CMPE 322/327 - Theory of Computation

Week 5: State Minimization & Myhill-Nerode Relations

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March 21-25, 2022

Outline

1. A Quick Recap
2. State Minimization
3. Myhill-Nerode Relations

Definition

regular expressions are restricted patterns which use only

$$\boldsymbol{a} \in \Sigma \quad \boldsymbol{\varepsilon} \quad \boldsymbol{\emptyset} \quad \alpha + \beta \quad \alpha^* \quad \alpha\beta$$

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- homomorphism $h\colon \Sigma^* \to \Gamma^*$
- $h^{-1}(L(M)) = L(M')$ for DFA $M' = (Q, \Sigma, \delta', s, F)$ with $\delta'(q, a) := \widehat{\delta}(q, h(a))$

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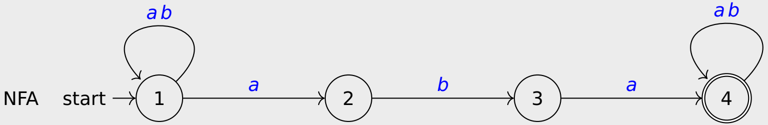
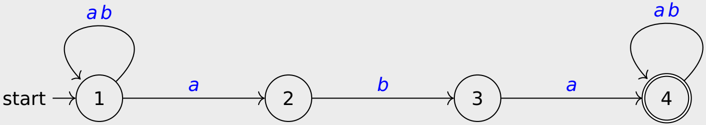
Outline

Example



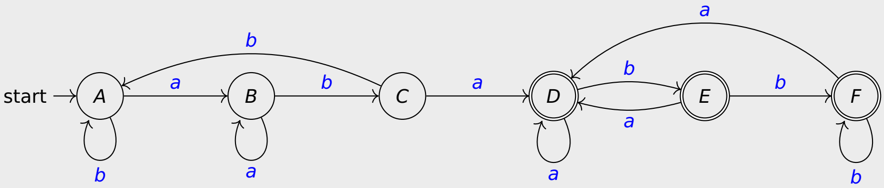
Example

NFA



A = {1} C = {1,3} E = {1,3,4}
B = {1,2} D = {1,2,4} F = {1,4}

DFA



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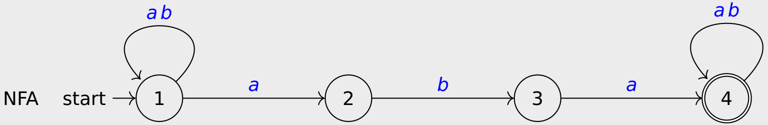

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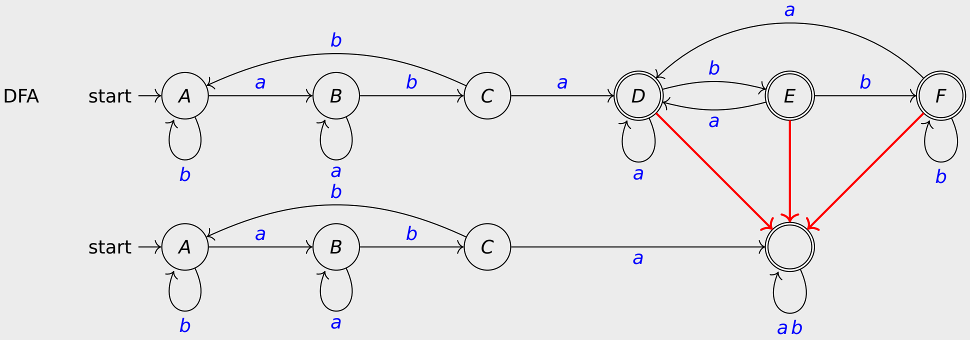

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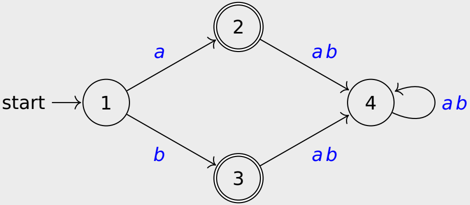

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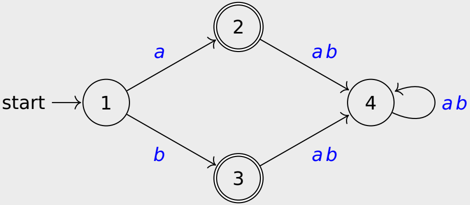
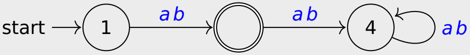


not minimal: states 2 and 3 can be collapsed



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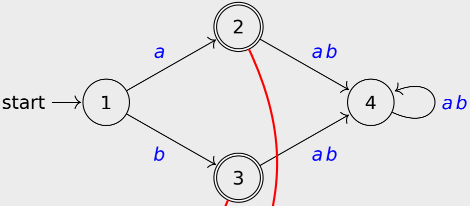

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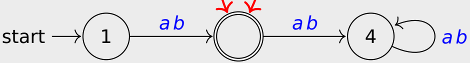

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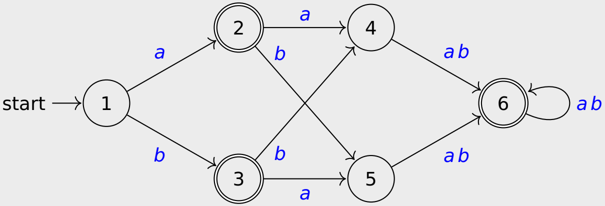

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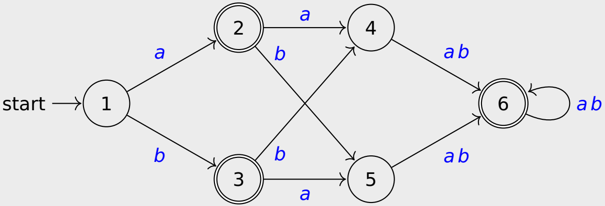

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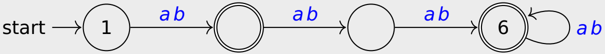

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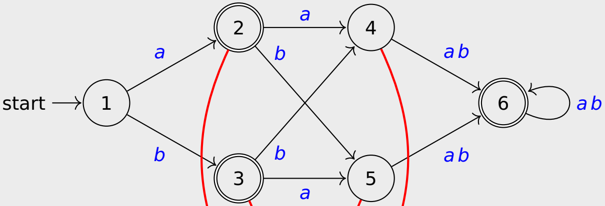

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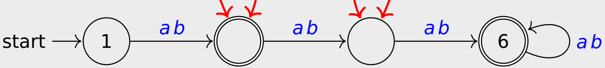

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3. collapse indistinguishable states

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given DFA $M = (Q, \Sigma, \delta, s, F)$ without inaccessible states

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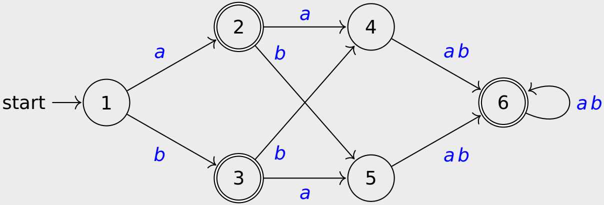

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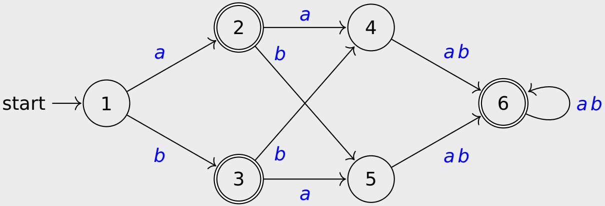

1					
✓	2				
✓		3			
	✓	✓	4		
	✓	✓		5	
✓			✓	✓	6

1. final/non-final states are distinguishable

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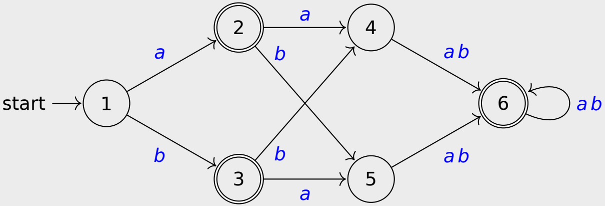

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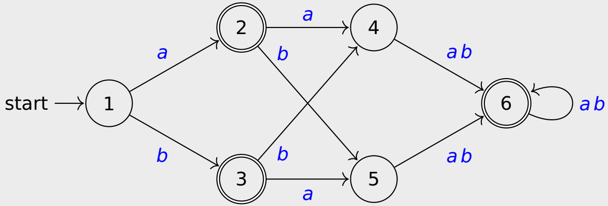

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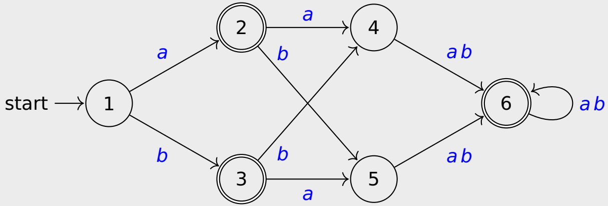

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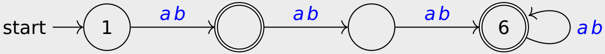
collapse states 2 and 3 and states 4 and 5: 

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3. $\forall p, q, r \in Q$ $\quad p \approx q \wedge q \approx r \implies p \approx r$ (transitivity)

Notation

$[p]_\approx := \{q \in Q \mid p \approx q\}$ denotes equivalence class of p

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2. $p \in F \iff [p]_\approx \in F'$

for all $p \in Q$

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Proof.

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\begin{aligned}
x \in L(M/\approx) \quad & \Longleftrightarrow \quad \widehat{\delta'}([s]_\approx, x) \in F' \\
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&\iff \quad \widehat{\delta}(s, x) \in F \\
&\iff \quad x \in L(M)
\end{aligned}
$$

Question

is $M/\approx$ minimum-state DFA for $L(M)$?

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Lemma

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&\Longleftrightarrow \quad \forall x \in \Sigma^* \quad ([\widehat{\delta}(p, x)]_\approx \in F' \quad \Longleftrightarrow \quad [\widehat{\delta}(q, x)]_\approx \in F')
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\end{aligned}
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&\iff \forall x \in \Sigma^* \quad (\widehat{\delta}(p, x) \in F \iff \widehat{\delta}(q, x) \in F) \\
&\iff \forall x \in \Sigma^* \quad p \approx q
\end{aligned}
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\begin{aligned}
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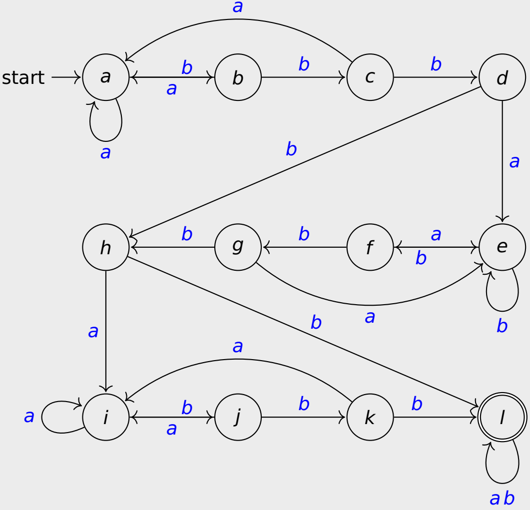
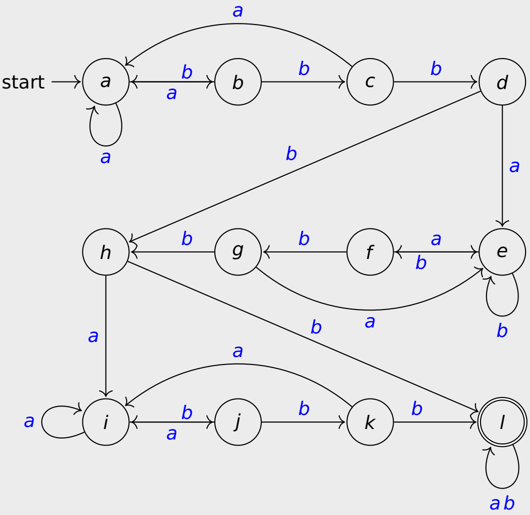
&\iff \quad \forall x \in \Sigma^* \quad p \approx q \\
&\iff \quad \forall x \in \Sigma^* \quad [p]_\approx = [q]_\approx
\end{aligned}
$$

$\square$

Example



Example



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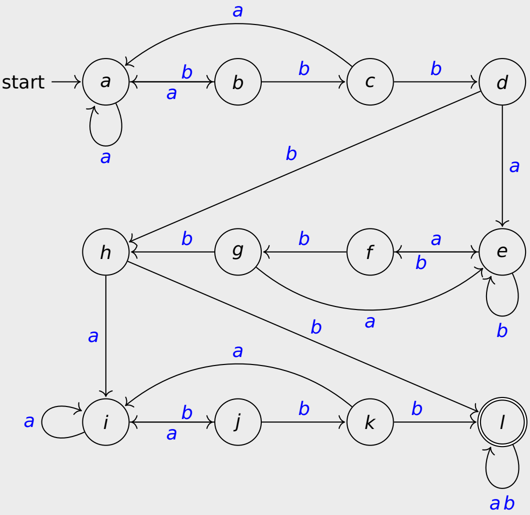
Example



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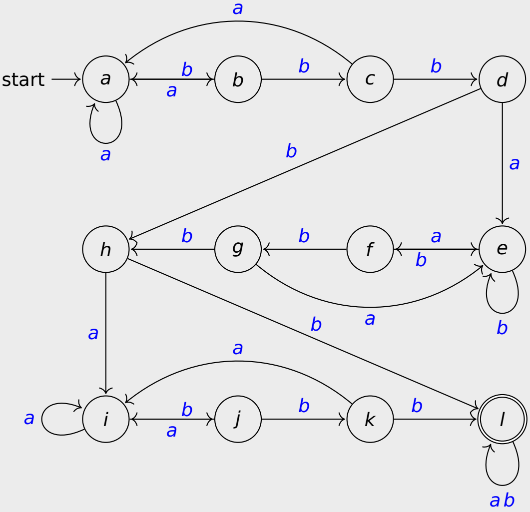
Example



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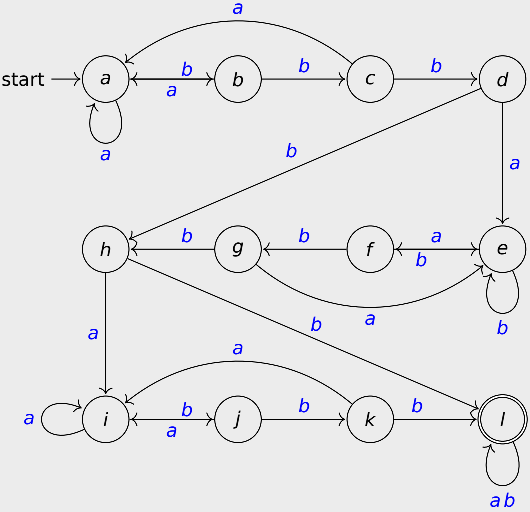
Example



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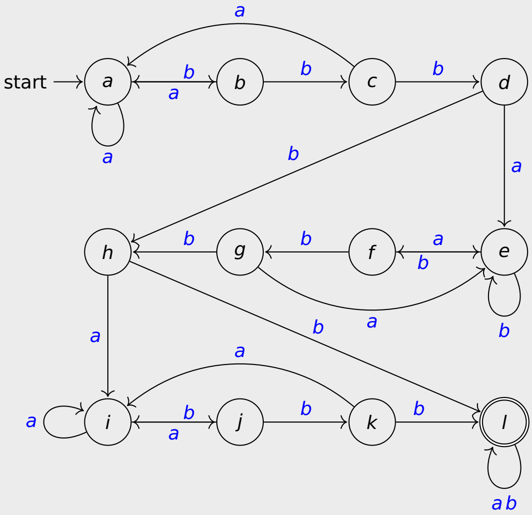
Example



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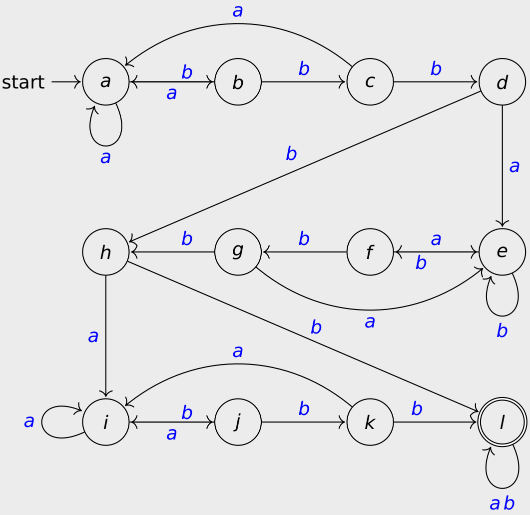
Example



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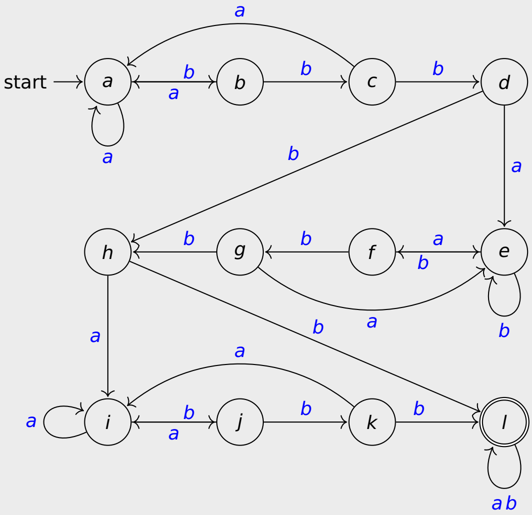
Example



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✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	*l*

states *d*, *g* and *h*, *k* can be merged

Outline

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Myhill-Nerode relation $\equiv$ for $L \subseteq \Sigma^*$ is an equivalence relation that

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equivalence relation $\equiv_M$ on Σ^* for DFA $M = (Q, \Sigma, \delta, s, F)$ is defined as follows:

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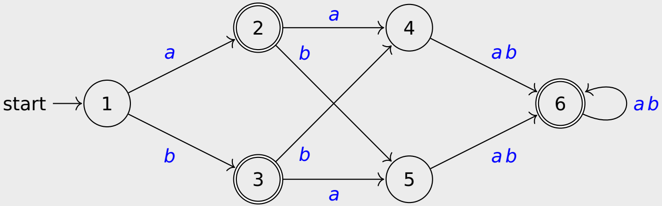
Corollary

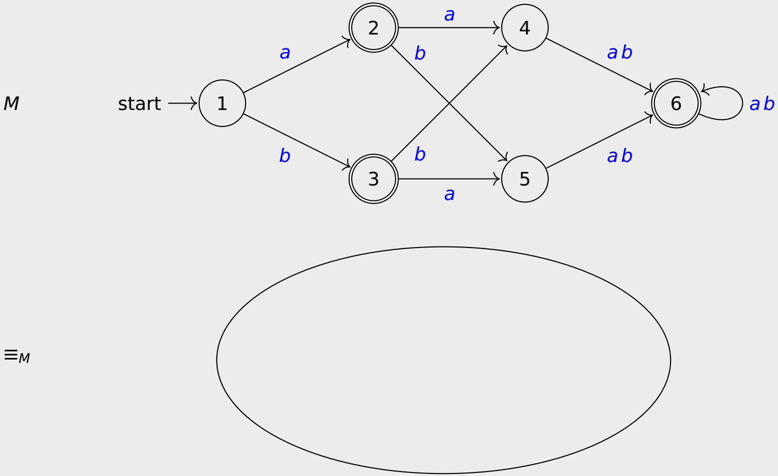
$\equiv_M$ is Myhill-Nerode relation for $L(M)$

Example

M



Example



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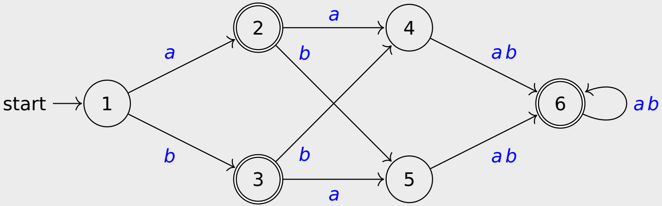
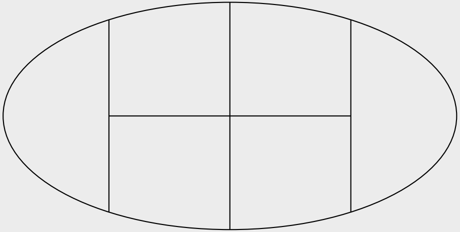


$\equiv_M$



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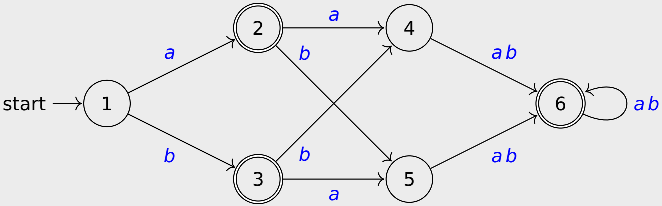

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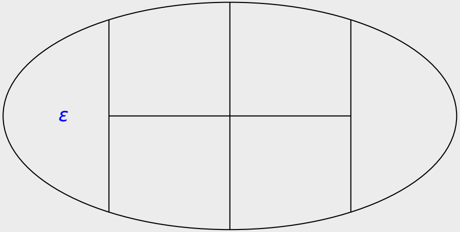

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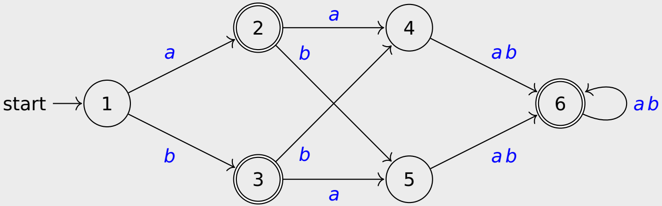

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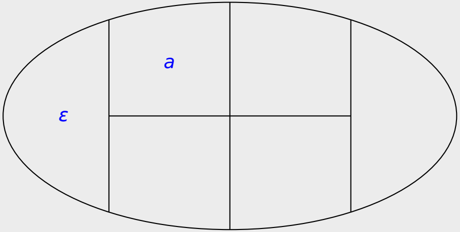

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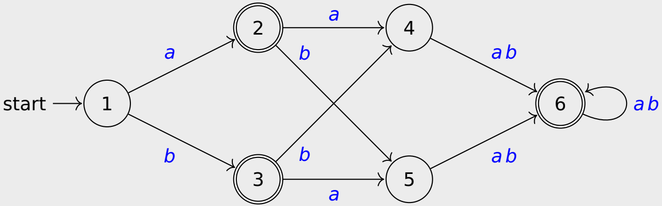

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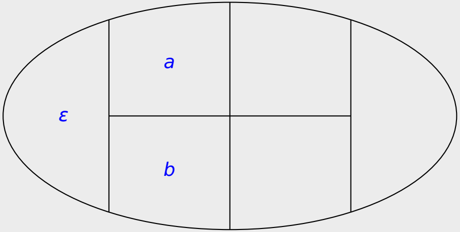

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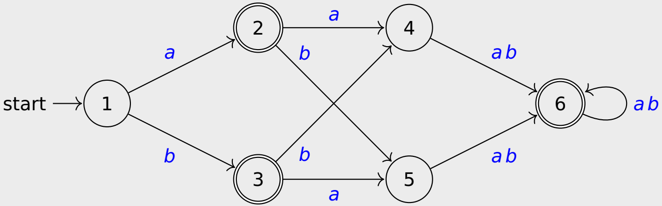

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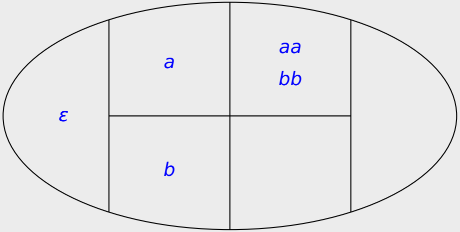

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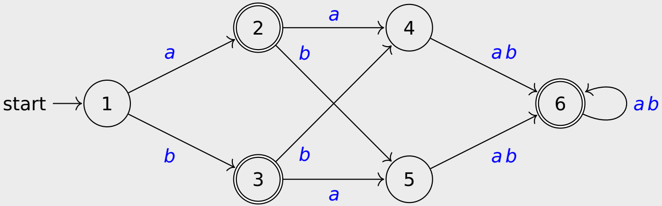

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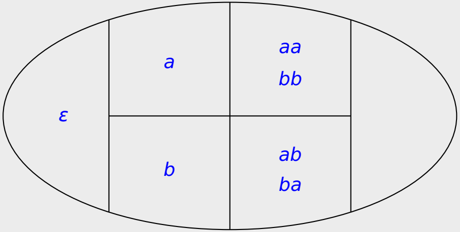

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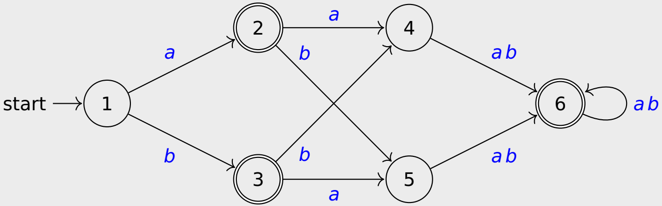

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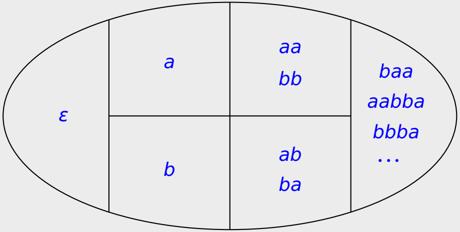

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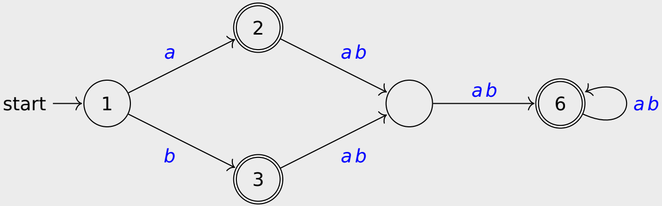

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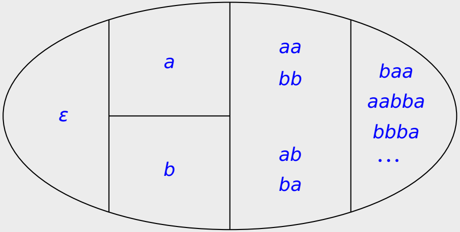

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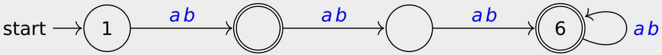

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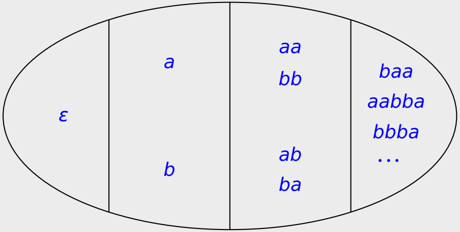

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1. $\widehat{\delta}([x]_\equiv, y) = [xy]_\equiv$ for all $y \in \Sigma^*$
2. $x \in L \iff [x]_\equiv \in F$

for all $x \in \Sigma^*$

Theorem

$L(M_\equiv) = L$

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Proof.

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Corollary

if L admits Myhill-Nerode relation then L is regular

Theorem

two mappings (for $L \subseteq \Sigma^*$)

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- $\approx \mapsto M_\approx$ from Myhill-Nerode relations for L to DFAs for L

are each others inverse (up to isomorphism of automata):

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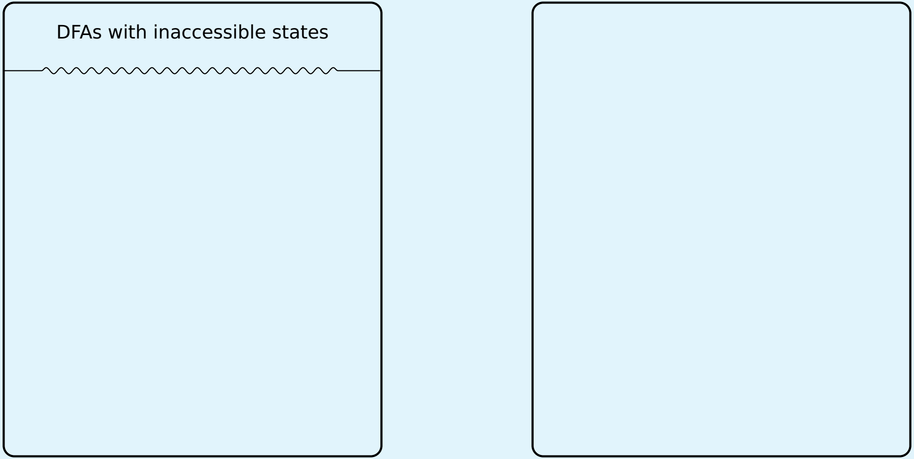
DFAs for L

Myhill-Nerode relations for L



DFAs for L

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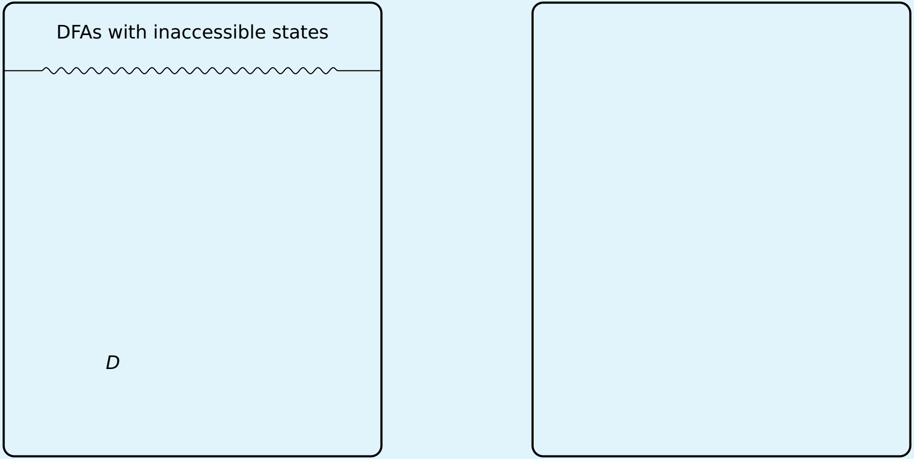

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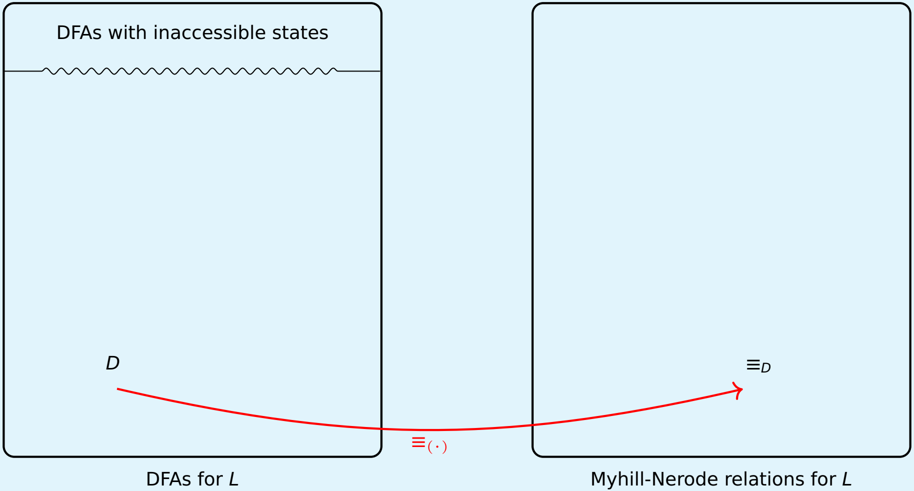
DFAs with inaccessible states
D
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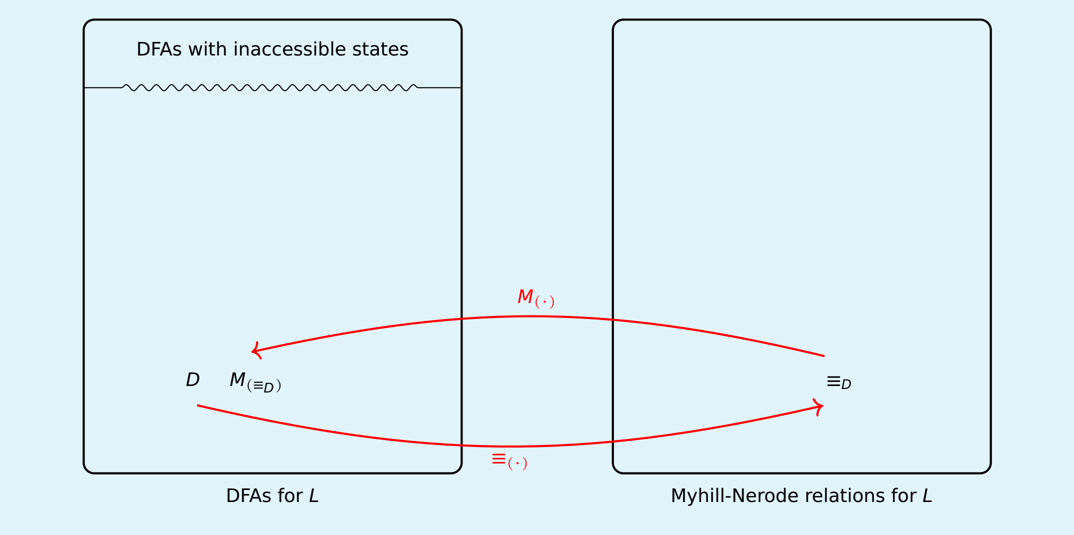
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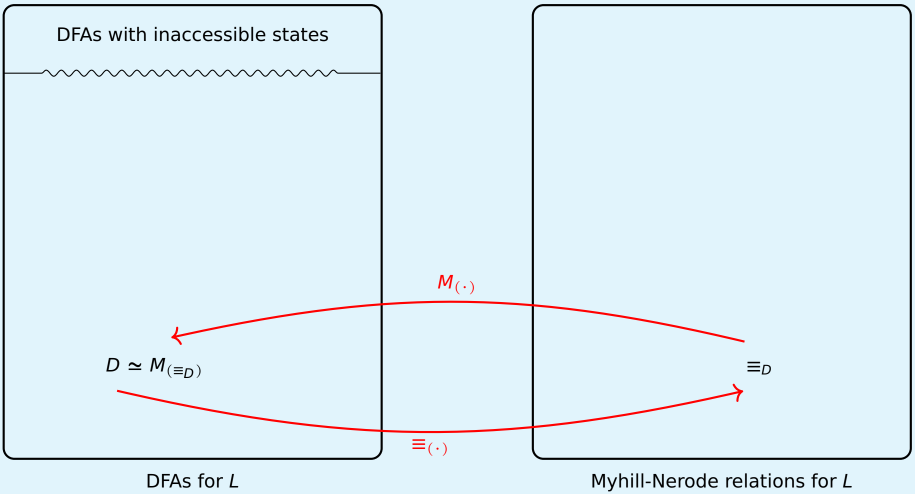
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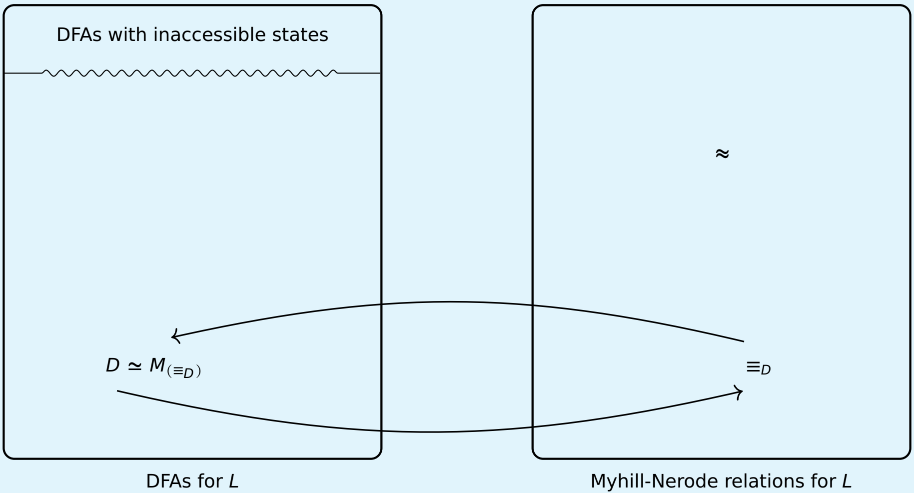
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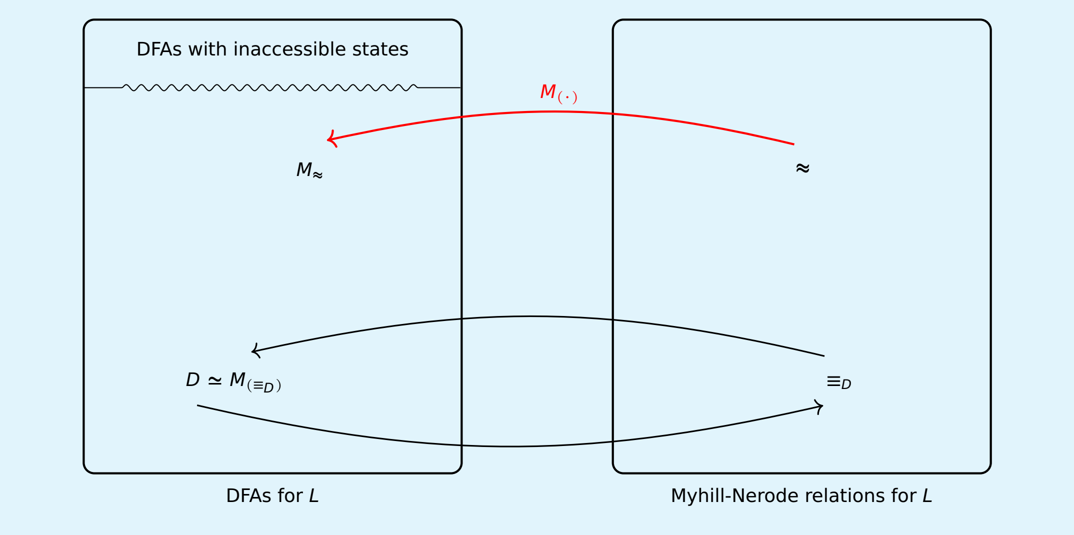
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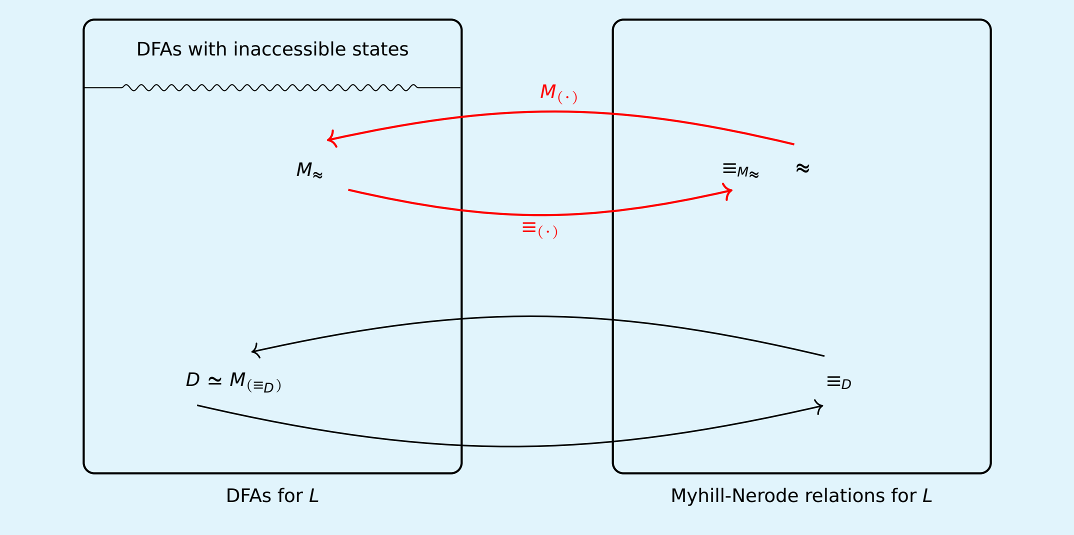
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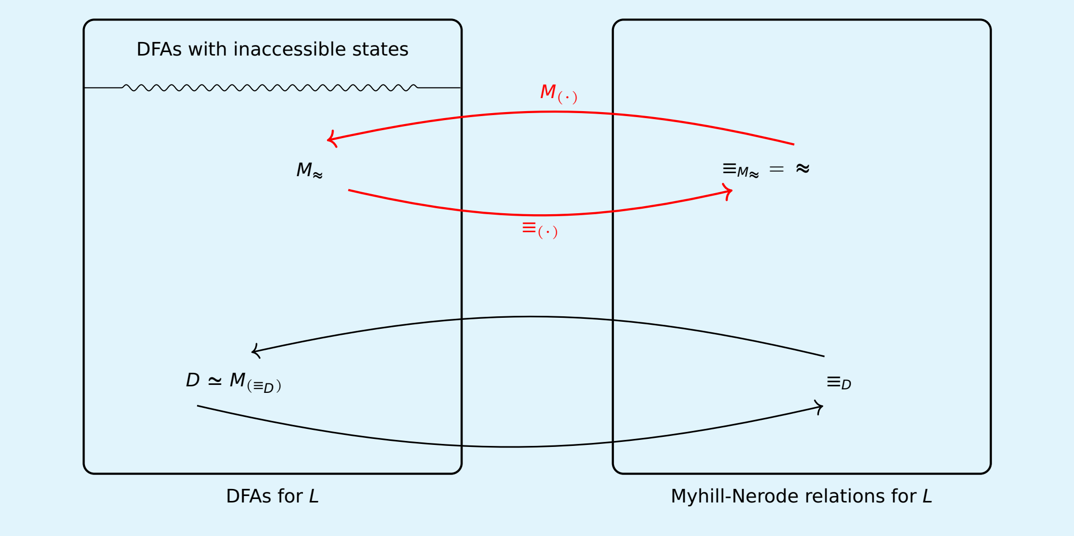
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Definition

for any set $L \subseteq \Sigma^*$, equivalence relation $\equiv_L$ on Σ^* is defined as follows:

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$\equiv_L$ has fewest equivalence classes

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following statements are equivalent for any set $L \subseteq \Sigma^*$:

- L is regular
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for every DFA M, $M/\!\approx\ \simeq\ M_{\equiv_L}$

Example

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- $a^i \in D$ and $a^{j(k+1)} \notin D$
- $\equiv_D$ does not refine D

Thanks! & Questions?