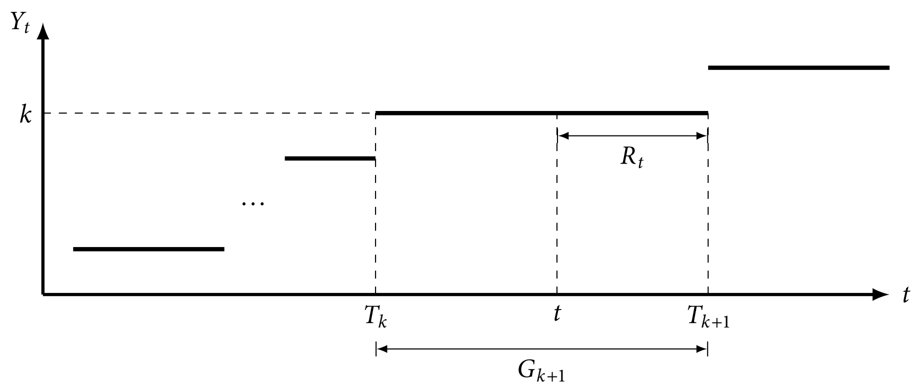
5 Why does the memoryless property hold?

- The memoryless property allows us to ignore when we start observing the Poisson process, since forward-recurrence times and interarrival times are distributed in the same way
- "Memoryless" $\longleftrightarrow$ how much time has passed doesn't matter
- Why is this true for Poisson processes?
- We want to show that $R_t \sim \text{Exponential}(\lambda)$, or equivalently, $\Pr\{R_t \le \Delta t\} = 1 - e^{-\lambda \Delta t}$
- Let's start by computing $\Pr\{R_t > \Delta t \mid Y_t = k\}$



- The event $\{R_t > \Delta t\}$ is equivalent to

$$\{G_{k+1} - (t - T_k) > \Delta t\} \iff \{G_{k+1} > t - T_k + \Delta t\}$$

- The event $\{Y_t = k\}$ is equivalent to

$$\{T_k \le t < T_{k+1}\} \iff \{G_{k+1} > t - T_k\}$$

- Therefore:

$$
\begin{aligned}
\Pr\{R_t > \Delta t \mid Y_t = k\} &= \Pr\{G_{k+1} > t - T_k + \Delta t \mid G_{k+1} > t - T_k\} && \text{(from above)} \\
&= \frac{\Pr\{G_{k+1} > t - T_k + \Delta t \text{ and } G_{k+1} > t - T_k\}}{\Pr\{G_{k+1} > t - T_k\}} && \text{(defn. of conditional prob.)} \\
&= \frac{\Pr\{G_{k+1} > t - T_k + \Delta t\}}{\Pr\{G_{k+1} > t - T_k\}} && (t - T_k + \Delta t > t - T_k) \\
&= \frac{e^{-\lambda(t - T_k + \Delta t)}}{e^{-\lambda(t - T_k)}} && (G_{k+1} \sim \text{Exp}(\lambda)) \\
&= e^{-\lambda \Delta t} && \text{(simplify)}
\end{aligned}
$$

- We can apply the law of total probability to "uncondition" this probability:

$$
\begin{aligned}
\Pr\{R_t > \Delta t\} &= \sum_{k=0}^{\infty} \Pr\{R_t > \Delta t \mid Y_t = k\} \Pr\{Y_t = k\} && \text{(law of total prob.)} \\
&= \sum_{k=0}^{\infty} e^{-\lambda \Delta t} \Pr\{Y_t = k\} && \text{(from above)} \\
&= e^{-\lambda \Delta t} \sum_{k=0}^{\infty} \Pr\{Y_t = k\} && (e^{-\lambda \Delta t} \text{ doesn't depend on } k) \\
&= e^{-\lambda \Delta t} && \Big(\sum_{\text{all } k} \Pr\{Y_t = k\} = 1\Big)
\end{aligned}
$$

- Therefore, $\Pr\{R_t \leq \Delta t\} = 1 - \Pr\{R_t > \Delta t\} = -e^{-\lambda \Delta t}$ as desired
- We also showed that R_t and Y_t are independent
- Note: This proof is a little sketchy – we actually need to condition on T_k instead of treating it as a constant
    - Works the same way, but with messier conditional statements and another use of the law of total probability
- The independent-increments and stationary-increments properties follow from the memoryless property and the fundamental relationship between Y_t and T_n (see Nelson pp. 110-111)