Solutions to Problem 1.

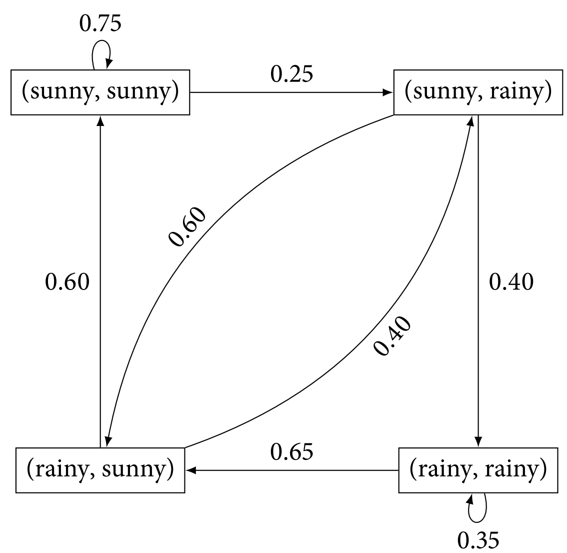
a. State space: $\mathcal{M} = \{(\text{sunny, sunny}), (\text{sunny, rainy}), (\text{rainy, sunny}), (\text{rainy, rainy})\}$

Meaning of state (x, y):

- x = weather two days ago
- y = weather yesterday

One time step = one day

Transition probabilities:



b. Transition probability matrix, assuming the states are ordered as follows: 1 ↔ (sunny, sunny), 2 ↔ (sunny, rainy), 3 ↔ (rainy, sunny), 4 ↔ (rainy, rainy):

$$\mathbf{P} = \begin{bmatrix} 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0.60 & 0.40 \\ 0.60 & 0.40 & 0 & 0 \\ 0 & 0 & 0.65 & 0.35 \end{bmatrix}$$

System of equations that give steady-state probabilities:

$$\begin{aligned} 0.75\pi_1 + 0.60\pi_3 &= \pi_1 \\ 0.25\pi_1 + 0.40\pi_3 &= \pi_2 \\ 0.60\pi_2 + 0.65\pi_4 &= \pi_3 \\ 0.40\pi_2 + 0.35\pi_4 &= \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

Solution to this system: $\pi_1 \approx 0.1227$, $\pi_2 \approx 0.4785$, $\pi_3 \approx 0.1994$, $\pi_4 \approx 0.1994$.

Long-run probability that the previous two days were sunny = long-run probability of being in (sunny, sunny) = $\pi_1 \approx 0.1227$

Solutions to Problem 2.

a. State space: $\mathcal{M} = \{0, 1, 2, \dots\} \leftarrow$ number of customers in the hair salon

$$\text{Arrival rates, } \lambda_i = \begin{cases} 5 & \text{if } i = 0,1,2,3 \\ 0 & \text{if } i = 4,5,\dots \end{cases} \qquad \text{Service rates, } \mu_i = 6 + (i-1)4 \quad \text{for } i = 1,2,\dots$$

Note that the system will never reach states $5, 6, 7, \dots$, so the service rates $\mu_5, \mu_6, \mu_7, \dots$ are not relevant.

b. First, determine steady-state probabilities:

$$\begin{aligned}
d_0 &= 1 \\
d_1 &= \frac{\lambda_0}{\mu_1} = \frac{5}{6} \\
d_2 &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2} = \frac{5}{6}\left(\frac{5}{10}\right) = \frac{5}{12} \\
d_3 &= \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} = \frac{5}{12}\left(\frac{5}{14}\right) = \frac{25}{168} \\
d_4 &= \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4} = \frac{25}{168}\left(\frac{5}{18}\right) = \frac{125}{3024} \\
d_5 &= d_6 = \cdots = 0
\end{aligned}$$

$$\text{Let } D = \sum_{j=0}^{\infty} d_j \approx 2.44$$

$$\begin{aligned}
\pi_0 &= \frac{d_0}{D} \approx 0.41, & \pi_1 &= \frac{d_1}{D} \approx 0.34, \\
\pi_2 &= \frac{d_2}{D} \approx 0.17, & \pi_3 &= \frac{d_3}{D} \approx 0.06, \\
\pi_4 &= \frac{d_4}{D} \approx 0.02, & \pi_5 &= \pi_6 = \cdots = 0
\end{aligned}$$

Expected number of customers in salon:

$$l = \sum_{j=0}^{\infty} j\pi_j \approx 0(0.41) + 1(0.34) + 2(0.17) + 3(0.06) + 4(0.02) = 0.94 \text{ customers}$$

c. Effective arrival rate:

$$\lambda_{\text{eff}} = \sum_{j=0}^{\infty} \lambda_j \pi_j = 4.9 \text{ customers/hour}$$

By Little's law: expected waiting time (includes customers who reneged):

$$w = \frac{l}{\lambda_{\text{eff}}} = \frac{0.94}{4.9} \approx 0.192 \text{ hours} = 11.51 \text{ minutes}$$

Solutions to Problem 3.

a. M/M/5 queue with arrival rate $\lambda = 6$ pairs/hour and service rate $\mu = \frac{3}{2}$ pairs/hour (based on an average court use rate of 40 minutes or 2/3 hour).

b. Traffic intensity:

$$\rho = \frac{\lambda}{s\mu} = \frac{6}{5\left(\frac{3}{2}\right)} = \frac{12}{15} = \frac{4}{5}$$

c.

$$\pi_0 = \left[\sum_{j=0}^{5} \frac{4^j}{j!} + \frac{5^5\left(\frac{4}{5}\right)^6}{5!\left(\frac{1}{5}\right)}\right]^{-1} = \frac{1}{77} \approx 0.01$$

d.

$$l_q = \frac{\pi_s \rho}{(1-\rho)^2} = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}\pi_0 \frac{\rho}{(1-\rho)^2} = \frac{4^5}{5!}\left(\frac{1}{77}\right)\frac{\frac{4}{5}}{\left(\frac{1}{5}\right)^2} = \frac{512}{231} \approx 2.22 \text{ pairs}$$

e.

$$w_q = \frac{l_q}{\lambda} = \frac{2.22}{6} \approx 0.37 \text{ hours} = 22 \text{ minutes}$$

Solutions to Problem 4.

Model as an M/M/∞ queue with arrival rate $\lambda = 12$ and service rate $\mu = 4$.

Let L = a random variable representing the number of call in the system in steady-state.
Since this is an M/M/∞ queue, $L \sim$ Poisson $(\frac{\lambda}{\mu} = 3)$.

We want to find the smallest value of c^* such that $\Pr\{L \leq c^*\} \geq 0.99$, or equivalently:

$$\sum_{j=0}^{c^*} \frac{e^{-3}(3)^j}{j!} \geq 0.99$$

By trial-and-error (starting with $c^* = 1$ and increasing c^* by 1 until the above expression is true), we find that $c^* = 8$.