CMPE 322/327 - Theory of Computation

Week 13: Rice's Theorem & Unrestricted Grammars

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Outline

Theorem

halting problem (HP) for TMs is undecidable (not recursive)

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Proof. (   )

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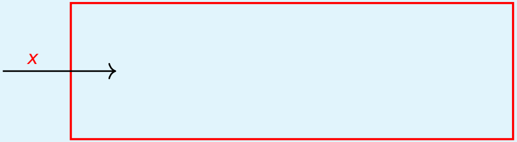
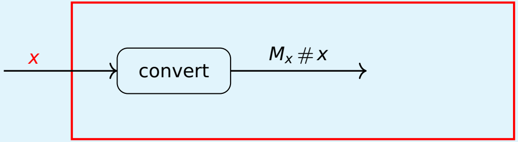


design of TM N



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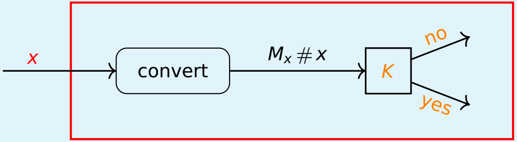

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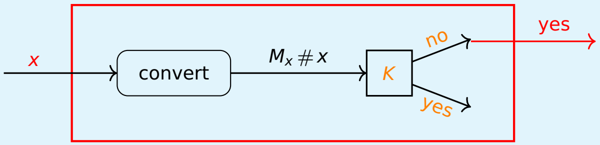

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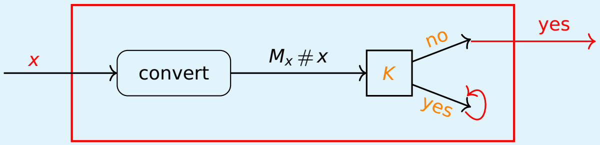

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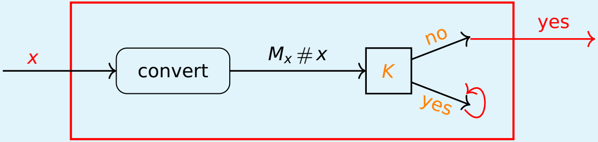
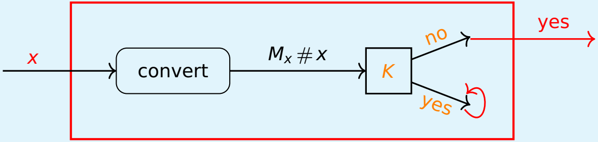


design of TM N

	ε	0	1	00	01	10	$\cdots$
M_ε	$\times$	$\checkmark$	$\times$	$\times$	$\checkmark$	$\times$	$\cdots$
M_0	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$	
M_1	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	
M_{00}	$\times$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$	
M_{01}	$\times$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$	
M_{10}	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	
$\vdots$	$\vdots$						

$\checkmark$: halts
$\times$: loops



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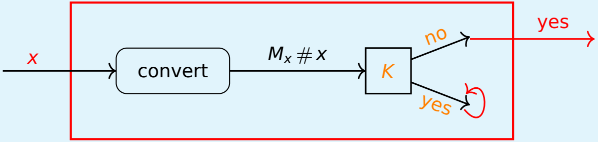
M_1	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	
M_{00}	$\times$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\times$	
M_{01}	$\times$	$\times$	$\checkmark$	$\times$	$\checkmark$	$\times$	
M_{10}	$\checkmark$	$\times$	$\checkmark$	$\times$	$\times$	$\times$	
$\vdots$	$\vdots$						

$\checkmark$: accepts
$\times$: rejects



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$\vdots$	$\vdots$						

$\checkmark$: halts

$\times$: loops

$L(N) = \{\varepsilon, 00, 10, \dots\}$

Theorem

halting problem (HP) for TMs is undecidable (not recursive)

Proof. ( )

- suppose $\text{HP} = \{M \# x \mid \text{TM } M \text{ halts on input } x\}$ is recursive
- $\text{HP} = L(K)$ for some total TM K
- construct TM N that on input x
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  - runs K on input $M_x \# x$
  - accepts if K rejects and loops if K accepts
- for all inputs x $\quad N$ halts on $x \iff K$ rejects $M_x \# x$

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Proof. (diagonalization)

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All Together

1. MP is r.e. (by corollary in w11.pdf on slide #29)

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5. $HP \le_m MP$ (by theorem in w12.pdf on slides #17 − 18)

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MP is not recursive

Proof.

by statements 5, contrapositive of 4 and 2

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Proof.

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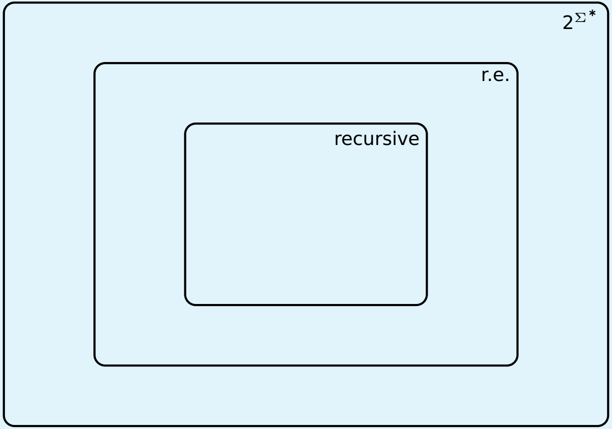
Theorem

HP is r.e.

Proof.

by statements 5, 3 and 1

Summary



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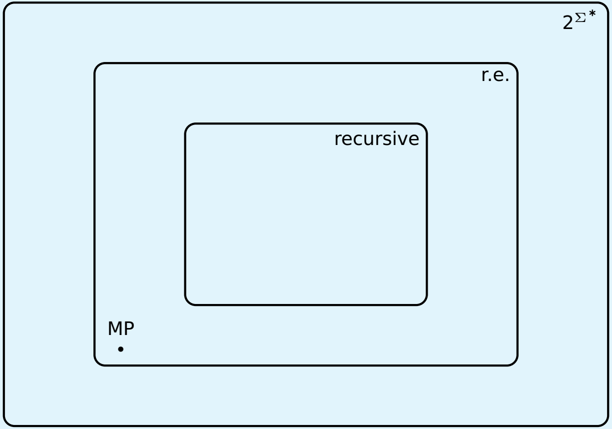

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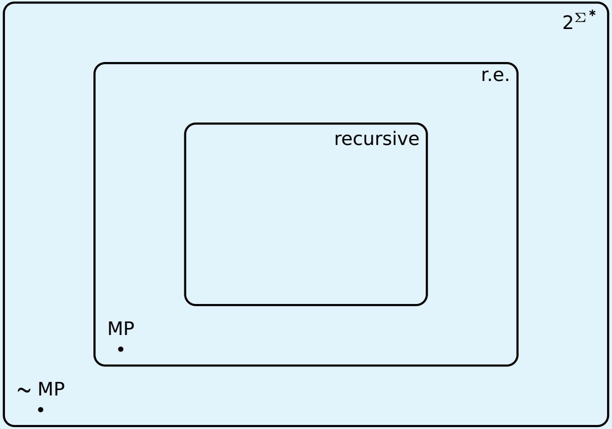

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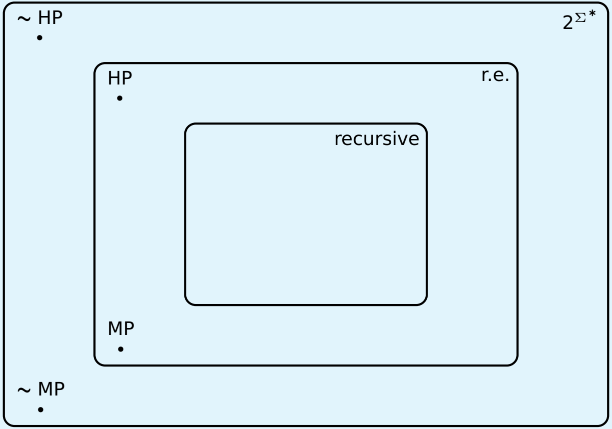

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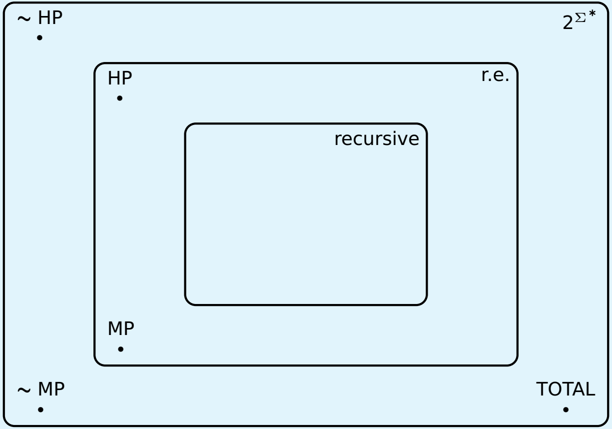

TOTAL $= \{M \mid \text{TM } M \text{ halts on all inputs}\}$

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instance: TM M
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- $$A := \sim \mathrm{HP}$$ $$B := \mathrm{FIN} = \{M \mid L(M) \text{ is finite}\}$$
- $\sim$ HP $\leq_m$ FIN and $\sim$ HP is not r.e $\implies$ FIN is not r.e.

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- $\sim$ HP $\leq_m$ FIN and True $\implies$ FIN is not r.e.

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reduction σ: $\sim \mathrm{HP} \rightarrow_m \mathrm{FIN}$

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- $$\begin{array}{lll} A & := & \sim \mathrm{HP} \\ B & := & \mathrm{FIN} = \{M \mid L(M) \text{ is finite}\} \end{array}$$
- $\sim$ HP $\leq_m$ FIN $\implies$ FIN is not r.e.

reduction σ: $\sim$ HP $\rightarrow_m$ FIN

- σ transforms $M\#x$ into TM N

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- $A := \sim \mathrm{HP}$
  $B := \mathrm{FIN} = \{M \mid L(M) \text{ is finite}\}$
- $\sim \mathrm{HP} \leq_m \mathrm{FIN} \implies$ FIN is not r.e.

reduction σ: $\sim \mathrm{HP} \to_m \mathrm{FIN}$

- σ transforms $M \# x$ into TM N that
  1. erases its input

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- $$A := \sim \mathrm{HP}$$ $$B := \mathrm{FIN} = \{M \mid L(M) \text{ is finite}\}$$
- $\sim \mathrm{HP} \leq_{\mathrm{m}} \mathrm{FIN} \implies \mathrm{FIN}$ is not r.e.

reduction σ: $\sim \mathrm{HP} \rightarrow_{\mathrm{m}} \mathrm{FIN}$

- σ transforms $M \# x$ into TM N that
  1. erases its input
  2. writes x on its tape

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  1. erases its input
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  3. runs M on input x

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- $$\begin{aligned} A &:= \sim \text{HP} \\ B &:= \text{FIN} = \{M \mid L(M) \text{ is finite}\} \end{aligned}$$
- $\sim$ HP $\leq_m$ FIN $\implies$ FIN is not r.e.

reduction σ: $\sim$ HP $\rightarrow_m$ FIN

- σ transforms $M \# x$ into TM N that
  1. erases its input
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  3. runs M on input x
  4. accepts if M halts on x

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- σ transforms $M \# x$ into TM N that
  1. erases its input
  2. writes x on its tape
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- $M \# x \in \sim \mathrm{HP} \iff M$ does not halt on x

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- $\sim$ HP $\leq_m$ FIN $\implies$ FIN is not r.e.

reduction σ: $\sim$ HP $\to_m$ FIN

- σ transforms $M\#x$ into TM N that
  1. erases its input
  2. writes x on its tape
  3. runs M on input x
  4. accepts if M halts on x
- $M\#x \in \sim$ HP $\iff$ M does not halt on x $\iff$ $L(N) = \varnothing$

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  1. erases its input
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  3. runs M on input x
  4. accepts if M halts on x
- $M \# x \in \sim \mathrm{HP} \iff M$ does not halt on $x \iff L(N) = \varnothing \iff N \in \mathrm{FIN}$

Outline

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every nontrivial property P of r.e. sets is undecidable ($\{M \mid P(L(M)) = \mathrm{T}\}$ is undecidable)

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design of TM N

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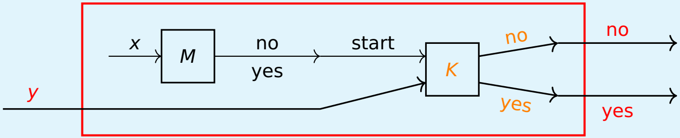

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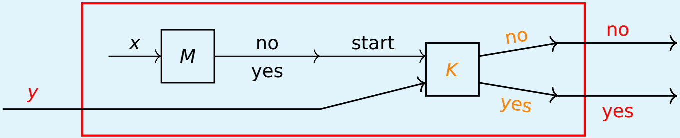


design of TM N

Corollary

emptiness, finiteness, regularity, context-freeness, recursiveness, ... are undecidable properties of r.e. sets

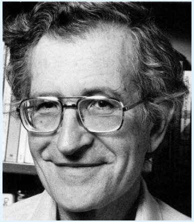
Some Decision Problems about TMs

- instance: TM M, state q
  question: does M enter state q on some input?
- instance: TM M
  question: does M take more than 100 steps on some input?
- instance: TM M
  question: does M take more than 100 steps on all inputs?
- instance: TM M
  question: does M take less than 100 steps on all inputs?
- instance: TM M
  question: does M accept all inputs?
- instance: TM M
  question: is there TM N with less states and $L(M) = L(N)$?

Outline

recursive enumerable

Turing machines
unrestricted grammars



$\{A \mid A \subseteq \Sigma^*\} = 2^{\Sigma^*}$

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- language generated by G: $L(G) = \{x \in \Sigma^* \mid S \xrightarrow[G]{*} x\}$

Example

$\{xx \mid x \in \{a, b\}^*\} = L(G)$ for unrestricted grammar $G = (N, \Sigma, P, S)$ with

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$$
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S \to ABC & DC \to B\,a\,C & D\,a \to a\,D & E\,a \to a\,E & a\,B \to B\,a & C \to \varepsilon \\
AB \to a\,AD \mid b\,AE \mid \varepsilon & EC \to B\,b\,C & D\,b \to b\,D & E\,b \to b\,E & b\,B \to B\,b &
\end{array}
$$

$abbabb \in L(G)$:

$$
\begin{aligned}
S &\xrightarrow[G]{1} ABC \xrightarrow[G]{1} a\,ADC \xrightarrow[G]{1} a\,AB\,a\,C \xrightarrow[G]{1} a\,b\,AE\,a\,C \xrightarrow[G]{1} ab\,A\,a\,EC \xrightarrow[G]{1} ab\,A\,a\,B\,b\,C \\
&\xrightarrow[G]{1} ab\,AB\,ab\,C \xrightarrow[G]{1} abb\,AE\,ab\,C \xrightarrow[G]{1} abb\,A\,a\,E\,b\,C \xrightarrow[G]{1} abb\,A\,ab\,EC \xrightarrow[G]{1} abb\,A\,ab\,B\,b\,C \\
&\xrightarrow[G]{1} abb\,A\,a\,B\,bb\,C \xrightarrow[G]{1} abb\,AB\,abb\,C
\end{aligned}
$$

Example

$\{xx \mid x \in \{a, b\}^*\} = L(G)$ for unrestricted grammar $G = (N, \Sigma, P, S)$ with

1. $N = \{S, A, B, C, D, E\}$
2. $\Sigma = \{\mathrm{a}, \mathrm{b}\}$
3. P consists of productions

$$
\begin{array}{llllll}
S \to ABC & DC \to B\,a\,C & D\,\mathrm{a} \to \mathrm{a}\,D & E\,a \to a\,E & a\,B \to B\,a & C \to \varepsilon \\
AB \to a\,AD \mid b\,AE \mid \varepsilon & EC \to B\,b\,C & D\,\mathrm{b} \to \mathrm{b}\,D & E\,b \to b\,E & b\,B \to B\,b &
\end{array}
$$

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$$
\begin{aligned}
S &\xrightarrow[G]{1} ABC \xrightarrow[G]{1} a\,ADC \xrightarrow[G]{1} \mathrm{a}\,AB\,a\,C \xrightarrow[G]{1} \mathrm{a}\,b\,AE\mathrm{a}\,C \xrightarrow[G]{1} ab\,A\mathrm{a}\,EC \xrightarrow[G]{1} ab\,A\,a\,B\,b\,C \\
&\xrightarrow[G]{1} ab\,AB\,\mathrm{ab}\,C \xrightarrow[G]{1} abb\,AE\,\mathrm{ab}\,C \xrightarrow[G]{1} abb\,A\,a\,E\,b\,C \xrightarrow[G]{1} abb\,A\,\mathrm{ab}\,EC \xrightarrow[G]{1} abb\,A\,\mathrm{ab}\,B\,b\,C \\
&\xrightarrow[G]{1} abb\,A\,a\,B\,\mathrm{bb}\,C \xrightarrow[G]{1} abb\,AB\,abb\,C \xrightarrow[G]{1} abbabb\,C
\end{aligned}
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&\xrightarrow[G]{1} ab\,AB\,ab\,C \xrightarrow[G]{1} abb\,AE\,ab\,C \xrightarrow[G]{1} abb\,A\,a\,E\,b\,C \xrightarrow[G]{1} abb\,A\,ab\,EC \xrightarrow[G]{1} abb\,A\,ab\,B\,b\,C \\
&\xrightarrow[G]{1} abb\,A\,a\,B\,bb\,C \xrightarrow[G]{1} abb\,AB\,abb\,C \xrightarrow[G]{1} abbabb\,C \xrightarrow[G]{1} abbabb
\end{aligned}
$$

Theorem

TMs and unrestricted grammars are equivalent:

1. $\forall$ TM M $\exists$ unrestricted grammar G such that $L(G) = L(M)$
2. $\forall$ unrestricted grammar G $\exists$ TM M such that $L(M) = L(G)$

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Proof.

1. given TM $M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)$

   construct unrestricted grammar $G = (N, \Sigma, P, S)$ with $N := ((\Sigma \cup \{\varepsilon\}) \times \Gamma) \cup Q \cup \{S, T, U\}$

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Basic Idea

two tracks

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- top track for input string of TM

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two tracks

- top track for input string of TM
- bottom track for simulating tape of TM

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two tracks

- top track for input string of TM
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Proof. (cont'd)

- P consists of

$$S \to s\begin{array}{|c|}\hline \varepsilon \\ \hline \vdash \\ \hline\end{array}T$$

Proof. (cont'd)

- P consists of

$$S \to s\boxed{\begin{array}{c}\varepsilon\\ \hline \vdash\end{array}}T \qquad T \to \boxed{\begin{array}{c}a\\ \hline a\end{array}}T \quad \text{for all } a \in \Sigma$$

Proof. (cont'd)

- P consists of

$$S \to s\begin{array}{|c|}\hline \varepsilon \\ \hline \vdash \\ \hline\end{array}T \qquad T \to \begin{array}{|c|}\hline a \\ \hline a \\ \hline\end{array}T \;\text{ for all } a \in \Sigma \qquad T \to U$$

Proof. (cont'd)

- P consists of

$$S \to s\begin{bmatrix}\varepsilon \\ \vdash\end{bmatrix}T \qquad T \to \begin{bmatrix}a \\ a\end{bmatrix}T \text{ for all } a \in \Sigma \qquad T \to U \qquad U \to \begin{bmatrix}\varepsilon \\ \sqcup\end{bmatrix}U$$

Proof. (cont'd)

- P consists of

$$S \to s\boxed{\begin{smallmatrix}\varepsilon\\ \vdash\end{smallmatrix}}T \qquad T \to \boxed{\begin{smallmatrix}a\\ a\end{smallmatrix}}T \text{ for all } a \in \Sigma \qquad T \to U \qquad U \to \boxed{\begin{smallmatrix}\varepsilon\\ \sqcup\end{smallmatrix}}U \qquad U \to \varepsilon$$

Proof. (cont'd)

- P consists of

$$S \to s\begin{array}{|c|}\hline \varepsilon \\ \hline \vdash \\ \hline\end{array}T \qquad T \to \begin{array}{|c|}\hline a \\ \hline a \\ \hline\end{array}T \text{ for all } a \in \Sigma \qquad T \to U \qquad U \to \begin{array}{|c|}\hline \varepsilon \\ \hline \sqcup \\ \hline\end{array}U \qquad U \to \varepsilon$$

$$p\begin{array}{|c|}\hline c \\ \hline a \\ \hline\end{array} \to \begin{array}{|c|}\hline c \\ \hline b \\ \hline\end{array}q \qquad \text{for all } \delta(p,a) = (q,b,R) \text{ and } c \in \Sigma \cup \{\varepsilon\}$$

Proof. (cont'd)

- P consists of

$$S \to s\boxed{\begin{array}{c}\varepsilon\\ \hline \vdash\end{array}}T \qquad T \to \boxed{\begin{array}{c}a\\ \hline a\end{array}}T \text{ for all } a \in \Sigma \qquad T \to U \qquad U \to \boxed{\begin{array}{c}\varepsilon\\ \hline \sqcup\end{array}}U \qquad U \to \varepsilon$$

$$p\boxed{\begin{array}{c}c\\ \hline a\end{array}} \to \boxed{\begin{array}{c}c\\ \hline b\end{array}}q \qquad \text{for all } \delta(p,a) = (q,b,R) \text{ and } c \in \Sigma \cup \{\varepsilon\}$$

$$\boxed{\begin{array}{c}e\\ \hline d\end{array}}p\boxed{\begin{array}{c}c\\ \hline a\end{array}} \to q\boxed{\begin{array}{c}e\\ \hline d\end{array}}\boxed{\begin{array}{c}c\\ \hline b\end{array}} \qquad \text{for all } \delta(p,a) = (q,b,L) \text{ and } d \in \Gamma \text{ and } c,e \in \Sigma \cup \{\varepsilon\}$$

Proof. (cont'd)

- P consists of

$$S \to s\begin{array}{|c|}\hline \varepsilon \\ \hline \vdash \\ \hline\end{array}T \qquad T \to \begin{array}{|c|}\hline a \\ \hline a \\ \hline\end{array}T \text{ for all } a \in \Sigma \qquad T \to U \qquad U \to \begin{array}{|c|}\hline \varepsilon \\ \hline \sqcup \\ \hline\end{array}U \qquad U \to \varepsilon$$

$$p\begin{array}{|c|}\hline c \\ \hline a \\ \hline\end{array} \to \begin{array}{|c|}\hline c \\ \hline b \\ \hline\end{array}q \qquad \text{for all } \delta(p,a) = (q,b,R) \text{ and } c \in \Sigma \cup \{\varepsilon\}$$

$$\begin{array}{|c|}\hline e \\ \hline d \\ \hline\end{array}p\begin{array}{|c|}\hline c \\ \hline a \\ \hline\end{array} \to q\begin{array}{|c|}\hline e \\ \hline d \\ \hline\end{array}\begin{array}{|c|}\hline c \\ \hline b \\ \hline\end{array} \qquad \text{for all } \delta(p,a) = (q,b,L) \text{ and } d \in \Gamma \text{ and } c,e \in \Sigma \cup \{\varepsilon\}$$

$$\begin{array}{|c|}\hline c \\ \hline d \\ \hline\end{array}t \to t\,c\,t \qquad t\begin{array}{|c|}\hline c \\ \hline d \\ \hline\end{array} \to t\,c\,t \qquad t \to \varepsilon \qquad \text{for all } d \in \Gamma \text{ and } c \in \Sigma \cup \{\varepsilon\}$$

Proof. (cont'd)

- P consists of

$$S \to s\boxed{\begin{array}{c}\varepsilon\\ \hline \vdash\end{array}}T \qquad T \to \boxed{\begin{array}{c}a\\ \hline a\end{array}}T \;\text{ for all } a \in \Sigma \qquad T \to U \qquad U \to \boxed{\begin{array}{c}\varepsilon\\ \hline \sqcup\end{array}}U \qquad U \to \varepsilon$$

$$p\boxed{\begin{array}{c}c\\ \hline a\end{array}} \to \boxed{\begin{array}{c}c\\ \hline b\end{array}}q \qquad \text{for all } \delta(p,a) = (q,b,R) \text{ and } c \in \Sigma \cup \{\varepsilon\}$$

$$\boxed{\begin{array}{c}e\\ \hline d\end{array}}p\boxed{\begin{array}{c}c\\ \hline a\end{array}} \to q\boxed{\begin{array}{c}e\\ \hline d\end{array}}\boxed{\begin{array}{c}c\\ \hline b\end{array}} \qquad \text{for all } \delta(p,a) = (q,b,L) \text{ and } d \in \Gamma \text{ and } c,e \in \Sigma \cup \{\varepsilon\}$$

$$\boxed{\begin{array}{c}c\\ \hline d\end{array}}t \to t\,c\,t \qquad t\boxed{\begin{array}{c}c\\ \hline d\end{array}} \to t\,c\,t \qquad t \to \varepsilon \qquad \text{for all } d \in \Gamma \text{ and } c \in \Sigma \cup \{\varepsilon\}$$

- $x \in L(M) \iff x \in L(G)$

Example

TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1, \vdash, R)$	$(2, b, R)$	$(4, a, L)$	$(2, a, L)$
2	$(1, \vdash, R)$	$(3, a, L)$	$(1, a, R)$	$(4, a, L)$

$$\underset{1}{\vdash} a\,b \xrightarrow[M]{1} \vdash \underset{1}{a}\,b \xrightarrow[M]{1} \vdash b\,\underset{2}{b} \xrightarrow[M]{1} \vdash b\,a\,\underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\,a\,\underset{2}{a} \xrightarrow[M]{1} \vdash b\,\underset{3}{a}\,a$$

Example

TM $M = (\{1,2,3,4\},\{a,b\},\{a,b,\vdash,\sqcup\},\vdash,\sqcup,\delta,1,3,4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1,\vdash,R)$	$(2,b,R)$	$(4,a,L)$	$(2,a,L)$
2	$(1,\vdash,R)$	$(3,a,L)$	$(1,a,R)$	$(4,a,L)$

$$\underset{1}{\vdash} a\,b \xrightarrow[M]{1} \vdash \underset{1}{a}\,b \xrightarrow[M]{1} \vdash b\,\underset{2}{b} \xrightarrow[M]{1} \vdash b\,a\,\underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\,\underset{2}{a}\,a \xrightarrow[M]{1} \vdash \underset{3}{b}\,a\,a$$

$$S \rightarrow 1 \boxed{\begin{array}{c}\varepsilon\\ \hline \vdash\end{array}} T$$

Example

TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
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$$\underset{1}{\vdash} a\, b \xrightarrow[M]{1} \vdash \underset{1}{a}\, b \xrightarrow[M]{1} \vdash b\, \underset{2}{b} \xrightarrow[M]{1} \vdash b\, a\, \underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\, \underset{2}{a}\, a \xrightarrow[M]{1} \vdash \underset{3}{b}\, a\, a$$

$$S \to 1 \begin{array}{|c|}\hline \varepsilon \\ \hline \vdash \\ \hline \end{array} T \to 1 \begin{array}{|c|}\hline \varepsilon \\ \hline \vdash \\ \hline \end{array}\begin{array}{|c|}\hline a \\ \hline a \\ \hline \end{array} T$$

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TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1, \vdash, R)$	$(2, b, R)$	$(4, a, L)$	$(2, a, L)$
2	$(1, \vdash, R)$	$(3, a, L)$	$(1, a, R)$	$(4, a, L)$

$$\underset{1}{\vdash} a\, b \xrightarrow[M]{1} \vdash \underset{1}{a}\, b \xrightarrow[M]{1} \vdash b\, \underset{2}{b} \xrightarrow[M]{1} \vdash b\, a\, \underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\, \underset{2}{a}\, a \xrightarrow[M]{1} \vdash \underset{3}{b}\, a\, a$$

$$S \to 1 \boxed{\begin{matrix} \varepsilon \\ \vdash \end{matrix}} T \to 1 \boxed{\begin{matrix} \varepsilon \\ \vdash \end{matrix}} \boxed{\begin{matrix} a \\ a \end{matrix}} T \to 1 \boxed{\begin{matrix} \varepsilon \\ \vdash \end{matrix}} \boxed{\begin{matrix} a \\ a \end{matrix}} \boxed{\begin{matrix} b \\ b \end{matrix}} T$$

Example

TM $M = (\{1,2,3,4\},\{a,b\},\{a,b,\vdash,\sqcup\},\vdash,\sqcup,\delta,1,3,4)$ with

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$$S \to 1\begin{array}{|c|}\hline \varepsilon \\ \hline \vdash \\ \hline\end{array}T \to 1\begin{array}{|c|c|}\hline \varepsilon & a \\ \hline \vdash & a \\ \hline\end{array}T \to 1\begin{array}{|c|c|c|}\hline \varepsilon & a & b \\ \hline \vdash & a & b \\ \hline\end{array}T \to 1\begin{array}{|c|c|c|}\hline \varepsilon & a & b \\ \hline \vdash & a & b \\ \hline\end{array}U$$

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$$S \to 1\boxed{\begin{array}{c}\varepsilon\\ \hline \vdash\end{array}}T \to 1\boxed{\begin{array}{c}\varepsilon\\ \hline \vdash\end{array}}\boxed{\begin{array}{c}a\\ \hline a\end{array}}T \to 1\boxed{\begin{array}{c}\varepsilon\\ \hline \vdash\end{array}}\boxed{\begin{array}{c}a\\ \hline a\end{array}}\boxed{\begin{array}{c}b\\ \hline b\end{array}}T \to 1\boxed{\begin{array}{c}\varepsilon\\ \hline \vdash\end{array}}\boxed{\begin{array}{c}a\\ \hline a\end{array}}\boxed{\begin{array}{c}b\\ \hline b\end{array}}U \to 1\boxed{\begin{array}{c}\varepsilon\\ \hline \vdash\end{array}}\boxed{\begin{array}{c}a\\ \hline a\end{array}}\boxed{\begin{array}{c}b\\ \hline b\end{array}}\boxed{\begin{array}{c}\varepsilon\\ \hline \sqcup\end{array}}U$$

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$$S \to 1 \boxed{\substack{\varepsilon \\ \vdash}} T \to 1 \boxed{\substack{\varepsilon \\ \vdash}}\boxed{\substack{a \\ a}} T \to 1 \boxed{\substack{\varepsilon \\ \vdash}}\boxed{\substack{a \\ a}}\boxed{\substack{b \\ b}} T \to 1 \boxed{\substack{\varepsilon \\ \vdash}}\boxed{\substack{a \\ a}}\boxed{\substack{b \\ b}} U \to 1 \boxed{\substack{\varepsilon \\ \vdash}}\boxed{\substack{a \\ a}}\boxed{\substack{b \\ b}}\boxed{\substack{\varepsilon \\ \sqcup}} U \to 1 \boxed{\substack{\varepsilon \\ \vdash}}\boxed{\substack{a \\ a}}\boxed{\substack{b \\ b}}\boxed{\substack{\varepsilon \\ \sqcup}}$$

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$$S \to 1\boxed{\begin{array}{c}\varepsilon\\\hline\vdash\end{array}}T \to 1\boxed{\begin{array}{c}\varepsilon\\\hline\vdash\end{array}}\boxed{\begin{array}{c}a\\\hline a\end{array}}T \to 1\boxed{\begin{array}{c}\varepsilon\\\hline\vdash\end{array}}\boxed{\begin{array}{c}a\\\hline a\end{array}}\boxed{\begin{array}{c}b\\\hline b\end{array}}T \to 1\boxed{\begin{array}{c}\varepsilon\\\hline\vdash\end{array}}\boxed{\begin{array}{c}a\\\hline a\end{array}}\boxed{\begin{array}{c}b\\\hline b\end{array}}U \to 1\boxed{\begin{array}{c}\varepsilon\\\hline\vdash\end{array}}\boxed{\begin{array}{c}a\\\hline a\end{array}}\boxed{\begin{array}{c}b\\\hline b\end{array}}\boxed{\begin{array}{c}\varepsilon\\\hline\sqcup\end{array}}U \to 1\boxed{\begin{array}{c}\varepsilon\\\hline\vdash\end{array}}\boxed{\begin{array}{c}a\\\hline a\end{array}}\boxed{\begin{array}{c}b\\\hline b\end{array}}\boxed{\begin{array}{c}\varepsilon\\\hline\sqcup\end{array}}$$

$$\to \boxed{\begin{array}{c}\varepsilon\\\hline\vdash\end{array}}\,1\,\boxed{\begin{array}{c}a\\\hline a\end{array}}\boxed{\begin{array}{c}b\\\hline b\end{array}}\boxed{\begin{array}{c}\varepsilon\\\hline\sqcup\end{array}}$$

Example

TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

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$$\underset{1}{\vdash} a\, b \xrightarrow[M]{1} \vdash \underset{1}{a}\, b \xrightarrow[M]{1} \vdash b\, \underset{2}{b} \xrightarrow[M]{1} \vdash b\, a\, \underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\, \underset{2}{a}\, a \xrightarrow[M]{1} \vdash \underset{3}{b}\, a\, a$$

$$S \to 1\begin{array}{|c|}\hline \varepsilon\\\hline \vdash\\\hline\end{array}T \to 1\begin{array}{|c|c|}\hline \varepsilon & a\\\hline \vdash & a\\\hline\end{array}T \to 1\begin{array}{|c|c|c|}\hline \varepsilon & a & b\\\hline \vdash & a & b\\\hline\end{array}T \to 1\begin{array}{|c|c|c|}\hline \varepsilon & a & b\\\hline \vdash & a & b\\\hline\end{array}U \to 1\begin{array}{|c|c|c|c|}\hline \varepsilon & a & b & \varepsilon\\\hline \vdash & a & b & \sqcup\\\hline\end{array}U \to 1\begin{array}{|c|c|c|c|}\hline \varepsilon & a & b & \varepsilon\\\hline \vdash & a & b & \sqcup\\\hline\end{array}$$

$$\to \begin{array}{|c|}\hline \varepsilon\\\hline \vdash\\\hline\end{array}1\begin{array}{|c|c|c|}\hline a & b & \varepsilon\\\hline a & b & \sqcup\\\hline\end{array} \to \begin{array}{|c|c|}\hline \varepsilon & a\\\hline \vdash & b\\\hline\end{array}2\begin{array}{|c|c|}\hline b & \varepsilon\\\hline b & \sqcup\\\hline\end{array}$$

Example

TM $M = (\{1,2,3,4\},\{a,b\},\{a,b,\vdash,\sqcup\},\vdash,\sqcup,\delta,1,3,4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1,\vdash,R)$	$(2,b,R)$	$(4,a,L)$	$(2,a,L)$
2	$(1,\vdash,R)$	$(3,a,L)$	$(1,a,R)$	$(4,a,L)$

$$\underset{1}{\vdash}\, a\, b \xrightarrow[M]{1} \vdash \underset{1}{a}\, b \xrightarrow[M]{1} \vdash b\, \underset{2}{b} \xrightarrow[M]{1} \vdash b\, a\, \underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\, \underset{2}{a}\, a \xrightarrow[M]{1} \vdash \underset{3}{b}\, a\, a$$

$$S \to 1 \begin{array}{|c|}\hline \varepsilon \\ \hline \vdash \\ \hline\end{array} T \to 1 \begin{array}{|c|c|}\hline \varepsilon & a \\ \hline \vdash & a \\ \hline\end{array} T \to 1 \begin{array}{|c|c|c|}\hline \varepsilon & a & b \\ \hline \vdash & a & b \\ \hline\end{array} T \to 1 \begin{array}{|c|c|c|}\hline \varepsilon & a & b \\ \hline \vdash & a & b \\ \hline\end{array} U \to 1 \begin{array}{|c|c|c|c|}\hline \varepsilon & a & b & \varepsilon \\ \hline \vdash & a & b & \sqcup \\ \hline\end{array} U \to 1 \begin{array}{|c|c|c|c|}\hline \varepsilon & a & b & \varepsilon \\ \hline \vdash & a & b & \sqcup \\ \hline\end{array}$$

$$\to \begin{array}{|c|}\hline \varepsilon \\ \hline \vdash \\ \hline\end{array} 1 \begin{array}{|c|c|c|}\hline a & b & \varepsilon \\ \hline a & b & \sqcup \\ \hline\end{array} \to \begin{array}{|c|c|}\hline \varepsilon & a \\ \hline \vdash & b \\ \hline\end{array} 2 \begin{array}{|c|c|}\hline b & \varepsilon \\ \hline b & \sqcup \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline \varepsilon & a & b \\ \hline \vdash & b & a \\ \hline\end{array} 1 \begin{array}{|c|}\hline \varepsilon \\ \hline \sqcup \\ \hline\end{array}$$

Example

TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1, \vdash, R)$	$(2, b, R)$	$(4, a, L)$	$(2, a, L)$
2	$(1, \vdash, R)$	$(3, a, L)$	$(1, a, R)$	$(4, a, L)$

$$\underset{1}{\vdash} a\, b \xrightarrow[M]{1} \vdash \underset{1}{a}\, b \xrightarrow[M]{1} \vdash b\, \underset{2}{b} \xrightarrow[M]{1} \vdash b\, a\, \underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\, a\, \underset{2}{a} \xrightarrow[M]{1} \vdash \underset{3}{b}\, a\, a$$

$$S \to 1\boxed{\substack{\varepsilon\\ \vdash}}T \to 1\boxed{\substack{\varepsilon\\ \vdash}}\boxed{\substack{a\\ a}}T \to 1\boxed{\substack{\varepsilon\\ \vdash}}\boxed{\substack{a\\ a}}\boxed{\substack{b\\ b}}T \to 1\boxed{\substack{\varepsilon\\ \vdash}}\boxed{\substack{a\\ a}}\boxed{\substack{b\\ b}}U \to 1\boxed{\substack{\varepsilon\\ \vdash}}\boxed{\substack{a\\ a}}\boxed{\substack{b\\ b}}\boxed{\substack{\varepsilon\\ \sqcup}}U \to 1\boxed{\substack{\varepsilon\\ \vdash}}\boxed{\substack{a\\ a}}\boxed{\substack{b\\ b}}\boxed{\substack{\varepsilon\\ \sqcup}}$$

$$\to \boxed{\substack{\varepsilon\\ \vdash}}1\boxed{\substack{a\\ a}}\boxed{\substack{b\\ b}}\boxed{\substack{\varepsilon\\ \sqcup}} \to \boxed{\substack{\varepsilon\\ \vdash}}\boxed{\substack{a\\ b}}2\boxed{\substack{b\\ b}}\boxed{\substack{\varepsilon\\ \sqcup}} \to \boxed{\substack{\varepsilon\\ \vdash}}\boxed{\substack{a\\ b}}\boxed{\substack{b\\ a}}1\boxed{\substack{\varepsilon\\ \sqcup}} \to \boxed{\substack{\varepsilon\\ \vdash}}\boxed{\substack{a\\ b}}2\boxed{\substack{b\\ a}}\boxed{\substack{\varepsilon\\ a}}$$

Example

TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1, \vdash, R)$	$(2, b, R)$	$(4, a, L)$	$(2, a, L)$
2	$(1, \vdash, R)$	$(3, a, L)$	$(1, a, R)$	$(4, a, L)$

$$\underset{1}{\vdash}\,a\,b \xrightarrow[M]{1} \vdash\,\underset{1}{a}\,b \xrightarrow[M]{1} \vdash\,b\,\underset{2}{b} \xrightarrow[M]{1} \vdash\,b\,a\,\underset{1}{\sqcup} \xrightarrow[M]{1} \vdash\,b\,\underset{2}{a}\,a \xrightarrow[M]{1} \vdash\,\underset{3}{b}\,a\,a$$

$$S \to 1\genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}T \to 1\genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}T \to 1\genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}T \to 1\genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}U \to 1\genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup}U \to 1\genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup}$$

$$\to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}1\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{b}2\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{b}\genfrac{[}{]}{0pt}{}{b}{a}1\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{b}2\genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}3\genfrac{[}{]}{0pt}{}{a}{b}\genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a}$$

Example

TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1, \vdash, R)$	$(2, b, R)$	$(4, a, L)$	$(2, a, L)$
2	$(1, \vdash, R)$	$(3, a, L)$	$(1, a, R)$	$(4, a, L)$

$$\underset{1}{\vdash} a\, b \xrightarrow[M]{1} \vdash \underset{1}{a}\, b \xrightarrow[M]{1} \vdash b\, \underset{2}{b} \xrightarrow[M]{1} \vdash b\, a\, \underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\, \underset{2}{a}\, a \xrightarrow[M]{1} \vdash \underset{3}{b}\, a\, a$$

$$S \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}T \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}T \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}T \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}U \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix}U \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix}$$

$$\to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}1\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix} \to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}2\begin{bmatrix}b\\ b\end{bmatrix}\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix} \to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}\begin{bmatrix}b\\ a\end{bmatrix}1\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix} \to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}2\begin{bmatrix}b\\ a\end{bmatrix}\begin{bmatrix}\varepsilon\\ a\end{bmatrix} \to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}3\begin{bmatrix}a\\ b\end{bmatrix}\begin{bmatrix}b\\ a\end{bmatrix}\begin{bmatrix}\varepsilon\\ a\end{bmatrix}$$

$$\to 3\,3\begin{bmatrix}a\\ b\end{bmatrix}\begin{bmatrix}b\\ a\end{bmatrix}\begin{bmatrix}\varepsilon\\ a\end{bmatrix}$$

Example

TM $M = (\{1,2,3,4\}, \{a,b\}, \{a,b,\vdash,\sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1,\vdash,R)$	$(2,b,R)$	$(4,a,L)$	$(2,a,L)$
2	$(1,\vdash,R)$	$(3,a,L)$	$(1,a,R)$	$(4,a,L)$

$$\underset{1}{\vdash}\,a\,b \xrightarrow[M]{1} \vdash\underset{1}{a}\,b \xrightarrow[M]{1} \vdash b\,\underset{2}{b} \xrightarrow[M]{1} \vdash b\,a\,\underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\,\underset{2}{a}\,a \xrightarrow[M]{1} \vdash\underset{3}{b}\,a\,a$$

$$S \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}T \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}T \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}T \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}U \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix}U \to 1\begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix}$$

$$\to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}1\begin{bmatrix}a\\ a\end{bmatrix}\begin{bmatrix}b\\ b\end{bmatrix}\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix} \to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}2\begin{bmatrix}b\\ b\end{bmatrix}\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix} \to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}\begin{bmatrix}b\\ a\end{bmatrix}1\begin{bmatrix}\varepsilon\\ \sqcup\end{bmatrix} \to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}2\begin{bmatrix}b\\ a\end{bmatrix}\begin{bmatrix}\varepsilon\\ a\end{bmatrix} \to \begin{bmatrix}\varepsilon\\ \vdash\end{bmatrix}3\begin{bmatrix}a\\ b\end{bmatrix}\begin{bmatrix}b\\ a\end{bmatrix}\begin{bmatrix}\varepsilon\\ a\end{bmatrix}$$

$$\to 3\,3\begin{bmatrix}a\\ b\end{bmatrix}\begin{bmatrix}b\\ a\end{bmatrix}\begin{bmatrix}\varepsilon\\ a\end{bmatrix} \to 3\,3\,a\,3\begin{bmatrix}b\\ a\end{bmatrix}\begin{bmatrix}\varepsilon\\ a\end{bmatrix}$$

Example

TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1, \vdash, R)$	$(2, b, R)$	$(4, a, L)$	$(2, a, L)$
2	$(1, \vdash, R)$	$(3, a, L)$	$(1, a, R)$	$(4, a, L)$

$$\underset{1}{\vdash} a\, b \xrightarrow[M]{1} \vdash \underset{1}{a}\, b \xrightarrow[M]{1} \vdash b\, \underset{2}{b} \xrightarrow[M]{1} \vdash b\, a\, \underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\, \underset{2}{a}\, a \xrightarrow[M]{1} \vdash \underset{3}{b}\, a\, a$$

$$S \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash} T \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a} T \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b} T \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b} U \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} U \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup}$$

$$\to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash} 1 \genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{b} 2 \genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{b}\genfrac{[}{]}{0pt}{}{b}{a} 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{b} 2 \genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash} 3 \genfrac{[}{]}{0pt}{}{a}{b}\genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a}$$

$$\to 3\, 3 \genfrac{[}{]}{0pt}{}{a}{b}\genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a} \to 3\, 3\, a\, 3 \genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a} \to 3\, 3\, a\, 3\, b\, 3 \genfrac{[}{]}{0pt}{}{\varepsilon}{a}$$

Example

TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1, \vdash, R)$	$(2, b, R)$	$(4, a, L)$	$(2, a, L)$
2	$(1, \vdash, R)$	$(3, a, L)$	$(1, a, R)$	$(4, a, L)$

$$\underset{1}{\vdash} a\, b \xrightarrow[M]{1} \vdash \underset{1}{a}\, b \xrightarrow[M]{1} \vdash b\, \underset{2}{b} \xrightarrow[M]{1} \vdash b\, a\, \underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\, \underset{2}{a}\, a \xrightarrow[M]{1} \vdash \underset{3}{b}\, a\, a$$

$$S \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash} T \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a} T \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b} T \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b} U \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} U \to 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup}$$

$$\to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash} 1 \genfrac{[}{]}{0pt}{}{a}{a}\genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{b} 2 \genfrac{[}{]}{0pt}{}{b}{b}\genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{b}\genfrac{[}{]}{0pt}{}{b}{a} 1 \genfrac{[}{]}{0pt}{}{\varepsilon}{\sqcup} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash}\genfrac{[}{]}{0pt}{}{a}{b} 2 \genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a} \to \genfrac{[}{]}{0pt}{}{\varepsilon}{\vdash} 3 \genfrac{[}{]}{0pt}{}{a}{b}\genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a}$$

$$\to 3\,3 \genfrac{[}{]}{0pt}{}{a}{b}\genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a} \to 3\,3\,a\,3 \genfrac{[}{]}{0pt}{}{b}{a}\genfrac{[}{]}{0pt}{}{\varepsilon}{a} \to 3\,3\,a\,3\,b\,3 \genfrac{[}{]}{0pt}{}{\varepsilon}{a} \to 3\,3\,a\,3\,b\,3\,3$$

Example

TM $M = (\{1, 2, 3, 4\}, \{a, b\}, \{a, b, \vdash, \sqcup\}, \vdash, \sqcup, \delta, 1, 3, 4)$ with

δ	$\vdash$	a	b	$\sqcup$
1	$(1, \vdash, R)$	$(2, b, R)$	$(4, a, L)$	$(2, a, L)$
2	$(1, \vdash, R)$	$(3, a, L)$	$(1, a, R)$	$(4, a, L)$

$$\underset{1}{\vdash} a\, b \xrightarrow[M]{1} \vdash \underset{1}{a}\, b \xrightarrow[M]{1} \vdash b\, \underset{2}{b} \xrightarrow[M]{1} \vdash b\, a\, \underset{1}{\sqcup} \xrightarrow[M]{1} \vdash b\, \underset{2}{a}\, a \xrightarrow[M]{1} \vdash \underset{3}{b}\, a\, a$$

$$S \to 1\begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}T \to 1\begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}\begin{bmatrix}a\\a\end{bmatrix}T \to 1\begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}\begin{bmatrix}a\\a\end{bmatrix}\begin{bmatrix}b\\b\end{bmatrix}T \to 1\begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}\begin{bmatrix}a\\a\end{bmatrix}\begin{bmatrix}b\\b\end{bmatrix}U \to 1\begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}\begin{bmatrix}a\\a\end{bmatrix}\begin{bmatrix}b\\b\end{bmatrix}\begin{bmatrix}\varepsilon\\\sqcup\end{bmatrix}U \to 1\begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}\begin{bmatrix}a\\a\end{bmatrix}\begin{bmatrix}b\\b\end{bmatrix}\begin{bmatrix}\varepsilon\\\sqcup\end{bmatrix}$$

$$\to \begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}1\begin{bmatrix}a\\a\end{bmatrix}\begin{bmatrix}b\\b\end{bmatrix}\begin{bmatrix}\varepsilon\\\sqcup\end{bmatrix} \to \begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}2\begin{bmatrix}b\\b\end{bmatrix}\begin{bmatrix}\varepsilon\\\sqcup\end{bmatrix} \to \begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}b\\a\end{bmatrix}1\begin{bmatrix}\varepsilon\\\sqcup\end{bmatrix} \to \begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}2\begin{bmatrix}b\\a\end{bmatrix}\begin{bmatrix}\varepsilon\\a\end{bmatrix} \to \begin{bmatrix}\varepsilon\\\vdash\end{bmatrix}3\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}b\\a\end{bmatrix}\begin{bmatrix}\varepsilon\\a\end{bmatrix}$$

$$\to 3\,3\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}b\\a\end{bmatrix}\begin{bmatrix}\varepsilon\\a\end{bmatrix} \to 3\,3\,a\,3\begin{bmatrix}b\\a\end{bmatrix}\begin{bmatrix}\varepsilon\\a\end{bmatrix} \to 3\,3\,a\,3\,b\,3\begin{bmatrix}\varepsilon\\a\end{bmatrix} \to 3\,3\,a\,3\,b\,3\,3 \to^* a\,b$$

Thanks! & Questions?