Computational Data mining

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Matrix Factorization

$$A = \boldsymbol{CR} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}\begin{bmatrix} & & \\ & & \end{bmatrix}$$ **Independent columns in** $\boldsymbol{C}$

$$A = \boldsymbol{LU} = \begin{bmatrix} \ddots & 0 \\ & \ddots \end{bmatrix}\begin{bmatrix} \ddots & \\ 0 & \ddots \end{bmatrix}$$ **Triangular matrices** $\boldsymbol{L}$ **and** $\boldsymbol{U}$

$$A = \boldsymbol{QR} = \begin{bmatrix} q_1 & q_n \end{bmatrix}\begin{bmatrix} \ddots & \\ 0 & \ddots \end{bmatrix}$$ **Orthogonal columns in** $\boldsymbol{Q}$

$$S = \boldsymbol{Q\Lambda Q^{\mathrm{T}}} \quad Q^{\mathrm{T}} = Q^{-1}$$ **Orthogonal eigenvectors** $\boldsymbol{Sq = \lambda q}$

$A = \boldsymbol{X\Lambda X^{-1}}$ Eigenvalues in Λ Eigenvectors in X $\boldsymbol{Ax = \lambda x}$

$A = \boldsymbol{U\Sigma V^{\mathrm{T}}}$ Diagonal $\Sigma =$ Singular values $\sigma = \sqrt{\boldsymbol{\lambda(A^{\mathrm{T}}A)}}$

Orthogonal vectors in $U^{\mathrm{T}}U = V^{\mathrm{T}}V = I$ $\boldsymbol{Av = \sigma u}$

Matrix Factorization

- five important factorizations

$$\boxed{A = LU \qquad A = QR \qquad S = Q\Lambda Q^{\mathrm{T}} \qquad A = X\Lambda X^{-1} \qquad A = U\Sigma V^{\mathrm{T}}}$$

- $A = LU$ comes from elimination
  - The matrix L is lower triangular
  - U is upper triangular as in equation
- $A = QR$ comes from
  - Orthogonalzing the columns a_1 to a_n
  - Q has orthonormal columns ($Q^T Q = I$)
  - R is upper triangular.

Matrix Factorization

- $S = Q\Lambda Q^{\mathrm{T}}$
  - comes from: **eigenvalues** $\lambda_1, \ldots, \lambda_n$ of a symmetric matrix $S = S^{\mathrm{T}}$
  - Eigenvalues on the diagonal of Λ
  - **Orthonormal eigenvectors** in the columns of Q.
- $A = X\Lambda X^{-1}$
  - Is diagonalization when A is n by n with n independent eigenvectors
  - Eigenvalues of A on the diagonal of Λ
  - Eigenvectors of A in the columns of X

Matrix Factorization

- $A = U\Sigma V^{\mathrm{T}}$
  - Is singular value decomposition of any matrix A
  - **Singular values** $\sigma_1, \ldots, \sigma_r$ in Σ.
  - Orthonormal singular values in U and V

LU decomposition

$$\textbf{Step 1} \begin{bmatrix} \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{x} \\ \mathbf{0} & x & x & x \\ \mathbf{0} & x & x & x \\ \mathbf{0} & x & x & x \end{bmatrix}. \qquad \textbf{Step 2} \begin{bmatrix} \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{x} \\ \mathbf{0} & \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{x} \\ \mathbf{0} & \mathbf{0} & x & x \\ \mathbf{0} & \mathbf{0} & x & x \end{bmatrix}. \qquad \textbf{Final } \boldsymbol{U} = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{x} \\ & \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{x} \\ & & \boldsymbol{x} & \boldsymbol{x} \\ & & & \boldsymbol{x} \end{bmatrix}.$$

- Row 1 is the first pivot row-it doesn't change
- multiplied that row by numbers $\ell_{21}, \ell_{31}, \ell_{41}$ and subtracted from rows 2, 3, 4 of *A*
- $$\textbf{Multipliers} \qquad \boldsymbol{\ell_{21}} = \frac{\boldsymbol{a_{21}}}{\boldsymbol{a_{11}}} \qquad \boldsymbol{\ell_{31}} = \frac{\boldsymbol{a_{31}}}{\boldsymbol{a_{11}}} \qquad \boldsymbol{\ell_{41}} = \frac{\boldsymbol{a_{41}}}{\boldsymbol{a_{11}}}$$

LU decomposition

Key idea : Step 1
removes $\boldsymbol{\ell_1 u_1^*}$

$$A = \begin{bmatrix} 1 \text{ times row } 1 \\ \ell_{21} \text{ times row } 1 \\ \ell_{31} \text{ times row } 1 \\ \ell_{41} \text{ times row } 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{\boldsymbol{A_2}} & \\ 0 & & & \end{bmatrix}.$$

3 by 3 example
Remove rank 1 matrix
Column / row to zero

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 2 & 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{3} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & \boldsymbol{A_2} & \end{bmatrix}$$

Solution to $Ax = b$

- direct way is to include bas an additional column
- work with the matrix [A b]

$$\textbf{Start from} \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} LU & \boldsymbol{b} \end{bmatrix} \quad \textbf{Elimination produces} \begin{bmatrix} U & L^{-1}\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} U & \boldsymbol{c} \end{bmatrix}$$

- steps from A to U (upper triangular) will change the right side b to c
- Elimination on $Ax = b$ produces the equations $Ux = c$ that are ready for back substitution

- Notice elimination steps required nonzero pivots
- first pivot is *a11*
- second pivot is in the comer of A2
- nth pivot is in the 1 by 1 matrix *An*
- What do we do if a11 = 0 ?
  - If there is a nonzero number lower down in column 1, its row can be the pivot row

Row Exchanges (Permutations)

Every invertible n by n matrix A leads to $PA = LU$: P = permutation.

The inverse of every permutation matrix P is its transpose P^{T}

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Question

Factor these matrices into $A = LU$:

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$