SA402 – Dynamic and Stochastic Models — Fall 2016
Assoc. Prof. Nelson Uhan

Lesson 15. A Quick Start Guide to Markov Processes

Course standards covered in this lesson: F1 – Generator matrices and transition rate diagrams, F2 – Steady-state probabilities of a Markov process

1 Overview

- Last few lessons: a Markov chain is a stochastic process that focuses on the state changes, ignoring the actual times at which the changes occur
- Today: let's look at **Markov processes**, which are similar to Markov chains, but also incorporate the time between state changes
- We will use Markov processes as a framework to study **queueing processes**

2 An important fact about exponential random variables

Fact 1. Let $H_j \sim \text{Exponential}(\lambda_j)$ for $j = 1, \ldots, k$ and $H = \min\{H_1, \ldots, H_k\}$.

Then $\boxed{}$.

3 Markov processes

- $\mathcal{M} = \{0, 1, 2, \ldots, m\}$ is the **state space**
    - By convention we include 0
    - For example, the state might represent number of customers in a queue
- S_n = nth state visited for $n = 0, 1, 2, \ldots$ (**state variable**)
- T_n = time when nth state is visited for $n = 0, 1, 2, \ldots$ (**event epochs**)
- $Y_t = S_n$ for $t \in [T_n, T_{n+1})$ (**output process**)
- p_j = probability of starting in state $j \in \mathcal{M}$
- H_{ij} = **transition time** from state i to state j ($i \neq j$)
    - $H_{ij} \sim \text{Exponential}(g_{ij})$
    - g_{ij} = **transition rate** from state i to state j
        - $1/g_{ij}$ = expected time between $i \to j$ transitions
    - H_{ij}'s are independent of each other
- We transition from state i to state j if the transition time for $i \to j$ is the smallest out of all the transition times from state i
    - In other words, H_{ij} is the smallest out of H_{ik} for $k = 0, \ldots, m$, $k \neq i$

- We can draw a **transition rate diagram** for a Markov process, in the same way as the transition probability diagram for a Markov chain, using the transition rates instead of transition probabilities as arc labels

Example 1. Simplexville College maintains a fleet of vans to be used by faculty and students for travel to conferences, field trips, etc. The time between requests is exponentially distributed with a mean of 1 day. The time a van is used is also exponentially distributed with a mean of 2 days. Requests to use a van occur at about 1 per day, and a van is used for an average of 2 days. If someone requests a van and one is not available, then the request is denied and other transportation, not provided by the motor pool, must be found. The motor pool currently has 3 vans.

a. Suppose there are no vans available. At what rate does the system transition to having one van available?
b. Model this system as a Markov process by (i) specifying the state space and (ii) drawing the transition rate diagram.

4 Time until next transition

- The **overall transition rate** out of state i is $g_{ii} = \sum_{j \neq i} g_{ij}$
- Suppose $S_n = i$ (at time T_n)
- The time of the next event (next transition) is:

- H_i is called the **holding time** in state i
- Because H_i is the minimum of $m - 1$ independent exponential random variables:

- In words, the time until the next transition is exponentially distributed with rate g_{ii}

5 The Markov and time stationarity properties

- The **Markov property**: only the state of the process at the current time t matters for probability statements about future times:

$$\Pr\{Y_{t+\Delta t} = j \mid Y_t = i \text{ and } Y_a \text{ for all } a < t\} = \Pr\{Y_{t+\Delta t} = j \mid Y_t = i\}$$

- The **time-stationarity property**: only the time increment matters, not the starting time:

$$\Pr\{Y_{t+\Delta t} = j \mid Y_t = i\} \text{ is the same for all } t \geq 0$$

- Powerful fact:
  - Let $\{Y_t; t \geq 0\}$ be a continuous-time stochastic process with discrete state space
  - Suppose $\{Y_t; t \geq 0\}$ satisfies the Markov and time stationarity properties

  $\Rightarrow$ $\{Y_t; t \geq 0\}$ must be a Markov process

6 Steady state probabilities

- (We assume in this section that the entire state space $\mathcal{M}$ is **irreducible**. For the definition of an irreducible set of states for a Markov process, see Chapter 7 of Nelson.)
- The **steady state probability** of being in state j:

$$\pi_j = \lim_{t \to \infty} \Pr\{Y_t = j\}$$

  - probability of finding the process in state j after a long period of time
  - long-run fraction of time the process is in state j

- How do we compute these probabilities?
  - Over the long run, the transition rate into state j is
  - Over the long run, the transition rate out of state j is
  - These quantities should be equal in steady state

- In matrix form:
  - **G** is the **generator matrix** of the Markov process: $\mathbf{G} = \begin{pmatrix} -g_{00} & g_{01} & \cdots & g_{0m} \\ g_{10} & -g_{11} & \cdots & g_{1m} \\ \vdots & & \ddots & \vdots \\ g_{m0} & g_{m1} & \cdots & -g_{mm} \end{pmatrix}$
  - Then the steady state probabilities can be found by solving

Example 2. Find the generator matrix of the Markov process described in Example 1.

Example 3. Consider Example 1. In the long run, what fraction of time are there 2 vans available? 1? 0?

Example 4. Consider Example 1. In the long run, what is the rate at which requests are denied? In the long run, what is the average number of vans in use?

Problem 1. Each customer service representative at Jungle.com spends his or her time answering e-mails and taking phone calls. Phone calls receive first priority, so a representative must interrupt tending to his or her e-mail whenever the phone rings. The time between phone calls is exponentially distributed with a mean of 5 minutes. The length of each phone call is exponentially distributed with a mean of 2 minutes.

a. Model how a representative switches between his or her two tasks as a Markov process by (i) specifying the state space and defining what the states mean, and (ii) specifying the transition rates, either by drawing the transition rate diagram or defining the generator matrix.
b. What is the long-run fraction of time each customer service representative spends answering e-mail? Taking phone calls?