Euclid's Algorithms

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Objectives

Your Objectives:

- Be able to calculate the GCD of two numbers using Euclid's algorithm.
- Use the extended Euclid's algorithm to solve Linear Diophantine equations.

Calculating the GCD

- Let $a > b > 0$.
- $gcd(a, b) = gcd(b, mod(a, b))$
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- That would be slow, so how about $gcd(a, b) = gcd(b, a - nb)$, where $n > 0 and a - nb > 0$ and minimal.
- Easy! Just let $n = mod(a, b)$

An example

$$
\begin{aligned}
gcd(a,b) = gcd(b, mod(a,b)) &= gcd(90,25) \\
&= gcd(25,15) \\
&= gcd(15,10) \\
&= gcd(10,5) \\
&= gcd(5,0) \\
&= 5
\end{aligned}
$$

Diophantine Equations

- A *Diophantine Equation* is a polynomial equation where we are only interested in integer solutions.
- Linear Diophantine equation: $ax + by = 1$,
- It doesn't have to be 1....
- Running example: Suppose you go to the store. You buy x apples at 72 cents each and y oranges at 33 cents each. You spend \$5.85. How many of each did you buy?

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- We want: $ax + by = g$, where $g = gcd(a, b)$. We know a, b, and we calculate g. How can we get x and y?

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- Then take $a \mod b = a - \lfloor \frac{a}{b} \rfloor * b$ This gives:

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- Rearrange a bit..

$$bx_1 + ay_1 - \left\lfloor \frac{a}{b} \right\rfloor by_1 = g \quad \Rightarrow \quad ay_1 + b(x_1 - \left\lfloor \frac{a}{b} \right\rfloor y_1) = g$$

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- This in turn gives us:

$$\begin{aligned} x &= y_1 \\ y &= x_1 - \left\lfloor \tfrac{a}{b} \right\rfloor y_1 \end{aligned}$$

The Code

$$
\begin{aligned}
x &= \quad y_1 \\
y &= \quad x_1 - \left\lfloor \frac{a}{b} \right\rfloor y_1
\end{aligned}
$$

```
// Stolen from Competitive Programming 3
// store x, y, and d as global variables
void extendedEuclid(int a, int b) {
   if (b == 0) { x = 1; y = 0; d = a; return; }
   extendedEuclid(b, a % b);
   // similar as the original gcd
   int x1 = y;
   int y1 = x - (a / b) * y;
   x = x1;
   y = y1;
}
```

An Example

- Suppose you go to the store. You buy x apples at 72 cents each and y oranges at 33 cents each. You spend \$5.85. How many of each did you buy?

a	**b**	**x**	**y**	$a \times x + b \times y = 3$
72	33			
33	6			
6	3			
3	0			

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6	3	0	1	$6 \times 0 + 3 \times 1 = 3$
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a	**b**	**x**	**y**	$a \times x + b \times y = 3$
72	33			
33	6	1	-5	$33 \times 1 + 6 \times -5 = 3$
6	3	0	1	$6 \times 0 + 3 \times 1 = 3$
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a	**b**	**x**	**y**	$a \times x + b \times y = 3$
72	33	-5	11	$72 \times -5 + 33 \times 11 = 3$
33	6	1	-5	$33 \times 1 + 6 \times -5 = 3$
6	3	0	1	$6 \times 0 + 3 \times 1 = 3$
3	0	1	0	$3 \times 1 + 0 \times 0 = 3$

An example, ctd.

- Suppose you go to the store. You buy x apples at 72 cents each and y oranges at 33 cents each. You spend \$5.85. How many of each did you buy?
- Running the algorithm, we get...

$$72 \times -5 + 33 \times 11 = 3$$

- We multiple both sides by 195 (since $585 = 3 \times 195$) This gives us...

$$72 \times -975 + 33 \times 2145 = 585$$

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- We can add $72(\frac{33}{3})n$ to the 72 term and subtract $33(\frac{72}{3})n$ from the second term and still have a valid equation.

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- Solve $-975 + 11n > 0$, this reduces to $n > 88.6$. So take $n = 89$.

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- We can add $72(\frac{33}{3})n$ to the 72 term and subtract $33(\frac{72}{3})n$ from the second term and still have a valid equation.
- Solve $-975 + 11n > 0$, this reduces to $n > 88.6$. So take $n = 89$.
- This gives us the final equation

$$72 \times 4 + 33 \times 9 = 585$$