Modeling in state-space

Modern control theory:

- applicable to MIMO systems
- Linear or non-linear time invariant or time varying

Classical control theory:

- applicable to LTI
- SISO systems

Definitions

* **State:** (of a dynamic system)
  smallest set of variables such that knowledge of these variables at $t=t_0$ together with knowledge of the input for $t \geq t_0$ completely determines the behavior of the system for any time $t \geq t_0$.

* **State variables:**
  variables making up the smallest set of variables that determine the state of the dynamic system.

Remarks:

- state variables need not be physically measurable or observable quantities
- variables that donot represent physical quantities and those that are neither meanurable nor observable can be chosen as state variables.

* **State vector:** if n state variables are needed to completely describe the behavior of a given system, then these n state variables can be considered the n components of state vector

A state vector is thus a vector that determines uniquely the system state $x(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the i/p $u(t)$ for $t \geq t_0$ is specified.

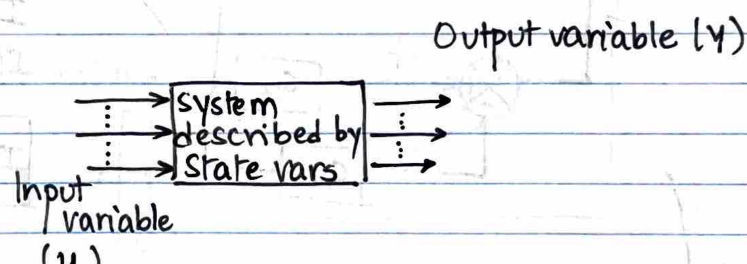
STATE Space:

The n-dimensional space whose coordinates axes consist of the x_1 axis, x_2 axis, . . . x_n axis where $x_1, x_2, \ldots x_n$ are state variables is called state space.

STATE-Space EQUATIONS (MIMO)

3 types of variables involved in modeling dynamic systems
- input variables
- output variables
- state variables

~~How~~



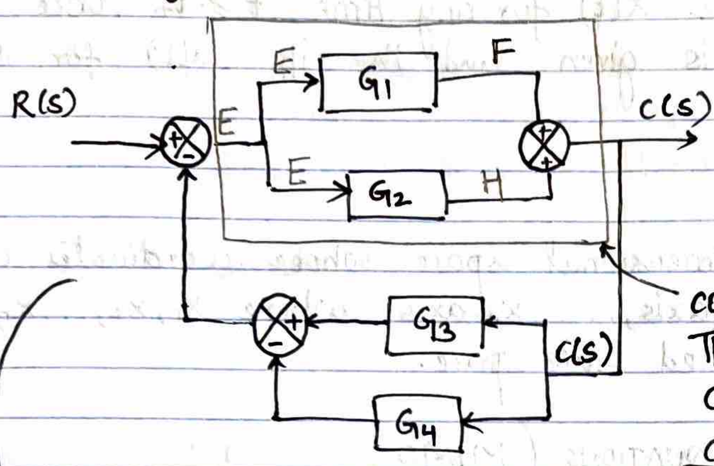
State Equations : Standard form

* The mathematical description of the system is expressed as a set of n coupled 1st order ODE. known as state equations.

* The time derivative of each state variable is expressed in terms of the state variables $x_1(t) \ldots x_n(t)$ and the system inputs $u_1(t), \ldots, u_r(t)$.

Block diagram examples

1.



Reduces to

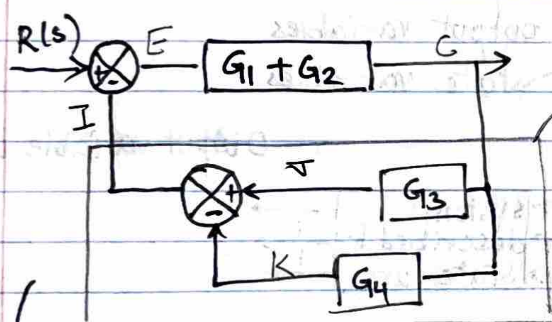
consider this

Then

$$C(s) = F(s) + H(s)$$

$$C(s) = E(s)G_1(s) + E(s)G_2(s)$$

$$(1) \; -- \; \boxed{C(s) = E(s)\left[G_1(s) + G_2(s)\right]}$$



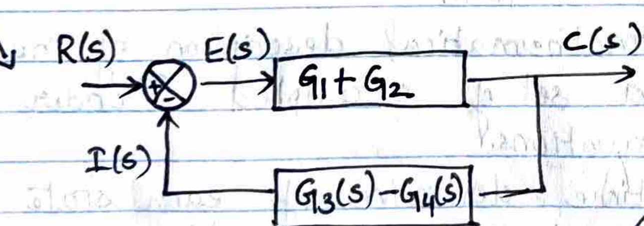
Consider this block

$$I(s) = J(s) - K(s)$$

$$I(s) = C(s)G_3(s) - C(s)G_4(s)$$

$$\boxed{I(s) = C(s)\left[G_3(s) - G_4(s)\right]}$$

Reduces to



Solve this complete block

$$E(s) = R(s) - I(s)$$

$$E(s) = R(s) - C(s)\left(G_3(s) - G_4(s)\right)$$

Substitute E(s) from (1) in

$$\frac{C(s)}{G_1(s) + G_2(s)} + C(s)\left(G_3(s) - G_4(s)\right) = R(s)$$

$$C(s)\left(1 + \left(G_3(s) - G_4(s)\right)\left(G_1(s) + G_2(s)\right)\right) = R(s)\left(G_1(s) + G_2(s)\right)$$

$$\frac{C(s)}{R(s)} = \frac{G_1(s) + G_2(s)}{1 + (G_1(s) + G_2(s))(G_3(s) - G_4(s))}$$

↑ Final solution

———— x ————

Modeling in state-space (continued):

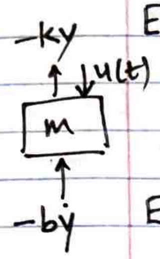
General case:

$$\dot{x}_1 = \boxed{f_1(x,u,t)}$$
$$\dot{x}_2 = f_2(x,u,t)$$
$$\vdots$$
$$\dot{x}_n = f_n(x,u,t)$$

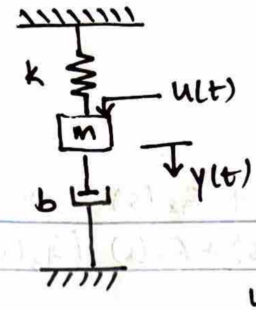
General, non-linear time vary func. of state variables, system inputs & time

Writing the state equations in a vector form

State vector $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$



Example:



* Assume system is linear

u(t) : external force / Input
y(t) : displacement / output

Equation of motion

$$m\ddot{y} = -ky - b\dot{y} + u(t)$$

$$m\ddot{y} + ky + b\dot{y} = u(t)$$

$$\boxed{m\ddot{y} + b\dot{y} + ky = u(t)} \quad — ①$$

2^{nd} order system:

Output equation :

Let $x_1(t) = y(t)$

$y = x_1$

$x_2(t) = \dot{x}_1(t) = \dot{y}(t)$

Then, equation ① becomes

$$m\dot{x}_2 + bx_2 + kx_1 = u(t)$$

$$\dot{x}_2(t) = \frac{-k}{m}x_1 \cancel{+} - \frac{b}{m}x_2 + \frac{1}{m}u$$

$$\dot{x}_1(t) = x_2$$

In vector-matrix form

STATE EQUATION
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

OUTPUT EQUATION.
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A : state matrix

B : input matrix

C : o/p matrix

D: direct transmission matrix

Standard form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Correlation b/w T.F and state-space Equations:

Transfer func: $G(s) = \frac{Y(s)}{U(s)}$

In state-space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Then, $\boxed{G(s) = C(sI-A)^{-1}B + D}$

Continuing the previous example:

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-b}{m} \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s+b/m \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$\boxed{G(s) = \frac{1}{ms^2+bs+k}}$$

Inverse of 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad ad-bc \neq 0$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

MATLAB:

- quite useful to transform from T.F to S.S & vice-versa

① T.F to SS

$$\frac{Y(s)}{U(s)} = \frac{num}{dem}$$

[A,B,C,D] = tf2ss[num,den]

$$A = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$D = 0$$

Example:

$$\frac{Y(s)}{U(s)} = \frac{5}{(s+10)(s^2+4s+16)}$$

expand the denominator

$$\frac{Y(s)}{U(s)} = \frac{5}{s^3+14s^2+56s+160}$$ ← Use Matlab

② SS to T.F:

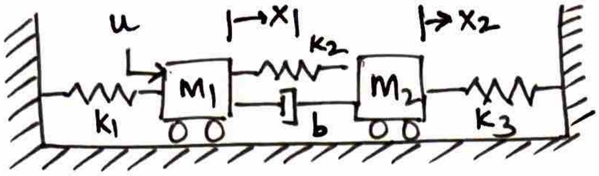
[num,den] = ss2tf(A,B,C,D)

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{25s+5}{s^3+5s^2+25s+5}$$



Example: Mass-spring damper

① Equations of motion for the system:

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1-x_2) - b(\dot{x}_1-\dot{x}_2) + u$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2-x_1) - b(\dot{x}_2-\dot{x}_1)$$

Simplifying:

$$m_1\ddot{x}_1 + b(\dot{x}_1-\dot{x}_2) + k_1x_1 + k_2(x_1-x_2) = u$$

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1+k_2)x_1 = b\dot{x}_2 + k_2x_2 + u$$

Take Laplace transform

$$\left[m_1s^2 + bs + (k_1+k_2)\right]x_1(s) = (bs+k_2)x_2(s) + U(s) \quad — ①$$

Simplifying:

$$m_2\ddot{x}_2 + k_3x_2 + k_2(x_2-x_1) + b(\dot{x}_2-\dot{x}_1) = 0$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2+k_3)x_2 = b\dot{x}_1 + k_2x_1$$

Applying L.T

$$(m_2s^2 + bs + (k_2+k_3))x_2(s) = (bs+k_2)x_1(s) \quad — ②$$

Solve ① & ②

$$x_2(s) = \frac{(bs+k_2)x_1(s)}{m_2s^2 + bs + (k_2+k_3)}$$

$$(m_1s^2 + bs + (k_1+k_2))x_1(s) = \frac{(bs+k_2)^2 x_1(s)}{m_2s^2+bs+(k_2+k_3)} + U(s)$$

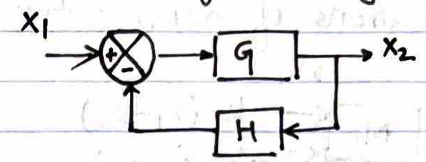
$$\frac{x_1(s)}{U(s)} = \frac{m_2s^2 + bs + (k_2+k_3)}{(m_1s^2+bs+(k_1+k_2))(m_2s^2+bs+(k_2+k_3)) - (bs+k_2)^2}$$

$$\frac{x_2(s)}{U(s)} = \frac{bs+k_2}{}$$

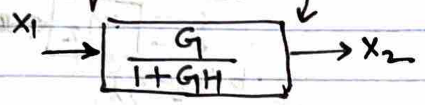
Block diagram Transformations

① Eliminating a feedback loop:

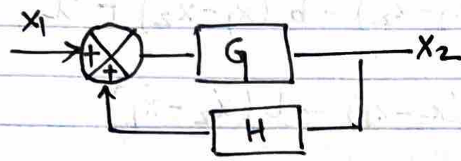
Original diagram



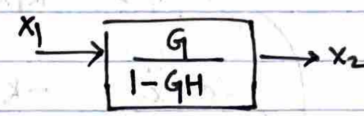
Equivalent



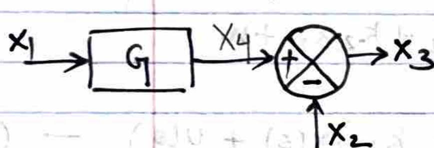
Similarly:



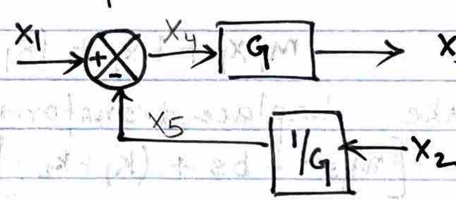
$\Rightarrow$



② moving a summing point ahead of a block:



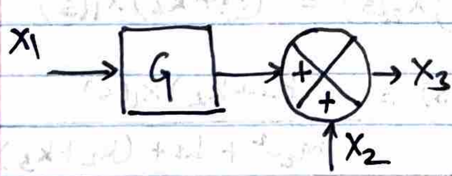
$\Rightarrow$



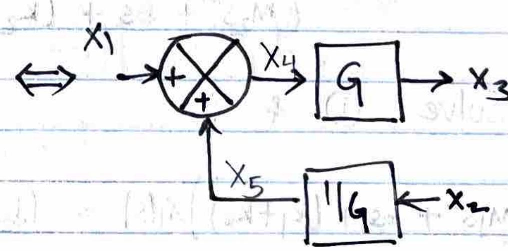
$$x_3 = x_4 - x_2$$
$$x_3 = x_1 G - x_2$$

$$x_4 = x_1 - x_5 = x_1 - \frac{x_2}{G}$$
$$x_3 = x_4 G = \left(x_1 - \frac{x_2}{G}\right) G = x_1 G - x_2$$

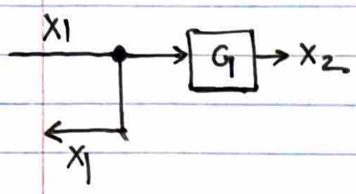
Similarly:



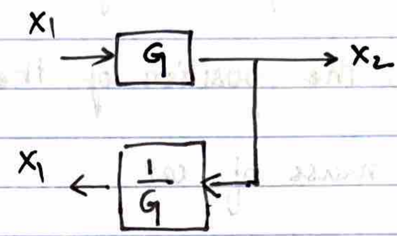
$\Leftrightarrow$



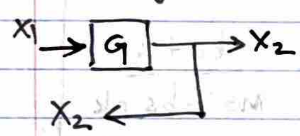
3. moving a branching point behind the block:



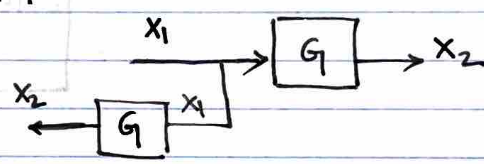
$X_2 = X_1 G$



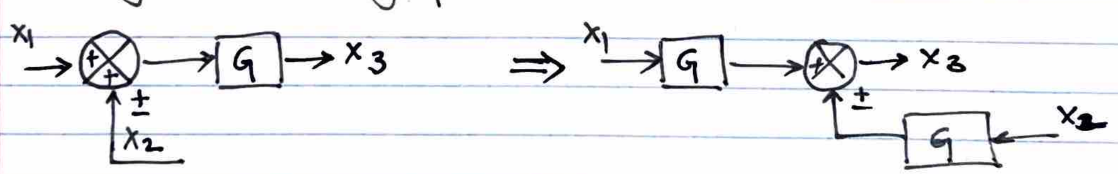
4. moving a branching point ahead of a block:

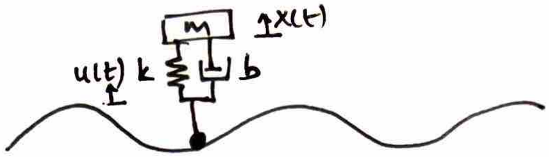


$X_2 = X_1 G$



5. moving a summing point behind a block:





Example: Simplified Quarter car model

u(t): end position of the system

x(t): the position of the mass

m: mass of car

b: damping coefficient

k: spring constant

Find $\frac{X(s)}{U(s)}$

Sol:

$$G(s) = \frac{X(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$$