Solutions to Problem 1.

a. Y is discrete, because the cdf $F_Y(a)$ is a step function.

b. Since Y is discrete, we want its pmf. We see from the cdf F_Y that Y takes values 1, 3, 5, 7 and 9 (where the jumps in the cdf occur). So, the pmf of Y is:

$$\begin{aligned}
p_Y(1) &= F_Y(1) - F_Y(-\infty) = 0.2 - 0 = 0.2\\
p_Y(3) &= F_Y(3) - F_Y(1) = 0.5 - 0.2 = 0.3\\
p_Y(5) &= F_Y(5) - F_Y(3) = 0.6 - 0.5 = 0.1\\
p_Y(7) &= F_Y(7) - F_Y(5) = 0.9 - 0.6 = 0.3\\
p_Y(9) &= F_Y(9) - F_Y(7) = 1 - 0.9 = 0.1
\end{aligned}$$

c. $E[Y] = 1 \cdot p_Y(1) + 3 \cdot p_Y(3) + 5 \cdot p_Y(5) + 7 \cdot p_Y(7) + 9 \cdot p_Y(9) = 4.6$

d. $$\begin{aligned}
\operatorname{Var}(Y) &= E[(Y - E[Y])^2]\\
&= (1-4.6)^2 \cdot p_Y(1) + (3-4.6)^2 \cdot p_Y(3) + (5-4.6)^2 \cdot p_Y(5) + (7-4.6)^2 \cdot p_Y(7) + (9-4.6)^2 \cdot p_Y(9)\\
&= 7.04
\end{aligned}$$

e. The maximum value of Y is 9. This can be seen from the cdf F_Y: the smallest value of a such that $F_Y(a) = 1$ is $a = 9$.

Solutions to Problem 2.

a. In general, $F_x(a) = \int_{-\infty}^{a} f_x(b)\,db$. Since the pdf comes in pieces, we need to find the cdf in pieces as well.

- If $a \le 0$, then $F_X(a) = \int_{-\infty}^{a} 0\,db = 0$.
- If $0 < a \le 1$, then $F_X(a) = \int_{-\infty}^{0} 0\,db + \int_{0}^{a} b\,db = \dfrac{a^2}{2}$.
- If $1 < a \le 2$, then $F_X(a) = \int_{-\infty}^{0} 0\,db + \int_{0}^{1} b\,db + \int_{1}^{a} (2-b)\,db = 2a - \dfrac{a^2}{2} - 1$.
- If $a > 2$, then $F_X(a) = \int_{-\infty}^{0} 0\,db + \int_{0}^{1} b\,db + \int_{1}^{2} (2-b)\,db + \int_{1}^{a} 0\,db = 1$.

Putting this all together, we get:

$$F_X(a) = \begin{cases} 0 & \text{if } a \le 0 \\ \frac{a^2}{2} & \text{if } 0 < a \le 1 \\ 2a - \frac{a^2}{2} - 1 & \text{if } 1 < a \le 2 \\ 1 & \text{if } a > 2 \end{cases}$$

b. $$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} a f_X(a)\,da \\ &= \int_{-\infty}^{0} a \cdot 0\,da + \int_{0}^{1} a \cdot a\,da + \int_{1}^{2} a \cdot (2-a)\,da + \int_{1}^{\infty} a \cdot 0\,da \\ &= 1 \end{aligned}$$

c. $$\begin{aligned} \mathrm{Var}(X) &= \int_{-\infty}^{\infty} (a - E[X])^2 f_X(a)\,da \\ &= \int_{-\infty}^{0} (a-1)^2 \cdot 0\,da + \int_{0}^{1} (a-1)^2 \cdot a\,da + \int_{1}^{2} (a-1)^2 \cdot (2-a)\,da + \int_{1}^{\infty} (a-1)^2 \cdot 0\,da \\ &= \frac{1}{6} \end{aligned}$$

d. $\Pr\{\frac{1}{2} \le X \le \frac{3}{4}\} = F_X(\frac{3}{4}) - F_X(\frac{1}{2}) = \frac{1}{2}(\frac{3}{4})^2 - \frac{1}{2}(\frac{1}{2})^2 = \frac{5}{32}$

e. The maximum possile value of X is 2. This can be seen from the pdf f_X: the largest value of a that has positive density (i.e., $f_x(a) > 0$) is $2 - \epsilon$ for arbitrarily small $\epsilon > 0$.

Solutions to Problem 3.

Since Y is discrete, the following is a random variate generator for Y:

$$Y = \begin{cases} 1 & \text{if } 0 \le U \le 0.2 \\ 3 & \text{if } 0.2 < U \le 0.5 \\ 5 & \text{if } 0.5 < U \le 0.6 \\ 7 & \text{if } 0.6 < U \le 0.9 \\ 9 & \text{if } 0.9 < U \le 1 \end{cases} \quad \text{where} \quad U \sim \text{Uniform}[0,1]$$

Therefore, here is an algorithm that outputs random variates of Y:

1: Set $u \leftarrow$ `random()`
2: **if** $0 \le u \le 0.2$ **then**
3:     Set $y \leftarrow 1$
4: **else if** $0.2 < u \le 0.5$ **then**
5:     Set $y \leftarrow 3$
6: **else if** $0.5 < u \le 0.6$ **then**
7:     Set $y \leftarrow 5$
8: **else if** $0.6 < u \le 0.9$ **then**
9:     Set $y \leftarrow 7$
10: **else if** $0.9 < u \le 1.0$ **then**
11:     Set $y \leftarrow 9$
12: **end if**
13: Output y as a random variate of Y

Note that lines 2-12 above are equivalent to lines 2-3 of the algorithm on page 3 of Lesson 4.

Solutions to Problem 4.

Define:

$$A = \begin{cases} 0 & \text{if walk-in} \\ 1 & \text{if ambulance} \\ 2 & \text{if public service vehicle} \end{cases} \qquad M = \begin{cases} 1 & \text{if MRI given} \\ 0 & \text{otherwise} \end{cases} \qquad I = \begin{cases} 1 & \text{if admitted to ICU} \\ 0 & \text{otherwise} \end{cases}$$

We are given:

$$\begin{aligned} \Pr\{A = 0\} &= 0.43 & \Pr\{A = 1\} &= 0.53 & \Pr\{A = 2\} &= 0.04 \\ \Pr\{M = 1 \mid A = 0\} &= 0.63 & \Pr\{M = 1 \mid A = 1\} &= 0.73 & \Pr\{M = 1 \mid A = 2\} &= 0.59 \\ \Pr\{I = 1 \mid A = 0\} &= 0.002 & \Pr\{I = 1 \mid A = 1\} &= 0.11 & \Pr\{I = 1 \mid A = 2\} &= 0.06 \end{aligned}$$

a. $\Pr\{A = 0 \text{ and } M = 1\} = \Pr\{M = 1 \mid A = 0\} \Pr\{A = 0\} \approx 0.2709$

b. $$\begin{aligned} \Pr\{I = 1\} &= \sum_{a=0}^{2} \Pr\{I = 1 \mid A = a\} \Pr\{A = a\} \\ &= (0.002)(0.43) + (0.11)(0.53) + (0.06)(0.04) \approx 0.062 \end{aligned}$$

Solutions to Problem 5.

- System events:

$$
\begin{aligned}
e_1 &= \text{phone arrival}\\
e_2 &= \text{phone departure}\\
e_0 &= \text{initialization}
\end{aligned}
$$

- State variables:

$$
\begin{aligned}
Q_n &= \text{number of phones in queue after } n\text{th system event}\\
A_n &= \begin{cases} 0 & \text{if cell is available} \\ 1 & \text{otherwise} \end{cases} \quad \text{after } n\text{th system event}
\end{aligned}
$$

- System event subroutines:

  - $e_0()$:

    $Q_n \leftarrow 0$ (start with empty queue)
    $A_n \leftarrow 0$ (cell available at start)
    $C_1 \leftarrow 30$ (first arrival)
    $C_2 \leftarrow +\infty$ (no pending departure)

  - $e_1()$:

    if $\{A_n = 0\}$ **then**
        $i \leftarrow F_T^{-1}(\texttt{random()})$ (get phone type)
        $C_2 \leftarrow T_{n+1} + F_{P_i}^{-1}(\texttt{random()})$ (set clock for next departure)
        $A_{n+1} \leftarrow 1$ (cell is not available)
    else
        $Q_{n+1} \leftarrow Q_n + 1$ (add job to queue)
    end if
    $C_1 \leftarrow T_{n+1} + 30$ (set clock for next arrival)

  - $e_2()$:

    if $\{Q_n = 0\}$ **then**
        $A_{n+1} \leftarrow 0$ (cell is available)
    else
        $Q_{n+1} \leftarrow Q_n - 1$ (remove job from queue)
        $i \leftarrow F_T^{-1}(\texttt{random()})$ (get next phone type)
        $C_2 \leftarrow T_{n+1} + F_{P_i}^{-1}(\texttt{random()})$ (set clock for next departure)
    end if