Quiz II (10 pts)

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Assigned : April the 21st, 20h15
Duration : 60 minutes

Q1. (7 pts) Let $\alpha = wz(xyw)^\star x$ and $\beta = wzx(ywx)^\star$ be a pair of regular expressions defined over the alphabet $\Sigma = \{x, y, z, w\}$. Decide whether $\alpha \equiv \beta$ employing derivatives and bisimulation. Justify your reasoning.

A1. We start with partially deriving the expression α with respect to the characters x, y, z and w until no new expression is generated:

$$\begin{aligned}(\alpha)_x &= \emptyset & (\alpha)_y &= \emptyset & (\alpha)_z &= \emptyset\\ (\alpha)_w &= \varepsilon z(xyw)^\star x \\ &= z(xyw)^\star x =: \alpha_1\end{aligned}$$

$$\begin{aligned}(\alpha_1)_x &= \emptyset & (\alpha_1)_y &= \emptyset & (\alpha_1)_z &= \varepsilon(xyw)^\star x\\ &&&& &= (xyw)^\star x =: \alpha_2\\ (\alpha_1)_w &= \emptyset\end{aligned}$$

$$\begin{aligned}(\alpha_2)_x &= \big((xyw)^\star\big)_x x + (x)_x & (\alpha_2)_y &= \emptyset & (\alpha_2)_z &= \emptyset\\ &= (xyw)_x(xyw)^\star x + \varepsilon \\ &= \varepsilon yw(xyw)^\star x + \varepsilon \\ &= yw(xyw)^\star x + \varepsilon =: \alpha_3 \\ (\alpha_2)_w &= \emptyset\end{aligned}$$

$$\begin{aligned}(\alpha_3)_x &= \emptyset & (\alpha_3)_y &= \big(yw(xyw)^\star x\big)_y + (\varepsilon)_y & (\alpha_3)_z &= \emptyset\\ && &= \varepsilon w(xyw)^\star x + \emptyset \\ && &= w(xyw)^\star x =: \alpha_4 \\ (\alpha_3)_w &= \emptyset\end{aligned}$$

$$\begin{aligned}(\alpha_4)_x &= \emptyset & (\alpha_4)_y &= \emptyset & (\alpha_4)_z &= \emptyset\\ (\alpha_4)_w &= \varepsilon(xyw)^\star x \\ &= (xyw)^\star x = \alpha_2\end{aligned}$$

$$\begin{aligned}(\emptyset)_x &= \emptyset & (\emptyset)_y &= \emptyset & (\emptyset)_z &= \emptyset\\ (\emptyset)_w &= \emptyset\end{aligned}$$

We apply the same procedure above for the expression β:

$$\begin{aligned}(\beta)_x &= \emptyset & (\beta)_y &= \emptyset & (\beta)_z &= \emptyset\\ (\beta)_w &= \varepsilon zx(ywx)^\star \\ &= zx(ywx)^\star =: \beta_1\end{aligned}$$

$$\begin{aligned}(\beta_1)_x &= \emptyset & (\beta_1)_y &= \emptyset & (\beta_1)_z &= \varepsilon x(ywx)^\star\\ & & & & &= x(ywx)^\star =: \beta_2\\ (\beta_1)_w &= \emptyset\end{aligned}$$

$$\begin{aligned}(\beta_2)_x &= \varepsilon(ywx)^\star & (\beta_2)_y &= \emptyset & (\beta_2)_z &= \emptyset\\ &= (ywx)^\star =: \beta_3\\ (\beta_2)_w &= \emptyset\end{aligned}$$

$$\begin{aligned}(\beta_3)_x &= (ywx)_x(ywx)^\star & (\beta_3)_y &= (ywx)_y(ywx)^\star & (\beta_3)_z &= (ywx)_z(ywx)^\star\\ &= \emptyset(ywx)^\star & &= \varepsilon wx(ywx)^\star & &= \emptyset(ywx)^\star\\ &= \emptyset & &= wx(ywx)^\star =: \beta_4 & &= \emptyset\\ (\beta_3)_w &= (ywx)_w(ywx)^\star\\ &= \emptyset(ywx)^\star\\ &= \emptyset\end{aligned}$$

$$\begin{aligned}(\beta_4)_x &= \emptyset & (\beta_4)_y &= \emptyset & (\beta_4)_z &= \emptyset\\ (\beta_4)_w &= \varepsilon x(ywx)^\star\\ &= x(ywx)^\star = \beta_2\end{aligned}$$

$$\begin{aligned}(\emptyset)_x &= \emptyset & (\emptyset)_y &= \emptyset & (\emptyset)_z &= \emptyset\\ (\emptyset)_w &= \emptyset\end{aligned}$$

We have the following derivative tables:

	x	y	z	w	
α	$\emptyset$	$\emptyset$	$\emptyset$	α_1	$\uparrow$
α_1	$\emptyset$	$\emptyset$	α_2	$\emptyset$	$\uparrow$
α_2	α_3	$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$
α_3	$\emptyset$	α_4	$\emptyset$	$\emptyset$	$\downarrow$
α_4	$\emptyset$	$\emptyset$	$\emptyset$	α_2	$\uparrow$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$

	x	y	z	w	
β	$\emptyset$	$\emptyset$	$\emptyset$	β_1	$\uparrow$
β_1	$\emptyset$	$\emptyset$	β_2	$\emptyset$	$\uparrow$
β_2	β_3	$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$
β_3	$\emptyset$	β_4	$\emptyset$	$\emptyset$	$\downarrow$
β_4	$\emptyset$	$\emptyset$	$\emptyset$	β_2	$\uparrow$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$

Therefore, the fact that $\alpha \equiv \beta$ follows by the bisimulation $\sim$ that satisfies

$$\begin{aligned}L(\alpha) &\sim L(\beta)\\ L(\alpha_1) &\sim L(\beta_1)\\ L(\alpha_2) &\sim L(\beta_2)\\ L(\alpha_3) &\sim L(\beta_3)\\ L(\alpha_4) &\sim L(\beta_4)\\ L(\emptyset) &\sim L(\emptyset).\end{aligned}$$

Q2. (3 pts) Simplify the regular expression

$$\alpha := (x^\star y)^\star + \big(x^\star (yx^\star)z\big)^\star (x^\star y)^\star xx^\star + \varepsilon$$

defined over the alphabet $\Sigma = \{x, y, z\}$ as much as possible benefiting Kleene Algebra axioms and rules (A.1) – (A.17). Clearly show simplification steps.

A2.

Lemma. $\forall x \in \Sigma, xx^\star = x^\star x.$

Proof.

$$\begin{aligned} xx^\star &= x(\varepsilon x)^\star && \text{(by A.7)} \\ &= (x\varepsilon)^\star x && \text{(by A.17)} \\ &= x^\star x && \text{(by A.8)} \end{aligned}$$

□

Therefore,

$$\begin{aligned} \alpha &:= (x^\star y)^\star + \big(x^\star (yx^\star)z\big)^\star (x^\star y)^\star xx^\star + \varepsilon \\ &= (x^\star y)^\star + \big(x^\star (yx^\star)z\big)^\star (x^\star y)^\star x^\star x + \varepsilon && \text{(by the lemma above)} \\ &= (x^\star y)^\star + \big(x^\star (yx^\star)z\big)^\star x^\star (yx^\star)^\star x + \varepsilon && \text{(by A.17)} \\ &= (x^\star y)^\star + \big(x + (yx^\star)z\big)^\star (yx^\star)^\star x + \varepsilon && \text{(by A.16)} \end{aligned}$$

Important Notice:

- Collaboration is strictly and positively prohibited; lowers your score to 0 if detected.
- Any submission after 60 minutes will NOT be accepted. Please beware and respect the deadline!
- All handwritten answers should somehow be scanned into a single pdf file, and only then submitted. Make sure that your handwriting is decent and readable.