Topology Qual Prep - Practice Qual 2 - 2023

Instructions Do eight problems, four from each part. That is four from part A and four from part B. This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

PART A

Problem 1 Let G be a finite group. Show that there exists a smooth manifold M with fundamental group G.

Proof. Let $G = \{g_1, ..., g_n\}$. We will assume that $n \geq 4$ (note that any group of smaller order is finitely generated abelian and we know how to realize these as manifolds). Using the Cayley-Yoneda lemma, we may embed G into the group of $n \times n$ permutation matrices over, say, $\mathbb{R}$. This is realized by forming the free vector space on G and letting G act linearly on this by multiplication. Thus, G has a continuous action on the sphere S^{n-1}. Further, observe that this action has precisely two fixed points: $\pm(1/\sqrt{n}, ..., 1/\sqrt{n})$. Let X denote S^{n-1} with these points removed. The action of G on X is now a covering action. Note that X is homotopy equivalent to S^{n-2} and is therefore simply connected. Thus $\pi_1(X/G) \cong G$ and X/G is a smooth manifold. QED

Problem 2 Let M and N be connected topological manifolds of equal positive dimension n.

(a) Define the connected sum $M\#N$.

(b) Assuming $n \geq 3$, find an explicit formula for $\pi_1(M\#N)$ in terms of the fundamental groups of M and N.

(c) Does your formula work for $n = 2$? Justify your answer.

Solution. (a) Let $B_M \cong B_N$ denote open balls in M, N contained in charts. Then $M\#N = (M \setminus B_M) \cup_{\partial B} (N \setminus B_N)$ where by this we mean that we glue $M \setminus B_M$ to $N \setminus B_N$ along the boundary of these balls.

(b) **Claim** If $n \geq 3$, there holds $\pi_1(M\#N) \cong \pi_1(M) * \pi_1(M)$.

Proof. Note that there is a neighborhood of ∂B_M in $M \setminus B_M$ which deformation retracts to ∂B_M. Let U_N denote the union of N and this neighborhood. Switching the roles of M, N, define U_M likewise. Then $M\#N = U_M \cup U_N$ and $U_M \cap U_N$ deformation retracts to the simply connected ∂B_M. The result follows by Van-Kampen. QED

(c) This is false. Note that the connected sum of two two-dimensional tori is the genus two orientable surface whose fundamental group is not $(\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z})$.

Problem 3 Prove or disprove the following conjecture.

> **Conjecture** If X is a connected, Hausdorff space such that every map from a connected, compact space to X is null homotopic, then X is homotopy equivalent to a point.

Solution. The conjecture is false. Let X denote the line with two origins. Just as in the proof that X is simply connected, any map of a connected compact space to X lies in a subspace homeomorphic to $[0, 1]$. So any such map is nullhomotopic. On the other hand, X is not path connected and so is not homotopy equivalent to a point.

Problem 4 If A is a finite subset of a real vector space, then the convex hull of A is the set

$$h(A) = \left\{ \sum_{a \in A} r_a a : r_a \in [0, 1] \text{ and } \sum_{a \in A} r_a = 1 \right\}.$$

Let $\{e_1, ..., e_{n+1}\}$ denote the standard basis of $\mathbb{R}^{n+1}$. Then for $n, i \in \mathbb{N}$ and $0 \leq i \leq n$ the standard topological n-simplex and (n, i)-horn are given by

$$|\Delta^n| = h(e_1, ..., e_{n+1}) \text{ and } |\Lambda^n_i| = \bigcup_{j \neq i} h(e_1, ..., \widehat{e_j}, ..., e_n).$$

(a) Draw, as best you are able, the spaces $|\Lambda_1^2|$ and $|\Delta^2|$ and $|\Lambda_1^3|$ and $|\Delta^3|$. (**Aside**: You don't need to show these sitting in some ambient $\mathbb{R}^n$).

(b) Prove that for any topological space X and any continuous map $|\Lambda_i^n| \to X$ there is a continuous extension $|\Delta^n| \to X$.

Problem 5 It is known that if $p : E \to B$ is a covering projection and Y is path connected and locally path connected then any continuous map $f : Y \to B$ lifts to E if $f_*\pi_1(Y)$ is trivial. Show that this result fails if the assumption that Y is locally path connected is removed. (**Hint**: Consider the *Warsaw circle/quasi-circle* which is obtained by connecting the two ends of the topologist's sine curve by a path)

Solution. Let $G = \{(x, \sin(1/x)) : x \in (0, 1/\pi]\}$ be the graph of $\sin(1/x)$ on the interval $(0, 1/\pi]$. Let G_ε denote the same graph but for $x \in [\varepsilon, 1/\pi]$. Let $Y = \{(0, y) : y \in [0, 1]\}$ denote a segment of the y-axis. Let $A = \{(x, 0) : x \in [0, -1]\}$ and $B = \{(-1, y) : y \in [0, 2]\}$ and $C = \{(x, 2) : x \in [-1, 1]\}$ and $D = \{(2, y) : y \in [0, 2]\}$ and lastly $E = \{(x, 0) : x \in [1/\pi, 1]\}$. Define the Warsaw circle

$$W = G \cup Y \cup A \cup B \cup C \cup D \cup E$$

and the partial Warsaw circle

$$W_\varepsilon = G_\varepsilon \cup Y \cup A \cup B \cup C \cup D \cup E.$$

Letting S denote square of side lengths 2 with opposing corners $(-1, 0), (1, 2)$ we see that W is obtained by removing part of the bottom edge of S and replacing it with the topologist's sine curve. Note that W is simply connected.

Define $f : W \to S$ by letting $f(x, y) = (x, 0)$ for $(x, y) \in G \cup Y$ and identity elsewhere. Note that $S \cong S^1$ and thus has universal cover $p : \mathbb{R} \to S^1$. We will show that f does not lift to $\mathbb{R}$. Indeed, suppose there were such a lift $F : W \to \mathbb{R}$. Note that $f_\varepsilon := f|_{W_\varepsilon}$ is not a surjection. Using the fact that we know what p is explicitly, we may find a lift $F_\varepsilon : W_\varepsilon \to \mathbb{R}$ of f_ε agreeing with F at (say) the point $(1, 2)$. Note, again by explicit knowledge of p, we have that $F_\varepsilon(Y) = \{t_0\}$ is a singleton and $t_0 < F_\varepsilon((\varepsilon, 0)) =: t_\varepsilon$ (here we choose the orientation with which p wraps $\mathbb{R}$ about S). Further, we see that t_ε is increasing as ε increase. Since W is path connected, lift that agree at a point are unique unique. That is, $F|_{W_\varepsilon} = F_\varepsilon$. for all $\varepsilon > 0$. Now, we observe that $(\varepsilon, 0) \to (0, 0)$ as $\varepsilon \to 0$. So, $F(\varepsilon, 0) = t_\varepsilon \to t_0 = F(0, 0)$. But this is not possible. So, F is not continuous and f does not lift.

Problem 6 For topological spaces X, Y, write $[X, Y]$ for the set of homotopy classes of maps $X \to Y$. Prove that if A, X, Y are topological spaces and $F \in [A, X]$ and $G \in [A, Y]$ then there exists a topological space Z equipped with $E_X \in [X, Z]$ and $E_Y \in [Y, Z]$ so that the following diagram commutes-up-to-homotopy

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "G"'] & X \arrow[d, "E_Z"] \\ Y \arrow[r, "E_Y"'] & Z \end{tikzcd}$$

and for any space T and commutative-up-to-homotopy

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "G"'] & X \arrow[d, "E_X"] \arrow[ddr, bend left, "H_X"] & \\ Y \arrow[r, "E_Y"'] \arrow[drr, bend right, "H_Y"'] & Z & \\ & & T \end{tikzcd}$$

there exists commutative-up-to-homotopy

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "G"'] & X \arrow[d, "E_X"] \arrow[ddr, bend left, "H_X"] & \\ Y \arrow[r, "E_Y"'] \arrow[drr, bend right, "H_Y"'] & Z \arrow[dr, dashed, "H"] & \\ & & T \end{tikzcd}$$

Definition Let $f : A \to X$ and $g : A \to Y$ be continuous mappings of topological spaces. We define the *double mapping cylinder* by

$$Z(f,g) = \big((A \times I) \sqcup X \sqcup Y\big)/\big((a,0) \sim f(a),\ (a,1) \sim g(a)\big).$$

That is, we glue $A \times I$ to X along f on the bottom of the cylinder $A \times I$ and to Y along g at the top of the cylinder. We denote by $e_X : X \to Z(f,g)$ and $e_Y : Y \to Z(f,g)$ the obvious inclusions. Let E_X and E_Y denote the homotopy classes of e_X and e_Y.

Claim If $f \in F$ and $g \in G$ are representatives of the given homotopy classes, then $Z(f,g)$ has the desired properties.

Proof. Let Ω be any space with homotopy classes of morphisms $H : X \to \Omega$ and $K : Y \to \Omega$ so that

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "G"'] & X \arrow[d, "E_X"] \arrow[ddr, bend left, "H"] & \\ Y \arrow[r, "E_Y"'] \arrow[drr, bend right, "K"'] & Z(f,g) & \\ & & \Omega \end{tikzcd}$$

commutes. Choose representative $h \in H$ and $k \in K$. There is a homotopy $T : hf = kg$. Define $\ell : Z(f,g) \to \Omega$ by

$$\begin{cases} \ell[a,t] = T(a,t) & (a,t) \in A \times I \\ \ell[x] = h(x) & x \in X \\ \ell[y] = k(y) & y \in Y \end{cases}$$

which one may show is continuous by the universal property of quotients and makes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "e_X"] \arrow[ddr, bend left, "h"] & \\ Y \arrow[r, "e_Y"'] \arrow[drr, bend right, "k"'] & Z(f,g) \arrow[dr, "\ell"] & \\ & & \Omega \end{tikzcd}$$

Pushing ℓ into the homotopy category proves the claim. QED

PART B

Problem 1 If M is a smooth manifold and $U \subseteq M$ is open, then extension by zero defines a linear map $\iota_U^M : \Omega_c(U) \to \Omega_c(M)$. Suppose that M has open cover $\{U, V\}$.

(a) Does pullback of forms provide a map $\Omega_c(M) \to \Omega_c(U)$?

(b) Prove that there exists a linear map $s : \Omega_c(U) \oplus \Omega_c(V) \to \Omega_c(M)$ so that the following sequence is exact

$$0 \longrightarrow \Omega_c(U \cap V) \xrightarrow{\omega \mapsto \iota_{U\cap V}^U \omega \oplus \iota_{U\cap V}^V(-\omega)} \Omega_c(U) \oplus \Omega_c(V) \xrightarrow{s} \Omega_c(M) \longrightarrow 0$$

Solution. (a) It does not. Let $M = \mathbb{R}$ and $U = (0,1)$. A function which does not vanish on U (and thus not compactly supported in U) may still be compactly supported in $\mathbb{R}$.

(b) We first define $s : \Omega_c(U) \oplus \Omega_c(V) \to \Omega_c(M)$ by

$$s(\omega, \tau) = \iota_U^M(\omega) + \iota_V^M(\tau).$$

Certainly, the composition of s with the prior map yields 0. Further, suppose that $s(\omega, \tau) = 0$. Then we see that $\omega = -\tau$ on $U \cap V$ and defined a form there. To see that (ω, τ) lies in the image of the prior map, it suffices to show that ω is compactly supported in $U \cap V$. Let $K \subseteq U$ denote the support of ω. Let $p \in K$. Then there is a sequence $p_n \to p$ so that $\omega(p_n) \neq 0$. As $s(\omega, \tau) = 0$, we have that $\tau(p_n) \neq 0$. So p is in the support of τ and thus in V. So, $K \subseteq U \cap V$.

It remains to show that s is surjective. Fix a differential form ω on M. Let ρ_U, ρ_V be a partition of unity on M with supp$(\rho_U) \subseteq U$ and supp$(\rho_V) \subseteq V$. Then $\omega_U = \rho_U \omega$ is a smooth form supported in U and $\omega_V = \rho_V \omega$ is a form supported in V. Note that $\omega_U + \omega_V = \omega$. Finally, these are compactly supported as

$$\text{supp}(\rho_U \omega) \subseteq \text{supp}(\rho_u) \cap \text{supp}(\omega)$$

is a closed subset of a compact set and thus compact. A similar statement holds for ω_V.

Problem 2 Give a definition of orientation on a manifold. From this definition[1]

(a) prove that if M is a manifold then TM and T^*M are orientable; and

(b) prove that there exist non-orientable manifolds.

Definition We define a manifold to be orientable when it has a non-vanishing top form.
It will be useful to have another characterization of orientability.

Proposition A manifold is orientable if it admits an atlas wherein all transition functions have positive Jacobian determinant.

Proof. Suppose M is an n-manifold with atlas $\{(U_\alpha, x_\alpha) : \alpha \in A\}$ so that each transition $\phi_{\beta\alpha}$ for $\alpha, \beta \in A$ and $U_\alpha \cap U_\beta \neq \emptyset$ has positive Jacobian determinant. Let ρ_α be a partition of unity subordinate to this atlas. Define the top form $\omega = \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$. Observe that withing a fixed chart U_{α_0} we have

$$\begin{aligned}
\omega &= \sum_\alpha \rho_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n \\
&= \sum_\alpha \rho_\alpha \det(J\phi_{\alpha_0\alpha}) dx_{\alpha_0}^1 \wedge \cdots \wedge dx_{\alpha_0}^n \\
&= \left(\sum_\alpha \rho_\alpha \det(J\phi_{\alpha_0\alpha}) \right) dx_{\alpha_0}^1 \wedge \cdots \wedge dx_{\alpha_0}^n
\end{aligned}$$

does not vanish. QED

[1] That is, if you want to use a property/characterization of orientation it must be proven from your given definition.

Claim For any manifold M, the tangent bundle TM is orientable.

Proof. Consider any pair of charts U, V for M with transition function ϕ. One shows (you should actually do this) that the associated transition function for TM

$$\begin{pmatrix} J\phi & 0 \\ 0 & J\phi \end{pmatrix}$$

which has determinant $\det(J\phi)^2 \neq 0$. QED

Claim For any n-manifold M, the cotangent bundle is orientable.

Proof. We will produce a non-vanishing top form. Cover M with an atlas $\{(U_\alpha, x_\alpha)\}$. Note that this gives rise to a chart on the cotangent bundle which shall be denoted $(V_\alpha, x_\alpha, c_\alpha)$ where $V_\alpha = T^*U_\alpha$ and

$$(x_\alpha, c_\alpha)\left(\sum_{i=1}^n a_i dx_p^i\right) = (p^1, ..., p^n, a_1, ..., a_n)$$

where $p \in U_\alpha$ gives a diffeomorphism $(x_\alpha, c_\alpha) : V_\alpha \to x_\alpha(U_\alpha) \times \mathbb{R}^n$.

Let $\pi : T^*M \to M$ be the natural projection. Recall that the Liouville form λ on T^*M is defined by

$$\lambda_{\omega_p}(X_{\omega_p}) = \omega_p(\pi_*(X_{\omega_p}))$$

for any $\omega_p \in T_p^*M$ and tangent vector $X_{\omega_p} \in T_{\omega_p}T^*M$. We know that there are maps $\{a_i, b_i : T^*U \to \mathbb{R}\}_{i=1}^n$ so that

$$\lambda = \sum_{i=1}^n a_i d\widetilde{x}^i + \sum_{i=1}^n b_i dc^i$$

on T^*U. We then see that for $\omega_p \in T^*U$ that

$$\begin{aligned}
a_i(\omega_p) &= \lambda_{\omega(p)}\left(\frac{\partial}{\partial \widetilde{x}^i}|_{\omega_p}\right) \\
&= \omega_p\left(\pi_* \frac{\partial}{\partial \widetilde{x}^i}|_{\omega_p}\right) \\
&= \omega_p\left(\sum_{j=1}^n \pi_* \frac{\partial}{\partial \widetilde{x}^i}|_{\omega_p}(x^j)\frac{\partial}{\partial x^j}\right) \\
&= \omega_p\left(\sum_{j=1}^n \frac{\partial \widetilde{x}^j}{\partial \widetilde{x}^i}(\omega_p)\frac{\partial}{\partial x^j}\right) \\
&= \omega_p\left(\frac{\partial}{\partial x^i}\right) \\
&= c^i(\omega_p).
\end{aligned}$$

Likewise,

$$\begin{aligned}
b_i(\omega_p) &= \lambda_{\omega(p)}\left(\frac{\partial}{\partial c^i}|_{\omega_p}\right) \\
&= \omega_p\left(\pi_*\frac{\partial}{\partial c^i}|_{\omega_p}\right) \\
&= \omega_p\left(\sum_{j=1}^n \pi_*\frac{\partial}{\partial c^i}|_{\omega_p}(x^j)\frac{\partial}{\partial x^j}\right) \\
&= \omega_p\left(\sum_{j=1}^n \frac{\partial \widetilde{x}^j}{\partial c^i}(\omega_p)\frac{\partial}{\partial x^j}\right) \\
&= 0.
\end{aligned}$$

Consequently, on T^*U, there holds

$$\lambda = \sum_{i=1}^n c^i d\widetilde{x}^i.$$

From this computation, we have that λ is smooth. One may then explicitly calculate in coordinates that the n-th fold wedge product of $d\lambda$ is non-vanishing. QED

To show that there are non-orientable manifolds, we may prove that the quotient of an orientable manifold by a covering action which does not preserve orientation results in a non-orientable manifold. Thus, $\mathbb{R}P^2$ is non orientable.

Problem 3 Let M be a smooth manifold, $i : S \hookrightarrow M$ an embedded submanifold. Prove that any smooth function $f : S \to \mathbb{R}^k$ may be extended to M.

Proof. By the rank theorem, we may cover S with charts U and M with matching charts V_U so that the inclusion $\iota : U \to V_U$ is as the inclusion of $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$. Clearly, we may find smooth $F_U : V_U \to \mathbb{R}^k$ which extend f along ι. Letting $\{\rho_U\}$ be a subordinate partition of unity, we define $F = \sum_U \rho_U F_U$. By construction, this extends F. QED

Problem 4 Consider the map $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^5$

$$f(x,y,z) = (xy, yz, zx, x^2 - y^2, x^2 + y^2 + z^2 - 1).$$

Find an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$

Solution. Note that the restriction of f to S^2 is constant on the fibres of the usual quotient map $S^2 \to \mathbb{R}P^2$ and has image in $\mathbb{R}^4 \subseteq \mathbb{R}^5$. Note that the jacobian of f (on $\mathbb{R}^3$) is

$$Jf = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \\ 2x & -2y & 0 \\ 2x & 2y & 2z \end{pmatrix}$$

and when $\pi : \mathbb{R}^5 \to \mathbb{R}^4$ is projection in the first four coordinates

$$J(\pi f) = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \\ 2x & -2y & 0 \end{pmatrix}$$

Passing to the quotient, we define $g : \mathbb{R}P^2 \to \mathbb{R}^4$ by $g[x : y : z] = (xy, yz, xz, x^2 - y^2)$. Note that πf may be recast as a map $h : (\mathbb{C} \times \mathbb{R}) \smallsetminus \{0\} \to \mathbb{C} \times \mathbb{R}^2$ given by

$$h(\mu, t) = (\mu^2, t\,\mathrm{Re}\,\mu, t\mathrm{Im}\mu).$$

Note that $h(\mu, t) = h(\nu, s)$ implies that $\mu = \pm\nu$ and $t = \pm s$ (signs chosen consistently) of $\mu \neq 0$. From this one see that $g[x : y : z] = g[x', y', z']$ only if $[x : y : z] = [x' : y' : z']$. So, g is a topological embedding.

Now, consider a tangent vector (a, b, c) at (x, y, z) in S^2. That is, $ax + by + cz = 0$. Then

$$J(\pi f)(a, b, c) = \begin{pmatrix} ay + bx \\ bz + cy \\ az + cx \\ 2ax - 2by \end{pmatrix}$$

If $J(\pi f)(a, b, c) = 0$ then much tedious algebra shows that $a = b = c = 0$. So, πf is an immersion. The quotient $S^2 \to \mathbb{R}P^2$ is a local diffeomorphism, so g is an immersion and thus a smooth embedding.

Problem 5 Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth positive function. Prove that the space obtained by rotating the graph of f about the x-axis is a smooth manifold. Show that this manifold is diffeomorphic to the cylinder $\mathbb{R} \times S^1$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, positive function. Let M be the set obtained by rotating the graph of f about the x-axis, that is

$$M = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = [f(x)]^2\}.$$

Define $U_y^\pm, U_z^\pm \subseteq M$ by

$$U_y^\pm = \{(x, y, z) \in M : \pm y > 0\} \text{ and } U_z^\pm = \{(x, y, z) \in M : \pm z > 0\}.$$

Let $\phi_y^\pm : U_y^\pm \to \mathbb{R}^2$ and $\phi_z^\pm : U_z^\pm \to \mathbb{R}^2$ be the projections onto the xz- and xy-planes respectively.

First note that since f does not vanish we have that M is covered by $U_y^\pm$ and $U_z^\pm$. Further, we have that each of the maps for these charts is continuous as a canonical projection out of a subspace of a product. Define

$$\begin{aligned} V_y^\pm &= \phi_y^\pm(U_y^\pm) \\ &= \{(x, z) \in \mathbb{R}^2 : f(x)]^2 - z^2 > 0\} \end{aligned}$$

and

$$\begin{aligned} V_z^\pm &= \phi_y^\pm(U_y^\pm) \\ &= \{(x, y) \in \mathbb{R}^2 : [f(x)]^2 - y^2 = 0\}. \end{aligned}$$

Note that all four of these is open in $\mathbb{R}^2$ as the preimage of the open set $(0, \infty)$ under a continuous map. Define $\psi_y^\pm : V_y^\pm \to U_y^\pm$ by

$$\psi_y^\pm(x, z) = (x, \pm\sqrt{[f(x)]^2 - z^2}, z)$$

and $\psi_z^\pm : V_z^\pm \to U_z^\pm$ by

$$\psi_y^\pm(x, y) = (x, y, \pm\sqrt{[f(x)]^2 - y^2}).$$

All four of these are clearly continuous and invert the appropriate ϕ. Thus, we have established that M is a topological manifold and it only remains to show that this atlas is smooth.

Note that if $s \in \{y, z\}$ then $\phi_s^+ \psi_s^- = \text{id}$ which is smooth. We then see see that (on its maximal domain of definition)

$$\phi_x^+ \psi_y^+(x, z) = (x, \sqrt{[f(x)]^2 - z^2})$$

which is smooth as the quantity under the radical cannot vanish. The other transitions are similar. It follows that the above atlas induces a smooth structure on M.

We now show that M is diffeomorphic to a cylinder. Define the map $\Phi : M \to \mathbb{R} \times S^1$ by

$$\Phi(x, y, z) = \left(x, \frac{(y, z)}{|(y, z)|} \right)$$

and $\Psi : \mathbb{R} \times S^1 \to M$ by

$$\Psi(x, y, z) = (x, f(x)y, f(x)z).$$

Observe that the codomain of Ψ is justified as if $x, y, z \in \mathbb{R}$ and $y^2 + z^2 = 1$ then

$$[f(x)y]^2 + [f(x)z]^2 = [f(x)]^2.$$

Both Φ and Ψ are continuous by inspection, we it remains to show that they are smooth. We note that Ψ has smooth projections to the xy- and xz-planes and so is smooth by definition (here we tacitly use that $\mathbb{R} \times S^1$ is a submanifold of $\mathbb{R}^3$). To see that Φ is smooth, observe that the inclusion $M \hookrightarrow \mathbb{R}^3 \smallsetminus \{(x, y, z) : y = z = 0\}$ is smooth, the ψ maps above having smooth component functions. Then Φ extends to a transparently smooth function $\mathbb{R}^3 \smallsetminus \{(x, y, z) : y = z = 0\} \to \mathbb{R} \times S^1$. Then Φ is a composition of smooth functions and is smooth. QED

Problem 6 Let M be a smooth manifold and $f : M \to M$ be a smooth function satisfying $f^2 = f$. Show that $f(M)$ is a submanifold of M.

Proof. First, note that if $p \in f(M)$ then $T_pM = \ker d_pf \oplus \ker(1 - d_pf)$ noting any tangent vector may be written $v = (v - d_pfv) + d_pfv$. By rank-nullity,

$$\dim M = \dim d_pf(T_pM) + \dim(1 - d_pf)T_pM.$$

Since rank is lower-semicontinuous, we may find a neighborhood of p in M where neither $\dim d_pf(T_pM), \dim(1- d_pf)T_pM$ may decrease. Thus, on this neighborhood, these quantities are constant. As $f(M)$ is connected, f has constant rank of k on all of $f(M)$ and indeed in a neighborhood U of $f(M)$.

By the rank theorem, f is locally the projection $\mathbb{R}^{k+\ell} \to \mathbb{R}^k$. This shows that $f(M)$ is a regular submanifold. QED