Minimum Spanning Trees

Connecting a Graph

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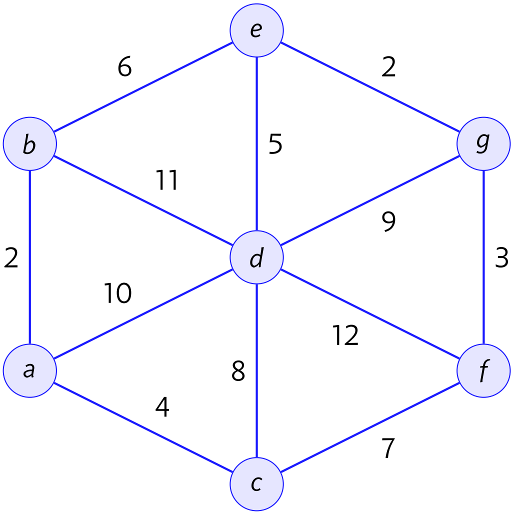
Fall, 2022

Outline

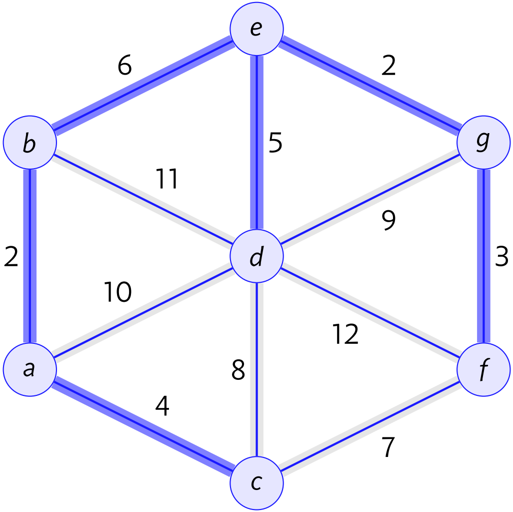
Objectives

- Define minimum spanning tree and explain its properties
- List some problems that MSTs solve
- Explain some variations of the MST
- Implement Kruscal's Algorithm

A Minimum Spanning Tree



A Minimum Spanning Tree



Properties of MSTs

- All connected graphs have one.
- May have more than one.
- $|E| = |V| - 1$
- Other tree properties hold…

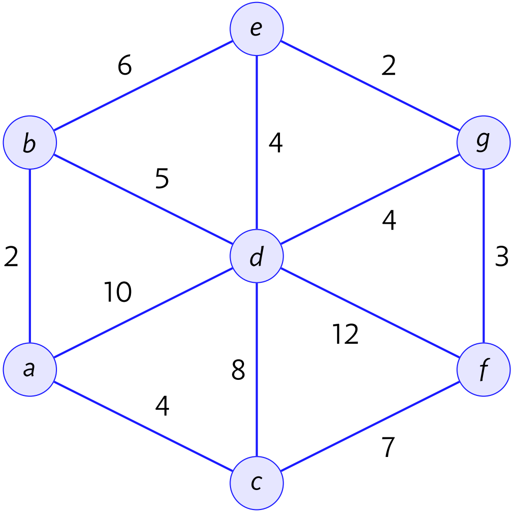
Variations

- Maximum spanning tree
- "minimum" spanning subgraph
- Minimum spanning forest
- Second minimum spanning tree: Compute MST, then try again $|E|$ times, removing a different edge from the MST each time.
- In contests: this algorithm is easy, so contest problems will try to disguise the fact that MST will solve it.

Outline

- Insert all edges into a priority queue
- Initialize a disjoint set with all the edges
- While there are fewer than $|V| - 1$ edges in your MST:
  - Dequeue an edge.
  - If the incident vertices are not both part of the MST already, add the edge. (Use the disjoint set to keep track)

Example Run



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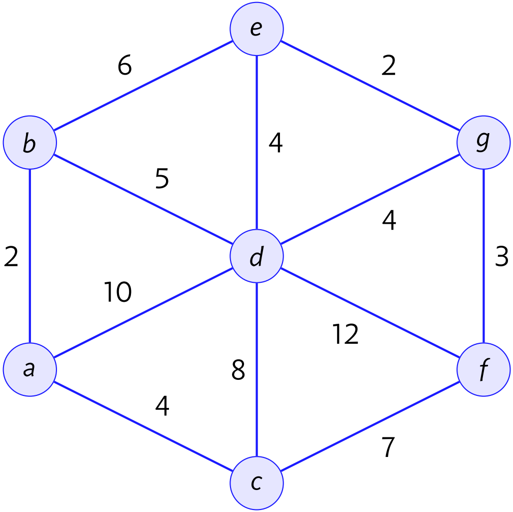

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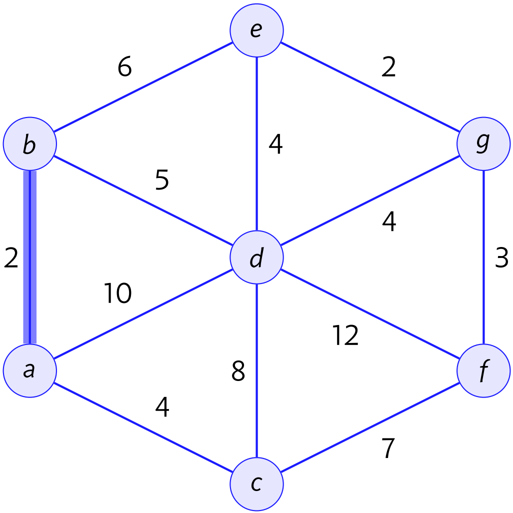

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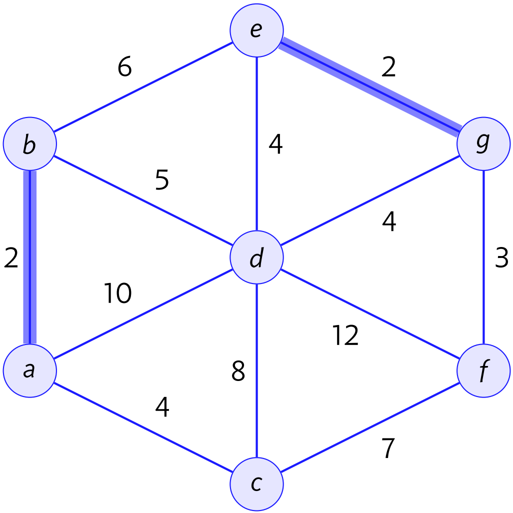

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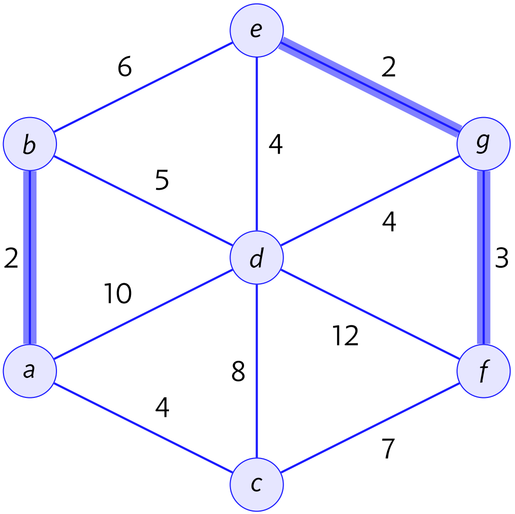

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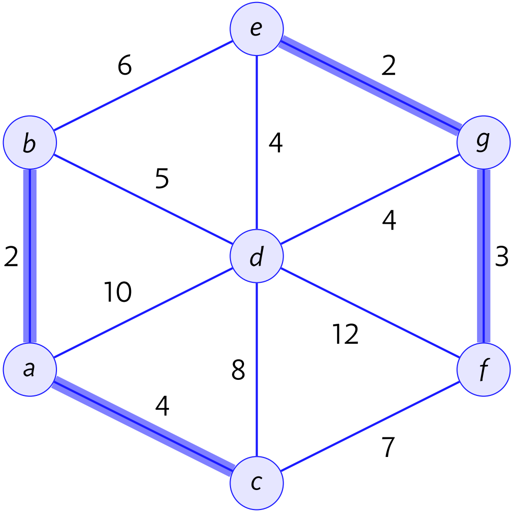

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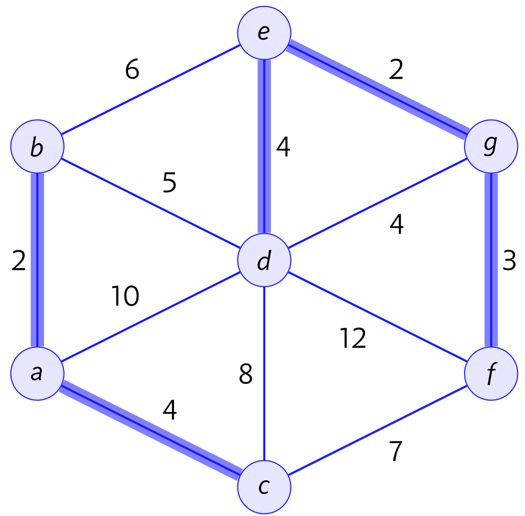

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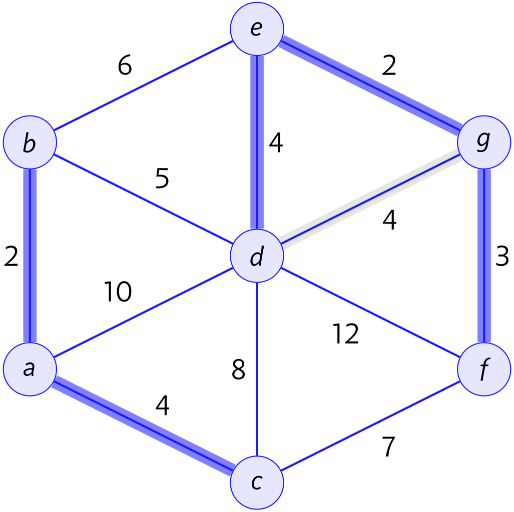

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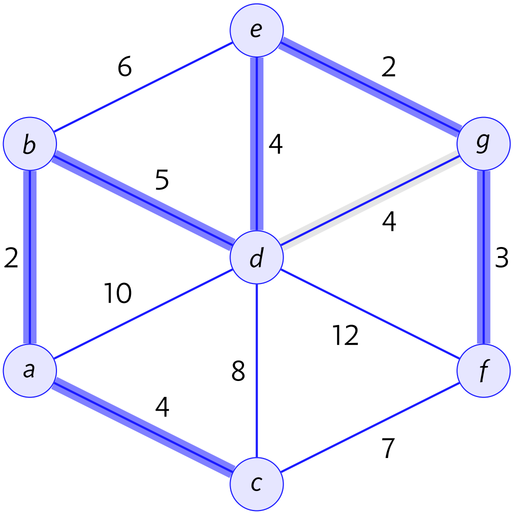

Code

```
vector< pair<int, ii> > EdgeList;
for (int i = 0; i < E; i++) {
    cin >> u >> v >> w;
    EdgeList.push_back(make_pair(w, ii(u, v)));
}
sort(EdgeList.begin(), EdgeList.end());
int mst_cost = 0;
UnionFind UF(V);
for (int i = 0; i < E; i++) {
    pair<int, ii> front = EdgeList[i];
    if (!UF.isSameSet(front.second.first
                      ,front.second.second)) {
        mst_cost += front.first;
        UF.unionSet(front.second.first, front.second.second)
} }
```

Idea

- In Kruscal's algorithm, you add edges by order of weight.
- In Prim's algorithm, you extend your current tree by adding a least-cost node.

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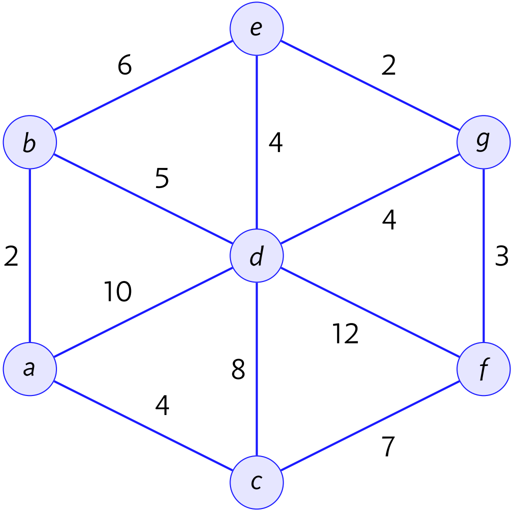

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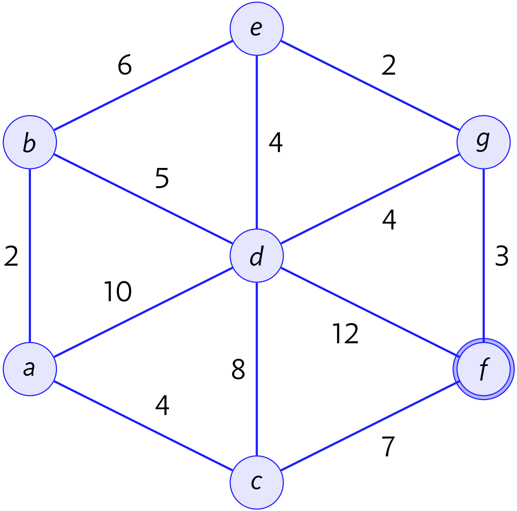

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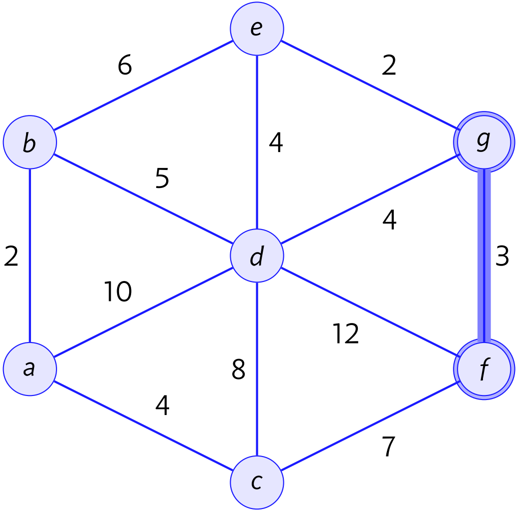

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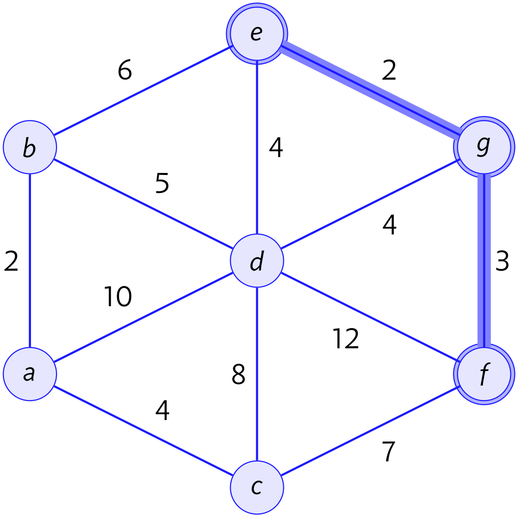

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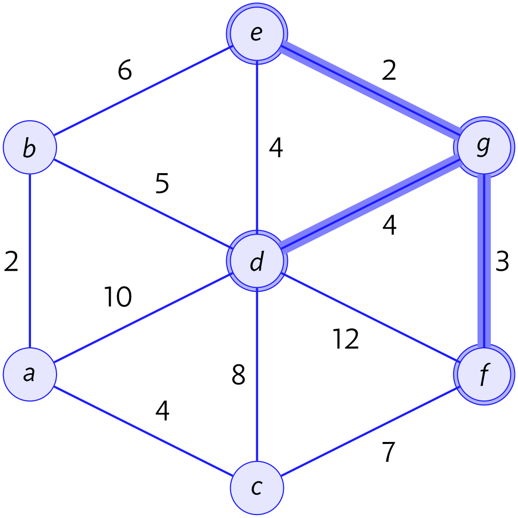

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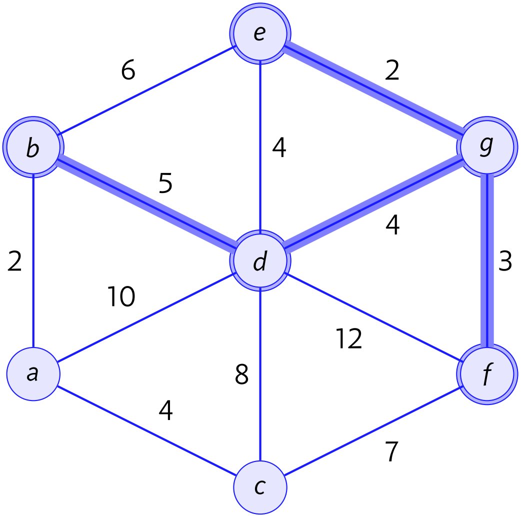

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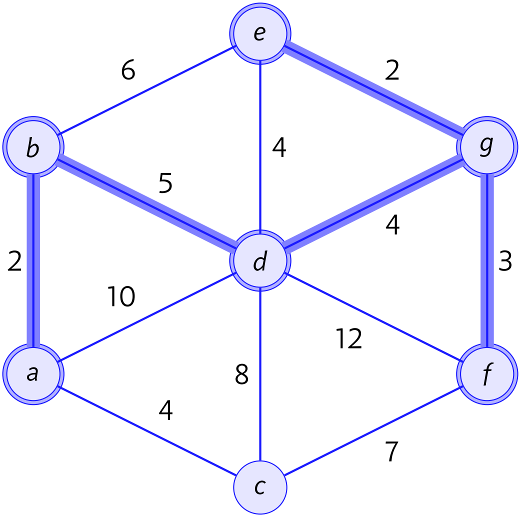

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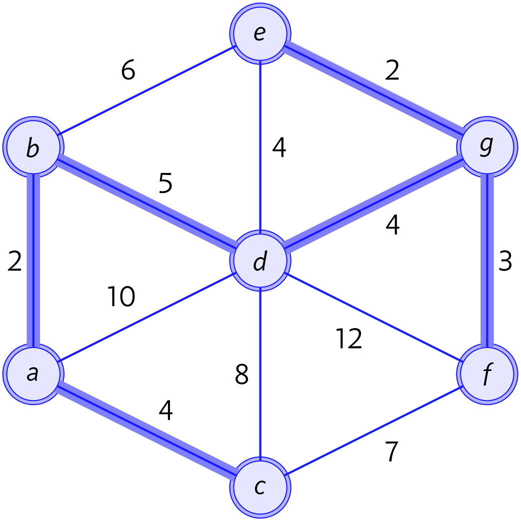

Code from CP book 4

```
vector<vii> AL; // the graph stored in AL
vi taken; // to avoid cycle
priority_queue<ii> pq; // to select shorter edges
// C++ STL priority_queue is a max heap, we use -ve sign to
void process(int u) { // set u as taken and enqueue neighbo
  taken[u] = 1;
  for (auto &[v, w] : AL[u])
    if (!taken[v]) pq.emplace(-w, -v); // sort by non-dec w
}
```

Code, part 2

- in main, setting up...

```
int V, E;
cin >> V >> E;
AL.assign(V, vii());
for (int i = 0; i < E; ++i) {
  int u, v, w;
  cin >> u >> v >> w; // read as (u, v, w)
  AL[u].emplace_back(v, w); AL[v].emplace_back(u, w);
}
taken.assign(V, 0); // no vertex is taken
```

Code, part 3

- The main loop

```
process(0); // take+process vertex 0
int mst_cost = 0, num_taken = 0; // no edge has been taken
while (!pq.empty()) { // up to O(E)
  auto [w, u] = pq.top(); pq.pop(); // C++17 style
  w = -w; u = -u; // negate to reverse order
  if (taken[u]) continue; // already taken, skipped
  mst_cost += w; // add w of this edge
  process(u); // take+process vertex u
  ++num_taken; // 1 more edge is taken
  if (num_taken == V-1) break; // optimization
}
cout << "MST cost = " << mst_cost << " (Prim's)" << endl;
```