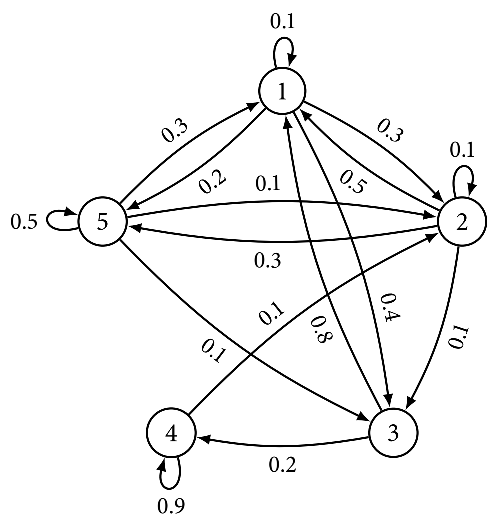
Solutions to Problem 1. Let $\mathcal{R} = \{2, 3, 4\}$. Note that

$$\mathbf{P}_{\mathcal{R}\mathcal{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0.4 & 0.1 \\ 1 & 0 & 0 \end{bmatrix}$$

is the transition probability matrix of a self-contained Markov chain, and no proper subset of $\mathcal{R}$ also forms a Markov chain. Therefore, states 2, 3, and 4 are recurrent, and state 1 is transient.

Solutions to Problem 2.

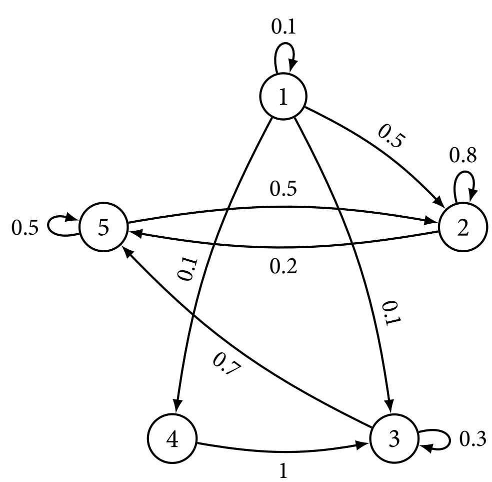
a. From the transition probability diagram (below), we see that all states communicate with each other, because we can find a sequence of positive probability transitions that starts at state 1, goes through all the other states, and then returns to state 1 (e.g., $1 - 2 - 3 - 4 - 2 - 5 - 1$). Therefore the entire state space $\mathcal{M} = \{1, 2, 3, 4, 5\}$ is a recurrent class, and so all states are recurrent.



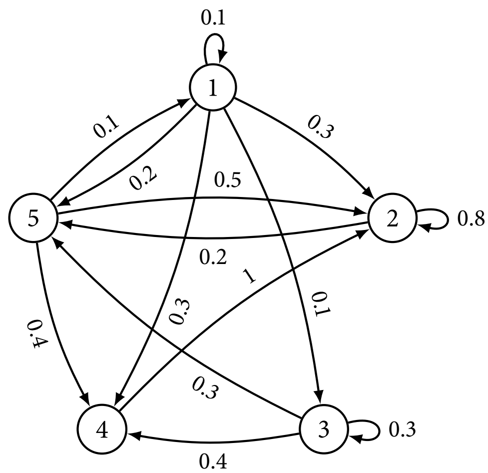
b. Let $\mathcal{R} = \{2, 5\}$. From the transition probability matrix, we see that $\mathcal{R}$ is a recurrent class because

$$\mathbf{P}_{\mathcal{R}\mathcal{R}} = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix},$$

indicating that $\mathcal{R}$ is a self-contained Markov chain and no proper subset of $\mathcal{R}$ is a Markov chain. Therefore, states 2 and 5 are recurrent. From the transition probability diagram (below), we can also see that states 1, 3, and 4 are transient, because the process will eventually leave each of these states and never return.



c. From the transition probability diagram (below), we see that all states communicate with each other, because we can find a sequence of positive probability transitions that starts at state 1, goes through all the other states, and then returns to state 1 (e.g., 1 – 3 – 4 – 2 – 5 – 1). Therefore the entire state space $\mathcal{M} = \{1, 2, 3, 4, 5\}$ is a recurrent class, and so all states are recurrent.



Solutions to Problem 3. Note that states 2, 5, and 6 are absorbing states, because $p_{ii} = 1$ for $i = 2, 5, 6$. In addition, note that the other states, 1, 3, and 4, are transient, because each of these states transitions to one of the absorbing states with positive probability. So, $\mathcal{T} = \{1, 3, 4\}$, and

$$\mathbf{N} = (\mathbf{I} - \mathbf{P}_{\mathcal{T}\mathcal{T}})^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0.4 \\ 0.2 & 0.4 & 0.3 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \approx \begin{bmatrix} 1 & 0 & 0.4 \\ 0.333 & 1.667 & 0.633 \\ 0 & 0 & 1 \end{bmatrix}$$

The absorption probabilities when $\mathcal{R} = \{2\}$ are:

$$\alpha_{\mathcal{T}\mathcal{R}} = \mathbf{N}\mathbf{P}_{\mathcal{T}\mathcal{R}} \approx \begin{bmatrix} 1 & 0 & 0.4 \\ 0.333 & 1.667 & 0.633 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \\ 0.9 \end{bmatrix} \approx \begin{bmatrix} 0.86 \\ 0.737 \\ 0.9 \end{bmatrix}$$

So $\alpha_{32} \approx 0.737$. Similarly, the absorption probabilities when $\mathcal{R} = \{5\}$ are:

$$\alpha_{\mathcal{T}\mathcal{R}} = \mathbf{N}\mathbf{P}_{\mathcal{T}\mathcal{R}} \approx \begin{bmatrix} 1 & 0 & 0.4 \\ 0.333 & 1.667 & 0.633 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix} \approx \begin{bmatrix} 0.14 \\ 0.097 \\ 0.1 \end{bmatrix}$$

So $\alpha_{35} \approx 0.097$. Finally, the absorption probabilities when $\mathcal{R} = \{6\}$ are:

$$\alpha_{\mathcal{T}\mathcal{R}} = \mathbf{N}\mathbf{P}_{\mathcal{T}\mathcal{R}} \approx \begin{bmatrix} 1 & 0 & 0.4 \\ 0.333 & 1.667 & 0.633 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.167 \\ 0 \end{bmatrix}$$

So $\alpha_{36} \approx 0.167$.

The expected times to absorption from $\mathcal{T}$ are:

$$\mu_{\mathcal{T}} = \mathbf{N}\mathbf{1} \approx \begin{bmatrix} 1 & 0 & 0.4 \\ 0.333 & 1.667 & 0.633 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 2.633 \\ 1 \end{bmatrix}$$

Therefore, $\mu_3 = 2.633$.

Solutions to Problem 4. Looking at the transition probability matrix, we see that the Markov chain is irreducible. Let $\mathcal{R} = \{1, 2, 3, 4\}$. We want π_4:

$$\begin{array}{c} \pi_{\mathcal{R}}^\top \mathbf{P}_{\mathcal{R}\mathcal{R}} = \pi^\top \\ \pi_{\mathcal{R}}^\top \mathbf{1} = 1 \end{array} \quad \Leftrightarrow \quad \begin{aligned} 0.70\pi_1 + 0.14\pi_2 + 0.14\pi_3 + 0.05\pi_4 &= \pi_1 \\ 0.14\pi_1 + 0.70\pi_2 + 0.14\pi_3 + 0.05\pi_4 &= \pi_2 \\ 0.14\pi_1 + 0.14\pi_2 + 0.70\pi_3 + 0.05\pi_4 &= \pi_3 \\ 0.02\pi_1 + 0.02\pi_2 + 0.02\pi_3 + 0.85\pi_4 &= \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} -0.30\pi_1 + 0.14\pi_2 + 0.14\pi_3 + 0.05\pi_4 &= 0 \\ 0.14\pi_1 - 0.30\pi_2 + 0.14\pi_3 + 0.05\pi_4 &= 0 \\ 0.14\pi_1 + 0.14\pi_2 - 0.30\pi_3 + 0.05\pi_4 &= 0 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

Note that we removed the last equation from $\pi_{\mathcal{R}}^\top \mathbf{P}_{\mathcal{R}\mathcal{R}} = \pi^\top$, because any one of them is redundant. Solving this system of equations, we get:

$$\pi_1 \approx 0.2941 \qquad \pi_2 \approx 0.2941 \qquad \pi_3 \approx 0.2941 \qquad \pi_4 \approx 0.1176$$

Therefore, the long-term market share for Poisson Puffs is 11.76%.

Solutions to Problem 5. Looking at the transition probability matrix, we see that the Markov chain is irreducible. Let $\mathcal{R} = \{1, 2, 3, 4\}$. We want π_4:

$$\begin{array}{c} \pi_{\mathcal{R}}^\top \mathbf{P}_{\mathcal{R}\mathcal{R}} = \pi^\top \\ \pi_{\mathcal{R}}^\top \mathbf{1} = 1 \end{array} \quad \Leftrightarrow \quad \begin{aligned} 0\pi_1 + \tfrac{1}{3}\pi_2 + \tfrac{1}{3}\pi_3 + \tfrac{1}{3}\pi_4 &= \pi_1 \\ \tfrac{1}{2}\pi_1 + 0\pi_2 + \tfrac{1}{3}\pi_3 + \tfrac{1}{3}\pi_4 &= \pi_2 \\ \tfrac{1}{2}\pi_1 + \tfrac{1}{3}\pi_2 + 0\pi_3 + \tfrac{1}{3}\pi_4 &= \pi_3 \\ 0\pi_1 + \tfrac{1}{3}\pi_2 + \tfrac{1}{3}\pi_3 + 0\pi_4 &= \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} -\pi_1 + \tfrac{1}{3}\pi_2 + \tfrac{1}{3}\pi_3 + \tfrac{1}{3}\pi_4 &= 0 \\ \tfrac{1}{2}\pi_1 - \pi_2 + \tfrac{1}{3}\pi_3 + \tfrac{1}{3}\pi_4 &= 0 \\ \tfrac{1}{2}\pi_1 + \tfrac{1}{3}\pi_2 - \pi_3 + \tfrac{1}{3}\pi_4 &= 0 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

Note that we removed the last equation from $\pi_{\mathcal{R}}^\top \mathbf{P}_{\mathcal{R}\mathcal{R}} = \pi^\top$, because any one of them is redundant. Solving this system of equations, we get:

$$\pi_1 = \frac{1}{4} \qquad \pi_2 = \frac{9}{32} \qquad \pi_3 = \frac{9}{32} \qquad \pi_4 = \frac{3}{16}$$

Therefore, the AGV spends 3/16 of the time at the output buffer in the long run.

Solutions to Problem 6.

a. Looking at the transition probability diagram, we can see that $\{1, 2\}$ and $\{3, 5\}$ form self-contained Markov chains, and no proper subsets of $\{1, 2\}$ or $\{3, 5\}$ form a self-contained Markov chain.

b. Recurrent states: 1, 2, 3, 5 (these are states that are part of an irreducible set, by part a)
Transient states: 4, 6 (these are states not part of an irreducible set)

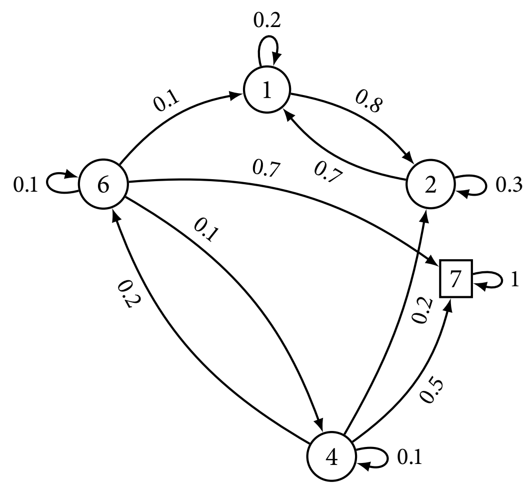
c. Let $\mathcal{R} = \{1, 2\}$. We want π_1. From the transition probability diagram, $\mathbf{P}_{\mathcal{R}\mathcal{R}} = \begin{bmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{bmatrix}$. Therefore,

$$\begin{array}{c} \pi_{\mathcal{R}}^\top \mathbf{P}_{\mathcal{R}\mathcal{R}} = \pi^\top \\ \pi_{\mathcal{R}}^\top \mathbf{1} = 1 \end{array} \quad \Leftrightarrow \quad \begin{aligned} 0.2\pi_1 + 0.7\pi_2 &= \pi_1 \\ 0.8\pi_1 + 0.3\pi_2 &= \pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned} \quad \Rightarrow \quad \pi_1 = \frac{7}{15}, \ \pi_2 = \frac{8}{15}$$

So, the long-run fraction of time the UAV spends in region 1 is 7/15.

d. This is a little tricky: the definition of an absorbing probability requires an absorbing state, that is, an irreducible set of states with only one state.

Let's replace states 3 and 5 with a "super state" called 7. We end up with the following transition probability diagram:



Now, let $\mathcal{T} = \{4, 6\}$ and $\mathcal{R} = \{7\}$. We want α_{47}:

$$\alpha_{\mathcal{TR}} = (\mathbf{I} - \mathbf{P}_{\mathcal{TT}})^{-1}\mathcal{P}_{\mathcal{TR}} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix} \approx \begin{bmatrix} 0.747 \\ 0.861 \end{bmatrix}$$

Therefore, $\alpha_{47} \approx 0.747$.