WARNING: the sample exam is to give you an idea of the length and difficulty level of the actual exam. It does not cover all the topics that might appear on the actual midterm.

1. (a) State the definition of a subspace of a vector space
   (b) One of the following sets is a subspace of $M_2(F)$ the other is not. Say which is which and justify your reasons for why it is not a subspace

   (a) $\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(F)|$ a = 0 and d = 0$\}$

   (b) $\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(F)$ a = 0 or d = 0 $\}$

2. (a) State the definition of what it means for a list $v_1, ..., v_n$ in a vector space to span, and what it means for the list to be linearly dependent.
   (b) Prove that if the list of elements $v_1, ..., v_n$ spans a vector space V and if v is any element of V then the list $v_1 ... v_n$ v is linearly independent.

3. (a) State the definition of the range of linear operator
   (b) Let $T \in L(F^n)$ be a linear operator and say the matrix of T with respect to the standard basis is $[a_{ij}]$. Show that the range of T equals the span of

   $$\{(a_{1i}, a_{2i}, ..., a_{ni}|i \in |\{1, ..., n\}\}$$

4. (a) State what ie means for an operator to have an upper triangular matrix with respect to some basis.
   (b) Let $S, T \in L(V)$ and $v_1, ..., v_n$ be a basis for V so that the matrices of S and T with respect to these bases are upper triangular. Prove that the matrix of ST with respect to this basis is also upper triangular.

5. Give examples of operators on C^2 with the following properties.
   (a) Invertible and not diagonalizable
   (b) Self adjoint non zero and not invertible
   (c) Normal and not self adjoint
   (d) Nilpotent and non-zero
   (e) Positive, inveritble, and not the identity
   (f) Self adjoint and neither T nor -T is positive

6. (a) State the Cauchy-Schwarz inequality and the Riesz representation theorem
   (b) Let V be finite dimensional inner product space and let $\phi : V \to F$ be a linear function. Prove that there is a constant $c > 0$ such that for all $v \in V$ $|\phi(v)| \le c||v||$

7. (a) State the spectral theorem for self-adjoint operators on real inner product space
   (b) Let V be a finite dimensional real inner product space and let T $\in L(V)$ be a self adjoint operator. Prove that if $T^3 = I$ then T = I.
   (c) Give an example of normal operator $T \in L(V)$ on a complex inner product space such that $T^3 = I$ but such that $T \neq I$

8. (a) State the definition of an isometry
   (b) Let V be a real inner product space, let $S \in L(V)$ be an isometry and let $\lambda \in \mathbb{R}$ be an eigenvalue of S. Prove that λ is either 1 or -1
   (c) Given an example of an isometry $S \in L(\mathbb{R}^{\ntrianglerighteq}$ that has no eigenvalues.

9. (a) State the definition of an eigenvector of an operator
   (b) Let $\lambda \in F$ be a scalar and $T \in L(V)$ be an operator. Assume that for some non-zero $v \in V, (T-)^n v = 0$ Prove that λ is an eigenvalue of T.

10. (a) State the definition of a nilpotent operator
    (b) Prove that a nilpotent has a non-trivial null space
    (c) Using the previous part (or otherwise) prove that a nilpotent operator on a finite dimesnional vector space cannot be surjective

11. (a) State the definition of a generalized eigenspace of an operator
    (b) Give examples of operators T $\in L(C^3)$ with the following properties or say if no operator exists
    (i) An operator with a three dimensional generalized eigenspace and a one dimensional eigenspace
    (ii) An operator with exactly 2 eigenvalues and each of whose generalized eigenspace equal the corresponding eigenspace
    (iii) An operator with exactly 2 eigenvalue and none of whose generalized eigensapces equas the corresponding eigenspace