Name:

SA402 – Dynamic and Stochastic Models

Assoc. Prof. Nelson Uhan

Fall 2016

Exam 2 – 4 November 2016

Instructions

- You have until the end of the class period to complete this exam.
- You may use a calculator.
- You may not consult any other outside materials (e.g. notes, textbooks, homework) except a printout of your scores on course standards.
- **Show all your work.** Your answers should be legible and clearly labeled. It is your responsibility to make sure that I understand what you are doing.
- **Cross out any problems that you did not attempt.**
- Keep this booklet intact.
- **Do not discuss the contents of this exam with any midshipmen until it is returned to you.**

Standard	Problems	Score
D1	1a	
D2	1b	
D3	1c	
D4	2a	
D5	2b	
D6	3	
E1	4	
E2	5a	
E3	5b	
E4	5c, 5d	
E5	6	
E6	7	

Problem 1. Passengers arrive at the Bellman Airlines check-in counter from 6:00 to 22:00 according to a stationary Poisson process at a rate of 40 per hour.

a. What is the probability that the 300th passenger arrives at or before 12:00, given that exactly 150 passengers arrive between 6:00 and 8:00?

b. What is the expected number of arrivals by the end of the day (6:00 - 22:00), given that exactly 250 passengers arrived between 6:00 and 12:00?

c. 15% of the passengers arriving at the check-in counter are traveling in First Class. What is the expected arrival time of the 60th First Class passenger?

Problem 2. Traffic engineers believe that the number of accidents at the intersection of College Avenue and King George Street follows a nonstationary Poisson process.

a. Using some preliminary data, they determined that the arrival rate function is:

$$\lambda(\tau) = \begin{cases} 10 & \text{if } 0 \leq \tau < 6 \\ 4 & \text{if } 6 \leq \tau \leq 12 \end{cases}$$

where the time τ is in months, and $\tau = 0$ corresponds to the beginning of January. What is the integrated rate function $\Lambda(\tau)$?

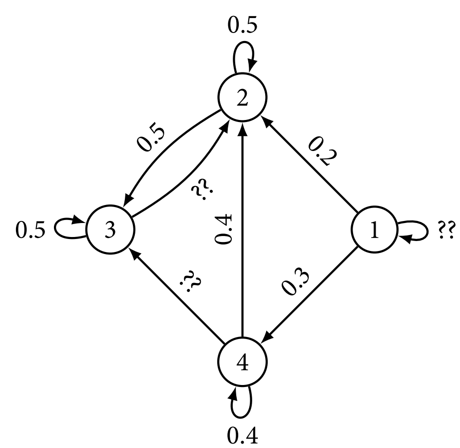
b. After collecting some more data, the traffic engineers have actually determined that the integrated rate function should be

$$\Lambda(\tau) = \begin{cases} 10\tau & \text{if } 0 \leq \tau < 3 \\ 6\tau + 2 & \text{if } 3 \leq \tau < 9 \\ 8\tau - 6 & \text{if } 9 \leq \tau \leq 12 \end{cases}$$

What is the probability that there are at most 15 accidents from the beginning of March to the end of May, if there are exactly 30 accidents from the beginning of January to the end of February?

Problem 3. You have been hired as a consultant for Markov Mobile Networks, a company that provides cellular phone service. Their engineers believe that phone calls arrive at a cell tower according to a stationary Poisson process. Describe what assumptions need to be made about phone call arrivals in order for this to be true. (You do not need to assess whether these assumptions are realistic.)

Problem 4. Consider the following transition-probability diagram for a Markov chain with four states – note that a few of the arc labels have missing values (??).



Write the corresponding one-step transition matrix. Your matrix must <u>not</u> have any missing values.

Problem 5. An autonomous UUV has been programmed to move randomly between five regions according to a Markov chain. Looking at the documentation written by the programmer, you find the following one-step transition matrix:

$$\mathbf{P} = \begin{bmatrix} 0.40 & 0.10 & 0.25 & 0 & 0.25 \\ 0.20 & 0.30 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.75 & 0.25 & 0 \\ 0 & 0 & 0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a. Suppose that the UUV starts in each of the five regions with equal probability. What is the probability that the UUV is in region 4 after 3 movements?

b. Do regions 2 and 3 form an irreducible set of states? Why or why not?

Here is the one-step transition matrix again:

$$\mathbf{P} = \begin{bmatrix} 0.40 & 0.10 & 0.25 & 0 & 0.25 \\ 0.20 & 0.30 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.75 & 0.25 & 0 \\ 0 & 0 & 0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c. Suppose the UUV reaches region 4. What is the long-run fraction of time it spends in region 4?

d. What is the probability that the UUV starts in region 1 and eventually ends up in region 5?

Problem 6. The statistician of the Simplexville Stars basketball team believes that players experience "hot hand" and "cold hand". According to her data, a player who made her last two shots has a 35% chance of making her next shot. On the other hand, a player who missed her last two shots has a 25% chance of making her next shot. Otherwise, a player has a 30% of making her next shot.

Model a player's shooting as a Markov chain by defining:

- the state space and the meaning of each state in the setting's context,
- the meaning of one time step in the setting's context, and
- the one-step transition probabilities.

Problem 7. Consider a model of officer promotion that defines the state of the system to be the rank of the officer (including separation and retirement) and the time index to be the number of years of service. Describe what assumptions need to be made in order for the Markov property and the time-stationarity property to hold. (You do not need to discuss whether these assumptions are realistic.)

Additional space for solutions or scratchwork

You may find the following information useful:

Exponential random variable with parameter λ: cdf $F(a) = \begin{cases} 1 - e^{-\lambda a} & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases}$ expected value $= 1/\lambda$

Erlang random variable with parameter λ and n phases: cdf $F(a) = \begin{cases} 1 - \sum_{j=0}^{n-1} \dfrac{e^{-\lambda a}(\lambda a)^j}{j!} & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases}$ expected value $= n/\lambda$

Poisson random variable with parameter λt: pmf $p(n) = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$ for $n = 0, 1, 2, \ldots$ expected value $= \lambda t$