Topology Qual Prep Problems - 2023

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Contents

Remark A Throughout, if a problem asks for an example one must also prove that that the example has the necessary properties unless one is explicitly instructed not to.

Remark B Some problems will reference ordinals and cardinals. If you are unfamiliar with these objects, feel free to assume that each is finite (*i.e.* just a natural number) or the set of all natural numbers, though this simplification should be stated explicitly in the solution.

Remark C The phrase "prove carefully" means to prove without *any* hand-waving. Your presentation of such a problem need not show all of the details, but you should be able to fill any gaps in the presented proof if questioned. The phrase "convince yourself" means to understand why the claim is true and be able to give at least an informal explanation during presentation.

Remark D When solving a problem, you may freely use the results of any other problem which has been assigned in a prior or current week or the results of any problem that the question statement says may be used without proof.

1 Algebraic Topology

1.1 Homotopy

Problem 1.1.1. Show that the general linear group $GL_n(\mathbb{R})$ deformation retracts onto the orthogonal group $O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : AA^T = I\}$.

Problem 1.1.2. Let $\mathbb{R}^\infty = \bigoplus_{n\geq 0} \mathbb{R}$. Equip $\mathbb{R}^\infty$ with the inner product $\langle x, y\rangle = \sum_n x_n y_n$ and the metric topology it induces. Show that the unit sphere S^∞ in this space is contractible.

Problem 1.1.3. State and prove the cellular approximation theorem. For simplicity, you may prove it for finite complexes only. [Stating the theorem seems fair for the qual but I would not expect one to be asked to prove it.]

Problem 1.1.4 (Tamu, Winter 2023)**.** Show that if $n > 1$ then every map $f : S^n \to S^1$ is nullhomotpic.

Problem 1.1.5 (Tamu, Fall 2022)**.** Show that any continuous maps $f : \mathbb{R}P^2 \to S^1 \times S^1$ is nullhomotpoic.

Problem 1.1.6. Prove carefully that $\mathbb{R}^2$ with n punctures is homotopy equivalent to a wedge of n-circles.

Problem 1.1.7. Let $q : X \to Y$ be a quotient map. Prove that $q \times I : X \times I \to Y \times I$ is also a quotient map. [**Addendum**: This is true if I is replaced by any locally compact, Hausdorff space. Feel free to prove the claim in that generality. **Hint**: This may be proven directly, but there is a slick proof utilizing 1.1.9 which may be used without proof.]

Problem 1.1.8. A *Serre fibration* is a continuous map $f : X \to Y$ so that for any commutative diagram

$$\begin{tikzcd} D^n \arrow[r, "u"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ D^n \times I \arrow[r, "v"'] & Y \end{tikzcd}$$

of topological spaces (here $i_0(x) := (x, 0)$) there exists a continuous map h so that the diagram

$$\begin{tikzcd} D^n \arrow[r, "u"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ D^n \times I \arrow[r, "v"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

commutes.

(a) Let $f : X \to Y$ be continuous. Assume that Y has an open cover $\mathcal{U}$ so that if $U \in \mathcal{U}$ then $f : f^{-1}(U) \to U$ is a Serre fibration. Prove f is a Serre fibration.

(b) Prove that any fibre bundle is a Serre fibration.

Problem 1.1.9 ($*$)**.** Let X and Y be topological spaces and write Y^X for the set of continuous maps from X to Y. For compact $K \subseteq X$ and open $U \subseteq Y$, let

$$W(K, U) = \{f \in Y^X : f(K) \subseteq U\}.$$

The topology on Y^X generated by the subbasis $\{W(K, U)\}_{K,U}$ is called the *compact-open* topology. Endow Y^X with this topology. Define further the *evaluation* map $\mathrm{ev}_{X,Y} : Y^X \times X \to Y$ given by $\mathrm{ev}(f, x) = f(x)$.

(a) Show that if X is locally compact[1] then ev is continuous.

(b) Let $f : A \times X \to Y$ a mapping of sets. Its *adjoint mapping* is $f^\wedge : Y \to Y^X$ specified by $f^\wedge(a)(x) = f(a, x)$ for all $a \in A$ and $x \in X$. Show that if f is continuous so is its adjoint. If $\mathrm{ev}_{X,Y}$ is continuous, show that $\alpha : Y^{A\times X} \to (Y^X)^A$ is a bijection. Denote its inverse mapping by $f \mapsto f^\vee$.[2]

[1] each neighborhood of each point contains a compact neighborhood of the point

[2] Moral exercises: (1) If A and X are locally compact, then α is a homeomorphism. (2) Verify that there are spaces X for which $(-) \times X$ does *not* have a right adjoint.

(c) Assume X is locally compact. If $g : A \to B$ is continuous, show that $g_* : A^X \to B^X$ is continuous where $g_*(f) := g \circ f$. Verify that the commutativity of the following pair of diagrams is equivalent (both when everything is a mapping of sets and when everything is a mapping of spaces)

$$\begin{tikzcd} W \times X \arrow[r, "k"] \arrow[d, "f \times 1_X"'] & A \arrow[d, "g"] \\ Y \times X \arrow[r, "h"'] & B \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "k^\wedge"] \arrow[d, "f"'] & A^X \arrow[d, "g_*"] \\ Y \arrow[r, "h^\wedge"'] & B^X \end{tikzcd}$$

(d) Make precise the informal statement "a homotopy from map f_0 to map f_1 is a path from f_0 to f_1" assuming the spacec involved are nice enough.

Problem 1.1.10 $(*)$**.** A *cofibration* is a map $i : A \to X$ of topological spaces which has the homotopy extension property (HEP) against all spaces: for any commutative diagram of solid arrows there is a dotted arrow for which the diagram commutes

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

(where here $\mathrm{ev}_0(f) := f(0)$). Prove that the class of cofibrations is *weakly saturated* where weakly saturated means

(i) Contains all isomorphisms.

(ii) Is closed under composition.

(iii) Is closed under transfinite composition: if λ is an ordinal and $X : \lambda \to \mathbf{TOP}$ is a cocontinuous functor valued in cofibrations then $X_0 \to \mathrm{colim}_\alpha X_\alpha$ is a cofibration. [**Addendum**: If ordinals and colimits are too unfamiliar, this may be skipped.]

(iv) Is closed under cobase change: if $i : A \to X$ is a fibration, $f : A \to A'$ is continuous, and

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ X \arrow[r] & X \sqcup_A A' \end{tikzcd}$$

is the canonical diagram, then i' is a cofibration.

(v) Is closed under retracts: for any commutative diagram

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r, "r"] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ X' \arrow[r, "g"'] \arrow[rr, bend right, "\mathrm{id}"'] & X \arrow[r, "s"'] & X' \end{tikzcd}$$

if i is a cofibration then so is i'.

(vi) Is closed under disjoint union.

Problem 1.1.11 $(*)$**.** Prove that a map $i : A \to X$ is a cofibration (see 1.1.10) if and only if it has HEP for its own mapping cone $Z(i)$.

Problem 1.1.12. Let X and Y be topological space, $A \subseteq X$, and $f, g : X \to Y$ define "f is homotopic to g relative to A."

Problem 1.1.13. The *standard topological n-simplex* $\Delta^n_{\mathbf{TOP}} \subseteq \mathbb{R}^{n+1}$ is defined to be

$$\Delta^n_{\mathbf{TOP}} = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ for all } i \geq 0 \text{ and } \sum_i x_i = 1\}.$$

Further, any non-decreasing map $\rho : \{0, 1, ..., n\} \to \{0, 1, ..., m\}$ defines a canonical continuous map $\rho_* : \Delta^n_{\mathbf{TOP}} \to \Delta^m_{\mathbf{TOP}}$ by $\rho_*(x_0, ..., x_n) = (y_0, ..., y_m)$ where

$$y_i = \sum_{\rho(j)=i} x_j.$$

For notational convenience, if $i \leq j$, then $\langle i, j\rangle : \{0, 1\} \to \{0, 1, 2\}$ is that map carrying 0 to i and 1 to j. Further, the map $\lambda : [0, 1] \to \Delta^1$ given by $\lambda(t) = (1 - t, t)$ is a homeomorphism.

Prove that if X is a topological space and γ, γ' are continuous paths in X with the same endpoints, then γ and γ' are homotopic if and only if there is a continuous map $\sigma : \Delta^2_{\mathbf{TOP}} \to X$ so that

$$\gamma = (\langle 0, 2\rangle^*\sigma) \circ \lambda \text{ and } \gamma\gamma' = (\langle 1, 2\rangle^*\sigma) \circ \lambda$$

and $\langle 0, 1\rangle^*\sigma$ is constant.

Problem 1.1.14 $(*)$**.** Recall the definition of Serre fibration from Problem 1.1.8. Further, say that a continuous map $f : X \to Y$ is a *weak homotopy equivalence* when it is bijective on path components and for all $x_0 \in X$ and $n \geq 0$ the natural map

$$f_* : [(S^n, p), (X, x_0)] \to [(S^n, p), (Y, f(x_0)]$$

of homotopy classes is bijective.[3] Show that if $f : X \to Y$ is both a weak homotopy equivalence and a Serre fibration then f has the right lifting property against the boundary inclusions $i : S^{n-1} \to D^n$ for all $n \geq 0$. That is, show that for every commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "u"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^n \arrow[r, "v"'] & Y \end{tikzcd}$$

of continuous maps, there exists a commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "u"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^n \arrow[r, "v"'] \arrow[ur, dashed, "K"] & Y \end{tikzcd}$$

of continuous maps.

Problem 1.1.15. Find a path connected space X which is not homotopy equivalent to a point but so that any continuous map $S^n \to X$ is null homotopic.

Problem 1.1.16. Show that the fundamental group of a CW complex depends only on the 2-skeleton of the complex.

Problem 1.1.17. Show that the Klein bottle may be obtained by gluing two Mobius bands along their boundary.

[3] Indeed, these are naturally group homomorphisms, but this is not needed here.

Problem 1.1.18 (Hatcher, Chapter 0, #28)**.** Suppose (X_1, A) is a CW-pair with the homotopy extension property and $f : A \to X_0$ is continuous. Show that $(X_1 \sqcup_f X_0, X_0)$ has the homotopy extension property.

Problem 1.1.19 (Boston College Qual)**.** Let $f : S^1 \vee S^1 \to T^2$ and $g : T^2 \to S^1 \vee S^1$ be continuous. Can $fg \simeq \mathrm{id}_{T^2}$? Can $gf \simeq \mathrm{id}_{S^1 \vee S^1}$.

Problem 1.1.20 (Hatcher, Chapter 0, #19)**.** Show that the space obtained by attaching n two cells to S^2 along circles is homotopy equivalent to $\bigvee_{i=1}^{n+1} S^2$.

Problem 1.1.21. Show that if (X, A) and (X, B) are CW-pairs with $A, B, A \cap B$ contractible, then X is contractible.

Problem 1.1.22. Show that any continuous map $f : S^3 \times S^3 \to \mathbb{R}P^3$ which is not surjective is homotopic to a constant map.

Problem 1.1.23. If α and β are paths in a topological space X and there is a continuous map $\varphi : I \to I$ with $\varphi(0) = 0$ and $\varphi(1) = 1$ so that

$$\begin{tikzcd} I \arrow[r, "\alpha"] & X \\ I \arrow[u, "\varphi"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes, then α and β are fixed endpoint homotopic.

Problem 1.1.24. Show that a space X is contractible if and only if (a) every map $f : X \to Y$ or (b) $f : Y \to X$ is nullhomotopic, where Y ranges over all spaces.

Problem 1.1.25. Fix a map $f : X \to Y$. Show that f is a homotopy equivalence if and only if there are maps $g, h : Y \to X$ so that fg and hf are homotopy equivalences.

Problem 1.1.26. Find a two dimensional CW complex containing both the Möbius band and the Klein bottle as deformation retracts.

Problem 1.1.27. Let X be a topological space and $A \subseteq X$. Suppose $f, g : X \times I \to X$ are deformation retracts of X onto A. Show that f and g are homotopic.

Problem 1.1.28. Call a space *well connected* when its path components and connected components coincide. Show that well connectedness is invariant under homotopy equivalence.

Problem 1.1.29. Does the Borsuk-Ulam theorem hold for the torus $S^1 \times S^1$. That is, if $f : S^1 \times S^1 \to \mathbb{R}^2$ if continuous, is there a point $(z, w) \in S^1 \times S^1$ so that $f(z, w) = f(-z, -w)$?

1.2 Fundamental Group

Problem 1.2.1. Compute the fundamental group of a genus g surface.

Problem 1.2.2 (5410 Homework)**.** Show that the fundamental group of a manifold is countable.

Problem 1.2.3. Show that if p, q lie in the same path component of a space X then $\pi_1(X, p) \cong \pi_1(X, q)$. Show that this is not necessarily true if p, q lie merely in the same connected component.

Problem 1.2.4. Find a space whose fundamental group is uncountable.

Problem 1.2.5 (Georgia, Fall 2022)**.** Let X be the topological space obtained from $S^1 \times [0, 1]$ by imposing the relation $(e^{i\theta}, 0) \sim (e^{3i\theta}, 1)$ for all θ. Compute the fundamental group of X.

Problem 1.2.6 (Georgia, Spring 2022)**.** Show that the spaces $S^3 \vee \mathbb{R}P^2$ and $S^2 \vee \mathbb{R}P^3$ have the same fundamental group.[4]

[4] Bonus problem: Show these spaces are not homotopy equivalent. This is a bonus because I am not sure this is doable without homology.

Problem 1.2.7 (Georgia, Spring 2022)**.** Let $T = S^1 \times D^2$ denote the solid torus and let p and q be relatively prime. Define $\phi : \partial T \to \partial T$ by

$$\phi(\psi, 1, \theta) = (p\psi + q\theta, 1, b\psi + a\theta)$$

where a and b are integers so $aq - bp = 1$. Compute the fundamental group of the *lens space* $L(p, q) = T \sqcup_\phi T$ in terms of p and q.

Problem 1.2.8 (UCLA, Spring 2001)**.** A homotopy class $[\alpha] \in \pi_1(X, x_0)$ is called *cyclic* when there is a homotopy $H : X \times I \to X$ with $H(-, 0) = H(-, 1) = \mathrm{id}_X$ and $H(x_0, -) = \alpha$. Let $J(X, x_0) \subseteq \pi_1(X, x_0)$ denote the set of such classes.

(a) Show that $J(X, x_0)$ is contained in the center of $\pi_1(X, x_0)$.

(b) Show that if X is a topological group then $J(X, x_0) = \pi_1(X, x_0)$

Problem 1.2.9 (Extension of Iowa, Fall 2022)**.** Prove that for any group homomorphism $f : G \to H$ there exists a continuous map $h : X \to Y$ of topological spaces and commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\cong"'] & H \arrow[d, "\cong"] \\ \pi_1(X) \arrow[r, "h_*"'] & \pi_1(Y) \end{tikzcd}$$

of group homomorphisms. [**Addendum**: To simplify matters, you may assume that the groups are finitely generated.]

Problem 1.2.10. Compute the fundamental group of S^2 with n punctures.

Problem 1.2.11. Define the complex projective space $\mathbb{C}P^n$. Find a cell structure for $\mathbb{C}P^n$. Prove that $\mathbb{C}P^n$ is simply connected for all n.

Problem 1.2.12 ($*$)**.** *For this problem, you may only use Van Kampen on finite open covers.* Fix a topological space X, a directed set D, and a family $\{U_\alpha : \alpha \in D\}$ of open subsets of X so that if $\alpha < \beta$ in D there holds $U_\alpha \subseteq U_\beta$.

(a) Suppose $\{G_\alpha : \alpha \in D\}$ is a family of groups and whenever $\alpha < \beta$ in D there is a group homomorphism $\phi_{\beta\alpha} : G_\alpha \to G_\beta$ satisfying $\phi_{\gamma\beta}\phi_{\beta\alpha} = \phi_{\gamma\alpha}$. Define[5] the directed limit/colimit of the G_α along the $\phi_{\beta\alpha}$.

(b) Assume that $X = \bigcup_{\alpha \in D} U_\alpha$ and prove that

$$\pi_1(X) = \mathrm{colim}_{\alpha \in D}\, \pi_1(U_\alpha).$$

(c) Prove that for any cardinal κ the wedge of κ many copies of S^1 has fundamental group free on κ many generators.

[**Addendum**: for ease of use, one may solve the simplified problem where $D = \mathbb{N}$ and $\kappa = |\mathbb{N}|$.]

Problem 1.2.13. For each $n \in \mathbb{N}$, let $X_n \subseteq \mathbb{R}^2$ denote the circle of radius $1/n$ centered on $1/n$. Define $X = \bigcup_n X_n$ and equip it with the subspace topology from $\mathbb{R}^2$. Prove that $\pi_1(X)$ is not countable.

Problem 1.2.14 (Generalized Van Kampen, ($*$))**.** A groupoid is a set G paired with a subset $C \subseteq G \times G$ of *composable elements* together with functions *composition* $* : C \to G$ (one writes $*(a, b) = ab$) and *inversion* $\iota : G \to G$ (one writes $\iota(a) = a^{-1}$) satisfying for $a, b, c \in G$

Associativity if ab and bc are defined then so are $(ab)c$ and $a(bc)$ which are equal;

Inverses aa^{-1} and $a^{-1}a$ are always defined;

[5] either directly (a typical element looks like this...) or abstractly (a group equipped with the data...)

Identity if ab is defined then $a^{-1}ab = b$ and $abb^{-1} = b$.

[Alternatively and equivalently, a groupoid is a category in which every morphism is invertible.]

(a) Convince yourself that any topological space X with subset A has a *fundamental groupoid* $\Pi(X, A)$ whose elements are fixed-endpoint homotopy classes of paths whose endpoints lie in A and whose composition is concatenation. [Notation: one writes $\Pi(X)$ for $\Pi(X, X)$]

(b) Write down a reasonable definition of groupoid homomorphism.[6] Prove that if $f : X \to Y$ is a continuous map and $f(A) \subseteq B$, then there is an induced homomorphism of groupoids $f_* : \Pi(X, A) \to \Pi(Y, B)$.

(c) Prove the following:

Van Kampen's Theorem If X is a topological space with open cover $U \cup V = X$ then for any groupoid G and groupoid homomorphisms making the diagram

$$\begin{tikzcd}
\Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(V) \arrow[d] \arrow[ddr, bend left, "f_V"] & \\
\Pi(U) \arrow[r] \arrow[drr, bend right, "f_U"'] & \Pi(X) & \\
& & G
\end{tikzcd}$$

commute, there exists a unique groupoid homomorphism $f : \Pi(X) \to G$ making

$$\begin{tikzcd}
\Pi(U \cap V) \arrow[r] \arrow[d] & \Pi(V) \arrow[d] \arrow[ddr, bend left, "f_V"] & \\
\Pi(U) \arrow[r] \arrow[drr, bend right, "f_U"'] & \Pi(X) \arrow[dr, dashed, "f"] & \\
& & G
\end{tikzcd}$$

commute.[7]

Problem 1.2.15 (Extension of Dieck, Ch. 3)**.** For each $n \geq 1$, we may define the *Euclidean space with two origins* $E^n = \mathbb{R}^n \sqcup \mathbb{R}^n / \sim$ where $\sim$ identifies all duplicate pairs other than the two origins.

(a) Compute $\pi_1(E^n)$ for $n \geq 1$.

(b) Let L_κ denote $\mathbb{R}$ with κ many origins. Compute $\pi_1(L_\kappa)$.

(c) Let $K = (\kappa_n)_{n \in \mathbb{N}}$ denote a sequence of cardinals. Let L_K denote $\mathbb{R}$ but with κ_n many copies of n for each $n \in \mathbb{N}$. Compute $\pi_1(L_K)$.

(d) Let P denote $\mathbb{R}^2$ with a doubled copy of S^1. Compute $\pi_1(P)$.

Problem 1.2.16. Compute the fundamental group of the very long line.

Problem 1.2.17. Find a space which is not homeomorphic to a CW complex. Find a space which is not homotopy equivalent to a Hausdorff space.

Problem 1.2.18 (Tulane, Fall 2012)**.** Prove that a connected space is simply connected if and only if every continuous map of S^1 into the space has a continuous extension to D^2.

[6] When all elements are composable a groupoid is just a group and your definition of groupoid homomorphism should reduce to that of group homomorphism.

[7] Moral exercises: (1) Extend this result to larger open covers. (2) Prove the usual version of Van Kampen's Theorem from the groupoid version [**Hint**: What can you say about a retract of a pushout diagram?].

Problem 1.2.19. Find the fundamental group of the space obtained from $\mathbb{R}^3$ by

(a) removing the unit circle in the xy-plane and the z-axis;

(b) removing the unit circle in the xy-plan and the x- and y-axes.

Problem 1.2.20. Find two spaces which are not homotopy equivalent but have the same fundamental group and universal cover.

Problem 1.2.21 (Iowa, Fall 2015)**.** Find the fundamental group of the torus with k-points removed.

Problem 1.2.22 (Iowa, Fall 2016)**.** Find the fundamental group of the space obtained by identifying the antipodal points of the two-dimensional unit disk.

Problem 1.2.23 (Iowa, Winter 2016)**.** Find the fundamental group of the space obtained from the three disk by identifying a finite collection of points. That is, find $\pi_1(D_3/A)$ where $A \subseteq D_3$ is finite.

Problem 1.2.24 (Iowa, Winter 2017)**.** Find the fundamental group of the space obtained from S^2 by identifying a finite collection of points. That is, find $\pi_1(S^2/A)$ where $A \subseteq S^2$ is finite.

Problem 1.2.25. Prove that S^n is simply connected for $n > 1$. You may not use Sard's theorem unless you provide a proof.

Problem 1.2.26. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

Problem 1.2.27. Find (pointed) spaces X and Y so that $\pi_1(X \vee Y) \neq \pi_1(X) * \pi_1(Y)$. [Proving this is hard and probably beyond the scope of the qual. It is still a good exercise to find such spaces.]

Problem 1.2.28. Fix a space X and $p \in X$. Define an associative monoid $L(X)$ whose elements a loops $\alpha : [0, a] \to X$ with $\alpha(0) = \alpha(a) = p$ and $a > 0$. The monoid operation is given by $\alpha * \beta : [0, a + b] \to X$ by

$$\alpha * \beta(t) = \begin{cases} \beta(t) & t \in [0, b] \\ \alpha(t - b) & t \in [b, b + a] \end{cases}$$

Let $\mathscr{L}(X)$ denote the group completion of $L(X)$. Find a surjective group homomorphism $\mathscr{L}(X) \to \pi_1(X, p)$ and describe its kernel.

Problem 1.2.29. Find a retraction which is not a deformation retraction but which induces isomorphism on the fundamental group.

Problem 1.2.30. Compute $\pi_1(T^*S^1)$.

Problem 1.2.31. Show that the fundamental group of the infinite genus orientable surface (see Hatcher, pg 54) has fundamental group free on infinitely many generators.

Problem 1.2.32. Consider the subset X of $\mathbb{R}^3$ consisting of the union of spheres of radius $1/n$ centered at $(1/n, 0, 0)$ for $n = 1, 2, ...$ Show X is simply connected.

Problem 1.2.33. Let $X \subseteq \mathbb{R}^3$ be the space obtained by taking the union of the circle of radius n centered at n for $n = 1, 2, ...$ Show that X is homotopy equivalent but not homeomorphic to the countable wedge of circles.

Problem 1.2.34. Show that if X and Y are simply connected, non-empty, and X path connected, then the join $X * Y$ is simply connected.

1.3 Covering Spaces

Problem 1.3.1 (Tamu, Fall 2022)**.** Show that if X is a path connected space whose universal cover is compact then $\pi_1(X)$ is finite.

Problem 1.3.2 (Tamu, Fall 2022)**.** Let X and Y be path connected and locally path connected spaces with universal covers $\widetilde{X}$ and $\widetilde{Y}$ respectively. Show that if X is homotopy equivalent to Y then $\widetilde{X}$ is homotopy equivalent to $\widetilde{Y}$.

Problem 1.3.3 (Factorization of Covering Maps)**.** Let X, Y, Z be path connected, locally path connected spaces and

$$\begin{tikzcd} Y \arrow[rr, "t"] \arrow[dr, "r"'] & & Z \arrow[dl, "s"] \\ & X & \end{tikzcd}$$

be a commutative diagram of continuous maps.[8] Show that if either r, s are covering maps or t, r are covering maps then the third map is a covering map.

Problem 1.3.4 (UCLA, Fall 2022)**.** Let B be path connected, locally path connected, and semilocally simply connected. Recall that a path connected covering $\pi : E \to B$ is called abelian when $\pi_1(E)$ is normal in $\pi_1(B)$ and the quotient is abelian. Show that there is a universal abelian cover: an abelian cover $\pi_0 : E_0 \to B$ so that any other abelian cover $\pi : E \to B$ factors as a covering map $E \to E_0$.

Problem 1.3.5. For each n, let F_n denote the free group on n generators. Prove that for each $n \in \mathbb{N}$ there is an injection $F_n \hookrightarrow F_2$.

Problem 1.3.6. Let $p : E \to B$ be a covering map of pointed topological spaces. State what it means for a map $f : X \to B$ to lift against p. Prove carefully that an path $\gamma : I \to B$ has a unique lift against p.

Problem 1.3.7 (Nielsen-Schreier Theorem)**.** Let Γ be a graph with vertices V and edges E.

(a) Prove that Γ has a maximal tree.[9] If Γ is connected, show that a maximal tree is exactly a tree containing each vertex.

(b) Recall how to view a graph as a 1-dimensional CW complex. Show that if Γ is a connected graph with maximal tree T then the space Γ/T obtained by collapsing T to a point is a wedge of circles which is homotopy equivalent to Γ.

(c) Prove that a covering space of a wedge of circles must be a graph.

(d) Prove

Nielsen-Schreier Theorem Any subgroup of a free group is free.

Problem 1.3.8 (5400 Final, 2021)**.** Find all connected covering spaces of the Möbius band.

Problem 1.3.9. Define covering space. Prove that if the base space is connected then each fibre has the same cardinality.

Problem 1.3.10 (Iowa, Fall 2015)**.** Find a non-trivial covering space of $\mathbb{R}^3$ without the z-axis.

Problem 1.3.11. Suppose $\pi : E \to B$ is a covering projection with finite fibres. Show that E is compact and Hausdorff if and only if B is compact and Hausdorff.

Problem 1.3.12. Find a universal cover for a sphere adjoined with its diameter. Describe it precisely.

Problem 1.3.13. Find all 3-sheeted normal covering maps of $S^1 \vee S^1$ and compute their corresponding group of deck transformations.

Problem 1.3.14. Show that if X is path connected, locally path connected then every map $X \to S^1$ is null-homotopic.

[8] Each surjective if you believe covering maps must be surjective.

[9] A tree is a connected subgraph without cycles.

1.4 CW Complexes

Problem 1.4.1 (Tulane, Winter 2019)**.** Define CW-complex[10] Give an example of a CW complex different from a sphere in each dimension.[11]. Produce a compact, locally connected space which is not homeomorphic to a CW complex.

Problem 1.4.2. Prove that any CW-complex X is perfectly normal Hausdorff; that is, Hausdorff and for any pair E, F of disjoint closed subsets of X there exists continuous $f : X \to \mathbb{R}$ so that $f^{-1}(\{1\}) = E$ and $f^{-1}(\{0\}) = F$.

Problem 1.4.3. Suppose that X is a CW-complex and $p \in X$. Show that there is a CW-structure on X with p as a zero cell.

Problem 1.4.4. Is the Hawaiian earrings of Problem 1.2.13 a CW complex?

Problem 1.4.5. Show that any CW complex is locally contractible (every point has a contractible neighborhood).

Problem 1.4.6. Show that a CW complex is path connected if and only if its 1-skeleton is path connected.

Problem 1.4.7. Show that a CW complex is locally compact if and only if each point has a neighborhood meeting only finitely many cells.

2 Manifolds

2.1 Foundations

Problem 2.1.1 ($*$)**.** State and prove the chain rule for Euclidean space.

Problem 2.1.2 ($*$)**.** State and prove Taylor's theorem.

Problem 2.1.3. Prove the existence of smooth functions which are not analytic.

Problem 2.1.4. Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$.

2.2 Point-Set

Problem 2.2.1. Show that every topological manifold is Lindelöf (every open cover admits a countable subcover)

Problem 2.2.2. Let M be a manifold. Show that every path in M is fixed-end-point homotopic to a smooth path. Using this fact, show that if $n > 1$ then S^n is simply connected. You may use

> **Sard's Theorem** The set of critical values of any smooth map has measure 0.

without proof.

Problem 2.2.3 (6410 Homework)**.** Find a smooth embedding $S^m \times S^n \hookrightarrow \mathbb{R}^{m+n+1}$.

Problem 2.2.4. Let M be a topological manifold. Show that there exists a family $\{A_n : n \in \mathbb{Z}\}$ of compact subsets of M and a family $\{V_n : n \in \mathbb{Z}\}$ of open sets in M so that each $A_n \subseteq V_n$, the A_n cover M, and each V_n intersects only finitely many V_k. [**Hint**: Find families U_n and K_n so that

$$U_n \subseteq K_n \subseteq U_{n+1}$$

of open U_n and compact K_n which cover M]. You may use Problem 2.2.1 without proof.

[10] Be specific about the topology placed on an infinite dimensional complex.
[11] Including infinite dimensions

Problem 2.2.5. An *exhaustion* of a topological space M is a continuous map $f : X \to \mathbb{R}$ so that for all $c \in \mathbb{R}$ the sets $f^{-1}((-\infty, c])$ is compact. Prove that every topological manifold admits a non-negative exhaustion and that if the manifold is smooth, the exhaustion may be taken to be smooth.

Problem 2.2.6. Prove that the product of a smooth manifold and a smooth manifold with boundary is a smooth manifold with boundary.

Problem 2.2.7. Let $K = [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2$. Show that there is a homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(S^1) = K$ but no diffeomorphism with this property.

2.3 Basics of Manifolds

Problem 2.3.1. Let $p : M \to N$ be a smooth map of smooth manifolds.

(a) Show that p is a surjective submersion if and only if p has local, smooth sections at each point in its image.

(b) Let X be a smooth manifold. Show that if p is a surjective submersion and the following diagram of continuous maps commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p"'] & X \\ N \arrow[ur, "h"'] & \end{tikzcd}$$

then f is smooth if and only if h is smooth.

Problem 2.3.2. Prove that an oriented, compact smooth manifold M with boundary cannot smoothly retract onto its boundary. Show that compactness is a necessary assumption.

Problem 2.3.3 (Stack of Records Theorem)**.** Let $f : X \to Y$ be a smooth map between smooth manifolds of equal dimensions. Let X be compact and y be a regular value of f. Prove that there is an open set $y \in U \subseteq Y$ so that $f^{-1}(U) = V_1 \cup \cdots \cup V_n$ and each $f : V_i \to U$ is a diffeomorphism.

Problem 2.3.4. Define Lie group. Prove that $GL_n(\mathbb{R})$ is a Lie group. How many connected components does it have? Prove your claim is correct.

Problem 2.3.5. In *Differential Topology* by CTC Wall, a smooth m-manifold M is defined to be a Hausdorff topological space which is a countable union of compact sets equipped with a family $\mathcal{F}$ of continuous real valued functions on M satisfying

(M I) *Locality:* if $f : M \to \mathbb{R}$ is such that for every point $p \in M$ there is a neighborhood $p \in V$ and $g \in \mathcal{F}$ so $f|_V = g|_V$, then $f \in \mathcal{F}$.

(M II) *Differential closure:* if $f_1, ..., f_k \in \mathcal{F}$ and F is a smooth real valued function on open $U \supseteq (f_1, ..., f_k)(M)$, then $F(f_1, ..., f_k) \in \mathcal{F}$.

(M III) *Locally Euclidean:* for each $p \in M$ there are $f_1, ..., f_m \in \mathcal{F}$ so that $(f_1, ..., f_m) : M \to \mathbb{R}^m$ gives a homeomorphism of an open neighborhood U of p to an open subset V of $\mathbb{R}^m$. Furthermore, every $f \in \mathcal{F}$ coincides on U with $F(f_1, ..., f_m)$ where $F : V \to \mathbb{R}$ is smooth.

Let M be a Hausdorff topological space which is a countable union of compact sets. Prove that the set of maximal atlases on M is in bijection with the set of families $\mathcal{F}$ satisfying (M I), (M II), and (M III).

Problem 2.3.6. Find a topological space with two different smooth structures which are still diffeomorphic.

Problem 2.3.7. Define $\mathbb{R}P^n$ and prove that it is a topological manifold. Give charts witnessing a smooth structure and prove that one of the transitions is smooth.

Problem 2.3.8. Prove that if A is a finitely generated abelian group then there is a connected manifold M so that $\pi_1(M) \cong A$. [**Hint**: Use the structure theorem for finitely generated abelian groups.]

Problem 2.3.9. For each of the following assertions, state whether it is true or false and provide a justification (can be a proof sketch).

(a) Every n-dimensional submanifold (without boundary) of $\mathbb{R}^n$ is an open set.

(b) Every submanifold of an orientable manifold is orientable.

(c) If G is a group, there is a manifold whose fundamental group is G.

(d) Every smooth submersion between manifolds of equal dimension is a covering projection.

Problem 2.3.10 $(*)$**.** Let M be a smooth manifold.

(a) Let $\Gamma \subseteq M \times M$ be the graph of an equivalence relation R on M. Prove that the following are equivalent

  (i) The quotient M/R is has a smooth manifold structure so that the quotient $q : M \to M/R$ is a submersion.

  (ii) The graph Γ is a smooth submanifold of $M \times M$ and $\pi_1 : \Gamma \to M$ by $\pi_1(x, y) = x$ is a submersion.

(b) If M is equipped with a smooth, free, proper action of a Lie group G then there is a unique-up-to-diffeomorphism smooth structure on M/G so that the quotient is a submersion.

Problem 2.3.11. Prove that there is a Lie group of each dimension.

Problem 2.3.12. Prove that any Lie group homomorphism has constant rank. Prove that the kernel of a Lie group homomorphism is a Lie subgroup

Problem 2.3.13. Let G be a Lie group. Let $W \subseteq G$ be a neighborhood of identity. Prove

(a) W generates an open subgroup of G.

(b) If W is connected, so is the subgroup it generates.

(c) If G is connected, then W generates G.

Problem 2.3.14. Let G be a Lie group which acts smoothly on N and smoothly and transitively on M. Prove that if $F : M \to N$ is smooth and equivariant then it has constant rank.

2.4 Tangent Spaces and Vector Bundles

Problem 2.4.1. Let M be a smooth manifold and $p \in M$.

(a) Show that the ring $C_p^\infty(M)$ is local (has a unique maximal ideal $\mathfrak{m}_p$).

(b) Let the field $k_p = C_p^\infty(M)/\mathfrak{m}_p$ and note that $\mathfrak{m}_p/\mathfrak{m}_p^2$ inherits the structure of a k_p vector space. Show that $k_p \cong \mathbb{R}$ as a field and that $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee \cong T_pM$ as $\mathbb{R}$ vector spaces.

Problem 2.4.2. Let M be a smooth manifold and $p \in M$. Show that the vector spaces of point derivations on $C_p^\infty(M)$ and $C^\infty(M)$ are isomorphic.

Problem 2.4.3. Let M be a smooth manifold and $\pi : E \to M$ a vector bundle. A *connection* on E is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E), \quad \nabla(\xi \otimes s) =: \nabla_\xi s$$

satisfying for all smooth vector fields $\xi, \zeta \in \Gamma(TM)$ and smooth $f : M \to \mathbb{R}$ and sections $s \in \Gamma(E)$

$$\nabla_{\xi + f\zeta} s = \nabla_\xi s + f\nabla\zeta \text{ and } \nabla_\xi(fs) = \xi(f)s + f\nabla_\xi s.$$

Show

(a) for any point $p \in M$ the quantity of $(\nabla_\xi s)_p$ depends only on the values of ξ and s in a neighborhood of p;

(b) any vector bundle $\pi : E \to M$ admits a connection.

Problem 2.4.4. Carefully define the tangent bundle of a manifold M and its structure as a smooth manifold. You may assume that the definition of tangent space at a point is known.

Problem 2.4.5 (UCLA, Spring 2021)**.** State Brouwer's fixed point theorem. Prove the theorem for D^2 without using homotopy groups. You may assume the hairy ball theorem: any vector field on S^2 vanishes.

Problem 2.4.6. Consider the Lie group $\Gamma = GL_n(\mathbb{R})$. When the tangent space of identity $e = I_n \in \Gamma$ is identified with $M_n(\mathbb{R})$, the differential of the determinant yields a map $\det_* : M_n(\mathbb{R}) \to \mathbb{R}$ which is a familiar operation on matrices. What is it?

Problem 2.4.7 (New Mexico, 2006)**.** Define local and global frames in the context of vector bundles. Prove from the definitions that a vector bundle is trivial if and only if it admits a global frame.

Problem 2.4.8 (New Mexico, 2007)**.** Show that TS^3 is a trivial vector bundle.

Problem 2.4.9 (New Mexico, 2008)**.** Find all integral curves to the vector field $x^2\partial_x + \partial_y$ on $\mathbb{R}^2$.

Problem 2.4.10 (New Mexico, 2009)**.** Prove that if X is a vector field on a manifold M which does not vanish at $p \in M$, then there are local coordinates $x^1, ..., x^n$ about p so that near p $X = \partial_{x^1}$.

Problem 2.4.11. Give an example of a non-trivial vector bundle.

Problem 2.4.12 ($*$)**.** A *smooth endofunctor on the category of finite dimensional vector spaces* is a gadget F so that

> For any $f : V \to W$ a linear map between finite dimensional vector spaces there is $F(f) : F(V) \to F(W)$ a linear map of finite dimensional vector spaces. Further, F distributes over composition and carries identity maps to identity maps. Moreover, F is smooth in the sense that the induced maps $\mathrm{Hom}(V, W) \to \mathrm{Hom}(F(V), F(W))$ is smooth.

(a) Convince yourself that direct sum $\oplus^n$, tensor product $\otimes^n$, and exterior power Λ^n are smooth endofunctors on the category of finite dimensional vector spaces.

(b) Show that any smooth endofunctor F on the category of finite dimensional vector spaces defines a functorial construction on smooth vector bundles fibre by fibre; that is

> Fix a base manifold B. Show that if $f : E \to E'$ is a vector bundle map over B then there is a vector bundle map $\widehat{F}(f) : \widehat{F}(E) \to \widehat{F}(E')$ so that on fibres $\widehat{F}(f)_p : \widehat{F}(E)_p \to \widehat{F}(E')$ is[12] $F(f_p) : F(E_p) \to F(E'_p)$.

[**Moral Exercise**: Extend the above to a functor F with domain $\textbf{FD-VECT}^n_{\mathbb{R}}$.]

Problem 2.4.13. Let $f : M \to N$ be a smooth map between smooth manifolds of equal dimension at least 2. Show that f is an open map if all but finitely many points of M are regular.

Problem 2.4.14. Prove the fundamental theorem of algebra.

Problem 2.4.15 (Lee, 16-6 (slightly modified))**.** Prove that the following statements are equivalent for a natural number n

(i) There exists a non-vanishing vector field on S^n;

(ii) There exists a smooth map $V : S^n \to S^n$ so that for all $p \in S^n$ there holds $V(p) \perp p$ with respect to the usual do product on $\mathbb{R}^{n+1}$;

(iii) The antipodal map $A : S^n \to S^n$ is smoothly homotopic to identity;

[12] at least up to some canonical identifications

(iv) The antipodal map is orientation preserving;

(v) The number n is odd.

Problem 2.4.16. Consider the map $F : \mathbb{R}P^2 \to \mathbb{R}^3$ given by

$$F[x : y : z] = \frac{(yz, xz, xy)}{x^2 + y^2 + z^2}.$$

1. Show that F is well-defined and smooth.
2. Is F an immersion?

Problem 2.4.17. Show that the space of vector fields on a smooth manifold of positive dimension is infinite dimensional (as a real vector space).

Problem 2.4.18. Prove that there is a smooth vector field on S^2 which vanishes at exactly one point.

Problem 2.4.19. Let $M \subseteq \mathbb{R}^n$ be a submanifold. Define $NM \subseteq M \times \mathbb{R}^n$ by

$$NM = \{(p, v) : v \perp T_pM\}$$

and prove that NM is a vector bundle over M (called the normal bundle). Show $T\mathbb{R}^n|_M \cong TM \oplus NM$.

Problem 2.4.20. Let $\pi : E \to M$ be a fibre bundle with fibre F. Prove

(a) π is an open quotient map.

(b) If π is smooth, then it is a submersion.

(c) π is proper iff F is compact.

(d) E is compact iff both F and M are.

2.5 Submanifolds

Problem 2.5.1 (6410 Midterm)**.** Identify the space $M_n(\mathbb{R})$ of real $n \times n$ matrices with $\mathbb{R}^{n^2}$.

(a) Show that $GL_n(\mathbb{R})$ is a submanifold of $M_n(\mathbb{R})$.

(b) Show that the space $S_n(\mathbb{R})$ of symmetric $n \times n$ matrices is a submanifold of $M_n(\mathbb{R})$.

(c) Show that $O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : AA^T = I\}$ is a submanifold of $M_n(\mathbb{R})$.

Problem 2.5.2. Let $F : M \to M'$ be a smooth map of smooth manifolds with regular value c. Let $N = F^{-1}(c)$ be a submanifold of M and let $\iota : N \to M$ be the inclusion. Show that for any $p \in N$ there holds $\iota_* T_pN = \ker F_* \subseteq T_pM$.

Problem 2.5.3. Show that if M and K are manifold of the same dimension with K compact and M not compact, then K may not be submersed in M.

Problem 2.5.4 (UCLA, Fall 2005)**.** Let N be an embedded submanifold of M. Show that if ξ, ζ are vector fields on M tangent to N then the vector field $[\xi, \zeta]$ on M is also tangent to N.

Problem 2.5.5. Define regular submanifold.

Problem 2.5.6. Let $M(n, m, k) \subseteq M(n, m)$ denote the collection of $n \times m$ matrices of rank k. Show that $M(n, m, k)$ is a regular submanifold of $M(n, m)$ of dimension $nm - (n - k)(m - k)$.

Problem 2.5.7. Let $M \subseteq \mathbb{R}^n$ be a submanifold. Let $UM \subseteq TM$ denote those tangent vector of unit length (under the usual identification $T_p\mathbb{R}^n \cong \mathbb{R}^n$). Show that UM a submanifold. What is its dimension?

Problem 2.5.8. Let M be a smooth manifold with boundary, N a smooth manifold, and $F : M \to N$ smooth. Show that if $c \in N$ is a regular value of F and $F|_{\partial M}$ then $S = F^{-1}(c) \subseteq M$ is a regular submanifold and $\partial S = S \cap \partial M$.

2.6 Partitions of Unity

Problem 2.6.1. Prove that any continuous function on a manifold is the uniform limit of smooth functions.

Problem 2.6.2 (∗)**.** Show that every open cover of a manifold admits a subordinate partition of unity. You may use the results of Problem 2.1.4 without proof.

Problem 2.6.3. Let M be a smooth manifold which is the union of open sets U, V. Show that the following is an exact sequence

$$0 \longrightarrow \Omega(M) \xrightarrow{\omega \mapsto (\omega|_U, \omega|_V)} \Omega(U) \oplus \Omega(V) \xrightarrow{(\omega,\eta) \mapsto \omega|_{U\cap V} - \eta|_{U\cap V}} \Omega(U \cap V) \longrightarrow 0$$

Problem 2.6.4. Show that a non-Hausdorff smooth manifold need not admit partitions of unity subordinate to arbitrary open covers.

Problem 2.6.5 (New Mexico, 2006)**.** Show that a smooth immersion is locally a smooth embedding.

Problem 2.6.6 (New Mexico, 2006)**.** Show that if a Lie group G has a smooth action on manifold M then each orbit is an immersed submanifold. Show that this need not be an embedded submanifold.

Problem 2.6.7 (∗)**.** Show that a connected locally Euclidean space is second countable and Hausdorff if and only if it admits partitions of unity subordinate to arbitrary open covers.[13]

Problem 2.6.8. A *Riemannian metric* on a smooth manifold M is, for each $p \in M$ an inner product $g : T_pM \times T_pM \to \mathbb{R}$ with the property that for any local coordinates (U, x) the maps $g_{i,j} : U \to \mathbb{R}$ given by

$$g_{i,j}(p) = g\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right)$$

is smooth. Show that any manifold admits a Riemanian metric.

Problem 2.6.9 (Very Weak Whitney Embedding Theorem)**.** Show that if M is a compact smooth manifold then M can be embedded in $\mathbb{R}^K$ for some K. [**Hint**: Cover the manifold by finitely many charts and proceed by "brute force".]

Problem 2.6.10. Let M be a Riemannian manifold, that is a smooth manifold equipped with a Riemannian metric as in 2.6.8. Let $f : M \to \mathbb{R}$ be a smooth function.

(a) Prove that there exists a unique smooth vector field X_f on M with the property that for any smooth vector field ξ on M there holds $\langle X_f, \xi\rangle = df(\xi)$.

(b) Prove that if c is a regular value of f and $S = f^{-1}(c)$ then$\iota_* T_pS = \{v \in T_pM : \langle X_{f,p}, v\rangle_p = 0\}$ where $\iota : S \to M$ is the inclusion.

(c) Prove that if M is orientable then so is S.

2.7 Orientation

Problem 2.7.1. Show that the product of orientable manifolds is orientable.

Problem 2.7.2. For what values of n is $\mathbb{R}P^n$ orientable.

Problem 2.7.3. Let $F : \mathbb{R}^n \to \mathbb{R}$ be smooth with regular value c. Prove that $F^{-1}(c)$ is orientable.

Problem 2.7.4 (Do Carmo, Ch 1)**.** Show that a regular surface $S \subseteq \mathbb{R}^3$ is orientable if and only if there is a smooth map $N : S \to \mathbb{R}^3$ so that for all $p \in S$ the vector $N(p)$ is perpendicular to T_pS and $|N(p)| = 1$.

Problem 2.7.5. Show that the tangent bundle and cotangent bundle of a smooth manifold are orientable.

[13]Thus, if you believe partitions of unity are important, you have to include Hasudorff and second countable in a definition of manifold.

Problem 2.7.6 (Do Carmo, Ch 1)**.** Let M be a smooth manifold equipped with a properly discontinuous smooth action of a group G.

1. Show that the orbit space M/G is orientable if and only if there is an orientation on M which is preserved by each $g \in G$.
2. Show that the Möbius band is not orientable.

Problem 2.7.7. Is the Klein bottle orientable?

Problem 2.7.8. Show that a simply connected manifold is orientable. Show that if a manifold is covered by two charts with connected intersection then the manifold is orientable.

Problem 2.7.9. Show that any 1-dimensional manifold is orientable.

Problem 2.7.10. Show that if M is an orientable smooth manifold then ∂M is orientable.

Problem 2.7.11. Show that every orientation reversing diffeomorphism of $\mathbb{R}$ has a fixed point.

2.8 Differential Forms

Problem 2.8.1 (6410 Midterm)**.** Consider the form $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on $\mathbb{R}^2 \smallsetminus \{0\}$.

(a) Compute the integral of ω along any circle of radius r centered on the origin.

(b) Is ω exact on $\mathbb{R}^2 \smallsetminus \{0\}$?

Problem 2.8.2. Prove the following generalization of integration by parts: If M is an n-manifold without boundary, $\omega \in \Omega_c^k(M)$, and $\eta \in \Omega_c^{n-k-1}(M)$ then

$$\int_M \omega \wedge d\eta = (-1)^k \int_M (d\omega) \wedge \eta.$$

Problem 2.8.3. Let M be a smooth manifold of dimension n. For any k, define the *k-th de Rham cohomology group* by

$$H^k(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k-1}(M))}{\operatorname{im}(d : \Omega^{k+1}(M) \to \Omega^k(M))}$$

and its compactly supported variant

$$H_c^k(M) = \frac{\ker(d : \Omega_c^k(M) \to \Omega_c^{k-1}(M))}{\operatorname{im}(d : \Omega_c^{k+1}(M) \to \Omega_c^k(M))}.$$

Show that the following is a well defined linear map:

$$I : H_c^k(M) \otimes H^{n-k}(M) \to \mathbb{R}, \quad I([\omega] \otimes [\eta]) = \int_M \omega \wedge \eta.$$

Problem 2.8.4 ($*$)**.** With notation as in Problem 2.7.3, show that $H^1(S^1) \cong \mathbb{R}$. [**Hint**: use the first isomorphism theorem.]

Problem 2.8.5. Find an n-manifold M without boundary and $\omega \in \Omega^{n-1}(M)$ so that $d\omega$ is compactly supported yet $\int_M d\omega = 1$. What is wrong with the following application of Stokes'

$$\int_M d\omega = \int_{\partial M} \omega = 0?$$

Problem 2.8.6. Let M be a connected smooth n-manifold. Define an equivalence relation on $\Lambda^n T^*M$ by $\omega_p \sim \eta_p$ iff $\omega_p = \lambda\eta_p$ for $\lambda \in \mathbb{R}_{>0}$. Denote the equivalence classes of this relation by $[\omega_p]$. Define the *oriented double cover* of M by

$$\widetilde{M} = \{(p, [\omega_p]) : p \in M \text{ and } \omega \in \Omega^n(M)\}.$$

Describe a topology on $\widetilde{M}$ so that

(i) The map $\pi : \widetilde{M} \to M$ by projection in the first coordinate is a two sheeted covering projection.

(ii) The manifold M is oriented if and only if $\widetilde{M}$ is disconnected.

Use this to show that M is orientable if $\pi_1(M)$ has no subgroup of index two.

Problem 2.8.7 (Tamu, Winter 2022)**.** Fix a smooth manifold with vector fields ξ, ζ and 1-form ω. Prove that

$$(d\omega)(\xi, \zeta) = \xi(\omega(\zeta)) - \zeta(\omega(\xi)) - \omega([\xi, \zeta]).$$

Problem 2.8.8 (UCLA, Spring 2022)**.** Let M be a closed n-manifold with volume form ω. For any vector field ξ on M, the *divergence* $\operatorname{div}(\xi)$ of ξ is that smooth function so that

$$\mathscr{L}_\xi \omega = \operatorname{div}(\xi)\omega$$

where $\mathscr{L}_\xi$ is the Lie derivative.

(a) Show that

$$\int_M \operatorname{div}(\xi)\omega = 0.$$

(b) Express $\operatorname{div}(\xi)$ in local coordinates.

Problem 2.8.9 (UCLA, Fall 2021)**.** Show that

$$\ker\left(\int : \Omega^n_c(\mathbb{R}^n) \to \mathbb{R}\right) = d\Omega^{n-1}_c(\mathbb{R}^n)$$

Problem 2.8.10 (UCLA, Fall 2002)**.** Let $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions which vanish outside of a bounded rectangle R. Prove directly (that is, without invoking Stokes') that

$$\int_R d(P\,dx \wedge dy + Q\,dx \wedge dz + R\,dy \wedge dz) = 0.$$

Problem 2.8.11. Show that if α is a non-vanishing smooth 1-form on a manifold M, then the only k-forms ω so that $\alpha \wedge \omega = 0$ are those for which there is a smooth $(k-1)$-form γ so that $\omega = \alpha \wedge \gamma$.

Problem 2.8.12 (Lee, Ex. 14.18)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(u, v) = (u, v, u^2 - v^2)$. Let $\omega \in \Omega^2(\mathbb{R}^3)$ be $\omega = ydx \wedge dz + xdy \wedge dz$. Compute $f^*\omega$

Problem 2.8.13 (Cartan's Lemma)**.** Let M be a smooth manifold and $\omega_1, ..., \omega_k$ a collection of 1-forms which are linearly independent at each point of M. Given 1-forms $\alpha_1, ..., \alpha_k$ with the property that

$$\sum_{i=1}^k \alpha_i \wedge \omega_i$$

show that each α_i may be written as a sooth linear combination of the ω_i.

Problem 2.8.14 (UCLA, Fall 2013, $(*)$)**.** Let M be a connected manifold. Show that if ω is a smooth 1-form so that for any piecewise smooth closed curve c in M there holds $\int_c \omega = 0$ then ω is exact.

Problem 2.8.15. Let M be a connected, oriented n-manifold. Prove that

$$\ker\left(\int : \Omega_c^n(M) \to \mathbb{R}\right) = d\Omega_c^{n-1}(M)$$

You may assume the case of $M = \mathbb{R}^n$. That is, a compactly supported top form is exact if and only if it is the exterior derivative of a compactly supported form.

Problem 2.8.16. Show that if M is a connected, oriented, non-compact smooth n-manifold, then ever closed n-form is exact. You may assume 2.8.15 and 2.2.5.

Problem 2.8.17 ($*$)**.** Let M be a smooth, oriented manifold with Riemannian metric g. That is, for each $p \in M$ an inner product $g : T_pM \times T_pM \to \mathbb{R}$ with the property that for any local coordinates (U, x) the maps $g_{i,j} : U \to \mathbb{R}$ given by

$$g_{i,j}(p) = g\left(\frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p\right)$$

is smooth.

(a) Prove that there exists a unique top form ω_g with the property that for any orthonormal, oriented, local frame $\xi_1, ..., \xi_n$ for TM there holds

$$\omega_g(\xi_1, ..., \xi_n) = 1.$$

(b) Prove that for any choice of local coordinate (U, x) there holds

$$\omega_g|_U = \sqrt{\det(g_{i,j})}dx^1 \wedge \cdots \wedge dx^n$$

Problem 2.8.18. Compute $\omega(X, Y)$ for $\omega = xdx \wedge dy + y^2 dy \wedge dz$ and $X = x\partial_x + y\partial_y$ and $Y = \sin(xy)\partial_z$.

Problem 2.8.19. Show that if $v_1, ..., v_n$ are n-linearly independent vectors in $\mathbb{R}^n$, the the volume of the n-parallelogram

$$P = \left\{\sum_i \lambda_i v_i : \lambda_i \in [0, 1]\right\}$$

they bound is given by

$$\mu(P) = |\det\{v_1, ..., v_n\}|.$$

Problem 2.8.20. Prove Green's theorem from Stokes' theorem.

Problem 2.8.21. Let ω be the 2-form on $\mathbb{C}^n$ given by

$$\omega = \sum_{i=1}^n dx^i \wedge dy^i.$$

Define $J \in GL_{2n}(\mathbb{R})$ to be the block diagonal matrix whose blocks are

$$J_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $A \in GL_{2n}\mathbb{R}$. Show that $A^*\omega = \omega$ if and only if $J = A^t JA$.

Problem 2.8.22 (Tu, 18.1)**.** Prove that a k-form ω on a manifold M is smooth if and only if for every $X_1, ..., X_k \in \mathfrak{X}(M)$ the function $\omega(X_1, ..., X_k)$ is smooth.

Problem 2.8.23 (Tu, 18.3)**.** Prove carefully that if $F : M \to N$ is smooth and α, β are k and ℓ forms on N then

$$F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta).$$

Problem 2.8.24. Let $\pi : M \to N$ be a smooth surjective submersion with connected fibres. A tangent vector $X_p \in T_pM$ is called *vertical* when $\pi_*(X_p) = 0$. Show that if $\omega \in \Omega^k(M)$ then $\omega = \pi^*\eta$ for $\eta \in \Omega^k(N)$ if and only if $i_{X_p}\omega_p = 0$ and $i_{X_p}d\omega_p = 0$ for all $p \in M$ and vertical X_p.

2.9 Linear Algebra

Problem 2.9.1. Let V be a finite dimensional real vector space. Let S_n act on $V^{\otimes n}$ by

$$\sigma \cdot v_1 \otimes \cdots \otimes v_n = (-1)^{|\sigma|} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Show that $V^{\otimes n}/S^n \cong A_n(V)$.

Problem 2.9.2. Let V and W be finite dimensional real vector spaces. Calculate $A_n(V \oplus W)$ in terms of $A_k(V)$ and $A_k(W)$ for various values k.

2.10 Vector Fields and Flows

Problem 2.10.1 ($*$)**.** Let M be a smooth manifold with boundary and let ν be a vector field on M which is inward pointing on ∂M. Prove that there exists a smooth function $\delta : \partial M \to \mathbb{R}_{\geq 0}$ and a smooth embedding $\Phi : P_\delta \to M$ where

$$P_\delta = \{(p,t) \in \partial M \times \mathbb{R} : 0 \leq t < \delta(p)\}$$

so that $\Phi(P_\delta)$ is a neighborhood of ∂M and for each $p \in \partial M$ the curve $\Phi(p,t)$ is an integral curve for ν.

3 Assigned Problems by Week

3.1 Week 1

- Monday: Algebraic Topology
  - Patrick 1.1.4
  - Blake 1.2.4
  - George 1.1.2
  - Marc 1.2.8
  - Nandita 1.3.1
  - Cole 1.1.12
  - Hemanth 1.2.3
  - Ashwin 1.3.9
  - Jacob 1.3.6
  - Merrick 1.4.1
  - Paria 1.1.26
  - Kevin 1.3.2
  - Fatemeh 1.1.5
  - Leslie 1.2.10
- Wednesday: Manifolds
  - Patrick 2.1.3
  - Marc 2.2.3
  - George 2.3.6
  - Blake 2.4.8
  - Nandita 2.2.1
  - Cole 2.2.4

- Hemanth 2.4.5
- Ashwin 2.5.5
- Jacob 2.8.1
- Merrick 2.6.9
- Paria 2.7.6
- Kevin 2.5.1
- Fatemeh 1.2.16
- Leslie 1.3.7

3.2 Week 2

- Monday: Algebraic Topology
  - Patrick 1.2.15
  - Blake 1.2.14
  - George 1.3.3
  - Marc 1.1.9
  - Nandita 1.2.13
  - Cole 1.2.9
  - Hemanth 1.3.2
  - Ashwin 1.2.2
  - Jacob 1.1.1
  - Merrick 1.1.6
  - Paria 1.3.5
  - Kevin 1.2.19
  - Fatemeh 2.3.7
  - Leslie 2.3.1
- Wednesday: Manifolds
  - Patrick 2.3.3
  - Marc 2.3.8
  - George 2.1.1
  - Blake 2.8.3
  - Nandita 2.6.10
  - Cole 2.3.2
  - Hemanth 2.2.5
  - Ashwin 2.6.8
  - Jacob 2.4.1
  - Merrick 2.2.2
  - Paria 2.8.11
  - Kevin 2.7.2
  - Fatemeh 2.7.5
  - Leslie 2.4.15

3.3 Week 3

- Monday: Algebraic Topology
  - Patrick 1.2.7
  - Blake 1.1.5
  - George 1.2.23
  - Marc 1.2.24
  - Nandita 1.2.21
  - Cole 1.3.8
  - Hemanth 1.1.13
  - Ashwin 1.1.8
  - Jacob 1.2.13
  - Merrick 1.2.11
  - Paria 1.1.10
  - Kevin 1.1.7
  - Fatemeh 1.3.10
  - Leslie 1.2.17
- Wednesday: Manifolds
  - Patrick 2.7.1
  - Marc 2.4.10
  - George 2.4.3
  - Blake 2.3.4
  - Nandita 2.4.12
  - Cole 2.5.4
  - Hemanth 2.4.7
  - Ashwin 2.4.13
  - Jacob 2.8.16
  - Merrick 2.6.10
  - Paria 2.6.7
  - Kevin 2.6.2
  - Fatemeh 2.6.6
  - Leslie 2.8.6

3.4 Week 4

- Monday: Algebraic Topology
  - Patrick 1.1.11
  - Blake 1.4.2
  - George 1.4.3
  - Marc 1.1.15
  - Nandita 1.4.4
  - Cole 1.1.16

  - Hemanth 1.2.16
  - Ashwin 1.2.1
  - Jacob 1.2.12
  - Merrick 1.2.27
  - Paria 1.2.25
  - Kevin 1.2.5
  - Fatemeh 1.2.22
  - Leslie 1.2.18
- Wednesday: Manifolds
  - Patrick 2.5.3
  - Marc 2.8.8
  - George 2.6.4
  - Blake 2.6.3
  - Nandita 2.3.5
  - Cole 2.4.6
  - Hemanth 2.8.5
  - Ashwin 2.7.4
  - Jacob 2.3.9
  - Merrick 2.8.11
  - Paria 2.4.4
  - Kevin 2.4.11
  - Fatemeh 2.8.2
  - Leslie 2.6.1

3.5 Week 5

- Monday: Algebraic Topology
  - Patrick 1.2.28
  - Blake 1.1.17
  - George 1.2.28
  - Marc 1.2.29
  - Nandita 1.2.30
  - Cole 1.1.18
  - Hemanth 1.1.19
  - Ashwin 1.1.20
  - Jacob 1.1.21
  - Merrick 1.3.11
  - Paria 1.2.22
  - Kevin 1.3.12
  - Fatemeh 1.1.23
  - Leslie 1.3.13

- Wednesday: Manifolds
  - Patrick 2.8.18
  - Marc 2.8.19
  - George 2.7.7
  - Blake 2.8.20
  - Nandita 2.7.8
  - Cole 2.9.1
  - Hemanth 2.9.2
  - Ashwin 2.4.16
  - Jacob 2.5.6
  - Merrick 2.7.9
  - Paria 2.7.10
  - Kevin 2.8.21
  - Fatemeh 2.8.22
  - Leslie 2.8.23

3.6 Week 6

- Monday: Algebraic Topology
  - Patrick 1.1.24
  - Blake 1.1.25
  - George 1.1.26
  - Marc 1.1.27
  - Nandita 1.1.28
  - Cole 1.1.29
  - Hemanth 1.2.31
  - Ashwin 1.2.32
  - Jacob 1.2.33
  - Merrick 1.2.34
  - Paria 1.3.14
  - Kevin 1.4.5
  - Fatemeh 1.4.6
  - Leslie 1.4.7
- Wednesday: Manifolds
  - Patrick 2.2.6
  - Marc 2.2.7
  - George 2.3.11
  - Blake 2.3.12
  - Nandita 2.3.13
  - Cole 2.3.14
  - Hemanth 2.5.7

– Ashwin 2.5.8
– Jacob 2.4.17
– Merrick 2.4.18
– Paria 2.4.19
– Kevin 2.4.20
– Fatemeh 2.8.24
– Leslie 2.7.11