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# Lecture 2: Vector Spaces

Today:

- Introduce the problem of **approximating functions**
- Introduce key concepts — **vector space, subspace**

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# Function approximation

# Recall our roadmap…

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} \qquad (1)$$

Start w/ this today!

Let's make some observations:

$\mathbf{u}$ is a function of $\mathbf{x}, t$ (space and time) → **Step 1**: we will learn how to approximate *prescribed* functions

Equation (1) is a PDE that depends on both space and time → **Step 2**: we will learn how to numerically solve *ODEs* in *time*

$$\frac{\partial \mathbf{u}}{\partial t} = f(\mathbf{u}, t) \approx f_a(\mathbf{u}, t)$$

→ **Step 3**: we will learn how to numerically solve *ODEs* in *space*

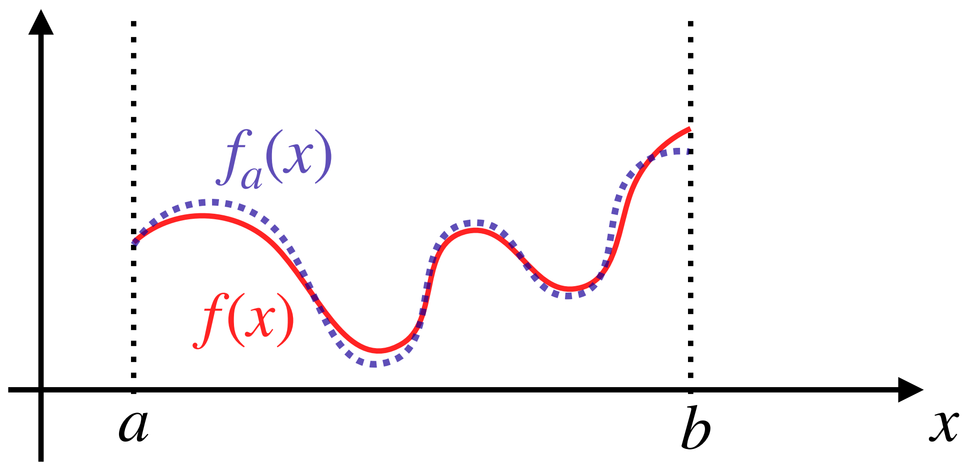
We can then use $f_a(\mathbf{u}, t)$ to solve an ODE in time → **Step 4**: we will learn how to numerically solve *PDEs* in *space and time*

This is our roadmap for the semester!

# Goal of function approximation

***Goal:*** find a function, $f_a(x)$, that approximates a given function, $f(x)$, *accurately* on $x \in [a, b]$

$x$ belonging to the interval $[a, b]$




How do we address these challenges?

What are the key ***challenges*** to this approach?

- There are infinitely many possible functions! Can't handle on a computer
- What does accurately mean?

Introduce key concept of ***norm*** to quantify the *size of the error*

Restrict the set of possible functions we are considering
Introduce key concepts of a ***vector space & subspace***

# Vector spaces and subspaces

# Vector spaces

**Definition:** A <u>vector space</u> is a set of elements where addition and scalar multiplication are well defined

**Ex:** $C[a, b]$ The space of all continuous functions defined on the interval $[a, b]$

Let's check that this is a vector space:

(1) *Additivity*:

- consider two continuous functions, $f(x), g(x)$
- $f(x) + g(x)$ makes sense (can add continuous functions)

(2) *Scalar multiplication*:

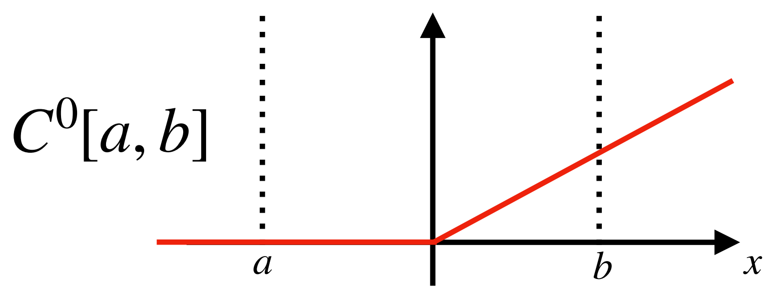
- consider a function $f(x)$ and a scalar number $\alpha \in \mathbb{R}$ $\alpha$ belongs to the set of real numbers
- $\alpha f(x)$ makes sense (can multiply continuous functions by a scalar number)

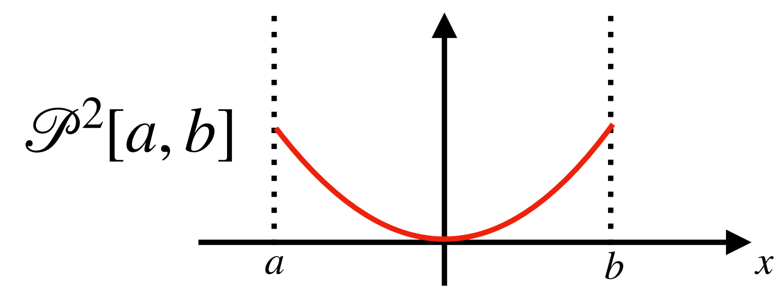
# Activity:
# Playing with vector spaces!

***Activity:***

(A) Is $C^k[a, b]$ (the space of k-times differential functions on the interval $[a, b]$) a vector space?

(B) What about $\mathscr{P}^n[a, b]$ (the space of all polynomials of degree n or less on the interval $[a, b]$)?




***Key takeaway:*** How did we know the answers to (1) are vector spaces??

Check that additivity and scalar multiplication makes sense

***Some questions you may have right now.***

A) Why are vector spaces important for approximating functions?

B) Why is it useful to approximate the desired function, $f(x)$, with another function, $f_a(x)$?

# Subspaces

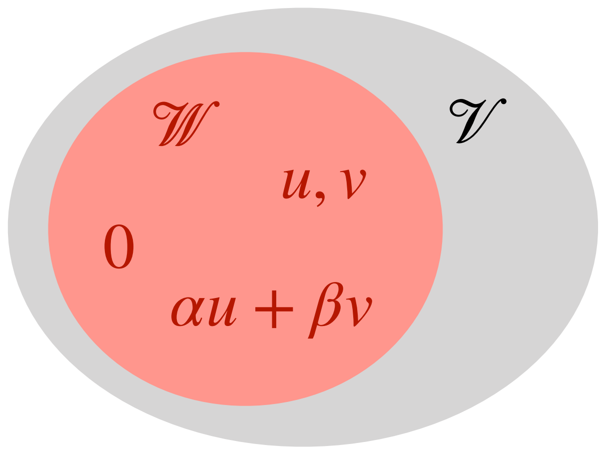
Don't forget about this!

**Definition:** A subset $\mathscr{W}$ of vector space $\mathscr{V}$ is called a <u>*subspace*</u> of $\mathscr{V}$ if:

(1) $0 \in \mathscr{W}$ The $0$ function, $f(x) = 0$, belongs to $\mathscr{W}$

(2) $\alpha u + \beta v \in \mathscr{W}$ for $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathscr{W}$

Elements of the subspace stay in the subspace after basic arithmetic operations




**Example:** $\mathscr{W} = C^1[a, b]$ is a *subspace* of $\mathscr{V} = C[a, b]$

(1) $\mathscr{W}$ is a subset of $\mathscr{V}$

(2) $0 \in C^1[a, b]$

(3) Multiply two differentiable functions by a scalar and add them together $\implies$ get a differentiable function

# Activity:
# Playing with subspaces!

***Activity:***

(A) Is $C^k[a, b]$ (the space of k-times differential functions on the interval $[a, b]$) a subspace of $C[a, b]$?

(B) $\mathscr{P}^n[a, b]$ (the space of all polynomials of degree n or less on the interval $[a, b]$) is a subspace of $C[a, b]$?

***Some questions you may have now:***

(A) How do subspaces help with approximating functions?

(B) Why is $C[a, b]$ not a subspace of $C^1[a, b]$ if $C[a, b]$ satisfies definitions (1) & (2) of a subspace?

# Vector spaces and subspaces:
# Why do we care?

***Motivate by example:***

(1) Let's say we know $f \in C[a, b]$

(2) I can pick a subspace $\mathscr{P}^n[a, b]$

(3) *Any* degree-n polynomial can be written as

$$f_a(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

(4) Turns the ***ambiguous goal*** of "approximate one of infinitely many possible $f(x)$" into the ***concrete, finite dimensional aim*** "solve for $n+1$ coefficients $c_0, \ldots, c_n$"

***So we care because…***

***vector space:*** useful concept because it helps us

(1) characterize the function we want to approximate

(2) decide which subspace to use to build our approximating function

***subspace:*** useful concept because it helps us

(1) limit the infinitely many possible functions to a finite-sized subset