STABILITY:

- most important problem in linear control systems.

Answer the question
→ under what conditions will a system become unstable?
→ if it is unstable, how should we stabilize the system?

Terminology:

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

→ a's, b's are constants
→ $m \leq n$
→ zeros of $G(s)$ = roots of $C(s)$
→ poles of $G(s)$ = roots of $R(s)$
→ characteristic polynomial = $R(s)$
→ characteristic equation = $R(s) = 0$

Example:

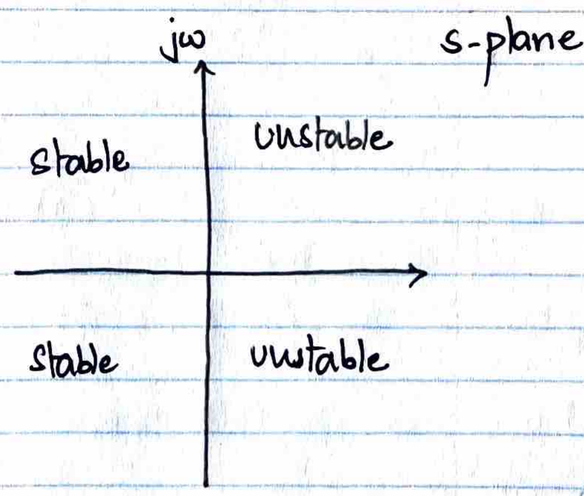
$$G(s) = \frac{(s-1)(s+1)}{(s+2)(s^2+1)}$$

zeros: $+1, -1$
poles: $-2, \pm j$

* Let s_i be the poles of $G(s)$
Then G is

(1) (BIBO, asymptotically stable)
- if $Re(s_i) < 0 \quad \forall i$

(2) (marginally stable)
- if $Re(s_i) \leq 0 \quad \forall i$
- simple root for $Re(s_i) = 0$



(3) (Unstable)
- it is neither stable nor marginally stable.

— enables us to determine the number of closed loop poles that lie in the right half s-plane.

ROUTH'S STABILITY Criterion:

— for LTI systems.

Procedure:

① Write the polynomial denominator

$$R(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0$$

where the coefficients are real quantities.

Assume $a_n \neq 0$:

— if this assumption does not hold. R(s) can be factored as

$$R(s) = s^m \underbrace{\left(\hat{a}_0 s^{n-m} + \dots + \hat{a}_{n-1} s + \hat{a}_n\right)}_{\hat{R}(s) \text{ where } \hat{a}_n \neq 0}$$

② If all coefficients are positive, arrange the coefficients of the polynomial in rows & columns according to the following pattern:

$$\begin{array}{llll}
s^n & a_0 & a_2 & a_4 \; \dots \\
s^{n-1} & a_1 & a_3 & a_5 \; \dots \\
s^{n-2} & b_1 & b_2 & b_3 \\
\vdots & c_1 & c_2 & c_3 \\
\vdots & \vdots & \vdots & \vdots \\
s^2 & & & \\
s^1 & & & \\
s^0 & & &
\end{array}$$

Total number of rows: $n+1$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

Continue untill all rows have been completed

* Routh's stability criterion states that the number of roots with +ve real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

Necessary and sufficient condition that all roots lie in the left-half s plane is that all the coefficient be positive and all terms in the 1st column of the array have positive signs.

Example:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6	0	
s^0	5		

$\left.\begin{array}{l}\text{Number of changes in} \\ \text{sign of the coefficients} \\ \text{in 1st column}\end{array}\right\} = 2$

Special case:

If a 1st column term in any row is zero, but the remaining terms are not zero, then the zero term is replaced by a very small +ve number ϵ and the rest of the array is evaluated.

Example:

$$s^3 + 2s^2 + s + 2 = 0$$

s^3	1	1
s^2	2	2
s^1	$0 \approx \epsilon$	
s^0	2	

* If the sign of the coefficient above the zero (ϵ) is the same as that below it, it indicates that there are are a pair of imaginary roots.

* If however, the sign of the coefficient above the zero (ϵ) is opposite that below it, it indicates there is 1st sign change

* The number of roots in the open right half-plane is equal to the number of sign changes in the first column of routh array.

Example 1:

$Q(s) = s^3 + s^2 + 2s + 8$

s^3	1	2
s^2	1	8
s^1	-6	
s^0	8	

* 2 sign changes
  $\Rightarrow$ 2 roots in RHP

Example 2:

$Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

s^5	1	2	11
s^4	2	4	10
s^3	$0 \approx \epsilon$	6	
s^2	$\frac{4\epsilon - 12}{\epsilon}$	10	
s^1	6		
s^0	10		

* If 0 appears in the first column of a non-zero row in routh array replace it with a small +ve number.

* 2 sign changes $\Rightarrow$ 2 roots in RHP

Example 3: $Q(s) = s^4 + s^3 + 3s^2 + 2s + 2$

s^4	1	3	2
s^3	1	2	
s^2	1	2	
s^1	$\cancel{0}$ 2		
s^0	2		

Take derivative of an auxiliary polynomial (which is a factor of $Q(s)$)

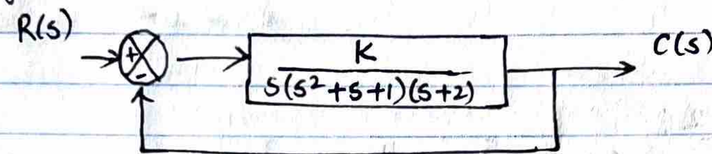
* If zero row appears in Routh array, Q has roots either on the imaginary axis or in RHP

No sign changes in 1st col
$\Downarrow$
No roots in RHP
(but some roots are on imaginary axis)

Application of Routh's stability criterion to control-system analysis.

* determine the effects of changing one or two parameters of a system by examining the values that cause instability

Example:



$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

Characteristic Equation:

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

s^4	1	3	K
s^3	3	2	0
s^2	$7/3$	K	
s^1	$2-9/7K$		
s^0	K		

* For stability, K must be +ve
* $2 - \frac{9}{7}K > 0$

$$K < \frac{14}{9}$$

$$0 < K < 14/9$$

K – design parameter

Example: $Q(s) = s^3 + 3Ks^2 + (K+2)s + 4$.
Find the range of K such that Q(s) has all roots in L.H.P

s^3	1	$K+2$
s^2	$3K$	4
s^1	$\frac{(K+2)3K-4}{3K}$	
s^0	4	

$$\begin{cases} 3K > 0 \\ (K+2)3K - 4 > 0 \end{cases} \qquad \therefore K > -1 + \sqrt{\frac{21}{3}}$$

can be attributed to many factors

STEADY-STATE errors in unity feedback control systems

* Any physical control system inherently suffers steady-state error in response to certain types of inputs.
* A system may have no steady-state error to a step input, but the same system may exhibit non-zero steady-state error to a ramp input.

Classification of control systems:

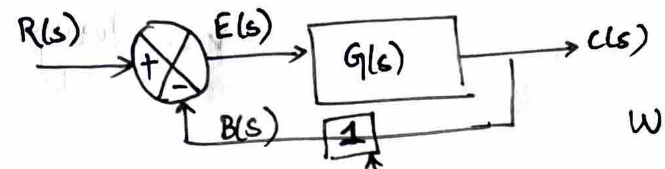
Consider unity feedback control system with the following open-loop T.F

$$G(s) = \frac{K(T_a s+1)(T_b s+1)\ldots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\ldots(T_p s+1)}$$

s^N represents a pole of multiplicity N at origin

$N=0 \Rightarrow$ Type 0 system
$N=1 \Rightarrow$ type 1 system
$N=2 \Rightarrow$ type 2 system

As Type ↑ increases,
→ accuracy is improved.
→ stability ↓



we want c(t) to track r(t)

unity feedback

Steady-state errors:

Closed loop T.F $\quad \dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

* Assuming that the closed-loop system is stable!

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1+G(s)}$$

Using final value theorem, steady state error $\quad e(t) = \cancel{E(s)}\ r(t) - c(t)$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\cdot E(s) = \lim_{s\to 0} \frac{s\cdot R(s)}{1+G(s)}$$

① **Static Position Error constant** (K_p): (STEP ERROR)

$$r(t) = 1 \Rightarrow R(s) = \frac{1}{s}$$

$$e_{ss} = \lim_{s\to 0} \frac{\cancel{s}}{1+G(s)}\cdot\frac{1}{\cancel{s}} = \frac{1}{1+G(0)} = \frac{1}{1+K_p}$$

* $K_p = \lim_{s\to 0} G(s) = G(0)$

For type 0 system; $\quad K_p = \lim_{s\to 0} \dfrac{K(T_a s+1)(T_b s+1)\dots}{(T_1 s+1)(T_2 s+1)\dots} = K$ (finite)

For type 1 or higher system

$$K_p = \lim_{s\to 0} \frac{K(T_a s+1)\dots}{s^N (T_1 s+1)\dots} = \infty \quad \forall\ N \geq 1$$

For a unit-step input, $e_{ss} = \frac{1}{1+K}$ for type 0

$e_{ss} = 0$ for type 1/higher

* If zero steady state error is desired for step input

② Static velocity error constant (K_v): (RAMP ERROR)

The steady state error of the system with unit-ramp input

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \lim_{s\to 0} \frac{1}{sG(s)}$$

Static velocity error constant $K_v = \lim_{s\to 0} s\,G(s)$

$$e_{ss} = \frac{1}{K_v}$$

For Type 0 system,

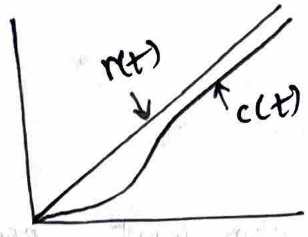
$$K_v = \lim_{s\to 0} \frac{sK(T_as+1)\ldots}{(T_1s+1)\ldots} = 0$$

For Type 1 system

$$K_v = \lim_{s\to 0} \frac{\cancel{s}K(T_as+1)\ldots}{\cancel{s}(T_1s+1)\ldots} = K$$

For type 2 or higher system

$$K_v = \lim_{s\to 0} \frac{s}{s^N} \frac{K(T_as+1)\ldots}{(T_1s+1)\ldots} = \infty$$



For unit ramp-input:

$e_{ss} = \frac{1}{K_v} = \infty$ for type 0

$e_{ss} = \frac{1}{K_v} = \frac{1}{K}$ for type 1

$e_{ss} = \frac{1}{K_v} = 0$ for type 2 or higher

Analysis:

* type 0 system is incapable of following a ramp input in the steady-state
* Type 1 system can follow ramp input with a finite error
* type 2 or higher system can follow a ramp input with zero error @ steady state.

③ Static Acceleration error constant (K_a): (PARABOLIC ERROR)

$r(t) = \frac{t^2}{2}$ (unit parabolic input) $R(s) =$

$$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s^3} = \lim_{s \to 0} \frac{1}{s^2 G(s)} = \frac{1}{K_a}$$

Let $K_a = \lim_{s \to 0} s^2 G(s)$

For type 0 - system

$$K_a = \lim_{s \to 0} \frac{s^2 K (T_a s + 1) \cdots}{(T_1 s + 1) \cdots} = 0$$

For type-1 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)\dots}{s(T_1 s+1)\dots} = 0$$

For type-2 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)\dots}{s^2 (T_1 s+1)\dots} = K$$

For type 3 or higher system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)\dots}{s^N (T_1 s+1)\dots} = \infty \quad \forall N \geq 3$$

Therefore steady-state error:

$e_{ss} = \infty$ for Type 0, type 1 systems

$e_{ss} = 1/K$ for type 2 systems (finite error)

$e_{ss} = 0$ for Type 3 systems

* Type 0, type 1 systems are incapable of following a parabolic input in steady state.

* K_p, K_v, K_a: ability to reduce steady-state error constants

Zero steady-state error:

* if error constant is infinite, we can achieve zero steady state error. (ACCURATE Tracking)

* For step $r(t)$

$$K_p = \lim_{s \to 0} G(s) = \infty \iff G(s) \text{ is of atleast type } 1$$

* For ramp $r(t)$

$$K_v = \lim_{s \to 0} sG(s) = \infty \iff G(s) \text{ is of atleast type } 2$$

* For parabolic $r(t)$

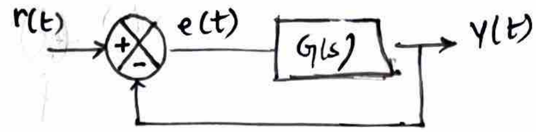
$$k_a = \lim_{s \to 0} s^2 G(s) = \infty \iff G(s) \text{ is of atleast type } 3$$

Example:

(1) $G(s) = \dfrac{K}{s^2(s+12)}$ $\qquad G(s) = \text{type } 2$

Characteristic eq: $1 + G(s) = s^3 + 12s^2 + K = 0$

s^3	1	0
s^2	12	K
s^1	$-K/12$	
s^0	0	



Example problem:

Consider the closed loop system where $G(s) = \dfrac{K(s+4)}{s(s+1)(s+2)}$

(a) Using Routh-Hurwitz criterion, range of K so that system is stable?

Sol: characteristic equation

$1 + G(s) = 0$

$s(s+1)(s+2) + K(s+4) = 0$

$s(s^2+3s+2) + K(s+4) = 0$

$s^3 + 3s^2 + 2s + Ks + 4K = 0$

$s^3 + 3s^2 + (2+K)s + 4K = 0$

s^3	1	$K+2$
s^2	3	$4K$
s	$\square$	
s^0	$4K$	

$\dfrac{3K+6-4K}{3} = \dfrac{-K+6}{3}$

$4K > 0$

$\boxed{K > 0}$

$\dfrac{-K+6}{3} > 0$

$-K+6 > 0$

$\boxed{6 > K}$

$\boxed{0 < K < 6}$

(b) Compute error signal $e(t) = r(t) - y(t)$. Compute the steady state error for

Sol: (i) $R(s) = 1/s$

$K_p = \lim_{s \to 0} G(s) = \infty$

$e_{ss} = \dfrac{1}{1+K_p} = 0$

(ii) $R(s) = 1/s^2$

$K_v = \lim_{s \to 0} s \cdot G(s) = 2K$

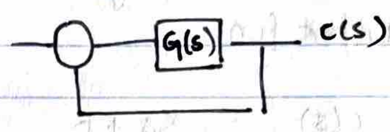
$e_{ss} = \dfrac{1}{2K}$

Openloop transfer function

B-5-2: $G(s) = \dfrac{1}{s(s+1)}$

* unit-step response
* unity feedback

$G(s) = \dfrac{1}{s^2+s}$



Closed loop T.F:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{1}{s^2+s+1}$$

compare to standard form

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

We get

$$\omega_n^2 = 1$$

$$\Rightarrow \boxed{\omega_n = 1}$$

$$2\zeta\omega_n = 1$$

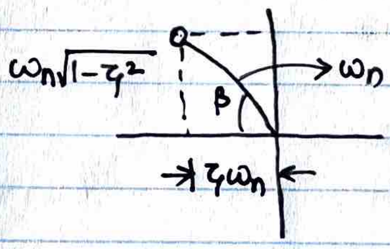
$$\zeta\omega_n = 1/2 \qquad \boxed{\zeta = 0.5}$$

underdamped system

Rise Time:

$$t_r = \frac{\pi-\beta}{\omega_d} = \frac{3.14-1.0472}{0.8660}$$

$$\boxed{t_r = 2.4166\,s}$$



$$\beta = \tan^{-1}\frac{\omega_d}{\zeta\omega_n}$$

$$\beta = 1.0472$$

$$\zeta\omega_n = 0.5$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{1-0.5^2} = \sqrt{0.75}$$

$$\omega_d = 0.8660$$

Peak Time: $t_p = \dfrac{\pi}{\omega_d} = \dfrac{3.14}{0.8660}$

$$\boxed{t_p = 3.6277\,s}$$

Maximum overshoot:

$$M_p = e^{-(\zeta\omega_n/\omega_d)\pi} = e^{-(0.5/0.8660)\pi}$$

$$M_p = 0.163 = 16.3\%$$

Settling time:

2% criterion $t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.5} = 8\,sec$

5% criterion $t_s = \frac{3}{\zeta\omega_n} = \frac{3}{0.5} = 6sec$

B-5-5: $G(s) = \frac{2s+1}{s^2}$ Closed-loop T.F

open-loop T.F

$$\frac{C(s)}{R(s)} = \frac{2s+1}{s^2+2s+1}$$

(unity-feedback)

(i) $R(s) = \frac{1}{s}$ (unitstep)

$$C(s) = \frac{(2s+1)}{(s^2+2s+1)s} = \frac{(2s+1)}{(s+1)^2 s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Apply P.F

$A(s^2+2s+1) + B(s)(s+1) + Cs$

$A(s^2+2s+1) + B(s^2+s) + Cs$

$C(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2}$

$2s+1 = (A+B)s^2 + s(2A+B+C) + A$

$A+B=0$ $\boxed{A=1}$

$\boxed{B=-1}$ $2A+B+C=2$

$\boxed{C=1}$

Apply I.L.T

~~C(s)=~~ $c(t) = 1 - e^{-t} + te^{-t}$

$A(s+1) + B = 2s+1$

$As + (A+B) = 2s+1$

$A = 2 \quad A+B = 1$

$B = -1$

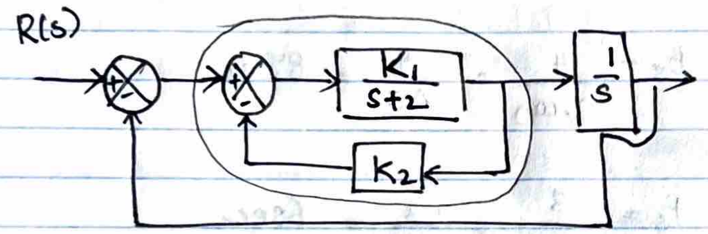
(ii) $R(s) = 1$

$$C(s) = \frac{2s+1}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2} = \frac{2}{s+1} - \frac{1}{(s+1)^2}$$

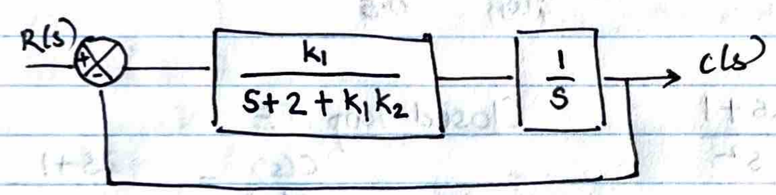
Apply I.L.T

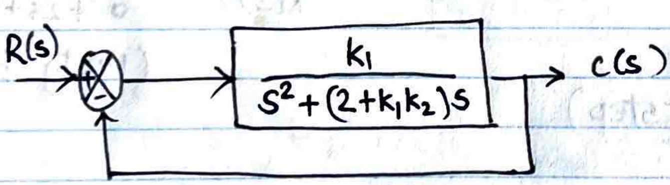
$$c(t) = 2e^{-t} - te^{-t}$$

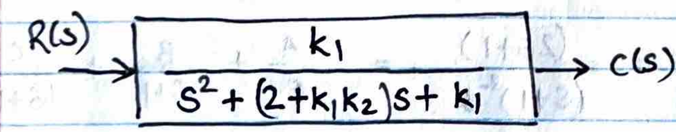
B-5-8:



$$\frac{K}{s+2+K+K}$$







Given:

$\zeta = 0.7$

$\omega_n = 4$

Standard form

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{16}{s^2 + 5.6s + 16}$$

Compare:

$k_1 = 16$

$2 + k_1 k_2 = 5.6$

$k_1 k_2 = 3.6$

$k_2 = \frac{3.6}{16} = 0.2250$

B-5-12: $\frac{C(s)}{R(s)} = \frac{36}{s^2+2s+36}$

compare it to standard form

$\omega_n^2 = 36 \Rightarrow \omega_n = 6$

$2\zeta\omega_n = 2 \Rightarrow \zeta = 1/6 = 0.1667$

Rise Time: $(t_r) = \frac{\pi - \beta}{\omega_d}$

$\beta = \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n}\right)$

$t_r = \frac{3.14 - 1.4033}{5.9161}$

$\beta = 1.4033$

$\boxed{t_r = 0.2936\ s}$

Peak Time: $(t_p) = \frac{\pi}{\omega_d} = \frac{3.14}{5.9161} = 0.5308\ s$

Maximum overshoot: (M_p)

$M_p = e^{-(\zeta\omega_n/\omega_d)\pi} = e^{-[1/5.9161]\pi} = e^{-0.5308} = 0.5881$

Settling time:

$t_s = \frac{4}{\zeta\omega_n} = 4\ sec$ (2% criterion)

$t_s = \frac{3}{\zeta\omega_n} = 3\ sec$ (5% criterion)

B-5-20: Determine the range of K for stability of a unity feedback control system

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

open-loop T.F

Closed-loop T.F : $\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+2)+K} = \frac{K}{s(s^2+3s+2)+K}$

$$\frac{C(s)}{R(s)} = \frac{K}{s^3+3s^2+2s+K}$$

Routh Array

s^3	1	2	0
s^2	3	K	0
s^1	$\frac{6-K}{3}$	0	
s^0	K		

$K > 0$

$\frac{6-K}{3} > 0$

$6-K > 0$

$6 > K$

$$\boxed{0 < K < 6}$$

B-5-21: $s^4 + 2s^3 + (4+K)s^2 + 9s + 25 = 0$

Routh Array:

s^4	1	4+K	25
s^3	2	9	0
s^2	b_1	b_2	0
s^1	c_1	0	
s^0	b_2		

$b_1 = \frac{2(4+K)-9}{2}$

$b_2 = 25$

$c_1 = \frac{b_1 9 - 2b_2}{b_1}$

$b_1 > 0$

$2(4+K)-9 > 0$

$(4+K) > 4.5$

$\boxed{K > 0.5}$

$c_1 > 0$

$9b_1 - 2b_2 > 0$

$9b_1 > 50$

$9\left(2(4+K)-9\right) > 100$

$2(4+K) > 20.1111$

$4+K > 10.0556$

$\boxed{K > 6.0556}$

$\therefore K > 6.0556$