MAE6292: Homework 1

Due date: January 31, 2020

Problem 1 Consider the following linear equation,

$$Ax = y,$$

where $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$.

(a) Suppose $m \geq n$, and $\mathrm{rank}(A) = n$. In this case, the above equation is overdetermined, i.e., in general, there is no x satisfying the above equation for a given y. Find the best approximate solution x that minimizes the error, $\|Ax - y\|^2 = (y - Ax)^T(y - Ax)$. Answer in terms of y and A.

(b) Suppose $n \geq m$, and $\mathrm{rank}(A) = m$. In this case, the above equation is underdetermined, i.e., there are infinite number of x satisfying the above equation for a given y. Find the particular solution x that has the smallest value of $\|x\|^2 = x^T x$. Answer in terms of y and A.

Problem 2

Consider a quadratic cost function,

$$J(x, u) = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $Q \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times m}$. Assume that Q is symmetric and positive-semidefinite, and R is symmetric and positive-definite. There is an equality constraint given by

$$f(x, u) = x + Bu + c = 0,$$

where $B \in \mathbb{R}^{n\times m}$ and $c \in \mathbb{R}^n$.

(a) Derive the conditions for a stationary point, i.e., necessary condition for local minimum.

(b) Show that the optimal control, state and multiplier values are given by

$$\begin{aligned}
u &= -(R + B^T QB)^{-1}B^T Qc,\\
x &= -(I - B(R + B^T QB)^{-1}B^T Q)c,\\
\lambda &= (Q^{-1} + BR^{-1}B^T)^{-1}c.
\end{aligned}$$

(You may have to use the matrix inversion lemma, `http://en.wikipedia.org/wiki/Woodbury_matrix_identity`).

(c) Suppose

$$Q = \begin{bmatrix} 3 & 1 & 1\\ 1 & 2 & 0\\ 1 & 0 & 1 \end{bmatrix}, \quad R = I_{2\times 2}, \quad B = \begin{bmatrix} 1 & 1\\ 1 & 0\\ 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1\\ -1\\ 0 \end{bmatrix}.$$

Write a Matlab code to find the optimal x, u, and λ using the steepest gradient method iteratively.

(d) Numerically compare your solution of (c) with (b).

Problem 3 Consider the following optimization problem,

$$\text{minimize } J(x) = -\sum_{i=1}^{n} \log(\alpha_i + x_i), \tag{1}$$

$$\text{subject to} \sum_{i=1}^{n} x_i = 1, \text{ and } x_i \geq 0, \text{ for all } 1 \leq i \leq n, \tag{2}$$

where the constants $\alpha_i > 0$ are given. It arises in information theory, to allocate the transmitter power x_i with the fixed sum to n channels, in an optimal fashion to maximize the total communication rate.

(a) Derive the KKT conditions for optimality.

(b) Solve the condition for stationary point for the multiplier μ_i of inequality constraints, and substitute it to the KKT conditions to rewrite them in terms of x_i, λ, α_i only.

(c) Show that the optimal state is given by

$$x_i = \max\{0, \frac{1}{\lambda} - \alpha_i\}.$$

(d) Suppose $n = 3$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$. Find the values of the optimal states Wx_i and multipliers λ, μ_i.