CMPE 322/327 - Theory of Computation

Week 10: Ogden's Lemma & Push Down Automata

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April 25-29, 2022

Outline

Definitions

CFG $G = (N, \Sigma, P, S)$ is in

- Chomsky normal form if for all $A \to \alpha$ in P $\quad \alpha = B\,C \in N^2 \quad$ or $\quad \alpha = a \in \Sigma$

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CFG G: $S \to X\text{b}S \mid XY\text{b} \mid YXZ \qquad X \to Z \mid \varepsilon \qquad Y \to \text{b}XY \mid \varepsilon \qquad Z \to \text{a}$
remove ε and unit productions

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split long right-hand sides

$$B \to \mathrm{b} \qquad S \to TS \mid UB \mid VZ \mid BS \mid YB \mid YZ \mid XB \mid XZ \mid \mathrm{b} \mid \mathrm{a}$$

$$X \to \mathrm{a} \qquad Y \to BU \mid BY \mid BX \mid \mathrm{b} \qquad Z \to \mathrm{a}$$

$$T \to XB \qquad U \to XY \qquad V \to YX$$

Pumping Lemma

$$A \text{ is context-free} \implies \begin{cases} \exists\, k & \\ \forall\, z \in A & \text{with } |z| \geqslant k \\ \exists\, u, v, w, x, y & \text{with } \begin{cases} z = uvwxy \\ |vwx| \leqslant k \\ vx \neq \varepsilon \end{cases} \\ \forall\, i \geqslant 0 & uv^i wx^i y \in A \end{cases}$$

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Proof. (Idea)

take $k = 2^{n+1}$ where n is number of nonterminals of any CFG in Chomsky normal form that accepts $A - \{\varepsilon\}$

Pumping Lemma (Contrapositive)

$$\left.\begin{array}{ll}
\forall\, k & \\
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\exists\, i \geqslant 0 & \text{with } uv^iwx^iy \notin A
\end{array}\right\} \implies A \text{ is not context-free}$$

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  2. vwx has no c's: uv^iwx^iy has more c's than b's or a's

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  compute T_{ij} by induction on $j - i$
- $x \in L(G) \iff S \in T_{0|x|}$

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Ogden's Lemma

$$
A \text{ is context-free} \quad \Longrightarrow \quad
\begin{cases}
\exists\, k & \\
\forall\, z \in A & \text{with at least } k \text{ marked letters in } z \\
\exists\, u, v, w, x, y & \text{with} \begin{cases} z = uvwxy \\ vwx \text{ contains at most } k \text{ marked letters} \\ vx \text{ contains at least one marked letter} \end{cases} \\
\forall\, i \geqslant 0 & uv^i wx^i y \in A
\end{cases}
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- suppose A is context-free
- let k be integer in Ogden's lemma
- choose: $z = a^k b^k c^{k+k!}$ mark all a's
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Outline

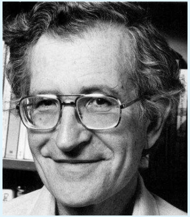
recursive enumerable

recursive

context-free

push-down automata

context-free grammars



$\{A \mid A \subseteq \Sigma^*\} = 2^{\Sigma^*}$

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  5. $s \in Q$: start state
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- next configuration relation is binary relation $\xrightarrow[M]{1}$ defined as: $(p, ay, A\beta) \xrightarrow[M]{1} (q, y, \gamma\beta)$
  for all $((p, a, A), (q, \gamma)) \in \delta$ with $a \in \Sigma \cup \{\epsilon\}$ and $y \in \Sigma^*$, $\beta \in \Gamma^*$

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input:	[	[	]	[	[	]	]	]
state:	1	1	1	1	1			
stack:	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$			
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state:	1	1	1	1	1	1		
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$$
\begin{array}{lcccccccc}
\text{input:} & [& [&] & [& [&] &] &] \\
\text{state:} & 1 & 1 & 1 & 1 & 1 & 1 & & \\
\text{stack:} & \perp & \perp & \perp & \perp & \perp & \perp & & \\
 & & [& [& [& [& [& & \\
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stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
		[	[	[	[	[	[
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state:	1		1		1		1		1		1		1		1	
stack:	$\perp$		$\perp$		$\perp$		$\perp$		$\perp$		$\perp$		$\perp$		$\perp$	
			[		[		[		[		[		[		[	
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input:		[	[	]	[	[	]	]
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state:	1	1	1	1	1	1	1	1
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input:	[	[	]	[	[	]	]	]	ε
state:	1	1	1	1	1	1	1	1	1
stack:	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
		[	[	[	[	[	[	[	
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state:	1	1	1	1	1	1	1	1	1	2
stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	
		[	[	[	[	[	[	[		
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input:		[	[	]	[	[	]	]	]	ϵ		[	[	]	]	]
state:	1	1	1	1	1	1	1	1	1	2						
stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$							
		[	[	[	[	[	[	[								
			[		[	[	[									
						[										

Example

$A = \{x \in \{[,]\}^* \mid x \text{ is balanced}\}$ is accepted by NPDA $M = (Q, \Sigma, \Gamma, \delta, 1, \perp, F)$ with

1. $Q = \{1, 2\}$
2. $\Sigma = \{[,]\}$
3. $\Gamma = \{\perp, [\}$
4. $F = \{2\}$
5. $s = 1$
6. $\delta = \{((1, [, \perp), (1, [\perp)),\ ((1,], [), (1, \varepsilon)),\ ((1, [, [), (1, [[)), ((1, \varepsilon, \perp), (2, \varepsilon))\}$

input:		[	[	]	[	[	]	]	]	ε
state:	1	1	1	1	1	1	1	1	1	2
stack:	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	
		[	[	[	[	[	[	[		
			[		[	[	[			
						[				

input:		[	[	]	]]
state:	1	1	1	1	1
stack:	⊥	⊥	⊥	⊥	⊥
		[	[	[	
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state:	1	1	1	1	1	1	1	1	1	2
stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	
		[	[	[	[	[	[	[		
			[		[	[	[			
						[				

input:	[	[	]	]	]
state:	1	1	1	1	1
stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
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state:	1	1	1	1	1	1	1	1	1	2
stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	
		[	[	[	[	[	[	[		
			[		[	[	[			
						[				

input:	[	[	]	]	]
state:	1	1	1	1	2
stack:	$\perp$	$\perp$	$\perp$	$\perp$	ε
		[	[	[	
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stack:	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	
		[	[	[	[	[	[	[		
			[		[	[	[			
						[				

input:	[	[	]	]	]
state:	1	1	1	1	2
stack:	$\perp$	$\perp$	$\perp$	$\perp$	ε
		[	[	[	
			[		

Definitions

NPDA $M = (Q, \Sigma, \Gamma, \delta, s, \perp, F)$

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Theorem

CFGs and NPDAs are equivalent:

1. $A = L(G)$ for some CFG G $\iff$
2. $A = L_f(M)$ for some NPDA M

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Determinism in PDAs (Informally)

ability to perform at most one transition (move)

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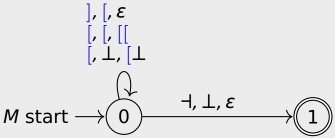
Definition

A deterministic pushdown automaton (DPDA) is an octuple $M = (Q, \Sigma, \Gamma, \delta, \perp, \dashv, s, F)$

1. $\dashv$ is a special symbol not in Σ, called the right endmarker
2. for any $p \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $A \in \Gamma$, the set $\delta \subseteq (Q \times (\Sigma \cup \{\dashv\} \cup \{\varepsilon\}) \times \Gamma) \times (Q \times \Gamma^*)$ contains
   - at most one element of the form $((p, a, A), (q, \beta))$
   - exactly one transition of the form $((p, a, A), (q, \beta))$ or $((p, \varepsilon, A), (q, \beta))$

Example

$A = \{x \in \{[,]\}^* \mid x \text{ is balanced}\}$ is accepted by DPDA $M = \big(\{0,1\}, \{[,]\}, \{[,\perp\}, \delta, \perp, \dashv, 0, \{1\}\big)$ with



the final state acceptance criterion.

Remark

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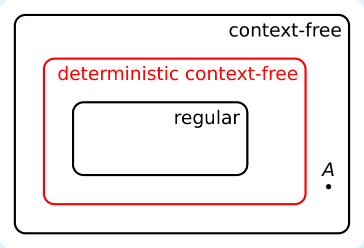

$$A = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$$

Outline

Theorem

context-free sets are effectively closed under

- union
- concatenation
- asterate
- homomorphic image
- homomorphic preimage

Theorem

context-free sets are effectively closed under

- union
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context-free sets are not closed under

- intersection
- complement

Proof. (union)

- $A = L(G_1)$ for CFG $G_1 = (N_1, \Sigma, P_1, S_1)$
  $B = L(G_2)$ for CFG $G_2 = (N_2, \Sigma, P_2, S_2)$

Proof. (union)

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- without loss of generality $N_1 \cap N_2 = \varnothing$

Context-Free Sets

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- $A \cup B = L(G)$ for CFG $G = (N, \Sigma, P, S)$ with
  1. $N := N_1 \cup N_2 \cup \{S\}$

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Example

- $A = \{a^n b^n a \mid n \geqslant 0\}$ $\quad S_1 \to Ta \quad T \to aTb \mid \varepsilon$
- $B = \{a^n b a^n \mid n \geqslant 0\}$ $\quad S_2 \to aS_2a \mid b$

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- $A \cup B$ $\qquad S \to S_1 \mid S_2$

Proof. (concatenation)

- $A = L(G_1)$ for CFG $G_1 = (N_1, \Sigma, P_1, S_1)$
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- $A = \{a^n b^n a \mid n \geqslant 0\}$ $\quad S_1 \to Ta \quad T \to aTb \mid \varepsilon$

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- $A = \{\mathtt{a}^n \mathtt{b}^n \mathtt{a} \mid n \geqslant 0\}$ $\quad S_1 \to T\mathtt{a} \quad T \to \mathtt{a}T\mathtt{b} \mid \varepsilon$
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- $h(A)$ $S_1 \to T\mathrm{b} \quad T \to \mathrm{b}T\mathrm{ac} \mid \varepsilon$

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- $h^{-1}(A) = L_f(N)$ for NPDA $N = (Q', \Delta, \Gamma, \delta', s', \bot, F')$ with
  1. $Q' := \{(q, x) \mid q \in Q$ and x is suffix of $h(a)$ for some $a \in \Delta\}$

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  2. $s' := (s, \varepsilon)$

Proof. (homomorphic preimage)

- $A = L_f(M)$ for NPDA $M = (Q, \Sigma, \Gamma, \delta, s, \perp, F)$
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Example

Let $\Sigma = \{c, d\}$, $\Delta = \{e, f\}$ and $h\colon \Delta^* \to \Sigma^*$ such that

$$h(e) = cdd \qquad h(f) = dcdd$$

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((p, cddcdddcdd), \varepsilon, A\gamma) & \xrightarrow[N]{1} ((q, ddcdddcdd), \varepsilon, B\gamma) & \text{if } ((p, c, A), (q, B)) \in \delta & \text{(by transition in 4.2)}
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((q, ddcdddcdd), \varepsilon, B\gamma) & \xrightarrow[N]{1} ((r, dcdddcdd), \varepsilon, C\gamma) & \text{if } ((q, d, B), (r, C)) \in \delta & \text{(by transition in 4.2)}
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((p', d), \varepsilon, A'\gamma) & \xrightarrow[N]{1} & ((q', \varepsilon), \varepsilon, B'\gamma) & \text{if } ((p', d, A'), (q', B')) \in \delta & \text{(by transition in 4.2)}
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- claim: $((s, \varepsilon), x, \perp) \xrightarrow[N]{*} ((q, \varepsilon), \varepsilon, \gamma) \iff (s, h(x), \perp) \xrightarrow[M]{*} (q, \varepsilon, \gamma)$ for all $x \in \Delta^*$

Theorem

context-free sets are not closed under intersection

Proof.

- $A = \{a^i b^i c^j \mid i, j \geqslant 0\}$
  $B = \{a^i b^j c^j \mid i, j \geqslant 0\}$

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intersection of context-free set and regular set is context-free

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Proof.

- $A = L_f(M_1)$ for NPDA $M_1 = (Q_1, \Sigma, \Delta, \delta_1, s_1, \perp, F_1)$

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- $A = L_f(M_1)$ for NPDA $M_1 = (Q_1, \Sigma, \Delta, \delta_1, s_1, \perp, F_1)$
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- define NPDA $N = (Q, \Sigma, \Delta, \delta, s, \perp, F)$ with
  - $Q := Q_1 \times Q_2$
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  - δ consists of transitions $(\forall p \in Q_1\ \forall q \in Q_2\ \forall A \in \Gamma)$

$$(((p,q),a,A),\ ((p',q'),\gamma)) \qquad \text{for all } a \in \Sigma,\ ((p,a,A),(p',\gamma)) \in \delta_1 \text{ and } q' = \delta_2(q,a)$$

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Proof. (cont'd)

- claim $(\forall\, p \in Q_1\ \forall\, q \in Q_2\ \forall\, x \in \Sigma^*\,)$

$$((p,q),x,\perp) \xrightarrow[N]{*} ((p',q'),\varepsilon,\gamma) \iff (p,x,\perp) \xrightarrow[M_1]{*} (p',\varepsilon,\gamma) \text{ and } \widehat{\delta}_2(q,x) = q'$$

  is proved by induction

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- hence

$$L_f(N) = \{x \in \Sigma^* \mid ((s_1,s_2),x,\perp) \xrightarrow[N]{*} ((p,q),\varepsilon,\gamma) \text{ and } (p,q) \in F\}$$

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Theorem

context-free sets are not closed under complement

Proof.

- $A = \{xx \mid x \in \{a, b\}^*\}$
- A is not context-free because
$$A \cap L(a^*b^*a^*b^*) = \{a^n b^m a^n b^m \mid m, n \geqslant 0\}$$
is not context-free (by pumping lemma)

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- $\sim A = \{a, b\}^* - \{ xx \mid x \in \{a, b\}^* \}$ is context-free

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$$S \to AB \mid BA \mid A \mid B \qquad A \to CAC \mid a$$
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deterministic context-free sets are effectively closed under

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- union
- intersection
- concatenation
- asterate
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- $A = \{a^i b^j c^k \mid i \neq j\}$ and $B = \{a^i b^j c^k \mid j \neq k\}$
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Thanks! & Questions?