Sets, Logic, Indicators

Narayana Santhanam

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Contents

Review the material from Section 1.1 of the textbook for basics of sets, operations on them (complements, unions, intersections). Venn diagram, cartesian prodcut of sets, and the algebra on sets (particularly how to figure out and prove basic stuff like De Morgan's laws).

1 Functions

A function f is a mapping from a set $\mathcal{D}$ to a set $\mathcal{R}$ that takes every element x of $\mathcal{D}$ and assigns to it exactly one element, denoted by $f(x)$ of $\mathcal{R}$. The set $\mathcal{D}$ is called the domain, and the set $\mathcal{R}$ is called the range. We denote a function like this as $f : \mathcal{D} \to \mathcal{R}$. Note that every element of $\mathcal{D}$ has an image in $\mathcal{R}$, but there could be elements of $\mathcal{R}$ that no element of $\mathcal{D}$ maps to. A function has the same sense that a programming language uses:

```
def f(input):
   do something with input
   return output
```

The notation $f : \mathcal{D} \to \mathcal{R}$ just means that the inputs will be the elements of the set $\mathcal{D}$ and the outputs will be from set $\mathcal{R}$.

Note also the distinction in notation between a function f and $f(x)$: for a function $f : \mathcal{D} \to \mathcal{R}$, for each $x \in \mathcal{D}$, $f(x)$ is just an element of $\mathcal{R}$ (and not a function). You can liken the function f to the function definition above, and $f(x)$ to a particular call of the function.

Let $\mathbb{R}$ be the set of all real numbers. Unfortunately, it is very common to write "Let $f(x) = \sin(x)$", as a shorthand for

"Let $f : \mathbb{R} \to \mathbb{R}$ be a function that assigns to each real number x, the real number $\sin(x)$."

Some authors write $f(\cdot) = \sin(\cdot)$, but such use can also get clunky. Generally, you need to infer from context if $\sin(x)$ refers to a specific number or the function $\sin(\cdot)$.

While $\mathcal{D}$ and $\mathcal{R}$ could be arbitrary sets in general, we will consider the case where the range is the set of real numbers, *i.e.,* $\mathcal{R} = \mathbb{R}$. We call such functions as real valued functions. A further special case of real-valued functions we will encounter often is when the range is just the set containing 0 and 1, namely $\mathcal{R} = \{0, 1\}$, which we call binary valued functions.

Let $f : \mathcal{D} \to \mathbb{R}$ and $g : \mathcal{D} \to \mathbb{R}$ be two functions from $\mathcal{D}$ to the set of all real numbers. The function $f + g$ is a function from $\mathcal{D} \to \mathbb{R}$ that assigns to each element $x \in \mathcal{D}$ the real number $f(x) + g(x)$. Similarly, the product fg is a function from $\mathcal{D} \to \mathbb{R}$ that assigns to each element $x \in \mathcal{D}$ the real number $f(x)g(x)$. For a real number c, it is also common to write $f + c$ (respectively $c - f$) for functions from $\mathcal{D}$ to $\mathbb{R}$ that assign every $x \in \mathcal{D}$ the number $f(x) + c$ (respectively $c - f(x)$). Similarly for cf. We can also make functions such as f^2, e^f, $1 - (1 - f)(1 - g)$ and so on, and manipulate such functions as usual, for example $1 - (1 - f)(1 - g) = f + g - fg$. Can you convince yourself all the above are valid shortcuts in notation?

We also overload the notations $\leq$, $\geq$ and $=$. We say $f \leq g$ if for all $x \in \mathcal{D}$, $f(x) \leq g(x)$ (and similarly for $\geq$ and $=$). Note that the comparisons between functions is only a partial order: it is quite possible that none of $f \leq g$, $f \geq g$ and $f = g$ hold (for example, let $f(x) = \cos(x)$ and $g(x) = x^2$). But if $f \leq g$ and $f \geq g$, then $f = g$.

1.1 Indicator functions

A particularly useful function we will pay attention to is the indicator function. Suppose Ω is some set, and let $A \subset \Omega$. The indicator function of A is a function $\mathbf{1}_A : \Omega \to \{0, 1\}$ (what is the domain and range?) assigns to every $x \in A$ the value 1, and to every $x \in A^c$ the value 0.

You can also think of indicator functions as logical statements. The indicator function $\mathbf{1}_A$ is the logical statement "Element belongs to the set A", and can be True (1) or False (0). Of course, from the context (*i.e.,* depending on what A and Ω are), there may be other logical statements that might describe $\mathbf{1}_A$ perfectly. For example, let $\Omega = \{1, \ldots ,6\}$ and $A = \{2, 3, 5\}$. Ω could have the interpretation in some context as all the outcomes of a dice throw, in which case the logical statement "The throw of dice yields a prime number" evaluates to True at exactly the elements of A.

The reason we are introducing this explicitly is that you will often make statements and assign them probabilities. What is really happening under the hood is that you are mapping your logical statement to a set (or subset of your universe), and assigning that set a probability. In common-speak, we don't think so carefully (and historically this is how pitchfork mobs usually form). But this course is generally an exercise in thinking carefully more than anything else, and it is useful to develop formalisms that will help towards that.

2 Sets

Review the material from Section 1.1 of the textbook for basics of sets, operations on them (complements, unions, intersections). Venn diagram, cartesian prodcut of sets, and the algebra on sets (particularly how to figure out and prove basic stuff like De Morgan's laws).

In addition, we will need to focus on a few things. The first is basic notation (which is sometimes called the set builder/set comprehension/set abstraction notation). The second concerns the connection between set operations and arithmetic operations on indicator functions. The last is a connection between set and logical operations that you may already be familiar with.

2.1 Notation

Sets could be specified explicitly as follows using ellipses {}

$$A = \{0, 1\}$$

and sometimes as

$$\mathbb{N} = \{1, 2, \ldots\}$$

to specify all natural numbers.

But a more common approach is the set comprehension/set builder/set abstraction. This is exactly equivalent to set comprehension in python (though the parent of python set comprehension is the math convention). For example consider the following piece of python code:

```
P = { x for x in range(2, 101)
      if not any(x % y == 0 for y in range (2,x))}
```

or writing $a \not| b$ for a does not divide b, and $\mathbb{N}$ for the set of natural numbers,

$$P = \{x \in \mathbb{N} : 2 \leq x \leq 100 \text{ and } y \not| x \text{ for } y \in \mathbb{N} \text{ and } 2 \leq y < x\}.$$

You would read the above as "x in $\mathbb{N}$ such that $2 \leq x \leq 100$ and $y \not| x$ for y in $\mathbb{N}$ and $2 \leq y < x$".

Both describe a set of all primes ≤ 100. You should get familiar with the set comprehension description since that is the dominant way we will describe sets.

2.2 Sets operations and arithmetic of indicator functions

Given the close relation between sets and indicator functions, we should expect that operations on sets are reflected in the arithmetic of indicator functions. Let Ω be a set and let A and B be subsets of Ω. Can you show that the following indicator functions (all taking inputs from Ω and outputing an element of the set $\{0, 1\}$)

$$\begin{aligned} \mathbf{1}_{A\cap B} &= \mathbf{1}_A \mathbf{1}_B \\ \mathbf{1}_{A^c} &= 1 - \mathbf{1}_A, \end{aligned}$$

and finally, since

$$A \cup B = (A^c \cap B^c)^c,$$

the above relations imply

$$\mathbf{1}_{A\cup B} = 1 - \mathbf{1}_{A^c\cap B^c} = 1 - \mathbf{1}_{A^c}\mathbf{1}_{B^c} = 1 - (1 - \mathbf{1}_A)(1 - \mathbf{1}_B).$$

Similarly, let $A_1, \ldots, A_n$ be subsets of Ω. Can you express the indicator functions $\mathbf{1}_{A_1\cap A_2\cap\cdots A_n}$ and $\mathbf{1}_{A_1\cup A_2\cup\cdots A_n}$ in terms of the indicator functions $\mathbf{1}_{A_1}$, $\mathbf{1}_{A_2}$ $\cdots$ $\mathbf{1}_{A_n}$?

2.3 Set and logical operations

Consider the set A of all elements $\omega \in \Omega$ on which a logical proposition a evaluates to True. Similarly, let set B correspond to logical proposition b. Then

- $A \cup B$ corresponds to $a \vee b$ (a OR b);
- $A \cup B$ corresponds to $a \wedge b$ (a AND b);
- A^c corresponds to $!a$ (NOT a);
- "if a then b" corresponds to $A \subset B$.

Clearly "if a then b" ($A \subset B$) is very different from "if b then a" ($B \subset A$). The two statements are called *converses* of each other. Once again, if a statement is True, there is no basis to believe the converse is True as well. But if both the statement and its converse happen to be True, we will have that $a = b$ (or $A = B$), and in this case we write a iff b.

Note that $A \subset B$ iff $B^c \subset A^c$. Therefore "if a then b" is completely equivalent to the statement "if $!b$ then $!a$". "if $!b$ then $!a$" is called the *contrapositive* of "if a then b", and they are two completely equivalent ways of saying the same thing.