Spectral Analysis of Koopman Operator and Nonlinear Optimal Control

Umesh Vaidya, *IEEE Senior Member*

***Abstract*— In this paper, we present an approach based on the spectral analysis of the Koopman operator for the approximate solution of the Hamilton Jacobi equation that arises while solving the optimal control problem. It is well-known that one can associate a Hamiltonian dynamical system with the Hamilton Jacobi equation. Furthermore, the Lagrangian submanifold of the Hamiltonian dynamical system play a fundamental role in solving the Hamilton Jacobi equation. We show that the principal eigenfunctions of the Koopman operator associated with the Hamiltonian dynamical system can be used in constructing the Lagrangian submanifold, thereby approximating the solution of the Hamilton Jacobi equation. We present simulation results to verify the main findings of the paper.**

Koopman Operator, Hamilton Jacobi Equation, Optimal Control.

I. Introduction

Hamilton Jacobi (HJ) equation is at the heart of several problems of interest in systems and control theory [1]. HJ equation and its discrete-time counterpart HJ Bellman (HJB) equation have attracted renewed attention due to the significance of these equations in data-driven control and Reinforcement learning [2], [3]. HJ equation is a nonlinear partial differential equation (PDE), and given the significance of the HJ equation in systems and control theory variety of methods are developed for the approximation of its solution [4].

In the development of numerical methods, the complexity associated with the nonlinear nature of the HJ is broken down by providing an iterative process for solving the HJB equation [5]. The iterative approach for solving SOCP via the HJB equation and the Bellman equation plays a fundamental role in various RL algorithms, including policy iteration, value iteration, and the actor-critic method.

An alternate approach for analyzing and approximating the HJ equation is based on a differential geometric-based interpretation of its solution. It is well known that a Hamiltonian dynamical system is associated with the HJ equation. The Lagrangian submanifolds of the Hamiltonian dynamical system are invariant manifolds used to construct the HJ equation solution. In [6], the authors have exploited this differential geometric approach in developing computational methods for approximating the HJ solution. The technique we discovered in this paper for the approximation of the HJ solution draw parallels to the methods found in [6]. In this paper, we show a strong connection between the differential geometric viewpoint of the HJ equation and the spectral analysis of the Koopman operator.

The recent work involving the Koopman operator for dynamical systems with dissipation provides a way of characterizing stable, unstable manifolds of nonlinear dynamical systems in terms of the zero level sets of Koopman eigenfunctions [7]. The explosion of research activities in Koopman theory provides for systematic data-driven and model-based methods for the computation of Koopman eigenfunctions which can be used to compute Lagrangian submanifolds. This idea is exploited in [8], where the primary focus is to prove the symplectic structure of the lifted Hamiltonian system in the function space. On the other hand, there is also extensive literature on the use of Koopman theory for control [8]–[17] [18]–[20]. However, one of the fundamental challenges with the current approaches to using Koopman theory for control is the bilinear nature of the Koopman-based lifting of control dynamical system.

This paper's main contribution is proposing a novel approach for the computation of Lagrangian submanifold based on the spectral analysis of the Koopman operator. We use the Koopman lifting of a Hamiltonian dynamical system associated with the HJ equation to compute the stable and unstable manifolds as zero-level sets of Koopman eigenfunctions. These eigenfunctions are used to provide a least square-based optimization solution to approximate the Lagrangian submanifold. It is known that the Riccatti equation arises as the linearization of the HJ equation. We show that the solution of the Lagrangian submanifolds admits a decomposition where the Riccatti solution appears from the coefficients of the linear basis function used in the

Financial support from of NSF CPS award and NSF 1932458 is greatly acknowledged. The author is with the Department of Mechanical Engineering, Clemson University, Clemson SC, 29631.

expansion of Koopman eigenfunctions. Hence, the proposed Koopman-based approach for analyzing the HJ equation provides a natural extension of linear system results to nonlinear systems. Finally, we demonstrate the application of the developed framework to optimal control and robust control design problems.

The organization of the paper is as follows. In Section II, we present some preliminaries on Hamiltonian dynamics, the HJ equation, and the spectral theory of the Koopman operator. In Section III, we provide algorithm for the computation of principal eigenfunctions. The main results are presented in Section IV followed by simulation results in Section V. Conclusions are presented in Section VI.

II. Preliminaries and Notations

In this section, we present some preliminaries on Hamiltonian dynamical system, HJ equation, and spectral theory of Koopman operator. We refer the readers to [1], [7], [21], [22] for further details.

Notations: $\mathbb{R}^n$ denotes the n dimensional Euclidean space. We denote by $\mathcal{L}_\infty(M)$, $\mathcal{C}^1(M)$ the space of all essentially bounded real-valued functions and continuously differentiable functions on $M \subset \mathbb{R}^n$ respectively.

A. Hamiltonian dynamics and Lagrangian manifolds

Consider a n-dimensional manifold M with local coordinates $\mathbf{x} = (x_1, \ldots, x_n)^\top$. The cotangent bundle $T^\star M$ is a $2n$-dimensional manifold, with natural local coordinates $(\mathbf{x}, \mathbf{p}) = (x_1 \ldots, x_n, p_1, \ldots, p_n)$. Given the natural coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ for $T^\star M$ we may locally define the canonical two-form ω on $T^\star M$ as $\omega = \sum_i dp_i \wedge dx_i$. The two-form ω is called the (canonical) symplectic form on the cotangent bundle $T^\star M$.

Definition 1 (Lagrangian submanifold)**.** *An n-dimensional submanifold N of $T^\star M$ is Lagrangian if ω restricted to N is zero.*

Now consider any C^2 function $S : M \to \mathbb{R}$, and the n-dimensional submanifold $N_S \subset T^\star M$, in local coordinates given as

$$N_S = \{(\mathbf{x}, \mathbf{p}) \in T^\star M : \; p_i = \frac{\partial S}{\partial x_i} \;\; i = 1, \ldots, n\}. \tag{1}$$

It immediately follows that N_S is Lagrangian. Given a Hamiltonian function $H : T^\star M \to \mathbb{R}$, the associated Hamiltonian vector field X_H on $T^\star M$ is defined in natural coordinates as

$$\dot{\mathbf{x}} = \frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}}, \;\; \dot{\mathbf{p}} = -\frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}}. \tag{2}$$

A submanifold $N \subset T^\star M$ is said to be invariant under the Hamiltonian vector field X_H if any solution of X_H starting in N remains in N. For a given function $S : M \to \mathbb{R}$ consider a Lagrangian submanifold as in Def. (1) and in local coordinates given by (1), then

$$H(\mathbf{x}, S_{\mathbf{x}}^\top(\mathbf{x})) = constant, \quad \forall \mathbf{x} \in M,$$

if and only if N_S is invariant submanifold of X_H. For more details on geometric aspects of Hamiltonian systems and Lagrangian manifolds refer to [22].

B. HJ equation in control theory

HJ equation is central to a variety of problems in control theory, including dissipativity theory, optimal control, robust control, and input-output $\mathcal{L}_2$ gain analysis. The HJ equation is a nonlinear partial differential equation of the form

$$\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) - \frac{1}{2} \frac{\partial V}{\partial \mathbf{x}} \mathbf{R}(\mathbf{x}) \frac{\partial V}{\partial \mathbf{x}}^\top + q(\mathbf{x}) = 0. \tag{3}$$

The HJ equation is to be solved for the unknown $V(\mathbf{x})$. Let $\mathbf{f}$, $\mathbf{R}(\mathbf{x})$, and $q(\mathbf{x})$ satisfies following assumption.

Assumption 1. *We assume that $\mathbf{f} : M \to \mathbb{R}^n$, $\mathbf{R} : M \to \mathbb{R}^{n \times n}$, $q : M \to \mathbb{R}$ are $\mathcal{C}^\infty$ functions. $\mathbf{R}$ is a symmetric matrix for all $x \in M$. Furthermore, $\mathbf{f}(0) = 0, q(0) = 0$, and $\frac{\partial q}{\partial \mathbf{x}}(0) = 0$, $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + O(|\mathbf{x}|^2), q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + O(|\mathbf{x}|^2)$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{Q}$ is a symmetric positive definite matrix.*

There is intimate connection between the HJ equation and the Hamiltonian dynamical system. To see this connection, we define a pre-Hamiltonian obtained by replacing $\frac{\partial V}{\partial \mathbf{x}}$ by $\mathbf{p}^\top$

$$H(\mathbf{x}, \mathbf{p}) = \mathbf{p}^\top \mathbf{f}(\mathbf{x}) - \frac{1}{2} \mathbf{p}^\top \mathbf{R}(\mathbf{x}) \mathbf{p} + q(\mathbf{x}). \tag{4}$$

The pre-Hamiltonian (4) can be used to construct a Hamiltonian dynamical system as in (2).

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \mathbf{R}(\mathbf{x})\mathbf{p} \tag{5}$$

$$\dot{\mathbf{p}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) \mathbf{p} + \frac{1}{2} \left(\frac{\partial \mathbf{p}^\top \mathbf{R}(\mathbf{x}) \mathbf{p}}{\partial \mathbf{x}}\right)^\top - \frac{\partial q}{\partial \mathbf{x}}^\top. \tag{6}$$

The solutions of the HJ equation (3) and the Hamiltonian system (6) are closely connected. In particular, the Lagrangian submanifold of the Hamiltonian vector field is used to construct the solution of the HJ equation.

C. Spectral theory of Koopman operator

Consider a dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in M \subset \mathbb{R}^n. \tag{7}$$

Let $\mathbf{s}_t(\mathbf{x})$ be the solution of the system at time t with initial condition $\mathbf{x}$.

Definition 2 (Koopman Operator)**.** *$\mathbb{U}_t : \mathcal{L}_\infty(M) \to \mathcal{L}_\infty(M)$ for dynamical system (7) is defined as*

$$[\mathbb{U}_t\psi](\mathbf{x}) = \psi(\mathbf{s}_t(\mathbf{x})). \tag{8}$$

The infinitesimal generator of the Koopman operator is

$$\lim_{t\to 0} \frac{(\mathbb{U}_t - I)\psi}{t} = \frac{\partial \psi}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) =: \mathcal{K}_{\mathbf{f}}\psi, \quad t \geq 0. \tag{9}$$

Definition 3 (Eigenvalues and Eigenfunctions of Koopman)**.** *A function $\phi(\mathbf{x}) \in \mathcal{C}^1(M)$ is said to be eigenfunction of the Koopman operator associated with eigenvalue λ if*

$$[\mathbb{U}_t\phi](\mathbf{x}) = e^{\lambda t}\phi(\mathbf{x}). \tag{10}$$

Using the Koopman generator, (10) can be written as

$$\frac{\partial \phi}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = \lambda\phi(\mathbf{x}). \tag{11}$$

The spectrum of the Koopman operator, in general, could be very complicated and could consist of continuous part and discrete or point spectrum. For example, a system with a non-hyperbolic equilibrium point, such as a conservative Hamiltonian system, will have a continuous spectrum. However, it is known that the Koopman operator associated with a system having hyperbolic (i.e., the real part of eigenvalues is not equal to zero) equilibrium point is discrete. It is known that for a system with a hyperbolic equilibrium point, the eigenvalues of the linearization will also form the eigenvalues of the Koopman operator. In the following, we will refer to the eigenfunctions of the Koopman operator associated with the eigenvalues of the Jacobian at linearization as *principal eigenfunctions*. The principal eigenfunctions are also used to characterize the domain of attraction of the equilibrium point [7]. This paper uses these eigenfunctions to identify stable and unstable invariant manifolds of an equilibrium point. We have the following proposition [7].

Proposition 1. *Let x_0 be the hyperbolic equilibrium point of the system (7). Let $\lambda_1, \ldots, \lambda_u$ be eigenvalues with positive real part with associated eigenfunctions $\phi_{\lambda_1}, \ldots, \phi_{\lambda_u}$ and $\lambda_{u+1}, \ldots, \lambda_n$ be eigenvalues with negative real part with eigenfunctions $\phi_{\lambda_{u+1}}, \ldots, \phi_{\lambda_n}$. Then the joint level set of the eigenfunctions*

$$\mathcal{M}_s = \{\mathbf{x} \in M : \phi_{\lambda_1}(\mathbf{x}) = \ldots = \phi_u(\mathbf{x}) = 0\},$$

forms the stable manifold and the joint level set of the eigenfunctions

$$\mathcal{M}_u = \{\mathbf{x} \in M : \phi_{\lambda_{u+1}}(\mathbf{x}) = \ldots = \phi_n(\mathbf{x}) = 0\},$$

is the unstable manifold of the equilibrium point x_0.

Local eigenfunctions can also be defined where (10) is valid on a particular region of the state space and over a finite time interval. [7].

Remark 1. *[7] The Koopman eigenfunctions corresponding to linear dynamical system, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, are given by $\phi_{\lambda_i}(\mathbf{x}) = \mathbf{v}_j^\top \mathbf{x}$, where $\mathbf{v}_j^\top$ is the left eigenvector of $\mathbf{A}$ with eigenvalue λ_j i.e., $\mathbf{v}_j^\top \mathbf{A} = \lambda_j \mathbf{v}_j^\top$.*

III. Computation of Principal Eigenfunctions of Koopman Operator

Consider a dynamical system

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}), \quad \mathbf{z} \in Z \subset \mathbb{R}^m. \tag{12}$$

We make following assumption on the vector field.

Assumption 2. *We assume that the vector field $\mathbf{F} : Z \to \mathbb{R}^m$ is a $\mathcal{C}^\infty$ function. Furthermore, $\mathbf{F}(0) = 0$ and $\mathbf{F}(\mathbf{x}) = \mathbf{E}\mathbf{x} + O(|\mathbf{x}|^2)$ where $\mathbf{E}$ is assumed to be hyperbolic with no eigenvalues on the $j\omega$ axis.*

Let $\phi_k(\mathbf{z})$ for $k = 1, \ldots, m$ be the principal eigenfunctions associated with the eigenvalues λ_k at the linearization, $\mathbf{E}$. The principal eigenfunction will also serve as a change of coordinates, where the system dynamics (12) will be transformed in the eigenfunction coordinates to $\dot{\phi}_k = \lambda_k \phi_k, \quad k = 1, \ldots, m$. Instead of seeking to find eigenfunction as coordinates, one could seek to find a nonlinear change of near identity coordinates of the form

$$\tilde{\mathbf{z}} = \mathbf{z} + \mathbf{h}(\mathbf{z}), \quad \frac{\partial \mathbf{h}}{\partial \mathbf{z}}(0) = 0, \tag{13}$$

that will transform the system equation (12) to linear equation of the form $\dot{\tilde{\mathbf{z}}} = \mathbf{E}\tilde{\mathbf{z}}$. Let $\hat{\mathbf{z}} = \mathbf{W}^\top \tilde{\mathbf{z}}$ be the linear change of coordinates such that rows of $\mathbf{W}^\top$ rows form the left eigenvectors of $\mathbf{E}$ and hence $\mathbf{W}^\top \mathbf{E}(\mathbf{W}^\top)^{-1} = \Lambda_{\mathbf{F}}$ is a real Jordan canonical form. Hence, the principal eigenfunctions, $\Phi(\mathbf{z})$, can be found from the transformed coordinates $\tilde{\mathbf{z}}$ as

$$\Phi(\mathbf{z}) = \mathbf{W}^\top(\mathbf{z} + \mathbf{h}(\mathbf{z})) = \mathbf{W}^\top \tilde{\mathbf{z}}, \ \mathbf{W}^\top \mathbf{E} = \mathbf{W}^\top \Lambda_{\mathbf{F}} \tag{14}$$

Let $\Gamma(\mathbf{z}) = [\gamma_1(\mathbf{x}), \ldots, \gamma_M(\mathbf{x})]^\top$ be the basis function used in the identification of the coordinates. Since the objective is to find purely nonlinear change of coordinates satisfying $\frac{\partial \mathbf{h}}{\partial \mathbf{z}}(0) = 0$, we consider the basis function, Γ, that satisfy $\frac{\partial \gamma_k}{\partial \mathbf{z}}(0) = 0$ and the parameterization for the unknown function $\mathbf{h}(\mathbf{z})$ as $\mathbf{h}(\mathbf{z}) = \mathbf{U}\Gamma(\mathbf{z})$, where $\mathbf{U} \in \mathbb{R}^{m \times M}$. Using this parameterization, we can write

$$\dot{\mathbf{z}} + \frac{\partial \mathbf{h}}{\partial \mathbf{z}}\dot{\mathbf{z}} = \mathbf{E}(\mathbf{z} + \mathbf{h}(\mathbf{z})) \rightarrow \left(I + \mathbf{U}\frac{\partial \Gamma}{\partial \mathbf{z}}\right)\dot{\mathbf{z}} - \mathbf{E}\mathbf{z} - \mathbf{E}\mathbf{U}\Gamma(\mathbf{z}).$$

Let $\{\mathbf{z}_k\}_{k=1}^L$ be the data-set, the problem of determining the matrix $\mathbf{U}$ can be written as following convex optimization problem

$$\min_{\mathbf{U}} \parallel \frac{1}{L}\sum_{k=1}^{L}\left(\mathbf{U}\frac{\partial \Gamma}{\partial \mathbf{z}}(\mathbf{z}_k) + I\right)\mathbf{F}(\mathbf{z}_k) - \mathbf{E}\mathbf{U}\Gamma(\mathbf{z}_k) - \mathbf{E}\mathbf{z}_k \parallel \tag{15}$$

The solution of the above convex optimization problem is used to construct the change of coordinate function $\mathbf{h}(\mathbf{z}) = \mathbf{U}\Gamma(\mathbf{z})$, which is then used to determine the principal eigenfunctions $\Phi(\mathbf{z})$ using Eq. (14).

IV. Koopman Eigenfunctions, Optimal Control, and Computation of Lagrangian Submanifold

For the simplicity of presentation, we consider the optimal control problem for system with single input of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \;\; \mathbf{x} \in M \subseteq \mathbb{R}^n, \; u \in U \subseteq \mathbb{R} \tag{16}$$

with the associated cost function $J = \int_0^\infty q(\mathbf{x}) + u^2 dt$. The optimal control input will be of the form $u^\star(\mathbf{x}) = -\frac{\partial V}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})$. where $V(\mathbf{x})$ is the solution of following HJ Bellman (HJB) equation.

$$\frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{1}{2}\frac{\partial V}{\partial \mathbf{x}}\mathbf{R}(\mathbf{x})\frac{\partial V}{\partial bx}^\top + q(\mathbf{x}) = 0 \tag{17}$$

where $\mathbf{R}(\mathbf{x}) = \mathbf{g}(\mathbf{x})\mathbf{g}^\top(\mathbf{x})$. The pre-Hamiltonian associated with the HJ equation can be obtained by replacing $\frac{\partial V}{\partial \mathbf{x}}$ by $\mathbf{p}$ as follows.

$$H(\mathbf{x}, \mathbf{p}) := \mathbf{p}^\top \mathbf{f}(\mathbf{x}) - \mathbf{p}^\top \mathbf{R}(\mathbf{x})\mathbf{p} + q(\mathbf{x}) = 0. \tag{18}$$

The Hamiltonian dynamical system associated with the HJ equation is written as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) - \mathbf{R}(\mathbf{x})\mathbf{p} \\ \dot{\mathbf{p}} &= -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\top \mathbf{p} + \frac{1}{2}\left(\frac{\partial \mathbf{p}^\top \mathbf{R}(\mathbf{x})\mathbf{p}}{\partial \mathbf{x}}\right)^\top - \frac{\partial q}{\partial \mathbf{x}}^\top, \end{aligned} \tag{19}$$

and the associated linearization of the nonlinear Hamiltonian at the origin as

$$\begin{pmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{p}}} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & -\bar{\mathbf{R}} \\ -\bar{\mathbf{Q}} & -\mathbf{A}^\top \end{pmatrix}\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}} \end{pmatrix} = \mathcal{H}_1 \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{p}} \end{pmatrix} \tag{20}$$

where $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0), \bar{\mathbf{R}} = \mathbf{R}(0)$, and $\bar{\mathbf{Q}} = \frac{\partial^2 q}{\partial \mathbf{x}^2}(0)$. The proposed approach for approximating the Lagrangian submanifold can be divided into two steps. 1) First, we construct principal eigenfunctions of the Koopman operator corresponding to the Hamiltonian dynamical system (19) based on the construction procedure outlined in Section III. 2) The second step is to use the zero level set of Koopman eigenfunctions to identify the Lagrangian submanifold. We have already discussed our convex approach for the computation of principal eigenfunction in Section III. We now proceed with step 2 on the computation of the Lagrangian submanifold.

A. Computation of Lagrangian Submanifold

We will use the procedure outlined in Section III for the computation of the Koopman eigenfunctions. Borrowing notations from Section III, $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$ system now corresponds to system equation (19) with $\mathbf{z} = (\mathbf{x}^\top, \mathbf{p}^\top)^\top$. We decompose $\mathbf{W}^\top$ from (14) and which for this case constitutes the left eigenvector of $\mathcal{H}_1$ (Eq. (20)) matrix as

$$\mathbf{W}^\top = \begin{pmatrix} \mathbf{W}_u & \mathbf{W}_s \end{pmatrix}^\top \in \mathbb{R}^{2n \times 2n}, \tag{21}$$

where $\mathbf{W}_u^\top \in \mathbb{R}^{n \times 2n}, \mathbf{W}_s^\top \in \mathbb{R}^{n \times 2n}$ corresponds to unstable and stable linear subspace respectively and satisfies $\mathbf{W}^\top \mathcal{H}_1 = \Lambda_{\mathbf{F}} \mathbf{W}^\top$ with $\mathcal{H}_1$ given in (20) and $\Lambda_{\mathbf{F}}$ corresponds to the matrix of eigenvalues of $\mathcal{H}_1$ in real Jordan canonical form. The eigenfunctions corresponding to the unstable eigenvalues is given by

$$\mathbf{W}_u^\top \left(\begin{pmatrix} \mathbf{x}^\top & \mathbf{p}^\top \end{pmatrix}^\top + \mathbf{U}\Gamma(\mathbf{z})\right), \tag{22}$$

where the matrix $\mathbf{U} \in \mathbb{R}^{2n \times M}$ is identified using the procedure outlined in Section III with the following choice of nonlinear basis function $\Gamma(\mathbf{z})$

$$\Gamma(\mathbf{z}) = \begin{pmatrix} \Xi(\mathbf{x})^\top & (\mathbf{x} \otimes \mathbf{p})^\top & (\Xi \otimes \mathbf{p})^\top \end{pmatrix}^\top, \tag{23}$$

where $\Xi : M \rightarrow \mathbb{R}^N$ is vector valued function and satisfies $\frac{\partial \Gamma}{\partial \mathbf{z}}(0) = 0$ and hence $M = N + n^2 + Nn$. Furthermore, the basis functions are chosen to be linear in variable $\mathbf{p}$. Given the form of $\mathbf{W}_u$ and $\Gamma(\mathbf{z})$, we can decompose (22) to determine the stable manifold as the zero level set of unstable eigenfunctions as

$$0 = \mathbf{W}_u^\top \left(\begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix} + \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{pmatrix}\begin{pmatrix} \Xi(\mathbf{x}) \\ \mathbf{x} \otimes \mathbf{p} \\ \Xi(\mathbf{x}) \otimes \mathbf{p} \end{pmatrix}\right), \tag{24}$$

where $[\mathbf{U}]_{ij} = U_{ij}$ is the splitting of the $\mathbf{U}$ matrix. The zero level set of the unstable eigenfunctions is decomposed into linear terms $\mathbf{W}_u^\top \mathbf{z}$ and nonlinear terms $\mathbf{W}_u^\top \mathbf{U}\Gamma(\mathbf{z})$. The zero level sets of $\Psi(\mathbf{z})$ can be identified with the zero level sets of the linear and nonlinear terms, namely

$$\{\mathbf{z} : \Psi(\mathbf{z}) = 0\} \equiv \{\mathbf{z} : \mathbf{W}_u^\top \mathbf{z} = 0\} \cap \{\mathbf{z} : \mathbf{W}_u^\top \mathbf{U}\Gamma(\mathbf{z}) = 0\}$$

The objective is to find Lagrangian submanifold of the form $\mathbf{p} = \frac{\partial V}{\partial \mathbf{x}}$ for some unknown function V such that $\Psi(\mathbf{x}, \mathbf{p} = \frac{\partial V}{\partial \mathbf{x}}) = 0$. Based of the decomposition of the zero level set, we consider following decomposition and paramaterization for the unknown function $V(\mathbf{x})$

$$V(\mathbf{x}) = V_l(\mathbf{x}) + V_n(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{J}_l \mathbf{x} + \frac{1}{2}\Xi(\mathbf{x})^\top \mathbf{J}_n \Xi(\mathbf{x})$$

where $\mathbf{J}_l \in \mathbb{R}^{n\times n}$ and $\mathbf{J}_n \in \mathbb{R}^{N\times N}$ corresponds to the paramaterization of the linear and nonlinear terms respectively. We have

$$\mathbf{p} = \frac{\partial V}{\partial \mathbf{x}}^\top = \mathbf{J}_l \mathbf{x} + \Xi_{\mathbf{x}}(\mathbf{x})^\top \mathbf{J}_n \Xi(\mathbf{x}) =: \mathbf{p}_l + \mathbf{p}_n. \tag{25}$$

Using the decomposition, $\Psi(\mathbf{x}, \mathbf{p}) = 0$ if and only if

$$\mathbf{W}_u \mathbf{z} = 0, \quad \& \quad \mathbf{W}_u^\top \mathbf{U}\Gamma(\mathbf{z}) = 0.$$

Substituting, the linear paramaterization, $\mathbf{p}_l = \mathbf{J}_l \mathbf{x}$ in $\mathbf{W}_u^\top \mathbf{z} = 0$ we obtain

$$\mathbf{W}_u^\top \begin{pmatrix} \mathbf{x}^\top & \mathbf{x}^\top \mathbf{J}_l \end{pmatrix}^\top = 0 \implies (\mathbf{W}_{u1}^\top + \mathbf{W}_{u2}^\top \mathbf{J}_l)\mathbf{x} = 0, \tag{26}$$

where we have assumed $\mathbf{W}_u^\top = (\mathbf{W}_{u1}^\top \quad \mathbf{W}_{u2}^\top)$ with $\mathbf{W}_{uk} \in \mathbb{R}^{n\times n}$ for $k = 1, 2$. (26) gives us

$$\mathbf{J}_l^\star = -(\mathbf{W}_{u2}^\top)^{-1} \mathbf{W}_{u1}^\top. \tag{27}$$

It is relatively easy to prove that the $\mathbf{J}_l^\star$ matches with the solution of Riccatti equation. In particular $\mathbf{J}_l^\star$ satisfies following Riccatti equation

$$\mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\bar{\mathbf{R}}\mathbf{P} + \bar{\mathbf{Q}} = 0, \tag{28}$$

where $\mathbf{V}\mathbf{J}_l^\star \mathbf{V}^\top =: \mathbf{P}$ with rows of $\mathbf{V}^\top$ consisting of left eigenvectors of $\mathbf{A}$ and hence satisfy $\mathbf{V}^\top \mathbf{A} = \Lambda \mathbf{V}^\top$. Now substituting the nonlinear paramaterization of manifold in $\mathbf{W}_u^\top \mathbf{U}\Gamma(\mathbf{z}) = 0$, we obtain

$$\mathbf{W}_u^\top \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{pmatrix} \begin{pmatrix} \Xi(\mathbf{x}) \\ \mathbf{x} \otimes \Xi_{\mathbf{x}}^\top \mathbf{J}_n \Xi \\ \Xi(\mathbf{x}) \otimes \Xi_{\mathbf{x}}^\top \mathbf{J}_n \Xi \end{pmatrix} = 0, \tag{29}$$

Since the above is true for all $\Xi(\mathbf{x})$, it is equivalent to

$$\mathcal{J}(\mathbf{x}, \mathbf{J}_n) := \mathcal{B}(\mathbf{x})\mathbf{J}_n - \bar{\mathcal{A}} = 0,$$

$$\text{where,} \quad \bar{\mathcal{A}} = -\mathbf{W}_{u1}^\top U_{11} - \mathbf{W}_{u2}^\top U_{21},$$

$$\mathcal{B}(\mathbf{x}) := (\mathbf{W}_{u1}^\top U_{12} + \mathbf{W}_{u2}^\top U_{22})\mathbf{x} \otimes \Xi_{\mathbf{x}}^\top + (\mathbf{W}_{u1}^\top U_{13} + \mathbf{W}_{u2}^\top U_{23})\Xi \otimes \Xi_{\mathbf{x}}^\top.$$

The objective is to determine $\mathbf{J}_n$ so that the above equality is satisfied. Since the equation is linear in $\mathbf{J}_n$, and function of $\mathbf{x}$, we can find $\mathbf{J}_n$ as a solution of following least square optimization problem

$$\min_{\mathbf{J}} \parallel \bar{\mathcal{B}}\mathbf{J} - \bar{\mathcal{A}} \parallel, \tag{30}$$

$$\text{where,} \quad \bar{\mathcal{B}} := \frac{1}{D}\sum_{k=1}^{D} \mathcal{B}(\mathbf{x}_k), \tag{31}$$

and $\{\mathbf{x}_k\}_{k=1}^D$ are the data points. The above optimization admits analytical solution given by

$$\mathbf{J}_n^\star = \bar{\mathcal{B}}^\dagger \bar{\mathcal{A}}, \tag{32}$$

where $\dagger$ stands for pseudo-inverse. This leads to the following approximation of the Lagrangian submanifold

$$\Lambda_V := \{(\mathbf{x}, \mathbf{p}) : \mathbf{p} = \mathbf{J}_l^\star \mathbf{x} + \Xi_{\mathbf{x}}^\top(\mathbf{x})\mathbf{J}_n^\star \Xi(\mathbf{x})\}, \tag{33}$$

where $\mathbf{J}_l^\star$ and $\mathbf{J}_n^\star$ are as given in (27) and (32).

Remark 2. *The expression for the Lagrangian submanifold in (33) admits a decomposition consisting of linear and nonlinear terms. Hence, when only linear basis functions are used in the approximation, the expression for the Lagrangian manifold will reduce to the solution obtained using the Riccatti equation.*

V. Simulation Results

All the simulations codes are developed on MATLAB and ran on a computer consisting of 16 GB of RAM and a 3.8 GHz Intel Core i7 processor. We approximated the principal eigenfunctions using monomial basis functions (i.e., $\{\mathbf{x}^K\}$).

We consider controlling a 2-D nonlinear oscillator.

$$\dot{x}_1 = x_2 \tag{34}$$

$$\dot{x}_2 = -x_1 + 0.105x_2 + \frac{1}{2}x_2^2 x_1 + 1.1x_1 x_2 + (1.1 + x_1)u$$

We assume quadratic cost as $q(\mathbf{x}) = x_2^2$ and control u^2. For this example the HJ equation admits an analytical solution [23]. The optimal control is given by $u^* = -(1.1 + x_1)x_2$ and the optimal cost function is $V^* = x_1^2 + x_2^2$. To approximate the eigenfunctions from data we used the methodology developed in section III. The system was discretized using Euler method to obtain

one-step time-series data with $1e^4$ initial conditions. In Fig. 1, we show the comparison between the true optimal control and Koopman spectrum-based optimal control. In Fig. 2, we compare the cost with linearized LQR control.

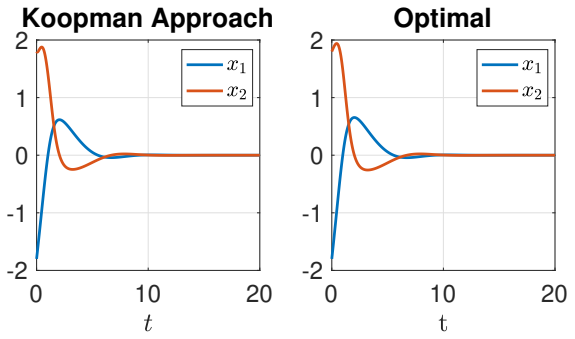

Fig. 1. Comparison of close-loop trajectories between Koopman approach and Optimal controllers.

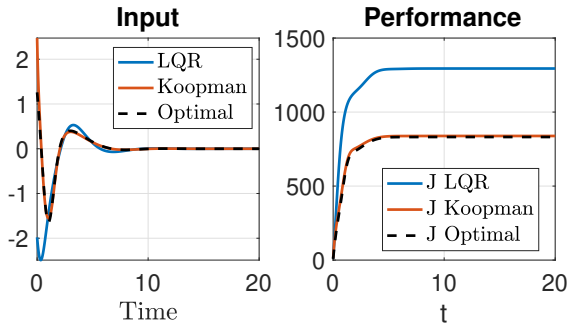

Fig. 2. Comparison of control inputs, and empirical cost function with LQR control input, Koopman approach, and optimal.

VI. Conclusions

The paper presents novel apporoach for approximating the solution of HJ equation based on the spectral analysis of Koopman operator. The eigenfunctions of the Koopman operator associated with the Hamiltonian system is used in the computation of the solution to the HJ equation. The HJ solution is used for solving the optimal control problem.

References

[1] A. Vandershaft, *L_2-Gain and passivity techniques in nonlinear control*. London: Springer, 1999.

[2] D. P. Bertsekas and S. E. Shreve, *Stochastic optimal control: the discrete-time case*. Athena Scientific, 1996, vol. 5.

[3] R. S. Sutton and A. G. Barto, *Reinforcement learning: An introduction*. MIT press, 2018.

[4] Y. Achdou, G. Barles, H. Ishii, and G. L. Litvinov, "Hamilton-jacobi equations: approximations, numerical analysis and applications," 2013.

[5] R. W. Beard, G. N. Saridis, and J. T. Wen, "Galerkin approximations of the generalized hamilton-jacobi-bellman equation," *Automatica*, vol. 33, no. 12, pp. 2159–2177, 1997.

[6] N. Sakamoto and A. J. van der Schaft, "Analytical approximation methods for the stabilizing solution of the hamilton–jacobi equation," *IEEE Transactions on Automatic Control*, vol. 53, no. 10, pp. 2335–2350, 2008.

[7] I. Mezić, "Spectrum of the Koopman operator, spectral expansions in functional spaces, and state-space geometry," *Journal of Nonlinear Science*, vol. 30, no. 5, pp. 2091–2145, 2020.

[8] M. E. Villanueva, C. N. Jones, and B. Houska, "Towards global optimal control via koopman lifts," *Automatica*, vol. 132, p. 109610, 2021.

[9] S. L. Brunton, B. W. Brunton, J. L. Proctor, and J. N. Kutz, "Koopman invariant subspaces and finite linear representations of nonlinear dynamical systems for control," *PloS one*, vol. 11, no. 2, p. e0150171, 2016.

[10] S. Peitz and S. Klus, "Koopman operator-based model reduction for switched-system control of pdes," *Automatica*, vol. 106, pp. 184–191, 2019.

[11] J. Borggaard, J. A. Burns, A. Surana, and L. Zietsman, "Control, estimation and optimization of energy efficient buildings," in *2009 American Control Conference*. IEEE, 2009, pp. 837–841.

[12] I. Abraham and T. D. Murphey, "Active learning of dynamics for data-driven control using koopman operators," *IEEE Transactions on Robotics*, vol. 35, no. 5, pp. 1071–1083, 2019.

[13] M. Korda and I. Mezić, "Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control," *Automatica*, vol. 93, pp. 149–160, 2018.

[14] A. Sootla, A. Mauroy, and D. Ernst, "Optimal control formulation of pulse-based control using koopman operator," *Automatica*, vol. 91, pp. 217–224, 2018.

[15] S. E. Otto and C. W. Rowley, "Koopman operators for estimation and control of dynamical systems," *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 4, pp. 59–87, 2021.

[16] K. Fackeldey, M. Oster, L. Sallandt, and R. Schneider, "Approximative policy iteration for exit time feedback control problems driven by stochastic differential equations using tensor train format," *arXiv preprint arXiv:2010.04465*, 2020.

[17] E. Kaiser, J. N. Kutz, and S. L. Brunton, "Data-driven discovery of koopman eigenfunctions for control," *Machine Learning: Science and Technology*, vol. 2, no. 3, p. 035023, 2021.

[18] B. Huang and U. Vaidya, "A convex approach to data-driven optimal control via perron-frobenius and koopman operators," *IEEE Transactions on Automatic Control*, 2022.

[19] H. Yu, J. Moyalan, U. Vaidya, and Y. Chen, "Data-driven optimal control of nonlinear dynamics under safety constraints," *IEEE Control Systems Letters*, 2022.

[20] J. Moyalan, H. Choi, Y. Chen, and U. Vaidya, "Sum of squares based convex approach for optimal control synthesis," in *2021 29th Mediterranean Conference on Control and Automation (MED)*. IEEE, 2021, pp. 1270–1275.

[21] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics*. New York: Springer-Verlag, 1994.

[22] R. Abraham and J. E. Marsden, *Foundations of mechanics*. American Mathematical Soc., 2008, no. 364.

[23] V. Nevistić and J. A. Primbs, "Constrained nonlinear optimal control: a converse hjb approach," 1996.