Recitation Notes : GAIN MARGIN (GM)

*important when designing controllers using bode plots

* The gain margin is the reciprocal of the magnitude $|G(j\omega)|$ at the frequency at which the phase angle is $-180°$.

ω_{pc} : phase cross-over frequency (at which phase angle of the open loop transfer function $= -180°$)

K_g : Gain margin

$$K_g = \frac{1}{|G(j\omega_{pc})|}$$

In terms of decibels;

$$K_g \text{ (dB)} = 20 \log K_g = -20 \log |G(j\omega_{pc})|$$

* Positive Gain margin (dB) ⇒ system is stable
Negative Gain margin (dB) ⇒ system is unstable

* For a stable minimum phase, the ~~gin~~ gain margin indicates how much the gain can be increased before the system becomes unstable

* For an unstable system, the gain margin is indicative of how much gain must be decreased to make the system stable.

Example:

Calculate phase cross-over frequency (ω_{pc}) and then calculate Gain margin in dB.

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}$$

Solution:

Step 1: Substitute $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(j\omega+1)(j\omega+2)}$$

Step 2: Obtain phase cross-over frequency

$$\angle G(j\omega)H(j\omega) = -180^\circ$$

$$-180^\circ = -\left[\tan^{-1}\left(\frac{\omega}{0}\right) + \tan^{-1}\left(\frac{\omega}{1}\right) + \tan^{-1}\left(\frac{\omega}{2}\right)\right]$$

$$-180^\circ = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right)$$

$$+90^\circ = +\left[\tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{2}\right)\right]$$

* $\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

* $\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$

$$90^\circ = \tan^{-1}\left(\frac{\omega + \frac{\omega}{2}}{1 - \frac{\omega^2}{2}}\right)$$

$$\frac{1}{0} = \frac{\frac{3}{2}\omega}{\frac{2-\omega^2}{2}} \Rightarrow 2-\omega^2 = 0$$

$$\omega^2 = 2$$

$$\boxed{\omega_{pc} = \sqrt{2}}$$

Step 3: Compute Magnitude @ $\omega = \omega_{pc}$

$$|G(j\omega)H(j\omega)| = \frac{1}{\sqrt{\omega^2}\sqrt{1+\omega^2}\sqrt{\omega^2+4}}$$

$$|G(j\omega_{pc})H(j\omega_{pc})|_{\omega=\omega_{pc}} = \frac{1}{\sqrt{2}\sqrt{1+2}\sqrt{2+4}} = \frac{1}{\sqrt{2}\sqrt{3}\sqrt{6}} = \frac{1}{\sqrt{36}} = \frac{1}{6}$$

Step 4: Compute Gain margin

$$K_g = \frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|} = \frac{1}{1/6} = 6$$

Step 5: Compute Gain margin in dB

$$K_g \text{ (in dB)} = -20\log|G(j\omega_{pc})H(j\omega_{pc})| = -20\log(1/6)$$

$$\boxed{K_g \text{ (in dB)} = 15.56 \text{ dB}}$$

PHASE MARGIN

* The phase margin is that amount of additional phase lag at the gain cross-over frequency required to bring the system to the verge of instability. The gain crossover frequency is the frequency at which $|G(j\omega)| = 1$

$|G(j\omega)|$: magnitude of open-loop transfer function.

$$\boxed{\text{P.M : phase margin} = 180^\circ + \phi}$$

ϕ = phase angle of the open-loop T.F @ gain cross-over frequency.

ω_{gc} : Gain cros-over frequency.

Example:

Calculate phase-margin for the following T.F

$$G(s) = \frac{1}{s(1+2s)(1+s)}$$

Solution:

Step 1: Substitute $s = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)(1+j2\omega)(1+j\omega)}$$

Step 2: Obtain ω when $|G(j\omega)| = 1$

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2}\sqrt{1+4\omega^2}\sqrt{1+\omega^2}} = 1$$

$$\boxed{\frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}} = 1}$$

Solve this equation to get ω_{gc}

* Use trial & error
* Use calculator

$\omega_{gc} = 0.57160$

Step 3: Compute ϕ_{gc} (phase @ ω_{gc})

$$\phi = \angle G(j\omega) = \cancel{\tan^{-1}(1)}^{\,0} - \left\{\tan^{-1}\left(\frac{\omega}{0}\right) + \tan^{-1}\left(\frac{2\omega}{1}\right) + \tan^{-1}\left(\frac{\omega}{1}\right)\right\}$$

$$\phi_{gc} = - \left\{90^\circ + \tan^{-1}(2\omega) + \tan^{-1}(\omega)\right\}$$

@ $\omega = \omega_{gc}$

$$\phi_{gc} = -\left\{90^\circ + 48.82 + 29.75\right\}$$

$$\boxed{\phi_{gc} = -168.57^\circ}$$

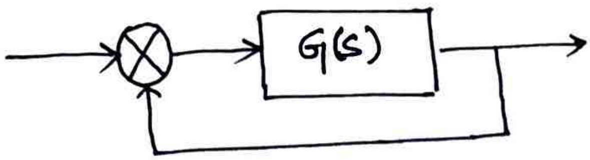
Step 4: Calculate phase margin

$$\text{Phase margin (PM)} = 180^\circ + \phi_{gc}$$

$$P.M = 180^\circ - 168.57^\circ$$

$$\boxed{PM = 11.43^\circ}$$

Example:



$$G(s) = \frac{K}{s(s^2+s+4)}$$

For the system shown in the figure, assign gain K such that the phase margin is 50°. For this particular gain (K) obtain gain margin.

Solution:

Step 1: substitute $P.M = 50°$ & find $\angle G(j\omega_{gc})$

$$P.M = 180° + \phi_{gc}$$

$$\phi_{gc} = \angle G(j\omega_{gc}) = -130°$$

Step 2: Obtain gain cross-over frequency:

$$\angle G(j\omega_{gc}) = -130°$$

$$G(j\omega) = \frac{K}{j\omega(j^2\omega^2 + j\omega + 4)}$$

$$G(j\omega) = \frac{K}{j\omega(-\omega^2 + j\omega + 4)}$$

$$G(j\omega) = \frac{K}{-\omega^3 j + j^2\omega^2 + j4\omega}$$

$$G(j\omega) = \frac{K}{-\omega^2 + j(4\omega - \omega^3)}$$

$$\cancel{\angle K}^{0} - \angle\left(-\omega^2 + j(4\omega - \omega^3)\right) = -130°$$

$$+\tan^{-1}\left(\frac{4\omega - \omega^3}{-\omega^2}\right) = +130°$$

$$\frac{4\cancel{\omega} - \omega^{\cancel{3}2}}{-\omega^{\cancel{2}}} = -1.19$$

$$4 - \omega^2 = 1.19\omega$$

$$\boxed{\omega^2 + 1.19\omega - 4 = 0} \qquad \omega = \omega_{gc} = 1.49$$

Step 3: $|G(j\omega_{gc})| = 1$

(ω_{gc} is the frequency @ which $|G(j\omega_{gc})| = 1$

$$G(j\omega_{gc}) = \frac{K}{-(1.49)^2 + j(4(1.49) - (1.49)^3)}$$

$$G(j\omega_{gc}) = \frac{K}{-2.22 + j(2.652)}$$

$$|G(j\omega_{gc})| = 1$$

$$\frac{K}{\sqrt{2.22^2 + 2.652^2}} = 1$$

$$\boxed{K = 3.46}$$

Step 4: Obtain phase cross-over frequency (ω_{pc})

$$\angle G(j\omega_{pc}) = -180^\circ$$

$$\cancel{\angle K}^{0} - \{\angle -\omega_{pc}^2 + j(4\omega_{pc} - \omega_{pc}^3)\} = -180^\circ$$

$$+\left\{ \tan^{-1}\left(\frac{4\omega_{pc} - \omega_{pc}^3}{-\omega_{pc}^2}\right)\right\} = +180^\circ$$

$$\frac{4\omega_{pc} - \omega_{pc}^3}{-\omega_{pc}^2} = 0$$

$$4\omega_{pc} - \omega_{pc}^3 = 0$$

$$\omega_{pc}(4 - \omega_{pc}^2) = 0$$

$\omega_{pc} = 0$

Ignore

Trivial solution

$\omega_{pc}^2 = 4$

$$\boxed{\omega_{pc} = 2}$$

Step 5: Compute magnitude @ $\omega = \omega_{pc}$

$$|G(j\omega_{pc})| = \left|\frac{3.46}{-2^2 + j(4(2) - 2^3)}\right| = +\frac{3.46}{4} = 0.865$$

Step 6: Compute gain margin in dB

$$G.M.\ (K_g) \text{ in dB} = -20 \log |G(j\omega_{pc})|$$

$$= -20 \log 0.865$$

$$\boxed{G.M = 1.26\,\text{dB}}$$