When Koopman Meets Hamilton and Jacobi

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***Abstract*— In this paper, we establish a connection between the spectral theory of the Koopman operator and the solution of Hamilton Jacobi (HJ) equation. The HJ equation occupies central place in system theory and its solution is of interest in various control problems including optimal control, robust control, and input-output gain analysis. One of the main contributions of this paper is to show that the Lagrangian submanifolds, which are fundamental objects in the solution of the Hamilton Jacobi equation, can be obtained using the spectrum analysis of the Koopman operator. We present two different approaches for the computation of Lagrangian submanifolds. Our first approach relies on Koopman eigenfunctions to decompose the Hamiltonian dynamical system associated with the HJ equation into an integrable and non-integrable form. As a result, the integrable part of the Hamiltonian system is resolved exactly, and the non-integrable part is approximately to construct the Lagrangian submanifold. In our second approach, we rely on the Koopman-based lifting of the Hamiltonian dynamical system to approximate the stable manifold and Lagrangian submanifold of the Hamiltonian system using the principal eigenfunctions of the Koopman operator. We present convex optimization-based approach for the computation of Lagrangian submanifold. Our solution approach to HJ equation using Koopman theory provides for a natural extension of results from linear systems to nonlinear system where the solution of linearized HJ equation i.e., Riccatti solution is obtained as a special case for a linear choice of basis function used for lifting the Koopman operator. The application of this work is demonstrated for solving the optimal control and robust control problems. Finally, we present simulation results to validate the paper's main findings.**

***Index Terms*— Koopman Operator, Hamilton Jacobi Equation, Optimal and Robust Control.**

I. Introduction

Hamilton Jacobi (HJ) equation is at the heart of several problems of interest in systems and control theory. HJ equation arises in optimal control, robust H_∞ control, $\mathcal{L}_2$ gain and dissipativity-based analysis of an input-output system, and control of systems with an adversary or min-max dynamic games [1]. HJ equation and its discrete-time counterpart HJ Bellman (HJB) equation has attracted renewed attention due to the significance of these equations in data-driven control and Reinforcement learning [2], [3]. HJ equation is a nonlinear partial differential equation (PDE), and given the significance of the HJ equation in systems and control theory variety of methods are developed for the approximation of its solution [4].

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Literature review: Given the nonlinear nature of the HJ PDE, the analytical solution is not possible, and one has to resort to a numerical scheme for its approximation. In the development of numerical methods, the complexity associated with the nonlinear nature of the HJB is broken down by providing an iterative process for solving the HJB equation. The iterative approach involves solving a linear PDE for the value function with a given control input. The value function is then used to update the control input. Finally, the linear PDE is solved approximately for the value function using Galerkin-type projection scheme [5]. The iterative approach for solving SOCP via HJB equation and also Bellman equation plays a fundamental role in the variety of RL algorithms, including policy iteration, value iteration, and actor-critic method [6]. The convergence of such iterative algorithm is proven in [7]. Another line of research involving viscosity-based approximate solution to the HJ equation is proposed in [8], [9]. The viscosity-based solution is weaker than the classical, differentiable solution of the HJ equation. An approximate suboptimal solution of the HJ equation based on the series expansion of higher-order nonlinear terms is proposed in [10]–[13]. In [14], a method based on the data-driven approximation of the Koopman operator is proposed to solve the HJ equation approximately.

Differential geometric viewpoint of HJ equation and Koopman theory: An alternate approach for the analysis and approximation of the HJ equation is based on differential geometric-based interpretation of its solution. It is well known that a Hamiltonian dynamical system is associated with the HJ equation. The Lagrangian submanifolds of the Hamiltonian dynamical system, which are invariant manifolds of the Hamiltonian system, are used to construct the solution of the HJ equation. This differential geometric viewpoint to the solution of the HJ equation is prevalent in mechanics literature. This viewpoint is also exploited to provide a solution to the optimal control, H_∞ control, and $\mathcal{L}_2$ gain analysis and synthesis problems in [1]. In [15], the authors have exploited this differential geometric approach in the development of computational methods for the approximation of the HJ solution. The methods we discovered in this paper for the approximation of HJ solution draw parallels to the techniques found in [15]. In this paper, we show a strong connection between the differential geometric viewpoint of the HJ equation and the spectral analysis of the Koopman operator. The Koopman operator from the ergodic theory of dynamical system provides a linear representation of a nonlinear system by lifting the point-wise behavior of system dynamics from the state space to linear evolution

of observables or functions in functional space. While originally developed for the ensemble or statistical analysis of conservative dynamical system [16], the spectral analysis of the Koopman operator has an intimate connection with the system geometry [17]. The spectrum of the Koopman operator involving eigenfunctions and eigenvalues are natural invariant of dynamical system and reveal information about the underlying system geometry. The recent work involving the Koopman operator for dynamical systems with dissipation provides a way of characterizing stable, unstable manifolds of nonlinear dynamical systems in terms of the zero level sets of Koopman eigenfunctions [18]. The explosion of research activities in Koopman theory provides for systematic data-driven and model-based methods for the computation of Koopman eigenfunctions which can be used to compute Lagrangian submanifolds [19]–[22]. This idea is exploited to some extent in [23], where the primary focus is to prove the symplectic structure of the lifted Hamiltonian system in the function space. However, the results presented in this paper go beyond the one developed in [23] by using spectral analysis of the uncontrolled dynamical system in the solution of the HJ equation. On the other hand there is also extensive literature on the use of Koopman theory for the purpose of control [23]–[32]. However, one of the fundamental challenges with the current approaches to using Koopman theory for control is the bilinear nature of the Koopman-based lifting of control dynamical system. The bilinear lifting is one of the main hurdles in the extension of linear system tools as they inhibit the development of convex or linear methods for nonlinear control. This is in contrast to the convex framework for control design using Perron-Frobenius operator, dual to the Koopman operator [33]–[35]. The framework presented in this paper using Koopman theory to solve HJ equation provides a natural approach to extend the control design methods from a linear system to a nonlinear system with convexity at the heart of computation. The complete list of contributions are stated as follow.

Main Contributions: The main contributions of this paper are as follows. We provide two different approaches for the construction of Lagrangian submanifolds based on spectral analysis of the Koopman operator. In our first approach, we show that the Koopman eigenfunctions of the uncontrolled system can be used to decompose the Hamiltonian associated with the HJ equation into an integrable and non-integrable parts. While the integrable part of the Hamiltonian can be resolved exactly, the non-integrable part admits a structure that can be exploited in the construction of Lagrangian submanifolds. The decomposition of Hamiltonian dynamical system into integrable and non-integrable parts using integral of motion is discovered in [15]. However, unlike Koopman eigenfunctions, no systematic approach exists for computing the integrals of motion. Furthermore, the approximate Lagrangian manifold obtained using integral of motion leads to a time parameterized family of the Lagrangian manifold. The approximation of the Lagrangian submanifold is suitable only for a small time, and determining the appropriate time is a challenge. In contrast, we provide an analytical expression for the approximate Lagrangian submanifold, which involves knowledge of Koopman eigenfunctions and is time independent. Since the Koopman eigenfunctions can be computed using data without knowing system dynamics, our proposed approach is ideally suited for data-driven computation of Lagrangian submanifold. For the particular case when the Koopman eigenfunctions are approximated as a linear function of the state, we prove that Lagrangian submanifold construction matches with the solution obtained using the linearized HJ equation i.e., Riccatti equation. In our second approach for the computation of Lagrangian submanifold, we use the Koopman lifting of a Hamiltonian dynamical system associated with the HJ equation to compute the stable and unstable manifolds as zero-level sets of Koopman eigenfunctions. These eigenfunctions are used to provide least square-based optimization solution to approximate the Lagrangian submanifold. It is known that the Riccatti equation arise as the linearization of the HJ equation. We show that the solution of the Lagrangian submanifolds obtained using procedure 1 and 2 admits a decomposition where Riccatti solution arise as the coefficients of the linear basis function used in the expansion of Koopman eigenfunctions. Hence the proposed Koopman based approach for the analysis of HJ equation provides for a natural extension of linear system results to nonlinear systems. Finally, we demonstrate the application of the developed framework to optimal control and robust control design problems. The results presented in this paper are extended version of the results that will appear in [36]. In particular, the focus of [36] is restricted to Koopman spectrum and optimal control problem. In [36], we mainly present computational aspects associated with our second approach for approximating HJ solution with specific focus to optimal control problem and without the rigorous proofs on the approximation of Koopman eigenfunctions. The first approach for the approximation of HJ solution based on integrable structure of Hamiltonian system is new to this paper. Finally, the application of the developed framework to robust control problem and the comparison of the two approaches for approximating the HJ solution is new to this paper.

Organization: The organization of the paper is as follows. In Section II, we present some preliminaries on Hamiltonian dynamics, HJ equation, and spectral theory of Koopman operator. In Section IV, V, and VI we present the problem formulation and the main results on the two different procedures for the computation of Lagrangian submanifolds respectively. In Section III, we discuss a new algorithm for the computation of principal eigenfunctions of the Koopman operator. The application of the main results for optimal control and robust control are shown in Section VIII followed by simulation results and conclusions in Sections IX and X respectively.

II. Preliminaries and Notations

In this section, we present some preliminaries on Hamiltonian dynamical system, HJ equation, and spectral theory of Koopman operator. The preliminaries will also

establish connection between the Hamiltonian dynamics based symplectic geometry framework to the solution of the HJ equation. We refer the readers to [1], [18], [37], [38] for further details on the preliminaries.

Notations: $\mathbb{R}^n$ denotes the n dimensional Euclidean space. We denote by $\mathcal{L}_\infty(\mathcal{M})$, $\mathcal{C}^1(\mathcal{M})$ the space of all essentially bounded real-valued functions and continuously differentiable functions on n-dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^n$ respectively.

A. Hamiltonian dynamics and Lagrangian manifolds

Consider a n-dimensional manifold $\mathcal{M}$ with local coordinates $\mathbf{x} = (x_1, \ldots, x_n)^\top$. The cotangent bundle $T^\star\mathcal{M}$ is a $2n$-dimensional manifold, with natural local coordinates $(\mathbf{x}, \mathbf{p}) = (x_1 \ldots, x_n, p_1, \ldots, p_n)$. Given the natural coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ for $T^\star\mathcal{M}$ we may locally define the canonical two-form ω on $T^\star\mathcal{M}$ as

$$\omega = \sum_i dp_i \wedge dx_i.$$

The two-form ω is called the (canonical) symplectic form on the cotangent bundle $T^\star\mathcal{M}$.

Definition 1 (Lagrangian submanifold)**.** *An n-dimensional submanifold $\mathcal{N}$ of $T^\star\mathcal{M}$ is Lagrangian if ω restricted to N is zero.*

Now consider any C^2 function $S : \mathcal{M} \to \mathbb{R}$, and the n-dimensional submanifold $N_S \subset T^\star\mathcal{M}$, in local coordinates given as

$$\mathcal{N}_S = \{(\mathbf{x}, \mathbf{p}) \in T^\star\mathcal{M} : \ \ p_i = \frac{\partial S}{\partial x_i} \ \ i = 1, \ldots, n\}. \tag{1}$$

It immediately follows that $\mathcal{N}_S$ is Lagrangian. Given a Hamiltonian function $H : T^\star\mathcal{M} \to \mathbb{R}$, the associated Hamiltonian vector field X_H on $T^\star\mathcal{M}$ is defined in natural coordinates as

$$\dot{\mathbf{x}} = \frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}}, \ \ \dot{\mathbf{p}} = -\frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}}. \tag{2}$$

A submanifold $N \subset T^\star\mathcal{M}$ is said to be invariant under the Hamiltonian vector field X_H if any solution of X_H starting in $\mathcal{N}$ remains in N. For a given function $S : \mathcal{M} \to \mathbb{R}$ consider a Lagrangian submanifold as in Def. (1) and in local coordinates given by (1), then

$$H(\mathbf{x}, S_{\mathbf{x}}^\top(\mathbf{x})) = constant, \quad \forall \mathbf{x} \in \mathcal{M},$$

if and only if N_S is invariant submanifold of X_H. For more details on symplectic geometric aspects of Hamiltonian dynamical systems and Lagrangian manifolds refer to [38].

B. HJ equation in control theory

HJ equation is central to a variety of problems in control theory, including dissipativity theory, optimal control, robust control, and input-output $\mathcal{L}_2$ gain analysis. The HJ equation is a nonlinear partial differential equation of the form

$$\frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{1}{2}\frac{\partial V}{\partial \mathbf{x}}\mathbf{R}(\mathbf{x})\frac{\partial V}{\partial \mathbf{x}}^\top + q(\mathbf{x}) = 0. \tag{3}$$

The HJ equation is to be solved for the unknown $V(\mathbf{x})$. Let $\mathbf{f}$, $\mathbf{R}(\mathbf{x})$, and $q(\mathbf{x})$ satisfies following assumption.

Assumption 1. *We assume that $\mathbf{f} : \mathcal{M} \to \mathbb{R}^n$, $\mathbf{R} : \mathcal{M} \to \mathbb{R}^{n\times n}$, $q : \mathcal{M} \to \mathbb{R}$ are $\mathcal{C}^\infty$ functions. $\mathbf{R}(\mathbf{x})$ is a symmetric matrix for all $\mathbf{x} \in \mathcal{M}$. Furthermore, $\mathbf{f}(0) = 0, q(0) = 0$, and $\frac{\partial q}{\partial \mathbf{x}}(0) = 0$, $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0) = \mathbf{A}$, $\frac{\partial^2 q}{\partial \mathbf{x}^2}(0) = \mathbf{Q}$ where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{Q}$ is a symmetric positive definite matrix.*

There is intimate connection between the HJ equation and the Hamiltonian dynamical system. To see this connection, we define a pre-Hamiltonian as

$$H(\mathbf{x}, \mathbf{p}) = \mathbf{p}^\top\mathbf{f}(\mathbf{x}) - \frac{1}{2}\mathbf{p}^\top\mathbf{R}(\mathbf{x})\mathbf{p} + q(\mathbf{x}). \tag{4}$$

Note that the pre-Hamiltonian is obtained from the HJ equation by replacing $\frac{\partial V}{\partial \mathbf{x}}$ by $\mathbf{p}^\top$. The pre-Hamiltonian (4) can be used to construct a Hamiltonian dynamical system as in (2). In particular, we have

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \mathbf{R}(\mathbf{x})\mathbf{p} \tag{5}$$

$$\dot{\mathbf{p}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\mathbf{p} + \frac{1}{2}\left(\frac{\partial \mathbf{p}^\top\mathbf{R}(\mathbf{x})\mathbf{p}}{\partial \mathbf{x}}\right)^\top - \frac{\partial q}{\partial \mathbf{x}}^\top. \tag{6}$$

The Hamiltonian system consists of closed loop system dynamics Eq. (5) and the controller dynamics corresponding to Eq. (6). The solutions of the HJ equation (3) and the Hamiltonian system (6) are closely connected. In particular, the Lagrangian submanifold of the Hamiltonian vector field is used to construct the solution of the HJ equation. Moreover, the following notion of stabilizing solution to the HJ equation is defined [15].

Definition 2. *A solution $V(\mathbf{x})$ of the HJ equation (3) is said to be the stabilizing solution if $\mathbf{p}(0) = 0$ and 0 is an asymptotically stable equilibrium of the vector field, $\mathbf{f}(\mathbf{x}) - \mathbf{R}(\mathbf{x})\mathbf{p}(\mathbf{x})$ where $\mathbf{p}(\mathbf{x}) = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}^\top$.*

A Lagrangian submanifold corresponding to a stabilizing solution $V(\mathbf{x})$ is given by

$$\Lambda_V = \left\{(\mathbf{x}, \mathbf{p}) : \mathbf{p} = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right\} \subset T^\star\mathcal{M}. \tag{7}$$

On the other hand, the Riccatti equation can be viewed as the linearization of the nonlinear HJ equation. In particular, following Assumption 3 the nonlinear dynamics can be replaced with its linearization and it can be shown that Lagrangian submanifold admits a parameterization of the form i.e., $\mathbf{p} = \frac{1}{2}\frac{\partial \mathbf{x}^\top\mathbf{P}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{P}\mathbf{x}$ for some symmetric matrix $\mathbf{P}$. The matrix $\mathbf{P}$ is obtained as the solution of following Riccatti equation

$$\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{R}(0)\mathbf{P} + \mathbf{Q} = 0. \tag{8}$$

Just like we associate Hamiltonian dynamical system (6) with the HJ equation (3), we can associate Hamiltonian matrix $\mathcal{H}$ with Riccati equation.

$$\mathcal{H} := \begin{pmatrix} \mathbf{A} & -\mathbf{R}(0) \\ -\mathbf{Q} & -\mathbf{A}^\top \end{pmatrix} \in \mathbb{R}^{2n\times 2n}. \tag{9}$$

The Hamiltonian matrix corresponds to the linearization of the nonlinear Hamiltonian system at the origin. The necessary and sufficient conditions for the existence of a positive definite

solution to the Riccatti equation (8) is that the Hamiltonian matrix has no eigenvalues on the imaginary axis and the generalized stable eigenspace $\mathbb{E}_s$ corresponding to stable eigenvalues of $\mathcal{H}$ satisfies the following condition

$$\mathbb{E}_s \oplus \begin{pmatrix} 0 \\ I \end{pmatrix} = \mathbb{R}^{2n}. \tag{10}$$

C. Spectral theory of Koopman operator

Consider a autonomous dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \;\; \mathbf{x} \in \mathcal{M} \subset \mathbb{R}^n. \tag{11}$$

Let $\mathbf{s}_t(\mathbf{x})$ be the solution of the system at time t with initial condition $\mathbf{x}$. We have following definitions.

Definition 3 (Koopman Operator)**.** *$\mathbb{U}_t : \mathcal{L}_\infty(\mathcal{M}) \to \mathcal{L}_\infty(\mathcal{M})$ for dynamical system* (28) *is defined as*

$$[\mathbb{U}_t\psi](\mathbf{x}) = \psi(\mathbf{s}_t(\mathbf{x})), \quad \psi(\mathbf{x}) \in \mathcal{L}_\infty(\mathcal{M}). \tag{12}$$

The infinitesimal generator for the Koopman operator is given by

$$\lim_{t\to 0} \frac{(\mathbb{U}_t - I)\psi}{t} = \frac{\partial \psi}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) =: \mathcal{K}_{\mathbf{f}}\psi, \;\; t \geq 0. \tag{13}$$

Definition 4 (Eigenvalues and Eigenfunctions of Koopman)**.** *A function $\phi(\mathbf{x}) \in \mathcal{C}^1(\mathcal{M})$ is said to be eigenfunction of the Koopman operator associated with eigenvalue λ if*

$$[\mathbb{U}_t\phi](\mathbf{x}) = e^{\lambda t}\phi(\mathbf{x}). \tag{14}$$

Using the Koopman generator, the (14) can be written as

$$\frac{\partial \phi}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = \lambda\phi(\mathbf{x}). \tag{15}$$

The eigenfunctions and eigenvalues of the Koopman operator enjoys the following property [18].

Property 1. *Let ϕ_{λ_1} and ϕ_{λ_2} are the eigenfunctions of the Koopman operator associated with eigenvalues λ_1 and λ_2 respectively. If $\phi^{k_1}_{\lambda_1}\phi^{k_2}_{\lambda_2} \in \mathcal{C}^1(\mathcal{M})$, for $k_1, k_2 \in \mathbb{R}$, then it is an eigenfunctions of Koopman operator with eigenvalue $k_1\lambda_1 + k_2\lambda_2$.*

Consider a scalar valued function $g : \mathcal{M} \to \mathbb{R}$, and assume that the function g can be expanded in terms of Koopman eigenfunctions as follows.

$$g(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{N}^n}^{N} \bar{g}_{\mathbf{k}} \prod_{i=1}^{n} \phi_i^{k_i}(\mathbf{x}) \tag{16}$$

where $\mathbf{k} = (k_1, \ldots, k_n)$ and $\bar{g}_{\mathbf{k}}$ are the Koopman modes and corresponds to the projection of function $g(\mathbf{x})$ on the eigenfunctions, $\phi^{k_1}_{\lambda_1}(\mathbf{x}), \ldots, \phi^{k_n}_{\lambda_n}(\mathbf{x})$. The scalar valued function will propagate under system dynamics as follows

$$g(\mathbf{s}_t(\mathbf{x})) = [\mathbb{U}_t g](\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{N}^n}^{N} \bar{g}_{\mathbf{k}} \prod_{i=1}^{n} \phi_i^{k_i}(\mathbf{x}) e^{k_i\lambda_i t} \tag{17}$$

The spectrum of the Koopman operator, in general, could be very complicated and could consist of continuous part and discrete or point spectrum. For example, a system with a non-hyperbolic equilibrium point, such as a conservative Hamiltonian system, will have a continuous spectrum. However, it is known that the Koopman operator associated with a system having hyperbolic (i.e., the real part of eigenvalues is not equal to zero) equilibrium point is discrete. It is known that for a system with a hyperbolic equilibrium point, the eigenvalues of the linearization will also form the eigenvalues of the Koopman operator. In the following, we will refer to the eigenfunctions of the Koopman operator associated with the eigenvalues of the Jacobian at linearization as *principal eigenfunctions*. The other eigenfunctions of the Koopman operator can be constructed from the principal eigenfunctions using Property 1. The principal eigenfunctions can be used as a change of coordinates for the linear representation of a nonlinear system and draw a connection to the famous Hartman-Grobman theorem on linearization and Poincare normal form. For more details on these connections, refer to [18], [39]. The principal eigenfunctions will be defined over a subset of the state space and can be extended to open eigenfunctions over the largest set P [Corollary 5.8, [18]]. The conjugacy between the linear and nonlinear system provided by the principal eigenfunction through the Hartman-Grobman theorem is exploited in this paper to compute these eigenfunctions. This paper uses these eigenfunctions to identify stable and unstable invariant manifolds of an equilibrium point. We have the following proposition.

Proposition 1. *Let $\mathbf{x}_0$ be the hyperbolic equilibrium point of the system (28). Let $\lambda_1, \ldots, \lambda_u$ be eigenvalues with positive real part with associated eigenfunctions $\phi_{\lambda_1}, \ldots, \phi_{\lambda_u}$ and $\lambda_{u+1}, \ldots, \lambda_n$ be eigenvalues with negative real part with associated eigenfunctions $\phi_{\lambda_{u+1}}, \ldots, \phi_{\lambda_n}$. Then the joint level set of the eigenfunctions*

$$\mathcal{M}_s = \{\mathbf{x} \in M : \phi_{\lambda_1}(\mathbf{x}) = \ldots = \phi_u(\mathbf{x}) = 0\}, \tag{18}$$

forms the stable manifold and the joint level set of the eigenfunctions

$$\mathcal{M}_u = \{\mathbf{x} \in M : \phi_{\lambda_{u+1}}(\mathbf{x}) = \ldots = \phi_n(\mathbf{x}) = 0\}, \tag{19}$$

is the unstable manifold of the equilibrium point x_0.

Local eigenfunctions can also be defined where (14) is valid on a particular region of the state space and over a finite time interval. It is easy to prove the following statement about the Koopman eigenfunction of the linear system [18].

Remark 1. *The Koopman eigenfunctions corresponding to linear dynamical system, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, are given by $\phi_{\lambda_i}(\mathbf{x}) = \mathbf{v}_j^\top \mathbf{x}$, where $\mathbf{v}_j^\top$ is the left eigenvector of $\mathbf{A}$ with eigenvalue λ_j i.e., $\mathbf{v}_j^\top \mathbf{A} = \lambda_j \mathbf{v}_j^\top$.*

III. Computation of Principal Eigenfunctions of Koopman Operator

One of the main contributions of this paper is in providing two different procedures for computing the Lagrangian submanifolds. Both these procedures rely on the computation of principal eigenfunctions of the Koopman operator. We effectively reduce the problem of solving HJ equation to the computation of principal eigenfunctions. There are several approaches available for the computation of Koopman operator

including results that provide sample complexity-based error bounds in the computation [19]–[22]. Modification of these algorithms can be proposed for the computation of principal eigenfunctions. In the following, we present computation framework based on the solution of linear equations for the computation of Koopman eigenfunctions. The approach does not rely on the explicit computation of Koopman operator itself. Consider a dynamical system

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}), \;\; \mathbf{z} \in \mathcal{Z} \subseteq \mathbb{R}^m. \tag{20}$$

We make following assumption on the vector field.

Assumption 2. *We assume that the vector field $\mathbf{F} : \mathcal{Z} \to \mathbb{R}^m$ is atleast $\mathcal{C}^2$ function of $\mathbf{z}$. Furthermore, $\mathbf{F}(0) = 0$ and $\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(0) = \mathbf{E}$ where $\mathbf{E}$ is assumed to be hyperbolic i.e., no eigenvalues on the $j\omega$ axis.*

We propose a modification of the approach presented in [40] for the computation of principal eigenfunctions. Furthermore, while the approach developed in [40] relies on Deep Neural Network for its computation, we propose a convex optimization-based formulation for the computation of principal eigenfunctions.

Using the results from [39] (Theorem 2.3, Corollary 2.1, and Corollary 2.3), it follows that under Assumption 2, Koopman operator admits an eigenfunction with eigenvalue λ of the form

$$\phi_\lambda(\mathbf{z}) = \bar{\mathbf{w}}^\top \mathbf{z} + H(\mathbf{z}) \tag{21}$$

where $\lambda \in \mathbb{C}$ is the eigenvalue of $\mathbf{E}$ and $\bar{\mathbf{w}}$ is the left eigenvector of the matrix $\mathbf{E}$ i.e., $\bar{\mathbf{w}}^\top \mathbf{E} = \lambda \bar{\mathbf{w}}^\top$. The mapping H is $\mathcal{C}^1$ diffeomorphism and is purely nonlinear i.e., $H(\mathbf{z}) \sim O(|\mathbf{z}|^2)$. For system with stable equilibrium i.e., all the eigenvalues of $\mathbf{E}$ with real part negative, the mapping H is defined in the domain of attraction of the equilibrium at the origin. For system with saddle equilibrium point, the mapping H is defined on the positive (negative) invariant region that is homeomorphic to stable (unstable) manifold of the equilibrium. Since the system model is assumed to be known, one can compute the eigenvalues and the associated left eigenvector of $\mathbf{E}$ and hence we only need to determine the nonlinear term . In the following we provide conditions for the finite dimensional approximation of the nonlinear term $H(\mathbf{z})$ of the eigenfunction corresponding to eigenvalue λ.

Proposition 2. *Let $\bar{\Gamma}_M(\mathbf{z}) = (\bar{\gamma}_1(\mathbf{z}), \ldots, \bar{\gamma}_M(\mathbf{z}))^\top$ be the finite dimensional basis function set with $\bar{\gamma}_k : \mathcal{Z} \to \mathbb{C}$ and $\frac{\partial \bar{\gamma}_k}{\partial \mathbf{z}}(0) = 0$ for $k = 1, \ldots, M$. Let $\mathcal{F}_M = \text{span}\{\bar{\gamma}_1, \ldots, \bar{\gamma}_M\}$ and $\mathcal{F} = \mathcal{L}_2(\mu)$ for some positive measure μ on $\mathcal{Z}$ with inner product $\langle f, g \rangle_{\mathcal{L}_2(\mu)} = \int_{\mathcal{Z}} f(\mathbf{z}) g(\mathbf{z}) d\mu(\mathbf{z})$. The optimal Galkerian projection for the nonlinear term of the eigenfunction i.e., $H(\mathbf{z})$ corresponding to eigenvalue λ is given by $\bar{\Gamma}_M(\mathbf{z})^\top \bar{\mathbf{u}}_M^\star$, where $\bar{\mathbf{u}}_N^\star$ is obtained as the solution of following linear equations.*

$$\left\langle \mathbf{F}^\top \frac{\partial \bar{\Gamma}_M}{\partial \mathbf{z}}^\top - \lambda \bar{\Gamma}_M^\top, \bar{\gamma}_k \right\rangle_{\mathcal{L}_2(\mu)} \bar{\mathbf{u}}_M \tag{22}$$

$$= -\left\langle \bar{\mathbf{w}}^\top \mathbf{F}_n, \bar{\gamma}_k \right\rangle_{\mathcal{L}_2(\mu)}, \;\; k = 1, \ldots, M \tag{23}$$

Proof: The eigenfunction $\phi_\lambda(\mathbf{z})$ and in particular, $H(\mathbf{z})$ corresponding to eigenvalue λ from (21) satisfies

$$\frac{\partial \phi_\lambda}{\partial \mathbf{z}} \mathbf{F}(\mathbf{z}) - \lambda \phi_\lambda(\mathbf{z}) = 0 \tag{24}$$

which implies

$$\bar{\mathbf{w}}^\top \mathbf{F}_n(\mathbf{z}) + \frac{\partial H}{\partial \mathbf{z}} \mathbf{F}(\mathbf{z}) - \lambda H(\mathbf{z}) = 0 \tag{25}$$

We seek to find a finite dimensional approximation of $H(\mathbf{z})$ using the finite basis function $\bar{\Gamma}_N(\mathbf{z})$ as

$$H(\mathbf{z}) \approx \sum_k^M \bar{u}_k \bar{\gamma}_k(\mathbf{z}) = \bar{\Gamma}_M^\top \bar{\mathbf{u}}_M$$

Substituting the above in (25), we can write the approximation error as

$$\text{error} = \left(\frac{\partial \bar{\Gamma}_M}{\partial \mathbf{z}} \mathbf{F} - \lambda \bar{\Gamma}_M(\mathbf{z}) \right)^\top \bar{\mathbf{u}}_M + \bar{\mathbf{w}}^\top \mathbf{F}_n(\mathbf{z}) \tag{26}$$

The objective is to determine the coefficient vector $\bar{\mathbf{u}}_N$ such that the projection of the error on the finite basis $\{\bar{\gamma}_k\}_{k=1}^N$ is equal to zero for all $\mathbf{z}$. Hence, we obtain

$$\left\langle \left(\frac{\partial \bar{\Gamma}_M}{\partial \mathbf{z}} \mathbf{F} - \lambda \bar{\Gamma}_M(\mathbf{z}) \right)^\top, \bar{\gamma}_k \right\rangle \bar{\mathbf{u}}_M = -\left\langle \mathbf{w}^\top \mathbf{F}_n(\mathbf{z}), \bar{\gamma}_k \right\rangle$$

where the inner product are in $\mathcal{L}_2(\mu)$.

□

In general computing the integral and inner product in $\mathcal{L}_2(\mu)$ would be computationally expensive. Instead, the inner product can be replaced by dot product using empirical measure. Let $\{\mathbf{z}_1, \ldots, \mathbf{z}_L\}$ be the finite data points uniformly sampled from $\mathbf{Z}$ with $\hat{\mu}_L$ the associated empirical measure i.e., $\hat{\mu}_L = \frac{1}{L} \sum_k^L \delta_{\mathbf{z}_k}$. We then have $\int_{\mathcal{Z}} f(\mathbf{z}) d\hat{\mu}_M = \frac{1}{L} \sum_k^L f(\mathbf{z}_k)$. With the use of this empirical measure, the equation (23) can be written to determine the approximate coefficient vector $\bar{\mathbf{u}}_N^M$ as the solution of finite linear equations

$$\underbrace{\frac{1}{M} \sum_{k=1}^L \bar{\Gamma}_M(\mathbf{z}_k) \left(\frac{\partial \bar{\Gamma}_N}{\partial \mathbf{z}}(\mathbf{z}_k) \mathbf{F}(\mathbf{z}_k) - \lambda \bar{\Gamma}(\mathbf{z}_k) \right)^\dagger}_{\mathcal{A}_M^L} \bar{\mathbf{u}}_M^L$$

$$= \underbrace{-\frac{1}{L} \sum_{k=1}^L \mathbf{w}^\top \mathbf{F}(\mathbf{z}_k) \bar{\Gamma}_M(\mathbf{z}_k)}_{b_M^L} \tag{27}$$

The solvalablity of these equation depends upon the invertibility of the matrix $\mathcal{A}_M^L$. Let $\phi_\lambda^{ML}(\mathbf{z}) = \mathbf{w}^\top \mathbf{z} + \bar{\Gamma}_M^\top(\mathbf{z}) (\bar{\mathbf{u}}_M^L)^\star$ and $\phi_\lambda^M(\mathbf{z}) = \mathbf{w}^\top \mathbf{z} + \bar{\Gamma}_M^\top(\mathbf{z}) \bar{\mathbf{u}}_M^\star$, where $\bar{\mathbf{u}}_M^\star$ and $(\bar{\mathbf{u}}_M^L)^\star$ are the solutions of (23) and (27) respectively. Then using proof techniques similar to the one introduced in [21] it can be shown that $\lim_{L \to \infty} \|\phi_\lambda^{ML} - \phi_\lambda^M\| = 0$ with probability one. In this paper, we use small modification of Eq. (27) to compute the finite dimensional approximation of Koopman eigenfunction. These modifications are introduced to deal with the complex eigenvalues while working with real basis functions. Given the linear structure of the problem it is also possible to derive sample complexity-based error bounds

provide error bounds with finite data. This will be the focus of future publication.

IV. Problem Set-up

We consider a autonomous system and the associated HJ equation (3) written in terms of pre-Hamiltonian obtained from HJ equation (3) by replacing $\mathbf{p}$ with $\frac{\partial V}{\partial \mathbf{x}}$ as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{M} \subseteq \mathbb{R}^n, \tag{28}$$

$$H(\mathbf{x}, \mathbf{p}) := \mathbf{p}^\top \mathbf{f}(\mathbf{x}) - \mathbf{p}^\top \mathbf{R}(\mathbf{x})\mathbf{p} + q(\mathbf{x}) = 0. \tag{29}$$

The HJ equation arise in various control problem including optimal and robust control problems (Section II-B). The control system associated with (28) and (29) is written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \mathbf{R}(\mathbf{x})\mathbf{p}. \tag{30}$$

The HJ equation is to be solved for unknown $V : \mathcal{M} \to \mathbb{R}$ and with the substitution of $\mathbf{p} = \frac{\partial V}{\partial \mathbf{x}}$ in (30) we obtain feedback control system. Following assumption is made on $\mathbf{f}$, $\mathbf{R}(\mathbf{x})$, and $q(\mathbf{x})$ in the rest of the paper.

Assumption 3. *We assume that $\mathbf{f} : \mathcal{M} \to \mathbb{R}^n$, $\mathbf{R} : \mathcal{M} \to \mathbb{R}^{n\times n}$, $q : \mathcal{M} \to \mathbb{R}$ are $\mathcal{C}^\infty$ functions. $\mathbf{R}(\mathbf{x})$ is a symmetric matrix for all $x \in M$. Furthermore, $\mathbf{f}(0) = 0, q(0) = 0$, and $\frac{\partial q}{\partial \mathbf{x}}(0) = 0$, $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0) = \mathbf{A}$, $\frac{\partial^2 q}{\partial \mathbf{x}^2}(0) = \mathbf{Q}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{Q}$ is a symmetric positive definite matrix. We assume that all the eigenvalues of $\mathbf{A}$ are distinct and not on the imaginary axis. Furthermore, λ_i for $i = 1, \ldots, n$ are assumed to be the eigenvalues of $\mathbf{A}$ and matrix Λ is a block diagonal matrix consisting of eigenvalues of $\mathbf{A}$ in real Jordan form.*

The assumption of hyperbolic equilibrium point ensures that the Koopman operator has a discrete spectrum as it is known that systems with non-hyperbolic equilibrium points can have a continuous spectrum. Therefore, we make the following assumption on the spectrum of the Koopman operator associated with the system (28).

Assumption 4. *Let λ_i be the eigenvalues of the linearization of $\mathbf{f}(\mathbf{x})$ i.e., $\mathbf{A}$ matrix and $\phi_{\lambda_i}(\mathbf{x}) \in \mathcal{C}^1(\mathcal{M})$ for $i = 1, \ldots, n$ are the associated principal eigenfunctions of the Koopman operator for the system (28) i.e.,*

$$\left(\frac{\partial \phi_{\lambda_i}}{\partial \mathbf{x}}\right)\mathbf{f}(\mathbf{x}) = \lambda_i \phi_{\lambda_i}(\mathbf{x}). \tag{31}$$

The next two sections present two different approaches for solving the HJ equation via spectral analysis of the Koopman operator.

V. Koopman Eigenfunctions and Integrable Structure of HJ Equation

Consider the pre-Hamiltonian in Eq. (29) written as sum of nominal Hamiltonian, H_0, and its perturbation, H_1,

$$H(\mathbf{x}, \mathbf{p}) = H_0(\mathbf{x}, \mathbf{p}) + H_1(\mathbf{x}, \mathbf{p}), \tag{32}$$

where

$$H_0(\mathbf{x}, \mathbf{p}) = \mathbf{p}^\top \mathbf{f}(\mathbf{x}), \quad H_1(\mathbf{x}, \mathbf{p}) = -\mathbf{p}^\top \mathbf{R}(\mathbf{x})\mathbf{p} + q(\mathbf{x}).$$

The nominal Hamiltonian arise from the uncontrolled part and the perturbation term is due to the control term. Consider the Hamiltonian system constructed using the nominal Hamiltonian $H_0(\mathbf{x}, \mathbf{p}) = \mathbf{p}^\top \mathbf{f}(\mathbf{x})$ and denoted by $\mathcal{X}_{H_0}$

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\partial H_0(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} = \mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{p}} &= -\frac{\partial H_0(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\top \mathbf{p} \end{aligned} \tag{33}$$

We show that the Koopman eigenfunctions corresponding to the dynamical system (28) can be used to provide integrable structure for the Hamiltonian system (33).

Proposition 3. *Let $\{\phi_{\lambda_k}(\mathbf{x}), \lambda_k\}$ for $k = 1, \ldots, n$ be the eigenfunctions eigenvalues pair of the Koopman operator corresponding to the system (28) (Assumption 4). Then, $\{\phi_{\lambda_k}(\mathbf{x}), \lambda_k\}$ and $\{H_0(\mathbf{x}, \mathbf{p}), 0\}$ are the eigenfunctions eigenvalues pair of the Koopman operator associated with Hamiltonian system (33).*

Proof: The Koopman generator corresponding to the lifted Hamiltonian system (33) is given by

$$\mathcal{K}_{\mathcal{X}_{H_0}} \psi(\mathbf{x}, \mathbf{p}) = \frac{\partial \psi}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) - \frac{\partial \psi}{\partial \mathbf{p}} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\top \mathbf{p}. \tag{34}$$

The fact that $\{\phi_k, \lambda_k\}$ for $k = 1, \ldots, n$ form the eigenfunctions, eigenvalues pair of the Koopman generator corresponding to system (33) follows from the fact that $\phi_{\lambda_i}(\mathbf{x})$ are not function of $\mathbf{p}$ and hence following Assumption 4 satisfies

$$\mathcal{K}_{X_{H_0}} \phi_{\lambda_i} = \lambda_i \phi_{\lambda_i}.$$

Similarly, we have

$$\begin{aligned} \mathcal{K}_{X_{H_0}} H_0 &= \left(\frac{\partial H_0}{\partial \mathbf{x}}\right)\mathbf{f} - \left(\frac{\partial H_0}{\partial \mathbf{p}}\right)\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\top \mathbf{p} \\ &= \mathbf{p}^\top \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\mathbf{f} - \mathbf{f}^\top \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\top \mathbf{p} = 0. \end{aligned} \tag{35}$$

□

The Hamiltonian Jacobi equation associated with (33) is given by

$$H_0\left(\mathbf{x}, \frac{\partial W}{\partial \mathbf{x}}\right) + \frac{\partial W}{\partial t} = 0. \tag{36}$$

Proposition 4. *Let the initial state of the HJ equation (36) $W_0(\mathbf{x}) := W(\mathbf{x}, 0)$ admits the following decomposition in Koopman eigenfunctions associated with system (28)*

$$W_0(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{N}^n} \bar{w}_{\mathbf{k}} \prod_{i=1}^{N} \phi_{\lambda_i}^{k_i}(\mathbf{x}) \tag{37}$$

where $\mathbf{k} = (k_1, \ldots, k_n)$ and $\bar{w}_{\mathbf{k}}$ are the Koopman modes associated with eigenfunctions $\phi_{\lambda_1}^{k_1}(\mathbf{x}) \ldots \phi_{\lambda_n}^{k_n}(\mathbf{x})$. Then, the solution, $W(\mathbf{x}, t)$, of the HJ (36) can be expressed as

$$W(\mathbf{x}, t) = \sum_{\mathbf{k}\in\mathbb{N}^n} \bar{w}_{\mathbf{k}} \prod_{i=1}^{N} \phi_{\lambda_i}^{k_i}(\mathbf{x}) e^{-k_i \lambda_i t} \tag{38}$$

,

Proof: The HJ equation (36) can be written as

$$\left(\frac{\partial W}{\partial \mathbf{x}}\right)\mathbf{f}(\mathbf{x}) + \frac{\partial W}{\partial t} = 0. \tag{39}$$

This linear partial differential equation (PDE) in terms of variable W can be written as

$$\frac{\partial W}{\partial t} = -\left(\frac{\partial W}{\partial \mathbf{x}}\right)\mathbf{f}(\mathbf{x}). \tag{40}$$

Above is a Koopman PDE generated by vector field $\dot{\mathbf{x}} = -\mathbf{f}(\mathbf{x})$ [37]. Hence the solution of the PDE can be written as

$$W(\mathbf{x}, t) = W_0(\mathbf{s}_{-t}(\mathbf{x})), \tag{41}$$

where $\mathbf{s}_{-t}(\mathbf{x})$ is the solution of differential equation $\dot{\mathbf{x}} = -\mathbf{f}(\mathbf{x})$ (i.e., time reversed flow of vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$). Above solution can also be written in terms of the Koopman operator as

$$W(\mathbf{x}, t) = W_0(\mathbf{s}_{-t}(\mathbf{x})) = [\mathbb{U}_{-t} W_0](\mathbf{x}), \tag{42}$$

where $\mathbb{U}_{-t}$ is the Koopman operator associated with $\dot{\mathbf{x}} = -\mathbf{f}(\mathbf{x})$. Since $\{\phi_{\lambda_k}(\mathbf{x}), \lambda_k\}$ are the principal eigenfunctions, eigenvalues pair of the Koopman operator corresponding to system (28), it follows that $\{\phi_{\lambda_k}, -\lambda_k\}$ are the principal eigenfunctions and eigenvalues pair of the Koopman operator corresponding to the system $\dot{\mathbf{x}} = -\mathbf{f}(\mathbf{x})$ as following (31) we obtain

$$\frac{\partial \phi_{\lambda_k}}{\partial \mathbf{x}}(-\mathbf{f}(\mathbf{x})) = -\lambda_k \phi_k(\mathbf{x}).$$

Since, $W_0(\mathbf{x})$ admits the decomposition (37), it follows that

$$W(\mathbf{x}, t) = [\mathbb{U}_{-t} W_0](\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}^n} \bar{w}_{\mathbf{k}} \prod_{i=1}^{N} \phi_{\lambda_i}^{k_i}(\mathbf{x}) e^{-k_i \lambda_i t} \tag{43}$$

□

We can restrict the initial condition of the HJ equality to be linear combinations of the principal eigenfunctions of the Koopman operator associated with (28) i.e.,

$$W_0(\mathbf{x}) = \sum_{j=1}^{n} P_j \phi_{\lambda_j}(\mathbf{x}). \tag{44}$$

The above expansion of the initial state corresponds to the Koopman modes $\bar{w}_{0,\ldots,1_j,\ldots,0} = P_j$ with higher order modes being equal to zero. Using the results of Proposition 4 it follows that

$$W(\mathbf{x}, t) = \sum_{j=1}^{n} P_j \phi_{\lambda_j}(\mathbf{x}) e^{-\lambda_j t} =: W(\mathbf{x}, P, t).$$

Define a canonical coordinates,

$$p_j = \frac{\partial W}{\partial x_j}, \;\; X_j = \frac{\partial W}{\partial P_j}. \tag{45}$$

Then we obtain

$$p_j(t) = \sum_{k=1}^{n} P_k \frac{\partial \phi_{\lambda_k}(\mathbf{x})}{\partial x_j} e^{-\lambda_k t}$$

$$X_j(t) = \phi_{\lambda_j}(\mathbf{x}) e^{-\lambda_j t},$$

Note that

$$\dot{X}_j = \frac{\partial \phi_{\lambda_j}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) e^{-\lambda_j t} - \lambda_j e^{-\lambda_j t}\phi_{\lambda_j}(\mathbf{x}) = 0, \tag{46}$$

where we have use the fact that ϕ_j is the eigenfunction with eigenvalue λ_j i.e.,

$$\frac{\partial \phi_{\lambda_j}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = \lambda_j \phi_{\lambda_j}(\mathbf{x}). \tag{47}$$

With some abuse of notation we write the above in compact form as

$$\mathbf{X} = e^{-\Lambda t}\Phi(\mathbf{x}) \tag{48}$$

$$\mathbf{x}(\mathbf{X}, t) = \Phi^{-1}(e^{\Lambda t}\mathbf{X}), \;\; \mathbf{p}(\mathbf{X}, \mathbf{P}, t) = \left(\frac{\partial \Phi}{\partial \mathbf{x}}\right)^{\top} e^{-\Lambda^{\top} t}\mathbf{P}, \tag{49}$$

where Λ is the block diagonal matrix in real Jordan canonical form and $\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \ldots, \phi_n(\mathbf{x}))^{\top}$ is the vector of real eigenfunctions. Following (46) and using the fact that P_j are constant coefficient used in the expansion of W_0, we obtain

$$\dot{\mathbf{X}} = 0, \;\; \dot{\mathbf{P}} = 0.$$

We can write the perturbed Hamiltonian $H_1(\mathbf{x}, \mathbf{p})$ in terms of the new variables $(\mathbf{X}, \mathbf{P})$ as

$$H_1(\mathbf{x}, \mathbf{p}) = H_1(\mathbf{x}(\mathbf{X}, t), \mathbf{p}(\mathbf{X}, \mathbf{P}, t)) =: \bar{H}_1(\mathbf{X}, \mathbf{P}, t). \tag{50}$$

$$\bar{H}_1(\mathbf{X}, \mathbf{P}, t) = -\frac{1}{2}\mathbf{P}^{\top} e^{-\Lambda t}\left(\frac{\partial \Phi}{\partial \mathbf{x}}\right) R(\mathbf{x}) \left(\frac{\partial \Phi}{\partial \mathbf{x}}\right)^{\top} e^{-\Lambda^{\top} t}\mathbf{P} + q(\Phi^{-1}(e^{\Lambda t}\mathbf{X})). \tag{51}$$

The Hamiltonian system is given by

$$\dot{\mathbf{X}} = \frac{\partial \bar{H}_1}{\partial \mathbf{P}} = -e^{-\Lambda t}\left(\frac{\partial \Phi}{\partial \mathbf{x}}\right) \mathbf{R}(\Phi^{-1}(e^{\Lambda t}\mathbf{X})) \left(\frac{\partial \Phi}{\partial \mathbf{x}}\right)^{\top} e^{-\Lambda^{\top} t}\mathbf{P}$$

$$\dot{\mathbf{P}} = -\frac{\partial \bar{H}_1}{\partial \mathbf{X}} = -\frac{\partial q(\Phi^{-1}(e^{\Lambda t}\mathbf{X}))}{\partial \mathbf{X}}, \tag{52}$$

where the first equation follows since $\Phi(\mathbf{x}) = e^{\Lambda t}\mathbf{X}(t)$. For second equation, we have used

$$e^{-\Lambda t}\left(\frac{\partial \Phi}{\partial \mathbf{x}}\right) R \left(\frac{\partial \Phi}{\partial \mathbf{x}}\right)^{\top} e^{-\Lambda^{\top} t} = \left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right) R \left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right)^{\top}$$

$$\frac{\partial}{\partial \mathbf{X}}\left[\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right) R \left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right)^{\top}\right]$$

$$= \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right) R \left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right)^{\top} + \left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right) R \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right)^{\top} = 0$$

The above discussion can be summarized in the form of following theorem

Theorem 2. *Let $(\mathbf{X}(t), \mathbf{P}(t))$ be the solution of the Hamiltonian dynamics (52). Substituting $(\mathbf{X}(t), \mathbf{P}(t))$ in (49), we*

obtain $(\mathbf{x}(t), \mathbf{p}(t))$ *as the solution of the following Hamiltonian system*

$$\begin{aligned}\dot{\mathbf{x}} =& \frac{\partial H}{\partial \mathbf{p}} = \mathbf{f}(\mathbf{x}) - \mathbf{R}(\mathbf{x})\mathbf{p} \\ \dot{\mathbf{p}} =& \frac{\partial H}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\top \mathbf{p} + \frac{1}{2}\frac{\partial (\mathbf{p}^\top \mathbf{R}(\mathbf{x})\mathbf{p})}{\partial \mathbf{x}} - \frac{\partial q(\mathbf{x})}{\partial \mathbf{x}},\end{aligned} \quad (53)$$

corresponding to HJ equation

$$H(\mathbf{x}, \mathbf{p}) = \mathbf{p}^\top \mathbf{f}(\mathbf{x}) - \mathbf{p}^\top \mathbf{R}(\mathbf{x})\mathbf{p} + q(\mathbf{x}) \quad (54)$$

Refer to the Appendix for the proof.

Following the results of Theorem 2, the solution of (52) can be used to parameterize the Lagrangian submanifold as

$$\Lambda_t = \{(\mathbf{x}, \mathbf{p}) : \mathbf{p} = \left(\frac{\partial \Phi}{\partial \mathbf{x}}\right)^\top e^{-\Lambda^\top t}\mathbf{P}(t)\}. \quad (55)$$

However, the paramaterization is time-dependent and the Hamiltonian dynamical system (52) is nonlinear and hence difficult to solve. To solve the Hamiltonian system, we make following approximation leading to an approximation of the Lagrangian submanifold.

Approximation: Consider the following approximation for the $\mathbf{R}(\mathbf{x}), q(\mathbf{x})$, and Koopman eigenfunctions $\Phi(\mathbf{x})$ as

$$\mathbf{R}(\mathbf{x}) \approx \mathbf{R} = \text{constant}, \quad q(\mathbf{x}) \approx \mathbf{x}^\top \mathbf{Q}\mathbf{x}, \quad \Phi(\mathbf{x}) \approx \mathbf{V}^\top \mathbf{x} \quad (56)$$

where the $\mathbf{V}^\top$ consists of left eigenvectors of the linearized system matrix $\mathbf{A}$ and hence satisfies $\mathbf{V}^\top \mathbf{A} = \mathbf{V}^\top \Lambda$. Using the above approximation, the system equation (52) can be written

$$\begin{pmatrix} \dot{\hat{\mathbf{X}}} \\ \dot{\hat{\mathbf{P}}} \end{pmatrix} = \begin{pmatrix} 0 & -e^{-\Lambda t}\hat{\mathbf{R}}e^{-\Lambda^\top t} \\ -e^{\Lambda^\top t}\hat{\mathbf{Q}}e^{\Lambda t} & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{X}} \\ \hat{\mathbf{P}} \end{pmatrix} \quad (57)$$

where

$$\hat{\mathbf{R}} = \mathbf{V}^\top \mathbf{R}\mathbf{V}, \quad \hat{\mathbf{Q}} = \mathbf{V}^{-1}\mathbf{Q}(\mathbf{V}^{-1})^\top \quad (58)$$

Remark 3. *It is important to emphasize that the "nonlinear" Koopman eigenfunctions are still involved in transforming* $(\mathbf{x}, \mathbf{p})$ *to* $(\mathbf{X}, \mathbf{P})$*. However, the local linear approximation of Koopman eigenfunction is used to convert the nonlinear Hamiltonian system (52) to linear system (57). The* $\mathbf{V}^\top \mathbf{x}$ *corresponds to the local linear approximation of the nonlinear principal Koopman eigenfunction and it follows from Remark 1 as nonlinear vector field can be expanded as* $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + O(\|\mathbf{x}\|^2)$ *in linear and nonlinear terms so are the Koopman eigenfunctions* $\Phi(\mathbf{x}) = \mathbf{V}^\top \mathbf{x} + O(\|\mathbf{x}\|^2)$ *in linear and nonlinear terms.*

The linear equation (57) can be solved analytically and the solution of the linear equation $(\hat{\mathbf{X}}, \hat{\mathbf{P}})$ leads to the following approximation of (55) for the Lagrangian submanifold

$$\hat{\Lambda}_t = \{(\mathbf{x}, \mathbf{p}) : \mathbf{p} = \left(\frac{\partial \Phi}{\partial \mathbf{x}}\right)^\top e^{-\Lambda^\top t}\hat{\mathbf{P}}(t)\}. \quad (59)$$

The above expression for the Lagrangian submanifold is parameterized by time, and determining the appropriate time over which the approximation is good is a challenge. We will show that the time parameterization is unnecessary and can be eliminated as follows. It is convenient to write the above system in the new coordinates to prove the subsequent results. Define time-varying change of coordinates as

$$\bar{\mathbf{X}} = e^{\Lambda t}\hat{\mathbf{X}}, \quad \bar{\mathbf{P}} = e^{-\Lambda^\top t}\hat{\mathbf{P}}. \quad (60)$$

and write (57) in $(\bar{\mathbf{X}}, \bar{\mathbf{P}})$ coordinates as

$$\begin{pmatrix} \dot{\bar{\mathbf{X}}} \\ \dot{\bar{\mathbf{P}}} \end{pmatrix} = \begin{pmatrix} \Lambda & -\hat{\mathbf{R}} \\ -\hat{\mathbf{Q}} & -\Lambda^\top \end{pmatrix} \begin{pmatrix} \bar{\mathbf{X}} \\ \bar{\mathbf{P}} \end{pmatrix} =: \mathcal{H} \begin{pmatrix} \bar{\mathbf{X}} \\ \bar{\mathbf{P}} \end{pmatrix}. \quad (61)$$

Since the system equations are linear, an analytical expression for the Lagrangian subspace can be found. We propose a new procedure for the construction of Lagrangian subspace for linear Hamiltonian system (61), which explicitly brings forward the power of Koopman theory. The proposed construction is also amicable to extension in the nonlinear setting and is essential at the heart of the computation behind Procedure 2. We make the following Assumption on the Hamiltonian matrix $\mathcal{H}$ as defined in (61).

Assumption 5. *We assume that the Hamiltonian matrix* $\mathcal{H}$ *has no eigenvalues on the imaginary axis. The generalized eigenspace,* $\mathbb{E}_s$ *for stable eigenvalues satisfies the following complementarity condition*

$$\mathbb{E}_s \oplus \begin{pmatrix} 0 \\ I \end{pmatrix} = \mathbb{R}^{2n}. \quad (62)$$

The Assumption 5 is the same assumption that is made on the Hamiltonian matrix associated with the Riccatti equation (9) (Refer to discussion following Eq. (9)). The main result of this construction procedure is summarized in the form of following proposition.

Proposition 5. *For the linear Hamiltonian system (61) under Assumption 5 the Lagrangian subspace is given by*

$$\Lambda = \{(\bar{\mathbf{X}}, \bar{\mathbf{P}}) : \bar{\mathbf{P}} = \mathbf{L}\bar{\mathbf{X}}, \ \ \mathbf{L} = -\mathbf{D}_2^{-1}\mathbf{D}_1\},$$

where

$$\mathbf{D} = \begin{pmatrix} \mathbf{d}_1^\top \\ \vdots \\ \mathbf{d}_n^\top \end{pmatrix} = \begin{pmatrix} \mathbf{D}_1 & \mathbf{D}_2 \end{pmatrix} \in \mathbb{R}^{n \times 2n}, \ \ \mathbf{D}_i \in \mathbb{R}^{n \times n}, \ i = 1, 2. \quad (63)$$

and $\mathbf{d}_j^\top$ *for* $j = 1, \ldots, n$ *are the left eigenvectors associated with eigenvalues with positive real part of the following matrix*

$$\mathcal{H} := \begin{pmatrix} \Lambda & -\hat{\mathbf{R}} \\ -\hat{\mathbf{Q}} & -\Lambda^\top \end{pmatrix}.$$

Refer to the Appendix for the proof. The results of the following Proposition are known in different form [41], but we present here for the sake of completeness.

Proposition 6. *The matrix* $\mathbf{L}$ *from Proposition 5 satisfies the Riccatti equation*

$$\Lambda^\top \mathbf{L} + \mathbf{L}\Lambda - \mathbf{L}\hat{\mathbf{R}}\mathbf{L} + \hat{\mathbf{Q}} = 0. \quad (64)$$

and the spectrum of the closed loop linear system $\Lambda - \hat{\mathbf{R}}\mathbf{L}$ *is in the left half plane.*

Refer to Appendix for the proof. Using (61) and change of coordinates (60), the solution to (57) can be written as

$$\begin{pmatrix}\hat{\mathbf{X}}(t,\hat{\mathbf{X}}_0,\hat{\mathbf{P}}_0)\\ \hat{\mathbf{P}}(t,\hat{\mathbf{X}}_0,\hat{\mathbf{P}}_0)\end{pmatrix}=\begin{pmatrix}e^{-\Lambda t} & 0\\ 0 & e^{\Lambda^\top t}\end{pmatrix}e^{\mathcal{H}t}\begin{pmatrix}\hat{\mathbf{X}}_0\\ \hat{\mathbf{P}}_0\end{pmatrix}. \tag{65}$$

Proposition 7. *For sufficiently small $|\hat{\mathbf{X}}_0|$, we have*

$$\mathbf{x}(t,\hat{\mathbf{X}}_0)=\mathbf{x}(\hat{\mathbf{X}}(t,\hat{\mathbf{X}}_0,\mathbf{L}\hat{\mathbf{X}}_0),t)=\Phi^{-1}(e^{(\Lambda-\hat{\mathbf{R}}\mathbf{L})t}\hat{\mathbf{X}}_0) \tag{66}$$

$$\mathbf{p}(t,\hat{\mathbf{X}}_0)=\mathbf{p}(\hat{\mathbf{X}}_0,\mathbf{L}\hat{\mathbf{X}}_0,t)=\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)^\top\mathbf{L}e^{(\Lambda-\hat{\mathbf{R}}\mathbf{L})t}\hat{\mathbf{X}}_0. \tag{67}$$

converge to origin as $t\to\infty$.

Proof: Since, $\mathbf{L}$ satisfies Riccatti equation, we obtain

$$\begin{pmatrix}\Lambda & -\hat{\mathbf{R}}\\ -\hat{\mathbf{Q}} & -\Lambda^\top\end{pmatrix}\begin{pmatrix}I\\ \mathbf{L}\end{pmatrix}=\begin{pmatrix}\Lambda-\hat{\mathbf{R}}\mathbf{L}\\ -\hat{\mathbf{Q}}-\Lambda^\top\mathbf{L}\end{pmatrix}=\begin{pmatrix}I\\ \mathbf{L}\end{pmatrix}\Lambda-\hat{\mathbf{R}}\mathbf{L} \tag{68}$$

hence

$$e^{\mathcal{H}t}\begin{pmatrix}I\\ \mathbf{L}\end{pmatrix}=\begin{pmatrix}I\\ \mathbf{L}\end{pmatrix}e^{(\Lambda-\hat{\mathbf{R}}\mathbf{L})t}. \tag{69}$$

If we restrict the initial condition to the Lagrangian subspace i.e.,

$$\begin{pmatrix}\hat{\mathbf{X}}_0\\ \hat{\mathbf{P}}_0\end{pmatrix}=\begin{pmatrix}I\\ \mathbf{L}\end{pmatrix}\hat{\mathbf{X}}_0.$$

Then following (65), (69), and (49) we obtain

$$\begin{pmatrix}\hat{\mathbf{X}}(t,\hat{\mathbf{X}}_0,\hat{\mathbf{P}}_0)\\ \hat{\mathbf{P}}(t,\hat{\mathbf{X}}_0,\hat{\mathbf{P}}_0)\end{pmatrix}=\begin{pmatrix}e^{-\Lambda t}e^{(\Lambda-\hat{\mathbf{R}}\mathbf{L})t}\hat{\mathbf{X}}_0\\ e^{\Lambda^\top t}\mathbf{L}e^{(\Lambda-\hat{\mathbf{R}}\mathbf{L})t}\hat{\mathbf{X}}_0\end{pmatrix}$$

$$\mathbf{x}(t,\hat{\mathbf{X}}_0)=\mathbf{x}(\hat{\mathbf{X}}(t,\hat{\mathbf{X}}_0,\mathbf{L}\hat{\mathbf{X}}_0),t)=\Phi^{-1}(e^{(\Lambda-\hat{\mathbf{R}}\mathbf{L})t}\hat{\mathbf{X}}_0) \tag{70}$$

$$\mathbf{p}(t,\hat{\mathbf{X}}_0)=\mathbf{p}(\hat{\mathbf{X}}_0,\mathbf{L}\hat{\mathbf{X}}_0,t)=\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)^\top\mathbf{L}e^{(\Lambda-\hat{\mathbf{R}}\mathbf{L})t}\hat{\mathbf{X}}_0. \tag{71}$$

Following the proof of Proposition 6, we know that the spectrum of $\Lambda^\top-\hat{\mathbf{R}}\mathbf{L}$ is in the left half plane and hence the results of the proposition follows. □

Combining the results of Proposition 5, change of coordinates (60), and expression for Lagrangian subspace (59), we obtain after eliminating the $\hat{\mathbf{X}}_0$ from (71) using (70) i.e., $\hat{\mathbf{X}}_0=e^{-(\Lambda-\hat{\mathbf{R}}\mathbf{L})t}\Phi(\mathbf{x}(t,\hat{\mathbf{X}}_0))$ the following expression for the approximate Lagrangian submanifolds.

$$\hat{\Lambda}=\left\{(\mathbf{x},\mathbf{p}):\mathbf{p}=\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)^\top\mathbf{L}\Phi(\mathbf{x})\right\}. \tag{72}$$

It is important to emphasize that the unlike (59), the above expression for the approximate Lagrangian submanifold is not parameterized by time. We know that $\mathbf{p}=\frac{\partial V}{\partial\mathbf{x}}$, and hence from (72), we obtain $V(\mathbf{x})=\Phi(\mathbf{x})^\top\mathbf{L}\Phi(\mathbf{x})$. This suggest that the approximated cost is quadratic in Koopman eigenfunction coordinates where following the results of Proposition 6, the matrix $\mathbf{L}$ is obtained as a solution of Riccatti equation.

Remark 4. *We observe from (72) that the Riccatti solution coming from the linearized HJ equation is obtained as a special case of the Lagrangian submanifold construction as proposed in procedure 1. In particular, the special case will correspond to the eigenfunctions $\Phi(\mathbf{x})$ being approximated as linear function of $\mathbf{x}$ i.e., $\Phi(\mathbf{x})=\mathbf{V}^\top\mathbf{x}$ with $\mathbf{V}^\top\mathbf{A}=\Lambda\mathbf{V}^\top$. In this case, we have*

$$V(\mathbf{x})=\Phi^\top(\mathbf{x})\mathbf{L}\Phi(\mathbf{x})=\mathbf{x}^\top\mathbf{V}\mathbf{L}\mathbf{V}^\top\mathbf{x}=\mathbf{x}^\top\mathbf{P}\mathbf{x}^\top.$$

Since we know that the matrix $\mathbf{L}$ satisfies the Riccatti equation (64) then it follows that the matrix $\mathbf{P}$ satisfies following Riccatti equation

$$\mathbf{A}^\top\mathbf{P}+\mathbf{P}\mathbf{A}-\mathbf{P}\mathbf{R}\mathbf{P}+\mathbf{Q}=0.$$

where $\mathbf{A}=(\mathbf{V}^{-1})^\top\Lambda\mathbf{V}^\top$ and $\mathbf{Q},\mathbf{R}$ are as given in (56).

The procedure for the construction of Lagrangian manifold can be outlined as follows.

---

1 Procedure 1

1) For the uncontrolled dynamical system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$, compute the principal eigenfunctions of the Koopman operator associated with the eigenvalues of the linearization i.e., $\mathbf{A}=\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(0)$ using the construction procedure outlined in Section III.
2) Construct the Hamiltonian matrix as given in Eq. (61) with $\Lambda,\hat{\mathbf{R}},\hat{\mathbf{Q}}$ as defined in Assumption 3 and Eq. (58).
3) Compute the left eigenvector of the Hamiltonian matrix associated with eigenvalues with positive real part.
4) Use the results of Proposition 5 to compute the matrix $\mathbf{L}$ and following Proposition 6, $\mathbf{L}$ is same as the solution of Riccatti equation.
5) Use the eigenfunctions and $\mathbf{L}$ to compute the approximate Lagrangian submanifolds using the formula given in Eq. (72).

Alternatively, we can simply the system equation (52) with the following change of coordinates

$$\mathbf{x}_1=e^{\Lambda t}\mathbf{X},\quad \mathbf{p}_1=e^{-\Lambda^\top t}\mathbf{P}, \tag{73}$$

where we know that

$$\mathbf{X}=e^{-\Lambda t}\Phi(\mathbf{x}),\ \ \mathbf{p}(\mathbf{X},\mathbf{P},t)=\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)^\top e^{-\Lambda^\top t}\mathbf{P}$$

and hence

$$\mathbf{x}_1=\Phi(\mathbf{x}),\ \ \mathbf{p}=\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)^\top\mathbf{p}_1.$$

Under the assumption that $q(\mathbf{x})=\mathbf{x}_1^\top\mathbf{Q}\mathbf{x}_1$, the system equation (52) can be written as

$$\begin{pmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{p}}_1\end{pmatrix}=\begin{pmatrix}\Lambda & \left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)\mathbf{R}(\Phi^{-1}(\mathbf{x}_1))\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)^\top\\ -\mathbf{Q} & -\Lambda^\top\end{pmatrix}\begin{pmatrix}\mathbf{x}_1\\ \mathbf{p}_1\end{pmatrix} \tag{74}$$

The $\mathbf{p}_1$ dynamics of the above system can be solved explicitly, in particular we have

$$\mathbf{p}_1(t)=e^{-\Lambda^\top t}\mathbf{p}_1(0)-\int_0^t e^{-\Lambda^\top(t-\tau)}\mathbf{Q}\mathbf{x}_1(\tau)d\tau \tag{75}$$

and hence

$$\mathbf{p}(t)=\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)e^{-\Lambda^\top t}\left(\mathbf{L}\Phi(\mathbf{x}(0))-\int_0^t e^{\Lambda\tau}\mathbf{Q}\Phi(\mathbf{x}(\tau))d\tau\right) \tag{76}$$

where we have restricted the initial conditions as follows $\mathbf{P} = \mathbf{L}\mathbf{X}$ then $\mathbf{p}_1 = e^{-\Lambda t}\mathbf{L}\mathbf{X} = e^{-\Lambda t}\mathbf{L}e^{-\Lambda t}\mathbf{x}_1$ and hence $\mathbf{p}_1(0) = \mathbf{L}\mathbf{x}_1(0) = \mathbf{L}\Phi(\mathbf{x}(0))$. So the feedback control system is given as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{R}(\mathbf{x})\mathbf{p}(t), \tag{77}$$

with $\mathbf{p}(t)$ given by (76).

VI. Koopman Eigenfunctions for Computation of Lagrangian Submanifold

The results of Proposition 5 suggest an alternate approach to approximate the Lagrangian submanifold for the original Hamiltonian system in $(\mathbf{x}, \mathbf{p})$ coordinates directly. The Hamiltonian dynamical system is repeated here for convenience.

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) - \mathbf{R}(\mathbf{x})\mathbf{p}\\ \dot{\mathbf{p}} &= -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\top \mathbf{p} + \frac{1}{2}\left(\frac{\partial \mathbf{p}^\top \mathbf{R}(\mathbf{x})\mathbf{p}}{\partial \mathbf{x}}\right)^\top - \frac{\partial q}{\partial \mathbf{x}}^\top,\end{aligned} \tag{78}$$

and the associated linearization of the nonlinear Hamiltonian at the origin as

$$\begin{pmatrix}\dot{\hat{\mathbf{x}}}\\ \dot{\hat{\mathbf{p}}}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & -\bar{\mathbf{R}}\\ -\bar{\mathbf{Q}} & -\mathbf{A}^\top\end{pmatrix}\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{p}}\end{pmatrix} = \mathcal{H}_1\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{p}}\end{pmatrix} \tag{79}$$

where $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0), \bar{\mathbf{R}} = \mathbf{R}(0)$, and $\bar{\mathbf{Q}} = \frac{\partial^2 q}{\partial \mathbf{x}^2}(0)$. The proposed approach of procedure two for approximating the Lagrangian submanifold can be divided into two steps.

1) First, we construct principal eigenfunctions of the Koopman operator corresponding to the Hamiltonian dynamical system (78) based on the construction procedure outlined in Section III.
2) The second step is to use the zero level set of Koopman eigenfunctions to identify the Lagrangian submanifold.

The first step of Procedure 2 of computing the principal eigenfunction of a dynamical system is common to both Procedure 1 and 2. The difference is that while in Procedure 1, the principal eigenfunctions are computed for uncontrolled system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, the principal eigenfunctions in Procedure 2 are computed for Hamiltonian dynamical system (78). We have already discussed our contribution towards the computation of principal eigenfunction in Section III. We now proceed with step 2 of Procedure 2 on the computation of the Lagrangian submanifold.

A. Computation of Lagrangian Submanifold

We will use the procedure outlined in Section III for the computation of the Koopman eigenfunctions. Borrowing notations from Section III, except for the change that we drop the over-bar and subscript notation from the basis function $\bar{\Gamma}_M$ and matrix $\bar{W}_M$ to distinguish from real valued basis function and matrices. $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$ system now corresponds to system equation (78) with $\mathbf{z} = (\mathbf{x}^\top, \mathbf{p}^\top)^\top$. We decompose $\mathbf{W}^\top$ which constitutes the left eigenvector of $\mathcal{H}_1$ (Eq. (79)) matrix as

$$\mathbf{W}^\top = \begin{pmatrix}\mathbf{W}_u^\top\\ \mathbf{W}_s^\top\end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{80}$$

where $\mathbf{W}_u^\top \in \mathbb{R}^{n\times 2n}, \mathbf{W}_s^\top \in \mathbb{R}^{n\times 2n}$ corresponds to unstable and stable linear subspace respectively and satisfies $\mathbf{W}^\top\mathcal{H}_1 = \Lambda_{\mathbf{F}}\mathbf{W}^\top$ with $\mathcal{H}_1$ given in (79) and $\Lambda_{\mathbf{F}}$ corresponds to the matrix of eigenvalues of $\mathcal{H}_1$ in real Jordan canonical form. The eigenfunctions corresponding to the unstable eigenvalues of the Hamiltonian system is given by

$$\mathbf{W}_u^\top\left(\begin{pmatrix}\mathbf{x}^\top & \mathbf{p}^\top\end{pmatrix}^\top + \mathbf{U}\Gamma(\mathbf{z})\right), \tag{81}$$

where the matrix $\mathbf{U} \in \mathbb{R}^{2n\times M}$ is identified using the procedure outlined in Section III with the following choice of nonlinear basis function $\Gamma(\mathbf{z})$

$$\Gamma(\mathbf{z}) = \begin{pmatrix}\Xi(\mathbf{x})^\top & (\mathbf{x}\otimes\mathbf{p})^\top & (\Xi\otimes\mathbf{p})^\top\end{pmatrix}^\top, \tag{82}$$

where $\Xi : \mathcal{M} \to \mathbb{R}^N$ is vector valued function and satisfies $\frac{\partial\Gamma}{\partial\mathbf{z}}(0) = 0$ and hence $M = N + n^2 + Nn$. Furthermore, the basis functions are chosen to be linear in variable $\mathbf{p}$. We will comment on the assumed form of basis function which is linear in variable $\mathbf{p}$ in Remark 6. Given the form of $\mathbf{W}_u$ and $\Gamma(\mathbf{z})$, we can decompose (81) to determine the stable manifold as the zero level set of unstable eigenfunctions as

$$0 = \Psi(\mathbf{x},\mathbf{p}) := \mathbf{W}_u^\top\left(\begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix} + \begin{pmatrix}U_{11} & U_{12} & U_{13}\\ U_{21} & U_{22} & U_{23}\end{pmatrix}\begin{pmatrix}\Xi(\mathbf{x})\\ \mathbf{x}\otimes\mathbf{p}\\ \Xi(\mathbf{x})\otimes\mathbf{p}\end{pmatrix}\right), \tag{83}$$

where $[\mathbf{U}]_{ij} = U_{ij}$ is the splitting of the $\mathbf{U}$ matrix. The zero level set of the unstable eigenfunctions is decomposed into linear terms $\mathbf{W}_u^\top\mathbf{z}$ and nonlinear terms $\mathbf{W}_u^\top\mathbf{U}\Gamma(\mathbf{z})$. The zero level sets of $\Psi(\mathbf{z})$ can be identified with the zero level sets of the linear and nonlinear terms, namely

$$\{\mathbf{z} : \Psi(\mathbf{z}) = 0\} \equiv \{\mathbf{z} : \mathbf{W}_u^\top\mathbf{z} = 0\} \cap \{\mathbf{z} : \mathbf{W}_u^\top\mathbf{U}\Gamma(\mathbf{z}) = 0\}.$$

The objective is to find Lagrangian submanifold of the form $\mathbf{p} = \frac{\partial V}{\partial \mathbf{x}}$ for some unknown function V such that $\Psi(\mathbf{x}, \mathbf{p} = \frac{\partial V}{\partial \mathbf{x}}) = 0$. Based of the decomposition of the zero level set, we consider following decomposition and paramaterization for the unknown function $V(\mathbf{x})$

$$V(\mathbf{x}) = V_l(\mathbf{x}) + V_n(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top\mathbf{J}_l\mathbf{x} + \frac{1}{2}\Xi(\mathbf{x})^\top\mathbf{J}_n\Xi(\mathbf{x}). \tag{84}$$

where $\mathbf{J}_l \in \mathbb{R}^{n\times n}$ and $\mathbf{J}_n \in \mathbb{R}^{N\times N}$ corresponds to the paramaterization of the linear and nonlinear terms respectively. We have

$$\mathbf{p} = \frac{\partial V}{\partial \mathbf{x}}^\top = \mathbf{J}_l\mathbf{x} + \Xi_{\mathbf{x}}(\mathbf{x})^\top\mathbf{J}_n\Xi(\mathbf{x}) =: \mathbf{p}_l + \mathbf{p}_n. \tag{85}$$

As the basis functions are split into linear and nonlinear terms where $\Gamma(\mathbf{z})$ is assumed to be purely nonlinear function of $(\mathbf{x}, \mathbf{p})$, we obtain $\Psi(\mathbf{x}, \mathbf{p}) = 0$ if and only if

$$\mathbf{W}_u\mathbf{z} = 0, \quad \& \quad \mathbf{W}_u^\top\mathbf{U}\Gamma(\mathbf{z}) = 0.$$

Substituting, the linear paramaterization, $\mathbf{p}_l = \mathbf{J}_l\mathbf{x}$ in $\mathbf{W}_u^\top\mathbf{z} = 0$ we obtain

$$\mathbf{W}_u^\top\begin{pmatrix}\mathbf{x}^\top & \mathbf{x}^\top\mathbf{J}_l\end{pmatrix}^\top = 0 \implies (\mathbf{W}_{u1}^\top + \mathbf{W}_{u2}^\top\mathbf{J}_l)\mathbf{x} = 0, \tag{86}$$

where we have assumed $\mathbf{W}_u^\top = (\mathbf{W}_{u1}^\top \ \ \mathbf{W}_{u2}^\top)$ with $\mathbf{W}_{uk} \in \mathbb{R}^{n\times n}$ for $k = 1, 2$. (86) gives us

$$\mathbf{J}_l^\star = -(\mathbf{W}_{u2}^\top)^{-1}\mathbf{W}_{u1}^\top. \tag{87}$$

It is important to emphasize that, following results of Propositions 5 and 6, $\mathbf{J}_l^\star$ matches with the solution of Riccatti equation. In particular $\mathbf{J}_l^\star$ satisfies following Riccatti equation

$$\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\bar{\mathbf{R}}\mathbf{P} + \bar{\mathbf{Q}} = 0, \tag{88}$$

where $\mathbf{V}\mathbf{J}_l^\star\mathbf{V}^\top =: \mathbf{P}$ with rows of $\mathbf{V}^\top$ consisting of left eigenvectors of $\mathbf{A}$ and hence satisfy $\mathbf{V}^\top\mathbf{A} = \Lambda\mathbf{V}^\top$. Now substituting the nonlinear paramaterization of manifold in $\mathbf{W}_u^\top\mathbf{U}\Gamma(\mathbf{z}) = 0$, we obtain

$$\mathbf{W}_u^\top\mathbf{U}\Gamma(\mathbf{x}, \mathbf{p}_n) = 0, \tag{89}$$

which is equivalent to

$$\mathbf{W}_u^\top \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{pmatrix} \begin{pmatrix} \Xi(\mathbf{x}) \\ \mathbf{x}\otimes\Xi_{\mathbf{x}}^\top\mathbf{J}_n\Xi \\ \Xi(\mathbf{x})\otimes\Xi_{\mathbf{x}}^\top\mathbf{J}_n\Xi \end{pmatrix} = 0, \tag{90}$$

$$\mathbf{W}_{u1}^\top\left(U_{11}\Xi + U_{12}\mathbf{x}\otimes\Xi_{\mathbf{x}}^\top\mathbf{J}_n\Xi + U_{13}\Xi\otimes\Xi_{\mathbf{x}}^\top\mathbf{J}_n\Xi\right)$$

$$+\mathbf{W}_{u2}^\top\left(U_{21}\Xi + U_{22}\mathbf{x}\otimes\Xi_{\mathbf{x}}^\top\mathbf{J}_n\Xi + U_{23}\Xi\otimes\Xi_{\mathbf{x}}^\top\mathbf{J}_n\Xi\right) = 0.$$

Since the above has to be true for all $\Xi(\mathbf{x})$, this is equivalent to

$$\mathcal{J}(\mathbf{x}, \mathbf{J}_n) := \mathcal{B}(\mathbf{x})\mathbf{J}_n - \bar{\mathcal{A}} = 0,$$

where

$$\bar{\mathcal{A}} = -\mathbf{W}_{u1}^\top U_{11} - \mathbf{W}_{u2}^\top U_{21},$$

$$\mathcal{B}(\mathbf{x}) := (\mathbf{W}_{u1}^\top U_{12} + \mathbf{W}_{u2}^\top U_{22})\mathbf{x}\otimes\Xi_{\mathbf{x}}^\top + (\mathbf{W}_{u1}^\top U_{13} + \mathbf{W}_{u2}^\top U_{23})\Xi\otimes\Xi_{\mathbf{x}}^\top.$$

The objective is to determine $\mathbf{J}_n$ so that the above equality is satisfied. Since the equation is linear in $\mathbf{J}_n$, and function of $\mathbf{x}$, we can find $\mathbf{J}_n$ as a solution of following least square optimization problem

$$\min_{\mathbf{J}} \parallel \bar{\mathcal{B}}\mathbf{J} - \bar{\mathcal{A}} \parallel, \tag{91}$$

where

$$\bar{\mathcal{B}} := \frac{1}{D}\sum_{k=1}^{D}\mathcal{B}(\mathbf{x}_k), \tag{92}$$

and $\{\mathbf{x}_k\}_{k=1}^D$ are the data points. The above optimization admits analytical solution given by

$$\mathbf{J}_n^\star = \bar{\mathcal{B}}^\dagger\bar{\mathcal{A}}, \tag{93}$$

where $\dagger$ stands for pseudo-inverse. This leads to the following approximation of the Lagrangian submanifold

$$\Lambda_V := \{(\mathbf{x}, \mathbf{p}) : \mathbf{p} = \mathbf{J}_l^\star\mathbf{x} + \Xi_{\mathbf{x}}^\top(\mathbf{x})\mathbf{J}_n^\star\Xi(\mathbf{x})\}, \tag{94}$$

where $\mathbf{J}_l^\star$ and $\mathbf{J}_n^\star$ are as given in (87) and (93) respectively.

Remark 5. *The expression for the Lagrangian submanifold in (94) admits a decomposition consisting of linear and nonlinear terms. Hence, when only linear basis functions are used in the approximation, the expression for the Lagrangian manifold will reduce to the solution obtained using the Riccatti equation (refer to Eq. (88) and discussion following it).*

The above construction procedure can be summarized in the following proposition.

Proposition 8. *For the Hamiltonian dynamical system (78), with $\mathbf{f}, \mathbf{R}, q$ satisfying Assumption 3, let*

1) *$\Gamma(\mathbf{x}, \mathbf{p}) = [\Xi(\mathbf{x})^\top, (\mathbf{x}\otimes\mathbf{p})^\top, (\Xi(\mathbf{x})\otimes\mathbf{p})^\top]^\top$ be the choice of basis function and $\Psi(\mathbf{x}, \mathbf{p}) = \mathbf{W}_u^\top(\mathbf{z} + \mathbf{U}\Gamma(\mathbf{x}, \mathbf{p}))$ be the finite dimensional approximation of the principal eigenfunctions of the Hamiltonian dynamical system (78) according the procedure outline in Section III.*
2) *$\mathbf{p} = \mathbf{p}_n + \mathbf{p}_l$, with $\mathbf{p}_l = \mathbf{J}_l\mathbf{x}$ and $\mathbf{p}_n = \Xi_{\mathbf{x}}(\mathbf{x})^\top\mathbf{J}_n\Xi(\mathbf{x})$ be the respective linear and nonlinear paramaterization of the Lagrangian submanifold for the Hamiltonian dynamical system (78).*

Then the optimal $\mathbf{J}_l^\star$ and $\mathbf{J}_n^\star$ are given by Eqs. (87) and (93) respectively.

Our approach for the construction of the Lagrangian submanifold can be summarized as following procedure.

---

1 Procedure 2

1) Choose a finite dimensional basis function, $\Gamma(\mathbf{x}, \mathbf{p})$, as given in Eq. (82).
2) Using the procedure from Section III, construct the approximation of unstable eigenfunction of the Hamiltonian dynamical system (52). This approximation is given in Eq. (81).
3) Consider a zero level set of the unstable eigenfunction for the computation of the stable manifold of the Hamiltonian system (83).
4) Assume a parameterization for the Lagrangian submanifold consisting of linear and nonlinear terms as given in Eq. (85).
5) The optimal parameterization for the linear and nonlinear terms can be obtained by solving Eq. (87) and (93) respectively.

Remark 6. *If we consider the basis function, $\Gamma(\mathbf{x}, \mathbf{p})$, containing nonlinear terms in the variable $\mathbf{p}$, then the problem of constructing Lagrangian submanifold in step 2 will no longer reduce to a least-square problem. The problem, in general, will be nonlinear and non-convex. One will have to resort to an iterative numerical algorithm to approximate the submanifold in such a case. However, the spectrum analysis of the Koopman operator will continue to play a central role in the approximation.*

In the above discussion, we described how the Koopman operator and its eigenfunctions could be used in the approximation of the Lagrangian submanifold of the original Hamiltonian system in the $(\mathbf{x}, \mathbf{p})$ coordinates, i.e., Eq. (78). An alternate approach will be to approximate the principal eigenfunctions of the Koopman operator corresponding to the Hamiltonian system in $(\mathbf{X}, \mathbf{P})$ coordinates, i.e., working with system Eq.

(52) instead of Eq. (78). This could be advantageous as system Eq. (52) has more structure compared to system Eq. (78). In particular, Eq. (78) is linear along $\mathbf{p}$ coordinates. We will explore this idea in our future publications.

VII. Comments on the two procedures and Hamilton Jacobi Koopman (HJK) Equation

There are some similarities and difference between the two procedures. Using procedure 1, the solution, $V(\mathbf{x})$ of HJ equation is written as

$$V(\mathbf{x}) = \frac{1}{2}\Phi^\top(\mathbf{x})\mathbf{L}\Phi(\mathbf{x}). \tag{95}$$

where $\Phi(\mathbf{x})$ are the principal eigenfunctions of the open loop system and $\mathbf{L}$ is the solution of linearized Riccati equation solved in the transformed eigenfunction coordinates i.e., Eq. (64). The $V(\mathbf{x})$ can be expanded as

$$V(\mathbf{x}) = \left(\mathbf{V}^\top\mathbf{x} + \mathbf{V}^\top\mathbf{h}(\mathbf{x})\right)^\top \mathbf{L}\left(\mathbf{V}^\top\mathbf{x} + \mathbf{V}^\top\mathbf{h}(\mathbf{x})\right)$$
$$= \mathbf{x}^\top\mathbf{V}\mathbf{L}\mathbf{V}^\top\mathbf{x} + 2\mathbf{x}^\top\mathbf{V}\mathbf{L}\mathbf{V}^\top\mathbf{h}(\mathbf{x}) + \mathbf{h}^\top(\mathbf{x})\mathbf{V}\mathbf{L}\mathbf{V}^\top\mathbf{h}(\mathbf{x}).$$

The first term in the above expansion can be identified with the solution of the Riccatti equation (Remark 4). The second and third terms corresponds to the contribution coming from the nonlinear part of the eigenfunctions to the solution of HJ equation. The solution obtained using procedure 2 for the HJ equation has the form

$$V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top\mathbf{J}_l\mathbf{x} + \Xi(\mathbf{x})^\top\mathbf{J}_n\Xi(\mathbf{x}),$$

where $\mathbf{J}_l$ satisfies the Riccatti equation. $\mathbf{J}_n$ is obtained as a solution of optimization problem and captures the contribution of the nonlinear terms to the HJ solution. Comparing the procedure 1 and procedure 2, we notice that the HJ solutions in both the cases admit an expansion where the quadratic terms are common to both the expansions and matches with the Riccatti solution. The procedure 1 and 2 suggests an alternate approach for the computation of HJ solution. In particular, we can consider following parameteization for the HJ solution, $V(\mathbf{x}) = \Phi^\top(\mathbf{x})\mathbf{P}\Phi(\mathbf{x})$. Under the assumption that $q(\mathbf{x}) = \Phi^\top(\mathbf{x})\mathbf{Q}\Phi(\mathbf{x})$ and substituting the assumed form of $V(\mathbf{x})$ in HJ equation (3), we obtain

$$\Phi^\top(\mathbf{x})\left(\Lambda\mathbf{P} + \mathbf{P}\Lambda - \mathbf{P}\bar{\mathbf{R}}(\mathbf{x})\mathbf{P} + \mathbf{Q}\right)\Phi(\mathbf{x}) = 0$$

where $\bar{\mathbf{R}}(\mathbf{x}) = \frac{\partial\Phi}{\partial\mathbf{x}}\mathbf{R}(\mathbf{x})(\frac{\partial\Phi}{\partial\mathbf{x}})^\top$. The objective is to find a matrix $\mathbf{P}$ so that following equality is satisfied

$$\Lambda\mathbf{P} + \mathbf{P}\Lambda - \mathbf{P}\bar{\mathbf{R}}(\mathbf{x})\mathbf{P} + \mathbf{Q} = (\leq)0. \tag{96}$$

We will refer to the above equation as *Hamilton Jacobi Koopman (HJK)* equation (inequality). For the particular case when $\mathbf{x}$ in (96) is replaced with $\mathbf{x} = 0$, it is easy to show that the solution obtained using the HJK equation matches with the one obtained using Procedure 1. However, the dependence of $\bar{\mathbf{R}}$ and $\mathbf{Q}$ on $\mathbf{x}$ presents a challenge and opens up the possibility of discovering methods for approximately solving (96). One possible approach is to use random sampling of $\mathbf{x}$ to convert (96) into finite (in)equalities at the sampled points.

Remark 7. *The two procedures for the computation of Lagrangian submanifold is well suited for the development of data-driven optimal and robust control. In particular, the time-series data of the open-loop system uncontrolled system can be used to compute the principal eigenfunction of Koopman Φ. Using the linearity property of the Koopman generator in terms of vector field one can identify Koopman generator corresponding to the vector field $\mathbf{g}$ and $\mathbf{d}$ with zero input and step input. The details of this procedure on how to identify control vector field using data is given in [33], [42], [43].*

VIII. Optimal Control and Robust Control

In this section, the demonstrate the application of results developed on the approximate construction of Lagrangian submanifold for solving optimal control and robust control problems.

A. Optimal Control Problem

For simplicity of presentation we consider a control dynamical system with single input

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \tag{97}$$

and the associated cost function of the form

$$J = \int_0^\infty q(\mathbf{x}) + u^2 dt \tag{98}$$

The optimal control input will be of the form

$$u^\star(\mathbf{x}) = -\frac{\partial V}{\partial\mathbf{x}}\mathbf{g}(\mathbf{x}) \tag{99}$$

where $V(\mathbf{x})$ is the solution of following HJ Bellman (HJB) equation.

$$\frac{\partial V}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{1}{2}\frac{\partial V}{\partial\mathbf{x}}\mathbf{g}(\mathbf{x})\mathbf{g}^\top(\mathbf{x})\frac{\partial V}{\partial bx}^\top + q(\mathbf{x}) = 0 \tag{100}$$

Comparing the above HJB equation with HJ equation (29), we identify $\mathbf{R}(\mathbf{x}) = \frac{1}{2}\mathbf{g}(\mathbf{x})\mathbf{g}^\top(\mathbf{x})$ and $\mathbf{p} = \frac{\partial V}{\partial\mathbf{x}}$ is the Lagrangian submanifold. The Lagrangian submanifold can now be approximated using either Procedure 1 or Procedure 2. For the case involving Procedure 1, the optimal feedback control input is given by

$$u^\star = -\Phi^\top\mathbf{L}\Phi_{\mathbf{x}}\mathbf{g}(\mathbf{x})$$

and the feedback control system is given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})\mathbf{g}^\top(\mathbf{x})\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)^\top\mathbf{L}\Phi(\mathbf{x}), \tag{101}$$

B. Robust $\mathcal{H}_\infty$ Control Problem

For robust optimal control, we consider single-input single-output control system with disturbance,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u + \mathbf{d}(\mathbf{x})w \tag{102}$$
$$y = h(\mathbf{x}) \tag{103}$$

where $u \in \mathbb{R}$ is the control input and $w \in \mathbb{R}$ is the disturbance input. The objective is to find the optimal control input such that the origin of the closed loop system is asymptotically

stable and the $\mathcal{L}_2$ gain from the disturbance to the output y is less than or equal to $\gamma > 0$. The solution to the H_∞ control problem relies on the following HJ Issac equation.

$$\frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{1}{2}\frac{\partial V}{\partial \mathbf{x}}\left(\mathbf{g}(\mathbf{x})\mathbf{g}^\top(\mathbf{x}) - \frac{1}{\gamma^2}\mathbf{d}(\mathbf{x})\mathbf{d}^\top(\mathbf{x})\right)\frac{\partial V}{\partial \mathbf{x}}^\top + \frac{1}{2}h^2(\mathbf{x}) = 0. \tag{104}$$

The optimal feedback control and disturbance input can be expressed in terms of the solution of the HJI equation as

$$u^\star(\mathbf{x}) = -\frac{\partial V}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x}), \quad w^\star(\mathbf{x}) = \frac{1}{\gamma^2}\frac{\partial V}{\partial \mathbf{x}}\mathbf{d}(\mathbf{x}) \tag{105}$$

If we use procedure 1, then the optimal input $u^\star$ and adversarial input $w^\star$ are written as

$$u^\star(\mathbf{x}) = -\Phi^\top \mathbf{L}\Phi_{\mathbf{x}}\mathbf{g}(\mathbf{x}), \quad w^\star(\mathbf{x}) = \frac{1}{\gamma^2}\Phi^\top \mathbf{L}\Phi_{\mathbf{x}}\mathbf{d}(\mathbf{x}) \tag{106}$$

where $\mathbf{L}$ is obtained as the solution of following Riccatti equation.

$$\Lambda^\top \mathbf{L} + \mathbf{L}\Lambda - \frac{1}{2}\mathbf{L}\left(\mathbf{B}\mathbf{B}^\top - \frac{1}{\gamma^2}\mathbf{D}\mathbf{D}^\top\right)\mathbf{L} + \mathbf{Q} = 0 \tag{107}$$

where $\mathbf{B} = \mathbf{g}(0)$, $\mathbf{D} = \mathbf{d}(0)$, and $\mathbf{Q} = \frac{1}{2}\frac{\partial^2 h}{\partial \mathbf{x}^2}(0)$.

IX. Simulation Results

This section presents numerical examples based on the procedure 1 and 2 for the approximation of Lagrangian manifolds. All the simulations codes are developed on MATLAB and ran on a computer consisting of 16 GB of RAM and a 3.8 GHz Intel Core i7 processor. Both the procedures 1 and 2 for solving the HJ equation has reduced the problem to the computation of principal eigenfunctions of Koopman operators.

A. Example 1

The example is the system for which the Koopman eigenfunctions can be determined analytically

$$\begin{aligned}\dot{x}_1 &= \mu x_1 + u \\ \dot{x}_2 &= \lambda(x_2 - x_1^2) + u\end{aligned} \tag{108}$$

where the eigenfunctions are given by

$$\Phi(\mathbf{x}) = \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = \begin{bmatrix}x_1 \\ x_2 + \beta x_1^2\end{bmatrix}. \tag{109}$$

where $\mu = -1, \lambda = 0.6$, and $\beta = 0.23$. We assume the quadratic cost $q(\mathbf{x}) = \mathbf{x}^\top\mathbf{x}$ and control u^2. For this example, there exists set of initial conditions where the linear quadratic regulator (LQR) controller designed based on the linearized dynamics fails to stabilize the origin. However, the controller designed using Procedure 1 and 2 are able to stabilize. In Fig. 1, we show the norm of the state response corresponding to an initial condition where LQR fails to stabilize. In Fig. 2 we show the plot for optimal cost functions obtained using linear Riccati equation i.e., $V_{lqr}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top\mathbf{P}\mathbf{x} = 0.25x_1^2 - 0.41x_1x_2 + 1.2x_2^2$, using the Procedure 1 i.e., $V_{P1} = \frac{1}{2}\mathbf{\Phi}(\mathbf{x})^\top\mathbf{L}\mathbf{\Phi}(\mathbf{x}) = (0.25x_1^2 - 0.41x_1x_2 + 1.2x_2^2) + (0.07x_1^4 + 0.05x_1^3)$, and finally, using Procedure 2 $V_{P2} = \frac{1}{2}\mathbf{x}^\top\mathbf{J}_l\mathbf{x} + \frac{1}{2}\Xi(\mathbf{x})^\top\mathbf{J}_n\Xi(\mathbf{x}) = 0.25x_1^2 - 0.41x_1x_2 + 1.2x_2^2 + \frac{1}{2}\Xi(\mathbf{x})^\top\mathbf{J}_n\Xi(\mathbf{x})$, where $\Xi(\mathbf{x}) \in \mathbb{R}^{7\times 7}$ are the order 3 monomial basis (i.e$\{\mathbf{x}^3\}$). We notice that the LQR controller is embedded in the controller obtained using procedure 1 and 2.

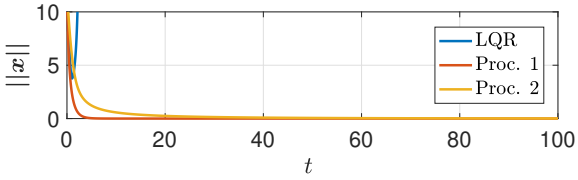

Fig. 1. Comparison of norm of state response with controller designed using LQR, Procedure 1, and Procedure 2.

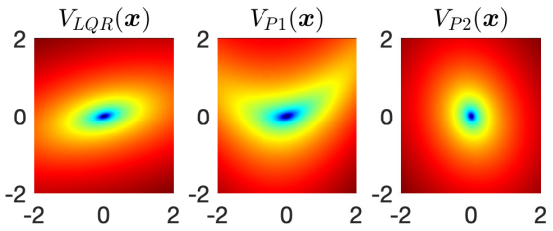

Fig. 2. Optimal cost function using LQR (left), Procedure 1 (middle), and Procedure 2 (right) .

B. Example 2

Our second example is about controlling a 2-D nonlinear oscillator.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 0.105x_2 + \frac{1}{2}x_2^2x_1 + 1.1x_1x_2 + (1.1 + x_1)u\end{aligned} \tag{110}$$

We again assume quadratic cost as $q(\mathbf{x}) = x_2^2$ and control u^2. For this example the HJ equation admits an analytical solution [44]. The optimal control is given by $u^* = -(1.1+x_1)x_2$ and the optimal cost function is $V^* = x_1^2 + x_2^2$. To approximate the eigenfunctions from data, we used the methodology developed in section III. The system was discretized using Euler method to obtain one-step time-series data with $1e^4$ initial conditions. Fig. 3. presents a comparison between linear eigenfunctions and nonlinear principal eigenfunctions. Similarly, in Fig. 4 we show the comparison of the feedback controller using LQR, Procedure 1, and Procedure 2. Comparison of the closed-loop trajectories for $(-1.8, 1.8)$ initial condition is shown in the Fig. 5. In Fig. 6, we plot the control inputs and performance in terms of the cost. The optimal cost obtained using the Procedure 1 and Procedure 2 shows a good match.

C. Example 3

For this example, we consider the problem of control of an electrostatically actuated micro-electromechanical systems

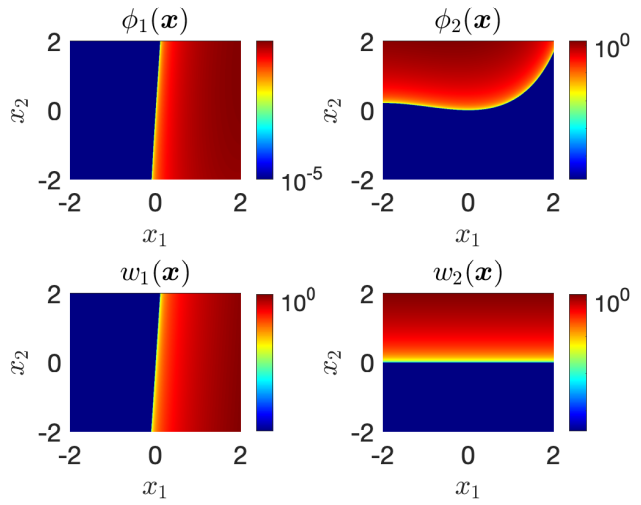

Fig. 3. Nonlinear eigenfunctions, $\boldsymbol{\phi_1, \phi_2}$ and linear eigenfunctions $\boldsymbol{w_1, w_2}$.

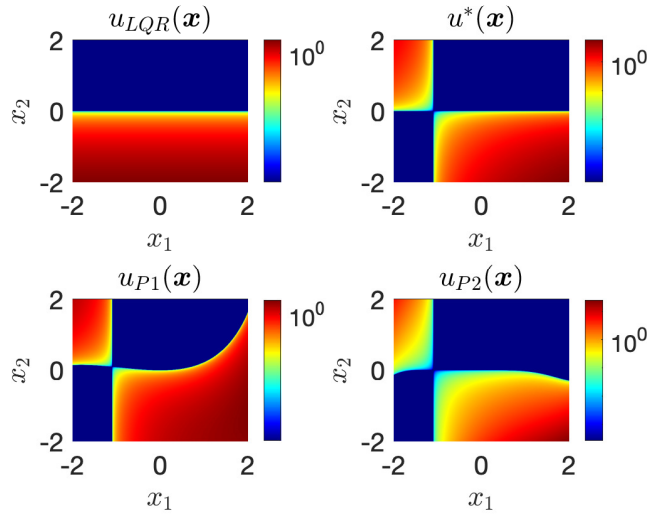

Fig. 4. Comparison of optimal feedback controllers.

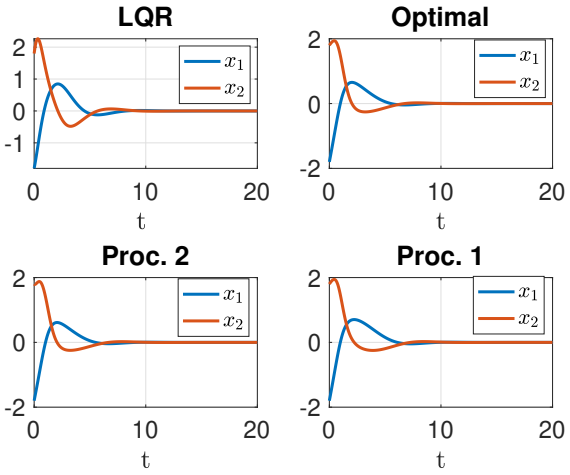

Fig. 5. Comparison of state trajectories with LQR control input, Procedure 1, and Procedure 2.

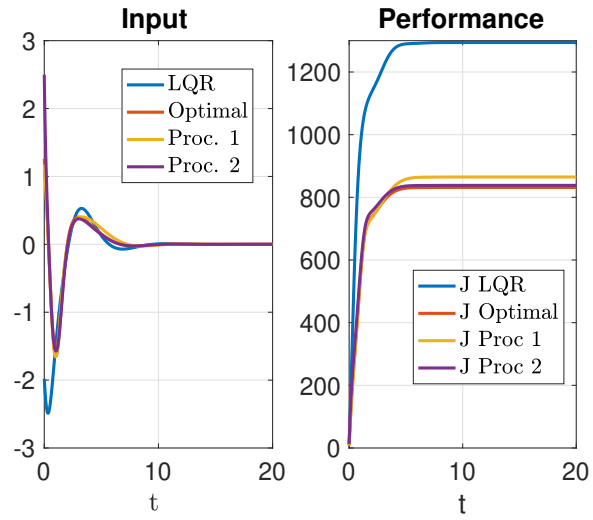

Fig. 6. Comparison of control inputs, and empirical cost function with LQR control input, procedure 1, and procedure 2.

with following dynamics

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\gamma_1(x_1^2 - 1)x_2 + (\omega_2/\omega_1)^2 x_1 - k(x_3 - x_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\gamma_2 x_4 - \delta x_3^2 + k(x_1 - x_3) + \cos\left(\frac{\omega_d}{\omega_1}\tau\right) u
\end{aligned} \tag{111}$$

we set all parameters as 1 (Chapter 6, [45]).

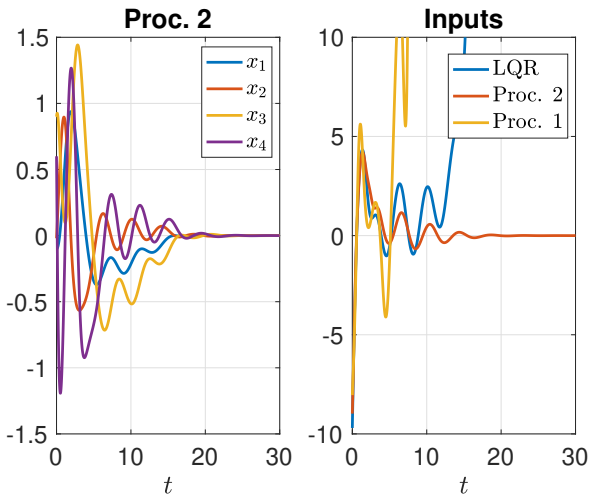

Fig. 7. Closed-Loop trajectories Procedure 2 and inputs.

In our numerical simulations, we observed that with the initial conditions close to the origin all the controller, LQR, Proedure 1 and Procedure 2 were able to stabilize the system at the equilibrium point. However when the initial conditions were chosen away from the origin then only the controller designed using Procedure 2 is able to stabilize the origin. In Fig. 7 shows the closed-loop states for the optimal control using Procedure 2, and the inputs for LQR, Procedure 1, and Procedure 2. We show that LQR and Procedure 1 inputs

diverge, which mean that both controllers are not able to stabilize the system.

X. Conclusions

This paper brings together tools from linear operator Koopman theory and differential geometry to provide a novel approach for the computation of the HJ solution. We present two different procedures for the computation of the Lagrangian submanifold, which are instrumental in solving the HJ equation. The two procedures also lead to the introduction of Hamilton Jaboci Koopman (HJK) equation, which arises by writing the HJ equation using the principal eigenfunctions of the Koopman operator. Furthermore, given the significance of the HJ equation in various problems in system theory, the proposed methodology involving the Koopman operator can be extended in several different directions, including analysis and synthesis of control system and robust control. The proposed approach is also the most promising direction for extending existing results in linear control theory to the nonlinear control system by exploiting the linearity of the Koopman. The future research efforts will focus on the implementation of the two procedures in the data-driven setting.

Appendix

Proof: (*Proof of Theorem 2*) *The* $(\dot{\mathbf{x}}, \dot{\mathbf{p}})$ *dynamics are given by*

$$\dot{\mathbf{x}} = \frac{\partial H_0}{\partial \mathbf{p}} + \frac{\partial H_1}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H_0}{\partial \mathbf{x}} - \frac{\partial H_1}{\partial \mathbf{x}} \tag{112}$$

where $H(\mathbf{x}, \mathbf{p}) = H_0(\mathbf{x}, \mathbf{p}) + H_1(\mathbf{x}, \mathbf{p})$. *Since the nominal Hamiltonian is integrable, the nominal Hamilton Jacobi equation has a solution* $W(\mathbf{x}, \mathbf{P}, t)$ *that satisfies*

$$H_0\left(\mathbf{x}, \frac{\partial W}{\partial \mathbf{x}}\right) + \frac{\partial W}{\partial t} = 0. \tag{113}$$

The canonical change of coordinates $(\mathbf{x}, \mathbf{p}) \to (\mathbf{X}, \mathbf{P})$ *given by*

$$\mathbf{p} = \frac{\partial W}{\partial \mathbf{x}}, \quad \mathbf{X} = \frac{\partial W}{\partial \mathbf{P}}$$

transform the unperturbed Hamiltonian system to the form

$$\dot{\mathbf{X}} = 0, \quad \dot{\mathbf{P}} = 0$$

Since W *satisfies (113), we have*

$$H\left(\mathbf{x}, \frac{\partial W}{\partial \mathbf{x}}\right) + \frac{\partial W}{\partial t} = H_1(\mathbf{x}, \mathbf{p})$$

the perturbed equation in the canonical coordinates $(\mathbf{X}, \mathbf{P})$ *becomes*

$$\dot{\mathbf{X}} = \frac{\partial H_1}{\partial \mathbf{P}}(\mathbf{x}, \mathbf{p}), \quad \dot{\mathbf{P}} = -\frac{\partial H_1}{\partial \mathbf{X}}(\mathbf{x}, \mathbf{p})$$

where $x_j = x_j(t, \mathbf{X}, \mathbf{P}), \quad p_j = p_j(t, \mathbf{X}, \mathbf{P})$

□

Proof: (*Proof of Proposition 5*). Following Assumption 5, we know that the Hamiltonian matrix, $\mathcal{H}$, does not have any eigenvalues on the $j\omega$ axis. Let $\mathbf{d}_1^\top, \ldots, \mathbf{d}_n^\top$ be the left eigenvectors associated with eigenvalues with positive real part. Following Remark 1, the stable subspace of the linear system is characterized by the zero level sets of Koopman eigenfunctions corresponding to eigenvalues with positive real part and hence the stable subspace is characterized as

$$\mathbb{E}_s = \{(\bar{\mathbf{X}}, \bar{\mathbf{P}}) : \mathbf{D}\begin{pmatrix}\bar{\mathbf{X}} \\ \bar{\mathbf{P}}\end{pmatrix} = \mathbf{D}_1\bar{\mathbf{X}} + \mathbf{D}_2\bar{\mathbf{P}} = 0\}$$

where the matrix $\mathbf{D}, \mathbf{D}_1, \mathbf{D}_2$ are as defined in (63). We seek the parameterization of the stable subspace in terms of $\bar{\mathbf{X}}$ and hence we write $\bar{\mathbf{P}} = \mathbf{L}\bar{\mathbf{X}}$, for some symmetric matrix $\mathbf{L}$. Symmetric $\mathbf{L}$ follows from the fact that, we seek parameterization of the form $\bar{\mathbf{P}} = \frac{\partial V(\bar{\mathbf{X}})}{\partial \bar{\mathbf{X}}}$ i.e., gradient of a scalar value function $V = \frac{1}{2}\bar{\mathbf{X}}^\top\mathbf{L}\bar{\mathbf{X}}$. Substituting for $\bar{\mathbf{P}}$ in the stable subspace equation, we obtain

$$\mathbf{D}_1\bar{\mathbf{X}} + \mathbf{D}_2\mathbf{L}\bar{\mathbf{X}} = (\mathbf{D}_1 + \mathbf{D}_2\mathbf{L})\bar{\mathbf{X}} = 0 \tag{114}$$

Since the above is true for all $\bar{\mathbf{X}}$, we have $\mathbf{D}_1 + \mathbf{D}_2\mathbf{L} = 0$. Following Assumption 5, the condition (62) is equivalent to

$$\mathbb{E}_s^\perp \oplus \begin{pmatrix}I \\ 0\end{pmatrix} = \mathbb{R}^{2n}. \tag{115}$$

Now since matrix $\mathbf{D}$ span the space orthogonal to $\mathbb{E}_s$ i.e., $\mathbb{E}_s^\perp$. Now (115) is equivalent to the invertability of matrix

$$\begin{pmatrix}\mathbf{D}_1 & \mathbf{D}_2 \\ I & 0\end{pmatrix} \implies \mathbf{D}_2 \text{ is invertible.} \tag{116}$$

Hence, we obtain $\mathbf{L} = \mathbf{D}_2^{-1}\mathbf{D}_1$. This gives us the required expression for the Lagrangian subspace in $\bar{\mathbf{X}}, \bar{\mathbf{P}}$ coordinates. □

Proof: (*Proof of Proposition 6*) Since, the matrix $\mathbf{D}$ from Proposition 5 consists of left eigenvectors with unstable eigenvalues of the Hamiltonian, it forms an invariant subspace and satisfies

$$\begin{pmatrix}\mathbf{D}_1 & \mathbf{D}_2\end{pmatrix}\begin{pmatrix}\Lambda & -\hat{\mathbf{R}} \\ -\hat{\mathbf{Q}} & -\Lambda^\top\end{pmatrix} = \bar{\Lambda}\begin{pmatrix}\mathbf{D}_1 & \mathbf{D}_2\end{pmatrix} \tag{117}$$

Premultiplying by $\mathbf{D}_2$ inverse, we obtain

$$\begin{pmatrix}-\mathbf{L} & I\end{pmatrix}\begin{pmatrix}\Lambda & -\hat{\mathbf{R}} \\ -\hat{\mathbf{Q}} & -\Lambda^\top\end{pmatrix} = \mathbf{D}_2^{-1}\bar{\Lambda}\mathbf{D}_2\begin{pmatrix}-\mathbf{L} & I\end{pmatrix} \tag{118}$$

Postmultiply by $\begin{pmatrix}I \\ \mathbf{L}\end{pmatrix}$, we obtain

$$\mathbf{L}\Lambda + \Lambda^\top\mathbf{L} - \mathbf{L}\hat{\mathbf{R}}\mathbf{L} + \hat{\mathbf{Q}} = 0 \tag{119}$$

and

$$\Lambda - \hat{\mathbf{R}}\mathbf{L} = -\mathbf{D}_2^\top\bar{\Lambda}^\top\mathbf{D}_2^{-1\top} \implies \sigma(\Lambda - \hat{\mathbf{R}}\mathbf{L}) = \sigma(-\bar{\Lambda})$$

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