Problem 1. Professor I. M. Right often has his facts wrong. Let X be a random variable that represents the number of questions he is asked during one class, and let Y be the number of questions that he answers incorrectly during one class. The joint pmf p_{XY} of X and Y is:

	p_{XY}	Y = 0	1	2
X	1	1/3	0	0
	2	1/4	1/12	0
	3	3/16	1/8	1/48

a. What is the probability that Professor Right answers all questions correctly during one class?

b. What is the probability that Professor Right answers 1 question incorrectly during one class, given that he is asked two questions?

c. Explain why $p_{XY}(1,2) = 0$.

Problem 2. The Simplex Company uses three machines to produce a large batch of similar manufactured items. 20% of the items were produced by machine 1, 30% by machine 2, and 50% by machine 3. In addition, 1% of the items produced by machine 1 are defective, 2% by machine 2 are defective, and 3% by machine 3 are defective. Suppose you select 1 item at random from the entire batch.

a. Define the random variable M as the machine used ($M \in \{1,2,3\}$) to produce this item. Write the pmf p_M of M.

b. Define another random variable D that is equal to 1 if this item is defective, and 0 otherwise. Find the probability that $D = 1$ given $M = m$, for $m = 1,2,3$.

c. Find the probability that $D = 1$; that is, the probability that the randomly selected item is defective.

Problem 3. Simplex Pizza sells pizza (of course) and muffins (that's weird). Let Z and M be random variables that represent the number of pizzas and muffins in one order, respectively. Based on historical data, the company has determined the joint pmf p_{ZM} for Z and M:

	p_{ZM}	M = 0	1	2
Z	0	0	0.09	0.06
	1	0.25	0.11	0.05
	2	0.10	0.08	0.07
	3	0.08	0.07	0.04

a. What is the conditional pmf of M, given that $Z = 2$?

b. What is the expected number of muffins in an order, given that it contains 2 pizzas?

c. It turns out that $\Pr\{M = 1\} = 0.35$ and $\Pr\{M = 1 \mid Z = 3\} \approx 0.368$. Based on this information, are M and Z independent? Why or why not?

Solutions to Problem 1.

a. $$\Pr\{Y=0\} = \Pr\{Y=0 \text{ and } X=1\} + \Pr\{Y=0 \text{ and } X=2\} + \Pr\{Y=0 \text{ and } X=3\}$$
$$= \frac{1}{3} + \frac{1}{4} + \frac{3}{16} = \frac{37}{48} \approx 0.7708$$

b. $\Pr\{Y = 1 \mid X = 2\} = \dfrac{\Pr\{Y = 1 \text{ and } X = 2\}}{\Pr\{X = 2\}} = \dfrac{\Pr\{Y = 1 \text{ and } X = 2\}}{\Pr\{Y = 0 \text{ and } X = 2\} + \Pr\{Y = 1 \text{ and } X = 2\} + \Pr\{Y = 2 \text{ and } X = 2\}}$

$$= \frac{\frac{1}{12}}{\frac{1}{4} + \frac{1}{12} + 0} = \frac{1}{4}$$

c. $p_{XY}(1, 2)$ is the probability that Professor Right is asked 1 question and answers 2 questions incorrectly, which is impossible.

Solutions to Problem 2.

a. The pmf of M is

$$p_M(a) = \begin{cases} 0.20 & \text{if } a = 1 \\ 0.30 & \text{if } a = 2 \\ 0.50 & \text{if } a = 3 \\ 0 & \text{otherwise} \end{cases}$$

b. These probabilities are given to us in the problem:

$$\Pr\{D = 1 \mid M = 1\} = 0.01 \qquad \Pr\{D = 1 \mid M = 2\} = 0.02 \qquad \Pr\{D = 1 \mid M = 3\} = 0.03$$

c. Using the law of total probability:

$$\begin{aligned} \Pr\{D = 1\} &= \Pr\{D = 1 \mid M = 1\} \Pr\{M = 1\} + \Pr\{D = 1 \mid M = 2\} \Pr\{M = 2\} + \Pr\{D = 1 \mid M = 3\} \Pr\{M = 3\} \\ &= 0.01(0.20) + 0.02(0.30) + 0.03(0.50) = 0.023 \end{aligned}$$

Solutions to Problem 3.

a. First, let's compute

$$\Pr\{Z = 2\} = \Pr\{Z = 2 \text{ and } M = 0\} + \Pr\{Z = 2 \text{ and } M = 1\} + \Pr\{Z = 2 \text{ and } M = 2\} = 0.25$$

The conditional pmf of M given $Z = 2$ is:

$$\begin{aligned} p_{M|Z=2}(0) = \Pr\{M = 0 \mid Z = 2\} &= \frac{\Pr\{M = 0 \text{ and } Z = 2\}}{\Pr\{Z = 2\}} = \frac{0.10}{0.25} = \frac{2}{5} \\ p_{M|Z=2}(1) = \Pr\{M = 1 \mid Z = 2\} &= \frac{\Pr\{M = 1 \text{ and } Z = 2\}}{\Pr\{Z = 2\}} = \frac{0.08}{0.25} = \frac{8}{25} \\ p_{M|Z=2}(2) = \Pr\{M = 2 \mid Z = 2\} &= \frac{\Pr\{M = 2 \text{ and } Z = 2\}}{\Pr\{Z = 2\}} = \frac{0.07}{0.25} = \frac{7}{25} \end{aligned}$$

b. $E[M|Z = 2] = 0 \cdot p_{M|Z=2}(0) + 1 \cdot p_{M|Z=2}(1) + 2 \cdot p_{M|Z=2}(2) = \dfrac{22}{25}$

c. M and Z are <u>not</u> independent: if they were, we would have $\Pr\{M = 1\} = \Pr\{M = 1 \mid Z = 3\}$.