Math 242 Final

Name:

Question	Points	Score
1	7	
2	10	
3	12	
4	40	
5	5	
6	32	
7	10	
8	6	
9	6	
10	6	
11	16	
Total:	150	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. (7 points) Say $g(x) = e^{3x-2}$. Find, and simplify, a formula for $g^{-1}(x)$.

2. Compute the derivatives of the following functions. You do not have to simplify your answers.

(a) (5 points) $f(x) = 2^{2x} + \ln(7)$.

(b) (5 points) $f(x) = (\sin x)^x$.

3. Compute the following limits. You must justify your solution using algebraic manipulations and / or l'Hôpital's rule for full credit.

   (a) (6 points) $\lim_{n\to\infty} \frac{\ln n}{e^n}$.

   (b) (6 points) $\lim_{x\to 0^+} (3/x)^x$.

4. Compute the following integrals, or say if they diverge.

   (a) (10 points) $\displaystyle\int \sin^5(x)dx$.

(b) (10 points) $\int x \ln(\sqrt{x}) dx$.

(c) (10 points) $\displaystyle\int_{3}^{\infty} \frac{1}{2x^2 - x - 1} dx.$

(d) (10 points) $\int_0^{\frac{1}{4}} \sqrt{1-4t^2}\, dt$.

5. (5 points) Compute the sum of the convergent series $\sum_{n=0}^{\infty} \frac{3^{n-1}}{4^n}$. Simplify your answer.

6. For each of the following series, say whether they converge or diverge. For full credit, you must justify your solutions, and state clearly which test(s) you are using (if any).

   (a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^{4/3}}$.

   (b) (8 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2+n}$.

(c) (8 points) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(n+2)!}$.

(d) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n!)}{1+n^2}$.

7. (10 points) Find the values of x for which the power series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}2^n}(x-1)^n$$

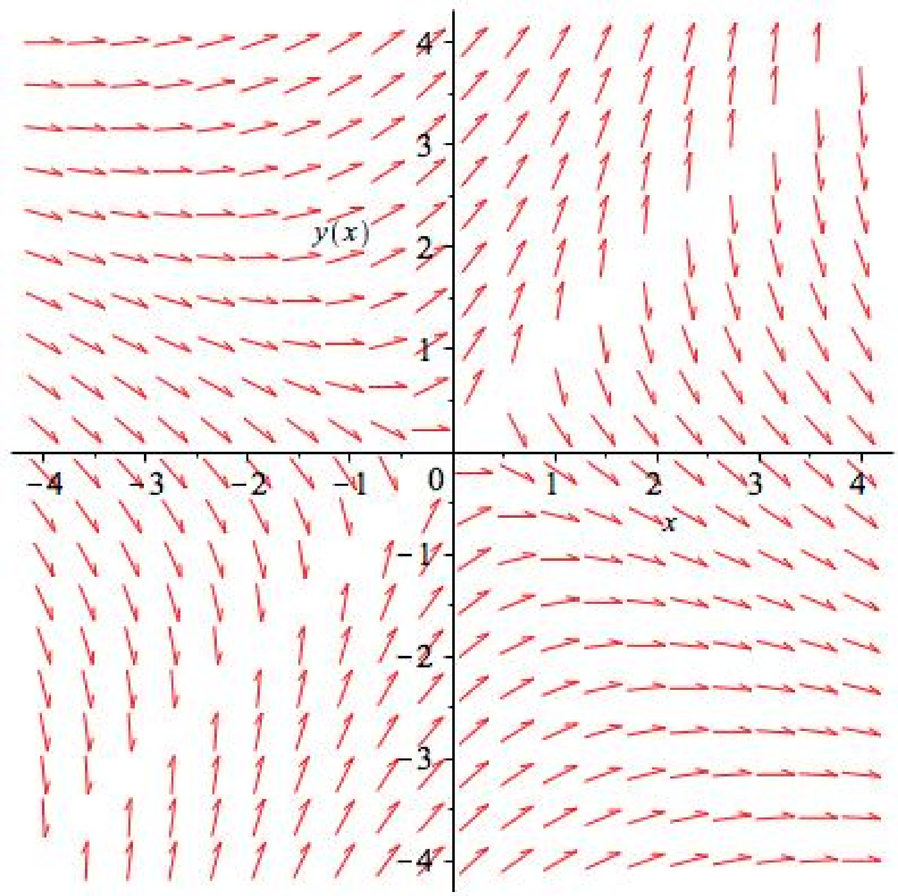
(a) converges absolutely; (b) converges conditionally; (c) diverges. Justify each answer.

8. (6 points) Find the order two Taylor polynomial for $\sqrt{x}$, centered at $a = 1$.

9. (6 points) The function $f(x) = \cos(3x)$ is sometimes approximated by its second-order Taylor polynomial, $P_2(x) = 1 - 9x^2/2$, for small values of x.

   Use a technique from the course to give a bound to the error of this estimate on the interval $[0, 0.1]$.

10. Consider the slope field pictured below.



(a) (3 points) Which of the differential equations below matches this slope field?

(a) $\dfrac{dy}{dx} = \dfrac{x}{y}$, (b) $\dfrac{dy}{dx} = x^2$, (c) $\dfrac{dy}{dx} = \dfrac{y+x}{y-x}$, (d) $\dfrac{dy}{dx} = \sin y$.

(b) (3 points) Sketch the solution to this differential equation that satisfies $y(0) = 1$ on the slope field.

11. Solve the following differential equations. Either give the general solution, or solve for a particular solution satisfying the given initial conditions. Your solution must give an explicit formula for y for full credit.

    (a) (8 points) $\dfrac{dy}{dx} = x\sqrt{1-y^2}, \quad y(0) = 1.$

(b) (8 points) $y' + \frac{1}{x}y = \cos x$.

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Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{(true for } -1 < x < 1\text{)}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \quad \text{(true for all } x\text{)}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \quad \text{(true for } x < -1 \text{ and } x > 1\text{)}$$

- Pythagorean identities (true for all x where the functions involved are defined).

$$\sin^2(x) + \cos^2(x) = 1, \quad \tan^2(x) + 1 = \sec^2(x), \quad 1 + \cot^2(x) = \csc^2(x).$$

- Reduction of power formulas / double angle formulas for sine and cosine (true for all x).

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x)), \quad \sin^2(x) = \frac{1}{2}(1-\cos(2x)), \quad 2\sin(x)\cos(x) = \sin(2x).$$

- Addition formulas for sine and cosine (true for all x and y).

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

- Integrals of tangent and secant.

$$\int \tan(x)dx = -\ln|\cos(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C.$$

- Standard power series expansions (centered at $a = 0$).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(valid for all } x\text{).}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{(valid for all } x\text{).}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{(valid for all } x\text{).}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad \text{(valid for } |x| < 1\text{).}$$

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n \qquad \text{(valid for } |x| < 1\text{).}$$

- Error estimate for approximations by Taylor polynomials.
  Say $f(x)$ is a function with derivatives of all orders on an interval $[b, c]$, and a is a point in $[b, c]$. Say $P_N(x)$ is the N^{th} Taylor polynomial for $f(x)$ centered at a, and $R_N(x) = f(x) - P_N(x)$ is the error when approximating $f(x)$ by $P_N(x)$. Then for all x in $[b, c]$

$$|R_N(x)| \leq \frac{M|x-a|^{N+1}}{(N+1)!},$$

  where M is the largest value taken by the $(N+1)^{\text{st}}$ derivative of $f(x)$ on $[b, c]$.