Final Exam (100 pts)

Burak Ekici

Assigned : June the 15th, 09h00
Duration : 120 minutes

Q1. (40 pts) Design Turing Machine (TM)

$$M = (\{\ldots, \text{halt-accept}, \text{halt-reject}\}, \{0,1,2\}, \{0,1,2,\vdash,\_,\cdots\}, \vdash, \_, \delta, s, \text{halt-accept}, \text{halt-reject})$$

accepts every member of the set

$$A := \{0^{a \times b}1^a2^b \mid a, b \geqslant 1\}$$

rejecting every non-member. Explain your code in a few lines.

Below are a few examples to the input-output harmony of the intended TM:

Input	Output
$\vdash \_^{\omega}$	reject
$\vdash 112\_^{\omega}$	reject
$\vdash 212\_^{\omega}$	reject
$\vdash 012\_^{\omega}$	accept
$\vdash 000011112\_^{\omega}$	accept
$\vdash 000012222\_^{\omega}$	accept
$\vdash 00001122\_^{\omega}$	accept
$\vdash 00000011122\_^{\omega}$	accept
$\vdash 000000000111222\_^{\omega}$	accept
$\vdash 0000000000001112222\_^{\omega}$	accept
$\vdash a00112\_^{\omega}$	reject
$\vdash 001b12\_^{\omega}$	reject
$\vdash 00112c\_^{\omega}$	reject
$\vdots$	$\vdots$

Important. Implement the machine M in Morphett's TM simulator (unless the simulator crashes), and explain your implementation in a few comment-out lines. Note that TMs designated elsewise will be graded zero.

A1. Turing Machine

$$\begin{aligned} M \;=\; & (\{\text{preprocess}, \text{preprocess2}, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \text{halt-accept}, \text{halt-reject}\}, \\ & \{0,1,2\}, \{0,1,2,\vdash,\_,m,n,g,x\}, \vdash, \_, \delta, \text{preprocess}, \text{halt-accept}, \text{halt-reject}) \end{aligned}$$

with transition function δ available here decides the set $A := \{0^{a \times b}1^a2^b \mid a, b \geqslant 1\}$.

Q2. (30 pts) Design non-deterministic push down automaton (NPDA) $N = (Q, \{x, y\}, \{\perp, \cdots\}, \delta, s, \perp, F)$ that accepts every member of the set

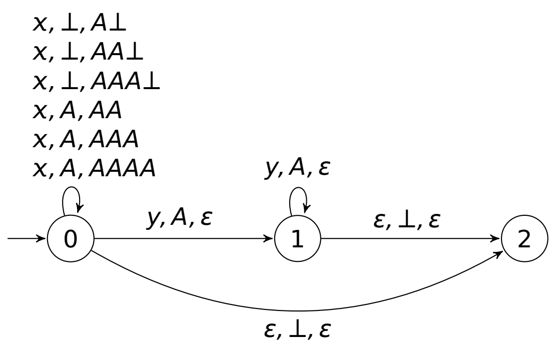
$$B := \{x^a y^b \mid 0 \leqslant a \leqslant b \leqslant 3a\}$$

rejecting every non-member. Justify your design in a few lines.

Below are a few examples to the input-output harmony of the intended NPDA:

Input	Output
y	reject
xxy	reject
$xyyyy$	reject
ε	accept
xy	accept
xyy	accept
$xyyy$	accept
$xxyyy$	accept
$xxxyyy$	accept
$xxxyyyy$	accept
$xxyyyyy$	accept
$xxyyyyyy$	accept
$xxyyyyyyy$	reject
$xxxyyyyyyyyyy$	reject
$\vdots$	$\vdots$

A2. The NPDA $N = (\{0, 1, 2\}, \{x, y\}, \{\perp, A\}, \delta, 0, \perp, \varnothing)$ with set of transitions δ depicted in below state diagram



accepts the set B by empty stack. In state 0, the machine N non-deterministically pushes either of $A\perp$, $AA\perp$ and $AAA\perp$ into the stack upon reading the first x from the input sting and popping the $\perp$ symbol off the stack. Similarly, benefiting from non-deterministic choice, N pushes one of AA, AAA and $AAAA$ into the stack if x is read and A is popped off. The machine passes the control to the state 1 if a y is read and A is popped off. It repeats this process until consuming every single y in the input string, and accepts, moving into state 2, if the stack becomes empty at the same time. Note also that the empty string needs to be accepted as $\varepsilon \in B$. This is performed with the transition from the state 0 into the state 2.

Q3. (30 pts) Which of the following sets are context free and which are not?

(a) **(10 pts)** $C := \{(ab^m)^n \mid m, n \geqslant 0\}$

(b) **(10 pts)** $D := \{a^n b^m a^m b^n \mid m, n \geqslant 0\}$

(c) **(10 pts)** $E := \{a^k b^m c^n \mid k = n \text{ and } m \text{ is odd}\}$

Give grammars for those that are context free and proofs by contrapositive of the Pumping Lemma for those that are not.

A3.

(a) The set $C := \{(ab^m)^n \mid m, n \geqslant 0\}$ is not context free. For a proof by contradiction assume that C is context-free. Let $k > 0$ be the constant from the pumping lemma and consider the string $z = ab^k ab^k ab^k \in C$. For any decomposition $z = uvwxy$ such that $|vwx| \leqslant k$ and $|vx| > 0$ the substring vwx can contain letters from at most two of the three equal blocks. When choosing $i = 0$ the string $uv^i wx^i y$ consists of blocks with different numbers of b's or does not begin with an a. Hence $uv^0 wx^0 y \notin C$ and therefore C is not context-free.

(b) The set $D := \{a^n b^m a^m b^n \mid m, n \geqslant 0\}$ is context-free since it is generated by the context-free grammar $G_D := (\{S, T\}, \{a, b\}, P, S)$ with below production rules in P:

$$\begin{aligned} S &\rightarrow aSb \mid T \\ T &\rightarrow bTa \mid \varepsilon \end{aligned}$$

Consider additionally the strings $x_i = a^i$ for $i \geqslant 0$. Let $i, j \geqslant 0$. If $i \neq j$ then $x_i b^i \in D$ and $x_j b^i \notin D$. It follows that the relation $\equiv_D$ has infinitely many equivalence classes. According to the Myhill–Nerode theorem, D cannot be regular.

(c) The set $E := \{a^k b^m c^n \mid k = n \text{ and } m \text{ is odd}\}$ is context-free since it is generated by the context-free grammar $G_E := (\{S, A, B\}, \{a, b, c\}, P, S)$ with below production rules in P:

$$\begin{aligned} S &\rightarrow A \\ A &\rightarrow aAc \mid B \\ B &\rightarrow bBb \mid b \end{aligned}$$

Consider additionally the strings $x_i = a^i b$ for $i \geqslant 0$. Let $i, j \geqslant 0$. If $i \neq j$ then $x_i c^i \in E$ and $x_j c^i \notin E$. It follows that the relation $\equiv_E$ has infinitely many equivalence classes. According to the Myhill–Nerode theorem, E cannot be regular.

Important Notice:

- Collaboration is strictly and positively prohibited; lowers your score to 0 if detected.
- Any submission after 120 minutes after the exam gets started will NOT be accepted. Please beware and respect the deadline!
- Submission policy:
  1. considering **Q1**, first implement a TM in Morphett's Simulator, then copy-and-paste your code in a text file named **A1.txt**;
  2. as for **Q2** and **Q3**, write your answers down on the paper, scan them into a PDF file named **A23.pdf**;
  3. and then submit both files **A1.txt** and **A23.pdf** in raw form. Please do not compress files!
- Make sure that your handwriting in **A23.pdf** is decent and readable.