Name:

SA402 – Dynamic and Stochastic Models — Fall 2016

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Quiz – 30 November 2016

Instructions. You have 30 minutes to complete this quiz. You may use your calculator. You may <u>not</u> use any other materials (e.g., notes, homework, books).

Standard	Problems	Score
F3	1a	
F4	2a	
F5	2b	
F6	1b	
F7	3	

Problem 1. Five Guys Burgers and Fries has three cashiers at its Simplexville location. Customers wait in a single queue and are served by the first available cashier, first-come first-served. The average service time is 2 minutes per customer, and customers arrive at a rate of 24 per hour. Assume the interarrival times and the service times are exponentially distributed.

a. Suppose the drive-thru lane is very short, and can only hold six cars (including the one receiving its order). Model this setting as a birth-death process by defining the arrival rate and service rate in each state.

b. Suppose the Simplexville location is enormous, and so for all intents and purposes, the queue has infinite capacity. What standard queueing model fits this setting best?

Problem 2. Customers call the reservation desk at Fluttering Duck Airlines at a rate of 4 customers per hour. There are 2 agents working at the reservation desk at any given time, and each phone call takes an average of 20 minutes. The phone system can only handle 5 customers at a time (2 with an agent, 3 waiting) – any phone calls arriving when there are 5 customers are simply lost. Some customers will renege if they are on hold too long – in particular, a customer is willing to spend 30 minutes waiting for service on average.

This setting can be modeled as a birth-death process with the following arrival and service rates:

$$\lambda_i = \begin{cases} 4 & \text{if } i = 0, 1, \dots, 5 \\ 0 & \text{if } i = 6, 7, \dots \end{cases} \qquad \mu_i = \begin{cases} 3i & \text{if } i = 1, 2 \\ 6 + 2(i-2) & \text{if } i = 3, 4, \dots \end{cases}$$

The resulting steady-state probabilities are:

$$\pi_0 \approx 0.26 \qquad \pi_1 \approx 0.34 \qquad \pi_2 \approx 0.23 \qquad \pi_3 \approx 0.11 \qquad \pi_4 \approx 0.05 \qquad \pi_5 \approx 0.01 \qquad \pi_j = 0 \text{ for } j = 6, 7, \dots$$

a. In the long run, what fraction of time are both agents busy?

b. In the long run, what is the expected waiting time?

Problem 3. Consider an M/M/2 queue with an arrival rate $\lambda = 8$ customers per hour and a service rate $\mu = 6$ customers per hour. What is the long-run expected delay?

You may find the following information useful:

- Steady-state probabilities of a birth-death process:

$$\pi_j = \frac{d_j}{D} \quad \text{for } j = 0, 1, 2, \dots \qquad \text{where} \qquad d_0 = 1 \qquad d_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j} \quad \text{for } j = 1, 2, \dots \qquad D = \sum_{i=0}^{\infty} d_i$$

- Steady state probabilities of an M/M/s system:

$$\pi_0 = \left[\left(\sum_{j=0}^{s} \frac{(s\rho)^j}{j!} \right) + \frac{s^s \rho^{s+1}}{s!(1-\rho)} \right]^{-1} \qquad \pi_j = \begin{cases} \dfrac{(\lambda/\mu)^j}{j!} \pi_0 & \text{for } j = 1, 2, \dots, s \\ \dfrac{(\lambda/\mu)^j}{s! s^{j-s}} \pi_0 & \text{for } j = s+1, s+2, \dots \end{cases} \qquad \text{where} \qquad \rho = \frac{\lambda}{s\mu}$$

- Expected number of customers in queue for an M/M/s system:

$$\ell_q = \frac{\pi_s \rho}{(1-\rho)^2} \qquad \text{where} \qquad \rho = \frac{\lambda}{s\mu}$$

- Poisson random variable L with parameter λ/μ:

$$L \sim \text{Poisson}(\lambda/\mu): \qquad p_L(n) = \frac{e^{-\lambda/\mu}(\lambda/\mu)^n}{n!} \quad \text{for } n = 0, 1, 2, \dots \qquad E[L] = \lambda/\mu$$