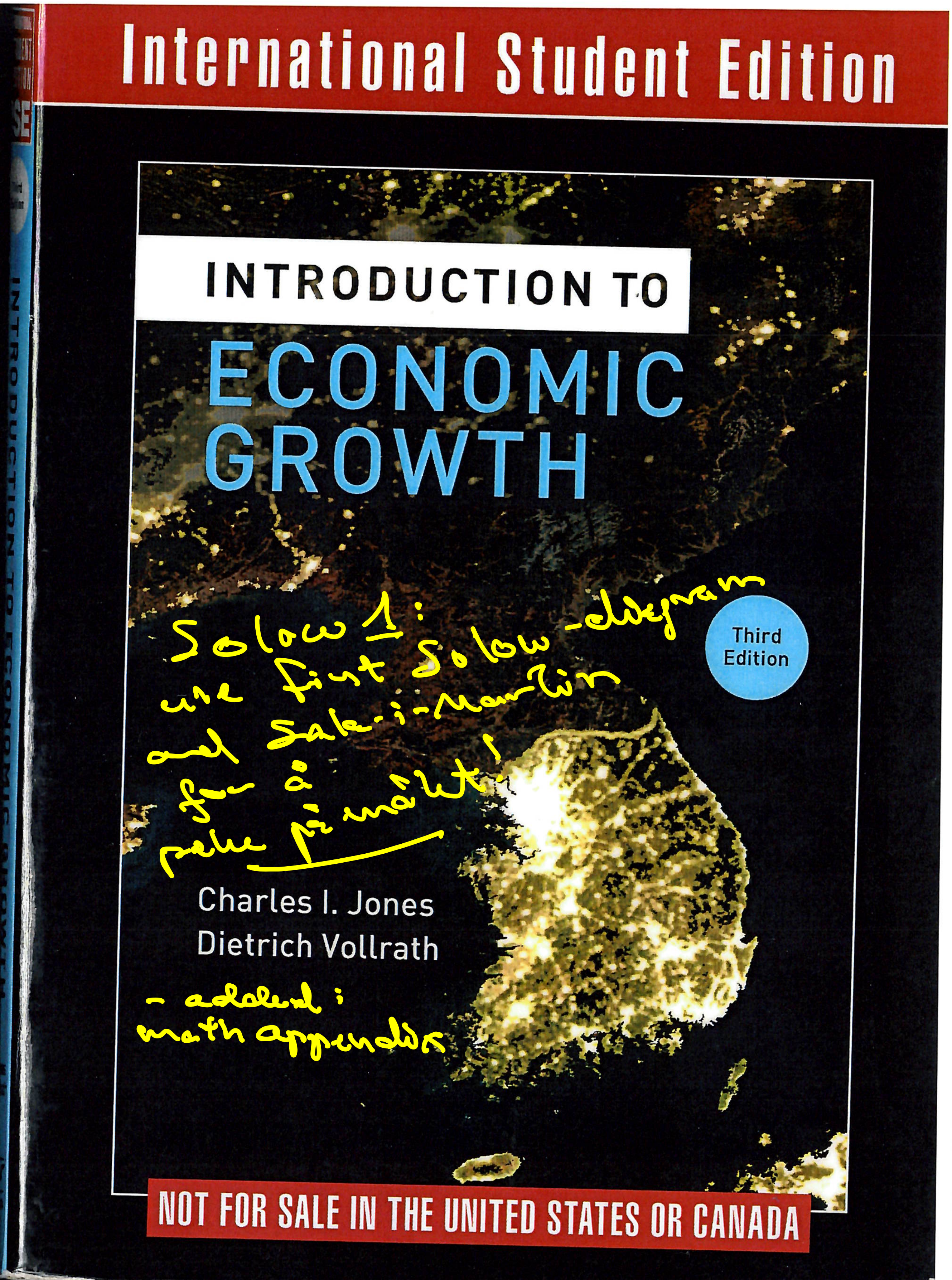



Verdens beste innføringsbok i økonomisk vekst:

Solow 1: the first Solow-diagram and Sala-i-Martin for å peke på målet!

– added: math appendix

Solow-diagrammet (Figure 2.2) ble brukt av Robert Solow i artikkelen 'A contribution to the Theory of Economic Growth' i Quarterly Journal of Economics fra 1956.

Det andre diagrammet som er spesielt nyttig er "Figure 2.8". Det ble første gang brukt av Sala-i-Martin i 1990 ("Lecture Notes on Economic Growth" trykket som NBER working Paper No 3563)

# 2 THE SOLOW MODEL

> All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive.
>
> —ROBERT SOLOW (1956), P. 65

In 1956, Robert Solow published a seminal paper on economic growth and development titled "A Contribution to the Theory of Economic Growth." For this work and for his subsequent contributions to our understanding of economic growth, Solow was awarded the Nobel Prize in economics in 1987. In this chapter, we develop the model proposed by Solow and explore its ability to explain the stylized facts of growth and development discussed in Chapter 1. As we will see, this model provides an important cornerstone for understanding why some countries flourish while others are impoverished.

Following the advice of Solow in the quotation above, we will make several assumptions that may seem to be heroic. Nevertheless, we hope that these are simplifying assumptions in that, for the purposes at hand, they do not terribly distort the picture of the world we create. For example, the world we consider in this chapter will consist of countries that produce and consume only a single, homogeneous good (*output*). Conceptually, as well as for testing the model using empirical data, it is convenient to think of this output as units of a country's gross domestic product, or GDP. One implication of this simplifying assumption is that there is no international trade in the model because there is only a single good: I'll give you a 1941 Joe DiMaggio autograph in exchange for . . . your 1941 Joe DiMaggio autograph? Another assumption of the model is that technology is *exogenous*—that is, the technology available to firms in this simple world is unaffected by the actions of the firms, including research and development (R&D). These are assumptions that we will relax later on, but for the moment, and for Solow, they serve well. Much progress in economics has been made by creating a very simple world and then seeing how it behaves and misbehaves.

Before presenting the Solow model, it is worth stepping back to consider exactly what a model is and what it is for. In modern economics, a model is a mathematical representation of some aspect of the economy. It is easiest to think of models as toy economies populated by robots. We specify exactly how the robots behave, which is typically to maximize their own utility. We also specify the constraints the robots face in seeking to maximize their utility. For example, the robots that populate our economy may want to consume as much output as possible, but they are limited in how much output they can produce by the techniques at their disposal. The best models are often very simple but convey enormous insight into how the world works. Consider the supply and demand framework in microeconomics. This basic tool is remarkably effective at predicting how the prices and quantities of goods as diverse as health care, computers, and nuclear weapons will respond to changes in the economic environment.

With this understanding of how and why economists develop models, we pause to highlight one of the important assumptions we will make until the final chapters of this book. Instead of writing down utility functions that the robots in our economy maximize, we will summarize the results of utility maximization with elementary rules that the robots obey. For example, a common problem in economics is for an individual to decide how much to consume today and how much to save for consumption in the future. Another is for individuals to decide how much time to spend going to school to accumulate skills and how much time to spend working in the labor market. Instead of writing these problems down formally, we will assume that individuals save a constant fraction of their income and spend a constant fraction of their time accumulating skills. These are extremely useful simplifications; without them, the models are difficult to solve without more advanced mathematical techniques. For many purposes, these are fine assumptions to make in our first pass at understanding economic growth. Rest assured, however, that we will relax these assumptions in Chapter 7.

## 2.1 THE BASIC SOLOW MODEL

The Solow model is built around two equations, a production function and a capital accumulation equation. The production function describes how inputs such as bulldozers, semiconductors, engineers, and steel-workers combine to produce output.[1] To simplify the model, we group these inputs into two categories, capital, $K$, and labor, $L$, and denote output as $Y$. The *production function* is assumed to have the Cobb-Douglas form and is given by

$$Y = F(K, L) = K^{\alpha}L^{1-\alpha} \tag{2.1}$$

where $\alpha$ is some number between 0 and 1.[2] Notice that this production function exhibits *constant returns to scale:* if all of the inputs are doubled, output will exactly double.[3]

Firms in this economy pay workers a wage, $w$, for each unit of labor and pay $r$ in order to rent a unit of capital for one period. We assume

[1]An important point to keep in mind is that even though the inputs to the production function are measured as physical quantities, for example, numbers of workers or units of capital goods, this does not mean that output consists only of tangible goods. For example, if you are reading this book, you are likely taking a class on economic growth. The class takes place in a classroom with desks and whiteboards and perhaps a video projector, all examples of physical capital. Your instructor constitutes the labor input. At the end of each class you will hopefully have learned something new, but you'll leave with nothing that you can touch, feel, or carry.

[2]Charles Cobb and Paul Douglas (1928) proposed this functional form in their analysis of U.S. manufacturing. Interestingly, they argued that this production function, with a value for $\alpha$ of 1/4 fit the data very well without allowing for technological progress.

[3]Recall that if $F(aK, aL) = aY$ for any number $a > 1$ then we say that the production function exhibits constant returns to scale. If $F(aK, aL) > aY$, then the production function exhibits *increasing returns to scale*, and if the inequality is reversed the production function exhibits *decreasing returns to scale*.

there are a large number of firms in the economy so that perfect competition prevails and the firms are price takers.[4] Normalizing the price of output in our economy to unity, profit-maximizing firms solve the following problem:

$$\max_{K,L} F(K, L) - rk - wL.$$

According to the first-order conditions for this problem, firms will hire labor until the marginal product of labor is equal to the wage and will rent capital until the marginal product of capital is equal to the rental price:

$$w = \frac{\partial F}{\partial L} = (1 - \alpha)\frac{Y}{L},$$

$$r = \frac{\partial F}{\partial K} = \alpha\frac{Y}{K}.$$

Notice that $wL + rK = Y$. That is, payments to the inputs ("factor payments") completely exhaust the value of output produced so that there are no economic profits to be earned. This important result is a general property of production functions with constant returns to scale. Notice also that the share of output paid to labor is $wL/Y = 1 - \alpha$ and the share paid to capital is $rK/Y = \alpha$. These factor shares are therefore constant over time, consistent with Fact 5 from Chapter 1.

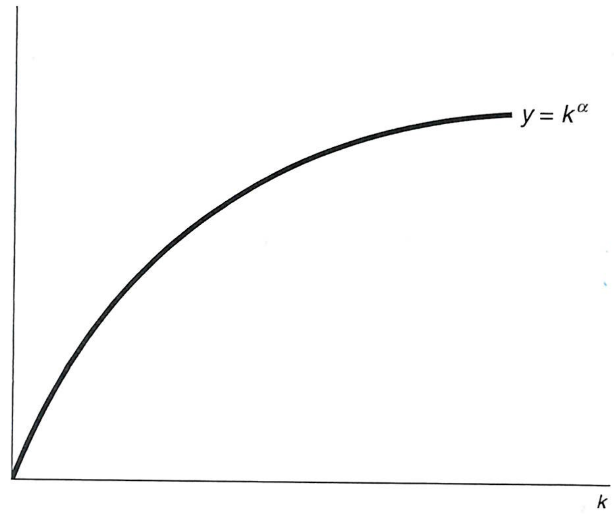
Recall from Chapter 1 that the stylized facts we are typically interested in explaining involve output per worker or per capita output. With this interest in mind, we can rewrite the production function in equation (2.1) in terms of output per worker, $y \equiv Y/L$, and capital per worker, $k \equiv K/L$:

$$y = k^{\alpha}. \tag{2.2}$$

This production function is graphed in Figure 2.1. With more capital per worker, firms produce more output per worker. However, there are diminishing returns to capital per worker: each additional unit of capital we give to a single worker increases the output of that worker by less and less.

[4]You may recall from microeconomics that with constant returns to scale the number of firms is indeterminate—that is, not pinned down by the model.

FIGURE 2.1 A COBB-DOUGLAS PRODUCTION FUNCTION




The second key equation of the Solow model is an equation that describes how capital accumulates. The capital accumulation equation is given by

$$\dot{K} = sY - \delta K. \tag{2.3}$$

This equation will be used throughout this book and is very important, so let's pause a moment to explain carefully what this equation says. According to this equation, the change in the capital stock, $\dot{K}$ is equal to the amount of gross investment, $sY$, less the amount of depreciation that occurs during the production process, $\delta K$. We'll now discuss these three terms in more detail.

The term on the left-hand side of equation (2.3) is the continuous time version of $K_{t+1} - K_t$, that is, the change in the capital stock per "period." We use the "dot" notation[5] to denote a derivative with respect to time:

$$\dot{K} = \frac{dK}{dt}.$$

[5]Appendix A discusses the meaning of this notation in more detail.

The second term of equation (2.3) represents gross investment. Following Solow, we assume that workers/consumers save a constant fraction, $s$, of their combined wage and rental income, $Y = wL + rK$. The economy is closed, so that saving equals investment, and the only use of investment in this economy is to accumulate capital. The consumers then rent this capital to firms for use in production, as discussed above.

The third term of equation (2.3) reflects the depreciation of the capital stock that occurs during production. The standard functional form used here implies that a constant fraction, $\delta$, of the capital stock depreciates every period (regardless of how much output is produced). For example, we often assume $\delta = .05$, so that 5 percent of the machines and factories in our model economy wear out each year.

To study the evolution of output per person in this economy, we rewrite the capital accumulation equation in terms of capital per person. Then the production function in equation (2.2) will tell us the amount of output per person produced for whatever capital stock per person is present in the economy. This rewriting is most easily accomplished by using a simple mathematical trick that is often used in the study of growth. The mathematical trick is to "take logs and then derivatives" (see Appendix A for further discussion). Two examples of this trick are given below.

*Example 1*:

$$k \equiv K/L \Rightarrow \log k = \log K - \log L$$

$$\Rightarrow \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}.$$

*Example 2*:

$$y = k^{\alpha} \Rightarrow \log y = \alpha \log k$$

$$\Rightarrow \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}.$$

Applying Example 1 to equation (2.3) will allow us to rewrite the capital accumulation equation in terms of capital per worker.

Before we proceed, let's first consider the growth rate of the labor force, $\dot{L}/L$. An important assumption that will be maintained throughout most of this book is that the labor force participation rate is constant and that the population growth rate is given by the parameter $n$.[6] This implies that the labor force growth rate, $\dot{L}/L$, is also given by $n$. If $n = .01$, then the population and the labor force are growing at 1 percent per year. This exponential growth can be seen from the relationship

$$L(t) = L_0 e^{nt}.$$

Take logs and differentiate this equation, and what do you get?

Now we are ready to combine Example 1 and equation (2.3):

$$\begin{aligned}\frac{\dot{k}}{k} &= \frac{sY}{K} - n - \delta \\ &= \frac{sy}{k} - n - \delta.\end{aligned}$$

This now yields the capital accumulation equation in per worker terms:

$$\dot{k} = sy - (n + \delta)k.$$

This equation says that the change in capital per worker each period is determined by three terms. Two of the terms are analogous to the original capital accumulation equation. Investment per worker, $sy$, increases $k$, while depreciation per worker, $\delta k$, reduces $k$. The term that is new in this equation is a reduction in $k$ because of population growth, the $nk$ term. Each period, there are $nL$ new workers around who were not there during the last period. If there were no new investment and no depreciation, capital *per worker* would decline because of the increase in the labor force. The amount by which it would decline is exactly $nk$, as can be seen by setting $\dot{K}$ to zero in Example 1.

### 2.1.1 SOLVING THE BASIC SOLOW MODEL

We have now laid out the basic elements of the Solow model, and it is time to begin solving the model. What does it mean to "solve" a model? To answer this question we need to explain exactly what a model is and to define some concepts.

[6]Often, it is convenient in describing the model to assume that the labor force participation rate is unity—that is, every member of the population is also a worker.

In general, a model consists of several equations that describe the relationships among a collection of *endogenous variables*—that is, among variables whose values are determined within the model itself. For example, equation (2.1) shows how output is produced from capital and labor, and equation (2.3) shows how capital is accumulated over time. Output, $Y$, and capital, $K$, are endogenous variables, as are the respective "per worker" versions of these variables, $y$ and $k$.

Notice that the equations describing the relationships among endogenous variables also involve *parameters* and *exogenous variables*. Parameters are terms such as, $\alpha$, $s$, $k_0$, and $n$ that stand in for single numbers. Exogenous variables are terms that may vary over time but whose values are determined outside of the model—that is, exogenously. The number of workers in this economy, $L$, is an example of an exogenous variable.

With these concepts explained, we are ready to tackle the question of what it means to solve a model. Solving a model means obtaining the values of each endogenous variable when given values for the exogenous variables and parameters. Ideally, one would like to be able to express each endogenous variable as a function only of exogenous variables and parameters. Sometimes this is possible; other times a diagram can provide insights into the nature of the solution, but a computer is needed for exact values.

For this purpose, it is helpful to think of the economist as a laboratory scientist. The economist sets up a model and has control over the parameters and exogenous variables. The "experiment" is the model itself. Once the model is set up, the economist starts the experiment and watches to see how the endogenous variables evolve over time. The economist is free to vary the parameters and exogenous variables in different experiments to see how this changes the evolution of the endogenous variables.

In the case of the Solow model, our solution will proceed in several steps. We begin with several diagrams that provide insight into the solution. Then, in Section 2.1.4, we provide an analytic solution for the long-run values of the key endogenous variables. A full solution of the model at every point in time is possible analytically, but this derivation is somewhat difficult and is relegated to the appendix of this chapter.

### 2.1.2 THE SOLOW DIAGRAM

At the beginning of this section we derived the two key equations of the Solow model in terms of output per worker and capital per worker. These equations are

$$y = k^{\alpha} \tag{2.4}$$

and

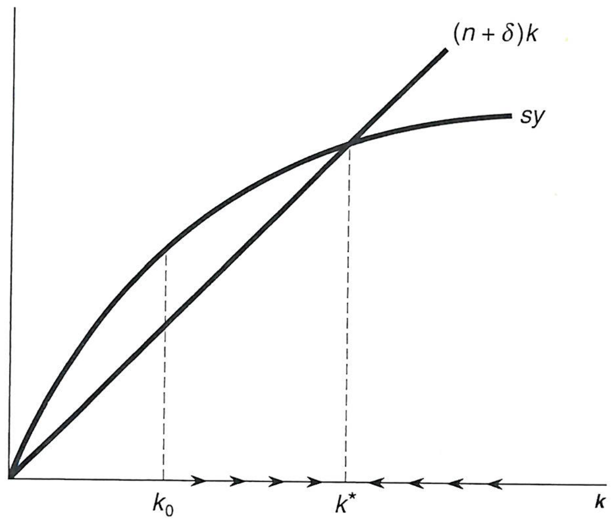
$$\dot{k} = sy - (n + \delta)k. \tag{2.5}$$

Now we are ready to ask fundamental questions of our model. For example, an economy starts out with a given stock of capital per worker, $k_0$, and a given population growth rate, depreciation rate, and investment rate. How does output per worker evolve over time in this economy—that is, how does the economy grow? How does output per worker compare in the long run between two economies that have different investment rates?

These questions are most easily analyzed in a Solow diagram, as shown in Figure 2.2. The Solow diagram consists of two curves, plotted as functions of the capital-labor ratio, $k$. The first curve is the amount of investment per person, $sy = sk^{\alpha}$. This curve has the same shape as the production function plotted in Figure 2.1, but it is translated down by the factor $s$. The second curve is the line $(n + \delta)k$, which represents the amount of new investment per person required to keep the amount of capital per worker constant—both depreciation and the growing workforce tend to reduce the amount of capital per person in the economy. By no coincidence, the difference between these two curves is the change in the amount of capital per worker. When this change is positive and the economy is increasing its capital per worker, we say that *capital deepening* is occurring. When this per worker change is zero but the actual capital stock $K$ is growing (because of population growth), we say that only *capital widening* is occurring.

To consider a specific example, suppose an economy has capital equal to the amount $k_0$ today, as drawn in Figure 2.2. What happens over time? At $k_0$, the amount of investment per worker exceeds the amount needed to keep capital per worker constant, so that capital deepening occurs—that is, $k$ increases over time. This capital deepen-

FIGURE 2.2 **THE BASIC SOLOW DIAGRAM**

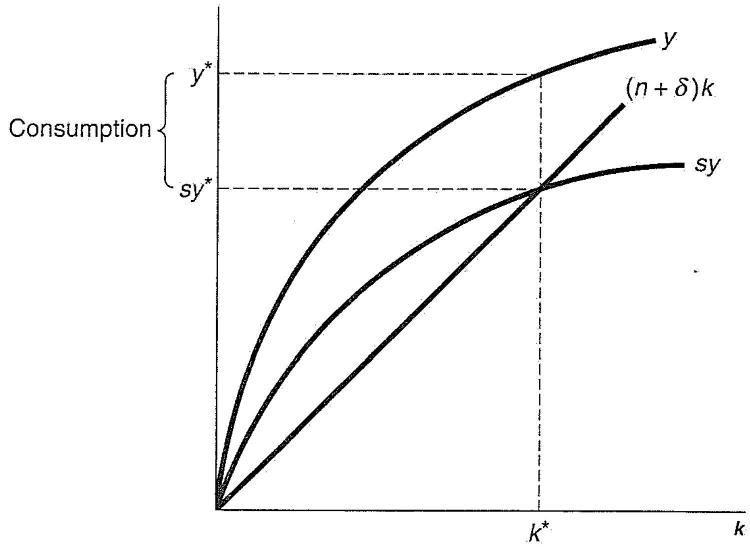



ing will continue until $k = k^*$, at which point $sy = (n + \delta)k$ so that $\dot{k} = 0$. At this point, the amount of capital per worker remains constant, and we call such a point a *steady state.*

What would happen if instead the economy began with a capital stock per worker larger than $k^*$? At points to the right of $k^*$ in Figure 2.2, the amount of investment per worker provided by the economy is less than the amount needed to keep the capital-labor ratio constant. The term $\dot{k}$ is negative, and therefore the amount of capital per worker begins to decline in this economy. This decline occurs until the amount of capital per worker falls to $k^*$.

Notice that the Solow diagram determines the steady-state value of capital per worker. The production function of equation (2.4) then determines the steady-state value of output per worker, $y^*$, as a function of $k^*$. It is sometimes convenient to include the production function in the Solow diagram itself to make this point clearly. This is done in Figure 2.3. Notice that steady-state consumption per worker is then given by the difference between steady-state output per worker, $y^*$, and steady-state investment per worker, $sy^*$.

FIGURE 2.3 THE SOLOW DIAGRAM AND THE PRODUCTION FUNCTION



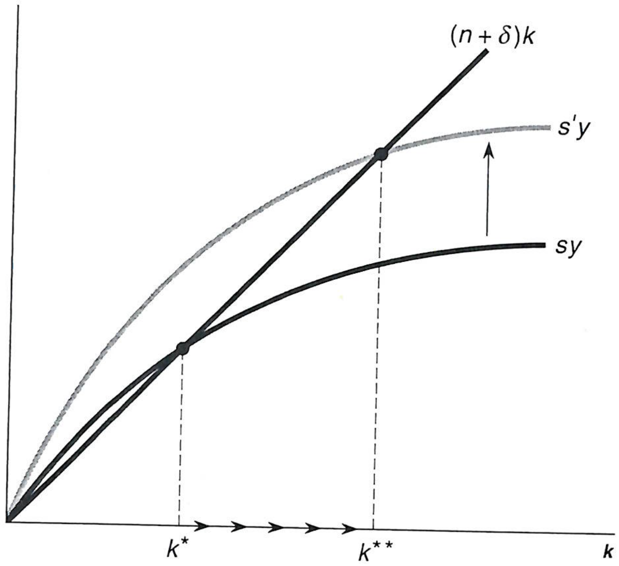

## 2.1.3 COMPARATIVE STATICS

Comparative statics are used to examine the response of the model to changes in the values of various parameters. In this section, we will consider what happens to per capita income in an economy that begins in steady state but then experiences a "shock." The shocks we will consider are an increase in the investment rate, $s$, and an increase in the population growth rate, $n$.

**AN INCREASE IN THE INVESTMENT RATE** Consider an economy that has arrived at its steady-state value of output per worker. Now suppose that the consumers in that economy decide to increase the investment rate permanently from $s$ to some value $s'$. What happens to $k$ and $y$ in this economy?

The answer is found in Figure 2.4. The increase in the investment rate shifts the $sy$ curve upward to $s'y$. At the current value of the capital stock, $k^*$, investment per worker now exceeds the amount required to keep capital per worker constant, and therefore the economy begins capital deepening again. This capital deepening continues until $s'y = (n + \delta)k$

FIGURE 2.4 **AN INCREASE IN THE INVESTMENT RATE**

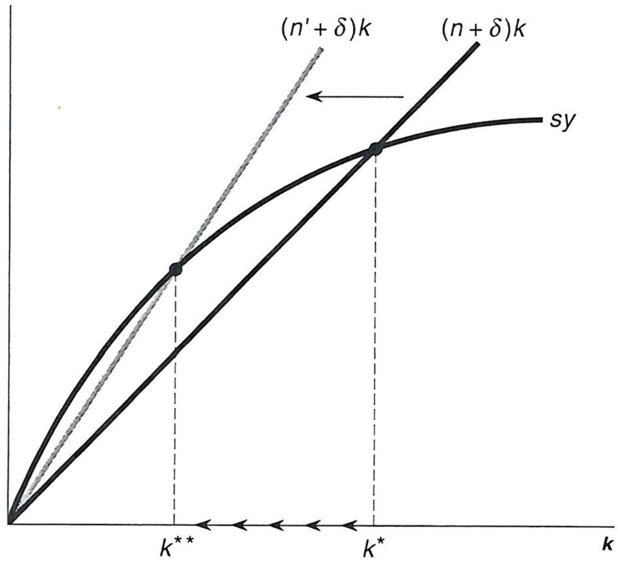



and the capital stock per worker reaches a higher value, indicated by the point $k^{**}$. From the production function, we know that this higher level of capital per worker will be associated with higher per capita output; the economy is now richer than it was before.

**AN INCREASE IN THE POPULATION GROWTH RATE** Now consider an alternative exercise. Suppose an economy has reached its steady state, but then because of immigration, for example, the population growth rate of the economy rises from $n$ to $n'$. What happens to $k$ and $y$ in this economy?

Figure 2.5 computes the answer graphically. The $(n + \delta)k$ curve rotates up and to the left to the new curve $(n' + \delta)k$. At the current value of the capital stock, $k^*$, investment per worker is now no longer high enough to keep the capital-labor ratio constant in the face of the rising population. Therefore the capital-labor ratio begins to fall. It continues to fall until the point at which $sy = (n' + \delta)k$, indicated by $k^{**}$ in Figure 2.5. At this point, the economy has less capital per worker than it began with and is therefore poorer: per capita output is ultimately lower after the increase in population growth in this example. Why?

FIGURE 2.5 AN INCREASE IN POPULATION GROWTH




## 2.1.4 PROPERTIES OF THE STEADY STATE

By definition, the steady-state quantity of capital per worker is determined by the condition that $\dot{k} = 0$. Equations (2.4) and (2.5) allow us to use this condition to solve for the steady-state quantities of capital per worker and output per worker. Substituting from (2.4) into (2.5),

$$\dot{k} = sk^{\alpha} - (n + \delta)k,$$

and setting this equation equal to zero yields

$$k^* = \left(\frac{s}{n + \delta}\right)^{1/(1-\alpha)}.$$

Substituting this into the production function reveals the steady-state quantity of output per worker, $y^*$:

$$y^* = \left(\frac{s}{n + \delta}\right)^{\alpha/(1-\alpha)}.$$

Notice that the endogenous variable $y^*$ is now written in terms of the parameters of the model. Thus, we have a "solution" for the model, at least in the steady state.

This equation reveals the Solow model's answer to the question "Why are we so rich and they so poor?" Countries that have high savings/investment rates will tend to be richer, *ceteris paribus*.[7] Such countries accumulate more capital per worker, and countries with more capital per worker have more output per worker. Countries that have high population growth rates, in contrast, will tend to be poorer, according to the Solow model. A higher fraction of savings in these economies must go simply to keep the capital-labor ratio constant in the face of a growing population. This capital-widening requirement makes capital deepening more difficult, and these economies tend to accumulate less capital per worker.

How well do these predictions of the Solow model hold up empirically? Figures 2.6 and 2.7 plot GDP per worker against gross investment as a share of GDP and against population growth rates, respectively. Broadly speaking, the predictions of the Solow model are borne out by the empirical evidence. Countries with high investment rates tend to be richer on average than countries with low investment rates, and countries with high population growth rates tend to be poorer on average. At this level, then, the general predictions of the Solow model seem to be supported by the data.[8]

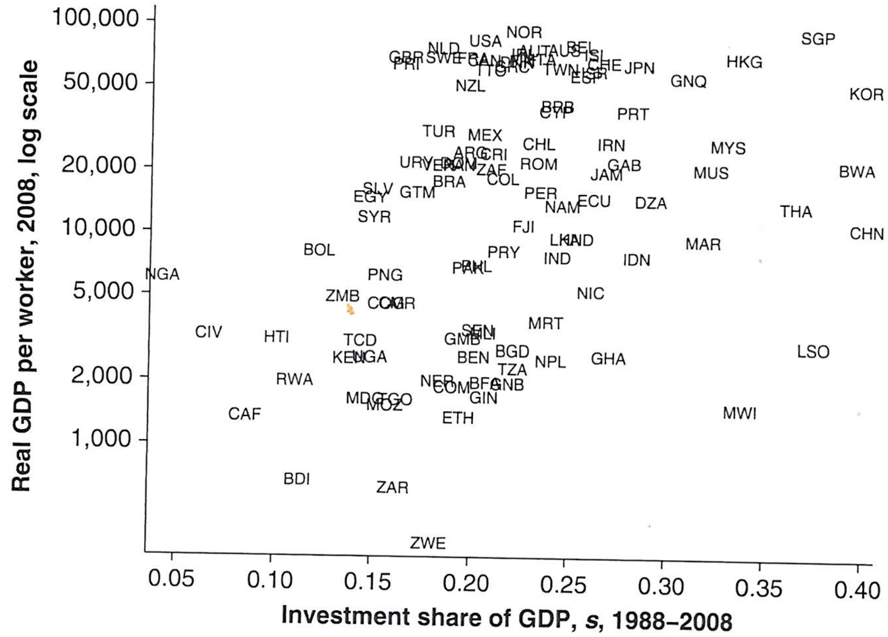
## 2.1.5 ECONOMIC GROWTH IN THE SIMPLE MODEL

What does economic growth look like in the steady state of this simple version of the Solow model? The answer is that there is *no* per capita growth in this version of the model! Output per worker (and therefore per person, since we've assumed the labor force participation rate is

[7] *Ceteris paribus* is Latin for "all other things being equal."

[8] Chang-Tai Hsieh and Pete Klenow (2007) highlight a very important observation regarding the relationship of investment rates and GDP per worker. Lower investment rates need not reflect a lower willingness to save or policies that tax investment. Rather, the low observed real investment rates in poor countries may represent low productivity in turning their savings into actual investment goods. We'll return to this possibility in Chapter 7 when we offer an explanation for investment rates.

FIGURE 2.6 GDP PER WORKER VERSUS THE INVESTMENT RATE




constant) is constant in the steady state. Output itself, $Y$, is growing, of course, but only at the rate of population growth.[9]

This version of the model fits several of the stylized facts discussed in Chapter 1. It generates differences in per capita income across countries. It generates a constant capital-output ratio (because both $k$ and $y$ are constant, implying that $K/Y$ is constant). It generates a constant interest rate, the marginal product of capital. However, it fails to predict a very important stylized fact: that economies exhibit sustained per capita income growth. In this model, economies may grow for a while, but not forever. For example, an economy that begins with a stock of capital per worker below its steady-state value will experience growth in $k$ and $y$ along the *transition path* to the steady state. Over time, however, growth slows down as the economy approaches its steady state, and eventually growth stops altogether.

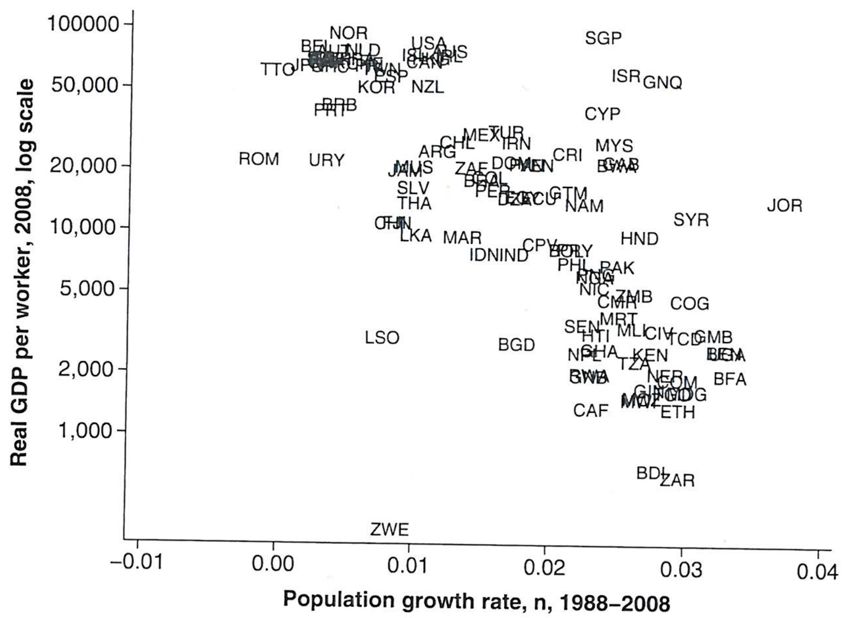
[9] This can be seen easily by applying the "take logs and differentiate" trick to $y \equiv Y/L$.

Det finnes lett tilgjengelige data av typen brukt i plottene her. Det mest oppdaterte er versjon 10.01 av Penn World Table som har sammenlignbare data for 183 land fra 1950 til 2019.

Du får tilgang gjennom Groningen Growth and Development Centre som også gir fyldig og god dokumentasjon



FIGURE 2.7 **GDP PER WORKER VERSUS POPULATION GROWTH RATES**




To see that growth slows down along the transition path, notice two things. First, from the capital accumulation equation (equation (2.5)), one can divide both sides by $k$ to get

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (n + \delta). \tag{2.6}$$

Because $\alpha$ is less than one, as $k$ rises, the growth rate of $k$ gradually declines. Second, from Example 2, the growth rate of $y$ is proportional to the growth rate of $k$, so that the same statement holds true for output per worker.

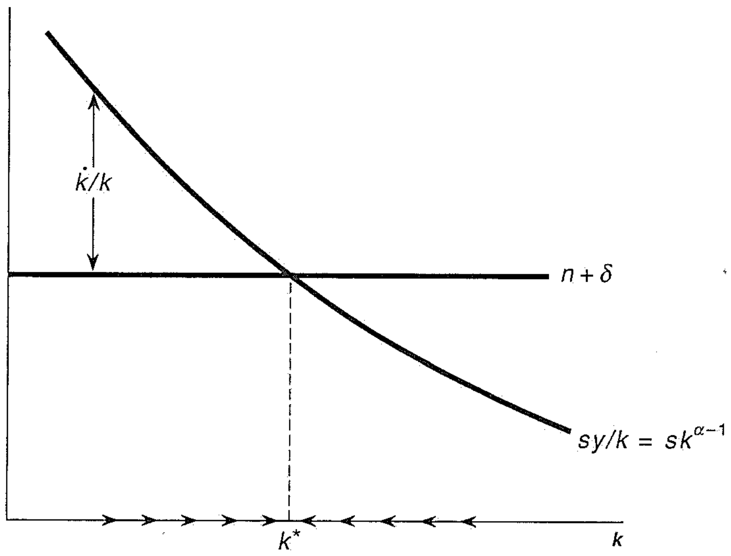
The transition dynamics implied by equation (2.6) are plotted in Figure 2.8.[10] The first term on the right-hand side of the equation is $sk^{\alpha-1}$, which is equal to $sy/k$. The higher the level of capital per worker,

!!

[10]This alternative version of the Solow diagram makes the growth implications of the Solow model much more transparent. Xavier Sala-i-Martin (1990) emphasizes this point.

Jfr. kommentaren min i begynnelsen av kapittelet (skrevet i blått)

FIGURE 2.8 TRANSITION DYNAMICS




the lower the average product of capital, $y/k$, because of diminishing returns to capital accumulation ($\alpha$ is less than one). Therefore, this curve slopes downward. The second term on the right-hand side of equation (2.6) is $n + \delta$, which doesn't depend on $k$, so it is plotted as a horizontal line. The difference between the two lines in Figure 2.8 is the growth rate of the capital stock, or $\dot{k}/k$. Thus, the figure clearly indicates that the further an economy is below its steady-state value of $k$, the faster the economy grows. Also, the further an economy is above its steady-state value of $k$, the faster $k$ declines.

## 2.2 TECHNOLOGY AND THE SOLOW MODEL

To generate sustained growth in per capita income in this model, we must follow Solow and introduce technological progress to the model. This is accomplished by adding a technology variable, $A$, to the production function:

$$Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}. \tag{2.7}$$

# APPENDIX A MATHEMATICAL REVIEW

This appendix presents a simple review of the mathematical tools used throughout the book. It assumes some basic familiarity with calculus and covers techniques that are commonly used in modeling economic growth and development. A special effort has been made to include more words than equations. We hope this will permit a quick and easy understanding of the mathematics used in this book. For additional details, please refer to an introductory calculus textbook.

## A.1 DERIVATIVES

The derivative of some function $f(x)$ with respect to $x$ reveals how $f(\cdot)$ changes when $x$ changes by a very small amount. If $f(\cdot)$ increases when $x$ increases, then $df/dx > 0$, and vice versa. For example, if $f(x) = 5x$, then $df/dx = 5$, or $df = 5\ dx$: For every small change in $x$, $f(\cdot)$ changes by five times that amount.

### A.1.1 WHAT DOES $\dot{K}$ MEAN?

In discussing economic growth, the most common derivative used is a derivative with respect to time. For example, the capital stock, $K$, is a function of time $t$, just like $f$ was a function of $x$ above. We can ask how the capital stock changes over time; this is fundamentally a question about the derivative $dK/dt$. If the capital stock is growing over time, then $dK/dt > 0$.

For derivatives with respect to time, it is conventional to use the "dot notation": $dK/dt$ is then written as $\dot{K}$—the two expressions are equivalent. For example, $\dot{K} = 5$, if then for each unit of time that passes, the capital stock increases by five units.

Notice that this derivative, $\dot{K}$, is very closely related to$K_{1997} - K_{1996}$. How does it differ? First, let's rewrite the change from 1996 to 1997 as $K_t - K_{t-1}$. This second expression is more general; we can evaluate it at $t = 1997$ or at $t = 1990$ or at $t = 1970$. Thus we can think of this change as a change per unit of time, where the unit of time is one period. Next, $\dot{K}$ is an *instantaneous* change rather than the change across an entire year. We could imagine calculating the change of the capital stock across one year, or across one quarter, or across one week, or across one day, or across one hour. *As the time interval across which we calculate the change shrinks, the expression $K_t - K_{t-1}$, expressed per unit of time, approaches the instantaneous change $\dot{K}$.* Formally, this is exactly the definition of a derivative. Let $\Delta t$ be our time interval (a year, a day, or an hour). Then,

$$\lim_{\Delta t \to 0} \frac{K_t - K_{t-\Delta t}}{\Delta t} = \frac{dK}{dt}.$$

## A.1.2 WHAT IS A GROWTH RATE?

Growth rates are used throughout economics, science, and finance. In economics, examples of growth rates include the inflation rate—if the inflation rate is 3 percent, then the price level is rising by 3 percent per year. The population growth rate is another example—population is increasing at something like 1 percent per year in the advanced economies of the world.

The easiest way to think about growth rates is as percentage changes. If the capital stock grew by 4 percent last year, then the change in the capital stock over the course of the last year was equal to 4 percent of its starting level. For example, if the capital stock began at \$10 trillion and rose to \$10.4 trillion, we might say that it grew by 4 percent. So one way of calculating a growth rate is as a percentage change:

$$\frac{K_t - K_{t-1}}{K_{t-1}}.$$

For mathematical reasons that we will explore below, it turns out to be easier in much of economics to think about the *instantaneous* growth rate. That is, we define the growth rate to be the derivative $dK/dt$ divided by its starting value, $K$. As discussed in the preceding section, we use $\dot{K}$ to represent $dK/dt$. Therefore, $\dot{K}/K$ is a growth rate. Whenever you see such a term, just think "*percentage change.*"

A couple of examples may help clarify this concept. First, suppose $\dot{K}/K = .05$; this says that the capital stock is growing at 5 percent per year. Second, suppose $\dot{L}/L = .01$; this says that the labor force is growing at 1 percent per year.

## A.1.3 GROWTH RATES AND NATURAL LOGS

The mathematical reason why this definition of growth rates is convenient can be seen by considering several properties of the natural logarithm:

1. If $z = xy$, then $\log z = \log x + \log y$.
2. If $z = x/y$, then $\log z = \log x - \log y$.
3. If $z = x^{\beta}$, then $\log z = \beta \log x$.
4. If $y = f(x) = \log x$, then $dy/dx = 1/x$.
5. If $y(t) = \log x(t)$, then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{x}\dot{x} = \frac{\dot{x}}{x}.$$

The first of these properties is that the natural log of the product of two (or more) variables is the sum of the logs of the variables. The second property is very similar, but relates the division of two variables to the difference of the logs. The third property allows us to convert exponents into multiplicative terms. The fourth property says that the derivative of the log of some variable $x$ is just $1/x$.

The fifth property is a key one. In effect, it says that *the derivative with respect to time of the log of some variable is the growth rate of that variable.* For example, consider the capital stock, $K$. According to property 5 above,

$$\frac{d \log K}{dt} = \frac{\dot{K}}{K},$$

which, as we saw in Section A.1.3, is the growth rate of $K$.

## A.1.4 "TAKE LOGS AND DERIVATIVES"

Each of the properties of the natural logarithm listed in the preceding section is used in the "take logs and derivatives" example below. Consider a simple Cobb-Douglas production function:

$$Y = K^{\alpha} L^{1-\alpha}.$$

If we take logs of both sides, then

$$\log Y = \log K^{\alpha} + \log L^{1-\alpha}.$$

Moreover, by property 3 discussed in Section A.1.3,

$$\log Y = \alpha \log K + (1 - \alpha) \log L.$$

Finally, by taking derivatives of both sides with respect to time, we can see how the growth rate of output is related to the growth rate of the inputs in this example:

$$\frac{d \log Y}{dt} = \alpha \frac{d \log K}{dt} + (1 - \alpha) \frac{d \log L}{dt},$$

which implies that

$$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1 - \alpha) \frac{\dot{L}}{L}.$$

This last equation says that the growth rate of output is a weighted average of the growth rates of capital and labor.

## A.1.5 RATIOS AND GROWTH RATES

Another very useful application of these properties is in situations in which the ratio of two variables is constant. First, notice that if a variable is constant, its growth rate is zero—it is not changing, so its time derivative is zero.

Now, suppose that $z = x/y$ and suppose we know that $z$ is constant over time—that is, $\dot{z} = 0$. Taking logs and derivatives of this relationship, one can see that

$$\frac{\dot{z}}{z} = \frac{\dot{x}}{x} - \frac{\dot{y}}{y} = 0 \Rightarrow \frac{\dot{x}}{x} = \frac{\dot{y}}{y}.$$

Therefore, if the ratio of two variables is constant, the growth rates of those two variables must be the same. Intuitively, this makes sense. If the numerator of the ratio were growing faster than the denominator, the ratio itself would have to be growing over time.

## A.1.6 $\Delta$log VERSUS PERCENTAGE CHANGE

Suppose a variable exhibits exponential growth:

$$y(t) = y_0 e^{gt}.$$

For example, $y(t)$ could measure per capita output for an economy. Then,

$$\log y(t) = \log y_0 + gt,$$

and therefore the growth rate, $g$, can be calculated as

$$g = \frac{1}{t}(\log y(t) - \log y_0).$$

Or, calculating the growth rate between time $t$ and time $t - 1$,

$$g = \log y(t) - \log y(t-1) \equiv \Delta \log y(t).$$

These last two equations provide the justification for calculating growth rates as the change in the log of a variable.

How does this calculation relate to the more familiar percentage change? The answer is straightforward:

$$\frac{y(t) - y(t-1)}{y(t-1)} = \frac{y(t)}{y(t-1)} - 1$$
$$= e^g - 1.$$

Recall that the Taylor approximation for the exponential function is $e^x \approx 1 + x$ for small values of $x$. Applying this to the last equation shows that the percentage change and the change in log calculations are approximately equivalent for small growth rates:

$$\frac{y(t) - y(t-1)}{y(t-1)} \approx g.$$

## A.2 INTEGRATION

Integration is the calculus equivalent of summation. For example, one could imagine a production function written as

$$Y = \sum_{i=1}^{10} x_i = x_1 + x_2 + \cdots + x_{10}, \tag{A.1}$$

that is, output is simply the sum of ten different inputs. One could also imagine a related production function

$$Y = \int_0^{10} x_i di. \tag{A.2}$$

In this production function, output is the weighted sum of a continuum of inputs $x_i$ that are indexed by the interval of the real line between zero and ten. Obviously, there are an infinite number of inputs in this second production function, because there are an infinite number of real numbers in this interval. However, each input is "weighted" by the average size of an interval, $di$, which is very small. This keeps production finite, even if each of our infinite number of inputs is used in positive amounts. Don't get too confused by this reasoning. Instead, think of integrals as sums, and think of the second production function in the