Quiz I (10 pts)

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Assigned : March the 17th, 20h00
Duration : 60 minutes

Q1 (7 pts). Design a deterministic finite automaton (DFA) $M = (Q, \Sigma = \{a, b\}, \delta, s, F)$ that recognizes the language $\mathcal{L} := \{x \in \Sigma^* \mid \#a(x) \geqslant 3 \;\wedge\; \#b(x) \leqslant 2\}$. Recall that $\#a(x)$ and $\#b(x)$ respectively denote the number of 'a's and number of 'b's contained in a given string x.

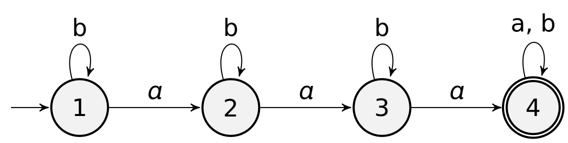
Below are a few examples to the input-output harmony of the intended DFA:

string	reaction of M
aaa	accepted
baaab	accepted
babaa	accepted
baaaba	accepted
bb	rejected
bab	rejected
baab	rejected
baabaab	rejected
⋮	⋮

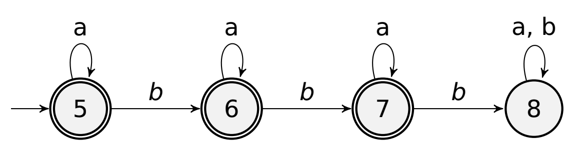
Hint: Individually design automata M_1 and M_2 recognizing languages $\{x \in \Sigma^* \mid \#a(x) \geqslant 3\}$ and $\{x \in \Sigma^* \mid \#b(x) \leqslant 2\}$ respectively, and only then compute the product automaton $M = M_1 \times M_2$ accepting the set of strings $\mathcal{L}$.

A1.

A transition function δ_1 of DFA $M_1 := \big(\{1, 2, 3, 4\}, \{a, b\}, \delta_1, 1, \{4\}\big)$ which recognizes the language $\{x \in \Sigma^* \mid \#a(x) \geqslant 3\}$ is pictured below.



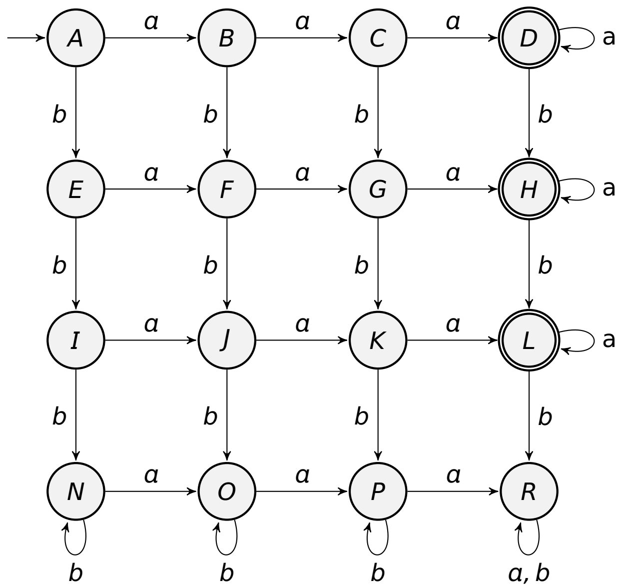
We then depict below, a transition function δ_2 of DFA $M_2 := \big(\{5, 6, 7, 8\}, \{a, b\}, \delta_2, 5, \{5, 6, 7\}\big)$ that recognizes the language $\{x \in \Sigma^* \mid \#b(x) \leqslant 2\}$.



Finally, we demonstrate a transition function δ of the product DFA

$$M = M_1 \times M_2 := \big(\{A, B, C, D, E, F, G, H, I, J, K, L, N, O, P, R\}, \{a, b\}, \delta, A, \{D, H, L\}\big)$$

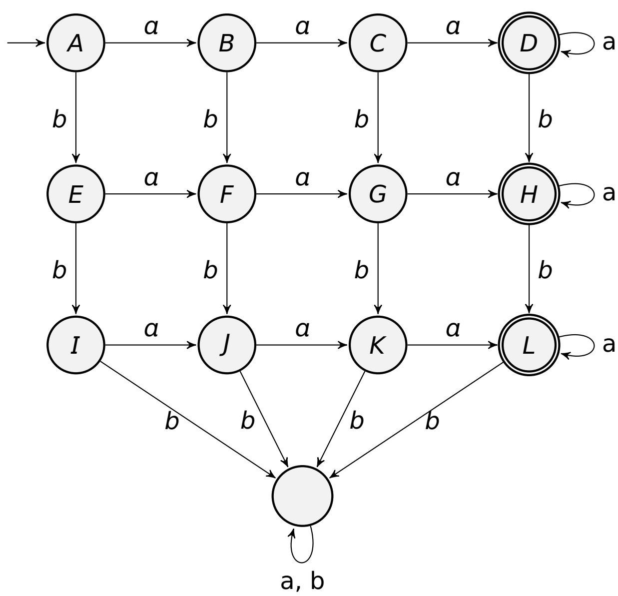
which recognizes the language $\mathcal{L}$ in the drawing below.



where

$$\begin{array}{lllllllllll} A & = & (1,5) & E & = & (1,6) & I & = & (1,7) & N & = & (1,8) \\ B & = & (2,5) & F & = & (2,6) & J & = & (2,7) & O & = & (2,8) \\ C & = & (3,5) & G & = & (3,6) & K & = & (3,7) & P & = & (3,8) \\ D & = & (4,5) & H & = & (4,6) & L & = & (4,7) & R & = & (4,8). \end{array}$$

As the states N, O, P and R are indistinguishable, the transition function δ could be simplified into the following.



Q2 (3 pts). We recursively define a function $|\cdot|$: String $\to \mathbb{N}$ that computes the length of a given string defined over some alphabet Σ as follows:

$$|\varepsilon| := 0 \qquad\qquad |xa| := |x| + 1 \quad \text{for all } a \in \Sigma.$$

Prove employing the mathematical induction principle that below equation

$$|xy| = |x| + |y|$$

holds for all strings $x, y \in \Sigma^*$.

Hint: Argue by induction over the length of the string y.

A2.

Proof. We argue by mathematical induction over the length of the string y:

1. base case: $|y| = 0$ thus $y = \varepsilon$

$$|x\varepsilon| = |x| = |x| + |\varepsilon|$$

2. step case: $|y| > 0$ thus $y = za$ such that $|z| = |y| - 1$ with $\begin{array}{lll} \text{given IH} & : & |xz| = |x| + |z| \\ \text{show} & : & |xza| = |x| + |za| \end{array}$

$$\begin{array}{rcll} |xy| = |xza| & = & |xz| + 1 & \text{by definition of } |\cdot| \\ & = & |x| + |z| + 1 & \text{by IH} \\ & = & |x| + |za| & \text{by definition of } |\cdot| \\ & = & |x| + |y| & \end{array}$$

□

Important Notice:

- Collaboration is strictly and positively prohibited; lowers your score to 0 if detected.
- Any submission after 60 minutes will NOT be accepted. Please beware and respect the deadline!
- All handwritten answers should somehow be scanned into a single pdf file, and only then submitted. Make sure that your handwriting is decent and readable.