Lesson 10. Introduction to Markov Chains

1 Overview

- Sample path of a stochastic process: sequence of state changes $(S_0, S_1, S_2, \dots)$ occurring at random points in time $(T_0, T_1, T_2, \dots)$
- In some settings, we care more about the state changes than the times at which the changes occur
- This lesson: a stochastic process model that focuses on the transitions between states

2 The Case of the Random Behavior

Jungle.com is an online retailer that sells everything from books to toothbrushes. Their data analytics group is currently evaluating changes to Jungle.com's computer architecture, and needs a model that describes customer behavior. The group has identified four key types of customer transactions:

(1) visit the Jungle.com home page to start shopping ("log on"),
(2) fetch the main page of a product,
(3) fetch and read the reviews of a product, and
(4) finish shopping by checking out or closing the browser ("log off").

The data analytics group believes that the next transaction a customer requests is strongly influenced by the last (most recent) transaction requested, and not significantly influenced by anything else. For a given customer, let

- N be a random variable representing the next transaction a customer requests, and
- L be a random variable representing the last transaction requested.

Based on its substantial historical data, it has determined the following conditional pmfs:

a	1	2	3	4
$p_{N\|L=1}(a)$	0	0.95	0.01	0.04
$p_{N\|L=2}(a)$	0	0.27	0.63	0.10
$p_{N\|L=3}(a)$	0	0.36	0.40	0.24
$p_{N\|L=4}(a)$	0	0	0	1

Denote the corresponding conditional cdfs as $F_{N|L=1}$, $F_{N|L=2}$, $F_{N|L=3}$, $F_{N|L=4}$.

- Let's model the transitions between customer transaction types as a stochastic process
- System events:

- State variables:

- System event subroutines:

- In this model, n =

- We are not keeping track of event epochs, so our algorithm Simulation can be simplified to:

algorithm Simulation:

1:	$n \leftarrow 0$	(initialize system-event counter)
	$e_0()$	(execute initial system event)
2:	$e_1()$	(update state of the system)
	$n \leftarrow n + 1$	(update system-event counter)
3:	go to line 2	

- Since $p_{N|L=4}(4) = 1$, once a sample path reaches state 4, it stays there

- The stochastic process model above – with the property that the probability distribution of the next state only depends on the last state – is called a **Markov chain**

- Let I be a random variable that represents the initial state with cdf F_I

- We can generalize this model so that the initial state is allowed to be random by redefining e_0:

    - In the Jungle.com case, we can think of I as a random variable with

$$\Pr\{I = 1\} = 1 \qquad \Pr\{I = i\} = 0 \quad \text{for } i = 2, 3, 4$$

    to model that a session always begins with a log on

3 Markov chains

- Discrete-time, discrete-state stochastic process $\{S_0, S_1, S_2, \dots\}$
- State space $\mathcal{M} = \{1, \dots, m\}$
- States evolve according to the algorithmic model above
- $\{S_0, S_1, S_2, \dots\}$ is a **Markov chain** if:

  - In other words, $\{S_0, S_1, S_2, \dots\}$ satisfies **the Markov property**: the conditional probability of the next state given the history of past states only depends on the last state
  - As a consequence:

Example 1. Recall that the performance-modeling group at Jungle.com believes that the next transaction a customer requests is essentially solely influenced by the last transaction requested. Compute the probability of the sequence of transactions 1, 2, 4.

- A Markov chain is **time-stationary** if:

  - In other words, the conditional probability of the next state given the last one does not depend on the number of time steps taken so far
  - As a consequence:

- In this course, we assume that Markov chains are time-stationary unless told otherwise

- The **one-step probabilities** p_{ij} are

- The **initial-state probabilities** p_i are

Example 2. Assuming time-stationarity, express the probability of the sequence of transactions 1, 2, 4 in terms of the one-step transition and initial-state probabilities.

- The sample paths of a time-stationary Markov chain are completely characterized by a corresponding sequence of one-step transition probabilities and initial-state probabilities

4 Representations of Markov chains

- We can organize the one-step transition probabilities into a **one-step transition matrix**:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

- We can also organize the initial-state probabilities into a **initial-state vector**:

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}$$

Example 3. Write the one-step transition matrix and initial-state vector for the Jungle.com Markov chain. Why do the rows of the one-step transition matrix sum up to 1?

- We can also draw a **transition probability diagram** where
  - each node represents a state of the system
  - a directed arc connects state i to state j if a one-step transition from i to j is possible
  - the one-step transition probability p_{ij} is written next to the arc from i to j

Example 4. Draw the transition probability diagram for the Jungle.com Markov chain.

5 Next time...

- Using the one-step transition matrix $\mathbf{P}$ and initial-state vector $\mathbf{p}$ to answer questions like:
  - Given that we are in state i right now, what is the probability we will be in state j after n time steps?
  - What is the unconditional probability we will be in state j after n time steps?