< Parameter Estimation >

Estimate $x \in \mathbb{R}^n$. (fixed).

① prior belief $x \sim N(\bar{x}, M)$

$M = M^T \in \mathbb{R}^{n\times n}$ cov.

② measurement

$$\underset{p\times 1}{z} = \underset{p\times n}{H}\,\underset{n\times 1}{x} + \underset{\substack{p\times 1\\ \text{noise}}}{v} \qquad v \sim N(0, R)$$

accurate → $R \downarrow$

noisy → $R \uparrow$

* x, v are indepen-dent ⇒ uncorrelated

Goal $x|z$

$$\begin{bmatrix} x \\ v \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} M & 0 \\ 0 & R \end{bmatrix}}_{\Sigma} \right)$$

$$\begin{bmatrix} x \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} I_n & 0 \\ H & I_p \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\sim N\left(A \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}, A\Sigma A^T \right) \qquad \begin{bmatrix} \Sigma_{xx} & \Sigma_{xz} \\ \Sigma_{zx} & \Sigma_{zz} \end{bmatrix}$$

$$\begin{bmatrix} x \\ z \end{bmatrix} = N\left(\begin{bmatrix} \bar{x} \\ H\bar{x} \end{bmatrix}, \begin{bmatrix} M & MH^T \\ HM^T & HMH^T + R \end{bmatrix} \right)$$

* cond. dist of jointly Gaussian r.v

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \right)$$

$$\Rightarrow x|y \sim N\left(a + CB^{-1}(y-b),\ A - CB^{-1}C^T\right)$$

$$x|z \sim N\left(\bar{x} + \boxed{MH^T(HMH^T+R)^{-1}}(z - H\bar{x}),\ M - \boxed{MH^T(HMH^T+R)^{-1}}HM \right)$$

Define $S = HMH^T + R$.

$K = MH^TS^{-1} = PHR^{-1} = I_{xz}I_{zz}^{-1}$

$P = M - KHM = (I - KH)M$

* matrix inversion lemma $= (M^{-1} + H^TR^{-1}H)^{-1}$

* joseph form $= (I-KH)M(I-KH)^T + KRK^T$

• estimation

$$\boxed{\hat{x} = E[x|z] = \bar{x} + K(z - H\bar{x})}$$

prior belief

actual z

guess of z

residual error

correction by z

very accurate sensor $\rightarrow R\downarrow \rightarrow K\uparrow \rightarrow$ rely more on z

confident about $\bar{x}$ $\rightarrow M\downarrow \rightarrow K\downarrow \rightarrow$ rely less on z

* properties.

Let $e = \hat{x} - x$. err in est.

$= (I - KH)(\bar{x} - x) + Kv$.

① $E[e] = (I - KH)\,E[\bar{x} - x] + K\,\cancel{E[v]}$

$= \text{"} \; (\bar{x} - \cancel{E[x]}) + \text{"} = 0$

$\Rightarrow$ est. is unbiased.

② $E[ee^T] = E[\{(I-KH)(\bar{x}-x) + Kv\}$

$\times \{(\bar{x}-x)^T(I-KH)^T + v^TK^T\}]$

$= E[\bigcirc\bigcirc^T + \bigcirc\triangle^T + \triangle\bigcirc^T + \triangle\triangle^T]$

$= (I-KH)M(I-KH)^T + KRK^T = P$

$= \mathrm{cov}[x|z]$.

accurate sensor / guess $\rightarrow$ $R\downarrow$ / $M\downarrow$ $\rightarrow P\downarrow \rightarrow$ more accurate $\hat{x}$.

③ $E[e\hat{x}^T] = E[\{(I-KH)(\bar{x}-x) + Kv\}$

$\times \{\bar{x} - KH(\bar{x}-x) + Kv\}^T]$

$= -(I-KH)MH^TK^T + KRK^T$

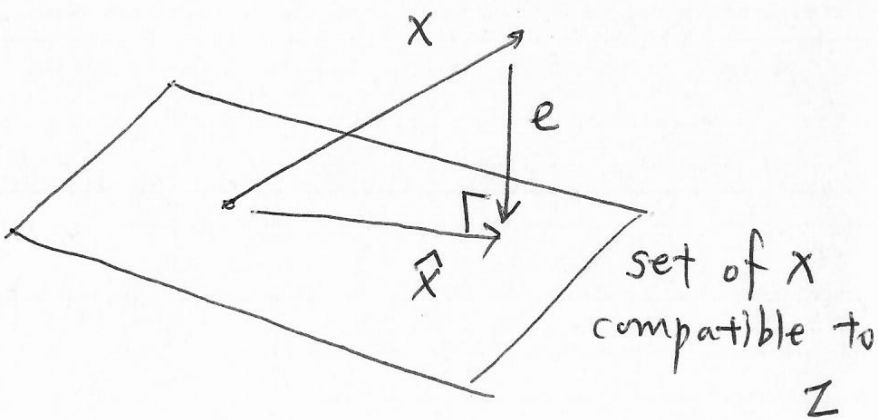
$= -PH^TK^T + PH^TK^T = 0.$

e and $\hat{x}$ are uncorrelated.

changing $\hat{x}$ does not change e.

* $\min J(x)$

necessary cond $\delta J = 0$ for any δx from x^*



* interpretation.

① Max. a posteriori est. (MAP

$$\hat{x}_{MAP} = \arg\max_{x} p(x|z)$$

$$= \quad '' \quad N(\hat{x}, P) = \hat{x}$$

② Min. mean squared err est (MMSE

$$\min J = E[\|x-\hat{x}\|^2 \mid z]$$

m m s. e

$$= \int \|x-\hat{x}\|^2 p(x|z)\, dx$$

$$\frac{\partial J}{\partial \hat{x}} = -2\int (x-\hat{x})\, p(x|z)\, d\cancel{x} = 0$$

$$= -2\left[E[x|z] - \hat{x}\cdot 1\right] = 0$$

$$\Rightarrow \hat{x} = E[x|z].$$

③ MAP

$$\max_x p(x|z) \Leftrightarrow \max_x \frac{p(x,z)}{p(z)} \Leftrightarrow \max_x p(x,z)$$

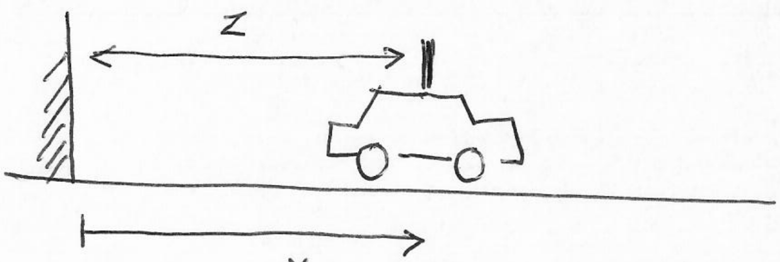
$$\Leftrightarrow \max_x p(z|x)\,p(x)$$

$$\Leftrightarrow \max_x \; -\boxed{\frac{1}{2}(z-Hx)^T R^{-1}(z-Hx)} - \frac{1}{2}(x-\bar{x})^T M^{-1}(x-\bar{x})$$

$$\Leftrightarrow \min_x \; + \boxed{\;''\;} \; + \; ''$$

dev. from mea.   dev. from $\bar{x}$

ex)



$x \in \mathbb{R}^1$ $x \sim N(\bar{x}, M)$

Lidar $z = x + v$ $v \sim N(0, R)$

$H = 1$.

$\Rightarrow \hat{x}, P$

* Nonlinear measurement.

$$z = h(x) + v.$$

$$x \sim N(\bar{x}, M), \qquad v \sim N(0, R)$$

x, v are independent

Let $\delta x = x - \bar{x}$

$$z = h(\bar{x}) + \left.\frac{\partial h}{\partial x}\right|_{x=\bar{x}} \delta x + H.O.T + v$$

$$z - h(\bar{x}) = \boxed{\delta z \cong \underbrace{\boxed{\left.\frac{\partial h}{\partial x}\right|_{\bar{x}}}}_{H.} \delta x + v} + \cancel{HOT}$$

assuming $H = \left.\frac{\partial h}{\partial x}\right|_{\bar{x}}$, apply lin. est.

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x̄, z
δx = 2*eps
while |δx| > eps
    δz = z − h(x̄)
    H = ∂h/∂x |x̄        ⇒ K, P
    δx = 0 + K δz
    x̄ = x̄ + δx.
end
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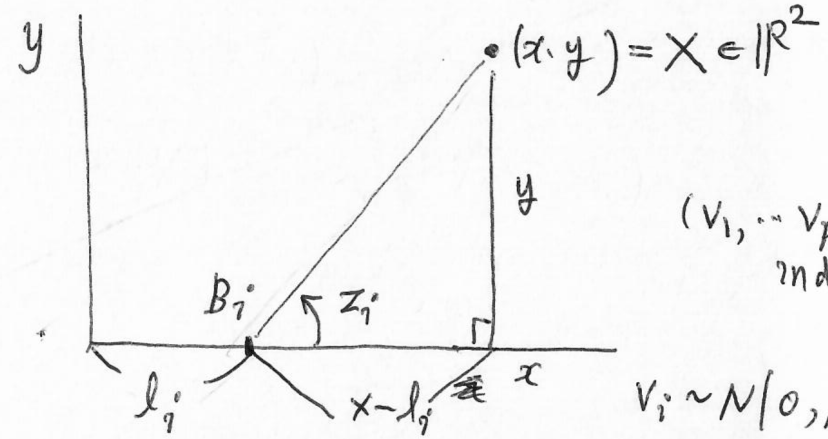
ex) pos. est. (2D) / angle measurement.



$(v_1, \cdots v_p)$, (x, y) independent

$v_i \sim N(0, R_i)$

$$z_i = h_i(x) = \frac{180}{\pi} \tan^{-1}\left(\frac{y}{x - l_i}\right) + v_i$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad Z = \begin{bmatrix} z_1 \\ \vdots \\ z_p \end{bmatrix} \quad R = \begin{bmatrix} R_1 & & & 0 \\ & R_2 & & \\ & & \ddots & \\ 0 & & & R_p \end{bmatrix}$$

- No prior knowledge. $M \to \infty$, $M^{-1} = 0$.

$$P = (H^T R^{-1} H)^{-1}, \quad K = P H^T R^{-1}.$$

- $$\underset{\substack{p \times n \\ \shortparallel \\ 2}}{H} = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \vdots & \vdots \\ \frac{\partial h_p}{\partial x} & \frac{\partial h_p}{\partial y} \end{bmatrix}$$

$$\frac{\partial z_i}{\partial x} = \frac{-y}{(x - l_i)^2 + y^2} \cdot \frac{180}{\pi}$$

$$\frac{\partial z_i}{\partial y} = \frac{x - l_i}{\text{''}} \quad \text{''}$$

$p=3$

i	l_i	R_i	Z_i
1	0	0.01	30.1°
2	500	0.01	45°
3	1000	0.04	73.6°

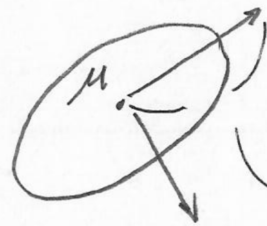
* Visualization of Gaussian.

$$c = p(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

contour : $p(x) = c$.

$$-2\ln\left(c\sqrt{\quad}\right) = \cancel{\tfrac{1}{2}}(x-\mu)^T \Sigma^{-1} (x-\mu)$$

: ellipsoid centered at μ



: major/minor axes are parallel to e/vec of Σ

$\propto$ e-val of Σ