Lecture 3: Approximation Error

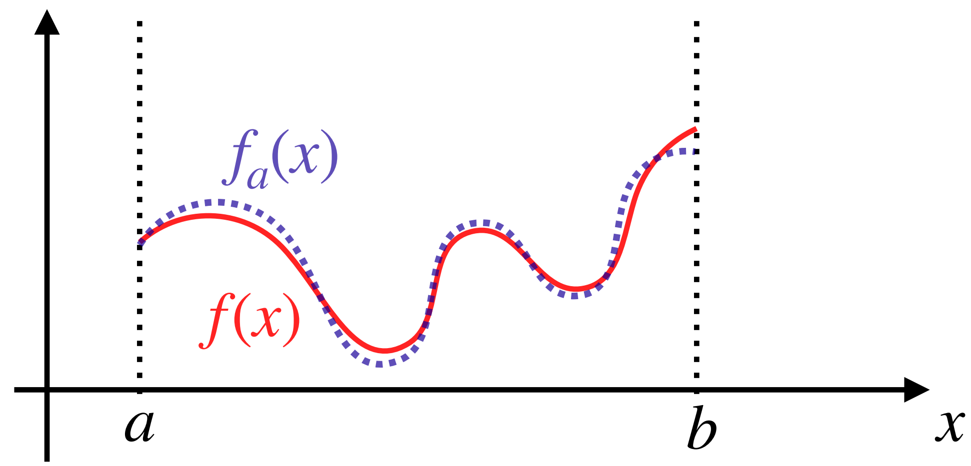
Today:

- Quantify ***error*** in function approximation
- More key concepts — ***inner product, norm***

Reminder from last week:
Where are we with approximating functions?

Goal: find a function, $f_a(x)$, that approximates a given function, $f(x)$, *accurately* on $x \in [a, b]$

x belonging to the interval $[a, b]$



How do we address these challenges?

What are the key ***challenges*** to this approach?

- There are infinitely many possible functions! Can't handle on a computer
- What does accurately mean?

Today!

Introduce key concept of ***norm*** to quantify the *size of the error*

Last lecture

Restrict the set of possible functions we are considering

Introduce key concepts of a ***vector space & subspace***

Stating where we are so far in words

Last time: We used the concepts of ***vector space*** and ***subspace*** to create a framework for posing the ***ambiguous aim*** "I want to be able to approximate any given function" into the ***concrete, finite-dimensional goal*** "I want to solve for a finite number of coefficients"

Today: We will use the concept of a ***norm*** to create a framework for characterizing how ***accurate*** our approximation is.

The norm will be defined in terms of an ***inner product***, so we'll start there.

Inner products

Inner product

Definition: A function $(\,\cdot\,,\cdot\,)$ defined on a vector space $\mathscr{V}$ is called an <u>inner product</u> if

(1) (u, v) returns a scalar number for any $u, v \in \mathscr{V}$

(2) $(u, v) = \overline{(v, u)}$ for any $u, v \in \mathscr{V}$

Complex conjugate

(3) $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$ for any $u, v, w \in \mathscr{V}$ and any $\alpha, \beta \in \mathbb{C}$

The set of complex numbers

(4) $(u, u) \geq 0$ for any $u \in \mathscr{V}$, with equality if and only if $u = 0$

This is a lot to take in at first blush, and is best learned through examples. So let's consider several different cases…

Inner product: example on $\mathbb{R}^2$

Consider the vector space $\mathscr{V} = \mathbb{R}^2$

The set of all 2x1 real-valued vectors

Example: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -13 \\ 212 \end{bmatrix} \in \mathbb{R}^2$

A permissible (and very common) inner product on this space is the function $(\,\cdot\,,\cdot\,)$ defined by

$$(u, v) = u^T v \text{ for any } u, v \in \mathbb{R}^2 \qquad (1)$$

$u^T v = u_1 v_1 + u_2 v_2$

Activity:

(A) Show that the proposed function (1) satisfies the properties of an inner product for vectors $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, w = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and your choice of α, β

(B) Give some physical intuition for what the inner product ***means***

Inner product: example on $\mathbb{R}^2$ (continued)

(A) We can work through each property one at a time:

(1) $u^T v = [1 \quad 1] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0$ is a scalar number

(2) $(v, u) = v^T u = [-1 \quad 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 = (u, v) = \overline{(u, v)}$

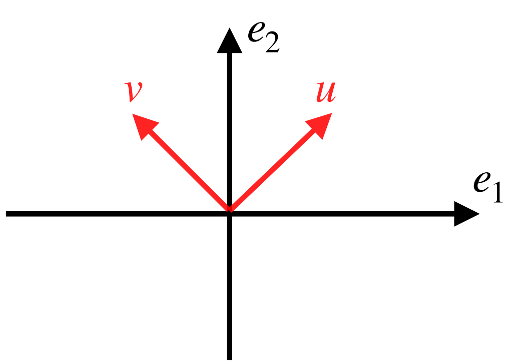
(3) Choosing $\alpha = 2$, $\beta = \dfrac{1}{2}$, we have that $(\alpha u + \beta v, w) = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 8$

But we also have that

$$\alpha(u, w) + \beta(v, w) = 2 \left([1 \quad 1] \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) + \frac{1}{2} \left([-1 \quad 1] \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) = 8 + 0 = 8$$

(4) $(u, u) = [1 \quad 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 > 0$

(B) The inner product is a measure of orthogonality between two vectors. The inner product (u, v) was zero for our choice of u and v above because they are perpendicular to one another.



Inner product: example on $\mathscr{C}[a, b]$

Consider the vector space $\mathscr{V} = C[a, b]$

Consider the candidate inner product defined by

$$(u, v) = \int_a^b u(x)v(x)dx \text{ for any } u, v \in \mathscr{V} \tag{2}$$

Activity:

(A) Argue for whether the function from (2) is an inner product or not

(B) If it is an inner product, use example 1 as an analogy to give some physical intuition for what the inner product ***means*** in this function setting

Inner product: example on $\mathscr{C}[a,b]$ (continued)

(A) We can work through each property one at a time:

(1) For any two continuous functions, taking the proposed integral will give a scalar real number

(2) For any $u, v \in \mathscr{V}$, $\int_a^b u(x)v(x)dx = \int_a^b v(x)u(x)dx = (v,u) = \overline{(v,u)}$

(3)

$$(\alpha u + \beta v, w) = \int_a^b (\alpha u(x) + \beta v(x))w(x)dx = \alpha \int_a^b u(x)w(x)dx + \beta \int_a^b v(x)w(x)dx = \alpha(u,w) + \beta(v,w)$$

(4) For any $u \in \mathscr{V}$, note that $\int_a^b u(x)u(x)dx = \int_a^b u(x)^2 dx \geq 0$

Moreover, the answer is only zero if $u(x) = 0$

(B) By analogy with example 1, the inner product gives us a generalized measure of orthogonality in this function setting. This ability to connect functions and vectors is powerful and beautiful!

Inner product: another example on $\mathscr{C}[a, b]$

Consider the vector space $\mathscr{V} = C[a, b]$

Consider the candidate inner product defined by

$$(u, v) = u(x)v(x) \text{ for any } u, v \in \mathscr{V} \tag{3}$$

Group activity:

(A) Is the proposed function in equation (3) an inner product?

Inner product: another example on $\mathscr{C}[a, b]$ (continued)

(A) No, it is not an inner product because it does not obey the first property:

For any $u, v \in \mathscr{V}$, $(u, v) = u(x)v(x)$ is a function of x, not a scalar number

Norm

Using the inner product to obtain a norm

We have built intuition about inner products, but our original aim was to be able to measure the size of the error in our approximation. That is, how big is $f - f_a$?

The ***norm*** gives us this answer, and we define the norm from the inner product:

Definition: A norm is a function $||\cdot||$ defined on a vector space $\mathscr{V}$ in terms of an inner product, as

$$||u|| = \sqrt{(u,u)} \text{ for any } u \in \mathscr{V}$$

We can now assess our approximation error, $e = f - f_a$, via $||e|| = \sqrt{(e,e)}$

Some notes:

- By property (4) of an inner product, the norm is non-negative, as desired (can't have a negative size!)
- Using different inner products to induce the norm can help emphasize different things (e.g., in compressible flow, there are debates about whether to induce a norm from mechanical energy or entropy)