Lesson 1. Sample Paths

1 Course overview

- In this course, we will learn how to model and analyze systems that evolve dynamically over time and whose behavior is stochastic, or uncertain
- The models we will use to analyze such systems are called **stochastic processes**
- Numerous applications:
  - manufacturing - capacity planning, inventory control
  - business and economics - consumer behavior, portfolio management
  - military - combat strategy, ordnance effectiveness
- Techniques used: probability, statistics, linear algebra, basic notions of simulation
- This lesson: one example illustrating some basic ideas

2 The Case of the Copy Enlargement

The Darker Image, a national chain of small photocopying shops, currently configures each store with one photocopying machine and one clerk. Arriving customers stand in a single line to wait for the clerk. The clerk completes the customers' photocopying jobs one at a time, first-come-first-served, including collecting payment for the job.

Business is good enough that company headquarters plans to enlarge some stores by adding a second photocopying machine. The second copier could be operated by a new clerk, but since some customers with small copying jobs have complained about having to wait a long time, the company is considering installing the second copier for self-service copying only. The company wants to know which option will deliver better service.

- Some things need to be clarified, including:

- Important observation: neither option will be the best option all the time, since the system (and its performance) is subject to uncertainty, such as:

- We have some data collected one morning at a Darker Image location:
  - Time = minutes elapsed after 9:00
  - Customer numbers assigned in order of their arrival

Time	What happened	Comment	Time	What happened	Comment
0		shop opens	76	customer 12 arrived	
12	customer 1 arrived		79	customer 13 arrived	
14	customer 2 arrived		84	customer 10 finished	two-sided, staple
17	customer 3 arrived		86	customer 11 finished	
19	customer 1 finished	collate, staple	88	customer 14 arrived	
21	customer 2 finished		96	customer 12 finished	
22	customer 3 finished		96	customer 15 arrived	
38	customer 4 arrived		98	customer 13 finished	
39	customer 5 arrived		99	customer 14 finished	
41	customer 4 finished		102	customer 15 finished	collate, staple
43	customer 6 arrived		105	customer 16 arrived	
45	customer 7 arrived		107	customer 16 finished	
52	customer 8 arrived		108	customer 17 arrived	
57	customer 5 finished	special paper	110	customer 17 finished	
58	customer 9 arrived		113	customer 18 arrived	
60	customer 6 finished		118	customer 19 arrived	
61	customer 7 finished		120	customer 20 arrived	
68	customer 8 finished		121	customer 18 finished	special paper
71	customer 9 finished	collate, covers	127	customer 19 finished	two-sided, staple
73	customer 10 arrived		129	customer 20 finished	collate, staple
74	customer 11 arrived				

3 Analyzing the service times

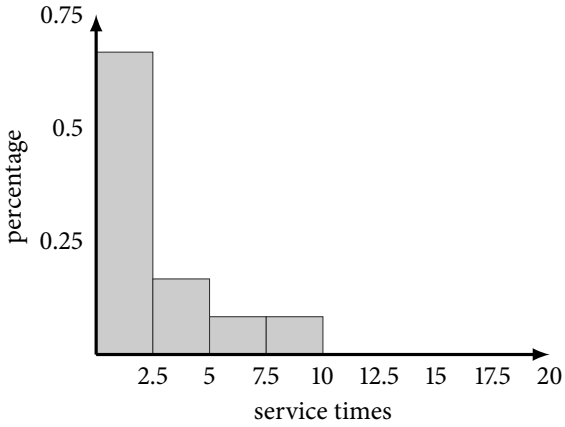
- Company HQ thinks any customer whose job doesn't need special handling (i.e. collating, stapling, two-sided, etc.) could have potentially used a self-service copier
- A reasonable first question: are the **service times** – the times required to complete the copying jobs – different for "regular" customers (jobs without special handling) compared to "special" customers (jobs with special handling)?
- Service time for customer 1:
- Service time for customer 2:
- More generally: for $i = 1, \ldots, 20$
  - Let a_i = arrival time for customer i
  - Let d_i = departure (finish) time for customer i
  - Service time for customer i:

- Service times for customers (* indicates special handling):

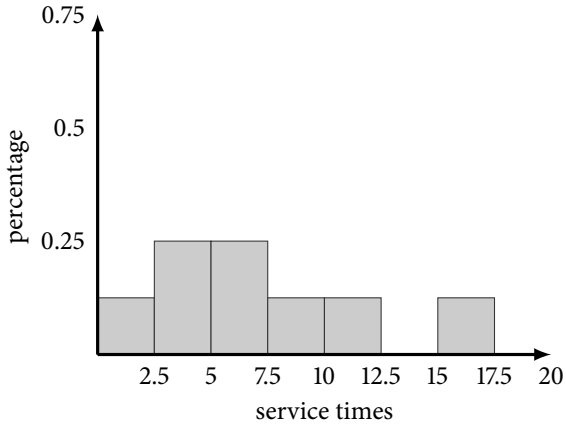
Customer number (i)	Service time (s_i)	Customer number (i)	Service time (s_i)
1*	7	11	2
2	2	12	10
3	1	13	2
4	3	14	1
5*	16	15*	3
6	3	16	2
7	1	17	2
8	7	18*	8
9*	3	19*	6
10*	11	20*	2

- Histograms of service times:

"Regular" customers
(Customers that do not require special handling)



"Special" customers
(Customers that require special handling)



- So, given the above analysis of the service times:
  - Are service times for "regular" and "special" customers different?
  - Should the company go with the self-service option?

- A better performance measure: **delay**, or the time a customer spends waiting in the **queue** (the line)
- How will adding a self-service copier or a second full-service copier affect the delay for regular and special customers?
- To measure delay, we need to consider the dynamics of the system
    - The delay of one customer depends on the actions of other customers

4 Sample path decomposition and analysis

- A **sample path** is a record of the time-dependent behavior of a system
    - For example, the time-study data on page 2
- A sample path can be decomposed into **inputs** and **logic**, for example:
    - Inputs = arrival times and service times of customers
    - Logic = how the shop operates: number of copiers, first-come-first-served, etc.
- **Simulation** generates new sample paths resulting from changes in the input or logic without building the new system
- **Sample-path analysis** extracts system performance measures from sample paths
- The Darker Image is interested in changing the logic of the system, while the inputs are out of its control
- To measure the change in delay when adding a self-service copier vs. a second full-service copier, we will simulate both systems (assuming the same customers arrive) and analyze the resulting sample paths

5 Interarrival times

- Working with interarrival times instead of arrival times is a common convention
- The **interarrival time** of customer i is the time gap between the arrival of customers $i-1$ and i:

- Interarrival times for customers (* indicates special handling):

Customer number (i)	Interarrival time (g_i)	Customer number (i)	Interarrival time (g_i)
1*	12	11	1
2	2	12	2
3	3	13	3
4	21	14	9
5*	1	15*	8
6	4	16	9
7	2	17	3
8	7	18*	5
9*	6	19*	5
10*	15	20*	2

6 Simulating the self-service system

- Let's start by simulating the proposed self-service system
- Assumptions on system logic:
  - 2 queues: full service queue, self-service queue
  - Regular customers always join the self-service queue, specials always join the full-service queue
  - Customers do not change queues after they join one
  - Service times are the same as in the original one-copier system
- To simulate the system, we need to keep track of the following **system events**:
  - a customer arrives
  - a job finishes at the full-service copier
  - a job finishes at the self-service copier
- Let's simulate the system until the arrival of customer 10
- After simulating all 20 customers (for homework), you should find that
  - the simulation ends after 129 minutes
  - customers 7, 13, 19 and 20 experience delays of 1, 7, 3 and 7 minutes, respectively
  - all other customers have delays of 0 minutes
- Therefore, the average delay per customer in this system is

7 Simulating the full-service system

- We can do something similar for the proposed full-service system
- Assumptions on system logic:
  - 1 queue for both copiers/clerks
  - Service delivered first-come-first-served by next available copier/clerk
- After simulating all 20 customers (for homework), you should find that
  - the simulation ends after 124 minutes
  - customers 7, 13 and 20 experience delays of 1, 5, and 1 minutes, respectively
  - all other customers have delays of 0 minutes
- Therefore, the average delay per customer in this system is

8 Comparisons

- Based on these simulations, which system is superior, self-service or full-service?
- How much confidence should we have in our conclusions?
- Concern:
  - We measured delay based on only one possible sample path for each proposed system
  - Better: a distribution of delay based on many possible sample paths
- One way to do this would be to obtain more data and perform more simulations – focus of SA421
- If we make certain reasonable assumptions, we can mathematically analyze sample paths without actually generating them – **focus of this course**

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service

Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service
Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service
Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service
Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service
Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service
Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

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Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service
Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service
Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service
Next system event	Time
customer arrival	
full-service finish	
self-service finish	

Current time:

Full Service	Self-Service
Next system event	Time
customer arrival	
full-service finish	
self-service finish	