4 – APT and Fama-French

Espen Sirnes

2024-09-18

Table of contents

1 Arbitrage Pricing Theory (APT)

1.1 The Capital Asset Pricing Model (CAPM)

The Capital Asset Pricing Model (CAPM) tells us that the expected risk premium of an asset is proportional to that of the market:

$$E[r_i|r_M] - r_f = \beta_i(r_M - r_f) \tag{1}$$

However, CAPM has several shortcomings. First, its theoretical derivation requires a number of assumptions that are quite restrictive. For example, for CAPM to hold in equilibrium, everyone must hold identical portfolios, which we know is not true in reality.

Second, empirical evidence suggests that expected returns depend on more than just the market factor.

1.2 The idea of the Arbitrage Pricing Theory (APT)

In 1976, Stephen Ross proposed the Arbitrage Pricing Theory (APT), which relies on the following assumptions:

1. Risky returns can be explained by a factor model.
2. There are sufficient assets to diversify away any idiosyncratic risk.
3. Well-functioning markets do not allow the persistence of arbitrage opportunities.

With the first assumption above, the return of asset i can be described by a one-factor model:

$$r_i = \mu_i + \beta_{i,F} F + e_i \tag{2}$$

where: - $ i $ is the unconditional expected return, - $ F $ is the factor risk premium, - $ {i,F} $ is the asset's exposure to the factor.

If we consider the factor F as the market return, we get the realized CAPM:

$$r_i - r_f = \beta_{i,F}(r_M - r_f) + e_i \tag{3}$$

This differs from the expected CAPM in that the left-hand term is not an expectation, and the right-hand term includes a residual term e_i.

The term e_i represents the unexplained asset return, or idiosyncratic risk, which is uncorrelated with any known factors affecting other assets. It is also uncorrelated with other assets. This independence means that the total variance of idiosyncratic risk in a portfolio is simply the sum of the idiosyncratic contributions from each asset.

For instance, let's assume we have a portfolio of n assets with equal weights $1/n$, and that the idiosyncratic variance is identical across assets: $\mathrm{var}[e_i] = \sigma_e^2$. The contribution of each asset to the total variance is:

$$\text{var}\left[\frac{1}{n}e_i\right] = \frac{1}{n^2}\sigma_e^2$$

Since there are \$ n \$ such contributions, the total idiosyncratic variance of the portfolio will be:

$$\text{var}\left(\sum_{i=1}^{n} e_i\right) = n \cdot \text{var}\left[\frac{1}{n}e_i\right] = \frac{1}{n}\sigma_e^2 \tag{4}$$

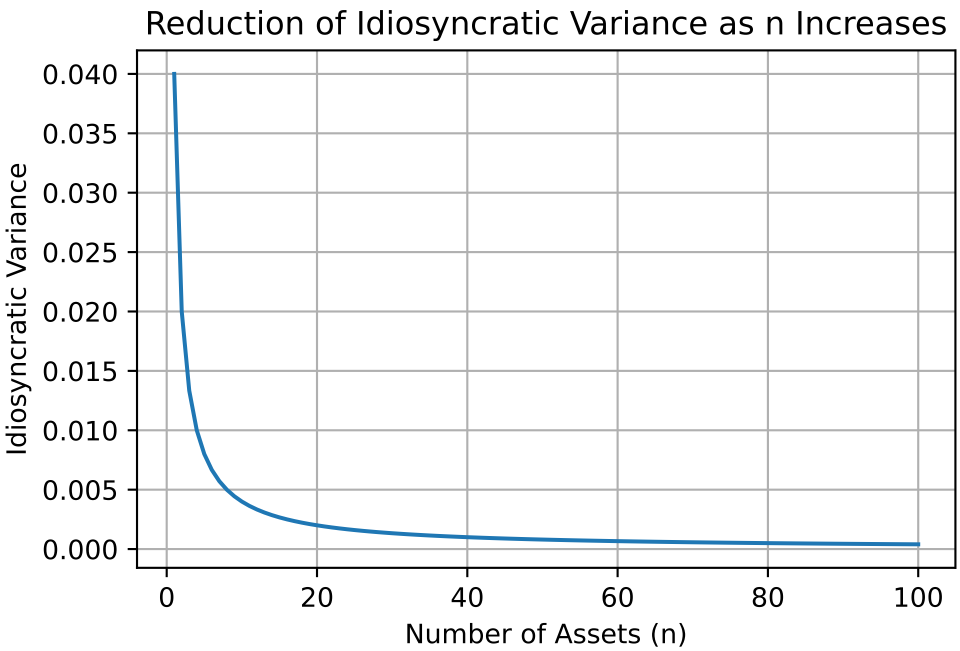
Hence, as the number of assets n increases, the total idiosyncratic variance diminishes. For large n, the idiosyncratic variance becomes negligible.

```
import numpy as np
import matplotlib.pyplot as plt

# Example: Plotting idiosyncratic variance reduction as n increases
n_values = np.arange(1, 101)  # Number of assets
sigma_e_squared = 0.04  # Example idiosyncratic variance

# Calculate idiosyncratic variance for each n
idiosyncratic_variance = sigma_e_squared / n_values

# Plot the results
plt.plot(n_values, idiosyncratic_variance)
plt.xlabel('Number of Assets (n)')
plt.ylabel('Idiosyncratic Variance')
plt.title('Reduction of Idiosyncratic Variance as n Increases')
plt.grid(True)
plt.show()
```



The plot above demonstrates how idiosyncratic variance decreases as the number of assets n increases, illustrating that a well-diversified portfolio can effectively eliminate idiosyncratic risk.

From now on, we assume that if a portfolio is well-diversified, we can ignore idiosyncratic risk.

1.3 Two Portfolios – One Factor

Assume that we have two large, well-diversified portfolios, A and B, with zero idiosyncratic risk. Both portfolios depend on the same factor:

$$r_A = \mu_A + \beta_A F$$

$$r_B = \mu_B + \beta_B F \tag{5}$$

Now consider buying $\frac{1}{\beta_A}$ of portfolio A and selling $\frac{1}{\beta_B}$ of portfolio B. The investor would then earn:

$$\frac{r_A}{\beta_A} - \frac{r_B}{\beta_B} = \frac{\mu_A}{\beta_A} - \frac{\mu_B}{\beta_B} \tag{6}$$

This strategy has zero exposure to the factor, meaning the investment is risk-free, even in the short run. If the return from this position is positive, there exists an arbitrage opportunity because all risk has been removed.

In this case, we would expect investors to buy stocks in portfolio A and sell stocks in portfolio B. Portfolio A's price would rise, and portfolio B's price would fall, until expected returns are equalized. Therefore, in equilibrium:

$$\frac{\mu_A}{\beta_A} = \frac{\mu_B}{\beta_B} \tag{7}$$

The risk-adjusted expected return must be the same for any pair of portfolios. Hence, with only one factor (e.g., the market index), all variation in returns should, in equilibrium, be explained by different exposures to that factor.

1.4 Three Portfolios – Two Factors

We can extend this idea by adding additional factors. These factors could be macroeconomic (GDP, inflation, etc.) or firm-specific characteristics (size, performance). These categories can be treated as factors.

Assume that portfolios A, B, and C each depend on two factors:

$$r_A = \mu_A + \beta_{A,1} F_1 + \beta_{A,2} F_2 \tag{8}$$

$$r_B = \mu_B + \beta_{B,1} F_1 + \beta_{B,2} F_2 \tag{9}$$

$$r_C = \mu_C + \beta_{C,1} F_1 + \beta_{C,2} F_2 \tag{10}$$

Let us divide equations (8) and (9) by $\beta_{A,1}$ and $\beta_{B,1}$ respectively, and calculate the difference:

$$\frac{r_A}{\beta_{A,1}} - \frac{r_B}{\beta_{B,1}} = \left(\frac{\mu_A}{\beta_{A,1}} - \frac{\mu_B}{\beta_{B,1}}\right) + \left(\frac{\beta_{A,2}}{\beta_{A,1}} - \frac{\beta_{B,2}}{\beta_{B,1}}\right) F_2 \tag{11}$$

We can follow the same procedure with equations (9) and (10) by dividing them by $\beta_{B,1}$ and $\beta_{C,1}$, respectively:

$$\frac{r_B}{\beta_{B,1}} - \frac{r_C}{\beta_{C,1}} = \left(\frac{\mu_B}{\beta_{B,1}} - \frac{\mu_C}{\beta_{C,1}}\right) + \left(\frac{\beta_{B,2}}{\beta_{B,1}} - \frac{\beta_{C,2}}{\beta_{C,1}}\right) F_2 \tag{12}$$

Now, the problem is reduced to a one-factor problem, and the result from the previous section (with only one factor) holds here as well. The difference is that the factor loadings and expected returns are now represented by:

- Factor loadings: $\frac{\beta_{A,2}}{\beta_{A,1}} - \frac{\beta_{B,2}}{\beta_{B,1}}$ and $\frac{\beta_{B,2}}{\beta_{B,1}} - \frac{\beta_{C,2}}{\beta_{C,1}}$
- Expected returns: $\frac{\mu_A}{\beta_{A,1}} - \frac{\mu_B}{\beta_{B,1}}$ and $\frac{\mu_B}{\beta_{B,1}} - \frac{\mu_C}{\beta_{C,1}}$

Let's define these new factor loadings as:

$$\beta_{AB} = \frac{\beta_{A,2}}{\beta_{A,1}} - \frac{\beta_{B,2}}{\beta_{B,1}} \tag{13}$$

$$\beta_{BC} = \frac{\beta_{B,2}}{\beta_{B,1}} - \frac{\beta_{C,2}}{\beta_{C,1}} \tag{14}$$

And the expected returns as:

$$\mu_{AB} = \frac{\mu_A}{\beta_{A,1}} - \frac{\mu_B}{\beta_{B,1}} \tag{15}$$

$$\mu_{BC} = \frac{\mu_B}{\beta_{B,1}} - \frac{\mu_C}{\beta_{C,1}} \tag{16}$$

We can cancel out the effect of factor F_2 the same way as in the previous section, by dividing each equation by the factor loading. As in the previous section, the resulting portfolio is a zero-cost, risk-free portfolio, which must return zero in the absence of arbitrage:

$$\frac{\mu_{AB}}{\beta_{AB}} = \frac{\mu_{BC}}{\beta_{BC}}$$

Since both these "portfolios of portfolios" depend on only one factor, the expected returns from equations (11) and (12) must be the same.

Thus, we can extend the simple case with one factor to any number of factors. The general conclusion is that:

> In equilibrium, all variation in expected returns is explained by different exposures to factors.

This finding is consistent with empirical observations. As we will see, asset returns systematically depend on a number of factors beyond just the market factor. This observation aligns with the Arbitrage Pricing Theory (APT) but contradicts the traditional CAPM, which assumes only one factor (the market).

1.5 The general case

The Arbitrage Pricing Theory (APT) tells us that portfolios generally depend on various factors. The return of a portfolio P can be written as:

$$r_P = \mu_P + \beta_1 F_1 + \beta_2 F_2 + \cdots + \beta_n F_n \tag{17}$$

If all investors held the same portfolio, there could only be one such factor: the market index. The fact that the APT model is more consistent with empirical evidence than the CAPM is a clear indication that investors do not hold identical portfolios.

APT is much more flexible than CAPM, allowing any factor to be used as an explanatory variable in addition to the market index. The factors do not need to be independent. The independent factor portfolios are merely a tool to explain why differences in return must be explained by differences in factor loadings.

One of APT's strengths is that it does not require strong assumptions. The most important assumption is the absence of arbitrage, a reasonable approximation in most financial markets.

1.6 Examples of Factors

Here are some examples of factors that can be used in APT:

- % change in industrial production
- % change in expected inflation
- % change in unanticipated inflation

- Excess return of corporate bonds over government bonds
- Excess return of long-term bonds over short-term bonds/certificates
- Return of small firms relative to big firms
- Return of firms with high book-to-market value relative to those with low B/M ratios.

2 The Fama-French Three-Factor Model

In 1993, Eugene Fama and Kenneth R. French introduced the following two additional factors to explain asset returns:

- SMB (Small Minus Big): The return difference between small and large companies.
- HML (High Minus Low): The return difference between companies with high and low book-to-market ratios.

Using these factors in addition to the market factor, the Fama-French model can be expressed as:

$$r_i = \alpha_i + \beta_M(r_M - r_f) + \beta_{SMB} \cdot SMB + \beta_{HML} \cdot HML + e_i \tag{18}$$

This formulation has been very successful in explaining excess returns. Below is a diagram that shows the efficient frontier constructed from the market, SMB, and HML. It shows that the market portfolio lies quite far inside the frontier, suggesting that the gains from betting on HML or SMB could be substantial.

2.1 Visualization of Efficient Frontiers

While the text references the efficient frontier for the Fama-French factors, we can visualize the potential benefits of betting on SMB and HML alongside the market portfolio.

```
# Code for visualizing performance of SMB and HML - the portfolio fr
```

2.2 Implications of Fama-French Model

The Fama-French model became a market standard for measuring the performance of fund managers, where r_i is the return of the fund. Historically, most evidence suggested that fund managers, as a group, were not able to outperform the index. However, some managers consistently beat the market.

Using the Fama-French three-factor model, it was revealed that many of these managers systematically bet on small companies or those with high book-to-market ratios—strategies that can easily be automated. However, it is not obvious that such a strategy will always succeed, as factor returns can vary substantially over time.

2.3 Simulation: Dynamic Factor Betting

Factor returns vary significantly over time, and this variation, along with transaction costs due to increased trading, may make short-term factor betting unattractive. Below are returns from dynamic factor betting on Oslo Stock Exchange (data from Titlon).

```
# Code for dynamic factor betting here
```

- Market Only: Sharpe ratio from holding only the market portfolio.
- Equal Weights: Strategy holding the market, SMB, and HML in equal weights.
- Ex Post: The strategy with perfect hindsight, knowing future factor performance.
- Short-term Dynamic: A strategy that rolls over the portfolio every year based on past information.
- Long-term Strategic: A strategy that rolls over the portfolio every 10 years, using only past information.

2.4 Insights from the Simulation

The results show that long-term strategic factor betting outperforms short-term dynamic strategies. The equal weights strategy performs almost as well as the long-term strategy, and given that we have not accounted for transaction costs, it may actually be the best alternative.

This suggests that if investors want to bet on factors, they should take a long-term perspective and avoid making frequent adjustments to their portfolios.