Computational Data Mining

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Course material

- https://github.com/uisf-course/CDM

Multiplication Ax Using rows of A

- Multiply *A* times *x* using the three rows of *A :*

$$\textbf{By rows} \quad \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix} = \begin{array}{c} \text{inner products} \\ \text{of the rows} \\ \text{with } \boldsymbol{x} = (x_1, x_2) \end{array}$$

- a row at a time
- are also known as "dot products" because of the dot notation

$$\textbf{row} \cdot \textbf{column} = (\mathbf{2},\ \mathbf{3}) \cdot (\boldsymbol{x}_1,\ \boldsymbol{x}_2) = \mathbf{2}\boldsymbol{x}_1 + \mathbf{3}\boldsymbol{x}_2$$

- use this for computing but not for understanding.

Multiplication Ax Using Columns of A

- For Understanding Ax using vector approach.

$$\textbf{By columns} \quad \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = \begin{array}{c} \text{combination} \\ \text{of the columns} \\ \boldsymbol{a}_1 \text{ and } \boldsymbol{a}_2 \end{array}$$

- vector approach sees Ax as a "linear combination" of a_1 and a_2
- linear combination of of a_1 and a_2 includes two steps :
  - Multiply the columns a_1 and a_2 by "scalars" x_1 and x_2
  - Add vectors x_1 a_1 + x_2 a_2 = *Ax.*
- *Ax* is a linear combination of the columns of *A*
- leads us to the column space of *A.*
  - all combinations of the columns
  - Space created by Ax for all vectors ***X***

Column space of A C(A)

- All combinations $x_1 a_1 + x_2 a_2 = Ax$ produce what part of the full 3D space?
- produce a plane
  - contains the complete line in the direction of $a_1 = (2, 2, 3)$
  - contains the complete line in the direction of $a_2 = (3, 4, 7)$
  - *sum* of any vector on one line plus any vector on the other line
  - This addition fills out an infinite plane containing the two lines.
  - does not fill out the whole 3-dimensional space
  - If $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the column space of A ?

Column space of A C(A)

- What are the column spaces of

$$A_2 = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 10 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix} ?$$

What is all possible column spaces inside R^3

- The zero vector (0, 0, 0)
- A line of all vectors x_1a_1
- A plane of all vectors $x_1a_1 + x_2a_2$
- The whole R3 with all vectors $x_1a_1 + x_2a_2 + x_3a_3$

Matrix C

- create a matrix *C* whose columns independence columns of A
- construction of *C* from the *n* columns of *A :*
  - If column 1 of *A* is not all zero, put it into the matrix *C.*
  - If column 2 of *A* is not a multiple of column 1, put it into *C.*
  - If column 3 of *A* is not a combination of columns 1 and 2, put it into *C. Continue.*
- At the end *C* will haver columns *(r <= n).*
- They will be a "basis" for the column space of *A.*
- A basis for a subspace is a full set of independent vectors
- All vectors in the space are combinations of the basis vectors

Example

- What is matrix C and R for the matrices blow :

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Rank of the matrix

The rank of a matrix is the dimension of its column space.

- Different basis, but always the same number of vectors
- That number r is the "dimension" of the column space of A and C

Matrix R

- The matrix C connects to A by a third matrix R: $A = CR$
- Their shapes are :
  - (m by n) = (m by r) (r by n).

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \boldsymbol{CR}$$

C multiplies the first column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of R, this produces column 1 of C and A.

C multiplies the second column $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of R, we get column 2 of C and A.

- C multiplies the third column $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ of R, we get 2(column 1) + 2(column 2).

Matrix R

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} = CR$$

All three matrices have rank $r = 1$

Column Rank = Row Rank

The number of *independent columns* equals the number of *independent rows*

Matrix-Matrix Multiplication *AB*

- **Inner products** *(rows times columns)* produce each of the numbers in *AB* = C:

$$\begin{array}{c}\textbf{row 2 of } \boldsymbol{A} \\ \textbf{column 3 of } \boldsymbol{B} \\ \textbf{give } \boldsymbol{c_{23}} \textbf{ in } \boldsymbol{C}\end{array} \qquad \begin{bmatrix} \cdot & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & b_{13} \\ \cdot & \cdot & b_{23} \\ \cdot & \cdot & b_{33} \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \boldsymbol{c_{23}} \\ \cdot & \cdot & \cdot \end{bmatrix}$$

dot product $c_{23} = (\text{row 2 of } A) \cdot (\text{column 3 of } B)$ is a sum of a's times b's :

$$\boldsymbol{c_{23}} = a_{21}\, b_{13} + a_{22}\, b_{23} + a_{23}\, b_{33} = \sum_{k=1}^{3} \boldsymbol{a_{2k}\, b_{k3}}$$

$$\boldsymbol{c_{ij}} = \sum_{k=1}^{n} \boldsymbol{a_{ik}\, b_{kj}}\,.$$

Matrix-Matrix Multiplication *AB*

- columns of *A* times rows of *B.*
- Outer product :
  - *one column u* times *one row* v^{T}

"Outer product" $$uv^{\mathrm{T}} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 12 \\ 6 & 8 & 12 \\ 3 & 4 & 6 \end{bmatrix} =$$ **"rank one matrix"**

- m by 1 matrix (a column *u)* times a 1 by *p* matrix (a row v^{T}) gives an m by *p* matrix

what is special about the rank one matrix uv^{T} :

AB multiplication

- All column of uv^{T} are multiply of $\boldsymbol{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$
- All rows are multiply of $\boldsymbol{v}^{\mathrm{T}} = \begin{bmatrix} 3 & 4 & 6 \end{bmatrix}$
- column space of uv^{T} is one-dimensional: *the line in the direction of u*
- All nonzero matrices uv^{T} have rank one

AB multiplication

AB = Sum of Rank One Matrices

$$AB = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & b_1^* & — \\ & \vdots & \\ — & b_n^* & — \end{bmatrix} = \underbrace{a_1 b_1^* + a_2 b_2^* + \cdots + a_n b_n^*}_{\textbf{sum of rank 1 matrices}}.$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix} =$$

Question

- Suppose the column space of an m by n matrix is all of **R3 .** What can you say about m ? What can you say about n ? What can you say about the rank r ?

- Suppose A is the 3 by 3 matrix ones(3, 3) of all ones. Find two independent vectors x and y that solve $Ax = 0$ and $Ay = 0$. Write that first equation $Ax = \mathbf{0}$ (with numbers) as a combination of the columns of A. Why don't I ask for a third independent vector with $Az = 0$?