Final Exam (100 pts)

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Assigned : June the 2nd, 13h00
Duration : 120 minutes

Q1. (40 pts)

a) (25 pts) Design a total Turing Machine (TM)

$$M = (Q, \{0\}, \{0, \vdash, \sqcup, x\}, \vdash, \sqcup, \delta, s, t, r)$$

accepts every member of the set

$$A = \{0^{4^n} \mid n \geqslant 0\}$$

rejecting every non-member. Explain your implementation in a few lines.

Below are a few examples to the input-output harmony of the intended TM:

Input	Output
$\vdash \sqcup^\omega$	reject
$\vdash 0\sqcup^\omega$	accept
$\vdash 0^2\sqcup^\omega$	reject
$\vdash 0^4\sqcup^\omega$	accept
$\vdash 0^8\sqcup^\omega$	reject
$\vdash 0^{10}\sqcup^\omega$	reject
$\vdash 0^{16}\sqcup^\omega$	accept
$\vdash 0^{30}\sqcup^\omega$	reject
$\vdash 0^{60}\sqcup^\omega$	reject
$\vdash 0^{64}\sqcup^\omega$	accept
$\vdash 0^{100}\sqcup^\omega$	reject
$\vdash 0^{256}\sqcup^\omega$	accept
$\vdash 0^{1000}\sqcup^\omega$	reject
$\vdash 0^{1024}\sqcup^\omega$	accept
$\vdash 00a00\sqcup^\omega$	reject
$\vdash 000b00\sqcup^\omega$	reject
$\vdash 00000c\sqcup^\omega$	reject
⋮	⋮

b) (15 pts) Design a total Turing Machine (TM)

$$M = (Q, \{a, b\}, \{a, b, 0, 1, 2, 3, 4, \vdash, \#, \sqcup\}, \vdash, \sqcup, \delta, s, t, r)$$

that inputs $x\#0$ with $x \in \{a, b\}^*$, computes and halts with the length of x in base 5 stored on its tape. Explain your implementation in a few lines.

Below are a few examples to the input-output harmony of the intended TM:

Input	Output
$\vdash \#0\sqcup^\omega$	$\vdash \#0\sqcup^\omega$
$\vdash a\#0\sqcup^\omega$	$\vdash \cdots\#1\sqcup^\omega$
$\vdash ba\#0\sqcup^\omega$	$\vdash \cdots\#2\sqcup^\omega$
$\vdash aba\#0\sqcup^\omega$	$\vdash \cdots\#3\sqcup^\omega$
$\vdash aaba\#0\sqcup^\omega$	$\vdash \cdots\#4\sqcup^\omega$
$\vdash abbba\#0\sqcup^\omega$	$\vdash \cdots\#10\sqcup^\omega$
$\vdash ababba\#0\sqcup^\omega$	$\vdash \cdots\#11\sqcup^\omega$
$\vdash abbbbbabab\#0\sqcup^\omega$	$\vdash \cdots\#20\sqcup^\omega$
$\vdash abaabbbbabab\#0\sqcup^\omega$	$\vdash \cdots\#22\sqcup^\omega$
$\vdash abbbaababbbbabab\#0\sqcup^\omega$	$\vdash \cdots\#30\sqcup^\omega$
$\vdash abbbaababababbbabab\#0\sqcup^\omega$	$\vdash \cdots\#34\sqcup^\omega$
$\vdash aabc\#0\sqcup^\omega$	reject
$\vdash axb\#0\sqcup^\omega$	reject
$\vdash ab\#e0\sqcup^\omega$	reject
⋮	⋮

Q2. (15 pts) Design non-deterministic push down automaton (NPDA)

$$N = (Q, \{a, b, c\}, \{\bot, \cdots\}, \delta, s, \bot, F)$$

that accepts every member of the set

$$A = \{a^n b^m c^k \mid n, k \geqslant 1 \text{ and } n + k = m\}$$

rejecting every non-member. Justify your design in a few lines.

Below are a few examples to the input-output harmony of the intended NPDA:

Input	Output
ε	reject
ab^2c	accept
$a^2b^4c^2$	accept
$a^2b^3c^2$	reject
a^3b^4c	accept
$a^3b^2c^2$	reject
$a^5b^9c^4$	accept
$a^5b^{12}c^7$	accept
$a^6b^3c^2$	reject
$a^8b^{20}c^{12}$	accept
bc	reject
cba	reject
a^2c	reject
⋮	⋮

Q3. (30 pts) Which of the following sets, constructed over the alphabet $\Sigma = \{a, b\}$, are context free and which are not?

(a) **(10 pts)** $A = \{a^n b^m \mid n = m^2\}$

(b) **(10 pts)** $B = \sim \{xx \mid x \in \Sigma^*\}$

(c) **(10 pts)** $C = \{a^n w^R w b^{3n} \mid w \in \Sigma^* \text{ and } n \geqslant 1\}$

Give grammars for those that are context free and proofs for those that are not.

Q4. (15 pts) Prove that the Halting Problem for Turing Machines

$$\mathrm{HP} = \{M\#x \mid \text{TM } M \text{ halts on input } x\}$$

is undecidable. Otherwise put, show that the set HP is recursively enumerable.