New Infinitely Differentiable Spline-like Basis Functions

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Abstract— New family of infinitely differentiable two-parametric spline-Like basis functions is presented. Basic properties of functions are investigated. Connection between obtained functions and known atomic functions (Kravchenko-Rvachev functions) is shown. Method of exact analytical computation of of new functions is demonstrated. Numerical experiment on solution of boundary value problems is performed.

1. INTRODUCTION

In interpolation and collocation methods basis of shifts of compactly supported function is used. Basis function have to be smooth to represent derivatives. The most known family of such functions is B-splines [1, 2]. They provide good degree of error but differentiable only several times. Another known family is atomic functions [3–5]. They are infinitely differentiable and compactly supported at the same time. The closest to b-splines family of atomic functions presents 2 less then B-splines degree of error. New family of infinitely differentiable and compactly supported functions is presented. They are closer to splines then atomic functions. They allow in special conditions achieve 1 less then B-splines degree of error.

2. *B*-SPLINES

B-spline of order n Θ_n is iterated convolution of $n+1$ rectangular pulse $\varphi(x)$

$$\Theta_n = \underbrace{\varphi * \varphi * \dots * \varphi}_{n+1}, \qquad \varphi(x) = \begin{cases} 1, & -\frac{1}{2} \leqslant x \leqslant \frac{1}{2}, \\ 0, & |x| > \frac{1}{2}. \end{cases}$$

It is supported on $[-\frac{n+1}{2}; \frac{n+1}{2}]$. Fourier transforms of rectangular pulse and B-spline are

$$\hat{\varphi}(t) = \operatorname{sinc}\left(\frac{t}{2}\right), \qquad \widehat{\Theta_n}(t) = \operatorname{sinc}^{n+1}\left(\frac{t}{2}\right). \tag{1}$$

From the form of the spectrum $\Theta_n(x)$ (1) follows that

$$\Theta_n'(x) = \Theta_{n-1}\left(x + \frac{1}{2}\right) - \Theta_{n-1}\left(x - \frac{1}{2}\right). \tag{2}$$

It is known that interpolation with shifts of $\Theta_n(x)$ on the grid with step h error is about h^{n+1} with matrix band n. Cubic B-spline $\Theta_3(x)$ is most common in applications. It presents 4-th order of error and allows to construct system of equations for unknown coefficients c_k in the form of tridiagonal matrix.

3. ATOMIC FUNCTIONS

Atomic functions [3] are compactly supported solutions of differential equations with linearly transformed argument of special form

$$L(f(x)) = \sum_{k=1}^{K} c_k f(ax + b_k), \quad a > 1, \tag{3}$$

where $L(f)$ is a linear differential operator. Atomic functions were originally presented by V. L. Rvachev in early 70-th. Large variety of atomic functions are constructed. Consider some of them.

3.1. Family of Atomic Functions $\mathrm{h}_a(x)$

For $\forall a \in \mathbb{R}$, $a > 1$ atomic function $\mathrm{h}_a(x)$ [3] is solution of

$$y'(x) = \frac{a^2}{2}\left(y(ax+1) - y(ax-1)\right) \tag{4}$$

supported on $[-\frac{1}{a-1}; \frac{1}{a-1}]$ and normalised to unit square

$$\int\limits_{-\frac{1}{a-1}}^{\frac{1}{a-1}} \mathrm{h}_a(x)dx = 1. \tag{5}$$

Shifts of $\mathrm{h}_a(x)$ by $\frac{2}{a}$ presents decomposition of unity

$$\sum_{k\in\mathbb{Z}} \mathrm{h}_a\left(x - \frac{2}{a}k\right) \equiv \frac{a}{2}. \tag{6}$$

If $a > 2$ function $\mathrm{h}_a(x)$ is constant $\mathrm{h}_a(x) = \frac{a}{2}$ on the segment $|x| < \frac{a-2}{a(a-1)}$. Then $\mathrm{h}_a(x)$ may be considered as smoothed rectangular pulse. Iterated convolutions of $\mathrm{h}_a(x)$ with itself are atomic functions $\mathrm{ch}_{a,n}(x)$. They may be respectively considered as atomic analogue of B-splines. They are tending to B-splines if a increase. Properties of $\mathrm{ch}_{a,n}(x)$ functions are discussed in [4–6]. With increase a the relative length of the segment of constancy increases. At the same time $\mathrm{h}_a(x)$ is infinite convolution of rectangular pulses of decreasing length. The spectrum of $\mathrm{h}_a(x)$ is presented in the form

$$\widehat{\mathrm{h}_a}(t) = \prod_{k=1}^{\infty} \mathrm{sinc}\left(\frac{t}{a^k}\right). \tag{7}$$

Moments of atomic function h_a are rational numbers. There are iterative formulas [3] for their exact computation. Due to $\mathrm{h}_a(x)$ is even function its odd symmetric moments vanishing. For even moments

$$\int\limits_{-\frac{1}{a-1}}^{\frac{1}{a-1}} x^{2n}\,\mathrm{h}_a(x)dx = a_{2n} = \frac{(2n)!}{a^{2n}-1}\sum_{k=0}^{n-1}\frac{a_{2k}}{(2k)!(2n-2k+1)!}. \tag{8}$$

If (5) then $a_0 = 1$. For odd asymmetric moments

$$\int\limits_{0}^{\frac{1}{a-1}} x^{2n-1}\,\mathrm{h}_a(x)dx = b_{2n-1} = \frac{1}{a^{2n-1}4n}\sum_{k=0}^{n} a_{2k}C_{2n}^{2k}. \tag{9}$$

Atomic function $\mathrm{up}(x)$ supported on $[-1; 1]$ is a special case of $\mathrm{h}_a(x)$ with $a = 2$.

3.2. Convolutions of Atomic Functions

Convolution of two atomic functions with same scale parameter a in (3) is an atomic function too with same scale parameter [6]. For convolution of two AFs $f(x)$ and $g(x)$ with the same scale parameter a and equations

$$f^{(n_f)} = \sum_{m=1}^{M} c_{f_m} f(ax - b_{f_m}) \qquad \text{and} \qquad g^{(n_g)} = \sum_{k=1}^{K} c_{g_k} f(ax - b_{g_k})$$

we have equation for convolution $h(x) = f(x) * g(x)$ in such form

$$h^{(n_f+n_g)} = \sum_m\sum_k \frac{c_{f_m}c_{g_k}}{a} h(ax - b_{f_m} - b_{g_k}). \tag{10}$$

This convolution exists as a convolution of smooth compactly supported functions. Equation (10) is a special case of (3) therefore $h(x)$ is atomic function. Note that in above statement not required $n_f > 0$ and $n_g > 0$. Then rectangular pulse and B-splines may be considered too.

3.3. Family of Atomic Functions $\mathrm{fup}_{n(x)}$

Atomic functions $\mathrm{fup}_n(x)$ [3] are defined as convolutions of $\mathrm{up}(x)$ and B-spline of order $n-1$

$$\mathrm{fup}_n(x) = \mathrm{up}(x) * \Theta_{n-1}(x).$$

Thereafter $\mathrm{fup}_0(x) \equiv \mathrm{up}(x)$.

Functions $\mathrm{fup}_n(x)$ are solutions of equations of the form

$$y'(x) = \frac{1}{2^{n-1}} \sum_{k=0}^{n+2} \left(C_{n+1}^k - C_{n+1}^{k-1}\right) y\left(2x + \frac{n+2}{2} - k\right).$$

Analogue to B-splines

$$\mathrm{fup}_n'(x) = \mathrm{fup}_{n-1}\left(x + \frac{1}{2}\right) - \mathrm{fup}_{n-1}\left(x - \frac{1}{2}\right). \tag{11}$$

The length of the support $\mathrm{fup}_n(x)$ is the sum of lengths of supports $\mathrm{up}(x)$ and $\Theta_{n-1}(x)$ which is equal to $n+2$. It is known that integer shifts of $\mathrm{fup}_n(x)$ exactly presents polynomials of degree $n-1$, providing error about h^n. Note that if the length of support $\mathrm{fup}_n(x)$ is $n+2$ then tridiagonal matrix allows application of $\mathrm{fup}_2(x)$ basis only with error about $O(h^2)$. To obtain error about h^4 the basis $\mathrm{fup}_4(x)$ and five-diagonal matrix is requred. Despite this disadvantage, atomic functions $\mathrm{fup}_n(x)$ are widely applied [3, 7].

4. NEW FAMILY OF INFINITELY SMOOTH COMPACTLY SUPPORTED FUNCTIONS

4.1. Definition of $\mathrm{fip}_{a,n(x)}$

For all real $a > 1$ and nonnegative integer $n \geq 0$ function $\mathrm{fip}_{a,n}(x)$ is defined as a convolution

$$\mathrm{fip}_{a,n}(x) = \mathrm{h}_a(x) * \Theta_{n-1}(x). \tag{12}$$

From definition (12) follows $\mathrm{fip}_{2,n}(x) = \mathrm{fup}_n(x)$, $\mathrm{fip}_{a,0}(x) = \mathrm{h}_a(x)$, $\mathrm{fip}_{2,0}(x) = \mathrm{up}(x)$. The length of support of $\mathrm{fip}_{a,n}(x)$ is sum of length of $\mathrm{h}_a(x)$ and $\Theta_{n-1}(x)$. Besides from $\mathrm{fup}_n(x)$ to n arbitrary small $\frac{2}{a-1}$ instead of 2 is added. Then for tridiagonal matrix is available for $n = 3$ and $a \geq 3$ presenting error about h^3. It is better then, for $\mathrm{fup}_2(x)$, but worse then B-spline. For big a quasi interpolation with $n = 4$ is possible.

4.2. Basic Properties

Consider some properties of $\mathrm{fip}_{a,n}(x)$. Function $\mathrm{fip}_{a,n}(x)$ is even as a convolution of even functions. The spectrum of $\mathrm{fip}_{a,n}(x)$ is equal to product of spectra of $\mathrm{h}_a(x)$ (7) and of B-spline (1)

$$\widehat{\mathrm{fip}_{a,n}}(t) = \mathrm{sinc}^n\left(\frac{t}{2}\right) \prod_{k=1}^{\infty} \mathrm{sinc}\left(\frac{t}{a^k}\right). \tag{13}$$

From (13) similar to (2) and (11) follows

$$\mathrm{fip}_{a,n}'(x) = \mathrm{fip}_{a,n-1}\left(x + \frac{1}{2}\right) - \mathrm{fip}_{a,n-1}\left(x - \frac{1}{2}\right). \tag{14}$$

By substitution (14) to its derivative we obtain

$$\mathrm{fip}_{a,n}'' = \mathrm{fip}_{a,n-2}(x+1) - 2\,\mathrm{fip}_{a,n-2}(x) + \mathrm{fip}_{a,n-2}(x-1).$$

Then in the similar way for any $m < n$

$$\mathrm{fip}_{a,n}^{(m)} = \sum_{k=0}^{m} C_m^k (-1)^k\, \mathrm{fip}_{a,n-m}\left(x + \frac{m}{2} - k\right). \tag{15}$$

If $m = n$ (15) transforms to

$$\mathrm{fip}_{a,n}^{(n)} = \sum_{k=0}^{n} C_n^k (-1)^k \,\mathrm{h}_a\left(x + \frac{n}{2} - k\right). \tag{16}$$

Derivatives $\mathrm{fip}_{a,n}(x)$ of order higher then n may be found by derivation of (16) according to (4).

Integer shifts of $\mathrm{fip}_{a,n}(x)$ presents decomposition of unity and exact representation of polynomials of degree up to $n-1$

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \mathrm{fip}_{a,n}(x-k) &\equiv 1, \\ \sum_{k\in\mathbb{Z}} c_k\, \mathrm{fip}_{a,n}(x-k) &\equiv P_m(x), \qquad m \leqslant n-1. \end{aligned} \tag{17}$$

If representation (17) is exact, it would stay correct for derivatives. Then

$$\sum_{k\in\mathbb{Z}} c_k\, \mathrm{fip}_{a,n}^{(l)}(x-k) \equiv P_m^{(l)}(x), \qquad m \le n-1, \quad l \le m.$$

This property allows to use shifts of $\mathrm{fip}_{a,n}(x)$ as effective interpolation and collocation basis. Note that from (6) follows that shifts of $\mathrm{fip}_{a,n}(x)$ by $\frac{2}{a}$ present decomposition of unity too but don't express polynomials.

$$\sum_{k\in\mathbb{Z}} \mathrm{fip}_{a,n}\left(x - \frac{2}{a}k\right) = \frac{a}{2}.$$

For computation $\mathrm{fip}_{a,n}(x)$ Fourier series may be used

$$\mathrm{fip}_{a,n}(x) = \frac{2}{l}\left(\frac{1}{2} + \sum_{k=1}^{\infty} \widehat{\mathrm{fip}_{a,n}}\left(\frac{2\pi k}{l}\right)\cos\left(\frac{2\pi k x}{l}\right)\right), \quad \text{where} \quad l = n + \frac{2}{a-1}.$$

4.3. $\mathrm{fip}_{a,n(x)}$ as Atomic Functions

For integer values of parameter a $\mathrm{fip}_{a,n}(x)$ are atomic functions. It is clear that $\forall m \in \mathbb{N}$ rectangular pulse $\varphi(x)$ meets equation

$$y(x) = \sum_{k=1}^{m} y\left(mx - k + \frac{m+1}{2}\right). \tag{18}$$

Then according to the theorem of convolution of atomic functions [6], equation for convolution $\mathrm{fip}_{m,1}(x) = \mathrm{h}_m(x) * \varphi(x)$ may be constructed. Then for the next convolution $\mathrm{fip}_{m,2}(x) = \mathrm{fip}_{m,1}(x) * \varphi(x)$, etc. Then for each integer $a > 1$, $\mathrm{fip}_{a,n}(x)$ are atomic functions.

Consider for example construction of equation for $\mathrm{fip}_{3,3}(x)$. If $a = 3$ we write out (18) with $m = 3$

$$y(x) = y(3x+1) + y(3x) + y(3x-1).$$

Equation for $\mathrm{fip}_{3,0}(x) = \mathrm{h}_3(x)$ is a special case of (4)

$$y'(x) = \frac{9}{2}\big(y(3x+1) - y(3x-1)\big).$$

According to the theorem $\mathrm{fip}_{3,1}(x)$ meets

$$y'(x) = \frac{3}{2}\big(y(3x+2) + y(3x+1) - y(3x-1) - y(3x-2)\big).$$

Then equation for $\mathrm{fip}_{3,2}(x)$ is

$$y'(x) = \frac{1}{2}\big(y(3x+3) + 2y(3x+2) + 2y(3x+1) - 2y(3x-1) - 2y(3x-2) - y(3x-3)\big).$$

And, finally, we obtain equation for $\mathrm{fip}_{3,3}(x)$

$$\begin{aligned} y'(x) \;=\; & \frac{1}{6}\big(y(3x+4)+3y(3x+3)+5y(3x+2)+4y(3x+1) \\ & -4y(3x-1)-5y(3x-2)-3y(3x-3)-y(3x-4)\big). \end{aligned} \tag{19}$$

This equation presents additional way for computation of derivatives $\mathrm{fip}_{3,3}(x)$ besides (14).

4.4. Exact Values of $\mathrm{fip}_{a,n(x)}$

For practical applications exact values of $\mathrm{fip}_{a,n}(x)$ in special points may be useful. Consider analytical computation of $\mathrm{fip}_{a,n}(x_0)$ as value of convolution (12). Substitution of explicit representation of B-spline as piecewise polynomial reduces convolution integral (12) to moments of $\mathrm{h}_a(x)$, which are computable by formulae (8) and (9).

Let's find $\mathrm{fip}_{3,3}(0)$ and $\mathrm{fip}_{3,3}(1)$ for example. Function h_3 is supported on $[-\frac{1}{2};\frac{1}{2}]$.

$$\mathrm{fip}_{3,3}(0) = \Theta_2(t) * \mathrm{h}_3(t)\big|_{x=0} = \int\limits_{-\infty}^{\infty}\Theta_2(t)\,\mathrm{h}_3(t-x)dt\bigg|_{x=0} = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\Theta_2(t)\,\mathrm{h}_3(t)dt.$$

It is known that B-spline $\Theta_2(x)$ may be presented in such form

$$\Theta_2 = \begin{cases} \frac{\left(\frac{3}{2}+x\right)^2}{2}, & -\frac{3}{2}\leqslant x<-\frac{1}{2}, \\ \frac{3}{4}-x^2, & -\frac{1}{2}\leqslant x\leqslant\frac{1}{2}, \\ \frac{\left(\frac{3}{2}-x\right)^2}{2}, & \frac{1}{2}\leqslant x\leqslant\frac{3}{2}, \\ 0, & |x|>\frac{3}{2}. \end{cases}$$

Then

$$\mathrm{fip}_{3,3}(0) = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\Theta_2(t)\,\mathrm{h}_3(t)dt = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{3}{4}-t^2\right)\mathrm{h}_3(t)dt = \frac{3}{4}-a_2 = \frac{17}{24},$$

where a_2 is the second moment of h_3, computed by formula (8) $a_2 = \frac{1}{24}$.

$$\begin{aligned} \mathrm{fip}_{3,3}(1) \;=\; & \Theta_2(t) * \mathrm{h}_3(t)\big|_{x=1} = \int\limits_{-\infty}^{\infty}\Theta_2(t)\,\mathrm{h}_3(t-x)dt\bigg|_{x=1} = \int\limits_{\frac{1}{2}}^{\frac{3}{2}}\Theta_2(t)\,\mathrm{h}_3(t-1)dt \\ =\; & \int\limits_{\frac{1}{2}}^{\frac{3}{2}}\frac{1}{2}\left(\frac{3}{2}-t\right)^2\mathrm{h}_3(t-1)dt = |t=y+1| = \frac{1}{2}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}-y\right)^2\mathrm{h}_3(y)dy \\ =\; & \frac{1}{2}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{4}-y+y^2\right)\mathrm{h}_3(y)dy = \frac{1}{2}\left(\frac{1}{4}+a_2\right) = \frac{1}{2}\left(\frac{1}{4}+\frac{1}{24}\right) = \frac{7}{48}. \end{aligned}$$

We use Equation (19) to find exact values of derivatives of $\mathrm{fip}_{3,3}(x)$ in the simplest way. Substituting $x=-1$ in (19) and omitting zero valued terms we obtain

$$\mathrm{fip}'_{3,3}(-1) = \frac{1}{6}\big(\mathrm{fip}_{3,3}(1)+3\,\mathrm{fip}_{3,3}(0)+5\,\mathrm{fip}_{3,3}(-1)\big) = \frac{1}{6}\Big(\frac{7}{48}+3\frac{17}{24}+5\frac{7}{48}\Big) = \frac{1}{6}\cdot\frac{144}{48} = \frac{1}{2}.$$

By derivation of (19) we get

$$\begin{aligned} y''(x) \;=\; & \frac{1}{2}\big(y'(3x+4)+3y'(3x+3)+5y'(3x+2)+4y'(3x+1) \\ & -4y'(3x-1)-5y'(3x-2)-3y'(3x-3)-y'(3x-4)\big). \end{aligned}$$

For $x = 0$ and $x = 1$ we obtain respectively

$$\begin{aligned}\mathrm{fip}''_{3,3}(0) &= 2\,\mathrm{fip}'_{3,3}(1) - 2\,\mathrm{fip}'_{3,3}(-1) = -2,\\ \mathrm{fip}''_{3,3}(1) &= \frac{1}{2}\big(- 5\,\mathrm{fip}'_{3,3}(1) - 3\,\mathrm{fip}'_{3,3}(0) - \mathrm{fip}'_{3,3}(-1)\big) = 1.\end{aligned}$$

We summarize the values obtained in the Table 1 taking in account that function is even.

Table 1: Exact values of function $\mathrm{fip}_{3,3}(x)$ and its derivatives.

$x =$	-1	0	1
$\mathrm{fip}_{3,3}(x) =$	$\frac{7}{48}$	$\frac{17}{24}$	$\frac{7}{48}$
$\mathrm{fip}'_{3,3}(x) =$	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\mathrm{fip}''_{3,3}(x) =$	1	-2	1

5. APPLICATIONS OF FUNCTIONS $\mathrm{fip}_{a,n(x)}$

5.1. Interpolation with Shifts of $\mathrm{fip}_{a,n(x)}$

Shifts by 1 of $\mathrm{fip}_{a,n}(x)$ present interpolation scheme exact for polynomials of degree $n-1$

$$S(x) = \sum_k c_k\ \mathrm{fip}_{a,n}(x-k).$$

The length of the support of $\mathrm{fip}_n(x)$ is $l(a,n) = n + \frac{2}{a-1}$. then for tridiagonal matrix $2 < l(a,n) \leqslant 4$ is required. Support of such length is obtained

$$\begin{aligned}&\text{for } n=1 \quad \text{if } \frac{5}{3} \le a < 3,\\ &\text{for } n=2 \quad \text{if } a \ge 2,\\ &\text{for } n=3 \quad \text{if } a \ge 3.\end{aligned}$$

Consider $\mathrm{fip}_{a,3}(x)$ interpolation exact for polynomials of 2 degree. If $a \ge 3$ then coefficients c_k meet the system with tridiagonal matrix of the form

$$\left(\begin{array}{cccccc|c} \mathrm{fip}_{a,3}(1) & \mathrm{fip}_{a,3}(0) & \mathrm{fip}_{a,3}(-1) & & & & y_0\\ & \mathrm{fip}_{a,3}(1) & \mathrm{fip}_{a,3}(0) & \mathrm{fip}_{a,3}(-1) & & & y_1\\ & & \ddots & \ddots & \ddots & & \vdots\\ & & & \mathrm{fip}_{a,3}(1) & \mathrm{fip}_{a,3}(0) & \mathrm{fip}_{a,3}(-1) & y_n \end{array}\right).$$

This system is incomplete. We add to the system conditions on the derivatives $S(x)$ at the ends of the segment

$$\begin{aligned}c_{-1}\mathrm{fip}'_{a,3}(1) + c_1\mathrm{fip}'_{a,3}(-1) &= y'(x_0),\\ c_{n-1}\mathrm{fip}'_{a,3}(1) + c_{n+1}\mathrm{fip}'_{a,3}(-1) &= y'(x_n).\end{aligned} \tag{20}$$

Solution of obtained system may be found by iterative algorithms. If step of the grid h not equal to 1 multiplication of $f'(x_i)$ by h in (20) is required.

5.2. Solving One-dimensional Boundary Value Problems

Expression (17) provides uniform approximation for derivatives

$$S^{(l)}(x) = \sum_{k\in\mathbb{Z}} c_k\,\mathrm{fip}^{(l)}_{a,n}(x+k).$$

This property allows the application of $\mathrm{fip}_{a,n}(x)$ to solution of differential equations. For equation $p(x)y'' + q(x)y' + r(x)y = f(x)$ with boundary conditions $y(x_0) = y_0$, $y(x_1) = y_1$. Tridiagonal matrix consists of the strings of the form

$$\begin{aligned}
&\left(p(x_0+ih)\frac{1}{h^2}\,\mathrm{fip}''_{a,n}(1)+q(x_0+ih)\frac{1}{h}\,\mathrm{fip}'_{a,n}(1)+r(x_0+ih)\,\mathrm{fip}_{a,n}(1)\right)c_{i-1}\\
&+\left(p(x_0+ih)\frac{1}{h^2}\,\mathrm{fip}''_{a,n}(0)+r(x_0+ih)\,\mathrm{fip}_{a,n}(0)\right)c_i\\
&+\left(p(x_0+ih)\frac{1}{h^2}\,\mathrm{fip}''_{a,n}(-1)+q(x_0+ih)\frac{1}{h}\,\mathrm{fip}'_{a,n}(-1)+r(x_0+ih)\,\mathrm{fip}_{a,n}(-1)\right)c_{i+1}\\
=\;& f(x_0+ih).
\end{aligned}$$

Boundary conditions are presented with two additional equations

$$\begin{aligned}
c_{-1}\,\mathrm{fip}_{a,n}(1)+c_0\,\mathrm{fip}_{a,n}(0)+c_1\,\mathrm{fip}_{a,n}(-1) &= y(x_0),\\
c_{n-1}\,\mathrm{fip}_{a,n}(1)+c_n\,\mathrm{fip}_{a,n}(0)+c_{n+1}\,\mathrm{fip}_{a,n}(-1) &= y(x_n).
\end{aligned}$$

Similar to [7, 4] consider model problems $-y''+r(x)y=f(x)$ with boundary conditions $y(0)=0$, $y(1)=0$ and analytical solution $\bar{y}(x)$. Explicit form of $r(x)$, $f(x)$ and $\bar{y}(x)$ is given in Table 2. Similar to [7, 4] we set the step of grid $h=0.05$. Results of solution model problems are presented in Table 3.

Table 2: Conditions of one-dimensional model problems.

No	$r(x)$	$f(x)$	$\bar{y}(x)$
1	x	$(4\pi^2+x)\sin 2\pi x$	$\sin 2\pi x$
2	e^x	$(\pi^2+e^x)\sin \pi x$	$\sin \pi x$
3	$\frac{1}{x-0{,}1\pi}$	$(\pi^2+r(x))\sin \pi x$	$\sin \pi x$
4	$2\tan^2 x+1$	$-(1+2\tan^2 x)\cdot\left(\frac{x}{\cos 1}+1-x\right)$	$\frac{1}{\cos x}-\frac{x}{\cos 1}+x-1$
5	$\sqrt{x}$	$-\frac{15}{4}\sqrt{x}+x^3-\sqrt{x^3}$	$\sqrt{x^5}-x$

5.3. BVP for Maxwell Equation

For two-dimensional collocation we use basis functions $\mathrm{fip}_{a,n}(x)\,\mathrm{fip}_{a,n}(y)$ scaled and shifted corresponding to grid. Such basis functions is supported on the square (or rectangle if steps of grid are different for x and y). If we consider $n=3$ and $a\geq 3$ as above, each basis function is non-zero in 9 points of grid leading to block-diagonal matrix. We are looking for the solution approximated by

$$S(x,y)=\sum_{i=1}^{N}\sum_{j=1}^{M}\mathrm{fip}_{a,n}\left(\frac{x-x_0}{h_x}-i\right)\mathrm{fip}_{a,n}\left(\frac{y-y_0}{h_y}+j\right)$$

with derivatives approximated by

$$\frac{\partial^{k+l}}{\partial x^k\partial y^l}S(x,y)=\frac{1}{h_x^k\,h_y^l}\sum_{i=1}^{N}\sum_{j=1}^{M}\mathrm{fip}^{(k)}_{a,n}\left(\frac{x-x_0}{h_x}-i\right)\mathrm{fip}^{(l)}_{a,n}\left(\frac{y-y_0}{h_y}+j\right).$$

Consider Maxwell equation [2] of the form

$$\frac{\partial^2 E}{\partial t^2}+\frac{\sigma}{\epsilon}\frac{\partial E}{\partial t}=\frac{1}{\mu\epsilon}\frac{\partial^2 E}{\partial x^2}$$

with two initial conditions

$$E(x,t=0)=\cos(\pi x),\qquad \frac{\partial E(x,t=0)}{\partial t}=Re\lambda\cos(\pi x)$$

and homogeneous Neumann boundary conditions

$$\frac{\partial E(x=0,t)}{\partial x}=0,\qquad \frac{\partial E(x=1,t)}{\partial x}=0.$$

Table 3: Results of solution of one-dimensional model problems.

No	basis	$\Delta_{\max}$	$y_{\max}$	$x_{\max}$	No	basis	$\Delta_{\max}$	$y_{\max}$	$x_{\max}$
1*	Θ_3	0.00845	0.75	-1	1	$\text{fip}_{3,3}$	0.00546	0.27	0.99
1*	fup_2	0.00559	0.75	-1	1	$\text{fip}_{4,3}$	0.00545	0.27	0.99
1**	$\text{ch}_{5,3}$	0.00461	0.275	0.983	1	$\text{fip}_{10,3}$	0.00545	0.27	0.99
1**	$\text{ch}_{10,3}$	0.00423	0.278	0.980	1	$\text{fip}_{20,3}$	0.00568	0.27	0.99
2*	Θ_3	0.00175	0.5	1	2	$\text{fip}_{3,3}$	0.00117	0.478	0.998
2*	fup_2	0.00164	0.5	1	2	$\text{fip}_{4,3}$	0.00119	0.478	0.997
2**	$\text{ch}_{5,3}$	0.00099	0.473	0.995	2	$\text{fip}_{10,3}$	0.00121	0.479	0.998
2**	$\text{ch}_{10,3}$	0.00091	0.471	0.995	2	$\text{fip}_{20,3}$	0.00142	0.481	0.998
3*	Θ_3	0.00282	0.45	0.985	3	$\text{fip}_{3,3}$	0.00204	0.306	0.82
3*	fup_2	0.00216	0.45	0.985	3	$\text{fip}_{4,3}$	0.00187	0.308	0.824
3**	$\text{ch}_{5,3}$	0.00172	0.309	0.824	3	$\text{fip}_{10,3}$	0.00174	0.310	0.827
3**	$\text{ch}_{10,3}$	0.00149	0.310	0.826	3	$\text{fip}_{20,3}$	0.00205	0.310	0.827
4*	Θ_3	0.00084	0.7	-0.288	4	$\text{fip}_{3,3}$	0.00069	0.737	-0.277
4*	fup_2	0.00056	0.7	-0.288	4	$\text{fip}_{4,3}$	0.00052	0.687	-0.291
4**	$\text{ch}_{5,3}$	0.00048	0.735	-0.278	4	$\text{fip}_{10,3}$	0.00035	0.637	-0.298
4**	$\text{ch}_{10,3}$	0.00045	0.735	-0.278	4	$\text{fip}_{20,3}$	0.00036	0.999	-0.002
5*	Θ_3	0.00023	0.2	-0.182	5	$\text{fip}_{3,3}$	0.00016	0.163	-0.152
5*	fup_2	0.00017	0.2	-0.182	5	$\text{fip}_{4,3}$	0.00017	0.213	-0.192
5**	$\text{ch}_{5,3}$	0.00015	0.164	-0.153	5	$\text{fip}_{10,3}$	0.0002	0.312	-0.258
5**	$\text{ch}_{10,3}$	0.00014	0.165	-0.153	5	$\text{fip}_{20,3}$	0.00026	0.462	-0.317

Results marked with * are taken from [7], results marked with ** are from [4].

This problem has known analytical solution [2]

$$\bar{E}(x,t) = e^{Re\lambda t}\cos(Im\lambda t)\cos(\pi x),$$

where

$$Re\lambda = -\frac{1}{2}\frac{\sigma}{\epsilon}, \qquad Im\lambda = \frac{1}{2}\left(\sqrt{\frac{4\pi^2}{\mu\epsilon} - \left(\frac{\sigma}{\epsilon}\right)^2}\right).$$

We set $\epsilon = 1$, $\mu = 1$, $\sigma = 1$ (damping case) and $\epsilon = 1$, $\mu = 1$, $\sigma = 1$ (no damping case).

For numerical experiment we use uniform grid with steps $h_x = h_y = h$ with $h = 0.2$ and $h = 0.1$. Obtained results are presented in Table 4.

Table 4: Values of error $\max|S - \bar{E}|$.

basis	damping case		no damping case	
	$h = 0.2$	$h = 0.1$	$h = 0.2$	$h = 0.1$
Θ_3	0.021	0.0056	0.0028	0.003
fup_2	0.019	0.005	0.004	0.0038
$\text{fip}_{3,3}$	0.020	0.0054	0.0038	0.0036
$\text{fip}_{4,3}$	0.019	0.0048	0.004	0.0038

6. CONCLUSION

New family of infinitely smooth compactly supported functions $\text{fip}_{a,n}(x)$ is presented. This functions are situated between B-splines and atomic functions. For integer a they are atomic functions meeting corresponding equations. Infinite differentiability allows application of $\text{fip}_{a,n}(x)$ functions to solution of differential equation of high order. Specially note that exact values of $\text{fip}_{a,n}(x)$ in grid points may be found. That allows to construct system if linear equations with exact coefficients and then improve the solution. To test practical use of new basis in comparison with known bases of B-splines and atomic functions $\mathrm{fup}_2(x)$ and $\mathrm{ch}_{a,n}(x)$ numerical experiment is performed. It shows that in some problems $\mathrm{fip}_{a,n}(x)$ gets less error then other basis functions.

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