Analysis Qualifying Exam Preparation

Adriana Fernández Quero

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This document contains problems that will be assigned to the people that will be taking the Spring 22 Qualifying exam for Analysis. The problems have been obtain from various sources, including:

- Homework problems and exam questions from Dr. Chifan on the academic year 2020-2021.
- Homework problems and exam questions from Dr. Lihe Wang on the academic year 2017-2018.
- Homework and exam problems from Dr. Stewart from his Theoretical Numerical Analysis (TNA) class on the academic year 2021-2022.
- Previous qualifying exams, both from The University of Iowa and other universities.
- Problems from other books, mainly Rudin and Royden.

UPDATE (Summer 2023): Homework and exam problems from Dr. Curto's Spring 2023 Complex Analysis class have been added to the document.

Real Analysis

Measure Theory

1. **(Qual Fall 2007** #3**)** Let A be a subset of $\mathbb{R}$ with the property that for each $\epsilon > 0$ there are Lebesgue measurable sets B and C such that $B \subset A \subset C$ and $m(C \cap B^c) < \epsilon$. Show that A is measurable.

2. **(Qual Summer 2017** #1**)** Show that if $f : \mathbb{R} \to \mathbb{R}$ is measurable then the set $A = \{x \in \mathbb{R} : m(f^{-1}(x)) > 0\}$ has measure zero.

3. If $\{f_n\}$ converges to f in L^1, then there exists a subsequence $\{f_{n_k}\}_k$ such that $\lim_{k\to\infty} f_{n_k} = f$ almost everywhere.

4. Assume $m(E) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise almost everywhere on E to f and f is finite almost everywhere on E. Then, $f_n \to f$ in measure on E.

5. **(Riesz Theorem)** If $f_n \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise almost everywhere on E to f.

6. **(Qual Fall 2007** #4**)** If $f \in L^1(\mathbb{R})$ is it true that $\lim_{|x|\to\infty} f(x) = 0$?

7. **(Qual Fall 2007** #5**)** True/False: If $\{f_n\}_n$ is a sequence of Lebesgue measurable functions such that $0 \le f_1 \le f_2 \le \ldots$, $\sup_n \int_{\mathbb{R}} f_n(x)dx < \infty$ and if $f_n \to f$ as $n \to \infty$ for all x, then $\{x : f(x) = \infty\}$ has measure zero.

8. **(Qual Summer 18** #5**)** Suppose $\{f_n\}_n$ is a non-negative sequence of Lebesgue measurable functions defined on $\mathbb{R}$ that converges almost everywhere to the function f. True/False: If $\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = 0$ then $f = 0$ almost everywhere.

9. **(Qual Spring 2018** #1(b)**)** Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Show that f has at most countably many discontinuity points. Conversely, if $A \subset \mathbb{R}$ is a countable set, show there is a function f whose discontinuity points coincide with A.

10. **(KU-S04-07)** Let $p \ge 1$ be a real number and let $\{f_n\}_n \subset L^p(\mathbb{R}, \lambda)$ be a sequence with $\lim_{n\to\infty} ||f_n||_p = 0$. Prove there exists integers $1 \le n_1 \le n_2 \le \ldots$ such that $\lim_{k\to\infty} f_{n_k} = 0$ λ-almost everywhere.

11. **(KU-S09-06)** Let E be a Lebesgue measurable set in $\mathbb{R}^n$. Prove that $E = A_1 \cup N_1 = A_2 \setminus N_2$ where A_1 is an F_σ set, A_2 is a G_δ set and $m(N_1) = m(N_2) = 0$.

12. **(KU-F14-06)** Let (X, M, μ) be a measure space. Suppose $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are measurable functions. Prove that the sets $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable.

13. **(Homework 3 - Chifan)** Let $A \subset [0, 1]$ be a set containing all numbers which do not have the digit 4 appearing in their decimal expression. Find $m(A)$.

14. **(Homework 3 - Chifan)** Suppose $\{E_k\}_k$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that $\sum_{k \in \mathbb{N}} m(E_k) < \infty$. Let $E = \{x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k\}$. Show that E is measurable and $m(E) = 0$.

15. **(Homework 3 - Chifan)** Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for every x. Show there exists a sequence $\{c_n\}$ of positive real numbers such that $\frac{f_n(x)}{c_n} \to 0$ as $n \to \infty$ for every x.

16. **(Homework 4 - Chifan)** Suppose E is a measurable subset of $\mathbb{R}$ with $m(E) > 0$. Prove that the difference set contains an open interval centered at the origin.

17. **(Homework 4 - Chifan)** Let N denote a non-measurable subset of $I = [0,1]$.

    (i) Prove that if $E \subset N$ is measurable, then $m(E) = 0$.

    (ii) If G is a subset of $\mathbb{R}$ with $m(G) > 0$, prove that G contains a non-measurable set.

18. **(UI Mid I Fall 13 - Chifan)** Prove that for a sequence of measurable sets $\{A_k\}_k$ we have

$$m\left(\lim_{n\to\infty} \inf_{k\geq n} A_k\right) \leq \lim_{n\to\infty} \inf_{k\geq n} m(A_k)$$

and

$$m\left(\lim_{n\to\infty} \inf_{k\geq n} A_k\right) \geq \lim_{n\to\infty} \sup_{k\geq n} m(A_k).$$

19. **(Fall 14 MT 1 - Chifan)** Let $E = [0,1] \setminus \mathbb{Q}$. Find $m(E)$.

20. **(Review MT 1 - Chifan)** Let E be the subset of all elements in $[0,1]$ which do not contain the digits 3 and 9 in their decimal expansion. Is E Lebesgue measurable? If yes, find its measure.

21. **(Review MT 1 - Chifan)** Let $\{E_k\}_k$ be a sequence of measurable subsets in $\mathbb{R}$ such that $\sum m(E_k) < \infty$. Consider the subsets

$$A_n = \{x \in \mathbb{R} : x \in E_k \text{ for exactly } n \text{ values of } k\}$$
$$B_n = \{x \in \mathbb{R} : x \in E_k \text{ for at least } n \text{ values of } k\}$$

Show that A_n and B_n are measurable and

$$\sum_n n \cdot m(A_n) = \sum_n m(B_n) = \sum_n m(E_n).$$

22. **(Midterm 1 - Chifan)** Let $E \subset [0,1]$ be the set of all numbers that have infinitely many 7 in its decimal decomposition. Show that E is Lebesgue measurable and find its measure.

23. **(Midterm 1 - Chifan)** Is there an uncountable closed set E with $\mathbb{Q} \subset E^c$?

24. **(Midterm 1 - Chifan)** Let $\{E_n\}_n$ be a sequence of measurable sets in $\mathbb{R}$ such that $m(E_n \cap E_{n+1}^c) = \frac{1}{n^2}$ and $m(E_n) = \frac{1}{n}$. Find

$$m\left(\bigcap_{n\in\mathbb{N}} \bigcup_{n\geq k} E_k\right).$$

25. **(Midterm 1 - Chifan)** Let $\{f_n\}$ be a sequence of measurable functions defined on $\mathbb{R}$. If for every $\alpha > 0$ we have that

$$\lim_{n\to 0} m(\{x \in \mathbb{R} : |f_n(x)| > \alpha\}) = 0,$$

then there exists a subsequence $(n_k)_k$ such that $\lim\limits_{k\to\infty} f_{n_k}(x) = 0$ for almost every $x \in \mathbb{R}$.

26. **(Rudin Theorem 1.36)** Let (X, M, μ) be a measure space, let M^* be the collection of all $E \subset X$ for which there exists sets A and B such that $A \subset E \subset B$ and $\mu(B - A) = 0$, and define $\mu(E) = \mu(A)$ in this situation. Show that M^* is a σ-algebra and μ is a measure on M^*.

27. **(Rudin Chapter 1 #6)** Let X be an uncountable set, let M be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that M is a σ-algebra in X and μ is a measure on M.

28. **(Rudin Chapter 1 #10)** Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

29. **(Rudin Chapter 6 Theorem 6.11)** Let μ and λ be positive measures on a σ-algebra M. We say λ is <u>absolutely continuous</u> with respect to μ (written as $\lambda << \mu$) if $\lambda(E) = 0$ for every $E \in M$ for which $\mu(E) = 0$. Prove the name given to this property is well-deserved, by proving the following are equivalent:

   (i) $\lambda << \mu$,

   (ii) For every $\epsilon > 0$, there exists $\delta > 0$ such that $\lambda(E) < \epsilon$ for all $E \in M$ with $\mu(E) < \delta$.

30. **(Review Final - Dr. Wang)** If $f \in L^1(\mathbb{R})$ show that $\lim_{n\to\infty} n^{-p} f(nx) = 0$ almost everywhere for any $p > 0$.

31. **(Harvard University - Qual 2011)** Let $A \subset [0,1]$ be the set of real numbers x that, when written as a decimal $x = 0.a_1a_2a_3...$ satisfy the rule $a_{n+2} \notin \{a_n, a_{n+1}\}$ for all $n \geq 1$. What is the Lebesgue measure of A?

32. **(TAM - Qual 2020)** Let μ be a Borel measure on $[0,1]$ such that $\mu([0,1]) = 1$. Show that if μ is atomless, then for every $0 < r < 1$, there exists a Borel measurable set A such that $0 < \mu(A) < r$. NOTE: A set A is an atom for the measure μ if $\mu(A) > 0$ and for every $E \subset A$ either $\mu(E) = 0$ or $\mu(E) = \mu(A)$.

33. **(UCLA Spring 2011 #3)** Let μ be a Borel measure on $\mathbb{R}$ and define $f(t) = \int e^{itx} d\mu(x)$. Suppose that

$$\lim_{t \to 0} \frac{f(0) - f(t)}{t^2} = 0.$$

Show that μ is supported at 0.

67. **(aTm-F02-Q2)** Suppose (X, M, μ) is a measure space, μ a σ finite measure and $f : X \to [0, \infty)$ is a measurable function. Suppose $\int_A f d\mu = \mu(A)$ for each measurable set A with $\mu(A) < \infty$. Prove that $f = 1$ almost everywhere.

Integration Theory

34. Let $\{a_n\}_n$ be a sequence of non-negative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x \leq n+1$. Show that

$$\int_E f = \sum_{n \in \mathbb{N}} a_n.$$

35. **(Qual Spring 2018 #4)** Let $f_n \in L^3([0,1])$ be non-negative functions such that $||f_n||_3 = 1$ for all $n \in \mathbb{N}$ and $f_n \to 0$ almost everywhere as $n \to \infty$. Show that

$$\int_0^1 f_n d\mu \to 0$$

as $n \to \infty$.

36. **(Qual Spring 2007 #5)** Let $F_k \subset [0,1]$ for all $k \in \mathbb{N}$ be measurable sets, such that there exists $\delta > 0$ with $m(F_k) \geq \delta$ for all k. Assume the sequence $a_k \geq 0$ satisfies

$$\sum_{k=1}^{\infty} a_k \chi_{F_k}(x) < \infty$$

for almost every $x \in [0,1]$. Show that $\sum\limits_{k=1}^{\infty} a_k < \infty$.

37. **(Qual Summer 19 #1)** Suppose f_n and f are measurable functions on $(0,1)$. Suppose $f_n \to f$. Is it true that $\int_0^1 f_n(x)dx \to \int_0^1 f(x)dx$?

38. **(Qual Summer 2019 #5)** If $f \in L^1(\mathbb{R})$, show that $\sum\limits_{n=-\infty}^{\infty} f(x+n)$ is convergent almost everywhere to a function which has period 1.

39. **(OSU - Qual 2020)** If $f : [0,1] \to [0, \infty)$ is non-negative and integrable, then

$$\lim_{n \to \infty} \int_0^1 \sqrt[n]{f} = m(\{x : f(x) > 0\}).$$

40. **(KU-S15-03)** Compute the following limit

$$\lim_{n \to \infty} \int_0^1 \frac{1 + \frac{n}{2}x}{(1+x)^{n/2}} dx.$$

41. **(KU-S13-05)** Let g be a bounded Lebesgue measurable function on $\mathbb{R}$ which has the property that $\lim\limits_{n \to \infty} \int_I g(nx)dx = 0$ for every interval $I \subset [0,1]$. Prove that for $f \in L^1([0,1])$,

$$\lim_{n \to \infty} \int_0^1 f(x)g(nx)dx = 0.$$

42. **(aTm-F09-Q4)** Let (X, M, μ) be a measure space with $\mu(X) < \infty$. Given sets $A_i \in M$ with $i \geq 1$, prove that

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu\left(\bigcap_{i=1}^{n} A_i\right).$$

43. **(UIUC-S15-06)** Let f_n be a sequence of Lebesgue measurable functions on the interval $[0, 1]$. Assume that f_n converges to a function f almost everywhere, and that

$$\int_{[0,1]} |f_n|^2 d\lambda \leq 1$$

for each n. Prove that f_n converges to f in L^1.

44. **(UIUC-M17-02)** Compute

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \frac{1}{1+e^{-kx}} \frac{1}{1+x^2} dx.$$

45. **(IU-S08-01)** Give an example of a function $f : [0, \infty) \to \mathbb{R}$ that satisfies

   (i) $f(x) \geq 0$ for all $x \geq 0$,

   (ii) for every $M > 0$, $\sup_{x>M} f(x) = \infty$, and

   (iii) $\int_0^\infty f(x)dx < \infty$

   or prove that no such function exists.

46. **(PSU-S17-04)** Suppose $g : [0, 1] \to [0, 1]$ is a Lebesgue measurable function. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with $f(0) \leq f(1)$. Show that the limit

$$\lim_{n\to\infty} \int_0^1 f(g(x)^n)dx$$

exists and lies in the interval $[f(0), f(1)]$.

47. **(Homework 5 - Chifan)** Prove that if f is integrable on $\mathbb{R}^d$ and $\delta > 0$, then $f(\delta x)$ converges to $f(x)$ in the L^1 norm as $\delta \to 1$.

48. **(Homework 5 - Chifan)** Suppose f is integrable on $(-\pi, \pi]$ and extend the function to $\mathbb{R}$ by making it of period 2π. Show that

$$\int_{-\pi}^{\pi} f(x)dx = \int_T f(x)dx$$

for any interval I in $\mathbb{R}$ of length 2π.

49. **(Homework 5 - Chifan)** If f is uniformly continuous and integrable on $\mathbb{R}$, then $\lim_{|x|\to\infty} f(x) = 0$. True or False?

50. **(Homework 5 - Chifan)** If f is integrable on $\mathbb{R}$ show that $F(x) = \int_{-\infty}^{x} f(t)dt$ is uniformly continuous.

51. **(Homework 5 - Chifan)** Suppose $f \geq 0$ and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$ prove that $m(E_\alpha) \leq \frac{1}{\alpha} \int_{\mathbb{R}} f(x)dx$.

52. **(Homework 5 - Chifan)** A sequence $\{f_k\}_k$ of measurable functions on $\mathbb{R}^d$ is Cauchy in measure if for every $\epsilon > 0$, $m(x : |f_k(x) - f_l(x)| > \epsilon) \to 0$ as $k, l \to \infty$. We say that $\{f_k\}$ converges in measure to a measurable function if for every $\epsilon > 0$, $m(\{x : |f_k(x) - f(x)| > \epsilon) \to 0$ as $k \to \infty$. Prove that if a sequence $\{f_k\}_k$ of integrable functions converges to f in L^1, then $\{f_k\}_k$ converges to f in measure. Is the converse true?

53. **(Homework 6 - Chifan)** Let $\{f_n\}_n$ be a collection of integrable functions in $L^1(\mathbb{R}^d)$ such that $f_n \to f$ pointwise as $n \to \infty$ almost everywhere. If $f \in L^1(\mathbb{R}^d)$, then

$$\lim_{n\to\infty} ||f - f_n||_1 = 0 \iff \lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n| = \int_{\mathbb{R}^d} |f|.$$

54. **(Homework 6 - Chifan)** Let g be a measurable function on $\mathbb{R}$. Show that for f an integrable function over $\mathbb{R}$ we have

$$\lim_{t\to 0} \int_{\mathbb{R}} g(x)(f(x) - f(x+t))dx = 0.$$

55. **(Previous Chifan Exam Problem)** Let $f, f_n : \mathbb{R} \to \mathbb{R}$ be integrable functions such that $\int_{\mathbb{R}} |f_n - f|dx \leq \frac{1}{n^2}$ for every n. Prove that $f_n \to f$ with respect to Lebesgue measure.

56. **(Previous Chifan Exam Problem)** For $x \in [1, \infty)$, find

$$\lim_{n\to\infty} \int_1^\infty \left(1 + \frac{x}{n}\right)^n e^{-2x} dx.$$

57. **(Previous Chifan Exam Problem)** Let $f : [0,1] \to [0,\infty)$ be a measurable function. Suppose there exists $M > 0$ such that

$$\int_0^{1-\frac{1}{n}} f(x)dx \leq M, \quad \forall n \geq 1.$$

Show f is Lebesgue integrable on $[0,1]$.

58. **(Review Midterm 2 - Chifan)** Let $f : \mathbb{R} \to [0,\infty)$ be an integrable function such that $\int_{\mathbb{R}} f^n(x)dx = \int_{\mathbb{R}} f(x)dx$ for all $n \in \mathbb{N}$. Show that there is a measurable set $E \subset \mathbb{R}$ such that $f(x) = \chi_E(x)$ for almost every $x \in \mathbb{R}$.

59. **(Review Midterm 2 - Chifan)** Let $m(E) < \infty$ for a measurable set $E \subset \mathbb{R}$. Suppose $\{f_k\}_k$ is a sequence of uniformly continuous functions over E. If $f_k \to f$ pointwise almost everywhere on E as $k \to \infty$ and if f is integrable over E, then

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

60. **(Midterm 2 - Chifan)** Let $f : [0,1] \to (0,\infty)$ be an integrable function and let $\{F_n\}_n \subset [0,1]$ be a sequence of measurable sets such that

$$\int_{F_n} f(x)dx \to 0 \text{ as } n \to \infty.$$

Show that $m(F_n) \to \infty$ as $n \to \infty$.

61. **(Midterm 2 - Chifan)** Let $f \in L^1(\mathbb{R})$, then

$$\lim_{n\to\infty} \int_0^2 f(x)\sin(x^n)dx = 0.$$

62. **(UCLA Spring 2011 #4)** Let $f_n : [0,1] \to [0,\infty)$ be Borel functions with

$$\sup_n \int_0^1 f_n(x)\log(2 + f_n(x))dx \le M < \infty.$$

Suppose $f_n \to f$ Lebesgue almost everywhere. Show that $f \in L^1$ and $f_n \to f$ in L^1. HINT: For $f_n \to f$ in L^1 use Egorov's Theorem and the absolute continuity of integrable functions.

63. **(UCLA Fall 2012 #5)** Let $f_n : \mathbb{R}^3 \to \mathbb{R}$ be a sequence of functions such that $\sup_n ||f_n||_{L^2} < \infty$. Show that if f_n converges almost everywhere to a function $f : \mathbb{R}^3 \to \mathbb{R}$, then

$$\int_{\mathbb{R}^3} \left| |f_n|^2 - |f_n - f|^2 - |f|^2 \right| dx \to 0.$$

64. **(UCLA Fall 2013 #8)** Compute the following:

$$\lim_{k\to\infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k dx, \text{ and } \lim_{k\to\infty} \int_0^\infty \left(1 + \frac{x}{k}\right)^{-k} \cos(x/k)dx$$

for $n \in \mathbb{N}$.

Differentiation Theory

65. **(Qual Fall 2020 #3)** Suppose E is a measurable set such that $m(E \cap (a,b)) \geq \frac{b-a}{2}$ for all rational numbers satisfying $a < b$. Show that E is the whole axis except for a set of measure zero.

66. **(Qual August 2019 #3)** Show that there is no measurable set such that $m(E \cap (a,b)) = \frac{b-a}{2}$ for all $a < b$.

68. **(UIUC-F15-04)** Fix $1 \leq p < \infty$.

    (i) Assume that f is absolutely continuous on every compact interval, and $f' \in L^p(\mathbb{R}, m)$. Prove that

    $$\sum_{n \in \mathbb{Z}} |f(n+1) - f(n)|^p < \infty.$$

    (ii) Prove or give a counterexample: The statement above remains valid if we instead assume that f is continuous, of bounded variation on every compact interval, and $f' \in L^p(\mathbb{R}, m)$.

69. **(Homework 7 - Chifan)** Prove that if a measurable subset E of $[0,1]$ satisfies $m(E \cap I) \geq \alpha m(I)$ for some $\alpha > 0$ and all intervals I in $[0,1]$, then E has measure 1.

70. **(Homework 8 - Chifan)** If f is of bounded variation on $[a,b]$, then

    (a) $\int_a^b |f'(x)| dx \leq TV_f(a,b)$.

    (b) $\int_a^b |f'(x)| dx = TV_f(a,b)$ if and only if f is absolutely continuous.

71. **(Homework 8 - Chifan)** Show that if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then

    (a) f maps sets of measure zero to sets of measure zero, and

    (b) f maps measurable sets to measurable sets.

72. **(Homework 8 - Chifan)** Suppose f is an increasing function on $[a,b]$.

    (a) Prove we can write $f = f_A + f_C + f_J$ where each f_A, f_C, f_J are increasing, f_A is an absolutely continuous function, f_C is continuous with $f_C'(x) = 0$ almost everywhere, and f_J is a jump function.

    (b) Prove that each component is uniquely determined up to an additive constant.

73. **(Homework 8 - Chifan)** Suppose $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove that f satisfies the Lipschitz condition if and only if f is absolutely continuous and $|f'(x)| \leq M$ for all $x \in \mathbb{R}$.

74. **(Homework 9 - Chifan)** Let f and g be absolutely continuous functions on $[a,b]$. Then,

    $$\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g.$$

75. **(Homework 9 - Chifan)** Let A be a measurable subset of $[0,2]$ and define $f : \mathbb{R} \to \mathbb{R}$ by letting $f(x) = m((-\infty, x] \cap A)$ for every $x \in \mathbb{R}$.

(a) Show that f is absolutely continuous on $\mathbb{R}$. Calculate f' and find $\int_0^2 f'(x)dx$.
(b) Show that for every $0 < b < m(A)$ there exists $x_0 \in \mathbb{R}$ such that $b = m((-\infty, x_0] \cap A)$.

76. **(Homework 9 - Chifan)** Construct an absolutely continuous function that is strictly continuous on $[0,1]$ for which $f' = 0$ on a set of positive measure.

77. **(Royden - Riesz Nagy)** Let E be a set of measure zero contained in the open interval (a,b). Assume there is a countable collection of open intervals contained in (a,b), $\{(c_k, d_k)\}_k$ for which each point in E belong to infinitely many intervals in the collection and $\sum_{k=1}^{\infty} |d_k - c_k| < \infty$. Define

$$f(x) = \sum_{k \geq 1} m((c_k, d_k) \cap (-\infty, x))$$

for every $x \in (a,b)$. Show that f is increasing and fails to be differentiable at each point in E.

78. **(Royden Section 6.4 #37)** Let f be a continuous function on $[0,1]$ that is absolutely continuous on $[\epsilon, 1]$ for each $0 < \epsilon < 1$. Show that f is absolutely continuous on $[0,1]$ if it is increasing.

79. **(Royden Section 6.4)** Prove:

(a) The sum of absolutely continuous functions is absolutely continuous; and
(b) If f is Lipschitz on $\mathbb{R}$ and g absolutely continuous on $[a,b]$, then $f \circ g$ is absolutely continuous on $[a,b]$.
(c) Let f, g be absolutely continuous functions and assume g is strictly monotone on $[a,b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a,b]$.

80. **(Royden Section 6.5 #49)** Let f be continuous on $[a,b]$ and differentiable almost everywhere on (a,b). Show

$$\int_a^b f'(t)dt = f(b) - f(a) \iff \int_a^b \lim_{n\to\infty} \frac{f(x+1/n) - f(x)}{1/n} dx = \lim_{n\to\infty} \int_a^b \frac{f(x+1/n) - f(x)}{1/n} dx.$$

81. **(Old Chifan Test Problem)** For every $x \in (0,1)$ and $\epsilon > 0$ there exists $0 < r < \epsilon$ such that $\int_{x-r}^{x+r} f(x)dx \geq 2r$. Show $f \geq 1$ for almost every $x \in [0,1]$.

82. **(Review Midterm 3 - Chifan)** Assume $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing such that

$$\int_{\mathbb{R}} f'(t)dt = 1, \quad \lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to\infty} f(x) = 1.$$

Show that f is absolutely continuous in any interval $[a,b]$.

83. **(Midterm 3 - Chifan)** Find all integrable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying that for every $I \subset \mathbb{R}$,

$$\left| \int_I f(t)dt \right| \leq (m(I))^2.$$

84. **(Midterm 3 - Chifan)** If f is increasing then f' exists almost everywhere. Moreover f' is measurable, non-negative and

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

If the equality holds, show f is absolutely continuous.

85. **(Midterm 3 - Chifan)** If $E + q = E$ for all rational numbers $q \in \mathbb{Q}$, then $m(E) = 0$ or $m(\mathbb{R} \setminus E) = 0$.

86. **(Review Final - Dr. Wang)** Let Ω be an open set in $\mathbb{R}^n$. Suppose that for all $x \in \partial\Omega$ we have $|\Omega \cap B_r(x)| \geq \frac{1}{2}|B_r|$ for all $0 < r < 1$. Show that $|\partial\Omega| = 0$.

87. **(Midterm 1 - Dr. Wang)** Let $E \subset [0, 1]$ measurable and assume that $m(E \cap (a, b)) \geq \frac{b-a}{4}$ for all $0 < a, b < 1$. Prove that $m(E) = 1$.

88. **(UCLA Spring 2013 #1)** Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, Lebesgue measurable, and

$$\lim_{h \to 0} \int_0^1 \frac{|f(x+h) - f(x)|}{h} dx = 0.$$

Show that f is a.e. constant on $[0, 1]$.

L^p spaces

89. **(Qual Fall 2020 #5)** Suppose (X, M, μ) is a measure space, μ positive measure, $f_n \in L^p(X)$ for all $n \in \mathbb{N}$ and $f \in L^p(X)$ for all $1 \leq p < \infty$. Prove

    (i) if $||f_n - f||_p \to 0$ as $n \to \infty$, then $||f_n||_p \to ||f||_p$ as $n \to \infty$; and

    (ii) if $f_n \to f$ almost everywhere and $||f_n||_p \to ||f||_p$ then $||f_n - f||_p \to 0$.

90. **(KU-F08-08)** Decide which space is bigger, $L^1([0,1])$ or $L^2([0,1])$. Explain why.

91. **(Kesavan Section 6.3 #1)** Let (X, S, μ) be a measure space, and let $1 \leq p, r, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, show that $fg \in L^r(\mu)$ and

$$||fg||_r \leq ||f||_p ||g||_q.$$

92. **(Kesavan Section 6.3 #2)** Let (X, S, μ) be a measure space and let $1 \leq p < \infty$. For $t > 0$ define $h_f(t) = \mu(\{|f| > t\})$. Show that

$$||f||_p^p = p \int_0^\infty t^{p-1} h_f(t) dt$$

93. **(Kesavan Section 6.3 #4)** Let (X, S, μ) be a measure space and let $1 \leq p < \infty$. Let $f_n \to f$ in $L^p(\mu)$. Let g_n be a sequence of measurable functions converging to a measurable function g almost everywhere in X. Assume further that g_n and g are all uniformly bounded by a constant $M > 0$ in X. Show that $f_n g_n \to fg$ in $L^p(\mu)$.

94. **(Kesavan Section 6.3 #5)** Let $f \in L^p(0, \infty)$ where $1 < p < \infty$. Define

$$g(x) = \frac{1}{x} \int_0^\infty f(t) dt$$

for $x \in (0, \infty)$. Show that $g \in L^p(0, \infty)$ and that

$$||g||_p \leq \frac{p}{p-1} ||f||_p.$$

95. **(Riemann-Lesbesgue Lemma)** Let $f \in L^1$. Then,

$$\lim_{n \to \infty} \int f(x) e^{inx} dx = 0.$$

96. **(TNA Homework 1 - Dr. Stewart)** Give two norms on infinite sequences $(x_1, x_2, ...)$ that are not equivalent.

97. **(TNA Homework 1 - Dr. Stewart)** Show that if two norms, $|| \cdot ||_a$ and $|| \cdot ||_b$, are equivalent then $x_k \to x$ as $k \to \infty$ in $|| \cdot ||_a$ if and only if $x_k \to x$ as $k \to \infty$ in $|| \cdot ||_b$.

98. **(TNA Homework 1 - Dr. Stewart)** Show that the space $Lip[a, b]$ of Lipschitz functions $[a, b] \to \mathbb{R}$ is a Banach space with the norm

$$||f||_{Lip} = ||f||_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

You may use the following lemma without proof:

Lemma. *Supposing that all limits exists,*

$$\lim_{k\to\infty} \sup_{a\in A} \psi_k(a) \geq \sup_{a\in A} \lim_{k\to\infty} \psi_k(a).$$

99. **(TNA Homework 2 - Dr. Stewart) Theorem.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then for every $\epsilon > 0$ there is a polynomial p where*

$$\max_{a\leq x\leq b} |f(x) - p(x)| < \epsilon.$$

To establish this theorem, you can use the following steps:

(a) Assume $[a, b] = [0, 1]$. For any positive integer n, define $x_{k,n} = k/n$,

$$\phi_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \quad \text{and}$$
$$f_n(x) = \sum_{k=0}^{n} f(x_{k,n})\phi_{k,n}(x).$$

Show that $\sum_{k=0}^{n} \phi_{k,n}(x) = 1$ for all x, and that $\phi_{k,n}(x) \geq 0$ for all $x \in [0, 1]$.

(b) Show that $\sum_{k=0}^{n} x_{k,n}\phi_{k,n}(x) = x$ for all x.

(c) Show that

$$\sum_{k=0}^{n} x_{k,n}^2 \phi_{k,n}(x) = x[(1 - 1/n)x + 1/n]$$

and that

$$\sum_{k=0}^{n} (x - x_{k,n})^2 \phi_{k,n}(x) = x(1 - x)/n.$$

(d) Let $\omega(f, \delta) = \max_{|x-y|\leq\delta} |f(x) - f(y)|$. Show that

$$|f(x) - f(y)| \leq \omega(f, \delta)[1 + |x - y|/\delta].$$

(e) Show that $f(x) - f_n(x) = \sum_{k=0}^{n} (f(x) - f(x_{k,n}))\phi_{k,n}(x)$ so that

$$|f(x) - f_n(x)| \le \sum_{k=0}^{n} \omega(f,\delta)\left[1 + \frac{|x - x_{k,n}|}{\delta}\right] \phi_{k,n}(x)^{1/2}\phi_{k,n}(x)^{1/2}.$$

Use the Cauchy-Schwart inequality to give the bound

$$|f(x) - f_n(x)| \le \omega(f,\delta)\left[1 + \frac{1}{\delta}\left(\frac{x(1-x)}{n}\right)^{1/2}\right].$$

Substitute $\delta = n^{-1/2}$ to bound $|f(x) - f_n(x)|$ in terms of $\omega(f, n^{-1/n})$.

(f) Use uniform continuity to show that $\omega(f,\delta) \to 0$ as $\delta \to 0^+$. Complete the proof.

100. **(TNA Homework 2 - Dr. Stewart)** Use the Weierstrass approximation theorem to show there is a countable dense subset of $C[a,b]$ in $||\cdot||_\infty$ norm; that is, $C[a,b]$ is a separable Banach space.

101. **(TNA Midterm - Dr. Stewart)** Show that the polynomials in $[a,b]$ are dense in $L^p[a,b]$ for all $1 \le p < \infty$. Does it hold when $p = \infty$?

102. **(TNA Homework 2 - Dr. Stewart)** Let $m : \ell^1 \to \mathbb{R}$ be the linear functional given by $m(x) = \sum_{i=1}^{\infty}(1 - 1/i)x_i$. Show $||m||_{(\ell^1)^*} = 1$, but there is no $x \in \ell^1$ with $||x||_{\ell^1}$ such that $|m(x)| = 1$.

103. **(Rudin Chapter 3 #4(a))** Suppose f is a measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p d\mu = ||f||_p^p$$

for $0 < p < \infty$. Let $E = \{p : \varphi(p) < \infty\}$. Show that if $r < p < s$ and $r, s \in E$, then $p \in E$.

104. **(Ruding Chapter 3 #4(e))** Suppose f is a measurable function on X and μ is a positive measure on X. Let f be a function with $||f||_\infty > 0$, and assume $||f||_r < \infty$ for some $r < \infty$. Prove that

$$||f||_p \to ||f||_\infty \quad \text{as } p \to \infty.$$

You may use this to start: If $||f||_\infty < \infty$, define $E_\alpha = \{x : |f(x)| > \alpha\}$ for $\alpha \in (0, ||f||_\infty)$. Show that $0 < \mu(E_\alpha) < \infty$, and show that $||f||_p \ge \alpha[\mu(E_\alpha)]^{1/p}$ for any $\alpha \in (0, ||f||_\infty)$.

105. **(Rudin Chapter #11)** Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \ge 1$. Prove that

$$\int_\Omega f d\mu \cdot \int_\Omega g d\mu \ge 1.$$

106. **(Rudin Chapter 3 #20)** Suppose φ is a real function on $\mathbb{R}$ such that

$$\varphi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 \varphi(f)dx$$

for every bounded measurable f. Prove that φ is convex.

107. **(Rudin Chapter 3 #23)** Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(\mu)$, $||f||_\infty > 0$ and

$$\alpha_n = \int_X |f|^n d\mu, \text{ for } n = 1, 2, 3, ...$$

Prove that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = ||f||_\infty.$$

108. **(Rudin Chapter 3 #24(a))** Suppose μ is a positive measure, $f \in L^p(\mu)$ and $g \in L^q(\mu)$. If $0 < p < 1$, prove that

$$\int \left||f|^p - |g|^p\right| d\mu \leq \int |f-g|^p d\mu$$

and that $\Lambda(f,g) = \int |f-g|^p d\mu$ defines a metric on $L^p(\mu)$.

109. **(Harvard University Qual 2020)**

(i) Specify the range of $1 \leq p < \infty$ for which

$$\varphi(f) = \int_0^1 \frac{f(t)}{\sqrt{t}} dt$$

defines a linear functional $\varphi : L^p([0,1]) \to \mathbb{R}$.

(ii) For those values of p, calculate the norm of the linear functional $\varphi : L^p([0,1]) \to \mathbb{R}$. The norm of the linear functional is defined as

$$||\varphi|| = \sup_{f \in L^p([0,1])\setminus\{0\}} \frac{|\varphi(f)|}{||f||_p}.$$

110. **(Harvard University - Qual 2018)** Let $K \subset \mathbb{R}^n$ be a compact set. Show that for any measurable function $f : K \to \mathbb{C}$, it holds that

$$\lim_{p\to\infty} ||f||_{L^p(K)} = ||f||_{L^\infty(K)}.$$

Recall: $||f||_{L^\infty(K)}$ is the essential supremum of f; i.e. the smallest upper bound if the behavior of f on null sets is ignored.

111. **(Harvard University - Qual 2016)** Let C^0 denote the vector space of continuous functions on the interval $[0,1]$. Define a norm on C^0 as follows: If $f \in C^0$, then its norm is

$$||f|| = \sup_{t\in[0,1]} |f(t)|.$$

Let C^∞ denote the space of smooth functions on $[0,1]$. View C^∞ as a normed, linear space with the norm defined as follows: if $f \in C^\infty$, then its norm is

$$||f||_* = \int_{[0,1]} \left(\left| \frac{d}{dt} f \right| + |f| \right) dt.$$

(a) Prove that C^0 is a Banach space with respect to the norm $||\cdot||$. In particular, prove that it is complete.

(b) Let ψ denote the 'forgetful' map from C^∞ to C^0 that sends f to f. Prove that ψ is a bounded map from C^∞ to C^0.

112. **(Harvard University - Qual 2012)** Suppose f_j and f are real functions on $[0,1]$. Suppose $f_j(x) \to f(x)$ almost everywhere for $x \in [0,1]$. Furthermore, assume that

$$\sup_{j\geq 1} ||f_j||_{L^2([0,1])} \leq 1, \quad ||f||_{L^2([0,1])} \leq 1.$$

(a) Is it always true that

$$\lim_{j\to\infty} ||f_j - f||_{L^2[0,1} = 0?$$

Prove it or give a counterexample.

(b) Is it always true that

$$\lim_{j\to\infty} ||f_j - f||_{L^1[0,1]} = 0?$$

Prove it or give a counterexample.

113. **(Harvard University - Qual 2011)** Prove that a linear operator between Banach spaces is continuous if and only if it is bounded.

114. **(Harvard University - Qual 2010)** Let (X,μ) be a measure space with $\mu(X) < \infty$. For $q > 0$, let $L^q(X,\mu)$ denote the Banach space completion of the space of bounded functions on X with the norm

$$||f||_q = \left(\int_X |f|^q d\mu \right)^{1/q}.$$

Suppose $0 < p \leq q$. Prove that all functions in L^q are in L^p, and that the inclusion map $L^q \to L^p$ is continuous.

115. **(UCLA Spring 2012 #4)** Let $S = \{f \in L^1(\mathbb{R}^3) : \int f(x)dx = 0\}$.

(a) Show that S is closed in the L^1 topology.

(b) Show that $S \cap L^2(\mathbb{R}^3)$ is a dense subset of $L^2(\mathbb{R}^3)$. HINT: Use the fact that the set of L^2 functions with compact support is dense in L^2.

116. **(UCLA Fall 2017 #2)** Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and let $a \in \mathbb{R}$ such that $a > 1 - 1/p$. Show that the series

$$\sum_{n=1}^{\infty} \int_n^{n+n^{-a}} |f(x+y)| dy$$

converges for almost all $x \in \mathbb{R}$.

117. **(UCLA Fall 2017 #3)** Let $f \in L^1_{loc}(\mathbb{R}^d)$ be such that for some $0 < p < 1$, we have

$$\left| \int f(x)g(x)dx \right| \leq \left(\int |g(x)|^p dx \right)^{1/p},$$

for all $g \in C_0(\mathbb{R}^d)$ (continuous functions with compact support). Show that $f(x) = 0$ a.e.

118. **(Halmos - A Hilbert Space Problem Book)** Is the space $A^2(D)$ of square integrable analytic functions on a region D a Hilbert space, or does it have to be completed before it becomes one?

Sobolev Spaces

Note: These problems may only be assigned to those who have taken PDE, since Sobolev spaces are covered in such class.

Definition. We define $W^{1,p}(\Omega) = \{f \in L^p(\Omega) : f' \in L^p(\Omega)\}$ where the derivative is understood as a weak derivative, that is f' is the derivative of f, if for all $\varphi \in C_0^\infty$ we have

$$\int_\Omega f'\varphi dx = -\int_\Omega f\varphi' dx.$$

119. **(Kesavan Def. 6.4.1)** Let $(a, b) \subset \mathbb{R}$ be a finite interval and $1 \leq p < \infty$. Consider $W^{1,p}(a, b)$ and show that this space is a Banach space with norm given by

$$||f||_{W^{1,p}(a,b)} = \left(||f||_p^p + ||f'||_p^p\right)^{1/p}.$$

120. **(Kesavan Proposition 6.4.4)** Let $1 \leq p < \infty$. Let $f \in W^{1,p}(a, b)$. Then, f is absolutely continuous.

121. **(Kesavan Thm 6.4.3)** The inclusion map from $W^{1,p}(a, b)$ to $C[a, b]$, the continuous functions on $[a, b]$, is continuous.

122. **(Kesavan Section 6.3 #20)** Let $(a, b) \subset \mathbb{R}$ be a finite interval and let $f : [a, b] \to \mathbb{R}$ be a Lipschitz continuous function; i.e. there exists $K > 0$ such that for all $x, y \in [a, b]$ we have

$$|f(x) - f(y)| \leq K|x - y|.$$

Show that $f \in W^{1,p}(a, b)$ for all $1 \leq p < \infty$.

123. **(TNA Homework 1 - Dr. Stewart)** We say a real or complex-valued function f is Hölder continuous if there exists non-negative real constants C and α such that

$$|f(x) - f(y)| \leq C||x - y||^{\alpha}$$

for all x, y in the domain of f.

Show that if $f \in W^{1,p}(\mathbb{R})$ for $1 \leq p < \infty$, then f is Hölder continuous.

Metric spaces

124. **(General Topology HW - Dr. Tehrani)** By means of an example, show that for $p < 1$, the equation $|x|^p = \left(\sum_{i=1}^{N} |x_i|^p\right)^{1/p}$ does not define a norm on $\mathbb{R}^N$.

125. **(General Topology HW - Dr. Tehrani)** If d is a metric on X, and X is given the metric topology, show that $d : X \times X \to X$ is continuous.

126. **(General Topology HW - Dr. Tehrani)** Let $f : X \to Y$ be a map between metric spaces with metrics d_X and d_Y. If $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$, show that f is an embedding (called isometric embedding).

127. **(General Topology HW - Dr. Tehrani)** Suppose $f : X \to X$ is an isometry on a compact metric space X. Show f is a homeomorphism.

128. **(Homework 3 - Dr. Curto)** Suppose Ω is a complete metric space and that $f : (D, d) \to (\Omega, \rho)$ is uniformly continuous, where D is dense in (X, d). Show that there is a uniformly continuous function $g : X \to \Omega$ with $g(x) = f(x)$ for every $x \in D$.

129. **(Homework 3 - Dr. Curto)** Let $(f_n)_n$ be a sequence of uniformly continuous functions from (X, d) into (Ω, ρ) and suppose that f_n converges to f uniformly. Prove that f is uniformly continuous. If each f_n is a Lipschitz function with constant M_n and $\sup M_n < \infty$, show that f is a Lipschitz function. If $\sup M_n = \infty$, show that f may fail to be Lipschitz.

130. **(Homework 9 - Dr. Curto)** Suppose $\{f_n\}$ is a sequence in $C(G, \Omega)$ which converges to f and $\{z_n\}$ is a sequence in G which converges to a point z in G. Show that $\lim_n f_n(z_n) = f(z)$.

131. **(Homework 9 - Dr. Curto)** (Dini's Theorem) Consider $C(G, \mathbb{R})$ and suppose that $\{f_n\}$ is a sequence in $C(G, \mathbb{R})$ which is monotonically increasing and $\lim f_n(z) = f(z)$ for all $z \in G$ where $f \in C(G, \mathbb{R})$. Show that $f_n \to f$.

Complex Analysis

Holomorphic Functions and Power Series

1. **(Review Final - Chifan)** Let f be an analytic function on the open disk $\mathbb{D}$. Assume that for every $z \in \mathbb{D}$ the power series expansion around z has a vanishing coefficient. Show that f is a polynomial function.

2. **(Review Final - Chifan)** Find the domain of convergence for the following function
$$f(z) = \sum_{n=1}^{\infty} \left(\sqrt[n]{3 + \frac{1}{n}} - 1 \right)^n (nz - n)^n.$$

3. **(Review Final - Chifan)** Find the domain of convergence for the following function
$$f(z) = \sum_{n=1}^{\infty} n! \left(\frac{z}{n} \right)^n.$$

4. **(Review Final - Chifan)** Assume $(a_n) = (1, 1, 2, 3, 5, 8, ...)$ is the Fibonacci sequence. Consider the power series $f(z) = \sum_n a_n z^n$. Find the radius of convergence for $f(z)$ and also determine the singularity on the circle of convergence (in case is finite).

5. **(Review Final - Chifan)** Find all entire functions f satisfying $|f(z)| \geq e^{2Im(z)}$ for every $z \in \mathbb{C}$.

6. **(Review Final - Chifan)** Let $f(z) = f(x, y) = u(x, y) + iv(x, y)$ be an entire function. Suppose that u is a function of x alone and v is a function of y alone. Find f.

7. **(Review Final - Chifan)** Find all entire functions satisfying $\lim_{|z| \to \infty} \frac{|f(z)|}{1+|z|^{5/2}} = 0$.

8. **(Review Final - Chifan)** Find all entire functions f satisfying
$$|f(z)| \leq 2020|z^2 - 3z + 2|^2, \quad \forall z \in \mathbb{C}.$$

9. **(Review Final - Chifan)** Find all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying
$$(4n^2 - 4n + 1) f \left(\frac{n+1}{2n-1} \right) = 3n^2 + 3n.$$

10. **(Review Final - Chifan)** Find all entire functions with the property that $|f(z)| = 1$ when $|z| = 1$. HINT: Let $g(z) = \frac{1}{\overline{f(1/\bar{z})}}$. What relationship is there between g and f? What can you say about the zeros of f?

11. **(aTm-S10-07)** Let entire functions f and g satisfy $e^f + e^g = 1$. Prove that both f and g are constants.

12. **(Stein Chapter 2 Thm 5.3)** Assume that $f_n : \mathbb{D} \to \mathbb{C}$ is a sequence of holomorphic functions such that f_n converges to f uniformly on the compact subsets of $\mathbb{D}$. Show that f is holomorphic and f_n' converges to f' uniformly in the compact subsets of $\mathbb{D}$.

13. **(Stein Chapter 3 Thm 6.2)** If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that $f(z) = e^{g(z)}$.

14. **(KU-S04-04)** Let $f : [0, \infty) \to \mathbb{C}$ be a Lebesgue measurable function. Assume there exists real numbers $a, k > 0$ such that

$$|f(x)| \leq ae^{-kx}, \quad \forall x \geq 0.$$

Consider the half plane $H = \{z \in \mathbb{C} : Im(z) > k\}$.

   (a) Prove that for every $z \in H$, the function $g : [0, \infty) \to \mathbb{C}$ defined by $g(t) = e^{itz} f(t)$ is Lebesgue integrable.

   (b) Prove that the function $F : H \to \mathbb{C}$ defined by $F(z) = \int_0^\infty e^{itz} f(t)dt$ is holomorphic.

15. **(KU-S13-03)** Let $\alpha > 0$, and define the set $S = \{re^{i\theta} : r > 0, 0 < \theta < \alpha\}$. Let f be a bounded holomorphic function on S. Show that

$$\lim_{r\to\infty} f'(re^{i\theta}) = 0$$

for each $0 < \theta < \alpha$.

16. **(aTm-F11-04)** Suppose f is a continuous function on $\{z \in \mathbb{C} : |z| \leq 1\}$ and f is holomorphic on the open unit disc. Prove that if $f(z)$ is real when $|z| = 1$, then f is a constant function.

17. **(UIUC-S15-01)** Find the radius of convergence for the following power series:

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{k^3} z^{k^2}.$$

18. Find all entire functions satisfying

$$f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 0, \quad \forall n \in \mathbb{N}.$$

19. **(Midterm 1 - Chifan)** Find the radius of convergence for the series

$$\sum_{n\geq 0} \sin(n) z^{n^2}.$$

20. **(Midterm 1 - Chifan)** Find all entire functions f satisfying $(|z| - 1)(Re(f(z)) - 1) \geq 0$ for all $z \in \mathbb{C}$.

21. **(Midterm 1 - Chifan)** Let f and g be entire such that g never vanishes. What is the relationship between f and g if

$$Re(f(z))Re(g(z)) \geq -Im(f(z))Im(g(z)), \quad \forall |z| \geq 1.$$

22. **(Homework 1 - Chifan)** Let $f(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ for $a_i \in \mathbb{C}$. Prove that

$$\sup_{|z|=1} |f(z)| \geq 1.$$

23. **(Homework 1 - Chifan)** Suppose that f is holomorphic in an open set Ω. Prove that in any of the following cases:

    (a) $Re(f)$ is constant;

    (b) $Im(f)$ is constant;

    (c) $|f|$ is constant;

    one can conclude that f is constant.

24. **(Homework 2 - Chifan)** Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{C_n^{2n}}{n^2} z^n.$$

25. **(Homework 3 - Chifan)** Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose f is a holomorphic function in Ω except possibly at a point $w \in T$. Prove that if f is bounded near w, then

$$\int_T f(z)dz = 0.$$

26. **(Homework 3 - Chifan)** Find all entire functions f satisfying $|Ref(z)| \leq |Im(f(z))|$ for all $|z| \geq 2021$.

27. **(Useful exercise)** If f is entire, then f is constant or $\overline{f(\mathbb{C})} = \mathbb{C}$.

28. **(Homework 3 - Chifan)** Suppose f and g are holomorphic on the unit disc $\mathbb{D}$ and

$$\frac{f'}{f}\left(\frac{1}{n}\right) = \frac{g'}{g}\left(\frac{1}{n}\right)$$

    for integers $n \in \mathbb{Z}^+$. If f, g never vanish on $\mathbb{D}$, what relationship exists between f and g?

29. **(Homework 5 - Chifan)** Show that there is no holomorphic function f in the unit disc $\mathbb{D}$ that extends continuously to $\partial\mathbb{D}$ such that $f(z) = 1/z$ for $z \in \partial\mathbb{D}$.

30. **(Homework 8 - Chifan)** Suppose $F(z)$ is holomorphic near $z = z_0$ and $F(z_0) = F'(z_0) = 0$, while $F''(z_0) \neq 0$. Show that there are two curves Γ_1 and Γ_2 that pass through z_0, are orthogonal at z_0 and that F restricted to Γ_1 is real and has a minimum at z_0, while F restricted to Γ_2 is also real but has a maximum at z_0.

31. **(Midterm - Dr. Wang)** Suppose f is an entire function with the condition that $f(0) = 1$ and $|f(z)f(1/z)| \leq 2$ for all $z \neq 0$. Show that $f = 1$.

32. **(Midterm Review - Dr. Wang)** Suppose f is entire with $|f''(z)| \leq 5(|z|^{2019}+1)$. Prove that f is a polynomial.

33. **(Final - Dr. Wang)** Is there a holomorphic function in the unit disk so that $|f(z)| \to \infty$ as $|z| \to 1$. Give an explicit example or a proof of non-existence.

34. **(Homework 2 - Dr. Wang)** Suppose u is a real-valued function defined on the unit disc that is a twice continuously and harmonic. Prove that there exists a holomorphic function f on the unit disc such that $Re(f) = 0$.

35. **(UCLA Fall 2015 #7)** Assume that f is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. If $f(z) = f(1/z)$ when $|z| = 1$, prove that f is constant.

36. **(Homework 5 - Dr. Curto)** Let $-\infty < a < b < \infty$ and $Mz = \frac{z-ia}{z-ib}$ and let log be the principal branch of the logarithm.

    (a) Show that $\log(Mz)$ is defined for all z except $z = ic$, $a \leq c \leq b$; and if $h(z) = Im(\log(Mz))$ then $0 < h(z) < \pi$ for $Re(z) > 0$.

    (b) Show that $\log(z - ic)$ is defined for $Re(z) > 0$ and any real number c; also prove that $|Im(\log(z - ic))| < \frac{\pi}{2}$ if $Re(z) > 0$.

    (c) Let h be as in (a) and prove that $h(z) = Im[\log(z - ia) - \log(z - ib)]$.

    (d) Show that

    $$\int_a^b \frac{dt}{z - it} = i(\log(z - ib) - \log(z - ia)).$$

    (e) Combine (c) and (d) to get that

    $$h(x + iy) = \int_a^b \frac{x}{x^2 + (y-t)^2} dt = arctan\left(\frac{y-a}{x}\right) - arctan\left(\frac{y-b}{x}\right).$$

37. **(Homework 6 - Dr. Curto)** Let $U : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $U(z) \geq 0$ for all $z \in \mathbb{C}$. Prove that U is constant.

38. **(Homework 6 - Dr. Curto)** Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p vanishes in $\{z : |z| \geq R\}$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, show that $\int_\gamma \frac{p'(z)}{p(z)} dz = 2\pi i n$.

39. **(Homework 7 - Dr. Curto)** Let $P : \mathbb{C} \to \mathbb{R}$ be defined by $P(z) = Re(z)$. Show that P is an open map but is not a closed map.

40. **(Review Midterm 1 - Dr. Curto)** Let G be a region in $\mathbb{C}$ and define $G^* = \{z : \overline{z} \in G\}$. Let $f : G \to \mathbb{C}$ be analytic, and define $f^*(z) := \overline{f(\overline{z})}$ for $z \in G^*$. Prove that $f^* : G^* \to \mathbb{C}$ is also analytic.

41. **(Review Midterm 1 - Dr. Curto)** Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all $z \in G$. Show that either $f = 0$ or $g = 0$.

42. **(Practice Midterm 2 - Curto)** Let f be holomorphic on and inside the unit circle, $|z| \leq 1$, except for a pole of order 1 at a point z_0 on the circle. Let

$$f = \sum a_n z^n$$

be the power series for f on the open disc $\mathbb{D}$. Prove that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

43. **(Homework 8 - Dr. Curto)** If f is meromorphic on G and $\tilde{f} : G \to \mathbb{C}_\infty$ is defined by $\tilde{f}(z) = \infty$ when z is a pole of f, and $\tilde{f}(z) = f(z)$ otherwise, show that $\tilde{f}$ is continuous.

44. **(Homework 10 - Dr. Curto)** Let $f, f_1, ...$ be elements of $H(G)$ and show that $f_n \to f$ iff for each closed rectifiable curve γ in G, $f_n(z) \to f(z)$ for $z \in \{\gamma\}$.

45. **(Final22 - Dr. Curto)** For $n \geq 1$ and $R > 0$, let $z_1, ..., z_n$ be distinct complex numbers contained in the disc $B(0, R)$ with boundary C_R. Let f be analytic on the closed disc $\overline{B(0, R)}$. Define

$$Q(z) := (z - z_1)...(z - z_n)$$

for $z \in \mathbb{C}$. Prove that

$$P(z) := \frac{1}{2\pi i} \int_{C_R} f(\xi) \frac{1 - \frac{Q(z)}{Q(\xi)}}{\xi - z} d\xi$$

is a polynomial of degree $n - 1$ having the same value as f at the points $z_1, ..., z_n$.

46. **(Final22 - Dr. Curto)** Let G be a region in $\mathbb{C}$, and let $f \in H(G)$ be such that $f(z) \neq 0$ for all $z \in G$, and let $b \in \mathbb{C}$. Suppose there is a function $g \in H(G)$ such that $f(z) = g(z)^2$ for all $z \in G$. Prove that there is a function $h \in H(G)$ such that $f(z) = h(z)^b$ for all $z \in G$.

47. **(Final23 - Dr. Curto)** Let $a_1, ..., a_k$ be complex numbers such that $Re a_j \leq 0$ for all $j = 1, ..., k$. On the open right half plane U, consider the analytic function

$$f(z) := \sum_{j=1}^{k} \frac{1}{z - a_j}, \quad \forall z \in U.$$

Prove that $f(U) \subset U$.

48. **(Final23 - Dr. Curto)** For $z \in \mathbb{D}$, consider the power series

$$f(z) = \sum_{n=1}^{\infty} z^{n!}.$$

(a) Prove that the radius of convergence, R, is equal to 1.

(b) Let $p/q \in \mathbb{R}$ be any rational number, expressed in minimal form. Prove that

$$\lim_{r \to 1} \left| f\left(r \cdot \exp\left(2\pi i \cdot \frac{p}{q} \right) \right) \right| = +\infty.$$

(c) Conclude that $\mathbb{T}$ is a natural boundary for f; that is, prove that f cannot be analytically extended beyond any point in $\mathbb{T}$. More precisely, a point $z_0 \in \mathbb{T}$ is called **regular** for f if there exists $\delta > 0$ and a function g analytic in $B(z_0; \delta)$ such that f and g agree on the set $B(z_0, \delta) \cap \mathbb{D}$. What is needed, therefore, is a proof that **no** point in the unit circle $\mathbb{T}$ is regular for f.

Integrals

49. **(Review Final - Chifan)** For a given number $a > 0$, compute integral $\int_0^\infty \frac{\cos(ax)}{(1+x^2)^2}$.
50. **(Review Final - Chifan)** Compute the following integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^3 dx.$$

51. **(Review Final - Chifan)** Compute $\int_{\infty}^{\infty} \frac{\sin(x)}{x} dx$.
52. **(Stein Chapter 3 Example 2)** If $0 < a < 1$, show that

$$\int_{\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(a\pi)}.$$

53. **(Stein Chapter 2 Example 1)** Prove that for all $\zeta \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \zeta} dx = e^{-\pi \zeta^2}.$$

54. **(Midterm 1 - Chifan)** If $m > n \geq 2$ are positive and $r > 0$ compute

$$\int_{|z-1|=r} \frac{z^m}{(z-1)^n} dz$$

and show $\binom{m}{n} \leq \frac{m^m n^{-n}}{(m-n)^{m-n}}$ for all $m \geq 2$.

55. **(Midterm 2 - Chifan)** Compute

$$\int_0^\infty \frac{\log(x)}{z^2 + a^2} dx$$

for $a > 0$. HINT: Choose the branch of the logarithm given by $\{z \in \mathbb{C} : -\pi/2 < arg(z) < 3\pi/2\}$.

56. **(Homework 2 - Chifan)**
    (a) Evalue the integral $\int_\gamma z^n dz$ for all integers n. Here γ is any circle centered at the origin with positive (counterclockwise) orientation.
    (b) Same question as before, but with γ any circle not containing the origin.
    (c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}.$$

57. **(Homework 3 - Chifan)** Prove that

$$\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \frac{\sqrt{2}}{4}.$$

58. **(Homework 3 - Chifan)** Prove that

$$\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}.$$

59. **(Homework 3 - Chifan)** Evaluate the integrals

$$\int_0^\infty e^{-ax}\cos(bx)dx \text{ and } \int_0^\infty e^{-ax}\sin(bx)dx$$

for $a > 0$ by integrating e^{-Az} where $A = \sqrt{a^2+b^2}$ over an appropriate sector with angle ω, with $\cos\omega = a/A$.

60. **(Homework 5 - Chifan)** Show that

$$\int_{-\infty}^\infty \frac{\cos(x)}{x^2+a^2}dx = \pi\frac{e^{-a}}{a}$$

for $a > 0$.

61. **(Homework 5 - Chifan)** Show that

$$\int_{-\infty}^\infty \frac{x\sin(x)}{x^2+a^2}dx = \pi e^{-a}$$

for $a > 0$.

62. **(Homework 5 - Chifan)** Show that

$$\int_0^1 \log(\sin(\pi x))dx = -\log(2).$$

63. **(Homework 5 - Chifan)** Show that if $|a| < 1$ then

$$\int_0^{2\pi} \log|1 - ae^{i\theta}|d\theta = 0.$$

64. **(Homework 6 - Chifan)** Compute

$$\int_{-\infty}^\infty \frac{\sin(\pi x)}{x(x^2-2x+2)}dx.$$

65. **(Stein Chapter 8 Prop 1)** If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$.

66. **(Midterm - Dr. Wang)** Compute $\int_{\partial D(0,1)} \overline{z}(\overline{z}+1)dz$.

67. **(Harvard University - Qual 2016)** Fix two positive real numbers $a, b > 0$. Calculate the value of the integral

$$\int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx.$$

HINT: Consider $F(z) = \frac{exp(iaz) - exp(ibz)}{z^2}$ and a keyhole contour.

68. **(Homework 5 - Dr. Curto)** Evaluate the following integrals:

(a) $\int_\gamma \frac{e^{iz}}{z^2} dz$, $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$

(b) $\int_\gamma \frac{dz}{z-a}$, $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$

(c) $\int_\gamma \frac{\sin(z)}{z^3} dz$, $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$

(d) $\int_\gamma \frac{\log(z)}{z^n} dz$, $\gamma(t) = 1 + \frac{1}{2}e^{it}$, $0 \leq t \leq 2\pi$ and $n \geq 0$

(e) $\int_\gamma \frac{z^2+1}{z(z^2+4)} dz$ where $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$ for all possible values of r, $0 < r < 2$ and $2 < r < \infty$.

69. **(Practice Midterm 2 - Curto)** Find

$$\int_\gamma \frac{z}{4z^2 + 1} dz$$

where γ is the unit circle traversed in the counterclockwise direction

70. **(Practice Midterm 2 - Curto)**

(a) Evaluate $\int_\gamma \frac{e^{-z^2}}{z^2} dz$ where γ is the ellipse defined by the equation $\frac{x^2}{4} + y^2 = 1$.

(b) Calculate $\int_\gamma \frac{\bar{z}}{z - \frac{1}{2}} dz$, where γ is the unit circle traversed in the counterclockwise direction.

71. **(Practice Midterm 2 - Curto)** Let f be a function which is analytic on the upper half-plane, and on the real line. Assume there exists numbers $B > 0$ and $c > 0$ such that

$$|f(\xi)| \leq \frac{B}{|\xi|^c}$$

for all ξ. Prove that for any z in the upper half plane, we have the integral formula

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt.$$

72. **(Homework 8 - Dr. Curto)** Calculate the following integrals:

(a) $\int_0^\infty \frac{x^2}{x^4 + x^2 + 1} dx$

(b) $\int_0^\pi \frac{d\theta}{(a + \cos(\theta))^2}$ where $a > 1$.

73. **(Homework 8 - Dr. Curto)** Suppose f has a simple pole at $z = a$, and let g be analytic in an open set containing a. Show that $Res(fg; a) = g(a)Res(f; a)$.

74. **(Homework 8 - Dr. Curto)** Let γ be the rectangular path $[n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni, n + \frac{1}{2} + ni]$ and evaluate the integral $\int_\gamma \pi(z+a)^{-2}\cot(\pi z)dz$ for $a \neq$ an integer. Show that $\lim_{n\to\infty} \int_\gamma \pi(z+a)^{-2}\cot(\pi z)dz = 0$ and, by using the first part, deduce that

$$\frac{\pi^2}{\sin^2(\pi a)} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}.$$

75. **(Practice Final - Dr. Curto)** Let f be analytic on the punctured open unit disc $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$, and assume that f has a simple pole at $z = 0$ with residue $c_1 \in \mathbb{Z}$. Show that there is an analytic function g on $\mathbb{D}$ such that $f = g'/g$, and

$$g(z) = (z - z_0)^n h(z)$$

where h is analytic and $h(z_0) \neq 0$.

76. **(Final22 - Dr. Curto)** Evaluate the integral

$$\int_\gamma \frac{e^{iz}}{e^z + e^{-z}} dz$$

where γ is the rectangle oriented counterclockwise, whose vertices are $(-\pi, 0)$, $(-\pi, \pi i)$, $(\pi, \pi i)$ and $(\pi, 0)$.

77. **(Final23 - Dr. Curto)** Evaluate the following integral

$$\int_0^\infty \frac{1}{1+x^n} dx$$

for $n = 2, 4, 6, ...$

Zeros and Singularities

78. **(Review Final - Chifan)** Let f be analytic in $\mathbb{D} \setminus \{0\}$ and suppose that there exists $M > 0$ and $m \geq 1$ such that $|f^{(m)}(z)| \leq \frac{M}{|z|^m}$ for all $0 < |z| < 1$. Show that f has a removable singularity at 0.

79. **(Review Final - Chifan)** Does there exists a holomorphic function $f : D(0,1) \setminus \{0\} \to \mathbb{C}$ such that
$$\lim_{z \to 0} \frac{f^2(z)}{z^3} = 1?$$

80. **(Stein Chapter 8 Prop 3.5)** Let Ω be a connected open subset of $\mathbb{C}$. Assume that $\{f_n\}$ is a sequence of injective holomorphic functions on Ω that converges uniformly on the compact subsets of Ω to a non-constant function f. Show that f is either injective or constant on Ω.

81. **(Qual Fall 2007 #2)** Suppose f is analytic on $0 < |z| < 1$ and suppose there is a constant K such that $|f(z)| \leq K|z|^{-1/2}$. What kind of isolated singularities does f have at zero?

82. **(aTm-S17-03)** Fix $R > 0$. Show that there exists $N \in \mathbb{N}$ such that for all $n > N$, the polynomial $f_m(z) = \sum\limits_{k=0}^{m} \frac{z^k}{k!}$ has no roots w with $|w| < R$.

83. **(Midterm 2 - Chifan)** Find all entire functions satisfying $|f(z)| \geq e^{Re(z)}|z-1|$ for all $z \in \mathbb{C}$.

84. **(Midterm 2 - Chifan)** Let $f \in H(\mathbb{C} \setminus \{0\})$ be injective. If $\lim\limits_{|z| \to \infty} f(z) = 0$ and $f(1) = 1$, show that $f(z) = 1/z$.

85. **(Rudin Chapter 10 #3)** Suppose f and g are entire functions, and $|f(z)| \leq |g(z)|$ for every z. What conclusion can we draw?

86. **(Rudin Chapter 10 #14)** Suppose Ω is a region, $\varphi \in H(\Omega)$, φ' has no zeros in Ω, $f \in H(\varphi(\Omega))$, $g = f \circ \varphi$, $z_0 \in \Omega$ and $w_0 = \varphi(z_0)$. Prove that if f has a zero of order m at w_0, then g also has a zero of order m at z_0.

87. Prove that there is no function f that is analytic on the punctured disc $\mathbb{D} \setminus \{0\}$ and f' has a simple pole at $z = 0$.

88. **(Homework 4 - Chifan)** Suppose f is holomorphic in a punctured disc $D(z_0, r) \setminus \{z_0\}$. Suppose also
$$|f(z)| \leq A|z - z_0|^{-1+\epsilon}$$
for some $\epsilon > 0$, and all z near z_0. What kind of singularity does f have at z_0?

89. **(Homework 5 - Chifan)** Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, $a \neq 0$. HINT: Start with $g(z) = f(1/z)$ and try to figure out what kind of singularity does g have at $z = 0$.

90. **(Midterm Review - Dr. Wang)** Let p and q be polynomials of degree 2000 and 2019, respectively. Show that $\int_\sigma \frac{p}{q} dz = 0$ if σ is a closed curve outside the disk $D(0, R)$ where R is large enough. Show the statement is not true if you change the degree of the polynomials to 2018 and 2019, respectively.

91. **(Final - Dr. Wang)** Suppose that a holomorphic function f has an isolated singularity at 0 in the unit disk and that its real part satisfies $Ref(z) \leq 1$. Prove that 0 is removable.

92. **(Harvard University Qual 2020)** Let $\Omega \subset \mathbb{C}$ be a connected open subset of the complex plane and $f_1, f_2, ...$ be a sequence of holomorphic functions on Ω converging uniformly on compacts sets to a function f. Suppose $f(z_0) = 0$ for some $z_0 \in \Omega$. Show that either $f \equiv 0$ or there exists a sequence $z_1, z_2, ... \in \Omega$ converging to z_0 with $f_n(z_n) = 0$.

93. **(Harvard University - Qual 2018)** Let f_n be a sequence of analytic functions on the unit disc $\Omega \subset \mathbb{C}$ such that $f_n \to f$ uniformly on compact sets and such that f is not identically zero. Prove that $f(0) = 0$ if and only if there is a sequence $z_n \to 0$ such that $f_n(z_n) = 0$ for n large enough.

94. **(Harvard University - Qual 2014)** Show that for any n, the roots of the polynomial

$$\sum_{i=0}^{n} z^i$$

all have absolute value less than 2.

95. **(Homework 7 - Dr. Curto)** If $f : G \to \mathbb{C}$ is analytic except for poles show that the poles of f cannot have a limit point in G.

96. **(Homework 7 - Dr. Curto)** Let $R > 0$ and $G = \{z : |z| > R\}$; a function $f : G \to \mathbb{C}$ has a *removable singularity, a pole or an essential singularity at infinity* if $f(z^{-1})$ has, respectively, a removable singularity, a pole, or an essential singularity at $z = 0$. If f has a pole at ∞ then the order of the pole is the order of the pole of $f(z^{-1})$ at $z = 0$.

   (a) Prove that an entire function has a removable singularity at infinity iff it is constant.
   (b) Prove that an entire function has a pole at infinity of order m iff it is a polynomial of order m.
   (c) Characterize those rational functions which have a removable singularity at infinity.
   (d) Characterize those rational functions which have a pole of order m at infinity.

97. **(Homework 10 - Dr. Curto)**

   (a) Let G be a region, let $a \in G$ and suppose that $f : G \setminus \{a\} \to \mathbb{C}$ is an analytic function such that $f(G \setminus \{a\}) = \Omega$ is bounded. Show that f has a removable singularity at $z = a$. If f is one-to-one, show that $f(a) \in \partial\Omega$.
   (b) Show that there is no one-to-one analytic function which maps $G = \{z : 0 < |z| < 1\}$ onto an annulus $\Omega = \{z : r < |z| < R\}$ where $r > 0$.

Maximum-Modulus Principle

98. **(Review Final - Chifan)** Suppose f is a non-vanishing continuous function on $\mathbb{D}$ that is holomorphic in $\mathbb{D}$. Prove that if $|f(z)| = 1$ whenever $|z| = 1$, then f is constant.

99. **(Review Final - Chifan)** Suppose that $f \in H(\Omega)$ where $\overline{D} \subset \Omega$, $|f(z)| > 2$ for all $|z| = 1$ and $f(0) = 1$. Must f have a zero in the unit disk?

100. **(Final Review - Dr. Wang)** Suppose f is analytic in the annular region $1 \leq |z| \leq 2$. Suppose also that $|f| = 1$ on $|z| = 1$ and $|f| = 4$ on $|z| = 2$. Show that $|f(z)| \leq |z|^2$ for all z with $1 \leq |z| \leq 2$.

101. **(Harvard University - Qual 2018)** Let f be an entire function such that

    (a) $f(z)$ vanishes at all points $z = n$ for $n \in \mathbb{Z}$;

    (b) $|f(z)| \leq e^{\pi|Imz|}$ for all $z \in \mathbb{C}$.

    Prove that $f(z) = c\sin(\pi z)$ with $|c| \leq 1$.

102. **(Harvard University - Qual 2010)** Let f be holomorphic function on a domain containing the closed unit disc $\{z : |z| \leq 3\}$, and suppose that

$$f(1) = f(i) = f(-1) = f(-i) = 0.$$

Show that

$$|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|$$

and find all such functions for which equality holds.

103. **(Homework 9 - Dr. Curto)** Let $0 < r < R$ and put $A = \{z : r \leq |z| \leq R\}$. Show that there is a positive number $\epsilon > 0$ such that for each polynomial p,

$$\sup\{|p(z) - z^{-1}| : z \in A\} \geq \epsilon.$$

This says that z^{-1} is not the uniform limit of polynomials on A.

104. **(Homework 9 - Dr. Curto)** Let f be analytic in the disc $G = B(0, R)$ and for $0 \leq r < R$ define $A(r) = \max\{Ref(z) : |z| = r\}$. Show that unless f is constant, $A(r)$ is strictly increasing function of r.

105. **(Homework 12 - Dr. Curto)** Let K be a compact subset of the open set G; then the following are equivalent:

    (a) If f is analytic in a neighborhood of K and $\epsilon > 0$ then there is a g in $H(G)$ with $|f(z) - g(z)| < \epsilon$ for all $z \in K$.

    (b) If D is a bounded component of $G \setminus K$ then $\overline{D} \cap \partial G \neq \emptyset$.

    (c) If z is any point in $G \setminus K$, then there is a function f in $H(G)$ with

$$|f(z)| > \sup\{|f(w)| : w \in K\}.$$

Rouché's Theorem

106. **(Review Final - Chifan)** Find the number of roots, counted with multiplicities of the following equation

$$z^6 - 5z^4 + 8z - 1 = 0$$

in the annular domain $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

107. **(Review Final - Chifan)** Suppose that f is holomorphic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and assume that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for $|z| < 1$. If f admits exactly 2020 roots (counted with multiplicities) in $\mathbb{D}$, then show that

$$\inf_{|z|=1} |f(z)| \leq |a_0| + |a_1| + ... + |a_{2020}|.$$

108. **(Qual Spring 17 #5)** Show that all roots of $e^z = 3z^2$ are on $\mathbb{D}$ and they are real.

109. **(Practice Midterm 2 - Chifan)** Let $f_n \in H(\Omega)$ where Ω is a region. Assume f_n has no zeros in Ω and f_n converges uniformly to f on compact subsets of Ω. Prove that either f has no zeros or $f(z) = 0$ for all $z \in \Omega$.

110. **(Midterm 2 - Chifan)** Let f be a non-identically zero continuous function on $\mathbb{C}$ and assume there exists a sequence of polynomials p_n that converge uniformly to f on any open disc of $\mathbb{C}$. If p_n has only real roots, show that f only has real roots.

111. **(Rudin Chapter 10 #21)** Suppose $f \in H(\Omega)$ and Ω contains the closed unit disc and $|f(z)| < 1$ whenever $|z| = 1$. How many fixed points must f have in the unit disc?

112. (a) If $\lambda > 1$, show the equation $z + e^{-z} = \lambda$ has exactly one solution in the positive real part.
   (b) Determine the number of zeros of $f(z) = z^4 + 5z + 3$ in $1 < |z| < 2$.
   (c) How many zeros does the function $f(z) = z^6 + 4z^2 e^{z+1} - 3$ have in the unit disc $D(0, 1)$?

113. **(Homework 5 - Chifan)** Suppose $f \in H(\Omega)$ and Ω is a region containing the closed unit disc of $\mathbb{C}$. If $|f(z)| \geq 2$ if $|z| = 1$ and $f(0) = 1$, must f have a zero in the unit disc?

114. **(Homework 5 - Chifan)**
   (a) How many roots (counted with multiplicity) does the function $g(z) = 6z^3 + e^z + 1$ have in the unit disc?
   (b) Determine the number of zeros of $P(z) = z^7 + z^3 + 1/16$ that lie in the closed disc $|z| \leq 1/2$.

115. **(Homework 6 - Chifan)** Show that the polynomial $P_n(z) = z^n + 3z + 1$ for $n \geq 2$ has exactly one zero in the unit disc and the rest of the $n - 1$ zeros are in the annulus $1 < |z| < 4^{\frac{1}{n-1}}$.

116. **(Midterm Review - Dr. Wang)** How many zeros does $z^{2019} + 8z^{2000} - 3z^{100} + 2z^2 - z + 1$ have in the unit disk, counting multiplicity?

Conformal Mappings

117. **(Review Final - Chifan)** Is there a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f'(1/2) = 2$? If yes, give a concrete example of such function. If no, prove there is no such function.

118. **(Review Final - Chifan)** Let $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. Then for any holomorphic function $f : \mathbb{H} \to \mathbb{H}$ and every $a \in \mathbb{H}$ we have

$$|f'(a)| \leq \frac{Im(f(a))}{Im(a)}.$$

119. **(Review Final - Chifan)** Find all the automorphisms of $\mathbb{D} \setminus \{0, 1/2\}$.

120. **(Review Final - Chifan)** Let Ω be a proper simply connected region in the plane. Show that any holomorphic non-identity function $f : \Omega \to \Omega$ can have at most one fixed point.

121. **(Review Final - Chifan)** Find a conformal mapping that maps the semi-disk $\{z \in \mathbb{D} : Re(z) > 0\}$ onto the infinite semi-strip $\{z \in \mathbb{C} : Re(z) > 2, 0 < Im(z) < 2\}$.

122. **(Review Final - Chifan)** Let $f : \mathbb{H} \to \mathbb{D}$ be holomorphic function such that $f(i) = 0$. Show that

$$|(i + z)f(z)| \leq |i - z|, \quad \forall z \in \mathbb{H}.$$

123. **(Review Final - Chifan)** Suppose that $f \in H(\mathbb{D})$, $|Re(f)| < 1$ on $\mathbb{D}$ and $f(0) = 0$. Prove that for all $z \in \mathbb{D}$ we have

$$|Im(f(z))| \leq \frac{2}{\pi} \log\left(\frac{1 + |z|}{1 - |z|}\right).$$

124. **(Review Final - Chifan)** Find a conformal mapping f which maps $\{z \in \mathbb{D} : Re(z) > 0\}$ onto $\mathbb{D}$.

125. **(Qual Spring 2018 #1(b))** Find a conformal map from the unit disc to $|Im(z)| < 1$.

126. **(Qual Summer 2020 #5)** Suppose $f : D(0, 1) \to P$ is a conformal mapping into a regular pentagonal region P with center at 0 such that $f(0) = 0$. Compute $f^{(2020)}(0)$.

127. **(Qual Summer 2017 #2)** If we denote by $\mathcal{H} = \{z \in \mathbb{C} : |z - i| > 1\}$, describe all analytic bijective maps $f : \mathcal{H} \to \mathcal{H}$.

128. **(Qual Spring 2018 #1)** Suppose f is a holomorphic function on $\mathbb{D}$ with the property that $Ref(z) > 0$ for all $z \in \mathbb{D}$. Prove that $|f'(0)| \leq 2Ref(0)$.

129. **(KU-S07-08)** Find a conformal map from the strip $\{z : 0 < Re(z) < 1\}$ onto the half disc $\{z : Im(z) > 0, 0 < |z| < 1\}$.

130. **(aTm-S15-04)** Prove that if f is holomorphic function that maps the open unit disc into itself, and z_1 and z_2 are zeros of f in the unit disc, then

$$|f(z)| \leq \left|\frac{(z - z_1)(z - z_2)}{(1 - \overline{z}_1 z)(1 - \overline{z_2} z)}\right|, \quad \text{when } |z| < 1.$$

131. **(Midterm 3 - Chifan)** If $\Omega = \mathbb{D} \setminus \{0, 1/2\}$ describe $Aut(\Omega)$.

132. **(Midterm 3 - Chifan)** Find a conformal mapping between $\{|z| < 2 : |arg(z)| < \pi/6\}$ and $\{z : Re(z) > 0, Im(z) < 0\}$.

133. **(Homework 8 - Chifan)** Does there exists a holomorphic surjection from the unit disc to $\mathbb{C}$?

134. **(Homework 8 - Chifan)** Prove that $f(z) = -1/2(z + 1/z)$ is a conformal map from the half disc $\mathbb{U} = \{z : |z| < 1, Im(z) > 0\}$ to the upper half-plane.

135. **(Homework 9 - Chifan)** Show that if $f : D(0, R) \to \mathbb{C}$ is holomorphic with $|f(z)| \leq M$ for some $M > 0$, then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)} \right| \leq \frac{|z|}{MR}.$$

136. **(Homework 9 - Chifan)**

(a) Prove that if $f : \mathbb{D} \to \mathbb{D}$ is analytic and has two distinct fixed points, then f is the identity.

(b) Must every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ have a fixed point?

137. **(Homework 9 - Chifan)** Prove that all conformal mappings from the upper half-plane to the unit disc take the form

$$e^{i\theta} \cdot \frac{z - \beta}{z - \overline{\beta}}$$

for $\theta \in \mathbb{R}$ and $\beta \in \mathbb{H}$.

138. **(Homework 9 - Chifan)** Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. If $a, b \in \mathbb{D}$, then show that

$$\left| \frac{f(a) - f(b)}{1 - \overline{f(a)}f(b)} \right| \leq \left| \frac{a - b}{1 - \overline{a}b} \right| \text{ and } |f'(a)| \leq \frac{1 - |f(a)|^2}{1 - |a|^2}.$$

139. **(Final Review - Dr. Wang)** Suppose $f(0) = 0$ and maps the unit disk to itself. Show that $|f(z) + f(-z)| \leq 2|z|^2$ and show that if the above inequality holds at a point $z_0 \neq 0$ in the disk then $f(z) = e^{i\theta}z^2$ for some $\theta \in \mathbb{R}$.

140. **(Harvard University - Qual 2015)** Let $S \subset \mathbb{C}$ be the open half-disc $\{x + iy : y > 0, x^2 + y^2 < 1\}$. Construct a surjective conformal mapping $f : S \to D$, where D is the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

141. **(Homework 4 - Dr. Curto)** If $Tz = \frac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\mathbb{T}) = \mathbb{T}$.

142. **(Homework 4 - Dr. Curto)** Suppose that one circle is contained inside another and that they are tangent at a point a. Let G be the region between the two circles. Map G conformally onto the open unit disc.

143. **(Homework 5 - Dr. Curto)**

(a) Show that a Möbius transformation has 0 and ∞ as its only fixed points if and only if it is a dilation, but not the identity.

(b) Show that a Möbius transformation has ∞ as its only fixed points if and only if it is a translation, but not the identity.

144. **(Midterm 1 - Dr. Curto)** Let $a, b \in \mathbb{R}$, with $0 < a < b$, and let $M(z) = \frac{z-ia}{z-ib}$ be a Möbius map. Let $G = \{z \in \mathbb{C} : Re(z) > 0 \text{ and } Im(z) > a\}$. Is it true that $M(G) = H^+$, where H^+ denotes the open half-plane?

145. **(Practice Midterm 2 - Curto)** Suppose f and g are two analytic functions mapping the open unit disc $\mathbb{D}$ into a region G. Suppose

(i) $f(0) = g(0)$;

(ii) f is one-to-one; and

(iii) f maps $\mathbb{D}$ onto G.

Let $r \in \mathbb{R}$ be such that $0 < r < 1$. Show that $g(B(0, r)) \subset f(B(0, r))$.

146. **(Homework 9 - Dr. Curto)** Suppose f is analytic in a region containing $\overline{B}(0; 1)$ and $|f(z)| = 1$ when $|z| = 1$. Suppose that f is a simple zero at $z = \frac{1}{4}(1+i)$ and a zero at $z = 1/2$. If $f(0) = 1/2$, find f.

147. **(Homework 10 - Dr. Curto)** Let G be a simply connected region which is not the whole plane and suppose that $\overline{z} \in G$ whenever $z \in G$. Let $a \in G \cap \mathbb{R}$ and suppose that $f : G \to \mathbb{D}$ is a one-to-one analytic function with $f(a) = 0$, $f'(a) > 0$ and $f(G) = \mathbb{D}$. Let $G_+ = \{z \in G : Im(z) > 0\}$. Show that $f(G_+)$ must lie entirely above or entirely below the real axis.

148. **(Practice Final - Dr. Curto)** Let h be an isomorphism of the disc $B(i; 2)$ with $\mathbb{D}$, and assume that $h(z_0) = 0$ for a (necessarily unique) point $z_0 \in B(i; 2)$. Show that

$$h(z) = \frac{2(z - z_0)}{4 - (z - i)(\overline{z_0} + i)} e^{i\theta}$$

for some real number θ and for all $z \in B(i; 2)$.

149. **(Final22 - Dr. Curto)** Let G be the open subset of $\mathbb{C}$ obtained by deleting the segment $(0, 1]$ from the right half plane.

(a) What is the image of G under the map $z \mapsto z^2$?

(b) What is the image of G under the map $z \mapsto z^2 - 1$?

(c) Find an isomorphism of G with the right half plane.

150. **(Final23 - Dr. Curto)** For $z \in \mathbb{C}$, consider the cross ratio $(z, 1, -i, i)$. Let $f : \mathbb{C} \to \mathbb{C}$ be the continuous function given by

$$(f(z), 1, -i, i) = \overline{(z, 1, -i, i)}.$$

Find an explicit formula for f. Is f analytic on $\mathbb{C}$?

Normal Families

151. **(Review Final - Chifan)** Let $\mathcal{F}$ be a family of holomorphic functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

such that $|a_n| \leq n^{2020}$ for all n. Is $\mathcal{F}$ a normal family?

152. **(Review Final - Chifan)** Let $\mathcal{F}$ be the class of all $f \in H(\mathbb{D})$ such that $Re(f) > 0$ and $f(0) = 1$. Show that $\mathcal{F}$ is a normal family.

153. **(Review Final - Chifan)** Show that if $\mathcal{F} \subset H(G)$ is normal then $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is also normal. Is the converse true? Can you add something to the hypothesis that $\mathcal{F}'$ is normal to ensure that $\mathcal{F}$ is normal?

154. **(Qual Fall 2017 #5)** True/False: Suppose $\mathcal{F}$ is a sequence of holomorphic functions on the open unit disc. Suppose each function $f \in \mathcal{F}$ maps in the upper half plane; i.e. $Imf > 0$. Then, $\mathcal{F}$ is a normal family.

155. **(Homework 9 - Chifan)** Let $\mathcal{F}$ be the collection of all functions $f \in H(\mathbb{D})$ such that

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})| dt = 1.$$

Show that $\mathcal{F}$ is a normal family.

156. **(Homework 10 - Dr. Curto)** Suppose $\mathcal{F}$ is normal in $H(G)$ and Ω is open in $\mathbb{C}$ such that $f(G) \subset \Omega$ for every $f \in \mathcal{F}$. Show that if g is analytic on Ω and is bounded on bounded sets, then $\{g \circ f : f \in \mathcal{F}\}$ is normal.

Entire Functions

157. **(Review Final - Chifan)**

    (a) Suppose $f : \mathbb{C}\setminus\{0\} \to \mathbb{R}$ is a non-constant, real-valued harmonic function on the puntured plane. Prove that the image of f is all of $\mathbb{R}$. HINT: Define $u : \mathbb{C} \to \mathbb{C}$ by $u(x, y) = f(e^x \cos(y), e^x \sin(y))$ and show it is harmonic and entire; so there exists entire function g such that $Re(g) = u$.

    (b) Let $f : \mathbb{C} \to \mathbb{R}$ be a non-constant, harmonic function. Use part (a) to show that for all $y \in \mathbb{R}$, the pre-image $f^{-1}(\{y\})$ is non-empty, closed, unbounded subset of $\mathbb{C}$.

158. **(Review Final - Chifan)** Let Ω be a region with $\overline{D} \subset \Omega$. Suppose $f \in H(\Omega)$ and $|f(e^{i\theta})| \geq 3$ for all θ real, $f(0) = 0$ and $z_1, ..., z_N$ are the zeros of $1-f$ in $\mathbb{D}$ counted with their multiplicities. Prove

$$|z_1 z_2 ... z_N| < \frac{1}{2}.$$

159. **(Review Final - Chifan)** Show that if f is an entire function of finite order that omits two values, then f is constant. You cannot cite Picard Theorem, but you can use other theorems.

160. **(Review Final - Chifan)** Suppose f is an entire functions such that the derivatives $f^{(n)}$ never vanish for any $n \geq 0$. Prove that if f is also of finite order, then $f(z) = e^{az+b}$ for some constants a and b.

161. **(Qual Fall 2020 #2)** Find all entire functions f of finite order such that f has 2020 roots and f' has 2022 counted with multiplicities.

162. **(Homework 7 - Chifan)** Show that the equation $e^z - z = 0$ has infinitely many solutions in $\mathbb{C}$.

163. **(Harvard University - Qual 2016)** Let $H \subset \mathbb{C}$ denote the open right half plane. Suppose $f : H \to \mathbb{C}$ is a bounded, analytic function such that $f(1/n) = 0$ for each positive integer n. Prove that $f(z) = 0$ for all z. HINT: Consider the behavior of the sequence of functions $\left\{ h_N(z) = \prod_{n=1}^{N} \frac{z-1/n}{z+1/n} \right\}_{N=1,2,...}$ on H and, in particular, on the positive real axis.

Harmonic Functions

164. **(Rudin Chapter 11 #1)** Suppose u and v are real harmonic functions in a plane region Ω. Under what conditions is uv harmonic? Show that u^2 cannot be harmonic in Ω unless u is constant. For which $f \in H(\Omega)$ is $|f|^2$ harmonic?

165. **(Rudin Chapter 11 #2)** Suppose f is a complex function in a region Ω, and both f and f^2 are harmonic in Ω. Prove that either f or $\overline{f}$ is holomorphic in Ω.

166. **(Rudin Chapter 11 #3)** If u is harmonic function in a region Ω, what can you say about the set of points at which the gradient of u is zero? (This is the set on which $u_x = u_y = 0$)

167. **(Rudin Chapter 11 #4)** Prove that every partial derivative of every harmonic function is harmonic.

168. **(Rudin Chapter 11 #5)** Suppose $f \in H(\Omega)$ and f has no zeros in Ω. Prove that $log|f|$ is harmonic in Ω, by computing its Laplacian. Is there an easier way?

169. **(Rudin Chapter 11 #8)** Suppose Ω is a region, $f_n \in H(\Omega)$ for $n = 1, 2, 3, ...,$ u_n is the real part of f_n, $\{u_n\}$ converges uniformly on compact subsets of Ω and $\{f_n(z)\}$ converges for at least $z \in \Omega$. Prove that then $\{f_n\}$ converges uniformly on compact subsets of Ω. HINT: First show that $\{f_n(z)\}$ converges for every $z \in \Omega$.

170. **(Rudin Chapter 11 #12)** (Harnack's inequalities) Suppose Ω is a region, K is a compact subset of Ω, $z_0 \in \Omega$. Prove that there exists positive numbers α and β such that $\alpha u(z_0) \leq u(z) \leq \beta u(z_0)$ for every positive harmonic function u in Ω and for all $z \in K$. If $\{u_n\}$ is a sequence of positive harmonic functions in Ω and $u_n(z_0) \to 0$, describe the behavior of $\{u_n\}$ in the rest of Ω. Do the same if $u_n(z_0) \to \infty$.

171. **(Rudin Chapter 11 #13)** Suppose u is a positive harmonic function in U and $u(0) = 1$. How large can $u(1/2)$ be?

172. **(Rudin Chapter 11 #15)** Suppose u is a positive harmonic function in U, and $u(re^{i\theta}) \to 0$ as $r \to 1$, for every $e^{i\theta} \neq 1$. Prove that there is a constant c such that $u(re^{i\theta}) = cP_r(\theta)$.

173. **(Homework 12 - Dr. Curto)** Let $f : \{z : Re(z) = 0\} \to \mathbb{R}$ be a bounded continuous function and define $u : \{z : Re(z) > 0\} \to \mathbb{R}$ by

$$u(z + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{xf(it)}{x^2 + (y-t)^2} dt.$$

Show that u is a bounded harmonic function on the right half plane such that for c in $\mathbb{R}$, $f(ic) = \lim_{z \to ic} u(z)$.

174. **(Practice Final - Dr. Curto)** True/False: In the open unit disc $\mathbb{D}$, the function

$$u(x, y) := \frac{y}{x^2 + y^2}$$

is not the real part of an analytic function in $\mathbb{D}$.

175. **(Practice Final - Dr. Curto)** Let u be a harmonic function in the open unit disc $\mathbb{D}$. Define $f : \mathbb{D} \to \mathbb{C}$ by

$$f(x + iy) := u_x(x, y) - iu_y(x, y).$$

Prove that f is analytic in $\mathbb{D}$.

176. **(Final22 - Dr. Curto)** On $\mathbb{R}^2$, consider the function u defined as

$$u(x, y) := 3x^2y - y^3,$$

for $x, y \in \mathbb{R}$.

(a) Prove that u is harmonic in $\mathbb{R}^2$.

(b) Find a harmonic conjugate v so that $f := u + iv$ is an entire function on $\mathbb{C}$.

Infinite Products

Let $E_0(z) = 1 - z$ and for $p = 1, 2, 3, ...$

$$E_p(z) = (1-z)exp\left\{z + \frac{z^2}{2} + ... + \frac{z^p}{p}\right\}.$$

177. **(Rudin Theorem 15.9)** Let $\{z_n\}$ be a sequence of complex numbers such that $z_n \neq 0$ and $|z_n| \to \infty$ as $n \to \infty$. If $\{p_n\}$ is a sequence of non-negative integers such that

$$\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{1+p_n} < \infty$$

for every positive $r_n = |z_n|$, then the infinite product

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

defines an entire function P which has a zero at each point z_n and which has no other zeros in the plane.

178. **(Rudin Theorem 15.11)** Let Ω be an open set in S^2, $\Omega \neq S^2$. Suppose $A \subset \Omega$ and A has no limit point in Ω. With each $\alpha \in A$ associate a positive integer $m(\alpha)$. Then, there exists an $f \in H(\Omega)$ all of whose zeros are in A, and such that f has a zero of order $m(\alpha)$ at each $\alpha \in A$.

179. **(Rudin Chapter 15 #1)** Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of complex numbers such that $\sum |a_n - b_n| < \infty$. On what sets will the product

$$\prod_{n=1}^{\infty} \frac{z - a_n}{z - b_n}$$

converge uniformly? Where will it define a holomorphic function?

180. **(Rudin Chapter 15 #2)** Suppose f is entire, λ is a positive number, and the inequality

$$|f(z)| < exp(|z|^\lambda)$$

holds for all large enough $|z|$. If $f(z) = \sum a_n z^n$, prove that the inequality

$$|a_n| \leq (e\lambda/n)^{m/\lambda}$$

holds for all large enough n.

181. **(Rudin Chapter 15 #5)** Suppose k is a positive integer, $\{z_n\}$ is a sequence of complex numbers such that $\sum |z_n|^{-k-1} < \infty$, and

$$f(z) = \prod_{n=1}^{\infty} E_k\left(\frac{z}{z_n}\right).$$

What can you say about the rate of growth of $M(r) = \max_\theta |f(re^{i\theta})|$?

182. **(Rudin Chapter 15 #6)** Suppose f is entire, $f(0) \neq 0$, $|f(z)| < exp(|z|^p)$ for large $|z|$, and $\{z_n\}$ is the sequence of zeros of f, counted with multiplicities. Prove that $\sum |z_n|^{-p-\epsilon} < \infty$ for every $\epsilon > 0$.

183. **(Rudin Chapter 15 #9)** Suppose $0 < \alpha < 1$, $0 < \beta < 1$, $f \in H(U)$, $f(U) \subset U$ and $f(0) = \alpha$. How many zeros can f have in the disc $\overline{D}(0, \beta)$? What is the answer if (a) $\alpha = 1/2$, $\beta = 1/2$; (b) $\alpha = 1/4, \beta = 1/2$; (c)$\alpha = 2/3$, $\beta = 1/3$; and (d) $\alpha = 1/1000$, $\beta = 1/10$?

184. **(Rudin Chapter 15 #12)** Suppose $0 < |\alpha_n| < 1$, $\sum(1 - |\alpha_n|) < \infty$, and B is the Blaschke product with zeros at the points α_n. Let E be the set of all points $1/\overline{\alpha_n}$ and let Ω be the complement of the closure of E. Prove that the product actually converges uniformly on every compact subset of Ω, so $B \in H(\Omega)$, and that B has a pole at each point of E.

185. **(Rudin Chapter 15 #13)** Let $\alpha_n = 1 - n^{-2}$ for $n = 1, 2, 3, ...$ let B be the Blaschke product with zeros at these points α, and prove that $\lim_{r \to 1} B(r) = 0$. More precisely, show that the estimate

$$|B(r)| < \prod_1^{N-1} \frac{r - \alpha_n}{1 - \alpha_n r} < \prod_1^{N-1} \frac{\alpha_N - \alpha_n}{1 - \alpha_n} < 2e^{-N/3}$$

is valid if $\alpha_{n-1} < r < \alpha_N$.

186. **(Rudin Chapter 15 #18)** Suppose B is a Blaschke product all of whose zeros lie on the segment $(0, 1)$ and $f(z) = (z - 1)^2 B(z)$. Prove that the derivative of f is bounded in the disc.

187. **(Homework 11 - Dr. Curto)**

   (a) Let $0 < |a| < 1$ and $|z| \leq r < 1$; show that

   $$\left| \frac{a + |a|z}{(1 - \overline{a}z)a} \right| \leq \frac{1 + r}{1 - r}.$$

   (b) Let $\{a_n\}$ be a sequence of complex numbers with $0 < |a_n| < 1$ and $\sum(1 - |a_n|) < \infty$. Show that the infinite product

   $$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \left(\frac{a_n - z}{1 - \overline{a_n}z} \right)$$

   converges in $H(\mathbb{D})$ and that $|B(z)| \leq 1$. What are the zeros of B?

   (c) Find a sequence $\{a_n\}$ in $B(0; 1)$ such that $\sum(1 - |a_n|) < \infty$ and every number $e^{i\theta}$ is a limit point of $\{a_n\}$.

188. **(Homework 11 - Dr. Curto)** For which values of z do the products $\prod(1-z^n)$ and $\prod(1+z^{2n})$ converge? Is there an open set G such that the product converges uniformly on each compact subset of G? If so, give the largest such open set.

189. **(Homework 11 - Dr. Curto)** Show that $\cos(\pi z) = \prod_{n=1}^{\infty} \left[1 - \frac{4z^4}{(2n-1)^2}\right]$

190. **(Homework 12 - Dr. Curto)** Let G be a region and let $\{a_n\}_n$ and $\{b_m\}_m$ be two sequences of distinct points in G without limit points in G such that $a_n \neq b_m$ for all n, m. Let $S_n(z)$ be a singular part at a_n and let p_m be a positive integer. Show that there is a meromorphic function f on G whose only poles and zeros are $\{a_n\}$ and $\{b_m\}$ respectively, the singular part at $z = a_n$ is $S_n(z)$ and $z = b_m$ is a zero of multiplicity p_m.

191. **(Homework 12 - Dr. Curto)** Let G be a region and let $\{a_n\}$ be a sequence of distinct points in G with no limit point in G. For each integer $n \geq 1$ choose integers $k_n \geq 0$ and constants $A_n^{(k)}$, $0 \leq k \leq k_n$. Show that there is an analytic function f on G such that $f^{(k)}(a_n) = k!A_n^{(k)}$.

192. **(Final22 - Dr. Curto)** Prove the existence of a meromorphic function $f \in M(\mathbb{C})$ with poles of order $1, \sqrt{2}, \sqrt{3}, ...$ such that:

   (i) the residue at each pole is 0, and

   (ii) $\lim_{z \to \sqrt{n}}(z - \sqrt{n})^2 f(z) = 1$ for all n.

193. **(Final23 - Dr. Curto)** Recall the sequence $\{a_n\}$ be a sequence of complex numbers with $0 < |a_n| < 1$ such that $\sum_n (1 - |a_n|) < \infty$ and every complex number $e^{i\theta} \in \mathbb{T}$ is a limit point of $\{a_n\}$. Prove that the Blaschke product B built from the sequence $\{a_n\}$ above is a bounded analytic function on the unit disc for which each point of the unit circle is a non-removable singularity.

Gamma Function

194. **(Homework 11 - Dr. Curto)** Show that $0 < \gamma < 1$.

195. **(Homework 11 - Dr. Curto)** Show that $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$ for z not an integer. Deduce from this that $\Gamma(1/2) = \sqrt{\pi}$.

196. **(Homework 11 - Dr. Curto)** Let f be analytic on the right half plane $Re(z) > 0$ and satisfy: $f(1) = 1$, $f(z+1) = zf(z)$ and $\lim_{n\to\infty} \frac{f(z+n)}{n^z f(n)} = 1$ for all z. Show that $f = \Gamma$.