В-26 ТРР1 Скрипець О.О. ІП-21

2.3 обчислити інтеграли

1) $\int_1^2 \ln(3x+2)dx = \int_5^8 \frac{1}{3}\ln(t)dt =$

$= \frac{1}{3}\int_5^8 \ln(t)dt = \frac{1}{3}\left(\ln(t)\cdot t - \int_5^8 t\frac{1}{t}dt\right) =$

$= \frac{1}{3}(\ln(t)\cdot t - t) = \frac{1}{3}(\ln(3x+2)(3x+2) - (3x+2)) =$

$= \left(\frac{1}{3}\ln(3x+2)(3x+2) - x - \frac{2}{3}\right)\Big|_1^2 =$

$= \frac{1}{3}\ln(3\cdot2+2)(3\cdot2+2) - 2 - \frac{2}{3} - \left(\frac{1}{3}\ln(3+2)(3+2) - 1 - \frac{2}{3}\right) =$

$= \frac{1}{3}\ln 8\cdot 8 - 2 - \frac{2}{3} - \left(\frac{1}{3}\ln 25 - 1 - \frac{2}{3}\right) = \frac{8}{3}\ln(8) - 2 - \frac{2}{3} - \frac{5}{3}\ln(5) + \frac{5}{3}$

$= \ln\left(8^{\frac{8}{3}}\cdot 5^{-\frac{5}{3}}\right) - 1 = \ln\left(\frac{256\sqrt[3]{5}}{25}\right) - 1$

2) $\int_1^2 \frac{dx}{x^3+1} = \int_1^2 \frac{1}{3(x+1)}dx + \int_1^2 \frac{-x+2}{3(x^2-x+1)}dx =$

$\int_1^2 \frac{1}{3(x+1)}dx + \int_1^2 \frac{-x+2}{3(x^2-x+1)}dx = \Bigg(\frac{1}{3}\ln(|x+1|) - \frac{1}{6}\ln(|x^2-x+1|) +$

$+ \frac{\sqrt{3}\,\mathrm{arctg}\left(\frac{2\sqrt{3}x-\sqrt{3}}{3}\right)}{3}\Bigg)\Big|_1^2 = \frac{1}{3}\ln(|3|) - \frac{1}{6}\ln(|3|) + \frac{\sqrt{3}\,\mathrm{arctg}\left(\frac{2\sqrt{3}\cdot2-\sqrt{3}}{3}\right)}{3} -$

$- \left(\frac{1}{3}\ln(|2|) - \frac{1}{6}\ln(|1|) + \frac{\sqrt{3}\,\mathrm{arctg}\left(\frac{2\sqrt{3}-\sqrt{3}}{3}\right)}{3}\right) = \frac{1}{3}\ln(3) - \frac{1}{6}\ln(3) + \frac{\sqrt{3}\frac{\pi}{3}}{3} - \left(\frac{1}{3}\ln(2) + \frac{\sqrt{3}\pi}{18}\right) =$

$= \frac{1}{3}\ln(3) - \frac{1}{6}\ln(3) + \frac{\sqrt{3}\pi}{9} - \frac{1}{3}\ln(2) - \frac{\sqrt{3}\pi}{18} = \frac{1}{6}\ln(3) + \frac{\sqrt{3}\pi}{18} - \frac{1}{3}\ln(2) =$

$$= \ln\left(3^{\frac{1}{6}}\right) + \frac{\sqrt{3}\pi}{18} + \ln\left(2^{-\frac{1}{3}}\right) = \ln\left(\frac{3}{4}\right)^{\frac{1}{6}} + \frac{\sqrt{3}\pi}{18} =$$

$$= \frac{1}{6}\ln\left(\frac{3}{4}\right) + \frac{\sqrt{3}\pi}{18}$$

3) $$\int_{-\frac{1}{3}}^{0} \frac{dx}{\sqrt{2-6x-9x^2}} = \int_{-\frac{1}{3}}^{0} \frac{1}{\sqrt{3-(3x+1)^2}}\,dx = \left[\begin{matrix} u=3x+1 \\ x=\frac{u-1}{3}, \; dx=\frac{1}{3}du \end{matrix}\right]$$

$$= \int_{-\frac{1}{3}}^{0} \frac{1}{3\sqrt{3-u^2}}\,du \left[\begin{matrix} v=\frac{u}{\sqrt{3}} \\ u=\sqrt{3}v, \; du=\sqrt{3}\,dv \end{matrix}\right] \frac{1}{3}\int_{-\frac{1}{3}}^{0} \frac{\sqrt{3}}{\sqrt{3-3v^2}}\,dv =$$

$$= \frac{1}{3}\int_{-\frac{1}{3}}^{0} \frac{1}{\sqrt{1-v^2}}\,dv = \int_{-\frac{1}{3}}^{0} \frac{1}{3}\cdot\frac{\arcsin(v)}{3} = \int_{-\frac{1}{3}}^{0} \frac{\arcsin\left(\frac{u}{\sqrt{3}}\right)}{3} =$$

$$= \int_{-\frac{1}{3}}^{0} \frac{\arcsin\left(\frac{3x+1}{\sqrt{3}}\right)}{3} = \frac{\arcsin\left(\frac{3x+1}{\sqrt{3}}\right)}{3}\Bigg|_{-\frac{1}{3}}^{0} = \frac{\arcsin\left(\frac{1}{\sqrt{3}}\right)}{3}$$

4) $$\int_{0}^{\pi} 2^4 \sin^8 x\,dx = \int_{0}^{\pi} 16\sin^8(x)\,dx = 16\int_{0}^{\pi} \sin^8 x\,dx =$$

$$= 16\left(-\frac{1}{8}\sin^7(x)\cos x + \frac{7}{8}\int_{0}^{\pi}\sin^6 x\,dx\right) =$$

$$= 16\left(-\frac{1}{8}\sin^7 x\cos x + \frac{7}{8}\left(-\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int_{0}^{\pi}\sin^4 x\,dx\right)\right) =$$

$$= 16\left(-\frac{1}{8}\sin^7 x\cos x + \frac{7}{8}\left(-\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3(x)\cos x + \frac{3}{4}\int_{0}^{\pi}\sin^2 x\,dx\right)\right)\right) =$$

$$= 16\left(-\frac{1}{8}\sin^7 x\cos x + \frac{7}{8}\left(-\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{8}\left(\int_{0}^{\pi}dx - \int_{0}^{\pi}\cos 2x\,dx\right)\right)\right)\right) =$$

$$= -2\sin^7 x\cos x - \frac{7\sin^5 x\cos x}{3} - \frac{35\sin^3 x\cos x}{12} + \frac{35}{8}x - \frac{35\sin 2x}{16}\Bigg|_{0}^{\pi} =$$

$$= -2\cdot 0^7\cdot(-1) - \frac{7\cdot 0(-1)}{3} - \frac{35\cdot 0\cdot(-1)}{12} + \frac{35\pi}{8} - \frac{35\cdot 0}{16} - \left(-2\cdot 0\cdot 1 - \frac{7\cdot 0\cdot 1}{3} - \frac{35\cdot 0\cdot 1}{12} + 0\cdot\frac{35\sin 0}{16}\right)$$

$$= \frac{35\pi}{8}$$

5) $\int\limits_{4\sqrt{\frac{2}{3}}}^{\sqrt{8}} \frac{\sqrt{x^2-8}}{x^4} =$ інтегрування частинами $=$

$\int fg' = fg - \int f'g$

$= g' = \frac{1}{x^4} \quad g = \int \frac{1}{x^4}dx = -\frac{1}{3x^3}$

$\left[\begin{matrix} u = \frac{1}{x}, \; x = \frac{1}{u} \\ dx = -\frac{1}{u^2}du \end{matrix}\right]$

$-\frac{\sqrt{x^2-8}}{3x^3} + \int\limits_{4\sqrt{\frac{2}{3}}}^{\sqrt{8}} \frac{1}{3x^2\sqrt{x^2-8}}\,dx = -\frac{\sqrt{x^2-8}}{3x^3} + \frac{1}{3}\int\limits_{4\sqrt{\frac{2}{3}}}^{\sqrt{8}} -\frac{u}{\sqrt{1-8u^2}}du =$

$= -\frac{\sqrt{x^2-8}}{3x^3} + \left[\begin{matrix} v = 1-8u^2 \\ -\frac{1}{16}dv = u\,du \end{matrix}\right] - \frac{1}{3}\int -\frac{1}{16\sqrt{v}}dv =$

$= -\frac{\sqrt{x^2-8}}{3x^3} + \frac{\sqrt{1-8u^2}}{24} = \left[u = \frac{1}{x}, \; x = \frac{1}{u}\right] = -\frac{\sqrt{x^2-8}}{3x^3} + \frac{\sqrt{(x^2-8)^3}}{24x} =$

$= \frac{(x^2-8)^{\frac{3}{2}}}{24x^3}\Bigg|_{4\sqrt{\frac{2}{3}}}^{\sqrt{8}} = \frac{4\sqrt{2}}{\sqrt{3}} - 2\sqrt{2} = \frac{1}{192}$

6) $\int\limits_4^9 \frac{\sqrt{x}\,dx}{\sqrt{x}-1} = \left[t = \sqrt{x}-1\right] = \int\limits_4^9 2\frac{t^2+4t+2}{t}dt =$

$= \int\limits_4^9 2t + 4 + \frac{2}{t}\,dt = \int\limits_4^9 2t\,dt + \int\limits_4^9 4\,dt + \int\limits_4^9 \frac{2}{t}dt =$

$= (\sqrt{x}-1)^2 + 4(\sqrt{x}-1) + 2\ln(|\sqrt{x}-1|) =$

$= x + 2\sqrt{x} - 3 + 2\ln(|\sqrt{x}-1|)\Big|_4^9 = 9 + 2\sqrt{9} - 3 + 2\ln(|\sqrt{9}-1|) -$

$-(4 + 2\sqrt{4} - 3 + 2\ln(|\sqrt{4}-1|)) = 9 + 6 - 3 + 2\ln(2) - (8 - 3 + 2\ln(1)) =$

$= 7 + 2\ln(2)$

(24) Обчислити інтеграли або довести їх розбіжність:

1) $\int\limits_1^\infty \frac{dx}{x^2(x+1)}$ — невластний інтеграл першого роду

$= \lim\limits_{b\to\infty}\int\limits_1^b \frac{1}{x^2(x+1)}\,dx = \left[\begin{matrix} u=\frac{x}{x+1},\ x=-\frac{1}{u-1}-1 \\ x=u(x+1),\ dx=\frac{1}{(u-1)^2}\,du\end{matrix}\right] =$

$= \int\limits_1^\infty -\frac{u-1}{u^2}\,du = -\int\limits_1^b \frac{1}{u} - \frac{1}{u^2}\,du = -\left(\int\limits_1^b \frac{1}{u}\,du - \int\limits_1^b \frac{1}{u^2}\,du\right) =$

$= \ln(|x+1|) - \ln(|x|) - \frac{x+1}{x}\Big|_1^\infty = 1-\ln(2)$

2) $\int\limits_1^5 \frac{x^2\,dx}{\sqrt{31(x^3-1)}}$ — невластний інтеграл другого роду з розривами

$= \lim\limits_{\delta\to 0}\int\limits_{1+\delta}^5 f(x)\,dx$

$f(x) = \frac{x^2}{\sqrt{31}\sqrt{x^3-1}}$ $\nexists$ $x=1$

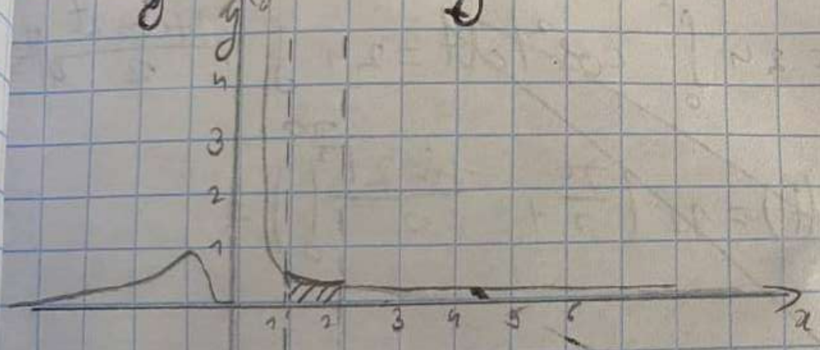
$\int\limits_1^5 \frac{x^2}{\sqrt{31(x^3-1)}} \left[\begin{matrix} u=x^3-1 \\ \frac{1}{3}du = x^2dx\end{matrix}\right] = \frac{1}{\sqrt{31}}\int\limits_1^5 \frac{1}{3\sqrt{u}}\,du =$

$= \left[\begin{matrix} \int u^n\,du = \frac{u^{n+1}}{n+1} \\ n=-\frac{1}{2}\end{matrix}\right] = \frac{1}{3\sqrt{31}}\cdot 2\sqrt{u} = \frac{2\sqrt{u}}{3\sqrt{31}} =$

$= \frac{2\sqrt{x^3-1}}{3\sqrt{31}}\Big|_1^5 = \frac{2\sqrt{5^3-1}}{3\sqrt{31}} = \frac{4}{3}$

15) обчислити площі фігур, обмежених кривими:

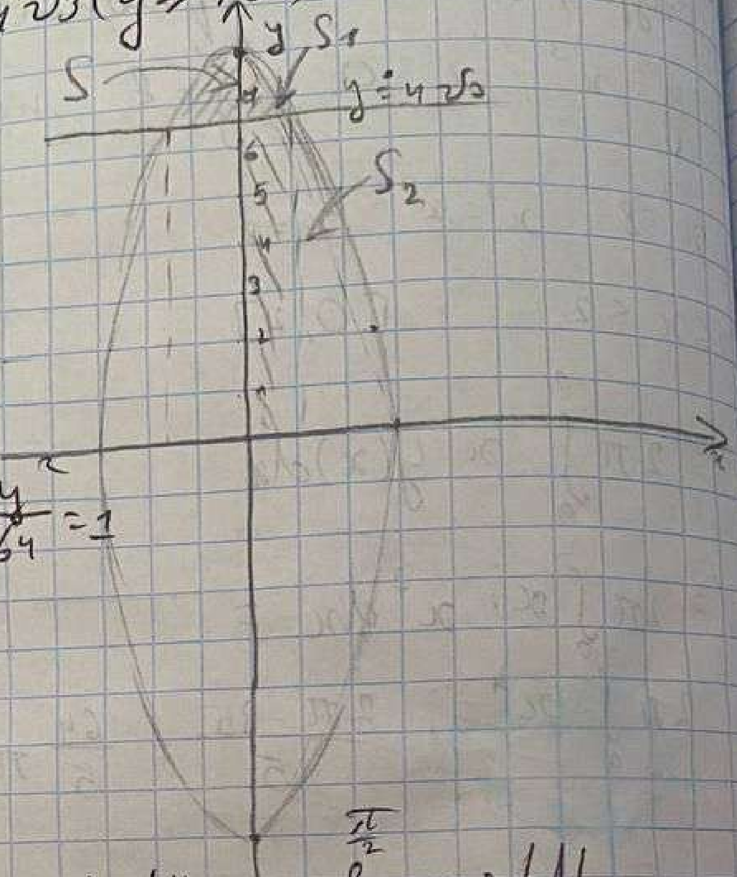
1) $y = x^{-2} e^{\frac{1}{x}}$, $y = 0$, $x = 2$, $x = 1$



$$\int_1^2 \frac{\sqrt[x]{e}}{x^2} = \left[\begin{matrix} u = \frac{1}{x} \\ -du = \frac{1}{x^2}dx \end{matrix}\right] = \int_1^2 -e^u = -\sqrt[x]{e}\,\Big|_1^2 =$$

$$= -\sqrt[2]{e} + \sqrt[1]{e} = e - \sqrt{e}$$

2) $\begin{cases} x = 3\cos t, & y = 4\sqrt{3}\ (y \geq 4\sqrt{3}) \\ y = 8\sin t, \end{cases}$



$y = 8\sin t$

$\sin t = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$

$t_1 = \frac{\pi}{3}$; $t_2 = \frac{\pi}{2}$

з еліпс буде $\frac{x^2}{9} + \frac{y^2}{64} = 1$

$S = 2S_1$

$S = \left| \int_{t_1}^{t_2} x'(t)\, y(t)\, dt \right|$

$S_1 + S_2 = \left| \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (-3\sin t \cdot 8\sin t)\, dt \right| = 24 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2 t\, dt =$

$= 24 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1 - \cos 2t}{2}\, dt = 12 \left(\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} dt - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos 2t\, dt \right) =$

$= 12 \left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sin 2t}{2} \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} \right) = 12 \left(\frac{\pi}{2} - \frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) =$

$= 6\pi - 4\pi + 3\sqrt{3} = 2\pi + 3\sqrt{3}$

$\frac{x^2}{9} + \frac{(4\sqrt{3})^2}{64} = 1 \Rightarrow \frac{x^2}{9} = 1 - \frac{48}{64} = \frac{1}{4}$

$x^2 = \frac{9}{4}$

$x = \pm \frac{3}{2}$

$$\sqrt{2} = 4\sqrt{3} \cdot \frac{3}{2} = 6\sqrt{3}$$

$$S_1 = 2\pi + 3\sqrt{3} - 6\sqrt{3} = 2\pi - 3\sqrt{3}$$

$$S = 2(2\pi - 3\sqrt{3}) = 4\pi - 6\sqrt{3}$$

3) $\rho = 2\sin 4\varphi$ – 4 пелюсткова троянда

$\rho \geq 0 \Rightarrow \sin 4\varphi \geq 0$

$\varphi \in [0; \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi)$ $\quad [0 + \frac{\pi k}{2}, \frac{\pi}{4} + \frac{\pi k}{2}]$

$$8 \cdot \frac{1}{2}\int_0^{\frac{\pi}{4}} (2\sin 4\varphi)^2 d\varphi = 4\int_0^{\frac{\pi}{4}} 4\sin^2 4\varphi \, d\varphi =$$

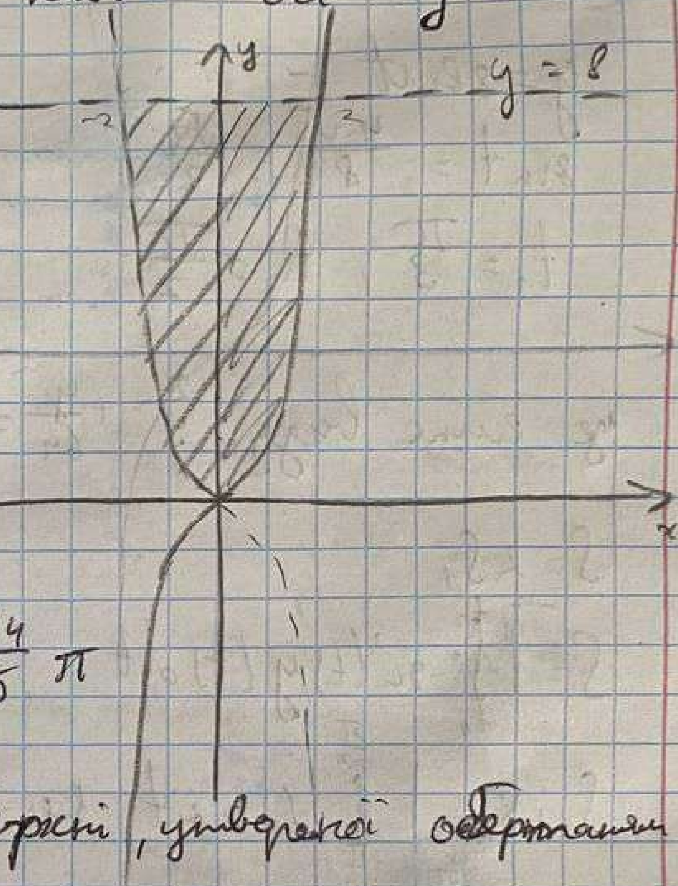
$$= \begin{bmatrix} u = 4\varphi & x = \frac{u}{4} \\ d\varphi = \frac{1}{4}du & \end{bmatrix} 4\int_0^{\frac{\pi}{4}} \sin^2(u)\,du = 4\int_0^{\frac{\pi}{4}} \frac{1-\cos(2u)}{2}\,du =$$

$$= 4\left(\frac{1}{2}\int_0^{\frac{\pi}{4}} 1\,du - \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2u\,du\right) = 4\cdot\left(2\varphi - \frac{\sin(8x)}{4}\right)\Big|_0^{\frac{\pi}{4}} =$$

$$= 4 \cdot \frac{\pi}{2} = 2\pi$$

26) Обчислити об'єм тіла, утвореного обертанням фігури, обмеженої кривими

$y = x^3$, $x = 0$ $y = 8$ навколо осі Oy



$x^3 = 8$, $x^3 = 2^3$; $x = 2$

$0 \le x \le 2$ $[0; 2]$

$$V = 2\pi \int_a^b x \cdot y(x) dx$$

$$V = 2\pi \int_0^2 x \cdot x^3 dx =$$

$$= 2\pi \int_0^2 x^4 = \frac{2\pi \cdot 35}{5} = \frac{64}{5}\pi$$

27) Обчислити площу поверхні, утвореної обертанням кривої $\rho^2 = 9 \cos 2\varphi$ навколо полярної осі

$\varphi \in [0; \pi]$ $\rho = 3\sqrt{\cos 2\varphi}$

$$\rho^2 = -3 \frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}$$

$$\rho^2 = 9 \frac{\sin^2 2\varphi}{\cos 2\varphi}$$

$$S = 2\pi \int_0^\pi 3\sqrt{\cos 2\varphi} \cdot |\sin \varphi| \cdot \sqrt{3\cos 2\varphi + 9 \cdot \frac{\sin^2 2\varphi}{\cos 2\varphi}} d\varphi =$$

$$= 18\pi \int_0^\pi \sin \varphi \, d\varphi = -18\pi \Big|_0^\pi = 36\pi$$