Lesson 3. Conditional Probability Review

1 Joint distributions

- Let X and Y be random variables
- X and Y could be **dependent**: for example,
  - X = service time of first customer in the shop
  - Y = delay (time in queue) of second customer in the shop
- If we want to determine the probability of an event that depends both on X and Y, we need their **joint distribution**
- The **joint cdf** of X and Y is:

  - Note: the textbook by Nelson uses $\Pr\{X = x, Y = y\}$ to mean $\Pr\{X = x \text{ and } Y = y\}$
- Suppose X and Y are discrete random variables:
  - X takes values $a_1, a_2, \dots$
  - Y takes values $b_1, b_2, \dots$
- The **joint probability mass function (pmf)** of discrete random variables X and Y is:

- We can obtain the **marginal pmfs** as follows:

- We can define a **joint probability density function (pdf)** and **marginal pdfs** for continuous random variables in a similar fashion

Example 1. The Markov Company sells three types of replacement wheels and two types of bearings for in-line skates. Wheels and bearings must be ordered as a set, but customers can decide which combination of wheel type and bearing type they want.

Let V and W be random variables that represent the type of bearing and wheel, respectively, in replacement sets ordered in the future. Based on historical data, the company has determined the probability that each wheels-bearings pair will be ordered:

		W		
	p_{VW}	1	2	3
V	1	2/10	1/10	1/10
	2	1/20	8/20	3/20

a. What is $\Pr\{V = 1 \text{ and } W = 2\}$?
b. What is $\Pr\{V = 1\}$?
c. What is $\Pr\{W = 2\}$?

2 Independence

- Let's consider events of the form $\{X \in \mathcal{A}\}$ and $\{Y \in \mathcal{B}\}$, for example:
  - $\mathcal{A} = (5, 27] \Rightarrow \{X \in \mathcal{A}\} =$
  - $\mathcal{B} = \{44, 73\} \Rightarrow \{Y \in \mathcal{B}\} =$
  - $\{X = a\}$ can be written as $\{X \in \mathcal{A}\}$ with
  - $\{Y \leq b\}$ can be written as $\{Y \in \mathcal{B}\}$ with
- Two random variables X and Y are **independent** if knowing the value of X does not change the probability of Y (and vice versa)
- Mathematically speaking, X and Y are independent if

Example 2. Are the random variables V and W in Example 1 independent? Why or why not?

3 Conditional probability

- **Conditional probability** addresses the question:

  How should we revise our probability statements about Y given that we have some knowledge of the value of X?

- The conditional probability that Y takes a value in $\mathcal{B}$ given that X takes a value in $\mathcal{A}$ is:

  - The revised probability is the probability of the joint event $\{Y \in \mathcal{B}, X \in \mathcal{A}\}$ normalized by the probability of the conditional event $\{X \in \mathcal{A}\}$

Example 3. In Example 1, what is the probability that a customer will order type 2 wheels, given that he or she orders type 1 bearings?

- If X and Y are independent, then:

- Let $\mathcal{A} \subseteq \mathcal{B}$. Then, if X and Y are perfectly dependent (i.e., $X = Y$), then:

4 Conditional distributions and expectations

- Let X and Y be discrete random variables
  - In particular, suppose Y takes on values $b_1, b_2, \ldots$
- The **conditional probability mass function (pmf)** of Y given $X \in \mathcal{A}$ is:

- The **conditional cumulative distribution function (cdf)** of Y given $X \in \mathcal{A}$ is:

- The **conditional expected value** of $g(Y)$ given $X \in \mathcal{A}$ is:

Example 4. In Example 1, find the conditional pmf of W given that $V = 1$.

Example 5. In Example 1, suppose that the profit from selling type 1, type 2, and type 3 wheels is \$4, \$6, and \$10, respectively. Find the expected profit from wheels, given that the customer ordered type 1 bearings.

5 Law of total probability

- We can write a joint probability as the product of a conditional probability and a marginal probability:

- Using this, we can also decompose a marginal probability into the products of conditional and marginal probabilities

- **The law of total probability.** Suppose X is a discrete random variable taking values $a_1, a_2, \dots$. Then:

- We have a similar law when X is a continuous random variable

Example 6. In Example 1, the conditional pmf of W given that $V = 2$ is:

b	1	2	3
$p_{W\|V=2}(b)$	1/12	8/12	3/12

Use this with your answer to Example 4 to find $\Pr\{W = 2\}$.

Example 7. Let Y be a random variable that models the time for you to get promoted from O-3 to O-4 in years. The cdf of Y is

$$F_Y(a) = \begin{cases} 1 - e^{-\frac{1}{5}a} & \text{if } a \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(Note this is the cdf of an exponential random variable with mean 5).

a. What is the probability that promotion takes longer than 5 years, given that it has already been 3 years?
b. What is the probability that promotion takes less than 6 years, given that it has already been 4 years?

6 Exercises

Problem 1. Professor I. M. Right often has his facts wrong. Let X be a random variable that represents the number of questions he is asked during one class, and let Y be the number of questions that he answers incorrectly during one class. The joint pmf p_{XY} of X and Y is:

		Y		
	p_{XY}	0	1	2
	1	1/3	0	0
X	2	1/4	1/12	0
	3	3/16	1/8	1/48

a. What is the probability that Professor Right answers all questions correctly during one class?

b. What is the probability that Professor Right answers 1 question incorrectly during one class, given that he is asked two questions?

c. Explain why $p_{XY}(1,2) = 0$.

Problem 2. The Simplex Company uses three machines to produce a large batch of similar manufactured items. 20% of the items were produced by machine 1, 30% by machine 2, and 50% by machine 3. In addition, 1% of the items produced by machine 1 are defective, 2% by machine 2 are defective, and 3% by machine 3 are defective. Suppose you select 1 item at random from the entire batch.

a. Define the random variable M as the machine used ($M \in \{1,2,3\}$) to produce this item. Write the pmf p_M of M.

b. Define another random variable D that is equal to 1 if this item is defective, and 0 otherwise. Find the probability that $D = 1$ given $M = m$, for $m = 1,2,3$.

c. Find the probability that $D = 1$; that is, the probability that the randomly selected item is defective.

Problem 3. Simplex Pizza sells pizza (of course) and muffins (that's weird). Let Z and M be random variables that represent the number of pizzas and muffins in one order, respectively. Based on historical data, the company has determined the joint pmf p_{ZM} for Z and M:

		M		
	p_{ZM}	0	1	2
	0	0	0.09	0.06
	1	0.25	0.11	0.05
Z	2	0.10	0.08	0.07
	3	0.08	0.07	0.04

a. What is the conditional pmf of M, given that $Z = 2$?

b. What is the expected number of muffins in an order, given that it contains 2 pizzas?

c. It turns out that $\Pr\{M = 1\} = 0.35$ and $\Pr\{M = 1 \mid Z = 3\} \approx 0.368$. Based on this information, are M and Z independent? Why or why not?