Midterm Solutions (100 pts)

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Assigned : April the 10th, 13h00
Duration : 150 minutes

Q1. (50 pts) Design deterministic finite automata, over the alphabet $\Sigma = \{a, b\}$, for each of the following sets.

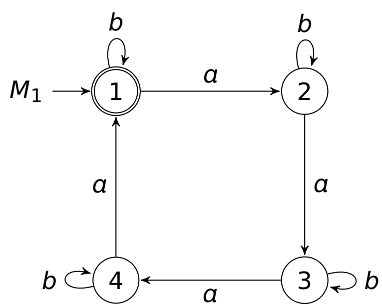
a) **(10 pts)** $\mathcal{L}_1 := \{x \in \Sigma^* \mid \#a(x) \text{ is a multiple of } 4\}$.

b) **(15 pts)** $\mathcal{L}_2 := \{a^m b^n \mid m \geqslant 0, n \geqslant 0\}$.

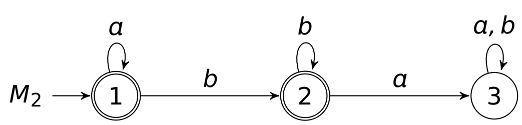
c) **(25 pts)** $\mathcal{L}_3 := \{x \in \Sigma^* \mid \#a(x) - \#b(x) \text{ is a multiple of } 2\}$.

A1.

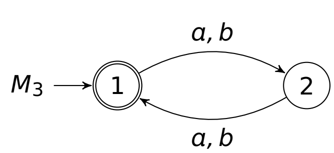
a) A transition function δ_1 of DFA $M_1 = \big(\{1, 2, 3, 4\}, \{a, b\}, \delta_1, 1, \{1\}\big)$ that recognizes the language $\mathcal{L}_1$ is pictured below.



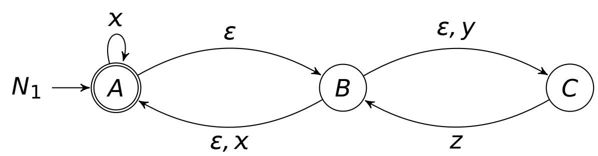
b) A transition function δ_2 of DFA $M_2 = \big(\{1, 2, 3\}, \{a, b\}, \delta_2, 1, \{1, 2\}\big)$ that recognizes the language $\mathcal{L}_2$ is pictured below.



c) A transition function δ_3 of DFA $M_3 = \big(\{1, 2\}, \{a, b\}, \delta_3, 1, \{1\}\big)$ that recognizes the language $\mathcal{L}_3$ is pictured below.



Q2. (30 pts) Given an NFA$_\varepsilon$ $N_1 = (\{A,B,C\}, \{x,y,z\}, \varepsilon, \Delta_1, \{A\}, \{A\})$ with the below state diagram



a) **(10 pts)** employ ε-elimination over N_1 to obtain an equivalent NFA $N_2 = (\{A,B,C\}, \{x,y,z\}, \Delta_2, \{A\}, F_2)$ with no ε-transitions. Clearly show intermediate steps.

b) **(10 pts)** apply subset construction algorithm to the NFA N_2 so as to get an equivalent DFA $D = (Q, \{x,y,z\}, \delta, s, F)$. Clearly show intermediate steps.

c) **(10 pts)** minimize the DFA D benefiting the marking algorithm. Justify your reasoning.

A2.

a) To start with, we compute ε-closure of below singleton sets:

$$C_\varepsilon(\{A\}) = \{A,B,C\} \qquad C_\varepsilon(\{B\}) = \{A,B,C\} \qquad C_\varepsilon(\{C\}) = \{C\}$$

We then apply ε-elimination to compute the transition function Δ_2 for the NFA N_2:

$$\begin{aligned}
\Delta_2(A,x) &= \widehat{\Delta}_1(\{A\},x) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,x)) \mid q \in \widehat{\Delta}_1(\{A\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(A,x)) \cup C_\varepsilon(\Delta_1(B,x)) \cup C_\varepsilon(\Delta_1(C,x)) \\
&= C_\varepsilon(\{A\}) \cup C_\varepsilon(\{A\}) \cup C_\varepsilon(\emptyset) \\
&= \{A,B,C\} \cup \{A,B,C\} \cup \emptyset \\
&= \{A,B,C\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(A,y) &= \widehat{\Delta}_1(\{A\},y) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,y)) \mid q \in \widehat{\Delta}_1(\{A\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(A,y)) \cup C_\varepsilon(\Delta_1(B,y)) \cup C_\varepsilon(\Delta_1(C,y)) \\
&= C_\varepsilon(\emptyset) \cup C_\varepsilon(\{C\}) \cup C_\varepsilon(\emptyset) \\
&= \emptyset \cup \{C\} \cup \emptyset \\
&= \{C\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(A,z) &= \widehat{\Delta}_1(\{A\},z) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,z)) \mid q \in \widehat{\Delta}_1(\{B\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(A,z)) \cup C_\varepsilon(\Delta_1(B,z)) \cup C_\varepsilon(\Delta_1(C,z)) \\
&= C_\varepsilon(\emptyset) \cup C_\varepsilon(\emptyset) \cup C_\varepsilon(\{B\}) \\
&= \emptyset \cup \emptyset \cup \{A,B,C\} \\
&= \{A,B,C\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(B,x) &= \widehat{\Delta}_1(\{B\},x) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,x)) \mid q \in \widehat{\Delta}_1(\{A\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(A,x)) \cup C_\varepsilon(\Delta_1(B,x)) \cup C_\varepsilon(\Delta_1(C,x)) \\
&= C_\varepsilon(\{A\}) \cup C_\varepsilon(\{A\}) \cup C_\varepsilon(\emptyset) \\
&= \{A,B,C\} \cup \{A,B,C\} \cup \emptyset \\
&= \{A,B,C\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(B,y) &= \widehat{\Delta}_1(\{B\},y) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,y)) \mid q \in \widehat{\Delta}_1(\{B\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(A,y)) \cup C_\varepsilon(\Delta_1(B,y)) \cup C_\varepsilon(\Delta_1(C,y)) \\
&= C_\varepsilon(\emptyset) \cup C_\varepsilon(\{C\}) \cup C_\varepsilon(\emptyset) \\
&= \emptyset \cup \{C\} \cup \emptyset \\
&= \{C\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(B,z) &= \widehat{\Delta}_1(\{B\},z) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,z)) \mid q \in \widehat{\Delta}_1(\{B\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(A,z)) \cup C_\varepsilon(\Delta_1(B,z)) \cup C_\varepsilon(\Delta_1(C,z)) \\
&= C_\varepsilon(\emptyset) \cup C_\varepsilon(\emptyset) \cup C_\varepsilon(\{B\}) \\
&= \emptyset \cup \emptyset \cup \{A,B,C\} \\
&= \{A,B,C\}
\end{aligned}$$

$$\begin{aligned}
\Delta_2(C,x) &= \widehat{\Delta}_1(\{C\},x) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,x)) \mid q \in \widehat{\Delta}_1(\{C\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(C,x)) \\
&= C_\varepsilon(\emptyset) \\
&= \emptyset
\end{aligned}$$

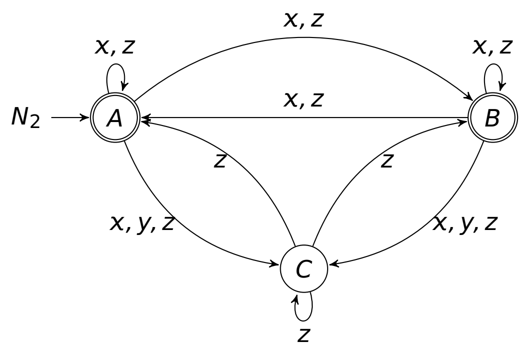
$$\begin{aligned}
\Delta_2(C,y) &= \widehat{\Delta}_1(\{C\},y) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,y)) \mid q \in \widehat{\Delta}_1(\{C\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(C,y)) \\
&= C_\varepsilon(\emptyset) \\
&= \emptyset
\end{aligned}$$

$$
\begin{aligned}
\Delta_2(C,z) &= \widehat{\Delta}_1(\{C\},z) \\
&= \bigcup\{C_\varepsilon(\Delta_1(q,z)) \mid q \in \widehat{\Delta}_1(\{C\},\varepsilon)\} \\
&= C_\varepsilon(\Delta_1(C,z)) \\
&= C_\varepsilon(\{B\}) \\
&= \{A,B,C\}
\end{aligned}
$$

The set of final states for N_2 is computed as follows:

$$F_2 \quad := \quad \{q \mid C_\varepsilon(\{q\}) \cap F_1 \neq \varnothing\} \quad = \quad \{A,B\}.$$

Therefore, the state diagram for N_2 looks like:



b) Let us now apply subset construction over the NFA N_2 to obtain an equivalent DFA $D = (Q, \{x,y,z\}, \delta, s, F)$:

$$
\begin{aligned}
\delta(\{A\},x) &= \widehat{\Delta}_2(\{A\},x) & \delta(\{A\},y) &= \widehat{\Delta}_2(\{A\},y) \\
&= \{A,B,C\} & &= \{C\} \\
\delta(\{A\},z) &= \widehat{\Delta}_2(\{A\},z) & \delta(\{C\},x) &= \widehat{\Delta}_2(\{C\},x) \\
&= \{A,B,C\} & &= \varnothing \\
\delta(\{C\},y) &= \widehat{\Delta}_2(\{C\},y) & \delta(\{C\},z) &= \widehat{\Delta}_2(\{C\},z) \\
&= \varnothing & &= \{A,B,C\} \\
\delta(\{A,B,C\},x) &= \widehat{\Delta}_2(\{A,B,C\},x) & \delta(\{A,B,C\},y) &= \widehat{\Delta}_2(\{A,B,C\},y) \\
&= \widehat{\Delta}_2(\{A\},x) \cup \widehat{\Delta}_2(\{B\},x) \cup & &= \widehat{\Delta}_2(\{A\},y) \cup \widehat{\Delta}_2(\{B\},y) \cup \\
&\quad\ \widehat{\Delta}_2(\{C\},x) & &= \widehat{\Delta}_2(\{C\},y) \\
&= \{A,B,C\} \cup \{A,B,C\} \cup \varnothing & &= \{C\} \cup \{C\} \cup \varnothing \\
&\quad\ \{A,B,C\} & &= \{C\} \\
\delta(\{A,B,C\},z) &= \widehat{\Delta}_2(\{A,B,C\},z) & \delta(\varnothing,x) &= \widehat{\Delta}_2(\varnothing,x) \\
&= \widehat{\Delta}_2(\{A\},z) \cup \widehat{\Delta}_2(\{B\},z) \cup & &= \varnothing \\
&= \widehat{\Delta}_2(\{C\},z) & & \\
&= \{A,B,C\} \cup \{A,B,C\} \cup \{A,B,C\} & & \\
&= \{A,B,C\} & & \\
\delta(\varnothing,y) &= \widehat{\Delta}_2(\varnothing,y) & \delta(\varnothing,z) &= \widehat{\Delta}_2(\varnothing,z) \\
&= \varnothing & &= \varnothing
\end{aligned}
$$

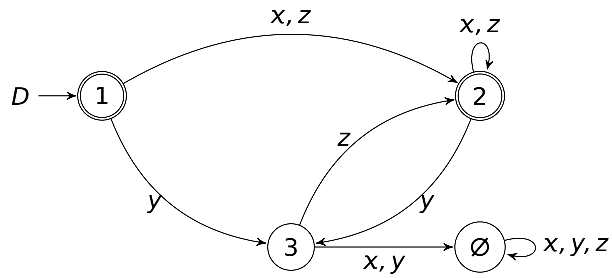
The set of final states F for the DFA D is given as follows:

$$F \quad := \quad \{A \subseteq Q_{N_2} \mid A \cap F_{N_2} \neq \varnothing\} \quad = \quad \{\{A\},\{A,B,C\}\}.$$

Obviously,

$$s \quad := \quad S_{N_2} \quad = \quad \{A\}.$$

Given all these, we depict below the state diagram for the DFA D:



where

$$1 \quad = \quad \{A\} \qquad\qquad 2 \quad = \quad \{A, B, C\} \qquad\qquad 3 \quad = \quad \{C\}.$$

c) We now check whether D is the minimal DFA with the above configuration. Observe that D has no inaccessible states. We can then employ the marking algorithm to perform the (in)distinguishability test for each pair of states.

As final and non-final states are distinguishable, we mark them in the below tabular right from the starch:

1			
	2		
$\checkmark$	$\checkmark$	3	
$\checkmark$	$\checkmark$		$\varnothing$

We then compare pairs of states in the below given order, and resume accordingly:

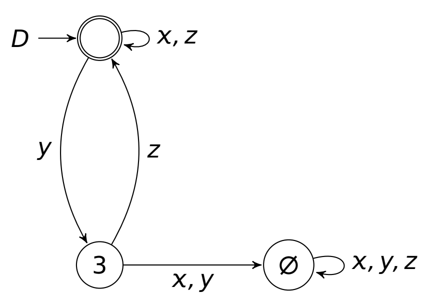
$$\{3, \varnothing\} \quad \xrightarrow{z} \quad \{2, \varnothing\} \quad \text{mark } (3, \varnothing) \quad \text{as } (2, \varnothing) \text{ is already marked}$$

1			
	2		
$\checkmark$	$\checkmark$	3	
$\checkmark$	$\checkmark$	$\checkmark$	$\varnothing$

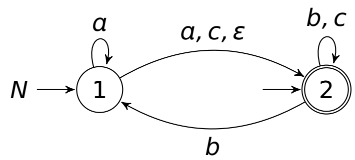
We cannot mark the pair $(1, 2)$ as the states 1 and 2 are indistinguishable:

$$\{1, 2\} \quad \xrightarrow{x} \quad \{2, 2\}$$
$$\{1, 2\} \quad \xrightarrow{y} \quad \{3, 3\}$$
$$\{1, 2\} \quad \xrightarrow{z} \quad \{2, 2\}$$

Therefore, we collapse states 1 and 2 to obtain the minimal DFA for D:



Q3. (20 pts) Given a NFA$_\varepsilon$ $N = (\{1,2\}, \{a,b,c\}, \varepsilon, \Delta, \{1,2\}, \{2\})$ with below depicted state diagram



compute the regular expression α such that $\mathcal{L}(\alpha) = \mathcal{L}(N)$ employing the algorithm (definition) given in w4.pdf, slide #18.

A3.

By specializing the theorem given in w4.pdf on slide #18, we obtain that $\mathcal{L}(N) = \alpha_{12}^{\{1,2\}} + \alpha_{22}^{\{1,2\}}$.

- The unfolding of the algorithm in computing the expression $\alpha_{12}^{\{1,2\}}$ is itemized as follows.
  1. 1st recursive call:
  $$\alpha_{12}^{\{1,2\}} \quad = \quad \alpha_{12}^{\{1\}} + \alpha_{12}^{\{1\}}(\alpha_{22}^{\{1\}})^* \alpha_{22}^{\{1\}} \qquad u = 1, \mathbf{q = 2}, v = 2$$
  2. 2nd recursive call:
  $$\begin{aligned} \alpha_{12}^{\{1\}} &= \alpha_{12}^{\varnothing} + \alpha_{11}^{\varnothing}(\alpha_{11}^{\varnothing})^* \alpha_{12}^{\varnothing} \qquad u = 1, \mathbf{q = 1}, v = 2 \\ \alpha_{22}^{\{1\}} &= \alpha_{22}^{\varnothing} + \alpha_{21}^{\varnothing}(\alpha_{11}^{\varnothing})^* \alpha_{12}^{\varnothing} \qquad u = 2, \mathbf{q = 1}, v = 2 \end{aligned}$$
  3. In the 3rd recursive call, the algorithm reaches the base case:
  $$\begin{aligned} \alpha_{12}^{\varnothing} &= \mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon} \\ \alpha_{11}^{\varnothing} &= \mathbf{a} + \boldsymbol{\varepsilon} \\ \alpha_{22}^{\varnothing} &= \mathbf{b} + \mathbf{c} + \boldsymbol{\varepsilon} \\ \alpha_{21}^{\varnothing} &= \mathbf{b} \end{aligned}$$
  4. At this stage, it folds back:
  $$\begin{aligned} \alpha_{12}^{\{1\}} &= (\mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon}) + \left[(\mathbf{a} + \boldsymbol{\varepsilon})(\mathbf{a} + \boldsymbol{\varepsilon})^*(\mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon})\right] \\ \alpha_{22}^{\{1\}} &= (\mathbf{b} + \mathbf{c} + \boldsymbol{\varepsilon}) + \left[(\mathbf{b})(\mathbf{a} + \boldsymbol{\varepsilon})^*(\mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon})\right] \end{aligned}$$

     Therefore,
  $$\begin{aligned} \alpha_{12}^{\{1,2\}} \quad = \quad & \Big(((\mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon}) + \left[(\mathbf{a} + \boldsymbol{\varepsilon})(\mathbf{a} + \boldsymbol{\varepsilon})^*(\mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon})\right]) + \\ & ((\mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon}) + \left[(\mathbf{a} + \boldsymbol{\varepsilon})(\mathbf{a} + \boldsymbol{\varepsilon})^*(\mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon})\right])((\mathbf{b} + \mathbf{c} + \boldsymbol{\varepsilon}) + \left[(\mathbf{b})(\mathbf{a} + \boldsymbol{\varepsilon})^*(\mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon})\right])^* \\ & ((\mathbf{b} + \mathbf{c} + \boldsymbol{\varepsilon}) + \left[(\mathbf{b})(\mathbf{a} + \boldsymbol{\varepsilon})^*(\mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon})\right])\Big) \end{aligned}$$
- The unfolding of the algorithm in computing the expression $\alpha_{22}^{\{1,2\}}$ is summarized in the following.
  1. 1st recursive call:
  $$\alpha_{22}^{\{1,2\}} \quad = \quad \alpha_{22}^{\{1\}} + \alpha_{22}^{\{1\}}(\alpha_{22}^{\{1\}})^* \alpha_{22}^{\{1\}} \qquad u = 2, \mathbf{q = 2}, v = 2$$
  2. 2nd recursive call:
  $$\alpha_{22}^{\{1\}} \quad = \quad \alpha_{22}^{\varnothing} + \alpha_{21}^{\varnothing}(\alpha_{11}^{\varnothing})^* \alpha_{12}^{\varnothing} \qquad u = 2, \mathbf{q = 1}, v = 2$$
  3. In the 3rd recursive call, the algorithm reaches the base case:
  $$\begin{aligned} \alpha_{12}^{\varnothing} &= \mathbf{a} + \mathbf{c} + \boldsymbol{\varepsilon} \\ \alpha_{11}^{\varnothing} &= \mathbf{a} + \boldsymbol{\varepsilon} \\ \alpha_{22}^{\varnothing} &= \mathbf{b} + \mathbf{c} + \boldsymbol{\varepsilon} \\ \alpha_{21}^{\varnothing} &= \mathbf{b} \end{aligned}$$

4. At this stage, it folds back:

$$\alpha_{22}^{\{1\}} \quad = \quad (\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right]$$

Therefore,

$$\begin{aligned}\alpha_{22}^{\{1,2\}} \quad = \quad & \Big(((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])+ \\ & ((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])^* \\ & ((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])\Big)\end{aligned}$$

- Finally,

$$\begin{aligned}\alpha_{12}^{\{1,2\}}+\alpha_{22}^{\{1,2\}} \quad = \quad & \Big(((\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{a}+\boldsymbol{\varepsilon})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])+ \\ & ((\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{a}+\boldsymbol{\varepsilon})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])^* \\ & ((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right]\Big)+ \\ & \Big(((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])+ \\ & ((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])^* \\ & ((\mathbf{b}+\mathbf{c}+\boldsymbol{\varepsilon})+\left[(\mathbf{b})(\mathbf{a}+\boldsymbol{\varepsilon})^*(\mathbf{a}+\mathbf{c}+\boldsymbol{\varepsilon})\right])\Big)\end{aligned}$$