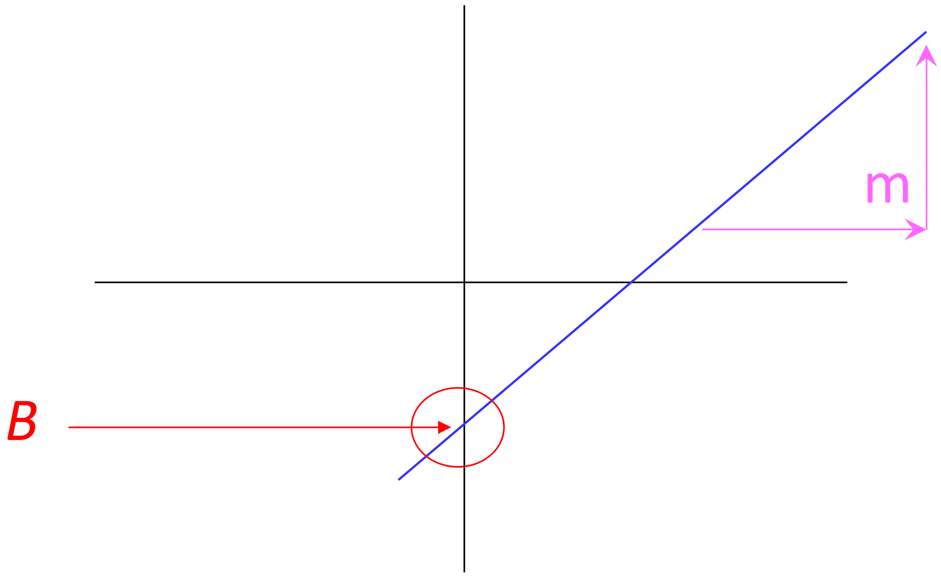
Linear Regression

What is linear ?

- Remember this:
  - *Y=mX+B?*

A slope of 2 means that every 1-unit change in X yields a 2-unit change in Y.



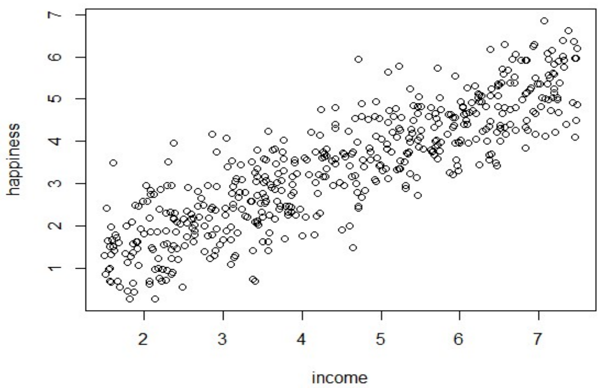
Linear regression example

- Housing prices prediction

$$h_\theta(x) = \theta_0 + \theta_1 \times frontage + \theta_2 \times depth$$

- Estimating body weight by person height

- Estimate the happiness score of a person using its income



Formulation

- We generally formulate the linear regression model in matrix form:
  - Y=Xw+ε
- the target value yi can be evaluated by
  - $y_i = \theta_0 + \theta_1 x_{i1} + \cdots + \theta_n x_{in} + \varepsilon_i$
  - Y represents a vector of length n containing the observed values Y=$(y_1, \dots, y_m)^T$
  - ε is a vector for errors $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)^T$
  - X is a matrix of the features in which the column of ones incorporate the intercept

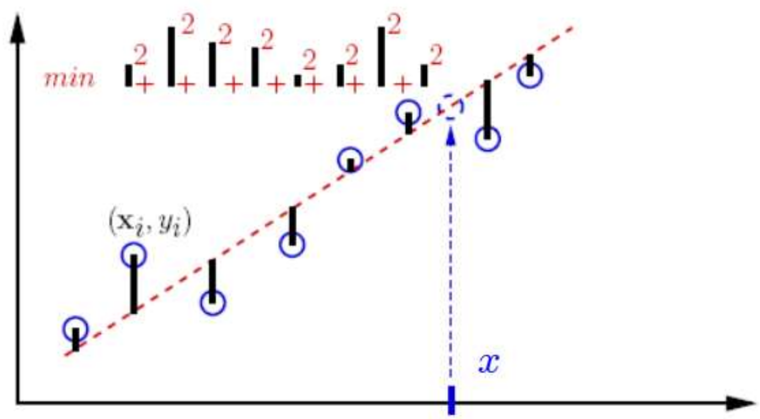
$$X = \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1n} \\ 1 & x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}$$

- Hypothesis:

$$y = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_d x_d = \sum_{j=0}^{d} \theta_j x_j$$

Assume $x_0 = 1$

- Fit model by minimizing sum of squared errors



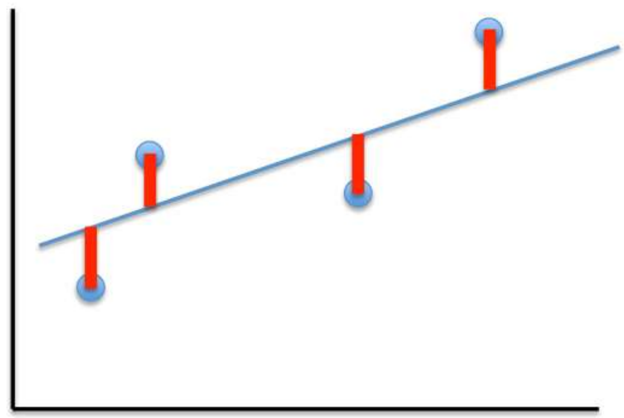
east squares (LSQ)
The fitted line is used as a predictor

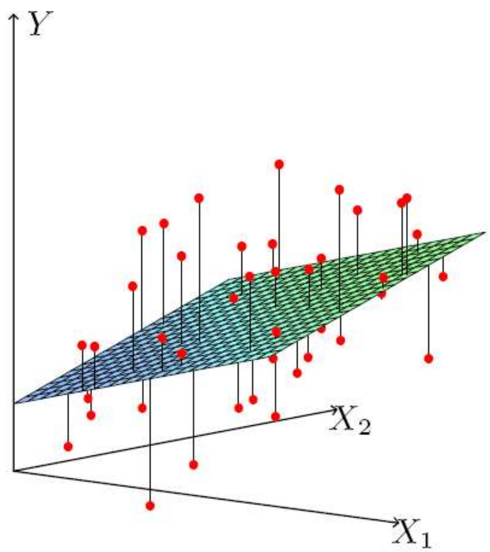
Least square linear regression

- Cost Function

$$J(\boldsymbol{\theta}) = \frac{1}{2n} \sum_{i=1}^{n} \left(h_{\boldsymbol{\theta}} \left(\boldsymbol{x}^{(i)} \right) - y^{(i)} \right)^2$$

- Fit by solving $\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$



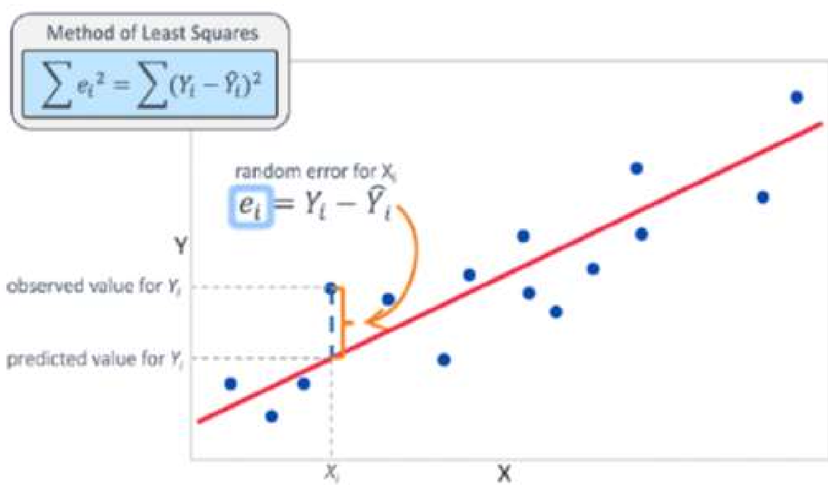


Optimization Problem

One suitable estimator of β should be the one minimizing the sum of the squared errors $\|\epsilon\|_2^2 = \sum_{i=1}^{m} \epsilon_i^2 = \epsilon^T \epsilon.$

- $\sum_{i=1}^{m} \varepsilon_i^2 = \varepsilon^T \varepsilon \quad = (Y - X\theta)^T (Y - X\theta)$

$$= Y^T Y - 2\theta^T X^T Y + \theta^T X^T X \theta$$



Vector derivative

$f(\mathbf{x})$	$\rightarrow$	$\frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}}$
$\mathbf{x}^T\mathbf{B}$	$\rightarrow$	$\mathbf{B}$
$\mathbf{x}^T\mathbf{b}$	$\rightarrow$	$\mathbf{b}$
$\mathbf{x}^T\mathbf{x}$	$\rightarrow$	$2\mathbf{x}$
$\mathbf{B}\mathbf{x}$	$\rightarrow$	$\mathbf{B}^T$

- Differentiating this term and setting it to zero, we find that the estimate for θ, which minimizes the squared error, satisfies the equation :

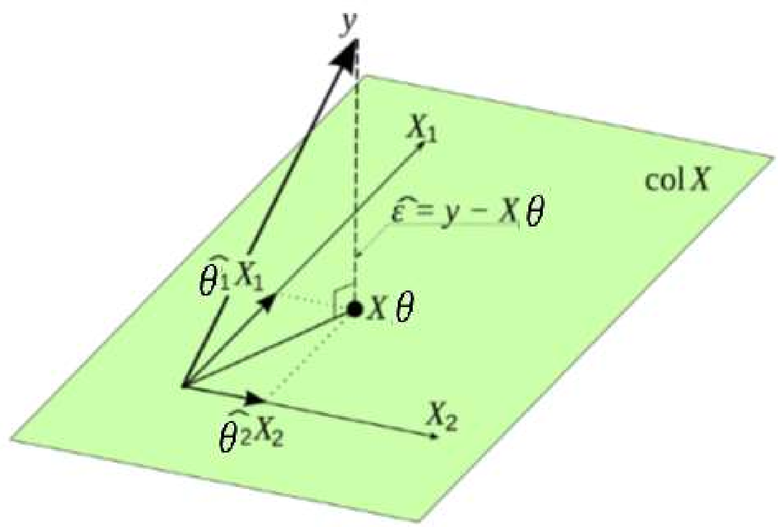
$$X^T X\theta = X^T Y$$

- Provided $X^T X$ is invertible :

$$\theta = (X^T X)^{-1} X^T Y$$

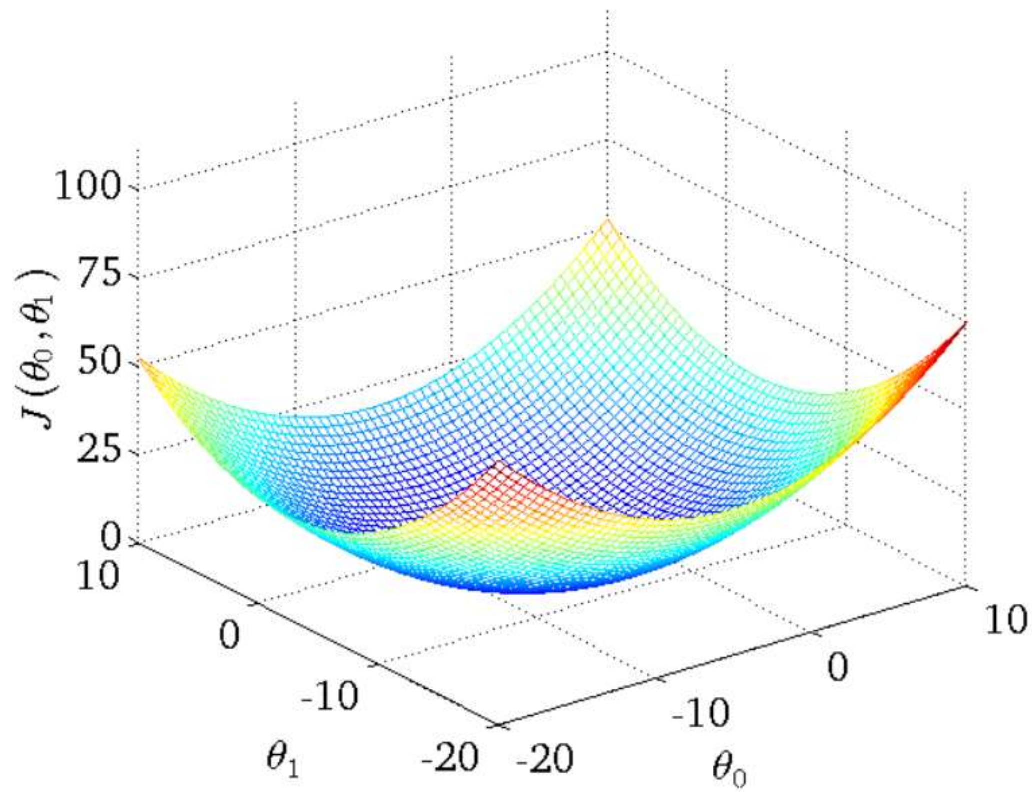
Geometric Approach

Another way to looking at this problem is to say we want a solution that lies in the space spanned by X become as close as possible to Y.
In this way, the systematic component $X\theta$ is projection of Y onto space spanned by X and residuals are Y- $X\theta$



Intuition Behind Cost Function

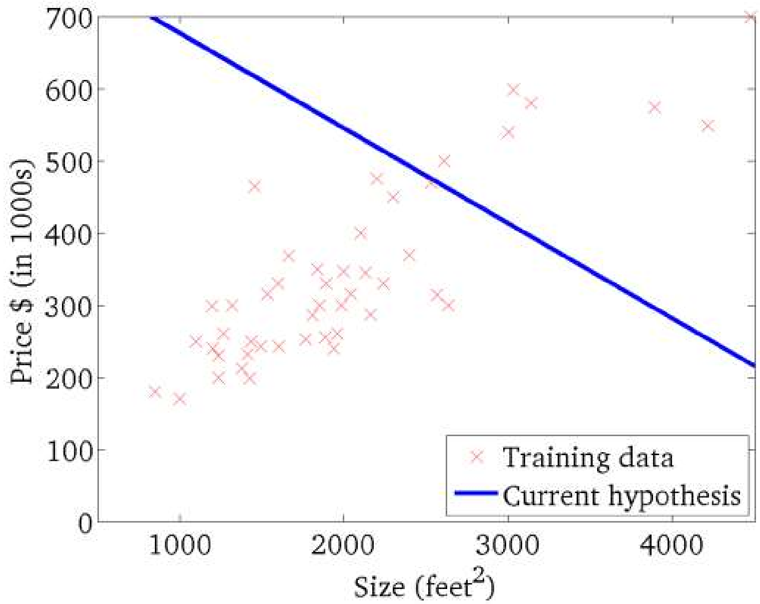
$$J(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\left(h_{\boldsymbol{\theta}}\left(\boldsymbol{x}^{(i)}\right) - y^{(i)}\right)^2$$



Intuition Behind Cost Function

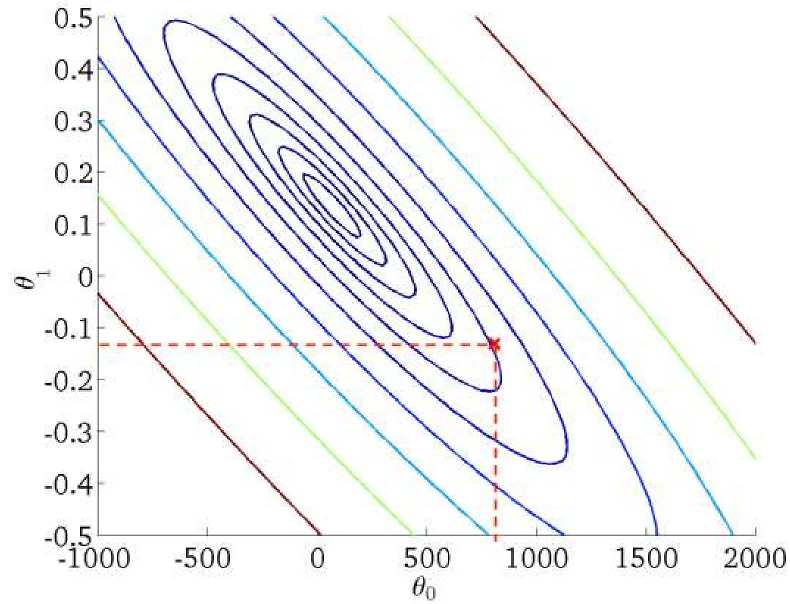
$h_\theta(x)$

(for fixed θ_0, θ_1, this is a function of x)



$J(\theta_0, \theta_1)$

(function of the parameters θ_0, θ_1)



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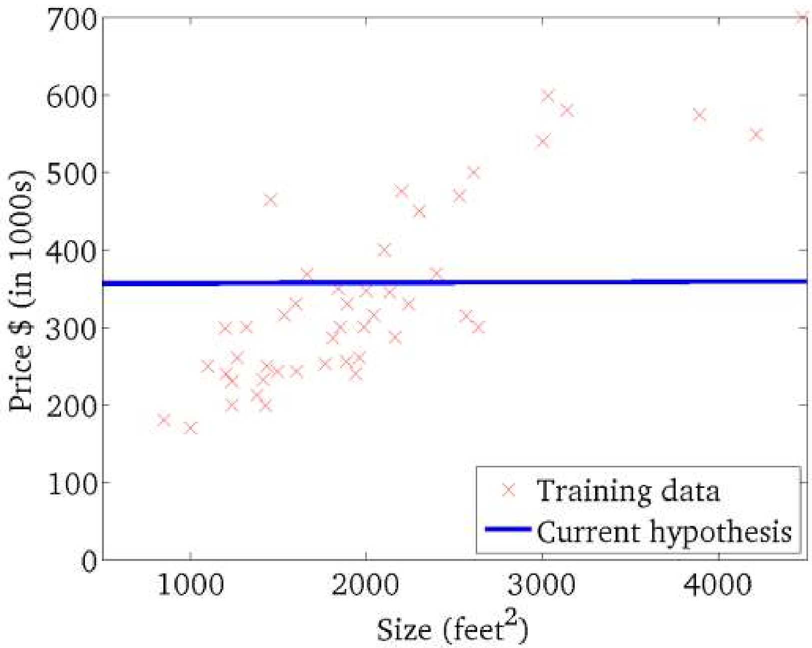

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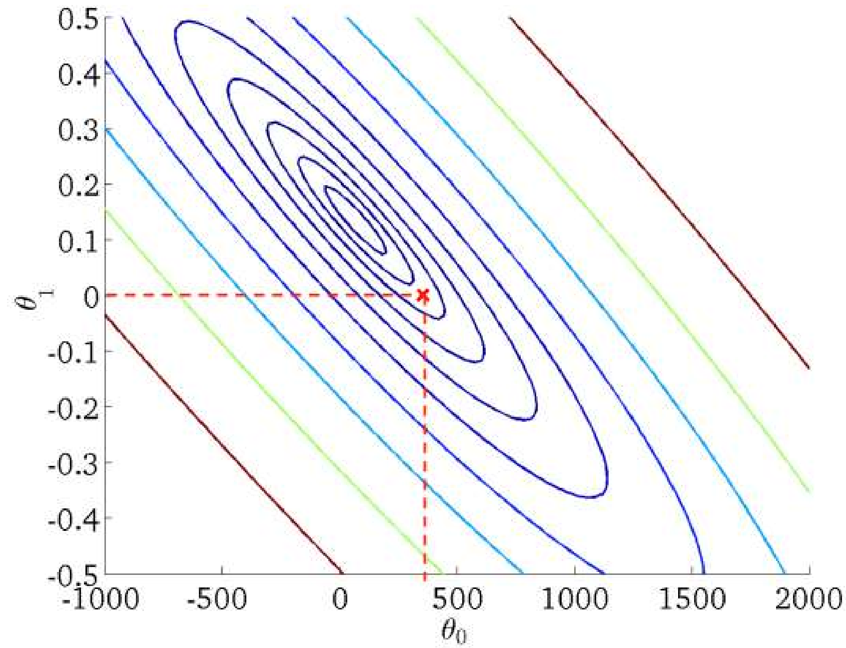

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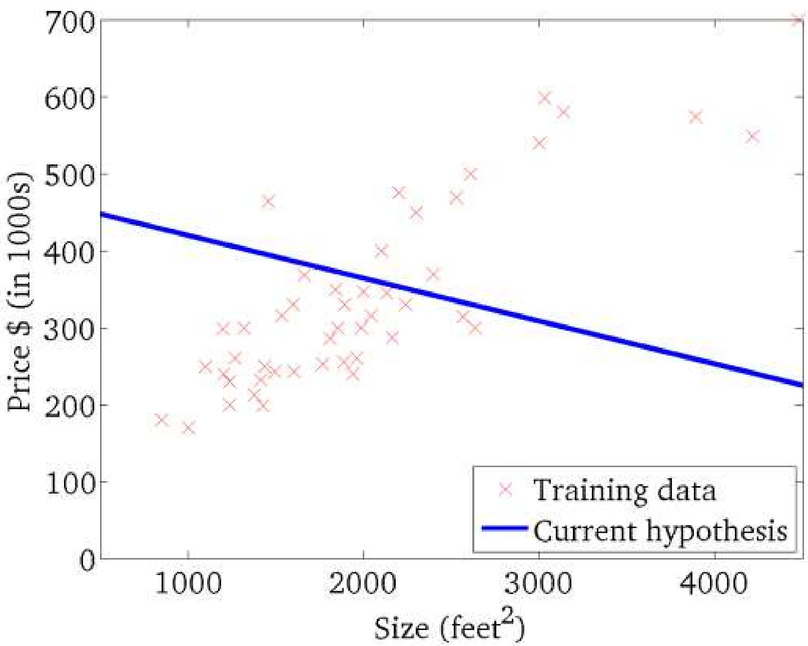

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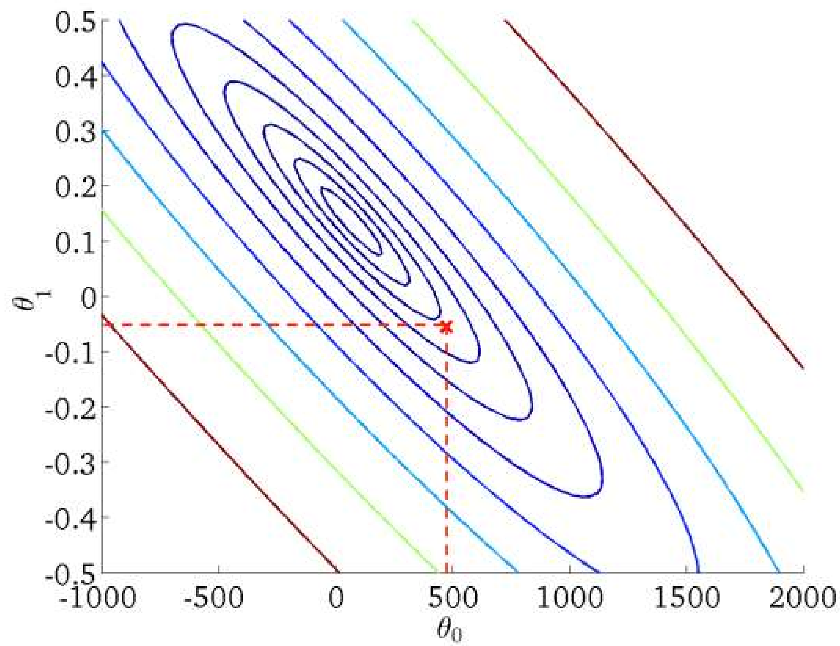

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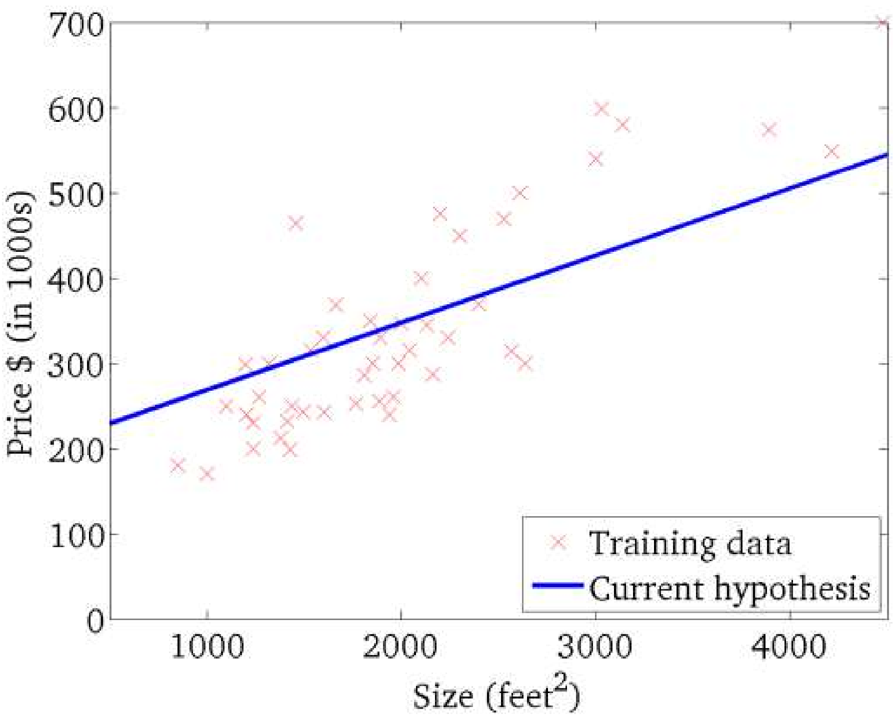

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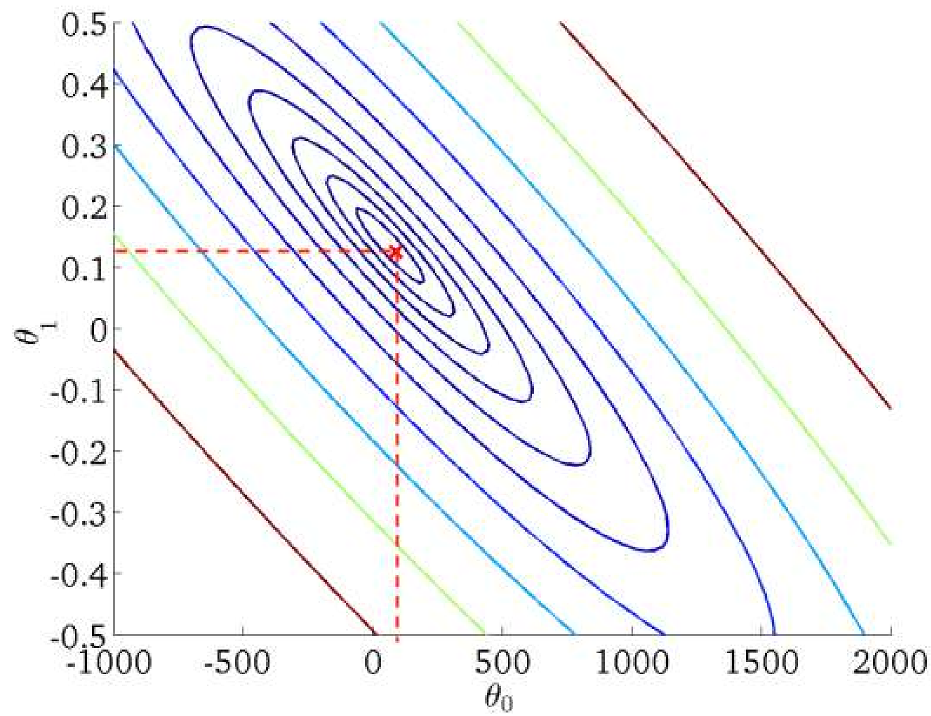
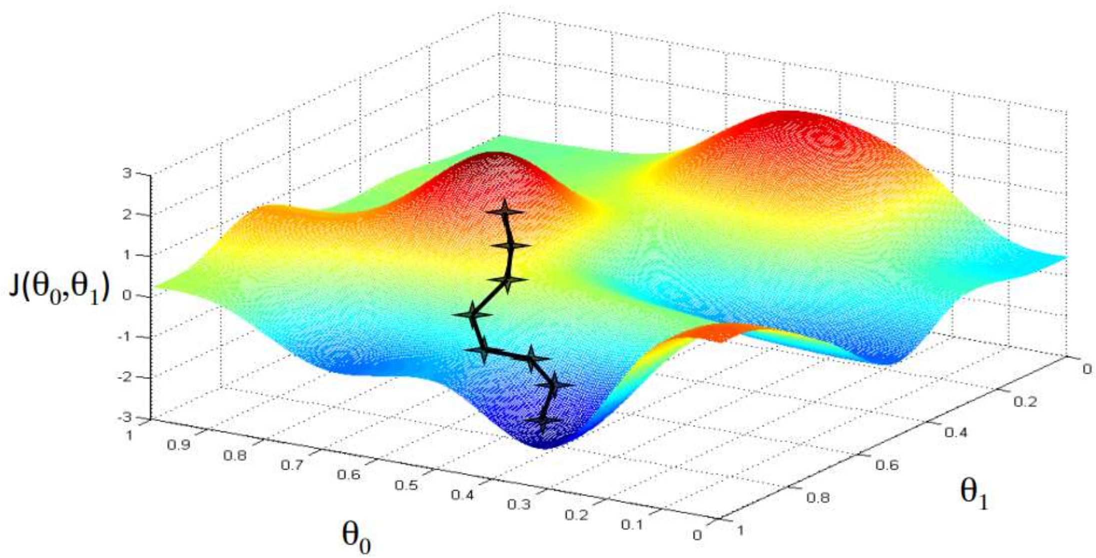


Basic search Procedure

- Choose initial value for $\boldsymbol{\theta}$
- Until we reach a minimum:
  - Choose a new value for $\boldsymbol{\theta}$ to reduce $J(\boldsymbol{\theta})$



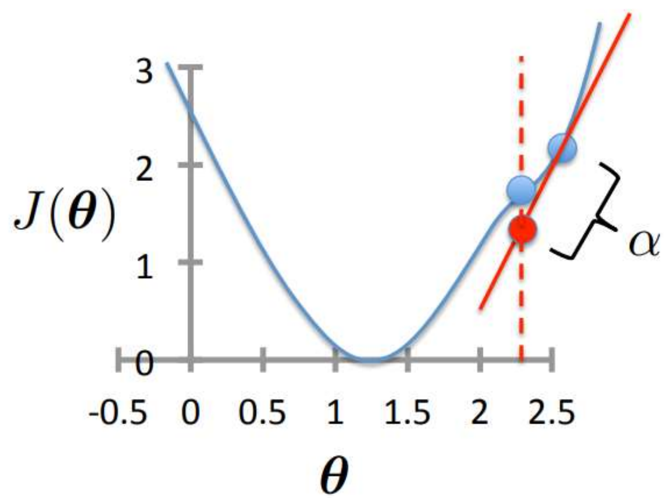
Gradient Descent

- Initialize $\boldsymbol{\theta}$
- Repeat until convergence

$$\theta_j \leftarrow \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\boldsymbol{\theta})$$

simultaneous update
for j = 0 ... d

learning rate (small)
e.g., α = 0.05



- Initialize $\boldsymbol{\theta}$
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$$\theta_j \leftarrow \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\boldsymbol{\theta})$$

simultaneous update for j = 0 ... d

For Linear Regression:

$$\begin{aligned}
\frac{\partial}{\partial \theta_j} J(\boldsymbol{\theta}) &= \frac{\partial}{\partial \theta_j} \frac{1}{2n} \sum_{i=1}^{n} \left(h_{\boldsymbol{\theta}} \left(\boldsymbol{x}^{(i)} \right) - y^{(i)} \right)^2 \\
&= \frac{\partial}{\partial \theta_j} \frac{1}{2n} \sum_{i=1}^{n} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right)^2 \\
&= \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right) \times \frac{\partial}{\partial \theta_j} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right) \\
&= \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right) x_j^{(i)}
\end{aligned}$$

- Initialize $\boldsymbol{\theta}$
- Repeat until convergence

$$\theta_j \leftarrow \theta_j - \alpha \frac{1}{n} \sum_{i=1}^{n} \left(h_{\boldsymbol{\theta}} \left(\boldsymbol{x}^{(i)} \right) - y^{(i)} \right) x_j^{(i)}$$

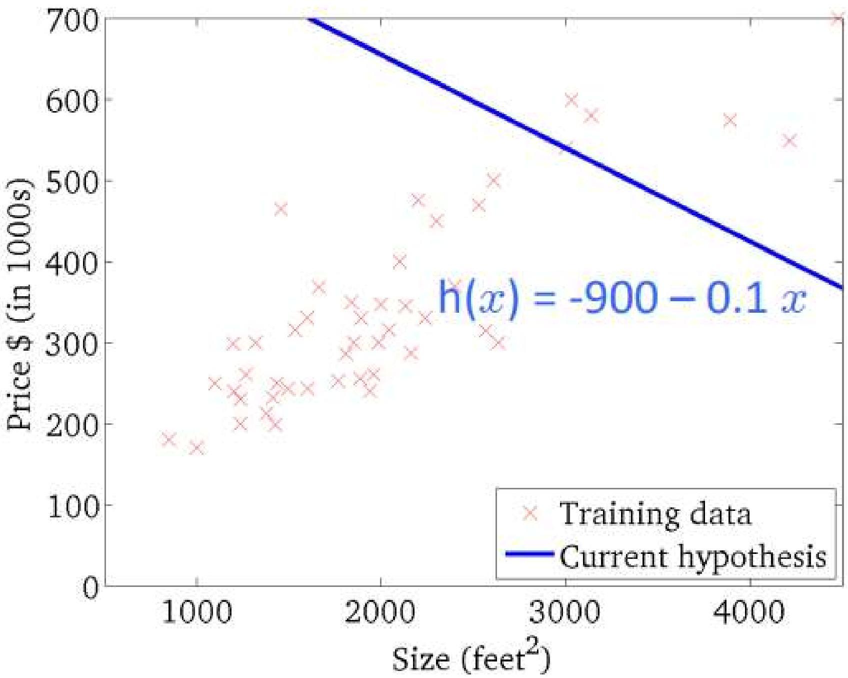
simultaneous update for j = 0 ... d

Assume convergence when $\|\boldsymbol{\theta}_{new} - \boldsymbol{\theta}_{old}\|_2 < \epsilon$

Gradient Descent

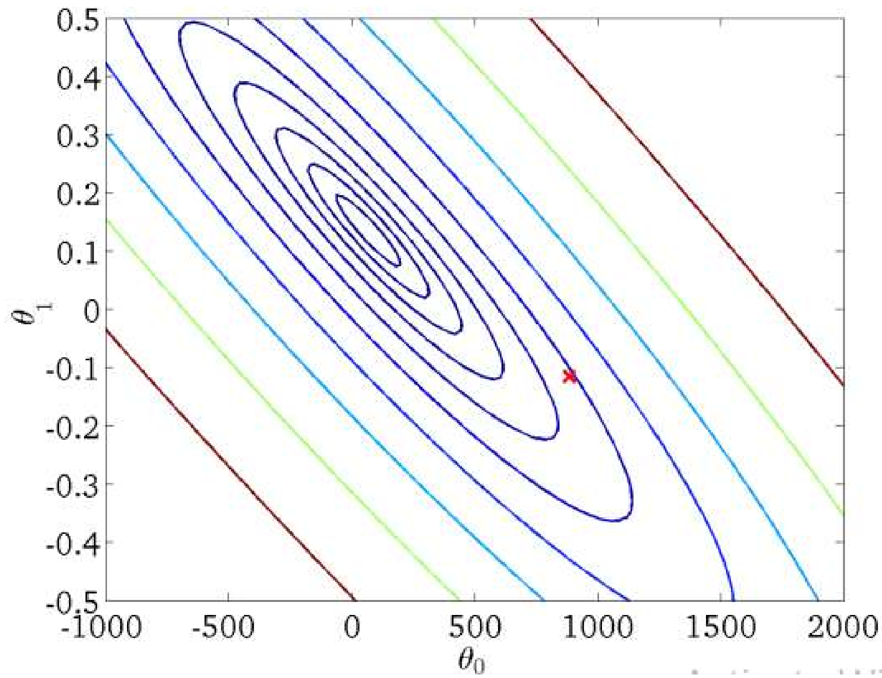
$$h_\theta(x)$$

(for fixed θ_0, θ_1, this is a function of x)



$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



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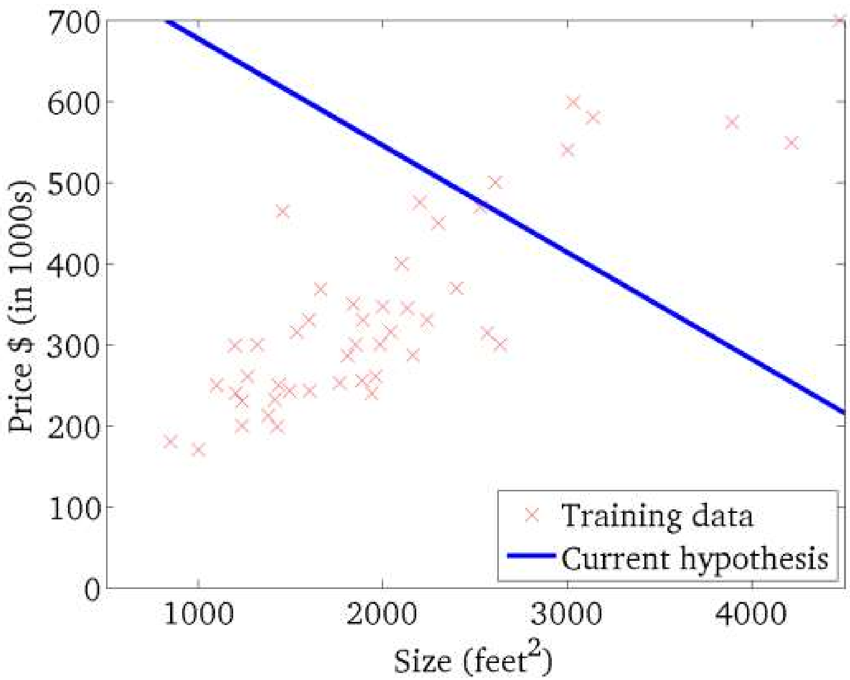

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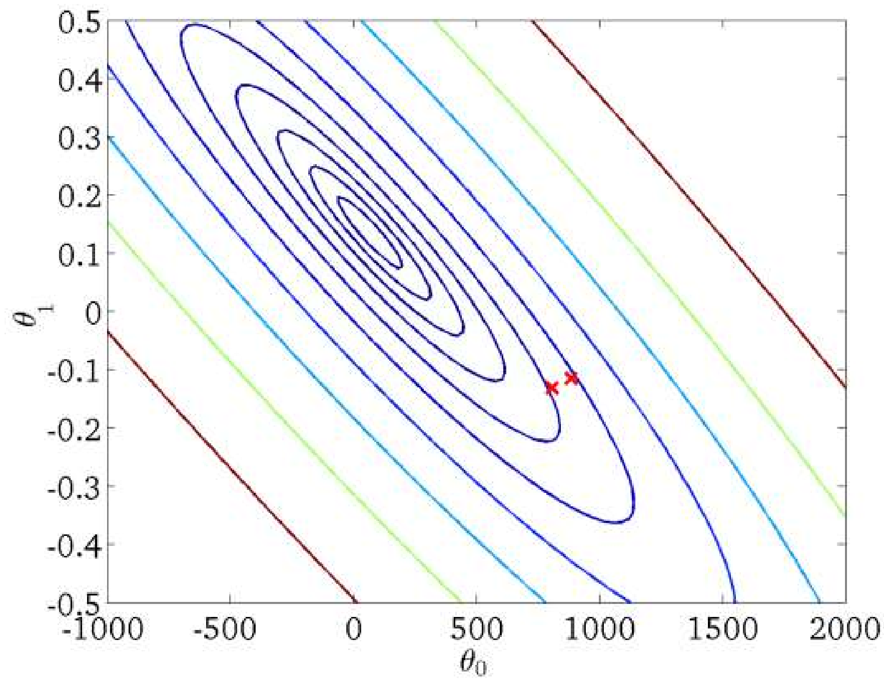

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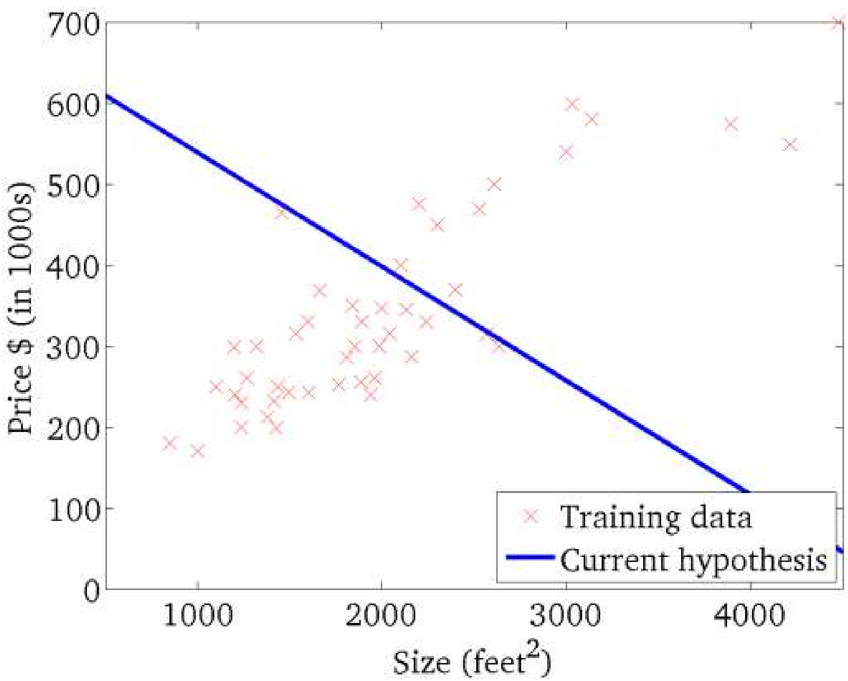

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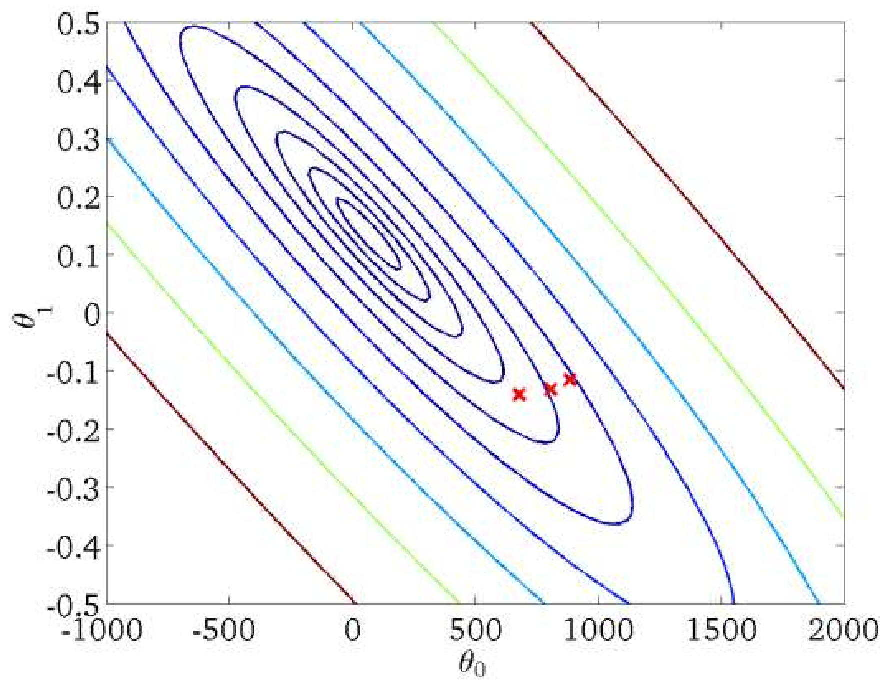

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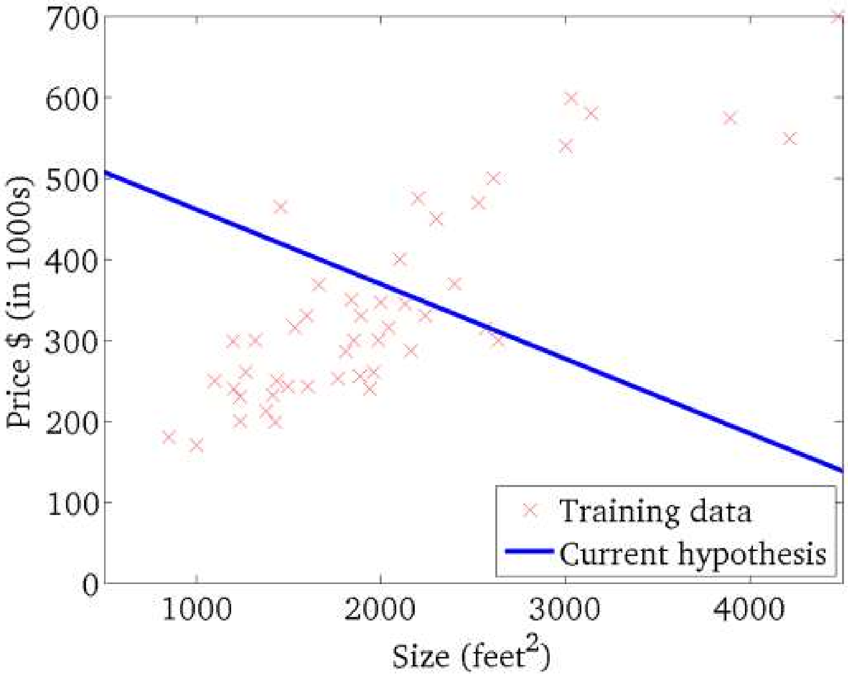

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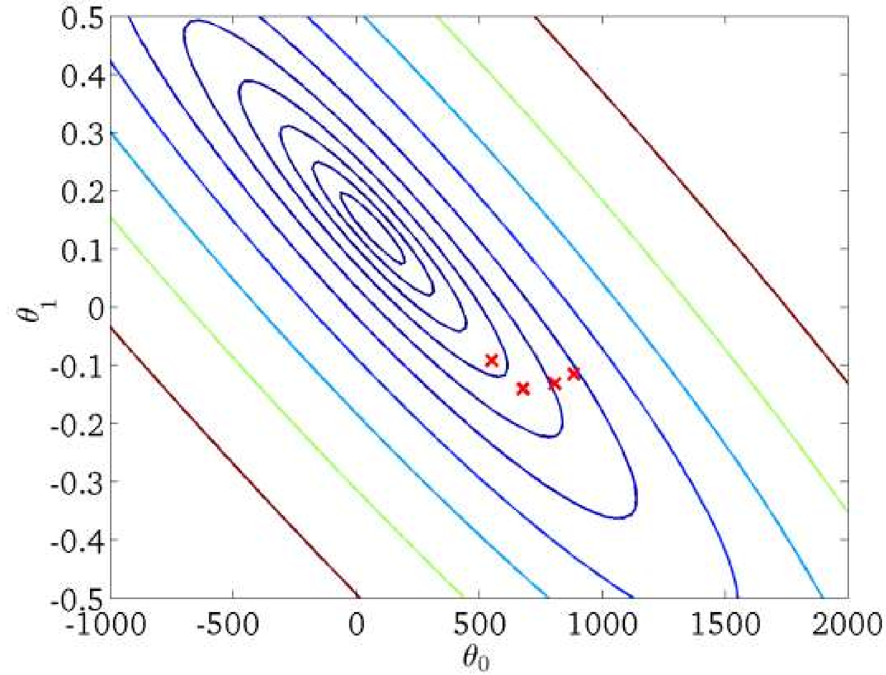

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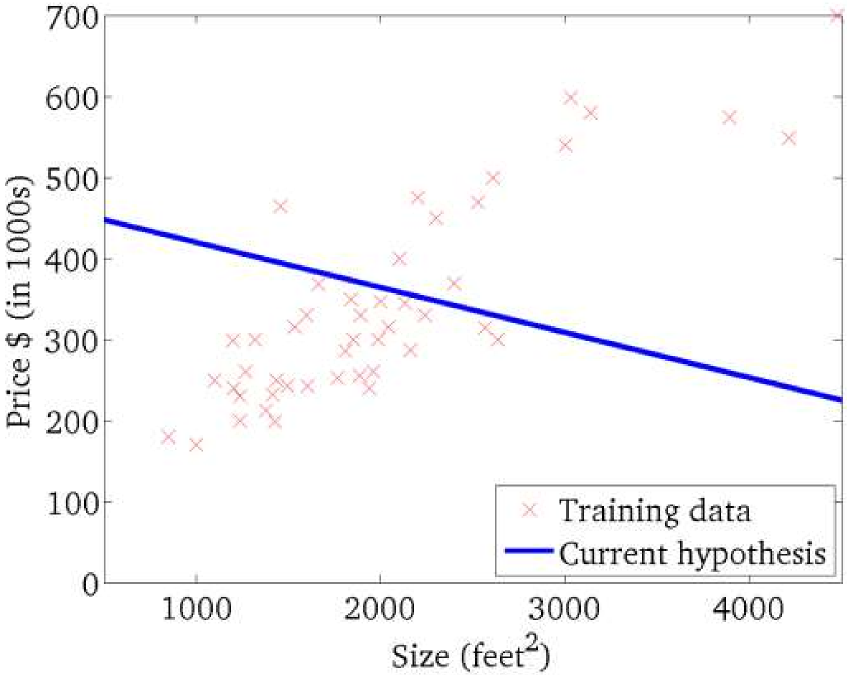

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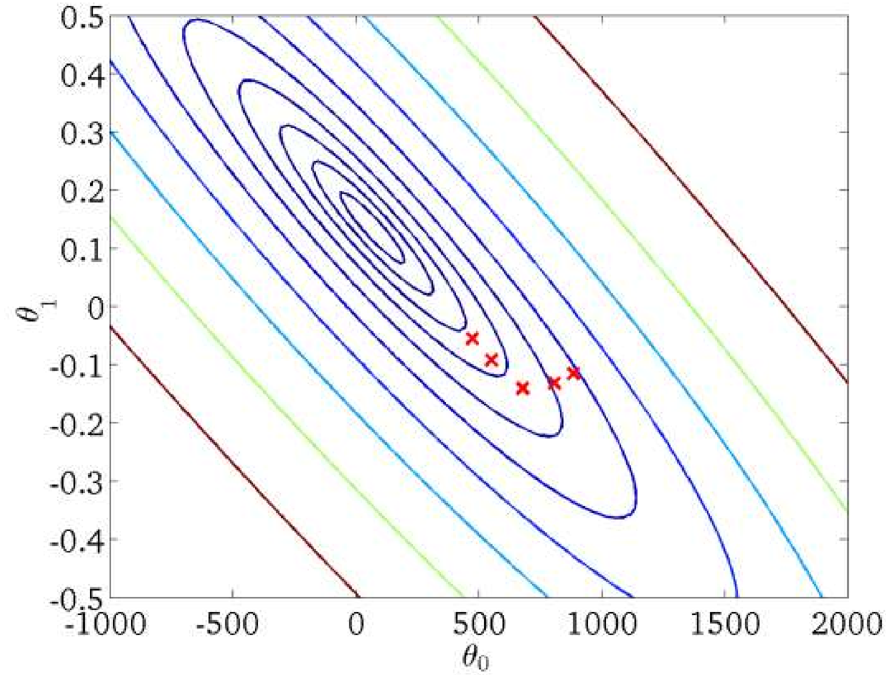

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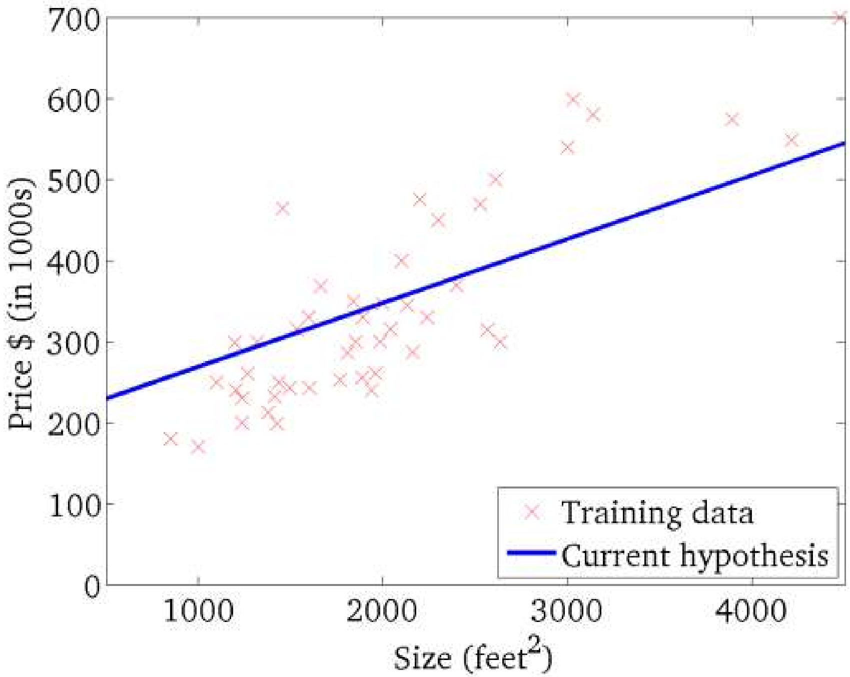

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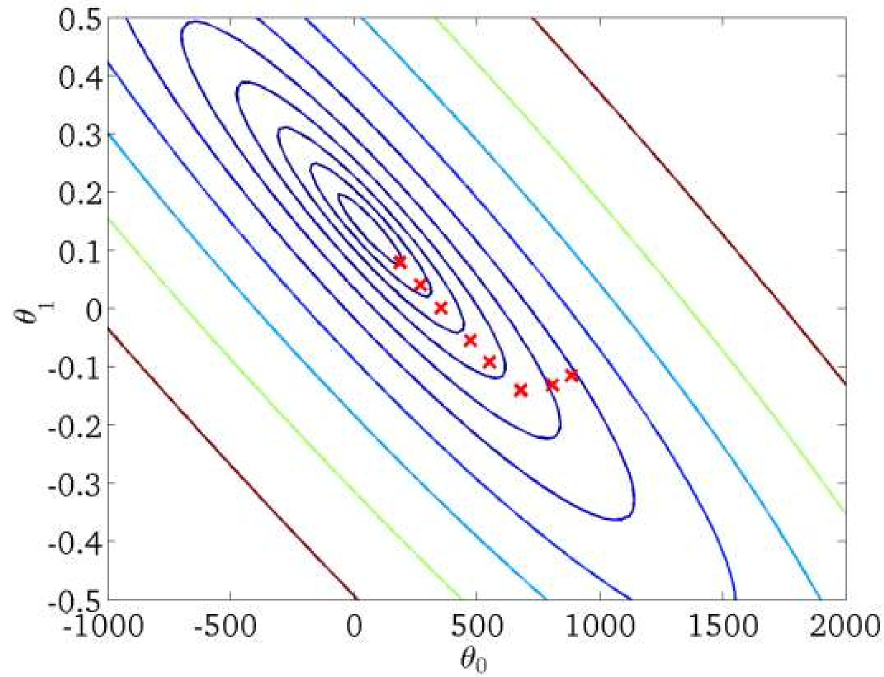

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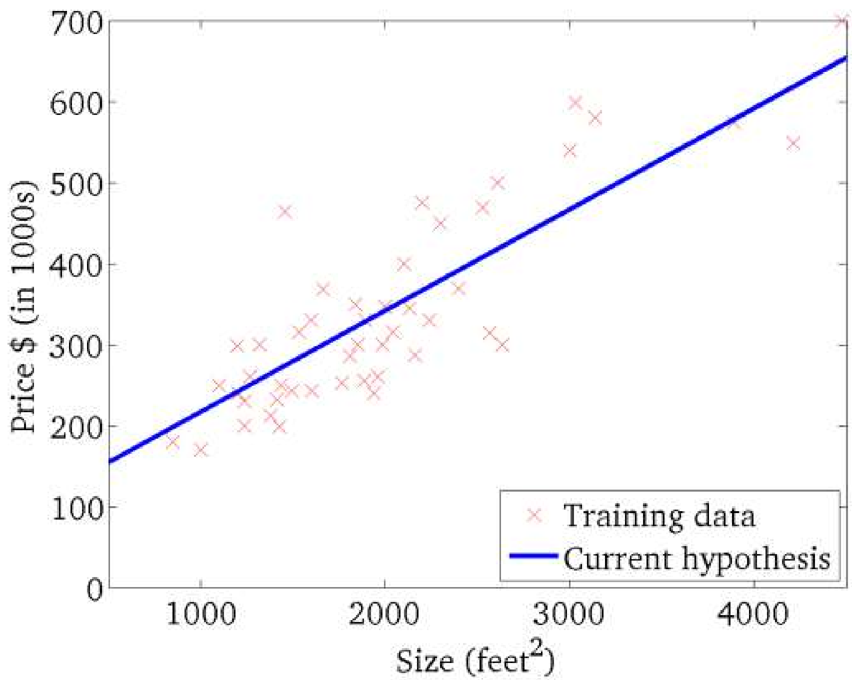

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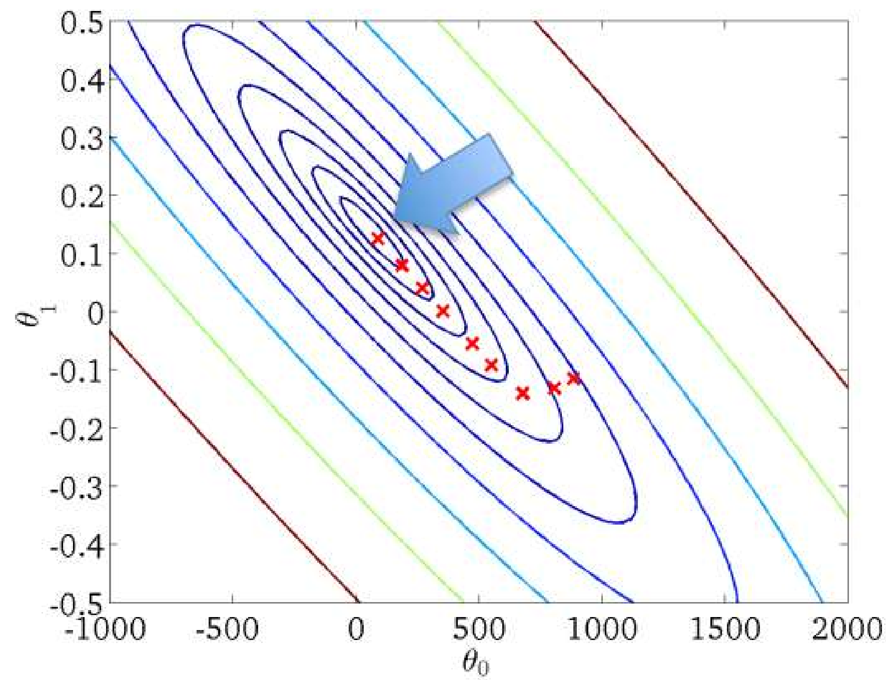
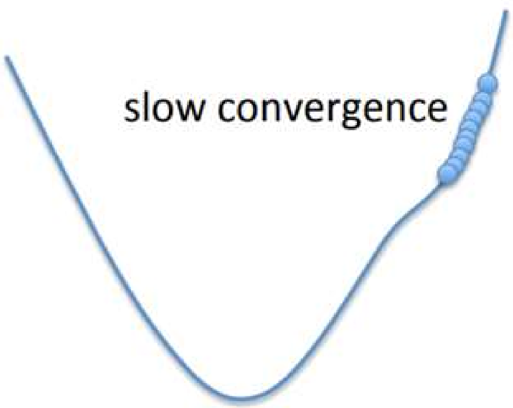
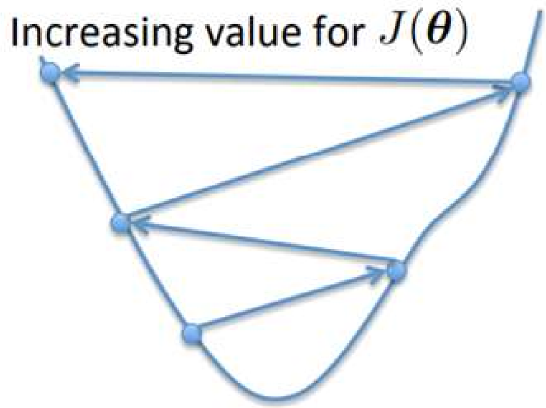


Choosing α

α too small



α too large



- May overshoot the minimum
- May fail to converge
- May even diverge

To see if gradient descent is working, print out $J(\boldsymbol{\theta})$ each iteration

- The value should decrease at each iteration
- If it doesn't, adjust α