MAE6292: Homework 3

Due date: February 28, 2020

Problem 1 In class, we covered an optimal orbital transfer of a spacecraft. Let the location of the spacecraft be defined by (r, α) in polar coordinates. The state vector is defined as $x = [r, \alpha, \dot{r}, r\dot{\alpha}]^T \in \mathbb{R}^4$. The equations of motion are given by

$$
\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= \frac{x_4}{x_1}, \\
\dot{x}_3 &= \frac{x_4^2}{x_1} - \frac{gR^2}{x_1^2} + \frac{T}{m}\sin u, \\
\dot{x}_4 &= -\frac{x_3}{x_4}x_1 + \frac{T}{m}\cos u.
\end{aligned}
$$

The control input corresponds to the angle of the fixed thrust from the local horizontal direction.

An optimal control problem to transfer the spacecraft from a surface of the Earth with the radius R to a circular orbit with the altitude D in a minimum time is considered in class, and we showed that the boundary conditions to solve the optimality conditions and to determine the terminal time are given by

$$
x_1(t_f) = R + D, \quad \lambda_2(t_f) = 0, \quad x_3(t_f) = 0, \quad x_4(t_f) = \sqrt{\frac{gR^2}{R+D}}, \quad H(t_f) = 0.
$$

In this problem, we derive boundary conditions for other optimal spacecraft missions.

(a) In this mission, the spacecraft is launched from $x(0) = [R, 0, 0, 0]^T$ and it is to rendezvous with another spacecraft that is on a fixed circular orbit with its altitude D. The orbital period of the target spacecraft is 2 hours, and both spacecraft are on the reference axis of $\alpha = 0$ initially when $t = 0$. This implies that the terminal state should satisfy

$$
\psi(t_f, x_f) = x_f - [R + D,\ \frac{\pi}{3600}t_f,\ 0,\ \frac{\pi}{3600}(R+D)]^T = 0_{4\times 1}.
$$

Find a boundary condition to determine the final time t_f.

(b) In this mission, the spacecraft should be transferred to a circle centered at $(R+E)(\cos\gamma, \sin\gamma)$ with its radius C, i.e., the terminal state constraint is given by

$$
\psi(t_f, x_f) = (x_1\cos x_2 - (R+E)\cos\gamma)^2 + (x_1\sin x_2 - (R+E)\sin\gamma)^2 - C^2 = 0.
$$

In addition to this, derive five more equations to determine four boundary conditions, one multiplier ν, and the terminal time t_f.

Problem 2 In class, we studied linear quadratic regulators. Here, we generalize it into linear quadratic *tracking* problems. Consider a linear dynamics given by

$$
\dot{x} = Ax + Bu.
$$

The cost function is defined as

$$
J = \frac{1}{2}(x(t_f) - r(t_f))^T Q_f (x(t_f) - r(t_f)) + \frac{1}{2}\int_{t_0}^{t_f} (x(t) - r(t))^T Q(t)(x(t) - r(t)) + u(t)^T R(t) u(t)\, dt,
$$

where $r(t) : \mathbb{R} \to \mathbb{R}^n$ is a reference trajectory to be followed.

(a) Derive necessary conditions for optimality to obtain

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ Qr \end{bmatrix},$$
$$u = -R^{-1}B^T\lambda.$$

(b) Show that the terminal boundary conditions are given by $\lambda(t_f) = Q_f(x(t_f) - r(t_f))$.

(c) We can show that the optimal state and the multiplier satisfying the above necessary conditions satisfy

$$\lambda = P(t)x + s(t),$$

for some $P(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $s(t) : \mathbb{R} \to \mathbb{R}^n$. Substitute it into the optimality condition, and derive differential equations that should be satisfied for $P(t)$ and $s(t)$. Also, find the boundary conditions for $P(t)$ and $s(t)$.
(Hint: the differential equation for $P(t)$ is identical to the Riccati equation for LQR).

(d) Suppose

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the cost function is given by

$$J = \frac{1}{2}\int_0^{20} (x_1 - \sin\frac{\pi}{2}t)^2 + 0.01u^2\,dt.$$

Numerically simulate the linear quadratic trackers derived above, and plot the corresponding optimal $x(t), r(t), u(t)$ and $K(t), s(t)$.