Lesson 11. Markov Chains – Time-Dependent Performance Measures

0 Warm up – The Case of The Defective Detective

Example 1. Quality control engineers at KRN Corporation are monitoring the performance of a manufacturing system that produces an electronic component. Components are inspected in the sequence they are produced. The engineers believe that there is some dependence between successively produced components, and so they model whether a component is acceptable or defective by a Markov chain with states $\mathcal{M} = \{1, 2\}$ (1 = acceptable, 2 = defective), and one-step transition matrix and initial-state vector

$$\mathbf{P} = \begin{bmatrix} 0.995 & 0.005 \\ 0.495 & 0.505 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 0.96 \\ 0.04 \end{bmatrix}$$

a. What does $p_{21} = 0.495$ represent in the context of this problem?
b. Draw the transition-probability diagram for this Markov chain.
c. What is the probability of the sequence of states $1, 1, 2, 2$?

1 n-step transition probabilities

- Consider a Markov chain with states $\mathcal{M} = \{1, \dots, m\}$
- The n-**step transition probability** $p_{ij}^{(n)}$ from state i to state j:
- Let $\mathbf{P}^{(n)}$ be the $m \times m$ matrix with elements $p_{ij}^{(n)}$
- We can compute n-step transition probabilities using:

Example 2. For the Defective Detective case, what is the probability that the third component is defective, given that the first one is not?

- Why does $\mathbf{P}^{(n)} = \mathbf{P}^n$? Let's reason why this works $n = 2$:

$$\begin{aligned} p_{ij}^{(2)} = \Pr\{i \to j \text{ in 2 steps}\} &= \sum_{h=1}^{m} \Pr\{i \to h, \text{ then } h \to j\} \\ &= \sum_{h=1}^{m} p_{ih} p_{hj} \\ &= (i\text{th row of } \mathbf{P}) \cdot (j\text{th column of } \mathbf{P}) \end{aligned}$$

- The proof for arbitrary n works in a similar fashion
- Also in a similar fashion, we can derive the **Chapman-Kolmogorov equation**:

$$p_{ij}^{(n)} = \sum_{h=1}^{m} p_{ih}^{(k)} p_{hj}^{(n-k)}$$

  - In other words,

$$\Pr\{i \to j \text{ in } n \text{ steps}\} = \sum_{h=1}^{m} \Pr\{i \to h \text{ in } k \text{ steps, then } h \to j \text{ in } n-k \text{ steps}\}$$

2 n-step state probabilities

- The **n-step state probability** $p_j^{(n)}$ for state j is

- Let $\mathbf{p}^{(n)}$ be the $m \times 1$ vector with elements $p_j^{(n)}$

- We can compute n-step state probabilities using:

> **Example 3.** For the Defective Detective case, what is the probability that the fourth component is defective?

- Why does $\mathbf{p}^{(n)\top} = \mathbf{p}^\top \mathbf{P}^n$?

$$p_j^{(n)} = \Pr\{S_n = j\} = \sum_{i=1}^{m} \Pr\{S_n = j \mid S_0 = i\} \Pr\{S_0 = i\} = \sum_{i=1}^{m} p_i p_{ij}^{(n)} = \mathbf{p} \cdot (j\text{th column of } \mathbf{P}^n)$$

3 First-passage probabilities

- Let $\mathcal{A}$ and $\mathcal{B}$ be two <u>disjoint</u> subsets of the state space $\mathcal{M}$
    - e.g. $\mathcal{M} = \{1, 2, 3, 4\}$, $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{B} = \{4\}$
- The **first-passage probability** $f_{ij}^{(n)}$ for initial state $i \in \mathcal{A}$ and final state $j \in \mathcal{B}$ in n time steps is:

- In other words,

$$f_{ij}^{(n)} = \Pr\{\text{start in } i \in \mathcal{A}, \text{ stay in states in } \mathcal{A} \text{ for } n-1 \text{ steps, end in } j \in \mathcal{B} \text{ at the } n\text{th step}\}$$

- Let $\mathbf{P}_{\mathcal{AB}}$ be the submatrix of $\mathbf{P}$ whose elements are p_{ij} with $i \in \mathcal{A}$ and $j \in \mathcal{B}$:
  - e.g. $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{B} = \{4\}$

$$\mathbf{P} = \begin{bmatrix} 0 & 0.95 & 0.01 & 0.04 \\ 0 & 0.27 & 0.63 & 0.10 \\ 0 & 0.36 & 0.40 & 0.24 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}_{\mathcal{AB}} = \qquad\qquad \mathbf{P}_{\mathcal{AA}} =$$

- Let $\mathbf{F}^{(n)}_{\mathcal{AB}}$ be the $|\mathcal{A}| \times |\mathcal{B}|$ matrix whose elements are $f^{(n)}_{ij}$

- We can compute first-passage probabilities using:

Example 4. The Markov chain in the Jungle.com case from the previous lesson had the following transition probability matrix:

$$\mathbf{P} = \begin{bmatrix} 0 & 0.95 & 0.01 & 0.04 \\ 0 & 0.27 & 0.63 & 0.10 \\ 0 & 0.36 & 0.40 & 0.24 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall that a session starts with a log on (state 1) and ends with a log off (state 4) Let X be the length of the session, excluding log on. Compute the probability that $X = 9$.

Example 5. Use the concept of first-passage probabilities to find a formula for the expected session length in the Jungle.com case.

4 Next time...

- What happens in the **long run**, i.e. when the number of time steps n approaches infinity?

Problem 1 (Nelson 6.11, modified)**.** A food manufacturer plans to introduce a new potato chip, Box O' Spuds, into a local market that already has two strong competitors. The marketing analysts would like to forecast the long-term market share for Box O' Spuds to determine whether it is worth entering the market.

Suppose the marketing analysts formulate a Markov chain model of customer brand switching in which the state space $\mathcal{M} = \{1, 2, 3\}$ corresponds to which of the two established brands or Box O' Spuds, respectively, that a customer currently purchases. The time index is the number of bags of chips purchased. Based on market research and experience with other products, the one-step transition matrix the marketing analysts anticipate is

$$\mathbf{P} = \begin{bmatrix} 0.70 & 0.28 & 0.02 \\ 0.28 & 0.70 & 0.02 \\ 0.10 & 0.10 & 0.80 \end{bmatrix}$$

a. Note that the diagonal entries of **P** are larger than the off-diagonal entries. What does this mean in the context of this problem?

b. Suppose that initially, a typical customer is equally likely to prefer one of the two existing brands. What is the probability that a typical customer prefers Box O' Spuds after he or she has bought 50 bags of chips?

c. What is the probability that a customer initially buys a bag of Brand 2 chips, buys only the two existing brands over the course of his or her next 9 bags of chips, and then purchases Box O' Spuds for his or her 11th bag of chips?