Problem 1. Suppose X is a discrete random variable with the following cdf:

$$F_X(a) = \begin{cases} 0 & \text{if } a < 2, \\ 0.4 & \text{if } 2 \leq a < 4, \\ 0.9 & \text{if } 4 \leq a < 5, \\ 1 & \text{if } a \geq 5. \end{cases}$$

a. What is the pmf of X?
b. What is the expected value of X?
c. What is the variance of X?
d. Professor I. M. Wright peeks over your shoulder and declares,

 "The probability that $X = 3$ is 0.4, since $F_X(3) = 0.4$."

 Is Professor Wright correct? Briefly explain.

Problem 2. Suppose X is a random variable with pdf

$$f_X(a) = \begin{cases} 0 & \text{if } a < 1, \\ \frac{1}{4}a - \frac{1}{4} & \text{if } 1 \leq a < 3, \\ \frac{1}{2} & \text{if } 3 \leq a < 4, \\ 0 & \text{if } a \geq 4. \end{cases}$$

a. What is the probability that $2 < X \leq 3$?
b. What is the expected value of X?
c. What is the probability that $X \leq 6$?
d. Professor I. M. Wright peeks over your shoulder and declares,

 "Since the maximum value of $f_X(a)$ is attained when $a = 3$, the maximum value that X can take is 3."

 Is Professor Wright correct? Briefly explain.

Solutions to Problem 1.

Note that the cdf F_X only changes value at 2, 4, 5. Therefore, X only takes values 2, 4, and 5 with positive probability.

Why is this true? Consider $X = 4$. Roughly speaking,

$$\Pr\{X = 4\} = \Pr\{4 - \varepsilon < X \leq 4\}.$$

for some very small positive value of ε.[1] Therefore,

$$\Pr\{X = 4\} = \Pr\{4 - \varepsilon < X \leq 4\} = F_X(4) - F_X(4 - \varepsilon) = 0.5.$$

Note that $F_X(4)$ and $F_X(4 - \varepsilon)$ are different because F_X changes value at 4.

On the other hand, consider $X = 3$. Again, roughly speaking,

$$\Pr\{X = 3\} = \Pr\{3 - \varepsilon < X \leq 3\}$$

and so

$$\Pr\{X = 3\} = \Pr\{3 - \varepsilon < X \leq 3\} = F_X(3) - F_X(3 - \varepsilon) = 0.$$

So, X does not take the value 3 with positive probability. Note that $F_X(3)$ and $F_X(3 - \varepsilon)$ are the same because F_X does not change value at 3.

[1] To be completely correct, $\Pr\{X = 4\} = \lim_{\varepsilon \to 0^+} \Pr\{4 - \varepsilon < 4 \leq 4\}$. Recall that $\lim_{\varepsilon \to 0^+}$ denotes the limit as ε approaches 0 from the right.

a. The pmf of X is

$$p_X(2) = \Pr\{X = 2\} = \Pr\{X \le 2\} = F_X(2) = 0.4$$
$$p_x(4) = \Pr\{X = 4\} = \Pr\{2 < X \le 4\} = F_X(4) - F_X(2) = 0.5$$
$$p_X(5) = \Pr\{X = 5\} = \Pr\{4 < X \le 5\} = F_X(5) - F_X(4) = 0.1$$

b. The expected value of X is

$$E[X] = 2(0.4) + 4(0.5) + 5(0.1) = 3.3$$

c. The variance of X is

$$\mathrm{Var}(X) = (2 - 3.3)^2(0.4) + (4 - 3.3)^2(0.5) + (5 - 3.3)^2(0.1) = 1.21$$

d. No, Professor Wright is not correct. $F_X(3)$ gives the probability that X is <u>less than or equal</u> to 3, not the probability that X is equal to 3. Furthermore, as we discussed above, $\Pr\{X = 3\} = 0$.

Solutions to Problem 2.

a. The probability that $2 < X \le 3$ is

$$\Pr\{2 < X \le 3\} = \int_2^3 f_x(a)\,da = \int_2^3 \left(\frac{1}{4}a - \frac{1}{4}\right) da = \frac{3}{8}$$

b. The expected value of X is

$$E[X] = \int_{-\infty}^{\infty} a f_X(a) da = \int_1^3 a\left(\frac{1}{4}a - \frac{1}{4}\right) da + \int_3^4 a\left(\frac{1}{2}\right) da \approx 2.92$$

c. $\Pr\{X \le 6\} = 1$, because the maximum value that X can take (with positive probability) is 4 (see part d).

d. No, Professor Wright is not correct. The maximum value that X can take (with positive probability) is 4, because $f_X(a) = 0$ for all $a > 4$.