Midterm

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Date : April the 18th, 13h00
Duration : 120 minutes

Q1. (40 pts) Design deterministic finite automata (DFA) that recognizes each of the following languages

a) **(15 pts)** $\mathcal{L}_1 := \{(w_1, w_2) \mid H(w_1, w_2) \leqslant 1\}$ where H is the Hamming distance function that counts number of places strings w_1 and w_2 differ (if $|w_1| \neq |w_2|$ then $H(w_1, w_2) = \infty$)

b) **(15 pts)** $\mathcal{L}_2 := \{(w_1, w_2) \mid w_1 > w_2 \text{ in base 2}\}$

c) **(10 pts)** $\mathcal{L}_3 := \{(w_1, w_2) \mid H(w_1, w_2) \leqslant 1 \text{ and } w_1 > w_2 \text{ in base 2}\}$

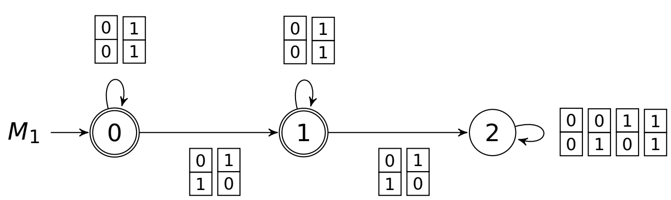
defined over the alphabet $\Sigma = \{0, 1\} \times \{0, 1\}$; namely over $\Sigma = \left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$

Nota Bene:

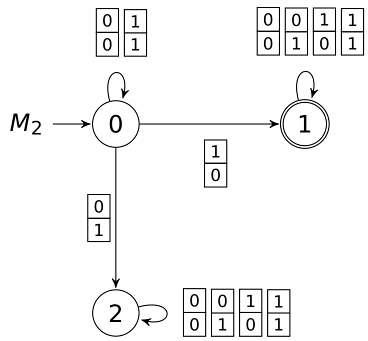
- A string over the alphabet Σ looks, for instance, like $\begin{bmatrix}1&0&1&0&1\\1&1&1&0&1\end{bmatrix}$ such that $w_1 = 10101$ and $w_2 = 11101$. Obviously, $w_2 > w_1$ in base 2, and $H(w_1, w_2) = 1$; thus $(w_1, w_2) \in \mathcal{L}_1$ but $(w_1, w_2) \notin \mathcal{L}_2$ and $(w_1, w_2) \notin \mathcal{L}_3$.
- Notice that strings w_1 and w_2 cannot be of different length.
- Also, be careful with your DFA designs in items *a*) and *b*) as the one in *c*) depends on them, and goes wrong if any of them is mistaken.

A1.

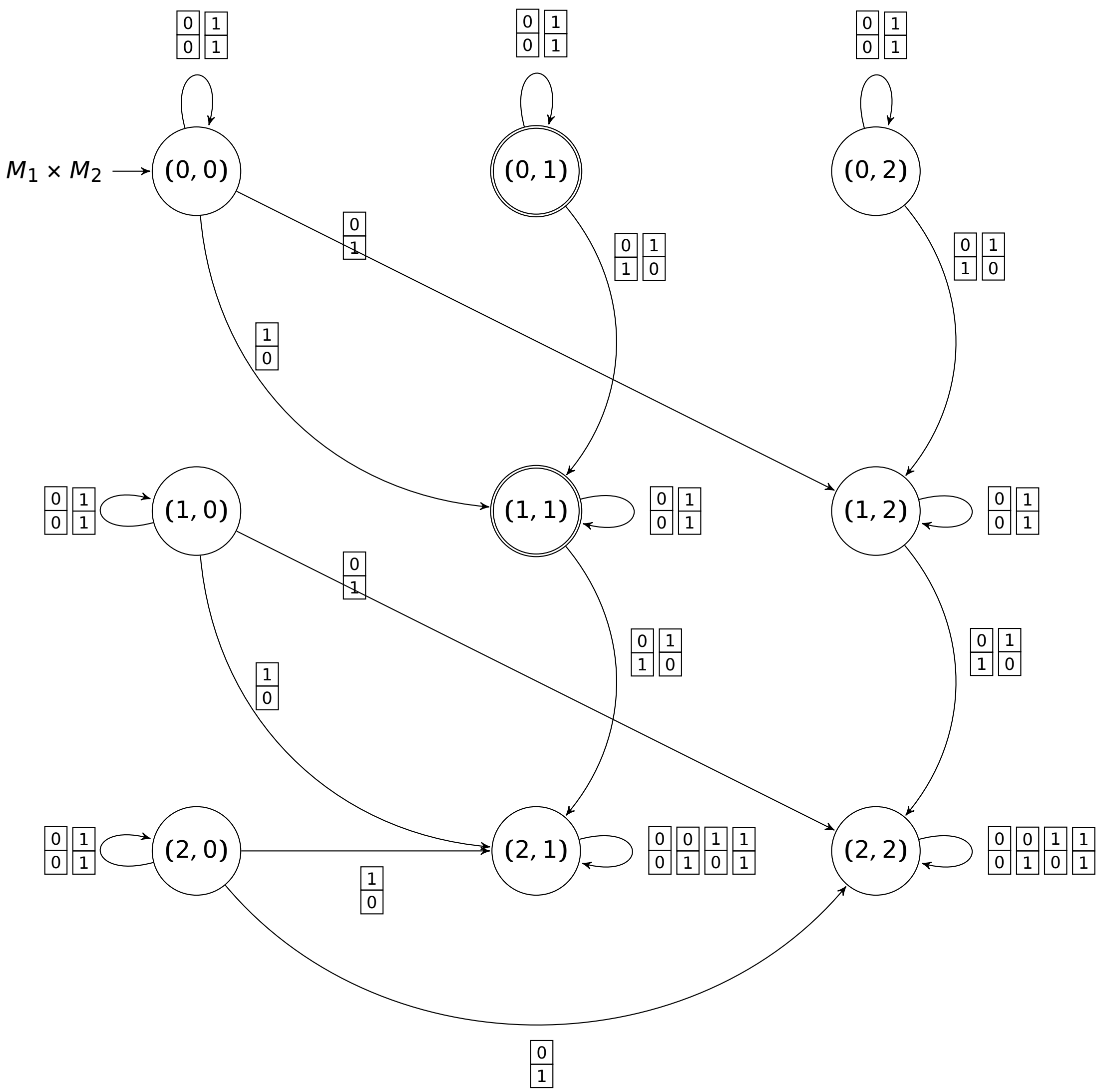
a) At state 0, we have $H(w_1, w_2) = 0$; namely $w_1 = w_2$. Being at state 1 means $H(w_1, w_2) = 1$. If $H(w_1, w_2) > 1$ then the machine needs to be at state 2.



b) If $w_1 = w_2$ the machine stays at state 0. Upon reading a $\begin{bmatrix}1\\0\end{bmatrix}$ pair, it moves to the state 1 and accepts whatever comes next as it is now definite that $w_1 > w2$. Unlikely, at state 0, if the $\begin{bmatrix}0\\1\end{bmatrix}$ pair is read, the machine goes to the dead state 2 and rejects the input pair of strings as $w_2 > w_1$.

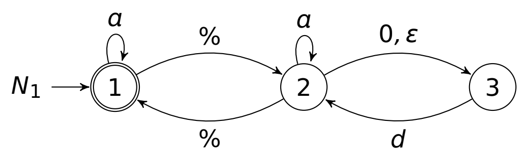


c) Just compute the product machine $M_1 \times M_2$.



Notice that states $(0,1), (0,2), (1,0)$ and $(2,0)$ are inaccessible. One can crop them out of the automata.

Q2. (40 pts) Given an NFA$_\varepsilon$ $N_1 = (\{1,2,3\}, \{\%, 0, a, d\}, \varepsilon, \Delta_1, \{1\}, \{1\})$ with the below state diagram



a) **(15 pts)** employ ε-elimination over N_1 to obtain an equivalent NFA $N_2 = (\{1,2,3\}, \{\%, 0, a, d\}, \Delta_2, \{1\}, F_2)$ with no ε-transitions. Clearly show intermediate steps.

b) **(15 pts)** apply subset construction algorithm to the NFA N_2 so as to get an equivalent DFA $D = (Q, \{\%, 0, a, d\}, \delta, s, F)$. Clearly show intermediate steps.

c) **(10 pts)** minimize the DFA D benefiting the marking algorithm. Justify your reasoning.

A2.

a) To start with, we compute ε-closure of below singleton sets:

$$C_\varepsilon(\{1\}) = \{1\} \qquad C_\varepsilon(\{2\}) = \{2,3\} \qquad C_\varepsilon(\{3\}) = \{3\}$$

We then apply ε-elimination to compute the transition function Δ_2 for the NFA N_2:

$$\begin{aligned}
\Delta_2(1,\%) &= \widehat{\Delta}_1(\{1\},\%) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,\%)) \mid q \in \widehat{\Delta}_1(\{1\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(1,\%)) \\
&= C_\varepsilon(\{2\}) \\
&= \{2,3\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(1,0) &= \widehat{\Delta}_1(\{1\},0) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,0)) \mid q \in \widehat{\Delta}_1(\{1\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(1,0)) \\
&= C_\varepsilon(\emptyset) \\
&= \emptyset
\end{aligned}$$

$$\begin{aligned}
\Delta_2(1,a) &= \widehat{\Delta}_1(\{1\},a) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,a)) \mid q \in \widehat{\Delta}_1(\{1\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(1,a)) \\
&= C_\varepsilon(\{1\}) \\
&= \{1\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(1,d) &= \widehat{\Delta}_1(\{1\},d) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,d)) \mid q \in \widehat{\Delta}_1(\{1\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(1,d)) \\
&= C_\varepsilon(\emptyset) \\
&= \emptyset
\end{aligned}$$

$$\begin{aligned}
\Delta_2(2,\%) &= \widehat{\Delta}_1(\{2\},\%) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,\%)) \mid q \in \widehat{\Delta}_1(\{2\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(2,\%)) \cup C_\varepsilon(\Delta_1(3,\%)) \\
&= C_\varepsilon(\{1\}) \cup C_\varepsilon(\emptyset) \\
&= \{1\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(2,0) &= \widehat{\Delta}_1(\{2\},0) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,0)) \mid q \in \widehat{\Delta}_1(\{2\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(2,0)) \cup C_\varepsilon(\Delta_1(3,0)) \\
&= C_\varepsilon(\{3\}) \cup C_\varepsilon(\emptyset) \\
&= \{3\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(2,a) &= \widehat{\Delta}_1(\{2\},a) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,a)) \mid q \in \widehat{\Delta}_1(\{2\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(2,a)) \cup C_\varepsilon(\Delta_1(3,a)) \\
&= C_\varepsilon(\{2\}) \cup C_\varepsilon(\emptyset) \\
&= \{2,3\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(2,d) &= \widehat{\Delta}_1(\{2\},d) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,d)) \mid q \in \widehat{\Delta}_1(\{2\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(2,d)) \cup C_\varepsilon(\Delta_1(3,d)) \\
&= C_\varepsilon(\emptyset) \cup C_\varepsilon(\{2\}) \\
&= \{2,3\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(3,\%) &= \widehat{\Delta}_1(\{3\},\%) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,\%)) \mid q \in \widehat{\Delta}_1(\{3\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(3,\%)) \\
&= C_\varepsilon(\varnothing) \\
&= \varnothing
\end{aligned}$$

$$\begin{aligned}
\Delta_2(3,0) &= \widehat{\Delta}_1(\{3\},0) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,0)) \mid q \in \widehat{\Delta}_1(\{3\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(3,0)) \\
&= C_\varepsilon(\varnothing) \\
&= \varnothing
\end{aligned}$$

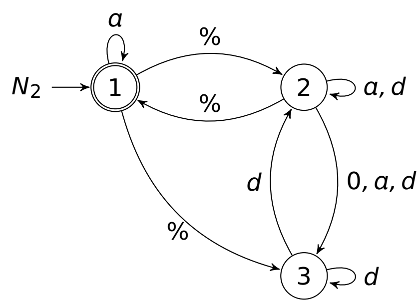
$$\begin{aligned}
\Delta_2(3,a) &= \widehat{\Delta}_1(\{3\},a) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,a)) \mid q \in \widehat{\Delta}_1(\{3\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(3,a)) \\
&= C_\varepsilon(\varnothing) \\
&= \varnothing
\end{aligned}$$

$$\begin{aligned}
\Delta_2(3,d) &= \widehat{\Delta}_1(\{3\},d) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,d)) \mid q \in \widehat{\Delta}_1(\{3\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(3,d)) \\
&= C_\varepsilon(\{2\}) \\
&= \{2,3\}
\end{aligned}$$

The set of final states for N_2 is computed as follows:

$$F_2 \quad := \quad \{q \mid C_\varepsilon(\{q\}) \cap F_1 \neq \varnothing\} \quad = \quad \{1\}.$$

Therefore, the state diagram for N_2 looks like:



$b)$ Let us now apply subset construction over the NFA N_2 to obtain an equivalent DFA $D = (Q, \{\%, 0, a, d\}, \delta, s, F)$:

$$\begin{aligned}
\delta(\{1\},\%) &= \widehat{\Delta}_2(\{1\},\%) \\
&= \{2,3\} \\
\delta(\{1\},a) &= \widehat{\Delta}_2(\{1\},a) \\
&= \{1\} \\
\delta(\{2,3\},\%) &= \widehat{\Delta}_2(\{2,3\},\%) \\
&= \widehat{\Delta}_2(\{2\},\%) \cup \widehat{\Delta}_2(\{3\},\%) \\
&= \{1\} \cup \varnothing \\
&= \{1\} \\
\delta(\{2,3\},a) &= \widehat{\Delta}_2(\{2,3\},a) \\
&= \widehat{\Delta}_2(\{2\},a) \cup \widehat{\Delta}_2(\{3\},a) \\
&= \{2,3\} \cup \varnothing \\
&= \{2,3\}
\end{aligned}$$

$$\begin{aligned}
\delta(\{1\},0) &= \widehat{\Delta}_2(\{1\},0) \\
&= \varnothing \\
\delta(\{1\},d) &= \widehat{\Delta}_2(\{1\},d) \\
&= \varnothing \\
\delta(\{2,3\},0) &= \widehat{\Delta}_2(\{2,3\},0) \\
&= \widehat{\Delta}_2(\{2\},0) \cup \widehat{\Delta}_2(\{3\},0) \\
&= \{3\} \cup \varnothing \\
&= \{3\} \\
\delta(\{2,3\},d) &= \widehat{\Delta}_2(\{2,3\},d) \\
&= \widehat{\Delta}_2(\{2\},d) \cup \widehat{\Delta}_2(\{3\},d) \\
&= \{2,3\} \cup \{2,3\} \\
&= \{2,3\}
\end{aligned}$$

$$\begin{aligned}
\delta(\{3\},\%) &= \widehat{\Delta}_2(\{3\},\%) \\
&= \varnothing \\
\delta(\{3\},a) &= \widehat{\Delta}_2(\{3\},a) \\
&= \varnothing \\
\delta(\varnothing,\%) &= \widehat{\Delta}_2(\varnothing,\%) \\
&= \varnothing \\
\delta(\varnothing,a) &= \widehat{\Delta}_2(\varnothing,a) \\
&= \varnothing
\end{aligned}$$

$$\begin{aligned}
\delta(\{3\},0) &= \widehat{\Delta}_2(\{3\},0) \\
&= \varnothing \\
\delta(\{3\},d) &= \widehat{\Delta}_2(\{3\},d) \\
&= \{2,3\} \\
\delta(\varnothing,0) &= \widehat{\Delta}_2(\varnothing,0) \\
&= \varnothing \\
\delta(\varnothing,d) &= \widehat{\Delta}_2(\varnothing,d) \\
&= \varnothing
\end{aligned}$$

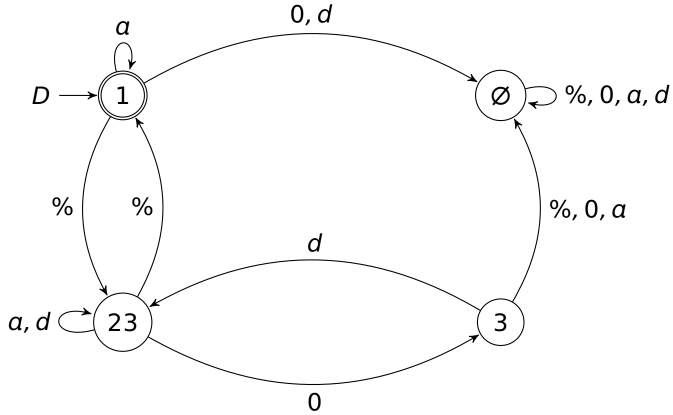
The set of final states F for the DFA D is given as follows:

$$F \quad := \quad \{A \subseteq Q_{N_2} \mid A \cap F_{N_2} \neq \varnothing\} \quad = \quad \{\{1\}\}.$$

Obviously,

$$s \quad := \quad S_{N_2} \quad = \quad \{1\}.$$

Given all these, we can now depict the state diagram for the DFA D as below:



Note that in the above diagram states 1 and 3 respectively represent singleton sets $\{1\}$ and $\{3\}$ while the state 23 stands to denote the set $\{2,3\}$.

c) We now check whether D is the minimal DFA with the above configuration. Observe that D has no inaccessible states. We can then employ the marking algorithm to perform the (in)distinguishability test for each pair of states.

As final and non-final states are distinguishable, we mark them in the below tabular right from the starch:

1			
✓	23		
✓		3	
✓			∅

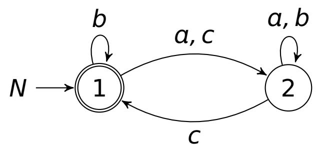
We then compare pairs of states in the below given order, and resume accordingly:

$\{23,\varnothing\}$	$\xrightarrow{\%}$	$\{1,\varnothing\}$	mark $(23,\varnothing)$	as $(1,\varnothing)$ is already marked
$\{23,3\}$	$\xrightarrow{\%}$	$\{1,\varnothing\}$	mark $(23,3)$	as $(1,\varnothing)$ is already marked
$\{3,\varnothing\}$	$\xrightarrow{d}$	$\{23,\varnothing\}$	mark $(3,\varnothing)$	as $(23,\varnothing)$ is already marked

1			
✓	23		
✓	✓	3	
✓	✓	✓	∅

Observe that all state pairs in D are distinguishable thus it is already the minimal DFA with respect to its local configuration.

Q3. (20 pts) Given a NFA$_\varepsilon$ $N = (\{1,2\}, \{a,b,c\}, \varepsilon, \Delta, \{1\}, \{1\})$ with below depicted state diagram



compute the regular expression α such that $\mathcal{L}(\alpha) = \mathcal{L}(N)$ employing the algorithm (definition) given in w4.pdf, slide #19.

A3.

In the first recursive call, the algorithm attempts to compute

$$\alpha_{11}^{\{1,2\}} \quad = \quad \alpha_{11}^{\{1\}} + \alpha_{12}^{\{1\}}(\alpha_{22}^{\{1\}})^* \alpha_{21}^{\{1\}} \qquad u = 1, \mathbf{q = 2}, v = 1$$

In the second recursive call, it computes

$$\begin{aligned}
\alpha_{11}^{\{1\}} &= \alpha_{11}^{\varnothing} + \alpha_{11}^{\varnothing}(\alpha_{11}^{\varnothing})^* \alpha_{11}^{\varnothing} && u = 1, \mathbf{q = 1}, v = 1 \\
\alpha_{12}^{\{1\}} &= \alpha_{12}^{\varnothing} + \alpha_{11}^{\varnothing}(\alpha_{11}^{\varnothing})^* \alpha_{12}^{\varnothing} && u = 1, \mathbf{q = 1}, v = 2 \\
\alpha_{22}^{\{1\}} &= \alpha_{22}^{\varnothing} + \alpha_{21}^{\varnothing}(\alpha_{11}^{\varnothing})^* \alpha_{12}^{\varnothing} && u = 2, \mathbf{q = 1}, v = 2 \\
\alpha_{21}^{\{1\}} &= \alpha_{21}^{\varnothing} + \alpha_{21}^{\varnothing}(\alpha_{11}^{\varnothing})^* \alpha_{11}^{\varnothing} && u = 2, \mathbf{q = 1}, v = 1
\end{aligned}$$

In the third recursive call, it hits the ground (namely reaches the base case), and computes

$$\begin{aligned}
\alpha_{11}^{\varnothing} &= \mathbf{b} + \boldsymbol{\varepsilon} \\
\alpha_{12}^{\varnothing} &= \mathbf{a} + \mathbf{c} \\
\alpha_{22}^{\varnothing} &= \mathbf{a} + \mathbf{b} + \boldsymbol{\varepsilon} \\
\alpha_{21}^{\varnothing} &= \mathbf{c}
\end{aligned}$$

At this stage, the algorithm folds back

$$\begin{aligned}
\alpha_{11}^{\{1\}} &= (\mathbf{b} + \boldsymbol{\varepsilon}) + (\mathbf{b} + \boldsymbol{\varepsilon})(\mathbf{b} + \boldsymbol{\varepsilon})^*(\mathbf{b} + \boldsymbol{\varepsilon}) \\
\alpha_{12}^{\{1\}} &= (\mathbf{a} + \mathbf{c}) + (\mathbf{b} + \boldsymbol{\varepsilon})(\mathbf{b} + \boldsymbol{\varepsilon})^*(\mathbf{a} + \mathbf{c}) \\
\alpha_{22}^{\{1\}} &= (\mathbf{a} + \mathbf{b} + \boldsymbol{\varepsilon}) + \mathbf{c}(\mathbf{b} + \boldsymbol{\varepsilon})^*(\mathbf{a} + \mathbf{c}) \\
\alpha_{21}^{\{1\}} &= \mathbf{c} + \mathbf{c}(\mathbf{b} + \boldsymbol{\varepsilon})^*(\mathbf{b} + \boldsymbol{\varepsilon})
\end{aligned}$$

Therefore,

$$\alpha_{11}^{\{1,2\}} = \Big((\mathbf{b}+\boldsymbol{\varepsilon}) + (\mathbf{b}+\boldsymbol{\varepsilon})(\mathbf{b}+\boldsymbol{\varepsilon})^*(\mathbf{b}+\boldsymbol{\varepsilon}) + \Big[\big((\mathbf{a}+\mathbf{c}) + (\mathbf{b}+\boldsymbol{\varepsilon})(\mathbf{b}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c})\big)\big((\mathbf{a}+\mathbf{b}+\boldsymbol{\varepsilon}) + \mathbf{c}(\mathbf{b}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c})\big)^*\big(\mathbf{c} + \mathbf{c}(\mathbf{b}+\boldsymbol{\varepsilon})^*(\mathbf{b}+\boldsymbol{\varepsilon})\big)\Big]\Big)$$