CMPE 322/327 - Theory of Computation

Week 2: Deterministic Finite Automata & Closure Properties

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Outline

Definitions

- alphabet is finite set; its elements are called symbols or letters
- string over alphabet Σ is finite sequence of elements of Σ
- length $|x|$ of string x is number of symbols in x
- empty string is unique string of length 0 and denoted by ϵ
- Σ^* is set of all strings over Σ ($\varnothing^* = \{\epsilon\}$)
- language over Σ is subset of Σ^*

Example

strings over $\Sigma = \{0, 1\}$: 0 0110
languages over Σ:

- $\{\epsilon, 0, 1, 00, 01, 10, 11\}$ (all strings having at most two symbols)
- $\{x \mid x$ is valid program in some machine language$\}$

Definitions

- string concatenation $x, y \in \Sigma^* \implies xy \in \Sigma^*$ is associative:

$$(xy)z = x(yz) \qquad \forall x, y, z \in \Sigma^*$$

- empty string is identity for concatenation:

$$\epsilon x = x\epsilon \qquad \forall x \in \Sigma^*$$

- x is substring (prefix, suffix) of y if $y = uxv$ $(y = xv, y = ux)$
- $x^n \, (x \in \Sigma^*, n \in \mathbb{N})$:

$$\begin{aligned} x^0 &= \epsilon \\ x^{n+1} &= x^n x \end{aligned}$$

- $\#a(x)(a \in \Sigma, x \in \Sigma^*)$ denotes number of a's in x

Definitions ($A, B \subseteq \Sigma^*$)

1. union $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$
2. intersection $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$
3. complement $\sim A := \Sigma^* - A := \{x \in \Sigma^* \mid x \notin A\}$
4. set concatenation $AB := \{xy \mid x \in A \text{ and } y \in B\}$
5. powers $A^n (n \in \mathbb{N})$ $A^0 = \{\varepsilon\}$ $A^{n+1} = AA^n$
6. asterate A^* is union of all finite powers of A

$$A^* := \bigcup_{n \geqslant 0} A^n = A^0 \cup A^1 \cup A^2 \cup A^3 \cup \cdots = \{x_1 x_2 \cdots x_n \text{ and } x_i \in A \text{ for all } 1 \leqslant i \leqslant n\}$$

7. plus A^+ is union of all finite powers of A except ε

$$A^+ = AA^* := \bigcup_{n \geqslant 1} A^n$$

8. power set $2^A := \{Q \mid Q \subseteq A\}$

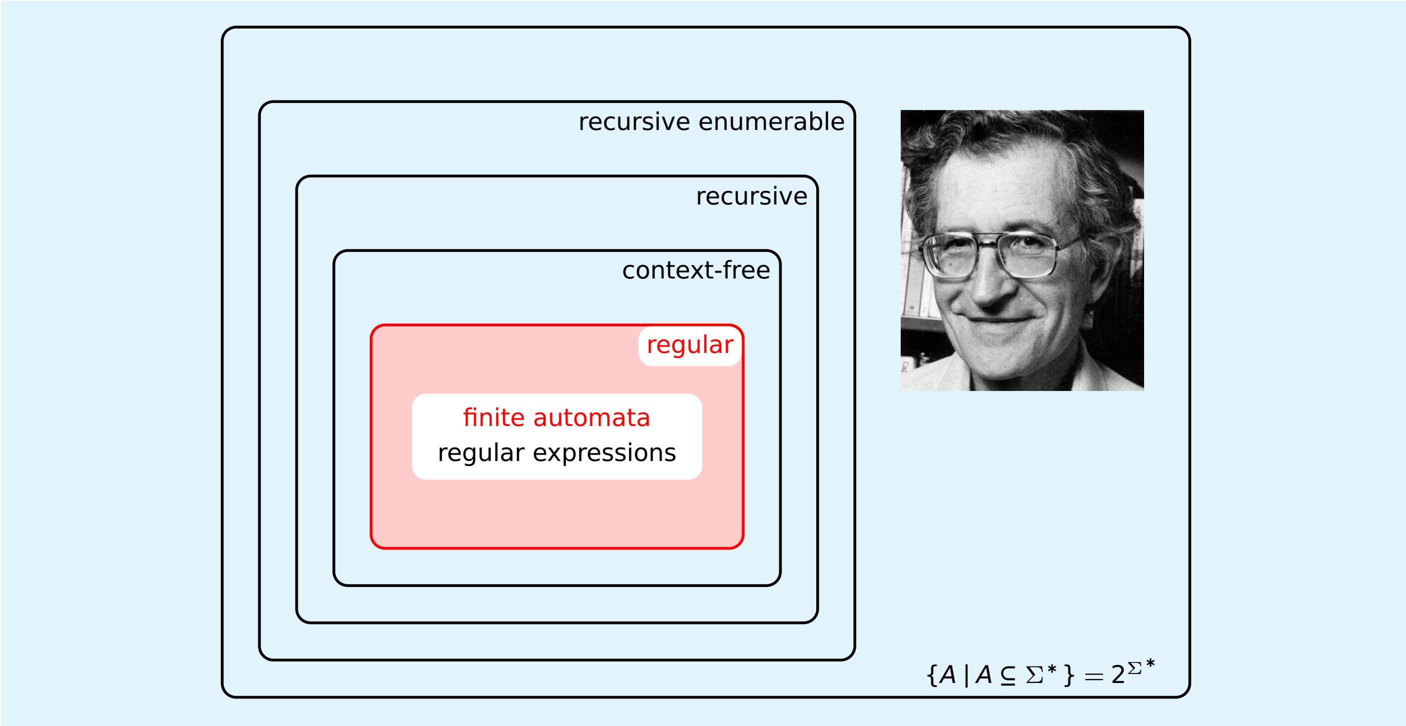
Example

- substrings of 011: 0, 1, 01, 11, 011, ε
- prefixes of 011: 0, 01, 011, ε
- suffixes of 011: 1, 11, 011, ε
- $(011)^3 = 011011011 \neq 011^3$
- $\#1(011011011) = 6 \quad \#0(\varepsilon) = 0$
- $\{0, 10, 111\}\{1, 11\} = \{01, 101, 1111, 011, 1011, 11111\}$
- $\{0, 01, 111\}\{1, 11\} = \{01, 011, 1111, 0111, 11111\}$
- $\{1, 01\}^3 = \{111, 0111, 1011, 01011, 1101, 01101, 10101, 010101\}$
- $\{1, 01\}^* = \{\varepsilon, 1, 01, 11, 011, 101, 0101, 111, 0111, 1011, 01011, \ldots\}$
- $2^{\{1,01\}} = \{\varnothing, \{1\}, \{01\}, \{1, 01\}\}$

Some Useful Properties

- $\{\varepsilon\}A = A\{\epsilon\} = A$
- $\emptyset A = A\emptyset = \emptyset$
- $\sim (A \cup B) = (\sim A) \cap (\sim B)$
- $\sim (A \cap B) = (\sim A) \cup (\sim B)$
- $A^{m+n} = A^m A^n$
- $A^* A^* = A^*$
- $A^{**} = A^*$
- $A^* = \{\varepsilon\} \cup AA^* = \{\varepsilon\} \cup A^* A$
- $\emptyset^* = \{\varepsilon\}$

Outline


recursive enumerable
recursive
context-free
regular
finite automata
regular expressions
$\{A \mid A \subseteq \Sigma^*\} = 2^{\Sigma^*}$

Outline

Definitions

- deterministic finite automaton (DFA) is quintuple $M = (Q, \Sigma, \delta, s, F)$ with
  1. Q : finite set of states
  2. Σ : input alphabet
  3. $\delta : Q \times \Sigma \to Q$: transition function
  4. $s \in Q$: start state
  5. $F \subseteq Q$: final (accept) states
- $\widehat{\delta}: Q \times \Sigma^* \to Q$ is inductively defined by

$$\widehat{\delta}(q, \varepsilon) := q \qquad \widehat{\delta}(q, xa) := \delta(\widehat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is accepted by M if $\widehat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is rejected by M if $\widehat{\delta}(s, x) \notin F$
- language accepted by M is given by $L(M) := \{x \mid \widehat{\delta}(s, x) \in F\}$
- set $A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M

Example (Unfolding of the multistep function $\widehat{\delta}$)

Let $x = abbaab$ over the alphabet $\Sigma = \{a, b\}$

$\delta(\widehat{\delta}(q_0, abbaa), b)$	first recursive call
$\delta(\delta(\widehat{\delta}(q_0, abba), a), b)$	second recursive call
$\delta(\delta(\delta(\widehat{\delta}(q_0, abb), a), a), b)$	third recursive call
$\delta(\delta(\delta(\delta(\widehat{\delta}(q_0, ab), b), a), a), b)$	fourth recursive call
$\delta(\delta(\delta(\delta(\delta(\widehat{\delta}(q_0, a), b), b), a), a), b)$	fifth recursive call
$\delta(\delta(\delta(\delta(\delta(\delta(\widehat{\delta}(q_0, \varepsilon), a), b), b), a), a), b)$	sixth recursive call
$\delta(\delta(\delta(\delta(\delta(\delta(q_0, a), b), b), a), a), b)$	
$\delta(\delta(\delta(\delta(\delta(q_1, b), b), a), a), b)$	assuming $\delta(q_0, a) = q_1$
$\delta(\delta(\delta(\delta(q_2, b), a), a), b)$	assuming $\delta(q_1, b) = q_2$
$\delta(\delta(\delta(q_3, a), a), b)$	assuming $\delta(q_2, b) = q_3$
$\delta(\delta(q_4, a), b)$	assuming $\delta(q_3, a) = q_4$
$\delta(q_5, b)$	assuming $\delta(q_4, a) = q_5$
q_6	assuming $\delta(q_5, b) = q_6$

Example (DFAs → Regular Sets)

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Example (DFA → Regular Sets)

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Theorem

The DFA M is correct with respect to predefined specs. Namely, M accepts every string of the form $a^n b$ s.t. $n \in \mathbb{N}$, rejecting all others.

Formally: $$\widehat{\delta}(1,x) = \begin{cases} 1 & \iff x \in L(a^*) \\ 2 & \iff x \in L(a^* b) \\ 3 & \iff x \in L\big(a^* b (a+b)^+\big) \end{cases}$$

Proof.

We argue by mathematical induction on the length of x.

1. Base Case: $|x| = 0 \iff x = \varepsilon \quad \widehat{\delta}(1,\varepsilon) = 1 \iff \varepsilon \in L(a^*)$
2. Step Case: Given IH : M is correct on every $x \in \Sigma^*$ such that $|x| = k$ with $k \geqslant 0$
   Show : M is correct on every xy for all $y \in \Sigma = \{a, b\}$ such that $|xy| = k+1$

Formally:

IH : $$\widehat{\delta}(1,x) = \begin{cases} 1 & \iff x \in L(a^*) \\ 2 & \iff x \in L(a^* b) \\ 3 & \iff x \in L\big(a^* b (a+b)^+\big) \end{cases}$$

Show : $$\widehat{\delta}(1,xy) = \begin{cases} 1 & \iff xy \in L(a^*) \\ 2 & \iff xy \in L(a^* b) \\ 3 & \iff xy \in L\big(a^* b (a+b)^+\big) \end{cases}$$

Proof. (cont'd)

1. $\widehat{\delta}(1,x) = 1$ and $y = a$
   $\widehat{\delta}(1,x) = 1 \iff x \in L(a^*)$ (by IH) $\quad \delta(\widehat{\delta}(1,x), a) = 1 \iff xa \in L(a^*)$
2. $\widehat{\delta}(1,x) = 1$ and $y = b$
   $\widehat{\delta}(1,x) = 1 \iff x \in L(a^*)$ (by IH) $\quad \delta(\widehat{\delta}(1,x), b) = 2 \iff xb \in L(a^* b)$
3. $\widehat{\delta}(1,x) = 2$ and $y = a$
   $\widehat{\delta}(1,x) = 2 \iff x \in L(a^* b)$ (by IH) $\quad \delta(\widehat{\delta}(1,x), a) = 3 \iff xa \in L\big(a^* b (a+b)^+\big)$
4. $\widehat{\delta}(1,x) = 2$ and $y = b$
   $\widehat{\delta}(1,x) = 2 \iff x \in L(a^* b)$ (by IH) $\quad \delta(\widehat{\delta}(1,x), b) = 3 \iff xb \in L\big(a^* b (a+b)^+\big)$
5. $\widehat{\delta}(1,x) = 3$ and $y = a$
   $\widehat{\delta}(1,x) = 3 \iff x \in L\big(a^* b (a+b)^+\big)$ (by IH) $\quad \delta(\widehat{\delta}(1,x), a) = 3 \iff xa \in L\big(a^* b (a+b)^+\big)$
6. $\widehat{\delta}(1,x) = 3$ and $y = b$
   $\widehat{\delta}(1,x) = 3 \iff x \in L\big(a^* b (a+b)^+\big)$ (by IH) $\quad \delta(\widehat{\delta}(1,x), b) = 3 \iff xb \in L\big(a^* b (a+b)^+\big)$

$\square$

Example (Regular Sets → DFA)

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Example (Regular Sets → DFA)

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Outline

Theorem

regular sets are effectively closed under intersection

Proof. (closure under intersection)

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- $A \cap B := L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$

  ➊ $Q_3 := Q_1 \times Q_2 \quad = \quad \{(p,q) \mid p \in Q_1 \text{ and } q \in Q_2\}$
  ➋ $F_3 \quad := \quad F_1 \times F_2$
  ➌ $s_3 \quad := \quad (s_1, s_2)$
  ➍ $\delta_3((p,q),a) \quad := \quad (\delta_1(p,a), \delta_2(q,a)) \qquad \forall p \in Q_1,\ \forall q \in Q_2,\ \forall a \in \Sigma$

- claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) \qquad \forall x \in \Sigma^*$
  proof: induction on $|x|$ next slide

proof of the claim

claim: $\widehat{\delta_3}((p,q),x) = (\widehat{\delta_1}(p,x), \widehat{\delta_2}(q,x)) \quad \forall x \in \Sigma^*$

- base case: $|x| = 0$ thus $x = \varepsilon$

$$\widehat{\delta_3}((p,q),\varepsilon) = (p,q) = (\widehat{\delta_1}(p,\varepsilon), \widehat{\delta_2}(q,\varepsilon))$$

- step case: $|x| > 0$ thus $x = ya$ s.t. $|y| = |x| - 1$ with IH : $\widehat{\delta_3}((p,q),y) = (\widehat{\delta_1}(p,y), \widehat{\delta_2}(q,y)))$

$$\begin{aligned}
\widehat{\delta_3}((p,q),ya) &= \delta_3(\widehat{\delta_3}((p,q),y),a) && \text{(by definition of } \widehat{\delta_3}\text{)}\\
&= \delta_3((\widehat{\delta_1}(p,y),\widehat{\delta_2}(q,y)),a) && \text{(by induction hypothesis IH)}\\
&= (\delta_1(\widehat{\delta_1}(p,y),a),\delta_2(\widehat{\delta_2}(q,y),a)) && \text{(by definition of } \delta_3\text{)}\\
&= (\widehat{\delta_1}(p,ya),\widehat{\delta_2}(q,ya)) && \text{(by definitions of } \widehat{\delta_1} \text{ and } \widehat{\delta_2}\text{)}\\
&= (\widehat{\delta_1}(p,x),\widehat{\delta_2}(q,x))
\end{aligned}$$

□

Proof. (closure under intersection (cont'd))

statement: $L(M_3) = L(M_1) \cap L(M_2)$

$$\begin{aligned}
\forall x \in \Sigma^*,\ x \in L(M_3) &\iff \widehat{\delta_3}(s_3,x) \in F_3 && \text{(by definition of acceptance)}\\
&\iff \widehat{\delta_3}((s_1,s_2),x) \in F_1 \times F_2 && \text{(by definition of } s_3 \text{ and } F_3\text{)}\\
&\iff (\widehat{\delta_1}(s_1,x),\widehat{\delta_2}(s_2,x)) \in F_1 \times F_2 && \text{(by claim proven in slide 21)}\\
&\iff \widehat{\delta_1}(s_1,x) \in F_1 \text{ and } \widehat{\delta_2}(s_2,x) \in F_2 && \text{(by definition of product)}\\
&\iff x \in L(M_1) \text{ and } x \in L(M_2) && \text{(by definition of acceptance)}\\
&\iff x \in L(M_1) \cap L(M_2) && \text{(by definition of intersection)}
\end{aligned}$$

□

Example (intersection)

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Theorem

regular sets are effectively closed under complement

Proof. (closure under complement)

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
- $\sim A := \Sigma^* - A$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

  1. $Q_2 := Q_1$
  2. $F_2 := Q_1 - F_1$
  3. $s_2 := s_1$
  4. $\delta_2(p, a) := \delta_1(p, a) \quad \forall p \in Q_1, \forall a \in \Sigma$

- obvious claim: $\widehat{\delta_2}(p, x) = \widehat{\delta_1}(p, x) \quad \forall x \in \Sigma^*$

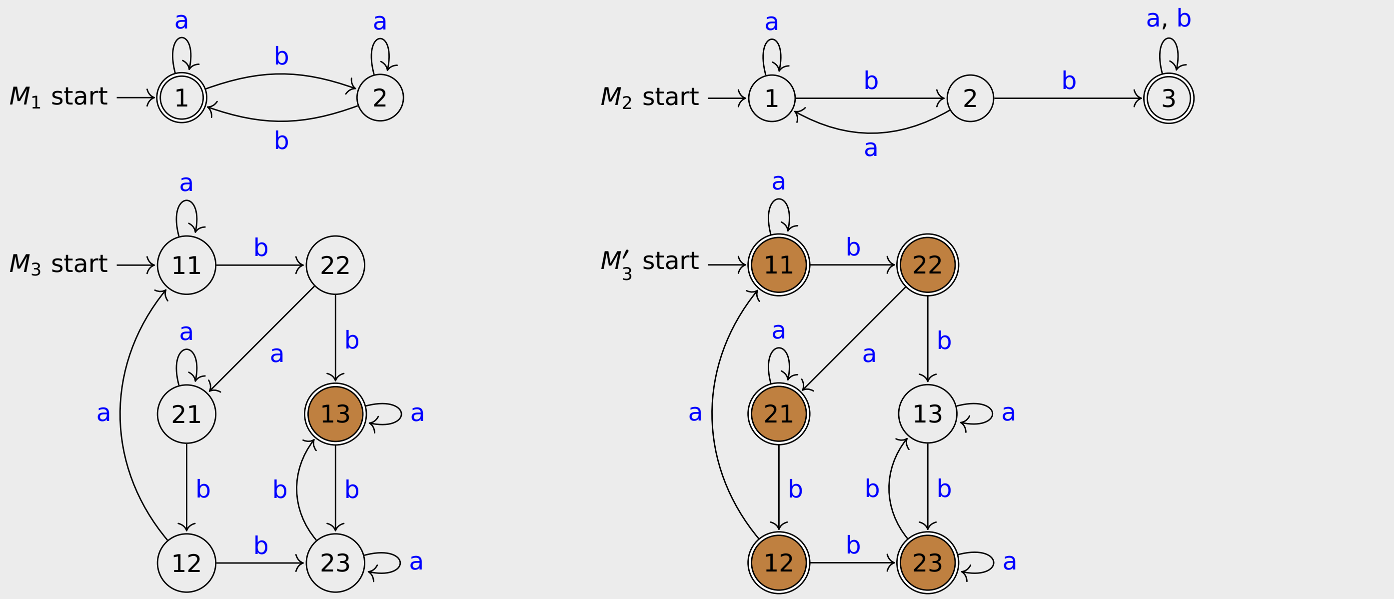
Proof. (closure under complement (cont'd))

statement: $L(M_2) = \Sigma^* - L(M_1)$

$$
\begin{array}{rcll}
\forall x \in \Sigma^*,\ x \in L(M_2) & \iff & \widehat{\delta_2}(s_2, x) \in F_2 & \text{(by definition of acceptance)} \\
& \iff & \widehat{\delta_1}(s_2, x) \in F_2 & \text{(by the obvious claim in slide 24)} \\
& \iff & \widehat{\delta_1}(s_1, x) \in Q_1 - F_1 & \text{(by definitions of } s_2 \text{ and } F_2\text{)} \\
& \iff & \widehat{\delta_1}(s_1, x) \in Q_1 \text{ and } \widehat{\delta_1}(s_1, x) \notin F_1 & \text{(by definition of set difference)} \\
& \iff & x \notin L(M_1) & \text{(by definition of acceptance)}
\end{array}
$$

□

Example (complement)



$L(M_1) \cap L(M_2) = L(M_3) := \{x \mid x$ contains even number of bs and bb as substring$\}$

$\sim L(M_3) := \{x \mid x$ contains odd number of bs or no bb as substring$\}$

Theorem

regular sets are effectively closed under union

Proof. (closure under union)

$A \cup B = \sim((\sim A) \cap (\sim B))$

A Quick Recap Chomsky Hierarchy Deterministic Finite State Automata Closure Properties

Theorem

regular sets are effectively closed under union

Proof. (closure under union – explicit construction)

- $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
  $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- $A \cup B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$

  1. $Q_3 = Q_1 \times Q_2 := \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$
  2. $F_3 := (F_1 \times Q_2) \cup (Q_1 \times F_2)$
  3. $s_3 := (s_1, s_2)$
  4. $\delta_3((p, q), a) := (\delta_1(p, a), \delta_2(q, a)) \quad \forall p \in Q_1, \forall q \in Q_2, \forall a \in \Sigma$

- claim: $\widehat{\delta_3}((p, q), x) = (\widehat{\delta_1}(p, x), \widehat{\delta_2}(q, x)) \quad \forall x \in \Sigma^*$
  proof: induction on $|x|$ – skipped (follows exact same steps with that is given at slide #21)

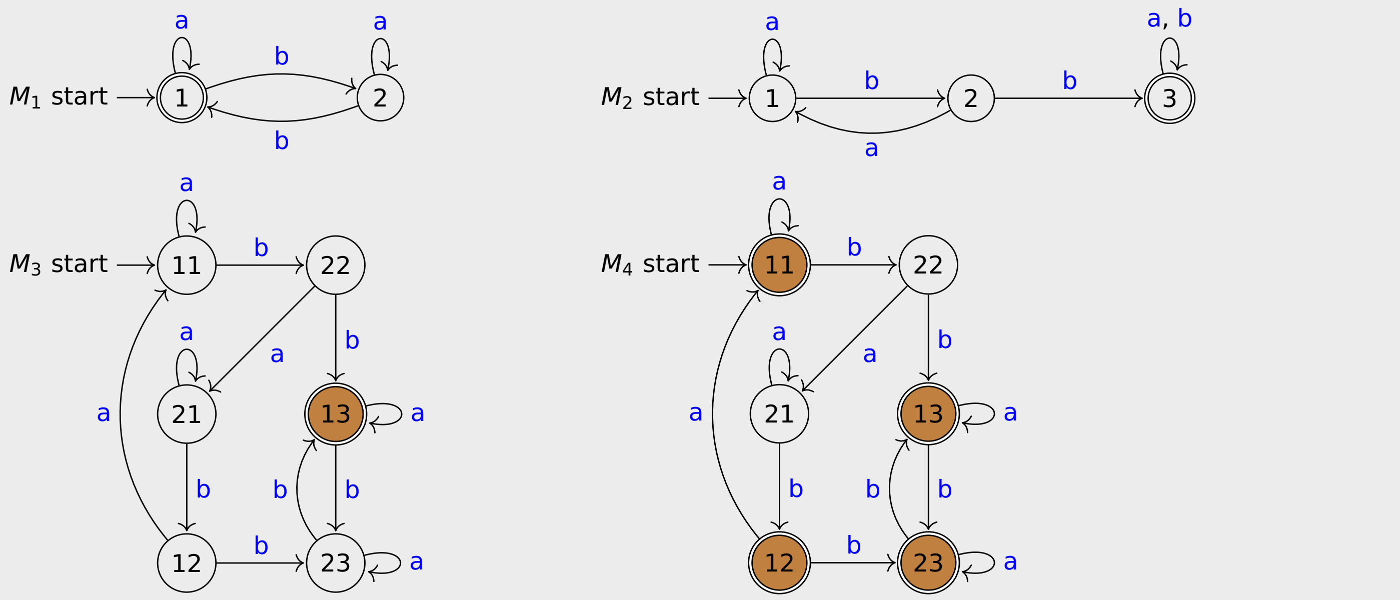
Proof. (closure under union – explicit construction (cont'd))

statement: $L(M_3) = L(M_1) \cup L(M_2)$

$$
\begin{aligned}
\forall x \in \Sigma^*,\ x \in L(M_3) \quad &\Longleftrightarrow \quad \widehat{\delta_3}(s_3, x) \in F_3 \\
&\Longleftrightarrow \quad \widehat{\delta_3}((s_1, s_2), x) \in (F_1 \times Q_2) \cup (Q_1 \times F_2) \\
&\Longleftrightarrow \quad (\widehat{\delta_1}(s_1, x), \widehat{\delta_2}(s_2, x)) \in (F_1 \times Q_2) \cup (Q_1 \times F_2) \\
&\Longleftrightarrow \quad (\widehat{\delta_1}(s_1, x), \widehat{\delta_2}(s_2, x)) \in (F_1 \times Q_2) \ \text{ or } \ (\widehat{\delta_1}(s_1, x), \widehat{\delta_2}(s_2, x)) \in (Q_1 \times F_2) \\
&\Longleftrightarrow \quad \left(\widehat{\delta_1}(s_1, x) \in F_1 \ \text{ and } \ \widehat{\delta_2}(s_2, x) \in Q_2\right) \ \text{ or } \ \left(\widehat{\delta_1}(s_1, x) \in Q_1 \ \text{ and } \ \widehat{\delta_2}(s_2, x) \in F_2\right) \\
&\Longleftrightarrow \quad x \in L(M_1) \text{ or } x \in L(M_2) \\
&\Longleftrightarrow \quad x \in L(M_1) \cup L(M_2)
\end{aligned}
$$

□

Example (union)



$L(M_1) \cap L(M_2) = L(M_3) := \{x \mid x$ contains even number of bs and bb as substring$\}$
$L(M_1) \cup L(M_2) = L(M_4) := \{x \mid x$ contains even number of bs or bb as substring$\}$

Thanks! & Questions?