Applied Production Analysis

SOK-3011—Part 1

Course content

- Focus on production theory & empirical applications
  - Theoretical readings: Varian (1992)
  - Empirical readings: Henningsen (2024, R-based)

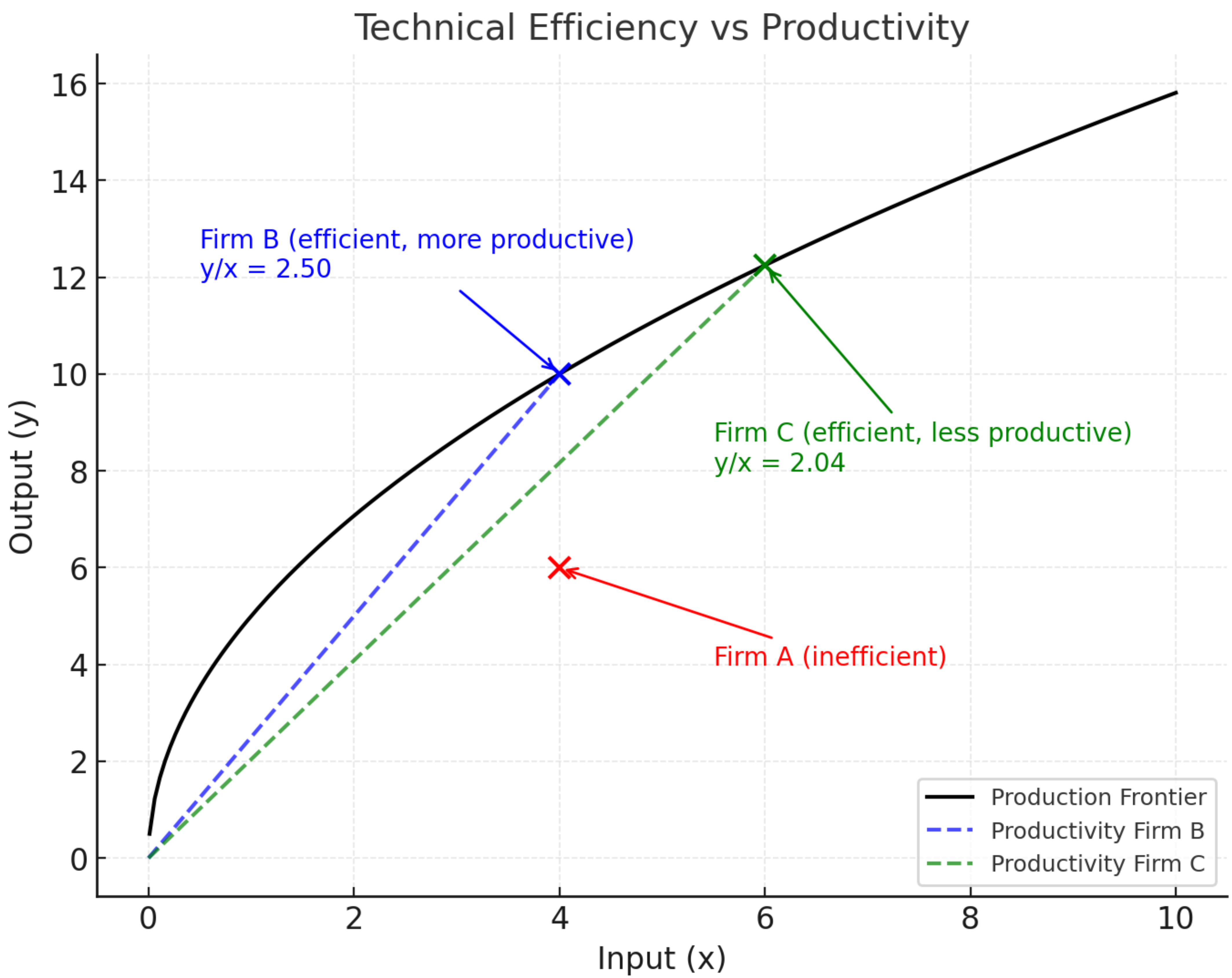
Objectives

- Estimating production technology from firm-level data
- Analyzing profit-maximizing, cost-minimizing behavior
- Applications:
    - profit firms, non-profits, industries (agriculture, manufacturing, services)

Method of Analysis

- Least-square method (LS)
- Total factor productictity (TFP)
- Data envelopment analysis (DEA)
- Stochastic frontier analysis (SFA)
  - We will focus on LS and TFP (which assume firms are technically efficient)
  - DEA and SFA can provide measures of relative efficiency

Technical efficiency vs productivity



Data

- I will primarily follow *appleProdFr86* data set, which can be found in Henningsen's *micEcon* R package
  - Additional datasets can be found in other R packages, for example, *sfaR*, *micEcon*, *rDEA*, *deaR*, *Benchmarking*, or others.
  - Another resourceful website: https://vincentarelbundock.github.io/Rdatasets/datasets.html

Dataset: appleProdFr86

- R code:

```
library(micEcon); library(psych); library(lmtest); library(car); library(miscTools)
options(scipen = 999)
data( "appleProdFr86", package = "micEcon" )
dat <- appleProdFr86
describe(dat)
```

	vars	n	mean	sd	median	trimmed	mad
vCap	1	140	102576.24	79992.28	84114.50	89202.88	55160.87
vLab	2	140	237199.39	194867.78	175871.00	199077.29	71095.12
vMat	3	140	201250.06	208054.52	136291.50	160457.37	92486.81
qApples	4	140	3.07	5.46	1.37	1.87	1.68
qOtherOut	5	140	1.50	1.32	1.07	1.29	0.95
qOut	6	140	2649825.38	3300778.29	1773989.17	2005998.00	1440430.19
pCap	7	140	1.30	0.79	1.11	1.20	0.56
pLab	8	140	1.01	0.20	0.96	1.00	0.20
pMat	9	140	6.77	2.64	6.25	6.55	2.75
pOut	10	140	1.01	0.53	0.83	0.91	0.31
adv	11	140	0.52	0.50	1.00	0.53	0.00

Dual approach to production analysis

- Duality is a fundamental concept in optimization, particularly in linear programming and game theory.
  —Every optimization problem (referred to as the primal problem) can be associated with a corresponding dual problem, where the solution to one provides bounds to the solution of the other.

- In the context of producer behavior, the primal approach involves studying how firms can optimally decide on the input mix for a given production technology to achieve an objective, such as minimizing expenses to produce a certain volume of output. The resulting optimal expense is referred to as the cost function.

- Duality tells us that the cost function is sufficiently informative, allowing us to confidently trace back the production technology under mild conditions.

Dual approach to production analysis

- Prior to 1970, economists mostly followed Samuelson's classic treatment of profit-maximizing firms, where firms face technological constraints, typically modeled with a smooth production function, and standard optimization techniques are used to infer producer responses to price perturbations.

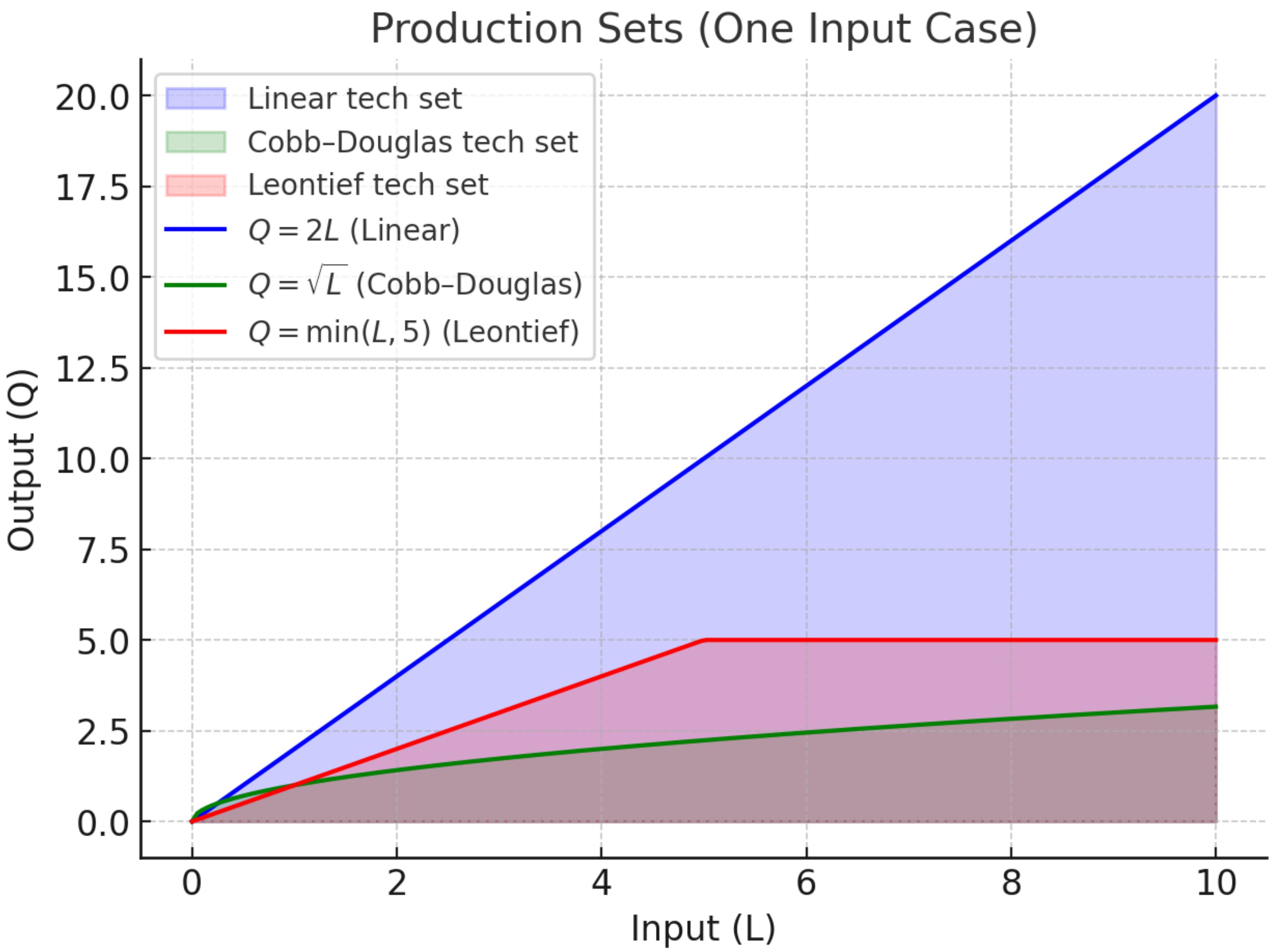
- This approach is often referred to as the primal approach.

- Later, the dual approach gained prominence, where exploring cost, profit, or revenue functions allows us to trace back the technological constraints.

Production technology

- The set of all combinations of inputs and outputs that comprise a technologically feasible way to produce is called a production (possibility) set.

- One-input case:

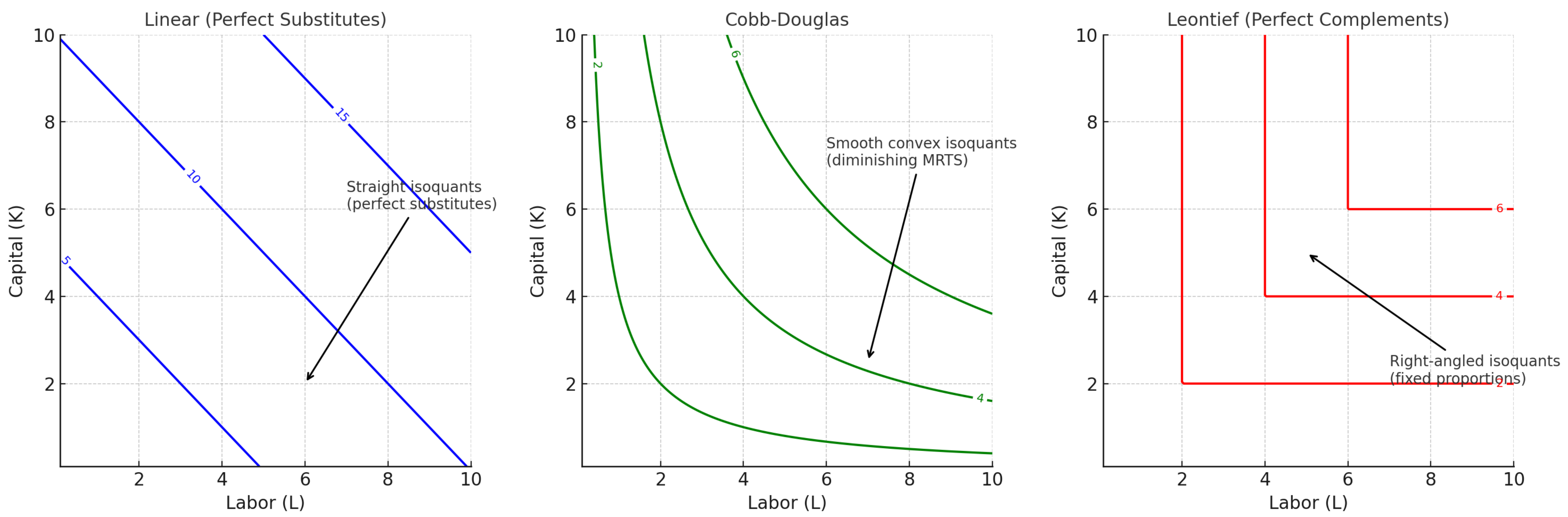


Production technology

- Two inputs—Presentation via isoquants

- The isoquants move in the top-right direction as y goes up, since we need more inputs to produce more output.

- The top-right section of the isoquants, and including the points on the isoquants, are often referred to as the input requirement set.



Production technology

- Convex technology—A technology is called convex if the input requirement set is convex.
    - For a convex technology, a convex combination of input choices increases the output volume.
- Monotone technology—A technology is called monotone if its input requirement set satisfies the monotonicity property, which suggests that for any input vector x belonging to the input requirement set, all input vectors weakly greater than x must belong to the input requirement set.
- A general representation of multi-output and multi-input production possibility is given by a transformation function T such that $T\,(\mathrm{x}, \mathrm{q}) = 0$ represents a relationship where an input vector x is used to produce an output vector q.

Production technology

- Some examples:

Linear: $y = \beta_0 + \sum_{i=1}^{N} \beta_i x_i$

Cobb-Douglas: $y = \beta_0 \prod_{i=1}^{N} {x_i}^{\beta_i}$, or equivalently, $\ln y = \beta_0 + \sum_{i=1}^{N} \beta_i \ln x_i$

Leontief: $y = \min_{i=1}^{N} \{\beta_i x_i\}$

CES: $y = \left[\sum_{i=1}^{N} \beta_i x_i^{\rho}\right]^{\frac{1}{\rho}}$

Quadratic: $y = \beta_0 + \sum_i \beta_i x_i + \frac{1}{2} \sum_i \sum_j \beta_{ij} x_i x_j$

Translog: $\ln y = \beta_0 + \sum_i \beta_i \ln x_i + \frac{1}{2} \sum_i \sum_j \beta_{ij} \ln x_i \ln x_j$

Production technology

- Returns to scale—How would the output change if we scale up or down the inputs?

- If the output goes up by the same factor k, we call it a constant returns to scale (CRS) technology. Mathematically, a CRS technology exhibits $f(kx) = kf(x)$.

- If the output increases less than k times, we call it a decreasing returns to scale (DRS) technology. Mathematically, a DRS technology exhibits $f(kx) < kf(x)$.

- If the output increases more than k times, we call it an increasing returns to scale (IRS) technology. Mathematically, an IRS technology exhibits $f(kx) > kf(x)$.

- *Exercise*: Consider a two-input Cobb-Douglas production function. Find conditions under which the technology exhibits different kinds of returns to scale.

Productivity

- Average and marginal product

- Single-input case: Consider a production relationship given by y = f(x) .

  The average productivity of the input x is defined by

  $$AP = f(x)/x$$

  The marginal productivity of the input x is defined by

  $$MP = \partial f(x)/\partial x$$

- Multi-input case:

$$AP_i = \frac{y}{x_i} = \frac{f(\mathbf{x})}{x_i}$$

$$MP_i = \frac{\partial y}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} = f_i$$

Output elasticity of an input

- The output elasticity of an input measures the percentage changes in output because of a percentage change in input.

$$\varepsilon_i = \frac{\partial f(\mathbf{x})/f(\mathbf{x})}{\partial x_i/x_i} = \frac{MP_i}{AP_i}$$

- The output elasticities are free of the unit of measurement.

- The elasticity of scale is the sum of output elasticities of all input:

$$\varepsilon = \textstyle\sum_i \varepsilon_i$$

- A technology exhibiting IRS, CRS, and DRS has the elasticity of scale

$$\varepsilon > 1, \varepsilon = 1, \text{ and } \varepsilon < 1,$$

respectively.

Total factor productivity (TFP)

- In multi-input production process, it is often desirable to calculate the total factor productivity (TFP) by aggregating inputs into an input index

$$TFP = \frac{y}{X},$$

where X is a quantity aggregating index of all inputs.

Indexing

- Indexing is used for measuring changes in a set of related variables.

- It can be used for comparison over time or space or both.

- Examples include price indices for measuring changes to consumer price, export or import prices, quantity indices measuring changes in output volume by a firm or industry over time or across firms.

- Consider a formula for measuring the change of the value of a basket consisting of n goods between the two period t and s can be measured by

$$X = \frac{\sum_{i=1}^{n} x_{it} p_{it}}{\sum_{i=1}^{n} x_{is} p_{is}}$$

  If we fix the prices (either to current or old prices), we get a measure due to changes in quantity, and it then reflects a quantity index. Similarly, if we fix the quantity (either to current or old quantity levels), we will get a price index.

Various indices

- Laspeyres quantity index:

$$X_j^L = \frac{\sum_i x_{ij} p_{i0}}{\sum_i x_{i0} p_{i0}}$$

- Paasche quantity index:

$$X_j^P = \frac{\sum_i x_{ij} p_{ij}}{\sum_i x_{i0} p_{ij}}$$

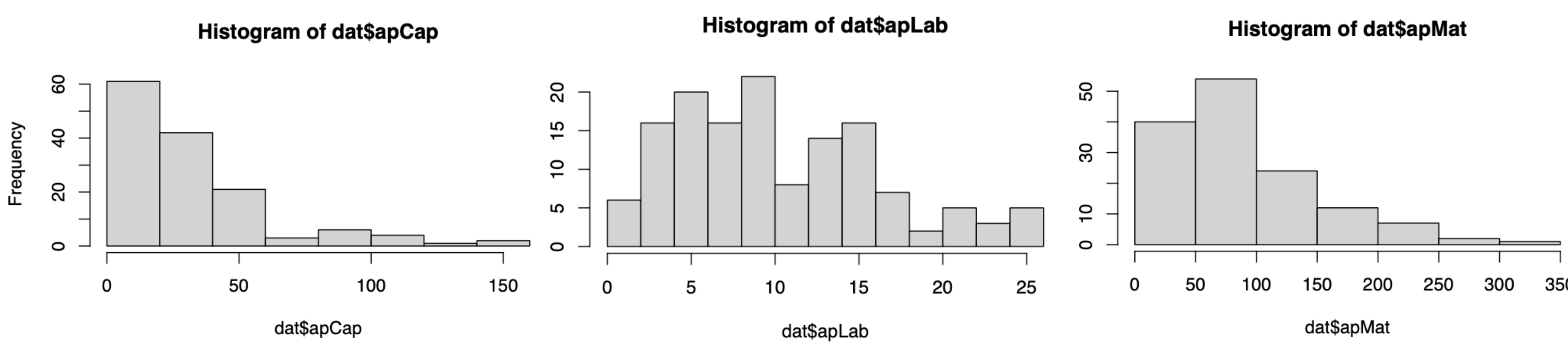
- Fisher's quantity index:

$$X_j^F = \sqrt{X_j^L \times X_j^P}$$

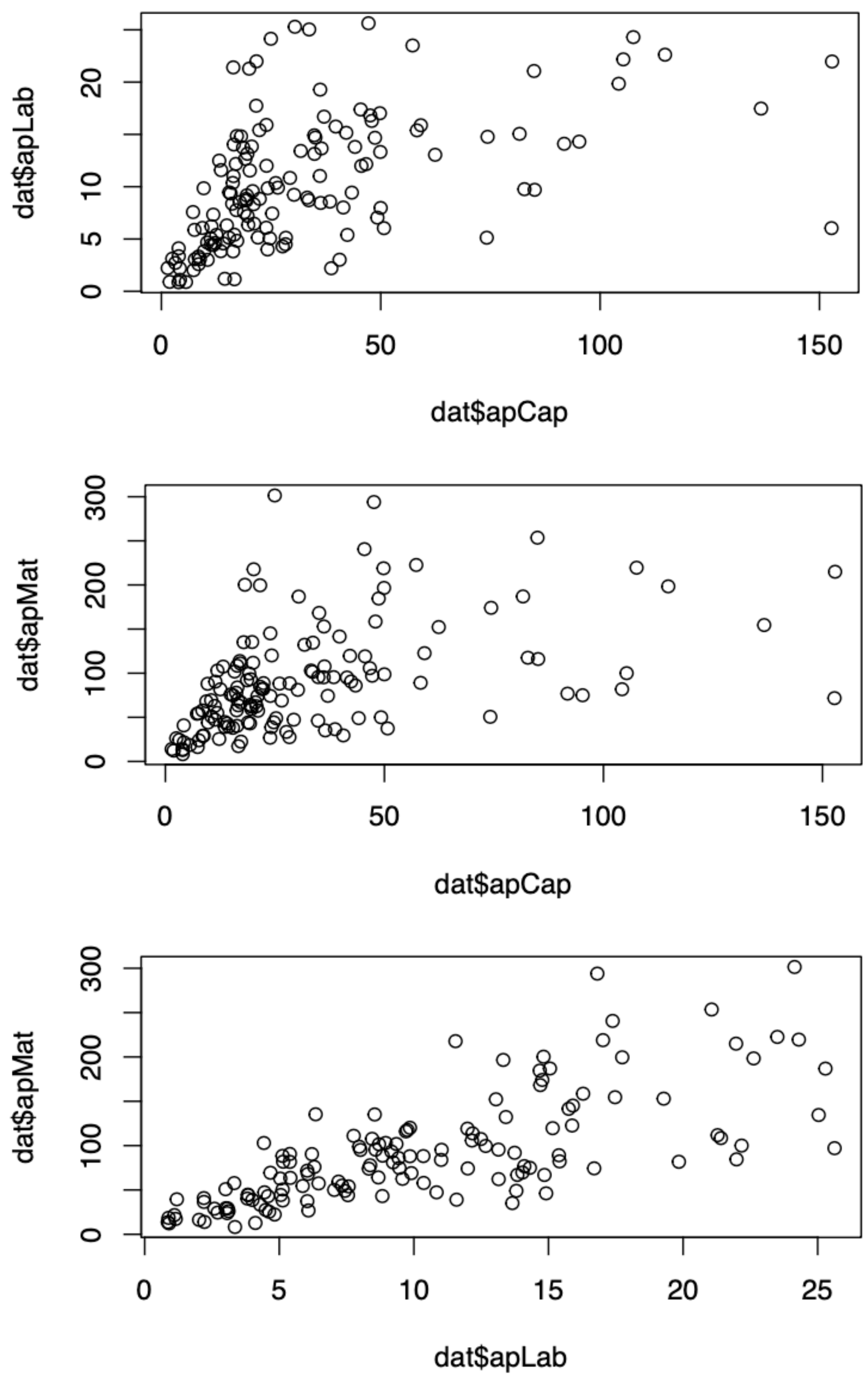
Productivity measures in our data set

- R code

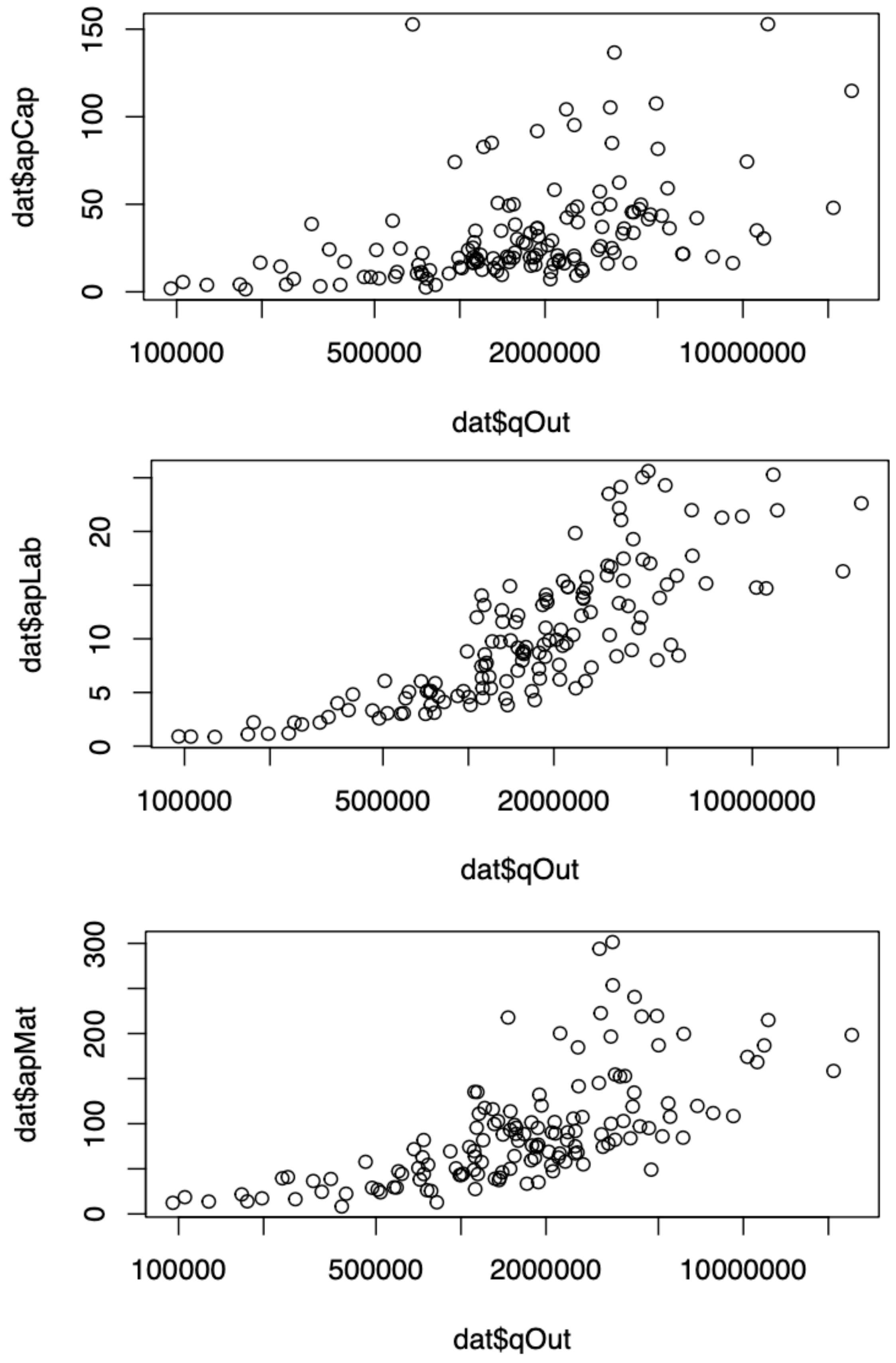
```
# Generate input quantities
dat$qCap <- dat$vCap / dat$pCap
dat$qLab <- dat$vLab / dat$pLab
dat$qMat <- dat$vMat / dat$pMat
# Creating quantity indices
dat$X <- sqrt( dat$XP * dat$XL ) # Fisher Index
# Measuring (partial) average product
dat$apCap <- dat$qOut / dat$qCap
dat$apLab <- dat$qOut / dat$qLab
dat$apMat <- dat$qOut / dat$qMat
```



Average products of the three inputs seem to be positively correlated



Firms producing more also exhibit higher output per unit of input used

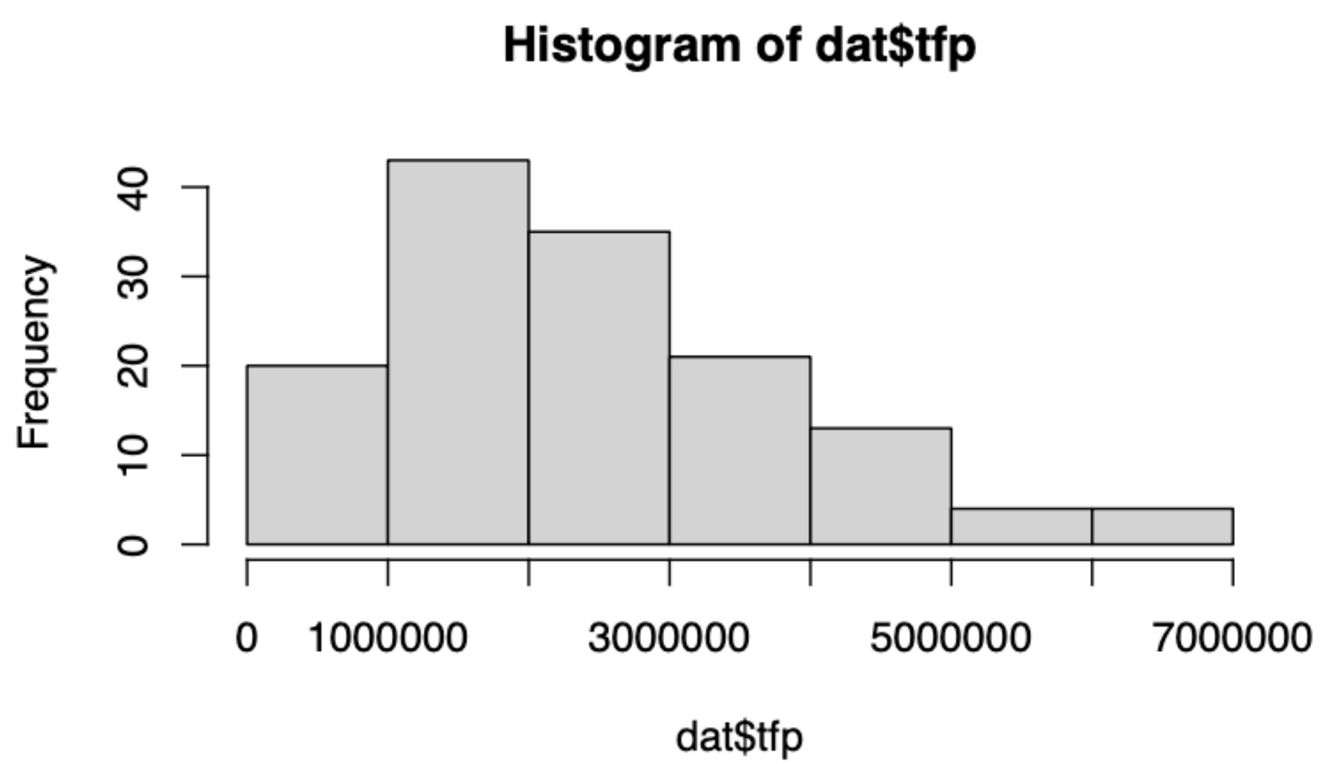


Total factor productivity

- R code

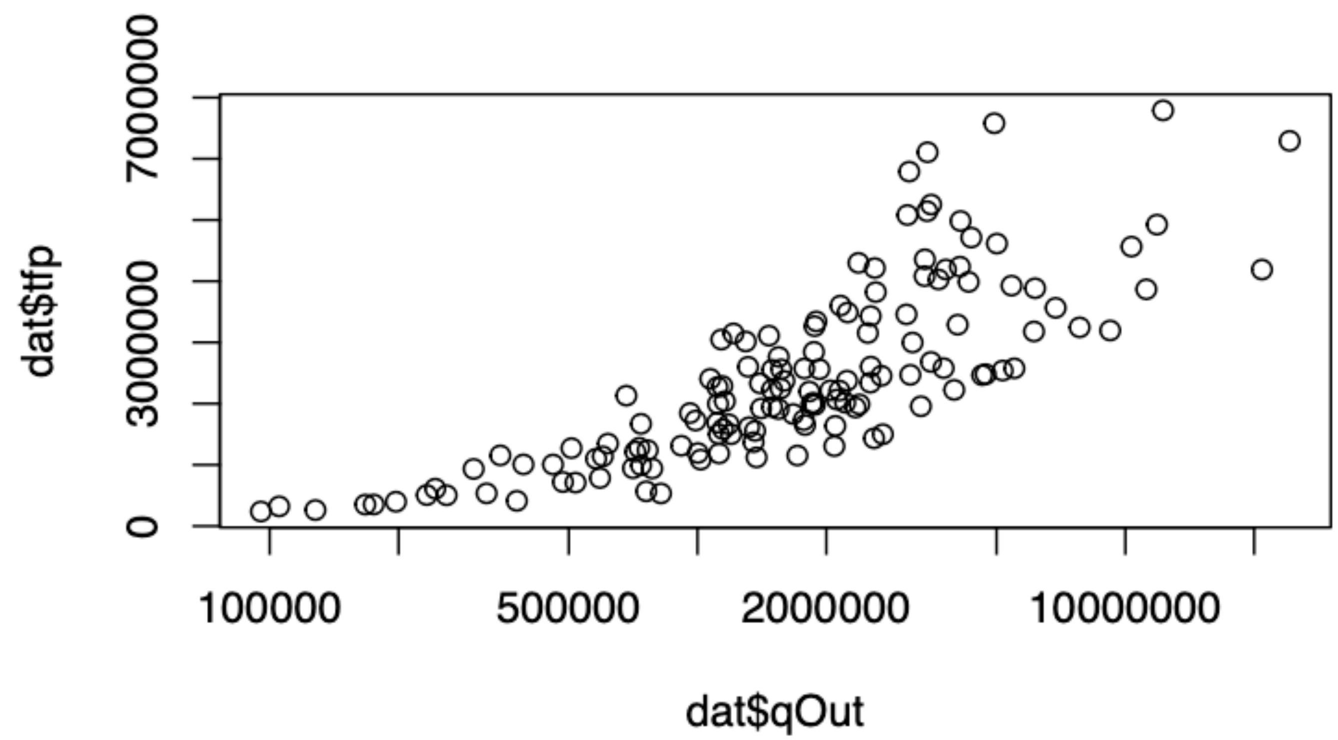
```
# Measuring total factor productivity
dat$tfp <- dat$qOut / dat$X # using Fisher index
```

- TFP varies considerably across firms, with the majority falling into the relatively low-TFP range.

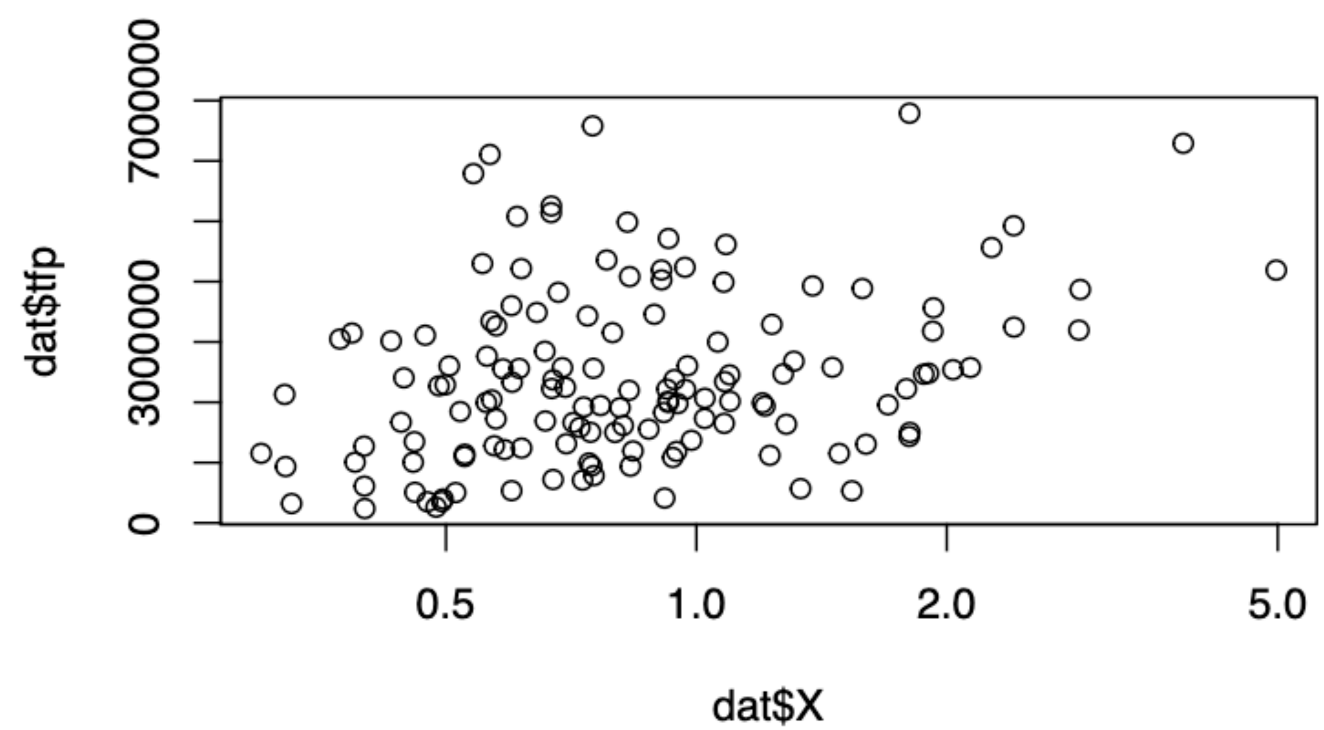


Total factor productivity

- Larger firms, characterized by higher output volumes, are typically associated with greater TFP.



- However, the plot of TFP against the aggregate input index shows only a mild positive association between the two.



Efficiency in resource allocation

- What might cause variation in the input mix chosen by different firms? Are all firms operating with allocative efficiency?

- If the nature of the production function were known, we could measure how a producer would optimally allocate its resources across various inputs.

- Let us begin by assuming a generic production function ***f*(…)** and developing the concepts of allocative efficiency.

- Subsequently, we will test these concepts using our estimated production function derived from the data.

Input substitution

- Marginal rate of technical substitution (MRTS)—How much of input 2 might we need if we like to substitute one unit of input 1 but keep producing the same amount of output.

$$MRTS = \frac{dx_2}{dx_1} = -\frac{f_1}{f_2} = -\frac{MP_1}{MP_2}$$

- Relative marginal rate of technical substitution (RMRTS)—The ratio of MRTS and input ratio. It measures the relative percentage change in one input (say, capital) needed to compensate for a relative percentage change in another input (say, labour).

$$RMRTS = \frac{MRTS}{x_2/x_1} = -\frac{MP_1}{MP_2}\frac{x_1}{x_2}$$

$$RMRTS = -\frac{MP_1}{y/x_1}\frac{y/x_2}{MP_2} = -\frac{MP_1}{AP_1}\frac{AP_2}{MP_2} = -\frac{\varepsilon_1}{\varepsilon_2}$$

Elasticity of substitution

- The importance of input substitution led to various definition of elasticities of substitutions.

- Hicks (1963) offers the following definition of elasticity between two inputs:

$$\sigma = \frac{d(x_2/x_1)}{d(f_1/f_2)}\frac{(f_1/f_2)}{(x_2/x_1)} = \frac{\%\text{ change in input ratio}}{\%\text{ change in MRTS}}$$

$$\sigma = \frac{MRTS}{(x_2/x_1)}(1/\frac{dMRTS}{d(x_2/x_1)}) = 1/\frac{d\ \ln MRTS}{d\ \ln(x_2/x_1)}$$

- An equivalent measure representation:

$$\sigma = \frac{-f_1 f_2(x_1 f_1 + x_2 f_2)}{x_1 x_2(f_{11}f_2^2 - 2f_{12}f_1 f_2 + f_{22}f_1^2)}$$

where f_i, f_{ii}, and f_{ij} are first-order, second-order, and cross derivatives.

Elasticity of substitution

A generalization of the above measure of elasticity of substitution is ***Allen partial elasticity of substitution***, which is defined as

$$\sigma_{ij} = \frac{\sum_i x_i f_i}{x_i x_j} \frac{F_{ij}}{F},$$

where F is the determinant of the bordered Hessian matrix:

$$F = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{12} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & f_{1n} & f_{2n} & \cdots & f_{nn} \end{vmatrix},$$

and F_{ij} is the co-factor of f_{ij}.

Elasticity of substitution

The final elasticity measure is the ***Morishima elasticity of substitution***, which is given by

$$\sigma_{ij}^{M} = \frac{f_i}{x_i}\frac{F_{ij}}{F} - \frac{f_j}{x_j}\frac{F_{jj}}{F} = \frac{x_j f_j}{\sum_i x_i f_i}(\sigma_{ij} - \sigma_{jj}),$$

where σ_{ij} (without the superscript) denote the Allen elasticity measure.

- The estimation of the rate and elasticity of substitution will be deferred until the production functional form is specified and estimated on the data.

Supply behavior of competitive firms

- **Core assumption**: Firms aim to maximize profits:

$$\pi(x) = \text{Revenue} - \text{Cost}$$

- Revenue:

$$TR = p \cdot y = p \cdot f(x)$$

- Cost:

$$TC = C(x) = \sum_{i=1}^{n} w_i x_i$$

- Profit:

$$\pi(x) = p \cdot f(x) - C(x)$$

- Notes—
  Firms take both input and output prices as given (price-takers).
  The assumption of profit maximization can be relaxed (e.g., consumer surplus, regulatory constraints) without changing the analytical framework.

Profit maximization

The firm's problem is to choose $\mathbf{x}$ to maximize profit:

$$\max_{\mathbf{x}} \pi(\mathbf{x}) = p \cdot f(\mathbf{x}) - \sum_{i=1}^{n} w_i x_i$$

First-order condition:

$$p \cdot \frac{\partial f(\mathbf{x}^*)}{\partial x_i} = w_i, \ \ \forall i = 1, 2, \dots, n$$

$$w_i = p \cdot MP_i = MVP_i$$

Implications of the F.O.C:

$$\frac{w_i}{w_j} = \frac{MP_i}{MP_j} = -MRTS_{ij}$$

Any statistical deviation from this relation indicates a misallocation of resources.

Profit maximization

In addition, denoting the cost share of input i by s_i where $s_i = w_i x_i / \sum_{k=1}^{n} w_k x_k$, we can express the ratio of cost share by a profit maximization firm as

$$\frac{s_i}{s_j} = \frac{w_i x_i}{w_j x_j} = -\frac{MRTS_{ij}}{x_j / x_i} = -RMRTS_{ij}$$

Recall the definition of $RMRTS = -\varepsilon_i / \varepsilon_j$, which gives us

$$\frac{s_i}{s_j} = \frac{\varepsilon_i}{\varepsilon_j}$$

The above relation along with the fact that $\sum_i s_i = 1$ implies that for a profit maximizing firm the cost share of each input must equal the ratio of output elasticity over the elasticity of scale:

$$s_i = \frac{\varepsilon_i}{\varepsilon}$$

Input demand function and supply function

- Input demand function:

  If we solve the simultaneous equations represented by the FOCs, we get the input demand function as a function of the output price and the input/factor prices:

$$x_i = x_i(p, \mathbf{w})$$

- Supply function:

  Replacing the inputs by the input demand functions in the production, we can derive the output supply function as a function of the output price and the input/factor prices:

$$y = f(x_1(p, \mathbf{w}), \ldots, x_n(p, \mathbf{w})) = y(p, \mathbf{w})$$

Deriving price elasticity

As the derived input demand and output functions are expressions of prices (output and input prices), we can derive the price elasticities of demand and supply:

$$\varepsilon_{ij}(p, \mathbf{w}) = \frac{\partial x_i(p, \mathbf{w})}{\partial w_j} \frac{w_j}{x_i(p, \mathbf{w})} \text{ : elasticity of input } i \text{ w.r.t the price of input } j$$

$$\varepsilon_{yi}(p, \mathbf{w}) = \frac{\partial y(p, \mathbf{w})}{\partial w_i} \frac{w_i}{y(p, \mathbf{w})} \text{ : elasticity of output } y \text{ w.r.t the price of input } i$$

$$\varepsilon_{yp}(p, \mathbf{w}) = \frac{\partial y(p, \mathbf{w})}{\partial p} \frac{p}{y(p, \mathbf{w})} \text{ : elasticity of output } y \text{ w.r.t the output price } p$$

$$\varepsilon_{ip}(p, \mathbf{w}) = \frac{\partial x_i(p, \mathbf{w})}{\partial p} \frac{p}{x_i(p, \mathbf{w})} \text{ : elasticity of input } i \text{ w.r.t the output price } p$$

Profit function

- Profit function:

  Replacing the inputs by the input demand functions and output by the supply function, we can determine the profit function, which characterizes the maximum profit as a function of the output price and the input/factor prices:

$$\pi(p, \mathbf{w}) = p \cdot y(p, \mathbf{w}) - \sum_{i=1}^{n} w_i x_i(p, \mathbf{w})$$

- We will later explore the properties of the profit function, which are useful for examining the dual approach.

Supply behavior with output constraint

- Consider a situation where the firm is required to produce a specific volume of output, may be because altering the output level is not feasible in the short run.
- In this case, the firm must make optimal input choices to maximize profit under these constraints.

Cost minimization

The firm's problem is to choose $\mathbf{x}$ to minimize costs

$$\min_{\mathbf{x}} \sum_{i=1}^{n} w_i x_i \text{ such that } y = f(\mathbf{x})$$

The constrained optimization problem can be solved by Lagrangian approach:

$$\min_{\mathbf{x},\lambda} \mathcal{L} = \sum_{i=1}^{n} w_i x_i + \lambda(y - f(\mathbf{x}))$$

Cost minimizing behavior

- First order condition gives

$$w_i = \lambda \cdot MP_i$$

and

$$\frac{w_i}{w_j} = \frac{MP_i}{MP_j} = -MRTS_{ij}$$

- Similar to the profit maximization problem, it can be shown here (in case of cost minimization) that the **ratio of cost shares** must equal the **absolute value of RMRTS**.

- The consistency arises because profit maximization implies producing the optimal output at minimum cost.

Conditional input demand and cost function

- The solutions to the cost minimization problem are called **conditional input demand functions**, which are expressed in terms of the output level y and the input (factor) prices.

$$x_i = x_i(y, \mathbf{w})$$

- By substituting the conditional input demand functions into the firm's cost expression, we obtain the **cost function**, which shows the minimum cost of producing a given output level as a function of output y and input (factor) prices.

$$c(y, \mathbf{w}) = \sum_{i=1}^{n} w_i x_i(y, \mathbf{w})$$

Reading materials

- Varian, Chapter 1, 2, and 4
- Henningsen, Chapter 1, 2