CMPE 322/327 - Theory of Computation

Week 4: Pattern Matching & Regular Expressions

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March 14-18, 2022

Outline

Definitions

- nondeterministic finite automaton (NFA) is quintuple $N = (Q, \Sigma, \Delta, s, F)$ with

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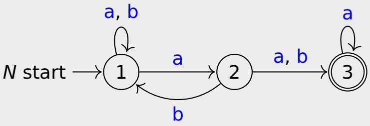

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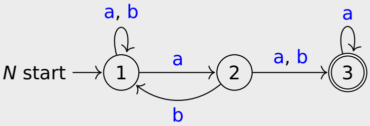

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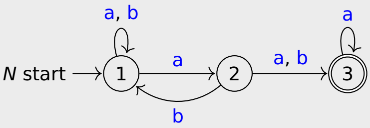

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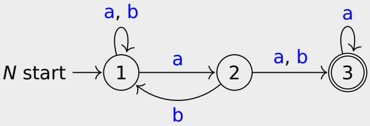

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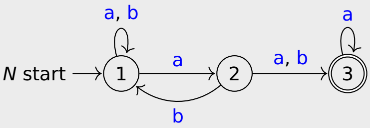

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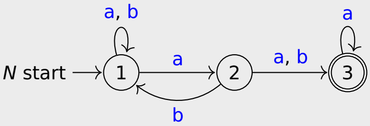

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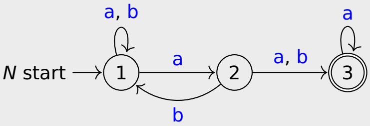

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- string $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \varnothing$

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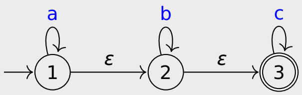

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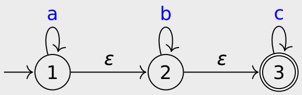

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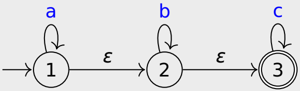

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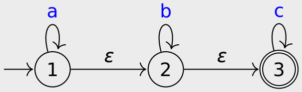

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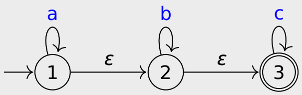

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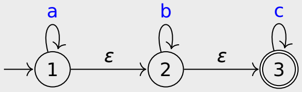

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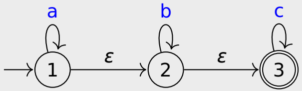

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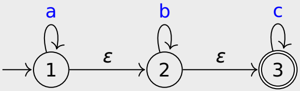

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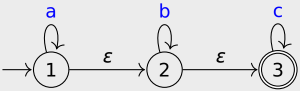

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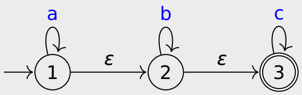

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`grep "b.g" file`	returns lines containing e.g. bag, big, bug, buggy

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$\beta \cap \gamma$	$L(\beta) \cap L(\gamma)$
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Pattern matching is important for

- lexical analysis of programs
- scripting languages (Perl, Ruby)
- search engines (Google Code Search)
- DNA analysis

Definitions

- pattern is string α that represents set of strings $L(\alpha) \subseteq \Sigma^*$
-

atomic pattern α	$L(\alpha)$
$\mathbf{a} \in \Sigma$	$\{a\}$
$\boldsymbol{\epsilon}$	$\{\epsilon\}$
$\boldsymbol{\varnothing}$	$\varnothing$
$\#$	Σ
$@$	Σ^*

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- string $x \in \Sigma^*$ matches pattern α if $x \in L(\alpha)$

Example

pattern	matched string
$@a@a@a@$	strings containing at least 3 occurrences of a

Example

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$@a@a@a@$	strings containing at least 3 occurrences of a
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$(\# \cap \sim a)^*$	strings without a

Questions

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\varepsilon & \equiv & \sim(\#@) \equiv \emptyset^* \\
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\# & \equiv & a_1 \ldots a_n \qquad \text{if } \Sigma = \{a_1 \ldots a_n\}
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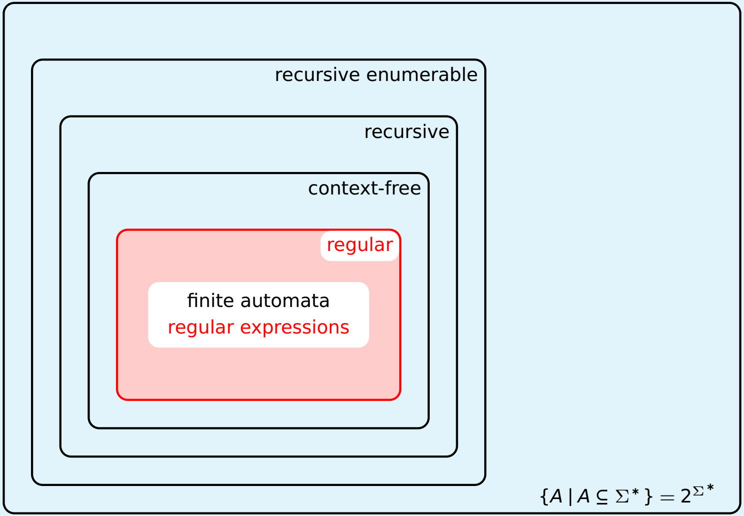
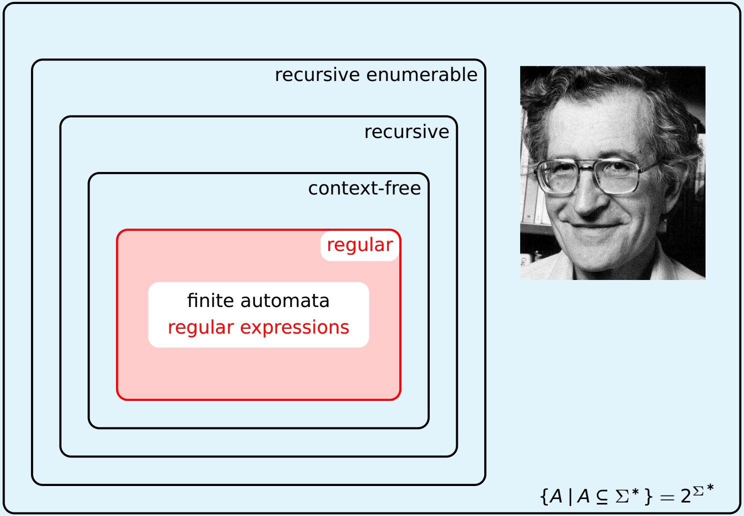
Notation

$\alpha \equiv \beta \quad$ if $L(\alpha) = L(\beta)$

Outline


recursive enumerable
recursive
context-free
regular
finite automata
regular expressions
$\{A \mid A \subseteq \Sigma^*\} = 2^{\Sigma^*}$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Definition

regular expressions are restricted patterns which use only

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for all $A \subseteq \Sigma^*$ ➊ A is regular

$\iff$ ➋ $A = L(\alpha)$ for some pattern α

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➁ $\implies$ ➀ induction on α (see slides $\#16-17$)

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➌	$\Longrightarrow$	➋	trivial	(every regular expression is a pattern)
➋	$\Longrightarrow$	➊	induction on α	(see slides $\#16-17$)
➊	$\Longrightarrow$	➌		(see slide $\#18$)

Proof. (2 $\implies$ 1)

statement: for any pattern α, $L(\alpha)$ is regular

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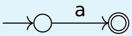
α	$L(\alpha)$	finite automaton
$\boldsymbol{a} \in \Sigma$	$\{a\}$	

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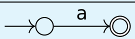
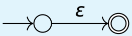
$\boldsymbol{a} \in \Sigma$	$\{a\}$	
$\boldsymbol{\varepsilon}$	$\{\varepsilon\}$	

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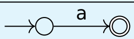
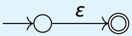
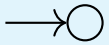
$\boldsymbol{a} \in \Sigma$	$\{a\}$	
$\boldsymbol{\varepsilon}$	$\{\varepsilon\}$	
$\boldsymbol{\emptyset}$	$\emptyset$	

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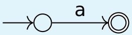
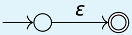
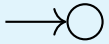
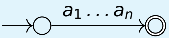
$\boldsymbol{a} \in \Sigma$	$\{a\}$	
$\boldsymbol{\varepsilon}$	$\{\varepsilon\}$	
$\boldsymbol{\varnothing}$	$\varnothing$	

α	$L(\alpha)$	finite automaton
$\#$	Σ	

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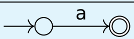
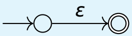
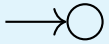
$\boldsymbol{a} \in \Sigma$	$\{a\}$	
$\boldsymbol{\varepsilon}$	$\{\varepsilon\}$	
$\boldsymbol{\varnothing}$	$\varnothing$	

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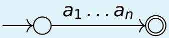
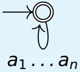
$\#$	Σ	
$@$	Σ^*	

Proof. (➋ $\Longrightarrow$ ➊)

statement: for any pattern α, $L(\alpha)$ is regular

induction on pattern α

➋ compound patterns

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2 compound patterns

α	$L(\alpha)$
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Proof. (② $\Longrightarrow$ ①)

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$L(\beta)$ and $L(\gamma)$ are regular according to induction hypothesis

Proof. (2 $\Longrightarrow$ 1)

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$L(\beta)$ and $L(\gamma)$ are regular according to induction hypothesis

hence $L(\alpha)$ is regular according to closure properties of regular sets

Proof. (1 $\implies$ 3 – An idea)

given NFA$_\varepsilon$ $N_\varepsilon = (Q, \Sigma, \varepsilon, \Delta, S, F)$

$\forall Y \subseteq Q \quad \forall u, v \in Q$ construct regular expression α_{uv}^{Y} such that

$x \in L(\alpha_{uv}^{Y}) \iff$ $\boxed{\exists$ a path from u to v labeled x $(v \in \widehat{\Delta}(\{u\}, x))$ such that all intermediate states belong to $Y}$

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Definitions

- $$\alpha_{uv}^{\emptyset} := \begin{cases} \mathbf{a_1} + \ldots + \mathbf{a_k} & \text{if } u \neq v \text{ and } k > 0 \\ \boldsymbol{\emptyset} & \text{if } u \neq v \text{ and } k = 0 \\ \mathbf{a_1} + \ldots + \mathbf{a_k} + \boldsymbol{\varepsilon} & \text{if } u = v \text{ and } k > 0 \\ \boldsymbol{\varepsilon} & \text{if } u = v \text{ and } k = 0 \end{cases}$$

$$\{a_1, \ldots, a_k\} := \{a \in \Sigma \cup \{\varepsilon\} \mid v \in \Delta(u, a)\}$$

Proof. (1 $\Longrightarrow$ 3 – An idea)

given NFA$_\varepsilon$ $N_\varepsilon = (Q, \Sigma, \varepsilon, \Delta, S, F)$

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- $\alpha_{uv}^{Y} := \alpha_{uv}^{Y-\{q\}} + \alpha_{uq}^{Y-\{q\}} (\alpha_{qq}^{Y-\{q\}})^{*} \alpha_{qv}^{Y-\{q\}}$ for some fixed $q \in Y$

Proof. (① $\implies$ ③ – An idea)

given NFA$_\varepsilon$ $N_\varepsilon = (Q, \Sigma, \varepsilon, \Delta, S, F)$

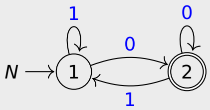
$\forall Y \subseteq Q \quad \forall u, v \in Q \quad$ construct regular expression α^Y_{uv} such that

$x \in L(\alpha^Y_{uv}) \iff$ $\exists$ a path from u to v labeled x ($v \in \widehat{\Delta}(\{u\}, x)$) such that all intermediate states belong to Y

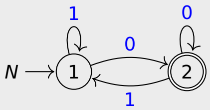
Theorem

$$L(N_\varepsilon) = L\left(\sum_{s\in S, t\in F} \alpha^Q_{st}\right)$$

Example



Example



$L(N) = L(\alpha)$ with

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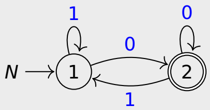
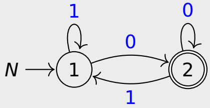


$L(N) = L(\alpha)$ with

$$\alpha \quad = \quad \alpha_{12}^{\{1,2\}}$$

Example



$L(N) = L(\alpha)$ with

$$\alpha \quad = \quad \alpha_{12}^{\{1,2\}} \quad = \quad \alpha_{12}^{\{1\}} + \alpha_{12}^{\{1\}}(\alpha_{22}^{\{1\}})^{*}\alpha_{22}^{\{1\}} \qquad (q = 2)$$

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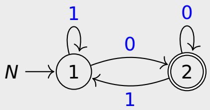

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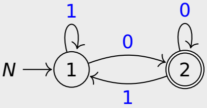

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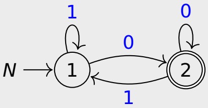

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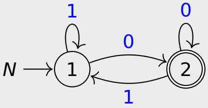
Example



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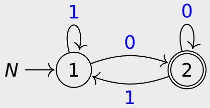

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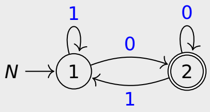

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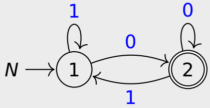

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$$
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\alpha &= \alpha_{12}^{\{1,2\}} = \alpha_{12}^{\{1\}} + \alpha_{12}^{\{1\}}(\alpha_{22}^{\{1\}})^* \alpha_{22}^{\{1\}} \quad (\text{q} = 2) \\
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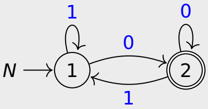

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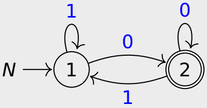

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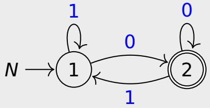

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$$\equiv (0+1)^*0$$

Outline

Theorem

regular sets are effectively closed under homomorphic image and preimage

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Definitions

- homomorphism is mapping $h\colon \Sigma^* \to \Gamma^*$ such that

$$h(\varepsilon) = \varepsilon \qquad\qquad h(xy) = h(x)h(y)$$

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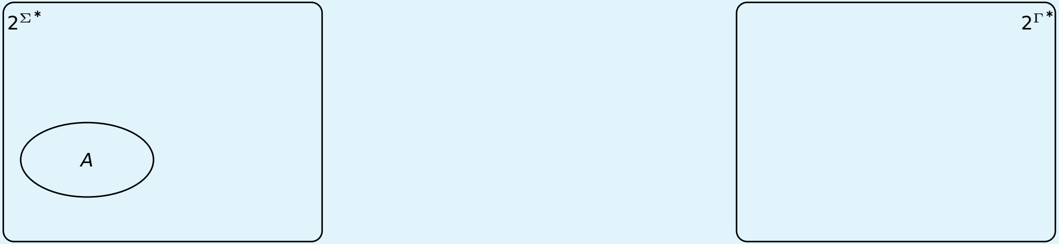
- if $A \subseteq \Sigma^*$ then $\quad h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$ "image of A under h"
  if $B \subseteq \Gamma^*$ then $\quad h^{-1}(B) = \{x \mid h(x) \in B\} \subseteq \Sigma^*$ "preimage of B under h"

2^{Σ^*}

2^{Γ^*}

- homomorphism $h\colon \Sigma^* \to \Gamma^*$



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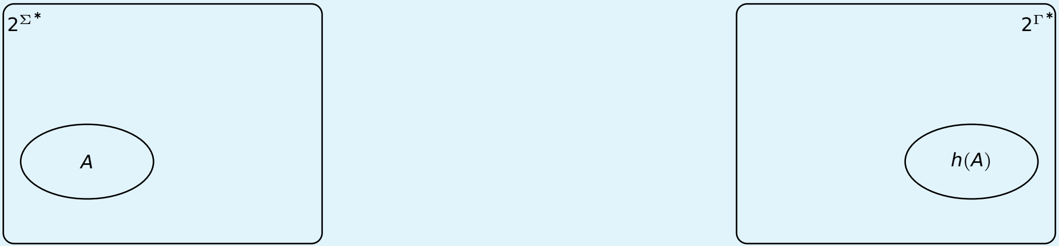

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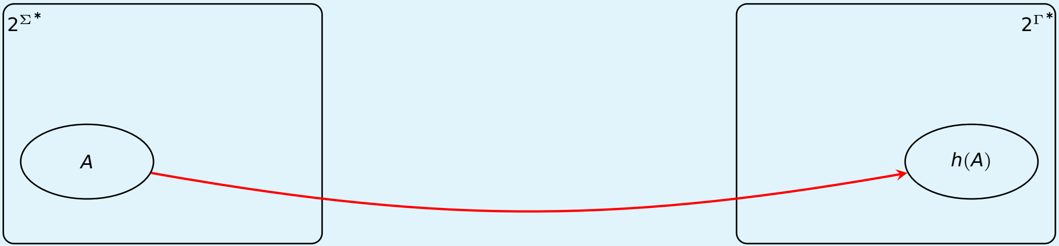

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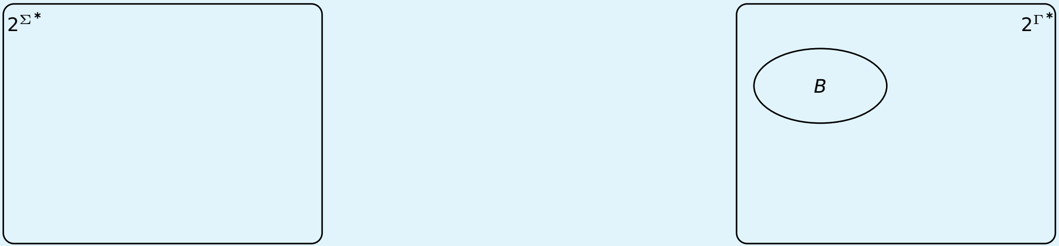
- $h^{-1}(h(A)) \supseteq A$



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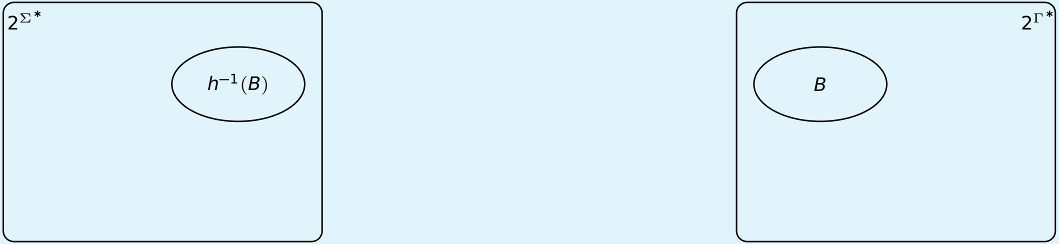

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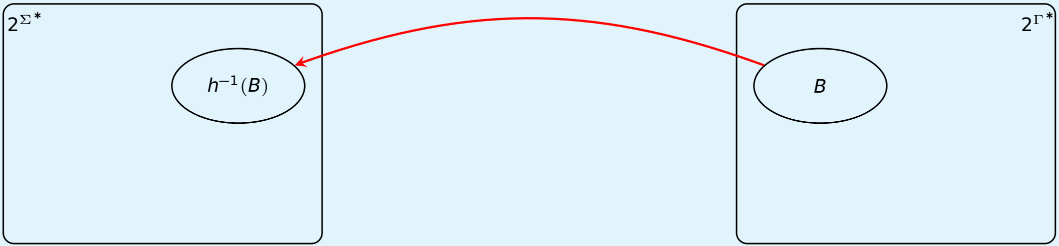

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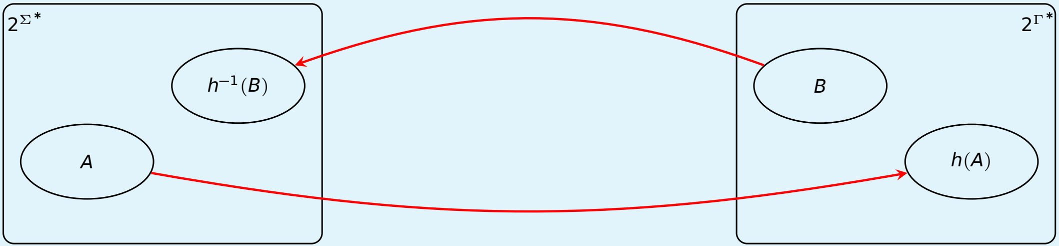

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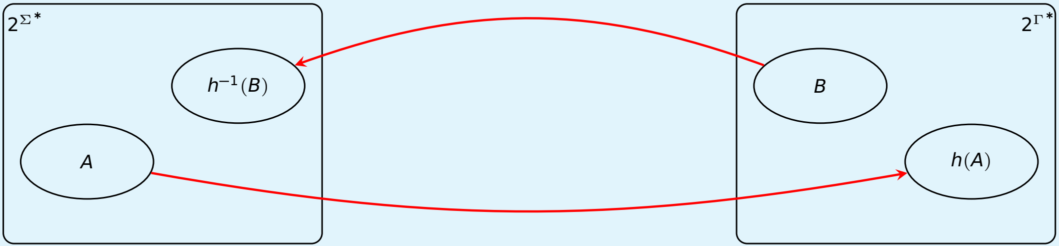

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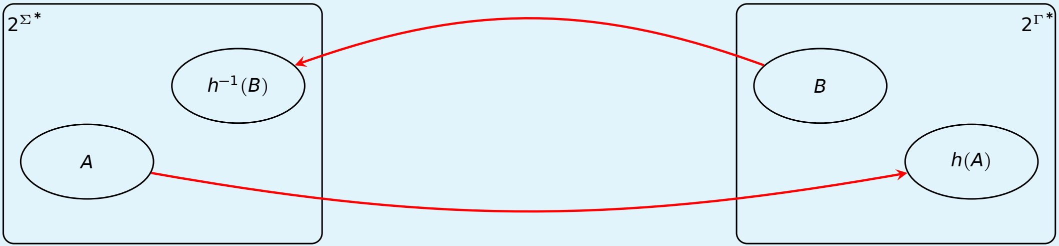
Example

$\Sigma = \Gamma = \{0, 1\} \quad h(0) = 11 \quad h(1) = 1$



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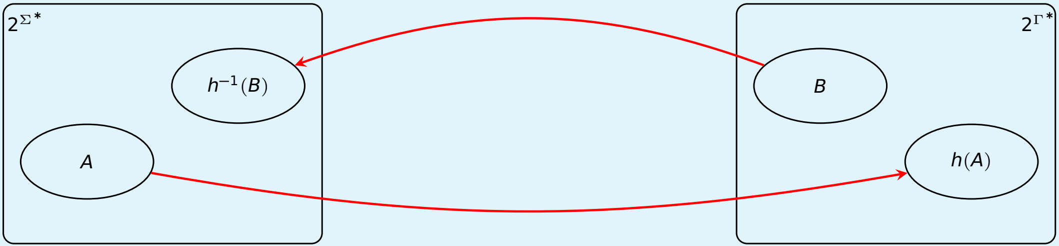

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$A \subseteq \{0,1\}^*$ is regular $\implies$ $\{xy \mid x1y \in A\}$ is regular

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Theorem

regular sets are effectively closed under homomorphic image and preimage

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Proof.

- DFA $M = (Q, \Gamma, \delta, s, F)$

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- $h^{-1}(L(M)) = L(M')$ for DFA $M' = (Q, \Sigma, \delta', s, F)$ with $\delta'(q, a) := \widehat{\delta}(q, h(a))$

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- claim: $\widehat{\delta'}(q, x) = \widehat{\delta}(q, h(x)) \quad \forall x \in \Sigma^* \ \forall q \in Q$

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- claim: $\widehat{\delta'}(q,x) = \widehat{\delta}(q,h(x))$ $\forall x \in \Sigma^*\ \forall q \in Q$
  proof of claim: induction on $|x|$ (see next slide)

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$$\widehat{\delta'}(q, \varepsilon) = q = \widehat{\delta}(q, h(\varepsilon))$$

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- step case: $|x| > 0$ thus $x = ya$ s.t. $|y| = |x| - 1$ with $\mathrm{IH} : \widehat{\delta'}(q,y) = \widehat{\delta}(q,h(y))$

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$$\begin{aligned} \widehat{\delta'}(q,ya) &= \delta'(\widehat{\delta'}(q,y),a) && \text{(by definition of } \widehat{\delta'}\text{)} \\ &= \delta'(\widehat{\delta}(q,h(y)),a) && \text{(by induction hypothesis IH)} \end{aligned}$$

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- step case: $|x| > 0$ thus $x = ya$ s.t. $|y| = |x| - 1$ with IH : $\widehat{\delta'}(q,y) = \widehat{\delta}(q,h(y))$

$$\begin{aligned}
\widehat{\delta'}(q,ya) &= \delta'(\widehat{\delta'}(q,y),a) && \text{(by definition of } \widehat{\delta'}\text{)}\\
&= \delta'(\widehat{\delta}(q,h(y)),a) && \text{(by induction hypothesis IH)}\\
&= \widehat{\delta}(\widehat{\delta}(q,h(y)),h(a)) && \text{(by definition of } \delta'\text{)}
\end{aligned}$$

proof of the claim

claim: $\widehat{\delta'}(q,x) = \widehat{\delta}(q,h(x)) \quad \forall x \in \Sigma^* \ \forall q \in Q$

- base case: $|x| = 0$ thus $x = \varepsilon$

$$\widehat{\delta'}(q,\varepsilon) = q = \widehat{\delta}(q,h(\varepsilon))$$

- step case: $|x| > 0$ thus $x = ya$ s.t. $|y| = |x| - 1$ with IH : $\widehat{\delta'}(q,y) = \widehat{\delta}(q,h(y))$

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&= \widehat{\delta}(q,h(y)h(a)) && \text{(by distributivity of } \widehat{\delta} \text{ – w3.pdf, slide 10)}
\end{aligned}$$

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&= \widehat{\delta}(q,h(y)h(a)) && \text{(by distributivity of } \widehat{\delta} \text{ – w3.pdf, slide 10)}\\
&= \widehat{\delta}(q,h(ya)) && \text{(by definition of homomorphism)}
\end{aligned}$$

proof of the claim

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&= \widehat{\delta}(\widehat{\delta}(q, h(y)), h(a)) && \text{(by definition of } \delta'\text{)} \\
&= \widehat{\delta}(q, h(y)h(a)) && \text{(by distributivity of } \widehat{\delta} \text{ – w3.pdf, slide 10)} \\
&= \widehat{\delta}(q, h(ya)) && \text{(by definition of homomorphism)} \\
&= \widehat{\delta}(q, h(x))
\end{aligned}$$

$\square$

Proof. (closedness under complement homomorphic preimage)

statement: $L(M') = h^{-1}(L(M))$

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$$\forall x \in \Sigma^*,\, x \in L(M') \quad \Longleftrightarrow \quad \widehat{\delta'}(s, x) \in F \qquad \text{(by definition of acceptance)}$$

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statement: $L(M') = h^{-1}(L(M))$

$$
\begin{aligned}
\forall x \in \Sigma^*,\ x \in L(M') \quad &\iff \quad \widehat{\delta'}(s,x) \in F && \text{(by definition of acceptance)} \\
&\iff \quad \widehat{\delta}(s,h(x)) \in F && \text{(by claim proven in slide 25)}
\end{aligned}
$$

Proof. (closedness under complement homomorphic preimage)

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&\iff h(x) \in L(M) && \text{(by definition of acceptance)}
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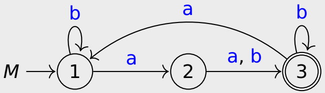
&\iff \widehat{\delta}(s, h(x)) \in F && \text{(by claim proven in slide 25)}\\
&\iff h(x) \in L(M) && \text{(by definition of acceptance)}\\
&\iff x \in h^{-1}(L(M)) && \text{(by definition of homomorphic preimage)}
\end{aligned}
$$

$\square$

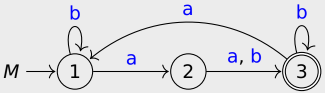
Example

- DFA M



Example

- DFA M

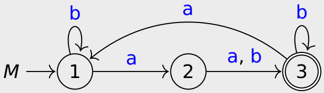


- homomorphism $h\colon \{a, b, c\}^* \to \{a, b\}^*$

$$h(a) = aa \quad h(b) = \varepsilon \quad h(c) = bab$$

Example

- DFA M



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Example

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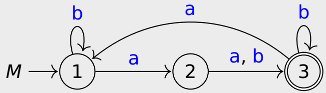

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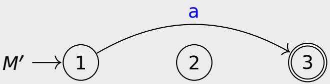
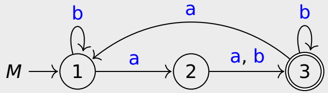


$$\delta'(1, a) = \widehat{\delta}(1, aa) = 3$$

Example

- DFA M



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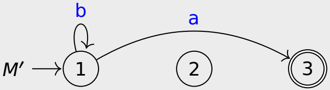
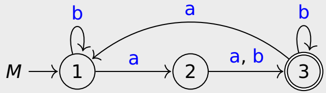


$$\delta'(1,b) = \widehat{\delta}(1,\varepsilon) = 1$$

Example

- DFA M



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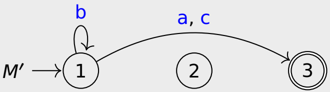
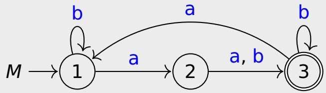


$$\delta'(1, c) = \widehat{\delta}(1, bab) = 3$$

Example

- DFA M



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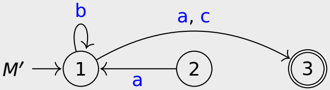
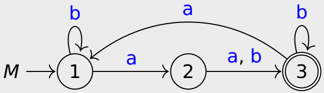


$\delta'(2, a) = \widehat{\delta}(2, aa) = 1$

Example

- DFA M



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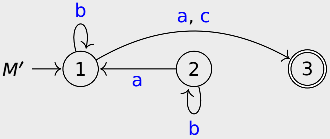

$\delta'(2, b) = \widehat{\delta}(2, \varepsilon) = 2$

Example

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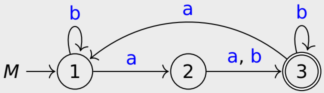

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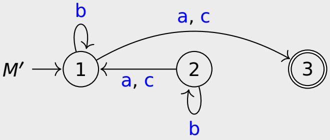

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Example

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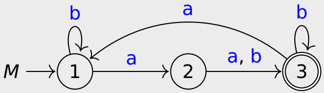

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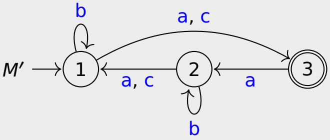
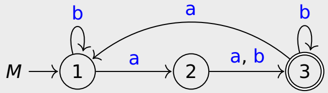


$\delta'(3, a) = \widehat{\delta}(3, aa) = 2$

Example

- DFA M



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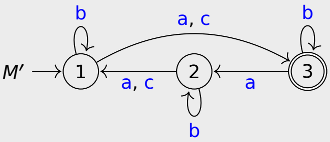

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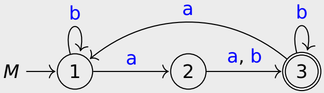

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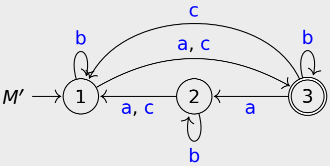

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Theorem

regular sets are effectively closed under homomorphic image and preimage

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$$
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\boldsymbol{\varnothing}' &= \boldsymbol{\varnothing}
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$$\begin{aligned} \mathbf{a}' &= h(\mathbf{a}) \quad \text{for } \mathbf{a} \in \Sigma & \qquad (\beta+\gamma)' &= \beta' + \gamma' \\ \boldsymbol{\varepsilon}' &= \boldsymbol{\varepsilon} & & \\ \boldsymbol{\varnothing}' &= \boldsymbol{\varnothing} & & \end{aligned}$$

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\boldsymbol{\varnothing}' &= \boldsymbol{\varnothing} & &
\end{aligned}
$$

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Definitions

- Hamming distance $H(x, y)$ is number of places where bit strings x and y differ (if $|x| \neq |y|$ then $H(x, y) = \infty$)
- $N_k(A) := \{x \in \{0, 1\}^* \mid H(x, y) \leq k \text{ for some } y \in A\}$

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Lemma

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$D_k = \{x \in (\{0,1\} \times \{0,1\})^* \mid x \text{ contains at most } k \text{ pairs } (0,1) \text{ or } (1,0)\}$ is regular

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&= \{x \in (\{0,1\} \times \{0,1\})^* \mid H(fst(x), snd(x)) \leq k\} \\
N_k(A) &= fst(snd^{-1}(A) \cap D_k)
\end{aligned}
$$

Example

- $A = \{0011\} \quad k = 2$
-

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- $N_k(A)$ consists of
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0	0	1	1

1	0	1	1

0	1	1	1

0	0	0	1

0	0	1	0

-

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- $N_k(A)$ consists of

0	0	1	1	1	0	1	1	0	1	1	1	0	0	0	1	0	0	1	0	1	1	1	1

-

Example

- $A = \{0011\} \quad k = 2$
- $N_k(A)$ consists of

0011	1011	0111	0001	0010	1111
1001					

-

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$\boxed{0}\boxed{0}\boxed{1}\boxed{1}$ $\boxed{1}\boxed{0}\boxed{1}\boxed{1}$ $\boxed{0}\boxed{1}\boxed{1}\boxed{1}$ $\boxed{0}\boxed{0}\boxed{0}\boxed{1}$ $\boxed{0}\boxed{0}\boxed{1}\boxed{0}$ $\boxed{1}\boxed{1}\boxed{1}\boxed{1}$

$\boxed{1}\boxed{0}\boxed{0}\boxed{1}$ $\boxed{1}\boxed{0}\boxed{1}\boxed{0}$

-

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0	0	1	1

1	0	1	1

0	1	1	1

0	0	0	1

0	0	1	0

1	1	1	1

1	0	0	1

1	0	1	0

0	1	0	1

-

Example

- $A = \{0011\} \quad k = 2$
- $N_k(A)$ consists of

0	0	1	1

1	0	1	1

0	1	1	1

0	0	0	1

0	0	1	0

1	1	1	1

1	0	0	1

1	0	1	0

0	1	0	1

0	1	1	0

-

Example

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$\boxed{0}\boxed{0}\boxed{1}\boxed{1}$ $\boxed{1}\boxed{0}\boxed{1}\boxed{1}$ $\boxed{0}\boxed{1}\boxed{1}\boxed{1}$ $\boxed{0}\boxed{0}\boxed{0}\boxed{1}$ $\boxed{0}\boxed{0}\boxed{1}\boxed{0}$ $\boxed{1}\boxed{1}\boxed{1}\boxed{1}$

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-

Example

- $A = \{0011\} \quad k = 2$
- $N_k(A)$ consists of

0011	1011	0111	0001	0010	1111
1001	1010	0101	0110	0000	

- $snd^{-1}(A)$ consists of

0000	0001	0010	0011	0100	0101
0011	0011	0011	0011	0011	0011
0110	0111	1000	1001	1010	1011
0011	0011	0011	0011	0011	0011
1100	1101	1110	1111		
0011	0011	0011	0011		

Example

- $A = \{0011\} \quad k = 2$
- $N_k(A)$ consists of

$\boxed{0|0|1|1}$ $\boxed{1|0|1|1}$ $\boxed{0|1|1|1}$ $\boxed{0|0|0|1}$ $\boxed{0|0|1|0}$ $\boxed{1|1|1|1}$

$\boxed{1|0|0|1}$ $\boxed{1|0|1|0}$ $\boxed{0|1|0|1}$ $\boxed{0|1|1|0}$ $\boxed{0|0|0|0}$

- $snd^{-1}(A) \cap D_k$ consists of

$\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 0&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 0&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 0&0&1&0\\\hline 0&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 0&0&1&1\\\hline 0&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 0&1&0&1\\\hline 0&0&1&1\\\hline\end{array}$

$\begin{array}{|c|c|c|c|}\hline 0&1&1&0\\\hline 0&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 0&1&1&1\\\hline 0&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 1&0&0&1\\\hline 0&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 1&0&1&0\\\hline 0&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 1&0&1&1\\\hline 0&0&1&1\\\hline\end{array}$

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- $fst(snd^{-1}(A) \cap D_k) = N_k(A)$

0000	0001	0010	0011		0101
0011	0011	0011	0011		0011
0110	0111		1001	1010	1011
0011	0011		0011	0011	0011
			1111		
			0011		

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0001	0010	0011	0101
0011	0011	0011	0011

0110	0111	1001	1010	1011
0011	0011	0011	0011	0011

1111
0011

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$\begin{array}{|c|c|c|c|}\hline 0&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 1&0&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 0&1&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 0&0&1&0\\\hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline\end{array}$

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		0011 / 0011		0101 / 0011
0110 / 0011	0111 / 0011	1001 / 0011	1010 / 0011	1011 / 0011
		1111 / 0011		

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$\boxed{1}\boxed{0}\boxed{1}\boxed{1}$ $\boxed{0}\boxed{1}\boxed{1}\boxed{1}$ $\boxed{1}\boxed{1}\boxed{1}\boxed{1}$

$\boxed{1}\boxed{0}\boxed{0}\boxed{1}$ $\boxed{1}\boxed{0}\boxed{1}\boxed{0}$ $\boxed{0}\boxed{1}\boxed{0}\boxed{1}$ $\boxed{0}\boxed{1}\boxed{1}\boxed{0}$

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0110	0111	1001	1010	1011	1111
0011	0011	0011	0011	0011	0011

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1	0	0	1
0	0	1	1

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0	0	1	1

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Thanks! & Questions?