Name:

SA402 – Dynamic and Stochastic Models
Assoc. Prof. Nelson Uhan

Fall 2016

Exam 1

Instructions

- You have until the end of the class period to complete this exam.
- You may use a calculator.
- You may not consult any other outside materials (e.g. notes, textbooks, homework) except a printout of your scores on course standards.
- **Show all your work.** Your answers should be legible and clearly labeled. It is your responsibility to make sure that I understand what you are doing. You will be awarded partial credit if your work merits it.
- **Cross out any problems that you did not attempt.**
- Keep this booklet intact.
- **Do not discuss the contents of this exam with any midshipmen until it is returned to you.**

Standard	Problems	Score
A1	1	
B1	2	
B2	3	
B3	4a, 4b	
B4	4d, 5	
B5	4c	
C1	6	
C2	7	

Problem 1. Customers arrive at the Simplexville Ice Cream Shop and form a single queue. There are two servers who serve customers from the single queue on a first-come-first-served basis. The interarrival and service times for the first 4 customers are:

customer	interarrival time	service time
1	3	12
2	1	6
3	5	4
4	7	10

a. Simulate the system for the first 4 customers below, starting at time 0.

Current time:

Server 1	Server 2
Queue	
Next system event	**Time**
customer arrival	
server 1 finish	
server 2 finish	

Current time:

Server 1	Server 2
Queue	
Next system event	**Time**
customer arrival	
server 1 finish	
server 2 finish	

Current time:

Server 1	Server 2
Queue	
Next system event	**Time**
customer arrival	
server 1 finish	
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Current time:

Server 1	Server 2
Queue	
Next system event	**Time**
customer arrival	
server 1 finish	
server 2 finish	

b. What is the average delay experienced by the first 4 customers?

Problem 2. Suppose X is a random variable with pdf

$$f_X(a) = \begin{cases} 0 & \text{if } a < 2, \\ \frac{9}{2}a - \frac{3}{4}a^2 - 6 & \text{if } 2 \le a < 4, \\ 0 & \text{if } a \ge 4. \end{cases}$$

a. What is the cdf F_X of X?

b. What is the expected value of X?

Problem 3. Suppose X is a random variable with cdf

$$F_X(a) = \begin{cases} 0 & \text{if } a < 1, \\ \frac{1}{2}\sqrt{x-1} & \text{if } 1 \le a < 5, \\ 1 & \text{if } a \ge 5. \end{cases}$$

a. Is X discrete or continuous? Briefly explain.

b. Professor May B. Wright peeks over your shoulder and says,

"Since $F_X(6) = 1$, X is always less than or equal to 6."

Is Professor Wright correct? Briefly explain.

Problem 4. As an analyst for the Markov Company, you have been tasked with better understanding the repair process of the servers that support the company's cloud IT infrastructure.

Let R be the number of repairs that a server has undergone in the past, and let T be the amount of time required to repair a server in hours. Based on historical data, you have determined the joint pmf between R and T:

	p_{RT}	T = 1	2	3
R	0	6/16	0	0
	1	2/16	2/16	0
	2	0	2/16	2/16
	3	0	0	2/16

a. What is the probability that a server takes 1 hour to repair?

b. A colleague of yours thinks that a server takes longer to repair if it has undergone more repairs in the past. Does your joint pmf above support this? Briefly explain.

c. Show that R and T are not independent with a numerical argument.

d. What is the probability that a server takes 2 hours to repair, given that the server has undergone 2 repairs in the past?

Problem 5. The Simplexville Safety Inspector suspects that the Primal Pralines factory has some serious issues. (Primal Pralines thinks that she is being paid by their competitor, Dual Doves, to give them a hard time.)

Based on her observations, there is a 30% chance that there are no accidents at the factory in any given week, a 40% chance of one accident, and a 20% chance of two accidents. There is a 5% chance that someone is injured in any given week if there were no accidents that week; that chance goes up to 20% if there was one accident that week, and 50% if there were two accidents that week.

What is the probability that someone is injured in a given week?

Problem 6. Suppose Y is a random variable with cdf

$$F_Y(a) = \begin{cases} 0 & \text{if } a < 0, \\ 0.18 & \text{if } 0 \le a < 2, \\ 0.42 & \text{if } 2 \le a < 7, \\ 0.77 & \text{if } 7 \le a < 8, \\ 1 & \text{if } a \ge 8. \end{cases}$$

a. Find a random variate generator for Y. Your solution should be in the form "$Y = \cdots$ where $U \sim \text{Uniform}[0,1]$."

b. Suppose you have access to a function `random()` that generates random variates of $U \sim \text{Uniform}[0,1]$. Briefly describe how you would use your random variate generator in part a to generate random variates of Y.

Problem 7. The Simplexville Ice Cream Shop is popular, but small and often overcrowded. If there are too many customers already in line at the shop, then arriving customers will go elsewhere.

There is one server who serves customers from a single queue on a first-come-first-served basis. Any customers who arrive when there are already 10 customers waiting in the queue simply leave without joining the queue. The interarrival time between customers is modeled by a random variable G, and the service time for customers is modeled by a random variable X. The interarrival times and service times are assumed to be independent.

Professor I. M. Wright is consulting for the Simplexville Ice Cream Shop, and has started to model the shop as a stochastic process using the algorithmic approach we discussed in class, as follows:

- System events:

$$\begin{aligned} e_0 &= \text{shop opens} \\ e_1 &= \text{customer arrives at shop} \\ e_2 &= \text{customer finishes being served and departs shop} \end{aligned}$$

- State variables:

$$\begin{aligned} Q_n &= \text{number of customers in the queue after the } n\text{th system event} \\ &\qquad \text{(not including the customer being served)} \\ A_n &= \begin{cases} 0 & \text{if the server is available} \\ 1 & \text{if the server is busy} \end{cases} \qquad \text{after the } n\text{th system event} \\ \mathbf{S}_n &= (Q_n, A_n) \end{aligned}$$

- System event subroutines – only for e_0 and e_2:

$e_0()$:

$Q_0 \leftarrow 0$
$A_0 \leftarrow 0$
$C_1 \leftarrow F_G^{-1}(\texttt{random()})$
$C_2 \leftarrow \infty$

$e_2()$:

if $\{Q_n = 0\}$ **then**
    $A_{n+1} \leftarrow 0$
else
    $Q_{n+1} \leftarrow Q_n - 1$
    $C_2 \leftarrow T_{n+1} + F_X^{-1}(\texttt{random()})$
end if

The general simulation algorithm is below for your reference. Recall that `random()` is a function that generates variates of Uniform$[0,1]$.

algorithm Simulation:

1:	$n \leftarrow 0$	(initialize system event counter)
	$T_0 \leftarrow 0$	(initialize event epoch)
	$e_0()$	(execute initial system event)
2:	$T_{n+1} \leftarrow \min\{C_1, \dots, C_k\}$	(advance time to next pending system event)
	$I \leftarrow \arg\min\{C_1, \dots, C_k\}$	(find index of next system event)
3:	$\mathbf{S}_{n+1} \leftarrow \mathbf{S}_n$	(temporarily maintain previous state)
	$C_I \leftarrow \infty$	(event I no longer pending)
4:	$e_I()$	(execute system event I)
	$n \leftarrow n + 1$	(update event counter)
5:	go to line 2	

(next page)

Write a subroutine for e_1. Annotate your code line-by-line.

Additional page for solutions or scratchwork