Math Graduate Board

PDEs Qualifying exam preparation guide

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PDE Group of Fall 2023

Introduction

This is an unofficial guide intended to serve as preparation for the PDEs qualifying exam at the University of Iowa. The document contains a very general description of the topics and typical problems that are covered in the course MATH:5700 and that are usually part of the corresponding qualifying exam in the area. It is worth to mention that the qualifying exam is completely determined by the topics that are covered in the class corresponding to the academic period of examination and that this guide may not include all the topics evaluated in the exam.

1 References and Material

Among the many resources related to PDEs the following are some textbooks that are normally used in MATH:5700:

- Partial Differential Equations in Action, by S. Salsa [Salsa, 2016]. This is the textbook used in MATH:5700, it contains all the topics covered in the Qualifying exam.
- Partial Differential Equations in Action, Complements and Exercises, by S. Salsa and G. Verzini [Salsa and Verzini, 2015]. This is a solution manual for some of the problems in [Salsa, 2016]. It also contains supplementary material and extra problems.
- Partial Differential Equations by L. Evans [Evans, 2010]. This is a more advanced book, it is used as textbook in MATH:6700 but it can be used as a reading reference here. In particular, the appendix is very well organized and useful.
- Introduction to Partial Differential Equations with MATLAB, by J. Cooper [Cooper, 2012]. This book has been used as textbook for MATH:5700 in the past.

2 Topics of the exam

This is a general overview of the topics that are normally covered in the MATH:5700 course. For the material related to Sobolev spaces, we would recommend using Evans for a cleaner presentation.

1. **Introduction. Chapter 1 of** [Salsa, 2016]. It is assumed that the student is familiar with the terminology and topics of this chapter.
2. **Diffusion: Chapter 2 in** [Salsa, 2016]. Sections 2.4, 2.5, 2.6, 2.7, 2.9 and 2.10 are sometimes excluded.
3. **The Laplace Equation: Chapter 3 in** [Salsa, 2016]. Sections 3.4, 3.9 are sometimes excluded.
4. **The Wave equation: Chapter 5 in** [Salsa, 2016]. Sections 5.1, 5.5, 5.7 - 5.11 are sometimes excluded.
5. **Scalar conservation laws: Chapter 4 in** [Salsa, 2016]. Sections 4.7, 4.8, 4.9 are sometimes excluded.
6. **Elements of Functional Analysis: Chapter 6 in** [Salsa, 2016]. Sections 6.9 and 6.10 are sometimes excluded. Usually it is assumed that the student is familiar with the topics in this chapter and this material is not covered in the class.
7. **Distributions and Sobolev spaces: Chapter 7 in** [Salsa, 2016]. Sections 7.5, 7.6 and 7.11 are sometimes excluded.
8. **Variational Formulation of Elliptic Problems: Chapter 8 in** [Salsa, 2016]. Sections 8.4, 8.5 and 8.6 are sometimes excluded.

3 Homework - Midterm problems in MATH:5700.

3.1 Spring 2022

- Homework 1: Chapter 2 in [Salsa, 2016] - 2.1,2.2,2.3.
- Homework 2: Chapter 2 in [Salsa, 2016] : 2.8, 2.9, 2.14, 2.16, 2.18, 2.19.
- Homework 3: Chapter 2 - Chapter 3 in [Salsa, 2016]: 2.13, 2.15,3.1, 3.2.
- Homework 4: Chapter 3 in [Salsa, 2016]: 3.5, 3.6, 3.10, 3.11, 3.13.
- Homework 5: Chapter 3 - Chapter 4 in [Salsa, 2016] : 3.15, 3.16. 4.1.
- Homework 6: Chapter 4 - Chapter 5 in [Salsa, 2016]: 4.3, 4.6, 4.7, 5.1, 5.2.
- Homework 7: Chapter 5 - Chapter 6 in [Salsa, 2016]: 5.3, 6.1, 6.3, 6.5, read chapter 6.
- Homework 8: Chapter 7 in [Salsa, 2016] : 7.1, 7.2,7.3, 7.6, 7.10.
- Homework 9 : Chapter 7 in [Salsa, 2016]: 7.18, 7.20, 7.22, 7.23, 7.24.
- Homework 10 : Chapter 8 in [Salsa, 2016]: 8.1, 8.2, 8.3, 8.4, 8.5.
- Homework 11 : Chapter 8 in [Salsa, 2016]: 8.8, 8.9, 8.11, 8.12.
- Homework 12 : Chapter 8 in [Salsa, 2016]: 8.13, 8.16, 8.17, 8.21, 8.23.

3.2 Spring 2023

- Homework 1: Chapter 2 in [Salsa, 2016] - 2.1,2.2,2.3.
- Homework 2: Chapter 2 in [Salsa, 2016] : 2.6, 2.8, 2.9, 2.14, 2.18, 2.19.
- Homework 3: Chapter 2 - Chapter 3 in [Salsa, 2016]: 2.15, 2.23, 3.1, 3.3, 3.7, 3.8.
- Homework 4: Chapter 3 in [Salsa, 2016]: 3.9, 3.11, 3.13, 3.15, 3.16.
- Homework 5: Chapter 4 - Chapter 5 in [Salsa, 2016] : 4.1, 4.3, 4.4, 4.7, 5.1, 5.2.
- Homework 6: Chapter 5 - Chapter 5 in [Salsa, 2016]: 5.3, 5.5, 5.7.
- Homework 7: Chapter 6 in [Salsa, 2016]: 6.1, 6.2, 6.6.
- Homework 8: Chapter 7 in [Salsa, 2016] : 7.2, 7.5 7.6, 7.7, 7.19.
- Homework 9 : Chapter 7 in [Salsa, 2016]: 7.23, 7.28-30, 7.32.
- Homework 10 : Chapter 8 in [Salsa, 2016]: 8.2, 8.4, 8.5, 8.6.
- Homework 11 : Chapter 8 in [Salsa, 2016]: 8.8,8.9,8.10, 8.11, 8.14, 8.17.

4 Main topics to consider

4.1 Diffusion

The main objective of this chapter is to study the existence and uniqueness of solutions for the Cauchy problem associated to the heat equation

$$\begin{cases} u_t - cu_{xx} = f(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$$

and some of its variants in higher dimensions or involving different boundary conditions. Some topics of special relevance in this direction are

- **Method of separation of variables**: This process is described step by step in section 2.1.4 of [Salsa, 2016] and it is a fundamental technique to construct solutions for some linear problems (not only restricted to the heat equation). A review of basic elements of Fourier series is very useful in this aspect.
- **Fundamental solution - Heat kernel:** This is covered in section 2.3 and it is an essential topic for this chapter. It allows to construct solutions for the global Cauchy problem. Section 2.8.3 illustrates the applications of the fundamental solution in the existence problems. Theorems 2.12 and 2.15 (Duhamel's method) are of special importance
- **Maximum principle:** This is one of the main tools to prove uniqueness of solutions in this context. The method is described in section 2.2.2 for bounded domains and in section 2.8.4 for the global case. Theorems 2.4 and 2.16 are of special relevance.
- **Energy methods for uniqueness:** The technique is described in section 2.2.1 of [Salsa, 2016].

4.2 Laplace Equation

- **Method of separation of variables** It follows similar ideas to those described in section 2.14 of [Salsa, 2016].
- **Mean value properties** Presented in section 3.3.2 of [Salsa, 2016]. This propery illustrates connections between complex analysis and harmonic functions.
- **Maximum principles - Comparison Principles** Presented in section 3.3.3. This is the main tool to prove uniqueness results for this kind of equations. With the construction of suitable barrier functions, it allows to prove uniqueness results in unbounded domains. See also Liouville theorem.
- **Energy methods for uniqueness:** The technique in this context is described in Theorem 3.1 , page 117 of [Salsa, 2016].
- **Poisson kernel for the ball** Presented in section 3.3.5
- **Fundamental solution** This is presented in section 3.6.1 in [Salsa, 2016].
- **Green's representation formula** This allows to construct solutions of the problem

  $$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases}$$

  it is pressented in section 3.7.3 of[Salsa, 2016]

4.3 Wave Equation

The main objective of this chapter is to study the existence and uniqueness of solutions for the Cauchy problem associated to the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

- **Method of separation of variables:** Described in section 5.3.2 of [Salsa, 2016].
- **d'Alembert formula** Presented in section 5.4.1 of [Salsa, 2016].
- **Domain of dependence - Range of influence** Pages 277 - 279 in [Salsa, 2016].
- **Duhamel's method** Described in section 5.4.4 of [Salsa, 2016].

4.4 Scalar conservation laws

The main objective of this chapter is to study the existence and uniqueness of solutions for the Cauchy problem associated to the scalar conservation law

$$\begin{cases} u_t + f(u)_x = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$$

For this kind of problems it is necessary to introduce the notion of weak solution. Some topics of special relevance are:

- **Introduction to shock waves and rarefaction waves** This material is covered in section 4.3 of [Salsa, 2016].
- **Notion of weak solution:** This is described in section 4.4.2 of [Salsa, 2016]. A motivation for the necessity of weak solutions is presented in section 4.3.2 through the method of characteristics.
- **Rankine - Hugoniot condition** Covered in section 4.4.3. [Salsa, 2016]
- **Entropy condition** Covered in section 4.5 of [Salsa, 2016]. This is the main tool to prove uniqueness of weak solutions for scalar conservation laws.
- **The Riemann Problem:** Covered in section 4.6.1. of [Salsa, 2016]. This is the most important topic to review for this part of the exam. It deals with the construction of a weak solution for the problem

$$\begin{cases} u_t + f(u)_x = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = h(x) \end{cases}$$

  where

$$h(x) = \begin{cases} u_- & \text{if } x < 0 \\ u_+ & \text{if } x \geqslant 0 \end{cases}$$

4.5 Sobolev Spaces

Some topics of special relevance:

1. **Introduction to distributions** Sections 7.3 and 7.4 of [Salsa, 2016].
2. **Sobolev spaces** Sections 7.7 to 7.10 of [Salsa, 2016]
3. **The space $H^1(\Omega)$ and $H^1_0(\Omega)$.** Sections 7.7.2 and 7.7.3 of [Salsa, 2016].
4. **The space $H^{-1}(\Omega)$** Presented in section 7.7.4.
5. **Extension Theorem** Presented in section 7.8.2
6. **Trace Theorem** Presented in section 7.9.1 of [Salsa, 2016]. This topic is very useful in abstract variational problems.
7. **Rellich's Theorem** Section 7.7.10.1
8. **Poincaré's inequalities** Section 7.10.2. Essential topic for the last part of the course.

Questions and topics to keep in mind when preparing this material:

- What is the definition of weak derivatives?

  Examples of weak derivatives that are not derivatives in the classic sense.

  Examples of functions that do not admit a weak derivative.

- How do you compute the weak derivative of a specific function?

  What function admits a weak derivative? If u has a weak derivative, under what conditions on f does $f(u)$ have a weak derivative?

  What does it mean for the weak derivative to equal to 0?

- Definition of Sobolev spaces.

  Sobolev spaces as closure of the space of smooth functions.

  What does it mean for a domain to be C^k? For $U \subset V$, is $W^{k,p}(U) \subset W^{k,p}(V)$?

  How do you extend a function in $W^{k,p}(U)$ to a function in $W^{k,p}(V)$?

- Sobolev Embedding Theorems. What are the conditions on the boundary?

  Equivalence of Lipschitz functions and $W^{1,\infty}$ functions on compact domains.

- Poincare inequalities. How do you prove them?

  There are many variations, most can be proved by the same method of contradiction.

- Definition of H^{-1}.

4.6 Variational formulation of elliptic problems

1. **Variational formulation of some elliptic PDEs** This method is described in sections 8.3.1 - 8.3.2 of [Salsa, 2016].

Questions and topics to keep in mind when preparing this material:

- What is the Riesz Representation Theorem.
- What is the Lax-Milgram Theorem.
- What is the definition of a weak solution? What space does it live in?

  How to define a weak solution under Neumann boundary condition? under Robin boundary condition? under Dirichlet boundary condition?

- How to prove that a bilinear form is coercive? Application of Poincaré's inequalities and the Trace Theorem in this context.

References

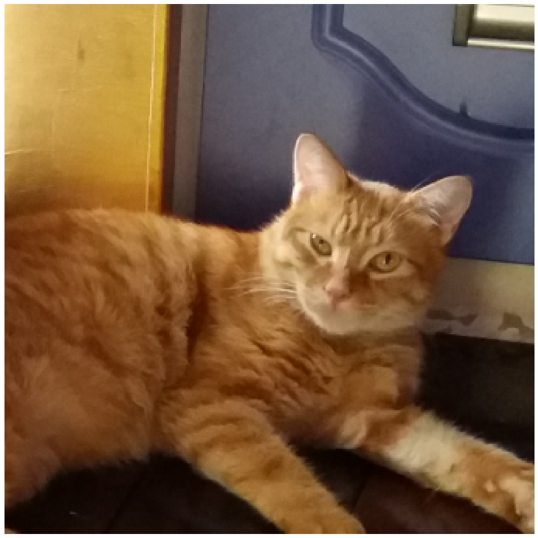
[Cooper, 2012] Cooper, J. (2012). *Introduction to Partial Differential Equations with MATLAB.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston. 1

[Evans, 2010] Evans, L. (2010). *Partial Differential Equations.* Graduate studies in mathematics. American Mathematical Society. 1

[Salsa, 2016] Salsa, S. (2016). *Partial Differential Equations in Action, From Modelling to Theory.* Springer, Switzerland. 1, 2, 3, 4, 5

[Salsa and Verzini, 2015] Salsa, S. and Verzini, G. (2015). *Partial Differential Equations in Action: Complements and Exercises.* UNITEXT. Springer International Publishing. 1

5 All the best and Good Luck!



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Wu Jinyang's Family Member: DuDu