Problem 1 (Nelson 6.11, modified)**.** A food manufacturer plans to introduce a new potato chip, Box O' Spuds, into a local market that already has three strong competitors. The marketing analysts would like to forecast the long-term market share for Box O' Spuds to determine whether it is worth entering the market.

Suppose the marketing analysts formulate a Markov chain model of customer brand switching in which the state space $\mathcal{M} = \{1, 2, 3, 4\}$ corresponds to which of the three established brands or Box O' Spuds, respectively, that a customer currently purchases. The time index is the number of bags of chips purchased. Based on market research and experience with other products, the one-step transition matrix the marketing analysts anticipate is

$$\mathbf{P} = \begin{pmatrix} 0.70 & 0.14 & 0.14 & 0.02 \\ 0.14 & 0.70 & 0.14 & 0.02 \\ 0.14 & 0.14 & 0.70 & 0.02 \\ 0.05 & 0.05 & 0.05 & 0.85 \end{pmatrix}$$

a. Note that the diagonal entries of $\mathbf{P}$ are larger than the off-diagonal entries. What does this mean in the context of this problem?

b. Suppose that initially, a typical customer is equally likely to prefer one of the three existing brands. What is the probability that a typical customer prefers Box O' Spuds after he or she has bought 50 bags of chips?

c. What is the probability that a customer initially buys a bag of Brand 2 chips, buys only the three existing brands over the course of his or her next 9 bags of chips, and then purchases Box O' Spuds for his or her 11th bag of chips?

a. The diagonal entries of $\mathbb{P}$ are the probabilities that a consumer purchases a brand, given that it previously purchased the same brand. The diagonal entries being higher than the off-diagonal entries indicates that a consumer is more likely to stick with a brand if he/she previously purchased that brand.

b. $\vec{p} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \end{pmatrix}$ We want $p_4^{(50)}$:

$$\vec{p}^{(50)T} = \vec{p}^T \mathbb{P}^{50} \approx \begin{pmatrix} 0.294 & 0.294 & 0.294 & \boxed{0.118} \end{pmatrix}$$

$$\Rightarrow p_4^{(50)} \approx 0.118$$

c. Let $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{B} = \{4\}$. We want $f_{24}^{(10)}$:

$$\mathbb{F}_{\mathcal{AB}}^{(10)} = \mathbb{P}_{\mathcal{AA}}^{9} \mathbb{P}_{\mathcal{AB}} = \begin{pmatrix} 0.70 & 0.14 & 0.14 \\ 0.14 & 0.70 & 0.14 \\ 0.14 & 0.14 & 0.70 \end{pmatrix}^9 \begin{pmatrix} 0.02 \\ 0.02 \\ 0.02 \end{pmatrix} \approx \begin{array}{c} \begin{pmatrix} 0.017 \\ \boxed{0.017} \\ 0.017 \end{pmatrix} \\ \underbrace{4}_{\mathcal{B}} \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \Big\} \mathcal{A}$$

$$f_{24}^{(10)} \approx 0.017$$