CMPE 322/327 - Theory of Computation

Week 3: Nondeterministic Finite State Automata & Epsilon Transitions

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Outline

1 A Quick Recap

2 Nondeterministic Finite Automata

3 Epsilon Transitions

4 Closure Properties

Definitions

- deterministic finite automaton (DFA) is quintuple $M = (Q, \Sigma, \delta, s, F)$ with
  1. Q : finite set of states
  2. Σ : input alphabet
  3. $\delta : Q \times \Sigma \to Q$: transition function
  4. $s \in Q$: start state
  5. $F \subseteq Q$: final (accept) states
- $\widehat{\delta} : Q \times \Sigma^* \to Q$ is inductively defined by

$$\widehat{\delta}(q, \varepsilon) := q \qquad \widehat{\delta}(q, xa) := \delta(\widehat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is accepted by M if $\widehat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is rejected by M if $\widehat{\delta}(s, x) \notin F$
- language accepted by M is given by $L(M) := \{x \mid \widehat{\delta}(x, s) \in F\}$

Example (DFA → Regular Sets)

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Definition

set $A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M

Theorem

regular sets are effectively closed under intersection, complement and union

Outline

1. A Quick Recap
2. Nondeterministic Finite Automata
3. Epsilon Transitions
4. Closure Properties

Definitions

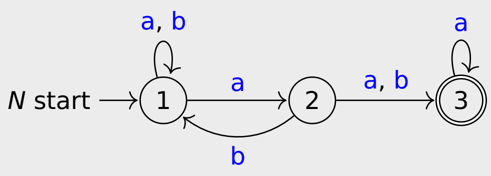
- nondeterministic finite automaton (NFA) is quintuple $N = (Q, \Sigma, \Delta, S, F)$ with
  1. Q : finite set of states
  2. Σ : input alphabet
  3. $\Delta : Q \times \Sigma \to 2^Q$: transition function
  4. $S \subseteq Q$: set of start states
  5. $F \subseteq Q$: final (accept) states
- $\widehat{\Delta}: 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}(A, \varepsilon) := A \qquad \widehat{\Delta}(A, xa) := \bigcup_{q \in \widehat{\Delta}(A,x)} \Delta(q, a)$$

- string $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \varnothing$

Example

$N = (Q, \Sigma, \Delta, S, F)$



1. $Q = \{1, 2, 3\}$
2. $\Sigma = \{a, b\}$
3. $\Delta : Q \times \Sigma \to 2^Q$
4. $S = \{1\}$
5. $F = \{3\}$

Δ	a	b
1	$\{1, 2\}$	$\{1\}$
2	$\{3\}$	$\{1, 3\}$
3	$\{3\}$	$\varnothing$

Example (Unfolding of the multistep function $\widehat{\Delta}$)

Let $x = ababba$ over the alphabet $\Sigma = \{a, b\}$

$\bigcup(q \in \widehat{\Delta}(A, ababb), a)$ — 1^{st} rec. call

$\bigcup(q \in \bigcup(q \in \widehat{\Delta}(A, abab)\,\Delta(q, b))\,\Delta(q, a))$ — 2^{nd} rec. call

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \widehat{\Delta}(A, aba)\,\Delta(q, b))\,\Delta(q, b))\,\Delta(q, a))$ — 3^{rd} rec. call

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \widehat{\Delta}(A, ab)\,\Delta(q, a))\,\Delta(q, b))\,\Delta(q, b))\,\Delta(q, a))$ — 4^{th} rec. call

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \widehat{\Delta}(A, a)\,\Delta(q, b))\,\Delta(q, a))\,\Delta(q, b))\,\Delta(q, b))\,\Delta(q, a))$ — 5^{th} rec. call

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \widehat{\Delta}(A, \varepsilon)\,\Delta(q, a))\,\Delta(q, b))\,\Delta(q, a))\,\Delta(q, b))\,\Delta(q, b))\,\Delta(q, a))$ — 6^{th} rec. call

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in A\,\Delta(q, a))\,\Delta(q, b))\,\Delta(q, a))\,\Delta(q, b))\,\Delta(q, b))\,\Delta(q, a))$

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in B\,\Delta(q, b))\,\Delta(q, a))\,\Delta(q, b))\,\Delta(q, b))\,\Delta(q, a))$ — assuming $\bigcup(q \in A\,\Delta(q, a)) = B$

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in C\,\Delta(q, a))\,\Delta(q, b))\,\Delta(q, b))\,\Delta(q, a))$ — assuming $\bigcup(q \in B\,\Delta(q, b)) = C$

$\bigcup(q \in \bigcup(q \in \bigcup(q \in D\,\Delta(q, b))\,\Delta(q, b))\,\Delta(q, a))$ — assuming $\bigcup(q \in C\,\Delta(q, a)) = D$

$\bigcup(q \in \bigcup(q \in E\,\Delta(q, b))\,\Delta(q, a))$ — assuming $\bigcup(q \in D\,\Delta(q, b)) = E$

$\bigcup(q \in F\,\Delta(q, a))$ — assuming $\bigcup(q \in E\,\Delta(q, b)) = F$

G — assuming $\bigcup(q \in F\,\Delta(q, a)) = G$

Lemma ($\widehat{\Delta}$ distributes)

$\widehat{\Delta}(A, xy) = \widehat{\Delta}(\widehat{\Delta}(A, x), y) \quad \forall A \subseteq Q_N$ and $x, y \in \Sigma^*$

Proof.

We argue by induction on $|y|$:

- base case: $|y| = 0$ thus $y = \varepsilon$

$$\widehat{\Delta}(A, x\varepsilon) = \widehat{\Delta}(A, x) = \widehat{\Delta}(\widehat{\Delta}(A, x), \varepsilon)$$

- step case: $|y| > 0$ thus $y = zb$ s.t. $|z| = |y| - 1$ with IH : $\widehat{\Delta}(A, xz) = \widehat{\Delta}(\widehat{\Delta}(A, x), z)$

$$\begin{aligned}
\widehat{\Delta}(A, xzb) &= \bigcup_{q \in \widehat{\Delta}(A, xz)} \Delta(q, b) && \text{(by definition of } \widehat{\Delta}\text{)} \\
&= \bigcup_{q \in \widehat{\Delta}(\widehat{\Delta}(A, x), z)} \Delta(q, b) && \text{(by IH)} \\
&= \widehat{\Delta}(\widehat{\Delta}(A, x), zb) && \text{(by definition of } \widehat{\Delta}\text{)} \\
&= \widehat{\Delta}(\widehat{\Delta}(A, x), y)
\end{aligned}$$

Theorem

every set accepted by NFA is regular

Proof.

- NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- $L(N) = L(M)$ for some DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ with
  1. $Q_M := 2^{Q_N}$
  2. $\delta_M(A, a) := \widehat{\Delta}(A, a) \quad \forall A \subseteq Q_N \;\; \forall a \in \Sigma$
  3. $s_M := S_N$
  4. $F_M := \{A \subseteq Q_N \mid A \cap F_N \neq \varnothing\}$
- claim: $\widehat{\delta_M}(A, x) = \widehat{\Delta}(A, x) \quad \forall A \subseteq Q$ and $x \in \Sigma^*$
  proof: by induction on $|x|$ see next slide

proof of the claim

claim: $\widehat{\delta_M}(A, x) = \widehat{\Delta_N}(A, x) \quad \forall A \subseteq Q_N$ and $x \in \Sigma^*$

- base case: $|x| = 0$ thus $x = \varepsilon$

$$\widehat{\delta_M}(A, \varepsilon) = A = \widehat{\Delta_N}(A, \varepsilon)$$

- step case: $|x| > 0$ thus $x = ya$ s.t. $|y| = |x| - 1$ with IH : $\widehat{\delta_M}(A, y) = \widehat{\Delta_N}(A, y)$

$$\begin{aligned}
\widehat{\delta_M}(A, ya) &= \delta_M(\widehat{\delta_M}(A, y), a) && \text{(by definition of } \widehat{\delta_M}\text{)} \\
&= \delta_M(\widehat{\Delta_N}(A, y), a) && \text{(by induction hypothesis IH)} \\
&= \widehat{\Delta_N}(\widehat{\Delta_N}(A, y), a) && \text{(by definition of } \delta_M\text{)} \\
&= \widehat{\Delta_N}(A, ya) && \text{(by distributivity of } \widehat{\Delta}\text{)} \\
&= \widehat{\Delta_N}(A, x)
\end{aligned}$$

□

Proof. (NFA regularity)

statement: $L(M) = L(N)$

$$
\begin{aligned}
\forall x \in \Sigma^*,\ x \in L(M) &\iff \widehat{\delta_M}(s_M, x) \in F_M && \text{(by definition of acceptance)}\\
&\iff \widehat{\delta_M}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\} && \text{(by definition of } s_M \text{ and } F_M\text{)}\\
&\iff \widehat{\Delta_N}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\} && \text{(by claim proven in slide 12)}\\
&\iff \widehat{\Delta_N}(S_N, x) \cap F_N \neq \emptyset && \text{(by set comprehension)}\\
&\iff x \in L(N) && \text{(by definition of acceptance)}
\end{aligned}
$$

□

Example

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Question

Every regular set is accepted by . . .

A . . . an NFA having exactly one final state,

B . . . a DFA having exactly one final state,

C . . . an NFA having exactly one start state.

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1 A Quick Recap

2 Nondeterministic Finite Automata

3 Epsilon Transitions

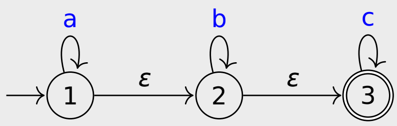
4 Closure Properties

Definitions

- NFA with ε-transitions (NFA$_\varepsilon$) is sextuple $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$ such that
  1. $\varepsilon \notin \Sigma$
  2. $N_\varepsilon = (Q, \Sigma \cup \{\varepsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\varepsilon\}$
- $\Delta : Q \times \Sigma \cup \{\varepsilon\} \to 2^Q$
- ε-closure of set $A \subseteq Q$ is defined as $C_\varepsilon(A) = \bigcup \{\widehat{\Delta}_{N_\varepsilon}(A, x) \mid x \in \{\varepsilon\}^*\}$
- $\widehat{\Delta}_N : 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}_N(A, \varepsilon) = C_\varepsilon(A) \qquad \widehat{\Delta}_N(A, xa) = \bigcup \{C_\varepsilon(\Delta(q, a)) \mid q \in \widehat{\Delta}_N(A, x)\}$$

Example



$$C_\varepsilon(\{1\}) = \{1, 2, 3\}$$
$$\widehat{\Delta}(\{1\}, b) = C_\varepsilon(\Delta(1, b)) \cup C_\varepsilon(\Delta(2, b)) \cup C_\varepsilon(\Delta(3, b))$$

Definitions

- NFA with ε-transitions (NFA$_\varepsilon$) is sextuple $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$ such that
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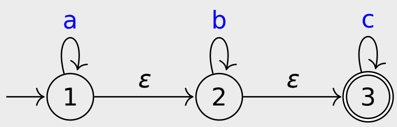
Example



$$C_\varepsilon(\{1\}) = \{1, 2, 3\}$$
$$\widehat{\Delta}(\{1\}, b) = C_\varepsilon(\varnothing) \cup C_\varepsilon(\{2\}) \cup C_\varepsilon(\varnothing)$$

Definitions

- NFA with ε-transitions (NFA$_\varepsilon$) is sextuple $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$ such that
  1. $\varepsilon \notin \Sigma$
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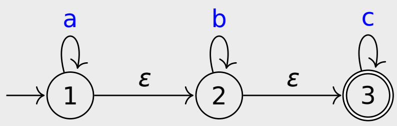
Example



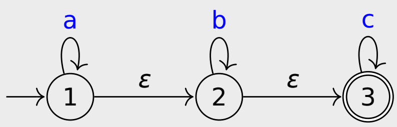
$$C_\varepsilon(\{1\}) = \{1, 2, 3\}$$
$$\widehat{\Delta}(\{1\}, b) = \emptyset \cup \{2, 3\} \cup \emptyset$$

Definitions

- NFA with ε-transitions (NFA$_\varepsilon$) is sextuple $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$ such that
  1. $\varepsilon \notin \Sigma$
  2. $N_\varepsilon = (Q, \Sigma \cup \{\varepsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\varepsilon\}$
- $\Delta : Q \times \Sigma \cup \{\varepsilon\} \to 2^Q$
- ε-closure of set $A \subseteq Q$ is defined as $C_\varepsilon(A) = \bigcup \{\widehat{\Delta}_{N_\varepsilon}(A, x) \mid x \in \{\varepsilon\}^*\}$
- $\widehat{\Delta}_N : 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}_N(A, \varepsilon) = C_\varepsilon(A) \qquad \widehat{\Delta}_N(A, xa) = \bigcup \{C_\varepsilon(\Delta(q, a)) \mid q \in \widehat{\Delta}_N(A, x)\}$$

Example



$$C_\varepsilon(\{1\}) = \{1, 2, 3\}$$
$$\widehat{\Delta}(\{1\}, b) = \{2, 3\}$$

Example (Unfolding of the multistep function $\widehat{\Delta}_N$)

Let $x = baa$ over the alphabet $\Sigma = \{a, b\}$

$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, ba)\}$	1st rec. call
$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, b)\}\}$	2nd rec. call
$\bigcup\Big\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,b)) \mid q \in \widehat{\Delta}_N(A, \varepsilon)\}\}\Big\}$	3rd rec. call
$\bigcup\Big\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,b)) \mid q \in C_\varepsilon(A)\}\}\Big\}$	
$\bigcup\Big\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in B\}\Big\}$	$\bigcup\{C_\varepsilon(\Delta(q,b)) \mid q \in C_\varepsilon(A)\} = B$
$\bigcup\Big\{C_\varepsilon(\Delta(q,a)) \mid q \in C\Big\}$	$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in B\} = C$
D	$\bigcup\Big\{C_\varepsilon(\Delta(q,a)) \mid q \in C\Big\} = D$

Lemma

$C_\varepsilon(A)$ is least extension of A that is closed under ε-transitions:

$$q \in C_\varepsilon(A) \implies \Delta_{N_\varepsilon}(q, \varepsilon) \subseteq C_\varepsilon(A)$$

Theorem

every set accepted by NFA$_\varepsilon$ is regular

Proof. (by construction)

- NFA$_\varepsilon$ $N_1 = (Q, \Sigma, \varepsilon, \Delta_1, S, F_1)$
- $L(N_1) = L(N_2)$ for NFA $N_2 = (Q, \Sigma, \Delta_2, S, F_2)$ with
  1. $\Delta_2(q, a) := \widehat{\Delta}_1(\{q\}, a) \qquad \forall q \in Q\ \forall a \in \Sigma$
  2. $F_2 := \{q \mid C_\varepsilon(\{q\}) \cap F_1 \neq \varnothing\}$

Example

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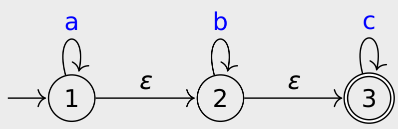
Example (cont'd)

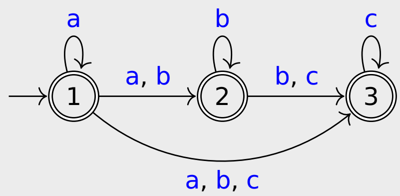
NFA$_\varepsilon$ $N_1 = (\{1,2,3\}, \{a,b,c\}, \varepsilon, \Delta_1, \{1\}, \{3\})$ with

Δ_1	a	b	c	ε
1	$\{1\}$	$\emptyset$	$\emptyset$	$\{2\}$
2	$\emptyset$	$\{2\}$	$\emptyset$	$\{3\}$
3	$\emptyset$	$\emptyset$	$\{3\}$	$\emptyset$



NFA $N_2 = (\{1,2,3\}, \{a,b,c\}, \Delta_2, \{1\}, F_2)$ with

- $F_2 = \{q \mid C_\varepsilon(\{q\}) \cap F_1 \neq \emptyset\}$
Δ_2	a	b	c
1	$\{1,2,3\}$	$\{2,3\}$	$\{3\}$
2	$\emptyset$	$\{2,3\}$	$\{3\}$
3	$\emptyset$	$\emptyset$	$\{3\}$



Outline

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2 Nondeterministic Finite Automata

3 Epsilon Transitions

4 Closure Properties

Theorem

regular sets are effectively closed under concatenation

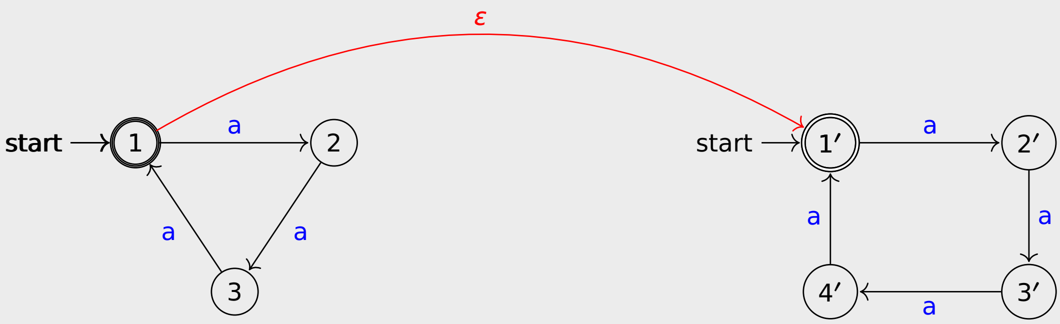
Proof. (by construction)

- $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
  $B = L(N_2)$ for NFA $N_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$
- without loss of generality $Q_1 \cap Q_2 = \varnothing$
- $AB = L(N)$ for NFA$_\varepsilon$ $N = (Q, \Sigma, \varepsilon, \Delta, S_1, F_2)$ with

  1. $Q \quad := \quad Q_1 \cup Q_2$
  2. $$\Delta(q,a) \quad := \quad \begin{cases} \Delta_1(q,a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma \\ \Delta_2(q,a) & \text{if } q \in Q_2 \text{ and } a \in \Sigma \\ S_2 & \text{if } q \in F_1 \text{ and } a = \varepsilon \\ \varnothing & \text{otherwise} \end{cases}$$

Example

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$ $\{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



$\{x \in \{a\}^* \mid |x| \notin \{1, 2, 5\}\}$

Theorem

regular sets are effectively closed under asterate

Proof. (by construction)

- $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
- $A^* = L(N)$ for NFA$_\varepsilon$ $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$ with

  ① $Q := Q_1 \cup \{s\}$

  ② $S := \{s\}$

  ③ $F := \{s\}$

  ④ $$\Delta(q, a) := \begin{cases} \Delta_1(q, a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma \\ S_1 & \text{if } q = s \text{ and } a = \varepsilon \\ S & \text{if } q \in F_1 \text{ and } a = \varepsilon \\ \varnothing & \text{otherwise} \end{cases}$$

Example

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A Quick Recap
Nondeterministic Finite Automata
Epsilon Transitions
Closure Properties

Thanks! $\&$ Questions?