3 The M/M/s queue

- Steady-state probabilities: using $\rho = \lambda/(s\mu)$,

$$\pi_0 = \left[\left(\sum_{j=0}^{s} \frac{(s\rho)^j}{j!}\right) + \frac{s^s \rho^{s+1}}{s!(1-\rho)}\right]^{-1} \qquad \pi_j = \begin{cases} \dfrac{(\lambda/\mu)^j}{j!}\pi_0 & \text{for } j = 1, 2, \ldots, s \\ \dfrac{(\lambda/\mu)^j}{s! s^{j-s}}\pi_0 & \text{for } j = s+1, s+2, \ldots \end{cases}$$

- Expected number of customers in queue and expected delay:

$$\ell_q = \frac{\pi_s \rho}{(1-\rho)^2} \qquad w_q = \frac{\ell_q}{\lambda}$$

- Expected number of customers in the system and expected waiting time:

$$\ell = \lambda w \qquad w = w_q + \frac{1}{\mu}$$

Example 4. Recall the Darker Image case: we considered adding a second photocopier to the copy shop. Suppose customers arrive according to a Poisson process with rate 4 customers per hour, and that the service time of each photocopier is exponentially distributed with a mean of 12 minutes. Compare the expected delay of customers when there is 1 copier vs. when there are 2 copiers.

$\lambda = 4$ customers/hr $\qquad \mu = 5$ customers/hr $\qquad$ Want w_q

$s=1$: $\rho = \frac{\lambda}{s\mu} = 0.8$

$$\pi_0 = \left[\left(\sum_{j=0}^{1} \frac{(0.8)^j}{j!}\right) + \frac{(0.8)^2}{(1-0.8)}\right]^{-1} = 0.2$$

$$\pi_1 = \frac{(4/5)^1}{1!}\pi_0 = 0.16$$

$$\Rightarrow \ell_q = \frac{\pi_1 \rho}{(1-\rho)^2} = \frac{(0.16)(0.8)}{(1-0.8)^2} = 3.2$$

$$w_q = \frac{3.2}{4} = 0.8$$

$s=2$: $\rho = \frac{\lambda}{s\mu} = 0.4$

$$\pi_0 = \left[\left(\sum_{j=0}^{2} \frac{(0.8)^j}{j!}\right) + \frac{2^2(0.4)^3}{2!(1-0.4)}\right]^{-1} \approx 0.4286$$

$$\pi_2 = \frac{(4/5)^2}{2!}\pi_0 \approx 0.1371$$

$$\Rightarrow \ell_q = \frac{\pi_2 \rho}{(1-\rho)^2} \approx 0.1524$$

$$w_q \approx 0.0381$$

4 Exercises

Problem 1. The Kalman Theater Group (KTG) is building a movie theater mega-complex. They have decided that there will automatic ticket kiosks in front of a single first-come-first-served queue, but they still need to decide how many kiosks to include in the complex design. Based on data that they have collected from their other theaters in similar markets, they have estimated that customers arrive at the kiosks at a rate of 5 per minute, and customers can be served in 2 minutes on average. KTG's standard configuration for mega-complexes in similar markets is 12 kiosks. You have been asked to evaluate this standard configuration.

Assume that the interarrival times and the service times are exponentially distributed.

a. What standard queueing model fits this setting best?
b. What is the traffic intensity in this queueing system?
c. What is the long-run expected fraction of time that all kiosks are <u>un</u>occupied?
d. Over the long-run, what is the expected time a customer waits in <ins>line</ins>?

a. $M/M/\overset{s}{12}$ with $\lambda = 5$ customers/min and $\mu = \frac{1}{2}$ customer/min

b. $\rho = \frac{\lambda}{s\mu} = \frac{5}{6} \approx 0.8333$

c. Want π_0: $$\pi_0 = \left[\left(\sum_{j=0}^{12} \frac{(\frac{5}{6}\cdot 12)^j}{j!} \right) + \frac{12^{12}(\frac{5}{6})^{13}}{12!(1-\frac{5}{6})} \right]^{-1} \approx 0.000036$$

d. Want w_q: $$\pi_{12} = \frac{10^{12}}{12!}\pi_0 \approx 0.0749$$

$$\Rightarrow l_q = \frac{\pi_{12}(\frac{5}{6})}{(1-\frac{5}{6})^2} \approx 2.2469$$

$$\Rightarrow w_q = \frac{l_q}{\lambda} \approx 0.4494 \text{ minutes}$$