Computational Data Mining

Fatemeh Mansoori

Orthogonal Vectors-matrix

- Orthogonal vectors *x* and *y* The test is $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = x_1y_1 + \cdots + x_ny_n = 0.$
- Orthogonal basis for a subspace: Every pair of basis vectors has $\boldsymbol{v}_i^{\mathrm{T}}\boldsymbol{v}_j = 0.$
- Orthonormal basis: Orthogonal basis of **unit vectors**: every $\boldsymbol{v}_i^{\mathrm{T}}\boldsymbol{v}_i = 1$ (length 1).
- Orthogonal subspaces R and N : Every vector in the space **R** is orthogonal to every vector in N (The row space and nullspace are orthogonal)

$$\begin{array}{c} Ax = 0 \text{ means} \\ \textbf{each row} \cdot \boldsymbol{x} = \mathbf{0} \end{array} \qquad \begin{bmatrix} \textbf{row 1 of } A \\ \vdots \\ \textbf{row } m \textbf{ of } A \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Every row (and every combination of rows) is orthogonal to all *x* in the nullspace

- Tall thin matrices Q with orthonormal columns: $Q^T Q = I$.

$$\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q} = \begin{bmatrix} \text{——} & \boldsymbol{q_1}^{\mathrm{T}} & \text{——} \\ & \vdots & \\ \text{——} & \boldsymbol{q_n^{\mathrm{T}}} & \text{——} \end{bmatrix} \begin{bmatrix} \Bigg| & & \Bigg| \\ \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_n \\ \Bigg| & & \Bigg| \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

If $m > n$ the m rows cannot be orthogonal in $\mathbf{R}^n$. Tall thin matrices have $Q\,Q^{\mathrm{T}} \neq I$.

$$Q = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \qquad Q^{\mathrm{T}}Q = I \quad \boxed{QQ^{\mathrm{T}} \neq I} \qquad \underset{\text{projection}}{QQ^{\mathrm{T}}QQ^{\mathrm{T}} = QQ^{\mathrm{T}}}$$

Orthogonal matrix

- Orthogonal matrices" are square with orthonormal columns
  For square matrices $Q^{\mathrm{T}}Q = I$ leads to $QQ^{\mathrm{T}} = I$
  $Q^{\mathrm{T}} = Q^{-1}$.
- columns of orthogonal *n* by *n* matrix are an orthonormal basis for R^n.

$$\boldsymbol{Q} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$ **is square**. Then $QQ^{\mathrm{T}} = I$ and $Q^{\mathrm{T}} = Q^{-1}$

If Q_1, Q_2 are orthogonal matrices, so are Q_1Q_2 and Q_2Q_1

$||Q\boldsymbol{x}||^2 = \boldsymbol{x}^{\mathrm{T}}Q^{\mathrm{T}}Q\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = ||\boldsymbol{x}||^2$ Length is preserved

Eigenvalues of Q $\quad Q\boldsymbol{x} = \lambda\boldsymbol{x} \quad ||Q\boldsymbol{x}||^2 = |\lambda|^2\,||\boldsymbol{x}||^2 \quad \boxed{|\lambda|^2 = 1}$

- **Haar wavelets are orthogonal vectors (columns of *W*)**

$n = 4$

$$W = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

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- Find $W^T W$ and $W W^T$ and W^{-1}

orthogonalizing

- Every subspace of R^n has an orthogonal basis
- Think of a plane in three-dimensional space R^3
  - plane has two independent vectors *a* and *b*
  - For an orthogonal basis, subtract away from *b* its component in the direction of *a* :

Orthogonal basis $\quad \boldsymbol{a}$ and $\boldsymbol{c}$ $\qquad\qquad \boldsymbol{c} = \boldsymbol{b} - \dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\,\boldsymbol{a}.$

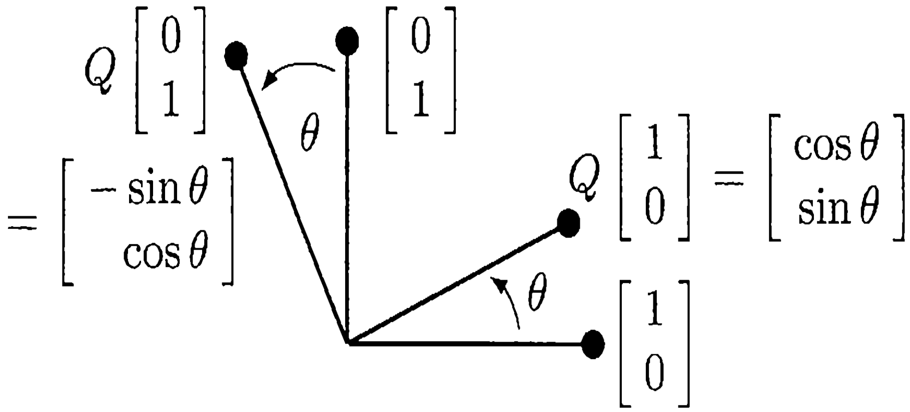
Rotation and Reflection

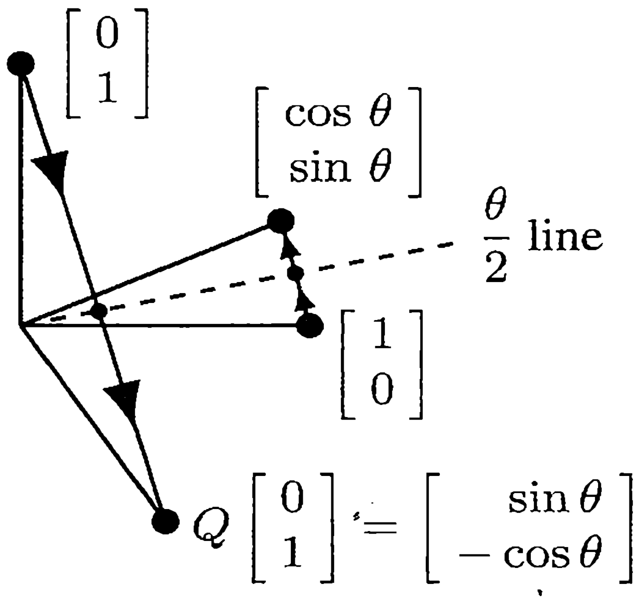
$$Q_{\text{rotate}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \textbf{rotation} \text{ through an angle } \theta.$$

- When the whole plane rotates around (0, 0)
  - lengths don't change
  - Angles between vectors don't change

$$Q_{\text{reflect}} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \textbf{reflection} \text{ across the } \frac{\theta}{2} - \text{ line}.$$

Rotation and Reflection





- multiplying orthogonal matrices produces an orthogonal matrix

$$\boxed{\boldsymbol{Q_1Q_2} \textbf{ is orthogonal} \qquad (Q_1Q_2)^{\mathrm{T}}(Q_1Q_2) = Q_2^{\mathrm{T}}Q_1^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_2 = I.}$$

- Rotation times rotation = rotation- Reflection times reflection = rotation - Rotation times reflection = *reflection*

n by n orthogonal matrix Q has columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$

- unit vectors are a basis for n-dimensional space R^n
- Every vector v can be written as a combination of the basis vectors

  $\boldsymbol{v} = c_1\boldsymbol{q}_1 + \cdots + c_n\boldsymbol{q}_n$
- $c_1 \ldots c_n$ are the projection of v onto a axes

$$\boxed{c_1 = \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{v} \qquad c_2 = \boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{v} \quad \cdots \quad c_n = \boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{v}}$$

- Proof:
  - Take dot products with q1 in equation
  - write equation as a matrix equation $v = Qc$

Key property of every orthogonal matrix

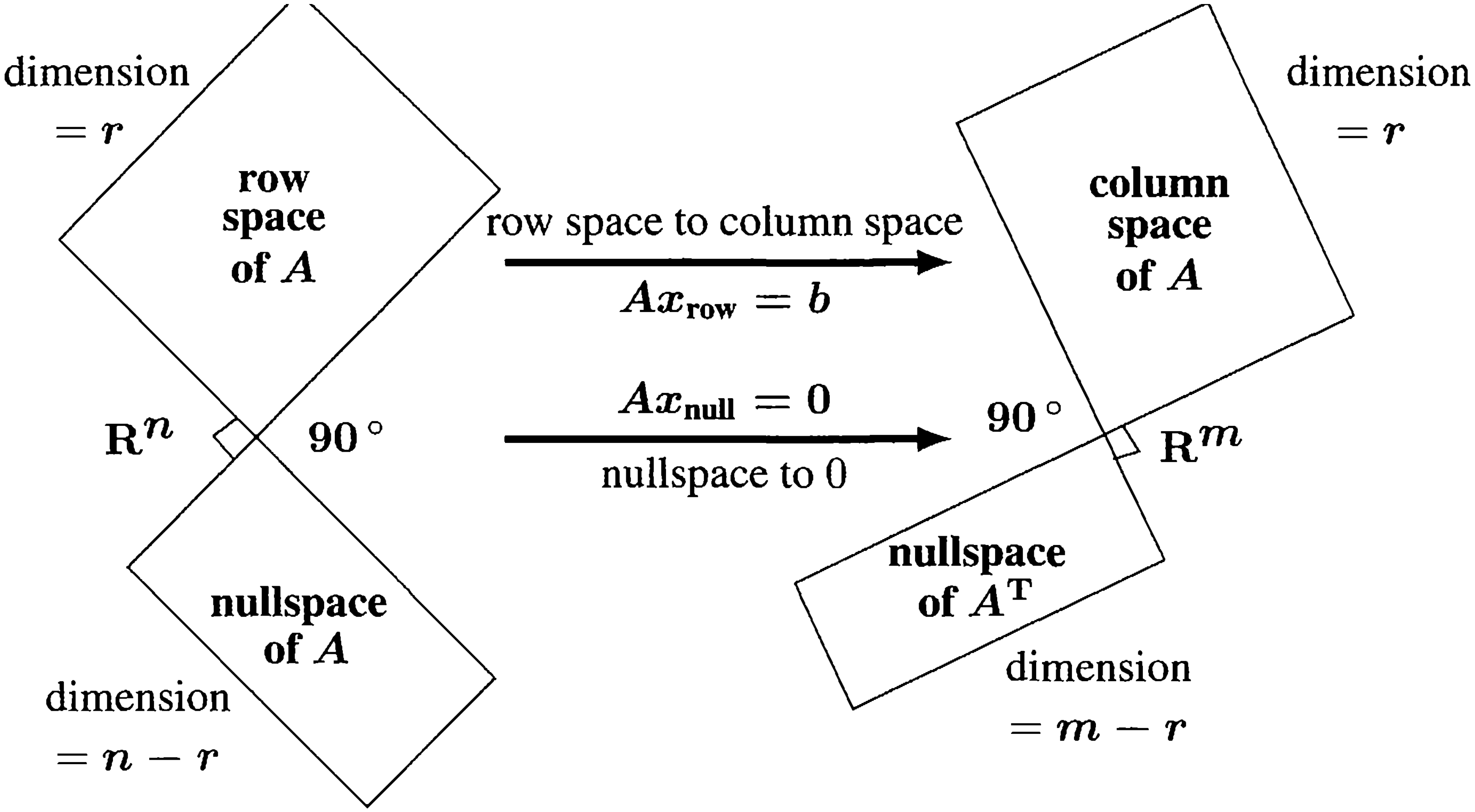
- Show that the lengths and angles are not changed by Q

- $(Qx)^T(Qy) = x^Ty$

$$||Q\boldsymbol{x}||^2 = ||\boldsymbol{x}||^2$$

- **Computations with *Q* never overflow**

Orthogonal subspaces



Tall thin Q with orthonormal columns

- three possible Q's, growing from (3 by 1) to (3 by 2) to an orthogonal matrix Q3

$$Q_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \qquad Q_2 = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad Q_3 = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}.$$

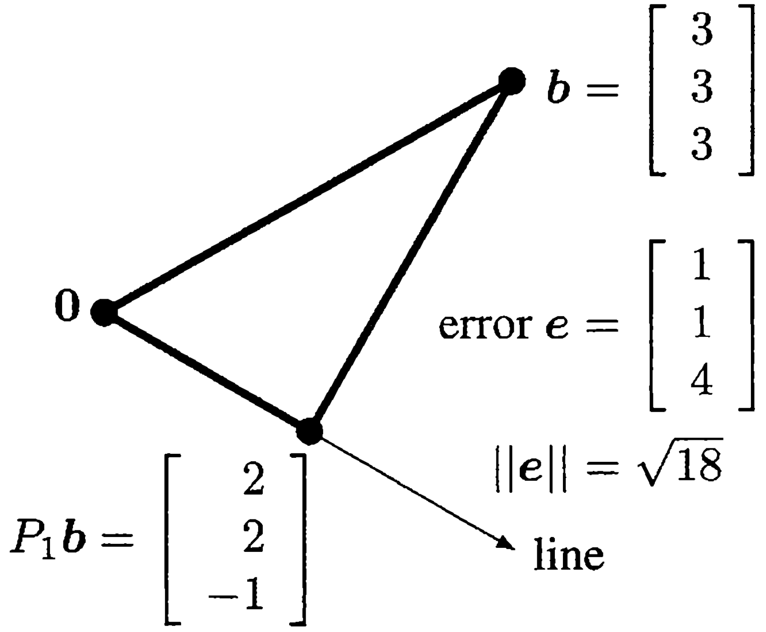
- all the matrices $P = QQ^T$ have $P^2 = P$

If $P^2 = P = P^{\mathrm{T}}$ then Pb is the orthogonal projection of b onto the column space of P.

Projection b on a line

Example 1 To project $b = (3,3,3)$ on the Q_1 line, multiply by $P_1 = Q_1 Q_1^{\mathrm{T}}$.
That matrix splits b in two perpendicular parts: projection $P_1 b$ and error $e = (I - P_1)b$.

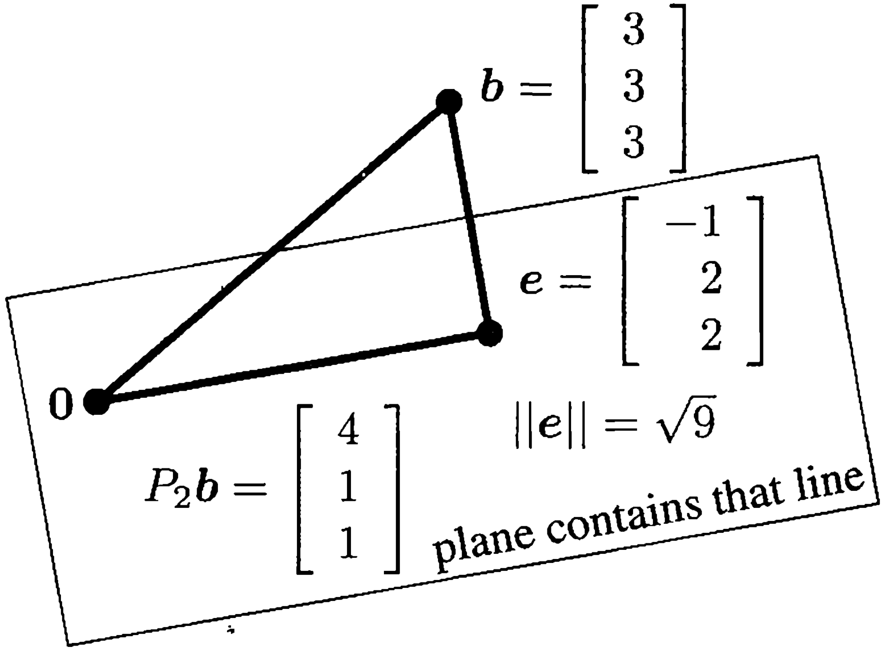
$$P_1 b = \frac{1}{9}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$



Projection b on a plane

$$Q_2 = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$P_2\,\boldsymbol{b} = \frac{1}{9}\begin{bmatrix} 2 & 2 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$



Question

- What is projecting *b* onto the whole space R^3 ?

$Q \Lambda Q^{\mathrm{T}}$

- starts with a symmetric matrix S
- Symmetric matrix S $S^T = S$ All $S_{ij} = S_{ji}$
- Orthogonal square matrix Q $\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{Q}^{-1}$
- Every real symmetric matrix S
  - has n orthonormal eigenvectors q1 to *qn.*
  - When multiplied by S the eigenvectors keep the same direction. They are just rescaled by the number λ

Eigenvector q and eigenvalue λ $S\boldsymbol{q} = \lambda\boldsymbol{q}$

$$SQ = S\begin{bmatrix} & & \\ q_1 & \cdots & q_n \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ \lambda_1 q_1 & \cdots & \lambda_n q_n \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ q_1 & \cdots & q_n \\ & & \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \boldsymbol{Q\Lambda}$$

Multiply $SQ = Q\Lambda$ by $Q^{-1} = Q^{\mathrm{T}}$ to get $\boldsymbol{S = Q\Lambda Q^{\mathrm{T}}}$ = a symmetric matrix

eigenvalue λ_k and each eigenvector q_k contribute a rank one piece $\lambda_k q_k q_k^{\mathrm{T}}$ to S.

Rank one pieces $\quad S = (Q\Lambda)Q^{\mathrm{T}} = (\lambda_1 q_1)q_1^{\mathrm{T}} + (\lambda_2 q_2)q_2^{\mathrm{T}} + \cdots + (\lambda_n q_n)q_n^{\mathrm{T}}$

Question

- Key property of every orthogonal matrix:
  - $||Q\boldsymbol{x}||^2 = ||\boldsymbol{x}||^2$ for every vector $\boldsymbol{x}$.
  - show that $(Q\boldsymbol{x})^T(Q\boldsymbol{y}) = \boldsymbol{x}^T\boldsymbol{y}$ for every vector $\boldsymbol{x}$ and $\boldsymbol{y}$