Computational Data Mining

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- The eigenvectors of *A* don't change direction when you multiply them by *A*
- *Ax* is on the same line as the input vector *x*

$$\boxed{\begin{aligned} \boldsymbol{x} &= \textbf{eigenvector of } \boldsymbol{A} \\ \boldsymbol{\lambda} &= \textbf{eigenvalue of } \boldsymbol{A} \end{aligned} \qquad\qquad \boldsymbol{Ax = \lambda x}}$$

$$A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}.$$

how eigenvalues and eigenvectors are useful

$A^2 x = \lambda^2 x.$

$A^k x = \lambda^k x$ for all $k = 1, 2, 3, \ldots$ And $A^{-1} x = \frac{1}{\lambda} x$ provided $\lambda \neq 0$.

- Most n by n matrices have n independent eigenvectors X_1 to X_n with n different eigenvalues AI to λ_1 to λ_n
- every n-dimensional vector v will be a combination of the eigenvectors

Every v	$v = c_1 x_1 + \cdots + c_n x_n$
Multiply by A	$Av = c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n$
Multiply by A^k	$A^k v = c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$

how eigenvalues and eigenvectors are useful

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Multiply by A	$Av = c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n$
Multiply by A^k	$A^k v = c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$

If $|\lambda_1| > 1$ then the component $c_1 \lambda_1^n x_1$ will grow as n increases

If $|\lambda_2| < 1$ then that component $c_2 \lambda_2^n x_2$ will steadily disappear

The powers of Q don't grow or decay

Eigenvalues and eigenvectors properties

Example $S = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$ has eigenvectors $S \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $S \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(**Trace of $\boldsymbol{S}$**) The sum $\lambda_1 + \lambda_2 = 3 + 1$ equals the diagonal sum $2 + 2 = 4$

(**Determinant**) The product $\lambda_1 \lambda_2 = (3)(1) = 3$ equals the determinant $4 - 1$

(**Real eigenvalues**) Symmetric matrices $S = S^{\mathrm{T}}$ always have real eigenvalues

(**Orthogonal eigenvectors**) If $\lambda_1 \neq \lambda_2$ then $\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = 0$.

Here $(1, 1) \cdot (1, -1) = 0$

Computing the Eigenvalues (by hand)

$A\boldsymbol{x} = \lambda\boldsymbol{x}$ is the same as $(A - \lambda I)\boldsymbol{x} = 0$

$A - \lambda I$ is not invertible:

determinant of $\boldsymbol{A - \lambda I}$ must be zero

equation $\det(A - \lambda I) = 0$ has n roots

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two eigenvalues

Question

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$

If A is shifted to $A + sI$, what happens to the $\boldsymbol{x}$'s and λ's ?

Similar Matrices

- For every invertible matrix B :
  - eigenvalues of BAB^{-1} are the same as the eigenvalues of A

$$\boxed{\text{If } A\boldsymbol{x} = \lambda\boldsymbol{x} \text{ then } (BAB^{-1})(B\boldsymbol{x}) = BA\boldsymbol{x} = B\lambda\boldsymbol{x} = \lambda(B\boldsymbol{x}).}$$

The matrices BAB^{-1} (for every invertible B) are **"similar"** to A: **same eigenvalues.**

Question

- Show that the eigenvalue of BAB^{-1} is equal to the eigenvalue of A

- Show that the eigenvalue of AB equals the eigenvalue of BA

The eigenvalues of any triangular matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ **are** $\lambda_1 = a$ **and** $\lambda_2 = d$.

Diagonalizing a Matrix

- *A* has a full set of *n* independent eigenvectors
- Put those eigenvectors x_1 , ... , x_n into an invertible matrix X
- Multiply AX column by column

$$A\begin{bmatrix} & & \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ A\boldsymbol{x}_1 & \cdots & A\boldsymbol{x}_n \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ & & \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

$$\boxed{\begin{array}{llll} \boldsymbol{\Lambda} & = \text{diagonal eigenvalue matrix} & A & = X\Lambda X^{-1} \\ \boldsymbol{X} & = \text{invertible eigenvector matrix} & A^2 & = (X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda^2 X^{-1} \end{array}}$$

$$A = \begin{bmatrix} \mathbf{0.8} & \mathbf{0.3} \\ \mathbf{0.2} & \mathbf{0.7} \end{bmatrix} \qquad \lambda_1 = 1 \text{ and } \lambda_2 = \frac{1}{2}$$

- *A* is a Markov matrix, with columns adding to 1

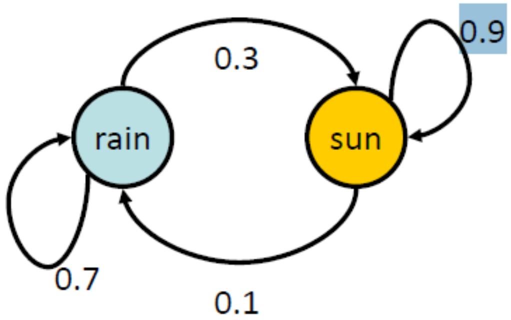
$$A^k \boldsymbol{v} = c_1(1)^k \boldsymbol{x}_1 + c_2\left(\tfrac{1}{2}\right)^k \boldsymbol{x}_2$$

As k increases, $A^k v$ approaches $c_1 x_1$ = steady state

- We can follow each eigenvector separately. Its growth or decay depends on the eigenvalue

- CPT $P(X_t \mid X_{t-1})$:

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7



- What is the probability distribution after one step?

$$P(X_2 = sun) = \sum_{x_1} P(x_1, X_2 = sun) = \sum_{x_1} P(X_2 = sun \mid x_1) P(x_1)$$

From initial observation of sun

$$\begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix}_{P(X_1)} \quad \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}_{P(X_2)} \quad \begin{pmatrix} 0.84 \\ 0.16 \end{pmatrix}_{P(X_3)} \quad \begin{pmatrix} 0.804 \\ 0.196 \end{pmatrix}_{P(X_4)} \Rightarrow \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}_{P(X_\infty)}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$

$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$

Forward simulation

From initial observation of rain

$$\begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix}_{P(X_1)} \quad \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}_{P(X_2)} \quad \begin{pmatrix} 0.48 \\ 0.52 \end{pmatrix}_{P(X_3)} \quad \begin{pmatrix} 0.588 \\ 0.412 \end{pmatrix}_{P(X_4)} \Rightarrow \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}_{P(X_\infty)}$$

From yet another initial distribution $P(X_1)$:

$$\begin{pmatrix} p \\ 1-p \end{pmatrix}_{P(X_1)} \quad \cdots \quad \Rightarrow \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}_{P(X_\infty)}$$

Factor these two matrices into $A = X\Lambda X^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

If $A = X\Lambda X^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.