Lesson 22. An Economic Interpretation of LP Duality

1   Overview

- An economic interpretation of duality

- Complementary slackness

2   Warm up

Example 1. The Fulkerson Furniture Company produces desks, tables, and chairs. Each type of furniture requires a certain amount of lumber, finishing, and carpentry:

Resource	Desk	Table	Chair	Available
Lumber (sq ft)	8	6	2	48
Finishing (hrs)	3	2	1	20
Carpentry (hrs)	2	2	1	8
Profit ($)	60	30	20	

Assume that all furniture produced is sold, and that fractional solutions are acceptable. Write a linear program to determine how much furniture Fulkerson should produce in order to maximize its profits.

3   Economic interpretation of the dual LP

- Suppose an entrepreneur wants to purchase all of Fulkerson's resources (lumber, finishing, carpentry)

- What prices should she offer for the resources that will entice Fulkerson to sell?

- Define decision variables:

$$y_1 = \text{price of 1 sq. ft. lumber}$$
$$y_2 = \text{price of 1 hour of finishing}$$
$$y_3 = \text{price of 1 hour of carpentry}$$

- To buy all of Fulkerson's resources, entrepreneur pays:

- Entrepreneur wants to minimize this cost

- Entrepreneur also needs to offer resource prices that will <u>entice Fulkerson to sell</u>

- One desk uses

  - 8 sq. ft. of lumber
  - 3 hours of finishing
  - 2 hours of carpentry

- One desk has profit of $60

$\Rightarrow$ Entrepreneur should pay <u>at least</u> $60 for this combination of resources:

- One table uses

  - 6 sq. ft. of lumber
  - 2 hours of finishing
  - 2 hours of carpentry

- One table has profit of $30

$\Rightarrow$ Entrepreneur should pay <u>at least</u> $30 for this combination of resources:

- One chair uses

  - 2 sq. ft. of lumber
  - 1 hours of finishing
  - 1 hours of carpentry

- One chair has profit of $20

$\Rightarrow$ Entrepreneur should pay <u>at least</u> $20 for this combination of resources:

- Increasing the availability of the resources potentially <u>increases</u> the maximum profits Fulkerson can achieve

⇒ Entrepreneur should pay nonnegative amounts for each resource:

- Putting this all together, we get:

$$
\begin{array}{rlll}
\min & 48y_1 + 20y_2 + 8y_3 & & \\
\text{s.t.} & 8y_1 + 3y_2 + 2y_3 \geq 60 & & (x_1\text{: desks}) \\
& 6y_1 + 2y_2 + 2y_3 \geq 30 & & (x_2\text{: tables}) \\
& 2y_1 + y_2 + y_3 \geq 20 & & (x_3\text{: chairs}) \\
& y_1, \quad y_2, \quad y_3 \geq 0 & &
\end{array}
$$

- This is the dual of Fulkerson's LP!

- In summary:

  - Optimal dual solution ⇔ "fair" prices for associated resources
  - Known as **marginal prices** or **shadow prices**

- Strong duality ⇒

$$
\begin{pmatrix} \text{Company's maximum revenue} \\ \text{from selling furniture} \end{pmatrix} = \begin{pmatrix} \text{Entrepreneur's minimum cost} \\ \text{of purchasing resources} \end{pmatrix}
$$

  - Equilibrium under perfect competition: company makes no excess profits

- This kind of economic interpretation is trickier for LPs with different types of constraints and variable bounds

4 Complementary slackness

- Optimal solution to Fulkerson's LP: $x_1 = 4$, $x_2 = 0$, $x_3 = 0$

- Resources used:

  lumber: $32 < 48$ finishing: $12 < 20$ carpentry: $8 = 8$

- How much would you pay for an extra sq. ft. of lumber?

- How much would you pay for an extra hour of finishing?

- Resource not fully utilized in optimal solution

  ⇒ marginal price = 0

- **Primal complementary slackness:** either

  - a primal constraint is active at a primal optimal solution, or
  - the corresponding dual variable at optimality = 0

- Same logic applies to the dual

- Dual constraints $\Leftrightarrow$ Primal decision variables

- **Dual complementary slackness:** either

  - a primal decision variable at optimality $= 0$, or
  - the corresponding dual constraint is active in a dual optimal solution

5 More duality practice

Example 2. Consider the following LP:

$$
\begin{aligned}
\text{minimize} \quad & 3x_1 - x_2 + 8x_3 \\
\text{subject to} \quad & -x_1 + 8x_3 \leq 6 \\
& 5x_1 - 3x_2 + 9x_3 \geq -2 \\
& x_1 \geq 0, x_2 \leq 0, x_3 \geq 0
\end{aligned}
$$

a. Write the dual.
b. Find a feasible solution to the primal and the dual.
c. Give a lower and an upper bound on the optimal value of the above LP.