Numerical Analysis Qual Prep Fall 2023

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This document contains the problems which will be assigned to the students taking Numerical Analysis qualifying exam in Fall 2023. The problems have been taken from various resources including:

- Homework problems and exams from Dr. Ayati for the academic year 2022-2023.
- Previous year's midterm and final exams by Dr. Ayati, Dr. Stewart and Dr. Jay.
- Previous qual problems posted on the Math department website
- Qual problems from different universities.
- PDEs (Math:5700) Spring 22/23

1 Math: 5800

1.1 Module 1: Basics of numerical computations

1. (George and Joe) If we try to solve $x^2+200x-1.5\times10^{-5}$ numerically using the direct quadratic equation, we will get an unnecessarily large error for one of the roots. Explain why this is and then describe a method that can be used to find the roots of this polynomial with less error.

2. (Nandita and Paria) Let $x \in \mathbb{R}^n$ and show that we have $\|x\|_\infty \leq \|x\|_2 \leq n^{1/2}\|x\|_\infty$

3. (Abdul and Bakhtiar) Derive the n-th order Taylor polynomial for e^x. What value of n will guarantee an error of no more than 10^{-10} for any x with $|x| < 1.2$?

4. (**UofI: F14 Qual** Claire and Yutian) The standard formula for computing roots of a quadratic $ax^2 + bx + c - 0$:
$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
is known to have problems computing the smaller root numerically if b^2 is *much* larger than $|ac|$. Explain the cause of this problem and propose a method for avoiding it.

5. (**UofI: F20 Qual** Victoria and Fatemeh) In many programming languages, the formulas
$cos\lambda = \frac{x}{\sqrt{x^2+y^2}}$ and $sin\lambda = \frac{y}{\sqrt{x^2+y^2}}$
give $cos\lambda = sin\lambda = 0$ for $x = y = 10^{200}$. Why does this occur? Can you rearrange the formulas to give equivalent formulas in exact arithmetic that work better numerically?

1.2 Module 2: Solution of equations by iteration

6. (Ashwin and Hemanth) Apply the bisection method to $xe^x = 4$ starting from the interval [0,2] to get an accuracy of 2×10^{-1}.

7. (Liz and James) The stopping criterion in the bisection method is $|b-a| < \epsilon$. For most other algorithms, the stopping criterion is $|f(x)| < \epsilon$. Show that if f is smooth and ϵ is small, then these two stopping criteria are within a factor of approximately $|f'(x^*)|$.

8. (Javier and Zhihua) Apply three steps of Newton's method to $e^x + x^2 - 4 = 0$ with starting point $x = 1$.

9. (George and Joe) A method for computing square roots is to solve $x^2 - a = 0$ for x.
i) Show that Newton's method applied to this equation is $x_{n-1} = \frac{1}{2}(x_n + \frac{a}{x_n})$.
ii) Show that the iteration function $g(x) = \frac{1}{2}(x + \frac{a}{x}$ has a minimum at $x = \sqrt{a}$ and $g(\sqrt{a}) = \sqrt{a}$, so that $x_{n+1} \geq \sqrt{a}$.
iii) Show that Newton's method is globally convergent for any positive starting point.

10. (Nandita and Paria) Here we analyze Newton's method applied to the equation $x^m = 0$ with $m > 1$. How quickly does Newton's method converge for this problem?

11. (**Dr. Ayati: F22 Midterm** Abdul and Bakhtiar) Apply two steps of Newton's method to find a root of $f(x) = cos(x)$ with initial guess $x_0 = \pi/4$.

12. (**Dr. Ayati: F22 Midterm** Claire and Yutian) Suppose that $f(\xi) = f'(\xi) = 0$, $f''(\xi) \neq 0$ so that f has a double root at ξ, and that f'' is defined and continuous in a neighborhood of ξ. If $\{x_k\}$ is a sequence obtained by Newton's method, show that $\xi - x_{k+1} = \frac{1}{2}(\xi - x_k)\frac{f''(\eta_k)}{f''(\chi_k)}$ where both η_k and χ_k lie between ξ and x_k.

13. (**UofI: F06 Qual** Victoria and Fatemeh) Give the Newton method for solving the equation $e^x + 4e^{-x} - 4 = 0$ and discuss the convergence order of the method.

14. (**UofI: S08 Qual** Ashwin and Hemanth) Write out a general step for the Newton and Secant methods for solving $f(x) = 0$. List all conditions needed in order for these methods to converge. Give the expected rates of convergence of these methods under the conditions you have stated. Finally, perform three steps of Newton's method to find the positive root of $x^3 - 3x - 3 = 0$ using the starting value $x_0 = 2$.

15. (**UofI: F14 Qual** Liz and James) Carry out two steps of the secant method for solving $x - cosx = 0$ starting with $x_0 = 0$ and $x_1 = 1$. What rate of convergence is expected and under what conditions is this rate of convergence obtained?

16. (**UofI: F20 Qual** Javier and Zhihua) Carry out two steps of the secant method for solving $xe^x = 3$ starting with $x_0 = 0$ and $x_1 = 1$. What rate of convergence is expected and under what conditions is this rate of convergence obtained?

17. (**Dr. Ayati: F12 Final** George and Joe) It can be shown that an iterative formula for determining the square root of a number Q based on Newton's method is $x_{k+1} = \frac{1}{2}(x_k + \frac{Q}{x_k})$.
Derive a similar formula based on the Secant method.

1.3 Module 3: Polynomial interpolation

18. (Nandita and Paria) Let $f(x) = \frac{e^{-x}}{1+x^2}$. Compute the quadratic interpolant $p(x)$ for this function using interpolation points $x_0 = 0$, $x_1 = \frac{1}{2}$, and $x_2 = 1$.

19. (Abdul and Bakhtiar) Show that if x_0, x_1, and x_2 are distinct, then $p(x) = x_0L_0(x) + x_1L_1(x) + x_2L_2(x) = x$ for all x. Here $L_j(x)$ are the quadratic Lagrange interpolation polynomials for these interpolation points.

20. (**Dr. Ayati: F22 Midterm** Claire and Yutian) Find the Lagrange interpolation polynomial $p_2(x)$ through the points (-1,1), (0,2), (1,0). For an arbitrary function $f \in C^3[-1, 1]$ that goes through these three points, find a (reasonably sharp) constant K such that $|f(x) - p_2(x)| \leq K * max_{\xi\in[-1,1]}|f'''(\xi)|$.

21. (**UofI: F06 Qual** Victoria and Fatemeh) Let $f(x) = sin(\pi x)$. Determine a function $p(x)$ such that $p(x)$ is a polynomial on $[0, 0.5]$ and $[0.5, 1]$, and satisfies the conditions
$p(x) = f(x), p'(x) = f'(x)$, for x=0, 0.5, 1.

22. (**UofI: S08 Qual** Ashwin and Hemanth) Given a function $f(x)$ on the interval $[a, b]$ and points $a \leq x_0 < x_1 < x_2 < ... < x_n \leq b$, give a method to construct a polynomial $p(x)$ of degree $\leq n$ where p interpolates f at $x_0, x_1, ...x_n$. Give a formula for estimating the interpolation error

$f(x) - p(x)$.
What is Chebyshev interpolation? From the formula for the interpolation error, explain how Chebyshev interpolation relates to minimax approximation.

23. (**UofI: F14 Qual** Liz and James) Using equally spaced interpolation points is known to result in Runge's phenomenon for the function $f(x) = \frac{1}{(1+x^2)}$ interpolated over [-5, 5]. What is this phenomenon? Can the use of a different set of interpolation points prevent this phenomenon? If so, how?

24. (**Dr. Ayati: F12 Final** Javier and Zhihua) Show that the Chebyshev polynomials on [-1,1], $T_n = cos(n*arccosx)$, $n = 0, ...$, have the recurrence relationship
$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, n=1,...
Use this to argue that the Chebyshev polynomials are indeed polynomials.
Note: $cos((n+1)\lambda) + cos((n-1)\lambda) = 2cos(\lambda)cos(n\lambda)$.

25. (**Dr. Jay: F20 Midterm** George and Joe) Consider the polynomial $P_6(x)$ of degree 6 interpolating the function $f(x) := sin(\pi x/2)$ on the interval [-3,7] at the 7 Chebyshev points $x_j = 2 + 5cos((13-2j)\pi/14)$ for $j = 0, 1, 2, 3, 4, 5, 6$. Estimate the maximum of the interpolation error for $x \in [-3, 7]$, i.e. give a sharp upper bound for
$max_{x\in[-3,7]}|sin(\pi x/2) - P_6(x)| \leq ?$

26. (**Dr. Stewart: F14 Final** Nandita and Paria) Consider using quadratic interpolation applies to $f(x) = 1/(1+x)$ with interpolation points $x_0 = 0$, $x_1 = x_0 + h$, and $x_2 = x_0 + 2h$. Estimate the largest value of $h > 0$ which will ensure that $max_{x_0 \leq x \leq x_2}|f(x) - p(x)| \leq 10^{-6}$ where p is the quadratic interpolant.

1.4 Module 4: Two dimensional interpolation

27. (Abdul and Bakhtiar) Show that for distinct points $x_1, x_2, ..., x_n \in \mathbb{R}^d$ the interpolation matrix $A(\alpha)$ with entries $a_{ij}(\alpha) = e^{-\alpha||x_i - x_j||_2^2}$ goes to the identity matrix as $\alpha \to \infty$, but goes to ee^T as α decreases to 0 where $e \in \mathbb{R}^n$ is the vector of ones.

1.5 Module 5: Approximation in infinity and two norms

28. (Claire and Yutian) Show that the three-term recurrence for a family of orthogonal polynomials can be re-written as $cp_k(x) = c_k p_{k-1}(x) + a_k p_k(x) + b_k p_{k+1}(x)$.
Furthermore, show that if the orthogonal polynomials are normalized (that is, $\langle p_j, p_j \rangle_w = 1$ for all j), then $c_k = b_{k-1}$.

29. (**Dr. Ayati: F12 Final** Victoria and Fatemeh) Construct orthogonal polynomials (in the L^2 sense) of degrees 0, 1, and 2 on the interval (0,1) with weight function $w(x) = -ln(x)$. Note that
$\int_0^1 -ln(x)x^k dx = \frac{1}{(k+1)^2}$.
Recall that Gram-Schmidt orthogonalization is an inductive process,
$q(x) = x^{n+1} - a_0\phi_0(x) - ... - a_n\phi_n(x)$, where $a_j = \frac{\int_0^1 w(x)x^{n+1}\phi_j(x)dx}{\int_0^1 w(x)\phi_j^2(x)dx}$.

30. (**Dr. Stewart: F14 Final** Ashwin and Hemanth) State the theorems of de la Valle Poussin and Chebyshev regarding minimax approximation: given $f : [a,b] \to \mathbb{R}$ find a polynomial p of degree $\leq$ n that minimizes $max_{a \leq x \leq b}|f(x) - p(x)|$. Use Chebyshev's theorem to show that $q(x) = T_{n+1}(x) - 2^n x^{n+1}$ minimizes
$max_{-1 \leq x \leq 1}|2^n x^{n+1} + q(x)|$ over all polynomials q of degree $\leq n$ where T_{n+1} is the Chebyshev polynomial defined by $T_{n+1}(cos\lambda) = cos((n+1)\lambda)$.

31. (**Dr. Stewart: F14 Final** Liz and James) Find the least squares approximation to $f(x) = 1/(1+x)$ on the interval [0,1] by linear functions $p(x) = ax + b$.

1.6 Module 6: Numerical integration

32. (Javier and Zhihua) Determine and show the degree of precision of Simpson's rule on the reference interval [-1,1], $\int_{-1}^1 f(x)dx \approx \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)$.

33. (**UofI: F06 Qual** George and Joe) Define, for function f, the function $I(f) = \int_{-h}^h f(x)dx \approx A_0 f(-h/2) + A_1 f(0) + A_2 f(h/2)$.
i) Find A_0, A_1, and A_2 such that this integration rule is exact for polynomials of degree ≤ 2.
ii) Show that the rule constructed above is in-fact exact for polynomials of degree ≤ 3.
iii) For the constructed rule, it can be proved that
$I(f) - (A_0 f(-h/2) + A_1 f(0) + A_2 f(h/2)) = c_0 f^{(4)}(\eta)h^5$, $\eta \in [-h, h]$
where c_0 is a constant independent of f. Find the constant c_0.

34. (**UofI: F14 Qual** Nandita and Paria) Use Simpson's method with five function evaluations to obtain an estimate of $\int_0^1 \frac{e^x}{1+x}dx$. What is the asymptotic order of the error of composite Simpson's method with $2n+1$ function evaluations? Give an example of a method that has an asymptotically faster rate of convergence than Simpson's method as the number of function evaluations goes to infinity.

35. (**Dr. Ayati: F12 Final** Abdul and Bakhtiar) Show that the weights for Gaussian quadrature of degree n, W_k, can be computed from
$W_k = \int_a^b w(x)L_k(x)dx$,

where $L_k(x)$ is a Lagrange shape function of degree n. Recall Gaussian quadrature of degree n is exact for polynomials of degree less than or equal to $2n - 1$.

36. (**Dr. Ayati: F12 Final** Claire and Yutian) Derive the two-point Gaussian quadrature formula on the interval [-1,1] using the method of undetermined coefficients.

37. (**Dr. Jay: F20 Midterm** Victoria and Fatemeh) Consider 2 nonsymmetric quadrature formulas of order 3 with s=2:
. The Radau IIA quadrature formula with weights $(b_1, b_2) = (3/4, 1/4)$ and nodes $(c_1, c_2) = (1/3, 1)$.
. The Radau IA quadrature formula with weights $(b_1, b_2) = (1/4, 3/4)$ and nodes $(c_1, c_2) = (0, 2/3)$.

    We now consider a composite quadrature formula defined as follows: on each subinterval $[x_j, x_j + h_j]$ we first apply the Radau IIA quadrature formula on the subinterval $[x_j, x_j + h_j/2]$ and then we apply the Radau IA quadrature formula on the subinterval $[x_j + h_j/2, x_j + h_j]$.
i) Express this composite quadrature formula as a standard quadrature formula on the subinterval $[x_j, x_j + h_j]$ and give its coefficients.
ii) Is this composite quadrature formula symmetric?
iii) What is the order of this composite quadrature formula?

38. (**Dr. Stewart: F14 Final** Ashwin and Hemanth) Let $T_n f = \frac{b-1}{n}[\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1/2}{f}(x_n)]$ (where $x_k = a + k(b-a)/n$) be the trapezoidal rule approximation to $\int_a^b f(x)dx$ with n+1 points. Show that one step of Richardson extrapolation:
$T_n^{(1)} f = \frac{2^2 T_{2n} f - T_n f}{2^2 - 1}$ is identical to Simpson's rule with 2n + 1 points.

1.7 Module 7: Multidimensional integration and numerical differentiation

39. (Liz and James) Show that for any region $R \subseteq \mathbb{R}^n$, if $\bar{x}_R = \frac{\int_R x dx}{\int_R dx}$ is the centroid of R, then the one-point rule $\int_R f(x)dx \approx vol_n(R) f(\bar{x}_R)$ is exact for all linear functions f. Note that $vol_n(R) = \int_R dx$ is the n-dimensional volume of R.

40. (**Dr. Stewart: F14 Final** Javier and Zhihua) Determine a symmetric 5-point method for approximating $f''(x)$ using the values $f(x)$, $f(x \pm h)$, and $f(x \pm 2h)$:
$f''(x) \approx af(x) + b[f(x+h) + f(x-h)] + c[f(x+2h) + f(x-2h)]$ that has an error of $O(h^4)$. Use Taylor series expansions with $O(h^6)$ remainders.

2 Math: 5810

2.1 Module 1: Direct methods for linear systems

1. (Victoria and Fatemeh) An $n \times n$ matrix is diagonally dominant $|a_{ii}| > \sum_{j:j\neq i} |a_{ij}|$ for all i. Show that strictly row dominant matrices are invertible. Give an example of a row dominant (but not strictly dominant) matrix that is not invertible: $|a_{ii}| \geq \sum_{j:j\neq i} |a_{ij}|$ for all i.

2. (**Dr. Ayati: S13 Midterm** Ashwin and Hemanth) Prove that if $A = MM^T$ for $M \in \mathbb{R}^{n\times n}$ is nonsingular, then A is symmetric positive definite.

3. (Liz and James) Develop a block LU factorization algorithm without pivoting by starting with the decomposition

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ & U_{22} \end{bmatrix}$$

with A_{11} be a $b \times b$ matrix. Use the standard LU factorization algorithm for factoring $A_{11} = L_{11}U_{11}$.

2.2 Module 2: Sparse matrices and least square problems

4. (Javier and Zhihua) Show that the LDL^T can be numerically unstable even when it succeeds, by considering the matrix

$$\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$$

with $0 \neq \epsilon \approx 0$.

5. (George and Joe) Show that if A is real, square and invertible, then the QR factorization is unique apart from a diagonal scaling by factors of ± 1. That is, if $A = Q_1R_1 = Q_2R_2$ with Q_1, Q_2 orthogonal, R_1, R_2 upper triangular, then there is a diagonal matrix D with the diagonal entries ± 1, where $Q_2 = Q_1D$ and $DR_2 = R_1$. In particular, show that if the diagonal entries of R_1, R_2 are all positive, then $Q_1 = Q_2$ and $R_1 = R_2$.

6. (Nandita and Paria) We can assign weights to the least square problem to emphasize the importance of certain components. Doing so can be generalized to minimizing $\|A\mathbf{x} - b\|_C$, where the "C"-norm corresponding to a symmetric positive definite matrix "C" is given by $\|\mathbf{x}\|_C = \sqrt{\mathbf{x}^T C\mathbf{x}}$. Derive the normal equations for this problem.

7. (**UIowa Qual F20** Abdul and Bakhtiar) Give the QR algorithm with or without shifting. Show that if the original matrix is symmetric, then every iterate of the QR factorization is symmetric. (*Additional* Explain how shifting is used to improve the rate of convergence of the QR algorithm.)

8. (**Dr. Ayati: S13 Final** Claire and Yutian) Derive the normal equations minimizer for the linear least square problem, $x = (A^T A)^{-1} A^T b$. *Hint*: use the limit definition of a gradient to find where $\nabla \|Ax - b\| = 0$.

9. (**Dr. Ayati: S13 Final** Victoria and Fatemeh) Derive the QR decomposition minimizer for the least square problem, $x = R^{-1} Q^T b$.

10. (**UIowa Qual S15** Ashwin and Hemanth) What is QR factorization? Explain how to use a QR factorization of a matrix to solve a least squares problem $\min_x \|A\mathbf{x} - \mathbf{b}\|_2$

11. (**UIowa Qual F14** Liz and James) The normal equations for solving least square problem $\min_x \|A\mathbf{x} - \mathbf{b}\|_2$ are $A^T A\mathbf{x} = A^T b$. Describe precisely what the Cholesky and QR factorizations are. Show how to solve the least square problem using either the QR factorization applied to the original system, or Cholesky factorization applied to the normal eqautions to the least squares problem. Also show that the matrix R in the QR factorization of A is one of the factors in a Cholesky factorization of $A^T A$.

12. (**UIowa Qual F05** Javier and Zhihua) What is the Cholesky factorization? Find the Cholesky factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{bmatrix}.$$

13. (George and Joe) Suppose we add a single symmetric entry to the top right and the bottom left corners to A. What is the graph of the resulting matrix? How much fill-in will be generated by Cholesky factorization? Assume that the resulting matrix is positive definite so that the Cholesky factorization exists.

14. (Nandita and Paria) The graph of an arrowhead matrix is a spokes graph. If we have a symmetric matrix that is the sum of a tridiagonal matrix and a matrix $\mathbf{u}\mathbf{e}_1^T + \mathbf{e}_1\mathbf{u}^T$ that fills in the first row and column, show that the corresponding graph is a wheel graph: a cycle of $n-1$ vertices together with a "hub vertex" that is connected to every node in the cycle. Show that there is an ordering that gives no fill-in for this graph.

15. (Abdul and Bakhtiar) Show that the graph of a symmetric matrix is a tree (a connected undirected graph with no cycles), then the matrix can be re-ordered so that the Cholesky factorization gives no fill-in.

16. (Claire and Yutian) Show that the symmetric banded matrix ($a_{ij} \neq 0$ only if $|i-j| \leq b$ where b is the 'bandwidth') the Cholesky factorization can be done with no fill-in outside the band. Note that a tridiagonal matrix is the special case of a banded matrix with bandwidth $b = 1$.

17. (**Dr. Ayati: S13 Midterm** Victoria and Fatemeh) Show the equivalence of 2-*norm* and ∞-*norm* by showing

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$$

2.3 Module 3: Iterative methods for linear systems

Note: No theoretical questions were asked in homework for this module. I will add few questions from the other quals/previous years as extra practice.

18. (**UIowa Qual S08** Ashwin and Hemanth) Define the Gauss-Jacobi and the Gauss-Seidel method for solving the linear system $Ax = b$, where $b \in \mathbb{R}^N$ is given and

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 3 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 3 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

For each method, determine a number of iterations $\|\cdot\|_\infty$ norm of the error by a factor of 0.01.

19. (**UIowa Qual F05** Liz and James) In iteratively solving the linear system $A\mathbf{x} = \mathbf{b}$ ($\det(A) \neq 0$), we write $A = P - N$, with P nonsingular, and generate a sequence $\{x^k\}$ by the formula

$$P\mathbf{x}^{(k+1)} = \mathbf{b} + N\mathbf{x}^{(k)},$$

starting with some initial guess $\mathbf{x}^{(0)}$. Denote the residual $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{x}^{(k)}$

(a) Show the the iteration formula can be equivalently expressed as

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + P^{-1}\mathbf{r}^{(k)}.$$

(b) Let $\alpha > 0$ be a constant. Define the stationary Richardson method by the formula

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha P^{-1}\mathbf{r}^{(k)}.$$

Show that the method converges if and only if $\alpha|\lambda|^2 < Re(\lambda)$ for any eigenvalue λ of $P^{-1}A$.

2.4 Module 4: Methods for eigenvalues and eigenvectors

20. (Javier and Zhihua) Matrix

$$\begin{bmatrix} 5 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

has eigenvalues $1, 2$ and 3. With the starting vector $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ perform one step of the power method. What value should the power method algorithm return if the stopping criteria is $\frac{\|Ax_k - \lambda_k x_k\|_2}{\|x\|_2} \leq 10^{-14}$.

21. (George and Joe) Show that the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 2 & & & \\ & & 1 & 3 & & \\ & & & & \ddots & \\ & & & & 1 & (n-1) \\ & & & & & 1 \end{bmatrix}$$

has one eigenvalue $\lambda = 1$ repeated n times. Symbolically compute the eigenvalues of $A + \epsilon e_n e_1^T$.

22. (Nandita and Paria) Show that the eigenvalues of the $n \times n$ matrix

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & -2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

are $\lambda_j = 2(1 - \cos(\frac{\pi j}{n+1}))$ for $j = 1, 2, \ldots, n$.

23. (**Dr. Stewart S15 Final** Abdul and Bakhtiar) Consider the QR algorithm

$$\text{Given } A$$

$$B_0 \leftarrow A$$

$$\text{for} k = 1, 2, 3, \ldots$$

$$B_k = Q_k R_k \quad \textit{QR Factorization}$$

$$B_{k+1} \leftarrow R_k Q_k$$

Show that B_{k+1} is orthogonally similar to B_k, and so have the same eigenvalues. Write $B_k = U_k^T A U_k$, where U_k is orthogonal. Obtain an updated formula for U_{k+1} in terms of U_k and Q_k. If $B_k \to B^*$ and $U_k \to U^*$ as $k \to \infty$. Show that B^* is an upper triangular and $A = U^* B^* (U^*)^T$ is a Schur decomposition of A. If A is symmetric, then show that B^* is diagonal.

24. (**Dr. Ayati: S22 HW** Claire and Yutian) Prove the following lemma

    Let A be $m \times m$, B be $n \times n$, and X be $m \times n$. Then the following properties hold:

    (a) $\vec{(AX)} = (I_n \bigotimes A) \cdot (\vec{X})$

    (b) $\vec{XB} = (B^T \bigotimes I_m) \cdot (\vec{X})$

2.5 Module 5: Methods for initial value problems

25. (Victoria and Fatemeh) Determine the symbol/stability of Heun's method.
26. (Ashwin and Hemanth) Determine the symbol/stability of the Trapezoidal rule.
27. (Liz and James) Determine tye symbol/stability of the Midpoint rule.
28. (**UIowa Qual S15** Javier and Zhihua) Show that the Heun's method

    $$z_{n+1} = y_n + hf(t_n.y_n),$$

    $$y_{n+1} = y_n + \frac{1}{2}h[f(t_n, y_n) + f(t_{n+1}, z_{n+1})]$$

    has a local truncation error of order $\mathcal{O}(h^3)$. What is its asymptotic global truncation error in the form $\mathcal{O}(h^m)$?

29. (**Dr. Ayati: S13 Midterm** George and Joe) Using "symbols" for the scalar case of the model problem, show that the Forward Euler is not absolutely stable and that the Backward Euler is absolutely stable.
30. (**Dr. Ayati: S13 Final** Nandita and Paria) Using "symbols" for the scalar case of the model problem, show that the Trapezoidal rule for ODEs is absolutely stable.
31. (**Dr. Ayati: S13 Midterm** Abdul and Bakhtiar) Show that the Forward Euler is only stable if $h \leq \frac{1}{s\delta^2}$, where h is the constant stepsize.

32. (**Dr. Stewart: S15 Midterm** Claire and Yutian) What are the consistency conditions or order conditions to determine the order of accuracy of a multistep method? Use this to construct a 3rd order Adam-Moulton method:

$$y_{n+1} = y_n + h \sum_{j=-1}^{p} b_j f(t_{n-j}, y_{n-j}),$$

with $p = 1$.

33. (**Dr. Stewart: S14 Midterm** Victoria and Fatemeh) Consider solving differential equation for smooth and Lipschitz $\mathbf{f}(t, \mathbf{y})$,

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0$$

with the explicit multistep method

$$\mathbf{y}_{n+1} = \sum_{j=0}^{p} a_j \mathbf{y}_{n-j} + h \sum_{j=0}^{p} b_j \mathbf{f}(t_{n-j}, \mathbf{y}_{n-j})$$

where $a_j \geq 0$ for all j and $\sum_{j=0}^{p} a_j = 1$. Suppose that

$$\mathbf{y}(t_{n+1}) = \sum_{j=0}^{p} a_j \mathbf{y}(t_{n-j}) + h \sum_{j=0}^{p} b_j \mathbf{f}(t_{n-j}, \mathbf{y}_{n-j}) + \tau_n(h)$$

with $\tau_n(h) = \mathcal{O}(h^{m+1})$. Show that the error $e_k = y(t_k) - y_k$ satisfies the inequality

$$\|e_{n+1}\| \leq (1 + h\beta L) \max_{0 \leq k \leq n} \|e_k\| + \|\tau_n(h)\|$$

where $\beta = sum_{j=0}^{p} |b_j|$ and L is the Lipschitz constant $\mathbf{f}(t, \mathbf{y})$.

34. (**UIowa Qual F05** Ashwin and Hemanth) Consider solving an initial value problem $y' = f(x, y)$ for $x \in [0, 1]$, $y(0) = Y_0$, f being a smooth function. Let $0 = x_1 < x_2 < \cdots < x_N = 1$ be a uniform partition of the interval $[0, 1]$ and denote h the step size. For a constant parameter $\theta \in [0, 1]$, introduce the following generalized mid-point method

$$y_{n+1} = y_n + h[(1 - \theta) f(x_n, y_n) + \theta f(x_{n+1}, y_{n+1})].$$

It is known that for h small enough, this relation defines a unique value y_{n+1}.

    (a) Determine the order of the method.
    (b) Show that the method is absolutely stable when $\theta \in [\frac{1}{2}, 1]$.

35. (**Dr. Jay: S21 Midterm** Liz and James) We consider the following implicit linear multistep method applied to $\dot{x} = f(t, x)$ with stepsize h (using the notation $f_j = f(t_j, x_j)$)

$$x_{n+1} = x_n + \frac{h}{12}(5 f_{n+1} + 8 f_n - f_{n-1}).$$

(a) What is the local error of this method?

(b) Is it 0-stable? Is it globally convergent?

2.6 Module 6: Methods for boundary value problems

36. (Javier and Zhihua) Blasius problem is related to the asymptotics of the viscous fluid flow over a plate, and is the third order differential equation $y''' + \frac{1}{2}yy'' = 0$ with the boundary conditions $y(0) = y'(0) = 0$ and $y'(\infty) = 1$. The boundary condition at infinity, $y'(\infty) = 1$ can be approximated by $y'(L)$, where L is chosen be large. Use the *shooting method* to solve the Blasius problem (the value of $y''(0)$ is the quantity to solve for). Choose different values of L and discuss the convergence of the solution as L becomes large.

37. (George and Joe) Implement the finite difference method for the problem $\nabla \cdot (a(\mathbf{x})\nabla u = f(\mathbf{x}))$ for $\mathbf{x} \in \Omega \subseteq \mathbb{R}^2$ and $u(\mathbf{x} = g(\mathbf{x})$ for $\mathbf{x} \in \delta\Omega$ using the approximation

$$\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}(x,y)\right) \approx \frac{1}{h}\left[a\left(x+\frac{1}{2}h, y\right)\frac{u(x+h,y)-u(x,y)}{h} + a\left(x-\frac{1}{2}h, y\right)\frac{u(x-h,y)-u(x,y)}{h}\right]$$

$$\frac{\partial}{\partial y}\left(a(x,y)\frac{\partial u}{\partial y}(x,y)\right) \approx \frac{1}{h}\left[a\left(x, y+\frac{h}{2}\right)\frac{u(x,y+h)-u(x,y)}{h} + a\left(x, y-\frac{h}{2}\right)\frac{u(x,y-h)-u(x,y)}{h}\right]$$

Show empirically that that the asymptotic error of this formula is $\mathcal{O}(h^m)$ as $h \to 0$ for some m assuming that both a and u are smooth. What is the value of m in general.

2.7 Module 7: Methods for partial differential equations

38. (Nandita and Paria) Consider the PDE boundary value problem

$$div(a(\mathbf{x})\nabla u) = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega$$

$$\beta(\mathbf{x})u(\mathbf{x}) + \frac{\delta u}{\delta \eta}(\mathbf{x}) = g(\mathbf{x}) \quad \text{for } \mathbf{x} \in \delta\Omega.$$

Write this PDE in the weak form. Show that the bilinear form in the weak form is coercive with respect to $H^1(\Omega)$ provided $\inf_{x\in\Omega} a(\mathbf{x}) > 0$ and $\inf_{x\in\partial\Omega} \beta(\mathbf{x}) > 0$. Assume that

$$\frac{\int_{\partial\Omega} u^2 dS(x)}{\int_\Omega (\|\nabla u\|^2 + u^2)d\mathbf{x}}$$

has a positive lower bound.

39. (**UNM S05** Abdul and Bakhtiar)) Consider the boundary value problem

$$-u'' + u = 0 \quad x \in (0, 1, \quad u(0) = u(1) = 1.$$

Write its weak form.

40. (**PDE S22/23** Claire and Yutian)) State *Lax-Milgram* Theorem. Consider the boundary value problem

$$\Delta u - a(x)u = 0 \text{ in } B_1$$

$$u = 2 \quad \text{on } \partial B_1,$$

where B_1 is a unit ball with center origin. Write its weak form and use *Lax-Milgram* to show that the weak solution is unique.

3 Note on extra problems:

There is an extra material posted on ICON for practice. You are advised to do them if you want more practice. As for qualifying exams posted from different universities, TAMU has a bit more harder problems and few of its topics were not taught by Dr. Ayati. You can skip them if you want. University of New Mexico (posted on ICON) has good problems, you can do them to strengthen your concepts. University of Colorado Boulder (UofCB) (posted on ICON) has pretty good questions as well, they are the classical problems even like University of Kansas qualifying exam-problems. I would suggest doing their (UofCB) qualifying exams will be extremely helpful in terms of many topics excluding their PDEs questions. Though we have posted solutions for UofCB's qualifying exams, please try the problems yourself first and just keep the solutions as a reference.
Good Luck!

4 Review Sheet Numerical Analysis Qual Prep 2023

4.1 Module 1.1: Basics of numerical computations (Abdul)

4.2 Module 1.2: Solution of equations by iteration (Bhaktiar)

4.3 Module 1.3: Polynomial interpolation (Liz)

4.4 Module 1.4: Two dimensional interpolation (Clair)

4.5 Module 1.5: Approximation in infinity and two norms (George)

4.6 Module 1.6: Numerical integration (Javier)

4.7 Module 1.7: Multidimensional integration and numerical differentiation (James)

4.8 Module 2.1: Direct methods for linear systems (Yutian)

4.9 Module 2.2: Sparse matrices and least square problems (Paria)

4.10 Module 2.3: Iterative methods for linear systems (Hemanth)

4.11 Module 2.4: Methods for eigenvalues and eigenvectors (Nandita)

4.12 Module 2.5: Methods for initial value problems (Fatemeh)

4.13 Module 2.6: Methods for boundary value problems (Joe)

4.14 Module 2.7: Methods for partial differential equations (Victoria)