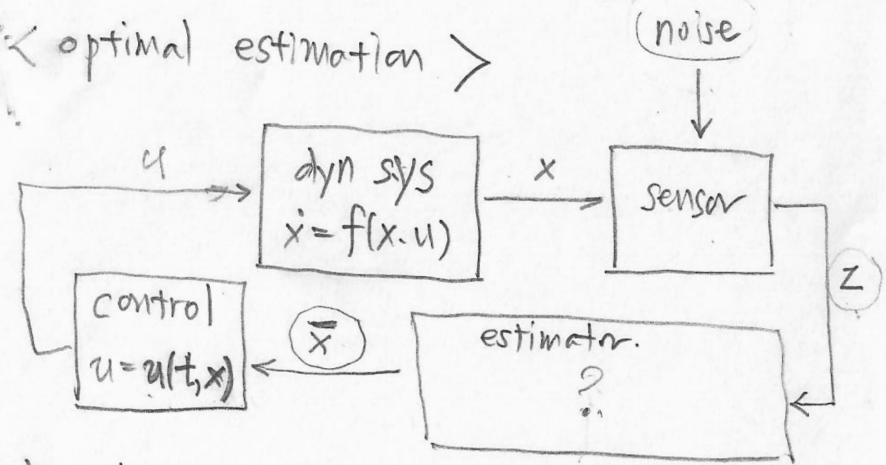
< optimal estimation >



- objective :
  estimate the state x.
  using measurement z.
  - func of x. $z = h(x)$
  - noise.
  - intermittent.

ex) UAV.

state : pos. vel. att. ang. vel.

sensor : GPS pos. $1 \sim 10$m. $1 \sim 10$Hz

IMU att. ang. vel

u : the complete state

$100 \sim 200$Hz.

< Probability >

3/27/20

Math study of random outcome.

ex) experiment. : toss a pair of dice.

one die 6.

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & \cdots & (1,6) \\ (2,1) & (2,2) & \cdots & (2,6) \\ \vdots & & & \vdots \\ (6,1) & \cdots & & (6,6) \end{matrix} \right.$$

outcome

$\mathcal{E}_1 = \{$ the sum of is 4 $\}$

$= \{(1,3), (2,2), (3,1)\}$

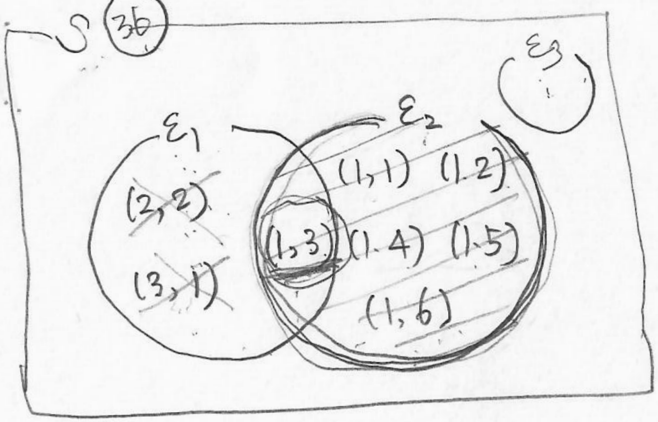
$\mathcal{E}_2 = \{$ the first is 1 $\}$

$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\}$

$\mathcal{E}_1 \cap \mathcal{E}_2 = \{(1,3)\}$

def. (sample space) set of all possible outcomes.

def. (event) a subset of S satisfying a certain condition.



def. sigma field : F

: collection of events satisfying

① $A \in F \Rightarrow A^c \in F$: complement

② $A, B \in F \Rightarrow A \cup B \in F$: union

③ $A_i \in F \Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$

De Morgan's rule : $A \cap B = (A^c \cup B^c)^c$: intersection

* F is closed under ^{c}, $\cup$, $\cap$ countable

ex) $F = \{$ all subsets of S $\}$

def. (probability space)

S : sample space

F : σ - field

P : prob. measure

① $P[\varepsilon] \geq 0$ for any $\varepsilon \in F$.

② $P[S] = 1$.

③ $P[\underbrace{\varepsilon_1 \cup \varepsilon_2}_{\text{event}}] = P[\varepsilon_1] + P[\varepsilon_2]$ if $\varepsilon_1 \cap \varepsilon_2 =$

Note $1 \overset{②}{=} P[\underbrace{\varepsilon \cup \varepsilon^c}_{S}] \overset{③}{=} P[\varepsilon] + P[\varepsilon^c]$ $\quad \varepsilon \cap \varepsilon^c = \phi$

$\Rightarrow \boxed{P[\varepsilon^c] = 1 - P[\varepsilon].}$

<classical probability>

$$P[\varepsilon] = \frac{(\#\text{ of outcomes in } \varepsilon)}{(\quad \text{〃} \quad S)}$$

ex).

$$P[\varepsilon_1] = \frac{3}{36} = \frac{1}{12}$$

$$P[\varepsilon_2] = \frac{6}{36} = \frac{1}{6}$$

def. relative freq.

$$P[\varepsilon] = \lim_{n\to\infty} \frac{n_\varepsilon}{n}$$

n : # of trials.

n_ε : # of ε.

def. conditional prob.

$P[\varepsilon_1 | \varepsilon_2]$: prob of ε_1 conditional on the knowledge that ε_2 has occured.

$$\boxed{P[\varepsilon_1|\varepsilon_2] = \frac{\#\text{ of } \varepsilon_1\cap\varepsilon_2}{\#\text{ of } \varepsilon_2}}$$

ex) $P[\varepsilon_1|\varepsilon_2] = \frac{1}{6}$, $P[\varepsilon_2|\varepsilon_1] = \frac{\#\text{ of } \varepsilon_1\cap\varepsilon_2}{\#\text{ of } \varepsilon_1} =$

ex) find x, given z , $P[X|Z]$.

- Bayes rule.

$$P[\varepsilon_2|\varepsilon_1] = \frac{P[\varepsilon_1\cap\varepsilon_2]}{P[\varepsilon_1]} = \frac{P[\varepsilon_1\cap\varepsilon_2]}{P[\varepsilon_2]} \times \frac{P[\varepsilon_2]}{P[\varepsilon_1]}$$

$$= P[\varepsilon_1|\varepsilon_2] \times \boxed{\frac{P[\varepsilon_2]}{P[\varepsilon_1]}}$$

$A\cap B$

$$\boxed{\begin{gathered} P[A,B] = P[A|B]\,P[B] = P[B|A]\,P[A]. \\ P[A|B] = P[B|A]\cdot\frac{P[A]}{P[B]} \end{gathered}$$

ex)

$$p[\varepsilon_2|\varepsilon_1] = \frac{p[\varepsilon_1|\varepsilon_2]\,p[\varepsilon_2]}{p[\varepsilon_1]} = \frac{\frac{1}{6} \times \frac{1}{6}}{\frac{1}{12}}$$

$$= \frac{12}{36} = \frac{1}{3}.$$

< Random variable >

def. consider (S, F, P), a r.v assigns a real number to each outcome.

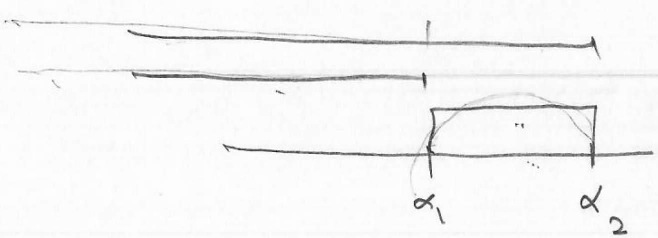
$$\boxed{\begin{array}{ll} * \; P[\varepsilon] & \varepsilon_1 = \{(1,3), (2,2), (3,1)\} \\ X(\text{outcome}) & (1,1), (1,2), \ldots \end{array}}$$

$X = \{\text{sum of two dice}\}$

* $(1,1) \longrightarrow X = 2$
  $(1,2) \longrightarrow X = 3$
  $\vdots$

$Y = \{\text{first die}\}$

$(1,1) \longrightarrow Y = 1$
$(1,2) \longrightarrow Y = 1$



def. distribution func of X

$$F_X(\alpha) = P[\underbrace{X \le \alpha}_{\text{event}}].$$

prob that $X \le \alpha$. ∵ increasing w.r.t α

$F_x[-\infty] = P[X \le -\infty] = 0$ ϕ

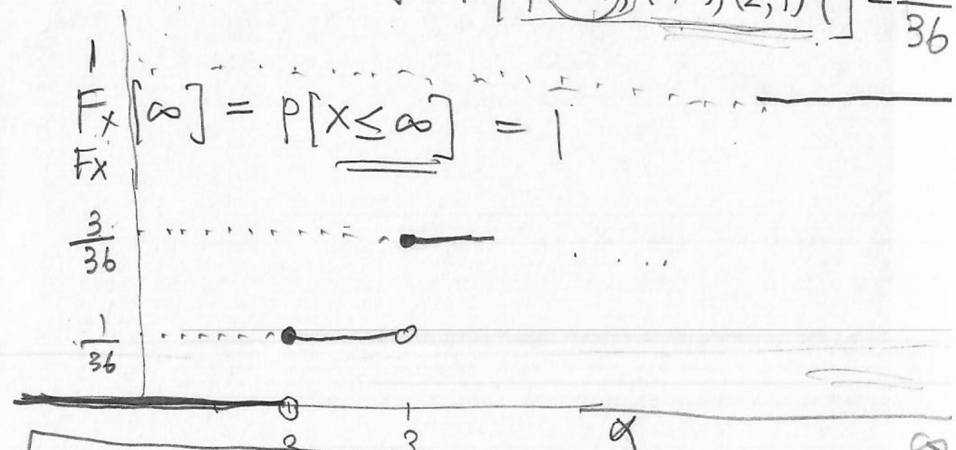
ex) $F_x[1] = P[X \le 1] = P[\{\ \}] = 0$.

$F_x[2] = P[X \le 2] = P[\{(1,1)\}] = \frac{1}{36}$

$F_x[2.5] = P[X \le 2.5] =$ // $= \frac{1}{36}$

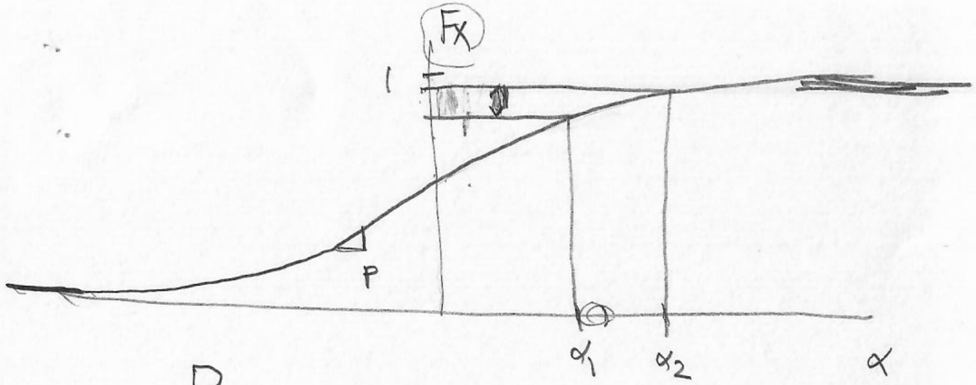
$F_x[3] = P[X \le 3] = P[\{(1,1), (1,2), (2,1)\}] = \frac{3}{36}$

$F_x[\infty] = P[X \le \infty] = 1$



$$\boxed{P[\alpha_1 < X \le \alpha_2] \overset{?}{=} F_x(\alpha_2) - F_x(\alpha_1)}$$

$$F_x(\alpha_2) = P[X \le \alpha_2] = P[X \le \alpha_1 \cup \alpha_1 < X \le \alpha_2]$$
$$= \underbrace{P[X \le \alpha_1]}_{F_x(\alpha_1)} + \boxed{P[\alpha_1 < X \le \alpha_2]}\ ?$$



P p

def. prob. density of X with F_X is a func defined s.t

$$F_X(\alpha) = \int_{-\infty}^{\alpha} p_X(x)\,dx.$$

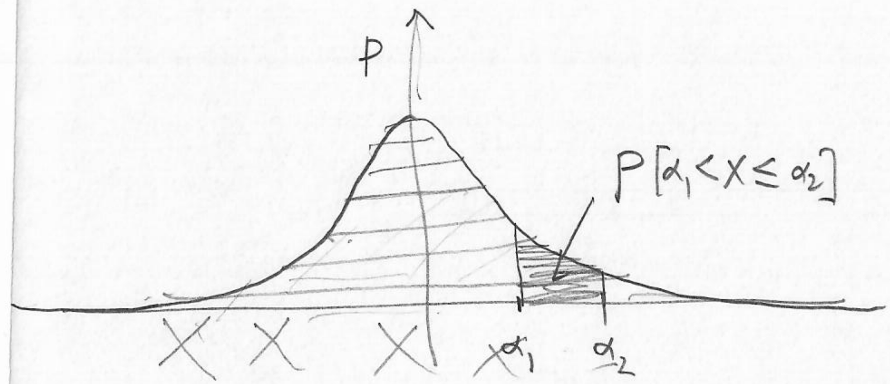
if X is continuous r.v.

$$p_X(\alpha) = \frac{dF_X(\alpha)}{d\alpha}$$

$$\boxed{P[\alpha_1 < X \le \alpha_2]} = F_X(\alpha_2) - F_X(\alpha_1)$$

$$= \int_{-\infty}^{\alpha_2} p(x)\,dx - \int_{-\infty}^{\alpha_1} p(x)\,dx$$

$$= \int_{\alpha_1}^{\alpha_2} p(x)\,dx$$



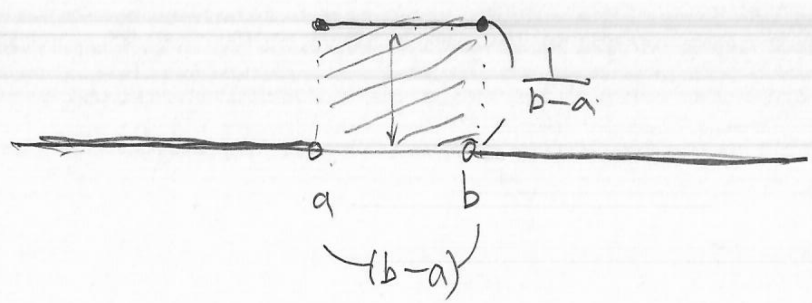
Note ① since F_X is increasing $p(x) \ge 0$.

② $\int_{-\infty}^{\infty} p(x)\,dx = P[-\infty < X < \infty] = 1$

③ $p_X(\alpha)$ is NOT the probability that $X = \alpha$.

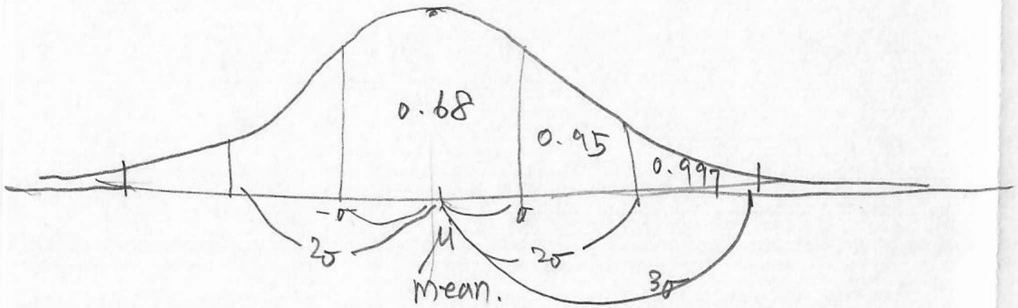
instead. $P[\alpha < X \le \alpha + d\alpha] = p(\alpha)\,d\alpha$

ex). uniform dist on $[a, b]$



def. normal (Gaussian) dist.

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[- \frac{(x-\mu)^2}{2\sigma^2} \right].$$



σ : s.t.d $\qquad x \sim N(\mu, \sigma^2)$

$x \in \mathbb{R}^n \quad \underset{\text{mean vector}}{\mu \in \mathbb{R}^{n\times 1}} \quad \Sigma = \Sigma^T \in \mathbb{R}^{n\times n}$ covariance matrix

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left[-\frac{1}{2} \underbrace{(x-\mu)^T}_{1\times n} \underbrace{\Sigma^{-1}}_{n\times n} \underbrace{(x-\mu)}_{n\times 1} \right]$$

$$x \sim N(\mu, \Sigma)$$

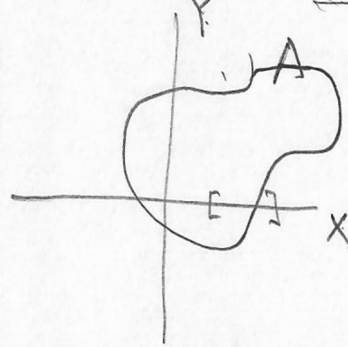
* Central limit thm.

: the sum of many identically distributed r.v is Gaussian.

$X_1 \; X_2 \; \cdots \; X_n$

$$\frac{1}{\sqrt{n}}(X_1 + X_2 + \cdots + X_n) \sim N$$

joint dist. $p(x, y)$



$$P[(x, y) \in A] = \int_A p(x, y)\, dx\, dy$$

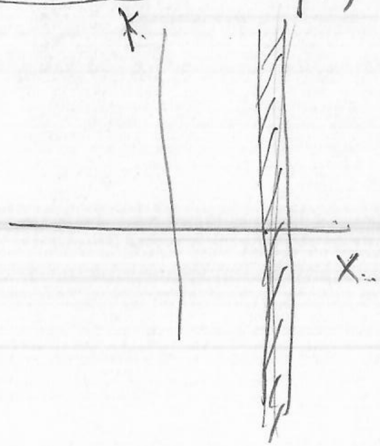
independent if $p(x,y) = p(x)\,p(y)$

cond. $\quad p(x|y) = \frac{p(x,y)}{p(y)}$

Bayes rule

$$p(x|y) = \frac{p(y|x)\,p(x)}{p(y)}$$

Marginal



$$p(x) = \int_{-\infty}^{\infty} p(x, y)\, dy$$

$$p(y) = \int_{-\infty}^{\infty} p(x, y)\, dx$$