CMPE 322/327 - Theory of Computation

Week 3: Nondeterministic Finite State Automata & Epsilon Transitions

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March 7-11, 2022

Outline

Definitions

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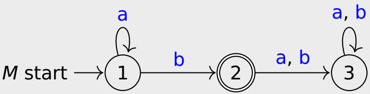
- language accepted by M is given by $L(M) := \{x \mid \widehat{\delta}(x, s) \in F\}$

Example (DFA → Regular Sets)

$M = (Q, \Sigma, \delta, s, F)$



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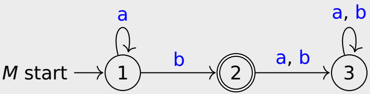

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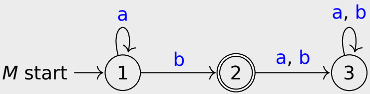

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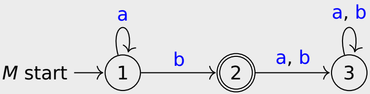

1. $Q = \{1, 2, 3\}$
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3. $\delta : Q \times \Sigma \to Q$

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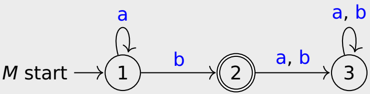

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2. $\Sigma = \{a, b\}$
3. $\delta : Q \times \Sigma \rightarrow Q$

δ	a	b
1	1	2
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3	3	3

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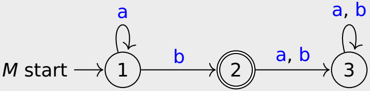

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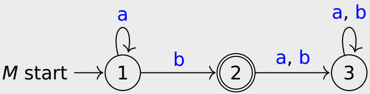

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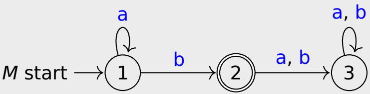

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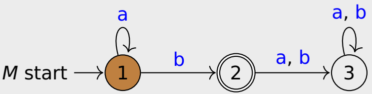

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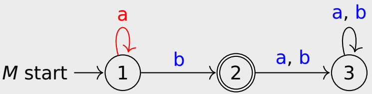

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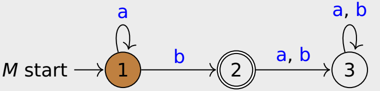

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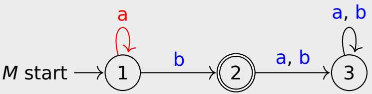

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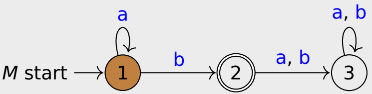

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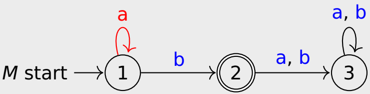

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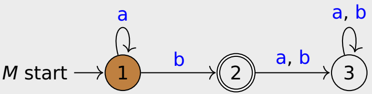

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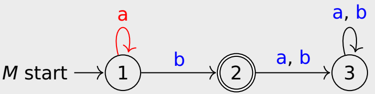

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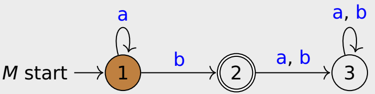

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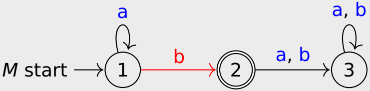

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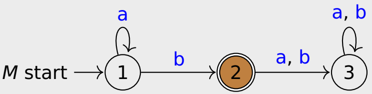

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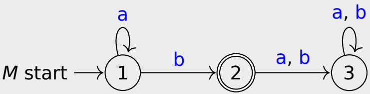

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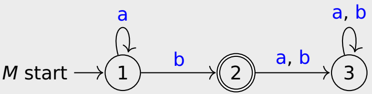

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a a b b a

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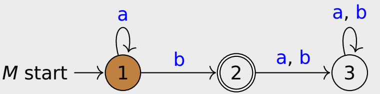

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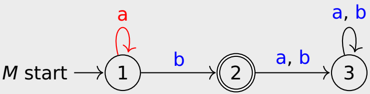

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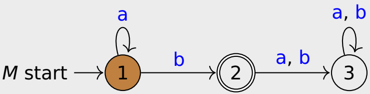

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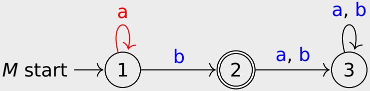

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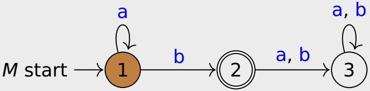

a a a a b $\in L(M)$

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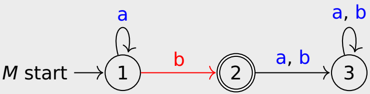

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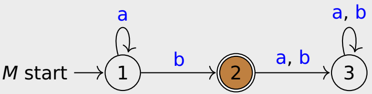

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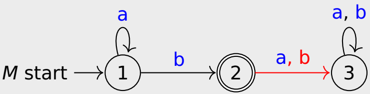

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1 $\quad$ 1 $\quad$ 1 $\quad$ 1 $\quad$ 1 $\quad$ 2

$\quad$ a $\quad$ a $\quad$ b $\quad$ b $\quad$ a

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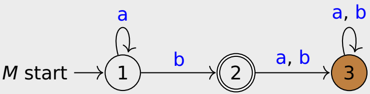

1. $Q = \{1, 2, 3\}$
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2	3	3
3	3	3

	a		a		a		a		b		$\in L(M)$
1		1		1		1		1		2	

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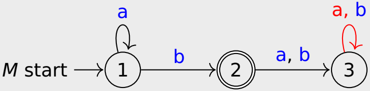

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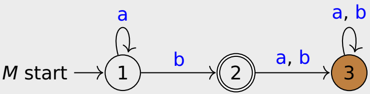

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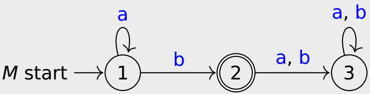

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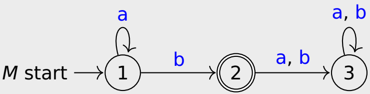

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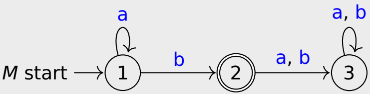

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$L(M) := \{x \mid x = a^n b, n \geq 0\}$

Definition

set $A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M

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Theorem

regular sets are effectively closed under intersection, complement and union

Outline

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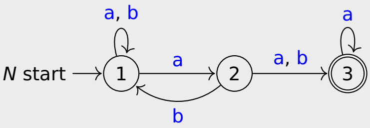

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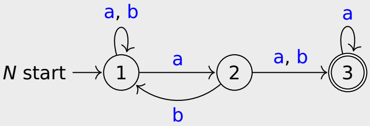

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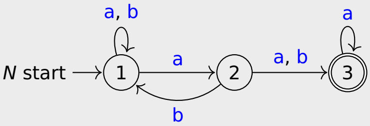

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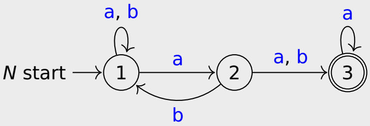

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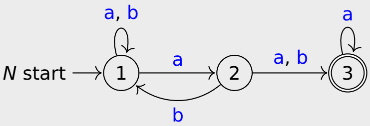

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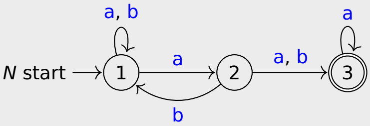

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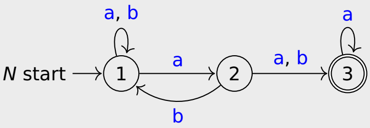

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$$\widehat{\Delta}(A, \varepsilon) := A \qquad \widehat{\Delta}(A, xa) := \bigcup_{q \in \widehat{\Delta}(A,x)} \Delta(q, a)$$

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- string $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \varnothing$

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Let $x = ababba$ over the alphabet $\Sigma = \{a, b\}$

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$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \widehat{\Delta}(A, \varepsilon)\, \Delta(q, a))\, \Delta(q, b))\, \Delta(q, a))\, \Delta(q, b))\, \Delta(q, b))\, \Delta(q, a))$ — 6^{th} rec. call

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in A\, \Delta(q, a))\, \Delta(q, b))\, \Delta(q, a))\, \Delta(q, b))\, \Delta(q, b))\, \Delta(q, a))$

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in B\, \Delta(q, b))\, \Delta(q, a))\, \Delta(q, b))\, \Delta(q, b))\, \Delta(q, a))$ — assuming $\bigcup(q \in A\, \Delta(q, a)) = B$

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in C\, \Delta(q, a))\, \Delta(q, b))\, \Delta(q, b))\, \Delta(q, a))$ — assuming $\bigcup(q \in B\, \Delta(q, b)) = C$

$\bigcup(q \in \bigcup(q \in \bigcup(q \in D\, \Delta(q, b))\, \Delta(q, b))\, \Delta(q, a))$ — assuming $\bigcup(q \in C\, \Delta(q, a)) = D$

$\bigcup(q \in \bigcup(q \in E\, \Delta(q, b))\, \Delta(q, a))$ — assuming $\bigcup(q \in D\, \Delta(q, b)) = E$

Example (Unfolding of the multistep function $\widehat{\Delta}$)

Let $x = ababba$ over the alphabet $\Sigma = \{a, b\}$

$\bigcup(q \in \widehat{\Delta}(A, ababb), a)$ — 1st rec. call

$\bigcup(q \in \bigcup(q \in \widehat{\Delta}(A, abab)\, \Delta(q, b))\, \Delta(q, a))$ — 2nd rec. call

$\bigcup(q \in \bigcup(q \in \bigcup(q \in \widehat{\Delta}(A, aba)\, \Delta(q, b))\, \Delta(q, b))\, \Delta(q, a))$ — 3rd rec. call

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Example (Unfolding of the multistep function $\widehat{\Delta}$)

Let $x = ababba$ over the alphabet $\Sigma = \{a, b\}$

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$\bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \bigcup(q \in \widehat{\Delta}(A, a)\, \Delta(q,b))\, \Delta(q,a))\, \Delta(q,b))\, \Delta(q,b))\, \Delta(q,a))$	5th rec. call
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Lemma ($\widehat{\Delta}$ distributes)

$\widehat{\Delta}(A, xy) = \widehat{\Delta}(\widehat{\Delta}(A, x), y) \quad \forall A \subseteq Q_N$ and $x, y \in \Sigma^*$

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We argue by induction on $|y|$:

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Theorem

every set accepted by NFA is regular

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Proof.

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  4. $F_M := \{A \subseteq Q_N \mid A \cap F_N \neq \varnothing\}$

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  proof: by induction on $|x|$ see next slide

proof of the claim

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$$\begin{aligned} \widehat{\delta_M}(A, ya) &= \delta_M(\widehat{\delta_M}(A, y), a) && \text{(by definition of } \widehat{\delta_M}\text{)} \\ &= \delta_M(\widehat{\Delta_N}(A, y), a) && \text{(by induction hypothesis IH)} \end{aligned}$$

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\end{aligned}$$

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\end{aligned}$$

□

Proof. (NFA regularity)

statement: $L(M) = L(N)$

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$\forall x \in \Sigma^*,\ x \in L(M) \iff \widehat{\delta_M}(s_M, x) \in F_M$ (by definition of acceptance)

Proof. (NFA regularity)

statement: $L(M) = L(N)$

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\forall x \in \Sigma^*,\ x \in L(M) &\iff \widehat{\delta_M}(s_M, x) \in F_M && \text{(by definition of acceptance)}\\
&\iff \widehat{\delta_M}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \varnothing\} && \text{(by definition of } s_M \text{ and } F_M\text{)}
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&\iff \widehat{\Delta_N}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \varnothing\} && \text{(by claim proven in slide 12)}
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&\iff \widehat{\Delta_N}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\} && \text{(by claim proven in slide 12)} \\
&\iff \widehat{\Delta_N}(S_N, x) \cap F_N \neq \emptyset && \text{(by set comprehension)}
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Proof. (NFA regularity)

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$$
\begin{aligned}
\forall x \in \Sigma^*,\ x \in L(M) &\iff \widehat{\delta_M}(s_M, x) \in F_M && \text{(by definition of acceptance)} \\
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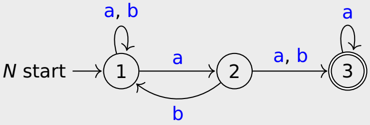
&\iff \widehat{\Delta_N}(S_N, x) \in \{A \subseteq Q_N \mid A \cap F_N \neq \varnothing\} && \text{(by claim proven in slide 12)} \\
&\iff \widehat{\Delta_N}(S_N, x) \cap F_N \neq \varnothing && \text{(by set comprehension)} \\
&\iff x \in L(N) && \text{(by definition of acceptance)}
\end{aligned}
$$

□

Example



$A = \varnothing$ $\quad E = \{1, 2\}$

$B = \{1\}$ $\quad F = \{1, 3\}$

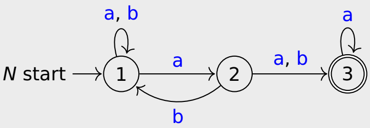
$C = \{2\}$ $\quad G = \{2, 3\}$

$D = \{3\}$ $\quad H = \{1, 2, 3\}$

Δ	a	b

Δ	a	b

Example



$A = \emptyset$ $\quad E = \{1, 2\}$

$B = \{1\}$ $\quad F = \{1, 3\}$

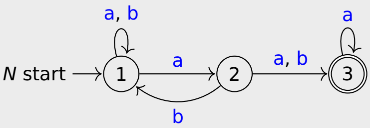
$C = \{2\}$ $\quad G = \{2, 3\}$

$D = \{3\}$ $\quad H = \{1, 2, 3\}$

Δ	a	b
A	A	A

Δ	a	b

Example



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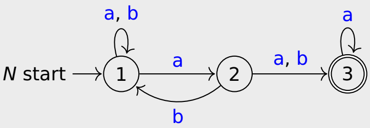
$D = \{3\}$ $\quad$ $H = \{1, 2, 3\}$

Δ	a	b
A	A	A
B	E	B

Δ	a	b

Example



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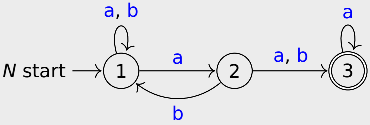
$D = \{3\} \qquad H = \{1,2,3\}$

Δ	a	b
A	A	A
B	E	B
C	D	F

Δ	a	b

Example



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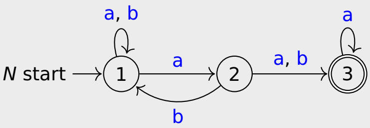
$C = \{2\}$ $G = \{2, 3\}$

$D = \{3\}$ $H = \{1, 2, 3\}$

Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b

Example



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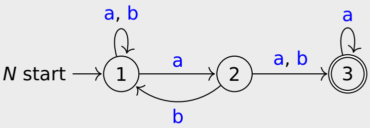
$B = \{1\}$ $F = \{1, 3\}$
$C = \{2\}$ $G = \{2, 3\}$
$D = \{3\}$ $H = \{1, 2, 3\}$

Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

Example



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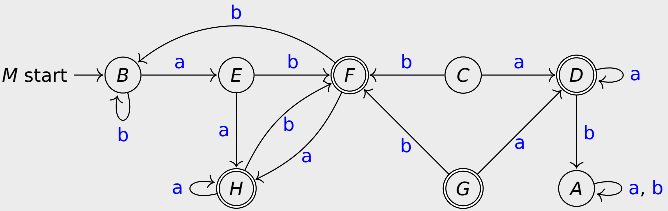
$C = \{2\}$ $\quad G = \{2, 3\}$

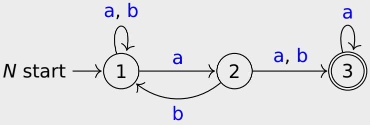
$D = \{3\}$ $\quad H = \{1, 2, 3\}$

Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F



Example



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$D = \{3\}$ $\quad$ $H = \{1, 2, 3\}$

Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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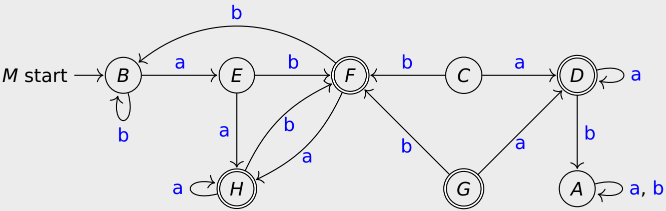
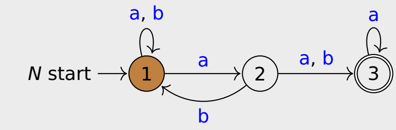


Example



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$D = \{3\}$ $\quad$ $H = \{1, 2, 3\}$

Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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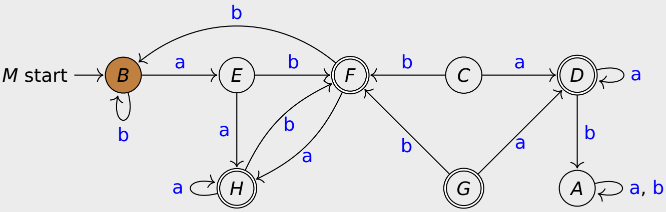
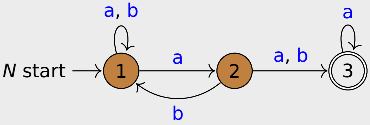


Example



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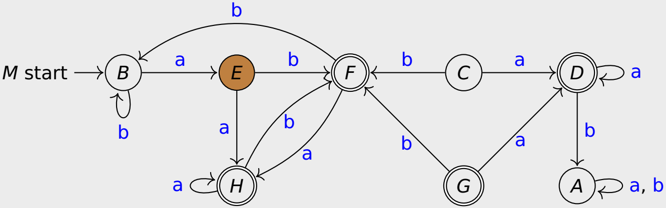
$B = \{1\}$ $\quad F = \{1, 3\}$
$C = \{2\}$ $\quad G = \{2, 3\}$
$D = \{3\}$ $\quad H = \{1, 2, 3\}$

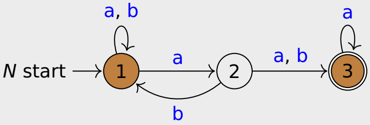
Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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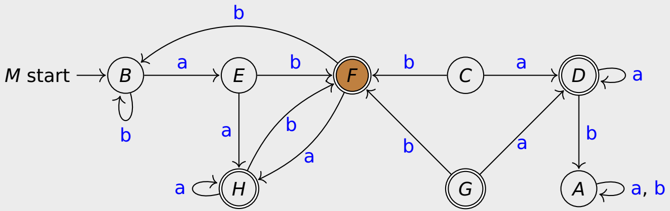
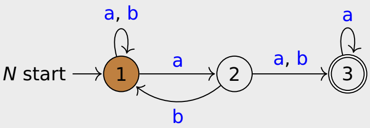


Example



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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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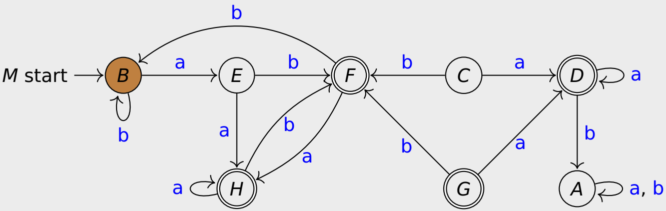
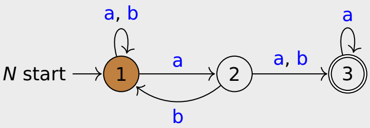


Example



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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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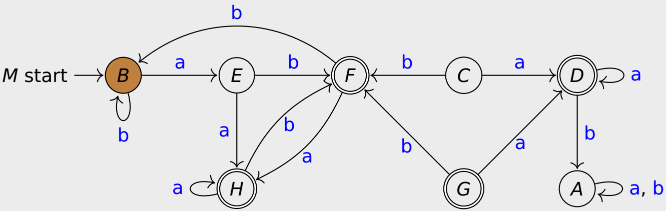
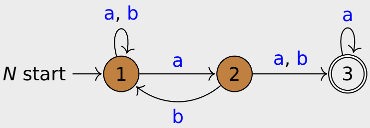


Example



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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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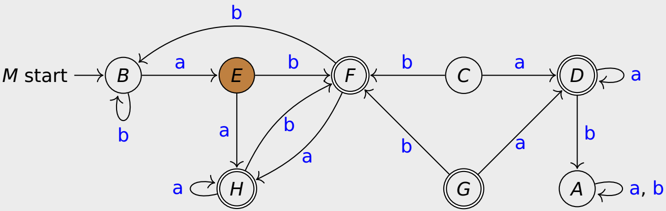
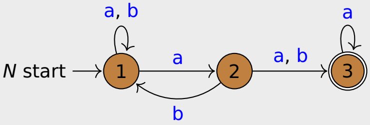


Example



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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
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F	H	B
G	D	F
H	H	F

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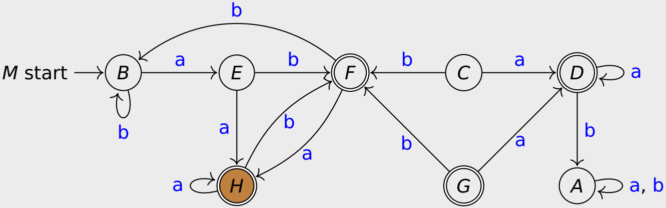
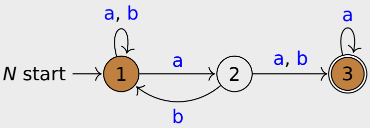


Example



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B	E	B
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H	H	F

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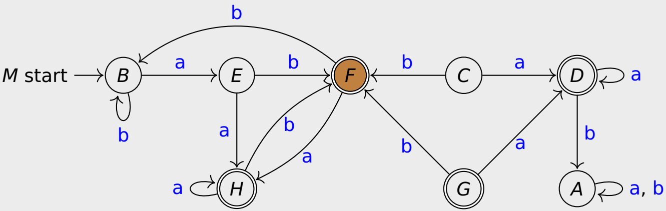

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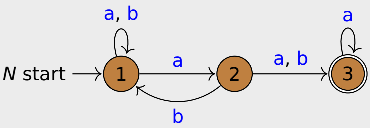

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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
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G	D	F
H	H	F

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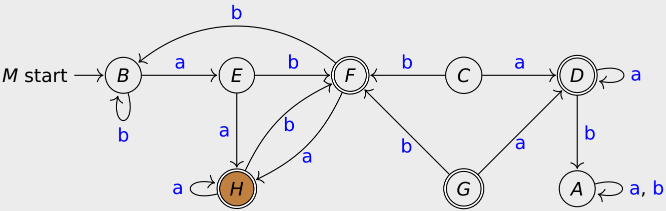

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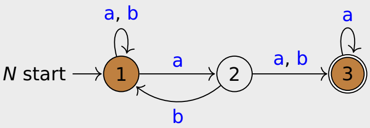

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A	A	A
B	E	B
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D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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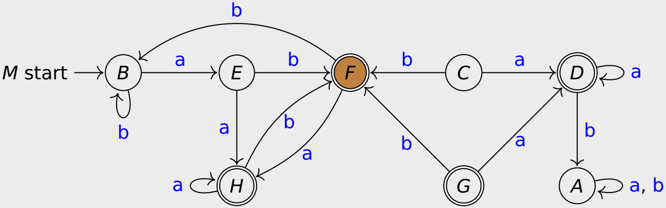

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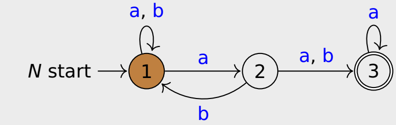

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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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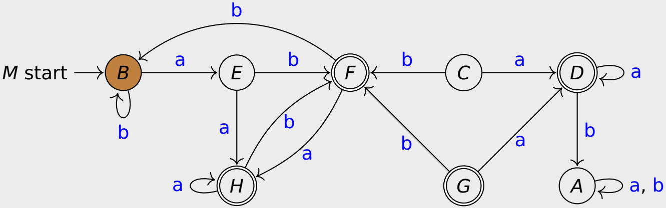

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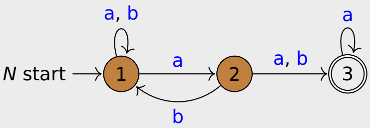

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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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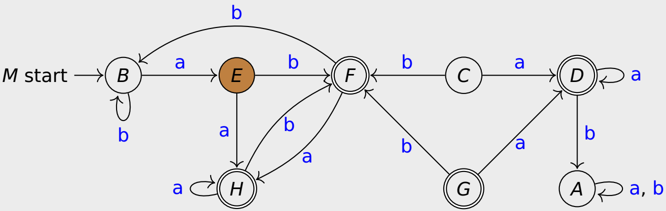

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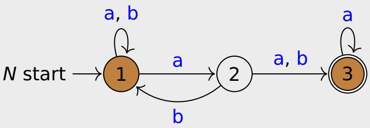

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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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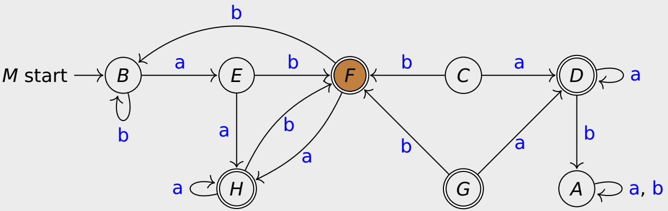

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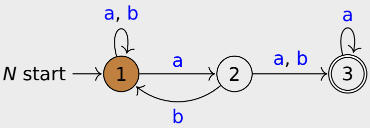

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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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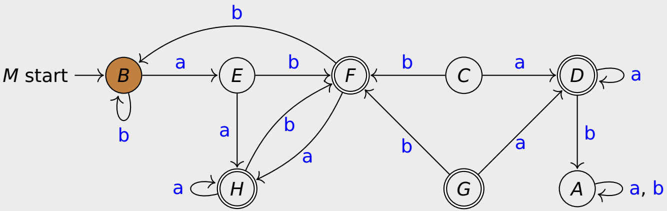

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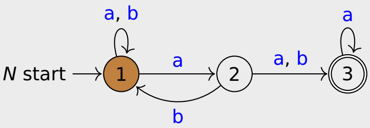

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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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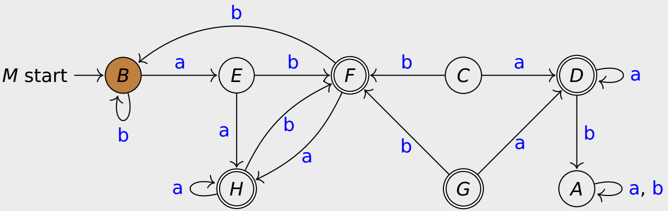
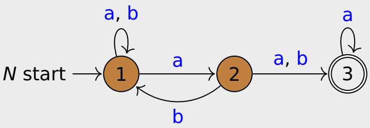


Example



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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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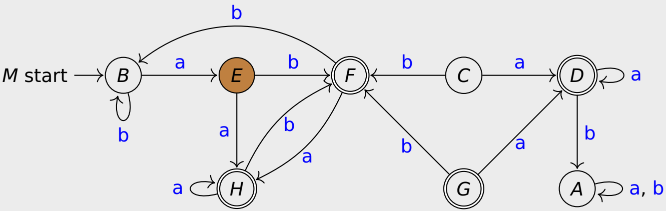

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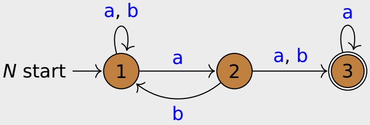

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Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

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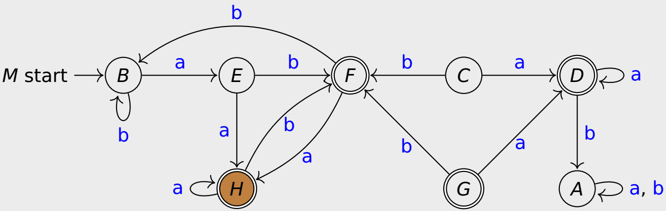
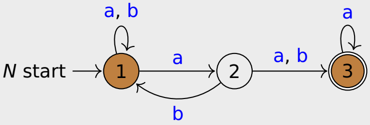


Example



$A = \emptyset$ $\quad E = \{1, 2\}$

$B = \{1\}$ $\quad F = \{1, 3\}$

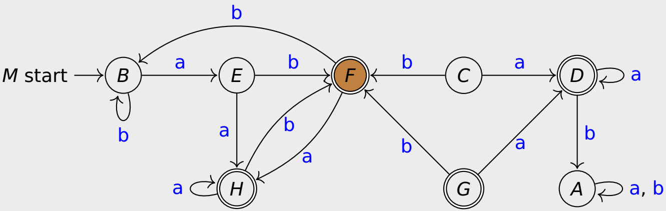
$C = \{2\}$ $\quad G = \{2, 3\}$

$D = \{3\}$ $\quad H = \{1, 2, 3\}$

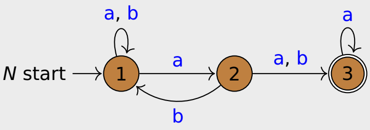
Δ	a	b
A	A	A
B	E	B
C	D	F
D	D	A

Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababba



Example



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Δ	a	b
A	A	A
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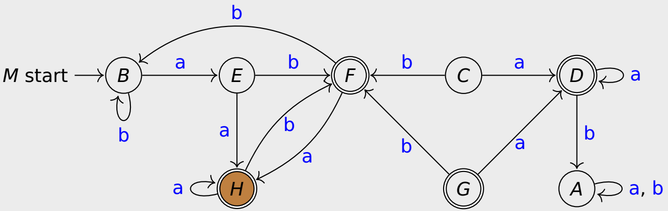
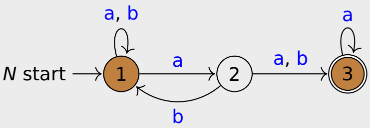
abbbaababbabbbaababba



Example



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A	A	A
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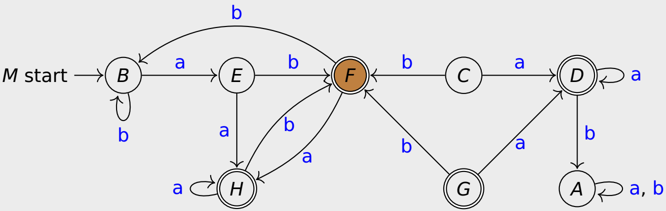
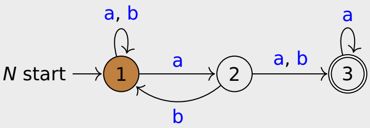
Δ	a	b
E	H	F
F	H	B
G	D	F
H	H	F

abbbaababbabbbaababbba



Example



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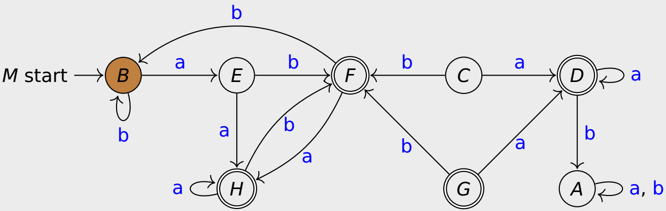
abbbaababbabbbaababba



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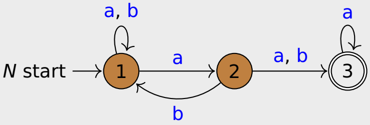

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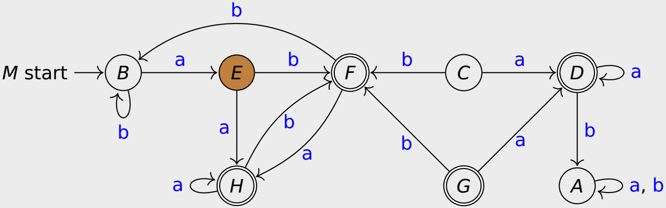
abbbaababbabbbaababba



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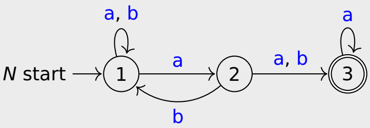

$B = \{1\}$

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Δ	a	b
B	E	B

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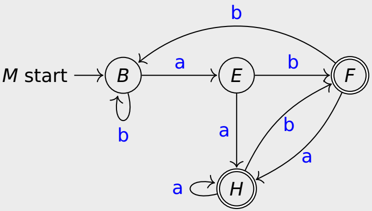
E	H	F
F	H	B
H	H	F

abbbaababbabbbaababba

remove inaccessible states



Question

Every regular set is accepted by ...

A ... an NFA having exactly one final state,

B ... a DFA having exactly one final state,

C ... an NFA having exactly one start state.

Outline

Definitions

- NFA with ε-transitions (NFA$_\varepsilon$) is sextuple $N = (Q, \Sigma, \varepsilon, \Delta, S, F)$ such that

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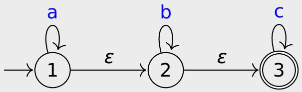

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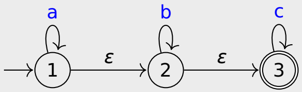

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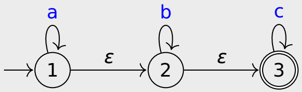

$$C_\varepsilon(\{1\}) \quad = \quad \{1, 2, 3\}$$

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$$\widehat{\Delta}_N(A, \varepsilon) = C_\varepsilon(A) \qquad \widehat{\Delta}_N(A, xa) = \bigcup \{C_\varepsilon(\Delta(q, a)) \mid q \in \widehat{\Delta}_N(A, x)\}$$

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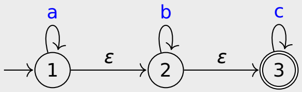

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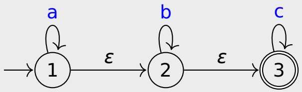

$$C_\varepsilon(\{1\}) = \{1, 2, 3\}$$
$$\widehat{\Delta}_N(\{1\}, b) =$$

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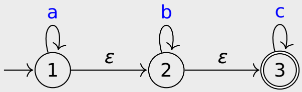

$$\begin{aligned} C_\varepsilon(\{1\}) &= \{1, 2, 3\} \\ \widehat{\Delta}_N(\{1\}, b) &= C_\varepsilon(\{1\}) \end{aligned}$$

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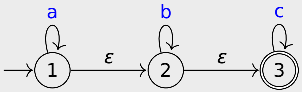

$$\begin{aligned} C_\varepsilon(\{1\}) &= \{1, 2, 3\} \\ \widehat{\Delta}_N(\{1\}, b) &= C_\varepsilon(\Delta(1, b)) \cup C_\varepsilon(\Delta(2, b)) \cup C_\varepsilon(\Delta(3, b)) \end{aligned}$$

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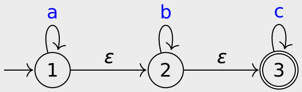

$$\begin{aligned} C_\varepsilon(\{1\}) &= \{1, 2, 3\} \\ \widehat{\Delta}_N(\{1\}, b) &= C_\varepsilon(\varnothing) \cup C_\varepsilon(\{2\}) \cup C_\varepsilon(\varnothing) \end{aligned}$$

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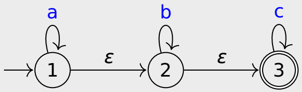

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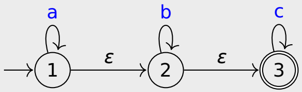

$$\begin{aligned} C_\varepsilon(\{1\}) &= \{1, 2, 3\} \\ \widehat{\Delta}_N(\{1\}, b) &= \{2, 3\} \end{aligned}$$

Example (Unfolding of the multistep function $\widehat{\Delta}_N$)

Let $x = baa$ over the alphabet $\Sigma = \{a, b\}$

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Example (Unfolding of the multistep function $\widehat{\Delta}_N$)

Let $x = baa$ over the alphabet $\Sigma = \{a, b\}$

$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, ba)\}$	1st rec. call
$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, b)\}\right\}$	2nd rec. call

Example (Unfolding of the multistep function $\widehat{\Delta}_N$)

Let $x = baa$ over the alphabet $\Sigma = \{a, b\}$

$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, ba)\}$	1^{st} rec. call
$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, b)\}\}$	2^{nd} rec. call
$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,b)) \mid q \in \widehat{\Delta}_N(A, \varepsilon)\}\}\right\}$	3^{rd} rec. call

Example (Unfolding of the multistep function $\widehat{\Delta}_N$)

Let $x = baa$ over the alphabet $\Sigma = \{a, b\}$

$$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, ba)\}$$ 1st rec. call

$$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, b)\}\right\}$$ 2nd rec. call

$$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,b)) \mid q \in \widehat{\Delta}_N(A, \varepsilon)\}\right\}\right\}$$ 3rd rec. call

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$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,b)) \mid q \in \widehat{\Delta}_N(A, \varepsilon)\}\right\}\right\}$ — 3rd rec. call

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$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in B\right\}\right\}$ — $\bigcup\{C_\varepsilon(\Delta(q,b)) \mid q \in C_\varepsilon(A)\} = B$

Example (Unfolding of the multistep function $\widehat{\Delta}_N$)

Let $x = baa$ over the alphabet $\Sigma = \{a, b\}$

$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, ba)\}$	1st rec. call
$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, b)\}\}$	2nd rec. call
$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,b)) \mid q \in \widehat{\Delta}_N(A, \varepsilon)\}\}\right\}$	3rd rec. call
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$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in C\right\}$	$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in B\} = C$

Example (Unfolding of the multistep function $\widehat{\Delta}_N$)

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$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in \widehat{\Delta}_N(A, b)\}\}$	2^{nd} rec. call
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$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in C\right\}$	$\bigcup\{C_\varepsilon(\Delta(q,a)) \mid q \in B\} = C$
D	$\bigcup\left\{C_\varepsilon(\Delta(q,a)) \mid q \in C\right\} = D$

Lemma

$C_\varepsilon(A)$ is least extension of A that is closed under ε-transitions:

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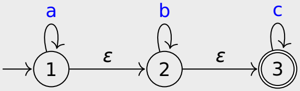
Example



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NFA$_\varepsilon$ $N_1 = (\{1,2,3\}, \{a,b,c\}, \varepsilon, \Delta_1, \{1\}, \{3\})$ with

Δ_1	a	b	c	ε
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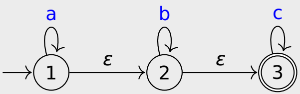

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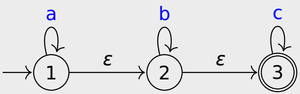

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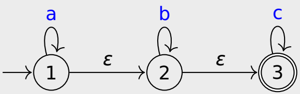

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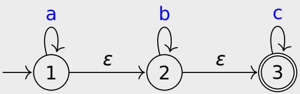

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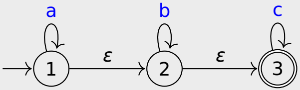

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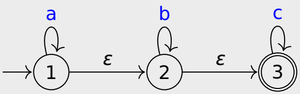

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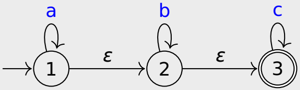

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1	$\{1,2,3\}$		

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\Delta_2(1,a) \quad = \quad \widehat{\Delta}_1(\{1\},a) &= \bigcup \{C_\varepsilon(\Delta_1(q,a)) \mid q \in \widehat{\Delta}_1(\{1\},\varepsilon)\} \\
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&= C_\varepsilon(\{1\}) \cup C_\varepsilon(\varnothing) \cup C_\varepsilon(\varnothing) \\
&= \{1,2,3\}
\end{aligned}$$

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NFA$_\varepsilon$ $N_1 = (\{1,2,3\},\{a,b,c\},\varepsilon,\Delta_1,\{1\},\{3\})$ with

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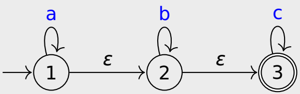

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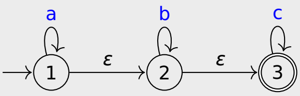

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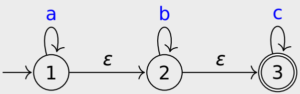

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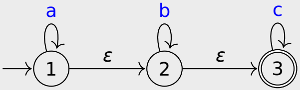

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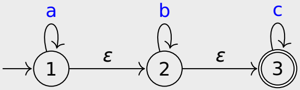

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&= C_\varepsilon(\emptyset) \cup C_\varepsilon(\{2\}) \cup C_\varepsilon(\emptyset) \\
&= \{2,3\}
\end{aligned}$$

Example (cont'd)

NFA$_\varepsilon$ $N_1 = (\{1,2,3\}, \{a,b,c\}, \varepsilon, \Delta_1, \{1\}, \{3\})$ with

-

Δ_1	a	b	c	ε
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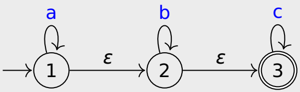

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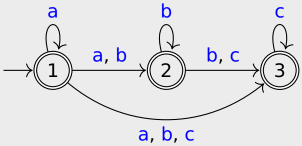

Outline

Theorem

regular sets are effectively closed under concatenation

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Proof. (by construction)

- $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
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  1. $Q \quad := \quad Q_1 \cup Q_2$

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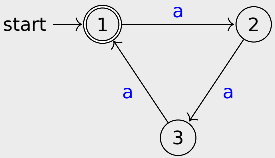
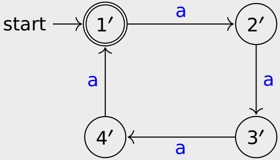
Example

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$

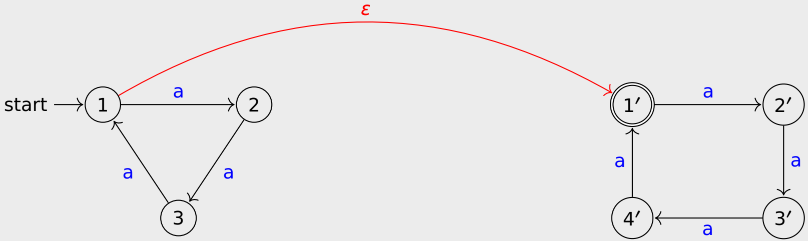


$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



Example

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$ $\{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



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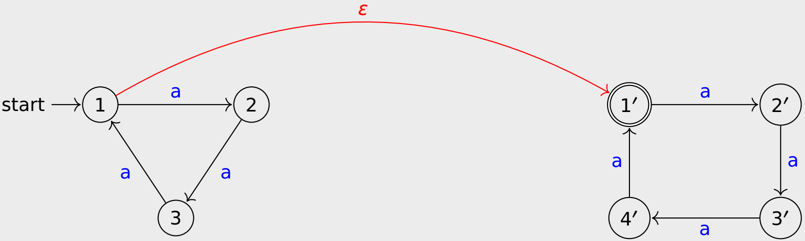

$\{x \in \{a\}^* \mid |x| \notin \{1, 2, 5\}\}$

Theorem

regular sets are effectively closed under asterate

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$$
\begin{aligned}
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&\text{3} \; F &:=\;& \{s\}\\
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4. $\Delta(q,a) := \begin{cases} \Delta_1(q,a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma \\ S_1 & \text{if } q = s \text{ and } a = \varepsilon \\ & \\ & \end{cases}$

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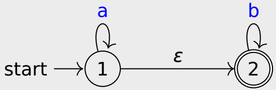
Example

$$\{a\}^*\{b\}^*$$



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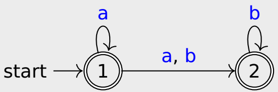
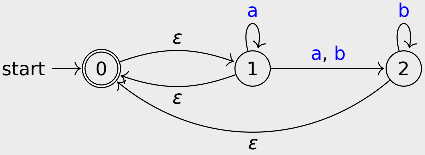


Example

$$(\ \{a\}^*\{b\}^*\ )^* = \{a,b\}^*$$



Thanks! & Questions?