Lesson 8. Poisson Arrival Processes, cont.

1 Overview

- Last lesson: a **Poisson process** is a renewal arrival counting process
  - arrivals one-at-a-time, interarrival times are independent and time-stationary
  - interarrival times $G \sim \text{Exponential}(\lambda)$ where λ is the **arrival rate**
  - Time of the nth arrival: $T_n \sim \text{Erlang}(\lambda, n)$
  - Number of arrivals by time t: $Y_t \sim \text{Poisson}(\lambda t)$
  - Properties: independent increments, stationary increments, memoryless
- This lesson:
  - When is the Poisson process a good model?
  - Decomposing a Poisson process into two arrival counting subprocesses
  - Superposing (combining) two Poisson processes into one arrival counting process

2 When is the Poisson process a good model?

- **Any arrival-counting process in which arrivals occur one-at-a-time and has independent and stationary increments must be a Poisson process**
  - If you can justify your arrivals having independent and stationary increments, then you can assume that the interarrival times are exponentially distributed
  - This is a very powerful result
- Independent increments $\Leftrightarrow$ number of arrivals in nonoverlapping intervals of time are independent
  - Reasonable when the arrival-counting process is formed by a large number of customers making individual, independent decisions about when to arrive
- Stationary increments $\Leftrightarrow$ expected number of arrivals = constant rate × length of time interval
  - Reasonable when arrival rate is approximately constant over time

Example 1. Discuss whether or not it is reasonable to approximate the following arrival processes as Poisson processes:

a. The arrival of cars at a toll booth during evening rush hour.
b. The arrival of students at a college football game.

3 Decomposition of Poisson processes

- Let's think back to Beehunter case
- Accidents (arrivals) occur according to a Poisson process with arrival rate λ accident/week
- Suppose that a fraction $(1-\gamma)$ of these accidents are major, γ are minor
- We can model each accident type as a **Bernoulli random variable** with success probability γ

$$B = \begin{cases} 0 & \text{with probability } 1-\gamma \quad \text{(major accident)} \\ 1 & \text{with probability } \gamma \quad \text{(minor accident)} \end{cases}$$

- Let's assume:
  - accident types for all accidents are independent and time stationary
  - accident types and interarrival times are independent
- The **decomposition property**:
  - Type 0 arrivals (e.g. major accidents) form a Poisson process with arrival rate $\lambda_0 = (1-\gamma)\lambda$
  - Type 1 arrivals (e.g. minor accidents) form a Poisson process with arrival rate $\lambda_1 = \gamma\lambda$
  - These two processes are independent
- This works because the Poisson process is decomposed by a independent Bernoulli variables
- Other methods of decomposition do not necessarily lead to Poisson subprocesses
- Proof on p. 111 of Nelson

Example 2. Suppose that in the Beehunter case, the accident rate is in fact 3/2 per week. In addition, 80% of the accidents are minor, and 20% are major.

a. What is the probability that fewer than 4 minor accidents occur during any 4-week period?
b. What is the expected number of major accidents in any 8-week period?

Example 3. You have been asked to conduct a study of the pedestrian crossing near Chick & Ruth's Delly in Downtown Annapolis. Assume the following behavior. Pedestrians approach the crossing at a rate of 6 pedestrians per minute; one-third of them are on the left side, and two-thirds of them are on the right side. Pedestrians wait until the "WALK" signal, at which time all waiting pedestrians cross instantaneously. Suppose the "WALK" signal occurs every 2 minutes.

a. What is the expected number of pedestrians crossing left to right on a given "WALK" signal?
b. What is the probability that at least one pedestrian crosses right to left on any particular signal?

4 Superposition of Poisson processes

- We can also combine Poisson processes
- Suppose that:
  - major accidents arrive at the intersection according to a Poisson process with arrival rate λ_0
  - minor accidents arrive at the intersection according to a Poisson process with arrival rate λ_1
  - these processes are independent of each other
- The **superposition property**: the arrivals from both processes (e.g. major and minor accidents) together form a Poisson process with arrival rate $\lambda = \lambda_0 + \lambda_1$
- This works because the two Poisson processes are independent
- Proof on pp. 111-112 of Nelson

Example 4. The Bank of Simplexville opens at 8 a.m. Customers arrive at the lobby at a rate of 10 per hour. The bank also has a separate ATM where customers arrive at a rate of 20 per hour. Approximate these arrival processes as Poisson processes. What is the probability that the 100th bank customer (at the lobby or the ATM) arrives before noon, given that 50 customers have arrived by 10 a.m.?

Example 5. The Markov Company has two salespeople, John and Louise. On average, John receives 8 orders per week, while Louise receives 12 per week. Suppose the orders arrive according to a Poisson process.

a. What is the probability that the total sales for two weeks will be more than 30 orders?
b. What is the expected number of orders in one month?