MAE 6292: Final Exam

Due: 5pm, Friday, May 8 2020

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Last Name First Name Student ID

Prob. 1	Prob. 2	Prob. 3	Prob. 4	Total

Honor Pledge

I pledge that I have neither given nor received unauthorized assistance on this work. I understand that any academic misconduct will be handled according to *GWU Code of Academic Integrity*.

Signature Date

Note

- Make all of your submission into a single pdf file, organized as
  - Cover page
  - Honor pledge signed by you
  - Written answer for Question 1
  - Matlab code and results for Question 1
  - Written answer for Question 2
  - Matlab code and results for Question 2
  - $\vdots$
- Use the *publish* feature of the Matlab to share the code and the results:
  `https://www.mathworks.com/help/matlab/matlab_prog/publishing-matlab-code.html`

Problem 1 (EKF for Localization) Consider a planar robot model whose configuration is described by its location (x, y) and the orientation θ. It translates with a fixed velocity V along its heading-direction that can be changed by a control input u, i.e., it may represent a wheeled planar robot. Assume that the velocity and the control input are perturbed by process noise w_{v_k}, w_{θ_k} respectively. The equations of motion are given by

$$\begin{aligned}
x_{k+1} &= x_k + h(V + w_{v_k})\cos\theta_k,\\
y_{k+1} &= y_k + h(V + w_{v_k})\sin\theta_k,\\
\theta_{k+1} &= \theta_k + h(u_k + w_{\theta_k}),
\end{aligned}$$

where h denotes the time step.

Consider several reference points around the robot. Suppose that the location of the i-th reference point is known, and it is given by (r_x^i, r_y^i). The robot is equipped with a range sensor to measure the distance between the robot and the reference point, i.e., the i-th range measurement is given by

$$z_r^i = \sqrt{\Delta x_i^2 + \Delta y_i^2} + v_r,$$

where the relative position of the i-th reference point from the robot is denoted by $(\Delta x_i, \Delta y_i) = (r_x^i - x, r_y^i - y)$, and v_r corresponds to the range measurement noise.

The robot is also equipped with a bearing sensor to measure the direction toward reference points. Since the bearing sensor is attached to the body of the robot, the sensor reading is given with respect to the body-fixed frame. The relative position is represented with respect to the body-fixed frame as

$$\begin{bmatrix} \Delta x_i^{body} \\ \Delta y_i^{body} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix}.$$

The corresponding i-th bearing measurement is

$$z_\theta^i = \tan^{-1}\frac{\Delta y_i^{body}}{\Delta x_i^{body}} + v_\theta,$$

where v_i denotes the measurement error. The complete measurement is given by $\mathbf{z} = [z_r^1, z_\theta^1, \ldots, z_r^{N_r}, z_\theta^{N_r}] \in \mathbb{R}^{2N_r}$ where N_r denotes the number of reference points.

(a) Let the state vector be $\mathbf{x}_k = [x_k, y_k, \theta_k]^T \in \mathbb{R}^3$. Show that the linearized equations of motion are given by

$$\delta\mathbf{x_{k+1}} = A_k\delta\mathbf{x_k} + B_k u_k + G_k w_k,$$

where

$$A_k = \begin{bmatrix} 1 & 0 & -hV\sin\theta_k \\ 0 & 1 & hV\cos\theta_k \\ 0 & 0 & 1 \end{bmatrix}, \quad B_k = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}, \quad G_k = \begin{bmatrix} h\cos\theta_k & 0 \\ h\sin\theta_k & 0 \\ 0 & h \end{bmatrix}.$$

(Note: the increase of P_k due to the process noise becomes $G_k Q_k G_k^T$.)

(b) Let the measurements from the i-th reference point be $\mathbf{z}^i = [z_r^i, z_\theta^i]^T \in \mathbb{R}^2$. Show that the linearized measurement equation can be written as

$$\delta\mathbf{z}^i = H^i\delta\mathbf{x},$$

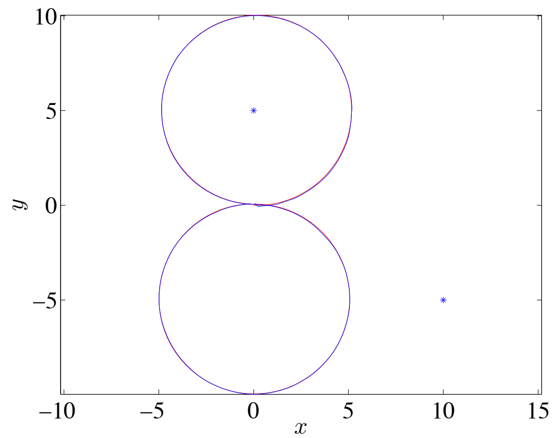
where the matrix $H^i \in \mathbb{R}^{2\times 3}$ are given by

$$H^i = \begin{bmatrix} \frac{\partial z_r^i}{\partial x} & \frac{\partial z_r^i}{\partial y} & \frac{\partial z_r^i}{\partial \theta} \\ \frac{\partial z_\theta^i}{\partial x} & \frac{\partial z_\theta^i}{\partial y} & \frac{\partial z_\theta^i}{\partial \theta} \end{bmatrix},$$

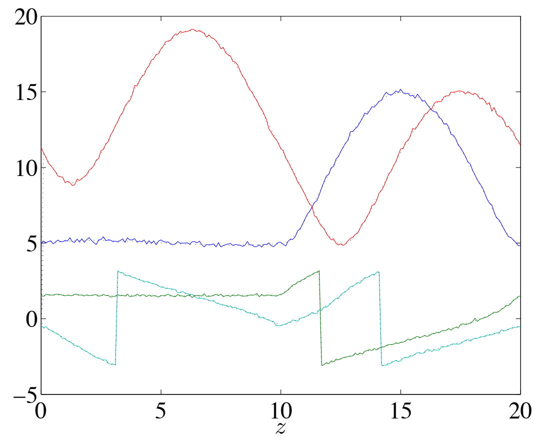
and the derivatives are given by

$$\frac{\partial z_r^i}{\partial x} = -\frac{\Delta x_i}{\sqrt{\Delta x_i^2 + \Delta y_i^2}}, \quad \frac{\partial z_r^i}{\partial y} = -\frac{\Delta y_i}{\sqrt{\Delta x_i^2 + \Delta y_i^2}}, \quad \frac{\partial z_r^i}{\partial \theta} = 0,$$
$$\frac{\partial z_\theta^i}{\partial x} = \frac{\Delta y_i}{\Delta x_i^2 + \Delta y_i^2}, \quad \frac{\partial z_\theta^i}{\partial y} = -\frac{\Delta x_i}{\Delta x_i^2 + \Delta y_i^2}, \quad \frac{\partial z_\theta^i}{\partial \theta} = -1.$$

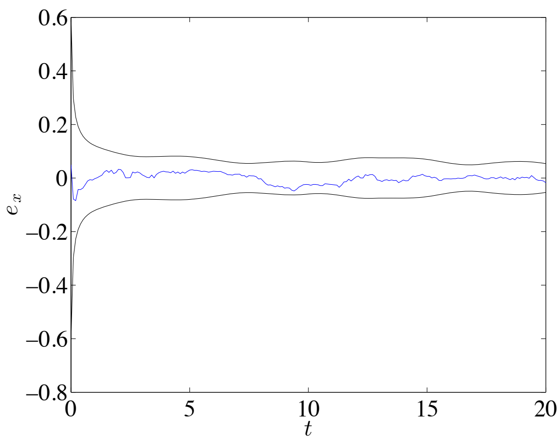
(c) All of the parameters for the robot model, control inputs, measurements, reference points, and initial guess are specified at `prob1.m`. Write a Matlab file for EKF to estimate the state $\mathbf{x}_k$, and generate 6 plots given at the next page.



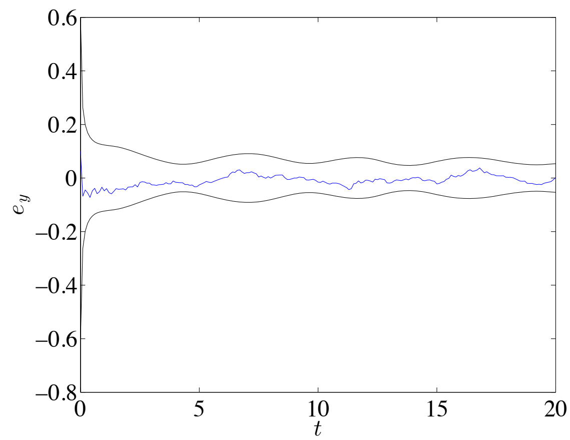
(a) Position



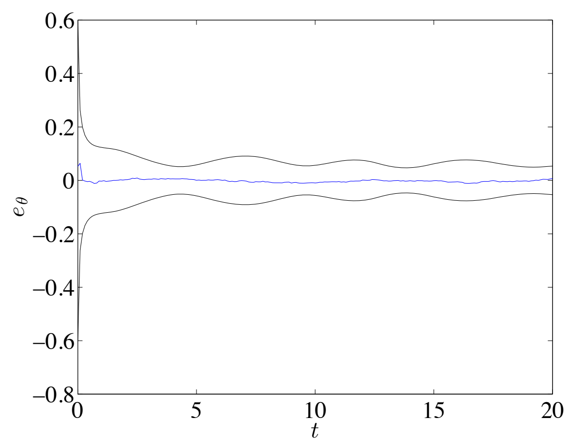
(b) Measurement



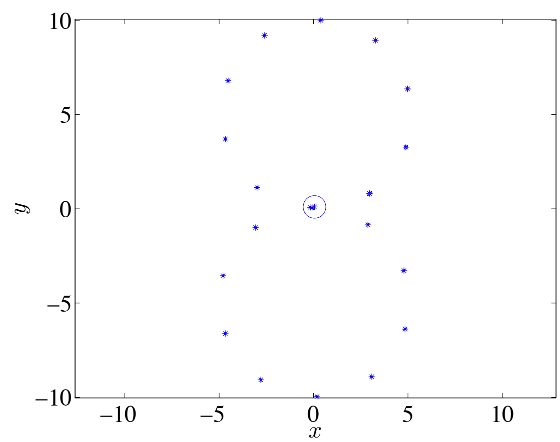
(c) Estimation error e_x



(d) Estimation error e_y



(e) Estimation error e_θ



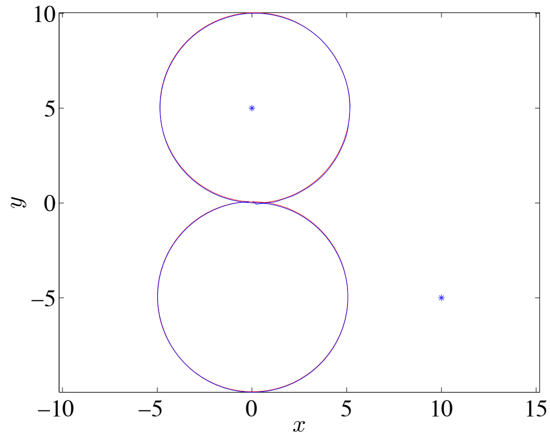
(f) Gaussian ellipsoid

Figure 1: EKF for Localization

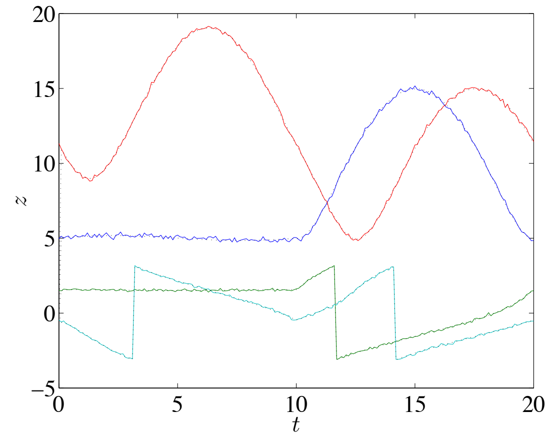
Problem 2 (UKF for Localization) Consider the robot model discussed at Problem 1.

(a) All of the parameters for the robot model, control inputs, measurements, reference points, and initial guess are specified at `prob2.m`. Write a Matlab file for UKF to estimate the state $\mathbf{x}_k$, and generate 6 plots given at the next page.

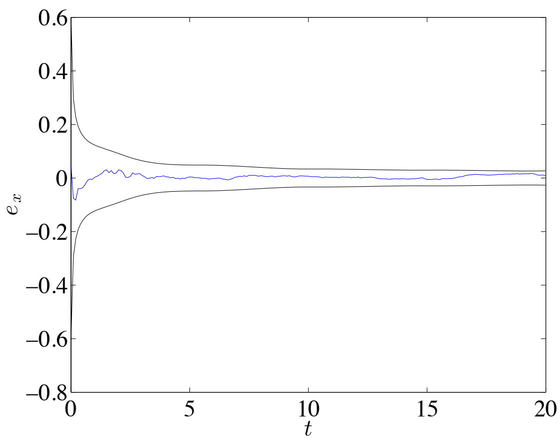
(b) Compare the performance of UKF with EKF for this robot model.



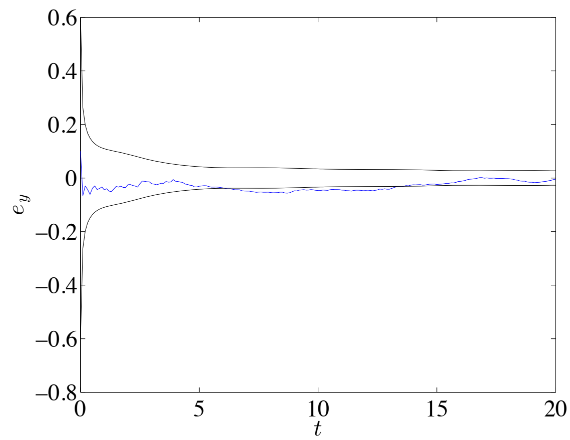
(a) Position



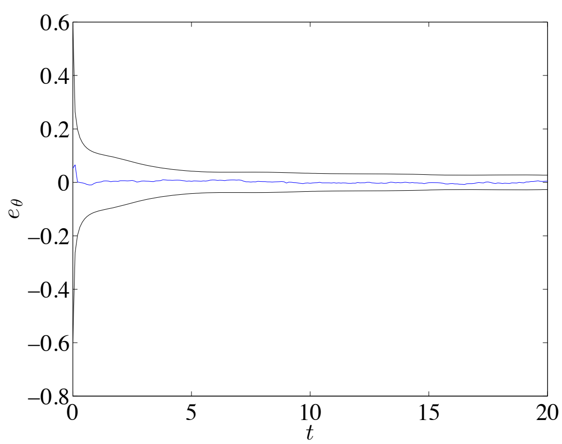
(b) Measurement



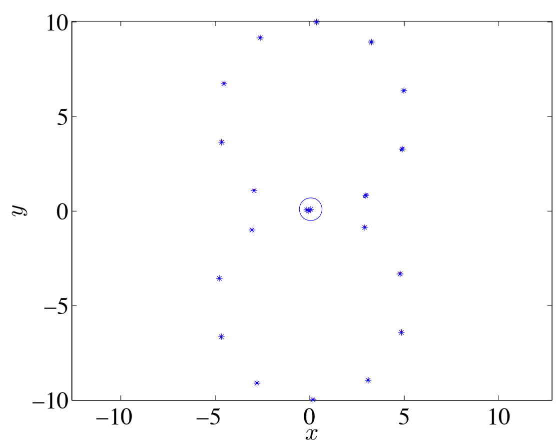
(c) Estimation error e_x



(d) Estimation error e_y



(e) Estimation error e_θ



(f) Gaussian ellipsoid

Figure 2: UKF for Localization

Problem 3 (EKF for SLAM) Consider the planar robot model discussed at Problem 1. In this question, we assume that the location of the reference points are unknown. The objective is to estimate the position and the orientation of the robot while estimating the location of the reference points as well. This is referred to as *simultaneous localization and mapping* (SLAM).

Since the location of the reference points are unknown. They are included to the state vector. When there are two reference points, the complete state vector for SLAM is given by

$$\mathbf{x} = [x, y, \theta, r_x^1, r_y^1, r_x^2, r_y^2]^T \in \mathbb{R}^7.$$

Assuming that the reference points are stationary, the equations of motion are given by

$$\begin{aligned}
x_{k+1} &= x_k + h(V + w_{v_k})\cos\theta_k,\\
y_{k+1} &= y_k + h(V + w_{v_k})\sin\theta_k,\\
\theta_{k+1} &= \theta_k + h(u_k + w_{\theta_k}),\\
r^1_{x_{k+1}} &= r^1_{x_k},\\
r^1_{y_{k+1}} &= r^1_{y_k},\\
r^2_{x_{k+1}} &= r^2_{x_k},\\
r^2_{y_{k+1}} &= r^2_{y_k}.
\end{aligned}$$

The measurements are identical to Problem 1, i.e., $\mathbf{z} = [z_r^1, z_\theta^1, z_r^2, z_\theta^2]^T \in \mathbb{R}^4$.

(a) Show that the linearized equations of motion are given by

$$\delta\mathbf{x_{k+1}} = A_k\delta\mathbf{x_k} + B_k u_k + G_k w_k,$$

where

$$A_k = \begin{bmatrix} 1 & 0 & -hV\sin\theta_k & 0 & 0 & 0 & 0\\ 0 & 1 & hV\cos\theta_k & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_k = \begin{bmatrix} 0\\0\\h\\0\\0\\0\\0 \end{bmatrix}, \quad G_k = \begin{bmatrix} h\cos\theta_k & 0\\ h\sin\theta_k & 0\\ 0 & h\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0 \end{bmatrix}.$$

(Note: the increase of P_k due to the process noise becomes $G_k Q_k G_k^T$.)

(b) Show that the linearized measurement equation can be written as

$$\delta\mathbf{z} = H\delta\mathbf{x},$$

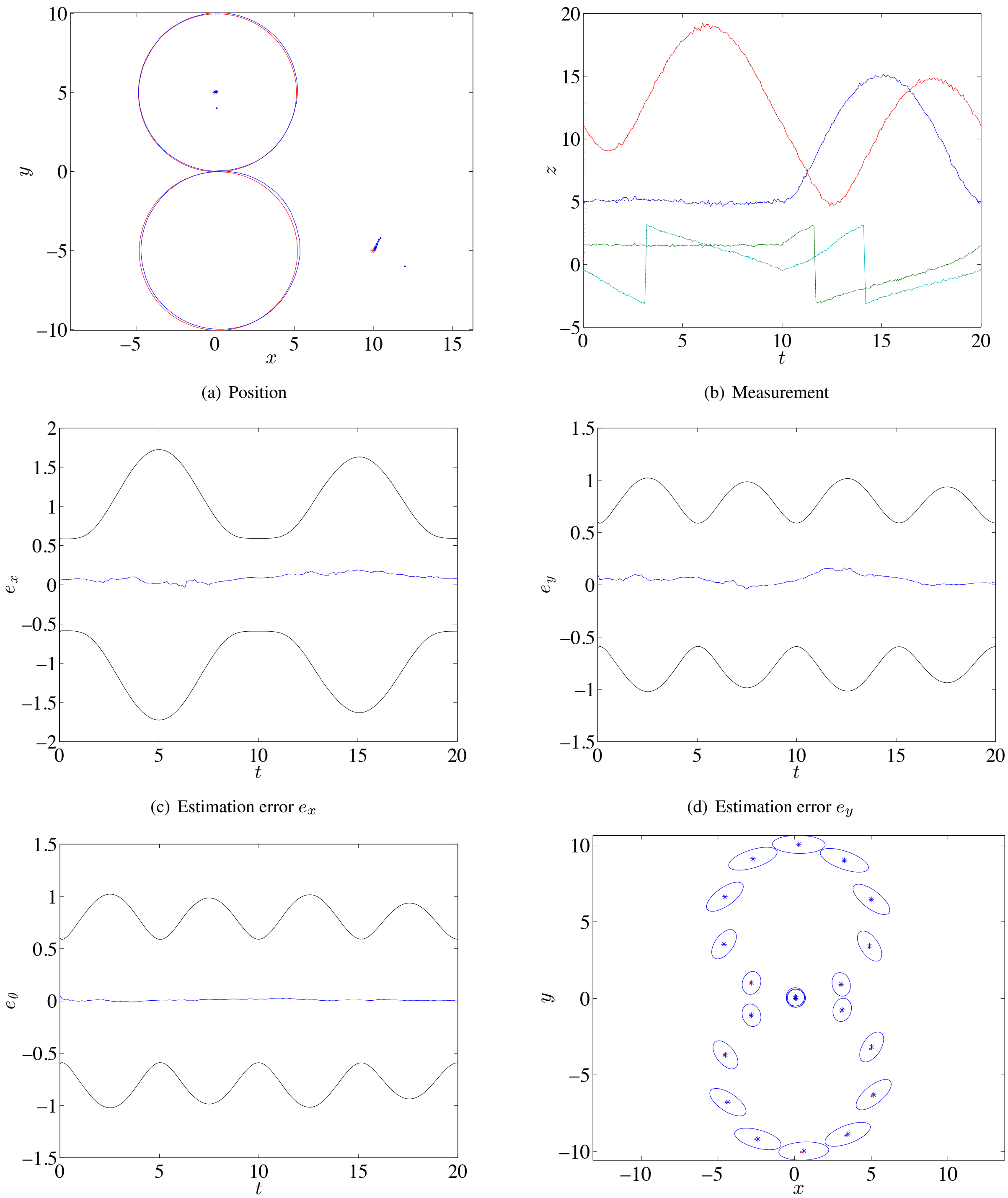
where the matrix $H \in \mathbb{R}^{4\times 7}$ are given by

$$H^i = \begin{bmatrix} \frac{\partial z_r^1}{\partial x} & \frac{\partial z_r^1}{\partial y} & \frac{\partial z_r^1}{\partial \theta} & \frac{\partial z_r^1}{\partial r_x^1} & \frac{\partial z_r^1}{\partial r_y^1} & 0 & 0\\ \frac{\partial z_\theta^i}{\partial x} & \frac{\partial z_\theta^i}{\partial y} & \frac{\partial z_\theta^i}{\partial \theta} & \frac{\partial z_\theta^1}{\partial r_x^1} & \frac{\partial z_\theta^1}{\partial r_y^1} & 0 & 0\\ \frac{\partial z_r^2}{\partial x} & \frac{\partial z_r^2}{\partial y} & \frac{\partial z_r^2}{\partial \theta} & 0 & 0 & \frac{\partial z_r^2}{\partial r_x^2} & \frac{\partial z_r^2}{\partial r_y^2}\\ \frac{\partial z_\theta^2}{\partial x} & \frac{\partial z_\theta^2}{\partial y} & \frac{\partial z_\theta^2}{\partial \theta} & 0 & 0 & \frac{\partial z_\theta^2}{\partial r_x^2} & \frac{\partial z_\theta^2}{\partial r_y^2} \end{bmatrix},$$

and the derivatives are given by

$$\frac{\partial z_r^i}{\partial r_x^i} = \frac{\Delta x_i}{\sqrt{\Delta x_i^2 + \Delta y_i^2}}, \quad \frac{\partial z_r^i}{\partial r_y^i} = \frac{\Delta y_i}{\sqrt{\Delta x_i^2 + \Delta y_i^2}},$$
$$\frac{\partial z_\theta^i}{\partial r_x^i} = -\frac{\Delta y_i}{\Delta x_i^2 + \Delta y_i^2}, \quad \frac{\partial z_\theta^i}{\partial r_y^i} = +\frac{\Delta x_i}{\Delta x_i^2 + \Delta y_i^2}.$$

(c) All of the parameters for the robot model, control inputs, measurements, reference points, and initial guess are specified at `prob3.m`. Write a Matlab file for EKF to estimate the state $\mathbf{x}_k$, and generate 6 plots given at the next page.



(a) Position

(b) Measurement

(c) Estimation error e_x

(d) Estimation error e_y

(e) Estimation error e_θ

(f) Gaussian ellipsoid

Figure 3: EKF for SLAM

Problem 4 (Particle Filter for Localization) Consider the robot model discussed at Problem 1. Here we implement a particle filter as follows.

(a) Show that the state transition probability density function is given by

$$p(\mathbf{x}_{k+1}|\mathbf{x_k}, u_k) \sim \mathcal{N}(\begin{bmatrix} x_k + hV\cos\theta_k \\ y_k + hV\sin\theta_k \\ \theta_k + hu_k \end{bmatrix}, G_k Q_k G_k^T)$$

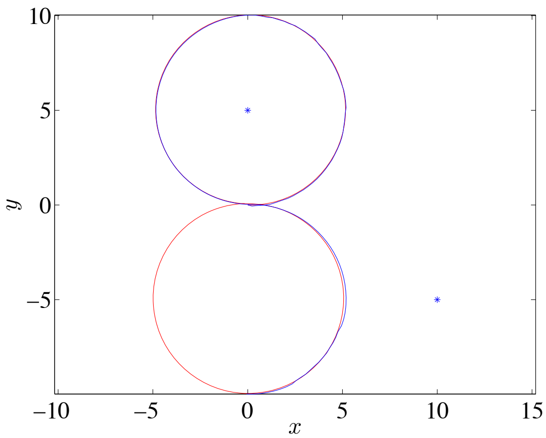
(b) Show that the sensor measurement probability density function for $\mathbf{z}^i = [z_r^i, z_\theta^i]^T \in \mathbb{R}^2$ is given by

$$p(\mathbf{z}^i|\mathbf{x}) = \mathcal{N}(\begin{bmatrix} \sqrt{\Delta x_i^2 + \Delta y_i^2} \\ \tan^{-1}\frac{\Delta y_i^{body}}{\Delta x_i^{body}} \end{bmatrix}, R_k)$$

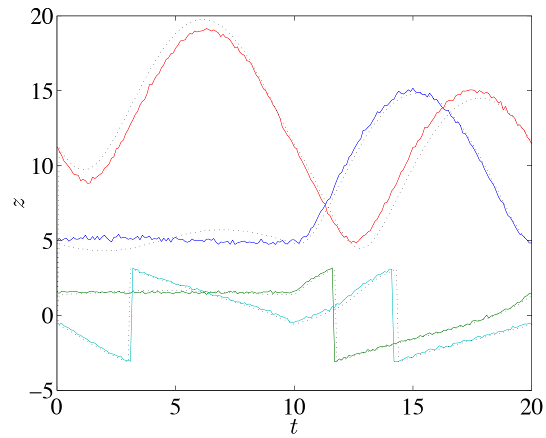
(c) All of the parameters for the robot model, control inputs, measurements, reference points, and initial guess are specified at `prob4.m`. This also includes Matlab codes for the prediction step of a particle filter. Complete the missing parts for the correction step, and 3 plots given at the next page.
(Hint: you may use the Matlab `mvnpdf` function to evaluate a multivariate Gaussian pdf).

(d) The mean value, `X_bar` becomes NaN when $t \geq 15$ or so. Discuss the reason for this.

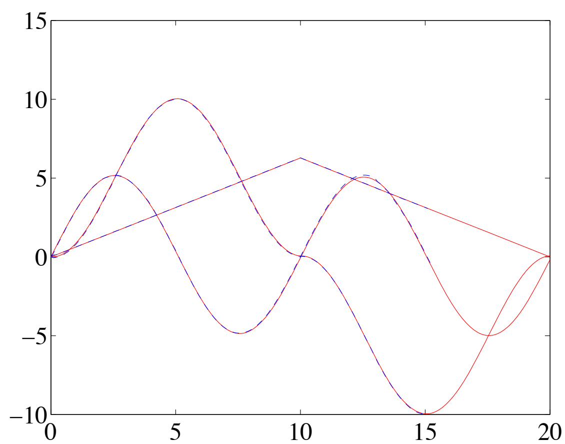
(e) Find, study, and summarize any resampling method for particle filters. Write a pseudo-code for the selected resampling algorithm.



(a) Position



(b) Measurement



(c) True state (red) and estimated state (blue)

Figure 4: Particle Filter for Localization