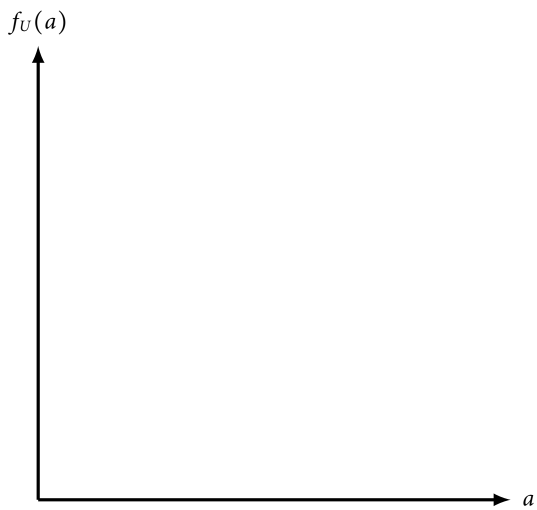
# Lesson 4. Random Variate Generation

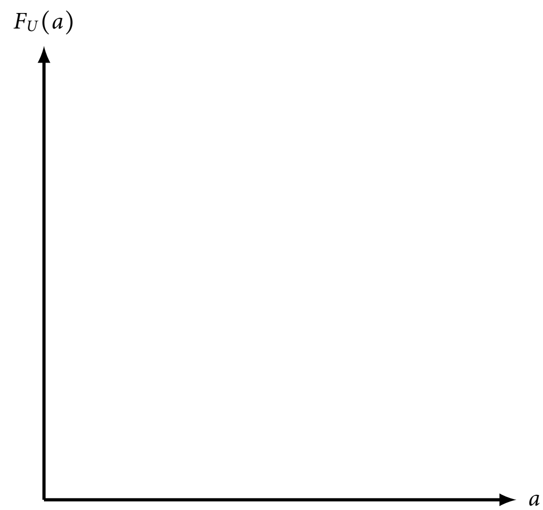
## 0 Warm up

**Example 1.** Let $U$ be a uniformly distributed random variable on $[0,1]$ (i.e. $U \sim \text{Uniform}[0,1]$). Recall that the pdf $f_U$ and cdf $F_U$ of $U$ are

$$f_U(a) = \begin{cases} 0 & \text{if } a < 0 \\ 1 & \text{if } 0 \le a \le 1 \\ 0 & \text{if } 1 < a \end{cases} \qquad\qquad F_U(a) = \begin{cases} 0 & \text{if } a < 0 \\ a & \text{if } 0 \le a \le 1 \\ 1 & \text{if } 1 < a \end{cases}$$

Graph $f_U$ and $F_U$ below.




## 1 Overview

- A **random variate** is a particular outcome of a random variable
- Given the cdf of a random variable, how can we generate random variates?
- One method: **the inverse transform method**
- Big picture:
  - We want to generate random variates of $X$ with cdf $F_X$
  - Assume we have a magic box that can generate random variates of $U \sim \text{Uniform}[0,1]$
  - We will transform random variates from this magic box into random variates of $X$
- How do we do this transformation? Need to define $X$ as a function of $U$

## 2 The discrete case

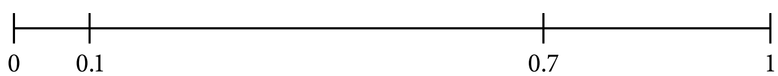
### First, an example

- Consider the discrete random variable $X$ with cdf

$$F_X(a) = \begin{cases} 0 & \text{if } a < 1, \\ 0.1 & \text{if } 1 \le a < 2, \\ 0.7 & \text{if } 2 \le a < 4, \\ 1 & \text{if } a \ge 4. \end{cases}$$

- Quick check: $p_X(2) =$

- Transformation idea:
  - Assign values of $X$ to values of $U$ (i.e. intervals on $[0, 1]$) according to the cdf




  - Mathematically speaking: set

- Does this transformation work? Let's check for $X = 2$:

- This also works for $X = 1$, $X = 3$, and $X = 4$

**More generally...**

- Let $X$ be a discrete random variable taking values $a_1 < a_2 < a_3 < \dots$
- Define $a_0 = -\infty$ so that

$$F_X(a_0) = \Pr\{X < a_0\} = \Pr\{X < -\infty\} = \boxed{\phantom{xxxxxxxx}}$$

- A **random variate generator** for $X$ is

$$X = a_i \quad \text{if } F_X(a_{i-1}) < U \le F_X(a_i) \qquad \text{for } i = 1, 2, \dots$$

- This works because for any $i$:

$$\begin{aligned}\Pr\{X = a_i\} &= \Pr\{F_X(a_{i-1}) < U \le F_X(a_i)\}\\ &= \Pr\{U \le F_X(a_i)\} - \Pr\{U \le F_X(a_{i-1})\}\\ &= F_X(a_i) - F_X(a_{i-1})\end{aligned}$$

  as desired!

- To generate a random variate $X$ with cdf $F_X$:

  1: Generate random variate $u$ of $U \sim \text{Uniform}[0, 1]$
  2: Find $a_i$ such that $F_X(a_{i-1}) < u \le F_X(a_i)$
  3: Set $x \leftarrow a_i$
  4: Output $x$ as a random variate of $X$

- Note: in the textbook by Nelson, the magic box that generates random variates of $U \sim \text{Uniform}[0, 1]$ is represented by the function `random()`
  - In other words, step 1 above can be written as: "Set $u \leftarrow$ `random()`"

## 3 The continuous case

- Now suppose $X$ is a continuous random variable
- We can't assign values of $X$ to intervals of $[0, 1]$ – $X$ takes on a continuum of values!
- New, related idea: set $X = F_X^{-1}(U)$
- Why does this transformation work?

- Therefore, $X = F_X^{-1}(U)$ is a **random variate generator** for $X$

- To generate a random variate of $X$ with cdf $F_X$:

  1: Generate random variate $u$ of $U \sim \text{Uniform}[0,1]$ (i.e. set $u \leftarrow$ `random()`)
  2: Set $x \leftarrow F_X^{-1}(u)$
  3: Output $x$ as a random variate of $X$

**Example 2.** Let $X$ be an exponential random variable with parameter $\lambda$. The cdf of $X$ is

$$F_X(a) = \begin{cases} 1 - e^{-\lambda a} & \text{if } a \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find a random variate generator for $X$.