Solving ODE (ordinary differential equation) using Laplace Transform:

Theory:

Let O·D·E be of the form

$$y''(t) + a_1 y'(t) + a_2 y(t) = u(t)$$

Subject to following initial conditions:

$$y(0) = y_0$$
$$y'(0) = v_0$$

Taking Laplace Transform of the ODE

$$L\{y''(t)\} + a_1 L\{y'(t)\} + a_2 L\{y(t)\} = L\{u(t)\}$$

Expanding ↑ & ↑

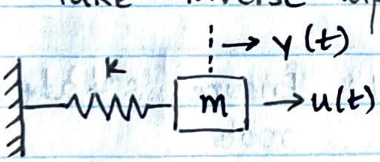
$$s^2Y(s) - sy(0) - y'(0) + a_1\big(sY(s) - y(0)\big) + a_2 Y(s) = U(s)$$

$$Y(s)\left[s^2 + a_1 s + a_2\right] = U(s) + sy(0) + v_0 + a_1 y(0)$$

$$Y(s) = \frac{U(s)}{s^2+a_1s+a_2} + \frac{(s+a_1)\,y_0}{s^2+a_1s+a_2} + \frac{v_0}{s^2+a_1s+a_2}$$

↑ Take inverse laplace transform to obtain $y(t)$

Example:



Initial conditions:

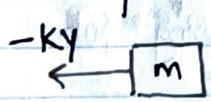
$y(0) = 0 \qquad y'(0) = 0$

$m = 1$

$k = 4$

$u(t) = 1$

Draw free-body diagram



$$m\ddot{y} = -ky + u(t)$$
$$m\ddot{y} + ky = u(t)$$

$$\ddot{y} + 4y = 1$$

Taking Laplace Transform

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s^2+4)}$$

To find $y(t)$: Apply Inverse Laplace Transform

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$$

Apply partial fraction : $\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$A = +\frac{1}{4}$ $\quad B = -\frac{1}{4}$

$$\therefore \frac{1}{s(s^2+4)} = \frac{1}{4s} - \frac{1(s)}{4(s^2+4)}$$

$$y(t) = \frac{1}{4}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}L^{-1}\left\{\frac{s}{s^2+2^2}\right\}$$

$$y(t) = \frac{1}{4} - \frac{1}{4}\cos 2t = \frac{1}{4}\{1-\cos 2t\}$$

Solution

Example:

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 5\,u_s(t)$$

Initial conditions:

$y(0) = -1$

$y'(0) = 2$

Apply Laplace Transform

$$s^2Y(s) - sy(0) - y'(0) + 3\{sY(s) - y(0)\} + 2Y(s) = \frac{5}{s}$$

$$(s^2+3s+2)Y(s) + s - 2 + 3 = \frac{5}{s}$$

$$Y(s)[s^2+3s+2] = \frac{5}{s} - s - 1 = \frac{5 - s^2 - s}{s}$$

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)}$$

- Apply partial fraction expansion

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$A = 5/2 \qquad B = -5 \qquad C = 3/2$

Therefore $Y(s) = \frac{5}{2s} - \frac{5}{s+1} + \frac{3}{2(s+2)}$

- Apply inverse Laplace transform

$$y(t) = L^{-1}\{Y(s)\} = \frac{5}{2} L^{-1}\left\{\frac{1}{s}\right\} - 5 L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{3}{2} L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$y(t) = \frac{5}{2} u_s(t) - 5e^{-t} + \frac{3}{2} e^{-2t}$$

Example: $\ddot{y}(t) - y(t) = t \qquad$ I.C's: $y(0)=1 \quad \dot{y}(0)=1$

Apply Laplace transform

$$s^2 Y(s) - sy(0) - \dot{y}(0) - Y(s) = \frac{1}{s^2}$$

$$(s^2 - 1) Y(s) - s - 1 = \frac{1}{s^2}$$

$$Y(s)(s^2 - 1) = \frac{1}{s^2} + s + 1 = \frac{s^3 + s^2 + 1}{s^2}$$

$$\cancel{Y(s) = \frac{s^3+s^2+1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}}$$

~~Comparing coefficients~~

$$\cancel{s^3 + s^2 + 1 = A(s^3 - s) + B(s^2 - 1) + (Cs+D)s^2}$$

$\cancel{A+C=1} \qquad \cancel{B+D=1} \qquad \cancel{-A=0}$

$\cancel{\boxed{A=0}}$

$$\cancel{s^3 + s^2 + 1 = As^3 - As + Bs^2 - B + Cs^3 + Ds^2}$$

$\cancel{A+C=1} \qquad \cancel{B+D=1} \qquad \cancel{-B=1} \qquad \cancel{-A=0}$

$\cancel{\boxed{B=-1}} \qquad \cancel{\boxed{A=0}}$

$$Y(s) = \frac{s^3+s^2+1}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-1}$$

$$s^3+s^2+1 = As(s^2-1) + B(s^2-1) + C(s-1)s^2 + Ds^2(s+1)$$

$$s^3+s^2+1 = A(s^3-s) + B(s^2-1) + C(s^3-s^2) + D(s^3+s^2)$$

$$s^3+s^2+1 = s^3(A+C) + s^2(B-C+D) + s(-A) + (-B)$$

comparing coefficients

$\boxed{B=-1}$ $\boxed{A=0}$

$A+C=1$

$\boxed{C=1}$

$B-C+D=1$

$-1-1+D=1$

$\boxed{D=3}$

$$Y(s) = \frac{-1}{s^2} + \frac{1}{s+1} + \frac{3}{s-1}$$

$$y(t) = -L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} + 3L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\boxed{y(t) = -t + e^{-t} + 3e^{t}}$$

* In this way, we can find a solution to ODE easily by using Laplace Transform table.

* Linear Systems: A system is called linear if the principle of superposition applies. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses.

- A control system may consist of a number of components. To show the functions performed by each component, we use block diagrams.

BLOCK DIAGRAMS

- A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.
- In a block diagram all system variables are linked to each other through functional blocks
- Functional block: a symbol for the mathematical operation on the input signal to the block that produces the o/p

Example: Input signal $\rightarrow$ $\boxed{\text{Transfer function } G(s)}$ $\rightarrow$ Output signal

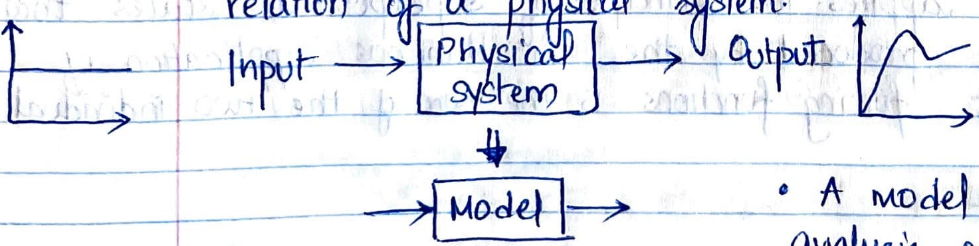
- Advantage: easy to form the overall block diagram for the entire system

Linear Time-Invariant Systems: A differential equation is linear if the coefficients are constants or functions only of the independent variable.

* If the coefficients are constants $\rightarrow$ Linear time-invariant systems.

* If the coefficients are functions of time $\rightarrow$ Linear time-varying systems.

Mathematical Model: representation of the input-output (signal) relation of a physical system.



- A model is used for the analysis and design of control systems.

- possible to represent system dynamics by algebraic equations in s.

TRANSFER Functions

*commonly used to characterize the input-output relationships of components or systems that can be described by linear, time invariant differential equations.

- The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Theory: Consider the linear time-invariant system defined by the following differential equation:

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \dots + a_{n-1}\dot{y} + a_n y \qquad \text{where } n \geq m$$

$$= b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \dots + b_{m-1}\dot{x} + b_m x$$

where y = output of the system
x = input

The transfer function of this system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero

$$\text{Transfer function} = G(s) = \left. \frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]} \right|_{\text{zero initial conditions}}$$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n}$$

- If the highest power of s in the denominator of the transfer function is equal to n, the system is called an n^{th} order system.

$$X(s) \longrightarrow \boxed{G(s)} \xrightarrow{Y(s)}$$

Properties of T.F:

① Limited to linear, time-invariant, differential equation systems.

② T.F of a system is a mathematical model (expresses the differential equation that relates output variable to the input variable)

③ is a property of the system itself

④ If the T.F of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system

⑤ If the T.F of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system.

Remarks on mathematical models:

- Modeling is the most important and difficult task in control system design.
- No mathematical model exactly represents a physical system.
  Math model $\neq$ Physical system
  Math model $\approx$ physical system
- System in the rest of the course means a mathematical model.

* For a linear, time-invariant system, the transfer function $G(s)$ is

$$G(s) = \frac{Y(s)}{X(s)} = \frac{\text{Laplace transform of the output of the system}}{\text{Laplace transform of the input to the system}}$$

where we assume that all initial conditions involved are 0.

$\therefore$ $Y(s) = G(s)\,X(s)$

from property ⑧ for Laplace Transform, we have

$$y(t) = \int_0^t x(\tau)\,g(t-\tau)\,d\tau = \int_0^t g(\tau)\,x(t-\tau)\,d\tau$$

where $g(t) = x(t) = 0$ for $t < 0$

IMPULSE-RESPONSE:

· what is the output (response) of a LTI system to a unit-impulse input when I.C's are 0?

Solution:

Laplace Transform of unit-impulse function is unity, then

$$Y(s) = G(s)$$

Taking inverse Laplace transform of $G(s)$

$$y(t) = L^{-1}\{G(s)\} = g(t)$$

a.k.a Impulse-response function (or) impulse response of the system.

* The impulse-response function $g(t)$ is thus the response of a linear time-invariant system to a unit-impulse input when the initial condition are zero.

$$L\{g(t)\} = G(s) \Rightarrow \text{T.F of the system}$$

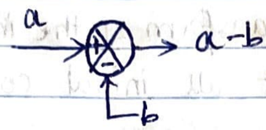
$\therefore$ The T.F and impulse response function of LTI system contain the same information about system dynamics.

· It is possible to obtain complete information about the dynamic characteristics of the system by exciting it with an impulse input and measuring the response.

* Any linear control system may be represented by a block diagram consisting of blocks, summing points and branching points.

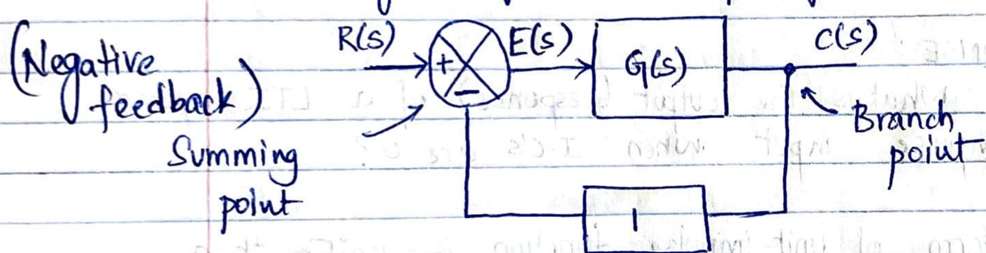
BLOCK DIAGRAM (continued):

Summing point:



Branching point: A branch point is a point from which the signal from a block goes concurrently to other blocks / summing points

Block diagram of a closed-loop system:



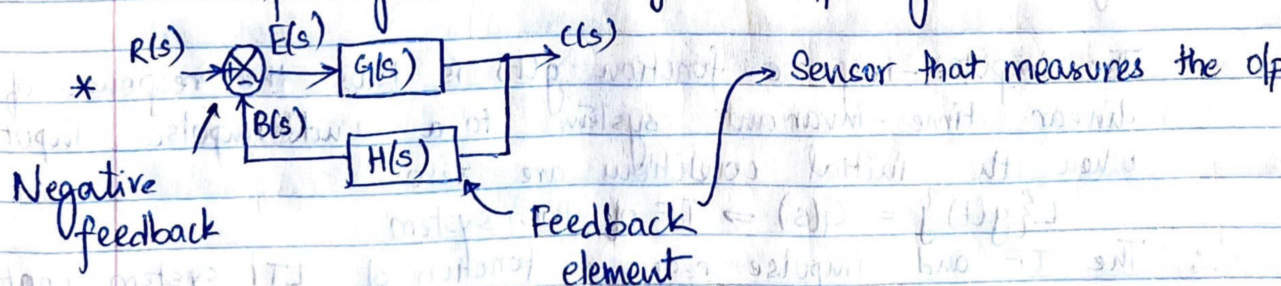
$$C(s) = E(s)\,G(s)$$

$$E(s) = R(s) - C(s)$$

$$E(s) = R(s) - E(s)\,G(s)$$

* When the o/p is fed back to the summing point for comparison with the input, it is necessary to convert the form of the o/p signal to that of the i/p signal.

*



$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$E(s) = R(s) - B(s)$$
$$E(s) = R(s) - C(s)H(s)$$
$$E(s) = R(s) - E(s)G(s)H(s)$$
$$E(s)[1+G(s)H(s)] = R(s)$$

* Open-Loop Transfer Function:

$$\text{Open-loop T.F} = \frac{B(s)}{E(s)} = \cancel{\frac{C(s)H(s)(1+G(s)H(s))}{R(s)}}$$

$$= \frac{C(s)H(s)}{E(s)} = \frac{E(s)G(s)H(s)}{E(s)}$$

$$\text{Open-loop Transfer Function} = \frac{B(s)}{E(s)} = G(s)H(s)$$

Ratio of feedback signal B(s) to the actuating error signal E(s)

* Feedforward Transfer function: Ratio of output C(s) to the actuating error signal E(s)

$$\text{Feed-forward Transfer function} = \frac{C(s)}{E(s)} = G(s)$$

Remark:

If the feedback transfer function H(s) is unity, then open-loop T.F and feedforward T.F are the same.

* Closed loop Transferfunction: $\frac{C(s)}{R(s)}$: Ratio of output c(s) to reference R(s)

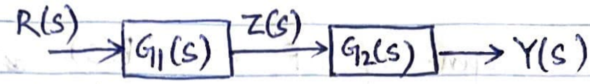
$$C(s) = G(s)E(s)$$
$$E(s) = R(s) - B(s)$$
$$E(s) = R(s) - H(s)C(s)$$
$$E(s) = R(s) - H(s)G(s)E(s)$$
$$E(s) = \frac{R(s)}{1+H(s)G(s)} \longrightarrow \text{substitute}$$

$$C(s) = \frac{G(s) \cdot R(s)}{1+H(s)G(s)}$$

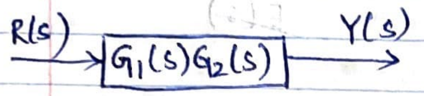
$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}}$$

Block Diagram Reduction

- Series connection

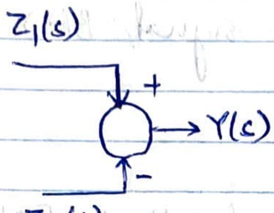


$$G_1(s) = \frac{Z(s)}{R(s)} \qquad G_2(s) = \frac{Y(s)}{Z(s)}$$



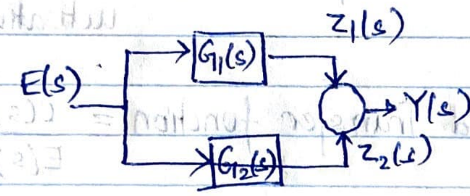
$$\frac{Y(s)}{R(s)} = G_1(s)\,G_2(s)$$

- Summing junction



$$Y(s) = Z_1(s) - Z_2(s)$$

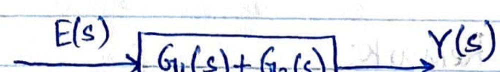
- Parallel connection:



$$Z_1(s) = G_1(s)\,E(s)$$

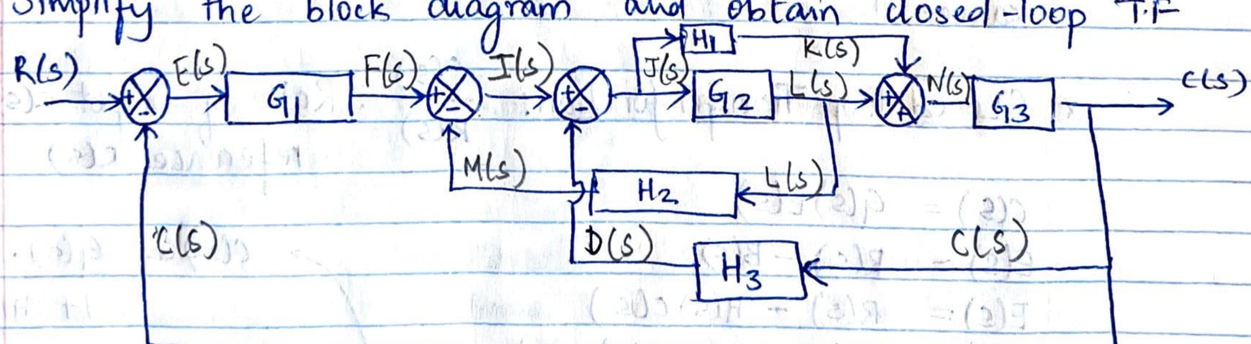
$$Z_2(s) = G_2(s)\,E(s)$$

$$Y(s) = Z_1(s) + Z_2(s)$$
$$= \big(G_1(s) + G_2(s)\big)E(s)$$



Example

Simplify the block diagram and obtain closed-loop T.F



$* \; C(s) = N(s)\,G_3(s)$

$* \; E(s) = R(s) - C(s)$

$E(s) = R(s) - N(s)\,G_3(s)$

$C(s) = \big(K(s) + L(s)\big)\,G_3(s)$

$C(s) = \big(J(s)H_1(s) + J(s)\,G_2(s)\big)\,G_3(s)$

$* \; C(s) = \big(J(s)H_1(s) + J(s)G_2(s)\big)\,G_3(s)$

$C(s) = \big(H_1(s) + G_2(s)\big)\,J(s)\,G_3(s)$

$* \; J(s) = I(s) - D(s)$

$J(s) = I(s) - C(s)H_3(s)$

$* \; I(s) = F(s) - M(s)$

$I(s) = F(s) - L(s)\,H_2(s)$

$I(s) = F(s) - J\,G_2H_2$

$J(s) = F(s) - J(s)G_2(s)H_2(s) - C(s)H_3(s)$

$J(s)\big(1 + G_2(s)H_2(s)\big) = F(s) - C(s)H_3(s)$

$$J(s) = \frac{F(s) - C(s)H_3(s)}{1 + G_2(s)H_2(s)} = \frac{E(s)G_1(s) - C(s)H_3(s)}{1 + G_2(s)H_2(s)}$$

$$J(s) = \frac{\big[R(s) - C(s)\big]G_1(s) - C(s)H_3(s)}{1 + G_2(s)H_2(s)}$$

$$J(s) = \frac{RG_1 - C(s)\big(H_3(s) + G_1(s)\big)}{1 + G_2H_2}$$

$$C(s) = \big(H_1(s) + G_2(s)\big)\left[\frac{RG_1 - C(H_3 + G_1)}{1 + G_2H_2}\right]G_3$$

$$C = \frac{(H_1 + G_2)RG_1G_3}{1 + G_2H_2} - \frac{C(H_3 + G_1)(H_1 + G_2)G_3}{1 + G_2H_2}$$

$$C\left(1 + \frac{G_3(H_3 + G_1)(H_1 + G_2)}{1 + G_2H_2}\right) = R\left(\frac{(H_1 + G_2)\,G_1G_3}{1 + G_2H_2}\right)$$

$$\boxed{\frac{C}{R} = \frac{G_1G_2G_3 + G_1G_3H_1}{1 + G_2H_2 + G_2G_3H_3 + G_3H_1H_3 + G_1G_2G_3 + G_1G_3H_1}}$$