Applied Production Analysis

SOK-3011—Part 1

Dual approach:
Estimation of cost and profit functions

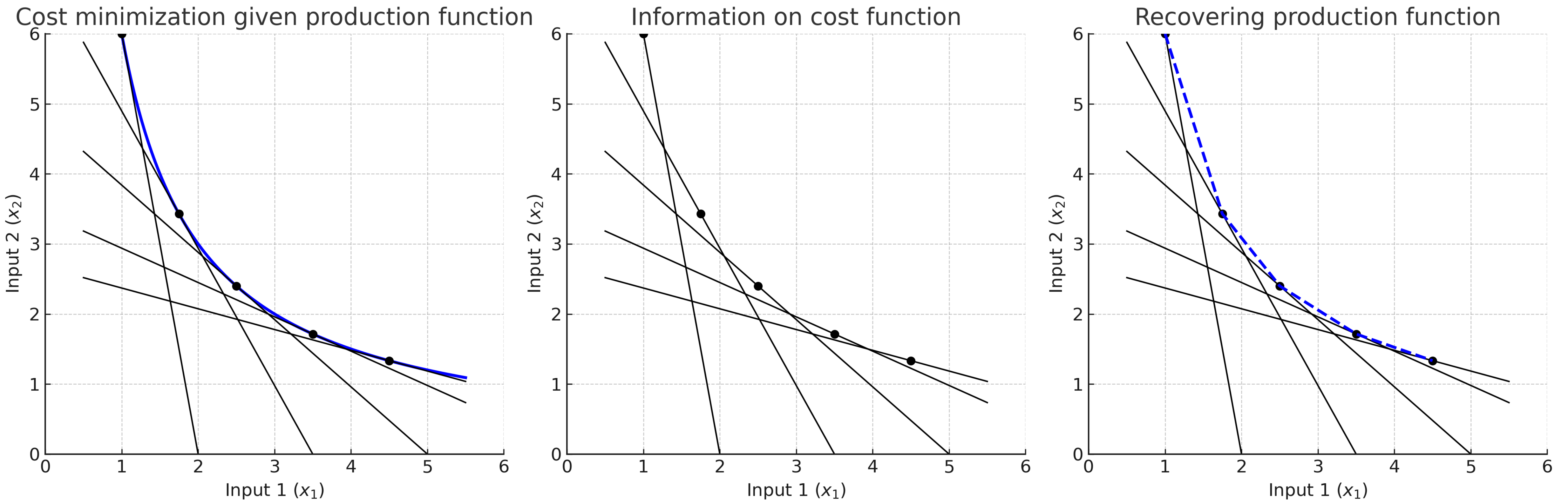
Overview

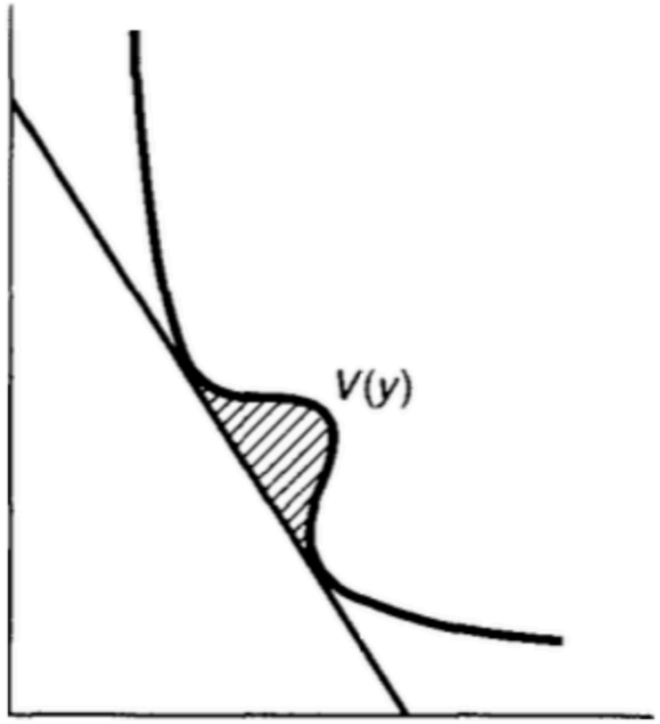
- Dual Approach
- Cost function
    - Recovery of production function
    - Properties
    - Estimation
- Profit function
    - Properties
    - Estimation

Dual Approach (vs. Primal Approach)

- **Primal approach**: Start with the **production function** → analyze how firms maximize profit or minimize cost given technology.
- **Dual approach**: Start with the **cost function** → analyze how much it costs to produce a given output at given input prices.
- Key duality insight:
  - From production function → we can derive the cost function.
  - From cost function → we can recover the production function (if technology is well-behaved, i.e., quasi-concave).

Recovery of production function





Recovery of production function

Theorem.

Let $c(y, w)$ be a differentiable function satisfying the regular properties (stated later) of a cost function.

Define a function f as

$$f(x) \equiv \max\{y \geq 0 \mid w \cdot x \geq c(y, w),\ \forall\, w \gg 0\}.$$

Then:

- f is increasing, unbounded above, and quasiconcave.
- Moreover, the cost function generated by f is precisely c.

Dual Approach (vs. Primal Approach)

- In the context of empirical estimation of technology, we can address it by studying the same problem from two mirrors:
    - *Primal*: What output can I get with these inputs?
    - *Dual*: What is the minimum cost of producing this output?
- In practice, the dual approach is often more useful because **prices and expenditures are visible in markets**, while internal technology is hidden.
- Applied advantage:
    - Easier to observe **input prices and costs** than to measure technical production details.
    - Researchers can estimate cost functions directly from market data and then infer the production side.

Cost function

Cost minimization and cost function

- **Cost Minimization Problem**:

$$\min_{x} \sum_{i=1}^{n} w_i x_i \quad \text{s.t. } f(x) \geq y$$

- **Cost Function**:

$$c(y, w) = \sum_{i=1}^{n} w_i x_i(y, w)$$

  where $x_i(y, w)$ is the cost-minimizing demand for input i.

- **Interpretation**: Minimum cost of producing output y given input prices w.

Properties of cost function

If production function f is continuous and strictly increasing:

1. $c(0, w) = 0$
2. Continuous on domain
3. Strictly increasing & unbounded in y (for $w \gg 0$)
4. Increasing in input prices w
5. Homogeneous of degree one in w
6. Concave in w

Properties of cost function

- An important result is the relationship between the cost function and the optimal input demand function

- **Shephard's Lemma**

$$\frac{\partial c(y,w)}{\partial w_i} = x_i(y, w)$$

- This is a direct application of the envelope theorem for constrained optimization

- We can use this result to check if firms' choices of inputs are consistent with cost minimization principle

Estimation of cost function

A Cobb-Douglas cost function

- We will estimate a Cobb-Douglas cost function using our data.
- Note that a Cobb-Douglas production function generates a cost function that exhibits a Cobb-Douglas form in input prices.
- If the original production technology exhibits constant returns to scale (CRS), the cost function is typically linear in y.
- We consider the following specification:

$$c = A\left(\prod_{i=1}^{N} {w_i}^{\alpha_i}\right) y^{\alpha_y}$$

and in its linearized form:

$$\ln c = \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln w_i + \alpha_y \ln y$$

Estimation

- Coefficients of log-input prices are **non-negative** → cost function is non-decreasing in input prices.
- Coefficient of log-output quantity is **non-negative** → cost function is non-decreasing in output.
- With a **positive intercept**, the cost function satisfies the **non-negativity condition**.

```
lm(formula = log(cost) ~ log(pCap) + log(pLab) + log(pMat) +
    log(qOut), data = dat)

Residuals:
     Min       1Q   Median       3Q      Max
-0.77663 -0.23243 -0.00031  0.24439  0.74339

Coefficients:
            Estimate Std. Error t value             Pr(>|t|)
(Intercept)  6.75383    0.40673  16.605 < 0.0000000000000002 ***
log(pCap)    0.07437    0.04878   1.525              0.12969
log(pLab)    0.46486    0.14694   3.164              0.00193 **
log(pMat)    0.48642    0.08112   5.996         0.0000000174 ***
log(qOut)    0.37341    0.03072  12.154 < 0.0000000000000002 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3395 on 135 degrees of freedom
Multiple R-squared:  0.6884,    Adjusted R-squared:  0.6792
F-statistic: 74.56 on 4 and 135 DF,  p-value: < 0.00000000000000022
```

Does it satisfy homogeneity?

- The **F-statistic** tests whether the null hypothesis (sum of coefficients = 1) holds.
- It compares the fit of a **restricted model** (with the null imposed) and an **unrestricted model**.
- A small difference in residuals → null likely true → **large Pr(>F)** (as in this case).

```
# Liner homogeneity condition check
linearHypothesis( costCD, "log(pCap) + log(pLab) + log(pMat) = 1"  )
```

```
Linear hypothesis test:
log(pCap)  + log(pLab)  + log(pMat) = 1

Model 1: restricted model
Model 2: log(cost) ~ log(pCap) + log(pLab) + log(pMat) + log(qOut)

  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1    136 15.563
2    135 15.560  1 0.0025751 0.0223 0.8814
```

Imposing homogeneity

- We can directly impose homogeneity of degree one by modifying the model.

- This involves setting one input's coefficient as **1 minus the sum of the others**, which changes the regression specification.

$$\ln c = \alpha_0 + \sum_{i=1}^{N-1} \alpha_i \ln w_i + (1 - \sum_{i=1}^{N-1} \alpha_i) \ln w_N + \alpha_y \ln y$$

or, equivalently,

$$\ln \frac{c}{w_N} = \alpha_0 + \sum_{i=1}^{N-1} \alpha_i \ln \frac{w_i}{w_N} + \alpha_y \ln y$$

Imposing homogeneity

```
lm(formula = log(cost/pMat) ~ log(pCap/pMat) + log(pLab/pMat) +
    log(qOut), data = dat)

Residuals:
     Min       1Q   Median       3Q      Max
-0.77096 -0.23022 -0.00154  0.24470  0.74688

Coefficients:
               Estimate Std. Error t value            Pr(>|t|)
(Intercept)     6.75288    0.40522  16.665 < 0.0000000000000002 ***
log(pCap/pMat)  0.07241    0.04683   1.546                0.124
log(pLab/pMat)  0.44642    0.07949   5.616          0.000000106 ***
log(qOut)       0.37415    0.03021  12.384 < 0.0000000000000002 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3383 on 136 degrees of freedom
Multiple R-squared:  0.5456,    Adjusted R-squared:  0.5355
F-statistic: 54.42 on 3 and 136 DF,  p-value: < 0.00000000000000022
```

- Coefficients change slightly but keep the **same interpretation**; the “materials” coefficient is derived as $\alpha_{MAT} = 0.48$.

Homogeneity-imposed model

```
Likelihood ratio test

Model 1: log(cost/pMat) ~ log(pCap/pMat) + log(pLab/pMat) + log(qOut)
Model 2: log(cost) ~ log(pCap) + log(pLab) + log(pMat) + log(qOut)
  #Df  LogLik Df  Chisq Pr(>Chisq)
1   5 -44.878
2   6 -44.867  1 0.0232      0.879
```

- A Likelihood-ratio (LR) test compares unrestricted vs. restricted models; here, the large Pr(>ChiSq) means the null holds → restricted model fits well.
- We adopt the homogeneity-imposed model for further analysis.
- Advantage: for Cobb-Douglas cost functions, **non-decreasing + linear homogeneity** automatically imply **concavity**.

Are firms' input choices consistent with cost-minimization principle?

- An implication of Shephard's lemma:

$$\alpha_i = \frac{\partial \ln c(y, \mathbf{w})}{\partial \ln w_i} = \frac{\partial c(y, \mathbf{w})}{\partial w_i} \cdot \frac{w_i}{c(y, \mathbf{w})} = \frac{w_i \cdot x_i(y, \mathbf{w})}{c(y, \mathbf{w})} = \text{ cost share of input i}$$

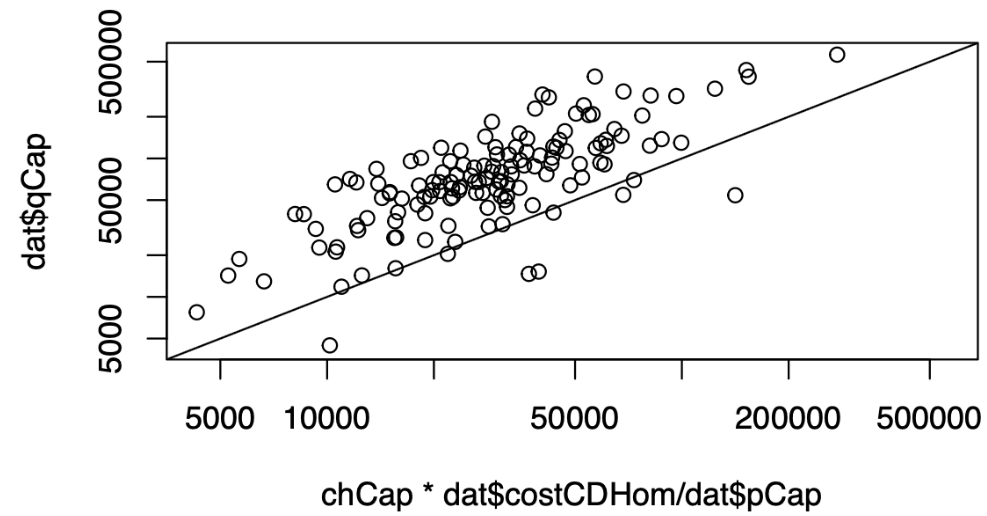
  where the first equality follows from Cobb-Douglas specification and the third follows from the Shephard's lemma.

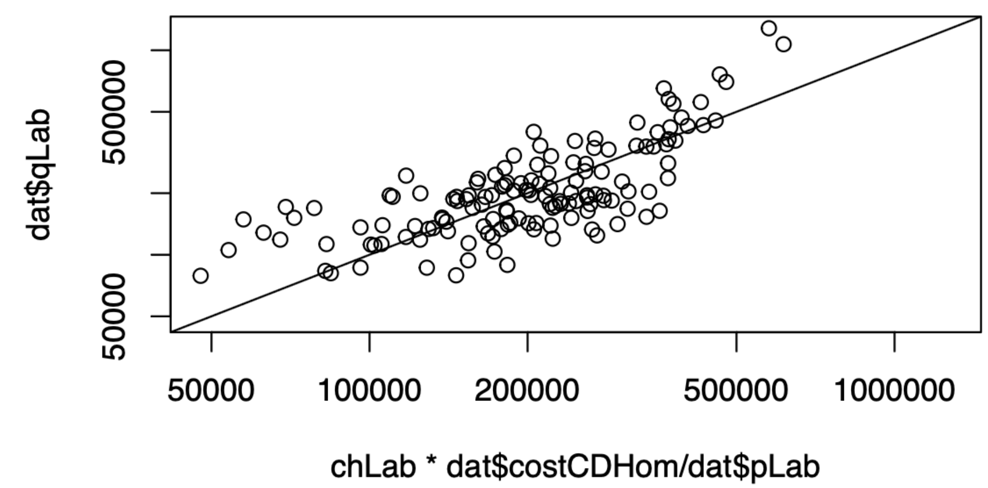
- This can also be rewritten as

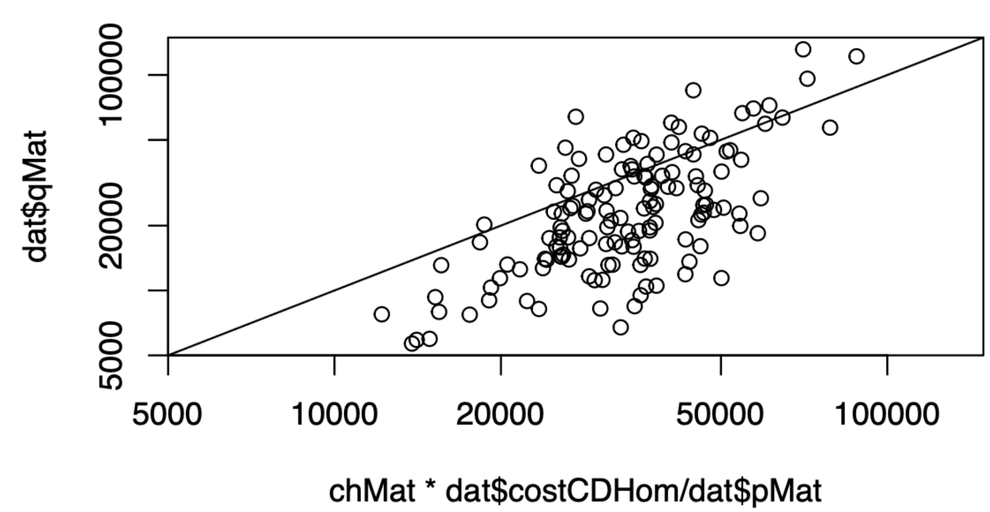
$$\alpha_i \frac{c(y, \mathbf{w})}{w_i} = x_i(y, \mathbf{w})$$

Are firms' input choices consistent with cost-minimization principle?

- If $(\alpha_i c)/w_i < x_i$, the firm is **overusing the input** relative to cost-minimization.
- The scatter plots show input use across firms.
- Findings: most firms **overuse capital** and **underuse materials**.

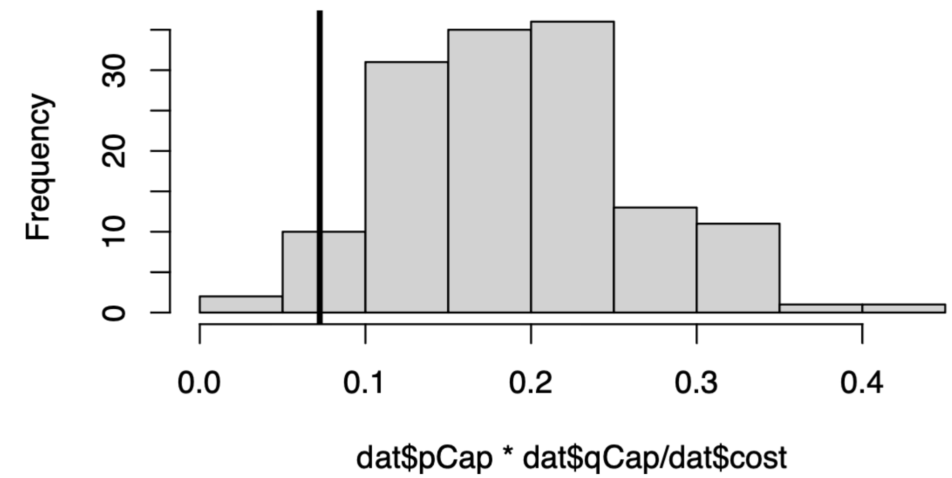


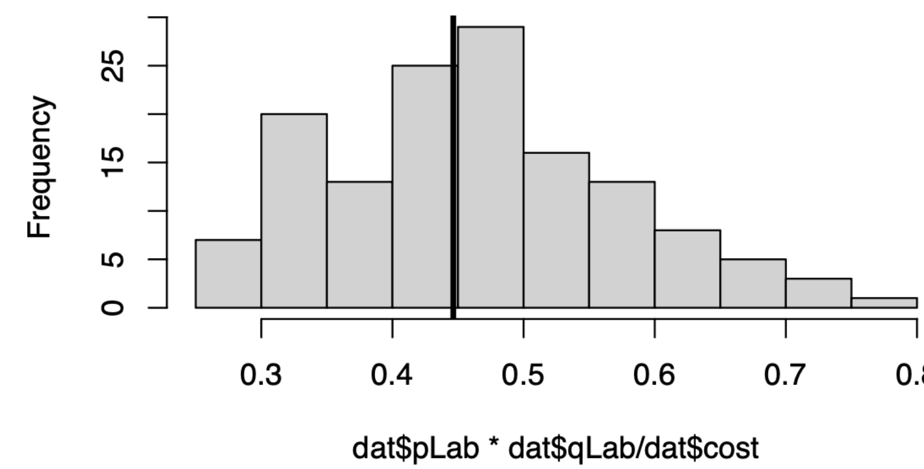


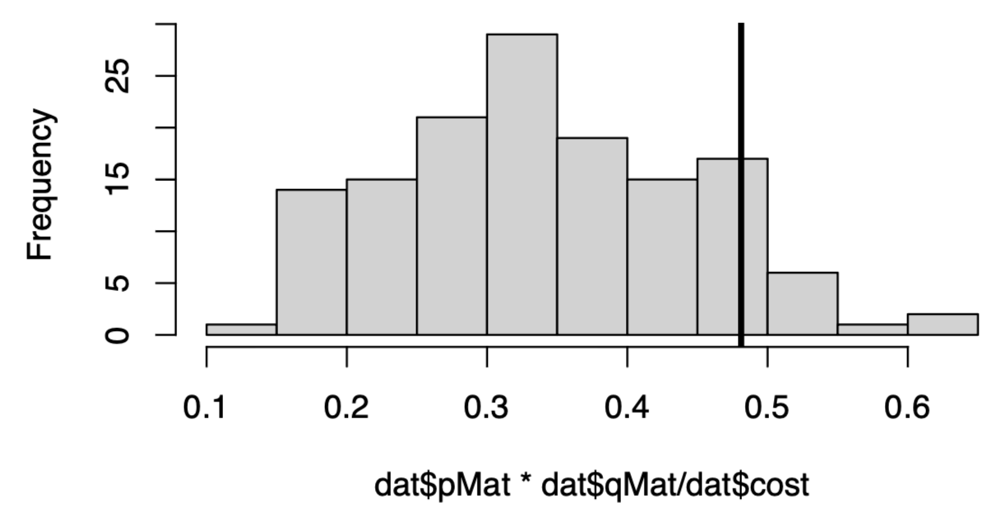


Are firms' input choices consistent with cost-minimization principle?

- We can also check the distribution of $x_i w_i / c$ across firms and compare it against α_i
- If α_i is sufficiently low, it implies firms are overusing the input.
- Findings: most firms **overuse capital** and **underuse materials**.







Profit function

Profit maximization and profit function

- **Profit Maximization Problem**:

$$\max_{x,y} \ \pi = p \cdot y - \sum_{i=1}^{n} w_i x_i \quad \text{s.t. } f(x) \geq y$$

- **Profit Function**:

$$\pi(p, w) = p \cdot y(p, w) - \sum_{i=1}^{n} w_i x_i(p, w)$$

  where $x_i(p, w)$ is the profit-maximizing demand for input i.

- **Interpretation**: Maximum profit attainable given output price p and input prices w.

Properties of profit function

If f is continuous, strictly increasing, and strictly quasiconcave:

1. Continuous
2. Increasing in output price p
3. Decreasing in input prices w
4. Homogeneous of degree one in (p, w)
5. Convex in (p, w)

Properties of profit function

- Like the Shephard's lemma, there is also an important result on the relationship between the profit function and the optimal output function

Hotelling's Lemma

- If $\pi(p, w)$ is differentiable at $(p, w) \gg 0$:

$$\frac{\partial \pi(p, w)}{\partial p} = y(p, w), \qquad -\frac{\partial \pi(p, w)}{\partial w_i} = x_i(p, w)$$

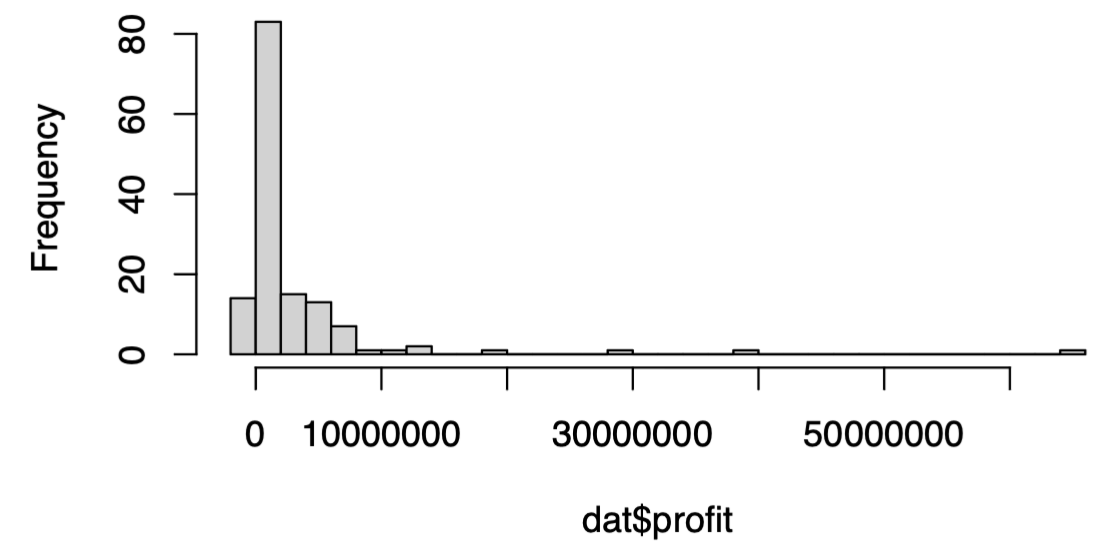
- The output and input demand functions can be recovered from the profit function. These can be derived via **envelope theorem** (analogous to Shephard's lemma).

- We can use this result to check if firms' choices of inputs and output volume are consistent with profit maximization principle

Estimation of profit function

A Cobb-Douglas cost function

- We define **Profit = Revenue – Costs**, assuming full information on both.
- In our data set, the constructed profit measure shows **negative profits** for several firms.
- Negative profits may reflect **short-run conditions**, where not all inputs can be adjusted (e.g., **capital fixed**).
- Even after treating capital costs as fixed, **8 observations** still show negative profits.
- For estimation, we **exclude 14 observations** with negative profits (see Henningsen 2024 for discussion).



A Cobb-Douglas profit function

- We estimate a Cobb-Douglas form using our data.
- We consider the following specification:

$$\pi = Ap^{\alpha_p}\left(\prod_{i=1}^{N} {w_i}^{\alpha_i}\right)$$

and in its linearized form:

$$\ln \pi = \alpha_0 + \alpha_p \ln p + \sum_{i=1}^{N} \alpha_i \ln w_i$$

Estimation

- Profit function is **decreasing in capital and labor prices**; coefficient for materials is positive but **not statistically significant**.
- Profit function is **increasing in output price** (positive, significant coefficient).
- With a **positive intercept**, the profit function satisfies the **non-negativity condition**.

```
lm(formula = log(profit) ~ log(pOut) + log(pCap) + log(pLab) +
    log(pMat), data = dat_clean)

Residuals:
    Min      1Q  Median      3Q     Max
-3.6183 -0.2778  0.1261  0.5986  2.0442

Coefficients:
            Estimate Std. Error t value             Pr(>|t|)
(Intercept)  13.9380     0.4921  28.321 < 0.0000000000000002 ***
log(pOut)     2.7117     0.2340  11.590 < 0.0000000000000002 ***
log(pCap)    -0.7298     0.1752  -4.165            0.0000586 ***
log(pLab)    -0.1940     0.4623  -0.420                0.676
log(pMat)     0.1612     0.2543   0.634                0.527
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.9815 on 121 degrees of freedom
Multiple R-squared:  0.5911,    Adjusted R-squared:  0.5776
F-statistic: 43.73 on 4 and 121 DF,  p-value: < 0.00000000000000022
```

Does it satisfy homogeneity?

```
Linear hypothesis test:
log(pOut)  + log(pCap)  + log(pLab)  + log(pMat) = 1

Model 1: restricted model
Model 2: log(profit) ~ log(pOut) + log(pCap) + log(pLab) + log(pMat)

  Res.Df    RSS Df Sum of Sq      F  Pr(>F)
1    122 119.78
2    121 116.57  1    3.2183 3.3407 0.07005 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- **F-test**: Null hypothesis of homogeneity of degree one is **rejected at 10%**, but **not at 5%** → may be critical if alternative functional forms offer better properties.

Imposing homogeneity

- We impose homogeneity of degree one by modifying the model.

- This involves setting one input's coefficient as **1 minus the sum of the others**, which changes the regression specification.

$$\ln \pi = \alpha_0 + (1 - \sum_{i=1}^{N} \alpha_i) \ln p + \sum_{i=1}^{N} \alpha_i \ln w_i$$

or, equivalently,

$$\ln \frac{\pi}{p} = \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln \frac{w_i}{p}$$

Imposing homogeneity

```
lm(formula = log(profit/pOut) ~ log(pCap/pOut) + log(pLab/pOut) +
    log(pMat/pOut), data = dat_clean)

Residuals:
    Min      1Q  Median      3Q     Max
-3.6045 -0.2724  0.0972  0.6013  2.0385

Coefficients:
               Estimate Std. Error t value             Pr(>|t|)
(Intercept)    14.27961    0.45962  31.068 < 0.0000000000000002 ***
log(pCap/pOut) -0.82114    0.16953  -4.844           0.00000378 ***
log(pLab/pOut) -0.90068    0.25591  -3.519             0.000609 ***
log(pMat/pOut) -0.02469    0.23530  -0.105             0.916610
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.9909 on 122 degrees of freedom
Multiple R-squared:  0.3568,    Adjusted R-squared:  0.341
F-statistic: 22.56 on 3 and 122 DF,  p-value: 0.00000000001091
```

```
Likelihood ratio test

Model 1: log(profit) ~ log(pOut) + log(pCap) + log(pLab) + log(pMat)
Model 2: log(profit/pOut) ~ log(pCap/pOut) + log(pLab/pOut) + log(pMat/pOut)
  #Df  LogLik Df  Chisq Pr(>Chisq)
1   6 -173.88
2   5 -175.60 -1 3.4316    0.06396 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- As this model is nested within the original model, we can do a likelihood ratio test.
- **Chi-square test**: Null rejected at **10%** but not at **5%**, similar to earlier homogeneity test.
- The **homogeneity-imposed model** ensures coefficients sum to one → used for further analysis.

Are firms' choices consistent with profit-maximization principle?

- An implication of Hotelling's lemma:

$$\alpha_p = \frac{\partial \ln \pi}{\partial \ln p} = \frac{\partial \pi}{\partial p} \cdot \frac{p}{\pi} = \frac{p \cdot y}{\pi} = \text{ profit share of output } (\geq 1)$$

$$\alpha_i = \frac{\partial \ln \pi}{\partial \ln w_i} = \frac{\partial \pi}{\partial w_i} \cdot \frac{w_i}{\pi} = -\frac{w_i \cdot x_i(p, \mathbf{w})}{\pi} = \text{ profit share of input i } (\leq 0)$$

- This can also be rewritten as **Profit shares** differ from cost shares → not bounded between 0 and 1.

- Example: profit share of **2 (output)** = revenue twice the realized profit; profit share of **–½ (input)** = input cost equals half of realized profit.

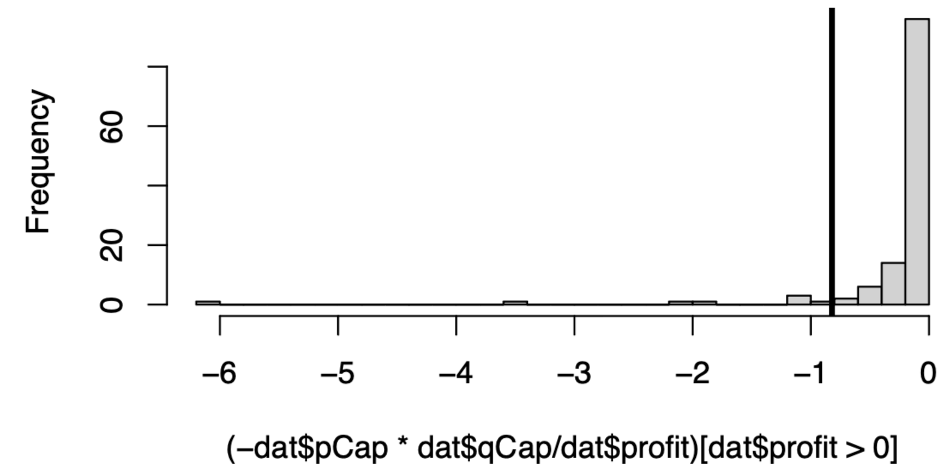
- Comparing actual profit shares with α_i shows whether firms overuse/underuse inputs or deviate from profit-maximizing output.

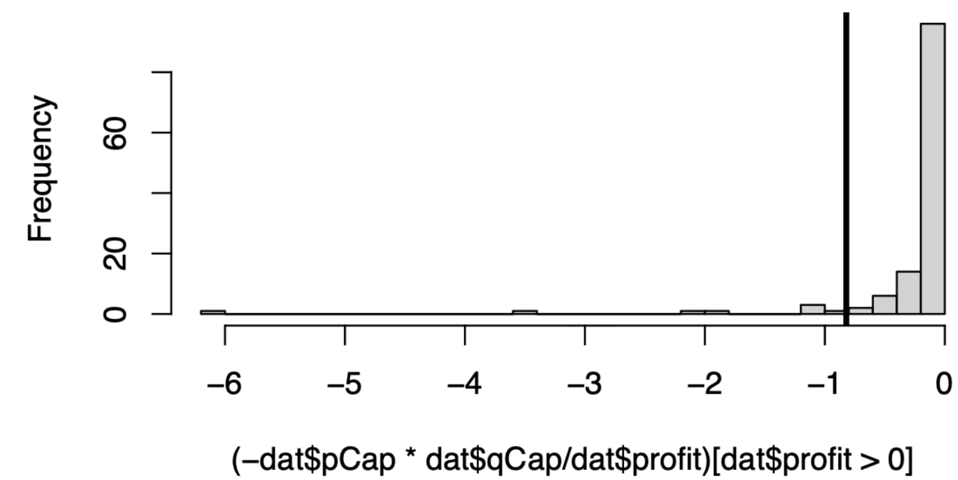
Are firms' choices consistent with profit-maximization principle?

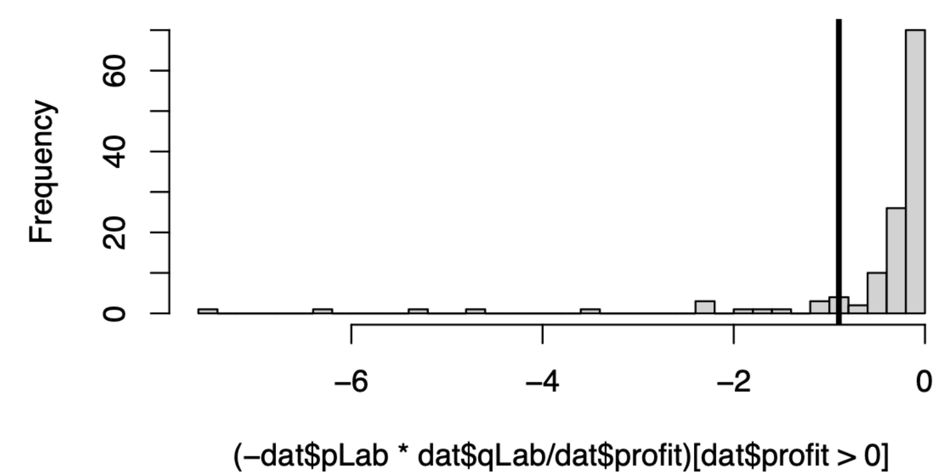
- Histograms show most firms operate below the optimal profit-maximizing level (similar to scale elasticity findings).

- Input use: firms could benefit from **more capital** and **less materials**, though this contrasts with earlier results.

- Possible reasons: short-run capital constraints or need to test alternative profit function forms (see Henningsen 2024).







Reading materials

- Varian, Chapter 3 and 5
- Henningsen, Chapter 3 and 4