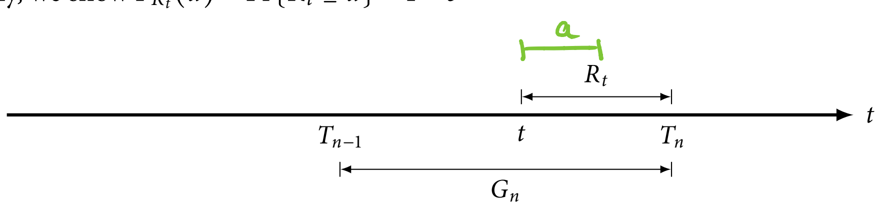
- **Any arrival-counting process in which arrivals occur one-at-a-time and has independent and stationary increments must be a Poisson process**
    - If you can justify your system having independent and stationary increments, then you can assume that interarrival times are exponentially distributed
    - This is a very deep and powerful result

5 Why does the memoryless property hold?

- The memoryless property allows us to ignore when we start observing the Poisson process, since forward-recurrence times and interarrival times are distributed in the same way
- "Memoryless" $\longleftrightarrow$ how much time has passed doesn't matter
- Why is this true for Poisson processes?
- Let's consider G_n, the interarrival time between the $(n-1)$th and nth arrival (between T_{n-1} and T_n)
    - Recall that $G_n \sim \text{Exponential}(\lambda)$
- Pick some t between T_{n-1} and T_n
- We want to show that the forward-recurrence time $R_t \sim \text{Exponential}(\lambda)$
    - Equivalently, we show $F_{R_t}(a) = \Pr\{R_t \le a\} = 1 - e^{-\lambda a}$



- Therefore:

$$
\begin{aligned}
\Pr\{R_t > a\} &= \Pr\{G_n > t - T_{n-1} + a \mid G_n > t - T_{n-1}\} && \text{(from the diagram)} \\
&= \frac{\Pr\{G_n > t - T_{n-1} + a \ \text{ and } \ G_n > t - T_{n-1}\}}{\Pr\{G_n > t - T_{n-1}\}} && \text{(def. of conditional prob.)} \\
&= \frac{\Pr\{G_n > t - T_{n-1} + a\}}{\Pr\{G_n > t - T_{n-1}\}} && (t - T_{n-1} + a > t - T_{n-1}) \\
&= \frac{e^{-\lambda(t - T_{n-1} + a)}}{e^{-\lambda(t - T_{n-1})}} = e^{-\lambda a} && (G_n \sim Exp(\lambda))
\end{aligned}
$$

$$\Rightarrow \quad \Pr\{R_t \le a\} = 1 - e^{-\lambda a}$$

- Note: This "proof" is rough and sketchy – we actually need to condition on T_{n-1} and Y_t
    - Repeated use of the law of total probability
- The independent-increments and stationary-increments properties follow from the memoryless property and the fundamental relationship between Y_t and T_n (see Nelson pp. 110-111)