Concepts Review

What's in this section? The following problems have been collected from various sources (e.g., algebra textbooks by Dummit and Foote, Aluffi, Hungerford, etc, and different online sources). Although there are no citations, I make no claims of originality. If needed, most of them can be found in most standard algebra texts, maybe as unsolved exercises or even as solved examples or proven results. The problems comprise several topics in group theory and shall help you test your understanding of the fundamentals. The general idea is to build intuition and boost confidence.

Though the problems span varying degrees of difficulty, they should still be easier than the ones that have appeared in the previous qualifying exams. You may treat these as warm-up exercises as you build yourself up to answering more complicated questions. Some of these exercises contain common examples/counterexamples and others are easy-to-prove results, all of which might come in handy when solving more involved problems.

In case some of the problems in the next section leave you clueless, it might be more productive to revisit this section before you take another stab at the next section. (Cleanse your palate, so to speak.) Take your mind off the difficult ones while still being productive and then make another attempt with a fresh perspective.

I've tried my best to keep this error-free yet you must keep an eye out for any typos that might have sneaked in. If you find any flaws, especially logical or grammatical, please let me know and I shall fix them.

[**1 — Basic Definitions**] Let G be a finite abelian group, with exactly one element f of order 2. Prove that $\prod_{g\in G} g = f$.

[**2 — Basic Definitions/Computation**] Let $g \in G$ be an element of finite order. Prove that g^m has finite order $\forall m \geq 0$, and that

$$|g^m| = \frac{\operatorname{lcm}(m, |g|)}{m} = \frac{|g|}{\gcd(m, |g|)}$$

(You can use the fact that for any two positive integers a and b, we have $\operatorname{lcm}(a, b) \cdot \gcd(a, b) = a \cdot b$, i.e., you only need to prove one of the above equalities and the other one will follow.)

[**3 — Basic Definitions/Computation**] Let G be a group and $g, h \in G$ be elements of finite order such that $gh = hg$. Show that there exists an element of order $\operatorname{lcm}(|g|, |h|)$.

[**4 — Basic Definitions/Computation**] Consider the matrices $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ with integer entries. Find the orders of the groups $\langle A\rangle$, $\langle B\rangle$ and $\langle A, B\rangle$, where the binary operation is matrix multiplication. (Hint: For $\langle A, B\rangle$ find the order of the product AB.)

[**5 — Homomorphisms**] To a permutation $\sigma \in S_n$ assign an $n \times n$ matrix M_σ with 1 in $(i, \sigma(i))$ and 0 everywhere else. Prove that $S_n \to GL_n : \sigma \mapsto M_\sigma^T$ is a group homomorphism.

[**6 — Homomorphisms**] Let G be a group, and let $g \in G$. Prove that the function $\gamma_g : G \to G : a \mapsto g^{-1}ag$ is an automorphism of G. Prove that the function $G \to \operatorname{Aut}(G)$ defined by $g \mapsto \gamma_g$ is a homomorphism. Prove that this homomorphism is trivial if and only if G is abelian.

[**7 — Homomorphisms**] Let G be a group. Show that $g \mapsto g^{-1}$ is a group automorphism (of order 2) if and only if G is abelian.

[**8 — Homomorphisms**] Prove that $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \simeq S_3$.

[**9 — Subgroups**] Let G and G' be groups and $\varphi : G \to G'$ be a map, then prove that the following are equivalent:

(a) φ is a monomorphism. (that is, for all groups H and all group homomorphisms $h_1, h_2 : H \to G$, $\varphi \circ h_1 = \varphi \circ h_2 \implies h_1 = h_2$.)

(b) $\ker(\varphi)$ is trivial.

(c) φ is injective.

[**10 — Homomorphisms**] Show that if G is a group of order n and k is an integer relatively prime to n, then the function $G \to G : g \mapsto g^k$ is surjective. Conclude that if G is a finite group of odd order then every element in G is a square.

[**11 — Basic Definitions**] If G is a group of even order then prove that there exists an element a (not the identity) such that $a = a^{-1}$.

[**12 — Homomorphisms**] Fermat's little theorem. Let $\varphi(n)$ be the Euler totient function. If $\gcd(a, n) = 1$, show that $a^{\varphi(n)} \cong 1 \bmod n$.

[**13 — Subgroups**] Assume G is a finite abelian group, and let p be a prime divisor of $|G|$. Prove that there exists an element in G of order p.

[**14 — Group Actions**] Let G be a finite group, and let H be a subgroup of index p, where p is the smallest prime dividing $|G|$. Prove that H is normal in G. (This is a stronger version of the statement: a subgroup of index 2 is normal.)

[**15 — Group Actions**] Let G be a group. Prove that $G/Z(G)$ is isomorphic to the group $\mathrm{Inn}(G)$ of inner automorphisms of G. Use this to show that if $G/Z(G)$ is cyclic then G is abelian.

[**16 — Normality**] Consider the alternating group $G = A_4$ and its subgroups $H = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$ and $K = \{\mathrm{id}, (12)(34)\}$. Show that H is normal in G and K is normal in H. Is K normal in G?

[**17 — Normality**] Let G be the dihedral group $D_4 = \langle r, s | r^4 = s^2 = 1, srs^{-1} = r^{-1} \rangle$. Consider the subgroups $H = \{1_G, r^2, s, sr^2\}$ and $K = \{1_G, s\}$. Show that H is normal in G and K is normal in H. Is K normal in G?

[**18 — Class Equation**] Find the number of finite groups, up to isomorphism, with exactly two conjugacy classes.

[**19 — Class Equation**] Let G be a finite group, and let $H \subseteq G$ be a subgroup of index 2. Then show that H is normal. For $a \in H$, denote by $[a]_H$, resp., $[a]_G$, the conjugacy class of a in H, resp., G. Prove that either $[a]_H = [a]_G$ or $[a]_H$ is half the size of $[a]_G$, according to whether the centralizer $Z_G(a)$ is not or is contained in H.

[**20 — Class Equation**] Let G be a finite group, let P be a p-Sylow subgroup, and let $H \subseteq G$ be a p-group. Then H is contained in a conjugate of P.

[**21 — Class Equation**] Classify all the groups of order 6 (up to isomorphism).

[**22 — Sylow**] Prove that there are no simple subgroups of order 12.

[**23 — Sylow**] Prove that there are no simple groups of order 24.

[**24 — Sylow**] Let q be an odd prime, and let G be a noncommutative group of order $2q$. Show that G is isomorphic to the dihedral group acting on a polygon with q sides.

[**25 — Sylow**] Let P be a p-Sylow subgroup of a finite group G. Prove that $N_G(N_G(P)) = N_G(P)$. (Where $N_G(P)$ is the normalizer of P in G.)

[**26 — Sylow**] Prove that any group of order 99 is abelian.

[**27 — Homomorphisms**] Prove that a nontrivial group G is simple if and only if its only homomorphic images (i.e., groups G' such that there is an onto homomorphism $G \to G'$) are the trivial group and G itself (up to isomorphism).

[**28 — Homomorphisms**] Let $\varphi : G_1 \to G_2$ be a group homomorphism. Prove that $\forall g, h \in G_1$ we have $\varphi([g, h]) = [\varphi(g), \varphi(h)]$, that is, $\varphi(G_1') \subseteq G_2'$. Use the fact that $x \mapsto gxg^{-1}$ is a group homomorphism to show that $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$.

[**29 — Solvability**] Let N be a normal subgroup of a group G. Then G is solvable if and only if both N and G/N are solvable.

[**30 — Symmetric Group**] Prove that S_n is generated by (12) and $(12...n)$.

[**31 — Semidirect Product**] Notice that the subgroup of order 3 in S_3 is normal. Identifying $\mathbb{Z}/3\mathbb{Z}$ with this subgroup gives us a short exact sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \to S_3 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

How is this different from the short exact sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0?$$

[**32 — Semidirect Product**] Show that $D_{2n} \simeq \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. Explicitly find the homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$.

[**33 — Semidirect Product**] Show that $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes S_3 \simeq S_4$.

[**34 — Finite Abelian Groups**] How many abelian groups of order 256 are there?

[**35 — Finite Abelian Groups**] Provide an example or disprove: There exists a finite abelian group G such that $\mathrm{Aut}(G)$ is isomorphic to $\mathbb{Z}/11\mathbb{Z}$.