Solutions to Problem 1.

a. $$\begin{aligned}\Pr\{T_{300} \le 6 \mid Y_2 = 150\} &= \Pr\{Y_6 \ge 300 \mid Y_2 = 150\} \\ &= \Pr\{Y_6 - Y_2 \ge 150 \mid Y_2 = 150\} \\ &= \Pr\{Y_6 - Y_2 \ge 150\} \\ &= \Pr\{Y_4 \ge 150\} \\ &= 1 - \sum_{j=0}^{149} \frac{e^{-40 \cdot 4}(40 \cdot 4)^j}{j!} \approx 0.7956\end{aligned}$$

b. $$\begin{aligned}E[Y_{16} \mid Y_6 = 250] &= 250 + E[Y_{16} - Y_6 \mid Y_6 = 250] \\ &= 250 + E[Y_{16} - Y_6] \\ &= 250 + E[Y_{10}] \\ &= 250 + 40(10) = 650\end{aligned}$$

c. The first class passengers arrive according to a Poisson process with arrival rate $\lambda_1 = 0.15(40) = 6$.

$$E[T_{1,60}] = \frac{60}{\lambda_1} = 10$$

Solutions to Problem 2.

a. If $0 \le \tau < 6$: $\quad \Lambda(\tau) = \int_0^\tau 10\,dt = 10\tau$

If $6 \le \tau \le 12$: $\quad \Lambda(\tau) = \int_0^6 10\,dt + \int_6^\tau 4\,dt = 4\tau + 36$

$$\Rightarrow \Lambda(\tau) = \begin{cases} 10\tau & \text{if } 0 \le \tau < 6 \\ 4\tau + 36 & \text{if } 6 \le \tau \le 12 \end{cases}$$

b. $$\begin{aligned}\Pr\{Z_5 - Z_2 \le 15 \mid Z_2 = 30\} &= \Pr\{Z_5 - Z_2 \le 15\} \qquad [Z_5 - Z_2 \sim \text{Poisson}(\Lambda(5) - \Lambda(2)) = \text{Poisson}(12)] \\ &= \sum_{j=0}^{15} \frac{e^{-12}(12)^j}{j!} \approx 0.8444\end{aligned}$$

Solutions to Problem 3.

In order for an arrival counting process (with arrivals one-at-a-time) to be Poisson, it must satisfy:

- Independent increments: in this context, the number of phone calls arriving at the cell tower in non-overlapping time intervals must be independent
- Stationary increments: in this context, the number of phone calls arriving at the cell tower only depends on the length of the interval, not when the time interval occurs

Solutions to Problem 4.

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.2 & 0 & 0.3 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.4 & 0.2 & 0.4 \end{bmatrix}$$

Solutions to Problem 5.

a. In this case, the initial state probabilities are

$$\mathbf{p} = \begin{bmatrix} 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \end{bmatrix}$$

We want $\Pr\{S_3 = 4\} = p_4^{(3)}$ = 4th element of $\mathbf{p}^{(3)} = \mathbf{p}^T \mathbf{P}^3$.

$$\mathbf{p}^T \mathbf{P}^3 = \begin{bmatrix} 0.0328 & 0.0172 & 0.37975 & 0.19525 & 0.375 \end{bmatrix}$$

Therefore, $\Pr\{S_3 = 4\} = 0.19525$.

b. Let $\mathcal{R} = \{2, 3\}$. So,

$$\mathbf{P}_{\mathcal{R}\mathcal{R}} = \begin{bmatrix} 0.30 & 0 \\ 0 & 0.75 \end{bmatrix}$$

Looking at $\mathbf{P}_{\mathcal{R}\mathcal{R}}$, we see that $\mathcal{R} = \{2, 3\}$ does not form a self-contained Markov chain since the rows of $\mathbf{P}_{\mathcal{R}\mathcal{R}}$ do not sum to 1. Therefore, $\mathcal{R} = \{2, 3\}$ is not an irreducible set of states.

c. Let $\mathcal{R} = \{3, 4\}$. We want π_4.

$$\begin{matrix} \pi_{\mathcal{R}} \mathbf{P}_{\mathcal{R}\mathcal{R}} = \pi_{\mathcal{R}} \\ \pi_{\mathcal{R}} \mathbf{1} = 1 \end{matrix} \quad \Leftrightarrow \quad \begin{aligned} 0.75\pi_3 + 0.50\pi_4 &= \pi_3 \\ 0.25\pi_3 + 0.50\pi_4 &= \pi_4 \\ \pi_3 + \pi_4 &= 1 \end{aligned} \quad \Rightarrow \quad \pi_3 = \frac{2}{3},\ \pi_4 = \frac{1}{3}$$

So, $\pi_4 = \frac{1}{3}$.

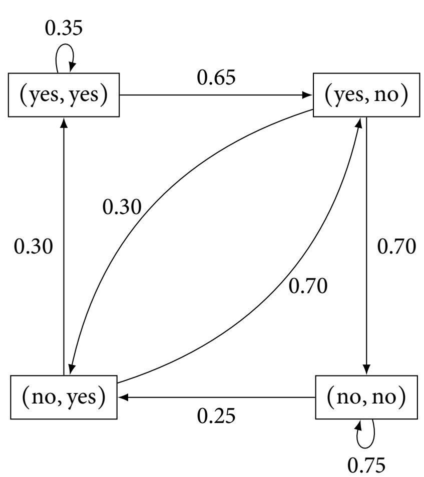
d. The set of all transient states is $\mathcal{T} = \{1, 2\}$. Let $\mathcal{R} = \{5\}$. Note that 5 is an absorbing state. We want α_{15}.

$$\alpha_{\mathcal{T}\mathcal{R}} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

So, $\alpha_{15} = 0.5$.

Solutions to Problem 6.

- State space: $\{(\text{yes}, \text{yes}), (\text{yes}, \text{no}), (\text{no}, \text{yes}), (\text{no}, \text{no})\}$
- Time step: 1 shot
- One-step transition probabilities:



Solutions to Problem 7.

- Markov property: the probability an officer is promoted, separated, or retired next year only depends on the officer's rank this year. For example, it does not matter how long the officer has held his or her current or previous ranks.
- Time-stationarity: the probability an officer is promoted, separated, or retired next year given the officer's rank this year stays constant from year to year.