04/17/20

MAE 6292:

Homework 4

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Problem 1:

$P[C_1] = 0.05 \qquad P[C_2] = 0.01 \qquad P[C_3] = 0.1$

$P[F|C_1] = 0.1 \qquad P[F|C_2] = 0.5 \qquad P[F|C_3] = 0.7$

(a) Law of Total probability

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$$P(F) = \sum_n P(F|C_n)\, P(C_n)$$

$$= P(F|C_1)\, P(C_1) + P(F|C_2)\cdot P(C_2) + P(F|C_3)\cdot P(C_3)$$

$$= 0.1 \times 0.05 + 0.5 \times 0.01 + 0.7 \times 0.1$$

$$\boxed{P(F) = 0.08}$$

(b) $P(F|C_1) = \dfrac{P(C_1|F)\cdot P(F)}{P(C_1)}$ from Bayes Theorem

$$P(C_1|F) = \frac{P(F|C_1)\cdot P(C_1)}{P(F)} = \frac{0.1 \times 0.05}{0.08} = 0.0625$$

Similarly

$$P(C_2|F) = \frac{P(F|C_2)\, P(C_2)}{P(F)} = \frac{0.5 \times 0.01}{0.08} = 0.0625$$

$$P(C_3|F) = \frac{P(F|C_3)\cdot P(C_3)}{P(F)} = \frac{0.7 \times 0.1}{0.08} = 0.875$$

Problem 2:

$$P_{XY}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Marginal probability densities:

$P_X(x)$

$$P_X(x) = \int_{-\infty}^{+\infty} P_{XY}(x,y)\,dy$$

$$P_X(x) = \int_{-\infty}^{\infty} \frac{1}{\pi}\,dy$$

$P_Y(y)$

$$P_Y(y) = \int_{-\infty}^{\infty} P_{XY}(x,y)\,dx$$

$$P_Y(y) = \int_{-\infty}^{\infty} \frac{1}{\pi}\,dx$$

only integrating over supported range

$$P_X(x) = \frac{1}{\pi}\int_{x^2+y^2=1} dy$$

$$P_X(x) = \frac{1}{\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy$$

$$\boxed{P_X(x) = \frac{2}{\pi}\sqrt{1-x^2}}$$

$$P_Y(y) = \frac{1}{\pi}\int_{x^2+y^2=1} dx$$

$$P_Y(y) = \frac{1}{\pi}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx$$

$$\boxed{P_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}}$$

(b) Conditional probability density

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$\Rightarrow$ We have

$$P_{X,Y}(x,y) = P_{X|Y}(x|y)\cdot P_Y(y)$$

$$= \frac{\frac{1}{\pi}}{\frac{2}{\pi}\sqrt{1-y^2}}$$

$$\boxed{P_{X|Y}(x|y) = \frac{1}{2\sqrt{1-y^2}}}$$

Add other cases for zero

(c) Conditional probability density (from Bayes rule)

Similar to the previous part, we have $P_{Y|X}(y|x) = \dfrac{P_{X,Y}(x,y)}{P_X(x)}$

$$\Rightarrow \quad P_{Y|X}(y|x) = \frac{P_{X|Y}(x|y) \cdot P_Y(y)}{P_X(x)}$$

$$= \frac{\dfrac{1}{\cancel{2}\cancel{\sqrt{1-y^2}}} \times \dfrac{\cancel{2}\cancel{\sqrt{1-y^2}}}{\cancel{\pi}}}{\dfrac{2}{\cancel{\pi}}\sqrt{1-x^2}}$$

$$\boxed{P_{Y|X}(y|x) = \frac{1}{2\sqrt{1-x^2}}}$$

(d) We know that, X and Y are independent variables if

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)$$

But as we can see from previous results

$$P_X(x) \cdot P_Y(y) \neq P_{X,Y}(x,y).$$

Hence X and Y are not independent random variables

(e) $$P\left[0 \le X \le \tfrac{1}{2}, 0 \le Y\right] = \int_0^{1/2}\int_0^{\sqrt{1-x^2}} dy\,dx$$ π

$$= \int_0^{1/2} \left(\sqrt{1-x^2}\right) dx$$

① Applying Trigonometry substitution $x = \sin(u)$

② $\cos^2(x) = \dfrac{1+\cos 2(x)}{2}$

$$= \frac{2\pi + 3\sqrt{3}}{24}$$

$$\boxed{P\left[0 \le X \le 1/2, 0 \le Y\right] = \frac{2\pi+3\sqrt{3}}{24}}$$ π

Problem 3:

Random variable X follows the Gaussian distribution

$$P_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

(a) Expected value of a continuous random variable:

$$E[x] = \int_{-\infty}^{\infty} x\, P_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} dx$$

$$E[x] = \frac{\sqrt{2}\,\sigma}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (\sqrt{2}\sigma z+\mu)\exp(-z^2)\,dz$$

Let $\frac{x-\mu}{\sqrt{2}\,\sigma} = z$

$x-\mu = \sqrt{2}\sigma z$

$dx = \sqrt{2}\sigma\, dz$

$$E[x] = \frac{1}{\sqrt{\pi}}\left(\sqrt{2}\sigma \int_{-\infty}^{\infty} z\exp(-z^2)\,dz + \mu\int_{-\infty}^{\infty}\exp(-z^2)\,dz\right)$$

$$E[x] = \frac{1}{\sqrt{\pi}}\left(\sqrt{2}\sigma\left[-\frac{1}{2}\exp(-z^2)\right]_{-\infty}^{\infty} + \mu\sqrt{\pi}\right)$$

$$E[x] = \frac{\mu\sqrt{\pi}}{\sqrt{\pi}}$$

$$\boxed{E[x] = \mu}$$

✓

(b) We know that

$$var(X) = \int_{-\infty}^{\infty} x^2 p_x(x)dx - (E[x])^2$$

Let $z = \frac{x-\mu}{\sqrt{2}\sigma}$

$x - \mu = \sqrt{2}\sigma z$

$dx = \sqrt{2}\sigma dz$

$$var(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx - \mu^2$$

$$var(x) = \frac{\sigma\sqrt{2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma z + \mu)^2 \exp(-z^2)dz - \mu^2$$

$$var(x) = \frac{1}{\sqrt{\pi}}\left(2\sigma^2 \int_{-\infty}^{\infty} z^2 \exp(-z^2)dz + 2\sqrt{2}\sigma\mu \int_{-\infty}^{\infty} z\exp(-z^2)dz + \mu^2 \int_{-\infty}^{\infty} \exp(-z^2)dz\right) - \mu^2$$

$$var(x) = \frac{1}{\sqrt{\pi}}\left(2\sigma^2 \int_{-\infty}^{\infty} z^2 \exp(-z^2)dz + 2\sqrt{2}\sigma\mu\left[-\frac{1}{2}\exp(-z^2)\right]_{-\infty}^{\infty} + \mu^2\sqrt{\pi}\right) - \mu^2$$

$$var(x) = \frac{1}{\sqrt{\pi}}\left(2\sigma^2 \int_{-\infty}^{\infty} z^2 \exp(-z^2)dz + 2\sqrt{2}\sigma\mu \cdot 0\right) + \mu^2 - \mu^2$$

$$var(x) = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 \exp(-z^2)dz$$

$$var(x) = \frac{2\sigma^2}{\sqrt{\pi}}\left(\left[-\frac{z}{2}\exp(-z^2)\right]_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} \exp(-z^2)dz\right)$$

$$var(x) = \frac{\cancel{2}\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{\cancel{2}} \int_{-\infty}^{\infty} \exp(-z^2)dz = \frac{\sigma^2}{\cancel{\sqrt{\pi}}}\cancel{\sqrt{\pi}} = \sigma^2$$

$$\boxed{var(x) = \sigma^2}$$

Problem 4:

Part (b):

Semi axes are given by $x_i = \sqrt{\lambda_i}$ i.e.

- eigenvectors determine direction of semi-axes
- eigenvalues determine length of semi-axes.

Direction that the estimate is most uncertain i.e.
semi-major axis of the Gaussian density ellipsoid

$\therefore$ It is in the direction of eigenvector v_1 corresponding to $\lambda_{max}(\Sigma)$

$$\therefore \quad v_1 = \begin{bmatrix} 0.9945 \\ 0.0899 \\ 0.0532 \end{bmatrix}$$

Part (c) Direction that the estimate is least uncertain i.e.
Semi-minor axis of the Gaussian density ellipsoid

$\therefore$ It is in the direction of eigenvector v_n corresponding to $\lambda_{min}(\Sigma) \rightarrow$ eigenvalue

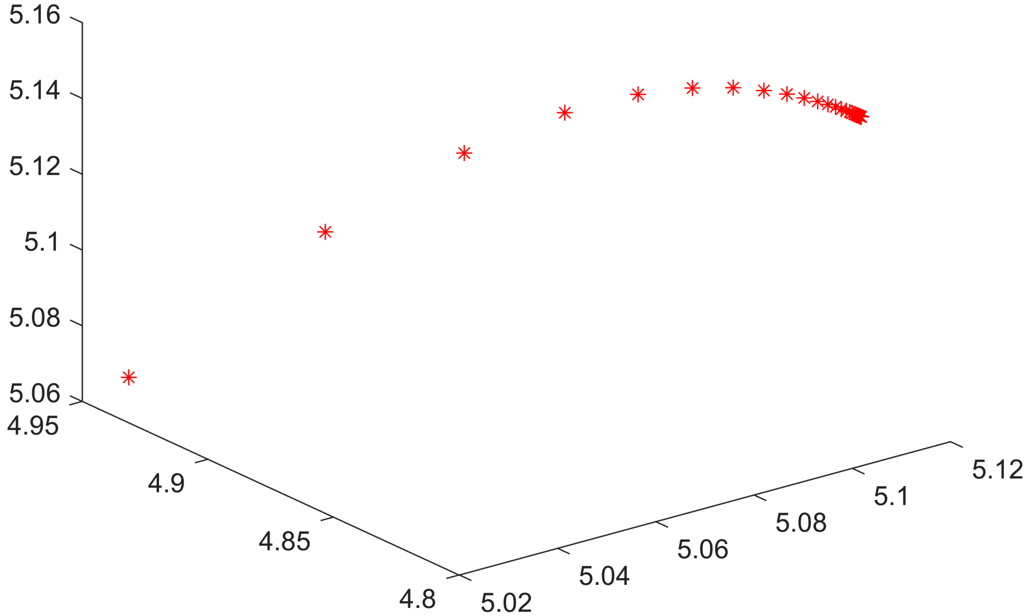
$$\therefore \quad v_n = \begin{bmatrix} -0.0668 \\ 0.1556 \\ 0.9856 \end{bmatrix}$$

(MATLAB code attached)

```
clc;
clear all;
close all;
load('prob4_data.mat'); % Load the data
```

```
% Given 4 Matrices used in calculation of Projection Matrix
A = C(1:2,1:3,:);
b = C(1:2,4,:);
c = C(3,1:3,:);
d = C(3,4,:);


R1 = diag(R(1:2,1:2));
R2 = diag(R(3:4,3:4));
R3 = diag(R(5:6,5:6));
R4 = diag(R(7:8,7:8));

R_ = 0*ones(2,1,4);
R_(:,:,1) = R1;
R_(:,:,2) = R2;
R_(:,:,3) = R3;
R_(:,:,4) = R4;

% Initial value of x
x = [5 5 5]';
% Initial value of dX
dX = [1 1 1]';

eps = 1e-6;

while norm(dX) > eps

    Z_bar = zeros(2,4);
    for i = 1:4
        Z_bar(:,i) = ((A(:,:,i)*x + b(:,:,i))/(c(:,:,i)*x + d(:,:,i))); %
    end
    Zbar = reshape(Z_bar,[8,1]); % reshape 8x1 matrix
    dZ = Z - Zbar;

    H = zeros(2,3,4);
    for j = 1:4
        H(:,:,j) = (A(:,:,j) * d(:,:,j) - b(:,:,j) * c(:,:,j))/(c(:,:,j)*x + d(:,:,j))^2;
    end
    H_bar = [H(:,:,1);H(:,:,2);H(:,:,3);H(:,:,4)]; % reshape the H - 8x3 matrix
    % 3x3 matrix
    P = inv(H_bar' * inv(R) * H_bar);
    % 3x8 matrix
    K = P * H_bar' * inv(R);

    dX = K * dZ;
    x = x + dX;
    % Plotting
    plot3(x(1),x(2),x(3), 'r*'); hold on
```

```
end
hold off
```




```
% Optimal estimate
disp("Optimal Estimate")
```

```
Optimal Estimate
```

```
disp(x)
```

```
    5.1132
    4.8224
    5.1413
```

```
% Error covariance matrix
disp("Error Covariance Matrix")
```

```
Error Covariance Matrix
```

```
disp(P)
```

```
    0.2123    0.0146    0.0097
    0.0146    0.0540   -0.0019
    0.0097   -0.0019    0.0365
```

```
% Part (b)
disp("Semi-major axis of Gaussian Density Ellipsoid")
```

```
Semi-major axis of Gaussian Density Ellipsoid
```

```
disp("The direction that the estimate is most uncertain is the ")
```

```
The direction that the estimate is most uncertain is the
```

```
% Calculate the eigenvalues and eigenvectors
[eigenvec, eigenval] = eig(P);

% Get the index of the largest eigenvector
[large_row, large_col] = find(eigenval== max(max(eigenval)));

largestEigenVec = eigenvec(:, large_row)
```

```
largestEigenVec = 3×1
   -0.9945
   -0.0899
   -0.0532
```

```
% The largest Eigenvalue
largestEigenVal = max(max(eigenval))
```

```
largestEigenVal = 0.2142
```

```
% Part (c)
disp("Semi-minor axis of Gaussian Density Ellipsoid")
```

```
Semi-minor axis of Gaussian Density Ellipsoid
```

```
disp("The direction that the estimate is least uncertain is the ")
```

```
The direction that the estimate is least uncertain is the
```

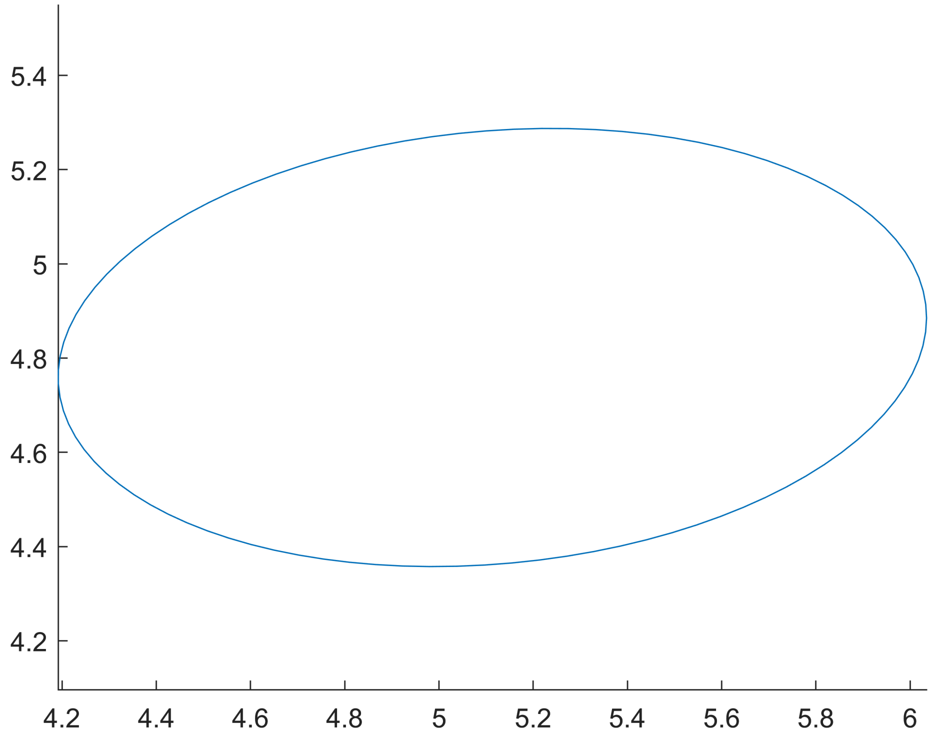
```
% Get the index of the smallest eigenvector
[small_row, small_col] = find(eigenval == min(max(eigenval)));

smallestEigenVec = eigenvec(:, small_row)
```

```
smallestEigenVec = 3×1
   -0.0668
    0.1556
    0.9856
```

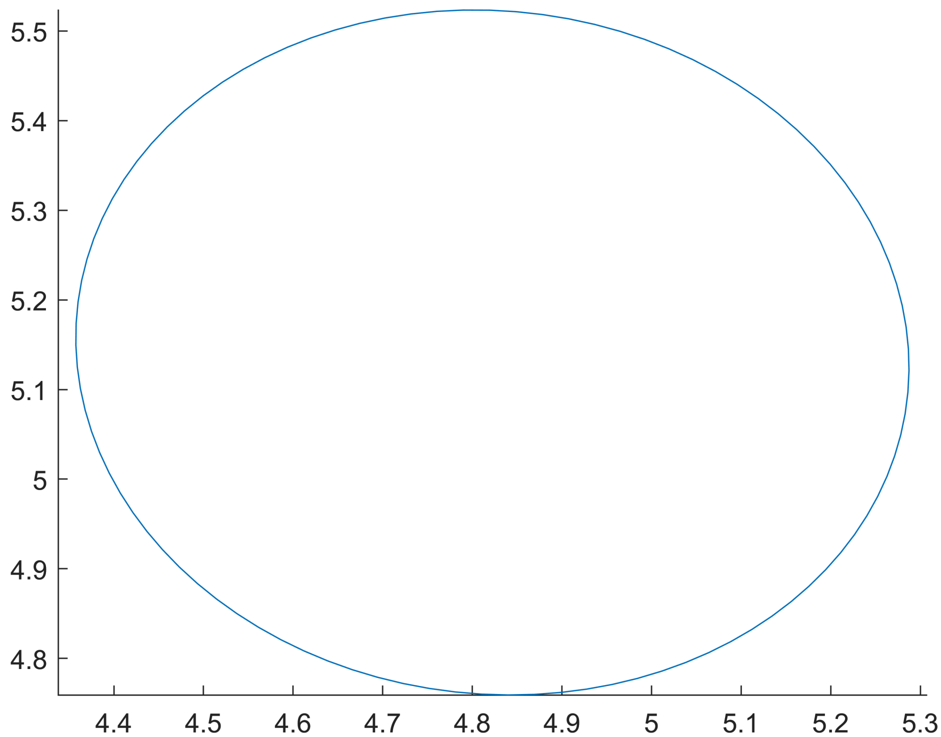
```
% Smallest Eigenvalue
smallestEigenVal = min(max(eigenval))
```

```
smallestEigenVal = 0.0355
```

```
% Part (d)
% 86% Gaussian ellipsoid for the last two components of x.
figure
plot_gaussian_ellipsoid(x(1:2),P(1:2,1:2),2);
axis equal;
```




```
% Part(e) -
% 86% Gaussian ellipsoid for the last two components of x.
figure
plot_gaussian_ellipsoid(x(2:3),P(2:3,2:3),2);
axis equal
```


5.5
5.4
5.3
5.2
5.1
5
4.9
4.8
4.4
4.5
4.6
4.7
4.8
4.9
5
5.1
5.2
5.3