

Combinatorics

CS 491 CAP

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Objectives

- ▶ Determine the next lexicographic permutation of an array
- ▶ Calculate and use Binomial Coefficients

Permutations

- ▶ A *permutation* is a rearrangement of elements of an array.
 - ▶ Some permutations of 1,2,3,4,5:
1 4 3 5 2
4 1 2 3 5
5 4 3 1 2
3 2 5 1 2
- ▶ There are $n!$ permutations of n distinct elements.

Permutations with Repetitions

- ▶ Suppose there are repeated elements
 - ▶ n total elements,
 - ▶ n_1 elements of class 1,
 - ▶ n_2 elements of class 2, etc...
 - ▶ n_j elements of class j .

There are $\frac{n!}{n_1!n_2!\cdots n_j!}$ total permutations.

- ▶ E.g., How many ways are there to line up 6 red balls and 3 white balls?
$$= \frac{9!}{6!3!}$$

Calculating Next Permutations

- ▶ C++ has a `next_permutation` function, but suppose you need to do this yourself?

- ▶ Find the highest index i such that $a[i] < a[i + 1]$ This is the *pivot*.
- ▶ Find the highest index j such that $a[j] > a[i]$.

1 4 3 5 2

- ▶ In the above array, $a[i] = 3$, $a[j] = 5$.
- ▶ Swap $a[j]$ and $a[i]$.

1 4 5 3 2

- ▶ Then sort the following elements.

1 4 5 2 3

Derangements

- ▶ A derangement is a permutation in which every element is relocated.

- ▶ Written $!n$

$$!0 = 0$$

$$!1 = 0$$

$$!n = (n - 1) * (! (n - 1) + ! (n - 2))$$

- ▶ $!2 = 1, !3 = 2, !4 = 9, !5 = 44, !6 = 265, \dots$

- ▶ Not that common, but easy to code with DP.

Binomial Coefficients

- ▶ Coefficients of the expansion of $(x + y)^n$
e.g. $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
- ▶ These are everywhere. E.g. Pascal's Triangle...

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

- ▶ Number of ways to chose k items from n objects. (k starts at 0...)
- ▶ The formula: $C(n, k) = \frac{n!}{k!(n-k)!}$
- ▶ The recurrence: "either take or ignore an item"
 $C(n, 0) = C(n, n) = 1$
 $C(n, k) = C(n-1, k-1) + C(n-1, k)$
- ▶ Use DP if you need a lot, but not all, of these numbers.