Midterm solutions

1.

(i) $$F(s) = \frac{5}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$5 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$$
$$5 = A(s^2+5s+6) + B(s^2+4s+3) + C(s^2+3s+2)$$
$$5 = (A+B+C)s^2 + (5A+4B+3C)s + (6A+3B+2C)$$

Comparing

$$A+B+C=0 \qquad 5A+4B+3C=0 \qquad 6A+3B+2C=5$$
$$A=-B-C \qquad 5(-B-C)+4B+3C=0 \qquad \cancel{6C}+3(\cancel{-2C})+2C=5$$
$$A=-(-2C)-C \qquad -5B-5C+4B+3C=0 \qquad 2C=5$$
$$A=2C-C \qquad -B-2C=0 \qquad \boxed{C=5/2}$$
$$\boxed{A=C} \qquad \boxed{B=-2C}$$

$$\boxed{A=5/2} \qquad \boxed{B=-5}$$

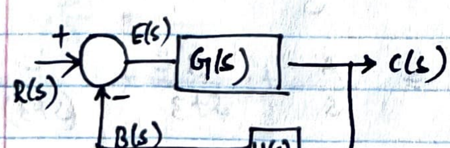
$$f(t) = L^{-1}\{F(s)\} = \frac{5}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - 5L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{5}{2}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$f(t) = \frac{5}{2}e^{-t} - 5e^{-2t} + \frac{5}{2}e^{-3t}$$

(ii) $f(t) = t^2e^{-3t}$

$$F(s) = L\{f(t)\} = \frac{2!}{(s+3)^3}$$

(iii)



(a) $\frac{B(s)}{E(s)}$ = open-loop T.F

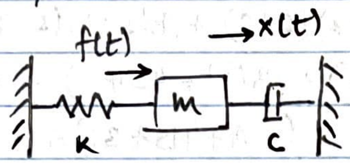
(b) $\frac{C(s)}{E(s)}$ = ~~closed~~ feedforward T.F

(c) $\frac{C(s)}{R(s)}$ = closed-loop T.F

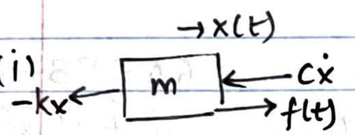
(iv) (a) Peak time
(b) Maximum % overshoot
(c) delay time
(d) Rise time.

(v) 63.2%

Problem 2:



(i)



$$m\ddot{x} = -c\dot{x} - kx + f(t)$$

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

(ii) Take Laplace transform (Assume zero initial conditions)

$$(ms^2 + cs + k)\, X(s) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

(iii) Order = 2

(iv) $C = 2$ $\quad m = \frac{1}{3}$ $\quad k = 3$ $\quad f(t) = 10$ $\quad F(s) = \frac{10}{s}$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{\frac{1}{3}s^2 + 2s + 3} = \frac{1}{\frac{s^2 + 6s + 9}{3}} = \frac{3}{s^2 + 6s + 9}$$

$$X(s) = \frac{10(3)}{s(s+3)^2} = \frac{30}{s(s+3)^2} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{(s+3)^2}$$

$$30 = A(s+3)^2 + Bs(s+3) + Cs$$

$$30 = A(s^2 + 6s + 9) + Bs^2 + 3Bs + Cs$$

$A + B = 0$ $\quad$ $6A + 3B + C = 0$ $\quad$ $9A = 30$

$$A = \frac{30}{9} = \frac{10}{3}$$

$B = -A$

$$\overset{2}{\cancel{6}}\left(\frac{10}{\cancel{3}}\right) + \cancel{3}\left(\frac{-10}{\cancel{3}}\right) + C = 0$$

$B = \frac{-10}{3}$

$$20 - 10 + C = 0$$

$\boxed{A = 10/3}$ $\quad$ $\boxed{C = -10}$ $\quad$ $\boxed{B = -10/3}$

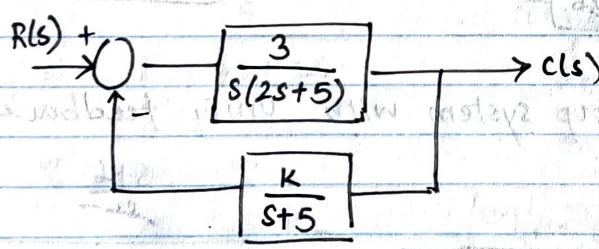
$$X(s) = \frac{10}{3}\left\{\frac{1}{s}\right\} + \left(\frac{-10}{3}\right)\left\{\frac{1}{s+3}\right\} - 10\left\{\frac{1}{(s+3)^2}\right\}$$

Applying inverse laplace transform

$$x(t) = \frac{10}{3} - \frac{10}{3}e^{-3t} - 10te^{-3t}$$

solution.

Problem 3:



Closed-loop T.F :
$$\frac{C(s)}{R(s)} = \frac{\frac{3}{s(2s+5)}}{1 + \frac{3}{s(2s+5)}\frac{K}{(s+5)}} = \frac{\frac{3}{s(2s+5)}}{\frac{s(2s+5)(s+5) + 3K}{s(2s+5)(s+5)}}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{3}{\cancel{s(2s+5)}}}{\frac{s(2s^2+10s+5s+25)+3K}{\cancel{s(2s+5)}(s+5)}} = \frac{3(s+5)}{2s^3+15s^2+25s+3K} = \frac{3/2\,(s+5)}{s^3+\frac{15}{2}s^2+\frac{25}{2}s+\frac{3}{2}K}$$

Routh Array

s^3	1	$25/2$
s^2	$15/2$	$3/2K$
s^1	b_1	
s^0	c_1	

$$b_1 = \frac{\left(\frac{15}{2}\right)\left(\frac{25}{2}\right) - \frac{3}{2}K}{15/2} = \frac{\cancel{186}}{\frac{15}{2}}$$

$$b_1 = \frac{375 - 6K}{15}$$

$$c_1 = \frac{3}{2}K$$

To maintain stability $\quad b_1 > 0 \quad\quad c_1 > 0$

$$\frac{375-6K}{15} > 0 \qquad\qquad \frac{3}{2}K > 0$$

$$\boxed{K > 0}$$

$$375 - 6K > 0$$

$$375 > 6K$$

$$\frac{375}{6} > K$$

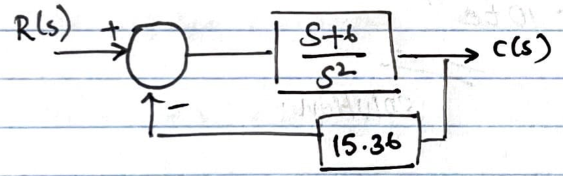
$$\boxed{62.5 > K}$$

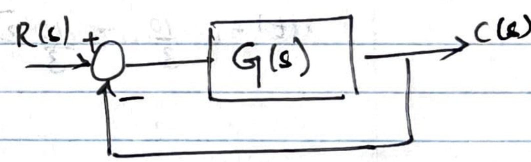
$$\boxed{62.5 > K > 0}$$

↑ condition for K.

Problem 4

(i)





(a) convert to closed-loop system with unity feedback

$$G(s) = \frac{\frac{s+6}{s^2}}{1 + \left(\frac{s+6}{s^2}\right)(15.36) - \left(\frac{s+6}{s^2}\right)} = \frac{\frac{s+6}{\cancel{s^2}}}{\frac{s^2+15.36s+92.16-s-6}{\cancel{s^2}}}$$

$$G(s) = \frac{s+6}{s^2+14.36s+86.16} = \frac{C(s)}{R(s)}$$

$$E(s) = R(s) - C(s) \Rightarrow \frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)}$$

$$\frac{E(s)}{R(s)} = \frac{s^2+14.36s+86.16-s-6}{s^2+14.36s+86.16} = \frac{s^2+13.36s+80.16}{s^2+14.36s+86.16}$$

(b) Steady state error due to unit-step input

$R(s) = 1/s$

$$e_{ss}(\text{step}) = \lim_{s\to 0} s \cdot E(s) = \lim_{s\to 0} \cancel{s} \cdot \frac{1}{\cancel{s}} \left[\frac{s^2 + 13.36s + 80.16}{s^2 + 14.36s + 86.16} \right]$$

$$e_{ss}(\text{step}) = \frac{80.16}{86.16} = 0.93$$

(c) Steady state error due to unit-ramp input $\quad R(s) = 1/s^2$

$$e_{ss}(\text{ramp}) = \lim_{s\to 0} s \cdot E(s) = \lim_{s\to 0} \cancel{s} \cdot \frac{1}{s^{\cancel{2}}} \left[\frac{s^2 + 13.36s + 80.16}{s^2 + 14.36s + 86.16} \right] = \infty$$

(ii) $$\frac{C(s)}{R(s)} = \frac{25}{s^2 + 4s + 25}$$

compare to standard form:

$\omega_n^2 = 25 \qquad 2\zeta\omega_n = 4$

$\omega_n = 5 \qquad \zeta\omega_n = 2$

$\zeta = 2/5 = 0.4$

underdamped.

(a) Max % overshoot:

$$= e^{-(\zeta/\sqrt{1-\zeta^2})\pi} \times 100 = e^{-(0.4/\sqrt{1-0.4^2})3.14} \times 100$$

$$= e^{-(0.4/0.91)\,3.14} \times 100$$

$$= \cancel{\ldots} \; 25.38\%$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

$= 5(0.91) = 4.55$

(b) Rise Time: $$\frac{\pi - \beta}{\omega_d} = \frac{3.14 - 1.15}{4.55} = 0.43$$

$$\beta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{0.91}{0.4}\right)$$

$= 1.15$

(c) ~~2%~~ Peak Time: $$\frac{\pi}{\omega_d} = \frac{3.14}{4.55} = 0.69\,s$$

(d) 2% Settling Time: $$\frac{4}{\zeta\omega_n} = \frac{4}{2} = 2\,s$$