CMPE 322/327 - Theory of Computation

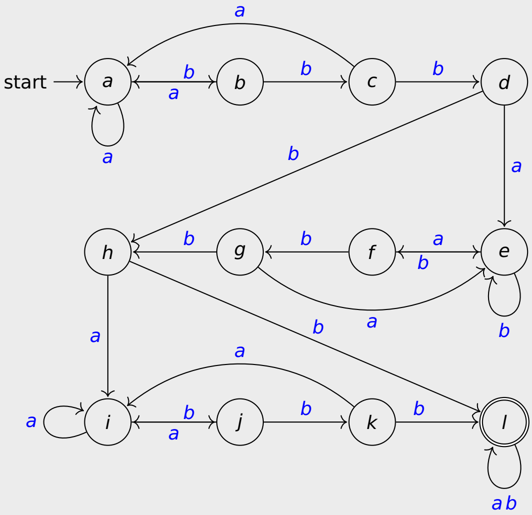
Week 6: Derivatives & Kleene Algebra & Equivalence of Regular Expressions

Burak Ekici

March 28 - April 1, 2022

Outline

Example (Marking Algorithm – Ordering)



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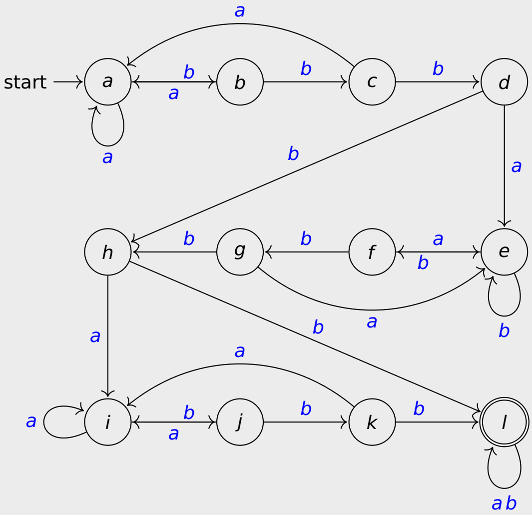

a	b	c	d	e	f	g	h	i	j	k	
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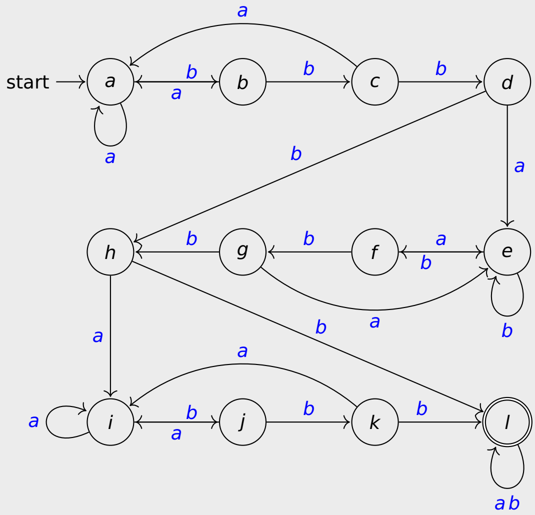

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l	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

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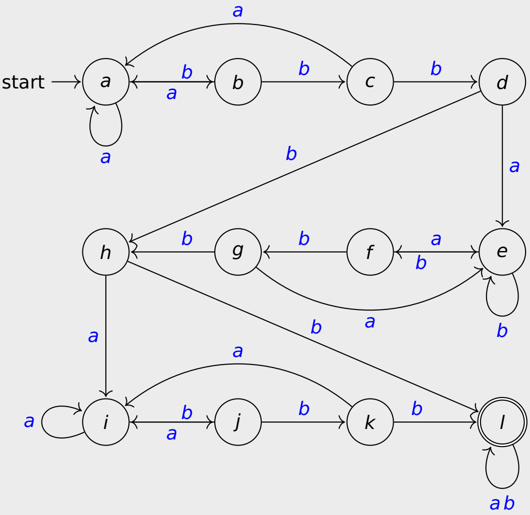

	a	b	c	d	e	f	g	h	i	j	k
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j	✓	✓	✓		✓	✓		✓	✓		
k	✓	✓	✓	✓	✓	✓	✓		✓	✓	
l	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

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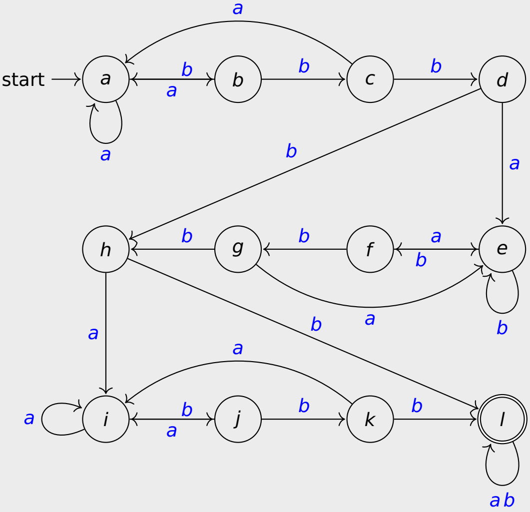

	a	b	c	d	e	f	g	h	i	j	k
a											
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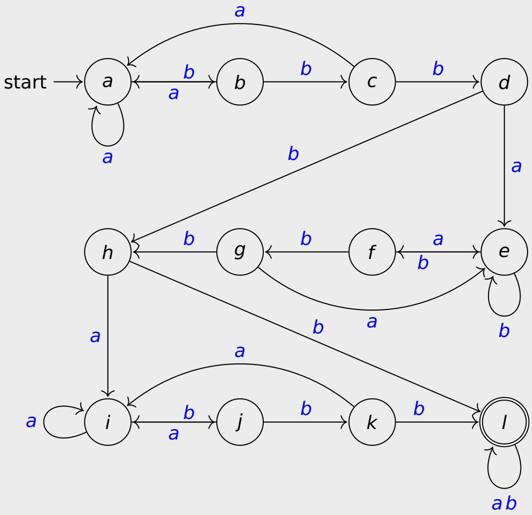

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✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	*l*

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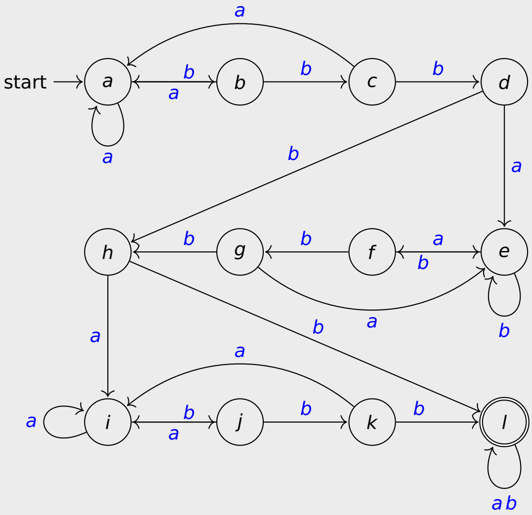

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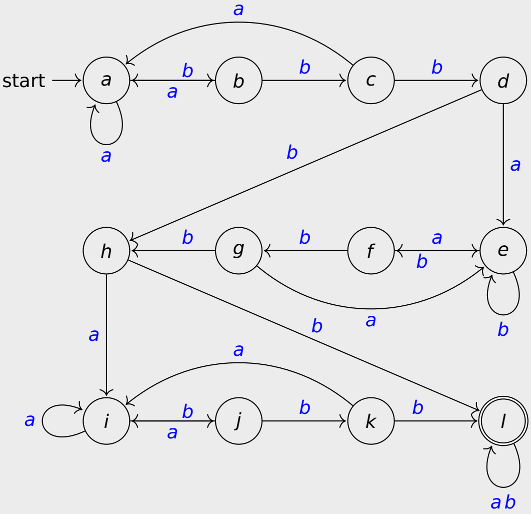

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states d, g and h, k can be merged

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equivalence relation $\equiv_M$ on Σ^* for DFA $M = (Q, \Sigma, \delta, s, F)$ is defined as follows:

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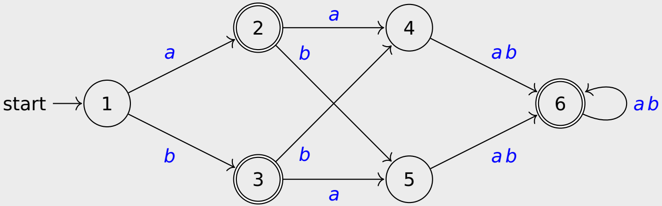
Definition

Myhill-Nerode relation for $L \subseteq \Sigma^*$ is right congruent equivalence relation of finite index on Σ^* that refines L

Example

M



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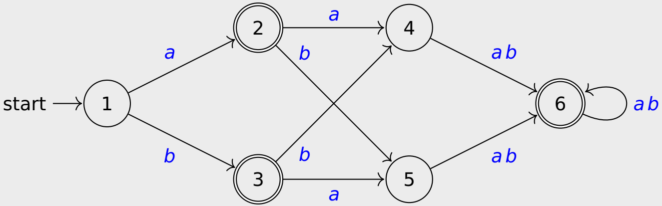
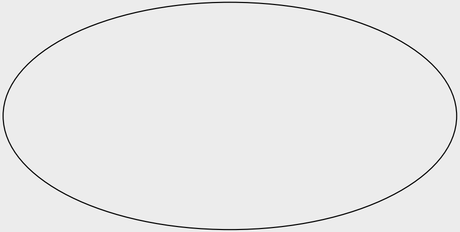


$\equiv_M$



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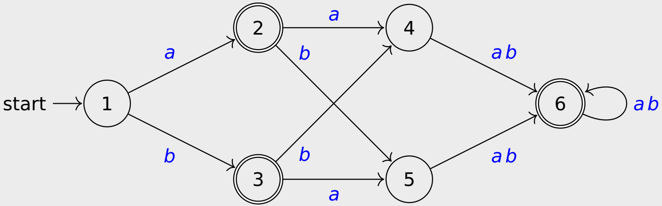

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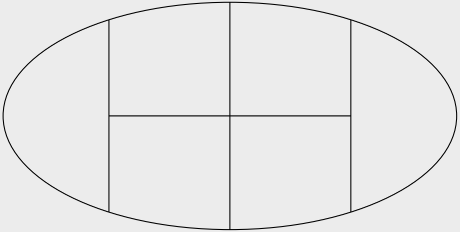

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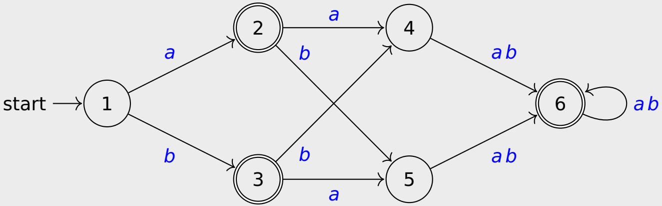

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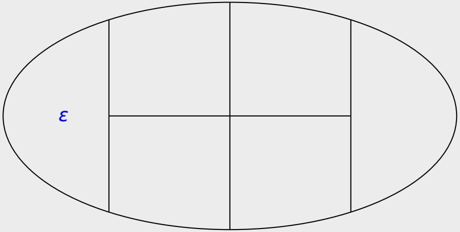

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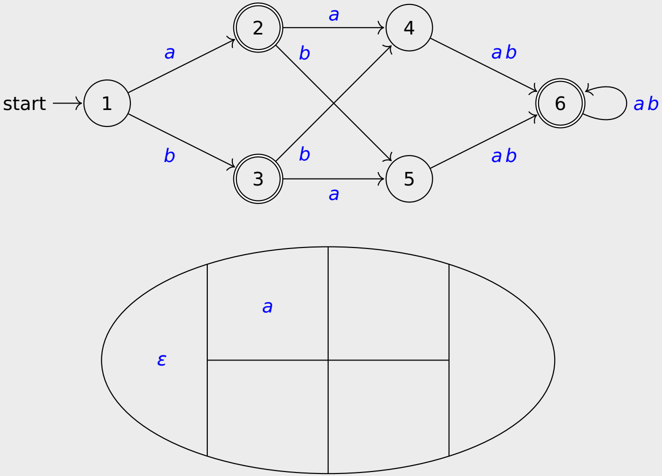

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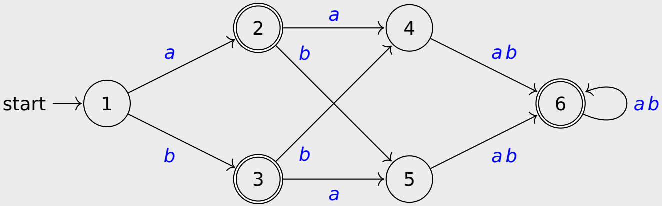

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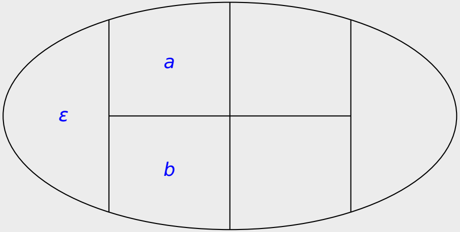

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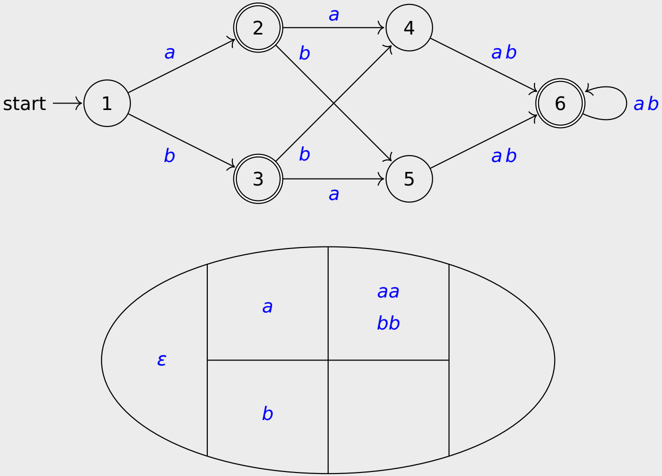

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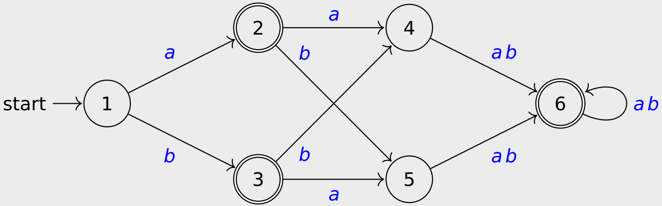

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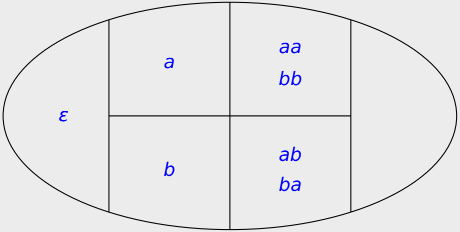

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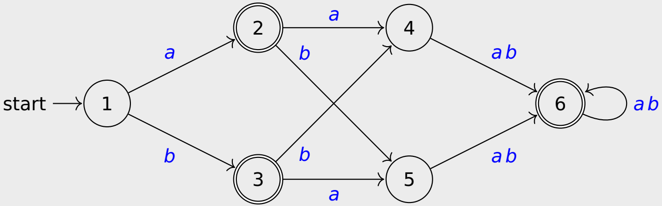

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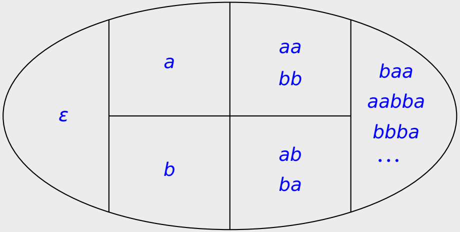

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$\equiv_L$ has fewest equivalence classes

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following statements are equivalent for any set $L \subseteq \Sigma^*$:

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Theorem

for every DFA M, $M/{\approx} \simeq M_{\equiv_L}$

Outline

$x \in \Sigma^*$ $\quad A \subseteq \Sigma^*$

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$x \in \Sigma^*$ $A \subseteq \Sigma^*$ $a \in \Sigma$ regular expression α over Σ

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$$\alpha_a := \begin{cases} \varnothing & \text{if } \alpha = \varnothing \\ & \\ & \\ & \end{cases}$$

$x \in \Sigma^*$ $\quad A \subseteq \Sigma^*$ $\quad a \in \Sigma$ $\quad$ regular expression α over Σ

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Example

- $\alpha = (a+b)^*$

$$\alpha_a = (a+b)_a(a+b)^* \qquad \alpha_b = (a+b)_b(a+b)^*$$

Example

- $\alpha = (a+b)^*$

$$\begin{aligned} \alpha_a &= (a+b)_a(a+b)^* \\ &= (a_a+b_a)(a+b)^* \end{aligned} \qquad \begin{aligned} \alpha_b &= (a+b)_b(a+b)^* \\ &= (a_b+b_b)(a+b)^* \end{aligned}$$

Example

- $\alpha = (a+b)^*$

$$\begin{aligned}\alpha_a &= (a+b)_a(a+b)^* \\ &= (a_a+b_a)(a+b)^* \\ &= (\varepsilon+\emptyset)(a+b)^*\end{aligned}$$

$$\begin{aligned}\alpha_b &= (a+b)_b(a+b)^* \\ &= (a_b+b_b)(a+b)^* \\ &= (\emptyset+\varepsilon)(a+b)^*\end{aligned}$$

Example

- $\alpha = (a+b)^*$

$$\begin{aligned} \alpha_a &= (a+b)_a(a+b)^* \\ &= (a_a+b_a)(a+b)^* \\ &= (\varepsilon+\emptyset)(a+b)^* \\ &\equiv (a+b)^* \end{aligned}$$

$$\begin{aligned} \alpha_b &= (a+b)_b(a+b)^* \\ &= (a_b+b_b)(a+b)^* \\ &= (\emptyset+\varepsilon)(a+b)^* \\ &\equiv (a+b)^* \end{aligned}$$

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- $\beta = (a^*b)^*a^*$

$$\beta_a = ((a^*b)^*)_a a^* + (a^*)_a \qquad \beta_b = ((a^*b)^*)_b a^* + (a^*)_b$$

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$$\begin{aligned} \beta_a &= ((a^*b)^*)_a a^* + (a^*)_a \\ &= (a^*b)_a(a^*b)^*a^* + a_a a^* \\ &= ((a^*)_a b + b_a)(a^*b)^*a^* + \varepsilon a^* \end{aligned} \qquad \begin{aligned} \beta_b &= ((a^*b)^*)_b a^* + (a^*)_b \\ &= (a^*b)_b(a^*b)^*a^* + a_b a^* \\ &= ((a^*)_b b + b_b)(a^*b)^*a^* + \emptyset a^* \end{aligned}$$

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$$\begin{aligned} \beta_b &= ((a^*b)^*)_b a^* + (a^*)_b \\ &= (a^*b)_b(a^*b)^*a^* + a_b a^* \\ &= ((a^*)_b b + b_b)(a^*b)^*a^* + \emptyset a^* \\ &= ((a_b)a^*b + \varepsilon)(a^*b)^*a^* + \emptyset a^* \\ &= (\emptyset a^*b + \varepsilon)(a^*b)^*a^* + \emptyset a^* \end{aligned}$$

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$$\begin{aligned}\beta_b &= ((a^*b)^*)_b a^* + (a^*)_b \\ &= (a^*b)_b(a^*b)^*a^* + a_b a^* \\ &= ((a^*)_b b + b_b)(a^*b)^*a^* + \varnothing a^* \\ &= ((a_b)a^*b + \varepsilon)(a^*b)^*a^* + \varnothing a^* \\ &= (\varnothing a^*b + \varepsilon)(a^*b)^*a^* + \varnothing a^* \\ &\equiv (a^*b)^*a^*\end{aligned}$$

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- $\alpha = (a+b)^*$

$$\begin{aligned} \alpha_a &= (a+b)_a(a+b)^* \\ &= (a_a+b_a)(a+b)^* \\ &= (\varepsilon+\varnothing)(a+b)^* \\ &\equiv (a+b)^* \end{aligned} \qquad \begin{aligned} \alpha_b &= (a+b)_b(a+b)^* \\ &= (a_b+b_b)(a+b)^* \\ &= (\varnothing+\varepsilon)(a+b)^* \\ &\equiv (a+b)^* \end{aligned}$$

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$$\begin{aligned} \beta_a &= ((a^*b)^*)_a a^* + (a^*)_a \\ &= (a^*b)_a(a^*b)^*a^* + a_a a^* \\ &= ((a^*)_a b + b_a)(a^*b)^*a^* + \varepsilon a^* \\ &= ((a_a)a^*b + \varnothing)(a^*b)^*a^* + \varepsilon a^* \\ &= (\varepsilon a^*b + \varnothing)(a^*b)^*a^* + \varepsilon a^* \\ &\equiv a^*b(a^*b)^*a^* + a^* \\ &\equiv (a^*b)^*a^* \end{aligned} \qquad \begin{aligned} \beta_b &= ((a^*b)^*)_b a^* + (a^*)_b \\ &= (a^*b)_b(a^*b)^*a^* + a_b a^* \\ &= ((a^*)_b b + b_b)(a^*b)^*a^* + \varnothing a^* \\ &= ((a_b)a^*b + \varepsilon)(a^*b)^*a^* + \varnothing a^* \\ &= (\varnothing a^*b + \varepsilon)(a^*b)^*a^* + \varnothing a^* \\ &\equiv (a^*b)^*a^* \end{aligned}$$

$x \in \Sigma^*$ $\quad A \subseteq \Sigma^*$ $\quad a \in \Sigma$ $\quad$ regular expression α over Σ

Definitions

- x-derivative of A: $\quad A_x := \{y \mid xy \in A\}$
- a-derivative of α is regular expression defined inductively as follows:

$$\alpha_a := \begin{cases} \varnothing & \text{if } \alpha = \varnothing \text{ or } \alpha = \epsilon \text{ or } \alpha = b \text{ with } b \neq a \\ \epsilon & \text{if } \alpha = a \\ \beta_a + \gamma_a & \text{if } \alpha = \beta + \gamma \\ \beta_a\gamma + \gamma_a & \text{if } \alpha = \beta\gamma \text{ and } \epsilon \in L(\beta) \\ \beta_a\gamma & \text{if } \alpha = \beta\gamma \text{ and } \epsilon \notin L(\beta) \\ \beta_a\beta^* & \text{if } \alpha = \beta^* \end{cases}$$

Lemma

$L(\alpha_a) = L(\alpha)_a$

Notation

$\alpha\downarrow$ for $\varepsilon \in L(\alpha)$ $\alpha\uparrow$ for $\varepsilon \notin L(\alpha)$

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- $\emptyset\uparrow$

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- $\varepsilon\downarrow$

Notation

$\alpha\downarrow$ for $\varepsilon \in L(\alpha)$ $\alpha\uparrow$ for $\varepsilon \notin L(\alpha)$

- $\emptyset\uparrow$
- $\varepsilon\downarrow$
- $a\uparrow$ for all $a \in \Sigma$

Notation

$\alpha\downarrow$ for $\varepsilon \in L(\alpha)$ $\quad$ $\alpha\uparrow$ for $\varepsilon \notin L(\alpha)$

- $\emptyset\uparrow$
- $\varepsilon\downarrow$
- $a\uparrow$ for all $a \in \Sigma$
- $(\alpha + \beta)\downarrow \iff \alpha\downarrow$ or $\beta\downarrow$

Notation

$\alpha\downarrow$ for $\varepsilon \in L(\alpha)$ $\alpha\uparrow$ for $\varepsilon \notin L(\alpha)$

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- $a\uparrow$ for all $a \in \Sigma$
- $(\alpha + \beta)\downarrow \iff \alpha\downarrow$ or $\beta\downarrow$
- $(\alpha\beta)\downarrow \iff \alpha\downarrow$ and $\beta\downarrow$

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- $a^*\downarrow$

Notation

$\alpha\downarrow$ for $\varepsilon \in L(\alpha)$ $\quad$ $\alpha\uparrow$ for $\varepsilon \notin L(\alpha)$

- $\varnothing\uparrow$
- $\varepsilon\downarrow$
- $a\uparrow$ for all $a \in \Sigma$
- $(\alpha + \beta)\downarrow \iff \alpha\downarrow$ or $\beta\downarrow$
- $(\alpha\beta)\downarrow \iff \alpha\downarrow$ and $\beta\downarrow$
- $a^*\downarrow$

Theorem

for every regular expression α over $\Sigma = \{a_1, \dots, a_n\}$ $\quad$ $\alpha \equiv \varepsilon(\alpha)$

Notation

$\alpha{\downarrow}$ for $\varepsilon \in L(\alpha)$ $\quad$ $\alpha{\uparrow}$ for $\varepsilon \notin L(\alpha)$ $\quad$ $\varepsilon(\alpha) = \varnothing$ if $\alpha{\uparrow}$ $\quad$ $\varepsilon(\alpha) = \varepsilon$ if $\alpha{\downarrow}$

- $\varnothing{\uparrow}$
- $\varepsilon{\downarrow}$
- $a{\uparrow}$ for all $a \in \Sigma$
- $(\alpha + \beta){\downarrow} \iff \alpha{\downarrow}$ or $\beta{\downarrow}$
- $(\alpha\beta){\downarrow} \iff \alpha{\downarrow}$ and $\beta{\downarrow}$
- $a^*{\downarrow}$

Theorem

for every regular expression α over $\Sigma = \{a_1, \ldots, a_n\}$ $\quad \alpha \equiv \varepsilon(\alpha) + a_1\alpha_{a_1} + \cdots + a_n\alpha_{a_n}$

Outline

Definition

Kleene Algebra consists of set K with distinguished elements $0, 1 \in K$ and operations $* : K \to K$ and $+, \times : K \times K \to K$

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- ab for $a \times b$ $\quad a^*$ for $*(a)$

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- ab for $a \times b$ $\quad a^*$ for $*(a)$
- binding precedence: $* > \times > +$

Definition

Kleene Algebra consists of set K with distinguished elements $0, 1 \in K$ and operations $* : K \to K$ and $+, \times : K \times K \to K$ such that

$$
\begin{aligned}
a + (b + c) &= (a + b) + c \\
a + b &= b + a \\
a + a &= a \\
a + 0 &= a
\end{aligned}
$$

for all $a, b, c \in K$

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$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl} a + (b + c) & = & (a + b) + c & a0 & = & 0 & a(bc) & = & (ab)c \\ a + b & = & b + a & 0a & = & 0 & & & \\ a + a & = & a & 1a & = & a & & & \\ a + 0 & = & a & a1 & = & a & & & \end{array}$$

for all $a, b, c \in K$

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a0 & = & 0 \\
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\end{array}
\qquad
\begin{array}{rcl}
a(bc) & = & (ab)c \\
(a + b)c & = & ac + bc \\
a(b + c) & = & ab + ac
\end{array}
$$

for all $a, b, c \in K$

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$$\begin{aligned} a + (b + c) &= (a + b) + c \\ a + b &= b + a \\ a + a &= a \\ a + 0 &= a \end{aligned} \qquad \begin{aligned} a0 &= 0 \\ 0a &= 0 \\ 1a &= a \\ a1 &= a \end{aligned} \qquad \begin{aligned} a(bc) &= (ab)c \\ (a + b)c &= ac + bc \\ a(b + c) &= ab + ac \end{aligned} \qquad \begin{aligned} 1 + aa^* &= a^* \\ 1 + a^* a &= a^* \end{aligned}$$

for all $a, b, c \in K$

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a + b &= b + a & 0a &= 0 & (a + b)c &= ac + bc & 1 + a^* a &= a^* \\
a + a &= a & 1a &= a & a(b + c) &= ab + ac & ac \leqslant c &\Longrightarrow a^* c \leqslant c \\
a + 0 &= a & a1 &= a & & & ca \leqslant c &\Longrightarrow ca^* \leqslant c
\end{aligned}
$$

for all $a, b, c \in K$

Notation

- ab for $a \times b$ $\quad$ a^* for $*(a)$ $\quad$ $a \leqslant b$ for $a + b = b$
- binding precedence: $* > \times > +$

Definition

Kleene Algebra consists of set K with distinguished elements $0, 1 \in K$ and operations $* : K \to K$ and $+, \times : K \times K \to K$ such that (A.1) – (A.13)

$$
\begin{array}{rclrclrclrcl}
a+(b+c) & = & (a+b)+c & a0 & = & 0 & a(bc) & = & (ab)c & 1+aa^* & = & a^* \\
a+b & = & b+a & 0a & = & 0 & (a+b)c & = & ac+bc & 1+a^*a & = & a^* \\
a+a & = & a & 1a & = & a & a(b+c) & = & ab+ac & ac \leqslant c & \Longrightarrow & a^*c \leqslant c \\
a+0 & = & a & a1 & = & a & & & & ca \leqslant c & \Longrightarrow & ca^* \leqslant c
\end{array}
$$

for all $a, b, c \in K$

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\end{array}
\qquad
\begin{array}{rcl}
1 + aa^* & = & a^* \\
1 + a^* a & = & a^*
\end{array}
$$

$$ac \leqslant c \implies a^* c \leqslant c$$
$$ca \leqslant c \implies ca^* \leqslant c$$

(A.14) $b + ac \leqslant c \implies a^* b \leqslant c$

for all $a, b, c \in K$

Notation

- ab for $a \times b$ $\quad$ a^* for $*(a)$ $\quad$ $a \leqslant b$ for $a + b = b$
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Kleene Algebra consists of set K with distinguished elements $0, 1 \in K$ and operations $* : K \to K$ and $+, \times : K \times K \to K$ such that (A.1) – (A.13)

$$\begin{aligned} a + (b + c) &= (a + b) + c & a0 &= 0 & a(bc) &= (ab)c & 1 + aa^* &= a^* \\ a + b &= b + a & 0a &= 0 & (a + b)c &= ac + bc & 1 + a^*a &= a^* \\ a + a &= a & 1a &= a & a(b + c) &= ab + ac & ac \leqslant c &\implies a^*c \leqslant c \\ a + 0 &= a & a1 &= a & & & ca \leqslant c &\implies ca^* \leqslant c \end{aligned}$$

(A.14) $b + ac \leqslant c \implies a^*b \leqslant c$ (A.15) $b + ca \leqslant c \implies ba^* \leqslant c$

for all $a, b, c \in K$

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a(bc) & = & (ab)c \\
(a + b)c & = & ac + bc \\
a(b + c) & = & ab + ac
\end{array}
\qquad
\begin{array}{rcl}
1 + aa^* & = & a^* \\
1 + a^*a & = & a^* \\
\multicolumn{3}{l}{ac \leqslant c \implies a^*c \leqslant c} \\
\multicolumn{3}{l}{ca \leqslant c \implies ca^* \leqslant c}
\end{array}
$$

(A.14) $b + ac \leqslant c \implies a^*b \leqslant c$ (A.15) $b + ca \leqslant c \implies ba^* \leqslant c$
(A.16) $(a + b)^* = (a^*b)^*a^*$

for all $a, b, c \in K$

Notation

- ab for $a \times b$ a^* for $*(a)$ $a \leqslant b$ for $a + b = b$
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Definition

Kleene Algebra consists of set K with distinguished elements $0, 1 \in K$ and operations $* : K \to K$ and $+, \times : K \times K \to K$ such that (A.1) – (A.13)

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1 + aa^* & = & a^* \\
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\end{array}
$$

$$ac \leqslant c \implies a^*c \leqslant c$$
$$ca \leqslant c \implies ca^* \leqslant c$$

(A.14) $b + ac \leqslant c \implies a^*b \leqslant c$ (A.15) $b + ca \leqslant c \implies ba^* \leqslant c$
(A.16) $(a + b)^* = (a^*b)^*a^*$ (A.17) $a(ba)^* = (ab)^*a$

for all $a, b, c \in K$

Notation

- ab for $a \times b$ $\quad a^*$ for $*(a)$ $\quad a \leqslant b$ for $a + b = b$
- binding precedence: $* > \times > +$

Example

- regular sets over alphabet Σ form Kleene algebra

Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0

Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0
  - $\{\varepsilon\}$ for 1

Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0
  - $\{\varepsilon\}$ for 1
  - union for $+$

Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0
  - $\{\varepsilon\}$ for 1
  - union for $+$
  - concatenation for $\times$

Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0
  - $\{\varepsilon\}$ for 1
  - union for $+$
  - concatenation for $\times$
  - asterate for $*$

Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0
  - $\{\varepsilon\}$ for 1
  - union for $+$
  - concatenation for $\times$
  - asterate for $*$
- binary relations over set A form Kleene algebra

Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0
  - $\{\varepsilon\}$ for 1
  - union for $+$
  - concatenation for $\times$
  - asterate for $*$
- binary relations over set A form Kleene algebra
  - empty relation $\varnothing$ for 0

Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0
  - $\{\varepsilon\}$ for 1
  - union for $+$
  - concatenation for $\times$
  - asterate for $*$
- binary relations over set A form Kleene algebra
  - empty relation $\varnothing$ for 0
  - identity relation $\{(a, a) \mid a \in A\}$ for 1

Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0
  - $\{\varepsilon\}$ for 1
  - union for $+$
  - concatenation for $\times$
  - asterate for $*$
- binary relations over set A form Kleene algebra
  - empty relation $\varnothing$ for 0
  - identity relation $\{(a, a) \mid a \in A\}$ for 1
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Example

- regular sets over alphabet Σ form Kleene algebra
  - $\varnothing$ for 0
  - $\{\epsilon\}$ for 1
  - union for $+$
  - concatenation for $\times$
  - asterate for $*$
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Example

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  - relational composition for $\times$
  - reflexive transitive closure for $*$

Theorem

for all regular expressions α and β

$$\alpha \equiv \beta \quad\Longleftrightarrow\quad \alpha = \beta \text{ can be proven from Kleene algebra axioms}$$

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Inference Rules

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$$\frac{}{\alpha = \alpha} \qquad \frac{\alpha = \beta}{\beta = \alpha}$$

Theorem

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Inference Rules

- equivalence

$$\frac{}{\alpha = \alpha} \qquad \frac{\alpha = \beta}{\beta = \alpha} \qquad \frac{\alpha = \beta \quad \beta = \gamma}{\alpha = \gamma}$$

Theorem

for all regular expressions α and β

$$\alpha \equiv \beta \quad \Longleftrightarrow \quad \alpha = \beta \text{ can be proven from Kleene algebra axioms}$$

Inference Rules

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$$\frac{}{\alpha = \alpha} \qquad \frac{\alpha = \beta}{\beta = \alpha} \qquad \frac{\alpha = \beta \quad \beta = \gamma}{\alpha = \gamma}$$

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$$\frac{\sigma(\gamma) = \sigma(\delta)}{\sigma(\alpha) = \sigma(\beta)} \qquad \forall \text{ axioms } \gamma = \delta \implies \alpha = \beta \quad \forall \text{ substitutions } \sigma$$

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Theorem

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Inference Rules

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$$\frac{}{\alpha = \alpha} \qquad \frac{\alpha = \beta}{\beta = \alpha} \qquad \frac{\alpha = \beta \quad \beta = \gamma}{\alpha = \gamma}$$

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$$\frac{\alpha = \gamma \quad \beta = \delta}{\alpha + \beta = \gamma + \delta} \qquad \frac{\alpha = \gamma \quad \beta = \delta}{\alpha\beta = \gamma\delta}$$

Theorem

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Inference Rules

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$$\frac{}{\alpha = \alpha} \qquad \frac{\alpha = \beta}{\beta = \alpha} \qquad \frac{\alpha = \beta \quad \beta = \gamma}{\alpha = \gamma}$$

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- congruence

$$\frac{\alpha = \gamma \quad \beta = \delta}{\alpha + \beta = \gamma + \delta} \qquad \frac{\alpha = \gamma \quad \beta = \delta}{\alpha\beta = \gamma\delta} \qquad \frac{\alpha = \beta}{\alpha^* = \beta^*}$$

Example (page 11)

$$\beta_a = a^*b(a^*b)^*a^* + a^*$$

Example (page 11)

$$\begin{aligned} \beta_a &= a^*b(a^*b)^*a^* + a^* \\ &= xx^*y + y \end{aligned}$$

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Example (page 11)

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\begin{aligned}
\beta_a &= a^*b(a^*b)^*a^* + a^* \\
&= xx^*y + y \qquad x := (a^*b) \qquad y := (a^*) \\
&= (xx^* + \boldsymbol{\varepsilon})y
\end{aligned}
$$

Example (page 11)

$$\begin{aligned}
\beta_a &= a^*b(a^*b)^*a^* + a^* \\
&= xx^*y + y \qquad x := (a^*b) \qquad y := (a^*) \\
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&= x^*y
\end{aligned}$$

Example (page 11)

$$
\begin{aligned}
\beta_a &= a^*b(a^*b)^*a^* + a^* \\
&= xx^*y + y \qquad x := (a^*b) \qquad y := (a^*) \\
&= (xx^* + \boldsymbol{\varepsilon})y \\
&= x^*y \\
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\end{aligned}
$$

Example (page 11)

$$
\begin{aligned}
\beta_a &= a^*b(a^*b)^*a^* + a^* \\
&= xx^*y + y \qquad x := (a^*b) \qquad y := (a^*) \\
&= (xx^* + \boldsymbol{\varepsilon})y \\
&= x^*y \\
&= (a^*b)^*a^* \\
&= (a+b)^*
\end{aligned}
$$

Example (w4.pdf – page 19)

$$\alpha \quad = \quad (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)$$

Example (w4.pdf – page 19)

$$\begin{aligned}\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\ &= x+xy^*y\end{aligned}$$

Example (w4.pdf – page 19)

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Example (w4.pdf – page 19)

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Example (w4.pdf – page 19)

$$
\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x := (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y := ((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}
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Example (w4.pdf – page 19)

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\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x := (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y := ((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^* \\
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0
\end{aligned}
$$

Example (w4.pdf – page 19)

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\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x+xy^*y \qquad x:=(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y:=((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x(\varepsilon+y^*y) \quad = \quad xy^*\\
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0\\
&= 0+(1+\boldsymbol{\varepsilon})1^*0
\end{aligned}
$$

Example (w4.pdf – page 19)

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\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x+xy^*y \qquad x:=(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y:=((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x(\varepsilon+y^*y) \quad = \quad xy^*\\
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0\\
&= 0+(1+\boldsymbol{\varepsilon})1^*0\\
&= 0+11^*0+1^*0
\end{aligned}
$$

Example (w4.pdf – page 19)

$$
\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x+xy^*y \qquad x := (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y := ((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}
$$

$$
\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0\\
&= 0+(1+\boldsymbol{\varepsilon})1^*0\\
&= 0+11^*0+1^*0\\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0
\end{aligned}
$$

Example (w4.pdf – page 19)

$$
\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x+xy^*y \qquad x:=(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y:=((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x(\varepsilon+y^*y) \quad = \quad xy^*\\
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0\\
&= 0+(1+\boldsymbol{\varepsilon})1^*0\\
&= 0+11^*0+1^*0\\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0\\
&= (1^*+1^*)0
\end{aligned}
$$

Example (w4.pdf – page 19)

$$
\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
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&= x(\varepsilon+y^*y) \quad = \quad xy^* \\
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 \\
&= 0+(1+\boldsymbol{\varepsilon})1^*0 \\
&= 0+11^*0+1^*0 \\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0 \\
&= (1^*+1^*)0 \\
&= 1^*0
\end{aligned}
$$

Example (w4.pdf – page 19)

$$\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x := (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y := ((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}$$

$$\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 \\
&= 0+(1+\boldsymbol{\varepsilon})1^*0 \\
&= 0+11^*0+1^*0 \\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0 \\
&= (1^*+1^*)0 \\
&= 1^*0
\end{aligned}
\qquad
y := (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0$$

Example (w4.pdf – page 19)

$$
\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x := (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y := ((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}
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$$
\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 \\
&= 0+(1+\boldsymbol{\varepsilon})1^*0 \\
&= 0+11^*0+1^*0 \\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0 \\
&= (1^*+1^*)0 \\
&= 1^*0
\end{aligned}
$$

$$
\begin{aligned}
y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0 \\
&= (0+\boldsymbol{\varepsilon})+11^*0
\end{aligned}
$$

Example (w4.pdf – page 19)

$$\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x := (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y := ((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}$$

$$\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 \\
&= 0+(1+\boldsymbol{\varepsilon})1^*0 \\
&= 0+11^*0+1^*0 \\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0 \\
&= (1^*+1^*)0 \\
&= 1^*0
\end{aligned}$$

$$\begin{aligned}
y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0 \\
&= (0+\boldsymbol{\varepsilon})+11^*0 \\
&= \boldsymbol{\varepsilon}+(0+11^*0)
\end{aligned}$$

Example (w4.pdf – page 19)

$$\alpha = (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)$$

$$= x+xy^*y \qquad x := (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y := ((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)$$

$$= x(\varepsilon+y^*y) \quad = \quad xy^*$$

$$\begin{aligned} x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 \\ &= 0+(1+\boldsymbol{\varepsilon})1^*0 \\ &= 0+11^*0+1^*0 \\ &= (\boldsymbol{\varepsilon}+11^*+1^*)0 \\ &= (1^*+1^*)0 \\ &= 1^*0 \end{aligned}$$

$$\begin{aligned} y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0 \\ &= (0+\boldsymbol{\varepsilon})+11^*0 \\ &= \boldsymbol{\varepsilon}+(0+11^*0) \\ &= \boldsymbol{\varepsilon}+(\boldsymbol{\varepsilon}+11^*)0 \end{aligned}$$

Example (w4.pdf – page 19)

$$\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x := (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y := ((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}$$

$$\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 \\
&= 0+(1+\boldsymbol{\varepsilon})1^*0 \\
&= 0+11^*0+1^*0 \\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0 \\
&= (1^*+1^*)0 \\
&= 1^*0
\end{aligned}$$

$$\begin{aligned}
y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0 \\
&= (0+\boldsymbol{\varepsilon})+11^*0 \\
&= \boldsymbol{\varepsilon}+(0+11^*0) \\
&= \boldsymbol{\varepsilon}+(\boldsymbol{\varepsilon}+11^*)0 \\
&= \boldsymbol{\varepsilon}+1^*0
\end{aligned}$$

Example (w4.pdf – page 19)

$$\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x := (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y := ((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}$$

$$\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 & y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0 \\
&= 0+(1+\boldsymbol{\varepsilon})1^*0 & &= (0+\boldsymbol{\varepsilon})+11^*0 \\
&= 0+11^*0+1^*0 & &= \boldsymbol{\varepsilon}+(0+11^*0) \\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0 & &= \boldsymbol{\varepsilon}+(\boldsymbol{\varepsilon}+11^*)0 \\
&= (1^*+1^*)0 & &= \boldsymbol{\varepsilon}+1^*0 \\
&= 1^*0 & &
\end{aligned}$$

$$xy^* := (1^*0)(\boldsymbol{\varepsilon}+1^*0)^*$$

Example (w4.pdf – page 19)

$$
\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x:=(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y:=((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}
$$

$$
\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 \\
&= 0+(1+\boldsymbol{\varepsilon})1^*0 \\
&= 0+11^*0+1^*0 \\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0 \\
&= (1^*+1^*)0 \\
&= 1^*0
\end{aligned}
$$

$$
\begin{aligned}
y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0 \\
&= (0+\boldsymbol{\varepsilon})+11^*0 \\
&= \boldsymbol{\varepsilon}+(0+11^*0) \\
&= \boldsymbol{\varepsilon}+(\boldsymbol{\varepsilon}+11^*)0 \\
&= \boldsymbol{\varepsilon}+1^*0
\end{aligned}
$$

$$
\begin{aligned}
xy^* &:= (1^*0)(\boldsymbol{\varepsilon}+1^*0)^* \\
&= (1^*0)(1^*0)^*
\end{aligned}
$$

Example (w4.pdf – page 19)

$$\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x:=(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y:=((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}$$

$$\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 \\
&= 0+(1+\boldsymbol{\varepsilon})1^*0 \\
&= 0+11^*0+1^*0 \\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0 \\
&= (1^*+1^*)0 \\
&= 1^*0
\end{aligned}$$

$$\begin{aligned}
y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0 \\
&= (0+\boldsymbol{\varepsilon})+11^*0 \\
&= \boldsymbol{\varepsilon}+(0+11^*0) \\
&= \boldsymbol{\varepsilon}+(\boldsymbol{\varepsilon}+11^*)0 \\
&= \boldsymbol{\varepsilon}+1^*0
\end{aligned}$$

$$\begin{aligned}
xy^* &:= (1^*0)(\boldsymbol{\varepsilon}+1^*0)^* \\
&= (1^*0)(1^*0)^* \\
&= (1^*0)^*(1^*0)
\end{aligned}$$

Example (w4.pdf – page 19)

$$\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x+xy^*y \qquad x:=(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y:=((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}$$

$$\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0\\
&= 0+(1+\boldsymbol{\varepsilon})1^*0\\
&= 0+11^*0+1^*0\\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0\\
&= (1^*+1^*)0\\
&= 1^*0
\end{aligned}$$

$$\begin{aligned}
y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0\\
&= (0+\boldsymbol{\varepsilon})+11^*0\\
&= \boldsymbol{\varepsilon}+(0+11^*0)\\
&= \boldsymbol{\varepsilon}+(\boldsymbol{\varepsilon}+11^*)0\\
&= \boldsymbol{\varepsilon}+1^*0
\end{aligned}$$

$$\begin{aligned}
xy^* &:= (1^*0)(\boldsymbol{\varepsilon}+1^*0)^*\\
&= (1^*0)(1^*0)^*\\
&= (1^*0)^*(1^*0)\\
&= ((1^*0)^*1^*)0
\end{aligned}$$

Example (w4.pdf – page 19)

$$
\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x+xy^*y \qquad x:=(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y:=((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0) \\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}
$$

$$
\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0 \\
&= 0+(1+\boldsymbol{\varepsilon})1^*0 \\
&= 0+11^*0+1^*0 \\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0 \\
&= (1^*+1^*)0 \\
&= 1^*0
\end{aligned}
\qquad
\begin{aligned}
y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0 \\
&= (0+\boldsymbol{\varepsilon})+11^*0 \\
&= \boldsymbol{\varepsilon}+(0+11^*0) \\
&= \boldsymbol{\varepsilon}+(\boldsymbol{\varepsilon}+11^*)0 \\
&= \boldsymbol{\varepsilon}+1^*0
\end{aligned}
$$

$$
\begin{aligned}
xy^* &:= (1^*0)(\boldsymbol{\varepsilon}+1^*0)^* \\
&= (1^*0)(1^*0)^* \\
&= (1^*0)^*(1^*0) \\
&= ((1^*0)^*1^*)0 \\
&= (1+0)^*0
\end{aligned}
$$

Example (w4.pdf – page 19)

$$\begin{aligned}
\alpha &= (0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)+(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0)((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)^*((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x+xy^*y \qquad x:=(0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0) \qquad y:=((0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0)\\
&= x(\varepsilon+y^*y) \quad = \quad xy^*
\end{aligned}$$

$$\begin{aligned}
x &:= 0+(1+\boldsymbol{\varepsilon})(1+\boldsymbol{\varepsilon})^*0\\
&= 0+(1+\boldsymbol{\varepsilon})1^*0\\
&= 0+11^*0+1^*0\\
&= (\boldsymbol{\varepsilon}+11^*+1^*)0\\
&= (1^*+1^*)0\\
&= 1^*0
\end{aligned}$$

$$\begin{aligned}
y &:= (0+\boldsymbol{\varepsilon})+1(1+\boldsymbol{\varepsilon})^*0\\
&= (0+\boldsymbol{\varepsilon})+11^*0\\
&= \boldsymbol{\varepsilon}+(0+11^*0)\\
&= \boldsymbol{\varepsilon}+(\boldsymbol{\varepsilon}+11^*)0\\
&= \boldsymbol{\varepsilon}+1^*0
\end{aligned}$$

$$\begin{aligned}
xy^* &:= (1^*0)(\boldsymbol{\varepsilon}+1^*0)^*\\
&= (1^*0)(1^*0)^*\\
&= (1^*0)^*(1^*0)\\
&= ((1^*0)^*1^*)0\\
&= (1+0)^*0\\
&= {\color{red}(0+1)^*0}
\end{aligned}$$

Outline

1. A Quick Recap
2. Derivatives
3. Kleene Algebra
4. Equivalence of Regular Expressions

Theorem

equivalence problem for regular expression

instance:	regular expressions α and β over alphabet Σ
question:	$L(\alpha) = L(\beta)$

is decidable

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Decision Procedure

1. convert α and β into equivalent finite automata N_α and N_β

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1. convert α and β into equivalent finite automata N_α and N_β
2. determinize and minimize N_α and N_β into D_α and D_β

Theorem

equivalence problem for regular expression

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Decision Procedure

1. convert α and β into equivalent finite automata N_α and N_β
2. determinize and minimize N_α and N_β into D_α and D_β
3. check whether D_α and D_β are identical (isomorphic):

   yes $\Longrightarrow$ $L(\alpha) = L(\beta)$ no $\Longrightarrow$ $L(\alpha) \neq L(\beta)$

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inefficient decision procedure

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1. derivatives:

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1. derivatives: build DFAs D_α and D_β

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Alternative Approaches (employing derivatives)

1. derivatives: build DFAs D_α and D_β then minimize and check whether $D_\alpha \simeq D_\beta$

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equivalence problem for regular expression

instance: regular expressions α and β over alphabet Σ

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Alternative Approaches (employing derivatives)

1. derivatives: build DFAs D_α and D_β then minimize and check whether $D_\alpha \simeq D_\beta$
2. derivatives + bisimulation:

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equivalence problem for regular expression

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3. check whether D_α and D_β are identical (isomorphic):

$$\text{yes} \implies L(\alpha) = L(\beta) \qquad \text{no} \implies L(\alpha) \neq L(\beta)$$

inefficient decision procedure

Alternative Approaches (employing derivatives)

1. derivatives: build DFAs D_α and D_β then minimize and check whether $D_\alpha \simeq D_\beta$
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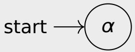
2. derivatives + bisimulation: check whether $L(\alpha) = L(\beta)$ (slides $\#23 - \#26$)

Example

- $\alpha = (a + b)^*$

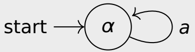


Lemma

every regular expression α can be transformed into equivalent DFA using derivatives

Example

- $\alpha = (a+b)^*, \ \alpha_a \equiv \alpha$



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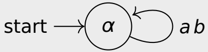

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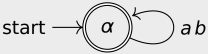

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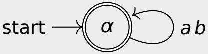
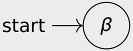


- $\beta = (a^* b)^* a^*$



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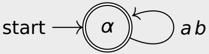
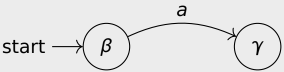


- $\beta = (a^*b)^*a^*\ , \beta_a \equiv a^*b(a^*b)^*a^* + a^* = \gamma$



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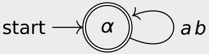

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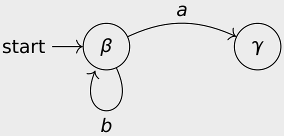

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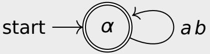

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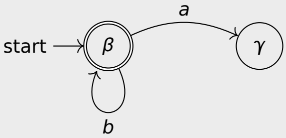

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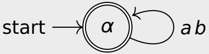

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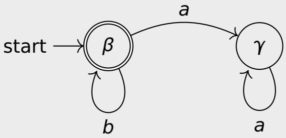

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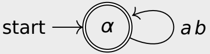

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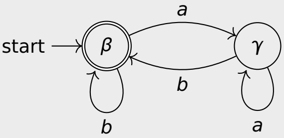

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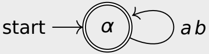

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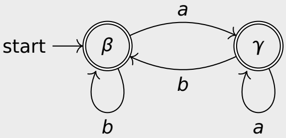

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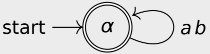

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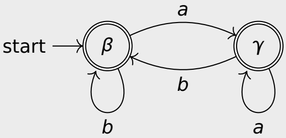

Lemma

every regular expression α can be transformed into equivalent DFA using derivatives
(and 'easy' Kleene algebra axioms for simplification)

Example

$\alpha = a^*$

Example

$\alpha = a^*$ $\alpha_a = \varepsilon a^*$

Example

$\alpha = a^*$ $\quad \alpha_a = \varepsilon a^*$ $\quad (\alpha_a)_a = \emptyset a^* + \varepsilon a^*$

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$((\alpha_a)_a)_a = \emptyset a^* + \emptyset a^* + \varepsilon a^* \equiv \emptyset a^* + \varepsilon a^* = (\alpha_a)_a$ modulo ACI of $+$

Example

$\alpha = a^*$ $\quad \alpha_a = \varepsilon a^*$ $\quad (\alpha_a)_a = \varnothing a^* + \varepsilon a^*$ $\quad ((\alpha_a)_a)_a = \varnothing a^* + \varnothing a^* + \varepsilon a^*$ $\quad (((\alpha_a)_a)_a)_a = \cdots$

$((\alpha_a)_a)_a = \varnothing a^* + \varnothing a^* + \varepsilon a^* \equiv \varnothing a^* + \varepsilon a^* = (\alpha_a)_a$ modulo ACI of $+$

Remark

- 'easy' Kleene algebra axioms: ACI of $+$

$$a + (b + c) = (a + b) + c \qquad a + b = b + a \qquad a + a = a$$

Example

$\alpha = a^*$ $\quad \alpha_a = \varepsilon a^*$ $\quad (\alpha_a)_a = \varnothing a^* + \varepsilon a^*$ $\quad ((\alpha_a)_a)_a = \varnothing a^* + \varnothing a^* + \varepsilon a^*$ $\quad (((\alpha_a)_a)_a)_a = \cdots$

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Remark

- 'easy' Kleene algebra axioms: ACI of $+$

$$a + (b + c) = (a + b) + c \qquad a + b = b + a \qquad a + a = a$$

Lemma

every regular expression has finitely many derivatives modulo ACI of $+$

Example

$\alpha = a^* \quad \alpha_a = \varepsilon a^* \quad (\alpha_a)_a = \varnothing a^* + \varepsilon a^* \quad ((\alpha_a)_a)_a = \varnothing a^* + \varnothing a^* + \varepsilon a^* \quad (((\alpha_a)_a)_a)_a = \cdots$

$((\alpha_a)_a)_a = \varnothing a^* + \varnothing a^* + \varepsilon a^* \equiv \varnothing a^* + \varepsilon a^* = (\alpha_a)_a$ modulo ACI of $+$

Remark

- 'easy' Kleene algebra axioms: ACI of $+$

$$a + (b + c) = (a + b) + c \qquad a + b = b + a \qquad a + a = a$$

- using more Kleene algebra axioms might speed up computation of equivalent DFA

Lemma

every regular expression has finitely many derivatives modulo ACI of $+$

Notation

$A\downarrow$ denotes $\varepsilon \in A$

Notation

$A{\downarrow}$ denotes $\varepsilon \in A$

Definition

bisimulation is binary relation $\sim$ between languages over alphabet Σ

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Definition

bisimulation is binary relation $\sim$ between languages over alphabet Σ such that if $A \sim B$ then

1. $A_a \sim B_a$ for all $a \in \Sigma$

Notation

$A{\downarrow}$ denotes $\varepsilon \in A$

Definition

bisimulation is binary relation $\sim$ between languages over alphabet Σ such that if $A \sim B$ then

1. $A_a \sim B_a$ for all $a \in \Sigma$
2. $A{\downarrow} \iff B{\downarrow}$

Example (bisimulation of languages)

$L = \{aa, ba\}$ and $M = \{aa, bb\}$ over $\Sigma = \{a, b\}$

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if $L \sim M$ then it must be that

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1. $L_a \sim M_a = \{a\} \sim \{a\}$

Example (bisimulation of languages)

$L = \{aa, ba\}$ and $M = \{aa, bb\}$ over $\Sigma = \{a, b\}$

if $L \sim M$ then it must be that

1. $L_a \sim M_a = \{a\} \sim \{a\}$
2. $L_b \sim M_b = \{a\} \sim \{b\}$

Example (bisimulation of languages)

$L = \{aa, ba\}$ and $M = \{aa, bb\}$ over $\Sigma = \{a, b\}$

if $L \sim M$ then it must be that

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3. $L\downarrow \iff M\downarrow \quad \checkmark$

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$L = \{aa, ba\}$ and $M = \{aa, bb\}$ over $\Sigma = \{a, b\}$

if $L \sim M$ then it must be that

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3. $L\downarrow \iff M\downarrow$ ✓

if $L_b \sim M_b$ then it must be that

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if $L_b \sim M_b$ then it must be that

1. $(L_b)_a \sim (M_b)_a = \{\varepsilon\} \sim \emptyset$

Example (bisimulation of languages)

$L = \{aa, ba\}$ and $M = \{aa, bb\}$ over $\Sigma = \{a, b\}$

if $L \sim M$ then it must be that

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if $L_b \sim M_b$ then it must be that

1. $(L_b)_a \sim (M_b)_a = \{\varepsilon\} \sim \emptyset$
2. $(L_b)_b \sim (M_b)_b = \emptyset \sim \{\varepsilon\}$

Example (bisimulation of languages)

$L = \{aa, ba\}$ and $M = \{aa, bb\}$ over $\Sigma = \{a, b\}$

if $L \sim M$ then it must be that

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if $L_b \sim M_b$ then it must be that

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3. $L_b\downarrow \iff M_b\downarrow$

Example (bisimulation of languages)

$L = \{aa, ba\}$ and $M = \{aa, bb\}$ over $\Sigma = \{a, b\}$

if $L \sim M$ then it must be that

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2. $L_b \sim M_b = \{a\} \sim \{b\}$
3. $L\downarrow \iff M\downarrow$ ✓

if $L_b \sim M_b$ then it must be that

1. $(L_b)_a \sim (M_b)_a = \{\varepsilon\} \sim \emptyset$
2. $(L_b)_b \sim (M_b)_b = \emptyset \sim \{\varepsilon\}$
3. $L_b\downarrow \iff M_b\downarrow$ ✓

Example (bisimulation of languages)

$L = \{aa, ba\}$ and $M = \{aa, bb\}$ over $\Sigma = \{a, b\}$

if $L \sim M$ then it must be that

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1. $(L_b)_a \sim (M_b)_a = \{\varepsilon\} \sim \varnothing$
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Remark

only equal languages are bisimilar (next slide)

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1. regular expressions α and β are equivalent $\iff L(\alpha) \sim L(\beta)$ for some bisimulation $\sim$

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$\Longrightarrow$ identity relation on languages is bisimulation that satisfies $L(\alpha) = L(\beta)$

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- if $x = \epsilon$ then $L(\alpha)\!\downarrow$
  $L(\beta)\!\downarrow$ because $L(\alpha) \sim L(\beta)$

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- if $x = \varepsilon$ then $L(\alpha)\downarrow$
  $L(\beta)\downarrow$ because $L(\alpha) \sim L(\beta)$ and thus $x \in L(\beta)$
- $x = ay$ for some $a \in \Sigma$ with IH: $\forall a \in \Sigma, y \in L(\alpha)_a \longleftrightarrow y \in L(\beta)_a$

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  given $x = ay$ then $y \in L(\alpha)_a$
  $y \in L(\beta)_a$ according to IH

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  therefore $L(\alpha)_a = L(\beta)_a \quad \forall a \in \Sigma$

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  $y \in L(\beta)_a$ according to IH
  therefore $L(\alpha)_a = L(\beta)_a \quad \forall a \in \Sigma$
  and thus $L(\alpha) = L(\beta)$

Example

$\alpha = (a+b)^*$ and $\beta = (a^*b)^*a^*$

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	a	b	
α	α	α	$\downarrow$

Example

$\alpha = (a+b)^*$ and $\beta = (a^*b)^*a^*$ and $\gamma = a^*b(a^*b)^*a^* + a^*$

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$\alpha = (a+b)^*$ and $\beta = (a^*b)^*a^*$ and $\gamma = a^*b(a^*b)^*a^* + a^*$

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2. $L(\alpha)_b \sim L(\gamma)_b = L(\alpha_b) \sim L(\gamma_b) = L(\alpha) \sim L(\beta)$ ✓
3. $L(\alpha)\downarrow \iff L(\gamma)\downarrow$ ✓

hence $\{(L(\alpha), L(\beta)), (L(\alpha), L(\gamma))\}$ is bisimulation and thus $L(\alpha) = L(\beta) = L(\gamma)$

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

derivatives

$$
\begin{aligned}
\alpha_a &= b^*(a+b)^*b =: \alpha_1 & \alpha_b &= \emptyset \\
(\alpha_1)_a &= (a+b)^*b =: \alpha_2 & (\alpha_1)_b &= b^*(a+b)^*b + (a+b)^*b + \varepsilon =: \alpha_3 \\
(\alpha_2)_a &= \alpha_2 & (\alpha_2)_b &= (a+b)^*b + \varepsilon =: \alpha_4 \\
(\alpha_3)_a &= \alpha_2 & (\alpha_3)_b &= \alpha_3 \\
(\alpha_4)_a &= \alpha_2 & (\alpha_4)_b &= \alpha_4
\end{aligned}
$$

$$
\begin{aligned}
\beta_a &= a^*(b^*a)^*b =: \beta_1 & \beta_b &= \emptyset \\
(\beta_1)_a &= a^*(b^*a)^*b + (b^*a)^*b =: \beta_2 & (\beta_1)_b &= b^*a(b^*a)^*b + \varepsilon =: \beta_3 \\
(\beta_2)_a &= \beta_2 & (\beta_2)_b &= \beta_3 \\
(\beta_3)_a &= (b^*a)^*b =: \beta_4 & (\beta_3)_b &= b^*a(b^*a)^*b =: \beta_5 \\
(\beta_4)_a &= \beta_4 & (\beta_4)_b &= \beta_3 \\
(\beta_5)_a &= \beta_4 & (\beta_5)_b &= \beta_5
\end{aligned}
$$

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

tables

	a	b	
α	α_1	$\emptyset$	$\uparrow$
α_1	α_2	α_3	$\uparrow$
α_2	α_2	α_4	$\uparrow$
α_3	α_2	α_3	$\downarrow$
α_4	α_2	α_4	$\downarrow$
$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$

	a	b	
β	β_1	$\emptyset$	$\uparrow$
β_1	β_2	β_3	$\uparrow$
β_2	β_2	β_3	$\uparrow$
β_3	β_4	β_5	$\downarrow$
β_4	β_4	β_3	$\uparrow$
β_5	β_4	β_5	$\uparrow$
$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

tables

	a	b	
α	α_1	$\emptyset$	$\uparrow$
α_1	α_2	α_3	$\uparrow$
α_2	α_2	α_4	$\uparrow$
α_3	α_2	α_3	$\downarrow$
α_4	α_2	α_4	$\downarrow$
$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$

	a	b	
β	β_1	$\emptyset$	$\uparrow$
β_1	β_2	β_3	$\uparrow$
β_2	β_2	β_3	$\uparrow$
β_3	β_4	β_5	$\downarrow$
β_4	β_4	β_3	$\uparrow$
β_5	β_4	β_5	$\uparrow$
$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$

- any bisimulation $\sim$ satisfying $L(\alpha) \sim L(\beta)$ requires $L(\alpha_1) \sim L(\beta_1)$

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

tables

	a	b	
α	α_1	$\varnothing$	$\uparrow$
α_1	α_2	α_3	$\uparrow$
α_2	α_2	α_4	$\uparrow$
α_3	α_2	α_3	$\downarrow$
α_4	α_2	α_4	$\downarrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

	a	b	
β	β_1	$\varnothing$	$\uparrow$
β_1	β_2	β_3	$\uparrow$
β_2	β_2	β_3	$\uparrow$
β_3	β_4	β_5	$\downarrow$
β_4	β_4	β_3	$\uparrow$
β_5	β_4	β_5	$\uparrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

- any bisimulation $\sim$ satisfying $L(\alpha) \sim L(\beta)$ requires $L(\alpha_3) \sim L(\beta_3)$

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

tables

	a	b	
α	α_1	$\emptyset$	$\uparrow$
α_1	α_2	α_3	$\uparrow$
α_2	α_2	α_4	$\uparrow$
α_3	α_2	α_3	$\downarrow$
α_4	α_2	α_4	$\downarrow$
$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$

	a	b	
β	β_1	$\emptyset$	$\uparrow$
β_1	β_2	β_3	$\uparrow$
β_2	β_2	β_3	$\uparrow$
β_3	β_4	β_5	$\downarrow$
β_4	β_4	β_3	$\uparrow$
β_5	β_4	β_5	$\uparrow$
$\emptyset$	$\emptyset$	$\emptyset$	$\uparrow$

- any bisimulation $\sim$ satisfying $L(\alpha) \sim L(\beta)$ requires $L(\alpha_3) \sim L(\beta_5)$

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

tables

	a	b	
α	α_1	$\varnothing$	$\uparrow$
α_1	α_2	α_3	$\uparrow$
α_2	α_2	α_4	$\uparrow$
α_3	α_2	α_3	$\downarrow$
α_4	α_2	α_4	$\downarrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

	a	b	
β	β_1	$\varnothing$	$\uparrow$
β_1	β_2	β_3	$\uparrow$
β_2	β_2	β_3	$\uparrow$
β_3	β_4	β_5	$\downarrow$
β_4	β_4	β_3	$\uparrow$
β_5	β_4	β_5	$\uparrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

- any bisimulation $\sim$ satisfying $L(\alpha) \sim L(\beta)$ requires $L(\alpha_3) \sim L(\beta_5)$
- $L(\alpha_3)\downarrow$ and $L(\beta_5)\uparrow$

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

tables

	a	b	
α	α_1	$\varnothing$	$\uparrow$
α_1	α_2	α_3	$\uparrow$
α_2	α_2	α_4	$\uparrow$
α_3	α_2	α_3	$\downarrow$
α_4	α_2	α_4	$\downarrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

	a	b	
β	β_1	$\varnothing$	$\uparrow$
β_1	β_2	β_3	$\uparrow$
β_2	β_2	β_3	$\uparrow$
β_3	β_4	β_5	$\downarrow$
β_4	β_4	β_3	$\uparrow$
β_5	β_4	β_5	$\uparrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

- any bisimulation $\sim$ satisfying $L(\alpha) \sim L(\beta)$ requires $L(\alpha_3) \sim L(\beta_5)$
- $L(\alpha_3)\downarrow$ and $L(\beta_5)\uparrow$ ↯

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

tables

	a	b	
α	α_1	$\varnothing$	$\uparrow$
α_1	α_2	α_3	$\uparrow$
α_2	α_2	α_4	$\uparrow$
α_3	α_2	α_3	$\downarrow$
α_4	α_2	α_4	$\downarrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

	a	b	
β	β_1	$\varnothing$	$\uparrow$
β_1	β_2	β_3	$\uparrow$
β_2	β_2	β_3	$\uparrow$
β_3	β_4	β_5	$\downarrow$
β_4	β_4	β_3	$\uparrow$
β_5	β_4	β_5	$\uparrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

- any bisimulation $\sim$ satisfying $L(\alpha) \sim L(\beta)$ requires $L(\alpha_3) \sim L(\beta_5)$
- $L(\alpha_3)\downarrow$ and $L(\beta_5)\uparrow$ $\lightning$
- $L(\alpha) \neq L(\beta)$

Example

$\alpha = ab^*(a+b)^*b$ and $\beta = aa^*(b^*a)^*b$

tables

	a	b	
α	α_1	$\varnothing$	$\uparrow$
α_1	α_2	α_3	$\uparrow$
α_2	α_2	α_4	$\uparrow$
α_3	α_2	α_3	$\downarrow$
α_4	α_2	α_4	$\downarrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

	a	b	
β	β_1	$\varnothing$	$\uparrow$
β_1	β_2	β_3	$\uparrow$
β_2	β_2	β_3	$\uparrow$
β_3	β_4	β_5	$\downarrow$
β_4	β_4	β_3	$\uparrow$
β_5	β_4	β_5	$\uparrow$
$\varnothing$	$\varnothing$	$\varnothing$	$\uparrow$

- any bisimulation $\sim$ satisfying $L(\alpha) \sim L(\beta)$ requires $L(\alpha_3) \sim L(\beta_5)$
- $L(\alpha_3)\downarrow$ and $L(\beta_5)\uparrow$ $\lightning$
- $L(\alpha) \neq L(\beta)$ $\quad\big($witness: $abb \in L(\alpha) \setminus L(\beta)\big)$

Thanks! & Questions?