graph (finite, no loops or multiple edges, undirected/directed)

$G = (V, E)$ where
- V (or $V(G)$) is a set of *vertices*
- E (or $E(G)$) is a set of *edges* each of which is
  - a set of two vertices (undirected), or
  - an ordered pair of vertices (directed)

Two vertices that are contained in an edge are *adjacent*;
two edges that share a vertex are *adjacent*;
an edge and a vertex contained in that edge are *incident*.

We often let $n = |V|$ and $m = |E|$.

For undirected graph $G = (V, E)$:
The *neighbourhood* of vertex v is $N(v) = \{u | uv \in E\}$
The *degree* of vertex v is $d(v) = |N(v)|$
$\delta(G)$: the minimum degree of a vertex of G
$\Delta(G)$: the maximum degree of a vertex of G
Note that $\sum_{v \in V} d(v) = 2|E|$.

subgraph

A *(partial) subgraph* of graph G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
The subgraph of $G = (V, E)$ *induced* by $V' \subseteq V$, denoted $G[V']$ or $G\langle V' \rangle$,
is the graph $(V', \{uv | uv \in E \text{ and } u, v \in V'\})$.

complement

The *complement* of graph $G = (V, E)$ is the graph $\overline{G} = (\{V, \{uv | u, v \in V, u \neq v, \text{ and } uv \notin E\})$.

clique

K_n: the complete undirected graph on n vertices (as a graph or subgraph)
A *maximum* clique of graph G is a complete subgraph of G with the maximum number of vertices.
A *maximal* clique of G is a complete subgraph of G that is not contained in any larger complete subgraph.

independent set (or *stable set*): a graph or subgraph having no edges

How many maximal independent sets can there be in a graph?

Give algorithms for the following problems:

- Given G, compute $\delta(G)$, $\Delta(G)$
- Given $G = (V, E)$, $V' \subseteq V$, does V' induce an independent set of G?
- Given G, does G have an independent set of size 4?
- Given G, k does G have an independent set of size k? $\geq k$?
- Given G, what is the maximum size of an independent set of G?
- Find an independent set (maximal independent set, maximum independent set) of G.

Basic Graph Theory Definitions and Notation continued

paths and cycles

walk $v_0e_1v_1e_2\dots e_kv_k$ where $e_i = v_{i-1}v_i$, $\forall 1 \le i \le k$
(often written as $v_0v_1\dots v_k$)

endpoints: v_0, v_k
length: k (or $\sum_{i=1}^{k} w(e_i)$ if G has edge weights $w : E \mapsto \mathcal{R}$)
closed if $v_0 = v_k$

Note: edges and vertices may be repeated

trail a walk with no repeated edge

path a trail with no repeated vertex (unless closed – then $v_0 = v_k$ but no other repetitions)

cycle closed path

A *chord* of a path/cycle is an edge between two vertices of the path/cycle that is not on the path/cycle.
P_n is the undirected chordless path on n vertices, $n \ge 1$ (graph or subgraph)
C_n is the undirected chordless cycle on n vertices, $n \ge 3$ (graph or subgraph)

For graph $G = (V, E)$:

- the *distance* between vertices u and v, denoted $d(u, v)$, is the length of a shortest u, v-path in G
- the *eccentricity* of vertex v is $\max_{u \in V} d(u, v)$
- the *diameter* of G is $\max_{u,v \in V} d(u, v)$
- the *girth* of G is the minimum length of a cycle in G

graph properties

Undirected graph $G = (V, E)$ is:

- *connected* if, between each pair of vertices, there is a path
- *acyclic* if it has no cycle
- a *tree* if it is connected and acyclic
- *bipartite* if V can be partitioned into (at most) two independent sets

isomorphism

Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic*, written $G_1 \cong G_2$
if there is a bijection $f : V_1 \mapsto V_2$ st
$\forall u, v \in V_1$, $uv \in E_1$ if and only if $f(u)f(v) \in E_2$.

Such a bijection f is called an *isomorphism* from G_1 to G_2.
An *automorphism* is an isomorphism from a graph to itself.