Example 2. Cantor's Car Repair is open from 9:00 ($\tau = 0$) to 15:00 ($\tau = 360$). Customers arrive according to a nonstationary Poisson process; the arrival rate at time τ is

$$\lambda(\tau) = \begin{cases} 1/6 & \text{if } 0 \le \tau < 180, \\ 1/5 & \text{if } 180 \le \tau < 360 \end{cases}$$

a. Find the integrated rate function $\Lambda(\tau)$. What does $\Lambda(\tau)$ mean in the context of the problem?
b. What is the probability that 5 customers arrive between 11:00 and 13:00?
c. What is the expected number of customers that arrive between 11:00 and 13:00?
d. If 15 customers have arrived by 11:00, what is the probability that more than 60 customers will have arrived throughout the course of the day?

a. If $0 \le \tau < 180$, $\Lambda(\tau) = \int_0^\tau \frac{1}{6}\, da = \frac{1}{6}\tau$

If $180 \le \tau < 360$, $\Lambda(\tau) = \int_0^{180} \frac{1}{6}\, da + \int_{180}^{\tau} \frac{1}{5}\, da$

$= 30 + \frac{1}{5}(\tau - 180) = \frac{1}{5}\tau - 6$

$$\Rightarrow \Lambda(\tau) = \begin{cases} \frac{1}{6}\tau & \text{if } 0 \le \tau < 180 \\ \frac{1}{5}\tau - 6 & \text{if } 180 \le \tau \le 360 \end{cases}$$

← $\Lambda(\tau)$ is the expected number of customers that arrive by time τ

b. $$\Pr\{\underbrace{Z_{240} - Z_{120}}_{\text{Poisson}(\underbrace{\Lambda(240) - \Lambda(120)}_{42 - 20 = 22})} = 5\} = \frac{e^{-22}(22)^5}{5!} \approx 0.000012$$

c. $E[Z_{240} - Z_{120}] = \Lambda(240) - \Lambda(120) = 22$

d. $\Pr\{Z_{360} > 60 \mid Z_{120} = 15\} = \Pr\{Z_{360} - Z_{120} > 45 \mid Z_{120} = 15\}$

$$= \Pr\{Z_{360} - Z_{120} > 45\} = 1 - \Pr\{\underbrace{Z_{360} - Z_{120}}_{\text{Poisson}(\underbrace{\Lambda(360) - \Lambda(120)}_{66 - 20 = 46})} \le 45\}$$

$$= 1 - \sum_{j=0}^{45} \frac{e^{-46}\, 46^j}{j!} \approx 0.5196$$