Computational Data Mining

Fatemeh Mansoori

Course material

- https://github.com/uisf-course/CDM

Multiplication Ax Using rows of A

- Multiply *A* times *x* using the three rows of *A :*

$$\textbf{By rows} \qquad \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix} = \begin{matrix} \text{inner products} \\ \text{of the rows} \\ \text{with } \boldsymbol{x} = (x_1, x_2) \end{matrix}$$

- a row at a time
- are also known as "dot products" because of the dot notation

$$\textbf{row} \cdot \textbf{column} = (\mathbf{2},\ \mathbf{3}) \cdot (\boldsymbol{x}_1,\ \boldsymbol{x}_2) = \mathbf{2}\boldsymbol{x}_1 + \mathbf{3}\boldsymbol{x}_2$$

- use this for computing but not for understanding.

Multiplication Ax Using Columns of A

- For Understanding Ax using vector approach.

$$\textbf{By columns} \quad \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = \begin{array}{c} \text{combination} \\ \text{of the columns} \\ \boldsymbol{a}_1 \text{ and } \boldsymbol{a}_2 \end{array}$$

- vector approach sees Ax as a "linear combination" of a_1 and a_2
- linear combination of of a_1 and a_2 includes two steps :
  - Multiply the columns a_1 and a_2 by "scalars" x_1 and x_2
  - Add vectors x_1 a_1 + x_2 a_2 = *Ax.*
- *Ax* is a linear combination of the columns of *A*
- leads us to the column space of *A.*
  - all combinations of the columns
  - Space created by Ax for all vectors ***X***

Column space of A C(A)

- All combinations $x_1 a_1 + x_2 a_2 = Ax$ produce what part of the full 3D space?
- produce a plane
  - contains the complete line in the direction of $a_1 =$ (2, 2, 3)
  - contains the complete line in the direction of $a_2 =$ (3, 4, 7)
  - *sum* of any vector on one line plus any vector on the other line
  - This addition fills out an infinite plane containing the two lines.
  - does not fill out the whole 3-dimensional space
  - If $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the column space of A ?

Column space of A C(A)

- What are the column spaces of

$$A_2 = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 10 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix} ?$$

What is all possible column spaces inside R^3

- The zero vector (0, 0, 0)
- A line of all vectors x_1a_1
- A plane of all vectors $x_1a_1 + x_2a_2$
- The whole R3 with all vectors $x_1a_1 + x_2a_2 + x_3a_3$

Matrix C

- create a matrix *C* whose columns independence columns of A
- construction of *C* from the *n* columns of *A :*
  - If column 1 of *A* is not all zero, put it into the matrix *C.*
  - If column 2 of *A* is not a multiple of column 1, put it into *C.*
  - If column 3 of *A* is not a combination of columns 1 and 2, put it into *C. Continue.*
- At the end *C* will haver columns *(r* <= *n).*
- They will be a "basis" for the column space of *A.*
- A basis for a subspace is a full set of independent vectors
- All vectors in the space are combinations of the basis vectors

Example

- What is matrix C and R for the matrices blow :

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Rank of the matrix

The rank of a matrix is the dimension of its column space.

- Different basis, but always the same number of vectors
- That number r is the "dimension" of the column space of A and C

Matrix R

- The matrix *C* connects to *A* by a third matrix R: *A* = *CR*
- Their shapes are :
  - (m by *n)* = (m by *r) (r* by *n).*

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \boldsymbol{CR}$$

C multiplies the first column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of R, this produces column 1 of C and A.

C multiplies the second column $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of R, we get column 2 of C and A.

C multiplies the third column $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ of R, we get 2(column 1) + 2(column 2).

Matrix R

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} = CR$$

All three matrices have rank $r = 1$
Column Rank $=$ **Row Rank**

The number of *independent columns* equals the number of *independent rows*

Column rank of A = row rank of A

- 1) The r column of C are independent (by their constraint)
- 2) Every column of A is combination of those r column (because A=CR)
- 3) The r rows of R are independent (They contain the r by r matrix)
- 4) Every row of A is a combination of those r rows (because A=CR)

- Key facts :
  - The r column of C are a basis for column space of A: dimension r
  - The r rows of R are a basis for row space of A : dimension r

Question

- Suppose the column space of an m by n matrix is all of $\mathbf{R}^3$ **.** What can you say about m ? What can you say about n ? What can you say about the rank r ?

- Suppose A is the 3 by 3 matrix ones(3, 3) of all ones. Find two independent vectors x and y that solve $Ax = 0$ and $Ay = 0$. Write that first equation $Ax = \mathbf{0}$ (with numbers) as a combination of the columns of A. Why don't I ask for a third independent vector with $Az = 0$?

- does these vectors have dependence column ?

$$A_2 = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

Counting Theorem

- A has $r = 2$ rank
- Counting Theorem
  - A has r rank
  - There are $n-r$ independent solution to $Ax = 0$

Matrix-Matrix Multiplication *AB*

- 4 **ways** to multiply matrices
- First way :
- **Inner products :**(**Row** i **of** A) · (**Column** j **of** B) produces one number : row i, column j of AB

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{17} & \cdot \\ \cdot & \cdot \end{bmatrix} \text{ because } \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \mathbf{17} \quad \textbf{Dot product}$$

row 2 of A
column 3 of B
give c_{23} in C

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & b_{13} \\ \cdot & \cdot & b_{23} \\ \cdot & \cdot & b_{33} \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathbf{c_{23}} \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\boldsymbol{c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}}\,.$$

Matrix-Matrix Multiplication *AB*

- Second way :

(**Matrix** $\boldsymbol{A}$)(**Column** $\boldsymbol{j}$ **of** $\boldsymbol{B}$) produces column j of AB : **Combine columns of** $\boldsymbol{A}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \mathbf{5} & 7 \\ \mathbf{6} & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{17} & \cdot \\ \mathbf{39} & \cdot \end{bmatrix} \text{ because } \mathbf{5} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \mathbf{6} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \mathbf{17} \\ \mathbf{39} \end{bmatrix}$$

- This is the best way for understanding : **Linear combinations**.

Matrix-Matrix Multiplication *AB*

- third way

(**Row** i **of** $\boldsymbol{A}$) (**Matrix** $\boldsymbol{B}$) produces row i of AB : **Combine rows of** $\boldsymbol{B}$

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{17} & \mathbf{23} \\ \cdot & \cdot \end{bmatrix} \text{ because } \mathbf{1}\begin{bmatrix} 5 & 7 \end{bmatrix} + \mathbf{2}\begin{bmatrix} 6 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{17} & \mathbf{23} \end{bmatrix}$$

Matrix-Matrix Multiplication *AB*

- 4^{th} way :
- (**Column** k **of** A) (**Row** k **of** B) produces a simple matrix : Add these simple matrices !

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 15 & 21 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 4 \end{bmatrix}\begin{bmatrix} 6 & 8 \end{bmatrix} = \begin{bmatrix} 12 & 16 \\ 24 & 32 \end{bmatrix} \quad \begin{matrix}\textbf{NOW} \\ \textbf{ADD}\end{matrix} \quad \begin{bmatrix} \mathbf{17} & \mathbf{23} \\ \mathbf{39} & \mathbf{53} \end{bmatrix} = \boldsymbol{AB}$$

Matrix-Matrix Multiplication *AB*

- Outer product :
  - *one column u* times *one row v* ^{T}

$$\text{“Outer product”} \quad uv^{\mathrm{T}} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 12 \\ 6 & 8 & 12 \\ 3 & 4 & 6 \end{bmatrix} = \text{“rank one matrix”}$$

- m by 1 matrix (a column *u)* times a 1 by *p* matrix (a row *v* ^{T}) gives an m by *p* matrix

what is special about the rank one matrix *uv* ^{T} :

AB multiplication

- All column of *uv* $^{\mathrm{T}}$ are multiply of $\boldsymbol{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$
- All rows are multiply of $\boldsymbol{v}^{\mathrm{T}} = \begin{bmatrix} 3 & 4 & 6 \end{bmatrix}$
- column space of *uv* $^{\mathrm{T}}$ is one-dimensional: *the line in the direction of u*
- All nonzero matrices *uv* $^{\mathrm{T}}$ have rank one

AB multiplication

AB = Sum of Rank One Matrices

$$AB = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & b_1^* & — \\ & \vdots & \\ — & b_n^* & — \end{bmatrix} = \underbrace{a_1 b_1^* + a_2 b_2^* + \cdots + a_n b_n^*}_{\textbf{sum of rank 1 matrices}}.$$

Rank 1 matrices are the building blocks of all matrices

Every rank r matrix is the sum of r rank one matrices.

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix} =$$

Block multiplication

$$\boxed{\begin{array}{c}\textbf{Block multiplication}\\ \text{Block sizes must fit}\end{array}} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} AE + BF \\ CE + DF \end{bmatrix}$$

Dot product

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \boldsymbol{ac + bd}$$

- v.w = w.v
- $v \cdot w = ||v||\ ||w|| \cos \theta$
- $v \cdot w = 0$: **Perpendicular**
- $||v + w|| \leq ||v|| + ||w||$
- $||v||^2 = v \cdot v$