Documentation GRDonuts

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March 2020

Contents

1 Effective potential

In this chapter, the theory of geometrically thick tori is discussed. In this context, the specific angular momentum is introduced, and the effective potential is established. In the end, there is a discussion on tori in Schwarzschild and Kerr spacetime.

1.1 von Zeipel cylinders and specific angular momentum

The subject of study are perfect fluids around compact stellar objects. Therefore, an expression of a generic, stationary and axially symmetric metric is to be considered:

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\vartheta\vartheta}d\vartheta^2 + g_{\varphi\varphi}d\varphi^2 \tag{1.1}$$

This metric admits two Killing vectors. When a Killing vector is a coordinate basis vector, the metric is independent on the corresponding coordinate. This coordinate is called a *cyclic coordinate*. For this case, there are two Killing vectors for t and ϕ:

$$\eta^\mu = (1,0,0,0), \qquad \xi^\mu = (0,0,0,1)$$

For circular motion ($u^r = u^\vartheta = 0$), the four-velocity can be written as

$$u^\mu = u^t(\eta^\mu + \Omega\xi^\mu),$$

where $\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$ is the coordinate angular velocity.
From the normalising condition $g_{\mu\nu}u^\mu u^\nu = -1$, the four-velocity can be expressed in terms of the metric in (1.1):

$$(u^t)^{-2} = -g_{tt} - 2\Omega g_{t\varphi} - \Omega^2 g_{\varphi\varphi}$$

Following the definition of the *specific angular momentum* $l = -\frac{u_\varphi}{u_t}$, one can find an expression for the Euler equation:

$$\partial_\mu \ln|u_t| - \left(\frac{\Omega}{1-\Omega l}\right)\partial_\mu l = -\frac{1}{\rho h}\partial_\mu p \tag{1.2}$$

1.2 Geometrically thick tori

If we assume barotropic fluids, the surfaces of constant l and Ω coincide, so the integration of the Euler equation (1.2) does not depend on the chosen path, thus

$$\mathcal{W} - \mathcal{W}_{in} := -\int_0^p \frac{\mathrm{d}p'}{\rho h} = \ln|u_t| - \ln|(u_t)_{in}| - \int_{l_{in}}^{l} \frac{\Omega}{1-\Omega l'}\mathrm{d}l' \ , \tag{1.3}$$

where $\mathcal{W}$ is the *effective potential* and $\mathcal{W}_{in}$ the potential at the inner edge of the torus.

The equation (1.3) simplifies if one chooses a fluid with constant specific angular momentum l. If one sets $\mathcal{W}_{in} = \ln|(u_t)_{in}|$, the equipotential surfaces are given by

$$\mathcal{W}(r,\vartheta) = \ln|u_t| \tag{1.4}$$

$$\text{with } (u_t)^{-2} = -\big(g^{tt} + 2lg^{t\varphi} + l^2 g^{\varphi\varphi}\big) \tag{1.5}$$

These two equations, (1.4) and (1.5), will be of most importance in the following computations. This will give insight in the dynamics and behavior of the fluid orbiting a compact stellar object.

2 Schwarzschild black holes - computation

2.1 Effective potential in Schwarzschild spacetime

In this section, geometrically thick tori are computated in Schwarzschild spacetime. For computation purposes, the contravariant metric elements are needed:

$$g^{tt} = \frac{1}{1 - \frac{r_s}{r}}, \quad g^{t\varphi} = 0, \quad g^{\varphi\varphi} = \frac{1}{r^2 \sin^2 \vartheta}$$

The $g^{t\varphi}$-term vanishes, as the Schwarzschild spacetime is static. In natural coordinates ($c = G = 1$), the Schwarzschild radius reduces to $rs = 2M$.
With this, computing the four-velocity in (1.5) gives:

$$u_t = \sqrt{\frac{(r-2)r^2 \sin^2 \vartheta}{r^3 \sin^2 \vartheta - l^2(r-2)}},$$

and so the equipotential surfaces (*cf.*, Eq. (1.4)) follow accordingly:

$$\mathcal{W}(r, \vartheta) = \frac{1}{2} \ln \left[\frac{(r-2)r^2 \sin^2 \vartheta}{r^3 \sin^2 \vartheta - l^2(r-2)} \right]$$

2.2 Inverse effective potential in Schwarzschild spacetime

The computation is straight forward. Setting $\mathcal{W}$ as constant, the angle ϑ can be expressed as

$$\vartheta = \arcsin \left(\pm \sqrt{\frac{(2-r)\omega l^2}{(1-\omega)r^3 - 2r^2}} \right), \quad \omega = e^{2 \cdot \mathcal{W}}$$

3 Kerr black holes - computation

3.1 Effective potential in Kerr spacetime

In the previous section, geometrically thick tori orbiting a Schwarzschild black holes are discussed. For astrophysically realistic settings, it is reasonable to assume that black holes are rotating. The general description of rotating black holes is carried out by the *Kerr* spacetime.
Again, for computing the four-velocity given in equation (1.5), one needs the contravariant time- and angular components of the metric:

$$g^{tt} = -\frac{1}{\Delta}\left[r^2 + a^2 + \frac{2ra^2}{\Sigma}\sin^2\vartheta\right], \quad g^{t\varphi} = -\frac{2ra}{\Sigma\Delta}, \quad g^{\varphi\varphi} = \frac{\Delta - a^2\sin^2\vartheta}{\Sigma\Delta\sin^2\vartheta},$$

where $\Delta = r^2 - 2r + a^2$ and $\Sigma = r^2 + a^2\cos^2\vartheta$. Given this, computing the four-velocity results in the following expression:

$$u_t = \sqrt{\frac{\Sigma\Delta}{\Sigma(r^2 + a^2) + 2ra(a\sin^2\vartheta + 2l) + a^2l^2 - \frac{\Delta l^2}{\sin^2\vartheta}}}$$

and so the equipotential surfaces (given in Eq. (1.4)) follow accordingly:

$$\mathcal{W} = \frac{1}{2}\ln\left[\frac{\Sigma\Delta}{\Sigma(r^2 + a^2) + 2ra(a\sin^2\vartheta + 2l) + a^2l^2 - \frac{\Delta l^2}{\sin^2\vartheta}}\right]$$

3.2 Inverse effective potential in Kerr spacetime

The computation for the polar dependency is rather complicated: For a given $\mathcal{W}$, this results in ($\Sigma = r^2 + a^2 \cos^2 \vartheta$):

$$\underbrace{e^{2\cdot\mathcal{W}}}_{=:\,\alpha} = \frac{\Sigma\Delta}{\Sigma(r^2+a^2) + 2ra(a\sin^2\vartheta + 2l) + a^2l^2 - \frac{\Delta l^2}{\sin^2\vartheta}}$$

$$\Leftrightarrow \Sigma(r^2+a^2) + 2ra(a\sin^2\vartheta + 2l) + a^2l^2 - \frac{\Delta l^2}{\sin^2\vartheta} = \frac{\Sigma\Delta}{\alpha} \quad \Big| \quad \text{use } \Sigma = r^2 + a^2\cos^2\vartheta$$

$$\Leftrightarrow \underbrace{r^2(r^2+a^2)}_{=:\,\beta} + \underbrace{a^2(r^2+a^2)}_{=:\,\gamma}\cos^2\vartheta + \underbrace{2ra^2}_{=:\,\delta}\sin^2\vartheta$$

$$+\underbrace{4ral}_{=:\,\epsilon} + a^2l^2 - \frac{\Delta l^2}{\sin^2\vartheta} = \underbrace{\frac{r^2\Delta}{\alpha}}_{=:\,\lambda} + \underbrace{\frac{a^2\Delta}{\alpha}}_{=:\,\nu}\cos^2\vartheta$$

$$\Leftrightarrow \beta + \gamma\cos^2\vartheta + \delta\sin^2\vartheta + \epsilon + a^2l^2 - \frac{\Delta l^2}{\sin^2\vartheta} = \lambda + \nu\cos^2\vartheta$$

$$\Leftrightarrow \beta + \gamma - \gamma\sin^2\vartheta + \delta\sin^2\vartheta + \epsilon + a^2l^2 - \frac{\Delta l^2}{\sin^2\vartheta} = \lambda + \nu - \nu\sin^2\vartheta$$

$$\Leftrightarrow \underbrace{(\delta + \nu - \gamma)}_{=:\,\sigma}\sin^2\theta - \frac{\Delta l^2}{\sin^2\vartheta} = \underbrace{\lambda + \nu - \beta - \gamma - \epsilon - a^2l^2}_{=:\,\mu}$$

$$\Leftrightarrow \sigma\sin^4\vartheta - \Delta l^2 = \mu\sin^2\vartheta$$

$$\Leftrightarrow \sigma\sin^4\vartheta - \mu\sin^2\vartheta - \Delta l^2 = 0$$

$$\Rightarrow \sin^2\vartheta = \frac{\mu + \sqrt{\mu^2 + 4\sigma\Delta l^2}}{2\sigma}$$

$$\Rightarrow \vartheta = \arcsin\left[\pm\sqrt{\frac{\mu + \sqrt{\mu^2 + 4\sigma\Delta l^2}}{2\sigma}}\right]_\square$$

4 Q-Metric - computation

4.1 Effective potential for the Q-Metric

Again, for computing the four-velocity given in equation (1.5), one needs the contravariant time- and angular components of the metric:

$$g^{tt} = -\left(\frac{1}{1-\frac{r_s}{r}}\right)^{-(1+q)} := -\frac{1}{\alpha^{1+q}}, \quad g^{t\varphi} = 0, \quad g^{\varphi\varphi} = \frac{\alpha^q}{r^2 \sin^2 \vartheta}$$

where q is related to the Geroch quadrupole moments. Given this, computing the four-velocity results in the following expression:

$$u_t = \sqrt{\frac{\alpha^{1+q} r^2 \sin^2 \vartheta}{r^2 \sin^2 \vartheta - l^2 \alpha^{1+2q}}}$$

and so the effective potential can be written as

$$\mathcal{W} = \frac{1}{2} \ln \left[\frac{\alpha^{1+q} r^2 \sin^2 \vartheta}{r^2 \sin^2 \vartheta - l^2 \alpha^{1+2q}}\right]$$

4.2 Inverse effective potential for the Q-Metric (Quevedo, 2011)

The computation is straight forward. Setting $\mathcal{W}$ as constant, the angle ϑ can be expressed as

$$\vartheta = \arcsin\left(\pm\sqrt{\frac{\omega l^2 \alpha^{1+2q}}{\omega r^2 + \alpha^{1+q} r^2}}\right)$$