MAE 6292: Midterm Exam

Due: 12:45pm, Friday March 27

, .

Last Name First Name Student ID

Prob. 1	Prob. 2	Prob. 3	Prob. 4	Total

Honor Pledge

I pledge that I have neither given nor received unauthorized assistance on this work. I understand that any academic misconduct will be handled according to *GWU Code of Academic Integrity*.

Signature Date

Problem 1 (Parameter Optimization) We wish to determine the location of a point in a three-dimensional space by observing it with multiple cameras. (For example, consider a VICON motion capture system.) Let $x \in \mathbb{R}^3$ be the location of the point, and let the image of the point on the i-th camera be located at $y_i \in \mathbb{R}^2$ at the image plane of the camera. They are related as

$$y_i = f_i(x) = \frac{1}{c_i^T x + d_i}(A_i x + b_i),$$

where $A_i \in \mathbb{R}^{2\times 3}$, $b_i \in \mathbb{R}^{2\times 1}$, $c_i \in \mathbb{R}^{3\times 1}$, and $d_i \in \mathbb{R}$ are sub-matrices of the *camera matrix*, $P_i \in \mathbb{R}^{3\times 4}$ defining the location, orientation and properties of the i-th camera,

$$P_i = \begin{bmatrix} A_i & b_i \\ c_i^T & d_i \end{bmatrix}.$$

(Google `camera matrix` for the formal definition of the camera matrix in computer vision.)

Suppose that the point is measured by N cameras. Due to various errors, such as imperfect camera calibration, we do not expect that the measured data y_i is exactly same as $f_i(x)$. To estimate the location x, we therefore minimize the weighted sum of the residual errors,

$$\mathcal{J}(x) = \sum_{i=1}^{N} w_i \|y_i - f_i(x)\|^2,$$

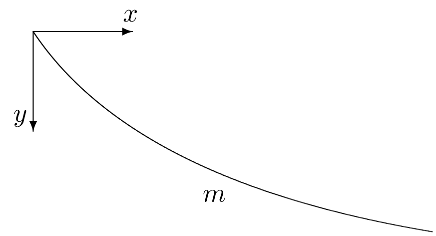
where $w_i \in \mathbb{R}$ are the weighting-parameters representing the degree of accuracy of the i-th camera.

(a) Derive the necessary conditions for optimality.

(b) For $N = 4$, the camera matrices P_i, the measurement y_i, and the weighting parameters w_i are stored at the Matlab MAT file, `prob1_data.mat`. Write a Matlab code to minimize the above cost function via the steepest descent method discussed in class.

(c) Find the optimal estimate $x^\star$, and the corresponding optimal cost $J(x^\star)$.

Problem 2 (Trajectory Optimization) Consider a mass particle m moving along a curve under the influence of the gravity.



The curve is defined by $y(x)$ as illustrated above, and it passes through the following two end points,

$$y(0) = 0, \quad y(x_f) = y_f, \tag{1}$$

for some positive constants x_f, y_f. We wish to find the optimal shape of the curve such that the mass particle reaches the terminal point (x_f, y_f) in a shortest time when released from the initial point $(0, 0)$ with zero velocity. Assume that there is no friction.

(a) From the conservation of the total energy, show that the velocity of the mass is given by $v(y) = \sqrt{2gy}$, where g denotes the gravitational acceleration.

(b) Using the facts that the infinitesimal distance travelled is $ds = \sqrt{dx^2 + dy^2}$ and $v = \frac{ds}{dt}$, show that the cost function can be written as

$$J(y, y') = \int_0^{t_f} \sqrt{\frac{1 + y'^2}{2gy}}\, dx,$$

where $y' = \frac{dy}{dx}$.

(c) Show that the following quantity is preserved along the optimal trajectory,

$$H(y, y') = y' \frac{\partial L(y, y')}{\partial y'} - L,$$

i.e., show $\frac{dH}{dt} = 0$.

(d) Show that the conservation of $H(y, y')$ implies

$$y(1 + y'^2) = c,$$

for some positive constant c.

(e) Show that the above equation has the following parametric solution,

$$x(\theta) = \frac{1}{2}c(\theta - \sin\theta), \quad y(\theta) = \frac{1}{2}c(1 - \cos\theta),$$

for $\theta \in [0, \theta_f]$.

(f) The constants c, θ_f can be determined by the boundary condition (1). Find c, θ_f when $(x_f, y_f) = (1, 1)$, and plot the corresponding optimal curve via Matlab
(You may use the Matlab function `solve` to find c and θ_f. Type `set(gca,'Ydir','reverse');axis equal;` after plotting $y(x)$.)

(g) The optimal curve is referred to as *cycloid*. Describe the geometric meaning of cycloids.

Problem 3 (Optimal Control) A boat is traveling in a river with a strong current, which makes the boat drift along the x_1 direction. It is moving with a constant speed v, and the control input corresponds to the steering angle θ. The equations of motion are given by

$$\dot{x}_1 = v\cos\theta + c,$$
$$\dot{x}_2 = v\sin\theta,$$

where $c \in \mathbb{R}$ represents the effects of the current.

We wish to control the steering angle such that the boat travels from $(x_1(0), x_2(0)) = (0, 0)$ to the shoreline of an island in a shortest time. The shoreline is described by the circle $S_f = \{x \in \mathbb{R}^2 \mid \|x - a\|^2 = r^2\}$ for $a = [a_1, a_2] \in \mathbb{R}^2$ and $r > 0$.

(a) Formulate this as an optimal control problem. Derive the optimal state equations and the co-state equations.

(b) Show that the optimal heading angle is fixed.

(c) Derive *algebraic* equations to determine the initial multiplier $(\lambda_{1_0}, \lambda_{2_0})$, the terminal time t_f, and the multiplier μ for the terminal state constraint. (You DO NOT have to solve these equations).

Problem 4 (Optimal Control) The dynamics of a spacecraft around the Earth are described by

$$\ddot{\mathbf{r}} = -\frac{1}{r^3}\mathbf{r} + u,$$

where $\mathbf{r} \in \mathbb{R}^2$ denotes the position of the spacecraft from the center of the Earth, and $r = \|\mathbf{r}\| \in \mathbb{R}$. The control thrust of the spacecraft is denoted by $u \in \mathbb{R}^2$. (Note that the units are normalized such that the gravitational parameter becomes one). Throughout this question, it is assumed that the spacecraft remains in a two-dimensional orbital plane. Let $\mathbf{r} = [x_1, x_2]$, $\dot{\mathbf{r}} = [x_3, x_4]$, and $u = [u_1, u_2] \in \mathbb{R}^2$. The equations of motion can be rewritten as

$$\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_3 &= -\frac{1}{r^3}x_1 + u_1, \\
\dot{x}_4 &= -\frac{1}{r^3}x_2 + u_2,
\end{aligned}$$

where $r = \sqrt{x_1^2 + x_2^2}$.

Initially, the spacecraft is on a circular orbit of the radius 1 with $x_0 = [1, 0, 0, 1]$. We wish that the spacecraft intercept with another spacecraft at $x_f = [-2, 0, 0, -\frac{1}{\sqrt{2}}]$ on a larger circular orbit when $t_f = \pi$, while minimizing the following cost,

$$\mathcal{J} = \frac{1}{2}\int_0^{\pi} (u_1^2 + u_2^2)\, dt.$$

Here, we solve this optimal control problem according to the quasi-linearization method discussed in class, as follows.

(a) The corresponding Hamiltonian is given by

$$H(x, \lambda, u) = \frac{1}{2}(u_1^2 + u_2^2) + \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3(-\frac{1}{r^3}x_1 + u_1) + \lambda_4(-\frac{1}{r^3}x_2 + u_2).$$

Find the optimal control as a function of x, λ, and substitute it back to H to obtain the *reduced* Hamiltonian,

$$H(x, \lambda) = -\frac{1}{2}(\lambda_3^2 + \lambda_4^2) + \lambda_1 x_3 + \lambda_2 x_4 - \lambda_3 \frac{1}{r^3}x_1 - \lambda_4 \frac{1}{r^3}x_2.$$

(b) The derivatives of the reduced Hamiltonian can be obtained by using the following Matlab commands:

```
syms x1 x2 x3 x4 l1 l2 l3 l4;

r=sqrt(x1^2+x2^2);
H=-1/2*(l3^2+l4^2)+l1*x3+l2*x4-l3*1/r^3*x1-l4*1/r^3*x2;

x=[x1 x2 x3 x4];
l=[l1 l2 l3 l4];
Hx=simplify(jacobian(H,x));
Hl=simplify(jacobian(H,l));
for i=1:4
    Hxx(i,:)=simplify(jacobian(Hx(i),x));
    Hxl(i,:)=simplify(jacobian(Hx(i),l));
    Hlx(i,:)=simplify(jacobian(Hl(i),x));
    Hll(i,:)=simplify(jacobian(Hl(i),l));
end
```

The results are copy/pasted to the Matlab file, `prob4_H_deriv.m`, which can be used to compute the derivatives of the Hamiltonian for given $(x(t), \lambda(t))$. Using this, compose another Matlab function, namely `[A e]=eomLin(x, lambda)` to compute the matrices $A(t) \in \mathbb{R}^{8\times 8}, e(t) \in \mathbb{R}^8$ for the linearized equations of motion for given $(x(t), \lambda(t))$.

(c) In this problem, when enforcing the boundary conditions, the initial multiplier should be chosen such that the terminal state satisfies $x(t_f) = x_f$. Find the expression of the initial multiplier to satisfy the terminal state, in terms of $\Phi_{x\lambda}^{-1}(t_f, t_0)$, $\Phi_{xx}(t_f, t_0)$, x_f, x_0, and $p(t_f)$.

(d) Using the above results, write a Matlab file to perform quasi-linearization. Plot the resulting optimal $x(t), \lambda(t), u(t)$, and show the values of the converged λ_0.
(Note: the initial guess of $x(t)$ should be chosen such that $r \neq 0$ at any t.)