Proposed Solution Homework 1 SOK-3008

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1 Question 1

Given $U(\mathbf{q}) = q_1^2 q_2^2$ and $y = p_1 q_1 + p_2 q_2$ we have the problem

$$\max_{q} \mathcal{L} = q_1^2 q_2^2 - \lambda(p_1 q_1 + p_2 q_2 - y) \tag{1}$$

FOCS

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_1} &= 2 q_1 q_2^2 - \lambda p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial q_2} &= 2 q_1^2 q_2 - \lambda p_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= y - p_1 q_1 - p_2 q_2 = 0
\end{aligned}$$

Solve both for λ such that

$$\begin{aligned}
\lambda &= \lambda \\
\implies \frac{2 q_1 q_2^2}{p_1} &= \frac{2 q_1^2 q_2}{p_2}
\end{aligned}$$

Simplify such that

$$\frac{q_2}{p_1} = \frac{q_1}{p_2} \implies \begin{cases} q_1 = q_2 \frac{p_2}{p_1} \\ q_2 = q_1 \frac{p_1}{p_2} \end{cases}$$

Insert into $y - p_1 q_1 - p_2 q_2 = 0$
For q_1

$$y = p_1 q_2 \frac{p_2}{p_1} + p_2 q_2 = 2 p_2 q_2 \tag{2}$$

Consequently optimal demand for q_2 (and q_1, same procedure just insert q_2 in constraint) is:

$$q_2 = \frac{y}{2p_2}$$
$$q_1 = \frac{y}{2p_1}$$

Elasticities are:

$$E_{11} = \frac{\partial q_1}{\partial p_1} \frac{p_1}{q_1} = -1 (= E_{22})$$
$$E_{12} = \frac{\partial q_1}{\partial p_2} \frac{p_2}{q_1} = 0 (= E_{21})$$
$$A_1 = \frac{\partial q_1}{\partial y} \frac{y}{q_1} = 1 (= A_2)$$

And the shares:

$$R_1 = \frac{p_1 q_1}{y} = \frac{p_1 \frac{y}{2p_1}}{y} = \frac{1}{2} (= R_2) \tag{3}$$

2 Question 2

Engel Aggregation:

$$R_1 A_1 + R_2 A_2 = 1$$
$$\frac{1}{2} \times 1 + \frac{1}{2} \times 1 = 1$$

Satisfied!
Cournot Aggregation:

$$R_1E_{11} + R_2E_{21} = -R_1$$
$$\text{and}\; R_1E_{12} + R_2E_{22} = -R_2$$

$$R_1E_{11} + R_2E_{21} = \frac{1}{2} \times (-1) + \frac{1}{2} \times 0 = -\frac{1}{2} = -R_1$$
$$R_1E_{12} + R_2E_{22} = \frac{1}{2} \times 0 + \frac{1}{2} \times (-1) = -\frac{1}{2} = -R_2$$

Satisfied!
Symmetry

$$E_{12} = \frac{R_2}{R_1}E_{21} + R_2(A_2 - A_1)$$
$$E_{12} = \frac{\frac{1}{2}}{\frac{1}{2}} \times 0 + \frac{1}{2}(1-1) = 0$$

Satisfied!
Homogeneity

$$E_{11} + E_{12} = -A_1$$
$$E_{21} + E_{22} = -A_2$$

$$E_{11} + E_{12} = -1 + 0 = -1 = -A_1$$
$$E_{21} + E_{22} = 0 + (-1) = -1 = -A_2$$

Satisfied!

3 Question 3

3.1 Min-cost problem for Hicksian

$$\min_q \mathcal{L} = p_1q_1 + p_2q_2 + \lambda(U - q_1^2q_2^2) \tag{4}$$

FOCs

$$\frac{\partial \mathcal{L}}{\partial q_1} = p_1 - 2\lambda q_2^2 q_1 = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = p_2 - 2\lambda q_1^2 q_2 = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = U - q_1^2 q_2^2 = 0 \tag{7}$$

Find $\lambda = \lambda$ from (5) and (6), i.e.

$$\begin{aligned} \lambda &= \lambda \\ \implies \frac{p_1}{2q_2^2 q_1} &= \frac{p_2}{2q_1^2 q_2} \\ \implies \frac{p_1}{q_2} &= \frac{p_2}{q_1} \implies \begin{cases} q_1 = \frac{p_2 q_2}{p_1} \\ q_2 = \frac{p_1 q_1}{p_2} \end{cases} \end{aligned}$$

Insert q_1 and q_2 into $U - q_1^2 q_2^2 = 0$ (separately). For q_1 inserted:

$$U = \frac{p_2^2 q_2^4}{p_1^2} \tag{8}$$

Solve for q_2:

$$q_2 = \sqrt{\sqrt{U}\frac{p_1}{p_2}} \tag{9}$$

Similar for inserting q_2:

$$q_1 = \sqrt{\sqrt{U}\frac{p_2}{p_1}} \tag{10}$$

Which is our optimal Hicksian demand for q_1 and q_2, respectively.

Rewrite a bit to get elasticities easier (I will only show all the steps for own-price elasticity. It's the same procedure for Cross-price and shares):

$$q_2 = U^{1/4} p_1^{1/2} p_2^{-1/2} \tag{11}$$

$$E_{22} = \frac{\partial q_2}{\partial p_2}\frac{p_2}{q_2} = \underbrace{-\frac{1}{2}U^{1/4}p_1^{1/2}p_2^{-3/2}}_{\frac{\partial q_2}{\partial p_2}}\underbrace{\frac{p_2}{U^{1/4}p_1^{1/2}p_2^{-1/2}}}_{\frac{p_2}{q_2}} = -\frac{1}{2} = E_{11}$$

$$E_{12} = E_{21} = \frac{1}{2}$$

$$R_1 = R_2 = \frac{1}{2}$$

3.2 Duality approach

From question 1 we have optimal marshallian demand:

$$q_2 = \frac{y}{2p_2}$$
$$q_1 = \frac{y}{2p_1}$$

Insert into $U = q_1^2 q_2^2$ for the indirect utility function, $v(\mathbf{p}, y)$

$$v(\mathbf{p}, y) = \left(\frac{y}{2p_2}\right)^2\left(\frac{y}{2p_1}\right)^2 = \frac{1}{(4p_1p_2)^2}y^4 \tag{12}$$

From microeconomic theory we have that $v(\mathbf{p}, e(\mathbf{p}, U)) = U$. I.e., we substitute $e(\mathbf{p}, U)$ for y and solve for $e(\mathbf{p}, U)$:

$$v(\mathbf{p}, e(\mathbf{p}, U)) = e(\mathbf{p}, U)^4\frac{1}{(4p_1p_2)^2} = U$$
$$e(\mathbf{p}, U) = \sqrt[4]{U}\sqrt{4p_1p_2}$$

We cna now use Sheppard's Lemma, $\frac{\partial e(\mathbf{p},U)}{\partial p_i} = q^h(\mathbf{p}, U)$, to get our Hicksian optimal demand.

$$\frac{\partial e(\mathbf{p}, U)}{\partial p_1} = \sqrt{\sqrt{U}\frac{p_2}{p_1}} = q_1^h(\mathbf{p}, U) = q_1$$

Exactly the same result as from the minimization problem.

For q_2 we also have

$$q_2 = \sqrt{\sqrt{U}\frac{p_1}{p_2}} \tag{13}$$

4 Question 4

Follow same procedure as for question 2, but only for the Hicksian restrictions this time (see the note on general restrictions.pdf on fronter).

5 Question 5

Think about Slutsky relationship, i.e.

$$\frac{\partial q_i}{\partial p_j} = \left(\frac{\partial q_i}{p_j}\right)^* - q_j \frac{\partial q_i}{\partial y} \tag{14}$$

And on elasticity form (we will derive this next lecture, but this should be quite straight forward for you to do by now. Follow same principles from the derivation of the Marshallian theoretical restrictions):

$$E_{ij} = E_{ij}^* - R_j A_i \tag{15}$$

Where E_{ij} is Marshallian elasticity and E_{ij}^* is Hicksian. I.e. The marshallian also includes the income effect (think about how a consumer reacts to a price change in a two-good diagram with indifference curve and budget constraint). The Hicksian is the compensated effect, i.e. substitution effect and is a more "pure" effect when measuring the effect of a price change. Policy makers vs industry etc.