Lecture 18: Finite Element Methods (1)

Today:

- *Finite element methods* for *boundary value problems (BVPs)*
  - This is a spectral method too!
  - But it is based on *locally* defined functions, not global ones

Where are we up to now?

Last time.

(A) We developed a framework for approximating the solution to a BVP using global spectral methods

(B) We arrived at a linear system to solve for the coefficients

NOTE: starting index at 1, not 0; ending index at $n+1$, not n. Either approach is ok as long as you are consistent!

$$\implies \begin{bmatrix} (b_1,b_1)_E & (b_2,b_1)_E & \cdots & (b_n,b_1)_E & (b_{n+1},b_1)_E \\ (b_1,b_2)_E & (b_2,b_2)_E & \cdots & (b_n,b_2)_E & (b_{n+1},b_2)_E \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (b_1,b_n)_E & (b_2,b_n)_E & \cdots & (b_n,b_n)_E & (b_{n+1},b_n)_E \\ (b_1,b_{n+1})_E & (b_2,b_{n+1})_E & \cdots & (b_n,b_{n+1})_E & (b_{n+1},b_{n+1})_E \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ c_{n+1} \end{bmatrix} = - \begin{bmatrix} (f,b_1)_s \\ (f,b_2)_s \\ \vdots \\ (f,b_n)_s \\ (f,b_{n+1})_s \end{bmatrix} \qquad (1)$$

Call this $\mathbf{Gc} = \mathbf{b}$

(C) Can back out our approximate solution via

$$u_a(x) = \sum_{j=1}^{n+1} c_j b_j(x)$$

Today. The ***finite element method (FEM):*** a locally-based spectral method

- What is the premise for the finite element method?
- How do we define a subspace that admits locally defined functions?
- How do we pick a basis for that subspace, and how does that affect (1)

The premise of FEMs

Philosophy behind the finite element method

The finite element method is a spectral method that uses locally defined functions to create a matrix G with predominately zero entries to facilitate a fast solution of $Gc = b$.
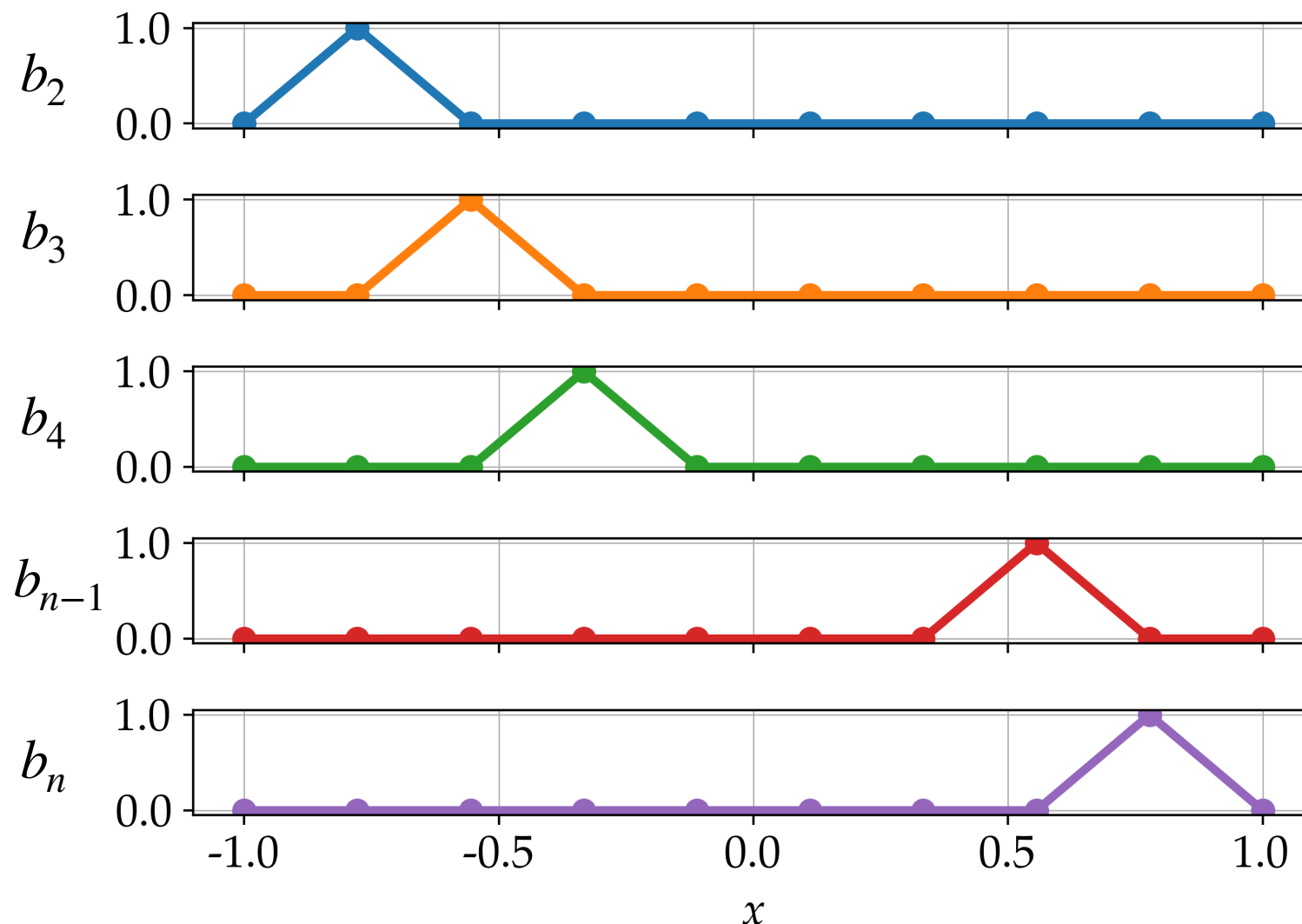
Remember that our energy inner product uses some variant of an integral, so using these locally defined functions will make most of the terms in G zero.
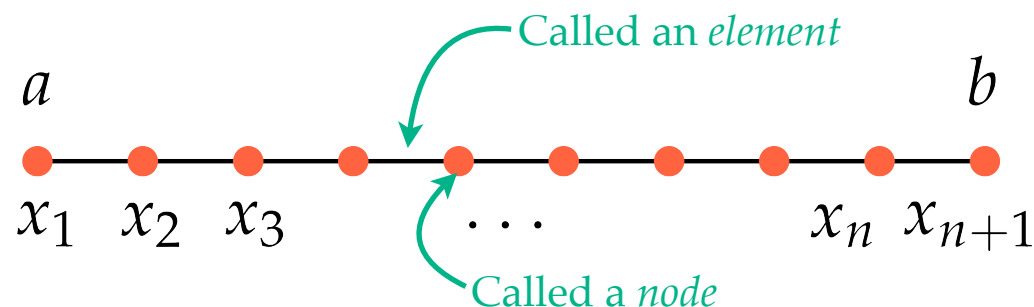
$\implies$ easier system to solve!

How do we define a subspace for these functions?

And how do we create a basis for that subspace?

Define the subspace using a discretized domain

$$x_j = a + \frac{(b-a)(j-1)}{n}, \quad j = 1, \ldots, n+1$$

a

$x_1 \quad x_2 \quad x_3 \qquad \cdots$

Called an *element*

a b

$x_1 \quad x_2 \quad x_3 \qquad \cdots \qquad x_n \quad x_{n+1}$

Called a *node*

Now that we have a set of points, we can define a function that is piecewise-linear using these points:

$$g(x) = a_i x + d_i \quad x \in [x_{i-1}, x_i], \quad i = 2, \ldots, n+1$$

This gives us intuition for the subspace we want! Let's define our subspace to be the collection of functions that

(A) Satisfy the BCs

(B) Are piecewise-linear on the set of discrete points:

$$\mathcal{V}_n^L = \{g(x) : g(a) = g(b) = 0 \text{ and}$$

$$g(x) = a_i x + d_i, \text{ for } a_i, d_i \in \mathbb{R}, \ x \in [x_{i-1}, x_i], \ i = 2, \ldots, n+1\}$$

Brain teaser — how would you go about showing this is a subspace?!

A basis for $\mathscr{V}_n^L$

We will show that the functions pictured in slide 3 form a basis for $\mathscr{V}_n^L$, define them mathematically, and then use them to build (1) for our FEM

The basis functions for $\mathscr{V}_n^L$ are defined as

$$b_i(x) = \begin{cases} \frac{1}{\Delta x}[x - a - (i-1)\Delta x] & x \in [x_{i-1}, x_i] \\ -\frac{1}{\Delta x}[x - a - (i+1)\Delta x] & x \in [x_i, x_{i+1}] \\ 0 & else \end{cases} \qquad i = 2, \ldots, n$$

$\mathscr{B} = \{b_2, \ldots, b_n\}$ forms a basis for $\mathscr{V}_n^L$:

The functions are clearly linearly independent (see the fig on slide 3 — can't write any one of the functions as a linear combo of the others!)

Take any piecewise linear function $q(x)$ defined as a set of lines over $[x_1, x_2], [x_2, x_3], \ldots, [x_n, x_{n+1}]$. Then this $q(x)$ can be written in terms of the basis functions and its *nodal values* $q(x_1), q(x_2), \ldots, q(x_n)$

$$q(x) = \sum_{j=2}^{n} q(x_j) b_j(x)$$

OK, so we have our basis functions! How does the matrix system (1) simplify for this basis? Let's figure that out for the 1D Poisson problem with zero-Dirichlet BCs!

Simplifying (1) for our FEM basis

Recall that our energy inner product is defined in terms of an integral of derivatives. So let's look at the derivative of the basis functions:

$$b_i'(x) = \begin{cases} \frac{1}{\Delta x} & x \in [x_{i-1}, x_i] \\ -\frac{1}{\Delta x} & x \in [x_i, x_{i+1}] \\ 0 & else \end{cases}$$

Because these derivatives are only nonzero for a small sub-interval of the domain, the only terms of row i of $\mathbf{G}$ that survive are $(b_i, b_{i-1})_E$, $(b_i, b_i)_E$, $(b_i, b_{i+1})_E$

These terms can be computed analytically:

$$(b_i, b_{i-1})_E = \int_{a+(i-1)\Delta x}^{a+i\Delta x} \left(\frac{-1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right) dx = -\frac{1}{\Delta x}$$

$$(b_i, b_i)_E = \int_{a+(i-1)\Delta x}^{a+(i+1)\Delta x} \left(\frac{1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right) dx = \frac{2}{\Delta x}$$

$$(b_i, b_{i+1})_E = \int_{a+i\Delta x}^{a+(i+1)\Delta x} \left(\frac{-1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right) dx = -\frac{1}{\Delta x}$$

Similarly, the $(f, b_i)_s$ term reduces to

$$(f, b_i)_s = \int_a^b f(x)b_i(x)dx = \int_{a+(i-1)\Delta x}^{a+(i+1)\Delta x} f(x)b_i(x)dx$$

Rewriting (1) for our FEM basis

$$\frac{1}{\Delta x} \begin{bmatrix} -2 & 1 & \cdots & 0 & 0 \\ 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} \int_a^{a+2\Delta x} f(x)b_2(x)dx \\ \int_{a+\Delta x}^{a+3\Delta x} f(x)b_3(x)dx \\ \vdots \\ \int_{a+(n-2)\Delta x}^{a+(n)\Delta x} f(x)b_{n-1}(x)dx \\ \int_{a+(n-1)\Delta x}^{a+(n+1)\Delta x} f(x)b_n(x)dx \end{bmatrix}$$

Punchline: lots of zeros in that matrix; easier to solve than the generic form of (1)

Punchline: Solve matrix system $\Longrightarrow$ get the u_1, u_2, …, u_n that approximate $u(x_1)$, $u(x_2)$, …, $u(x_n)$

Next time: some coding examples and concept discussions!