\sqcup^{\omega} \xrightarrow{1} \vdash$ $\vdash$ aXbcc $\sqcup^{\omega}$ $\frac{1}{M}$ ++ aXbcc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ ++ aXbxc $\coprod^{\omega}$ $\stackrel{1}{\longrightarrow}$ H- aXbxc $\sqcup^{\omega}$ $\stackrel{1}{\longrightarrow}$ H- aXbxc $\sqcup^{\omega}$ $\stackrel{1}{\longrightarrow}$ H- Xbxc $\sqcup^{\omega}$ $\frac{1}{M}$ +++ XXxc $\cup^{\omega}$ $\frac{1}{M}$ +++ XXxc $\cup^{\omega}$ $\frac{1}{M}$ +++ XXxx $\cup^{\omega}$ $\frac{1}{M}$ +++ XXxx $\cup^{\omega}$ $\frac{1}{M}$ +++ XXxx $\sqcup^{\omega}$ $\frac{1}{M}$ +++ XXxx $\sqcup^{\omega}$ $\frac{1}{M}$ +++ XXxx $\sqcup^{\omega}$ $\frac{1}{M}$ HH- XXXX $\sqcup^{\omega}$ $\frac{1}{M}$ HH- XXXX $\sqcup^{\omega}$ $\frac{1}{M}$ HH- XXXXX $\sqcup^{\omega}$

A Quick Recap

δ	⊢	a	b	С	П	X	X
1	( <b>1</b> , ⊢, <i>R</i> )	(2, ⊢, <i>R</i> )	(r, -, -)	(r, -, -)	(t, -, -)	(5, X, R)	(r, -, -)
2	(r, -, -)	(2, a, R)	(3, X, R)	(r, c, R)	(r, -, -)	(2, X, R)	(r, -, -)
3	(r, -, -)	(r, -, -)	(3, b, R)	(4, x, L)	(r, -, -)	(r, -, -)	(3, x, R)
	$(1, \vdash, R)$						
	( <i>r</i> , -, -)						

acb

⊢acb⊔<sup>ω</sup>

A Ouick Recap

acb

$$\vdash$$
 acb $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash$  acb $\sqcup^{\omega}$ 

A Ouick Recap

acb

$$\vdash$$
 acb $\sqcup^{\omega} \xrightarrow{1 \atop M} \vdash$  acb $\sqcup^{\omega} \xrightarrow{1 \atop M} \vdash$   $\vdash$  cb $\sqcup^{\omega}$ 

A Ouick Recap

acb  $\notin$  L(M)

$$\frac{1}{1}$$
  $\frac{1}{M}$   $\frac{1}{M}$   $\frac{1}{1}$   $\frac{1}{M}$   $\frac{1}$ 

A Ouick Recap

acb  $\notin$  L(M)

$$\frac{1}{1}$$
  $\frac{1}{M}$   $\frac{1}{M}$   $\frac{1}{1}$   $\frac{1}{M}$   $\frac{1}$ 

abca

Fabca⊔<sup>ω</sup>

Encodina

## Example

A Ouick Recap

acb  $\notin$  L(M)

$$\frac{1}{1}$$
  $\frac{1}{M}$   $\frac{1}{M}$   $\frac{1}{1}$   $\frac{1}{M}$   $\frac{1}$ 

$$\vdash abca \sqcup^{\omega} \xrightarrow{1}_{M} \vdash abca \sqcup^{\omega}$$

A Ouick Recap

acb ∉ L(M)

$$\frac{1}{1}$$
  $\frac{1}{M}$   $\frac{1}{M}$   $\frac{1}{1}$   $\frac{1}{M}$   $\frac{1}$ 

$$\vdash$$
abca $\sqcup^{\omega} \xrightarrow{1 \atop M} \vdash$ abca $\sqcup^{\omega} \xrightarrow{1 \atop M} \vdash$ bca $\sqcup^{\omega}$ 

A Ouick Recap

acb ∉ L(M)

$$\vdash$$
acb $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash$ acb $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash$  $\vdash$ cb $\sqcup^{\omega} \xrightarrow{1}_{M} \vdash$  $\vdash$ cb $\sqcup^{\omega}$ 

$$\frac{1}{1}$$
  $\frac{1}{M}$   $\frac{1}{M}$   $\frac{1}{1}$   $\frac{1}{M}$   $\frac{1}$ 

A Ouick Recap

acb  $\notin$  L(M)

$$\frac{1}{1}$$
  $\frac{1}{M}$   $\frac{1}$ 

A Ouick Recap

acb  $\notin$  L(M)

$$\frac{1}{1}$$
  $\frac{1}{M}$   $\frac{1}{M}$   $\frac{1}{1}$   $\frac{1}{M}$   $\frac{1}$ 

A Ouick Recap

acb ∉ L(M)

$$\frac{1}{1}$$
  $\frac{1}{M}$   $\frac{1}{M}$   $\frac{1}{1}$   $\frac{1}{M}$   $\frac{1}$ 

A Ouick Recap

acb  $\notin$  L(M)

$$\frac{1}{1}$$
  $\frac{1}{M}$   $\frac{1}{M}$   $\frac{1}{1}$   $\frac{1}{M}$   $\frac{1}$ 

A Ouick Recap

**Decision Problems** 

acb ∉ L(M)

$$\frac{1}{M}$$
  $\frac{1}{M}$   $\frac{1}$ 

A Ouick Recap

acb  $\notin$  L(M)

$$-\operatorname{lacb} \sqcup^{\omega} \xrightarrow{1}_{M} + \operatorname{lacb} \sqcup^{\omega} \xrightarrow{1}_{M} + \operatorname{lacb} \sqcup^{\omega} \xrightarrow{1}_{M} + \operatorname{lacb} \sqcup^{\omega}$$

abca ∉ L(M)

A Quick Recap

• configuration: element of  $Q \times \{y \sqcup^{\omega} \mid y \in \Gamma^*\} \times \mathbb{N}$ 

Decision Problems

# Definitions

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$$(p,z,n) \xrightarrow{1} \begin{cases} (q,z',n-1) & \text{if } \delta(p,z_n) = (q,b,L) \\ (q,z',n+1) & \text{if } \delta(p,z_n) = (q,b,R) \end{cases}$$

with

•  $z_n$ : n-th symbol of z

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Encodina

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Encodina

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Decision Problems

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  - recursive if A = L(M) for some total TM M

Turing Machines 000000000●00 Decision Problems

Encoding 0000000 Diagonalization 0000

### Theoren

• every recursive set is r.e.

### Theoren

- every recursive set is r.e.
- not every r.e. set is recursive

### Theorem

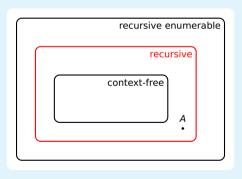
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 $A = \{ \mathbf{a}^n \mathbf{b}^n \mathbf{c}^n \mid n \ge 0 \}$ 

#### Theorer

A Quick Recap

if A and  $\sim A$  are r.e. then A is recursive

### Theorem

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- $A = L(M_1)$  for TM  $M_1 = TM(Q_1, \Sigma, \Gamma_1, \vdash, \sqcup, \delta_1, s_1, t_1, r_1)$
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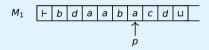
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- define TM M that simulates  $M_1$  and  $M_2$  on separate tracks of its single tape

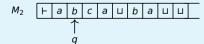
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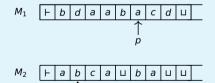


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A Ouick Recap

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extending TMs with

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A Ouick Recap

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Decision Problems

A Ouick Recap

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# Outline

- 1 A Quick Recap
- 3 Decision Problems

A Ouick Recap

halting problem for TMs

instance: TM M, string x question: does M halt on x?

A Ouick Recap

• halting problem for TMs

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TMMinstance:

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A Ouick Recap

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• membership problem for CFGs

instance: CFG G, string x

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## Decision Problems as Membership Problems

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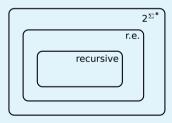
- code instance of problem as string over some alphabet
- language is set of all strings that correspond to yes instances

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A Ouick Recap

for every set A exactly one of following alternatives holds

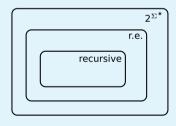
 $\bigcirc$  A and  $\sim$  A are recursive



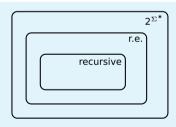
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A Ouick Recap

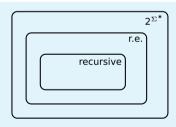
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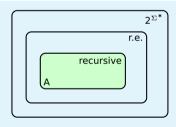
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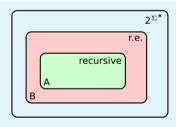
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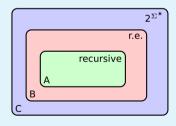
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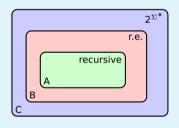
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Α	1	×	×
В	×	×	3
C	×	3	2

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## Definitions

A Ouick Recap

### decision problem P is

- decidable if there is algorithm (TM that always halts) for P:

  - accepts for every yes instancerejects for every no instance

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### Remark

decision problem P is (semi-)decidable  $\iff$ encoding of *P* is recursive(ly enumerable)

Decision Problems

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Decision Problems

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# Outline

- 1 A Quick Recap
- 2 Turing Machines
- 3 Decision Problems
- 4 Encoding
- Diagonalization

• 3 computable mappings

 $N \mapsto enc(N)$  from TMs to bit strings

### Theorem

A Ouick Recap

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 $x \mapsto \operatorname{dec}(x)$  from bit strings to TMs

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A Ouick Recap

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A LW N

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A LW N

$$enc(dec(x)) = \begin{cases} x \\ x \end{cases}$$

if 
$$x = enc(N)$$
 for some TM  $N$ 

$$\forall \, x \in \left\{0,1\right\}^*$$

• 3 computable mappings

$$N \mapsto \operatorname{enc}(N)$$
 from TMs to bit strings

 $x \mapsto dec(x)$  from bit strings to TMs

Encodina 0000000

such that

$$dec(enc(N)) = N$$

A LW N

$$enc(dec(x)) = \begin{cases} x & \text{if } x = enc(N) \text{ for some TM } N \\ enc(N_{trivial}) & \text{otherwise} \end{cases}$$

 $\forall x \in \{0,1\}^*$ 

where  $N_{\text{trivial}}$  is some fixed TM with  $L(N_{\text{trivial}}) = \emptyset$ 

A Quick Recap

• TM  $M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)$ 

A Ouick Recap

• TM  $M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)$  without loss of generality

• 
$$Q = \{0, 1, ..., n-1\}$$

$$enc(M) = 0^n$$

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  - **-**= 11

 $enc(M) = 0^n 10^m 10^k 10^s 10^t 10^r 10^u$ 

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  - **⊢**= *u*
  - $\sqcup = v$

 $enc(M) = 0^n 10^m 10^k 10^s 10^t 10^r 10^u 10^v 1$ 

```
• TM M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)
                                                  without loss of generality
```

- $Q = \{0, 1, \dots, n-1\}$
- $\Gamma = \{0, 1, ..., m-1\}$
- $\Sigma = \{0, 1, ..., k-1\}$
- ⊢= !!
- $\sqcup = v$

$$enc(M) = 0^{n}10^{m}10^{k}10^{s}10^{t}10^{r}10^{u}10^{v}1 \cdots 0^{p}10^{a}10^{q}10^{b}10^{c}1 \cdots$$

for all 
$$((p, a), (q, b, d)) \in \delta$$
 with  $c = \begin{cases} 0 & \text{if } d = L \\ 1 & \text{if } d = R \end{cases}$ 

### Example (Encoding)

A Ouick Recap

```
• TM M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)
                                                  without loss of generality
```

- $Q = \{0, 1, \dots, n-1\}$
- $\Gamma = \{0, 1, \dots, m-1\}$
- $\Sigma = \{0, 1, \dots, k-1\}$
- ⊢= !!
- $\sqcup = v$

$$\mathrm{enc}(\mathit{M}) = 0^{n} 10^{m} 10^{k} 10^{s} 10^{t} 10^{r} 10^{u} 10^{v} 1 \qquad \cdots \qquad 0^{p} 10^{a} 10^{q} 10^{b} 10^{c} 1 \qquad \text{(order is not important)}$$

for all 
$$((p, a), (q, b, d)) \in \delta$$
 with  $c = \begin{cases} 0 & \text{if } d = L \\ 1 & \text{if } d = R \end{cases}$ 

• string 
$$x = a_1 a_2 \cdots a_n \in \Sigma^*$$
  
enc(x) =  $0^{a_1} 10^{a_2} 1 \cdots 0^{a_n} 1$ 

•  $N_{\text{trivial}} = (\{0, 1\}, \{0, 1\}, \{0, 1, 2, 3\}, 2, 3, \delta, 0, 1, 0)$  with

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A Ouick Recap

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#### Theoren

A Ouick Recap

• 3 computable mappings

$$N \mapsto enc(N)$$
 from TMs to bit strings

 $x \mapsto dec(x)$  from bit strings to TMs

such that

$$dec(enc(N)) = N$$

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$$enc(dec(x)) = \begin{cases} x & \text{if } x = enc(N) \text{ for some TM } N \\ enc(N_{trivial}) & \text{otherwise} \end{cases}$$

 $\forall x \in \{0,1\}^*$ 

where  $N_{\text{trivial}}$  is some fixed TM with  $L(N_{\text{trivial}}) = \emptyset$ 

• 3 computable enumeration of all TMs with input alphabet {0, 1}

$$M_{\varepsilon}$$
,  $M_0$ ,  $M_1$ ,  $M_{00}$ ,  $M_{01}$ ,  $M_{10}$ , ...

 $\mathsf{MP} = \{\mathsf{enc}(M) \# \mathsf{enc}(x) \mid x \in L(M)\}$ 

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 $\mathsf{MP} = \{ \qquad M \ \# \qquad x \mid x \in L(M) \}$ 

A Ouick Recap

 $MP = {$ 

A Ouick Recap

M #

 $x \mid x \in L(M)$ 

### Theorem (Universal Turing Machine)

 $\exists \mathsf{TM} \mathsf{U} \mathsf{such} \mathsf{that} \mathsf{L}(\mathsf{U}) = \mathsf{MP}$ 

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# How to Prove Undecidability?

- diagonalization
- reduction

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# Outline

A Ouick Recap

- 1 A Quick Recap

- 5 Diagonalization

A Ouick Recap

• set A is countably infinite if there exists bijection  $f: \mathbb{N} \to A$ 

A Ouick Recap

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 $\exists$  bijection  $A \rightarrow B \iff \exists$  injective functions  $f: A \rightarrow B$  and  $g: B \rightarrow A$ 

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A Quick Recap

 $\Sigma \neq \emptyset \implies 2^{\Sigma^*}$  is uncountable

A Ouick Recap

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• suppose  $2^{\Sigma^*} = \{A_0, A_1, A_2, \dots\}$  is countable

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$$A_0$$
  $A_1$   $A_2$  ...
 $X_0$   $\times$ 
 $X_1$   $\checkmark$ 
 $X_2$   $\times$ 
 $X_3$   $\times$ 

A Ouick Recap

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  $A_1 = \{x_0, x_2, x_7, x_8, \dots\}$   $A_2 = \{x_2, x_5\}$ 

• define 
$$B = \{x_i \mid x_i \notin A_i\} \in 2^{\Sigma^*}$$

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Thanks! & Questions?