Nearest-Neighbors

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Introduction

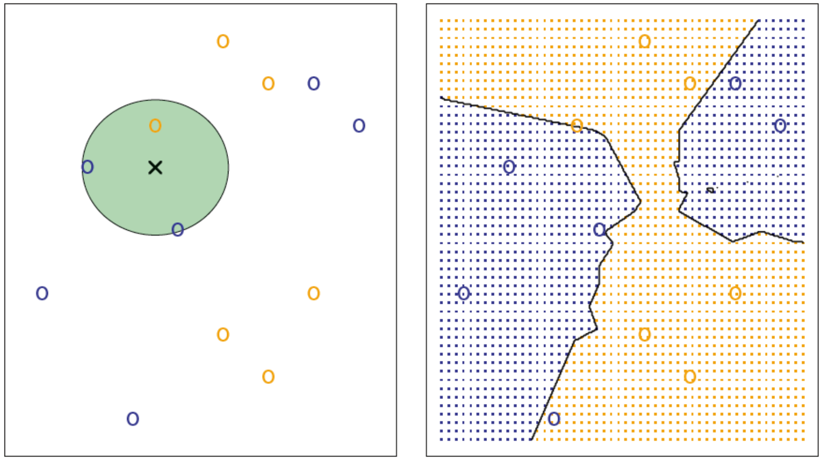
- Model-free method: memory-based
  - Fitting is not required
- Very effective for classification
- Works reasonably well for low-dimensional regression problems
  - For high-dimensional regression the bias-variance trade-off is not so good
- Bayes classifier: gold standard
  - Real data: we do not know the conditional distribution $\Pr(Y|X)$
- K-nearest neighbors (KNN)
  - Estimates the conditional distribution $\Pr(Y|X)$
  - Classifies an observation to the class with highest estimated probability

KNN

- Given K and x_0 (test observation):
  - Identify the K training points closest to x_0
  - Estimate the conditional probability for class j:

$$\Pr(Y = j | X = x_0) = \frac{1}{K} \sum_{i \in \mathcal{N}_0} I(y_i = j)$$

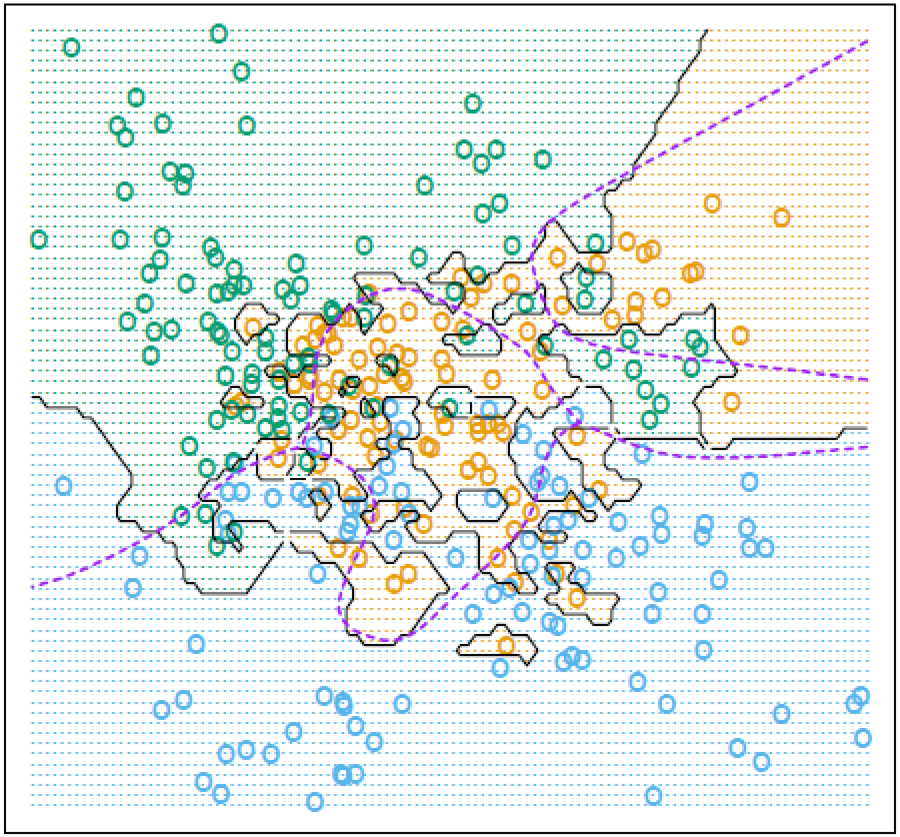
  - Apply Bayes rule: classify x_0 to the class with largest probability
    - Ties are broken at random

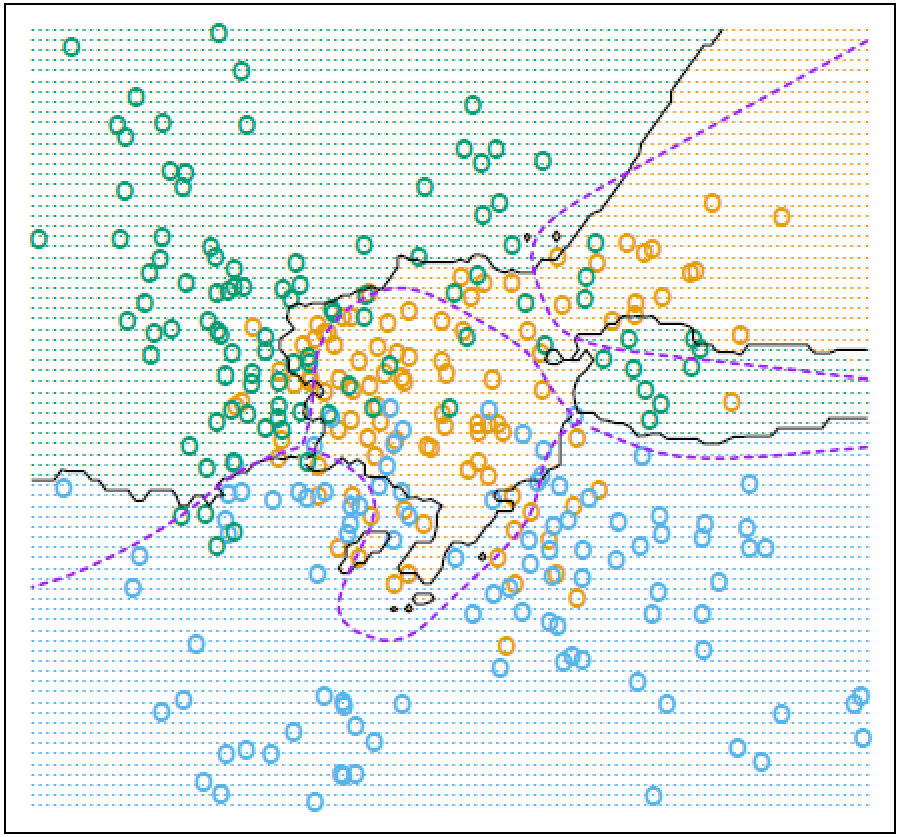


- Closest points:
  - For real-valued features, typically the Euclidean distance in the feature space
  - First standardize each of the features: mean zero, variance 1

KNN (ii)

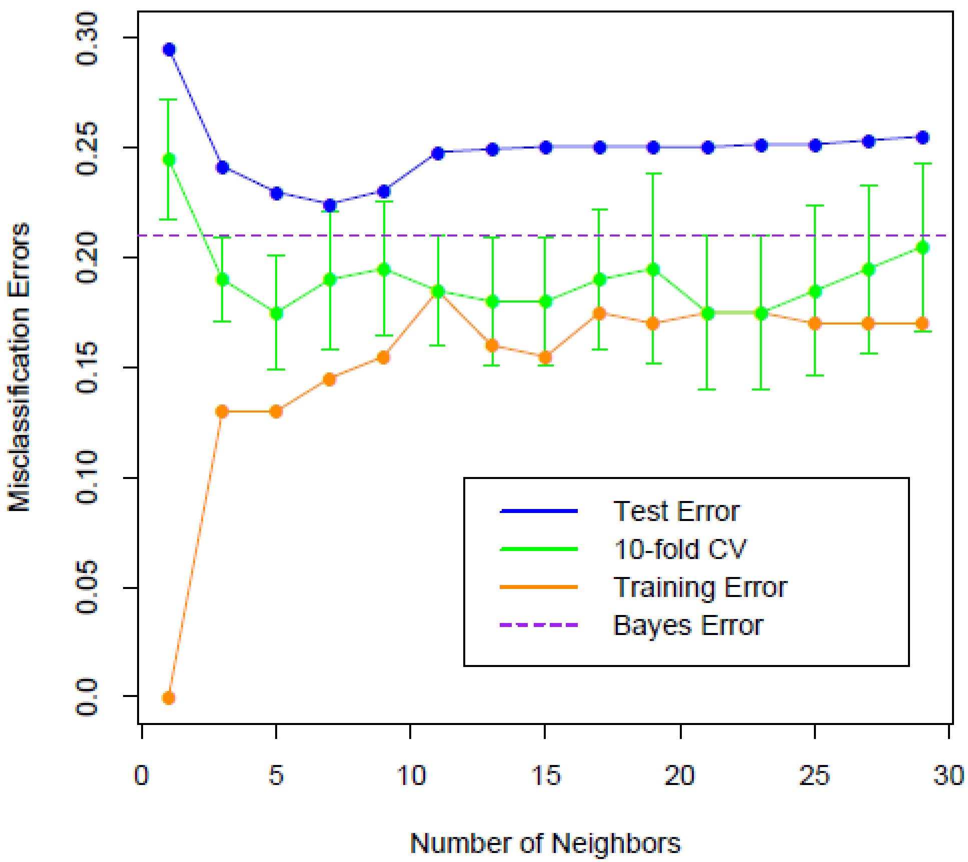
- KNN is successful with very irregular decision boundaries
- Example: simulated, three classes

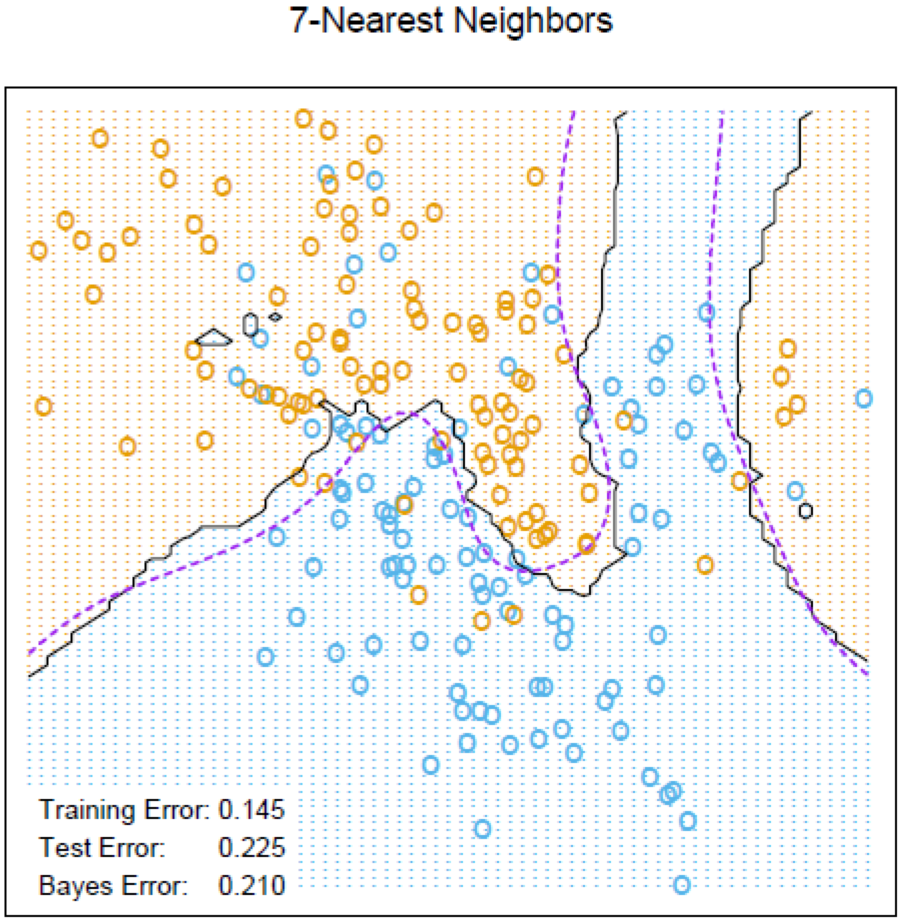




KNN (iii)

- Example: simulated, two classes



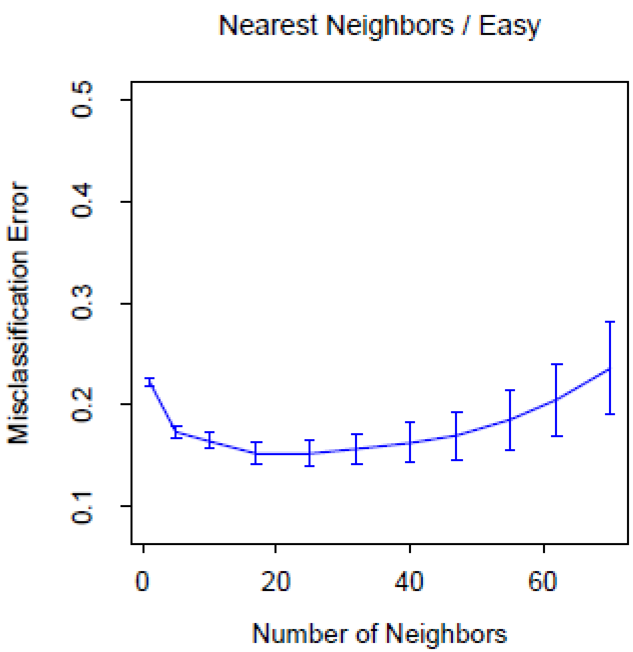


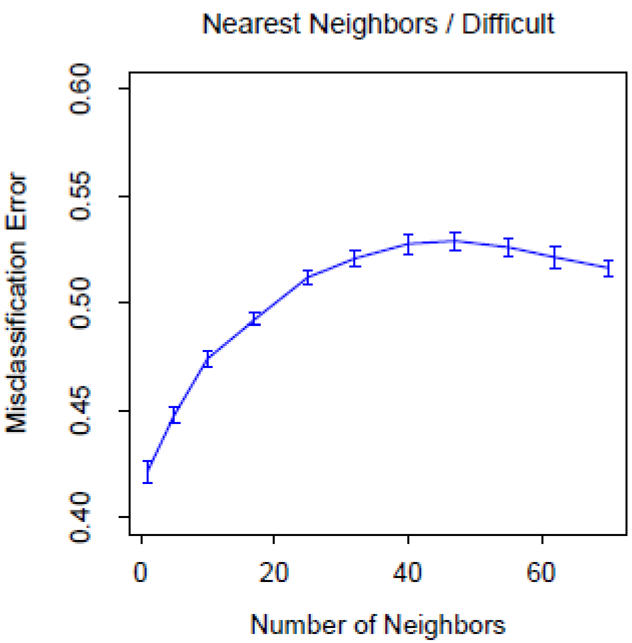
KNN (iv)

- Example: ten independent features, two classes

$$Y = I\left(X_1 > \frac{1}{2}\right); \quad \text{problem 1: “easy”},$$

$$Y = I\left(\text{sign}\left\{\prod_{j=1}^{3}\left(X_j - \frac{1}{2}\right)\right\} > 0\right); \quad \text{problem 2: “difficult.”}$$



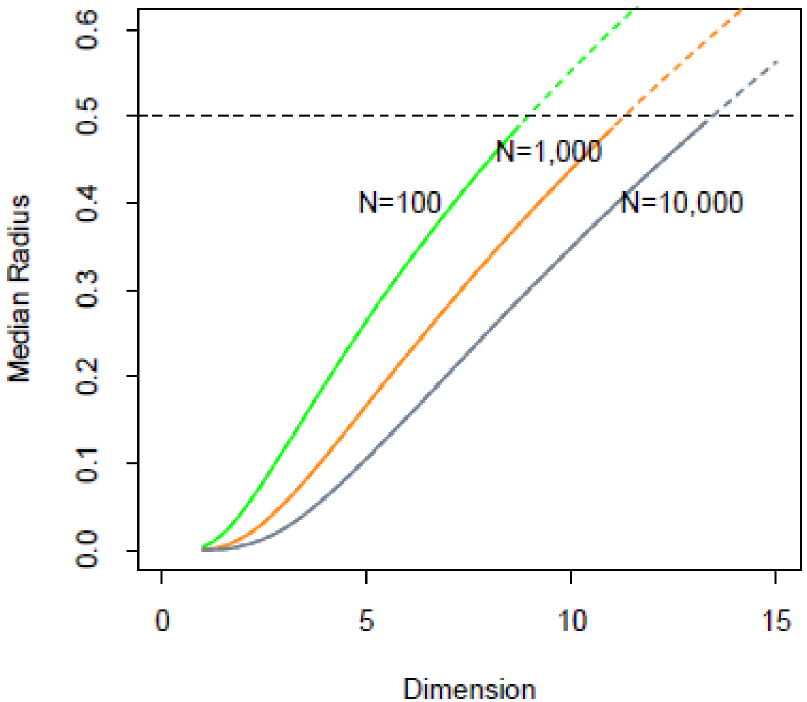


High-dimensional Feature Spaces

- The nearest neighbors can be very far away
  - Increases the bias of KNN
- Radius of 1-NN for N data points in the unit cube $[-0,5, 0,5]^p$

$$\mathrm{median}(R) = v_p^{-1/p}\left(1 - \frac{1}{2}^{1/N}\right)^{1/p}$$

  - The median quickly approaches 0.5 (the distance to the edge of the cube)



Computational Considerations

- Computational load:
  - Finding the neighbors
  - Storing the entire training set
- With N observations and p predictors, $N \times p$ operations for finding the neighbors
  - Fast algorithms for finding nearest-neighbors
- Reducing the storage requirements: instances selection
  - Keep the most important points: near the decision boundaries and on the correct side of those boundaries

Exercise

- Given the following classification data set with 6 examples, 3 input variables and one output variable, assuming that we want to make the prediction of the output variable for X1=0, X2=0, X3=0 by KNN:
  - Compute the distance between each observation and the test point.
  - What is the prediction for K=1? Why?
  - What is the prediction for K=3? Why?

Example	X_1	X_2	X_3	Y
1	0	3	2	1
2	3	0	3	0
3	0	3	-1	0
4	3	0	0	1
5	1	2	1	1
6	2	1	0	0

Bibliography

- G. James, D. Witten, T. Hastie, y R. Tibshirani, An Introduction to Statistical Learning with Applications in R. Springer, 2021.
  - Chapter 2, pp. 39-42
- T. Hastie, R. Tibshirani, y J. Friedman, The elements of statistical learning. Springer, 2009.
  - Chapter 13, Sec. 13.3-13.5