Final Exam (100 pts)

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Assigned : June the 2nd, 13h00
Duration : 120 minutes

Q1. (40 pts)

a) (25 pts) Design a total Turing Machine (TM)

$$M = (Q, \{0\}, \{0, \vdash, \sqcup, x\}, \vdash, \sqcup, \delta, s, t, r)$$

accepts every member of the set

$$A = \{0^{4^n} \mid n \geqslant 0\}$$

rejecting every non-member. Explain your implementation in a few lines.

Below are a few examples to the input-output harmony of the intended TM:

Input	Output
$\vdash \sqcup^\omega$	reject
$\vdash 0\sqcup^\omega$	accept
$\vdash 0^2\sqcup^\omega$	reject
$\vdash 0^4\sqcup^\omega$	accept
$\vdash 0^8\sqcup^\omega$	reject
$\vdash 0^{10}\sqcup^\omega$	reject
$\vdash 0^{16}\sqcup^\omega$	accept
$\vdash 0^{30}\sqcup^\omega$	reject
$\vdash 0^{60}\sqcup^\omega$	reject
$\vdash 0^{64}\sqcup^\omega$	accept
$\vdash 0^{100}\sqcup^\omega$	reject
$\vdash 0^{256}\sqcup^\omega$	accept
$\vdash 0^{1000}\sqcup^\omega$	reject
$\vdash 0^{1024}\sqcup^\omega$	accept
$\vdash 00a00\sqcup^\omega$	reject
$\vdash 000b00\sqcup^\omega$	reject
$\vdash 00000c\sqcup^\omega$	reject
$\vdots$	$\vdots$

b) (15 pts) Design a total Turing Machine (TM)

$$M = (Q, \{a, b\}, \{a, b, 0, 1, 2, 3, 4, \vdash, \#, \sqcup\}, \vdash, \sqcup, \delta, s, t, r)$$

that inputs $x\#0$ with $x \in \{a, b\}^*$, computes and halts with the length of x in base 5 stored on its tape. Explain your implementation in a few lines.

Below are a few examples to the input-output harmony of the intended TM:

Input	Output
$\vdash \#0\sqcup^\omega$	$\vdash \#0\sqcup^\omega$
$\vdash a\#0\sqcup^\omega$	$\vdash \cdots \#1\sqcup^\omega$
$\vdash ba\#0\sqcup^\omega$	$\vdash \cdots \#2\sqcup^\omega$
$\vdash aba\#0\sqcup^\omega$	$\vdash \cdots \#3\sqcup^\omega$
$\vdash aaba\#0\sqcup^\omega$	$\vdash \cdots \#4\sqcup^\omega$
$\vdash abbba\#0\sqcup^\omega$	$\vdash \cdots \#10\sqcup^\omega$
$\vdash ababba\#0\sqcup^\omega$	$\vdash \cdots \#11\sqcup^\omega$
$\vdash abbbbbabab\#0\sqcup^\omega$	$\vdash \cdots \#20\sqcup^\omega$
$\vdash abaabbbbabab\#0\sqcup^\omega$	$\vdash \cdots \#22\sqcup^\omega$
$\vdash abbbaabababbbabab\#0\sqcup^\omega$	$\vdash \cdots \#30\sqcup^\omega$
$\vdash abbbaababababbbabab\#0\sqcup^\omega$	$\vdash \cdots \#34\sqcup^\omega$
$\vdash aabc\#0\sqcup^\omega$	reject
$\vdash axb\#0\sqcup^\omega$	reject
$\vdash ab\#e0\sqcup^\omega$	reject
$\vdots$	$\vdots$

A1.

a) Turing Machine

$$\begin{aligned} M \quad = \quad & \big(\{1,2,3,4,5,6,7,8,9,10,\text{halt-accept},\text{halt-reject}\}, \\ & \{0\}, \{0, \vdash, \sqcup, x\}, \vdash, \sqcup, \delta, 1, \text{halt-accept}, \text{halt-reject}\big) \end{aligned}$$

with transition function δ available here decides the set A.

b) Turing Machine

$$\begin{aligned} M \quad = \quad & \big(\{1,2,3,4,\text{inc},\text{overflow},\text{zeros},\text{halt},\text{halt-reject}\}, \\ & \{a, b\}, \{a, b, 0, 1, 2, 3, 4, \vdash, \sqcup, x\}, \vdash, \sqcup, \delta, 1, \text{halt}, \text{halt-reject}\big) \end{aligned}$$

with transition function δ available here decides the intended set.

Q2. (15 pts) Design non-deterministic push down automaton (NPDA)

$$N = (Q, \{a, b, c\}, \{\perp, \cdots\}, \delta, s, \perp, F)$$

that accepts every member of the set

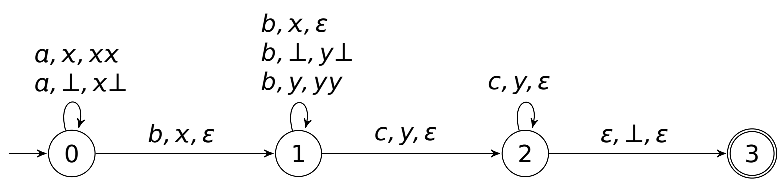
$$A = \{a^n b^m c^k \mid n, k \geqslant 1 \text{ and } n + k = m\}$$

rejecting every non-member. Justify your design in a few lines.

Below are a few examples to the input-output harmony of the intended NPDA:

Input	Output
ε	reject
ab^2c	accept
$a^2b^4c^2$	accept
$a^2b^3c^2$	reject
a^3b^4c	accept
$a^3b^2c^2$	reject
$a^5b^9c^4$	accept
$a^5b^{12}c^7$	accept
$a^6b^3c^2$	reject
$a^8b^{20}c^{12}$	accept
bc	reject
cba	reject
a^2c	reject
$\vdots$	$\vdots$

A2. The NPDA $N = (\{0,1,2,3\}, \{a,b,c\}, \{\perp, x, y\}, \delta, 0, \perp, \{3\})$ with set of transitions δ depicted in below state diagram



accepts the set A both by final state and empty stack. In state 0, the machine N pushes x's onto the stack upon reading a's from the input string. Then, it evolves into the state 1 on consuming the first b and popping an x off the stack. It keeps popping x's off, on b consumption, until reaching the bottom of the stack. At this stage, it pushes the bottom symbol and y's onto the stack on reading b's. This is obviously to keep track of the difference between the number of b's and a's in the input string: the number of y's in the stack amounts to the mentioned difference. Afterwards, on reading the first c, the machine advances into the state 2, popping a y off. It loops there, and pops y's off for every single c consumption until the bottom of the stack is handled. In this case, if the end of the string is reached, the machine evolves into the state 3, accepts the input string, and empties the stack.

Q3. (30 pts) Which of the following sets, constructed over the alphabet $\Sigma = \{a, b\}$, are context free and which are not?

(a) **(10 pts)** $A = \{a^n b^m \mid n = m^2\}$

(b) **(10 pts)** $B = \sim \{xx \mid x \in \Sigma^*\}$

(c) **(10 pts)** $C = \{a^n w^R w b^{3n} \mid w \in \Sigma^* \text{ and } n \geqslant 1\}$

Give grammars for those that are context free and proofs for those that are not.

A3.

(a) The set $A = \{a^n b^m \mid n = m^2\}$ is not context free.

Proof. For a proof by contradiction assume that A is context-free. Let $k > 0$ be the constant from the pumping lemma and consider the string $z = a^{k^2} b^k \in A$, and $k^2 + k \geqslant k$. For any decomposition $z = uvwxy$ such that $|vwx| \leqslant k$ and $|vx| > 0$ the substring vwx can contain only a's, only b's and a's and b's together such that a's are followed by b's:

- vwx has only a's: we have $0 < |vwx| = n \leqslant k$. Choosing $i = 2$, we get $uv^2wx^2y = a^{k^2+n}b^k \notin A$
- vwx has only b's: we have $0 < |vwx| = n \leqslant k$. Choosing $i = 2$, we get $uv^2wx^2y = a^{k^2}b^{k+n} \notin A$
- vwx has a's and b's together: let n be the number of a's, and m be the number of b's in vwx. Therefore $0 < n < k$ and $0 < m < k$ hold as we have $0 < n + m \leqslant k$. Choosing $i = 2$, we get $uv^2wx^2y = a^{k^2+n}b^{k+m} \notin A$. This is because $k^2 + n$ cannot be the perfect square of $k + m$. Here is a quick proof:

$$\begin{aligned}
\forall n > 0, \quad n < k \quad &\Longrightarrow \quad k^2 + n < k^2 + 2k + 1 \\
&\Longrightarrow \quad k^2 + n < (k+1)^2 \\
&\Longrightarrow \quad k^2 + n < (k+1)^2 \leqslant (k+m)^2, \quad \forall m > 0
\end{aligned}$$

□

(b) The set $B = \sim \{xx \mid x \in \Sigma^*\}$ is context-free since it is generated by the context-free grammar $G_B = \big(\{S, A, B, C\}, \{a, b\}, P, S\big)$ with below production rules in P:

$$\begin{aligned}
S &\rightarrow AB \mid BA \mid A \mid B & \qquad A &\rightarrow CAC \mid a \\
C &\rightarrow a \mid b & \qquad B &\rightarrow CBC \mid b
\end{aligned}$$

(c) The set $C = \{a^n w^R w b^{3n} \mid w \in \Sigma^* \text{ and } n \geqslant 1\}$ is context-free since it is generated by the context-free grammar $G_C = \big(\{S, T\}, \{a, b\}, P, S\big)$ with below production rules in P:

$$\begin{aligned}
S &\rightarrow aSbbb \mid T \\
T &\rightarrow aTa \mid bTb \mid \varepsilon
\end{aligned}$$

Q4. (15 pts) Prove (by contradiction) that the Halting Problem for Turing Machines

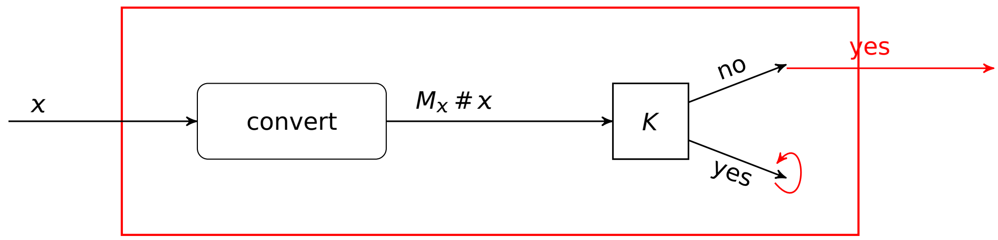
$$\mathrm{HP} = \{M\#x \mid \text{TM } M \text{ halts on input } x\}$$

is undecidable. Otherwise put, show that the set HP is recursively enumerable.

A4.

Proof.

1. suppose $\mathrm{HP} = \{M \# x \mid \text{TM } M \text{ halts on input } x\}$ is decidable (recursive)
2. $\mathrm{HP} = L(K)$ for some total TM K
3. construct TM N that on input x such that N exists $\iff$ K exists
   - 3.1. constructs M_x from x
   - 3.2. runs K on input $M_x \# x$
   - 3.3. accepts if K rejects and loops if K accepts



design of TM N

4. for all inputs x $\quad N$ halts on x $\iff$ K rejects $M_x \# x$ $\iff$ M_x does not halt on x
5. N is different from all M_x
6. N cannot exists thus K

$\square$