Lesson 11. Nonstationary Poisson Processes

1 Overview

- We've been studying Poisson processes with a **stationary** arrival rate λ
  - In other words, λ doesn't change over time
- This lesson: what happens when the arrival rate is **nonstationary**?
  - In other words, the arrival rate $\lambda(\tau)$ is a function of time τ
- Main idea: we transform a stationary Poisson process with arrival rate $\lambda = 1$ into a **nonstationary Poisson process** with a time-dependent arrival rate $\Lambda(\tau)$

2 Integrated rate functions

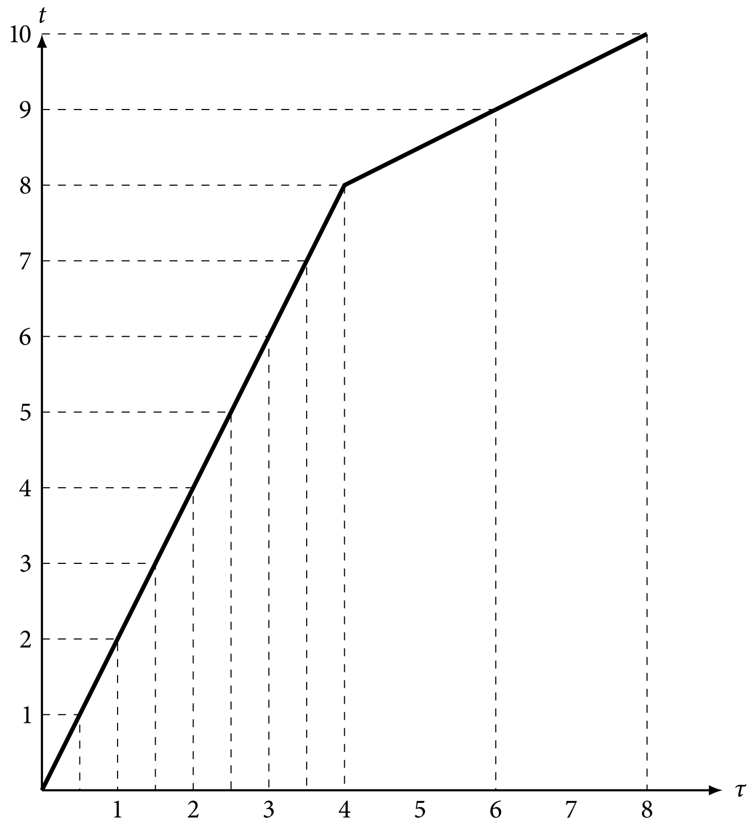
You have been put in charge of studying the operations at a helicopter maintenance facility. The data indicates that the facility is busier in the morning than in the afternoon. In the morning (8:00 - 12:00), the average time between helicopters arrivals is 0.5 hours. On the other hand, in the afternoon (12:00 - 16:00), the average time between helicopter arrivals is 2 hours.

- Let's say that $\tau = 0$ corresponds to 8:00
- Therefore, the (expected) arrival rate $\lambda(\tau)$ as a function of τ (in hours) is:

- We can compute the expected number of arrivals by time τ:

- $\Lambda(\tau)$ is called the **integrated-rate function**
- For the arrival rate $\lambda(\tau)$ given above, the integrated-rate function is

- A graph of the integrated-rate function $\Lambda(\tau)$:



- The inverse of the integrated-rate function $\Lambda(\tau)$:

- Key idea: τ and t represent <u>different time scales</u> connected by $t = \Lambda(\tau)$ or $\tau = \Lambda^{-1}(t)$
    - t represents the time scale for a stationary Poisson process with arrival rate 1
    - τ represents the time scale of a nonstationary Poisson process

- Why does this work? Intuitively, can be seen from the graph above

3 Nonstationary Poisson processes, formally

- Consider a Poisson process with arrival rate 1 with:
  - Y_t = number of arrivals by time t
  - T_n = time of nth arrival
- We can transform this into a **nonstationary Poisson process** with integrated-rate function $\Lambda(\tau)$:
  - Z_τ = ______ = number of arrivals by time τ
  - U_n = ______ = time of nth arrival
- The number of arrivals in the interval $(\tau, \tau + \Delta\tau]$ is
- Therefore, $E[Z_{\tau+\Delta\tau} - Z_\tau] =$
- A nonstationary Poisson process satisfies the independent-increments property:
- The probability distribution of the number of arrivals in $(\tau, \tau + \Delta\tau]$ depends on both $\Delta\tau$ and τ

  $\Rightarrow$ The stationary-increments and memoryless properties no longer apply
- For more details, see SMAS page 112

Example 1. In the maintenance facility example above:

a. What is the probability that 7 helicopters arrive between 8:00 and 13:00, given that 5 arrived between 8:00 and 11:00?
b. What is the expected number of helicopters to arrive between 10:00 and 14:00?

Example 2. Cantor's Car Repair is open from 9:00 ($\tau = 0$) to 15:00 ($\tau = 360$). Customers arrive according to a nonstationary Poisson process; the arrival rate at time τ is

$$\lambda(\tau) = \begin{cases} 1/6 & \text{if } 0 \leq \tau < 180, \\ 1/5 & \text{if } 180 \leq \tau < 360 \end{cases}$$

a. Find the integrated rate function $\Lambda(\tau)$. What does $\Lambda(\tau)$ mean in the context of the problem?
b. What is the probability that 5 customers arrive between 11:00 and 13:00?
c. What is the expected number of customers that arrive between 11:00 and 13:00?
d. If 15 customers have arrived by 11:00, what is the probability that more than 60 customers will have arrived throughout the course of the day?

4 Exercises

Problem 1 (SMAS Exercise 5.20)**.** Patients arrive at a hospital emergency room according to a nonstationary Poisson process with arrival rate function

$$\lambda(\tau) = \begin{cases} 1 & \text{if } 0 \le \tau < 6 \\ 2 & \text{if } 6 \le \tau < 13 \\ \frac{1}{2} & \text{if } 13 \le \tau < 24 \end{cases}$$

where time is measured in hours and time 0 is 6 a.m.

a. Derive the integrated rate function for this model.
b. What is the probability that the doctor will see more than 12 patients between 8 a.m. and 2 p.m.? What is the expected number of patients the doctor will see during that time?
c. If the doctor has seen 6 patients by 8 a.m., what is the probability that the doctor will see a total of 9 patients by 10 a.m.?
d. What is the probability that she will see her first patient in 15 minutes or less after coming on duty?
e. What is the probability that the doctor will see her thirteenth patient before 1 p.m.?

Problem 2 (SMAS Exercise 5.21, modified)**.** Traffic engineers in Simplexville are interested in the number of cars that pass the eastbound entrance to Primal Parkway on Main Street. After collecting data during a 4-hour period for 5 days, they have determined that the cars that pass the entrance approximately follow a nonstationary Poisson process during those 4 hours, with arrival rate function

$$\lambda(\tau) = \begin{cases} 144 & \text{if } 0 \le \tau < 1 \\ 229 & \text{if } 1 \le \tau < 2 \\ 383 & \text{if } 2 \le \tau < 3 \\ 96 & \text{if } 3 \le \tau \le 4 \end{cases}$$

a. Derive the integrated rate function for this model.
b. What is the expected number of cars passing the entrance between times 1.75 and 3.4?
c. What is the probability of more than 700 cars passing this location between times 1.7 and 3.4?

Problem 3. The Simplexville Emergency Dispatch receives phone calls according to a nonstationary Poisson arrival process with integrated rate function

$$\Lambda(\tau) = \begin{cases} 3\tau & \text{if } 0 \le t < 8 \\ 5\tau - 16 & \text{if } 8 \le t < 20 \\ \frac{3}{2}\tau + 54 & \text{if } 20 \le t \le 24 \end{cases}$$

where τ is in hours and $\tau = 0$ corresponds to 0:00.

a. What is the probability that 12 or fewer phone calls have been received between 18:00 and 22:00?
b. If exactly 40 phone calls have been received between 0:00 and 12:00, what is the probability that 80 or more phone calls have been received over the course of the entire day (0:00 - 24:00)?
c. In words, briefly describe the meaning of $\Lambda(24)$ in the context of this problem.