Param. est. ① $x \sim N(\bar{x}, M)$.

② $z = Hx + v$

$v \sim N(0, R)$

$$\hat{x} = \bar{x} + K(z - H\bar{x}).$$

Est. of Dyn. Sys. $x(t)$.

< Kalman Filter >. $\{t_0, t_1, \dots, t_N\}$

- Linear Gaussian sys.

$$x_{k+1} = x(t_{k+1})$$

$$= A_k x_k + \cancel{B_k u_k} + w_k.$$

$$z_k = H_k x_k + v_k.$$

where

$x \in \mathbb{R}^n$: state.

$u \in \mathbb{R}^m$: control input. deterministic / prescribed.

$w_k \in \mathbb{R}^n$: process noise. $\sim N(0, Q_k)$.

$z_k \in \mathbb{R}^p$: sensor measurement.

$v_k \in \mathbb{R}^p$: 〃 noise $\sim N(0, R_k)$

Goal: estimate x_k.

① prior knowledge : $x_0 \sim N(\bar{x}_0, P_0)$

② sensor.

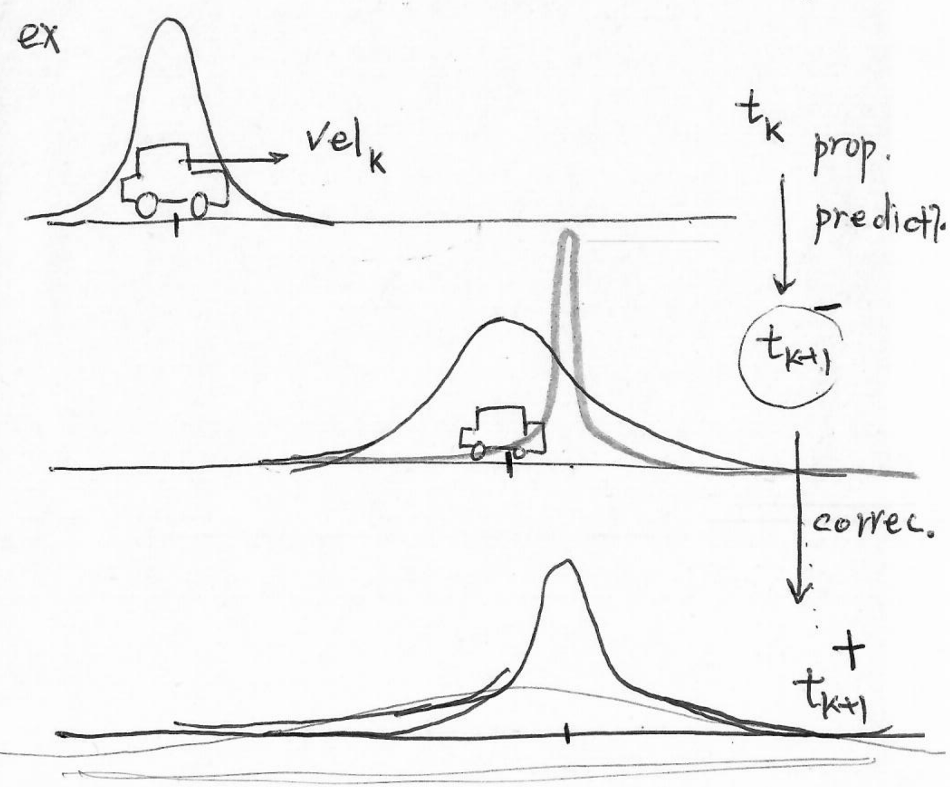
③ EOM.

w_k, v_k, x_0. are mutually <u>uncorrelated</u>

$(w_0, w_1, \dots, w_N), (v_0, v_1, \dots, v_N), x_0$

$\mathrm{cov}[\ ,\ ] = 0$. $\mathrm{cov}[w_{10}, w_{11}] = 0$.

Idea: prediction and correction.



• Prediction. $x_k \sim N(\bar{x}_k, P_k)$

$$\Rightarrow \quad x_{k+1} \sim N(\bar{x}_{k+1}, P_{k+1}),$$

Since x_{k+1} is a linear func. of x_k

x_{k+1} is a G. r.v.

$$E[x_{k+1}] = A_k E[x_k] + \cancel{B_k u_k} + 0$$

$$\begin{aligned}
\mathrm{cov}[x_{k+1}] &= E[(x_{k+1} - \bar{x}_{k+1})(\quad '' \quad)^T] \\
&= E[\{A_k(x_k - \bar{x}_k) + w_k\}\{(x - \bar{x})^T A^T + w^T\}] \\
&= E[\bigcirc \bigcirc^T + \cancel{\triangle \bigcirc^T} + \cancel{\bigcirc \triangle^T} + \triangle \triangle^T] \\
&= A_k P_k A_k^T \\
&\quad + Q_k
\end{aligned}$$

x, w are un.correl.

$$\boxed{\begin{aligned} \bar{x}_{k+1} &= A_k \bar{x}_k + B_k u_k \\ P_{k+1} &= A_k P_k A_k^T + Q_k \end{aligned}}$$

• Correction. $(\bar{x}_{k+1}^-, P_{k+1}^-) \xrightarrow{} (\bar{x}_{k+1}^+, P_{k+1}^+)$

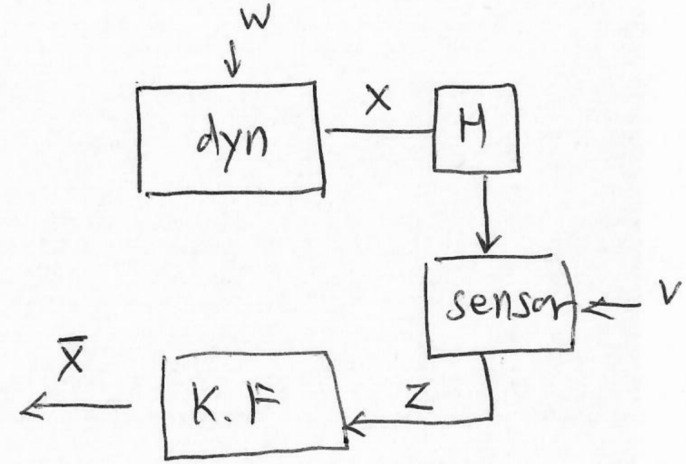
prior "a priori" $\qquad z_{k+1} \qquad$ posterior "a posteriori"

$$z = p = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H} \begin{bmatrix} p \\ vel \end{bmatrix} + v_z$$

$$v_z \sim N(0, \sigma_z^2).$$

* same as param. est.

$$\boxed{\begin{aligned} \bar{x}^+ &= \bar{x}^- + K(z - H\bar{x}^-) \\ K &= P^- H^T (H P^- H^T + R)^{-1} = P^+ H^T R^{-1} \\ P^+ &= (I - KH) P^- = \cdots = \cdots \end{aligned}}$$



ex)

EOM $\qquad \dot{p} = v$

$\qquad \dot{v} = u$

$$x = \begin{bmatrix} p \\ v \end{bmatrix}. \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

$$x_{k+1} = A_k x_k + B_k u_k + w_k.$$

$$A_k = \text{expm}(A_k \Delta t) = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}. \qquad w_k \sim N(0, Q)$$

$$B_k = \int_0^{\Delta t} e^{At} B \, dt = \begin{bmatrix} \Delta t^2/2 \\ \Delta t \end{bmatrix}. \qquad Q = \begin{bmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}.$$

< Extended Kalman Filter >

EOM $\quad x_{k+1} = f(x_k, u_k) + w_k$

$$z_k = h(x_k) + v_k$$

$x_0 \sim N(\bar{x}_0, P_0)$, $\quad w_k \sim N(0, Q_k)$, $\quad v_k \sim N(0, R_k)$ } mutually uncorrelated.

Idea: linearization.

- Prediction. $\quad x_k \sim N(\bar{x}_k, P_k)$

$\Rightarrow$ x_{k+1} is not Gaussian in general.

We assume all r.v are Gaussian.

"assumed density filter"

$$\Rightarrow x_{k+1} \sim N(\bar{x}_{k+1}, P_{k+1})$$

$$E[x_{k+1}] = E[f(x_k, u_k)] + E[\cancelto{0}{w_k}] = f(E[x], u_k)$$

not true!

$$= E\left[f(\bar{x}_k, u_k) + \boxed{\left.\frac{\partial f}{\partial x}\right|_{\bar{x}, u}}(x - \bar{x}) + H.O.T\right]$$

$$= f(\bar{x}_k, u_k) + \boxed{A_k}\Big(\underbrace{E[x] - \bar{x}}_{0}\Big) + \underbrace{H.O.T}_{E[(x-\bar{x})('')^T] \; P_k}$$

$$\approx f(\bar{x}_k, u_k) \quad \text{(first-order approx)}$$

$$\mathrm{cov}[x_{k+1}] = A_k P_k A_k^T + Q_k$$

$*$ identical to K.F except

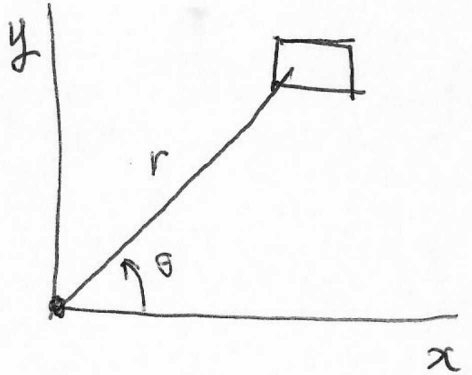
$$\bar{x}_{k+1} = f(\bar{x}_k, u_k)$$

$$A_k = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}_k, u_k}$$

• Correction

$$\boxed{\begin{aligned} &\text{Let} \quad H = \frac{\partial h}{\partial x}\Big|_{\bar{x}} \\ &\bar{x}^+ = \bar{x}^- + K(z - h(\bar{x})) \\ &\text{other eqns. for } K,\ P^+ \\ &\qquad \text{are identical to K.F} \end{aligned}}$$

ex) planar robot



state

$$X = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix}$$

$$X_{k+1} = \underbrace{\begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A_k} X_k + \underbrace{\begin{bmatrix} 0 \\ w_x \\ 0 \\ w_y \end{bmatrix}}_{w_k}$$

$$w_k \sim N(0, Q = \mathrm{diag}(0, \sigma_w^2, 0, \sigma_w^2))$$

$$Z = \begin{bmatrix} r \\ \theta \end{bmatrix} = \overbrace{\begin{bmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1} \frac{y}{x} \end{bmatrix}}^{h(x)} + \begin{bmatrix} v_r \\ v_\theta \end{bmatrix}$$ nolinear mea.

$$v \sim N(0, R = \mathrm{diag}(\sigma_r^2, \sigma_\theta^2))$$

dynamics is linear. $\left(A = \frac{\partial f}{\partial x}\right)$

mea is nonlinear

$$\underset{2\times 4}{H} = \frac{\partial h}{\partial x}\Big|_{\bar{x}} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial \dot{x}} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial \dot{y}} \\ \frac{\partial \theta}{\partial x} & " & \cdots & \cdots \end{bmatrix}_{\bar{x}}$$

$$= \begin{bmatrix} \frac{x}{r} & 0 & \frac{y}{r} & 0 \\ -\frac{y}{r^2} & 0 & \frac{x}{r^2} & 0 \end{bmatrix}_{\bar{x}}$$

< Unscented Kalman Filter >

$x \sim N(\bar{x}, P) \longrightarrow y = f(x) + w$

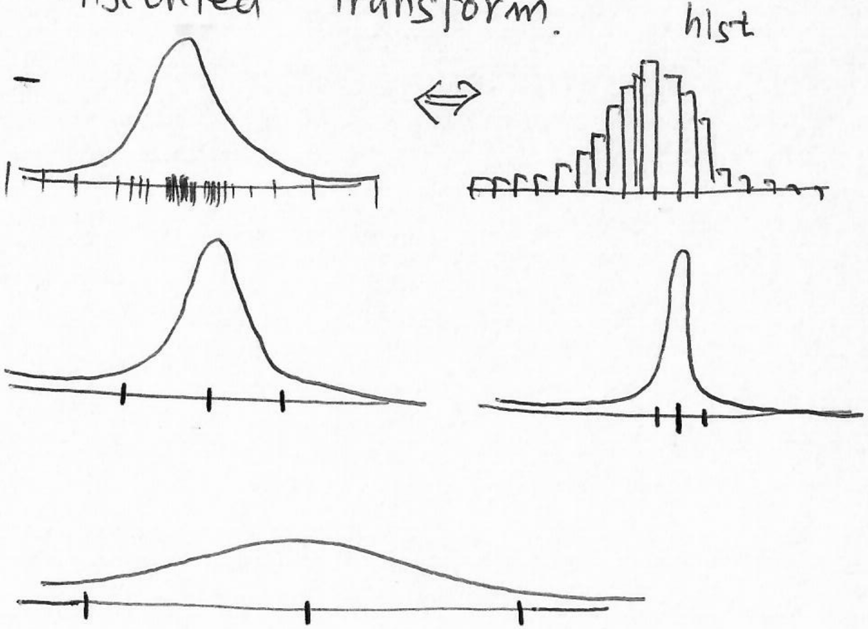
① $f(x)$ is linear. y is Gau.

② approx. it

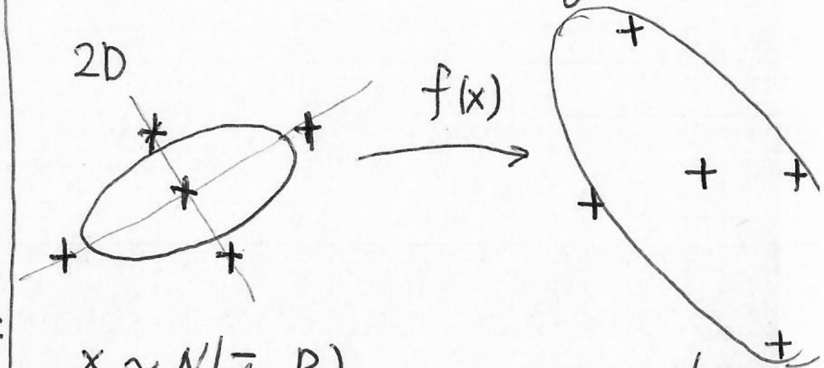
$$f(\bar{x}) + \frac{\partial f}{\partial x}\Big|_{\bar{x}} \delta x + \cancel{H.O.T}$$

s.t y is Gaussian. : EKF

• Unscented Transform



- find a set of samples characterising $N(\bar{x}, P)$



$x \sim N(\bar{x}, P)$ → $x^0, x^1, \cdots x^5$: unscented transform

$y \sim N(\bar{y}, P_y)$: $y^0, \cdots y^5$: inv. U.T

$$\begin{tikzcd} x \sim N(\bar{x}, P) \arrow[r, "?"] \arrow[d, "U.T"'] & \boxed{y \sim N(\bar{y}, P_y)} \\ \{x^0, x^1 \dots\} \arrow[r, "f(x)"] & \{y^0, y^1 \cdots\} \arrow[u, "inv.\ U.T"'] \end{tikzcd}$$

sigma point

no approx.