Key Notes

- Linear Transformation
  Given vector spaces V and W, the transformation F: $V \to W$, which assigns a single vector F($\overrightarrow{u}$) in W to every vector $\overrightarrow{u} \in V$, is said to be a linear transformation if

  1. **Closed Under Addition**
     for all vectors $\overrightarrow{u}, \overrightarrow{v} \in V$
     $$F(\overrightarrow{v} + \overrightarrow{u}) = F(\overrightarrow{v}) + F(\overrightarrow{u})$$

  2. **Closed Under Scalar Product**
     $\forall \overrightarrow{u} \in V$ and real number λ
     $$F(\lambda \overrightarrow{u}) = \lambda F(\overrightarrow{u})$$

  Theorem. *If V_1, V_2, V_3 are vector spaces and $F : V_1 \to V_2, G : V_2 \to V_3$ are linear transformation then $H : V_1 \to V_3$ defined by*
  $$H(\overrightarrow{u}) = G(F(\overrightarrow{u}))$$
  is also a linear transformation.

Example Problems

1. $F_1(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y \\ 2z \end{bmatrix}$ is a linear transformation because

   - It is closed under addition.
   $$F_1(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}) = \begin{bmatrix} u_1 + u_2 + v_1 + v_2 \\ 2(u_3 + v_3) \end{bmatrix} = F_1(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}) + F_2(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix})$$

   - It is closed under scalar product, let c be a constant
   $$F_1(c \begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} cx + cy \\ c2z \end{bmatrix} = c \begin{bmatrix} x+y \\ 2z \end{bmatrix} = cF_1(\begin{bmatrix} x \\ y \\ z \end{bmatrix})$$

1. For each $F : R^2 \to R^2$ determine whether F is a linear transformation

   a) $F(\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$

   b) $F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2 \\ x - y \end{bmatrix}$

2. For each $G : R^3 \to R^2$ determine whether G is a linear transformation

   a) $G(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x \\ yz \end{bmatrix}$

   b) $G(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x + 2z \\ 3y - 2 \end{bmatrix}$