5 Some more food for thought...

Problem 1. Consider the setting described in Example 1.

a. Over the long run, at what rate are customers lost?
b. Suppose that the shop makes a revenue of \$2 per customer served and pays the server \$4 per hour. What is the shop's long-run expected profit per hour?

Problem 2. Turingtown has a small urgent care center. Due to the recent increase in patients visiting the center, Turingtown is considering expanding the center. You have been asked to evaluate the current configuration.

The center has two doctors. You have estimated that the time a doctor spends with a patient is exponentially distributed with a mean of 30 minutes. Based on your observations and interviews, you have also estimated that patients arrive outside the center according to a Poisson process with a rate of 2 per hour. However, a patient will only enter the center if there is at most 1 patient already waiting (in other words, at most three patients total in the center); otherwise, the patient will opt to go to the hospital emergency room in nearby Gaussville.

a. Model this setting as a birth-death process by defining
   - the state space and what each state means,
   - the arrival rate in each state, and
   - the service rate in each state.
b. Find the steady-state distribution of the number of patients in the urgent care center.
c. What is the long-run expected number of patients waiting for a doctor?
d. What is the long-run expected time a patient waits for a doctor?
e. What fraction of the arriving patients opt to go to Gaussville instead?

$\boxed{1}$ a. When there are j customers in the shop, customers are lost at a rate of $20\left(\frac{j}{5}\right) = 4j$ per hour.

$\Rightarrow$ Lost customer rate $= \pi_0(0) + \pi_1(4) + \pi_2(8) + \pi_3(12) + \pi_4(16) + \pi_5(20)$

≈ 10.7 customers per hour

b. Expected profit per hour $= \left(\text{Expected \# customers}/\text{hr}\right)\left(\text{revenue}/\text{customer}\right) - \text{cost}/\text{hr}$

$= \lambda_{eff} \cdot 2 - 4 = 14.6$

$\boxed{2}$ a. state space $\mathcal{M} = \{\overbrace{0, 1, 2, \ldots}\}$ → number of patients in urgent care center

Arrival rates: $\lambda_i = \begin{cases} 2 & \text{if } i = 0,1,2,3 \\ 0 & \text{if } i = 4,5,\ldots \end{cases}$ Service rates: $\mu_i = \begin{cases} 2 & \text{if } i = 1 \\ 4 & \text{if } i = 2,3,\ldots \end{cases}$

b. $d_0 = 1$

$d_1 = \frac{\lambda_0}{\mu_1} = 1$

$d_2 = d_1 \frac{\lambda_1}{\mu_2} = 1\left(\frac{2}{4}\right) = \frac{1}{2}$

$d_3 = d_2 \frac{\lambda_2}{\mu_3} = \left(\frac{1}{2}\right)\left(\frac{2}{4}\right) = \frac{1}{4}$

$d_4 = d_3 \frac{\lambda_3}{\mu_4} = \frac{1}{4}\left(\frac{2}{4}\right) = \frac{1}{8}$

$d_5 = d_4 \frac{\lambda_4}{\mu_5} = 0$

$d_6 = d_7 = \cdots = 0$

$\Rightarrow D = \sum_{i=0}^{\infty} d_i = \frac{23}{8}$

$\Rightarrow \pi_0 = \frac{d_0}{D} = \frac{8}{23}$

$\pi_1 = \frac{d_1}{D} = \frac{8}{23}$

$\pi_2 = \frac{d_2}{D} = \frac{4}{23}$

$\pi_3 = \frac{d_3}{D} = \frac{2}{23}$

$\pi_4 = \frac{d_4}{D} = \frac{1}{23}$

$\pi_5 = \pi_6 = \cdots = 0$

c. $\ell_q = \sum_{i=s+1}^{\infty} (i - \overset{s=2}{s})\, \pi_i = (3-2)\pi_3 + (4-2)\pi_4 = \frac{4}{23}$ customers

d. $\lambda_{\text{eff}} = \sum_{i=0}^{\infty} \lambda_i \pi_i = 2\pi_0 + 2\pi_1 + 2\pi_2 + 2\pi_3 + 0\pi_4 = \frac{44}{23}$ customers/hr

$\Rightarrow w_q = \frac{\ell_q}{\lambda_{\text{eff}}} = \frac{1}{11}$ hours

e. Fraction of arriving customers going to Gaussville $= \pi_4 = \frac{1}{23}$