Topology Qual Prep - Practice Qual 1 - 2023

Instructions Do eight problems, four from each part. That is four from part A and four from part B. This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

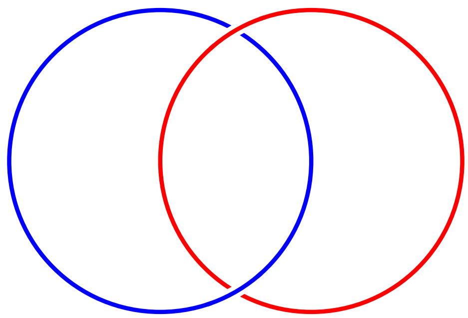
Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

PART A

Problem 1 Define real projective space and compute the fundamental group of $\mathbb{R}P^2$.

Solution. Let $\mathbb{Z}_2$ act on S^n as the antipodal map. We may then define the orbit space $\mathbb{R}P^n := S^n/\mathbb{Z}_2$. Note that this is a free action of a finite group and therefore a covering action. If $n \geq 2$, we have that S^n is simply connected and so covering space theory tells us $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$.

Problem 2 Compute the fundamental group of space obtained by removing a Hopf link from $\mathbb{R}^3$.



Let H denote the complement of the Hopf link in $\mathbb{R}^3$.

Claim There holds $\pi_1(H) = \mathbb{Z} \times \mathbb{Z}$.

Proof. In the above picture, draw a thickened open line from the south pole of the blue circle to its north pole so that the line does not intersect the red circle. Write H as the union of two half spaces U, V where U contains everything to the left of the line and V everything to the right. One sees that U is homeomorphic to $\mathbb{R}^3$ minus an axis and has fundamental group $\mathbb{Z}$. Call the generator a Likewise, V is homeomorphic to $\mathbb{R}^3$ minus S^1 in the xy-plane and the z-axis. So $\pi_1(V) = \mathbb{Z} \times \mathbb{Z}$, the free abelian group on generators we'll call c, d. One observes that $U \cap V$ deformation retracts to the twice punctured strip $I \times \mathbb{R}$ and thus has fundamental group free on generators we'll call x, y. With $i : U \cap V \to U$ and $j : U \cap V \to V$, we may choose x, y so that $i_* x = a$ and $j_* x = c$ and $j_* y = d$. At once, by Van Kampen,

$$\pi_1(H) = \mathbb{Z} *_{\mathbb{Z} * \mathbb{Z}} \mathbb{Z}^2 = \mathbb{Z}^2.$$

QED

Problem 3

(a) Define regular covering. Give two connected coverings of the wedge of two circles, one regular, one not.

(b) If X is a topological space, define what is meant by the suspension $\Sigma(X)$ of X. Prove that if $\pi_1(\Sigma(X))$ is trivial if X is path connected. Does this remain true when X is not path connected?

Solution. (a) A covering projection $p : E \to B$ is called regular when for all $b \in B$ and $e, e' \in E_b$ there is a deck transformation $\phi : E \to E$ so that $\phi(e) = e'$. For the rest, see the discussion in Hatcher (pg. 70 - 71).

(b) If X is a topological space, the suspension of X is the space

$$\Sigma(X) = (X \times I)/\sim \text{ where } (x,0) \sim (x',0) \text{ and } (x,1) \sim (x',1) \text{ for all } x, x' \in X.$$

Let U denote the image of $X \times [0, 3/4)$ in $\Sigma(X)$ and V the image of $X \times (1/4, 1]$. Note that $\{U, V\}$ is an open cover of X. Further, note that that both U and V deformation retract to a point (this from the deformation retraction $X \times [0, 3/4)$ to $X \times \{0\}$). Further, note that $U \cap V$ deformation retracts to $X \cong X \times \{1/2\}$ and so is connected if and only if X is. By Van Kampen, we have that $\pi_1(\Sigma(X))$ is trivial if X is path connected.

On the other hand, we have that $\Sigma(\{0, 1\}) \cong S^1$ whence $\pi_1(\Sigma(X))$ need not be trivial.

Problem 4 Let M denote the Möbius band and K the Klein bottle.

(a) Show that M does does not retract onto its boundary.

(b) Show that there exist homotopically non-trivial curves γ, σ in K so that K retracts to σ but not to γ.

(c) Define universal cover. Give a space which is not simply connected which has compact universal cover.

Proof. (a) Let σ denote a generator of the fundamental group of the central circle of M (and thus of $\pi_1(M)$ as well). Let β denote the generator of $\pi_1(\partial M)$. With $c : M \to M$ the retraction of M to its central circle, we have $c_*\beta = \pm 2\sigma$. Suppose for contradiction that there is a retraction $r : M \to M$ of M onto ∂M, then there is some $k \in \mathbb{Z}$ so that $r_*\sigma = k\beta$. Then projecting onto the central circle, $(cr)_*\sigma = \pm 2k\beta$ and $(cr)_*$ cannot be surjective. But, retractions act on the fundamental group by surjection, a contradiction.

(b) View K as a quotient of the square in the standard way; that is, $K = [-1, 1]^2/\sim$ where $(-1, t) \sim (1, t)$ for all $t \in [-1, 1]$ and $(t, -1) \sim (-t, 1)$ for all $t \in [-1, 1]$. Let $\sigma = \{(0, t) : t \in [-1, 1]\}$ and let $\gamma = \{(\pm 1/2, t) : t \in [-1, 1]\}$. To see that K deformation retracts onto σ, consider the quotient of the straight line homotopy.[1] The argument that K does not retract to γ is the same as that M does not retract onto ∂M.

(c) A universal cover of X is a covering space which is simply connected. Observe that $\mathbb{R}P^2$ is not simply connected and has compact universal cover: S^2. QED

Problem 5 Consider the space Y obtained from $S^2 \times [0, 1]$ by identifying $(x, 0)$ with $(-x, 0)$ and also $(x, 1)$ with $(-x, 1)$ for all $x \in S^2$.

(a) Show that Y is homeomorphic to the connected sum of $\mathbb{R}P^3$ with itself.

(b) Show that $S^2 \times S^1$ is a double cover of Y.

Proof. (a) View $\mathbb{R}P^3$ as the quotient of the upper hemisphere of S^3 be the antipodal relation on its boundary, *i.e.* the equatorial S^2. It is then clear that the space obtained by removing a small open disk about the north pole is homeomorphic to $S^2 \times [0, 1/2]/\sim$ where $(x, 0)$ with $(-x, 0)$ for all $x \in S^2$. Gluing two such objects together along the boundary of the removed circle, one obtains Y and the connected sum of $\mathbb{R}P^3$ with itself (by definition of connected sum).

(b) Parameterize $S^1 = \{e^{i\theta\pi/2} : \theta \in [0, 4]\}$. In the above definition of Y, exchange the unit interval I for the interval $J = [-1, 1]$ so that $Y \cong (S^2 \times J)/\sim$ for the appropriate choice of relation $\sim$. Define $p : S^2 \times S^1 \to Y$ by

$$p(x, e^{i\theta}) = \begin{cases} (x, \theta) & \theta \in [0, 1] \\ (-x, 2 - \theta) & \theta \in [1, 3] \\ (x, \theta - 4) & \theta \in [3, 4] \end{cases}$$

One may check that this is a covering map. QED

Problem 6 Let B be a path connected space and $p : E \to B$ a regular n-sheeted cover. Show that if E is path connected then $\pi_1(B)$ has order at least n. Show that if B is simply connected then E has n path components.

[1] The argument is identical to the Möbius band retracting to its central circle.

Proof. Assume that E is path connected. We then know that $n = [\pi_1(B), p_*\pi_1(E)]$. Thus, n is a quotient of $|\pi_1(B)|$ by a positive integer and we must have $n \leq |\pi_1(B)|$.

Next, assume that B is simply connected. Let $E_1, ..., E_k$ denote the path components of E and note that since p is n-sheeted we have $k \leq n$. Let n_i denote the number of sheets of the restricted cover $p : E_i \to B$. We have

$$n_k = [\pi_1(B), p_*\pi_1(E_i)] = [\{e\}, p_*\pi_1(E_i)] = 1.$$

The n lifts of a point in B must live in some E_i for $i = 1, ..., n$ and the above computation of $n_i = 1$ shows each lift belongs to a unique E_i. So, $k = n$. QED

PART B

Problem 1 Let M be a smooth manifold equipped with a non-vanishing smooth 1-form α so that $\alpha \wedge d\alpha = 0$. Prove that in a neighborhood U of any point of M there exist smooth functions $f, \lambda : U \to \mathbb{R}$ so that $\alpha = f d\lambda$ on U. You may use the following consequence of the Frobenius Theorem without proof.

> **Corollary** If $\xi \subseteq TM$ is a sub-bundle sections of which are closed under the Lie bracket then any point $p \in M$ has local coordinates (U, x) so that $\xi|_U$ is spanned by $\{\partial_{x_1}, ..., \partial_{x_k}\}$.

Proof. Suppose that $\dim M = n$. We recall the fact that $\alpha \wedge d\alpha = 0$ and α non-vanishing implies the existence of a one form ω so that $d\alpha = \alpha \wedge \omega$. Further, since α is non-vanishing, the pointwise defined $\ker \alpha$ is a sub-bundle of TM. Consider two vector field X, Y valued in $\ker \alpha$. Observe that

$$\begin{aligned}
\alpha([X,Y]) &= X(\alpha(Y)) - Y(\alpha(X)) - d\alpha(X,Y) \\
&= -d\alpha(X,Y) \\
&= -(\omega \wedge \alpha)(X,Y) \\
&= -\omega(X)\alpha(Y) + \alpha(X)\omega(Y) \\
&= 0.
\end{aligned}$$

Fix $p \in M$. By the Frobenius theorem, we therefore have local coordinates (U, x) about p so that $\ker \alpha|_U$ is spanned by $\{\partial_{x^1}, ..., \partial_{x^k}\}$. Note that $\alpha_p : T_pM \to \mathbb{R}$ is non-zero and thus $k = n-1$. Write $\alpha = \sum_{i=1}^n f_i dx^i$. Observe that for $i = 1, ..., n-1$ there holds

$$0 = \alpha(\partial_{x_i}) = f_i$$

so that $\alpha = f_n dx^n$ which proves the claim. QED

Problem 2 Let M be a smooth manifold and $p \in M$. Prove that if $\xi_1, ..., \xi_k$ are linearly independent elements of T_pM then there exists a chart (U, x) about p so that for $i = 1, ..., k$ there holds $\frac{\partial}{\partial x^i}|_p = \xi_i$.

Proof. Let $\dim M = n$. Choose your favorite chart (U, y) centered at p; that is, with $y(p) = 0$. Let $V = y(U)$ be an open subset of $\mathbb{R}^n$. For $i = 1, ..., k$ set $\eta_i = y_* \xi_i$. Choose $\eta_{k+1}, ..., \eta_n$ so that $\eta_1, ..., \eta_n$ is a basis for T_0V. Let L be a change of basis matrix from the η basis to the standard basis. Let $W = L(V)$ which is open in $\mathbb{R}^n$ and set $x = L \circ y$. Note that $x : U \to W$ is a diffeomorphism and $x_* \xi_i$ is the i-th standard basis vector. Thus, $\xi_i = \partial_{x_i}$ as desired. QED

Problem 3 Prove that every Lie group is parallelizable (that is, has trivial tangent bundle).

Proof. Let G be a Lie group with identity e. We know that if X_e is a tangent vector in T_eG then the vector field $X_g = g_* X_e$ is smooth on G. Fix any basis $X_e^1, ..., X_e^n$ for T_eG. And define $X_g^i = g_* X_e^i$ for all $g \in G$ and $i = 1, ..., n$. The resulting vector fields $X^1, ..., X^n$ are then a smooth frame for G. QED

Problem 4 Let M be a topological manifold with open cover $\mathcal{U}$ and basis $\mathcal{B}$. Prove that there exists a countable, locally finite refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$.

We will establish two lemmas before proving the claim. Fix $M, \mathcal{U}$, and $\mathcal{B}$ as in the problem.

Lemma 4.1 We may find a collection $\{V_i, K_i \subseteq M : i \in \mathbb{N}_{\geq 0}$ with each V_i open, each K_i compact, $M = \bigcup_i V_i$, and $V_i \subseteq K_i \subseteq V_{i+1}$ for all i.

Proof. Given any point $p \in M$ we may restrict to a neighborhood of the point which is homeomorphic to $\mathbb{R}^n$ and thus find open balls

$$p \in B^0 \subseteq B^1 \subseteq B^2 \subseteq \cdots \text{ and with } \overline{B^0} \text{ compactly contained in } B^1.$$

As topological manifolds are Lindelöf, we may find a countable collection $\{B_n^0\}_{n\in\mathbb{N}}$ of such B^0 balls which covers M. One may then simply define

$$\begin{aligned}
V_0 &= B_0^0 \\
K_0 &= \overline{B_0^0} \\
V_1 &= B_0^1 \cup B_1^0 \\
K_1 &= \overline{B_0^1} \cup \overline{B_1^0} \\
&\vdots \\
V_i &= B_0^i \cup B_1^{i-1} \cup \cdots \cup B_i^0 \\
K_i &= \overline{B_0^i} \cup \overline{B_1^{i-1}} \cup \cdots \overline{\cup B_i^0}.
\end{aligned}$$

QED

Lemma 4.2 There exists a family $\{A_n : n \in \mathbb{Z}\}$ of compact subsets of M and a family $\{U_n : n \in \mathbb{Z}\}$ of open sets in M so that each $A_n \subseteq U_n$, the A_n cover M, and each U_n intersects only finitely many U_k.

Proof. Let V_i, K_i be as in Lemma 4.2. For $i < 0$ we define $V_i = K_i = \emptyset$. Now, choose

$$A_n = K_n \smallsetminus V_{n-1} \text{ and } U_n = V_{n+1} \smallsetminus K_{n-1}.$$

QED

Now we may address Problem 4.

Proof. Let A_n, U_n be as in Lemma 4.2. Since $\mathcal{B}$ is a basis and A_n is compact, we may find for each $n \in \mathbb{Z}$ some $\mathcal{B}_n \subseteq \mathcal{B}$. So that each $\mathcal{B}_n$ is a finite cover for A_n and if $B \in \mathcal{B}_n$ then $B \subseteq U_n \cap U$ for some $U \in \mathcal{U}$. At once, $\bigcup_n \mathcal{B}_n$ is the desired refinement of $\mathcal{U}$. QED

Problem 5 Prove that homotopic diffeomorphisms between compact, connected, oriented smooth manifolds are either both orientation preserving or orientation reversing. You may use without proof that homotopic smooth maps are smoothly homotopic.

Proof. Let $f, g : M \to N$ be homotopic diffeomorphisms between compact, connected, oriented smooth manifolds. Let $H : M \times I \to N$ be a smooth homotopy from f to g. Let ω be an orientation form for N. Observe that

$$\begin{aligned}
\int_M g^*\omega - \int_M f^*\omega &= \int_{\partial(M\times I)} H^*\omega \\
&= \int_{M\times I} dH^*\omega \\
&= \int_{M\times I} H^*d\omega \\
&= \int_{M\times I} H^*0 \\
&= 0.
\end{aligned}$$

So, $\int_M g^*\omega = \int_M f^*\omega$. Since both $f^*\omega$ and $g^*\omega$ are orientation forms for M which is connected, they differ by a multiple of a smooth function of constant sign. As the above integrals coincide, that function is positive. Thus, $f^*\omega$ and $f^*\eta$ define the same orientation, either the given orientation for M or its opposite. QED

Problem 6 Consider the function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by $f(x, y) = x^2 + xy + y^2$. Let $S = f^{-1}(\{0\})$.

(a) Show that $S \subseteq \mathbb{R}^2 \setminus \{0\}$ is a regular submanifold.

(b) Show that

$$\xi = y(2\partial_x - \partial_y) + x(\partial_x - 2\partial_y)$$

is a vector field on S.

(c) Let η denote the radial vector field $\eta = x\partial_x + y\partial_y$ on $\mathbb{R}^2$. Prove/Disprove: η is tangent to S. Let $\widetilde{\eta}$ denote the orthogonal projection of η to TS (using the inner product derived from $T\mathbb{R}^2 \cong \mathbb{R}^4$) and compute $\mathscr{L}_\xi \widetilde{\eta}$.

(d) Let $\omega = xdy + ydx$ and compute $\mathscr{L}_\xi \omega$.

Solution. (a) Observe that $\nabla f = (2x + y, x + 2y)$ which vanishes precisely when $(x, y) = 0$ which does not lie on S. So, by the regular level set theorem, S is a submanifold.

(b) We need merely verify that

$$\nabla f \cdot \xi = (2x + y, x + 2y) \cdot (x + 2y, -y - 2x) = 0$$

(c) We compute

$$\nabla f \cdot (x, y) = 2x^2 + xy + xy + 2y^2 = 2f(x, y) = 0$$

whence η is tangent to S and $\eta = \widetilde{\eta}$. Then

$$\begin{aligned}
\mathscr{L}_\xi \eta &= [\xi, \eta] \\
&= [(x + 2y)\partial_x - (y + 2x)\partial_y, x\partial_x + y\partial_y] \\
&= [[(x + 2y)\partial_x - (y + 2x)\partial_y, x\partial_x + y\partial_y]](x)\partial_x + [(x + 2y)\partial_x - (y + 2x)\partial_y, x\partial_x + y\partial_y](y)\partial_y \\
&= (\xi(\eta(x)) - \eta(\xi(x)))\partial_x + (\xi(\eta(y)) - \eta(\xi(y)))\partial_y \\
&= 0\partial_x + 0\partial_y \\
&= 0.
\end{aligned}$$

(d) Lastly, we compute

$$\begin{aligned}
\mathscr{L}_\xi \omega &= \mathscr{L}_\xi(xdy) + \mathscr{L}_\xi(ydx) \\
&= \mathscr{L}_\xi(x)dx + xd\mathscr{L}_\xi(y) + \mathscr{L}_\xi(y)dx + yd\mathscr{L}_\xi x \\
&= (x + 2y)dx - xd(y + 2x) - (y + 2x)dx + yd(x + 2y) \\
&= (x + 2y)dx - xdy - 2xdx - ydx - 2xdx + ydx + 2ydy \\
&= 4ydy - 4xdx
\end{aligned}$$