Lesson 13. Markov Chains – Modeling and Assumptions

0 Quick summary

- Algorithmic representation of a Markov chain:
  - State space $\mathcal{M} = \{1, \dots, m\}$
  - One-step transition probabilities p_{ij} represented by cdfs $F_{N \mid L=i}(j)$ for all $i, j \in \mathcal{M}$
  - Initial state probabilities p_j represented by cdf F_{S_0}
  - S_n = state at time step n

$e_0()$: (generate initial state)
1: $S_0 \leftarrow F_{S_0}^{-1}(\texttt{random()})$ (initialize state of the process)

$e_1()$: (go to next state)
1: $S_{n+1} \leftarrow F_{N \mid L=S_n}^{-1}(\texttt{random()})$ (next state depends on current state)

algorithm Simulation:
1: $n \leftarrow 0$ (initialize system-event counter)
$e_0()$ (execute initial system event)
2: $e_1()$ (update state of the system)
$n \leftarrow n+1$ (update system-event counter)
3: go to line 2

- Two assumptions:
  1. The **Markov property**: only the last state influences the next state
$$\Pr\{S_{n+1} = j \mid S_n = i, S_{n-1} = a, \dots, S_0 = z\} = \Pr\{S_{n+1} = j \mid S_n = i\}$$
  2. The **time stationarity property**: one-step transition probabilities don't depend on when the transition happens
$$\Pr\{S_{n+1} = j \mid S_n = i\} \text{ is the same for all } n = 0, 1, 2, \dots$$

- Time-dependent performance measures
  - n-step transition probabilities
  - n-step state probabilities
  - first-passage probabilities
- Time-independent performance measures
  - Steady-state probabilities
  - Absorption probabilities

- This lesson: when is a Markov chain an appropriate model?

Example 1. For each of the following cases, discuss what assumptions need to be made in order for the Markov property and time stationarity property hold, and whether these assumptions are plausible.

a. A model of a taxi's movement defines the state of the system to be the region of the city that is the destination of the current ride, and the time index to be the number of riders the taxi has transported. When the taxi delivers a rider, it stays in the destination region until it picks up another rider.

b. A model of computer keyboard usage defines the state of the system to be the key that a person is currently typing, and the time index to be the number of keys typed.

c. A model of the weather in Annapolis defines the state of the system to be the high temperature, in whole degrees, and the time index to be the number of days.

Example 2. Arrow Auto Insurance wants to model the likelihood that a policyholder will be in an accident in a given year. The insurance company believes that after having an accident, a policyholder is a more careful driver for the next 2 years. In particular, actuarial records show that a customer who had at least 1 accident in the past two years has a 5% chance of having an accident during the current year, while a customer who did not have an accident during the past two years has a 10% chance of having an accident during the current year.

Model this setting as a Markov chain by defining:

- the state space and the meaning of each state in the setting's context,
- the meaning of one time step in the setting's context,
- the meaning of the state visited in the nth time step in the setting's context, and
- the one-step transition probabilities and initial state probabilities.

Problem 1. Consider a model of a consumer's preferences for toothpaste brands that defines the state of the system to be the brand that the consumer currently uses, and the time index to be the number of tubes of toothpaste purchased. Discuss what assumptions need to be made in order for the Markov property and time stationarity property hold, and whether these assumptions are plausible.

Problem 2. Bit Bucket Computers specializes in installing and maintaining computer systems. Unfortunately, the reliability of their computer systems is somewhat questionable. One of its standard configurations is to install two computers. When both are working at the start of business, there is a 30% chance that one will fail by the close of business, and a 10% chance that both will fail. If only one computer is working at the start of business, then there is a 20% chance it will fail by the close of business. Computers that fail during the day are picked up the following morning, repaired, and then returned the next day after that, before the start of business. Suppose we start observing an office with this configuration on a day with two working computers at the start of business.

Model this setting as a Markov chain by defining:

- the state space and the meaning of each state in the setting's context,
- the meaning of one time step in the setting's context,
- the meaning of the state visited in the nth time step in the setting's context, and
- the one-step transition probabilities and initial state probabilities.