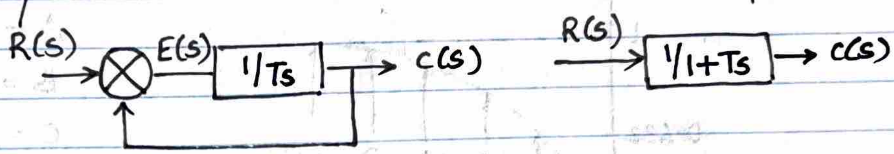
* The 1st step in analyzing a control system was to derive a mathematical model of the system.
* Once such a model has been obtained, various methods are available for the analysis of system performance.

First order systems: systems described by a single first order differential equation.

examples of such systems include RC circuits, thermal systems

Block diagram of 1st order system



$$\boxed{\frac{C(s)}{R(s)} = \frac{1}{Ts+1}}$$

* Assuming all initial conditions are zero, we will analyze the system response to different inputs:

① UNIT-STEP : $r(t) = 1$ $R(s) = 1/s$

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s}$$

Apply Partial fractions

$$C(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{(s+1/T)}$$

Take inverse L.T $\Rightarrow$ $\boxed{c(t) = 1 - e^{-t/T}}$ $t \geq 0$

known as exponential response curve

What is error?

$$e(t) = r(t) - c(t) = 1 - 1 + e^{-t/T}$$

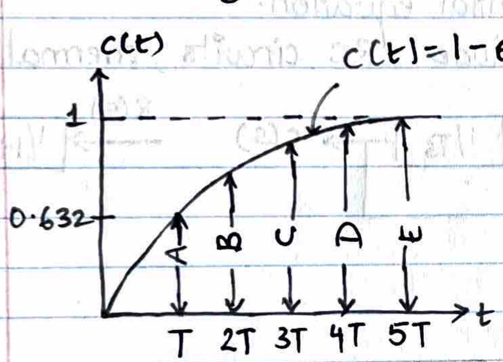
$$\boxed{e(t) = e^{-t/T}}$$

Steady-state error: $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$

At $t=0$ $\quad e(t) = \frac{1}{e^{t/T}} = \frac{1}{e^0} = 1 \qquad c(t) = 1 - \frac{1}{e^{t/T}} = 1-1=0$

∴ Initially the output $c(t)=0$ and error is 1

As $t\to\infty$ $\quad e(t) = \frac{1}{e^{\infty}} = 0 \qquad c(t) = 1 - \frac{1}{e^{\infty}} = 1$



A = 63.2%
B = 86.5%
C = 95%
D = 98.2%
E = 99.3%

At $t=T$ $\quad c(t) = 1 - \frac{1}{e^1} = 1 - 0.3678 = 0.6321$

i.e. the response/output has reached 63.21% of its total change

The smaller the time constant T, the faster the system response

T: time when response rises to 63% of final value.

Slope of the tangent line to c(t):

$$\left.\frac{dc}{dt}\right|_{t=0} = \left.\frac{1}{T} e^{-t/T}\right|_{t=0} = \frac{1}{T}$$

* Slope of response curve c(t) decreases monotonically from $1/T$ (@ $t=0$) to 0 (@ $t=\infty$)

* In one time constant T, the exponential response curve has gone from 0 to 63.2% of the final value.
* In $t=2T$, the response reaches 86.5% of the final value
* In $t=3T$, 95%.
* In $t=4T$, 98.2%.
* In $t=5T$, 99.3%.

For $t \geq 4T$, the response remains within 2% of final value.
Used in practice.

② UNIT- RAMP: $r(t) = t$ $\quad R(s) = \frac{1}{s^2}$

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2} \qquad \frac{1}{s^2(Ts+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{Ts+1}$$

Apply partial fractions

$$1 = A(Ts^2+s) + B(Ts+1) + Cs^2$$

$$1 = (AT+C)s^2 + (BT+A)s + B$$

$$C(s) = \frac{-T}{s} + \frac{1}{s^2} + \frac{T^2}{Ts+1} \qquad B = 1 \quad A = -T \quad C = T^2$$

Take laplace inverse transform:

$$L^{-1}\left\{\frac{T^2}{s+1/T}\right\}$$

$$c(t) = -T(1) + t + Te^{-t/T}$$

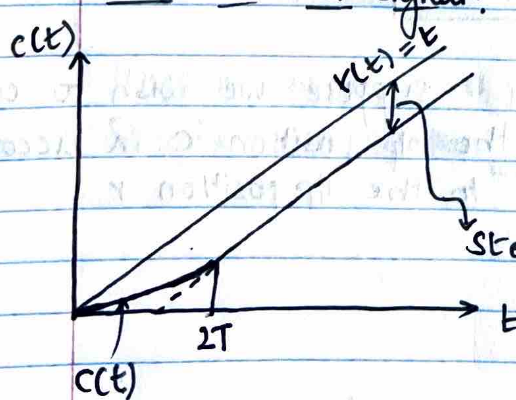
$$\boxed{c(t) = t - T + Te^{-t/T}}$$

What is error signal?

$$e(t) = r(t) - c(t)$$

$$= \cancel{t} - \cancel{t} + T - Te^{-t/T}$$

$$\boxed{e(t) = T(1-e^{-t/T})}$$



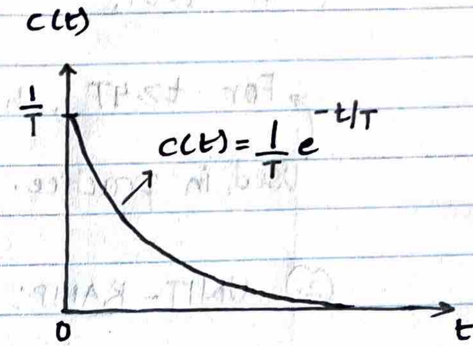
As $t \to \infty$ $\quad e(t) = T$

Steady-state error

* Error is equal to T for sufficiently large t.
* Smaller the time constant T, the smaller the steady-state error.

③ UNIT-IMPULSE: $r(t) = \delta(t)$ $\quad R(s) = 1$

$$C(s) = \frac{1}{Ts+1} = \frac{1}{s+\frac{1}{T}} \cdot \frac{1}{T}$$



Taking inverse laplace transform

$$\boxed{c(t) = \frac{1}{T} e^{-t/T}} \qquad t \geq 0$$

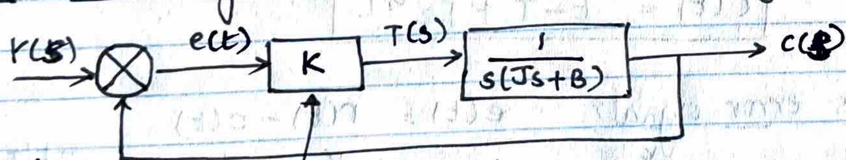
$t = 0 \qquad c(t) = 1/T$

$t = T \qquad c(t) = \frac{0.3678}{T}$

$t = 2T \qquad c(t) = \frac{0.1353}{T}$

$\vdots$

Second-order System: consider a servo system as an example



• Assume zero initial conditions

T(s) : Torque produced by proportional controller whose gain is K

proportional controller

$$\frac{C(s)}{T(s)} = \frac{1}{s(Js+B)}$$

Suppose we wish to control the o/p position c in accordance to the i/p position r

* Transient response occurs in the 1st period of time after the input is applied and reflects the mismatch b/w initial condition & steady-state solution.

TIME Response

$$\xrightarrow{r(t)} \boxed{\text{System}} \xrightarrow{c(t)}$$

- We would like to analyze a system property by applying a test input $r(t)$ and observing a time response $c(t)$

Time responce: $c(t) = \boxed{c_t(t)} + \boxed{c_{ss}(t)}$

Transient response: $\lim_{t\to\infty} c_t(t) = 0$

Steady-state responce (after $c_t(t)$ dies out)

* Steady state response is the portion of the output response that reflects the long-term behavior of the system under given inputs.

Usage of time responses:

1. Modeling:
   Some parameters in the system may be estimated by time responses
2. Analysis:
   evaluate steady-state and transient responses and evaluate if they are satisfactory or not.
3. Design:
   given design specs in terms of transient and steady-state responses, design controllers satisfying all the design specs.

Example: $r(t)=u_s(t) \rightarrow \boxed{\dfrac{3}{2s+1}} \rightarrow c(t) = 3-3e^{-t/2}$

Transient response: $c_t(t) = -3e^{-t/2}$

Steady-state response: $c_{ss}(t) = 3$

closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K}$$

Such a system where closed-loop T.F possesses 2 poles is called 2^{nd} order system

Standard form of 2^{nd} order system:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n: undamped natural frequency

ζ: damping ratio.

* The behavior of 2^{nd} order system can be described in-terms of ζ, ω_n

* $s^2 + 2\zeta\omega_n s + \omega_n^2$: roots

$$= \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4(1)\omega_n^2}}{2}$$

$$= \frac{-2\zeta\omega_n \pm \sqrt{4(\zeta^2-1)\omega_n^2}}{2}$$

$$= \frac{-\cancel{2}\zeta\omega_n \pm \cancel{2}\omega_n\sqrt{\zeta^2-1}}{\cancel{2}}$$

roots are

$-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

Case 1: if ζ is between $0<\zeta<1$ [UNDERDAMPed CASE]

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})}$$

Let $\omega_d = \omega_n\sqrt{1-\zeta^2}$ where ω_d: damped natural frequency

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}$$

Let $R(s) = 1/s \Rightarrow r(t) = 1$

$$\frac{As+B}{s^2+2\zeta\omega_n s+\omega_n^2} + \frac{C}{s} = \omega_n^2$$

$$As^2+Bs + Cs^2 + C2\zeta\omega_n s + C\omega_n^2$$

$A+C=0$ $\quad$ $C=1$

$A=-C$ $\quad$ $B+2\zeta\omega_n C=0$

$\boxed{A=-1}$ $\quad$ $B=-2\zeta\omega_n$

$$C(s) = \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)s}$$

$$C(s) = \frac{\omega_n^2}{s(s+\zeta\omega_n 2s + \omega_n^2)}$$

Apply partial

Take inverse laplace transform

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \left(\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right)\frac{\omega_d}{\omega_d}$$

$$L^{-1}\left\{\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right\} = e^{-\zeta\omega_n t}\cos\omega_d t$$

$$L^{-1}\left\{\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\} = e^{-\zeta\omega_n t}\sin\omega_d t$$

$$\boxed{c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - e^{-\zeta\omega_n t}\sin\omega_d t \cdot \frac{\zeta}{\sqrt{1-\zeta^2}}}$$

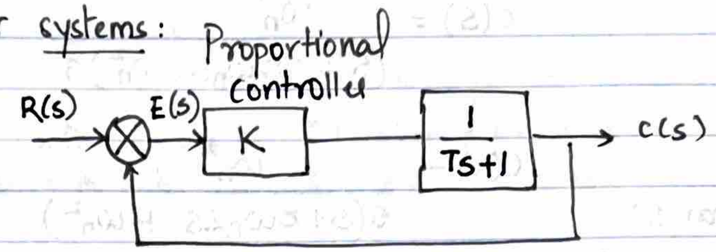
Error signal for this system: $e(t) = r(t) - c(t)$

$$e(t) = e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \qquad t \geq 0$$

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\cos\omega_d t\sqrt{1-\zeta^2} + \zeta\sin\omega_d t\right]$$

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Analysis of 1^{st} order systems:



Proportional control of a system without an integrator will result in a steady-state error with a step input

$$G(s) = \frac{K}{Ts+1} \qquad \frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)}$$

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = 1 - \frac{G(s)}{1+G(s)} = \frac{1}{1+G(s)}$$

$$\boxed{E(s) = \frac{R(s)}{1+G(s)}}$$

What if $R(s) = 1/s$?

Transient-response specifications:

* The performance characteristics of a control system are specified in terms of the transient response to a unit-step input.
* If the response to a step-input is known, it is mathematically possible to compute the response to any input.

* Transient response of a practical control system often exhibits damped oscillations before reaching steady state.

Definitions:

(1) Delay Time: (t_d)

The delay time is the time required for the response to reach half the final value the very first time.

(2) Rise Time: (t_r)

The rise time is the time required for the response to rise from 10% to 90%, 5% to 95% or 0% to 100% of its final value.

For underdamped 2nd order systems, the 0% to 100% rise time is normally used.

For overdamped systems, the 10% to 90% rise time is commonly used.

(3) Peak time: (t_p)

The peak time is the time required for the response to reach the first peak of the overshoot.

* For overdamped systems : t_p, max % overshoot donot apply

4. Maximum (%) overshoot: (M_p)

The maximum overshoot is the maximum peak value of the responce curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot.

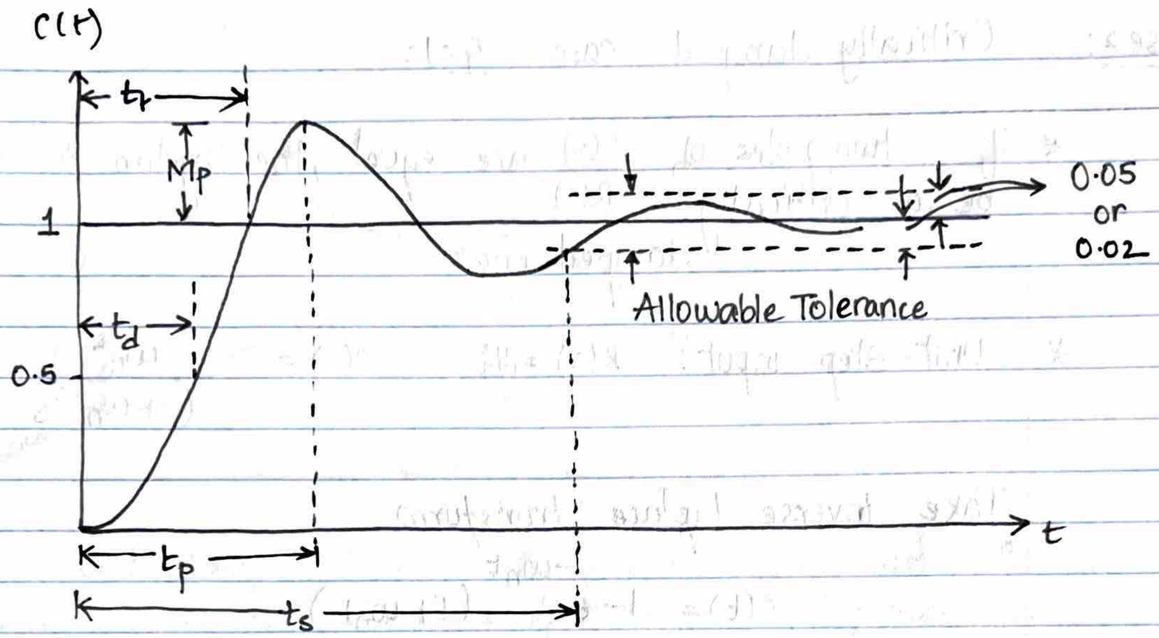
$$\text{Maximum \% overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum % overshoot directly indicates the relative stability of the system.

5. Settling time: (t_s)

The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute % of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the ~~constat~~ control system.

- Time-domain specifications are quite important, since most control systems are time-domain systems; they must exhibit acceptable time responses.
- Control system must be modified until the transient response is satisfactory.



Continuation of 2^{nd} order system:

$c(t)$ = damped sine function oscillating at the damped natural frequency ω_d with a phase shift $\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$

Similarily
$e(t)$ exhibits damped sinusoidal oscillations.

As $t \to \infty$ $e(t) \to 0$

What happens when $\xi = 0$?

- Undamped
- Oscillations continue indefinitely.

$c(t) = 1 - \cos\omega_n t$

ω_n ← undamped natural frequency i.e. frequency at which the system would oscillate if damping were 0.

Case 2: Critically damped case $\zeta = 1$:

* if two poles of $\frac{C(s)}{R(s)}$ are equal, the system is said to be a critically damped one.

* Unit-step input: $R(s) = 1/s$ $\quad C(s) = \dfrac{\omega_n^2}{(s+\omega_n)^2 s}$

Take inverse laplace transform

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

Case 3: Overdamped case ($\zeta > 1$):

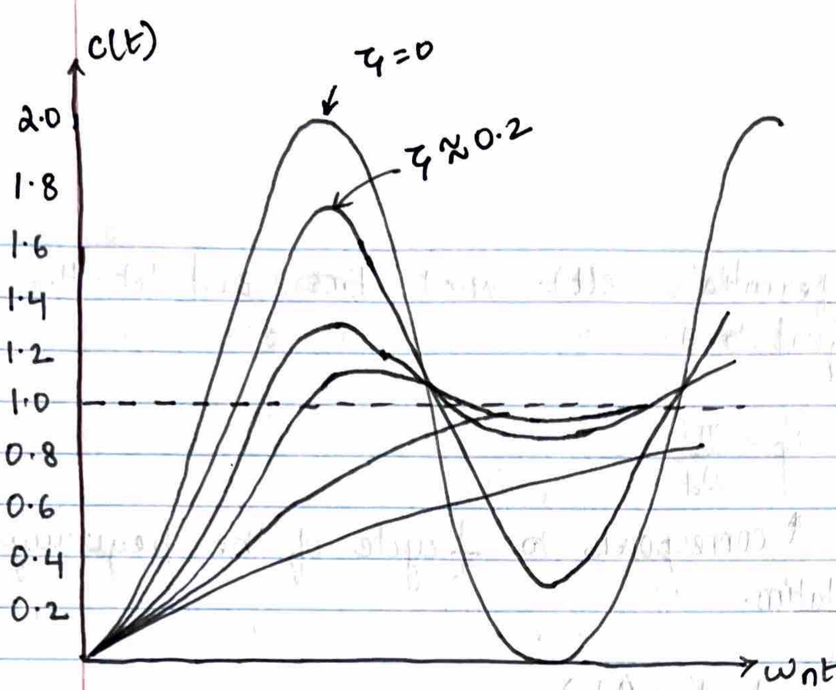
In this case, the two poles of $\frac{C(s)}{R(s)}$ are negative real and unequal.

Let $R(s) = 1/s$, then $C(s) = \dfrac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1})s}$

Taking Inverse laplace transform

$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)$$

where $s_1 = \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n$ $\quad s_2 = \left(\zeta - \sqrt{\zeta^2-1}\right)\omega_n$



* An underdamped system with ζ between 0.5 and 0.8 gets close to the final value more rapidly than a critically damped or overdamped system.

* Among the systems responding without oscillations, a critically damped system exhibits fastest responce.

* An overdamped system is always sluggish in responding to any inputs.

2^{nd} ORDER Systems and Transient Response Specifications:

Rise time

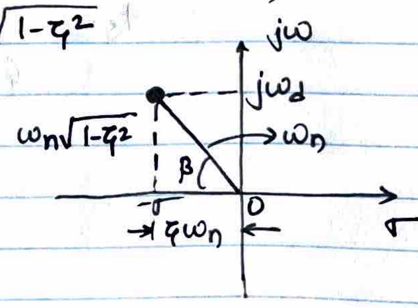
* t_r:

$$c(t_r) = 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right) = 1$$

Cannot be 0. Then

$$\cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r) = 0$$

$\zeta\omega_n = \sigma$



$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi-\beta}{\omega_d}$$

* Peak time: (t_p)

differentiate $c(t)$ w.r.t time and let the derivative be equal to 0.

$$t_p = \frac{\pi}{\omega_d}$$

↑ corresponds to $\frac{1}{2}$ cycle of the frequency of damped oscillation.

* Maximum (%) overshoot : (M_p)

- occurs at peak time (t_p)

$$M_p = c(t_p) - 1$$ final value of o/p

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi} \times 100\%$$

* Settling time (t_s):

- underdamped second-order system

$$t_s = \frac{4}{\zeta\omega_n} \quad (2\% \text{ criterion})$$

$$t_s = \frac{3}{\zeta\omega_n} \quad (5\% \text{ criterion})$$

Example:

Let a second-order system have $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Find transient-responce specifications.

Sol:

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{\omega_n\sqrt{1-\zeta^2}} = \frac{3.14 - \beta}{4} = 0.55\text{se}$$

$$\beta = \tan^{-1}\frac{\omega_d}{\zeta\omega_n} = \tan^{-1}\frac{4}{3} = 0.93$$

$$t_p : \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

$$M_p = e^{-(\zeta\omega_n/\omega_d)\pi} = 9.5\%$$

$$t_s \text{ for } 2\% \text{ criterion} = \frac{4}{3} = 1.33 \text{ sec}$$

$$t_s \text{ for } 5\% \text{ criterion} = \frac{3}{3} = 1 \text{ sec}$$

STABILITY:

- most important problem in linear control systems.

Answer the question
- → under what conditions will a system become unstable?
- → if it is unstable, how should we stabilize the system?

Terminology:

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

- → a's, b's are constants
- → $m \leq n$
- → zeros of $G(s)$ = roots of $C(s)$
- → poles of $G(s)$ = roots of $R(s)$
- → characteristic polynomial = $R(s)$
- → characteristic equation = $R(s) = 0$

Example:

$$G(s) = \frac{(s-1)(s+1)}{(s+2)(s^2+1)}$$

zeros: $+1, -1$

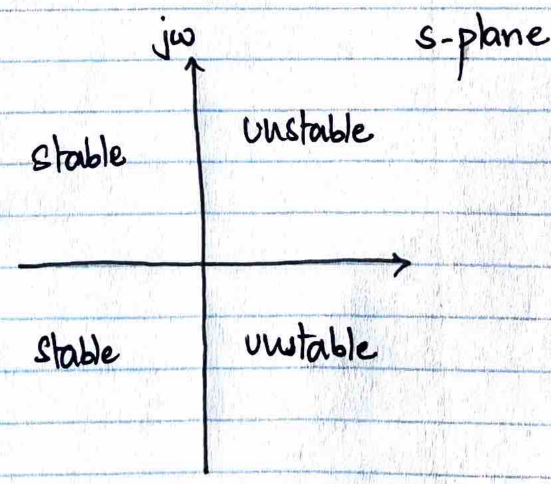
poles: $-2, \pm j$

* Let s_i be the poles of $G(s)$

Then G is

(1) (BIBO, asymptotically stable)
- if $Re(s_i) < 0 \ \forall i$

(2) (marginally stable)
- if $Re(s_i) \leq 0 \ \forall i$
- simple root for $Re(s_i) = 0$



(3) (Unstable)
- it is neither stable nor marginally stable.

– enables us to determine the number of closed loop poles that lie in the right half s-plane.

ROUTH'S STABILITY Criterion:

– for LTI systems.

Procedure:

① Write the polynomial denominator

$R(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0$

where the coefficients are real quantities.

Assume $a_n \neq 0$:

– if this assumption does not hold. $R(s)$ can be factored as

$$R(s) = s^m \underbrace{(\hat{a}_0 s^{n-m} + \dots + \hat{a}_{n-1} s + \hat{a}_n)}_{\hat{R}(s) \text{ where } \hat{a}_n \neq 0}$$

② If all coefficients are positive, arrange the coefficients of the polynomial in rows & columns according to the following pattern:

$$\begin{array}{llll} s^n & a_0 & a_2 & a_4 \; \dots \\ s^{n-1} & a_1 & a_3 & a_5 \; \dots \\ s^{n-2} & b_1 & b_2 & b_3 \\ \vdots & c_1 & c_2 & c_3 \\ \vdots & \vdots & \vdots & \vdots \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array}$$

Total number of rows: $n+1$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

Continue untill all rows have been completed

* Routh's stability criterion states that the number of roots with +ve real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

Necessary and sufficient condition that all roots lie in the left-half s plane is that all the coefficients be positive and all terms in the 1^{st} column of the array have positive signs.

Example:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6	0	
s^0	5		

$\left.\begin{array}{c}\text{Number of changes in}\\ \text{sign of the coefficients}\\ \text{in } 1^{st} \text{ column}\end{array}\right\} = 2$

Special case:

If a 1^{st} column term in any row is zero, but the remaining terms are not zero, then the zero term is replaced by a very small +ve number ϵ and the rest of the array is evaluated