Generalization of Orthogonal Kravchenko Wavelets with Convolutions of Rectangular Pulse and Atomic Functions

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Abstract— Construction of orthogonal Kravchenko wavelets is similar to Meyer wavelets. It begins from Fourier transform of scaling function. It must be supported on the segment $[-4\pi/3; 4\pi/3]$ and equal to one on the segment $[-2\pi/3; 2\pi/3]$. Sum of its shifted squares must present partition of unity. If Fourier transform of scaling function is selected, then scaling function and wavelet may be found by known formulae. Meyer wavelets use trigonometric polynomials as Fourier transform of scaling function. For Kravchenko wavelets square root of sums of scaled and shifted atomic functions are applied.

Atomic functions are compactly supported solutions of differential equations with linearly transformed argument of special form. They are infinitely smooth. Their shifts present partition of unity, which is used for wavelet construction. Note that requirements of support and constant segment length impose strict restrictions on selected atomic functions. If atomic functions $\mathrm{h}_a(x)$ or $\mathrm{ch}_{a,n}(x)$ is used then among continuum of real values of parameter a only countable set of rational values defined by number of summed shifts and positive integer n (for $\mathrm{ch}_{a,n}(x)$) is applicable.

In the report another approach to construction of squared spectrum of scaling function is demonstrated. Convolution of rectangular pulse of length 3 and arbitrary partly continuous function supported on segment of length 1 would have support of length 4 and constant segment of length 2. Shifts of obtained function would present partition of unity according to properties of convolution. Then, after appropriate scaling such function may be considered as squared Fourier transform of scaling function of Meyer-like wavelet. Convolutions of atomic functions and rectangular pulses may be simply computed by means of Fourier transform. If we consider atomic functions in convolutions of rectangular pulses we obtain generalization of Kravchenko wavelets. Some of special cases of such convolutions are equivalent to sums of scaled shifts of atomic functions. This approach allows us to overcome limitations on values of atomic function parameters.

1. INTRODUCTION

Wavelets are widely applied in modern science and technology. Family of Meyer wavelets is well-known [1–3]. Their construction consists of several steps. The first one is construction of the spectrum of scaling function as trigonometric polynomial of special form. In the next steps Fourier transform of wavelet and wavelet itself is constructed. Family of Kravchenko wavelets [4–9] is constructed in the similar way. For these wavelets square root of sums of scaled shifts of atomic functions is selected as Fourier transform of scaling function. Other steps are similar to the Meyer construction. In this report new approach to construction of scaling function spectrum is presented. Square root of convolution of rectangular pulse and arbitrary even partly continuous compactly supported function may be considered as Fourier transform of scaling function. In the report convolutions of rectangular pulse with some atomic functions is considered. Comparison with previously constructed Kravchenko wavelets is performed. Results of numerical computation of new wavelets are presented.

2. GENERAL CONSTRUCTION OF MEYER AND KRAVCHENKO WAVELETS

Wavelets present orthogonal (or Riesz) basis of $L^2(\mathbb{R})$ which consists of scaled shifts of one function $\psi(x)$ called mother wavelet. Construction of wavelets starts from construction of multiresolution analysis [1–3], a system of closed embedded subspaces $V_j \subset L^2(\mathbb{R})$ with following properties:

1. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

2. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
3. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$,
4. $\exists$ scaling function $\varphi(x) \in V_0$ which shifts form a Riesz basis in V_0.

Let V_0 be a subspace of $L^2(\mathbb{R})$, generated by shifts of function $\varphi(x)$. In order that shifts of function $\varphi(x-n)$ generate a Riesz basis [1–3] in V_0 conditions of this theorem must be satisfied.

Theorem 1. Shifts of function $\{\varphi(x-n)\}$ is a Riesz basis in subspace $V_0 \subset L^2(\mathbb{R})$ if and only if there exist such positive constants A and B that

$$A \leqslant \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2 \leqslant B.$$

With additional condition orthonormal basis is available.

Theorem 2. Functions $\varphi_n(x) = \varphi(x-n)$ form orthonormal basis in subspace $V_0 \subset L^2(\mathbb{R})$ if and only if

$$\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2 = 1$$

almost everywhere.

For more clear consideration we introduce function $\chi(\omega)$ with following properties:

1. $\chi(\omega) = \chi(-\omega)$,
2. $\mathrm{supp}(\chi(\omega)) = \left[-\frac{4\pi}{3}; \frac{4\pi}{3}\right]$,
3. $\chi(\omega) = 1$ for $\omega \in \left[-\frac{2\pi}{3}; \frac{2\pi}{3}\right]$,
4. $\sum_{n\in\mathbb{Z}} \chi(\omega + 2\pi n) = 1$.

According to the *Theorem 2* $\hat{\varphi}(\omega) = \sqrt{\chi(\omega)}$ generates a multiresolution analysis. Refinement equation [1–3] presented in the form

$$\hat{\varphi}(\omega) = H_0\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right)$$

leads to the conjugation filter presented as

$$H_0(\omega) = \frac{\hat{\varphi}(2\omega)}{\hat{\varphi}(\omega)} = \hat{\varphi}(2\omega)$$

due to $\hat{\varphi}(\omega) = 1$ on the whole support of $\hat{\varphi}(2\omega)$. Then we periodically continue H_0 with period 2π

$$H_0(\omega) = \sum_{n\in\mathbb{Z}} \hat{\varphi}(2(\omega + 2\pi n)) \tag{1}$$

to meet conditions of the *Theorem 3*.

Theorem 3. If shifts $\varphi_n(x) = \varphi(x-n)$ form orthonormal basis in subspace $V_0 \subset L^2(\mathbb{R})$ then conjugation filter $H_0(\omega)$ has the property

$$|H_0(\omega)|^2 + |H_0(\omega + \pi)|^2 = 1.$$

If *Theorem 3* is satisfied then according to the wavelet theory $\varphi(x)$ is a scaling function which provides orthonormal multiresolution analysis and Fourier transform of wavelet function may be presented in such form

$$\hat{\psi}(\omega) = e^{\frac{i\omega}{2}}\overline{H_0\left(\frac{\omega}{2} + \pi\right)}\hat{\varphi}\left(\frac{\omega}{2}\right). \tag{2}$$

If $H_0(\omega)$ is a sum of shifted compactly supported functions and supports of only two terms of sum (1) are intersected with support of $\hat{\varphi}\left(\frac{\omega}{2}\right)$ then expression (2) for $\hat{\psi}(\omega)$ can be simplified

$$\hat{\psi}(\omega) = e^{\frac{i\omega}{2}}\left(\hat{\varphi}(\omega - 2\pi) + \hat{\varphi}(\omega + 2\pi)\right)\hat{\varphi}\left(\frac{\omega}{2}\right). \tag{3}$$

Depending on the form of $\chi(\omega)$ different wavelets may be obtained. In this work convolutions of atomic functions with rectangular pulse considered as $\chi(\omega)$.

3. BASIC PROPERTIES OF CONVOLUTION WITH RECTANGULAR PULSE

Consider rectangular pulse $\phi(x)$ [10–12]

$$\phi(x) = \begin{cases} 1 & |x| < 0.5, \\ 0.5 & |x| = 0.5, \\ 0 & |x| > 0.5. \end{cases} \tag{4}$$

It is clear that $\phi(x)$ is supported on the segment $\left[-\frac{1}{2}; \frac{1}{2}\right]$ and $f(x) = \phi\left(\frac{x}{2a}\right)$ is supported on $[-a; a]$. Then consider arbitrary nonnegative partly continuous function $g(x)$ supported on the segment $[-b; b]$ and positive on the interval $(-b; b)$. We formulate some propositions on their convolution

$$h(x) = f(x) * g(x) = \int\limits_{-\infty}^{\infty} f(x-t)g(t)dt. \tag{5}$$

Proposition 1. Convolution of $f(x)$ and $g(x)$ supported on the segment $[-a-b; a+b]$ and positive on the interval $(-a-b; a+b)$.

Proof. Consider convolution integral (5). If $|x| > a+b$ then supports of $f(x-t)$ and $g(t)$ are not intersected and $f(x-t)g(t) = 0$ for each t and integral vanishes. For $x = \pm(a+b)$ intersection of supports consists of only one point and integral vanishes too. If $|x| < a+b$ then intersection of supports is segment from $\max(-a-x, -b)$ to $\min(a-x, b)$. This is segment of the positive length. $f(x-t)$ and $g(t) = 0$ are both positive at the interior points of the segment then $f(x-t)g(t) > 0$ and value of integral is positive too.

Proposition 2. If $a > b$ then for $|x| \le a-b$ convolution $h(x) = f(x) * g(x)$ equals to constant.

Proof. From $a > b$ and $|x| \le a-b$ follows that $\max(-a-x, -b) = -b$ and $\min(a-x, b) = b$. Then for convolution integral (5) the chain of equalities holds

$$f(x) * g(x) = \int\limits_{-\infty}^{\infty} f(x-t)g(t)dt = \int\limits_{\max(-a-x,-b)}^{\min(a-x,b)} f(x-t)g(t)dt = \int\limits_{-b}^{b} f(x-t)g(t)dt = \int\limits_{-b}^{b} g(t)dt. \tag{6}$$

The last equality takes into account (4).

Proposition 3. Shifts of convolution $h(x) = f(x) * g(x)$ forms partition of constant

$$\sum_{k=-\infty}^{\infty} h(x-2ak) = c. \tag{7}$$

Proof. Rectangular pulse provides partition of unity

$$\sum_{k=-\infty}^{\infty} f(x-2ak) \equiv 1. \tag{8}$$

Note that for each value of argument sums (7) and (8) contains limited number of nonzero terms. Then for (7) we have

$$\sum_{k=-\infty}^{\infty} h(x-2ak) = \sum_{k=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x-2ak-t)g(t)dt = \int\limits_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(x-2ak-t)g(t)dt = \int\limits_{-\infty}^{\infty} g(t)dt.$$

If $g(x)$ has unit square than (7) transforms into partition of unity and value of convolution in (6) equals to one.

Proposition 4. If additionally $g(x)$ is even then convolution (5) is even too as convolution of two even functions.

4. PRACTICAL PROCEDURE OF WAVELETS CONSTRUCTION

Consider $\chi(\omega)$ defined as

$$\chi(\omega) = \phi\left(\frac{\omega}{2\pi}\right) * g(\omega), \tag{9}$$

where $g(\omega)$ satisfies following conditions:

$$\mathrm{supp}\,\big(g(\omega)\big) = \left[-\frac{\pi}{3}; \frac{\pi}{3}\right], \qquad \int\limits_{-\frac{\pi}{3}}^{\frac{\pi}{3}} g(\omega)d\omega = 1, \qquad g(\omega) = g(-\omega). \tag{10}$$

According to *Propositions 1–4* function (9) meets all requirements and may be used for orthogonal wavelets construction. Fourier transform of (9) equals to

$$\hat{\chi}(t) = 2\pi\,\mathrm{sinc}(\pi t)\hat{g}(t) \tag{11}$$

then $\chi(\omega)$ may be approximated by partial sum of Fourier series

$$\chi(\omega) = \frac{3}{4\pi}\left(\pi + \sum_{k=1}^{\infty}\hat{\chi}\left(\frac{3}{4}k\right)\cos\left(\frac{3}{4}kx\right)\right) \approx \frac{3}{4\pi}\left(\pi + \sum_{k=1}^{K}\hat{\chi}\left(\frac{3}{4}k\right)\cos\left(\frac{3}{4}kx\right)\right).$$

If we have $\chi(\omega)$ then Fourier transform of scaling function $\hat{\varphi}(\omega) = \sqrt{\chi(\omega)}$ is obtained. Scaling function can be found by inversion of Fourier transform

$$\varphi(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\hat{\varphi}(\omega)e^{i\omega x}d\omega = \frac{1}{\pi}\int\limits_{0}^{\frac{4\pi}{3}}\hat{\varphi}(\omega)\cos(\omega x)d\omega. \tag{12}$$

Note that if $\hat{\varphi}(\omega)$ compactly supported then Fourier integral (12) is perfomed on the finite segment and may be computed for each x with standard methods of numerical integration.

Then we construct $\hat{\psi}(\omega)$ according to (3). Inverse Fourier transform of wavelet function can be simplified.

$$\psi(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{\frac{i\omega}{2}}\big(\hat{\varphi}(\omega-2\pi)+\hat{\varphi}(\omega+2\pi)\big)\hat{\varphi}\left(\frac{\omega}{2}\right)e^{i\omega x}d\omega.$$

After multiplication of exponents and taking into account supports of factors we have

$$\psi(x) = \frac{1}{\pi}\int\limits_{\frac{2\pi}{3}}^{\frac{8\pi}{3}}\hat{\varphi}(\omega-2\pi)\hat{\varphi}\left(\frac{\omega}{2}\right)\cos\omega\left(x+\frac{1}{2}\right)d\omega. \tag{13}$$

If we note that $\hat{\varphi}(\omega-2\pi) = 1$ for $\omega \in \left[\frac{4\pi}{3}; \frac{8\pi}{3}\right]$ and $\hat{\varphi}\left(\frac{\omega}{2}\right) = 1$ for $\omega \in \left[-\frac{4\pi}{3}; \frac{4\pi}{3}\right]$ then we can present integral (13) as a sum of two integrals

$$\psi(x) = \frac{1}{\pi}\int\limits_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\hat{\varphi}(\omega-2\pi)\cos\omega\left(x+\frac{1}{2}\right)d\omega + \frac{1}{\pi}\int\limits_{\frac{4\pi}{3}}^{\frac{8\pi}{3}}\hat{\varphi}\left(\frac{\omega}{2}\right)\cos\omega\left(x+\frac{1}{2}\right)d\omega. \tag{14}$$

Representation of wavelet in such form decreases computation cost and error of numerical integration.

5. UNCERTAINTY CONSTANTS

Uncertainty constants [2,3] are widely used as characteristics of wavelets quality. They are invariant from shifts and scaling of function. For function $f(t)$

$$\Delta_f = \frac{1}{\|f(t)\|}\sqrt{\int\limits_{-\infty}^{\infty}\left(t-t_f^*\right)^2|f(t)|^2dt},$$

where

$$t_f^* = \frac{1}{\|f(t)\|^2}\int\limits_{-\infty}^{\infty} t|f(t)|^2dt.$$

We make some notes on norms and centers of wavelets, scaling functions and their Fourier transforms. According to (11) the norm of Fourier transform of scaling function equals to

$$\|\hat{\varphi}(\omega)\|^2 = \int\limits_{-\infty}^{\infty}\chi(\omega)d\omega = \hat{\chi}(0) = 2\pi$$

Then according to Parseval's theorem $\|\varphi(x)\| = 1$. Centers of even functions $\hat{\varphi}(\omega)$ and $\varphi(x)$ are zero.

Consider Fourier transform of wavelet in form (3). We transform integral similar to (14)

$$\|\hat{\psi}(\omega)\|^2 = \int\limits_{-\infty}^{\infty}|e^{\frac{i\omega}{2}}|^2\big(\hat{\varphi}(\omega-2\pi)+\hat{\varphi}(\omega+2\pi)\big)^2\hat{\varphi}^2\left(\frac{\omega}{2}\right) =$$

$$= 2\int\limits_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\hat{\varphi}^2(\omega-2\pi)d\omega + 2\int\limits_{\frac{4\pi}{3}}^{\frac{8\pi}{3}}\hat{\varphi}^2\left(\frac{\omega}{2}\right)d\omega = 2\int\limits_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\chi(t)dt + 4\int\limits_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\chi(t)dt = 2\pi.$$

Then $\|\psi(x)\| = 1$. Center of $\hat{\psi}(\omega)$ is zero too due to $|e^{\frac{i\omega}{2}}|^2 = 1$. Center of wavelet function $\psi(x)$ is $-\frac{1}{2}$ that clearly consequent from (13).

6. ATOMIC FUNCTIONS

Atomic functions [8–10, 13] are compactly supported solutions of differential equations of special form. Their Fourier transforms are presented as fast convergent infinite products of scaled sinc functions and may be well approximated with finite products.

6.1. Atomic function up(x)

Atomic function up(x) [8–10, 13] is the simplest of atomic functions. It is solution of equation

$$y'(x) = 2\big(y(2x+1) - y(2x-1)\big)$$

supported on $[-1;1]$. It has unit square and Fourier transform

$$\widehat{\mathrm{up}}(t) = \prod_{k=1}^{\infty}\mathrm{sinc}\left(\frac{t}{2^k}\right).$$

To meet requirements (10) consider $g(\omega) = \frac{3}{\pi}\,\mathrm{up}\left(\frac{3}{\pi}\omega\right)$. Then according to (11)

$$\hat{\chi}(t) = 2\pi\,\mathrm{sinc}(\pi t)\widehat{\mathrm{up}}\left(\frac{\pi}{3}t\right). \qquad (15)$$

Multiplication and division by $\sin\left(\frac{\pi t}{3}\right)$ and application of Euler formula transforms (15) into

$$\hat{\chi}(t) = 2\pi\,\mathrm{sinc}(\pi t)\prod_{k=1}^{\infty}\mathrm{sinc}\left(\frac{\pi}{3}\cdot\frac{t}{2^k}\right) = \frac{2\pi}{3}\cdot\frac{\sin(\pi t)}{\sin\left(\frac{\pi t}{3}\right)}\,\mathrm{sinc}\left(\frac{\pi t}{3}\right)\prod_{k=1}^{\infty}\mathrm{sinc}\left(\frac{\pi}{3}\cdot\frac{t}{2^k}\right) =$$
$$= \frac{2\pi}{3}\cdot\frac{e^{i\pi t}-e^{-i\pi t}}{e^{i\frac{\pi}{3}t}-e^{-i\frac{\pi}{3}t}}\prod_{k=1}^{\infty}\mathrm{sinc}\left(\frac{2\pi}{3}\cdot\frac{t}{2^k}\right) = \left(e^{i\frac{2\pi}{3}t}+1+e^{-i\frac{2\pi}{3}t}\right)\frac{2\pi}{3}\widehat{\mathrm{up}}\left(\frac{2\pi}{3}t\right). \quad (16)$$

The last representation of $\hat{\chi}(t)$ in (16) corresponds to the sum of scaled shifts of $\mathrm{up}(\omega)$

$$\chi(\omega) = \mathrm{up}\left(\frac{3\omega}{2\pi}+1\right) + \mathrm{up}\left(\frac{3\omega}{2\pi}\right) + \mathrm{up}\left(\frac{3\omega}{2\pi}-1\right). \quad (17)$$

Such form of $\chi(\omega)$ and corresponding wavelet was considered in [6–9].

6.2. Family of atomic functions $\mathrm{h}_a(x)$

Family of atomic functions $\mathrm{h}_a(x)$ [8–13] presents native generalization of $\mathrm{up}(x)$ function. For each real $a > 1$ $\mathrm{h}_a(x)$ is solution of equation

$$y'(x) = \frac{a^2}{2}\big(y(ax+1) - y(ax-1)\big)$$

supported on $\left[-\frac{1}{a-1}, \frac{1}{a-1}\right]$ and normalized to unit square. Its Fourier transform has such form

$$\widehat{\mathrm{h}_a}(t) = \prod_{k=1}^{\infty}\mathrm{sinc}\left(\frac{t}{a^k}\right).$$

According to (10) consider $g(\omega) = \frac{3}{\pi(a-1)}\,\mathrm{h}_a\left(\frac{3}{\pi(a-1)}\omega\right)$ and

$$\hat{\chi}(t) = 2\pi\,\mathrm{sinc}(\pi t)\widehat{\mathrm{h}_a}\left(\frac{\pi(a-1)}{3}t\right). \quad (18)$$

To find if constructed $\chi(\omega)$ is presented by sum of scaled shifts of $\mathrm{h}_a(\omega)$ we attempt to transform (18) similar to (16). Multiplication and division of (18) by $\sin\left(\frac{\pi(a-1)}{3}t\right)$ leads to

$$\hat{\chi}(t) = 2\pi\,\mathrm{sinc}(\pi t)\prod_{k=1}^{\infty}\mathrm{sinc}\left(\frac{\pi(a-1)}{3}\cdot\frac{t}{a^k}\right) = \frac{2}{a}\cdot\frac{\sin(\pi t)}{\sin\left(\frac{\pi(a-1)}{3}t\right)}\cdot\frac{a\pi(a-1)}{3}\widehat{\mathrm{h}_a}\left(\frac{a\pi(a-1)}{3}t\right). \quad (19)$$

Then $\chi(\omega)$ may be presented by sum of scaled shifts of $\mathrm{h}_a(\omega)$ if and only if $\frac{\sin(\pi t)}{\sin\left(\frac{\pi(a-1)}{3}t\right)}$ may be presented by sum of phase factors. This transformation is based on the equality

$$\frac{q^{r+1}-q^{-r-1}}{q-q^{-1}} = \sum_{k=0}^{r} q^{r-2k}, \quad (20)$$

where r is nonnegative integer and $q = e^{i\frac{\pi(a-1)}{3}t}$. To perform the transformation $q^{r+1} = e^{i\pi t}$ is required. That holds if and only if

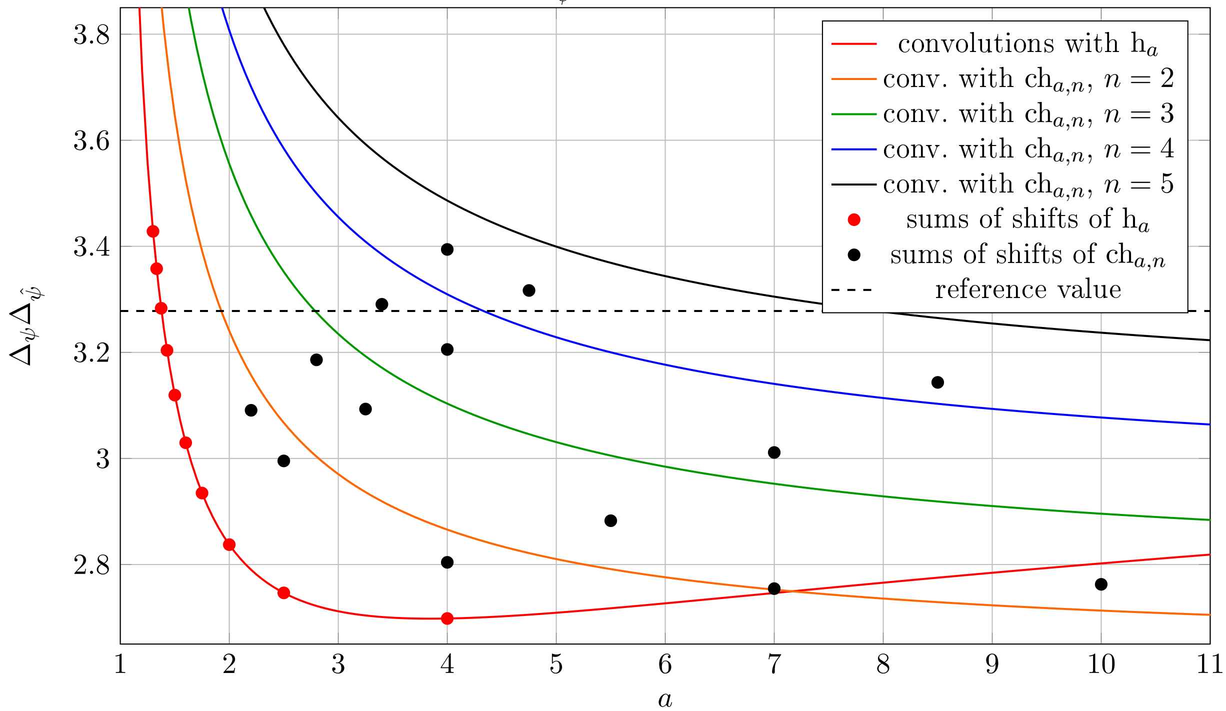
$$a = \frac{r+4}{r+1}. \quad (21)$$

For such a $\chi(\omega)$ is presented as sum of scaled shifts

$$\chi(\omega) = \frac{2}{a}\sum_{k=0}^{r}\mathrm{h}_a\left(\frac{3}{a\pi(a-1)}\omega + \frac{r-2k}{a}\right). \quad (22)$$

Then obtained wavelets for such a are equivalent to wavelets considered in [4,7,9]. For increasing r corresponding a form decreasing tending to 1 sequence. Among them if $r = 2$ then $a = 2$, previously discussed $\mathrm{up}(x)$ obtained as special case of $\mathrm{h}_a(x)$ and (22) turns to (17). For each a which can not be presented by (21) new wavelets are obtained. Previously constructed wavelets form countable subset in continuous family of new wavelets.

Figure 1: Values of uncertainty constants $\Delta_\psi \Delta_{\hat{\psi}}$ for wavelets based on $\mathrm{h}_a(x)$ and $\mathrm{ch}_{a,n}(x)$ depending on a



6.3. Family of atomic functions $\mathrm{ch}_{a,n}(x)$
Atomic function $\mathrm{ch}_{a,n}(x)$ [5, 10, 14] is a solution of equation

$$y^{(n)} = a^{n+1} 2^{-n} \sum_{k=0}^{n} C_n^k (-1)^k y(ax + n - 2k)$$

supported on $\left[-\frac{n}{a-1}, \frac{n}{a-1}\right]$ and normalized to unit square. Atomic function $\mathrm{ch}_{a,n}(x)$ is n-times iterated convolution of atomic function $\mathrm{h}_a(x)$ then it's Fourier transform is Fourier transform of atomic function $\mathrm{h}_a(x)$ raised to the power n

$$\widehat{\mathrm{ch}_{a,n}}(t) = \prod_{k=1}^{\infty} \mathrm{sinc}^n \left(\frac{t}{a^k}\right) = \left(\widehat{\mathrm{h}_a}(t)\right)^n.$$

Consider $g(\omega) = \frac{3n}{\pi(a-1)} \mathrm{ch}_{a,n}\left(\frac{3n\omega}{\pi(a-1)}\right)$ and

$$\hat{\chi}(t) = 2\pi\, \mathrm{sinc}(\pi t) \widehat{\mathrm{ch}_{a,n}}\left(\frac{\pi(a-1)t}{3n}\right). \tag{23}$$

Compare obtained wavelets to previously constructed countable family of wavelets [5] based on sums of scaled shifts of $\mathrm{ch}_{a,n}(x)$. Note that if $n = 1$ than $\mathrm{ch}_{a,1}(x) = \mathrm{h}_a(x)$. This case is considered in previous subsection. If $n > 1$ then transform of (23) like (16) and (19) is impossible due to different degree of t. Then all constructed wavelets are new and different from previously constructed.

In fig. 1 values of $\Delta_\psi \Delta_{\hat{\psi}}$ for wavelets based on $\mathrm{h}_a(x)$ and $\mathrm{ch}_{a,n}(x)$ depending on a is presented. Lower values generally corresponds to better wavelets. Reference value $\Delta_\psi \Delta_{\hat{\psi}} = 3.27802$ corresponds to Meyer wavelet. Wavelets with lower value may be considered as good and higher value as bad ones. Red line corresponds to wavelets based on h_a discussed in previous subsection. It has minimum about $a \approx 3.85$ increasing for higher values. Red points present wavelets constructed in [4, 7, 9]. Point $a = 4$ lays near minimum, point $a = 2$ corresponds to wavelet based on $\mathrm{up}(x)$. Orange, green, blue and black lines corresponds to wavelets constructed in this section. All four curves demonstrates decrease with increasing a. We note orange line $\mathrm{ch}_{a,1}$, that demonstrates minim values among four curves. Black points present wavelets constructed in [5]. Due to these wavelets are different from wavelets presented in this section points corresponding to them do not lie on curves.

6.4. Family of atomic functions $\mathrm{fup}_n(x)$

Atomic functions $\mathrm{fup}_n(x)$ [8–13] are defined as $\mathrm{fup}_n(x) = \mathrm{up}(x) * \Theta_{n-1}(x)$, where Θ_{n-1} is B-spline of $n-1$ order which is equal to convolution of n rectangular pulses $\phi(x)$. They are solutions of equation

$$y'(x) = \frac{1}{2^{n-1}} \sum_{k=0}^{n+2} \left(C_{n+1}^{k} - C_{n+1}^{k-1} \right) y\left(2x + \frac{n+2}{2} - k \right) \tag{24}$$

supported on $\left[-\frac{n+2}{2}; \frac{n+2}{2}\right]$ and normalized to unit square. Fourier transform of $\mathrm{fup}_n(x)$ is presented in the form

$$\widehat{\mathrm{fup}_n}(t) = \mathrm{sinc}^n \left(\frac{t}{2}\right) \cdot \prod_{k=1}^{\infty} \mathrm{sinc}\left(\frac{t}{2^k}\right) = \mathrm{sinc}^n\left(\frac{t}{2}\right) \cdot \widehat{\mathrm{up}}(t).$$

According to (10) and support of function consider $g(\omega) = \frac{3(n+2)}{2\pi} \mathrm{fup}_n \left(\frac{3(n+2)}{2\pi}\omega\right)$. Then

$$\hat{\chi}(t) = 2\pi \,\mathrm{sinc}(\pi t) \,\mathrm{sinc}^n \left(\frac{\pi t}{3(n+2)}\right) \cdot \widehat{\mathrm{up}} \left(\frac{2\pi t}{3(n+2)}\right). \tag{25}$$

We multiply and divide (25) by $\sin\left(\frac{\pi t}{3(n+2)}\right)$ and obtain

$$\begin{aligned} \hat{\chi}(t) = \frac{2\pi}{3(n+2)} \cdot \frac{\sin(\pi t)}{\sin\left(\frac{\pi t}{3(n+2)}\right)} \cdot \mathrm{sinc}^{n+1}\left(\frac{\pi t}{3(n+2)}\right) \cdot \widehat{\mathrm{up}}\left(\frac{2\pi t}{3(n+2)}\right) = \\ = \left(\sum_{k=0}^{3n+5} e^{i \frac{3n+5-2k}{3n+2} \pi t} \right) \frac{2\pi}{3(n+2)} \widehat{\mathrm{fup}_{n+1}} \left(\frac{2\pi t}{3(n+2)}\right). \end{aligned} \tag{26}$$

Then $\chi(\omega)$ is presented as sum of $3n+6$ scaled shifts of $\mathrm{fup}_{n+1}(\omega)$

$$\chi(\omega) = \sum_{k=0}^{3n+5} \mathrm{fup}_{n+1} \left(\frac{3(n+2)}{2\pi}\omega + \frac{3n+5-2k}{2} \right) \tag{27}$$

and obtained wavelets are similar to wavelets presented in [8,9]. We don't obtain new wavelets here, but we note that new technique allows us to derive expression (27) by simple and clear consideration (26) avoiding complicated analysis of (24).

7. FAMILY OF INFINITELY SMOOTH FUNCTIONS $\mathrm{fip}_{a,n}(x)$

Consider new family of infinitely smooth functions $\mathrm{fip}_{a,n}(x)$ [10–12], which are defined for each real $a > 1$ and nonegative integer n as convolutions of atomic function $\mathrm{h}_a(x)$ and $B-$spline

$$\mathrm{fip}_{a,n}(x) = \mathrm{h}_a(x) * \Theta_{n-1}(x).$$

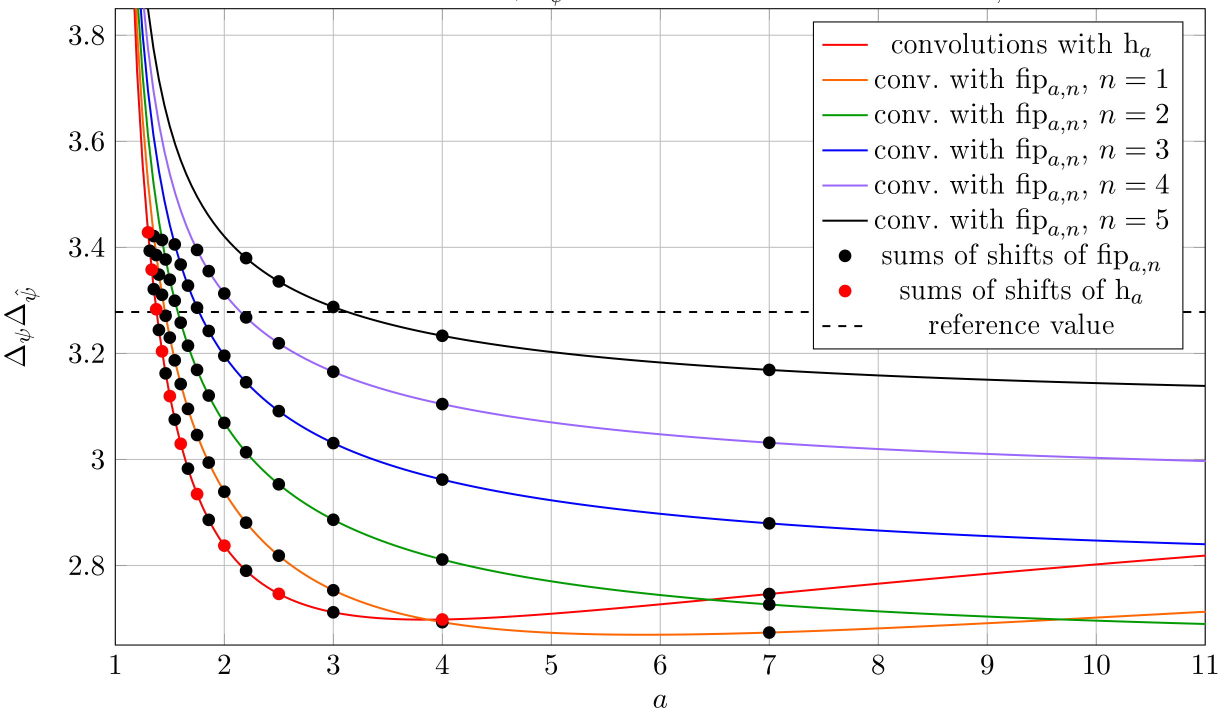
Then for each $\mathrm{fip}_{a,n}(x)$ Fourier transform is

$$\widehat{\mathrm{fip}_{a,n}}(t) = \mathrm{sinc}^n\left(\frac{t}{2}\right) \cdot \prod_{k=1}^{\infty} \mathrm{sinc}\left(\frac{t}{a^k}\right) = \mathrm{sinc}^n\left(\frac{t}{2}\right) \cdot \widehat{\mathrm{h}_a}(t).$$

These functions are not atomic functions due to in general case they are not presented as compactly supported solutions of functional-differential equations. Our construction is based only on properties of convolution and does not involve functional-differential equations. Then it is suitable for $\mathrm{fip}_{a,n}(x)$ whether the functions are atomic or not. $\mathrm{fip}_{a,n}(x)$ is supported on the segment $[-l, l]$, where $l = \frac{1}{a-1} + \frac{n}{2}$. Then we consider $g(\omega) = \frac{3l}{\pi} \mathrm{fip}_{a,n}(\frac{3l}{\pi}\omega)$ and

$$\hat{\chi}(t) = 2\pi \,\mathrm{sinc}(\pi t) \,\mathrm{sinc}^n \left(\frac{\pi t}{6l}\right) \cdot \widehat{\mathrm{h}_a}\left(\frac{\pi t}{3l}\right). \tag{28}$$

Figure 2: Values of uncertainty constants $\Delta_\psi \Delta_{\hat{\psi}}$ for wavelets based on $\mathrm{h}_a(x)$ and $\mathrm{fip}_{a,n}(x)$ depending on a



In fig. 2 values of $\Delta_\psi \Delta_{\hat{\psi}}$ of wavelets based on convolutions of $\mathrm{h}_a(x)$ and $\mathrm{fip}_{a,n}(x)$ depending on a are presented. Red line corresponds to wavelets based on $\mathrm{h}_a(x)$. Red points present wavelets based on sums of shifts $\mathrm{h}_a(x)$. Orange, green, blue and black lines present wavelets constructed from $\mathrm{fip}_{a,n}(x)$. We specially note orange line corresponding to $\mathrm{fip}_{a,1}(x)$. This line lies below red line if $a > 4$ and below its minimum if $a \in [4; 9]$. Minimum value of orange line is reached at $a \approx 5.85$. Green, blue and black lines decreases monotonically.

Consider if (28) can be presented in form similar to (27). We multiply and divide (28) by $\sin\left(\frac{\pi t}{6l}\right)$

$$\hat{\chi}(t) = \frac{\pi}{3l} \cdot \frac{\sin(\pi t)}{\sin\left(\frac{\pi t}{6l}\right)} \operatorname{sinc}^{n+1}\left(\frac{\pi t}{6l}\right) \cdot \widehat{\mathrm{h}_a}\left(\frac{\pi t}{3l}\right). \tag{29}$$

For application of equality (20) to perform transformation $\pi t = (r+1)\frac{\pi t}{6l}$ is required. Then $l = \frac{r+1}{6}$. Returning to a and n we obtain that for each n, $r > 3n - 1$ and $a = \frac{6}{r+1-3n} + 1$ transformation is allowed and (29) transforms to

$$\hat{\chi}(t) = \frac{\pi}{3l} \left(\sum_{k=0}^{r} e^{\frac{i\pi t}{6l}(r-2k)}\right) \operatorname{sinc}^{n+1}\left(\frac{\pi t}{6l}\right) \cdot \widehat{\mathrm{h}_a}\left(\frac{\pi t}{3l}\right).$$

Inverse Fourier transform leads to

$$\chi(\omega) = \sum_{k=0}^{r} \mathrm{fip}_{a,n+1}\left(\frac{3l}{\pi}\omega + \frac{r-2k}{2}\right). \tag{30}$$

Such cases are presented in fig. 2 with black points. Note that a depends on $r - 3n$. We specially note $a = 2$ if $r = 3n+5$. In this case wavelets constructed from sums of scaled shifts of $\mathrm{fup}_n(x)$ are obtained and (30) transforms to (27). Integer values $a = 7$, $a = 4$ and $a = 3$ allow construction of functional-differential equations for $\mathrm{fip}_{a,n}(x)$ [11,12]. In this case iterative algorithms can be applied for accurate computation of $\mathrm{fip}_{a,n}(x)$. Then $\chi(\omega)$ is obtained according to (30). This approach is also allowed for $\mathrm{fup}_n(x)$ if $a = 2$.

8. CONCLUSION

New approach to construction of wavelets is presented. It consists of several steps. At first convolution of rectangular pulse and given function is performed. Then square root of that convolution is considered as Fourier transform of scaling function and wavelet is constructed according to process similar to Meyer wavelet construction. In general case application of arbitrary nonegative compactly supported function with countable Fourier transform to construction of wavelet is allowed.

Computation of convolution in initial step may be derived to sum of Fourier series. Inverse Fourier transforms for computation of scaling function and wavelet comes down to numerical integration. That allows to obtain new wavelets with reasonable computation cost.

Application of atomic functions makes process clear and effective due to the form of their Fourier transform. Continuous families of atomic functions lead to continuous families of wavelets. Previously constructed wavelets based on sums of scaled shifts of $\mathrm{h}_a(x)$ form countable subset in continuous families of wavelets based on atomic functions $\mathrm{h}_a(x)$, $\mathrm{ch}_{a,n}(x)$ and $\mathrm{fip}_{a,n}(x)$. Wavelets constructed from $\mathrm{fup}_n(x)$ form subset in family of wavelets based of $\mathrm{fip}_{a,n}(x)$. Wavelets constructed from sums of scaled shifts of $\mathrm{ch}_{a,n}(x)$ for $n > 1$ are different from obtained wavelets.

Analysis of uncertainty constants allows suggestion of good localization and numerical efficiency of some of constructed wavelets.

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