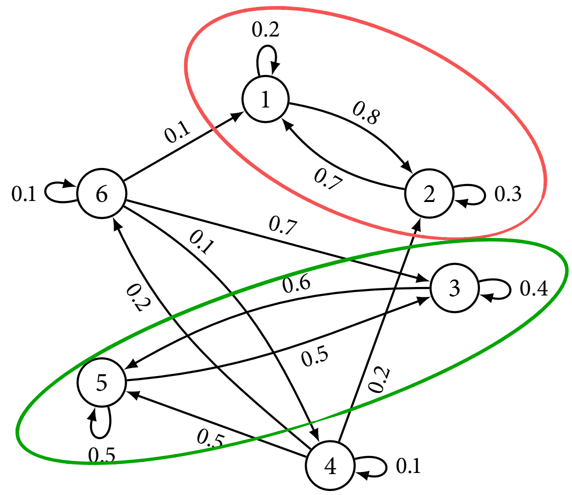
Problem 1. An autonomous UAV has been programmed to move between six regions to perform surveillance. The movements of the UAV follow a Markov chain with 6 states (1 for each region), and the following transition probability diagram:



a. There are two irreducible sets of states: $\{1, 2\}$ and $\{3, 5\}$. Briefly explain why these sets are irreducible.
b. Which states are transient? Which states are recurrent? Briefly explain.
c. Suppose the UAV starts in region 1. What is the long-run fraction of time that the UAV spends in region 1?
d. What is the probability that the UAV is absorbed into states 3 or 5, given that it starts in region 4?

a. Looking at the transition probability diagram, we can see that $\{1, 2\}$ and $\{3, 5\}$ form self-contained Markov chains, and no proper subsets of $\{1, 2\}$ or $\{3, 5\}$ form a self-contained Markov chain.

b. Recurrent states: 1, 2, 3, 5
(states part of an irreducible set by part a)

Transient states: 4, 6
(states not part of an irreducible set)

c. Let $\mathcal{R} = \{1, 2\}$. We want π_1.

From the transition probability diagram, $\mathbb{P}_{\mathcal{R}\mathcal{R}} = \begin{bmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{bmatrix}$

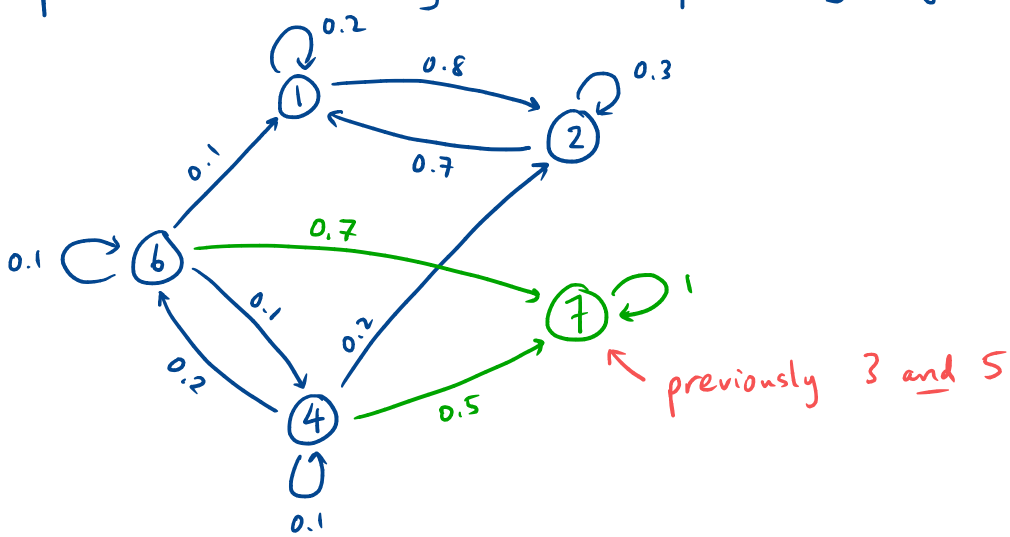
So:
$$\begin{aligned} \pi_{\mathcal{R}}^T \mathbb{P}_{\mathcal{R}\mathcal{R}} &= \pi_{\mathcal{R}} \\ \pi_{\mathcal{R}}^T \mathbb{1} &= 1 \end{aligned} \iff \begin{aligned} 0.2\pi_1 + 0.7\pi_2 &= \pi_1 \\ 0.8\pi_1 + 0.3\pi_2 &= \pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned}$$

$$\Rightarrow \quad \underline{\pi_1 = \frac{7}{15}}, \quad \pi_2 = \frac{8}{15}$$

Long-run fraction of time the UAV spends in region 1

d. This is a little tricky – the definition of an absorbing probability requires an absorbing state – an irreducible set of states with only one state.

Let's replace states 3 and 5 with a "super state" called 7. We end up with the following transition probability diagram:



Let $T = \{4, 6\}$ and $\mathcal{R} = \{7\}$. We want α_{47}:

$$\alpha_{T\mathcal{R}} = (I - \mathbb{P}_{TT})^{-1}\mathbb{P}_{T\mathcal{R}} = \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}0.1 & 0.2\\ 0.1 & 0.1\end{bmatrix}\right)^{-1}\begin{bmatrix}0.5\\ 0.7\end{bmatrix} \approx \begin{array}{cc} 7 & \\ \begin{bmatrix}0.747\\ 0.861\end{bmatrix} & \begin{matrix}4\\ 6\end{matrix}\end{array}$$

$$\Rightarrow \alpha_{47} \approx 0.747$$