Name: Notes

SA402 · Dynamic and Stochastic Models Fall 2019 · Uhan

Exam 1 – 18 September 2019

Instructions

- You have until the end of the class period to complete this exam.
- You may use your calculator.
- You may not consult any outside materials (e.g. notes, textbooks, homework, computer).
- **Show all your work.** To receive full credit, your solution must be completely correct, sufficiently justified, and easy to follow.
- Keep this booklet intact.

Problem	Weight	Score
1	1	
2	1	
3	1	
4	1	
5	1	
6	1	
7	1	
8	1	
9	1	
10	1	
Total		/ 100

For Problems 1 and 2, consider the random variable X with the following pdf:

$$f_X(a) = \begin{cases} 0 & \text{if } a < 0, \\ \frac{2}{3} - \frac{2}{9}a & \text{if } 0 \leq a \leq 3, \\ 0 & \text{if } a > 3. \end{cases}$$

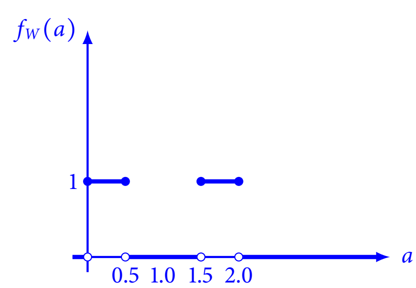
Problem 1. Find the expected value of X.

- Nearly all of you answered this correctly.
- Figure out whether X is discrete or continuous, and then apply the appropriate definition of expected value (Lesson 2).

Problem 2. For the random variable X, which is more likely: a value near 1, or a value near 2? Briefly explain.

- Many of you said that a value near 1 is more likely because 1 is the expected value. A value near 1 is more likely, but this is not the reason why. Here is an example of a random variable with expected value of 1, but a value near 1 is not likely at all:

$$f_W(a) = \begin{cases} 0 & \text{if } a < 0, \\ 1 & \text{if } 0 \leq a \leq 0.5, \\ 0 & \text{if } 0.5 < a < 1.5, \\ 1 & \text{if } 1.5 \leq a \leq 2, \\ 0 & \text{if } a > 2 \end{cases}$$



For Problems 3, 4 and 5, consider the random variable Y with the following cdf:

$$F_Y(a) = \begin{cases} 0 & \text{if } a < 2, \\ 0.2 & \text{if } 2 \le a < 4, \\ 0.7 & \text{if } 4 \le a < 6, \\ 1 & \text{if } a \ge 6. \end{cases}$$

Problem 3. What is the probability that $Y > 5$?

- Nearly all of you answered this correctly.
- Use the definition of a cdf: $F_Y(a) = \Pr\{X \dots\}$ (Lesson 2).

Problem 4. Using the inverse transform method, construct a random variate generator for Y. Your solution should be in the form: "$Y = \cdots$ where $U \sim \text{Uniform}[0,1]$".

- Be careful about which values Y takes, and the corresponding intervals of U.

Problem 5. Suppose you have access to a function `random()` that generates random variates of $\text{Uniform}[0,1]$. Say that `random()` returns the value 0.8372. What value of Y does the random variate generator you constructed in Problem 4 generate? Briefly explain.

- Nearly all of you had the right idea here.

For Problems 6 and 7, consider the following setting.

As an analyst for the Primal Pizza Company, you have determined that the delivery times (in hours) are best modeled using a random variable Z with the following cdf:

$$F_Z(a) = \begin{cases} 0 & \text{if } a < 0, \\ 1 - e^{-4a} & \text{if } a \geq 0. \end{cases}$$

The company promises delivery within 0.5 hours or the pizza is free.

Problem 6. What is the probability that delivery takes less than 0.5 hours?

- Note that you were given the cdf of Z, not the pdf.
- Use the definition of a cdf: $F_Y(a) = \Pr\{X \dots\}$ (Lesson 2).

Problem 7. What is the probability that the delivery takes more than 0.5 hours, given that a customer has already waited 0.25 hours?

- See Example 7 of Lesson 3 for a similar example.

For Problems 8 and 9, consider the following setting.

The Orange Company was having problems with its automated manufacturing cells yesterday: sometimes a tablet came out of a cell defective. 50% of the tablets were produced in cell 1, 30% in cell 2, and 20% in cell 3. 2% of the tablets produced in cell 1 came out defective, 3% in cell 2, and 5% in cell 3.

Suppose you select 1 tablet made yesterday at random. Let C be a random variable that represents the cell it was produced in (i.e. $C = 1, 2$ or 3). In addition, let D represent a random variable indicating whether the tablet came out defective (i.e. 1 if defective, 0 otherwise).

Problem 8. What is the probability that the randomly selected tablet came out defective, i.e. $\Pr\{D = 1\}$?

- Nearly all of you answered this correctly.
- Be careful with how you translate the probabilities given in the problem into probability statements $\Pr\{\dots\}$.
- See Problem 4 of the Review Problems for Exam 1 for a similar example.

Problem 9. Are C and D independent? Give a numerical argument for why or why not.

- Make sure you use the definition of independence, or one of the statements about independence, correctly (Lesson 3).

Problem 10. At the Markov Butcher Shop, there is one server who serves customers from a single queue on a first-come-first-served basis. The shop is small and the customers are impatient: any customers who arrive when there are already 5 customers waiting in the queue simply leave without joining the queue.

The interarrival time between customers is modeled by a random variable G, and the service time for customers is modeled by a random variable X. The interarrival times and service times are assumed to be independent.

Professor I. M. Wright is consulting for the shop, and has started to model the shop as a stochastic process using the algorithmic approach we discussed in class, as follows:

- System events:

$$
\begin{aligned}
e_0 &= \text{shop opens} \\
e_1 &= \text{customer arrives at shop} \\
e_2 &= \text{customer finishes being served and departs shop}
\end{aligned}
$$

- State variables:

$$
\begin{aligned}
Q_n &= \text{number of customers in the queue after the } n\text{th system event} \\
&\qquad \text{(not including the customer being served)} \\
A_n &= \begin{cases} 0 & \text{if the server is available} \\ 1 & \text{if the server is busy} \end{cases} \qquad \text{after the } n\text{th system event} \\
\mathbf{S}_n &= (Q_n, A_n)
\end{aligned}
$$

- System event subroutines – <u>only for e_0 and e_2</u>:

$e_0()$:

1: $Q_0 \leftarrow 0$
2: $A_0 \leftarrow 0$
3: $C_1 \leftarrow F_G^{-1}(\texttt{random()})$
4: $C_2 \leftarrow \infty$

$e_2()$:

1: **if** $\{Q_n = 0\}$ **then**
2: $\quad A_{n+1} \leftarrow 0$
3: **else**
4: $\quad Q_{n+1} \leftarrow Q_n - 1$
5: $\quad C_2 \leftarrow T_{n+1} + F_X^{-1}(\texttt{random()})$
6: **end if**

The general simulation algorithm is below for your reference. Recall that `random()` is a function that generates variates of Uniform$[0, 1]$.

algorithm Simulation:

1: $n \leftarrow 0$ (initialize system event counter)
$T_0 \leftarrow 0$ (initialize event epoch)
$e_0()$ (execute initial system event)

2: $T_{n+1} \leftarrow \min\{C_1, \dots, C_k\}$ (advance time to next pending system event)
$I \leftarrow \arg\min\{C_1, \dots, C_k\}$ (find index of next system event)

3: $\mathbf{S}_{n+1} \leftarrow \mathbf{S}_n$ (temporarily maintain previous state)
$C_I \leftarrow \infty$ (event I no longer pending)

4: $e_I()$ (execute system event I)
$n \leftarrow n + 1$ (update event counter)

5: go to line 2

(next page)

Help Professor Wright finish the model by writing a subroutine for e_1. Annotate your code line-by-line.

- Please make sure the beginning and end of your if-else if-else blocks are clear. Be careful with how you indent your code.
- The butcher shop setting is here is identical to the copy shop in Lesson 6, except that customers don't enter the butcher shop if there are 5 or more customers already waiting in the queue.
- Start with the subroutine $e_1()$ we came up with for the copy shop, and modify it for this setting. In particular, how can you prevent the state variables from updating if there are already 5 or more customers waiting in the queue? (Equivalently, how can you update the state variables only if there are fewer than 5 customers waiting in the queue?)

Additional page for scratchwork or solutions