Name: Notes

SA402 · Dynamic and Stochastic Models Fall 2019 · Uhan

# Exam 2 – 6 November 2019

## Instructions

- You have until the end of the class period to complete this exam.
- You may use your calculator.
- You may not consult any outside materials (e.g. notes, textbooks, homework, computer).
- **Show all your work.** To receive full credit, your solution must be completely correct, sufficiently justified, and easy to follow.
- Keep this booklet intact.

| Problem | Weight | Score |
|---|---|---|
| 1 | 1 | |
| 2 | 1 | |
| 3 | 1 | |
| 4 | 1 | |
| 5 | 1 | |
| 6 | $\frac{1}{2}$ | |
| 7 | 1 | |
| 8 | 1 | |
| 9 | $\frac{1}{2}$ | |
| 10 | 1 | |
| 11 | 1 | |
| Total | | / 100 |

For the problems on this page, consider the following setting.

Customers arrive at Four Guys Burgers and Fries from 5:00 to 23:00 according to a stationary Poisson process at a rate of 15 per hour.

**Problem 1.** What is the probability that the 100th customer arrives at or before 12:00, given that exactly 60 customers arrive between 5:00 and 9:00?

Take a look at Problem 1a from the Review Problems for Exam 2 for a similar problem.

**Problem 2.** What is the expected number of customers by the end of the day (5:00 - 23:00), given that exactly 120 customers arrive between 5:00 and 13:00?

Take a look at Problem 1b from the Review Problems for Exam 2 for a similar problem.

**Problem 3.** 60% of the customers order from the drive-thru window. What is the probability that 5 or fewer customers arrive at the drive-thru window in 1 hour?

Take a look at Example 2 from Lesson 8 for a similar problem.

**Problem 4.** After collecting some more data, the data scientists at Four Guys have determined that customers actually arrive according to a nonstationary Poisson process with the following integrated rate function:

$$\Lambda(\tau) = \begin{cases} 10\tau & \text{if } 0 \le \tau < 6 \\ 20\tau - 60 & \text{if } 6 \le \tau < 8 \\ 10\tau + 20 & \text{if } 8 \le \tau \le 18 \end{cases}$$

where $\tau = 0$ corresponds to 5:00. What is the expected number of arrivals between 10:00 and 14:00?

Take a look at Example 1b from Lesson 9 for a similar problem.

**Problem 5.** Professor I. M. Wright is consulting for the data science group at Four Guys, and insists that the customer arrivals can be modeled as a stationary Poisson process. Describe what assumptions need to be made about customer arrivals in order for this to be true. (You do not need to discuss whether these assumptions are realistic.)

Take a look at Problem 3 from the Review Problems for Exam 2 for a similar problem.

For the problems on this page, consider the following setting.

Simplexville Bikeshare is studying the movement of its bikes between three regions: Downtown, North, and South. The company's data science team has modeled the movement of a bike as a Markov chain with 3 states. The states 1, 2, 3 correspond to Downtown, North, and South, respectively, and each time step corresponds to one bike trip. When a bike reaches its destination, it stays in the destination region until it is used again. The one-step transition probability matrix for a bike is:

$$\mathbf{P} = \begin{bmatrix} 0.20 & 0.50 & 0.30 \\ 0.45 & 0.30 & 0.25 \\ 0.25 & 0.40 & 0.35 \end{bmatrix}$$

**Problem 6.** Draw the transition-probability diagram for this Markov chain.

**Problem 7.** Suppose at the beginning of the day, 50% of the bikes are in the Downtown region, 25% in the North region, and 25% in the South region. What is the probability that a bike randomly chosen at the beginning of the day will be in the Downtown region after 4 trips?

- Take a look at Problem 5a from the Review Problems for Exam 2 for a similar problem.
- Note that $\mathbf{p}^\top \mathbf{P}^n$ is <u>not</u> equal to $\mathbf{P}^n \mathbf{p}$.

**Problem 8.** What is the probability that a bike starts in the North region, stays in either the North or South regions for 4 trips, and then goes to the Downtown region in the 5th trip?

Take a look at Example 4 from Lesson 11 for a similar problem.

For the problems on this page, consider the following setting.

An autonomous UAV has been programmed to move randomly between four regions according to a Markov chain. Looking at the documentation written by the programmer, you find the following one-step transition matrix:

$$\mathbf{P} = \begin{bmatrix} 0.30 & 0.40 & 0.20 & 0.10 \\ 0.30 & 0.20 & 0.20 & 0.30 \\ 0 & 0 & 0.40 & 0.60 \\ 0 & 0 & 0.70 & 0.30 \end{bmatrix}$$

**Problem 9.** Do regions 1 and 2 form an irreducible set of states? Why or why not?

Take a look at Problem 5b from the Review Problems for Exam 2 for a similar problem.

**Problem 10.** Suppose the UAV reaches region 4. What is the long-run fraction of time it spends in region 4?

- Take a look at Problem 5c from the Review Problems for Exam 2 for a similar problem.
- Also note that $\pi_{\mathcal{R}}^{\top} \mathbf{P}_{\mathcal{R}\mathcal{R}}$ is not the same as $\mathbf{P}_{\mathcal{R}\mathcal{R}} \pi_{\mathcal{R}}$.

**Problem 11.** Consider a model of consumer preferences for breakfast cereal that defines the state of the system to be the cereal brand that the consumer most recently purchased, and the time index to be the number of boxes of cereal purchased. Describe what assumptions need to be made in order for the Markov property and the time-stationarity property to hold. (You do not need to discuss whether these assumptions are realistic.)

Take a look at Problem 7 from the Review Problems for Exam 2 for a similar problem.