A Ouick Recap

# CMPE 322/327 - Theory of Computation

Week 3: Nondeterministic Finite State Automata & Epsilon Transitions

Burak Ekici

March 7-11, 2022

**Epsilon Transitions** 

#### Outline

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- 1 A Quick Recap
- 2 Nondeterministic Finite Automata

# Definitions

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• deterministic finite automaton (DFA) is quintuple  $M = (Q, \Sigma, \delta, s, F)$  with

**Epsilon Transitions** 

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  - **1 Q**: finite set of states

**Epsilon Transitions** 

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① Q: finite set of states

**②** Σ: input alphabet A Ouick Recap

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**⑤**  $\delta$  :  $Q \times \Sigma \rightarrow Q$  : transition function

**Epsilon Transitions** 

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•  $\hat{\delta}: Q \times \Sigma^* \to Q$  is inductively defined by

$$\widehat{\delta}(q, \varepsilon) := q$$
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A Quick Recap

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A Ouick Recap

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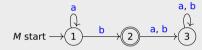
$$\widehat{\delta}(q, \varepsilon) := q$$
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- string  $x \in \Sigma^*$  is accepted by M if  $\widehat{\delta}(s, x) \in F$
- string  $x \in \Sigma^*$  is rejected by M if  $\widehat{\delta}(s,x) \notin F$
- language accepted by M is given by  $L(M) := \{x \mid \widehat{\delta}(x,s) \in F\}$

$$M = (Q, \Sigma, \delta, s, F)$$

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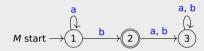


**Epsilon Transitions** 

$$M = (Q, \Sigma, \delta, s, F)$$

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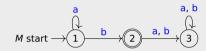
**Epsilon Transitions** 

① 
$$Q = \{1, 2, 3\}$$

$$M = (Q, \Sigma, \delta, s, F)$$

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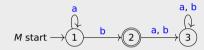
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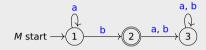
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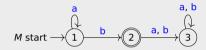


**Epsilon Transitions** 

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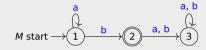
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**Epsilon Transitions** 

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- **4** s = 1

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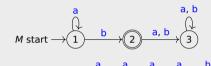
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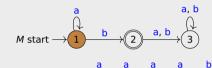
**6** 
$$F = \{2\}$$

$$M = (Q, \Sigma, \delta, s, F)$$



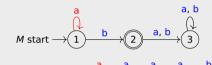
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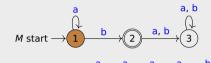
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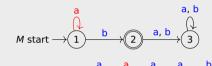
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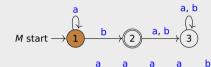
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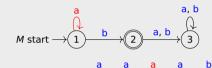
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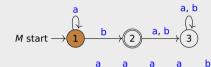
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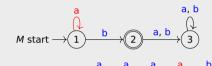
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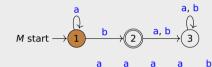
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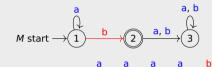
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$$M = (Q, \Sigma, \delta, s, F)$$



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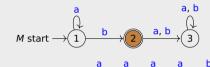


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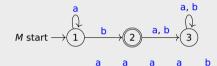




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 $\in L(M)$ 

$$M = (Q, \Sigma, \delta, s, F)$$

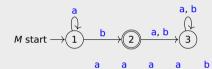


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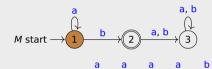


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**Epsilon Transitions** 

- b b а





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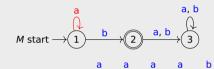
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  1 1 2
- 2 3 3 3 3 3

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- a a b b
- 1 1 1 2 3

$$M = (Q, \Sigma, \delta, s, F)$$

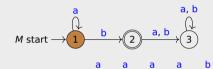


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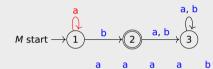


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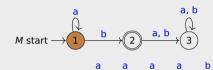
- b b а





A Ouick Recap

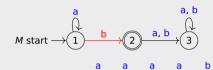
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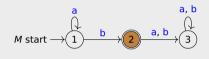


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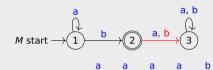
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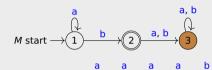


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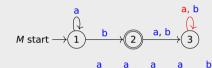
**Epsilon Transitions** 

- b b а

 $\in L(M)$ 

# Example (DFA → Regular Sets)

$$M = (Q, \Sigma, \delta, s, F)$$



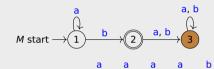
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  1 1 2
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- 1 1 1 2 3



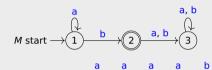


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**Epsilon Transitions** 

- b b а

$$M = (Q, \Sigma, \delta, s, F)$$



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- **a** s = 1
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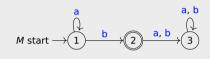
- $\begin{array}{c|cccc} \delta & a & b \\ \hline 1 & 1 & 2 \end{array}$
- 2 3 3 3 3 3

- 1 1 1 1 1
  - a a b b a  $\not\in L(M)$

 $\in L(M)$ 

1 1 1 2 3

$$M = (Q, \Sigma, \delta, s, F)$$



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- **a** s = 1
- ⑤  $F = \{2\}$

- δ a b
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- 2 3 3 3 3 3

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b

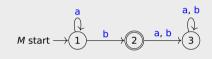
1 1 1 2 3 3

$$L(M) := \{x \mid$$

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## Example (DFA → Regular Sets)





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- $\bigcirc Q = \{1, 2, 3\}$
- $\Sigma = \{a, b\}$
- **4** s = 1
- **⑤**  $F = \{2\}$

- - b  $\notin L(M)$ а b

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 $\in L(M)$ 

3 3

 $L(M) := \{x \mid x = a^n b, n \ge 0\}$ 

A Quick Recap

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set  $A \subseteq \Sigma^*$  is regular if A = L(M) for some DFA M

#### Definition

A Ouick Recap

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set  $A \subseteq \Sigma^*$  is regular if A = L(M) for some DFA M

#### Theorem

regular sets are effectively closed under intersection, complement and union

# Outline

A Ouick Recap

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- 2 Nondeterministic Finite Automata
- 3 Epsilon Transitions

A Ouick Recap



A Ouick Recap

• nondeterministic finite automaton (NFA) is quintuple  $N=(Q,\Sigma,\Delta,S,F)$  with Q:

# Definitions

A Ouick Recap

 nondeterministic finite automaton (NFA) is quintuple N =  $(Q, \Sigma, \Delta, S, F)$ with

① Q:  $2 \Sigma$ : finite set of states input alphabet

(NFA)

is

quintuple

N =

 $(Q, \Sigma, \Delta, S, F)$ 

with

A Ouick Recap

• nondeterministic finite automaton •  $\mathbf{0}$  Q: finite set of states •  $\mathbf{\Sigma}$ : input alphabet •  $\mathbf{\Delta}: Q \times \Sigma \rightarrow \mathbf{2}^Q$ : transition function

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with

A Ouick Recap

• nondeterministic finite automaton (NFA) is quintuple  $N=(Q,\Sigma,\Delta,S,F)$  finite set of states

**⑤**  $\Delta: Q \times \Sigma \rightarrow {\color{red} 2^Q}:$  transition function

 $4 S \subseteq Q$ : set of start states

#### **Definitions**

A Ouick Recap

 nondeterministic finite (NFA) quintuple N =  $(Q, \Sigma, \Delta, S, F)$ with automaton is

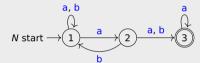
**1** Q: finite set of states  $\Sigma$ : input alphabet

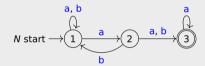
transition function

 $\bigcirc S \subseteq Q$ : set of start states

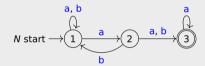
 $\bigcirc$   $F \subseteq Q$ : final (accept) states

A Quick Recap



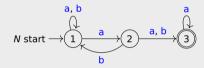


① 
$$Q = \{1, 2, 3\}$$

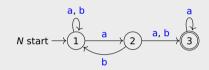


$$\bigcirc Q = \{1, 2, 3\}$$

$$\Sigma = \{a, b\}$$



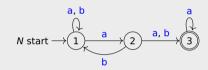
- $\bigcirc Q = \{1, 2, 3\}$
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① 
$$Q = \{1, 2, 3\}$$

$\Delta$	a	b
1	{1,2}	{1}
2	{3}	{1,3}
3	{3}	Ø

A Ouick Recap

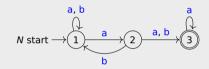


① 
$$Q = \{1, 2, 3\}$$

$$\mathbf{Q} \Sigma = \{\mathbf{a}, \mathbf{b}\}$$

$$\P$$
  $S = \{1\}$ 

$\Delta$	a	b
1	{1,2}	{1}
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$$\bigcirc Q = \{1, 2, 3\}$$

$$\mathbf{Q} \Sigma = \{\mathbf{a}, \mathbf{b}\}$$

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**6** 
$$F = \{3\}$$

A Ouick Recap

• nondeterministic finite automaton (NFA) is quintuple  $N=(Q,\Sigma,\Delta,S,F)$  with

**1)** Q: finite set of states Q: input alphabet

**6)**  $\Delta: Q \times \Sigma \rightarrow 2^Q:$  transition function

 $4 \le Q$ : set of start states

⑤  $F \subseteq Q$ : final (accept) states

•  $\widehat{\Delta} \colon 2^Q \times \Sigma^* \to 2^Q$  is inductively defined by

 $\widehat{\Delta}(A, \varepsilon) := A$   $\widehat{\Delta}(A, xa) := \bigcup_{q \in \widehat{\Delta}(A, x)} \Delta(q, a)$ 

A Ouick Recap

#### **Definitions** nondeterministic finite automaton (NFA) is quintuple $(Q, \Sigma, \Delta, S, F)$ with **(1)** O: finite set of states $\Sigma$ : input alphabet **⑤** $\Delta : O \times \Sigma \rightarrow 2^{Q} :$ transition function $\bigcirc S \subseteq O$ : set of start states ⑤ $F \subseteq Q$ : final (accept) states • $\widehat{\Delta}: 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by $\widehat{\Delta}(A, \varepsilon) := A$ $\widehat{\Delta}(A, xa) :=$ $\Delta(q,a)$ $a \in \widehat{\Delta}(A, x)$ • string $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \emptyset$

# Example (Unfolding of the multistep function $\widehat{\Delta}$ )

Let x = ababba over the alphabet  $\Sigma = \{a, b\}$ 

A Ouick Recap

# Example (Unfolding of the multistep function $\widehat{\Delta}$ )

Let 
$$x = ababba$$
 over the alphabet  $\Sigma = \{a, b\}$   $\bigcup (q \in \widehat{\Delta}(A, ababb), a)$ 

A Ouick Recap

1<sup>st</sup> rec. call

**Epsilon Transitions** 

# Example (Unfolding of the multistep function $\widehat{\Delta}$ )

Let 
$$x = ababba$$
 over the alphabet  $\Sigma = \{a, b\}$   $\bigcup (q \in \widehat{\Delta}(A, ababb), a)$   $\bigcup (q \in \bigcup (q \in \widehat{\Delta}(A, abab) \Delta(q, b)) \Delta(q, a))$ 

A Ouick Recap

1<sup>st</sup> rec. call 2<sup>nd</sup> rec. call

# Example (Unfolding of the multistep function $\widehat{\Delta}$ )

A Ouick Recap

Let x = ababba over the alphabet  $\Sigma = \{a, b\}$  $\bigcup (q \in \widehat{\Delta}(A, ababb), a)$  $\bigcup (q \in \bigcup (q \in \widehat{\Delta}(A, abab) \, \Delta(q, b)) \, \Delta(q, a))$  $\bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A, aba) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))$ 

1<sup>st</sup> rec. call 2<sup>nd</sup> rec. call 3<sup>rd</sup> rec. call

**Epsilon Transitions** 

# Example (Unfolding of the multistep function $\widehat{\Delta}$ )

```
 \begin{array}{ll} \text{Let } x = ababba \text{ over the alphabet } \Sigma = \{a,b\} \\ & \bigcup (q \in \widehat{\Delta}(A,ababb),a) & 1^{\text{st}} \text{ rec. call} \\ & \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,abab) \, \Delta(q,b)) \, \Delta(q,a)) & 2^{\text{nd}} \text{ rec. call} \\ & \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,aba) \, \Delta(q,b)) \, \Delta(q,b)) \, \Delta(q,a)) & 3^{\text{rd}} \text{ rec. call} \\ & \bigcup (q \in \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,ab) \, \Delta(q,a)) \, \Delta(q,b)) \, \Delta(q,a)) & 4^{\text{th}} \text{ rec. call} \\ \end{array}
```

```
Example (Unfolding of the multistep function \widehat{\Delta})
```

A Ouick Recap

```
Let x = ababba over the alphabet \Sigma = \{a,b\} \bigcup (q \in \widehat{\Delta}(A,ababb),a) \qquad \qquad 1^{\text{st}} \text{ rec. call } \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,abab) \, \Delta(q,b)) \, \Delta(q,a)) \qquad \qquad 2^{\text{nd}} \text{ rec. call } \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,aba) \, \Delta(q,b)) \, \Delta(q,b)) \, \Delta(q,a)) \qquad \qquad 3^{\text{rd}} \text{ rec. call } \bigcup (q \in \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,aba) \, \Delta(q,b)) \, \Delta(q,b)) \, \Delta(q,b)) \, \Delta(q,a)) \qquad \qquad 4^{\text{th}} \text{ rec. call } \bigcup (q \in \widehat{\Delta}(A,ab) \, \Delta(q,b)) \, \Delta(q,b)) \, \Delta(q,b)) \, \Delta(q,b)) \, \Delta(q,a)) \qquad \qquad 5^{\text{th}} \text{ rec. call }
```

```
Example (Unfolding of the multistep function \widehat{\Delta})
```

```
Let x = ababba over the alphabet \Sigma = \{a,b\} \bigcup (q \in \widehat{\Delta}(A,ababb),a) \qquad \qquad 1^{\text{st}} \text{ rec. call } \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,abab) \triangle(q,b)) \triangle(q,a)) \qquad \qquad 2^{\text{nd}} \text{ rec. call } \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,aba) \triangle(q,b)) \triangle(q,b)) \triangle(q,a)) \qquad \qquad 3^{\text{rd}} \text{ rec. call } \bigcup (q \in \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,aba) \triangle(q,a)) \triangle(q,b)) \triangle(q,a)) \qquad \qquad 4^{\text{th}} \text{ rec. call } \bigcup (q \in \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,ab) \triangle(q,a)) \triangle(q,b)) \triangle(q,b)) \triangle(q,a)) \qquad \qquad 5^{\text{th}} \text{ rec. call } \bigcup (q \in \widehat{\Delta}(A,a) \triangle(q,b)) \triangle(q,a)) \triangle(q,b)) \triangle(q,a)) \triangle(q,b) \triangle(q,a)) \qquad 6^{\text{th}} \text{ rec. call }
```

```
Example (Unfolding of the multistep function \widehat{\Delta})
```

```
Let x = ababba over the alphabet \Sigma = \{a,b\}  \bigcup (q \in \widehat{\Delta}(A,ababb),a) \qquad \qquad 1^{\text{st}} \text{ rec. call }   \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,abab) \triangle(q,b)) \triangle(q,a)) \qquad \qquad 2^{\text{nd}} \text{ rec. call }   \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,aba) \triangle(q,b)) \triangle(q,b)) \triangle(q,a)) \qquad \qquad 3^{\text{rd}} \text{ rec. call }   \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,aba) \triangle(q,b)) \triangle(q,b)) \triangle(q,b)) \triangle(q,a)) \qquad \qquad 4^{\text{th}} \text{ rec. call }   \bigcup (q \in \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,ab) \triangle(q,a)) \triangle(q,b)) \triangle(q,b)) \triangle(q,a)) \qquad \qquad 5^{\text{th}} \text{ rec. call }   \bigcup (q \in \widehat{\Delta}(A,a) \triangle(q,a)) \triangle(q,b)) \triangle(q,a)) \triangle(q,b)) \triangle(q,a)) \qquad 6^{\text{th}} \text{ rec. call }   \bigcup (q \in A(A,a) \triangle(q,a)) \triangle(q,b)) \triangle(q,a)) \triangle(q,b)) \triangle(q,a)) \qquad 6^{\text{th}} \text{ rec. call }
```

## Example (Unfolding of the multistep function $\widehat{\Delta}$ )

```
Let x = ababba over the alphabet \Sigma = \{a,b\}  \bigcup (q \in \widehat{\Delta}(A,ababb),a) \qquad \qquad 1^{\text{st}} \text{ rec. call }   \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,abab),\Delta(q,b)) \Delta(q,a)) \qquad \qquad 2^{\text{nd}} \text{ rec. call }   \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,aba)) \Delta(q,b)) \Delta(q,b)) \Delta(q,a)) \qquad \qquad 3^{\text{rd}} \text{ rec. call }   \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,aba)) \Delta(q,b)) \Delta(q,b)) \Delta(q,b)) \Delta(q,a)) \qquad \qquad 4^{\text{th}} \text{ rec. call }   \bigcup (q \in \bigcup (q \in \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A,ab)) \Delta(q,b)) \Delta(q,b)) \Delta(q,b)) \Delta(q,b)) \Delta(q,a)) \qquad \qquad 5^{\text{th}} \text{ rec. call }   \bigcup (q \in \widehat{\Delta}(A,a)) \Delta(q,b)) \Delta(q,b)) \Delta(q,b)) \Delta(q,b)) \Delta(q,a)) \qquad 6^{\text{th}} \text{ rec. call }   \bigcup (q \in A\Delta(q,a)) \Delta(q,b)) \Delta(q,b)) \Delta(q,b)) \Delta(q,b)) \Delta(q,a))   \bigcup (q \in B\Delta(q,b)) \Delta(q,a)) \Delta(q,b)) \Delta(q,a)) \qquad \text{assuming } \bigcup (q \in A\Delta(q,a)) = B
```

## Example (Unfolding of the multistep function $\widehat{\Delta}$ )

A Ouick Recap

```
Let x = ababba over the alphabet \Sigma = \{a, b\}
      | | (q \in \widehat{\Delta}(A, ababb), a) |
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       1st rec. call
      \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A, abab) \Delta(q, b)) \Delta(q, a))
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       2<sup>nd</sup> rec. call
      ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, aba) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       3<sup>rd</sup> rec. call
      ||(q \in ||(q \in ||(q \in \Delta(A, ab) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       4<sup>th</sup> rec. call
      ||(q \in ||(q \in ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, a) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       5th rec. call
      \bigcup(q \in \bigcap(A, \varepsilon) \Delta(q, a)) \Delta(q, b)) \Delta(q, 
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       6th rec. call
      \bigcup (q \in A \Delta(q, a)) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))
      \bigcup (q \in B \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a)) \quad \text{assuming } \bigcup (q \in A \Delta(q, a)) = B
      ||(q \in ||(q \in ||(q \in ||(q \in C\Delta(q, a))\Delta(q, b))\Delta(q, b))\Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                  assuming | |(q \in B \Delta(q, b)) = C
```

# Example (Unfolding of the multistep function $\widehat{\Delta}$ )

```
Let x = ababba over the alphabet \Sigma = \{a, b\}
     | | (q \in \widehat{\Delta}(A, ababb), a) |
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        1st rec. call
     \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A, abab) \Delta(q, b)) \Delta(q, a))
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        2<sup>nd</sup> rec. call
     ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, aba) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        3<sup>rd</sup> rec. call
     ||(q \in ||(q \in ||(q \in \Delta(A, ab) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        4<sup>th</sup> rec. call
     ||(q \in ||(q \in ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, a) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        5th rec. call
     | | (a \in \widehat{\Delta}(A, \varepsilon) \Delta(q, a)) \Delta(q, b)) \Delta(q, b))
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        6th rec. call
     \bigcup (q \in A \Delta(q, a)) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))
     \bigcup (q \in B \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))
                                                                                                                                                                                                                                                                                                                                                                                                                       assuming \bigcup (q \in A \Delta(q, a)) = B
     ||(q \in ||(q \in ||(q \in ||(q \in C\Delta(q, a))\Delta(q, b))\Delta(q, b))\Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                             assuming | |(q \in B \Delta(q, b)) = C
     ||(q \in ||(q \in ||(q \in D\Delta(q, b))\Delta(q, b))\Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                              assuming | |(q \in C \Delta(q, a)) = D
```

## Example (Unfolding of the multistep function $\widehat{\Delta}$ )

```
Let x = ababba over the alphabet \Sigma = \{a, b\}
     | | (q \in \widehat{\Delta}(A, ababb), a) |
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               1st rec. call
     \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A, abab) \Delta(q, b)) \Delta(q, a))
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               2<sup>nd</sup> rec. call
     ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, aba) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               3<sup>rd</sup> rec. call
     ||(q \in ||(q \in ||(q \in \Delta(A, ab) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               4<sup>th</sup> rec. call
     ||(q \in ||(q \in ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, a) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               5th rec. call
     | | (a \in \widehat{\Delta}(A, \varepsilon) \Delta(q, a)) \Delta(q, b)) \Delta(q, b))
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               6th rec. call
     \bigcup (q \in A \Delta(q, a)) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))
     \bigcup (q \in B \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))
                                                                                                                                                                                                                                                                                                                                                                                                    assuming \bigcup (q \in A \Delta(q, a)) = B
     ||(q \in ||(q \in ||(q \in ||(q \in C\Delta(q, a))\Delta(q, b))\Delta(q, b))\Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                         assuming | |(q \in B \Delta(q, b)) = C
     ||(q \in ||(q \in ||(q \in D\Delta(q, b))\Delta(q, b))\Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                           assuming | |(q \in C \Delta(q, a)) = D
                                                                                                                                                                                                                                                                                                                                                                                                           assuming | |(q \in D \Delta(q, b))| = E
     \bigcup (q \in \bigcup (q \in E \Delta(q, b)) \Delta(q, a))
```

A Ouick Recap

```
Let x = ababba over the alphabet \Sigma = \{a, b\}
     | | (q \in \widehat{\Delta}(A, ababb), a) |
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   1st rec. call
     \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A, abab) \Delta(q, b)) \Delta(q, a))
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   2<sup>nd</sup> rec. call
     ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, aba) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   3<sup>rd</sup> rec. call
     ||(q \in ||(q \in ||(q \in \Delta(A, ab) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   4<sup>th</sup> rec. call
     ||(q \in ||(q \in ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, a) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   5th rec. call
     \bigcup (q \in \widehat{\Delta}(A, \varepsilon) \Delta(q, a)) \Delta(q, b)) \Delta(
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   6th rec. call
     \bigcup (q \in A \Delta(q, a)) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))
     \bigcup (q \in B \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))
                                                                                                                                                                                                                                                                                                                                                                                             assuming \bigcup (q \in A \Delta(q, a)) = B
     ||(q \in ||(q \in ||(q \in ||(q \in C\Delta(q, a))\Delta(q, b))\Delta(q, b))\Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                assuming | |(q \in B \Delta(q, b)) = C
     ||(q \in ||(q \in ||(q \in D\Delta(q, b))\Delta(q, b))\Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                 assuming | |(q \in C \Delta(q, a)) = D
     | | (q \in | | (q \in E \Delta(q, b)) \Delta(q, a)) |
                                                                                                                                                                                                                                                                                                                                                                                                 assuming \bigcup (q \in D \Delta(q, b)) = E
     ||(q \in F \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                 assuming ||(q \in E \Delta(q, b))| = F
```

## Example (Unfolding of the multistep function $\widehat{\Delta}$ )

```
Let x = ababba over the alphabet \Sigma = \{a, b\}
     | | (q \in \widehat{\Delta}(A, ababb), a) |
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         1st rec. call
     \bigcup (q \in \bigcup (q \in \widehat{\Delta}(A, abab) \Delta(q, b)) \Delta(q, a))
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         2<sup>nd</sup> rec. call
     ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, aba) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         3<sup>rd</sup> rec. call
     ||(q \in ||(q \in ||(q \in \Delta(A, ab) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         4<sup>th</sup> rec. call
     ||(q \in ||(q \in ||(q \in ||(q \in ||(q \in \widehat{\Delta}(A, a) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         5th rec. call
     \bigcup (q \in \widehat{\Delta}(A, \varepsilon) \Delta(q, a)) \Delta(q, b)) \Delta(
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       6<sup>th</sup> rec. call
     \bigcup (q \in A \Delta(q, a)) \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))
     \bigcup (q \in B \Delta(q, b)) \Delta(q, a)) \Delta(q, b)) \Delta(q, b)) \Delta(q, a))
                                                                                                                                                                                                                                                                                                                                                                                   assuming \bigcup (q \in A \Delta(q, a)) = B
     ||(q \in ||(q \in ||(q \in ||(q \in C\Delta(q, a))\Delta(q, b))\Delta(q, b))\Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                       assuming | |(q \in B \Delta(q, b)) = C
     ||(q \in ||(q \in ||(q \in D\Delta(q, b))\Delta(q, b))\Delta(q, a))||
                                                                                                                                                                                                                                                                                                                                                                                        assuming | |(q \in C \Delta(q, a)) = D
     | | (q \in | | (q \in E \Delta(q, b)) \Delta(q, a)) |
                                                                                                                                                                                                                                                                                                                                                                                        assuming \bigcup (q \in D \Delta(q, b)) = E
     | | (q \in F \Delta(q, a))|
                                                                                                                                                                                                                                                                                                                                                                                        assuming ||(q \in E \Delta(q, b))| = F
                                                                                                                                                                                                                                                                                                                                                                                       assuming \bigcup (q \in F \Delta(q, a)) = G
     G
```

Lemma ( $\widehat{\Delta}$  distributes)

A Ouick Recap

$$\widehat{\Delta}(A, xy) = \widehat{\Delta}(\widehat{\Delta}(A, x), y) \quad \forall A \subseteq Q_N \text{ and } x, y \in \Sigma^*$$

$$\widehat{\Delta}(A, xy) = \widehat{\Delta}(\widehat{\Delta}(A, x), y) \quad \forall A \subseteq Q_N \text{ and } x, y \in \Sigma^*$$

## Proof.

A Ouick Recap

We argue by induction on |y|:

## Lemma ( $\widehat{\Delta}$ distributes)

$$\widehat{\Delta}(A, xy) = \widehat{\Delta}(\widehat{\Delta}(A, x), y) \quad \forall A \subseteq Q_N \text{ and } x, y \in \Sigma^*$$

### Proof.

We argue by induction on |y|:

• base case: |y| = 0 thus  $y = \varepsilon$ 

$$\widehat{\Delta}(A, x\varepsilon) = \widehat{\Delta}(A, x) = \widehat{\Delta}(\widehat{\Delta}(A, x), \varepsilon)$$

## Lemma ( $\widehat{\Delta}$ distributes)

$$\widehat{\Delta}(A, xy) = \widehat{\Delta}(\widehat{\Delta}(A, x), y) \quad \forall A \subseteq Q_N \text{ and } x, y \in \Sigma^*$$

### Proof.

We argue by induction on |y|:

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A Ouick Recap

We argue by induction on |y|:

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$$\widehat{\Delta}(A, x\varepsilon) = \widehat{\Delta}(A, x) = \widehat{\Delta}(\widehat{\Delta}(A, x), \varepsilon)$$

**Epsilon Transitions** 

$$\widehat{\Delta}(A,xzb) = \bigcup_{q \in \widehat{\Delta}(A,xz)} \Delta(q,b)$$
 (by definition of  $\widehat{\Delta}$ )

$$\widehat{\Delta}(A, xy) = \widehat{\Delta}(\widehat{\Delta}(A, x), y) \quad \forall A \subseteq Q_N \text{ and } x, y \in \Sigma^*$$

A Ouick Recap

We argue by induction on |y|:

• base case: |y| = 0 thus  $y = \varepsilon$ 

$$\widehat{\Delta}(A, x\varepsilon) = \widehat{\Delta}(A, x) = \widehat{\Delta}(\widehat{\Delta}(A, x), \varepsilon)$$

**Epsilon Transitions** 

$$\begin{array}{lll} \widehat{\Delta}(\textit{A},\textit{xzb}) & = & \bigcup\limits_{q \in \widehat{\Delta}(\textit{A},\textit{xz})} \Delta(q,b) & \text{(by definition of } \widehat{\Delta}) \\ & = & \bigcup\limits_{q \in \widehat{\Delta}(\widehat{\Delta}(\textit{A},\textit{x}),\textit{z})} \Delta(q,b) & \text{(by IH)} \end{array}$$

$$\widehat{\Delta}(A, xy) = \widehat{\Delta}(\widehat{\Delta}(A, x), y) \quad \forall A \subseteq Q_N \text{ and } x, y \in \Sigma^*$$

A Ouick Recap

We argue by induction on |y|:

• base case: |y| = 0 thus  $y = \varepsilon$ 

$$\widehat{\Delta}(A, x\varepsilon) = \widehat{\Delta}(A, x) = \widehat{\Delta}(\widehat{\Delta}(A, x), \varepsilon)$$

**Epsilon Transitions** 

$$\begin{array}{lll} \widehat{\Delta}(\textit{A},\textit{xzb}) & = & \bigcup_{q \in \widehat{\Delta}(\textit{A},\textit{xz})} \Delta(q,b) & \text{(by definition of } \widehat{\Delta}) \\ & = & \bigcup_{q \in \widehat{\Delta}(\widehat{\Delta}(\textit{A},\textit{x}),z)} \Delta(q,b) & \text{(by IH)} \\ & = & \widehat{\Delta}(\widehat{\Delta}(\textit{A},\textit{x}),zb) & \text{(by definition of } \widehat{\Delta}) \end{array}$$

$$\widehat{\Delta}(A, xy) = \widehat{\Delta}(\widehat{\Delta}(A, x), y) \quad \forall A \subseteq Q_N \text{ and } x, y \in \Sigma^*$$

A Ouick Recap

We argue by induction on |y|:

• base case: |y| = 0 thus  $y = \varepsilon$ 

$$\widehat{\Delta}(A, x\varepsilon) = \widehat{\Delta}(A, x) = \widehat{\Delta}(\widehat{\Delta}(A, x), \varepsilon)$$

**Epsilon Transitions** 

$$\begin{array}{lll} \widehat{\Delta}(\textit{A},\textit{xzb}) & = & \bigcup_{q \in \widehat{\Delta}(\textit{A},\textit{xz})} \Delta(q,b) & \text{(by definition of } \widehat{\Delta}) \\ & = & \bigcup_{q \in \widehat{\Delta}(\widehat{\Delta}(\textit{A},\textit{x}),z)} \Delta(q,b) & \text{(by IH)} \\ & = & \widehat{\Delta}(\widehat{\Delta}(\textit{A},\textit{x}),zb) & \text{(by definition of } \widehat{\Delta}) \\ & = & \widehat{\Delta}(\widehat{\Delta}(\textit{A},\textit{x}),y) & \end{array}$$

A Quick Recap

every set accepted by NFA is regular

### Theorem

A Ouick Recap

every set accepted by NFA is regular

## Proof.

• NFA  $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$ 

### Theorem

A Ouick Recap

every set accepted by NFA is regular

- NFA  $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- L(N) = L(M) for some DFA  $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$  with

A Ouick Recap

every set accepted by NFA is regular

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A Ouick Recap

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A Ouick Recap

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### Proof.

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- L(N) = L(M) for some DFA  $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$  with

  - $\bigcirc \delta_M(A,a) := \widehat{\Delta}(A,a)$

 $\forall A \subseteq Q_N \ \forall a \in \Sigma$ 

**Epsilon Transitions** 

every set accepted by NFA is regular

### Proof.

A Ouick Recap

- NFA  $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- L(N) = L(M) for some DFA  $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$  with

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A Ouick Recap

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claim:  $\widehat{\delta_M}(A, x) = \widehat{\Delta}(A, x) \quad \forall A \subseteq Q \text{ and } x \in \Sigma^*$ 

A Ouick Recap

every set accepted by NFA is regular

- NFA  $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- L(N) = L(M) for some DFA  $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$  with

  - $\bigcirc \delta_M(A,a) := \widehat{\Delta}(A,a)$  $\forall A \subseteq Q_N \ \forall a \in \Sigma$

  - claim:  $\widehat{\delta_M}(A, x) = \widehat{\Delta}(A, x) \quad \forall A \subseteq Q \text{ and } x \in \Sigma^*$ by induction on |x| see next slide proof:

claim:  $\widehat{\delta_M}(A, x) = \widehat{\Delta_N}(A, x) \quad \forall A \subseteq Q_N \text{ and } x \in \Sigma^*$ 

claim:  $\widehat{\delta_M}(A, x) = \widehat{\Delta_N}(A, x) \quad \forall A \subseteq Q_N \text{ and } x \in \Sigma^*$ 

• base case: |x| = 0 thus  $x = \varepsilon$ 

$$\widehat{\delta_M}(A, \varepsilon) = A = \widehat{\Delta_N}(A, \varepsilon)$$

A Ouick Recap

claim:  $\widehat{\delta_M}(A, x) = \widehat{\Delta_N}(A, x) \quad \forall A \subseteq Q_N \text{ and } x \in \Sigma^*$ 

• base case: |x| = 0 thus  $x = \varepsilon$ 

$$\widehat{\delta_M}(A, \varepsilon) = A = \widehat{\Delta_N}(A, \varepsilon)$$

• step case: |x| > 0 thus x = ya s.t. |y| = |x| - 1 with  $H : \widehat{\delta_M}(A, y) = \widehat{\Delta_N}(A, y)$ 

A Ouick Recap

claim:  $\widehat{\delta_M}(A, x) = \widehat{\Delta_N}(A, x) \quad \forall A \subseteq Q_N \text{ and } x \in \Sigma^*$ 

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$$\widehat{\delta_M}(A, ya) = \delta_M(\widehat{\delta_M}(A, y), a)$$
 (by definition of  $\widehat{\delta_M}$ )

A Ouick Recap

claim:  $\widehat{\delta_M}(A, x) = \widehat{\Delta_N}(A, x) \quad \forall A \subseteq Q_N \text{ and } x \in \Sigma^*$ 

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 (by definition of  $\widehat{\delta_M}$ )
$$= \delta_M(\widehat{\Delta_N}(A,y),a)$$
 (by induction hypothesis IH)

12/27

A Ouick Recap

claim:  $\widehat{\delta_M}(A, x) = \widehat{\Delta_N}(A, x) \quad \forall A \subseteq Q_N \text{ and } x \in \Sigma^*$ 

• base case: |x| = 0 thus  $x = \varepsilon$ 

$$\widehat{\delta_M}(A, \varepsilon) = A = \widehat{\Delta_N}(A, \varepsilon)$$

**Epsilon Transitions** 

• step case: |x| > 0 thus x = ya s.t. |y| = |x| - 1 with  $H : \widehat{\delta_M}(A, y) = \widehat{\Delta_N}(A, y)$ 

$$\widehat{\delta_M}(A,ya) = \delta_M(\widehat{\delta_M}(A,y),a)$$
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$$= \delta_M(\widehat{\Delta_N}(A,y),a)$$
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$$= \widehat{\Delta_N}(\widehat{\Delta_N}(A,y),a)$$
 (by definition of  $\delta_M$ )

A Ouick Recap

claim:  $\widehat{\delta_M}(A, x) = \widehat{\Delta_N}(A, x) \quad \forall A \subseteq Q_N \text{ and } x \in \Sigma^*$ 

• base case: |x| = 0 thus  $x = \varepsilon$ 

$$\widehat{\delta_M}(A, \varepsilon) = A = \widehat{\Delta_N}(A, \varepsilon)$$

**Epsilon Transitions** 

• step case: |x| > 0 thus x = ya s.t. |y| = |x| - 1 with  $H : \widehat{\delta_M}(A, y) = \widehat{\Delta_N}(A, y)$ 

$$\begin{array}{lll} \widehat{\delta_M}(A,ya) & = & \delta_M(\widehat{\delta_M}(A,y),a) & \text{(by definition of } \widehat{\delta_M}) \\ & = & \delta_M(\widehat{\Delta_N}(A,y),a) & \text{(by induction hypothesis IH)} \\ & = & \widehat{\Delta_N}(\widehat{\Delta_N}(A,y),a) & \text{(by definition of } \delta_M) \\ & = & \widehat{\Delta_N}(A,ya) & \text{(by distributivity of } \widehat{\Delta}) \end{array}$$

A Ouick Recap

 $\widehat{\delta_M}(A, x) = \widehat{\Delta_N}(A, x) \quad \forall A \subseteq O_N \text{ and } x \in \Sigma^*$ claim:

• base case: |x| = 0 thus  $x = \varepsilon$ 

$$\widehat{\delta_M}(A,\varepsilon) = A = \widehat{\Delta_N}(A,\varepsilon)$$

• step case: |x| > 0 thus x = ya s.t. |y| = |x| - 1 with  $IH : \widehat{\delta_M}(A, y) = \widehat{\Delta_N}(A, y)$ 

$$\begin{array}{lll} \widehat{\delta_M}(A,ya) & = & \delta_M(\widehat{\delta_M}(A,y),a) & \text{(by definition of } \widehat{\delta_M}) \\ & = & \delta_M(\widehat{\Delta_N}(A,y),a) & \text{(by induction hypothesis IH)} \\ & = & \widehat{\Delta_N}(\widehat{\Delta_N}(A,y),a) & \text{(by definition of } \delta_M) \\ & = & \widehat{\Delta_N}(A,ya) & \text{(by distributivity of } \widehat{\Delta}) \\ & = & \widehat{\Delta_N}(A,x) & \end{array}$$

# Proof. (NFA regularity)

A Ouick Recap

statement: L(M) = L(N)

## Proof. (NFA regularity)

statement: L(M) = L(N)

 $\forall x \in \Sigma^*, x \in L(M) \iff \widehat{\delta_M}(s_M, x) \in F_M$ 

(by definition of acceptance)

## Proof. (NFA regularity)

A Ouick Recap

statement: L(M) = L(N)

 $\forall x \in \Sigma^*, x \in L(M)$  $\iff \widehat{\delta_M}(S_M, X) \in F_M$ 

 $\widehat{\delta_M}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\}$ 

(by definition of acceptance) (by definition of  $s_M$  and  $F_M$ )

## Proof. (NFA regularity)

A Ouick Recap

statement: L(M) = L(N)

 $\forall x \in \Sigma^*, \, x \in L(M) \quad \iff \quad \widehat{\delta_M}(s_M, x) \in F_M \qquad \qquad \text{(by definition of acceptance)}$ 

 $\iff$   $\widehat{\delta_M}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\}$  (by definition of  $s_M$  and  $F_M$ )

 $\iff$   $\widehat{\Delta_N}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\}$  (by claim proven in slide 12)

# Proof. (NFA regularity)

A Ouick Recap

statement: L(M) = L(N)

$$\forall x \in \Sigma^*, x \in L(M) \iff \widehat{\delta_M}(s_M, x) \in F_M$$

$$\iff \widehat{\delta_M}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\}$$

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$$\iff \widehat{\Delta_N}(S_N,x) \cap F_N \neq \emptyset$$

(by definition of acceptance)

**Epsilon Transitions** 

(by definition of  $s_M$  and  $F_M$ ) (by claim proven in slide 12)

(by set comprehension)

A Ouick Recap

statement: 
$$L(M) = L(N)$$

ement: 
$$L(M) = L(N)$$

$$\forall x \in \Sigma^*, x \in L(M) \iff \widehat{\delta_M}(s_M, x) \in F_M$$

$$\iff \widehat{\delta_M}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\}$$

$$\iff \widehat{\Delta_N}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\}$$

$$\iff \widehat{\Delta_N}(S_N, x) \cap F_N \neq \emptyset$$

$$\iff x \in L(N)$$

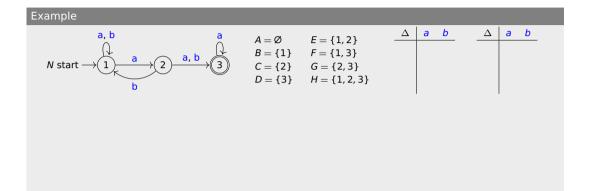
**Epsilon Transitions** 

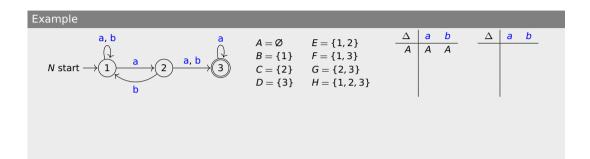
(by definition of acceptance) (by definition of  $s_M$  and  $F_M$ )

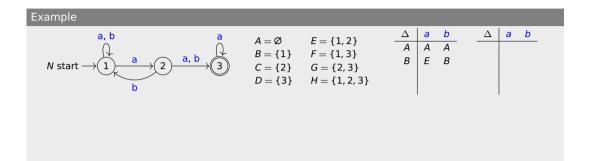
(by claim proven in slide 12)

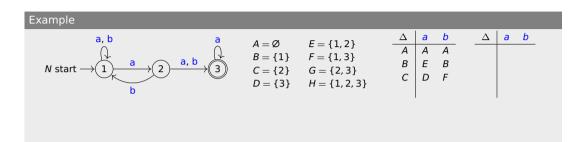
(by set comprehension)

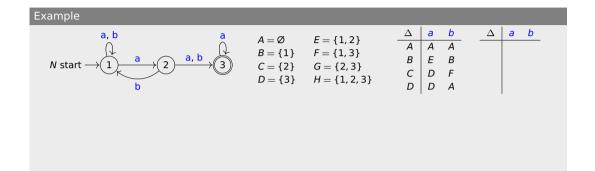
(by definition of acceptance)

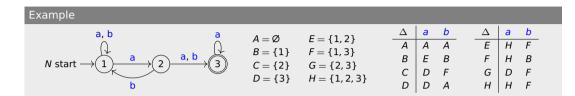


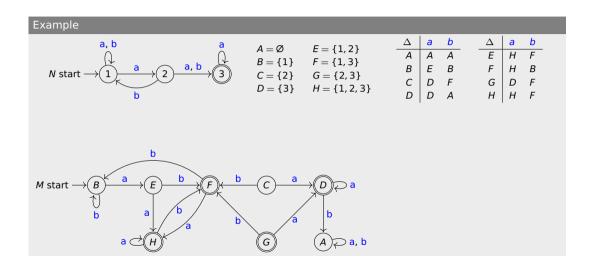




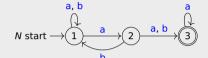








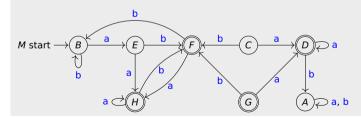




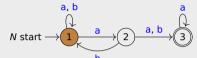
$$A = \emptyset$$
  
 $B = \{1\}$ 

$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 

b





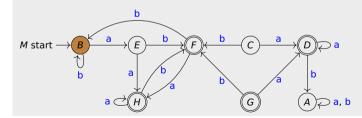


$$A = \emptyset$$

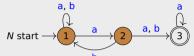
$$A = \emptyset$$
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 $B = \{1\}$   $F = \{1, 3\}$   
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D

Δ	а	b
Ε	Н	F
F	Н	В
G	D	F
Н	Н	F

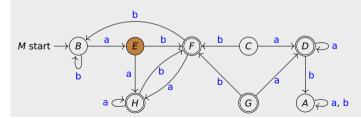




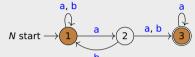


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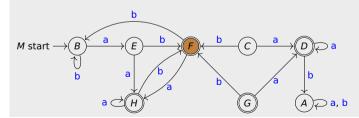




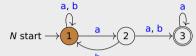


$$A = \emptyset$$
  
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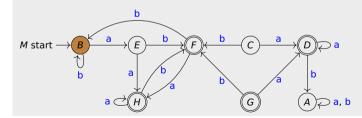
$$A = \emptyset$$
  $E = \{1, 2\}$   
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 $C = \{2\}$   $G = \{2, 3\}$   
 $D = \{3\}$   $H = \{1, 2, 3\}$ 

$$\begin{array}{c|cccc} \Delta & a & b \\ \hline A & A & A \\ B & E & B \end{array}$$

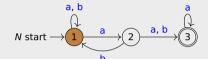
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Н







$$A = \emptyset$$

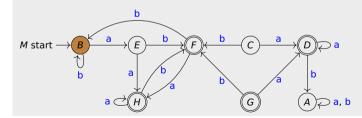
$$A = \emptyset$$
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D

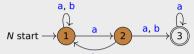
$$\begin{array}{c|cccc} \Delta & a & b \\ \hline E & H & F \\ F & H & B \\ G & D & F \\ \end{array}$$

Н

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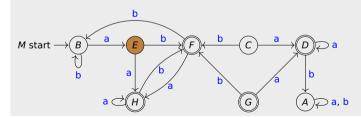


$$A = \emptyset$$

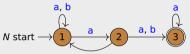
$$A = \emptyset$$
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$$B = \{1\}$$
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 $C = \{2\}$   $G = \{2, 3\}$   
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$$\begin{array}{c|cccc}
\Delta & a & b \\
\hline
A & A & A \\
\hline
B & 5 & B
\end{array}$$



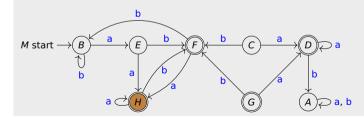




$$A = \emptyset$$
  
 $B = \{1\}$ 

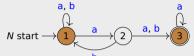
$$A = \emptyset$$
  $E = \{1, 2\}$   
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D





A Ouick Recap

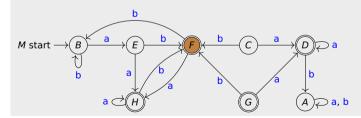


$$A = \emptyset$$

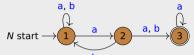
$$A = \emptyset$$
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 $D = \{3\}$   $H = \{1, 2, 3\}$ 

D





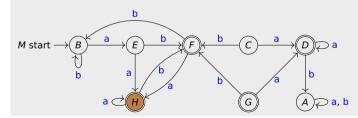


$$A = \emptyset$$

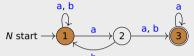
$$A = \emptyset$$
  $E = \{1, 2\}$   
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D

b



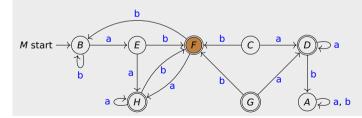




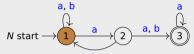
$$A = Q$$

$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$ 

$$B = \{1\}$$
  $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 



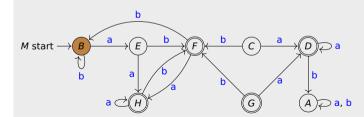




$$A = \emptyset$$
 $B = \{1\}$ 

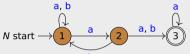
$$A = \emptyset$$
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 $B = \{1\}$   $F = \{1, 3\}$   
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D





A Ouick Recap

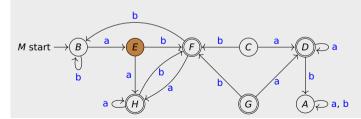


$$A = \emptyset$$
  
 $B = \{1\}$ 

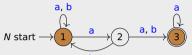
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 $D = \{3\}$   $H = \{1, 2, 3\}$ 

D







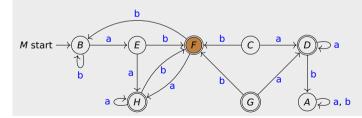
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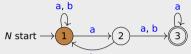
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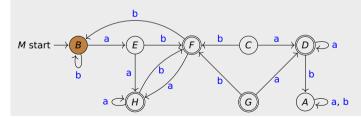




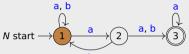
$$A = \emptyset$$

$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 

D





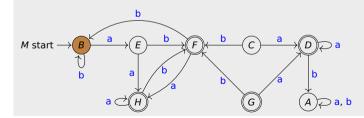


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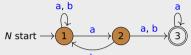
$$A = \emptyset$$
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b

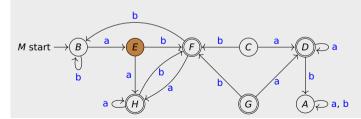




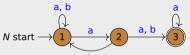
$$A = \emptyset$$

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  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 

 $D = \{3\}$   $H = \{1, 2, 3\}$ 



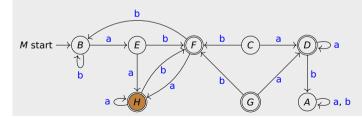






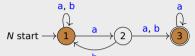
$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 

D





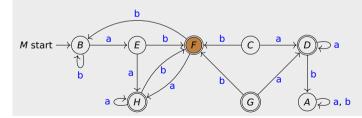
A Ouick Recap



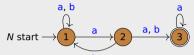
$$A = \emptyset$$

$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 

 $D = \{3\}$   $H = \{1, 2, 3\}$ 





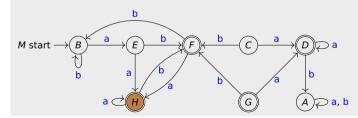


$$A = \emptyset$$

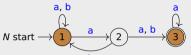
$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$ 

$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 

D





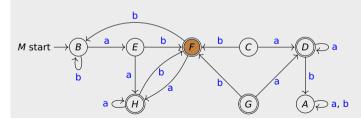


$$A = \emptyset$$

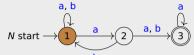
$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 

$$\begin{array}{c|cccc} \Delta & a & b \\ \hline E & H & F \\ F & H & B \\ G & D & F \\ \end{array}$$

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$$A = \emptyset$$

$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 

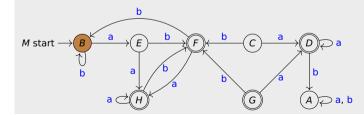
$$B = \{1\}$$
  $F = \{1, 2\}$   
 $C = \{2\}$   $G = \{2, 3\}$   
 $C = \{3\}$   $C = \{3\}$   $C = \{4\}$ 

D

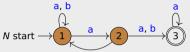
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abbbaababbaababb<mark>a</mark>

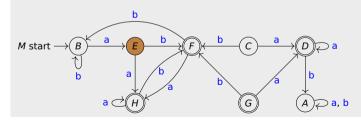


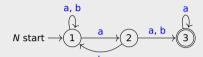




$$A = \emptyset$$
 $B = \{1\}$ 

$$A = \emptyset$$
  $E = \{1, 2\}$   
 $B = \{1\}$   $F = \{1, 3\}$   
 $C = \{2\}$   $G = \{2, 3\}$ 





$$E = \{1, 2\}$$
  
 $B = \{1\}$   $F = \{1, 3\}$ 

$$E = \{1, 2\}$$
  
 $F = \{1, 3\}$ 

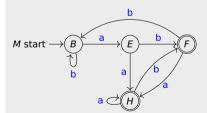
 $H = \{1, 2, 3\}$ 

$$\Delta$$
 a b

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abbbaababbaababba

remove inaccessible states



**Epsilon Transitions** 

## Question

A Ouick Recap

Every regular set is accepted by ...

A ... an NFA having exactly one final state,

B ... a DFA having exactly one final state,

C ... an NFA having exactly one start state.

**Epsilon Transitions** 

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A Ouick Recap

- 1 A Quick Recap
- 2 Nondeterministic Finite Automata
- 3 Epsilon Transitions

# Definitions

A Ouick Recap

• NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  such that

- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  such that
  - $\mathbf{0} \ \boldsymbol{\varepsilon} \notin \Sigma$

**Epsilon Transitions** 

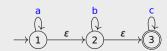
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- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  such that
  - $\bullet$   $\varepsilon \notin \Sigma$

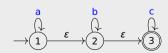
**Epsilon Transitions** 

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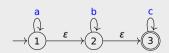
- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  such that
  - $\bullet$   $\varepsilon \notin \Sigma$
  - $\mathbb{Q}$   $N_{\varepsilon} = (Q, \Sigma \cup \{\varepsilon\}, \Delta, S, F)$  is NFA over alphabet  $\Sigma \cup \{\varepsilon\}$



- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  such that
  - $\mathbf{0}$   $\varepsilon \notin \Sigma$
  - Q  $N_{\varepsilon} = (Q, \Sigma \cup \{\varepsilon\}, \Delta, S, F)$  is NFA over alphabet  $\Sigma \cup \{\varepsilon\}$
- $\Delta: Q \times \Sigma \cup \{\varepsilon\} \rightarrow 2^Q$



- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N=(Q,\Sigma,\varepsilon,\Delta,S,F)$  such that
  - $\blacksquare \ \varepsilon \notin \Sigma$
  - $Q N_{\varepsilon} = (Q, \Sigma \cup {\varepsilon}, \Delta, S, F)$  is NFA over alphabet  $\Sigma \cup {\varepsilon}$
- $\Delta: O \times \Sigma \cup \{\varepsilon\} \rightarrow 2^Q$
- $\varepsilon$ -closure of set  $A \subseteq Q$  is defined as  $C_{\varepsilon}(A) = \bigcup \{\widehat{\Delta}_{N_{\varepsilon}}(A, x) \mid x \in \{\varepsilon\}^*\}$

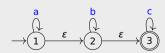


$$C_{\varepsilon}(\{1\}) \qquad = \qquad \{1,2,3\}$$

- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N=(Q,\Sigma,\varepsilon,\Delta,S,F)$  such that
  - $\mathbf{1} \quad \boldsymbol{\varepsilon} \notin \Sigma$
  - $\bigcirc$   $N_{\varepsilon} = (Q, \Sigma \cup \{\varepsilon\}, \Delta, S, F)$  is NFA over alphabet  $\Sigma \cup \{\varepsilon\}$
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- $\widehat{\Delta}_N : 2^Q \times \Sigma^* \to 2^Q$  is inductively defined by

$$\widehat{\Delta}_N(A, \varepsilon) = C_{\varepsilon}(A)$$

$$\widehat{\Delta}_{N}(A, xa) = \left\{ \int \left\{ C_{\varepsilon}(\Delta(q, a)) \mid q \in \widehat{\Delta}_{N}(A, x) \right\} \right\}$$



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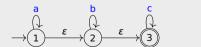
$$C_{\varepsilon}(\{1\}) = \{1, 2, 3\}$$

$$\widehat{\Delta}_{N}(\{1\}, b) =$$

- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N=(Q,\Sigma,\varepsilon,\Delta,S,F)$  such that
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$$\widehat{\Delta}_{N}(A, xa) = \bigcup \{ \underline{C}_{\mathcal{E}}(\Delta(q, a)) \mid q \in \widehat{\Delta}_{N}(A, x) \}$$



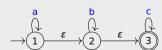
$$C_{\varepsilon}(\{1\})$$
 =  $\{1, 2, 3\}$   
 $\widehat{\Delta}_{N}(\{1\}, b)$  =  $C_{\varepsilon}(\{1\})$ 

A Ouick Recap

- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N=(Q,\Sigma,\varepsilon,\Delta,S,F)$  such that
  - **①** ε ∉ Σ
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$$\widehat{\Delta}_{N}(A, xa) = \left\{ \int \left\{ \frac{C_{\varepsilon}(\Delta(q, a))}{C_{\varepsilon}(\Delta(q, a))} \mid q \in \widehat{\Delta}_{N}(A, x) \right\} \right\}$$

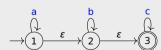


$$\begin{array}{lcl} C_{\varepsilon}(\{1\}) & = & \{1,2,3\} \\ \widehat{\Delta}_{N}(\{1\},b) & = & C_{\varepsilon}(\Delta(1,b)) \cup C_{\varepsilon}(\Delta(2,b)) \cup C_{\varepsilon}(\Delta(3,b)) \end{array}$$

- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N=(Q,\Sigma,\varepsilon,\Delta,S,F)$  such that
  - $\mathbf{1} \quad \boldsymbol{\varepsilon} \notin \Sigma$
  - $Q N_{\varepsilon} = (Q, \Sigma \cup {\varepsilon}, \Delta, S, F)$  is NFA over alphabet  $\Sigma \cup {\varepsilon}$
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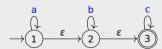
$$C_{\varepsilon}(\{1\}) = \{1, 2, 3\}$$

$$\widehat{\Delta}_{N}(\{1\}, b) = C_{\varepsilon}(\emptyset) \cup C_{\varepsilon}(\{2\}) \cup C_{\varepsilon}(\emptyset)$$

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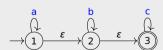


$$\begin{array}{lcl} C_{\varepsilon}(\{1\}) & = & \{1,2,3\} \\ \widehat{\Delta}_{N}(\{1\}, {\color{red}b}) & = & \varnothing \cup \{2,3\} \cup \varnothing \end{array}$$

- NFA with  $\varepsilon$ -transitions (NFA $_{\varepsilon}$ ) is sextuple  $N=(Q,\Sigma,\varepsilon,\Delta,S,F)$  such that
  - **①** ε ∉ Σ
  - $\bigcirc$   $N_{\varepsilon} = (Q, \Sigma \cup \{\varepsilon\}, \Delta, S, F)$  is NFA over alphabet  $\Sigma \cup \{\varepsilon\}$
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$$C_{\varepsilon}(\{1\}) = \{1, 2, 3\}$$
  
 $\widehat{\Delta}_{N}(\{1\}, b) = \{2, 3\}$ 

**Epsilon Transitions** 

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Let x = baa over the alphabet  $\Sigma = \{a, b\}$ 

# Example (Unfolding of the multistep function $\widehat{\Delta}_N$ )

Let x = baa over the alphabet  $\Sigma = \{a, b\}$ 

$$\bigcup \{C_{\varepsilon}(\Delta(q,a)) \mid q \in \widehat{\Delta}_{N}(A,ba)\}$$

1<sup>st</sup> rec. call

Let x = baa over the alphabet  $\Sigma = \{a, b\}$ 

A Ouick Recap

$$\bigcup \{ C_{\varepsilon}(\Delta(q, a)) \mid q \in \widehat{\Delta}_{N}(A, ba) \}$$
$$\bigcup \{ C_{\varepsilon}(\Delta(q, a)) \mid q \in \bigcup \{ C_{\varepsilon}(\Delta(q, a)) \mid q \in \widehat{\Delta}_{N}(A, b) \} \}$$

1<sup>st</sup> rec. call 2<sup>nd</sup> rec. call

**Epsilon Transitions** 

**Epsilon Transitions** 

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```
Let x = baa over the alphabet \Sigma = \{a, b\}
              | | \{ C_{\varepsilon}(\Delta(q, a)) | q \in \widehat{\Delta}_{N}(A, ba) \} |
                                                                                                                                                                                                                                                           1st rec. call
              \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \widehat{\Delta}_{N}(A,b) \right\} \right\} 
 \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,b)) \mid q \in \widehat{\Delta}_{N}(A,\varepsilon) \right\} \right\} \right\} 
                                                                                                                                                                                                                                                           2<sup>nd</sup> rec. call
                                                                                                                                                                                                                                                           3<sup>rd</sup> rec. call
```

**Epsilon Transitions** 

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# Example (Unfolding of the multistep function $\widehat{\Delta}_N$ )

```
Let x = baa over the alphabet \Sigma = \{a, b\}
               | | \{ C_{\varepsilon}(\Delta(q,a)) | q \in \widehat{\Delta}_{N}(A,ba) \} |
                                                                                                                                                                                                                                                                     1st rec. call
               \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \widehat{\Delta}_{N}(A,b) \right\} \right\}
                                                                                                                                                                                                                                                                     2<sup>nd</sup> rec. call
             \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,b)) \mid q \in \widehat{\Delta}_{N}(A,\varepsilon) \right\} \right\} 
\bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,b)) \mid q \in C_{\varepsilon}(A) \right\} \right\} 
                                                                                                                                                                                                                                                                     3<sup>rd</sup> rec. call
```

```
Let x = baa over the alphabet \Sigma = \{a, b\}
             | | \{ C_{\varepsilon}(\Delta(q,a)) | q \in \widehat{\Delta}_{N}(A,ba) \} |
                                                                                                                                                                                                                                                          1st rec. call
             \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \widehat{\Delta}_{N}(A,b) \right\} \right\}
                                                                                                                                                                                                                                                          2<sup>nd</sup> rec. call
             \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,b)) \mid q \in \widehat{\Delta}_{N}(A,\varepsilon) \right\} \right\} 
 \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,b)) \mid q \in C_{\varepsilon}(A) \right\} \right\} 
                                                                                                                                                                                                                                                          3<sup>rd</sup> rec. call
             \bigcup \Big\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \Big\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \mathcal{B} \Big\} \Big\}
                                                                                                                                                                                                                                                          | | \{ C_{\varepsilon}(\Delta(q,b)) | q \in C_{\varepsilon}(A) \} = B
```

**Epsilon Transitions** 

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# Example (Unfolding of the multistep function $\widehat{\Delta}_N$ )

```
Let x = baa over the alphabet \Sigma = \{a, b\}
           | | \{ C_{\varepsilon}(\Delta(q,a)) | q \in \widehat{\Delta}_{N}(A,ba) \} |
                                                                                                                                                                                                             1st rec. call
           \left\{ \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \left\{ \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \widehat{\Delta}_{N}(A,b) \right\} \right\} \right\}
                                                                                                                                                                                                             2<sup>nd</sup> rec. call
           \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,b)) \mid q \in \widehat{\Delta}_{N}(A,\varepsilon) \right\} \right\} \right\}
                                                                                                                                                                                                             3<sup>rd</sup> rec. call
           \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,b)) \mid q \in C_{\varepsilon}(A) \right\} \right\} \right\}
          \bigcup \Big\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \Big\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in B \Big\} \Big\}
                                                                                                                                                                                                            | | \{ C_{\varepsilon}(\Delta(q,b)) | q \in C_{\varepsilon}(A) \} = B
           \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in C \right\}
                                                                                                                                                                                                             | | \{C_{\varepsilon}(\Delta(q,a)) | q \in B\} = C
```

```
Let x = baa over the alphabet \Sigma = \{a, b\}
           | | \{ C_{\varepsilon}(\Delta(q,a)) | q \in \widehat{\Delta}_{N}(A,ba) \} |
                                                                                                                                                                                                          1st rec call
           \left\{ \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \left\{ \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \widehat{\Delta}_{N}(A,b) \right\} \right\} \right\}
                                                                                                                                                                                                          2<sup>nd</sup> rec. call
           \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \left\{ C_{\varepsilon}(\Delta(q,b)) \mid q \in \widehat{\Delta}_{N}(A,\varepsilon) \right\} \right\} \right\}
                                                                                                                                                                                                          3<sup>rd</sup> rec. call
           \bigcup \Big\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \Big\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \Big\{ C_{\varepsilon}(\Delta(q,b)) \mid q \in C_{\varepsilon}(A) \Big\} \Big\} \Big\}
           \bigcup \Big\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in \bigcup \Big\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in B \Big\} \Big\}
                                                                                                                                                                                                         | | \{ C_{\varepsilon}(\Delta(q,b)) | q \in C_{\varepsilon}(A) \} = B
           \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in C \right\}
                                                                                                                                                                                                         ||\{C_{\varepsilon}(\Delta(q,a))||q\in B\}=C
                                                                                                                                                                                                         \bigcup \left\{ C_{\varepsilon}(\Delta(q,a)) \mid q \in C \right\} = D
```

**Epsilon Transitions** 

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A Ouick Recap

 $C_{\varepsilon}(A)$  is least extension of A that is closed under  $\varepsilon$ -transitions:

$$q \in C_{\varepsilon}(A) \implies \Delta_{N_{\varepsilon}}(q, \varepsilon) \subseteq C_{\varepsilon}(A)$$

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**Epsilon Transitions** 

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every set accepted by NFA<sub>E</sub> is regular

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### Theorem

every set accepted by  $NFA_{\varepsilon}$  is regular

# Proof. (by construction)

• NFA<sub> $\varepsilon$ </sub>  $N_1 = (Q, \Sigma, \varepsilon, \Delta_1, S, F_1)$ 

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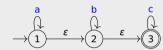
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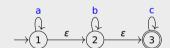
  - $P_2 := \{ q \mid C_{\varepsilon}(\{q\}) \cap F_1 \neq \emptyset \}$



**Epsilon Transitions** 

# NFA $_{\varepsilon}$ $N_1 = (\{1, 2, 3\}, \{a, b, c\}, \varepsilon, \Delta_1, \{1\}, \{3\})$ with

•	$\Delta_1$	a	b	C	ε
	1	{1}	Ø	Ø	{2}
	2	Ø	{2}	Ø	{3}
	3	Ø	Ø	{3}	Ø

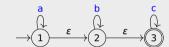


**Epsilon Transitions** 

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$$\Delta_2$$
 a b c



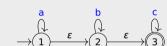
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$$\Delta_2(1,a) = \widehat{\Delta}_1(\{1\},a) = \bigcup \{C_{\varepsilon}(\Delta_1(q,a)) \mid q \in \widehat{\Delta}_1(\{1\},\varepsilon)\}$$



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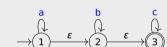
•	$\Delta_1$	a	~	C	ε
	1	{1}	Ø {2}	Ø	{2}
	2	Ø	{2}	Ø	{3}
	3	Ø	Ø	{3}	Ø

NFA  $N_2 = (\{1, 2, 3\}, \{a, b, c\}, \Delta_2, \{1\}, F_2)$  with

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 a b c

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**Epsilon Transitions** 

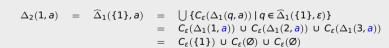
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•	$\Delta_1$	а		C	ε
	1	{1}	Ø {2} Ø	Ø	{2}
	2	Ø	{2}	Ø	{3}
	3	Ø	Ø	{3}	Ø

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$$\Delta_2$$
 a b c





**Epsilon Transitions** 

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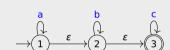
•	$\Delta_1$	a		С	ε
	1	{1}	Ø {2}	Ø	{2}
	2	Ø	{2}	Ø	{3}
	3	Ø	Ø	{3}	Ø

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• 
$$F_2 = \{q \mid C_{\varepsilon}(\{q\}) \cap F_1 \neq \emptyset\}$$

• 
$$\frac{\Delta_2}{1}$$
  $\frac{a}{\{1,2,3\}}$ 

$$\begin{array}{rcl} \Delta_2(1,a) & = & \widehat{\Delta}_1(\{1\},a) & = & \bigcup \left\{C_{\varepsilon}(\Delta_1(q,a)) \mid q \in \widehat{\Delta}_1(\{1\},\varepsilon)\right\} \\ & = & C_{\varepsilon}(\Delta_1(1,a)) \cup C_{\varepsilon}(\Delta_1(2,a)) \cup C_{\varepsilon}(\Delta_1(3,a)) \\ & = & C_{\varepsilon}(\{1\}) \cup C_{\varepsilon}(\emptyset) \cup C_{\varepsilon}(\emptyset) \\ & = & \{1,2,3\} \end{array}$$



**Epsilon Transitions** 

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	1	{1}	Ø	Ø	{2}
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	3	Ø	Ø	{3}	Ø

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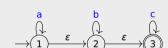
•	$\Delta_1$	a		С	ε
	1	{1} Ø Ø	Ø	Ø	{2}
	2	Ø	{2}	Ø	{3}
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**Epsilon Transitions** 

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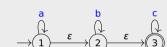
•	$\Delta_1$		b	С	ε
	1	{1}	Ø {2}	Ø	{2}
	2	Ø	{2}	Ø	{3}
	3	Ø	Ø	{3}	Ø

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**Epsilon Transitions** 

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#### Example

A Ouick Recap

NFA<sub> $\varepsilon$ </sub>  $N_1 = (\{1, 2, 3\}, \{a, b, c\}, \varepsilon, \Delta_1, \{1\}, \{3\})$  with

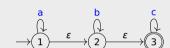
•	$\Delta_{1}$	a	b	C	ε
	1	{1}	Ø	Ø	{2}
	2	Ø	{2}	Ø	{3}
	3	Ø	Ø	{3}	Ø

NFA  $N_2 = (\{1, 2, 3\}, \{a, b, c\}, \Delta_2, \{1\}, F_2)$  with

• 
$$F_2 = \{ q \mid C_{\varepsilon}(\{q\}) \cap F_1 \neq \emptyset \}$$

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$$\begin{array}{rcl} \Delta_2(1,b) & = & \widehat{\Delta}_1(\{1\},b) & = & \bigcup \left\{C_{\varepsilon}(\Delta_1(q,b)) \mid q \in \widehat{\Delta}_1(\{1\},\varepsilon)\right\} \\ & = & C_{\varepsilon}(\Delta_1(1,b)) \cup C_{\varepsilon}(\Delta_1(2,b)) \cup C_{\varepsilon}(\Delta_1(3,b)) \\ & = & C_{\varepsilon}(\emptyset) \cup C_{\varepsilon}(\{2\}) \cup C_{\varepsilon}(\emptyset) \end{array}$$



NFA<sub> $\varepsilon$ </sub>  $N_1 = (\{1, 2, 3\}, \{a, b, c\}, \varepsilon, \Delta_1, \{1\}, \{3\})$  with

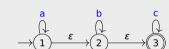
•	$\Delta_{1}$	a	b	C	ε
	1	{1}	Ø {2}	Ø	{2}
	2	Ø	{2}	Ø	{3}
	3	Ø	Ø	{3}	Ø

NFA  $N_2 = (\{1, 2, 3\}, \{a, b, c\}, \Delta_2, \{1\}, F_2)$  with

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• 
$$\frac{\Delta_2}{1}$$
 |  $\frac{a}{\{1,2,3\}}$  |  $\frac{b}{\{2,3\}}$ 

$$\begin{array}{rcl} \Delta_2(1,b) & = & \widehat{\Delta}_1(\{1\},b) & = & \bigcup \left\{C_{\varepsilon}(\Delta_1(q,b)) \mid q \in \widehat{\Delta}_1(\{1\},\varepsilon)\right\} \\ & = & C_{\varepsilon}(\Delta_1(1,b)) \cup C_{\varepsilon}(\Delta_1(2,b)) \cup C_{\varepsilon}(\Delta_1(3,b)) \\ & = & C_{\varepsilon}(\emptyset) \cup C_{\varepsilon}(\{2\}) \cup C_{\varepsilon}(\emptyset) \\ & = & \{2,3\} \end{array}$$



**Epsilon Transitions** 

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# Example (cont'd)

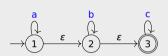
NFA $_{\varepsilon}$   $N_1 = (\{1, 2, 3\}, \{a, b, c\}, \varepsilon, \Delta_1, \{1\}, \{3\})$  with

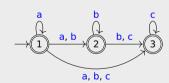
	$\Delta_1$	а	a	С	ε
•	1	{1}	Ø	Ø	{2}
	2	Ø	{2}	Ø	{3}
	3	Ø	Ø	{3}	Ø

NFA  $N_2 = (\{1, 2, 3\}, \{a, b, c\}, \Delta_2, \{1\}, F_2)$  with

• 
$$F_2 = \{q \mid C_{\varepsilon}(\{q\}) \cap F_1 \neq \emptyset\}$$

	$\Delta_{2}$	a	b	C
•	1	{1, 2, 3}	{2,3}	{3}
	2	Ø	{2,3}	{3}
	3	Ø	Ø	{3}





# Outline

A Ouick Recap

- 1 A Quick Recap
- 2 Nondeterministic Finite Automata
- 3 Epsilon Transitions
- 4 Closure Properties

#### Theoren

A Ouick Recap

regular sets are effectively closed under concatenation

A Ouick Recap

regular sets are effectively closed under concatenation

• 
$$A = L(N_1)$$
 for NFA  $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$   
•  $B = L(N_2)$  for NFA  $N_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$ 

regular sets are effectively closed under concatenation

# Proof. (by construction)

$$A = L(N_1)$$
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$$B = L(N_2)$$
 for NFA  $N_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$ 

• without loss of generality  $Q_1 \cap Q_2 = \emptyset$ 

A Ouick Recap

regular sets are effectively closed under concatenation

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- without loss of generality  $Q_1 \cap Q_2 = \emptyset$
- AB = L(N) for NFA<sub>E</sub>  $N = (Q, \Sigma, \varepsilon, \Delta, S_1, F_2)$  with

A Ouick Recap

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 $B = L(N_2)$  for NFA  $N_2 = (O_2, \Sigma, \Delta_2, S_2, F_2)$ 

$$\bigcirc Q :=$$

$$:= \quad Q_1 \cup Q_2$$

A Ouick Recap

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• 
$$A = L(N_1)$$
 for NFA  $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$   
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- without loss of generality  $Q_1 \cap Q_2 = \emptyset$
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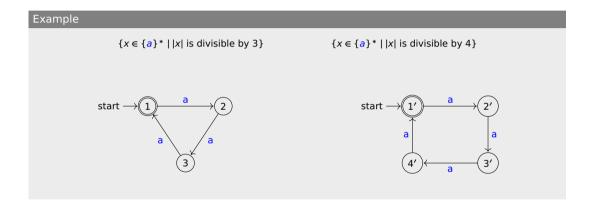
- without loss of generality  $Q_1 \cap Q_2 = \emptyset$
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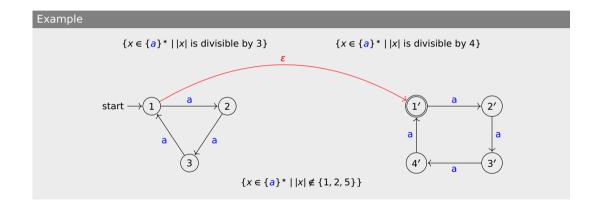
A Ouick Recap

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A Ouick Recap

regular sets are effectively closed under asterate

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A Ouick Recap

regular sets are effectively closed under asterate

- $A = L(N_1)$  for NFA  $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
- $A^* = L(N)$  for NFA<sub> $\varepsilon$ </sub>  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  with

A Ouick Recap

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- $A = L(N_1)$  for NFA  $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
- $A^* = L(N)$  for NFA<sub>E</sub>  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  with
  - **1** 0  $:= O_1 \cup \{s\}$

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  - ⑤ F  $:= \{s\}$

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  - **0**  $:= O_1 \cup \{s\}$
  - $\bigcirc S := \{s\}$
  - $:= \{s\}$

$$\Delta_1(q,a) \quad \text{if } q \in Q_1 \text{ and } a \in \Sigma$$

$$\triangle (q,a)$$
 :=

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- $A^* = L(N)$  for NFA<sub>E</sub>  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  with
  - **0**  $:= O_1 \cup \{s\}$
  - $\bigcirc S := \{s\}$
  - $:= \{s\}$

A Ouick Recap

regular sets are effectively closed under asterate

- $A = L(N_1)$  for NFA  $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
- $A^* = L(N)$  for NFA<sub>E</sub>  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  with
  - **1** 0  $:= O_1 \cup \{s\}$

  - $:= \{s\}$

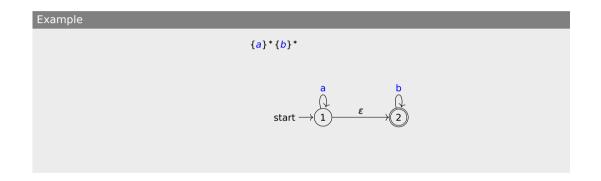
$$\Delta(q, a) := \begin{cases} \Delta_1(q, a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma \\ S_1 & \text{if } q = s \text{ and } a = \varepsilon \\ S & \text{if } q \in F_1 \text{ and } a = \varepsilon \end{cases}$$

regular sets are effectively closed under asterate

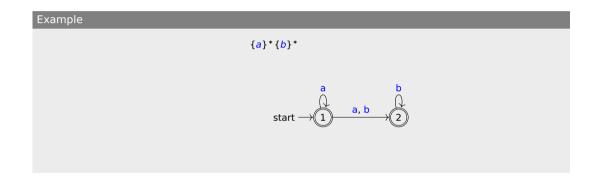
- $A = L(N_1)$  for NFA  $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
- $A^* = L(N)$  for NFA<sub>E</sub>  $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$  with

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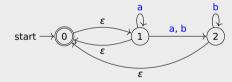
**Epsilon Transitions** 



# Example

A Quick Recap

$$( \{a\}^* \{b\}^*)^* = \{a,b\}^*$$



Thanks! & Questions?

**Epsilon Transitions**