CMPE 322/327 - Theory of Computation

Week 1: Central Concepts of Automata Theory & Mathematical Preliminaries

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Outline

1. Sets
2. Relations
3. Functions
4. Graphs
5. Trees
6. Proof Techniques
7. Alphabets & Strings
8. Languages

Definition (Sets)

- A set is a collection of objects

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Definition (Sets)

- A set is a collection of objects

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{\text{bicycle, bus, train, airplane}\} \\ 1 &\in A \qquad\qquad 1 \text{ is an element of the set } A \end{aligned}$$

Definition (Sets)

- A set is a collection of objects

$$\begin{array}{lll} A & = & \{1, 2, 3\} \\ B & = & \{\text{bicycle, bus, train, airplane}\} \\ 1 & \in & A \\ \text{ship} & \notin & B \end{array}$$

1 is an element of the set A

ship is not an element of the set B

Example (Representation of Sets)

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S &= \{2, 4, 6, \cdots\}
\end{aligned}
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C is a finite set

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C & = & \{a, b, \cdots, k\} & C \text{ is a finite set} \\
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S & := & \{j \in \mathbb{Z} \mid j > 0 \text{ and } j = 2k \text{ for some } k > 0\} &
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S & := & \{j \mid j \text{ is a positive and even integer}\} &
\end{array}
$$

Definition (Diagrammatic Representation of Sets (Venn Diagrams))

$$A = \{1, 2, 3, 4, 5\}$$

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U is a universal set (set of all elements under consideration)

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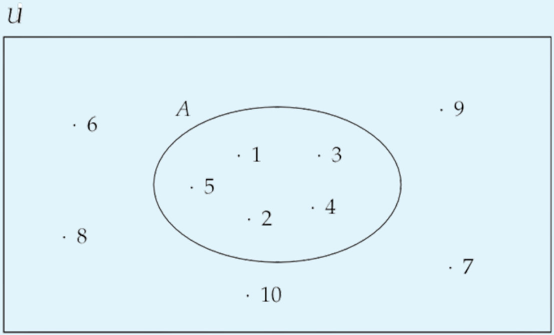

Definition (Basic Set Operations)

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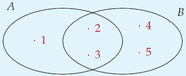
Union	$A \cup B := \{x \mid x \in A \vee x \in B\} = \{1, 2, 3, 4, 5\}$	

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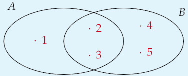
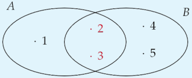
Union	$A \cup B := \{x \mid x \in A \vee x \in B\} = \{1, 2, 3, 4, 5\}$	
Intersection	$A \cap B := \{x \mid x \in A \wedge x \in B\} = \{2, 3\}$	

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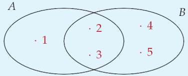
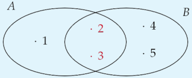
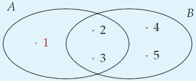
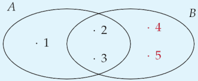
Union	$A \cup B := \{x \mid x \in A \vee x \in B\} = \{1, 2, 3, 4, 5\}$	
Intersection	$A \cap B := \{x \mid x \in A \wedge x \in B\} = \{2, 3\}$	
Difference	$A - B := \{x \mid x \in A \wedge x \notin B\} = \{1\}$	
	$B - A := \{x \mid x \in B \wedge x \notin A\} = \{4, 5\}$	

Definition (Basic Set Operations (cont'd))

$$U = \{1, 2, \cdots, 7\}$$
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$$\overline{A} := \{x \mid x \notin A \land x \in U\} = \{4, 5, 6, 7\}$$

$\overline{A}$ is the complement of A with respect to U

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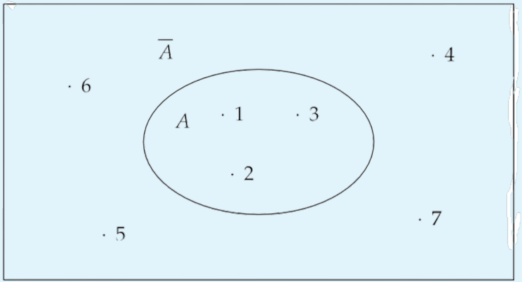
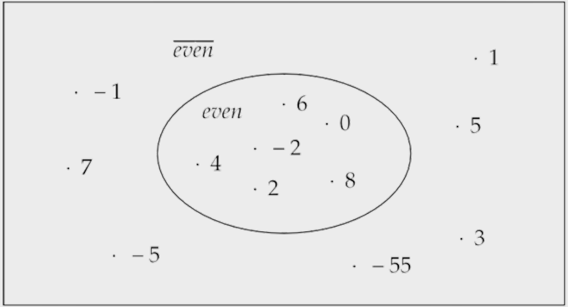


$$\overline{(\overline{A})} = A$$

Example (Complement)

- The complement set of even integers $\overline{\{\text{even integers}\}}$:

$\mathbb{Z}$



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Theorem (De Morgan Laws)

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

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&= \overline{A} \cap \overline{B} && \text{by definition of intersection}
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 & = \overline{A} \cup \overline{B} & \text{by definition of union}
\end{array}
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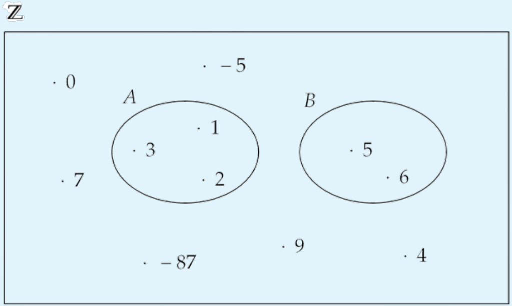

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- Observe that the number of elements in 2^S amount to the 2 to the number of elements in S:

$$|2^S| = 2^{|S|}$$

Definition (Cartesian Product of Sets)

The Cartesian product of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. That formally is

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- Remark also that Cartesian products generalize (to more than two sets)

$$A_1 \times A_2 \times \cdots \times A_n.$$

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3. $x \in x \longleftrightarrow x \notin x$ – Russell's Paradox

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$$\exists x \forall z [z \in x \leftrightarrow z \notin z]$$

3. $x \in x \leftrightarrow x \notin x$ – This is not a pipe

The Axiom of Unrestricted Comprehension
$\exists x \forall z [z \in x \leftrightarrow \phi(z)]$

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$\forall y \exists x \forall z [z \in x \leftrightarrow (z \in y \text{ and } \phi(z))]$

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Remarks

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Remarks

1. Given some set y, the axiom of restricted comprehension only guarantees the existence of the subset x consisting of those elements of y that satisfy ϕ
2. Impossible to construct the set of all sets satisfying certain property

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5. We silently consider sets in ZFC within the scope of this course (to avoid Russell-like paradoxes)

Outline

1. Sets
2. Relations
3. Functions
4. Graphs
5. Trees
6. Proof Techniques
7. Alphabets & Strings
8. Languages

Definition (Binary Relations)

A binary relation R over sets A and B is a subset of the Cartesian product $A \times B$

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A binary relation R over some set A ($R \subseteq A \times A$) is said to be an equivalence relation if and only if it is *reflexive*, *symmetric* and *transitive* such that

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$\forall a \in A,\ \forall b \in A,\ \forall c \in A,\ \big((a, b) \in R\ \wedge\ (b, c) \in R\big) \implies (a, c) \in R$	transitivity

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- In this lecture, the keyword "*function*" refers to "*total function*".

Example (Functions)

$A = \{1, 2, 3\} \quad B = \{a, b, c, d\}$

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$$\begin{array}{lcllcl} A & = & \{1, 2, 3\} & B & = & \{a, b, c, d\} \\ f & : & A \rightarrow B & f & = & \{(2, d), (3, c)\} \end{array}$$

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1. right-unique: for all $a \geq 10$, from $(a, b_1) \in f$ and $(a, b_2) \in f$, we obtain $b_1 = a + 1$ and $b_2 = a + 1$. It is then obvious that $b_1 = b_2$. Therefore, f obeys right-uniqueness.

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2. left-total: $\forall a \in \mathbb{N}$, $0 \leq a < 10$, $\nexists b \in \mathbb{N}$, $(a, b) \in f$. Thus, f does not satisfy left-totality.

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2. left-total: $\forall a \in \mathbb{N}$, there exists $b = a + 1$ such that $(a, a + 1) \in f$. This gives $a + 1 = a + 1$ which definitely holds. Thus, f does satisfy left-totality.

□

Definitions (Injection & Surjection)

- A function $f : A \to B$ is an injection (or one-to-one) if

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- A function $f : A \to B$ is a surjection (or onto) if

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- A function $f : A \to B$ is a bijection (or both one-to-one and onto) if

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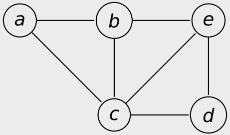

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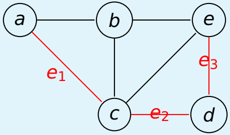

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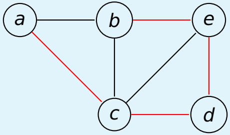

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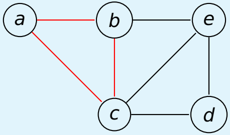

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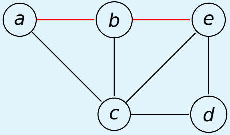

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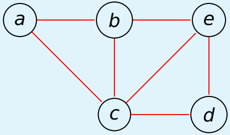

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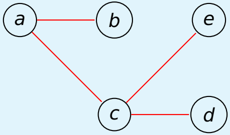

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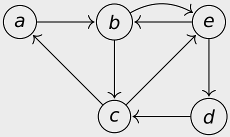

Outline

1. Sets
2. Relations
3. Functions
4. Graphs
5. Trees
6. Proof Techniques
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Definitions (Trees)

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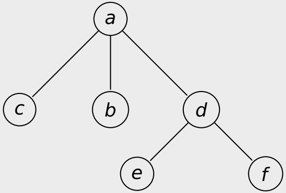

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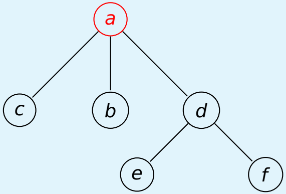

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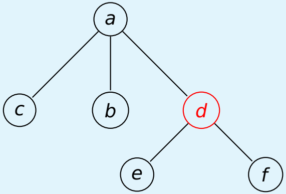

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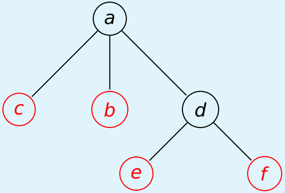

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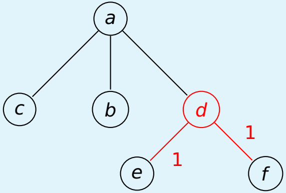

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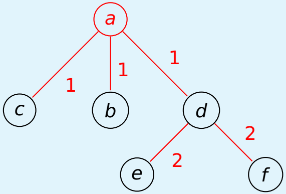

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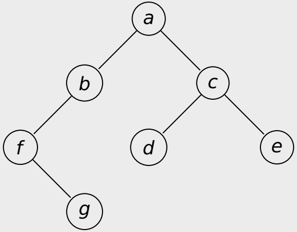

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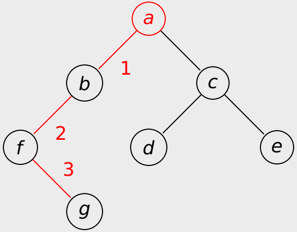

$height(T) = 3$

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We argue by mathematical induction over the cardinality k of A.

1. Base case: $k = 0 \iff A = \emptyset \iff |P(\emptyset)| = 1 = 2^0$
2. Step case: Given : $|A| = k$ such that $k \geqslant 0$ $\quad A = \{1, 2, 3, \cdots, k\}$ $\quad$ IH: $|P(A)| = 2^k$
   Show : $|P(A \cup \{p\})| = 2^{k+1}$

   By injecting p in A, we newly introduce

$\binom{k}{0}$	# of 1-element subset	$\{p\}$
$\binom{k}{1}$	# of 2-element subsets	$\{1,p\}, \{2,p\}, \cdots, \{k,p\}$
$\binom{k}{2}$	# of 3-element subsets	$\{1,2,p\}, \{1,3,p\}, \cdots, \{1,k,p\}, \cdots, \{k-1,k,p\}$
$\vdots$	$\vdots$	$\vdots$
$\binom{k}{k}$	# of (k+1)-element subset	$\{1,2,3,\cdots,k,p\}$

   It is provable (again by mathematical induction) that $\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k} = 2^k$.

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   Therefore, $|P(A \cup \{p\})| = |P(A)| + \#$ of new subsets $= 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

$\square$

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$L(k)$	$<$	2^{k+1}	by IH
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□

Outline

1. Sets
2. Relations
3. Functions
4. Graphs
5. Trees
6. Proof Techniques
7. Alphabets & Strings
8. Languages

Definitions (Alphabets & Strings)

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- Any subset of the set Σ^* for some alphabet Σ is called a language

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- Let $\mathcal{L}$ be the language of all strings w over the alphabet $\Sigma = \{a, b\}$ such that $w = a^n b^n$ for some $n \geq 0$. That, in set comprehension notation, is $\mathcal{L} := \{w | w \in \Sigma^* \text{ and } w = a^n b^n \text{ for some } n \geq 0\}$.

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- A *prime number* is a number $x \geq 1$ that is divided (with reminder 0) only by 1 and itself. Let $\mathcal{L}$ be the set of prime numbers defined over the alphabet $\Sigma = \{0, 1, 2, \ldots, 9\}$. Namely, $\mathcal{L} := \{x \mid x \in \Sigma^+ \text{ and } x \text{ is prime}\}$.

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$\Sigma = \{0, 1, 2, \ldots 9\}$	$\mathcal{L}_E := \{x \mid x \in \Sigma^+ \text{ and } x \text{ is even}\}$
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$\Sigma = \{1, \#\}$	$\mathcal{L}_S := \{x\#y \in \Sigma^+ \mid x = 1^n, y = 1^m, m = n^2 \text{ and } n \geq 1\}$
	$\mathcal{L}_S = \{1\#1, 11\#1111, 111\#111111111, \ldots\}$
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- Recall that $|\varepsilon| = 0$ which should not be confused with $|\{\varepsilon\}| = 1$

Definitions (Operations on Languages)

Let Σ be an alphabet and let $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ be languages over Σ.

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- Kleene star (similarly Kleene plus) can be viewed as an operation defined as

$$\Sigma^* = \mathcal{L} := \{x \mid x = \varepsilon \ \vee \ x \in \mathcal{L} \ \vee \ x \in \mathcal{L}\mathcal{L} \ \vee \ x \in \mathcal{L}\mathcal{L}\mathcal{L} \ \vee \ \ldots\}$$

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- The idea is to build automatons which help in solving such decision problems out

Thanks! & Questions?