

Problem 1. Suppose X is a discrete random variable with the following cdf:

$$F_X(a) = \begin{cases} 0 & \text{if } a < 2, \\ 0.4 & \text{if } 2 \leq a < 4, \\ 0.9 & \text{if } 4 \leq a < 5, \\ 1 & \text{if } a \geq 5. \end{cases}$$

- What is the pmf of X ?
- What is the expected value of X ?
- What is the variance of X ?
- Professor I. M. Wright peeks over your shoulder and declares,

“The probability that $X = 3$ is 0.4, since $F_X(3) = 0.4$.”

Is Professor Wright correct? Briefly explain.

Problem 2. Suppose X is a random variable with pdf

$$f_X(a) = \begin{cases} 0 & \text{if } a < 1, \\ \frac{1}{4}a - \frac{1}{4} & \text{if } 1 \leq a < 3, \\ \frac{1}{2} & \text{if } 3 \leq a < 4, \\ 0 & \text{if } a \geq 4. \end{cases}$$

- What is the probability that $2 < X \leq 3$?
- What is the expected value of X ?
- What is the probability that $X \leq 6$?
- Professor I. M. Wright peeks over your shoulder and declares,

“Since the maximum value of $f_X(a)$ is attained when $a = 3$, the maximum value that X can take is 3.”

Is Professor Wright correct? Briefly explain.

Solutions to Problem 1.

Note that the cdf F_X only changes value at 2, 4, 5. Therefore, X only takes values 2, 4, and 5 with positive probability.

Why is this true? Consider $X = 4$. Roughly speaking,

$$\Pr\{X = 4\} = \Pr\{4 - \varepsilon < X \leq 4\}.$$

for some very small positive value of ε .¹ Therefore,

$$\Pr\{X = 4\} = \Pr\{4 - \varepsilon < X \leq 4\} = F_X(4) - F_X(4 - \varepsilon) = 0.5.$$

Note that $F_X(4)$ and $F_X(4 - \varepsilon)$ are different because F_X changes value at 4.

On the other hand, consider $X = 3$. Again, roughly speaking,

$$\Pr\{X = 3\} = \Pr\{3 - \varepsilon < X \leq 3\}$$

¹To be completely correct, $\Pr\{X = 4\} = \lim_{\varepsilon \rightarrow 0^+} \Pr\{4 - \varepsilon < 4 \leq 4\}$. Recall that $\lim_{\varepsilon \rightarrow 0^+}$ denotes the limit as ε approaches 0 from the right.

and so

$$\Pr\{X = 3\} = \Pr\{3 - \varepsilon < X \leq 3\} = F_X(3) - F_X(3 - \varepsilon) = 0.$$

So, X does not take the value 3 with positive probability. Note that $F_X(3)$ and $F_X(3 - \varepsilon)$ are the same because F_X does not change value at 3.

a. The pmf of X is

$$p_X(2) = \Pr\{X = 2\} = \Pr\{X \leq 2\} = F_X(2) = 0.4$$

$$p_X(4) = \Pr\{X = 4\} = \Pr\{2 < X \leq 4\} = F_X(4) - F_X(2) = 0.5$$

$$p_X(5) = \Pr\{X = 5\} = \Pr\{4 < X \leq 5\} = F_X(5) - F_X(4) = 0.1$$

b. The expected value of X is

$$E[X] = 2(0.4) + 4(0.5) + 5(0.1) = 3.3$$

c. The variance of X is

$$\text{Var}(X) = (2 - 3.3)^2(0.4) + (4 - 3.3)^2(0.5) + (5 - 3.3)^2(0.1) = 1.21$$

d. No, Professor Wright is not correct. $F_X(3)$ gives the probability that X is less than or equal to 3, not the probability that X is equal to 3. Furthermore, as we discussed above, $\Pr\{X = 3\} = 0$.

Solutions to Problem 2.

a. The probability that $2 < X \leq 3$ is

$$\Pr\{2 < X \leq 3\} = \int_2^3 f_X(a) da = \int_2^3 \left(\frac{1}{4}a - \frac{1}{4}\right) da = \frac{3}{8}$$

b. The expected value of X is

$$E[X] = \int_{-\infty}^{\infty} a f_X(a) da = \int_1^3 a \left(\frac{1}{4}a - \frac{1}{4}\right) da + \int_3^4 a \left(\frac{1}{2}\right) da \approx 2.92$$

c. $\Pr\{X \leq 6\} = 1$, because the maximum value that X can take (with positive probability) is 4 (see part d).

d. No, Professor Wright is not correct. The maximum value that X can take (with positive probability) is 4, because $f_X(a) = 0$ for all $a > 4$.