# Lesson 14. A Very Brief Introduction to Markov Processes

## 1 Overview

- Last few lessons: a Markov chain is a stochastic process that focuses on the state changes, ignoring the actual times at which the changes occur
- Today: let's look at **Markov processes**, which are similar to Markov chains, but also incorporate the time between state changes
- We will use Markov processes as a framework to study **queueing processes**

## 2 Markov processes

- $\mathcal{M} = \{0, 1, 2, \ldots, m\}$ is the **state space**
  - By convention we include 0
  - For example, the state might represent number of customers in a queue
- $H_{ij}$ = **transition time** from state $i$ to state $j$ ($i \neq j$)
  - $H_{ij} \sim \text{Exponential}(g_{ij})$
  - $g_{ij}$ = **transition rate** from state $i$ to state $j$
    - $g_{ij}$ = expected number of transitions per unit time (e.g. 3 transitions per day)
    - $1/g_{ij}$ = expected time between $i \to j$ transitions (e.g. 1/3 day per transition)
  - $H_{ij}$'s are independent of each other
- We transition $i \to j$ if the transition time for $i \to j$ is the smallest out of all the transition times from state $i$
  - In other words, $H_{ij}$ is the smallest out of $H_{ik}$ for $k = 0, \ldots, m$, $k \neq i$
- We can draw a **transition rate diagram** for a Markov process, in the same way as the transition probability diagram for a Markov chain, using the transition rates instead of transition probabilities as arc labels

**Example 1.** Simplexville College maintains 2 vans to be used by faculty and students for travel to conferences, field trips, etc. The time between requests is exponentially distributed with a mean of 1 day. The time a van is used is also exponentially distributed with a mean of 2 days. If someone requests a van and one is not available, then the request is denied and other transportation, not provided by the motor pool, must be found.

a. Suppose two vans are in use. At what rate does the system transition to having one van in use?
b. Model this system as a Markov process by (i) specifying the state space and (ii) drawing the transition rate diagram.

## 3 Steady state probabilities

- The **overall transition rate** out of state $i$ is $g_{ii} = \sum_{j \neq i} g_{ij}$
- The **steady state probability** of being in state $j$:

$$\begin{aligned}\pi_j &= \text{probability of finding the process in state } j \text{ after a long period of time}\\ &= \text{long-run fraction of time the process is in state } j\end{aligned}$$

- How do we compute these probabilities?
  - Over the long run, the transition rate into state $j$ is
  - Over the long run, the transition rate out of state $j$ is
  - These quantities should be equal in steady state
- In matrix form:
  - **G** is the **generator matrix** of the Markov process: $\mathbf{G} = \begin{pmatrix} -g_{00} & g_{01} & \cdots & g_{0m} \\ g_{10} & -g_{11} & \cdots & g_{1m} \\ \vdots & & \ddots & \vdots \\ g_{m0} & g_{m1} & \cdots & -g_{mm} \end{pmatrix}$
  - Then the steady state probabilities can be found by solving

**Example 2.** Find the generator matrix of the Markov process described in Example 1.

**Example 3.** Consider Example 1. In the long run, what fraction of time are there 0 vans in use? 1? 2?

**Example 4.** Consider Example 1. In the long run, what is the average number of vans in use?

## 4 Notes

- This was a very brief introduction to Markov processes, but just enough to get started with queueing processes.
- We discussed why we can add rates intuitively. We can do this mathematically because of the following property:

$$\text{If } H_1 \sim \text{Exponential}(\lambda_1) \text{ and } H_2 \sim \text{Exponential}(\lambda_2), \text{ then } \min\{H_1, H_2\} \sim \text{Exponential}(\lambda_1 + \lambda_2).$$

- We assume in this lesson that the entire state space $\mathcal{M}$ is **irreducible**. The notion of irreducibility is similar to that for Markov chains.
- For more details, see Chapter 7 of Nelson.

## 5 Exercises

**Problem 1.** Each customer service representative at Jungle.com spends his or her time answering e-mails and taking phone calls. Phone calls receive first priority, so a representative must interrupt tending to his or her e-mail whenever the phone rings. The time between phone calls is exponentially distributed with a mean of 5 minutes. The length of each phone call is exponentially distributed with a mean of 2 minutes.

a. Model how a representative switches between his or her two tasks as a Markov process by (i) specifying the state space and defining what the states mean, and (ii) specifying the transition rates, either by drawing the transition rate diagram or defining the generator matrix.

b. What is the long-run fraction of time each customer service representative spends answering e-mail? Taking phone calls?