MAE6292: Homework 2

Due date: February 14, 2020

Problem 1 Find the minimum-length trajectory $x(t)$ that connects the point $x(0) = 5$ and the circle $x^2(t) + (t - 5)^2 - 4 = 0$ by using the necessary conditions for optimality with the end point constraint.

Problem 2 Here, we consider Queen DiDo's optimization problem. We consider the half of an area enclosed by the curve $x(t)$ and the t axis. Without loss of generality, we shift the curve along the t axis such that the curve intersects the t axis, at $t = -a$ and $t = a$ for some constant a, i.e., we have $x(-a) = 0$ and $x(a) = 0$ and the center of two intersecting points corresponds to the origin. The resulting enclosed area is given by

$$A = \int_{-a}^{a} x(t)dt, \tag{1}$$

and the isoperimetric constraint is given by

$$\frac{L}{2} = \int_{-a}^{a} \sqrt{1 + \dot{x}^2(t)}\, dt, \tag{2}$$

where L is the given perimeter. Note that L is prescribed, but a is unknown.

(a) Show that the constraint (2) can be converted into

$$\dot{z}(t) = \sqrt{1 + \dot{x}(t)^2}, \quad z(-a) = 0, \quad z(a) = \frac{L}{2}, \tag{3}$$

i.e., give the expression for $z(t)$ satisfying (2) and (3).

(b) Consider the augmented cost functional,

$$J_a = \int_{-a}^{a} x(t) + \lambda(t)(\sqrt{1 + \dot{x}^2(t)} - \dot{z}(t))\, dt.$$

Derive the differential equations for $x(t), \lambda(t), z(t)$ representing the necessary condition for optimality.

(c) Show that the differential equation for $x(t)$ derived above and its boundary conditions are satisfied by the following solution:

$$x(t) = \sqrt{a^2 - t^2}. \tag{4}$$

for a certain value of λ. Also, find λ in terms of a.

(d) Find the value of a satisfying the given isoperimetric constraint, i.e., find a in terms of L such that (2) or (3) is satisfied.

(e) By substituting the above a into (4), give the expression of the optimal $x^*(t)$, and describe the resulting geometric shape of the optimal curve.