Complex Analysis

URL: https://www.youtube.com/playlist?list=PLMrJAkhIeNNQBRslPb7I0yTnES981R8Cg

4. Complex Logarithm

$$z = x + iy$$
$$= R[\cos(\theta) + i\sin(\theta)]$$
$$= Re^{i\theta}$$

$$z = x + iy$$
$$w = u + iv$$
$$w = \log(z)$$

The inverse pair of w is $z = e^w$.

$$z = e^{u+iv} = e^u e^{iv}$$
$$= e^u[\cos(v) + i\sin(v)]$$

By applying polar coordinate form:

$$R = e^u \to u = \log(R) := \log(|z|)$$
$$v = \theta := \angle z$$

On the z-plane, the complex number of z is a point with multiple times of $2\pi n$. However, on w-plane the complex logarithm of z, $w = \log(z)$ can have infinite number of points on the line of $\log(R)$.

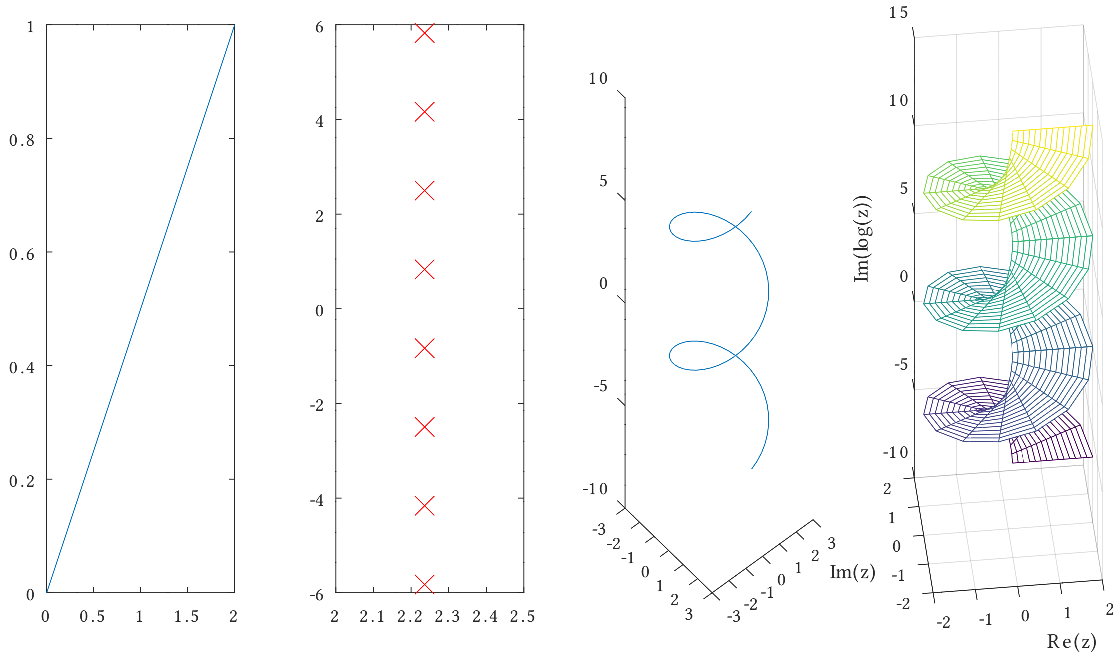
$v = \theta_p + 2\pi n := \angle z$ true for all integer n.

This idea comes down to as follows:

$$u = \log(|z|)$$
$$v = \theta_p + 2\pi n$$
$$w = u + iv$$
$$= \log(|z|) + i(\theta_p + 2\pi n)$$

Replace w as $\log(z)$ then,

$\log(z) = \log(|z|) + i(\theta_p + 2\pi n)$ for all integer n.



5. Roots of Unity and Rational Powers of z

$\log(z) = \log(|z|) + i(\theta_p + 2n\pi) n \in \mathbb{Z}$

z^a

$z = e^{\log(z)} \Rightarrow z^a = e^{a \log(z)}$

Rational $a = \frac{m}{n} \in \mathbb{Q} \subset \mathbb{R} \quad m, n \in \mathbb{Z}$

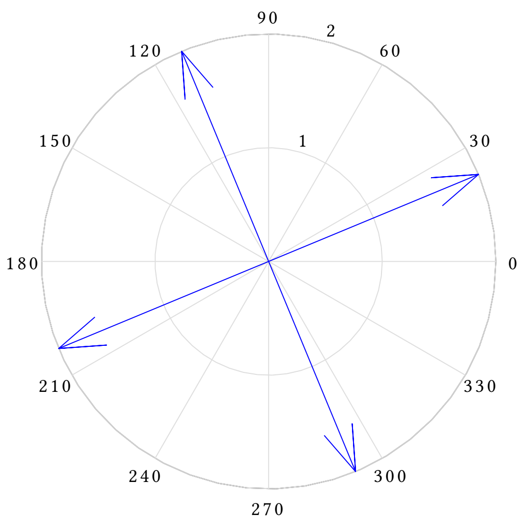
$$
\begin{aligned}
Z^{m/n} &= e^{m/n \log(Z)} \\
&= e^{\frac{m}{n}\log(R) + i\frac{m}{n}(\theta_p + 2\pi k)} \\
&= e^{\frac{m}{n}\log(R)} e^{i\frac{m}{n}\theta_p} e^{i\frac{m}{n}2\pi k}
\end{aligned}
$$

E.g. $\sqrt[4]{16i}$

$16i = 16e^{i\frac{\pi}{2}}$

$$
\begin{aligned}
16i^{\frac{1}{4}} &= 16^{\frac{1}{4}} e^{i\frac{\pi}{8}} e^{i\frac{2\pi k}{4}} \\
&= 2e^{i\frac{\pi}{8}} e^{i\frac{\pi}{2}k}
\end{aligned}
$$

k	k-th root
0	$2e^{i\frac{\pi}{8}}$
1	$2e^{i\frac{\pi}{8}} e^{i\frac{\pi}{2}} \to 2e^{i\frac{5\pi}{8}}$
2	$2e^{i\frac{\pi}{8}} e^{i\pi} \to 2e^{i\frac{9\pi}{8}}$
3	$2e^{i\frac{\pi}{8}} e^{i\frac{3\pi}{2}} \to 2e^{i\frac{13\pi}{8}}$



6. Analytic Functions and Cauchy-Riemann Condition

Keyword: well-behaved, analytic function

Analytic Function

A function $f(z)$ is **analytic** in a domain $\mathbb{D} \subset \mathbb{C}$ if $f(z)$ is single valued and has a finite derivative $f'(z)$ for every $Z \in \mathbb{D}$.

- Path Independent $\to$ derivative must be same in any direction

Non Analytic Function

$f(z) = \overline{z} = x - iy \to$ no derivative exist!

$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\overline{\Delta Z}}{\Delta Z}$

$= \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$

Case 1. Approach $\Delta z \to 0$ from real axis: $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

Case 2. Approach $\Delta z \to 0$ from imag axis: $\lim_{\Delta y \to 0} \frac{-i\Delta y}{\Delta y} = -1$

- Path Dependent

Examples of Analytic & Non-Analytic Functions

Analytic $f(z)$: Polynomial(z^n), functions w/ Taylor series ($e^z, \cos(z), \sin(z), \log(z), \ldots$)

Non-analytic $\overline{z}$: cliff, cusp, ...

Cauchy-Riemann for Analytic function

- For a function to be analytic the derivative must be the same for the two paths on real & imag axes.

$f(z) = u(x, y) + iv(x, y)$

$z = x + iy$

$\frac{df}{dz} = \lim_{\begin{cases} \Delta x \to 0 \\ \Delta y \to 0 \end{cases}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$

(1) Approach from real axis: $\Delta y = 0$

$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

(2) Approach from imag axis: $\Delta x = 0$

$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y}$

$= \frac{i(\Delta u + i\Delta v)}{i \cdot i\Delta y}$

$= \frac{\Delta v - i\Delta u}{\Delta y}$

$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

As a result, the real part of (1) must be equal to real part of (2). The imag part of (1) must be equal to image part of (2).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Furthermore, if a function $f(z)$ is analytic, the real and imag part satisfies Laplace's equation.

$\nabla^2 u = 0, \nabla^2 v = 0$ u and v are harmonic functions.

Example: Polynomial Z^n

$f(z) = z^2$

$= (x + iy)^2$

$= x^2 - y^2 + i2xy \to u = x^2 - y^2, v = 2xy$

(1) Real Axis

$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y = 2z$

(2) Imag Axis

$\frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial 2xy}{\partial y} - i\frac{\partial(x^2 - y^2)}{\partial y} = 2x + i2yi = 2z$

7. Analytic Functions Solve Laplace's Equation

Analytic Function:

- single valued function
- derivative exist (path independent)
- Real and Imag part satisfy Laplace Equation
- harmonic function

$f(z) = u(x, y) + iv(x, y), z = x + iy$

If $f(z)$ is analytic and u, v both twice differentiable, then

(1) check u

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$

$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$

$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \nabla^2 u = 0$

u is a harmonic function!

(2) check v

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial}{\partial y} \Rightarrow \frac{\partial u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y^2}$

$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow \nabla^2 v = 0$

v is a harmonic function!

Cauchy-Riemann Condition in Polar Coordinates

$f(z) = u(R, \theta) + iv(R, \theta)$ where $z = Re^{i\theta}$

Chain Rule Method

Total differential of z

$dz = \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta$

Radial Direction: if θ is a constant (θ not change), $d\theta = 0$:

$dz = \frac{\partial z}{\partial r} dr \to \frac{\partial z}{\partial r} = \frac{dz}{dr} = \frac{d}{dr} re^{i\theta} = e^{i\theta}$
$\frac{dz}{dr} = e^{i\theta} \to \frac{dr}{dz} = \frac{1}{e^{i\theta}}$

$$\begin{aligned} f'(z) &= \frac{\partial f}{\partial r}\frac{dr}{dz} \\ &= e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \end{aligned}$$

Tangential Direction: if r is a constant (r not changed), $dr = 0$:

$dz = \frac{\partial z}{\partial \theta} d\theta \to \frac{\partial z}{\partial \theta} = \frac{dz}{d\theta} = \frac{d}{d\theta} re^{i\theta} = rie^{i\theta}$

$$\begin{aligned} \frac{dz}{d\theta} = rie^{i\theta} \to \frac{d\theta}{dz} &= \frac{1}{rie^{i\theta}} \\ &= \frac{-ie^{-i\theta}}{r} \end{aligned}$$

$$\begin{aligned} f'(z) &= \frac{\partial f}{\partial \theta}\frac{d\theta}{dz} \\ &= \frac{-ie^{-i\theta}}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) \\ &= \frac{e^{-i\theta}}{r}\left(-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right) \end{aligned}$$

Equate the two $f'(z)$ in both directions:

$e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{e^{-i\theta}}{r}\left(-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right)$

Divide both sides by $e^{-i\theta}$:

$$\begin{aligned}\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} &= \frac{1}{r}\left(-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right)\\ &= \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)\end{aligned}$$

Finally, Cauchy-Riemann Equation as follows:

Real part: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

Imag part: $\frac{\partial v}{\partial r} = -\frac{i}{r}\frac{\partial u}{\partial \theta}$

Limit Method (ref. Gemini)

The standard definition of the complex derivative:

$f'(z) = \lim_{\Delta z \longrightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

For a complex function to be differentiable, this limit must be the same regardless of the direction in which Δz approaches to zero.

Path 1. The radial limit: a small amount of change in radial direction at z $(\Delta\theta = 0)$

- $z = re^{i\theta}$
- $$\begin{aligned} z + \Delta z &= (r + \Delta r)e^{i(\theta + \Delta\theta)}\\ &= (r + \Delta r)e^{i\theta}\end{aligned}$$
- $$\begin{aligned}\Delta z &= -re^{i\theta} + (r + \Delta r)e^{i\theta}\\ &= (-r + r + \Delta r)e^{i\theta}\\ &= \Delta r e^{i\theta}\end{aligned}$$

The limit becomes:

$$\begin{aligned} f'(z) &= \lim_{\Delta z \longrightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}\\ &= \lim_{\Delta r \to 0} \frac{f\left((r+\Delta r)e^{i\theta}\right) - f\left(re^{i\theta}\right)}{\Delta r e^{i\theta}}\\ &= \frac{1}{e^{i\theta}}\left[\lim_{\Delta r \to 0} \frac{f(r+\Delta r, \theta) - f(r, \theta)}{\Delta r}\right]\\ &= e^{-i\theta}\frac{\partial f}{\partial r}\\ &= e^{-i\theta}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r}\right)\end{aligned}$$

Path 2. The tangential limit $(\Delta r = 0)$

- $z = re^{i\theta}$
- $z + \Delta z = re^{i(\theta + \Delta\theta)}$
- $$\begin{aligned}\Delta z &= -z + re^{i(\theta+\Delta\theta)}\\ &= -re^{i\theta} + re^{i(\theta+\Delta\theta)}\\ &= -re^{i\theta} + re^{i\theta} \cdot e^{i\Delta\theta}\\ &= \left(e^{i\Delta\theta} - 1\right)re^{i\theta}\end{aligned}$$

Using Taylor Series for $e^{i\Delta\theta}$:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$e^{i\Delta\theta} = 1 + (i\Delta\theta) + \frac{(i\Delta\theta)^2}{2!} + \frac{(i\Delta\theta)^3}{3!} + \ldots$

We can drop the higher terms as $\Delta\theta \to 0$.

$e^{i\Delta\theta} \approx 1 + i\Delta\theta$
$\Delta z = \left(e^{i\Delta\theta} - 1\right)re^{i\theta}$
$\quad = (1 + i\Delta\theta - 1)re^{i\theta}$
$\quad = ire^{i\theta}\Delta\theta$

The limit becomes:

$$\begin{aligned}
f'(z) &= \lim_{\Delta z \longrightarrow 0} \tfrac{f(z+\Delta z)-f(z)}{\Delta z} \\
&= \lim_{\Delta\theta\to 0} \tfrac{f\left(re^{i(\theta+\Delta\theta)}\right)-f\left(re^{i\theta}\right)}{ire^{i\theta}\Delta\theta} \\
&= \tfrac{1}{ire^{i\theta}}\left[\lim_{\Delta\theta\to 0} \tfrac{f(r,\theta+\Delta\theta)-f(r,\theta)}{\Delta\theta}\right] \\
&= \tfrac{1}{ire^{i\theta}}\tfrac{\partial f}{\partial\theta} \\
&= \tfrac{-ie^{-i\theta}}{r}\tfrac{\partial f}{\partial\theta} \\
&= \tfrac{-ie^{-i\theta}}{r}\left(\tfrac{\partial u}{\partial\theta} + i\tfrac{\partial v}{\partial\theta}\right) \\
&= \tfrac{e^{-i\theta}}{r}\left(-i\tfrac{\partial u}{\partial\theta} + \tfrac{\partial v}{\partial\theta}\right)
\end{aligned}$$

If we equate the two $f'(z)$ in both directions, we can get the same Cauchy-Riemann equation with Chain-Rule.

Verify that $\log(z)$ is analytic away from $z = 0$ using Cauchy-Riemann condition with polar form.
$\log(z) = \log(R) + i\theta$
$u(R,\theta) = \log(R)$
$v(R,\theta) = \theta$

Clearly, $u_\theta = 0$ because u is a function of R and v_R is zero because v is a function of θ.

Real part: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}$
$\frac{\partial u}{\partial R} = \frac{1}{\partial R}\log(R) = \frac{1}{R}$
$\frac{\partial v}{\partial\theta} = \frac{1}{\partial\theta}\theta = 1$
$\frac{\partial u}{\partial R} = \frac{1}{r}\frac{\partial v}{\partial\theta}$ becomes $\frac{1}{R} = 1\frac{1}{R}$
Imag part: $\frac{\partial v}{\partial r} = -\frac{i}{r}\frac{\partial u}{\partial\theta}$
$\frac{\partial v}{\partial R} = \frac{\partial}{\partial R}\theta = 0$
$\frac{\partial u}{\partial\theta} = \frac{\partial}{\partial\theta}R = 0$

Complex Functions(Analytic)

Theorem 1. A function $f(z) = u + iv$ that is
- single valued, and
- has continuous $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

in a domain $\mathbb{D} \subset \mathbb{C}$ is analytic in $\mathbb{D}$ *iff* the Cauchy-Riemann conditions are satisfied at every point $z \in \mathbb{D}$.

Theorem 2. If a $f(z)$ is analytic at z then $f(z)$ has continuous derivatives of all orders! $f'(z), f''(z), \ldots f^{15}(z), \ldots$

The main application: $f(z)$ is analytic at z_0 *iff* its Taylor series exists and converges to $f(z)$ in a neighborhood of z_0.

8. Integrals in the Complex Plane (1/8/2025 Thu)

Given a function $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ then
$\oint f(z) = \oint(u + iv)(dx + idx) = \oint[(udx - vdy) + i(udy + vdx)]$

Cauchy-Goursat Theorem:

If $f(z)$ is analytic inside a simple closed curve $C \subset \mathbb{C}$ then $\oint f(z)dz = 0$.

By Green's Theorem:

$$\oint (Pdx + Qdy) = \iint\limits_{D} \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right) dA$$

$$\oint [(udx - vdy) + i(udy + vdx)] = \iint\limits_{S} \left(-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy + \iint\limits_{S} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$
$$= 0$$

By Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \text{ and } -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Fundamental Theorem of Complex Calculus: If f is analytic in $\mathbb{D}$ and $z_0, z_1 \in \mathbb{D}$ then,

$$\int\limits_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0) = 0$$

Integrals in the Complex Plance

Given a function $f(z) = u(x, y) + iv(x, y)$ then

$$\int\limits_{C} f(z)dz = \int\limits_{C} [(udx - vdy) + i(udy + vdx)]$$

$$\int\limits_{C_1} f(z)dz = \int\limits_{C_2} f(z)dz$$

$$\int\limits_{C_1} f(z)dz - \int\limits_{C_2} f(z)dz = \int\limits_{C_1} f(z)dz + \int\limits_{-C_2} f(z)dz = \oint\limits_{C} f(z)dz = 0$$

Fundamental Theorem Complex Calculus:

If f is analytic in $\mathbb{D}$ and $z_0, z_1 \in \mathbb{D}$ then,

$$\int\limits_{z_0}^{z_1} = F(z_1) - F(z_0)$$

Inside an analytic region $\mathbb{D}$, we can deform contours C continuously and not change the value of integral.

ML Bound: if $|f| \leq M$ on C and length of that curve

$$\int\limits_{C} dx = L$$

then

$$\left|\int\limits_{C} f(z)dz\right| \leq \text{ML}$$

8. Complex Residues

All polynomials and all convergent Taylor series are all analytic inside Radius of convergence.

Functions w/ singularities are not analytic at singularities.

Ex:

$$(z-a)^n \text{ is } \begin{cases} \text{analytic for} \quad n = 0, 1, 2, \ldots \\ \text{non-analytic for at } z = a \quad n = -1, -2, -3, \ldots \end{cases}$$

Taylor series:

$$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$$

Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n$$

Case 1. $n = +1, +2, +3, \ldots$ and n=-1, -2, -3, ...

$$\oint_C (z-a)^n dz = \left. \frac{(z-1)^{n+1}}{n+1} \right|_{z_0}^{z_0} = 0$$

Case 2. $n = -1$

$$\oint_C (z-a)^{-1} dz = \log(z-a)|_{z_0}^{z_1} = 2\pi i$$

$$\log(z) = \log(R) + i\left(\theta_p + 2\pi n\right)$$

Circle w/ radius: R.

$z - a = Re^{i\theta}, dz = iRe^{i\theta} d\theta$

$$\begin{aligned} \oint (z-a)^{-1} dz &= \oint \frac{1}{Re^{i\theta}} iRe^{i\theta} d\theta \\ &= \oint i d\theta \\ &= 2\pi i \end{aligned}$$

$$\oint (z-a)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

10. Cauchy Integral Formula (CIF)

If $f(z)$ is analytic inside and on a simple closed curve C, and if a point, 'a' inside C, then following integral formula is true.

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Proof.

$\oint \frac{f(z)}{z-a} dz = \oint \frac{f(a)}{z-a} dz + \oint \frac{f(z)-f(a)}{z-a} dz$

$I_1 = \oint \frac{f(a)}{z-a} dz$

$I_2 = \oint \frac{f(z)-f(a)}{z-a} dz$

Solve I_1 : $\oint \frac{f(a)}{z-a} dz$

$f(a)$ is a constant, $f(a)$ can be placed outside the integral.

$$f(a) \oint \frac{1}{z-a} dz = f(a) 2\pi i$$

Solve I_2

Deform contour C into C_s. Shrink C. Everywhere is analytic except a. We can use Cauchy-Goursat Theorem. Shrink the couture C very close to a. Since $f(z)$ is analytic around "a", we can choose δ such that $|f(z) - a| < \varepsilon$.

$$I_2 = \oint_{C_\delta} \frac{f(z) - f(a)}{z - a} dz \Rightarrow |I_2| \leq \oint \frac{\varepsilon}{z-a} dz = 2\pi i \varepsilon = 0$$

This is true for all $\varepsilon > 0$.

11. Examples of Cauchy-Integral Fromula

◆ $f(z)$ analytic inside and on closed curve C, and 'a' inside:

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Example. $f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-a)(z-2)}$ has poles at $z = 1, 2$

Note $\frac{1}{z-1}$ is analytic at $z = 2$ and $\frac{1}{z-1}$ is analytic at $z = 1$

Consider four contours. C_1 around at $z = 1$, C_2 around at $z = 2$, C_3 including $z = 1$ and $z = 2$, and C_4 away from the singularities excluding $z = 1$ and $z = 2$.

Solve C_1:

$$\oint_{C_1} f(z) dz = \oint_{C_1} \frac{\frac{1}{z-2}}{z-1} dz = -2\pi i \qquad \because f(z) = \frac{1}{x-2} \to f(1) = -1$$

Solve C_2:

$$\oint_{C_2} f(z) dz = \oint_{C_2} \frac{\frac{1}{z-1}}{z-2} dz = 2\pi i \qquad \because f(z) = \frac{1}{x-1} \to f(2) = 1$$

Solve C_3:

If we draw a closed loop surrounding the singularities ($z = 1, 2$) in counter clock wise, the two paths going from $z = 2$ to $z = 1$ and vise versa cancel out. Therefore, the integral of C_3 will end up with the addition of the integral of C_1 and C_2.

$$\begin{aligned}
\oint_{C_3} f(z) dz &= \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz \\
&= \oint_{C_1} \frac{\frac{1}{z-2}}{z-1} dz + \oint_{C_2} \frac{\frac{1}{z-1}}{z-2} dz \\
&= -2\pi i + 2\pi i \\
&= 0
\end{aligned}$$

Solve C_4:

Everywhere is analytic in C_4. Therefore, the integral around C_4 is zero.

$$\oint_{C_4} f(z)dz = 0$$

12. Examples of Complex Integrals

$\int_{-\infty}^{\infty} f(x)dx$ for real valued $f(x)$

Ex. $\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4+a^4}$

$f(z) = \frac{1}{z^4+a^4}$ has poles at $z^4 = -a^4 \rightarrow z = a^4\sqrt{(-1)}$

Solving $z^4 = -a^4$ (Ref. Gemini)

To solve the equation, we first write the constant in polar form:

$$-a^4 = a^4 e^{i(\pi+2k\pi)}$$

Taking the fourth root of both sides:

$$z = \left(a^4 e^{i(\pi+2k\pi)}\right)^{\frac{1}{4}} = a e^{i\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)}$$

For $k = 0, 1, 2, 3$, we find the roots:

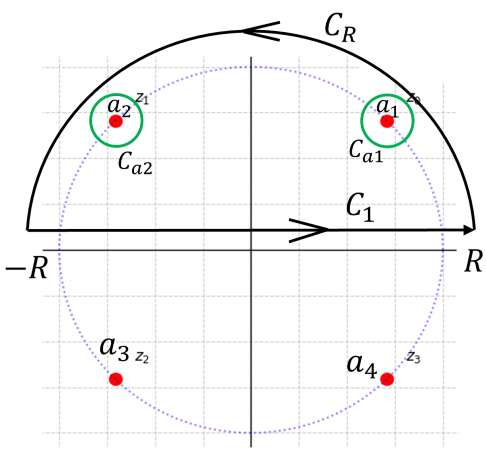
$$z_k = a\left(\cos\left(\frac{\pi}{4}+\frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)\right)$$

The four solutions are:

$$z_0, z_3 = \pm\frac{a}{\sqrt{2}} + i\frac{a}{\sqrt{2}}$$
$$z_1, z_2 = \pm\frac{a}{\sqrt{2}} - i\frac{a}{\sqrt{2}}$$

Final compact form:

$$z = \frac{a}{\sqrt{2}}(\pm 1 \pm i)$$



If I take the limit goes to infinity, C_1 becomes $-\infty$ to $+\infty$ and C_R becomes ∞. Then the contribution of $\int_{C_R}$ becomes zero, and we can calculate $\int_{C_1}$ using Cauchy Integral Formula.

$$\oint_C = \int_{C_1} + \int_{C_R} \quad \Rightarrow \quad \oint_C f(z)dz = \int_{-R}^{R} \frac{dx}{x^4 + a^4} + \int_{C_R} \frac{dz}{x^4 + a^4}$$

Factor out $z^4 + a^4 = (z - a_1)(z - a_2)(z - a_3)(z - a_4)$

$$\oint_C f(z)dz = \oint_C \frac{dz}{(z - a_1)(z - a_2)(z - a_3)(z - a_4)}$$

Be aware of the property of the factors of $f(z)$ except at $z = a_1$ and a_2. Away from $z = a_1$ all others terms are analytic at a_1. Similarly, away from $z = a_2$, all others terms are analytic at a_2. We can use Cauchy Integral Formula.

$$\begin{aligned}\oint_C \frac{dz}{(z - a_1)(z - a_2)(z - a_3)(z - a_4)} &= \oint_{C_{a1}} \frac{\frac{1}{(z-a_2)(z-a_3)(z-a_4)}}{z - a_1} + \oint_{C_{a2}} \frac{\frac{1}{(z-a_1)(z-a_3)(z-a_4)}}{z - a_2} \\ &= 2\pi i \frac{1}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)} + 2\pi i \frac{1}{(a_2 - a_1)(a_2 - a_3)(a_2 - a_4)} \\ &= \frac{2\pi i(a_3 + a_4 - a_1 - a_2)}{(a_1 - a_3)(a_1 - a_4)(a_2 - a_3)(a_2 - a_4)}\end{aligned}$$

Given the roots:

$$a_1 = \frac{a}{\sqrt{2}}(1 + i), \quad a_2 = \frac{a}{\sqrt{2}}(-1 + i), \quad a_3 = \frac{a}{\sqrt{2}}(-1 - i), \quad a_4 = \frac{a}{\sqrt{2}}(1 - i)$$

The numerator evaluates to:

$$a_1 + a_2 = \frac{a}{\sqrt{2}}(1 + i - 1 + i) = \frac{a}{\sqrt{2}}(2i) = \frac{ai}{\sqrt{2}}$$

$$a_3 + a_4 = \frac{a}{\sqrt{2}}(-1 - i + 1 - i) = \frac{a}{\sqrt{2}}(-2i) = -\frac{ai}{\sqrt{2}}$$

$$(a_3 + a_4) - (a_1 + a_2) = \frac{-ai}{\sqrt{2}} - \frac{ai}{\sqrt{2}} = \frac{-2ai}{\sqrt{2}}$$

$$2\pi i(a_3 + a_4 - a_1 - a_2) = 2\pi i\left(-2ai\sqrt{2}\right) = 4\pi a\sqrt{2}$$

The denominator evaluates to:

$$(a_1 - a_3) = \frac{a}{\sqrt{2}}(1 + i - (-1 - i)) = \frac{a}{\sqrt{2}}(2 + 2i) = a\sqrt{2}(1 + i)$$

$$(a_1 - a_4) = \frac{a}{\sqrt{2}}(1 + i - (1 - i)) = \frac{a}{\sqrt{2}}(2i) = ai\sqrt{2}$$

$$(a_2 - a_3) = \frac{a}{\sqrt{2}}(-1 + i - (-1 - i)) = \frac{a}{\sqrt{2}}(2i) = ai\sqrt{2}$$

$$(a_2 - a_4) = \frac{a}{\sqrt{2}}(-1 + i - (1 - i)) = \frac{a}{\sqrt{2}}(-2 + 2i) = a\sqrt{2}(-1 + i)$$

$$\begin{aligned}(a_1-a_3)(a_1-a_4)(a_2-a_3)(a_2-a_4) &= a\sqrt{2}(1+i)\cdot(ai)\sqrt{2}\cdot(ai)\sqrt{2}\cdot a\sqrt{2}(-1+i) \\ &= a^4\cdot\left(\sqrt{2}\right)^4\cdot i^2\cdot(1+i)(-1+i) \\ &= a^4\cdot 4\cdot(-1)\cdot(-1-1) \\ &= 8a^4\end{aligned}$$

Final result:

$$S = \frac{4\pi a\sqrt{2}}{8a^4} = \frac{\pi\sqrt{2}}{2a^3}$$

If R goes to ∞, $\oint\limits_{C_R}\frac{dz}{z^4+a^4}$ goes to zero.

Proof:

$$\lim_{R\to\infty}\oint\limits_{C_R}\frac{dz}{z^4+a^4} = 0$$

$$\left|\oint\limits_{C_R}\frac{dz}{z^4+a^4}\right| \leq \mathrm{ML}\text{ where }\begin{cases}\mathrm{L} = \pi R \\ m = \max\limits_{C_R}\frac{1}{z^4+a^4} \leq \frac{1}{R^4}\end{cases}$$

$$\left|\oint\limits_{C_R}\frac{dz}{z^4+a^4}\right| = \frac{\pi}{R^4} \to 0\text{ where }R\to\infty$$

Therefore,

$$\begin{aligned}\int\limits_0^\infty\frac{dx}{x^4+a^4} &= \frac{1}{2}\int\limits_{-\infty}^\infty\frac{dx}{x^4+a^4} \\ &= \frac{1}{2}\frac{\pi\sqrt{2}}{2a^3}\end{aligned}$$

13. Bromwich Integrals and the Inverse Laplace Transform

Laplace Transform: maps time domain to frequency domain.

Inverse Laplace Transform: from freq to time.

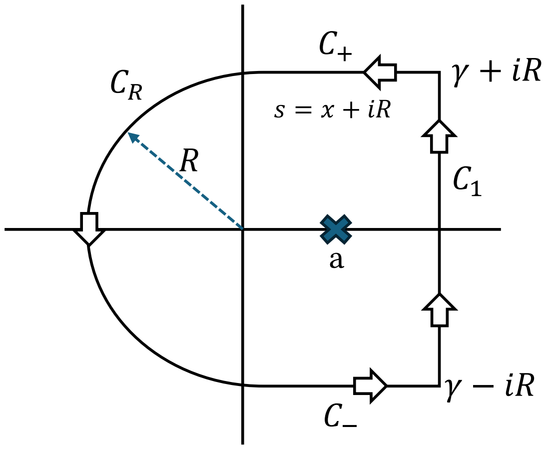
$$L^{-1}\left\{\hat{f}(s)\right\} \triangleq \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\hat{f}(s)e(st)ds$$

where $\gamma >$ real part of all poles of $\hat{f}$

Why Laplace transform? To solve ODE and PDE(such as heat equation). Convert ODE to algebraic equation and solve.

Specific example: $\hat{f} = \frac{1}{s-a}$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}}{s-a}ds$$



Using Cauchy Integral Formula.

$$\frac{1}{2\pi i}\oint_C \frac{e^{st}}{s-a}ds = e^{at}$$

$$= \oint_{C_1} + \oint_{C_+} + \oint_{C_-} + \oint_{C_R}$$

if $R \to \infty$, then

$$\oint_{C_1} + \oint_{C_+} + \oint_{C_-} + \oint_{C_R} = \oint_{C_1} + 0 + 0 + 0$$

$$= e^{at}$$

$$\oint_{C_+} + \oint_{C_-} \leftarrow \text{use ML bound}$$

Length is always $\gamma = L$

$$\oint_{C_+} \frac{e^{st}}{s-a}ds = \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s-a}ds = \int_{\gamma}^{0} \frac{e^{x+iR}t}{x+iR-a}dx \le \text{ML}$$

$$M = \max_{x \le [0,\gamma]}\left|\frac{e^{(x+iR)t}}{x+iR-a}\right| \le \frac{e^{\gamma t}}{R}$$

Numerator:

$$e^{(x+iR)t} = e^{xt} \cdot e^{iRt} = e^{xt}(\cos(Rt) + i\sin(Rt))$$

$\left|e^{(x+iR)t}\right| = e^{xt}\sqrt{\cos(Rt)^2 + \sin(Rt)^2} = e^{xt} \to x = \gamma \text{ then } e^{\gamma t}$

Denominator:

$|(x + iR - a)| = \sqrt{(x-a)^2 + R^2} \to x = a \text{ for smallest value} \to R$

Therefore,

$$M = \frac{e^{\gamma t}}{R} \qquad L = \gamma \qquad \text{when R} \to \infty, \qquad \text{ML} = \frac{e^{\gamma t}}{R}\gamma \to 0$$

Solve for $\int_{C_R}$:

ML bound doesn't work for C_R. Try polar coordinate.

Step 4: Show the arc contribution vanishes (Ref: Claude)

On the semicircular arc in the left half-plane where $s = \gamma + Re^{i\theta}$ with $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$:

$$|e^{st}| = |e^{(\gamma + Re^{i\theta})t}| = e^{\gamma t} \cdot e^{Rt\cos\theta}$$

Since $\cos\theta < 0$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, we have $e^{Rt\cos\theta} \to 0$ as $R \to \infty$ for $t > 0$.

By Jordan's lemma, the contribution from the arc vanishes.