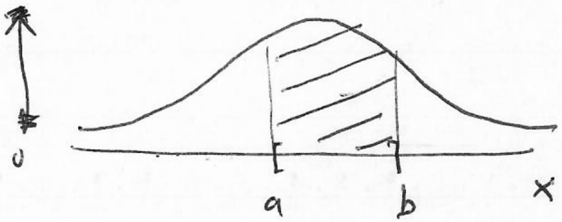
< Probability >

- random variable

  X : outcome $\rightarrow \mathbb{R}$

- density. $P[a \le X < b] = \int_a^b \underline{p(x)}\,dx$



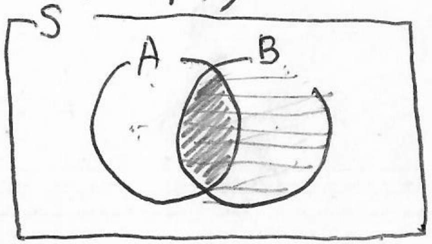
$p(x) \ge 0$

$\int_{-\infty}^{\infty} p(x)\,dx = 1$

- independent

  $p(A, B) = p(A)\,p(B)$

- cond. $p(A|B) = \frac{p(A, B)}{p(B)}$



* $p(A) = \frac{(\text{area of } A)}{(\text{'' } S)}$

$p(A, B) = p(A|B)\,p(B)$

$\quad = p(B|A)\,p(A)$.

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$p(A|B) = \frac{p(B|A)\,p(A)}{p(B)}$ Bayes rule

• Properties of r.v

- mode : value of X that max. $p(x)$
- median : 50% percentile

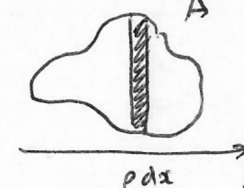
  $P[X \ge med] = P[X \le med] = 0.5$

- mean $E[\boxed{X}] = \bar{X} = \int_{-\infty}^{\infty} x\,p(x)\,dx$ (c.g of prob.

expected value

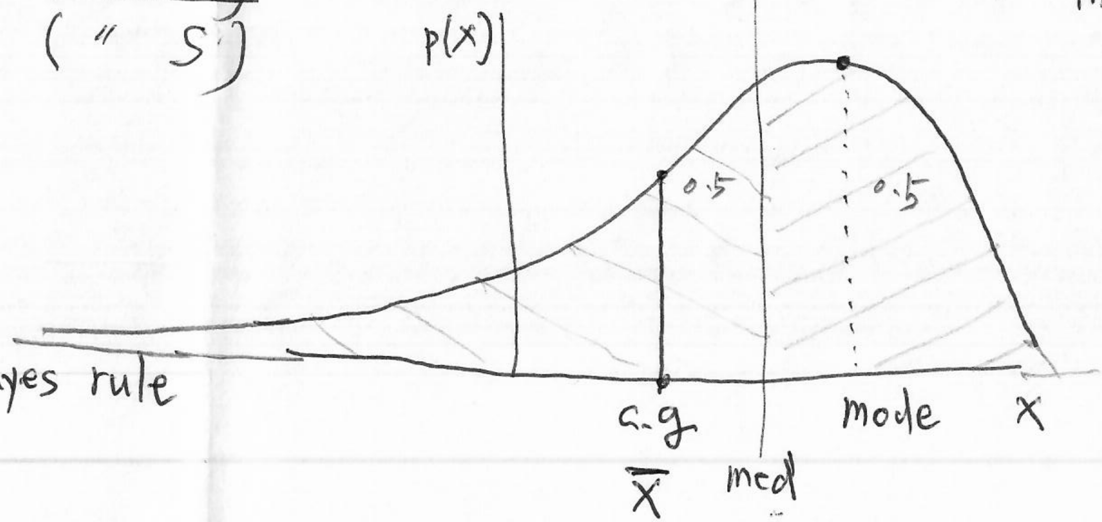
* $\int_A \rho(x)\,dx = $ mass

mass density, $\rho = \frac{ma}{are}$



$\int x\,\rho(x)\,dx = $ c.g

$\int (x - c.g)^2\,\rho\,dx = $ inerti



- variance.

$$V[X] = \int_{-\infty}^{\infty} \boxed{(x-\bar{X})^2}\, p(x)\, dx \qquad \left(\text{* moment of inertia}\right)$$

$$= E[(X-\bar{X})^2]$$

$X \in \mathbb{R}^n$ co-variance

$$Var[X] = E[(X-\bar{X})(X-\bar{X})^T]$$

* Mean is linear

$$E[aX+b] = \int_{-\infty}^{\infty} (aX+b)\, p(x)\, dx$$

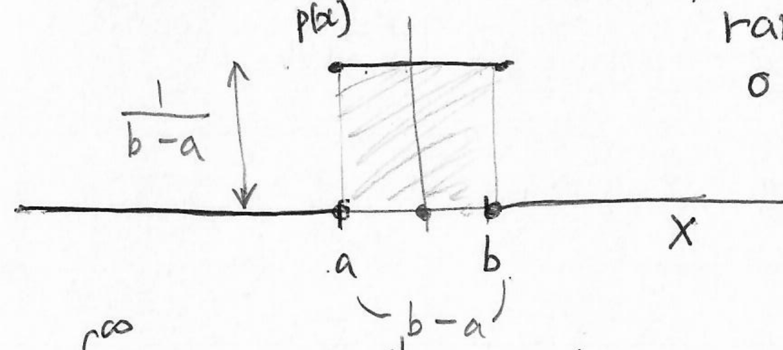
$$= a \boxed{\int_{-\infty}^{\infty} X\, p(x)\, dx} + b \int p(x)\, dx$$

$$= a\, E[X] + b$$

$$\boxed{\begin{aligned} Var[X] &= E[(X-\bar{X})(X-\bar{X})^T] \\ &= E[XX^T - X\bar{X}^T - \bar{X}X^T + \bar{X}\bar{X}^T] \\ &= E[XX^T] - E[X]\bar{X}^T - \cancel{\bar{X}E[X^T]} + \cancel{\bar{X}\bar{X}^T} \\ &= E[XX^T] - \bar{X}\bar{X}^T \end{aligned}}$$

ex) uniform dist on [a, b]

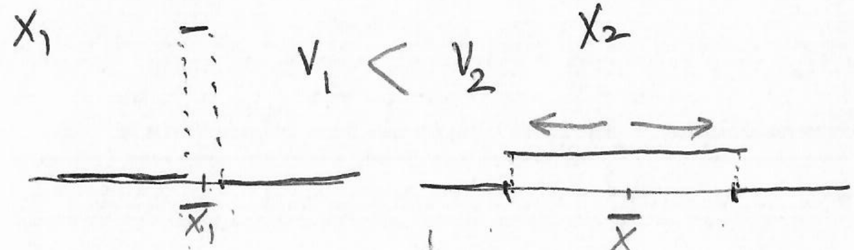
* rand 0, 1

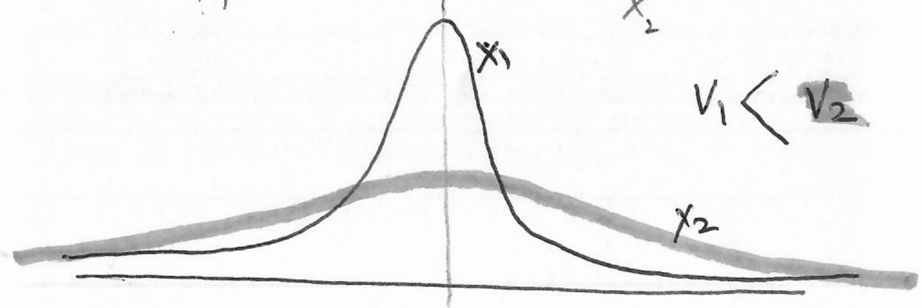


$$\bar{X} = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_a^b x \times \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \cdot \frac{x^2}{2} \Big|_a^b = \frac{b+a}{2}$$

$$V[X] = \frac{1}{3}\left(\frac{b-a}{2}\right)^3 = \frac{1}{3}\left(\frac{b-a}{2}\right)^3$$





covariance

$$cov(X,Y) = E[(X-\bar{X})(Y-\bar{Y})^T]$$
$$= E[XY^T] - \bar{X}\bar{Y}^T$$
$$cov(Y,X) = E[YX^T] - \bar{Y}\bar{X}^T$$
trans pos.

def. X, Y independent $p(X,Y) = p(X)p(Y)$

def. 〃 uncorellated if $cov(X,Y) = 0$

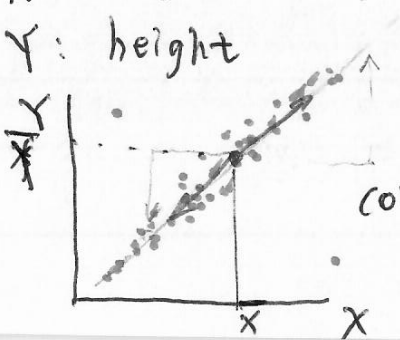
or $E[XY^T] = \bar{X}\bar{Y}^T$.

X, Y independent $\Rightarrow$ $E[X|Y] = E[X]$, $E[Y|X] = E[Y]$ $\Rightarrow$ $cov(X,Y) = 0$ un cor.

ave.) linear rel. btw X, Y

X: weight

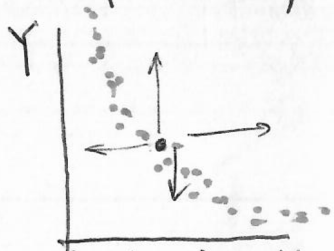
Y: height



$cov(X,Y) > 0$

X: alt

Y: density



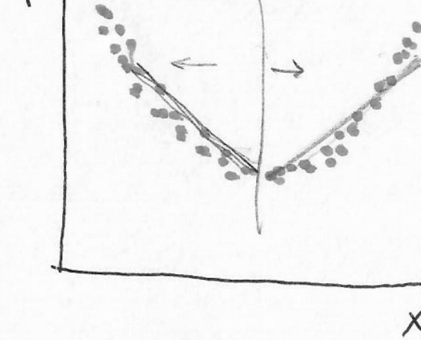
$cov(X,Y) < 0$

$(X-\bar{X})(Y-\bar{Y})$ $-$

$-$ $+$

$(X-\bar{X})(Y-\bar{Y})$ $+$

$+$ $+$

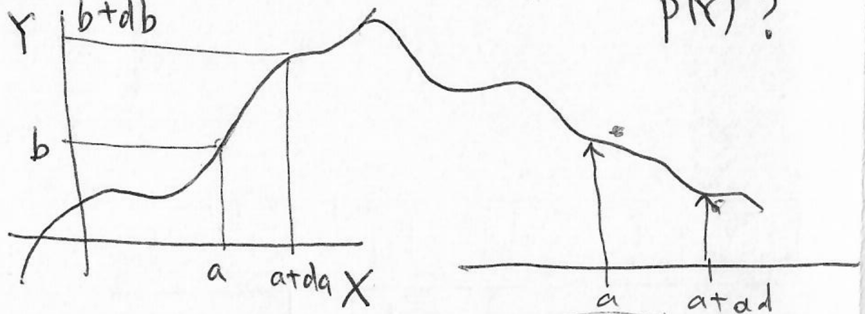


$cov(X,Y) = 0$.

- func. of r.v

Given X, $p(X)$ let $Y = f(X)$

$p(Y)$?



$$P[a \le X \le a + da] = P[b \le Y \le b + db]$$

$$p_X(x)\,dx = p_Y(y)\,dy$$

since $y = f(x)$ $\quad x = f^{-1}(y)$

$$dx = \frac{df^{-1}}{dy}\,dy$$

$$\boxed{p_Y(y)}\,dy = \boxed{p_X(f^{-1}(y)) \left|\frac{df^{-1}}{dy}\right|}\,dy$$

$X, Y \in \mathbb{R}^n$

$$\boxed{p_Y(y) = p_X(f^{-1}(Y)) \left|\det\left(\frac{df^{-1}}{dy}\right)\right|}$$

ex) $X \sim N(\mu, \Sigma)$

Quad. form $x^T Q x$

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left[-\frac{1}{2}(x-\mu)^T \boxed{\Sigma^{-1}} (x-\mu)\right]$$

$y = Ax$ $\qquad x = A^{-1}y$. $\qquad y$?

$$p_Y(y) = p_X(f^{-1}(Y)) \left|\det\left(\frac{df^{-1}}{\partial y}\right)\right|$$

$$= p_X(A^{-1}y)\,|\det(A^{-1})|$$

$$\propto \exp\left[-\frac{1}{2}(A^{-1}y-\mu)^T \Sigma^{-1} (A^{-1}y-\mu)\right]$$

$$[A^{-1}(y - A\mu)]^T$$

$$= \exp\left[-\frac{1}{2}(y - A\mu)^T \boxed{A^{-T}\Sigma^{-1}A^{-1}} (y - A\mu)\right]$$

$$\boxed{\begin{array}{l} Y \sim N(A\mu,\ A\Sigma A^T) \\ X \sim N(\mu, \Sigma) \end{array} \quad Y = AX}$$

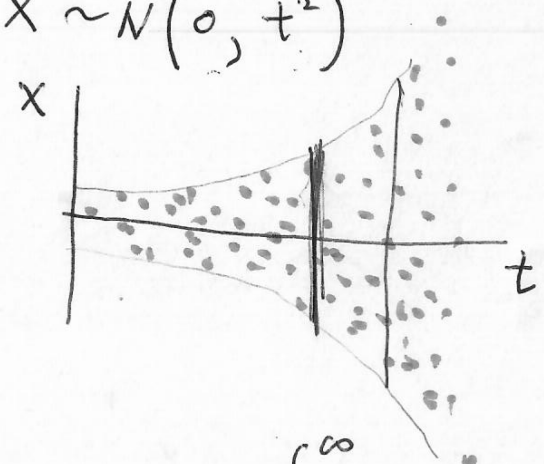
<Random Process>

det. X: (outcome) $\longrightarrow \mathbb{R}^n$ (r.v)

det. r.p. $X(t)$ is a family of r.v indexed by t.

ex) $X = tW$. $W \sim N(0, 1^2)$

$X \sim N(0, t^2)$



$$E[X(t)] = \bar{X}(t) = \int_{-\infty}^{\infty} x \, p_X(t, x) \, dx.$$

$Var[X(t)] =$ ....

$$Cov[X(t)Y(t)] = E\left[(X(t) - \bar{X}(t))(Y(t) - \bar{Y}(t))^T\right]$$

cross cov

$$E\left[(X(t) - \bar{X}(t))(Y(\tau) - \bar{Y}(\tau))^T\right]$$

def. stationary process.

$X(t)$ is stationary if $X(t)$ and $X(\tau)$ have the same statistics.

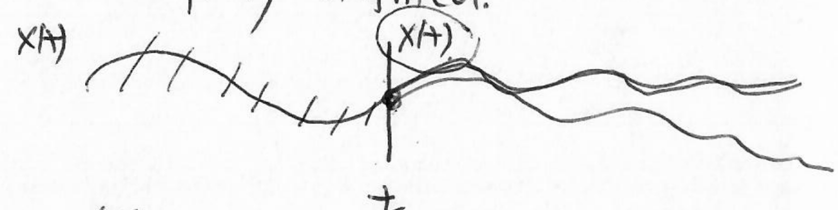
* density is invariant under time shift.

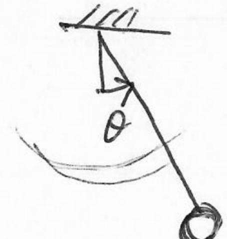
def. <u>Markov</u> process

* recall : state of a dyn. sys

$\dot{x} = f(t, x, u)$ └ control input

Given $X(t)$ at t, $u(t)$, the future behavior can be uniquely defined.





$[\theta, \dot{\theta}] = x.$

A random process. disc.

$X(0),\ X(1),\ \cdots\cdots\ X(N).$

is Markov if

$$\underbrace{p[X(k+1)}_{\text{future}} | \underbrace{X(k)}_{\text{current}}, \underbrace{X(k-1), \cdots X(0)}_{\text{past}}]$$

$$= p[X(k+1) | X(k)]$$

$$X_0 \to X_1 \to X_2 \cdots\cdots X_k \xrightarrow{p[X_{k+1}|X_k]} X_{k+1} \to \cdots$$

$p[X_N , X_{N-1}, \cdots, X_0]$

$\left({}^{*} p(A,B) = p(A|B) \times p(B) \right)$

$= p[X_N | X_{N-1} \cdots X_0]\ p[X_{N-1} \cdots X_0]$

$= p[X_N | X_{N-1}] \cdots\cdots$

$= p[X_N | X_{N-1}]\ p[X_{N-1}|X_{N-2}] \times \cdots \times p[X_1|X_0]\ p[X_0]$

state transition prob $p[X_{k+1} | X_k]$

< Parameter Estimation >

We wish to determine $x \in \mathbb{R}^n$ using

① prior guess $\quad x \sim N(\bar{x}, M)$

② sensor measurement

$$\underset{p\times 1}{z} = \underset{p\times n}{H}\ \underset{n\times 1}{x} + \underset{p\times 1}{v} \qquad H \in \mathbb{R}^{p\times n}$$

$v \sim N(0, R)$ (independent of x)

Objective : $x|z$

$$\begin{bmatrix} x \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} I_n & 0 \\ \underset{p\times n}{H} & I_p \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \end{bmatrix}$$

$\mathrm{cov}(x, v)$

$$\begin{bmatrix} x \\ v \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} M & 0 \\ 0 & R \end{bmatrix}}_{\Sigma} \right)$$

$$\begin{bmatrix} x \\ z \end{bmatrix} \sim N\left(A \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}, A \Sigma A^T \right)$$