What's in this section? The following problems have been collected from various sources (e.g., algebra textbooks by Dummit and Foote, Aluffi, Hungerford, etc, and different online sources). Although there are no citations, I make no claims of originality. If needed, most of them can be found in most standard algebra texts, maybe as unsolved exercises or even as solved examples or proven results. The problems comprise several topics in group theory and shall help you test your understanding of the fundamentals. The general idea is to build intuition and boost confidence.

Though the problems span varying degrees of difficulty, they should still be easier than the ones that have appeared in the previous qualifying exams. You may treat these as warm-up exercises as you build yourself up to answering more complicated questions. Some of these exercises contain common examples/counterexamples and others are easy-to-prove results, all of which might come in handy when solving more involved problems.

In case some of the problems in the next section leave you clueless, it might be more productive to revisit this section before you take another stab at the next section. (Cleanse your palate, so to speak.) Take your mind off the difficult ones while still being productive and then make another attempt with a fresh perspective.

I've tried my best to keep this error-free yet you must keep an eye out for any typos that might have sneaked in. If you find any flaws, especially logical or grammatical, please let me know and I shall fix them.

1. F is an extension field of K, then show that the following statements are equivalent.
   (a) F is algebraic and Galois over K;
   (b) F is separable over K and F is a splitting field over K of a set S of polynomials in $K[x]$;
   (c) F is a splitting field over K of a set T of separable polynomials in $K[x]$.
2. If F is a splitting field over K of S, then F is also a splitting field over K of the set T of all irreducible factors of polynomials in S.
3. If $f \in K[x]$ has degree n and F is a splitting field of f over K, then $[F : K]$ divides $n!$.
4. If $[F : K] = 2$, then F is normal over K.
5. Let K be a field and $f \in K[x]$ an irreducible polynomial of degree 2 with Galois group G. If f is separable (as is always the case when char $K \neq 2$), then $G \cong \mathbb{Z}_2$; otherwise $G = 1$.

Def: Let K be a field with char $K \neq 2$ and $f \in K[x]$ a polynomial of degree n with n distinct roots $u_1, \dots, u_n$ in some splitting field F of f over K. Let $\Delta = \prod_{i<j}(u_i - u_j) \in F$; the discriminant of f is the element $D = \Delta^2$.

6. (a) The discriminant Δ^2 of actually lies in K.
   (b) For each $\sigma \in \mathrm{Aut}_K(F) \subset S_n$, σ is an even [resp. odd] permutation if and only if $\sigma(\Delta) = \Delta$ [resp. $\sigma(\Delta) = -\Delta$].
7. Let f be a separable cubic with Galois group S_3 and roots $u_1, u_2, u_3 \in F$ (splitting field of f over K). Then the distinct intermediate fields of the extension of K by F are F, $K(\Delta)$, $K(u_1)$, $K(u_2)$, $K(u_3)$, K. The corresponding subgroups of the Galois group are 1,A_3, T_1, T_2, T_3 and S_3 where $T_i = \{(1), (jk) | j \neq i \neq k\}$.

8. Show that S_4 has no transitive subgroup of order 6.

9. Let f be an (irreducible) separable quartic over K and u a root of f. There is no field properly between K and $K(u)$ if and only if the Galois group of f is either A_4 or S_4.

10. Determine all the subgroups of the Galois group and all of the intermediate fields of the splitting field (over $\mathbb{Q}$) of the polynomial $(x^3-2)(x^2-3) \in \mathbb{Q}[x]$.

11. Let K be a subfield of the real numbers and $f \in K[x]$ an irreducible quartic. If f has exactly two real roots, the Galois group of f is S_4 or D_4.

12. Let K be a field, K an algebraic closure of K and $\sigma \in \mathrm{Aut}_K K$. Let

$$F = \{u \in \overline{K} \mid \sigma(u) = u\}.$$

Then F is a field and every finite dimensional extension of F is cyclic.

13. Which roots of unity are contained in the following fields: $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$?

14. Let F_n be a cyclotomic extension of $\mathbb{Q}$ of order n.

    (a) Determine $\mathrm{Aut}_{\mathbb{Q}} F_5$ and all intermediate fields.

    (b) Do the same for F_8.

    (c) Do the same for F_7; if f is a primitive 7th root of unity what is the irreducible polynomial over $\mathbb{Q}$ of $\zeta + \zeta^{-1}$?