Trees

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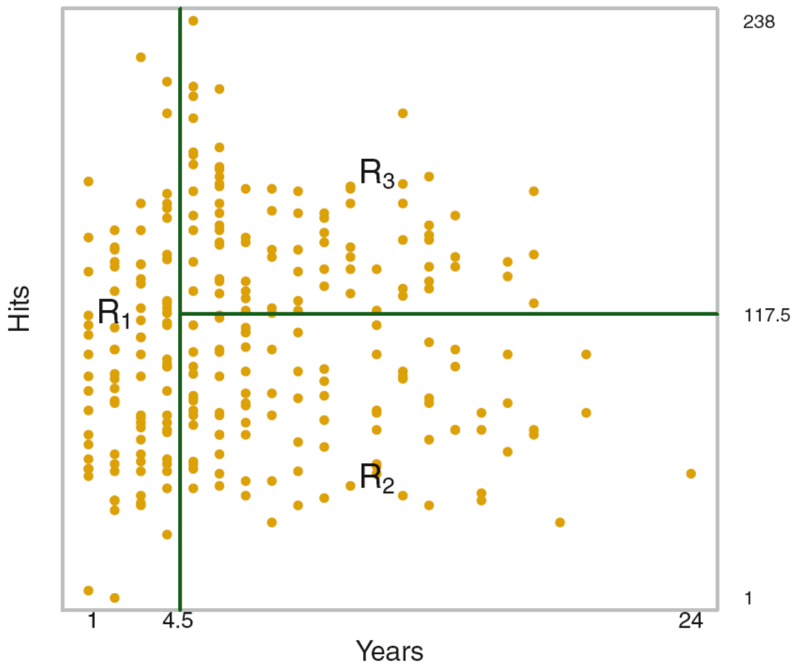
Background

- Segment input space into regions
- Predictions of new observations: mean or mode
- Simple and useful for interpretation
- Accuracy not competitive with best supervised learning approaches
- Dramatic improvements when used in combination with bagging or boosting
- We focus the discussion on CART: classification and regression tree

Example

- Predicting the baseball player's salaries using regression trees
  - Inputs: years, hits, etc.
  - Output: salary (log-transformed)
    - R_1 mean log salary ($1,000) of $165,174 (5.107); R_2 $402,834; R_3 $845,346



Regression Trees

- Two steps:
  - Divide the predictor space (set of possible values of X_1, …X_p) into J distinct and non-overlapping regions, R_1, …, R_J
  - For each observation in R_j, the prediction will be the mean of the output values of the training observations in R_j

- How do we construct the regions, R_1, …, R_J?
  - Regions are boxes (high dimensional rectangles)
  - Find boxes that minimize RSS:

$$\sum_{j=1}^{J} \sum_{i \in R_j} (y_i - \hat{y}_{R_j})^2$$

    - Computationally infeasible to consider every partition into J boxes

Regression Trees

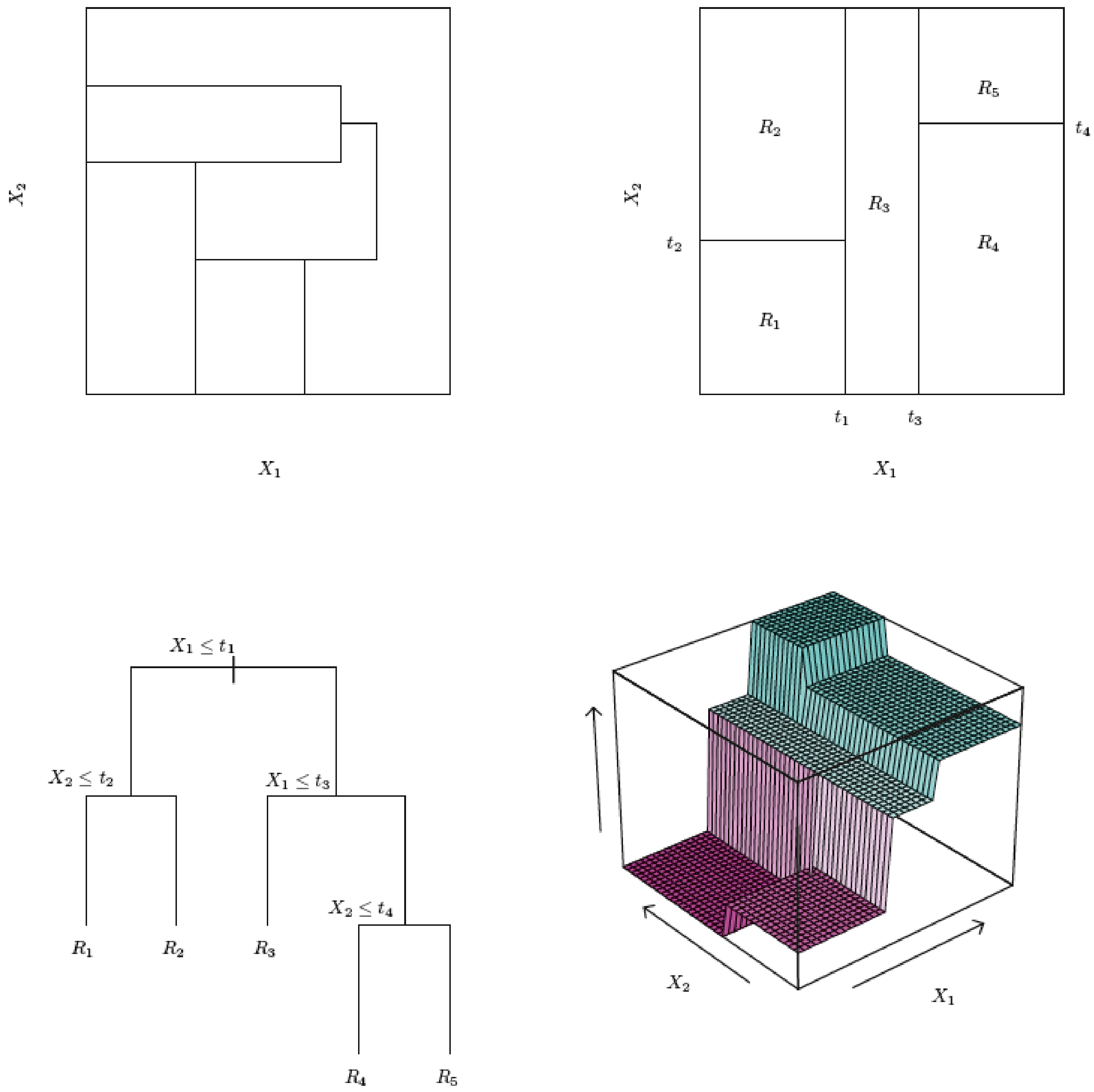
- Top-down greedy approach: recursive binary splitting
  - For each box, select the predictor X_j and the cutpoint *s* to minimize RSS within each region:

$$R_1(j,s) = \{X|X_j < s\} \text{ and } R_2(j,s) = \{X|X_j \geq s\}$$

$$\sum_{i:\, x_i \in R_1(j,s)} (y_i - \hat{y}_{R_1})^2 + \sum_{i:\, x_i \in R_2(j,s)} (y_i - \hat{y}_{R_2})^2$$

  - Stopping criterion: (overfitting/underfitting)
    - <u>Minimum node size</u>
    - Maximum depth
    - Decrease in error under some threshold
- Predict the response of a new test observation using the mean of the training observations in the region
  - Predict the confidence: standard deviation

Example

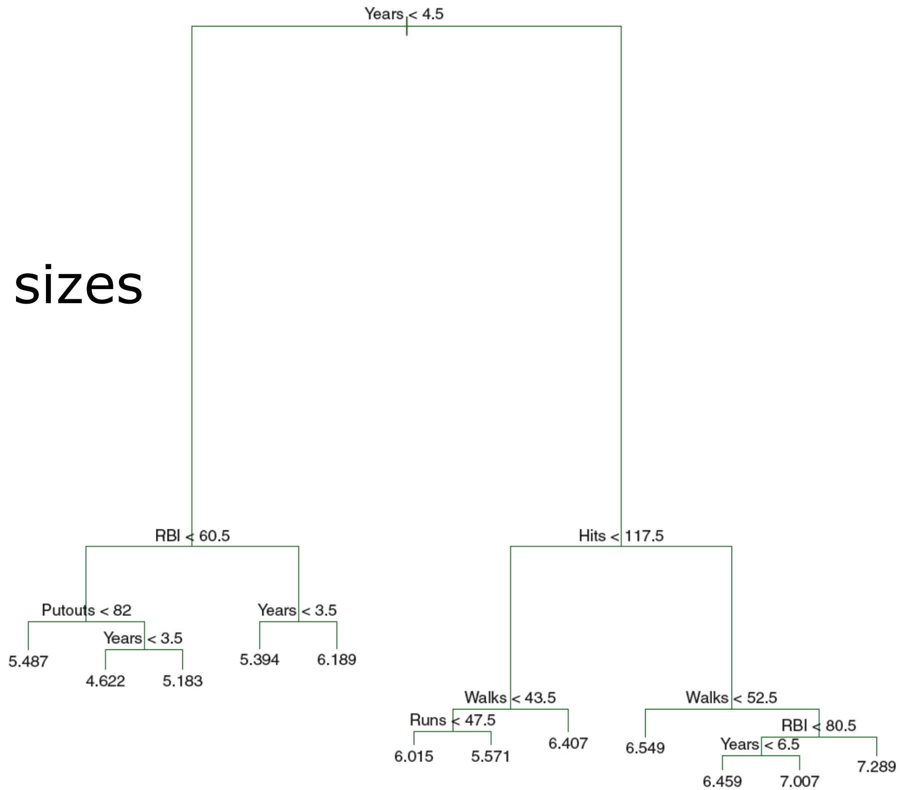


Tree Pruning

- Grow a large tree (relax stopping criteria), and prune it back
  - Best way to prune?
    - Try all combinations: extremely large number of subtrees
    - Cost complexity pruning (weakest link pruning)
      - Collapse the internal node that produces the smallest per-node increase in:

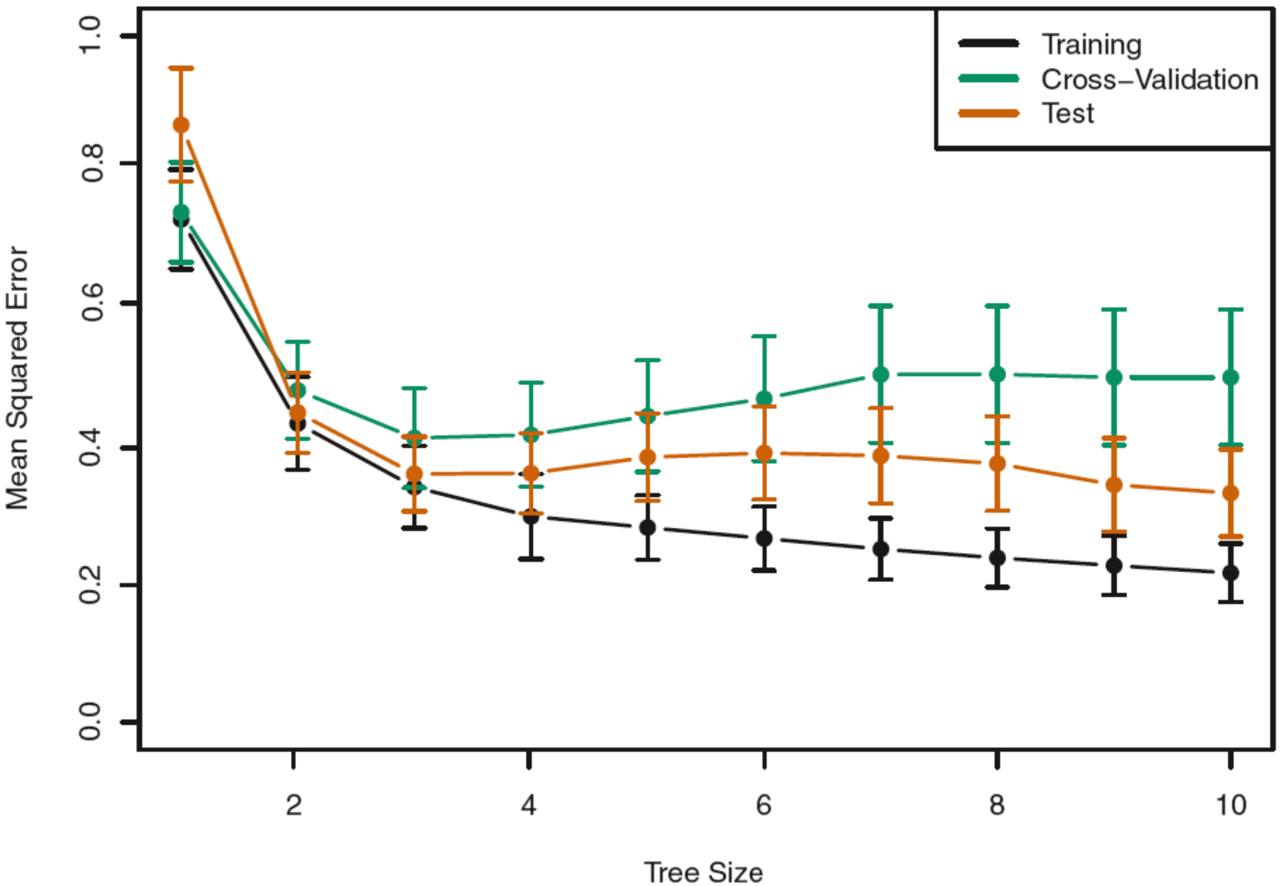
$$\sum_{m=1}^{|T|} \sum_{i:\, x_i \in R_m} (y_i - \hat{y}_{R_m})^2$$

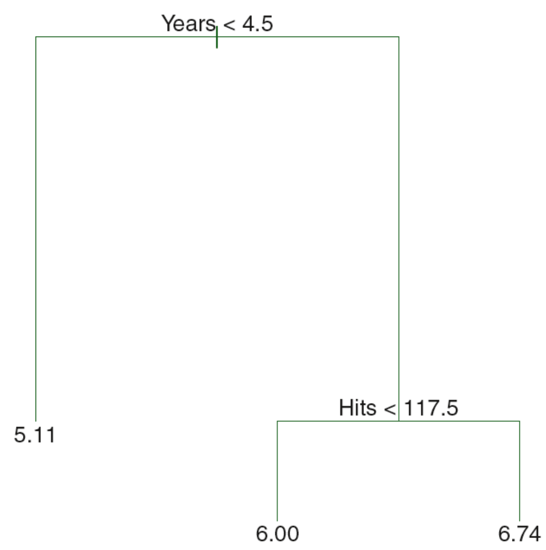
- Sequence of trees with different sizes
- Example: hitters data
  - Tree learned with 9 features
  - Training: 132 examples
  - Test: 131 examples
  - Unpruned tree



Example (hitters data)

- Six-fold cross-validation (number of examples multiple of 6)
- Minimum error: three node tree





Classification Trees

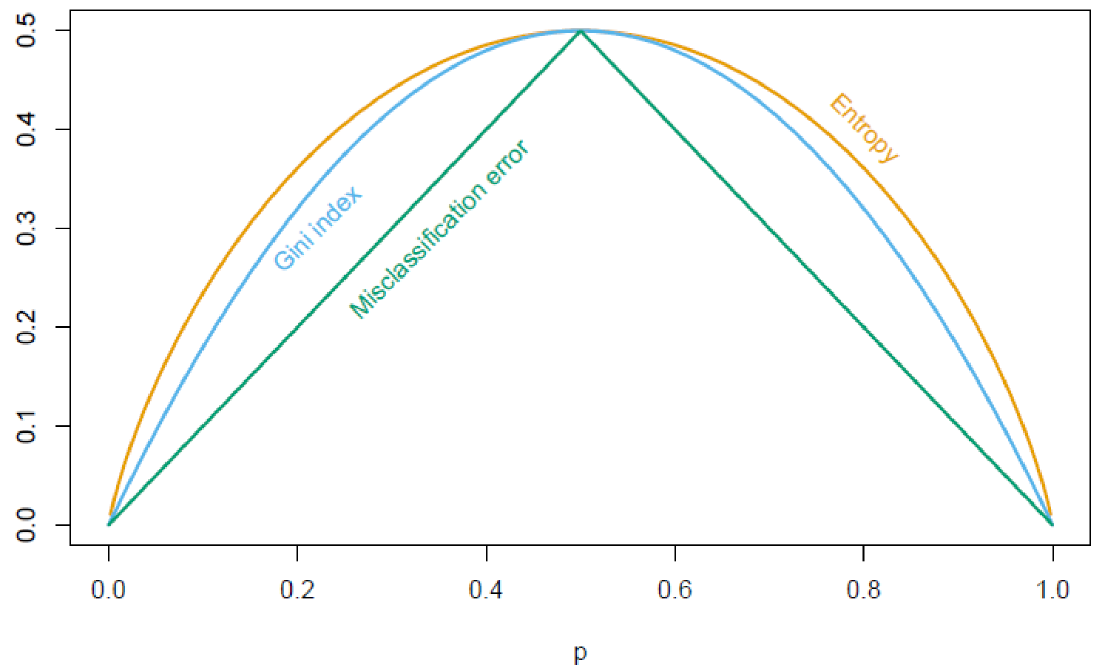
- Similar to a regression tree, but with qualitative response
- Predicted class: most commonly occurring class of training observations in the region to which it belongs
  - We are also interested in the class proportions among the training observations in that region
- Tree growing: recursive binary splitting (as in regression)
  - RSS cannot be used
  - A natural alternative: classification error rate
    - For the m-th region: $E = 1 - \max_k(\hat{p}_{mk})$
    - Not sufficiently sensitive for tree-growing

Classification Trees (ii)

- Gini index:
  - $G = \sum_{k=1}^{K} \hat{p}_{mk}(1 - \hat{p}_{mk})$
  - Total variance across the K classes
- Cross-entropy:
  - $D = -\sum_{k=1}^{K} \hat{p}_{mk} \log \hat{p}_{mk}$

Node impurity measures for two-class classification

Cross-entropy has been scaled



Classification Trees (iii)

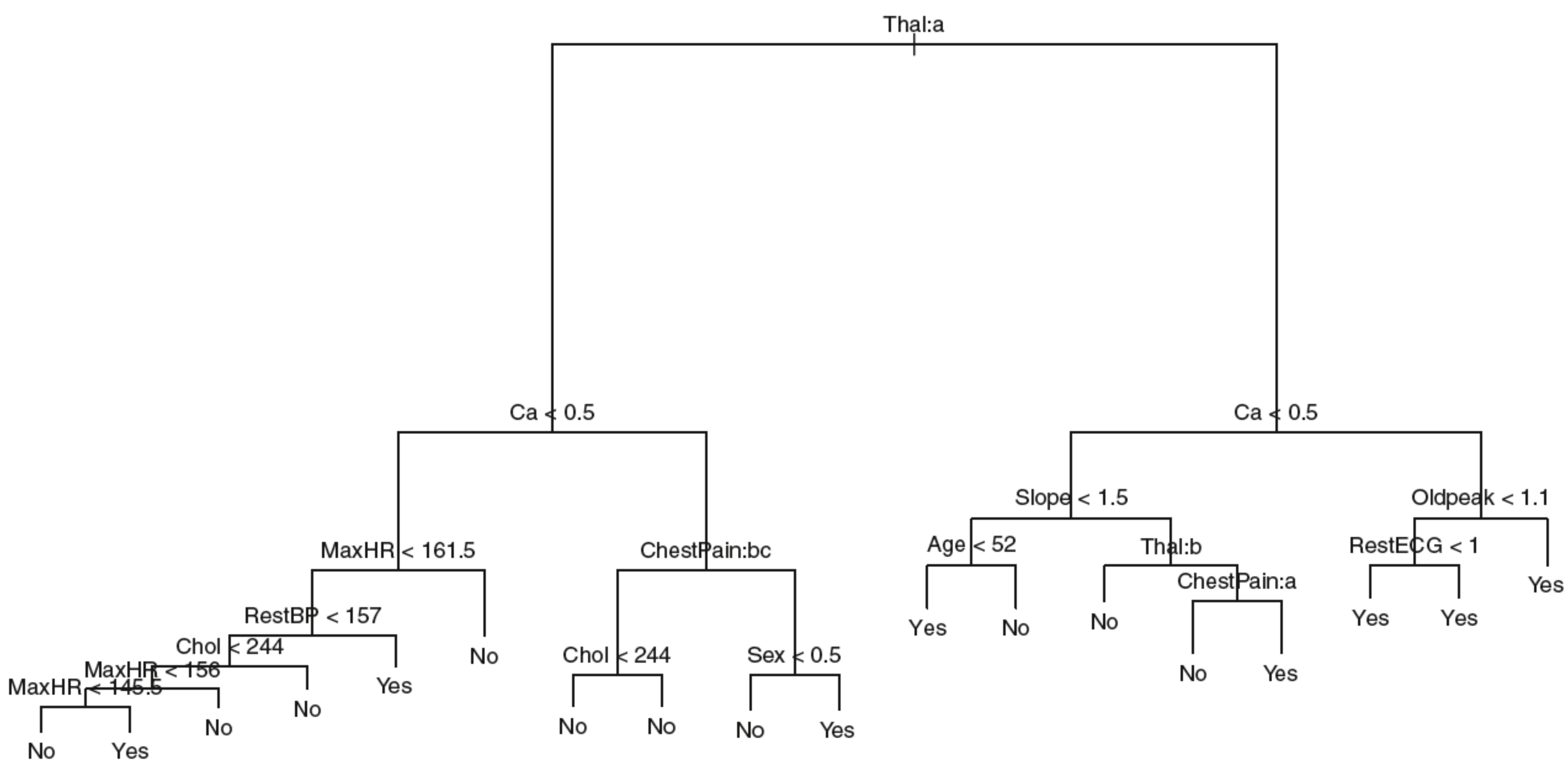
- Weight the node impurity measures by the number of observations in the two created child nodes

- For building the tree:
  - Gini index and cross-entropy are preferable: more sensitive to node purity

- For pruning the tree:
  - Any of the three approaches might be used
  - Classification error rate is preferable if prediction accuracy of the final pruned tree is the goal

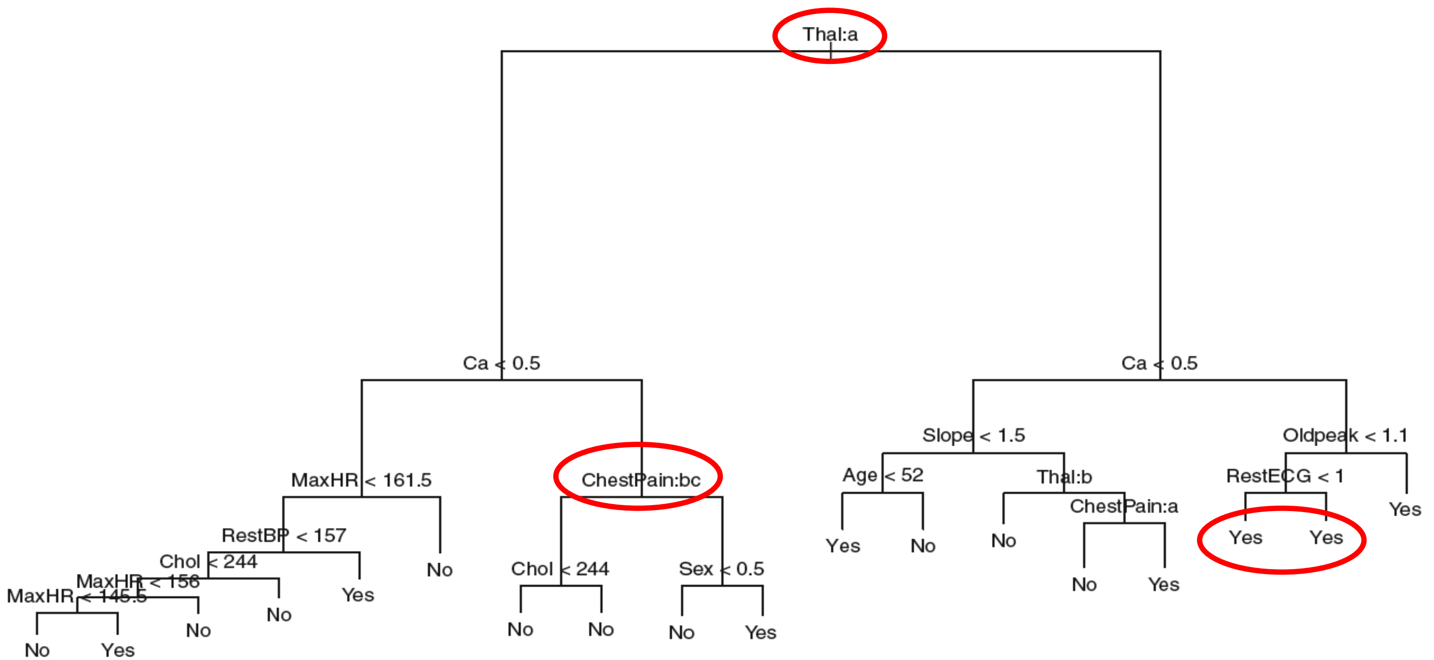
Example: Heart dataset

- Binary outcome (HD) for 303 patients
  - HD: Yes (heart disease) or No
- 13 predictors: Age, Sex, Chol (cholesterol measurement), Thal (Thalium stress test), ChestPain, …
  - Categorical (qualitative) predictors: Sex, Thal, ChestPain



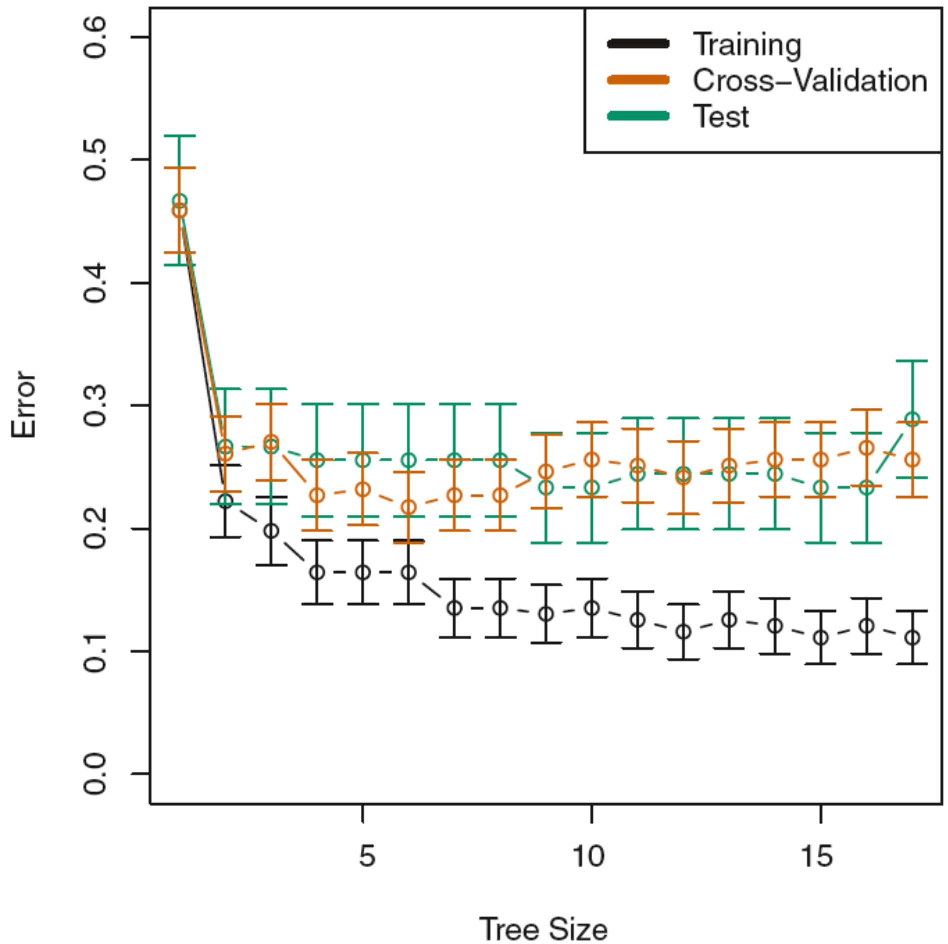
Example: Heart dataset (ii)

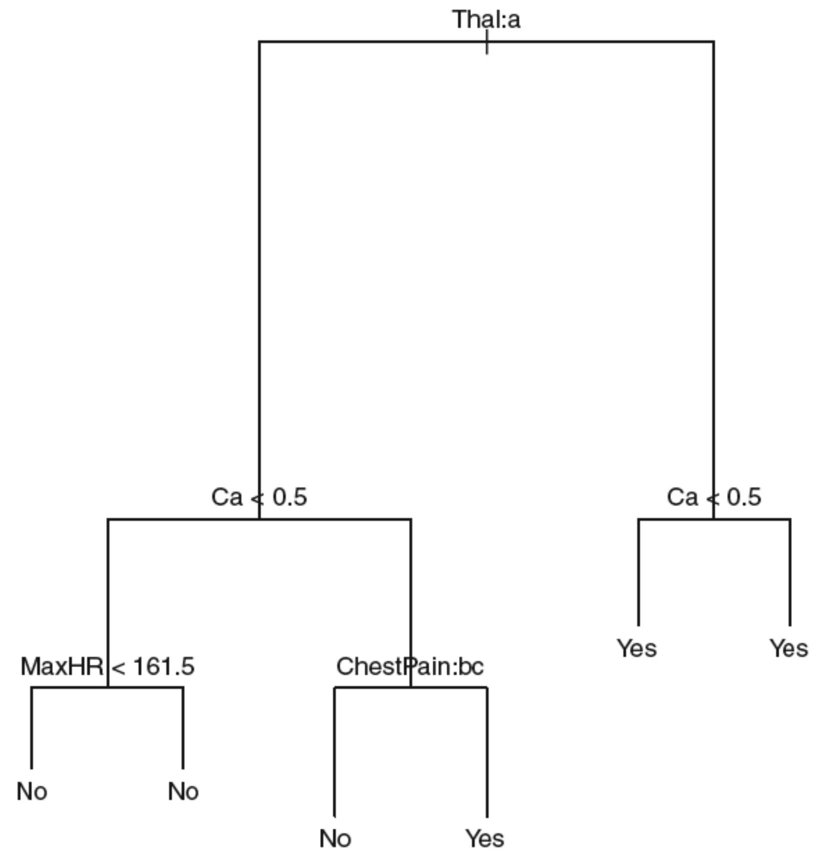
- ChestPain: (a) typical angina, (b) atypical angina, (c) non-anginal pain, (d) asymptomatic
- RestECG: increased node purity
  - Improves Gini index and cross-entropy
  - Classification error not improved
  - Right-hand leaf: 9/9 observations with response value Yes
  - Left-hand leaf: 7/11 observations with response value Yes



Example: Heart dataset (iii)

- Best tree: six nodes





Categorical predictors

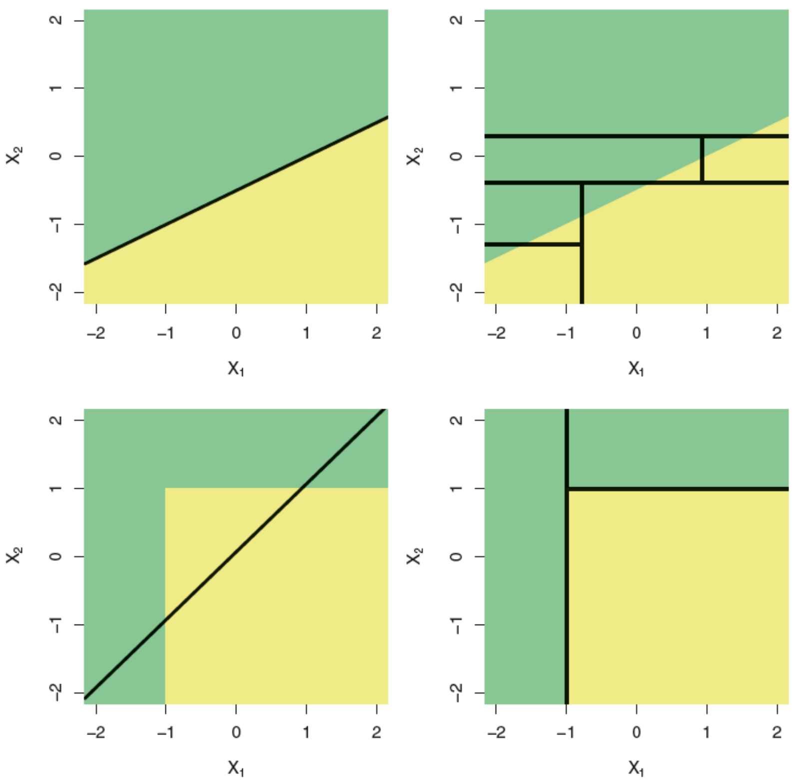
- q values: $2^{q-1}-1$ possible partitions into two groups
- In a two-class problem: order the predictor classes according to the proportion falling in outcome class 1
  - Then split as an ordered predictor
  - Optimal split in terms of Gini index and cross-entropy
- This also holds for regression (RSS)
  - Order the categories by increasing mean of the outcome
- For multi-category outcomes no such simplifications are possible
- Try to avoid variables with large q: overfitting

Trees vs. Linear Models

- Linear regression: $$f(X) = \beta_0 + \sum_{j=1}^{p} X_j \beta_j$$
- Regression trees: $$f(X) = \sum_{m=1}^{M} c_m \cdot 1_{(X \in R_m)}$$
- A two-dimensional classification example:

Linear model is better

Tree is better



Advantages and Disadvantages of trees

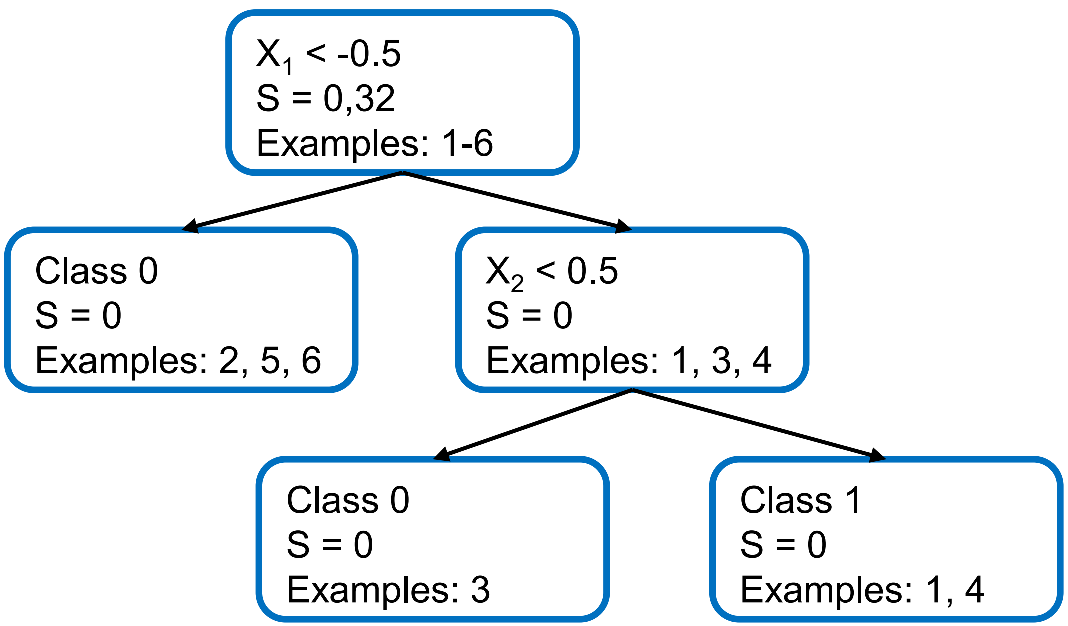
- Pros:
  - Easy to explain
  - More closely mirror human decision-making
  - Can be displayed graphically, and are easily interpreted
  - Can handle qualitative predictors without the need to create dummy variables

- Cons:
  - Predictive accuracy is lower than other approaches

Exercise

- Given the following classification dataset with 6 examples, 2 input variables and 1 output variable, build the classification tree by CART and using entropy as a criterion. The stop condition must be that the leaf nodes are pure (all examples are of the same class). In each node of the tree, you must indicate: (i) the variable and its threshold value; (ii) the corresponding entropy; (iii) the examples that belong to it; (iv) in the leaf nodes, the class of the node

Example	X_1	X_2	Y
1	4	3	1
2	-3	-1	0
3	3	-2	0
4	1	4	1
5	-2	3	0
6	-3	5	0



Bibliography

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  - Chapter 8, Sec. 8.1.

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  - Chapter 9, Sec. 9.2.