Pg 23 # 2, 3, 5, 6, 7

2. $W = \{ f(a,b) : a,b \in \mathbb{R} \}$

$$f(a,b) = ab + (a-b)X + (a+b)X^2$$

note: $ab = \dfrac{(a+b)^2 - (a-b)^2}{4}$

To show W is not a subspace of $\mathbb{R}_2[x]$ it is enough to show that W is not closed under scalar multiplication (or vector addition). To show it is not closed under scalar multiplication, it is enough to find a vector $w \in W$ and a real number c such that $cw \notin W$.

Let $w = f(1,1) = 1 + 2X^2$

look at $2w = 2 + 4X^2 \stackrel{?}{=} ab + (a-b)X + (a+b)X^2$

$2 + 4X^2 \in W \iff 2 = \dfrac{4^2 - 0^2}{4} = 4$

$\Rightarrow 2w \notin W$

3. $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ ; \ 3x - 2y + 4z = 0 \right\}$

For $w_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \ w_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in W$

$$\Rightarrow \ w_1 + w_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

with $3(x_1+x_2) - 2(y_1+y_2) + 4(z_1+z_2)$

$= (3x_1 - 2y_1 + 4z_1) + (3x_2 - 2y_2 + 4z_2)$

$= 0$

$\Rightarrow w_1 + w_2 \in W$

$\forall c \in \mathbb{R}, \ w \in W, \ cw = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix}$

with $3cx - 2cy + 4cz$

$= c(3x - 2y + 4z) = 0$

$\Rightarrow cw \in W$

$\Rightarrow$ subspace

5. $U + W = \{ u + w \ ; \ u \in U, \ w \in W \}$

$\forall \ y \in U+W \ \ \exists \ u_y \in U, \ w_y \in W$ s.t

$y = u_y + w_y$

$\Rightarrow \forall c \in \mathbb{F} \ \ cy = cu_y + cw_y \in U + W$

since U, W closed under scalar multiplication

6. V v.s. $U, W \subsetneq V$ U, W subspaces,

$U \not\subseteq V$, $V \not\subseteq U$.

$U \cup W = \{ v ; v \in U \text{ or } v \in V \text{ (or both)} \}$

if $v \in U \cup W \Rightarrow \forall c \in F$ $cv \in U \cup W$

b/c if $v \in U \Rightarrow cv \in U$ and if $v \in W \Rightarrow cv \in W$.

$\Rightarrow$ closed under scalar multiplication.

Let $u \in U, w \in W$ s.t. $u, w \notin U \cap W$

since U is a subspace, if $u + w \in U$

then $-u + u + w = w \in U$, which it isn't

$\Rightarrow u + w \notin U$. Similarly, $u + w \notin W$.

Thus, $u + w \notin U \cup W$ and $U \cup W$ is

not a subspace.

7. $\mathbb{R}^2 = \{ (x, 0) ; x \in \mathbb{R} \} \oplus \{ (0, y) ; y \in \mathbb{R} \}$

$= \{ (x, 0) ; x \in \mathbb{R} \} \oplus \{ (x, x) ; x \in \mathbb{R} \}$