What's in this section? The following problems have been collected from various sources (e.g., algebra textbooks by Dummit and Foote, Aluffi, Hungerford, etc, and different online sources). Although there are no citations, I make no claims of originality. If needed, most of them can be found in most standard algebra texts, maybe as unsolved exercises or even as solved examples or proven results. The problems comprise several topics in group theory and shall help you test your understanding of the fundamentals. The general idea is to build intuition and boost confidence.

Though the problems span varying degrees of difficulty, they should still be easier than the ones that have appeared in the previous qualifying exams. You may treat these as warm-up exercises as you build yourself up to answering more complicated questions. Some of these exercises contain common examples/counterexamples and others are easy-to-prove results, all of which might come in handy when solving more involved problems.

In case some of the problems in the next section leave you clueless, it might be more productive to revisit this section before you take another stab at the next section. (Cleanse your palate, so to speak.) Take your mind off the difficult ones while still being productive and then make another attempt with a fresh perspective.

I've tried my best to keep this error-free yet you must keep an eye out for any typos that might have sneaked in. If you find any flaws, especially logical or grammatical, please let me know and I shall fix them.

1. Let R. be a commutative ring, and let $F = \bigoplus_{1 \le i \le n} R$ be a free module over R. Let $\mathfrak{m}$ be a maximal ideal of R, and let $k = R/\mathfrak{m}$ be the quotient field. Prove that $F/\mathfrak{m}F \cong \bigoplus_{1 \le i \le n} k$ as k-vector space.

2. Let A be an abelian group such that $\mathrm{End}_{\mathrm{Ab}}(A)$ is a field of characteristic 0. Prove that $A \cong \mathbb{Q}$.

3. (The Short Five Lemma) Let R be a ring and

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

   a commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence. Then

   (a) α, γ injective $\implies$ β is injective;

   (b) α, γ surjective $\implies$ β is surjective;

   (c) α, γ isomorphisms $\implies$ β is an isomorphism.

4. Show that if $f : A \to A$ is an R-module homomorphism such that $f \circ f = f$ then

$$A = \ker f \oplus \mathrm{Im} f.$$

5. Show that if $f : A \to B$ and $g : B \to A$ are R-module homomorphisms such that $g \circ f = \mathrm{id}_A$, then $B = \mathrm{Im} f \oplus \ker f$.

6. Let A, B be sets. Prove that the free groups $F(A), F(B)$ are isomorphic if and only if there is a bijection $A \cong B$.

7. Let R be an integral domain, and let A, B be sets. If $F^R(A)$ (resp. $F^R(B)$) is a free R module generated by A (resp. B). Show that

$$F^R(A) \cong F^R(B) \iff \text{there is a bijection } A \cong B.$$

Def. Let R be a ring with identity such that for every free R-module F, any two bases of F have the same cardinality. Then R is said to have the invariant dimension property and the cardinal number of any basis of F is called the dimension (or rank) of F over R.

8. Prove that both $\mathrm{Mat}_n(R)$ and $\mathrm{Hom}_R(R^n, R^n)$ are R-algebras in a natural way and there is an isomorphism of R-algebras $\mathrm{Hom}_R(R^n, R^n) = \mathrm{Mat}_n(R)$. In particular, if the matrix M corresponds to the homomorphism $\varphi : R^n \to R^n$, then M is invertible in $\mathrm{Mat}_n(R)$ if and only if φ is an isomorphism.

9. (a) If $0 \to A \to B \xrightarrow{f} C \to 0$ and $0 \to C \xrightarrow{g} D \to E \to 0$ are short exact sequences of modules, then the sequence $0 \to A \to B \xrightarrow{gf} D \to E \to 0$ is exact.
   (b) Show that every exact sequence may be obtained by splicing together suitable short exact sequences as in (a).

10. Let R. he an integral domain, M an R module, and assume $M \cong R^r \oplus T$, with T a torsion module. Prove that $\mathrm{rank} M = r$ and $T \cong \mathrm{Tor}_R(M)$.

11. (a) If R has an identity and A is an R-module, then there are submodules B and C of R such that B is unitary, $RC = 0$ and $A = B \oplus C$.
    (b) Let A_1 be another R-module, with $A_1 = B_1 \oplus C_1$ (B_1 unitary, $RC_1 = 0$). If $f : A \to A_1$ is an R-module homomorphism then $f(B) \subseteq B_1$ and $f(C) \subseteq C_1$.
    (c) If the map f of part (b) is surjective (resp. isomorphism), then so are $f|_B : B \to B_1$ and $f|_C : C \to C_1$.

12. Let R be a principal ideal domain, A a unitary left R-module, and $p \in R$ a prime (=irreducible). Let $pA = \{pa | a \in A\}$ and $A[p] = \{a \in A | pa = 0\}$.
    (a) $R/(p)$ is a field.
    (b) pA and $A[p]$ are submodules of A.
    (c) A/pA is a vector space over $R/(p)$, with $(r + (p))(a + pA) = ra + pA$.
    (d) $A[p]$ is a vector space over $R/(p)$, with $(r + (p))a = ra$.

13. Let $f : V \to V'$ be a linear transformation of finite dimensional vector spaces V and V' such that $\dim V = \dim V'$. Then the following conditions are equivalent: (i) f is an isomorphism; (ii) f is an injection; (iii) f is a surjection.

14. If A, B are abelian groups and m, n integers such that $mA = 0 = nB$ then every element of $\mathrm{Hom}(A, B)$ has order dividing (m, n).

15. Let $\pi : \mathbb{Z} \to \mathbb{Z}_2$ be the canonical projection. The induced map
$\overline{\pi} : \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)$ is the zero map. Since $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \neq 0$, $\overline{\pi}$ is not an epimorphism.

16. Let R be a commutative ring and M an R-module.

(a) Prove that $\operatorname{Ann}(M)$ is an ideal of R.

(b) If R is an integral domain and M is finitely generated, prove that M is torsion if and only if $\operatorname{Ann}(M) \neq 0$.

(c) Give an example of a torsion module M over an integral domain, such that $\operatorname{Ann}(M) = 0$.

17. Let M be a module over a commutative ring R. Prove that an ideal I of R is the annihilator of an element of M if and only if M contains an isomorphic copy of R/I (viewed as an R-module).

18. Let N, P be submodules of a module M, such that $N \cap P = \{0\}$ and $M = N + P$. Prove that $M \cong N \oplus P$.

19. Let $\alpha \in \operatorname{End}_R(F)$ be a linear transformation on a free R-module F. Prove that the set of polynomials $f(t) \in R[t]$ such that $f(\alpha) = 0$ is an ideal of $R[t]$.

20. Let $A \in \operatorname{Mat}_n(R)$ be a square matrix and let A^t be its transpose. Prove that A and A^t have the same characteristic polynomial and the same annihilator ideals.

21. Let F_1, F_2 be free R-modules of finite rank, and let α_1, resp., α_2, be linear transformations of F_1, resp., F_2. Let $F = F_1 \oplus F_2$, and let $\alpha = \alpha_1 \oplus \alpha_2$ be the linear transformation of F restricting to α_1 on F_1 and α_2 on F_2.

    (a) Prove that $P_\alpha(t) = P_{\alpha_1}(t)P_{\alpha_2}(t)$. That is, the characteristic polynomial is multiplicative under direct sums.

    (b) Find an example showing that the minimal polynomial is not multiplicative under direct sums.

22. Let k be a field, and let K be a field containing k. Two square matrices $A, B \in \operatorname{Mat}_n(k)$ may be viewed as matrices with entries in the larger field K. Prove that A and B are similar over k if and only if they are similar over K.

23. The 0 linear transformation $E \to E$ has invariant factors (resp. elementary divisors) $q_1 = x, q_2 = x, ..., q_n = x$.