What's in this section? The following problems have been collected from various sources (e.g., algebra textbooks by Dummit and Foote, Aluffi, Hungerford, etc, and different online sources). Although there are no citations, I make no claims of originality. If needed, most of them can be found in most standard algebra texts, maybe as unsolved exercises or even as solved examples or proven results. The problems comprise several topics in group theory and shall help you test your understanding of the fundamentals. The general idea is to build intuition and boost confidence.

Though the problems span varying degrees of difficulty, they should still be easier than the ones that have appeared in the previous qualifying exams. You may treat these as warm-up exercises as you build yourself up to answering more complicated questions. Some of these exercises contain common examples/counterexamples and others are easy-to-prove results, all of which might come in handy when solving more involved problems.

In case some of the problems in the next section leave you clueless, it might be more productive to revisit this section before you take another stab at the next section. (Cleanse your palate, so to speak.) Take your mind off the difficult ones while still being productive and then make another attempt with a fresh perspective.

I've tried my best to keep this error-free yet you must keep an eye out for any typos that might have sneaked in. If you find any flaws, especially logical or grammatical, please let me know and I shall fix them.

1. Consider the set of 2×2 matrices $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ with $x \in \mathbb{R}$. Is this a ring, under the usual matrix multiplication and addition? Is this a subring of $\mathrm{Mat}_2(\mathbb{R})$? (Notice that your answer will depend on how you define a subring.)

2. An element a of a ring R is nilpotent if $a^n = 0$ for some n.
   - Prove that if a and b are nilpotent in R and $ab = ba$, then $a + b$ is also nilpotent.
   - Is the hypothesis $ab = ba$ in the previous statement necessary for its conclusion to hold?

3. Prove that $R[x]$ is an integral domain if and only if R is an integral domain.

4. Denote the ring of endomorphisms of $\mathbb{Z}$ as an abelian group by $\mathrm{End}_{\mathrm{Ab}}(\mathbb{Z})$. Show that $\mathrm{End}_{\mathrm{Ab}}(\mathbb{Z}) \cong \mathbb{Z}$ as rings.

5. Let A be an abelian group such that $\mathrm{End}_{\mathrm{Ab}}(A)$ is a commutative ring. What can you say about A?

6. Let R be a commutative ring, $a \in R$, and $f_1(x), ..., f_r(x) \in R[x]$.
   (a) Prove the equality of ideals $\langle f_1(x), ..., f_r(x), x - a\rangle = \langle f_1(a), ..., f_r(a)\rangle$.
   (b) Prove that
   $$\frac{R[x]}{\langle f_1(x), ..., f_r(x), x - a\rangle} \simeq \frac{R}{\langle f_1(a), ..., f_r(a)\rangle}$$
   .

7. Let R be a ring, and let I be an ideal of R. Then
   $$\frac{R[x]}{IR[x]} \simeq \frac{R}{I}[x].$$

8. Let R be a ring and $a, b \in R$. Denote by $\bar{b}$ the image of b in $R/\langle a \rangle$. Then show that

$$\frac{R}{\langle a, b \rangle} \simeq \frac{R/\langle a \rangle}{\langle \bar{b} \rangle}.$$

9. **Norm.** Let $d \in \mathbb{Z}$ and R be the subring $\{a + b\sqrt{d} | a, b \in \mathbb{Z}\}$ of the ring of complex numbers $\mathbb{C}$. Define the map $N : R \to \mathbb{Z}$ by $N(a + b\sqrt{d}) = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - b^2 d$. Prove the following:

   (a) The map N is multiplicative, i.e., $N(rs) = N(r)N(s)$ for all $r, s \in R$.

   (b) $N(r) = 0$ if and only if $r = 0$.

   (c) u is a unit in R if and only if $N(u) = \pm 1$.

10. Prove that if k is a field, then $k[x]$ is a PID.

11. The center of a ring R is the set $\{a \in R | ar = ra, \forall r \in R\}$. Let k be a field. Show that the center of $\mathrm{Mat}_2(k)$ (the ring of 2×2 matrices over k) consists of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. What if we replaced k with a division ring?

12. Let R be an integral domain. Check that $a \in R$ is irreducible if and only if

   - $a = bc$ implies that $\langle a \rangle = \langle b \rangle$ or $\langle a \rangle = \langle c \rangle$.
   - $\langle a \rangle \subseteq \langle b \rangle \implies \langle b \rangle = \langle a \rangle$ or $\langle b \rangle = \langle 1 \rangle$
   - $\langle a \rangle$ is maximal among proper principal ideals.

13. Let R be an integral domain, and let $a \in R$ be a nonzero prime element. Then a is irreducible.

14. Let R be an integral domain, and let r be a nonzero, nonunit element of R. Assume that every ascending chain of principal ideals

$$rR \subseteq r_1 R \subseteq r_2 R \subseteq \dots$$

   stabilizes. Then r has a factorization into irreducibles.

15. Let R be a UFD, and let a be an irreducible element of R. Then a is prime.

16. Suppose $R \subseteq S$ is an inclusion of integral domains, and assume that R is a PID. Let $a, b \in R$, and let $d \in R$ be a gcd for a and b in R. Prove that d is also a gcd for a and b in S.

17. **Zorn's lemma practice.** Let R be a commutative ring, and let $I \subseteq R$ be a proper ideal. Prove that the set of prime ideals containing I has minimal elements.

18. **Zorn's lemma practice.** The Jacobson radical of a commutative ring R is the intersection of the maximal ideals in R. Prove that r is in the Jacobson radical if and only if $1 + rs$ is invertible for every $s \in R$.

19. Let R be a commutative ring and M be a proper ideal of R. Show that M is maximal if and only if for every $r \in R \setminus M$, there exists $x \in R$ such that $1 - rx \in M$.

20. Let R be a commutative ring, M a maximal ideal of R and n a positive integer. Show that the ring R/M^n has a unique prime ideal.

21. Let R be a ring. An element $a \in R$ is called a nilpotent element if $a^n = 0$ for some n. Show that the following are equivalent:

    (a) R has no nonzero nilpotent elements.

    (b) If $a \in R$ and $a^2 = 0$, then $a = 0$.

22. Let R be a commutative ring and N be the set of all nilpotent elements of R. Show that N is an ideal and R/N is a ring with no nonzero nilpotent elements.

23. A subset S of a commutative ring R is called a multiplicative subset if

    (a) $1 \in S$

    (b) $s, t \in S \implies st \in S$.

    If P is a prime ideal of R then show that $R \setminus P$ is a multiplicative set.

24. Let R be a ring (not necessarily commutative) and P be an ideal of R. Then prove that the following statements are equivalent:

    (a) P is a prime ideal.

    (b) If U and V are left ideals in R such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

    (c) If U and V are right ideals in R such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

25. Let k be an algebraically closed field. Prove that k must be infinite.

26. Let F be a finite field. Prove that there are irreducible polynomials in $F[x]$ of arbitrarily high degree.

27. Prove that a finite integral domain is a field.

28. **Zorn's lemma.** Consider the abelian group $\mathbb{Q}$ under usual addition. Define multiplication by $qr = 0$ for all $q, r \in \mathbb{Q}$. This makes $\mathbb{Q}$ a ring with no identity. Prove that this ring has no maximal ideal.