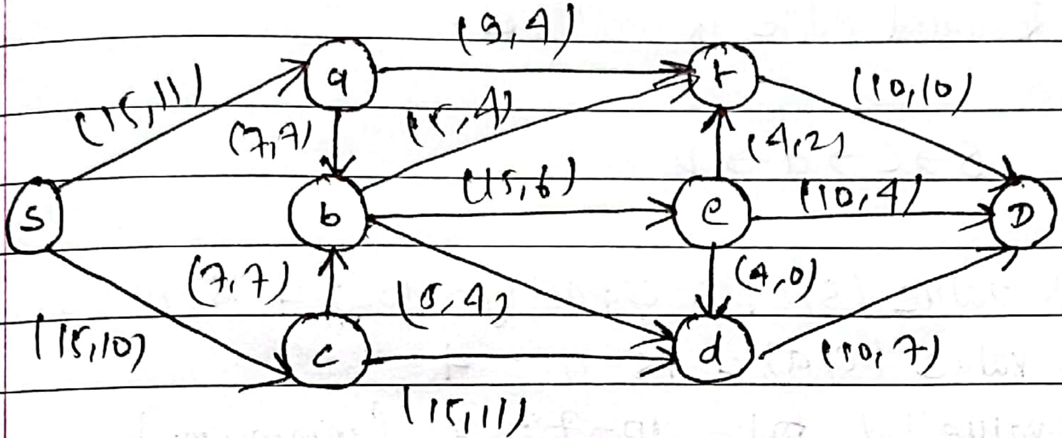
Practice Questions (Warm UP)

1) What is S-D cut? For the following network flow find the maximal flow from S to D.



$A \xrightarrow{(4,2)} B$

capacity = 4 slack = capacity - flow

flow = ~~4~~ 2

for forward direction:

~~[illegible]~~ saturated edge : capacity = flow

unsaturated edge : capacity ≠ flow

We can't take path of saturated edge.

for backward:

(4,0)

If flow = 0 then saturated edge.

~~Q. slack = flow. f : slack = [illegible]~~

flow = slack in backward.

→ Soln :-

First we try to find out the augmenting path with all forward edge if possible.

Then,

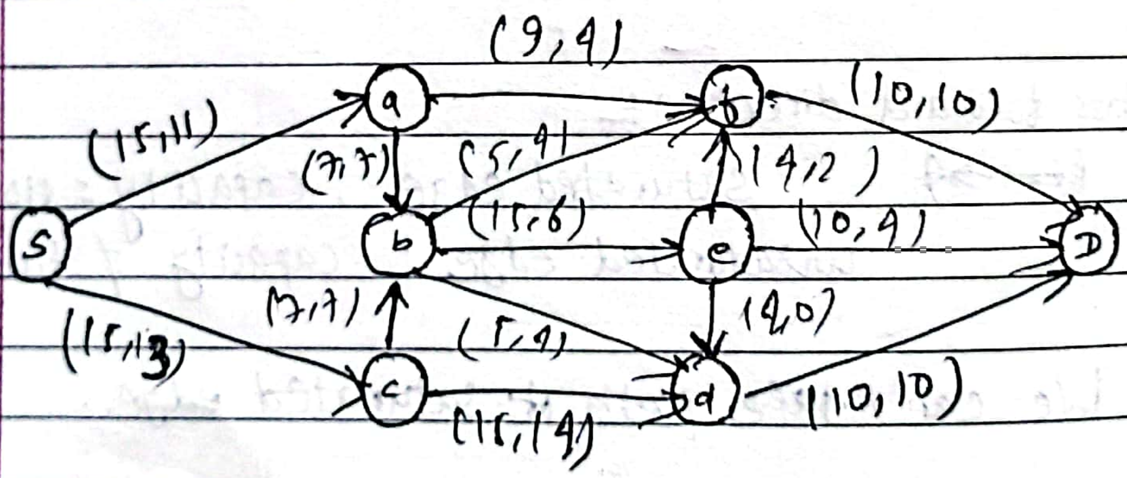
$s \to c \to d \to D$

slack value (s, c) = capacity - flow = 15-10 = 5

slack value (c, d) = 15-11 = 4

slack value (d, D) = 10-7 = 3 (minimum)

The minimum slack is 3. After increasing by 3 in flow we get,

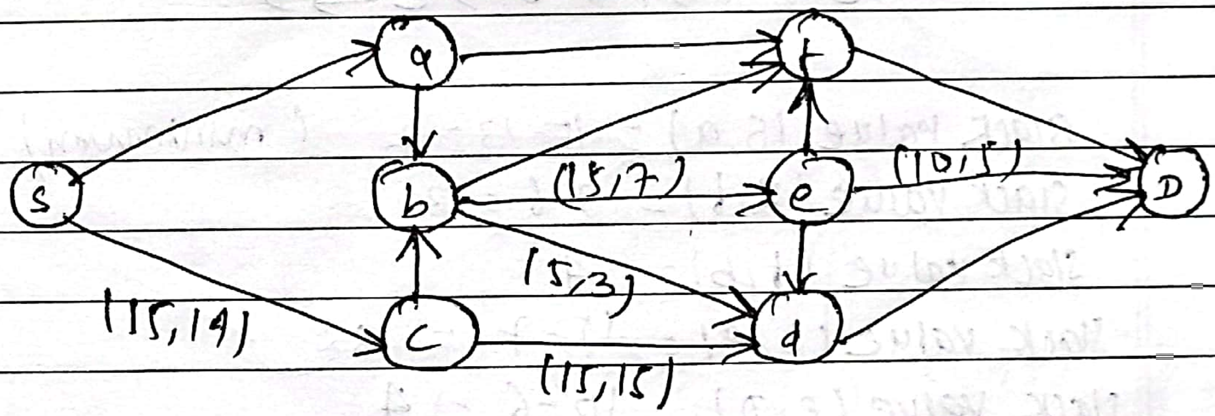


Now, there is no directed augmenting path. So, we look for augmenting path with some backward edge.

$s \to c \to d \to b \to e \to D$

slack value $(s,c) = 15-13 = 2$
slack value $(c,d) = 15-14 = 1$ (minimum).
slack value $(b,d) = 4$ (flow)
slack value $(b,e) = 15-6 = 9$
slack value $(e,D) = 10-4 = 6$

forward edge = increase ($\uparrow$)
backward edge = decrease ($\downarrow$).

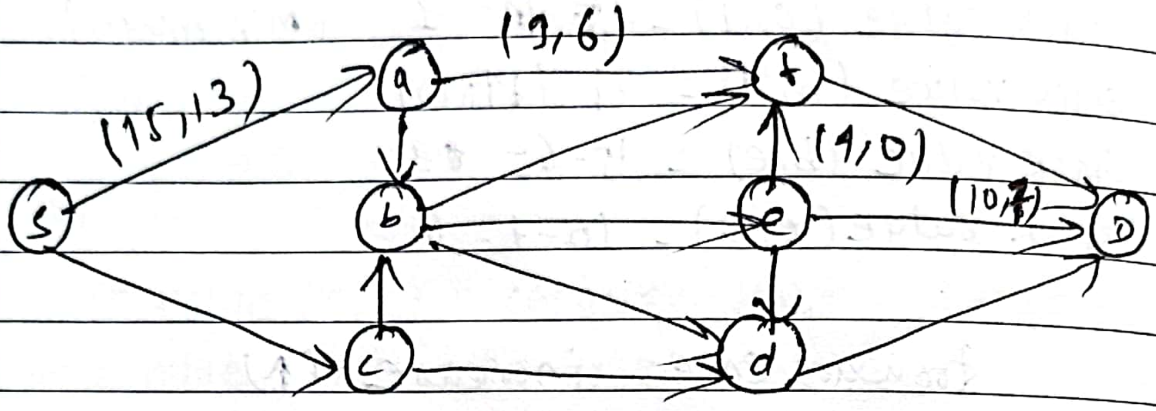


Again, we look for augmenting path with some backward edge.

$s \to a \to f \to e \to D$.

slack value $(s,a) = 15-11 = 9$
slack value $(a,f) = 9-4 = 5$
slack value $(f,e) = 2$ (minimum)
slack value $(e,D) = 10-5 = 5$

Adding 2 in forward and subtracting 2
we get,

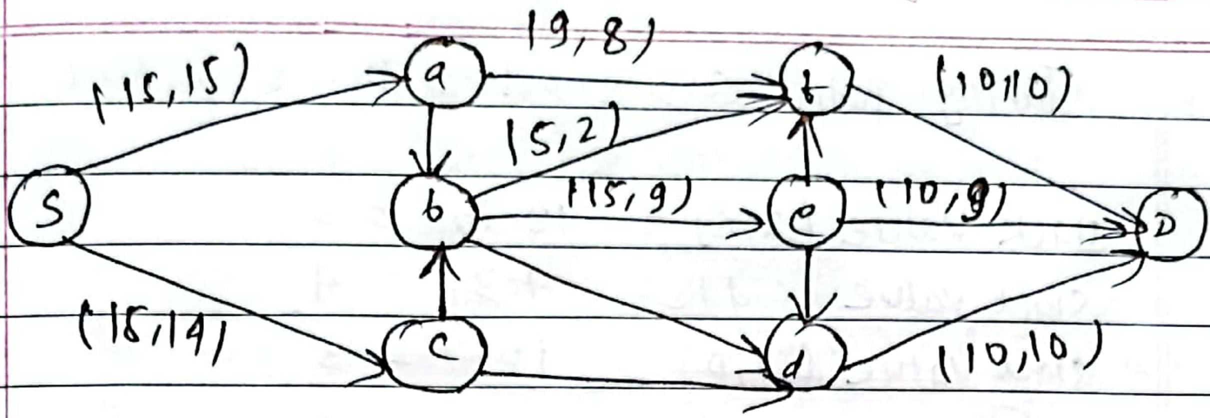


Again,

s → a → f → b → e → D

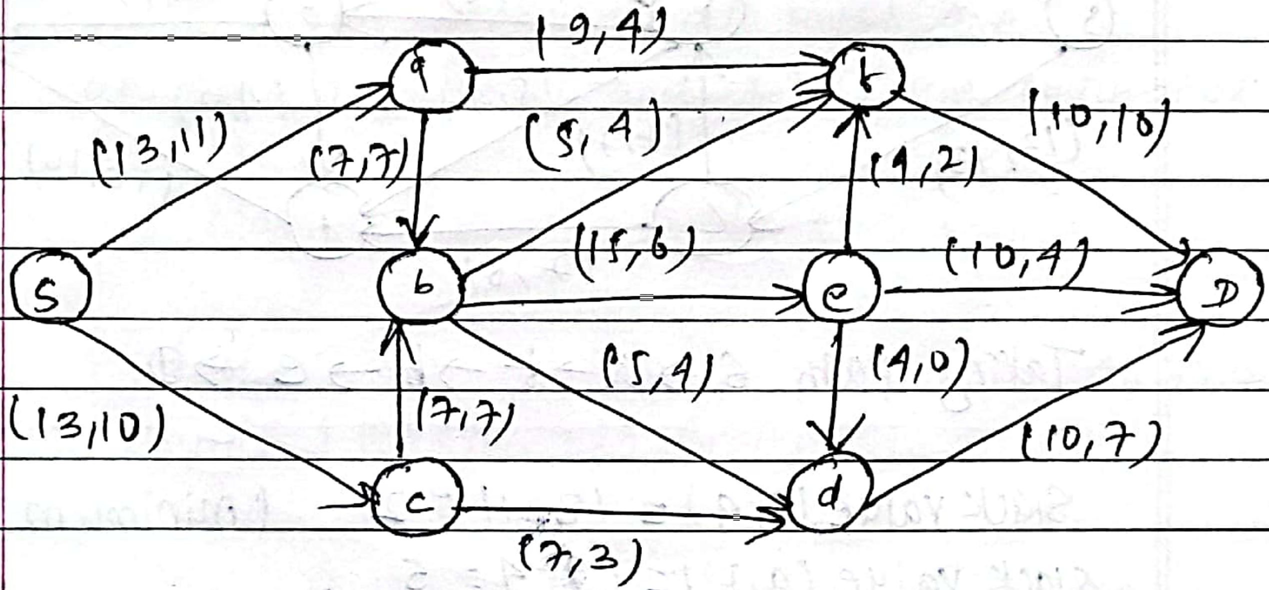
slack value (s,a) = 15 − 13 = 2 (minimum)
slack value (a,f) = 9 − 6 = 3
slack value (f,b) = 4
slack value (b,e) = 15 − 7 = 8
slack value (e,D) = 10 − 6 = 4

Adding 2 in forward and subtracting 2 in backward we get,



Maximum flow = 15+14
= 29

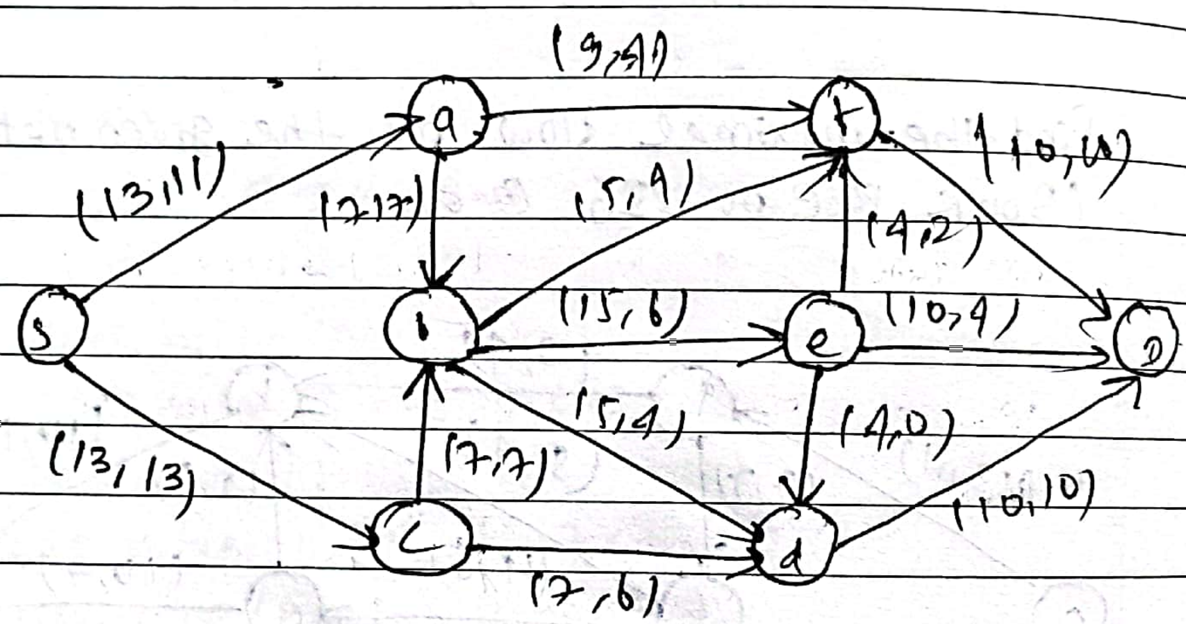
Find the maximal flow for the given network
(Book page no. 226 Q.8).



Taking path $s \to c \to d \to D$.

slack value (s,c) = 13-10 = 3
slack value (c,d) = 7-3 = 4
slack value (d,D) = 10-7 = 3

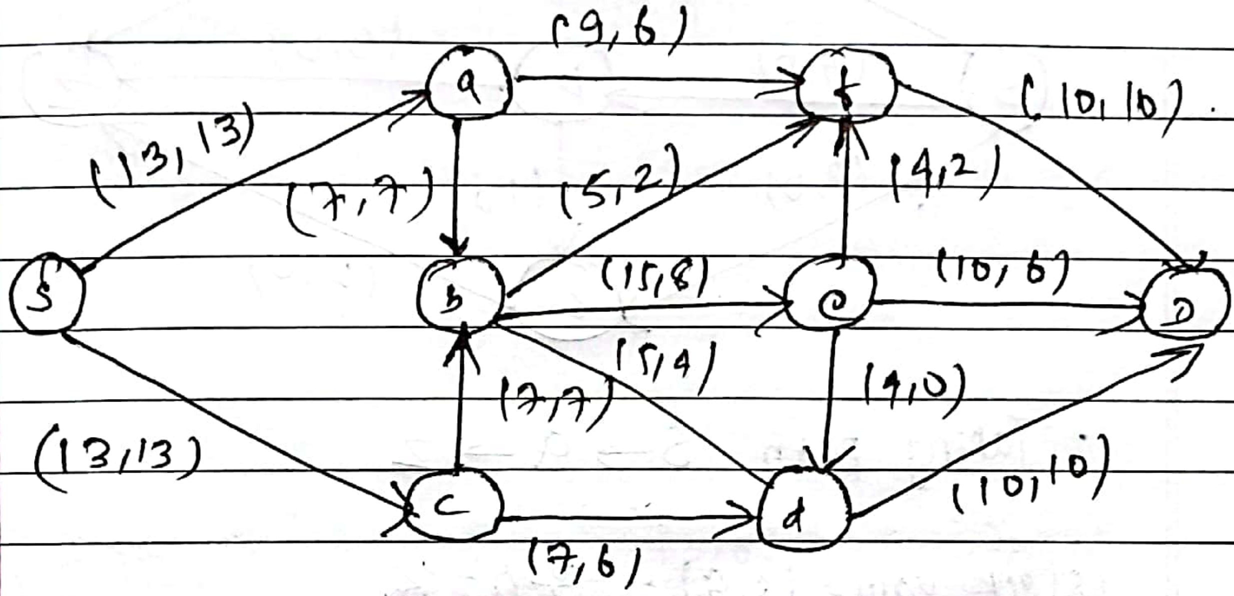
Since, 3 is minimum we add 3 in flow of the path chosen.



Taking path $s \to a \to t \to b \to e \to D$

Slack value (s,a) = 13-11 = 2 (minimum)
slack value (a,t) = 9-4 = 5
slack value (t,b) = 4 (flow)
Slack value (b,e) = 15-6 = 9
slack value (e,D) = 10-4 = 6

Adding 2 in flow of forward direction and subtracting 2 in flow of backward direction, we get,

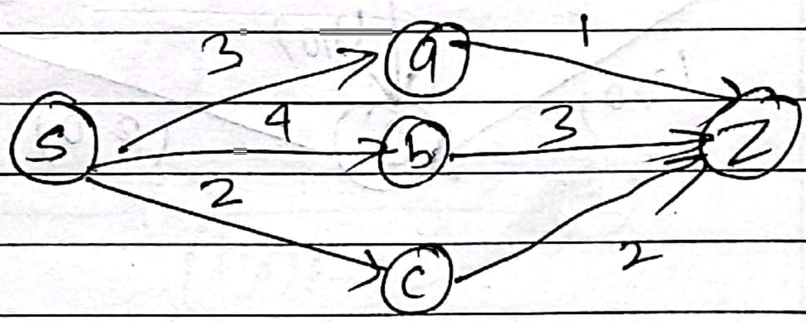


All paths are occupied and hence no path can be taken because all paths are saturated i.e. (S,a) and (S,C) are saturated.

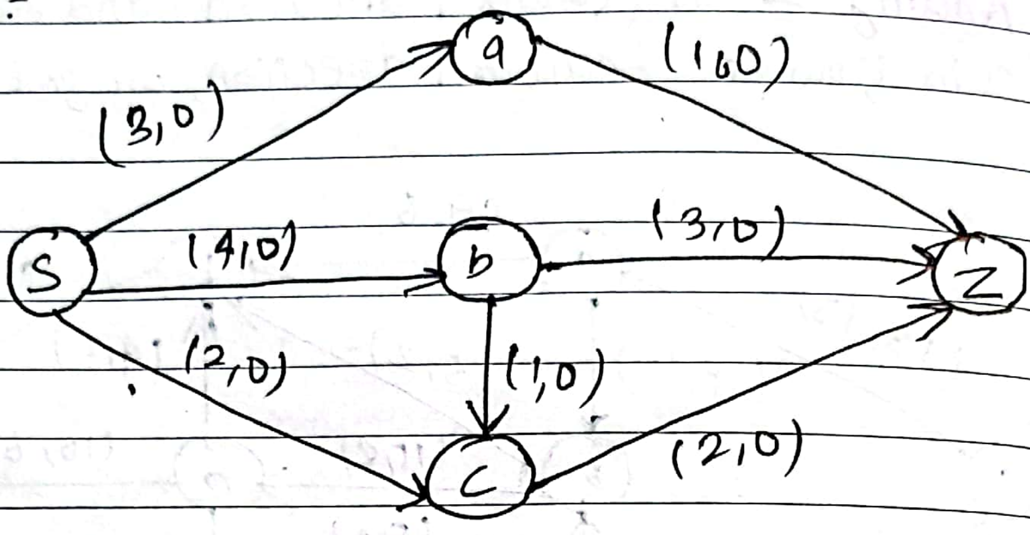
So,

Maximal flow = 13+13 = 26.

* Find the maximal flow in the network shown in figure. (page no. 229 Example).



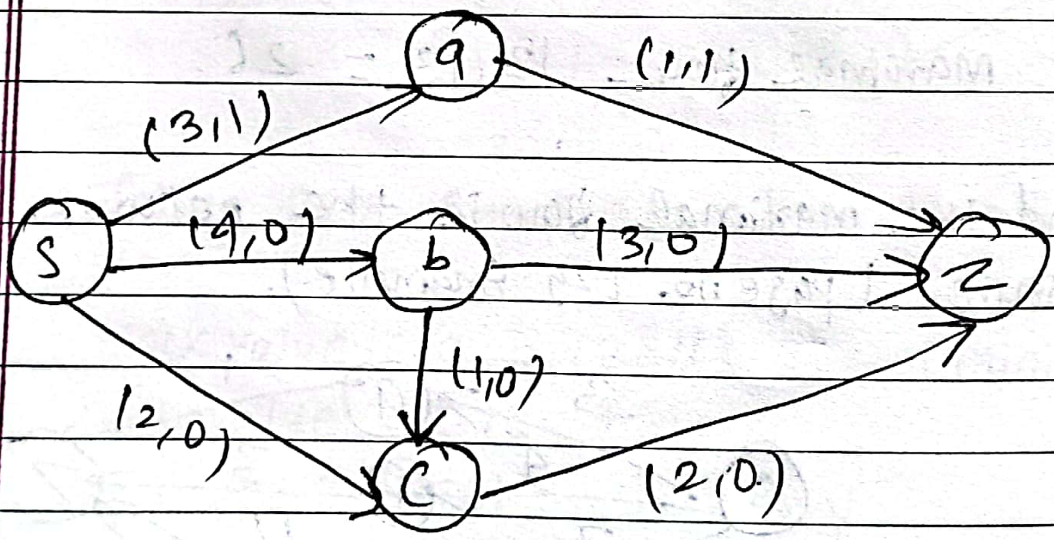
→ Soln:-



· Taking path, $S \to a \to z$

slack value $(s,a) = 3-0=3$
slack value $(a,z) = 1-0=1$ (minimum)

Adding 1 in flow of path chosen we set,

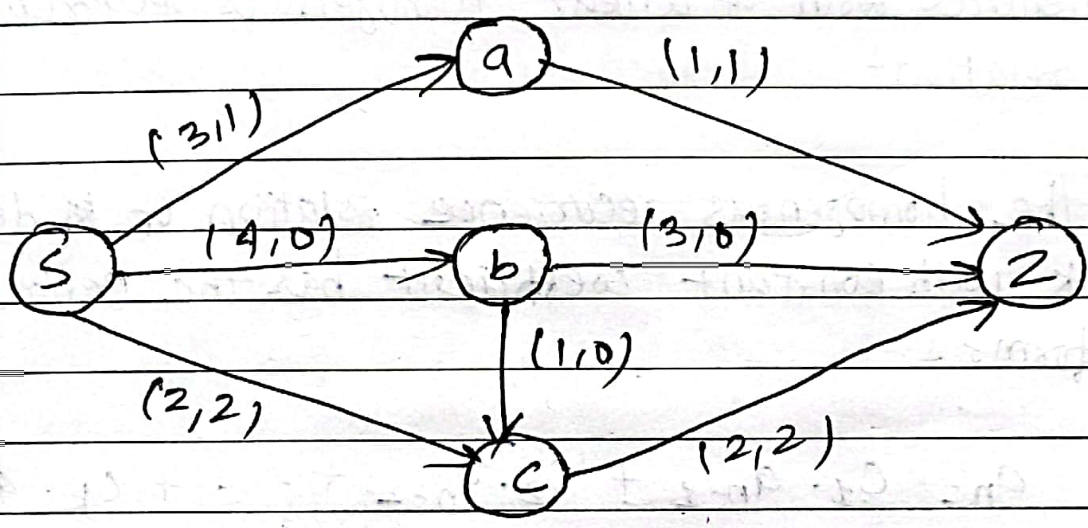


Taking path $S \to c \to z$

slack value $(s,c) = 2-0 = 2$

slack value $(c,z) = 2-0 = 2$

Adding 2 in flow of path chosen,

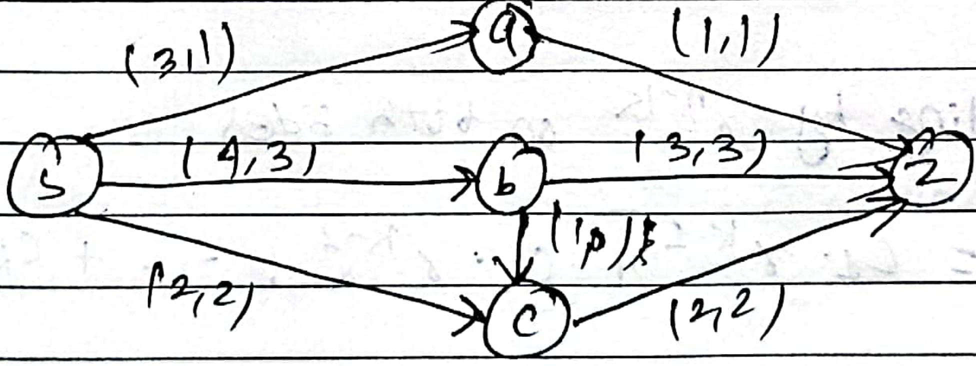


Taking path $s \to b \to z$

slack value $(s,b) = 4-0 = 4$

slack value $(b,z) = 3-0 = 3$ (min).

Adding 3 in flow of path chosen we get,



There is no path left now..

Hence, maximal flow = 1+3+2
= 6.

* Advance Counting:

* General form of linear Homogenous recurrence relation:

→ The homogenous recurrence relation of degree k with constant coefficient has the general form:

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k} \quad \text{---(i)}$$

If $a_n = r^n$ is solution of equ^n (i) then it must satisfy equation (i)
i.e.

$$r^n = c_1 \cdot r^{n-1} + c_2 \cdot r^{n-2} + \cdots + c_k r^{n-k}$$

Dividing by r^{n-k} on both sides

$$r^k = c_1 \cdot r^{k-1} + c_2 \cdot r^{k-2} + \cdots + c_k$$

classmate
Date
Page

$$r^k - c_1 \cdot r^{k-1} - c_2 r^{k-2} \dots - c_k = 0 \quad -(ii)$$

This eqn is known as characteristics equation of given recurrence relation and it provides characteristics roots of recurrence relation which are used to given on explicit formula for all the soln of recurrence relation.

Theorem: 1 (Without proof)

Let c_1 and c_2 are real numbers. Suppose that $r^2 - c_1 \cdot r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation.

$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$ if and only if
$a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n$ for $n = 0, 1, 2 \dots$
where α_1 and α_2 are constant.

Example:- Solve the recurrence relation
$a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

→ Given:- $a_n = 5a_{n-1} - 6a_{n-2}$
Comparing with $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \dots + c_k \cdot a_{n-k}$

$\therefore c_1 = 5 \quad c_2 = -6$

and characteristic eqn :-

$$r^K - C_1 \cdot r^{K-1} - C_2 r^{K-2} - \cdots - C_k = 0.$$

or, $r^2 - C_1 \cdot r - C_2 = 0$

or, $r^2 - 5r + 6 = 0$

or, $r^2 - (3+2)r + 6 = 0$

or, $r^2 - 3r - 2r + 6 = 0$

or, $r(r-3) - 2(r-3) = 0$

or, $(r-3)(r-2) = 0$

Either $r = 3$ or $r = 2$.

$r_1 = 3$ & $r_2 = 2$

Since two characteristics roots are distinct so, the general form of soln is :-

$$a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n$$

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n \quad \text{--- (i)}$$

Initial condition :-

$a_0 = 1$, $a_1 = 0$

put $n = 0$ then eqn (i) becomes

$$a_0 = \alpha_1 \cdot 3^0 + \alpha_2 \cdot 2^0$$

$$1 = \alpha_1 + \alpha_2$$

$$\alpha_1 = 1 - \alpha_2 \quad \text{--- } (*)$$

Again, put $n = 1$ in eqn (i)

$$a_1 = \alpha_1 \cdot 3^1 + \alpha_2 \cdot 2^1$$

or, $0 = 3\alpha_1 + 2\alpha_2$

or, $3(1-\alpha_2) + 2\alpha_2 = 0$

or, $3 - 3\alpha_2 + 2\alpha_2 = 0$

or, $3 - 1\alpha_2 = 0$

or, $3 = \alpha_2$

$\therefore \alpha_2 = 3$

$\therefore \alpha_1 = 1 - 3 = -2$

Hence, eqn (i) becomes :-

$$a_n = -2 \cdot 3^n + 3 \cdot 2^n$$

$= 3 \cdot (2)^n + (-2) \cdot (3)^n$ is req^rd sol^n.

* Solve the recurrence relation :-

$a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$ $a_1 = 10$

Comparing ~~eqn~~ with $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$

we set,

$c_1 = 6$ and $c_2 = -8$

and characteristic equ^n is:-

$$r^k - c_1 \cdot r^{k-1} - c_2 \cdot r^{k-2} \ldots - c_k = 0$$

or, $r^2 - c_1 \cdot r^1 - c_2 = 0$

or, $r^2 - 6r + 8 = 0$

or, $r^2 - (2+4)r + 8 = 0$

or, $r^2 - 2r - 4r + 8 = 0$

or, $r(r-2) - 4(r-2) = 0$

or, $(r-2)(r-4) = 0$

$\therefore r = 2$ ~~or~~ ~~and~~ 4.

$r_1 = 2$ and $r_2 = 4$

Since two characteristic roots are distinct so, the general form of sol^n is:-

$a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n$

or, $a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 4^n$ — (i)

Initial condition:-

$a_0 = 4$ and $a_1 = 10$.

put $n = 0$ then eqn (i) becomes:-

$a_0 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 4^0$

or, $4 = \alpha_1 + \alpha_2$

or, $\alpha_1 = 4 - \alpha_2$ — (*)

put $n=1$. eqn (i) becomes,

$$a_1 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot 4^1$$

or, $10 = 2\alpha_1 + 4\alpha_2$

or, $10 = 2(4-\alpha_2) + 4\alpha_2$

or, $10 = 8 - 2\alpha_2 + 4\alpha_2$

or, $10 = 8 + 2\alpha_2$

or, $2 = 2\alpha_2$

$\therefore \alpha_2 = 1$

$\therefore \alpha_1 = 4 - 1 = 3$.

Hence, eqn (i) becomes

$a_n = 3 \cdot (2)^n + 1 \cdot (4)^n$ is the required soln.

* Theorem 2:- Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$ if and only if $a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot n \cdot r_0^n$ for $n = 0, 1, 2 \ldots$ where α_1 and α_2 are constant.

Example:

Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 6$.

→ Soln:

Given,

$$a_n = 6a_{n-1} - 9a_{n-2}$$

Compare with $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$

$\therefore c_1 = 6 \quad c_2 = -9$

and characteristic eqn is:

$$r^2 - c_1 \cdot r - c_2 = 0$$

or, $r^2 - 6r + 9 = 0$

or, $r^2 - (3+3)r + 9 = 0$

or, $r^2 - 3r - 3r + 9 = 0$

or, $r(r-3) - 3(r-3) = 0$

or, $(r-3)(r-3) = 0$

$\therefore r = 3, 3.$

i.e. $r_0 = 3$

then soln is

$$a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot n \cdot r_0^n$$

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n \quad \text{— (i).}$$

Initial condition:

$a_0 = 1$ & $a_1 = 6$

put $n=0$,

$a_0 = \alpha_1 \cdot 3^0 + \alpha_2 \cdot 0 \cdot 3^0$

$\Rightarrow 1 = \alpha_1 + 0$

$\therefore \alpha_1 = 1$

put $n=1$,

$a_1 = 1 \cdot 3^1 + \alpha_2 \cdot 1 \cdot 3^1$

or, $6 = 3 + 3\alpha_2$

or, $3 = 3\alpha_2$

$\therefore \alpha_2 = 1$.

Hence, eqn (i) becomes

$a_n = 3^n + n \cdot 3^n$ is the required soln.

* ~~Theorem 3~~ :

* Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$ & $a_1 = 1$

→ Comparing with $a_n = C_1 \cdot a_{n-1} + C_2 \cdot a_{n-2}$

$C_1 = 2$ and $C_2 = -1$

and characteristic eqn is :-

$$\gamma^2 - C_1\gamma - C_2 = 0$$

or, $\gamma^2 - 2\gamma + 1 = 0$

or, $\gamma^2 - (1+1)\gamma + 1 = 0$

or, $\gamma^2 - 1\gamma - 1\gamma + 1 = 0$

or, $\gamma(\gamma-1) - 1(\gamma-1) = 0$

or, $(\gamma-1)(\gamma-1) = 0$

$\therefore \gamma = 1, 1.$

$\therefore \gamma_0 = 1.$

The general soln is :-

$$a_n = \alpha_1 \cdot \gamma_0^n + \alpha_2 \cdot n \cdot \gamma_0^n$$

or, $a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot n \cdot 1^n$ — (i)

Initial condition :-

$a_0 = 4$ & $a_1 = 1$

put $n = 0$, (i) becomes,

$a_0 = \alpha_1 \cdot 1^0 + \alpha_2 \cdot 0 \cdot 1^0$

$\Rightarrow 4 = \alpha_1$

$\therefore \alpha_1 = 4$

put $n = 1$, (i) becomes,

$a_1 = 4 \cdot 1^1 + \alpha_2 \cdot 1 \cdot 1^n$

$\Rightarrow 1 = 4 + \alpha_2$

$\therefore \alpha_2 = -3.$

Hence, Eqn (i) becomes

$a_n = 4 \cdot 1^n + (-3) \cdot n \cdot 1^n$ is the reqrd soln.

* Theorem 3:- Let $c_1, c_2, \ldots c_k$ be a real numbers. Suppose that $r^k - c_1 \cdot r^{k-1} \ldots - c_k = 0$ has k distinct root $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \ldots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n + \ldots + \alpha_k r_k^n$ for $n = 0, 1, 2, \ldots$ where $\alpha_1, \alpha_2, \ldots \alpha_k$ are constant.

Q) Find the solution of $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$ with $a_0 = 3$, $a_1 = 6$, $a_2 = 0$

Given,

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

Comparing $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + c_3 \cdot a_{n-3}$

$\therefore c_1 = 2$, $c_2 = 1$ and $c_3 = -2$

and characteristic equn is (obtained by $k=3$).

$$r^3 - c_1 \cdot r^2 - c_2 \cdot r - c_3 = 0$$

$\Rightarrow \quad r^3 - 2r^2 - r + 2 = 0$

$\Rightarrow \quad r^2(r-2) - 1(r-2) = 0$

$\Rightarrow \quad (r-2)(r^2-1) = 0$

$\Rightarrow \quad (r-2)(r-1)(r+1) = 0$

$\therefore r = 1, -1, 2$

$\therefore r_1 = 1,\ r_2 = -1\ \&\ r_3 = 2$

Solution is given by :-

$a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n + \alpha_3 \cdot r_3^n$

$\Rightarrow \quad a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot (-1)^n + \alpha_3 \cdot 2^n \longrightarrow (i)$

Initial condition :-

$a_0 = 3$

put $n=0$, (i) becomes,

$a_0 = \alpha_1 \cdot 1^0 + \alpha_2 \cdot (-1)^0 + \alpha_3 \cdot 2^0$

$\Rightarrow \quad 3 = \alpha_1 + \alpha_2 + \alpha_3$

$\Rightarrow \quad \alpha_1 = 3 - \alpha_2 - \alpha_3 \longrightarrow (ii)$

put $n = 1$, (i) becomes,

$a_1 = \alpha_1 \cdot 1^1 + \alpha_2 \cdot (-1)^1 + \alpha_3 \cdot 2^1$

$\Rightarrow \quad 6 = \alpha_1 + (-\alpha_2) + 2\alpha_3$

$\Rightarrow \quad 6 = 3 - \alpha_2 - \alpha_3 - \alpha_2 + 2\alpha_3$

classmate
Date
Page

$\Rightarrow \quad 6 = 3 + \alpha_3 - 2\alpha_2$

$\Rightarrow \quad 3 = \alpha_3 - 2\alpha_2 \quad$ — (iii)

Again, put $n=2$,

$a_2 = \alpha_1 \cdot 1^2 + \alpha_2 \cdot (-1)^2 + \alpha_3 \cdot 2^2$

$\Rightarrow \quad 0 = \alpha_1 + \alpha_2 + 4\alpha_3$

$\Rightarrow \quad 0 = 3 - \cancel{\alpha_2} - \alpha_3 + \cancel{\alpha_2} + 4\alpha_3$

$\Rightarrow \quad 0 = 3 + 3\alpha_3$

$\Rightarrow \quad -3 = 3\alpha_3$

$\Rightarrow \quad -1 = \alpha_3$

$\therefore \alpha_3 = -1$

putting α_3 in equ^n (iii)

$3 = -1 - 2\alpha_2$

$3 + 1 = -2\alpha_2$

$\frac{4}{-2} = \alpha_2 \quad \therefore \alpha_2 = -2$

putting α_2 and α_3 in (i) we get,

$\alpha_1 = 3 - (-2) - (-1)$

$= 3 + 2 + 1$

$= 6$

$\therefore \alpha_1, \alpha_2, \alpha_3 = 6, -2, -1$ respectively.

Hence, the equation (i) becomes :-

$a_n = \cancel{6} \; 6 \cdot 1^n + (-2) \cdot (-1)^n + (-1) \cdot 2^n$ is the

$\Rightarrow$ ~~$a_n = 6 \cdot 1^n$~~ required sol^n.

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classmate
Date
Page

* Theorem 4 :-

Let $c_1, c_2, \dots c_k$ be a real numbers. Suppose the characteristic equ^n is

$$r^k - c_1 \cdot r^{k-1} - \dots - c_k = 0$$

has t distinct roots $r_1, r_2, \dots, r_t$ with multiplicities $m_1, m_2, \dots, m_n$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \dots, t$ and $m_1 + m_2 + \dots + m_i = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \dots + c_k \cdot a_{n-k}$$

if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \dots + \alpha_{1,m_1-1} n^{m_i - 1}) r_1^n$$
$$+ (\alpha_{2,0} + \alpha_{2,1} n + \dots + \alpha_{2,m_2-1} n^{m_i - 1}) r_2^n$$
$$+ \dots + (\alpha_{t,0} + \alpha_{t,1} n + \dots + \alpha_{t,m_t-1} n^{m_t - 1}) r_t^n$$

for $n = 0, 1, 2 \dots$ where $a_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Q) Solve the recurrence relation $a_n = 5a_{n-1} - 7a_{n-2} + 3a_{n-3}$ for $n \geq 3$, $a_0 = 1$, $a_1 = 9$ and $a_2 = 15$.

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→ Given:

$$a_n = 5a_{n-1} - 7a_{n-2} + 3a_{n-3}$$

Comparing $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + c_3 \cdot a_{n-3}$

$\therefore c_1 = 5, \; c_2 = -7, \; c_3 = 3$

and characteristic equ^n is

$$r^3 - c_1 \cdot r^2 - c_2 \cdot r - c_3 = 0$$

$\Rightarrow r^3 - 5r^2 + 7r - 3 = 0$

$\Rightarrow$ On solving we get,

$r = 1, 3, 1$

$r_1 = 1$ $\quad\quad$ $m_1 = 2$ (repeat vako r_1)

$r_2 = 3$ $\quad\quad$ $m_2 = 1$ (repeat vako r_2).

The sol^n is :-

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n) r_1^n + (\alpha_{2,0}) r_2^n$$

$\Rightarrow a_n = (\alpha_{1,0} + \alpha_{1,1} n) 1^n + (\alpha_{2,0}) \cdot 3^n \quad — (i)$

Initial condition:-

put $n = 0$ (i) becomes,

$a_0 = (\alpha_{1,0} + \alpha_{1,1} \cdot 0) \cdot 1^0 + (\alpha_{2,0}) \cdot 3^0$

$\Rightarrow 1 = \alpha_{1,0} + \alpha_{2,0}$

$\Rightarrow \alpha_{1,0} = 1 - \alpha_{2,0} \quad — (ii)$

Again,

put $n = 1$,

$$a_1 = (\alpha_{1,0} + \alpha_{1,1} 1^1) 1^1 + (\alpha_{2,0}) 3^1$$

$$\Rightarrow 9 = \alpha_{1,0} + \alpha_{1,1} + 3\alpha_{2,0}$$

$$\Rightarrow 9 = 1 - \alpha_{2,0} + \alpha_{1,1} + 3\alpha_{2,0}$$

$$\Rightarrow 8 = \alpha_{1,1} + 2\alpha_{2,0}$$

$$\Rightarrow \alpha_{1,1} = 8 - 2\alpha_{2,0} \quad \text{--- (iii)}$$

~~put $n = 2$,~~

~~$a_2 = (\alpha_{1,0} + \alpha_{1,1} 2) 2$~~

put $n = 2$,

$$a_2 = (\alpha_{1,0} + \alpha_{1,1} 2) \cdot 1^2 + (\alpha_{2,0}) \cdot 3^2$$

$$\Rightarrow 15 = (\alpha_{1,0} + 2\alpha_{1,1}) \cdot 1 + (\alpha_{2,0}) \cdot 9$$

$$\Rightarrow 15 = \alpha_{1,0} + 2\alpha_{1,1} + 9\alpha_{2,0}$$

$$\Rightarrow 15 = 1 - \alpha_{2,0} + 2(8 - 2\alpha_{2,0}) + 9\alpha_{2,0}$$

$$\Rightarrow 15 = 1 - \alpha_{2,0} + 16 - 4\alpha_{2,0} + 9\alpha_{2,0}$$

$$\Rightarrow 15 - 17 = 4\alpha_{2,0}$$

$$\Rightarrow -2 = 4\alpha_{2,0}$$

$$\therefore \alpha_{2,0} = -\frac{1}{2}.$$

Again, from eqn (iii)

$$\alpha_{1,1} = 8 - 2 \times -\frac{1}{2}$$

$$= 8 + 1 = 9.$$

Again from (ii)

$$\alpha_{1,0} = 1 - \alpha_{2,0}$$
$$= 1 + 1/2 = 3/2$$

Hence equ^n (i) becomes:

$$a_n = (3/2 + 9 \cdot n)(1)^n + \left(-\frac{1}{2}\right)(3)^n$$ is

the required soln.

Imp ✱ Theorem 5 (of linear non-homogenous recurrence relation).

Q) Find all the solution of the recurrence relation $a_n = 4a_{n-1} + n^2$. Also find the solution of the relation with initial condition $a_1 = 1$.

→ Given,

$$a_n = 4a_{n-1} + n^2$$

We have, associated linear homogenous recurrence relation as:

$$a_n = 4a_{n-1}$$

Comparing with $a_n = C_1 \cdot a_{n-1}$

$$\therefore C_1 = 4.$$

and characteristic equation is:

$$r - C_1 = 0 \Rightarrow r - 4 = 0 \Rightarrow r = 4.$$

It degree 1
$an+b$.

denotes homogenous.

Solution is $a_n^{(h)} = \alpha \cdot (4)^n$ (Applying theorem 1)

$f(n) = n^2$ is a polynomial of degree 2.
Soln is a quadratic function in n say $P_n = an^2 + bn + c$ where a, b and c are constants. To determine whether there are any solns of this form, suppose that
$P_n = an^2 + bn + c$ is such a solution.

Then the equation becomes,

$$a_n = 4a_{n-1} + n^2$$

$$an^2 + bn + c = 4\{a(n-1)^2 + b(n-1) + c\} + n^2$$

$\Rightarrow$ $an^2 + bn + c = 4\{an^2 - 2an + a + bn - b + c\} + n^2$

$\Rightarrow$ $an^2 + bn + c = 4an^2 - 8an + 4a + 4bn - 4b + 4c + n^2$

$\Rightarrow$ $an^2 + bn + c = (4a+1)n^2 + (-8a+4b)n + (4a+4c-4b)$

Comparing we get,

$4a+1 = a$	$-8a+4b = b$	$4a+4c-4b = c$
$\Rightarrow 4a - a = -1$	$-8a = -3b$	$\therefore c = -\frac{20}{27}$
$\Rightarrow 3a = -1$	$-8 \times -\frac{1}{3} = -3b$	
$\therefore a = -\frac{1}{3}$	$\frac{8}{3} = -3b$	
	$\therefore b = -\frac{8}{9}$	

$\therefore \; a_n^{(p)} = an^2 + bn + c$

$\Rightarrow a_n^{(p)} = -\frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27}$

All soln is:-

$a_n = \{a_n^{(p)} + a_n^{(h)}\}$

$\Rightarrow a_n = -\frac{1}{3}n^2 - \frac{8}{9}n - \frac{20}{27} + \alpha.(4)^n \quad -(i)$

Initial condition:-

$a_1 = 1$

put $n=1$ in (i) we get,

$a_1 = -\frac{1}{3} \times 1^2 - \frac{8}{9} \times 1 - \frac{20}{27} + \alpha . 4^1$

$\Rightarrow 1 = -\frac{1}{3} - \frac{8}{9} - \frac{20}{27} + 4\alpha$

$\Rightarrow \alpha = \frac{20}{27}$

So, req soln is

$a_n = \left(-\frac{1}{3}\right)n^2 + \left(\frac{-8}{9}\right)n + \left(\frac{-20}{27}\right) + \left(\frac{20}{27}\right).4^n$

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classmate

Date

Page

* Theorem 6 :-

Q) Find the solution of recurrence relation
$a_n = 2a_{n-1} - n \cdot 2^n$

→ Given recurrence relation is :-

$$a_n = 2a_{n-1} + n \cdot 2^n$$

Comparing with

$$a_n = c_1 a_{n-1}$$

$$\therefore c_1 = 2$$

characteristic equn is :-

$$r - c_1 = 0$$

$$r - 2 = 0$$

$$\therefore r = 2$$

soln is :-

$$a_n^{(h)} = \alpha \cdot (2)^n$$

we have,

$F(n) = n \cdot 2^n$ (of the form $n \cdot s^n$

where S is a root of the characteristic equation ~~of the~~ and the multiplicity of 2 is 1. i.e. $m = 1$. so, particular soln has the form,

$$a_n^{(p)} = n^1 (P_1 n^1) 2^n$$

$$\therefore a_n = \{a_n^{(p)} + a_n^{(h)}\}$$

$$= P_1 n^2 2^n + \alpha \cdot (2)^n \quad \text{Ans}$$

Advance Counting Tips

Yedi c_1 and c_2 vayepar
Theorem 1 and 2 use garne

Yedi c_1, c_2, c_3 vayepar
Theorem 3 and 4 use garne

Yedi $c_1, c_2 + f(n)$ vayepar
Theorem 5 and 6 use garne.

Theorem 4 formula correction:-

When r is same or repeated we use theorem 4 and soln is given by:-

$$a_n = (\alpha_{1,0} + \alpha_{1,1} \cdot n + \dots + \alpha_{1,m_1-1} \times n^{m_1-1}) r_1^n$$
$$+ (\alpha_{2,0} + \alpha_{2,1} \times n + \dots + \alpha_{2,m_2-1} \times n^{m_2-1}) r_2^n$$
$$+ \dots$$
$$+ (\alpha_{t,0} + \alpha_{t,1} \cdot n + \dots + \alpha_{t,m_t-1} \times n^{m_t-1}) r_t^n.$$

for $n = 0, 1, 2, \dots$ where $a_{i,j}$ are constants.
for $1 \le i \le t$ and $0 \le j \le m_i - 1$.

Date ______
Page ______

* Chinese Remainder Theorem :-

Formula :-

$$x \equiv a_1 \pmod{n_1}$$
$$x \equiv a_2 \pmod{n_2}$$
$$x \equiv a_3 \pmod{n_3}$$
$$\vdots$$
$$x \equiv a_k \pmod{n_k}$$

Moreover,

$$X = a_1 M_1 Y_1 + a_2 M_2 Y_2 + \cdots + a_k M_k Y_k \pmod{M}$$

where,

$$M_i = \frac{M}{n_i}$$

and $M_i Y_i = 1 \pmod{n_i}$ for $i = 1, 2, \ldots k$

Q) find the value of x such that:

$$x \equiv 2 \pmod{3}$$
$$x \equiv 4 \pmod{5}$$
$$x \equiv 5 \pmod{7}$$

using chinese Remainder Theorem

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Comparing we get;

$a_1 = 2$ $m_1 = 3$

$a_2 = 4$ $m_2 = 5$

$a_3 = 5$ $m_3 = 7$

$$M = m_1 \times m_2 \times m_3$$
$$= 3 \times 5 \times 7$$
$$= 105$$

$$M_1 = \frac{M}{m_1} = \frac{105}{3} = 35$$

$$M_2 = \frac{M}{m_2} = \frac{105}{5} = 21$$

$$M_3 = \frac{M}{m_3} = \frac{105}{7} = 15$$

Now,

$$M_1 Y_1 = 1 \pmod{m_1}$$
$$35 Y_1 = 1 \pmod{3}$$
$$2Y_1 = 1 \pmod{3}$$

Multiplying by 2 on both sides:

$$4Y_1 = 2 \pmod{3}$$
$$1 \cdot Y_1 = 2 \pmod{3}$$
$$\therefore Y_1 = 2.$$

Again,

$$M_2 y_2 \equiv 1 \pmod{m_2}$$

$\Rightarrow 21 y_2 \equiv 1 \pmod 5$

$\Rightarrow 1 y_2 \equiv 1 \pmod 5$

$\therefore y_2 = 1$

$M_3 y_3 \equiv 1 \pmod{m_3}$

$15 y_3 \equiv 1 \pmod 7$

$1 y_3 \equiv 1 \pmod 7$

$\therefore y_3 = 1$

Then, equ^n is given by:-

$x \equiv a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3 \pmod M$

$x \equiv 2 \times 35 \times 2 + 4 \times 21 \times 1 + 5 \times 15 \times 1 \pmod{105}$

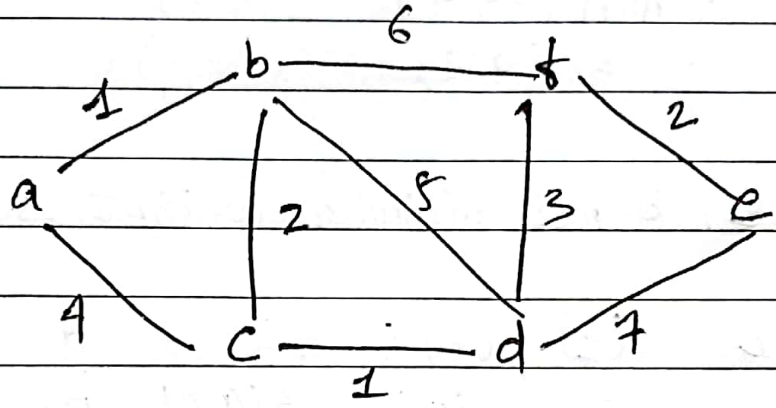
$x \equiv 140 + 84 + 75 \pmod{105}$

$x \equiv 299 \pmod{105}$

$x \equiv 89 \pmod{105}$

Dijkstra's Algorithm
(Shortest Path)

Q) Find the shortest path from a to e in the following figure. (Dijkstra's Algorithm).



Since we have to find the shortest path from vertex a to e. So, we assign weight 0 to a and ∞ to all remaining vertices.

Vertex	(a)	b	c	d	e	f
Label	0	∞	∞	∞	∞	∞

The vertices adjacent to a are b ~~or~~ and c. We calculated weight of b and c and lablee them with minimum weight.

$$wt(b) = \min(wt(b), wt(a) + wt(a,b))$$
$$= \min(\infty, 0+1)$$
$$= \min(\infty, 1)$$
$$= 1\ (a,b).$$

$wt(c) = \min(wt(c), wt(a) + wt(a,c))$
$= \min(\infty, 0+4)$
$= \min(\infty, 4)$
$= 4 \ (a,c).$

Since, b has minimum weight. So, we select it.

Vertex	(a)	(b)	c	d	e	f
Label	0	1 (a,b)	4 (a,c)	∞	∞	∞

Again, the vertices b adjacent to c, d, f.

$\therefore wt(c) = \min(wt(c), wt(b) + wt(b,c))$
$= \min(4, 1+2)$
$= \min(4, 3)$
$= 3 \ (a,b,c).$

$\therefore wt(d) = \min(wt(d), wt(b) + wt(b,d))$
$= \min(\infty, 1+5)$
$= \min(\infty, 6)$
$= 6 \ (a,b,d).$

$\therefore wt(f) = \min(wt(f), wt(b) + wt(b,f))$
$= \min(\infty, 1+6)$
$= 7 \ (a,b,f).$

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Since c has min wt so we have to select it

Vertex	(a)	(b)	(c)	d	e	f
Label	0	1(a,b)	3(a,b,c)	6(a,b,d)	∞	7(a,b,f)

The vertex c adjacent to a, b, d.

$\therefore$ wt(d) = min[wt(d), wt(c) + wt(c,d)]
= min(6, 3+1)
= min(6, 4)
= 4(a,b,c,d).

$\therefore$ select d.

Vertex	(a)	(b)	(c)	(d)	e	f
Label	0	1(a,b)	3(a,b,c)	4(a,b,c,d)	∞	7(a,b,f).

The vertices adjacent to d f, e, b, c.

$\therefore$ wt(e) = min[wt(e), wt(d) + wt(d,e)]
= min(∞, 4+7)
= min(∞, 11)
= 11(a,b,c,d,e)

$\therefore$ wt(f) = min(wt(f), wt(d) + wt(d,f))
= min(7, 4+3)
= 7(a,b,c,d,f).

Since f is min. wt. So we select it.

Vertex	(a)	(b)	(c)	(d)	e	(f)
Label	0	1(a,b)	3(a,b,c)	4(a,b,c,d)	11(a,b,c,d,e)	7(a,b,c,d,f)

The vertices f adjacent to e are : b, d, e.

$\therefore wt(e) = \min(wt(e), wt(f) + wt(f,e))$
$= \min(11, 7+2)$
$= 9(a,b,c,d,f,e)$.

Select e.

Vertex	(a)	(b)	(c)	(d)	(e)	(f)
Label	0	1(a,b)	3(a,b,c)	4(a,b,c,d)	9(a,b,c,d,f,e)	7(a,b,c,d,f)

If we go through a-b-c-d-f-e
= 9 ;

then, we get a shortest path.

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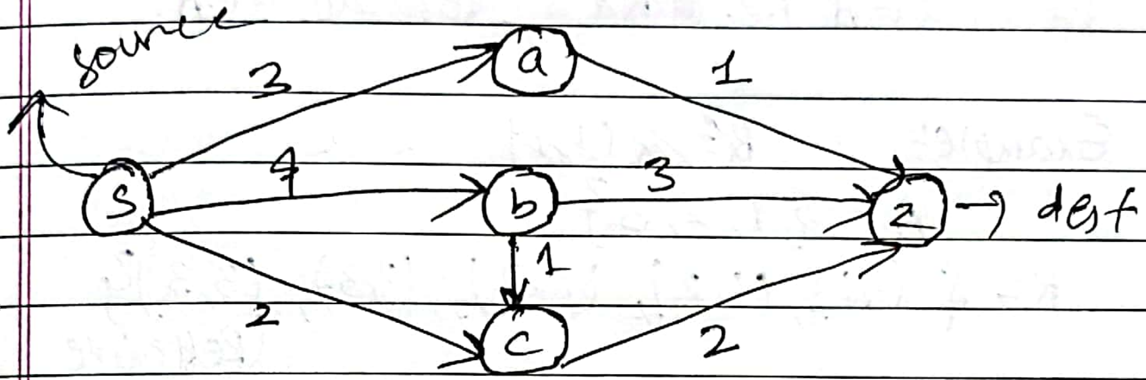
S-D Cut

* Find all the S-D cuts in given transport network and find maximal flow.



Maximum flow = Minimal cuts.

Sol^n:

	Source X	Destination $\bar{X}$	$(X, \bar{X})$	$K(X, \bar{X})$
1.	S	$\{a, b, c, z\}$	$\{(s,a), (s,b), (s,c)\}$	$3+4+2=9$
2.	$\{s, a\}$	$\{b, c, z\}$	$\{(s,b), (s,c), (a,z)\}$	$4+2+1=7$
3.	$\{s, b\}$	$\{a, c, z\}$	$\{(s,a), (s,c), (b,c), (b,z)\}$	$3+2+1+3=9$
4.	$\{s, c\}$	$\{a, b, z\}$	$\{(s,a), (s,b), (c,b), (c,z)\}$	$3+4+1+2=10$
5.	$\{s, a, b\}$	$\{c, z\}$	$\{(s,c), (a,z)\}$	

and so on.

The minimum value of $K(X, \bar{X})$ is the maximum flow.

classmate
Date
Page

Chapter - 6

(Relation and Graph)

Imp

* **Reflexive Relation:-**

A relation R on a set A is reflexive if $(a,a) \in R$ for all $a \in A$ i.e. aRa for all $a \in A$.

Example:- ~~R = {(1,1)}~~

$A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$

(Reflexive)

* $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (2,1), (2,2), (3,3), (3,4), (4,1)\}$

↳ Not reflexive.

* **Symmetric Relation:-**

A relation R on a set A is symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a, b \in A$.

Example:- $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (1,3), (2,3), (2,1), (3,2), (3,1)\}$

* Asymmetric Relation :-

A relation R on a set A is asymmetric if $(a,b) \in R$ then $(b,a) \notin R$ for all $a, b \in A$.

Example:

$$R = \{(1,1), (1,2), (2,3), (3,1)\}$$
$$A = \{1, 2, 3\}.$$

* Anti-symmetric Relation :-

A relation R on a set is Antisymmetric if $a = b$ whenever aRb and bRa.
The contrapositive of this definition is that R is antisymmetric if $a \not R b$ and $b \not R a$ whenever $a \neq b$.

$$R = \{(1,2), (2,3), (2,3)\}$$
$$A = \{1, 2, 3\}.$$

* Transitive Relation:-

A relation R on a set A is transitive if whenever aRb and bRc i.e. $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ for all $a, b, c \in A$.

Eg:- $A = \{1, 2, 3\}$.
$R = \{(1,2), (3,2), (2,3), (1,3), (2,2), (3,3)\}$.

Types of Relations:-

1/ Complementary Relations:-

Let R be a relation from a set A to set B. The complementary relation of R is denoted by $\bar{R}$ which consists of those ordered pairs which are not in R.

i.e. $\overline{R} = \{(a,b) \in A \times B : (a,b) \notin R\}$

Example:- Let R be a relation on set $A = \{1, 2, 6\}$ defined $R = \{(x,y) : x < y\}$. Find the complement relation of R.

→ $A = \{1, 2, 6\}$

$A \times A = \{1, 2, 6\} \times \{1, 2, 6\}$

$= \{(1,1), (1,2), (1,6), (2,1), (2,2), (2,6), (6,1), (6,2), (6,6)\}$

$R = \{(x,y) : x < y\}$

$= \{(1,2), (1,6), (2,6)\}$

$\bar{R} = A \times A - R$

$= \{(1,1), (2,1), (2,2), (6,1), (6,2), (6,6)\}$

2) Inverse relation :-

Let R be a relation from A to B. The inverse of R denoted by R^{-1} is the relation from B to A. which consists of those ordered pairs when reversed belong to R.

i.e. $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Eg :-

$R = \{(1,2), (1,3), (2,1), (2,2)\}$

$R^{-1} = \{(2,1), (3,1), (1,2), (2,2)\}$.

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classmate

Date

Page

3) Identity relation:-

A relation R in a set A i.e. a relation R from A to A is said to be a identity relation. Generally denoted by I_A, if $I_A = \{(x,x) : x \in A\}$

Eg:- $A = \{1,2,3\}$

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$.

$I_A = \{(1,1), (2,2), (3,3)\}$

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4) N-Ary Relation:-

Relationship among element of more than two sets often arise. The relationship among more than two sets are called n-ary relations.

Eg:- Let $A = \{1,2\}$ and Let R be the relation defined by the property '$x+y+z$' is even

Soln:- $A = \{1,2\}$.

$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$.

$A \times A \times A = \{(1,1), (1,2), (2,1), (2,2)\} \times \{1,2\}$.

$= \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,2,2)$

$(2,1,2), (2,1,2), (2,2,1),\}$

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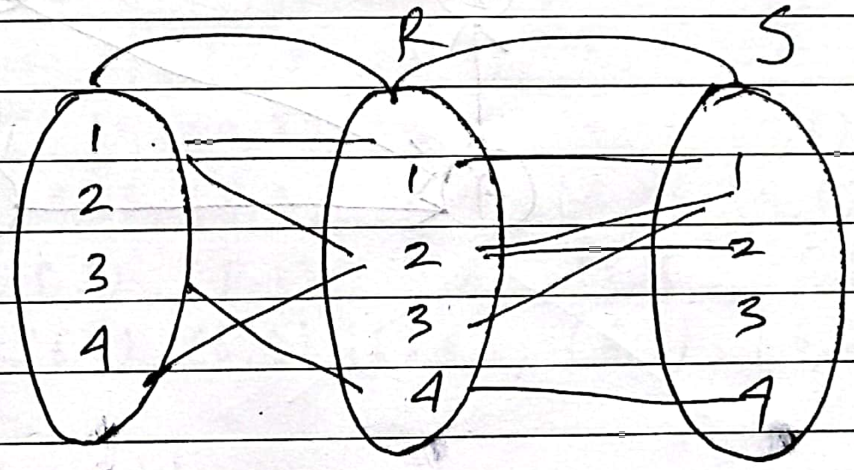
$R = \{(1,1,2), (1,2,1), (2,1,1), (2,2,2)\}$

5) Composition of Relation :-

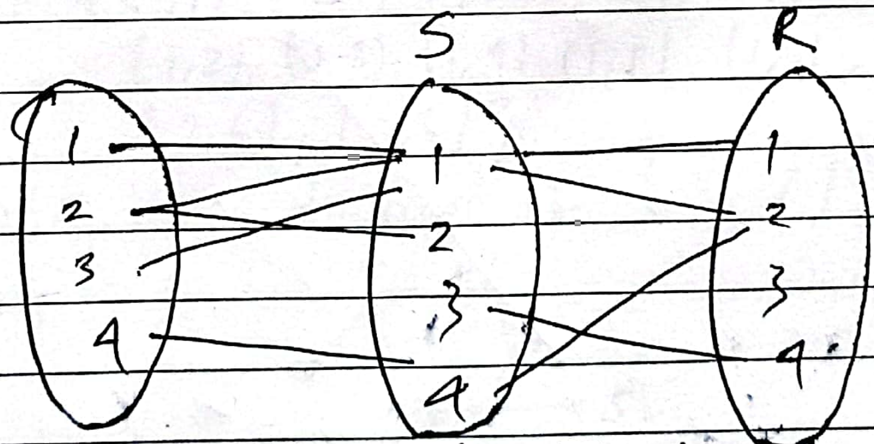
If R and S be relations on $A = \{1,2,3,4\}$ defined by :-

$R = \{(1,1), (1,2), (3,4), (4,2)\}$ and
$S = \{(1,1), (2,1), (3,1), (4,4), (2,2)\}$
find ROS and SOR.

ROS



$ROS = \{(1,1), (1,2), (3,4), (4,1), (4,2)\}$



• $SOR = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2), (4,2)\}$

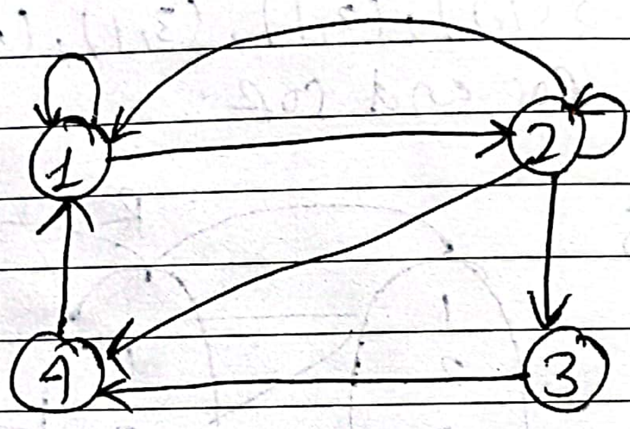
* Directed graphs of relations:

Eg:

$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1)$

then the diagram is shown in figure:



VVImp

Hasse Diagram

Q) Let $X = \{1,2,3,4,5,6\}$ then $/$ is a partial order relation on X. Draw Hasse diagram of $(X, /)$.

4

→ Soln :-

Given :-

$$X = \{1,2,3,4,5,6\}$$

$= \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

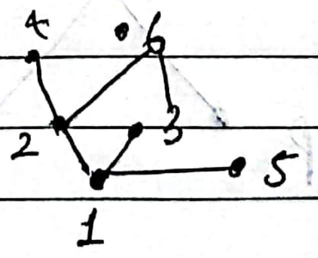
$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$/$ denotes $\frac{y}{x}$.

Now, $(X, /)$

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6),$
$(1,2), (1,3), (1,4), (1,5), (1,6), (2,4),$
$(2,6), (3,6)\}$.

Now in Hasse diagram of $(X, /)$



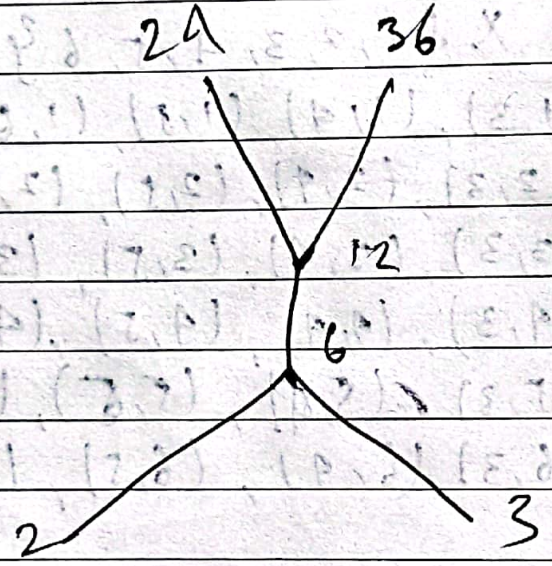
V.Imp

* Maximal element and Minimal element :
* Greatest element and least element
* upper bound and lower bound
* Least upper bound and greatest lower bound

1) for eg :

Top element = maximal element

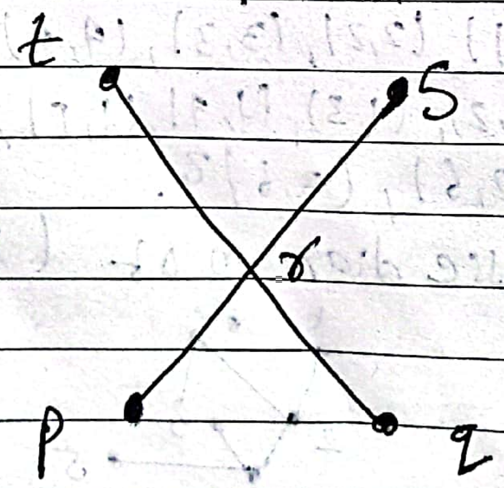
Low element = minimal element.



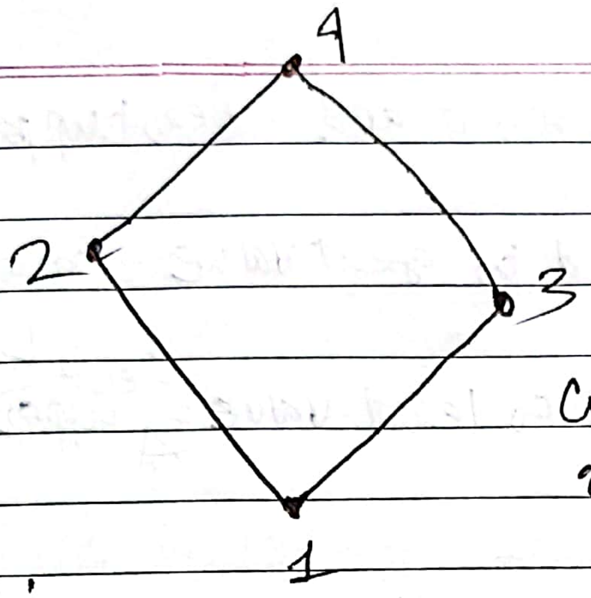
24 & 36 → maximal

2 & 3 → minimal

2) Greatest element and least element :



ii)



greatest element = 4
least element = 1

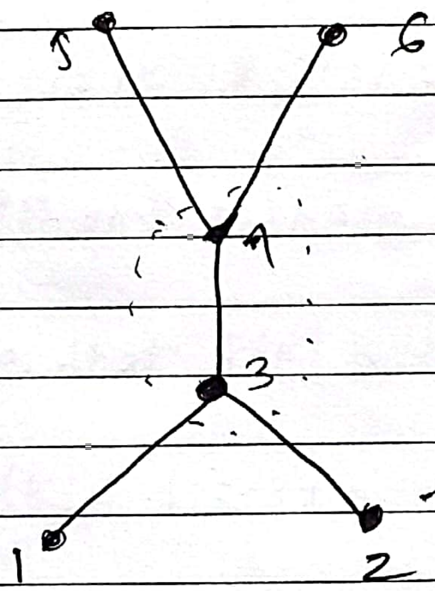
Connected huna parcha greatest rw least element हुन.

Tips :-

Greatest – one only i.e. single
Least – one only i.e. single.

3) upper bound and lower bound :-

Eg: (i)



$A = \{1, 2, 3, 4, 5, 6\}$
and
$B = \{3, 4\} \rightarrow$ subset

upper bound $= \{4, 5, 6\}$
lower bound $= \{3, 2, 1\}$.

Subset lai herne. Subset ma vako upper element rw tesko mathiko element is called upper bound.

lower bound is vice-versa.

4) greatest lower bound and least upper bound

lower bound ka great value = greatest lower bound

upper bound ka least value = least upper bound formula ÷

$$P(K) + x = P(K+1)$$

Mathematical Induction

1) Weak Mathematical Induction ÷

→ Same as 12/11.

eg ÷ $1+2+3+\dots+n = \frac{n(n+1)}{2}$

$$p(n) = 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$1+2+3+\dots+n-1+n$

Step 1 : (Basic step)

$$p(1) = \frac{1(1+1)}{2} = \frac{1\times 2}{2} = 1$$

and $p(2) = \frac{2(2+1)}{2} = \frac{2\times 3}{2} = 3$.

So, p(1) and p(2) are true.

Assume that P(K) is true. So, assumption

$$P(K) = 1+2+3+\dots+K-1+K = \frac{K(K+1)}{2}.$$

Step 3 ÷ $p(K+1) = 1+2+3+\dots+K+1-1+K+1$

$$= \frac{K+1(K+1+1)}{2}$$

$=$

My step:

Let $p(n) = 1+2+3+ \cdots + n = \frac{n(n+1)}{2}$

1) Basic step:

putting $n=1$ we get,

$p(1) = \frac{1(1+1)}{2} = \frac{1 \times 2}{2} = 1$ (True)

$p(2) = \frac{2(2+1)}{2} = \frac{2 \times 3}{2} = 3$ (True).

So, $p(1)$ and $p(2)$ are true.

2) Induction hypothesis:

Let $p(k)$ be true whenever $p(n)$ is true where $k \in n$. Then,

$p(k) = 1+2+3+ \cdots + k = \frac{k(k+1)}{2}$.

3) Induction step:

put $n = k+1$ so that,

$p(k+1) = 1+2+3+ \cdots + (k+1) = \cancel{\frac{n(n+1)}{2}} \frac{(k+1)(k+2)}{2}$

Taking L.H.S. $= 1+2+3+ \cdots + (k+1)$

$= 1+2+3+ \cdots + k + (k+1)$

$= \frac{k(k+1)}{2} + (k+1) =$ R.H.S

proved.

Q) $n^3 - n$ is divisible by 3.

Let $p(n) = n^3 - n$ (divisible by 3).

Basic step:-

Put $n = 1$

$p(1) = 1^3 - 1$

$= 0$ (True i.e. divisible by 3).

$p(2) = 2^3 - 2$

$= 8 - 2$

$= 6$ (True i.e. divisible by 3).

So, $p(1)$ and $p(2)$ are true.

Induction hypothesis:-

Let $p(k)$ be true whenever $p(n)$ is true for all $k \in n$.

$p(k) = k^3 - k$ (divisible by 3).

Induction Step:-

Put $n = k+1$ so that

$p(k+1) = (k+1)^3 - (k+1)$

$= k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$

$= k^3 - k + 3k^2 + 3k$

$= (k^3 - k) + 3k^2 + 3k$

Here, $3k^2+3k$ is divisible by 3 because it is the multiplication of 3 and (k^3-k) is divisible by 3 according to our inductive hypothesis.

Q) prove that 5^n-1 is divisible by 4 using mathematical induction.

→ Let $p(n)$ be the proposition which says 5^n-1 is divisible by 4.
i.e. $p(n): 5^n-1$ (divisible by 4)

1) Basic Step:-

putting $n=1$,

$p(1) = 5^1-1 = 4$ (True)

$p(2) = 5^2-1 = 24$ (True)

2) Induction hypothesis:-

Let $p(k)$ be true whenever $p(n)$ is true where $k \in n$. Then,

$p(k): 5^k-1$

Rough

3) Induction step:-

putting $n = k+1$,

$p(k+1) = 5^{k+1} - 1$

$= 5^k - 1 + 5^k . 4$

$= (5^k - 1) + 4 \cdot 5^k$

= divisible by 4.

$p(k) + x = p(k+1)$

$5^k - 1 + x = 5^{k+1} - 1$

$x = 5^{k+1} \cancel{-1} + \cancel{1} - 5^k$

$= 5^{k+1} - 5^k$

$= 5^k (5^1 - 1)$

$= 5^k . 4$

Since, $p(k) = 5^k - 1$ is divisible by 4 for all $k \in n$ and $4 \cdot 5^k$ is also divisible by 4. Hence, the statement is true and verified.

Q) prove that for every ~~integ~~ integer $n \geq 1$, $n^2 + n$ is even integer using mathematical induction.

→ Let $p(n)$ be the proposition which says $n^2 + n$ is even integer for $n \geq 1$ i.e.

$p(n) : n^2 + n$ (is even integer for $n \geq 1$).

1) Basic step:-

putting $n = 1$

$p(1) = 1^2 + 1 = 2$ (True)

$p(2) = 2^2 + 2 = 6$ (True)

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2) Induction hypothesis:

Let $p(k)$ be true whenever $p(n)$ is true for $k \in n$.

Then,

$p(k)$: $k^2 + k$ is even for $k \geq 1$.

3) Induction step:

putting $n = k+1$,

$$p(k+1) = (k+1)^2 + (k+1)$$
$$= k^2 + 2k + 1 + k + 1$$
$$= k^2 + k + 2k + 2$$
$$= \text{even}$$

Since, $k^2 + k$ is even from (2) and $2k$ is even and 2 is also even.

Hence, the given statement is true.

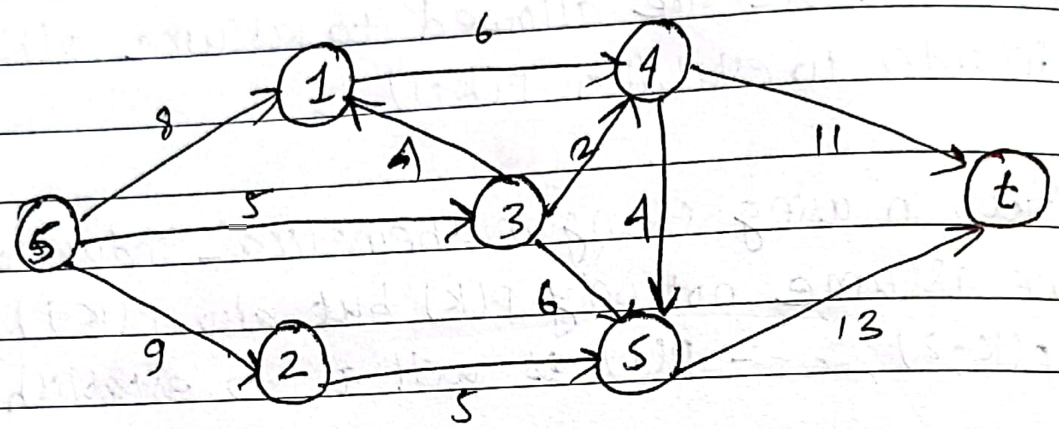
* Strong Mathematical Induction :-

→ In proof using the principle of mathematical induction we are allowed to assume p(K) in order to establish p(K+1).

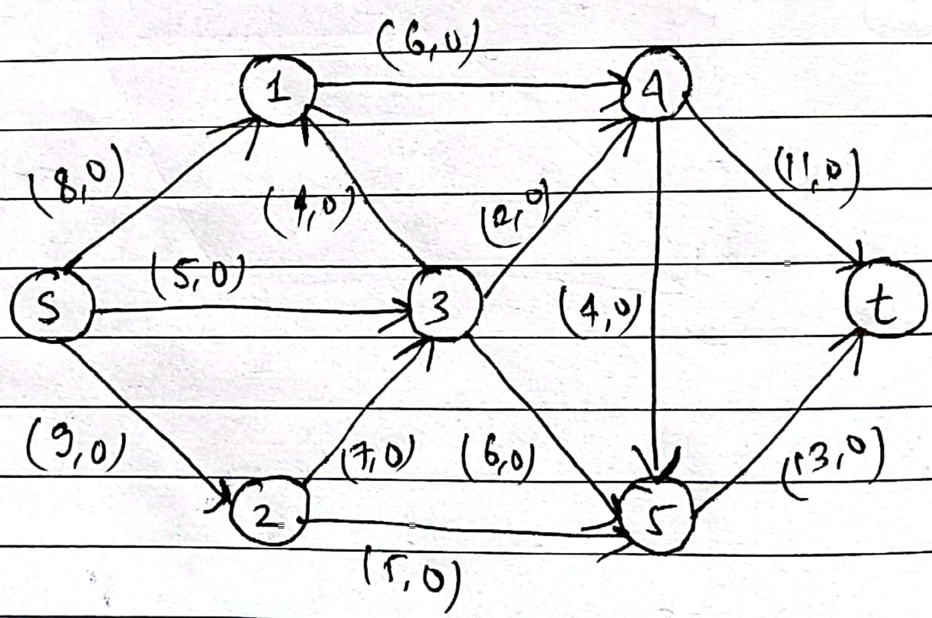
But, in using strong mathematical induction, we assume not only p(K) but also p(K-1), p(K-2), --- p(1) as well as to establish p(K+1).

process to solve strong mathematical induction is same as ~~of~~ weak mathematical induction.

Q) find the maximal flow from s to t from the given network.



Initializing all paths with 0 we get,



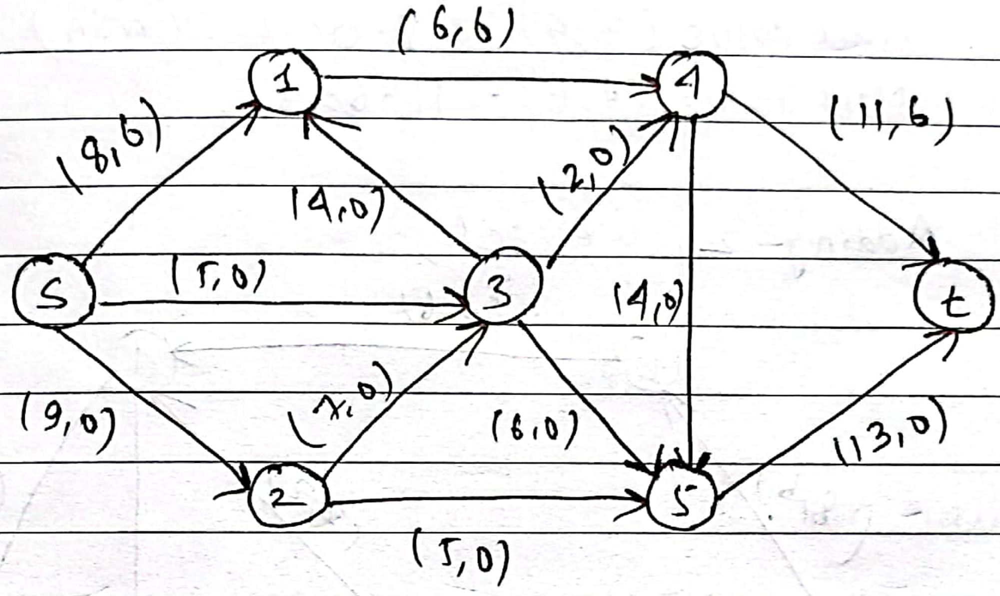
Taking path s – 1 – 4 – t we have :-

slack value (s, 1) = 8-0 = 8

slack value (1, 4) = 6-0 = 6 (min).

slack value (4, t) = 11-0 = 11

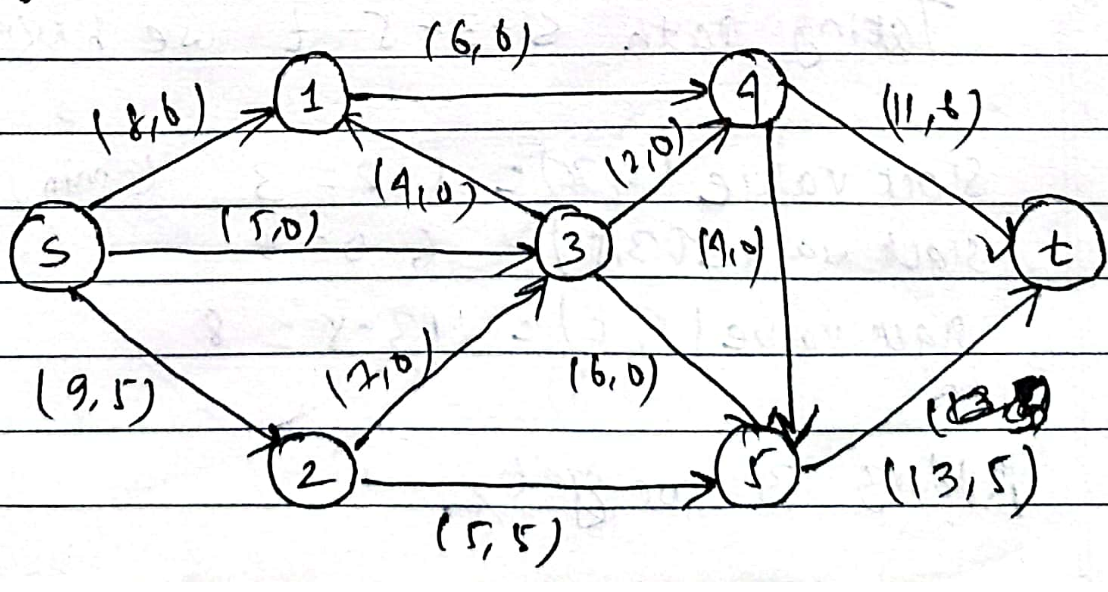
min. slack value is 6. After increasing by 6 in flow of the path we get,



Taking path S-2-5-t we have:

Slack value (S, 2) = 9-0 = 9
slack value (2,5) = 5-0 = 5 (min)
slack value (5,t) = 13-0 = 13

Adding 5 in flow of the path we get,



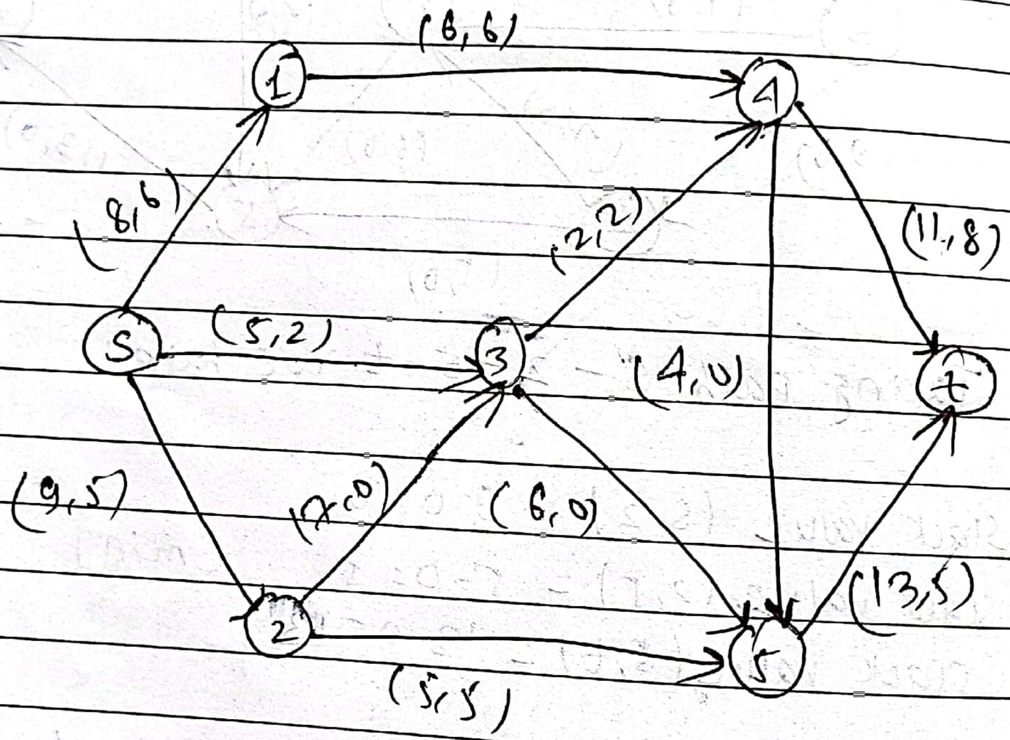
Taking path s-3-4-t we have:-

Slack value (s,3) = 5-0=5

slack value (3,4) = 2-0=2 (min)

slack value (4,t) = 11-6=5

Adding 2 we get,



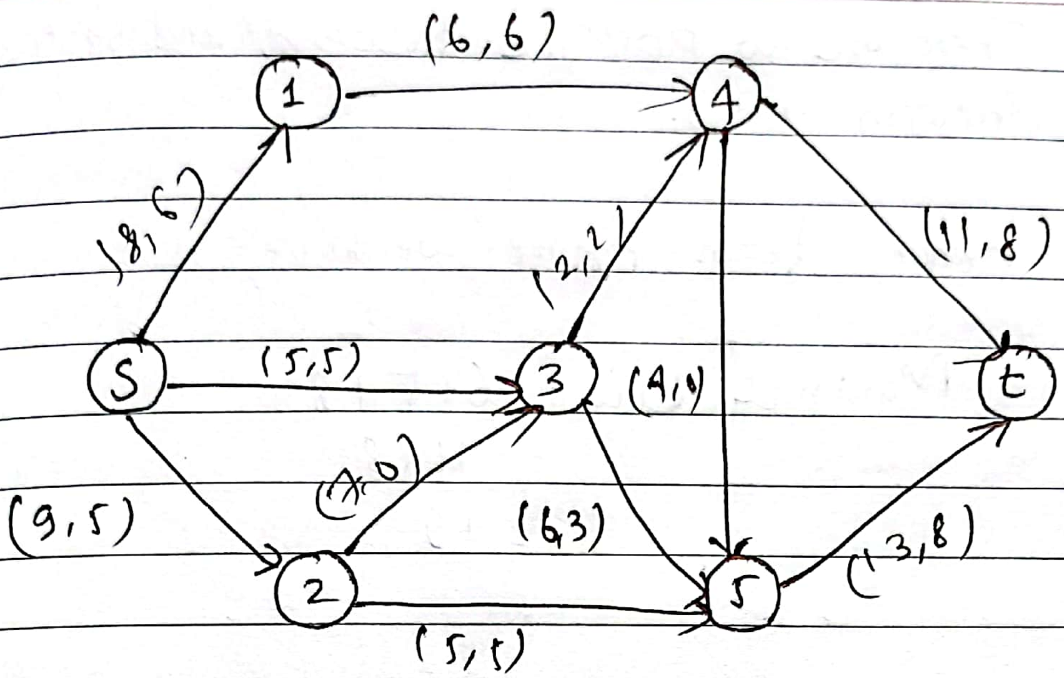
Taking path s-3-5-t we have,

Slack value (s,3) = 5-2=3 (min)

slack value (3,5) = 6-0=6

slack value (5,t) = 13-5 = 8

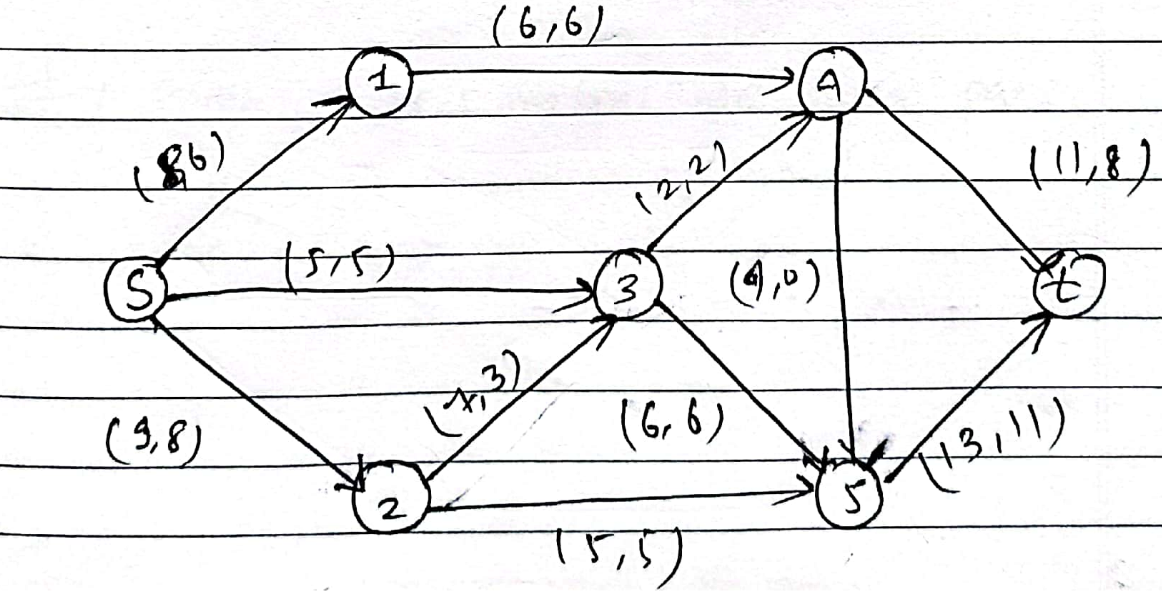
Adding 3 we get,



Taking path s-2-3-5-t we have:

slack value (s,2) = 9-5 = 4
slack value (2,3) = 7-0 = 7
slack value (3,5) = 6-3 = 3 (min).
slack value (5,t) = 13-8 = 5

Adding 3 we get,



There are no possible forward and backward direction now.

Hence, from figure we have,

Maximal flow $= 6+5+8$
$= 11+8$
$= 19$

Chapter-2

Part I

- Primes
- prime factorization
- LCM, GCD (using prime factorization).

- Primes :

for eg: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, .. etc

- Prime factorization :

Q) Find the prime factorization of 99, 110, 645 and 875.

$99 = 3 \cdot 3 \cdot 11 = 3^2 \cdot 11$

$110 = 2 \cdot 5 \cdot 11$

$645 = 3 \cdot 5 \cdot 43$

$875 = 5 \cdot 5 \cdot 5 \cdot 7 = 5^3 \cdot 7$

(Prime numbers matrai use huna paro)

* LCM, GCD (using prime factorization) :

formula :

$$gcd(a,b) = p_1^{min(a_1,b_1)} * p_2^{min(a_2,b_2)} * \cdots p_n^{min(a_n,}$$

$$LCM(a,b) = p_1^{max(a_1,b_1)} * p_2^{max(a_2,b_2)} * \cdots * p_n^{max(a}$$

eg: Use prime factorization to find the gcd of 12 and 3

→ soln :

$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3^1 \cdot 5^0$$

$$30 = 2 \cdot 3 \cdot 5 = 2^1 \cdot 3^1 \cdot 5^1$$

$$\therefore gcd(12,30) = 2^{min(2,1)} \times 3^{min(1,1)} \times 5^{min(0,1)}$$

$$= 2^1 \times 3^1 \times 5^0$$

$$= 2 \times 3 \times 1$$

$$= 6.$$

$$\therefore gcd(12,30) = 6.$$

eg2: $$LCM(12,30) = 2^{max(2,1)} \times 3^{max(1,1)} \times 5^{max(0,}$$

$$= 2^2 \times 3^1 \times 5^1$$

$$= 4 \times 3 \times 5$$

$$= 60$$

$$\therefore LCM(12,30) = 60$$

$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3^1$$

$$18 = 2 \cdot 3 \cdot 3 = 2^1 \cdot 3^2$$

$$\gcd(12,18) = 2^{\min(2,1)} \times 3^{\min(1,2)}$$

$$= 2^1 \times 3^1$$

$$= 2 \times 3$$

$$= 6$$

$\therefore \gcd(12,18) = 6.$

$$\mathrm{LCM}(12,18) = 2^{\max(2,1)} \times 3^{\max(1,2)}$$

$$= 2^2 \times 3^2$$

$$= 4 \times 9$$

$$= 36.$$

Prime factorization of 7007 :

7	7007
7	1001
11	143
13	13
	1

$\therefore 7007 = 7 \times 7 \times 11 \times 13$

$= 7^2 \times 11 \times 13$

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$s_1, s_2 \; \{1, 0$
$t_1, t_2 \; | \; 0, 1\}$ assumption.

$s = s_1 - Q s_2$, $t = t_1 - Q t_2$.

* Euclidean Algorithm :

See Next copy.

* Extended Euclidean Algorithm :

Q) Find the gcd of (161, 28) and the value of s and t.

Q	A	B	R	s_1	s_2	s	t_1	t_2	t
5	161	28	21	1	0	1	0	1	-5
1	28	21	7	0	1	1	1	-5	6
3	21	7	0	1	-1	4	-5	6	-23
	7	0		-1	4		6	-23	

$\therefore$ gcd (161, 28) = 7

$\therefore$ s = -1 and t = 6 Ans.

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Date ______

Page ______

* Permutation:

$$P(n,r) \text{ or } {}^nP_r = \frac{n!}{(n-r)!}$$

Eg: How many license plates consisting of 3 different digits can be made out of given integers 3, 4, 5, 6, 7?

Given:

$n = 5$

$r = 3$

$$P(n,r) = \frac{n!}{(n-r)!} = \frac{5!}{(5-3)!} = \frac{5\times4\times3\times\cancel{2}\times\cancel{1}}{\cancel{2\times1}}$$

$= 60$ plates.

Alternative way:

By using product rule:

$\overline{3}\ \ \overline{4}\ \ \overline{5}$

$5\times4\times3 = 60$

Q) How many number of 3 digits can be formed from the digits 3, 4, 5, 6, 7, 8 ? How many of these are divisible by 5.

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}$
4 5 6

$4 \times 5 \times 6$
$= 120$ numbers of 3 digits.

$\underline{\quad}\ \underline{\quad}\ \underline{5}$
4 5 1

$= 20$ numbers divisible.

Alternatively:-

$n = 6, \ r = 3$

$$p(n,r) = \frac{n!}{(n-r)!} = \frac{6!}{(6-3)!}$$

$$= \frac{6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}}$$

$= 120$

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To find the number divisible by 5, we fix the digit 5 on unit place.

$\therefore$ $n = 6-1 = 5$, $r = 3-1 = 2$

$$\therefore P(5,2) = \frac{5!}{(5-2)!} = \frac{5 \times 4 \times 3!}{3!} = 20$$

* Permutations with Repitition :-

$$\frac{n!}{p!\,q!\,r!}$$

The permutation of n objects taken all at a time, when there are p objects of one kind, q objects are of second kind, r objects are of a third kind, is $\frac{n!}{p!\,q!\,r!}$

Binomial Theorem

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} y^n$$

Q) Expand $(2a+5)^5$ by the binomial theorem.

Here, $n = 5$
$x = 2a$
$y = 5$

$$(2a+5)^5 = \binom{5}{0} \times (2a)^5 + \binom{5}{1} \times (2a)^4 \times 5 + \binom{5}{2} \times (2a)^3 \times 5^2 + \binom{5}{3} \times 2a^2 \times 5^3 + \binom{5}{4} \times 2a^1 \times 5^4 + \binom{5}{5} \times 5^5$$

$$= \cancel{\binom{5}{0} \times 2a^5 + \binom{5}{1} \times 2a^4 \times 5 + \binom{5}{2} \times 2a^3}$$

$$\frac{5!}{0!(5-0)!} \times 32a^5 + \frac{5!}{1!(5-1)!} \times 16a^4 \times 5$$

$$+ \frac{5!}{2!(5-2)!} \times 8a^3 \times 25 + \frac{5!}{3!(5-3)!} \text{ ———}$$

and so on

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Date

Page

* General term :-

General term in the expansion of $(x+y)^n$ is t_{r+1} which is :-

$C(n,r)\, x^{n-r}\, y^r$.

Q) Find the coefficients of x^{16} in the expansion of $\left(2x^2 - \frac{x}{2}\right)^{12}$.

→ page no. 108 Example (DS book).

→ Practice yourself if you see it Next time.

* Middle Term

~~When n is even.~~
~~When n is odd.~~

If n+1 odd one middle term
else vice-versa

One middle

$t_{n/2 + 1}$

Two middle

$t_{(n+1)/2}$ and $\cancel{t_{(n+1)/2+1}}$ $t_{(n+1)/2+1}$

* Pascal's Triangle ÷

1) First and last number in each row is 1.

2) Every other number in the array can be obtained by adding the two numbers appearing directly above it.

Q) Write down the expansion of $(1+y)^6$ using pascal's theorem.

$n = 0 - 6.$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Practice 2075, 2076, 2078, Model Internal paper

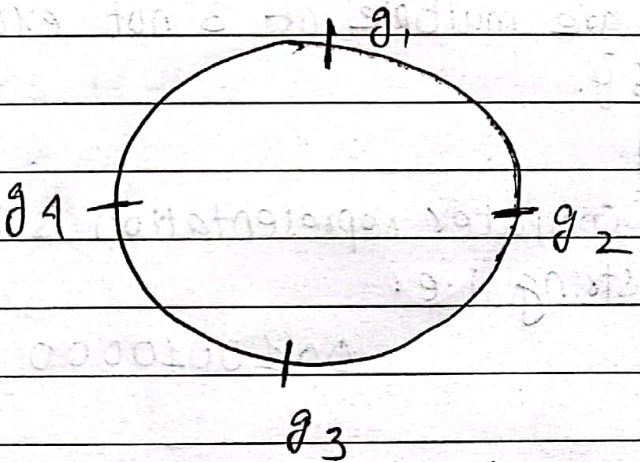
$$\therefore (1+y)^6 = 1\times(1)^6 + 6\times(1)^5.y + 15\times(1)^4 y^2 + 20\times(1)^3\times(y)^3 + 15\times(1)^2\times(y)^4 + 6\times(1)^1\times(y)^5 + 1\times(1)^0\times(y)^6$$

$$= 1 + 6y + 15y^2 + 20y^3 + 15y^4 + 6y^5 + y^6.$$

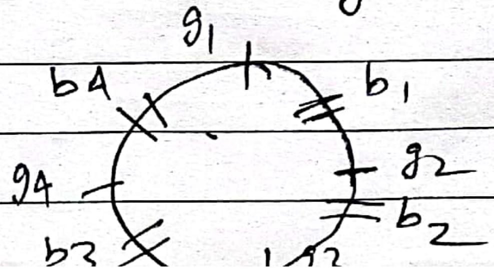
Definition also asked.

Q) In how many ways can 4 girls and 4 boys be arranged alternatively on a round table?

→



4 girls can be arranged on a table in $(4-1)!$ $= 3!$ ways. But 4 boys between the girls can be arranged in $4!$ ways, since a girl is already been fixed. Hence, 4 girls and 4 boys can be arranged alternatively on a round table is $3! \times 4!$ ways i.e. 144 ways.



2075 QSN

* Consider a set U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. What will be the computer representation for set containing the numbers which are multiple of 3 not exceeding 6? Describe injective, Subjective and bijective function with examples.

→ Given,

set U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Here, we know the set containing the numbers which are multiple of 3 not exceeding 6 is {3, 6}.

And

its computer representation is its corresponding bit string i.e.

0010010000

Hint ÷ 1,2 xaina so 00 vayo tespani 3 xa so 1 vayo ani 4,5 xaina so 0 0 vayo 6 xa so 1 vayo tespani 7 8 9 10 xaina so 0000 vayo

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Date

Page

Let A and B be two non-empty sets. A function f from A to B is a set of ordered pairs with the property that for each element $x \in A$ there is a unique element $y \in B$.

1) Injective function :-

A function $f: A \to B$ is said to be injective (or one-to-one, or 1-1) if for any $x, y \in A$, $f(x) = f(y)$ implies $x = y$.

All this means is that for a function to be one-to-one every distinct element in A has a distinct image in B.

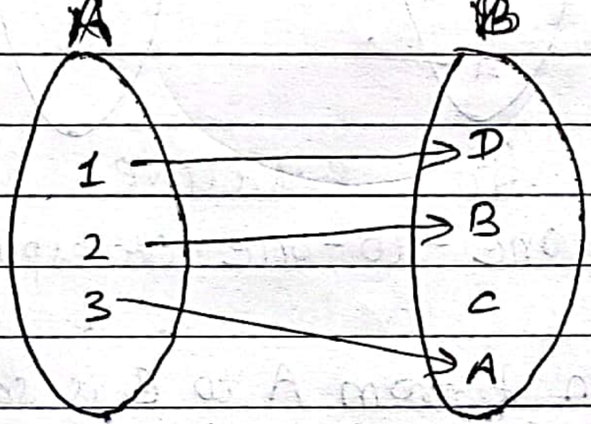

fig :- Injective

(If not understood watch trev tutor youtube).

2) Subjective function :-

If every element of B has a corresponding element in A such that $f(x) = y$. then it is subjective function or onto function.
It is not required that x is unique ; The function f may map one or more elements of A to the same element of B.

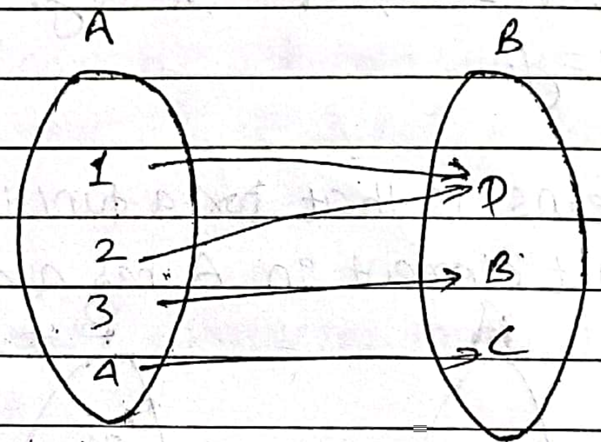

fig:- subjective

3) Bijective (one-to-one correspondence)

A function f from A to B is said to be bijective if it is both injective and subjective.

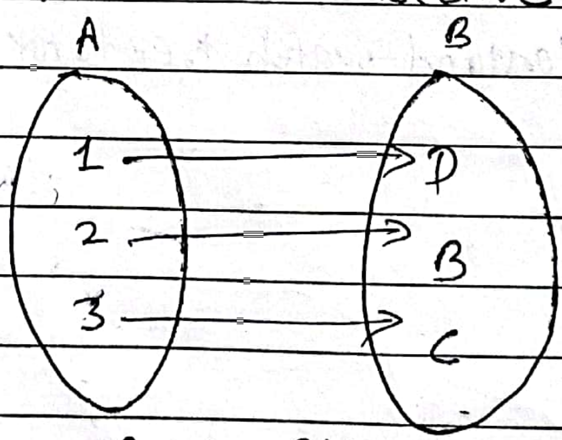

Fig:- Bijective

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Date
Page

If function is bijective, its inverse exists.

* Compute the following:-

a) 3 mod 4 b) 7 mod 5 c) -5 mod 3
d) 11 mod 5 e) -8 mod 6

Write down the recursive algorithm to find the value of b^n and prove its correctness using induction.

→ a) 3 mod 4

Here, $a = 3$ = dividend

$n = 4$ = divisor

$a = q \times n + r$ (By division algorithm) $(0 \leq r < n)$

$\Rightarrow 3 = (q \times 4) + r$

$\Rightarrow 3 = (0 \times 4) + 3$

$\therefore r = 3$ Hence, $3 \bmod 4 = 3$

b) 7 mod 5

Here, $a = 7$

$n = 5$

$a = q \times n + r$ (By division algorithm)

$\Rightarrow 7 = (q \times 5) + r$ $(0 \leq r < n)$

$\Rightarrow 7 = (1 \times 5) + 2$

$\therefore r = 2$ Hence, $7 \bmod 5 = 2$.

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c) $-5 \bmod 3$

Here, $a = -5$ = dividend
$n = 3$ = divisor

$a = q \times n + r$ (By division algorithm).
$(0 \le r < n)$

$\Rightarrow -5 = (q \times 3) + r$

$\Rightarrow -5 = (-2 \times 3) + 1$

$\therefore r = 1$ $\because (0 \le r < n)$.

$\therefore -5 \bmod 3 = 1$

d) $11 \bmod 5$

Here, $a = 11$
$n = 5$

$a = q \times n + r$ (By division algorithm)

$\Rightarrow 11 = (q \times 5) + r$

$\Rightarrow 11 = (2 \times 5) + 1$

$\therefore r = 1$

$\therefore 11 \bmod 5 = 1$.

e) $-8 \bmod 6$

$a = -8$
$n = 6$.

$a = q \times n + r$

$-8 = (q \times 6) + r$

$\Rightarrow -8 = (-2 \times 6) + 4$

$\therefore r = 4$

$\therefore -8 \bmod 6 = 4$

Recursive algorithm ~~for~~ for b^n.

→ procedure power (b is non zero real number, n is non-negative integer).

The power function can be defined recursively as:-

a) Base case : $b^0 = 1$

b) Recursive definition : $b^n = b \cdot b^{n-1}$ for $b > 1$.

Power (b, n)

1. Start
2. if (n==0)
   return 1;
   else
   return b * power (b, n-1)
3. End

Note: An algorithm ... a problem by reducing it into instance of the same problem with smaller input.

Example:

5^4

Here, $b=5$ $n=4$

$\because n \neq 0$ we get, $5 \times$ power $(5,3)$

Again, $b=5$, $n=3$, $\because n \neq 0$

we get, $5 \times 5 \times$ power $(5,2)$

and then $5 \times 5 \times 5 \times$ power $(5,1)$

and then $5 \times 5 \times 5 \times 5 \times$ Power$(5,0)$

and then $5 \times 5 \times 5 \times 5 \times 1$

and then 625 is the output / result.

Proof by using induction:

1) Base case:

If $n=0$, $b^0 = 1$. (for every non-zero real number b).

Here,

Power $(b,0) = 1$

2) Induction hypothesis:

~~Let power (b, k) =~~

Assume power $(b, k) = b^k$ for all $b \neq 0$ and k is +ve integer.

3) Inductive step :-

Now, To show: $power(b, k+1) = b^{k+1}$

By induction hypothesis (step 2) :-

$power(b, k+1) = b \cdot power(b, k)$
$= b \cdot b^k$
$= b^{k+1}$

proved.

2075
7) Here,
A = "Aldo is italian"
B = "Bob is English".

a) Aldo isn't italian
$\Rightarrow \neg A$

b) Aldo is Italian while Bob is English.
$\Rightarrow A \land B$ (while means AND.)

c) If Aldo is Italian then ~~BoB~~ Bob is not English.
$\Rightarrow A \rightarrow \neg B$

d) Aldo is Italian or if Aldo isn't Italian then Bob is English.

$\Rightarrow$ $A \vee (\neg A \rightarrow B)$.

e) Either Aldo is Italian and Bob is English, or neither Aldo is Italian nor Bob is English.

$\Rightarrow$ $(A \wedge B) \vee (\neg A \wedge \neg B)$

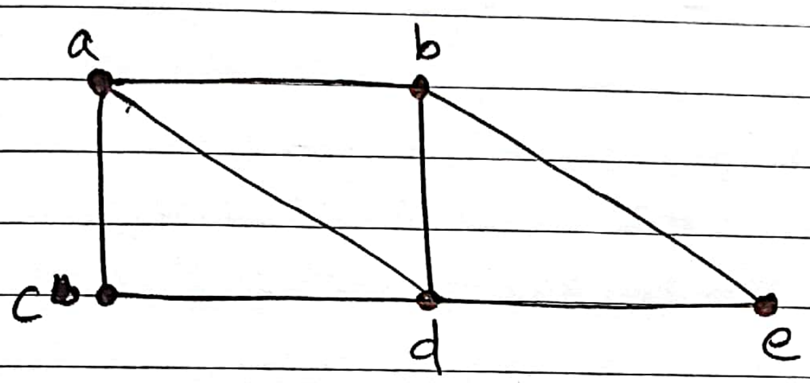
Logically equivalent to $A \Leftrightarrow B$.

Q8) Define Eular path and Hamiltonian path with examples. Draw the Hasse diagram for the divisible relation on the set $\{1, 2, 5, 8, 16, 32\}$ and find the maximal, minimal, greatest and least elements if exist.

$\rightarrow$ Eular path :-

A simple path in a graph G that passes through every edge once and only once is called eular path. An eular circuit is an Eular path that returns to its starting vertex.

A connected multigraph has an Eular path but not an eular circuit if and only if it has at most (max) two vertices of odd degree.

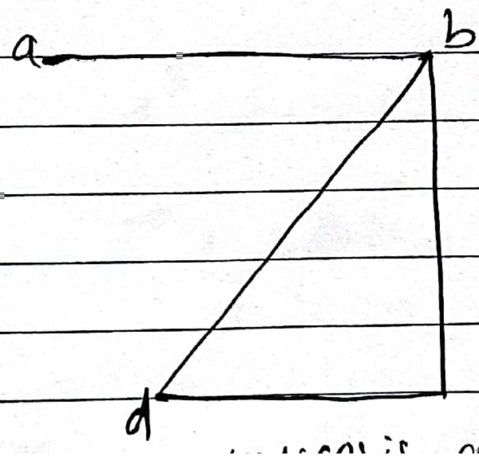


a → c → d → e → b → d → a → b.

Here, it passes through all edges only once.
No edge is repeated.

Hamiltonian path :-

A simple path in a graph G that passes through every vertex exactly once is called hamiltonian path.



a, b, c, d
or a, b, d, c.

Here, it passes through every vertices and no vertex is repeated. So it is H-path.

Watch this part in pdf of discrete 2075 solution.

Also see gupta tutorial.

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Date ______

Page ______

Q9) What does Primality testing means? Describe how Fermat's little theorem tests for a prime number with suitable example.

→ ~~When you~~ primality testing is the test for determining the given number if that is prime or not. To check a given number if it is prime or not there are several tests. Among them fermat's little theorem is one.

fermat's little theorem ÷
Is p prime?

Test: $a^p - a$ → 'p' is prime if this is a multiple of 'p' for all $1 \leq a < p$.

Example ÷

Is 5 prime?

Here, $p = 5$

$a = (1 \leq a < p)$

$= 1, 2, 3, 4$

for $a = 1$,

$1^5 - 1 = 0$ (multiple of 5)

for $a = 2$,

$2^5 - 2$

$= 30$ (multiple of 5)

for $a = 3$,

$3^5 - 3 = 240$ (multiple of 5)

for $a = 4$, $4^5 - 4 = 1020$ = multiple of 5

Q10) List any two applications of conditional probability. you have 9 families you would like to invite to a wedding. Unfortunately, you can only invite 6 families. How many different sets of invitations could you write?

→ The probability that an event A occurs given that event E has already occured written as p (A/E) and read as the conditional probability of A given E is

$$p\left(\frac{A}{E}\right) = \frac{P(A \cap E)}{P(E)}, \quad P(E) > 0.$$

Its two applications are :-

(i) Diagnosis of medical conditions (sensitivity / specificity).

(ii) Data analysis and model comparison.

(iii) In Baye's theorem and Markov process.

Here, given

Total families = 9

Families I can invite = 6

Total set of invitation I could write = $9C6$

$$= \frac{9!}{6!\,(9-6)!}$$

$$= \frac{9!}{6! \times 3!}$$

$$= \frac{9 \times 8 \times 7 \times \cancel{6!}}{\cancel{6!} \times 3!} = \frac{\overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}} \times 7}{\cancel{3} \times \cancel{2}} = 84$$

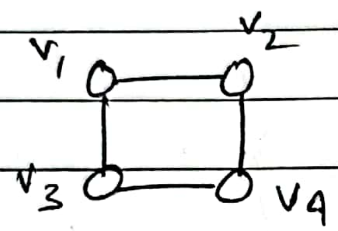
Therefore, 84 different set of invitations can be written by me.

Q11) Define spanning tree and minimum spanning tree. Mention the conditions for two graphs for being isomorphic with an example.

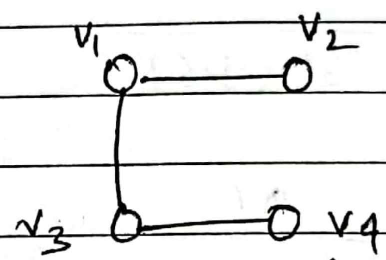
→ A connected subgraph 'S' of Graph G(V,E) is said to be spanning tree iff,

1) 'S' should contain all vertices of 'G'.
2) 'S' should contain (|V|-1) edges.

Example:



Graph G.

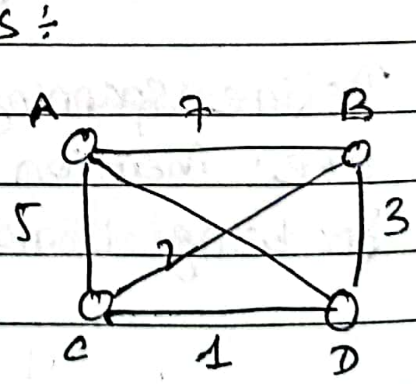


Spanning tree

A minimum spanning tree is a subgraph of a weighted graph G that includes all vertices of graph G and has minimum weight than all other spanning trees of the same graph.

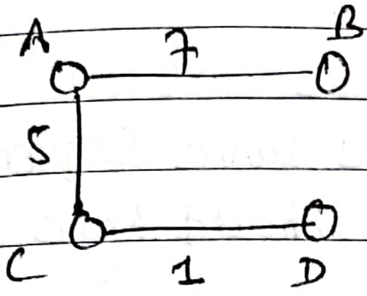
Suppose a weighted graph is:

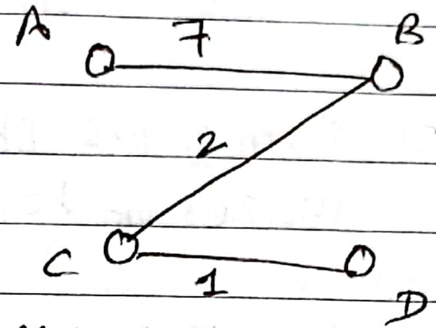
No. of vertices (n) = 4
So all spanning trees should have 3 edges.



possible spanning trees $= n^{n-2}$
$= 4^2 = 16$

Let us assume the ~~three~~ two spanning trees are:-





weight = 7+5+1
= 13

weight = 7+2+1
= 10

Similarly, other 14 spanning trees that are remaining are drawn and among them the tree with the minimum weight is called minimum spanning tree.

Two graphs are said to be isomorphic if they are perhaps the same graphs just drawn differently with different names i.e. they have identical behaviour for any graph theoretic property.

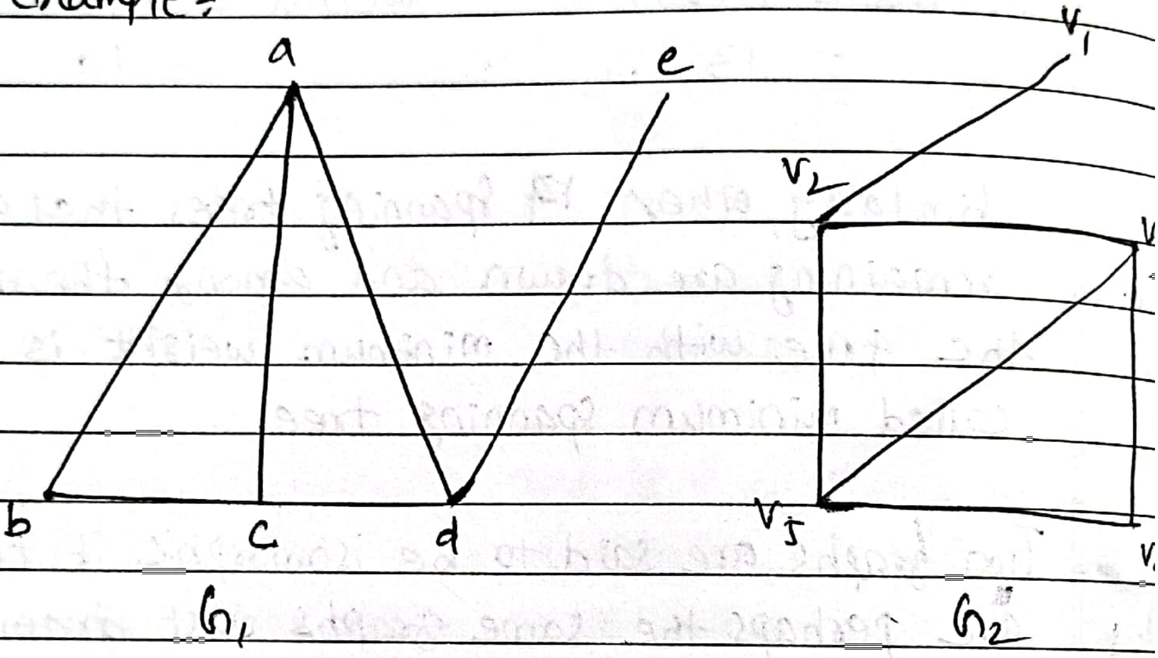
Some conditions for two graphs to be isomorphic are:-

1) Both graphs should have same no. of vertices i.e. $|V_1| = |V_2|$.

2) No. of edges of both graphs should be equa[l]
i.e. $|E_1| = |E_2|$

3) Both graphs should have same sequences and vertices of both graphs should have same degree

Example:



(i) Both have same no. of vertices.
(ii) Both have same no. of edges.

degrees

G_1		G_2	
deg (a) =	3	deg (v_2) =	3
deg (b) =	2	deg (v_4) =	2
deg (c) =	3	deg (v_5) =	3
deg (d) =	3	deg (v_3) =	3
deg (e) =	1	deg (v_1) =	1

Hence, they are isomorphic.

* Pigeon hole principle:-

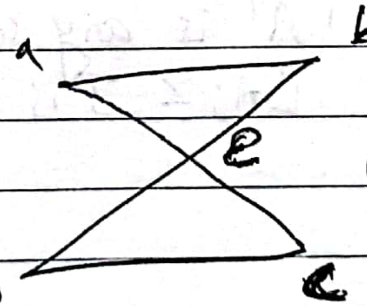
The pigeon hole principle states that, if $k+1$ or more pigeons are placed into k pigeonholes, then there is at least one pigeonhole containing two or more of the pigeons.

Eg:- If one has three gloves, then one must have at least two right-hand gloves or at least two left hand gloves.

* Euler circuit:-

An Euler circuit is a circuit made using a path where every edge of the graph G is used exactly one and only once and the path returns to its starting vertex. A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degrees.

Eg:-



(a-e-c-d-e-b-a)

It is an Euler circuit because starting and ending vertex are same. (a-e-c-d-e-b

* Logic and proof methods:-

learn from youtube if you get time.

2076 Q 5)

Hamro csit see

or Ankit pandeni

note see

pdf ps downloaded.

* Posets and Identification. (partial ordering Relation)

A relation R is said to be a partial order relation if R is reflexive, anti-symmetric and transitive.

Partially order set (POSET):- A set 'A' with partial ordering relation 'R' defined on 'A' is called POSET. denoted by [A; R]

Eg:- If 'A' is any set of real number then [A; ≤] is a poset.

Q) Which of the following are posets?

a) $(Z, =)$
b) $(Z, \neq)$
c) $(Z, \subseteq)$
d) $(Z, \geq)$

a) $(Z, =)$

1. When $a \in Z$ then $a = a$ and thus relation is reflexive (Here $a = a \; \forall a \in Z$)

2. When $a = b$ and $b = a \; \forall a, b \in Z$ then $a = b$ and thus relation is antisymmetric.

3. When $a = b$ and $b = c \; \forall a, b, c \in Z$ then $a = c$ and thus relation is transitive.

So, $(Z, =)$ is a poset.

b) $(Z, \neq)$

1. When $a \in Z$, then $a \neq a$ thus relation is not reflexive. (Here $a \neq a \; \forall a \in Z$).

2. When $a \neq b$ and $b \neq a \; \forall a, b \in Z$ then $a \neq b$ and thus relation is not antisymmetric.

3. When $a \neq b$ and $b \neq c \; \forall a, b, c \in Z$ then $a \neq c$ and thus relation is not transitive.

So, $(Z, \neq)$ is not a poset.

c) $(Z, \subseteq)$

1. When $a \in Z$, then $a \subseteq a$ thus relation is reflexive. (Here $a \subseteq a \ \forall a \in Z$).

2. When $a \subseteq b$ and $b \subseteq a \ \forall a, b \in Z$ then $a \subseteq b$ and thus relation is antisymmetric.

3. When $a \subseteq b$ and $b \subseteq c \ \forall a, b, c \in Z$ then $a \subseteq c$ and thus relation is transitive.

So, $(Z, \subseteq)$ is a poset.

d) $(Z, \geq)$

1. When $a \geq Z$, then $a \geq a$ thus relation is reflexive (Here, $a \geq a \ \forall a \in Z$).

2. When $a \geq b$ and $b \geq a \ \forall a, b \in Z$ then $a = b$ and thus relation is antisymmetric.

3. When $a \geq b$ and $b \geq c \ \forall a, b, c \geq Z$ then $a \geq c$ and thus relation is transitive.

So, $(Z, \geq)$ is a poset.

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Q7) Reflexive closure and symmetric closure.
Find the remainder when $4x^2 - x + 3$ is divided by $x+2$ using remainder theorem.

→ Suppose, for example, R is not reflexive. If so, we could add missing ordered pairs to this relation to make it reflexive which is called reflexive closure i.e. the reflexive closure of a relation R on set A is obtained by adding (a, a) to R to each $a \in A$.

It is denoted by $R \cup \Delta$, where $\Delta = \{(a, a) : a \in A\}$.

Similarly, the symmetric closure of R is obtained by adding (b, a) to R for each $(a, b) \in R$. It is denoted by $R \cup R^{-1}$, $R^{-1} = \{(b, a) :$ to each $a, b \in R\}$.

The transitive closure of R is obtained by repeatedly adding (a, c) to R for each $(a, b) \in R$ and $(b, c) \in R$..

Problem part :-

Let $p(x) = 4x^2 + x + 3$

putting $x + 2 = 0$

$x = -2$

Now, by remainder theorem,

$$\begin{aligned} \text{Remainder} &= p(-2) \\ &= 4 \times (-2)^2 + (-2) + 3 \\ &= 4 \times 4 - 2 + 3 \\ &= 16 - 2 + 3 \\ &= 19 - 2 \\ &= 17 \end{aligned}$$

$\therefore$ By remainder theorem, the remainder when $4x^2 + x + 3$ is divided by $x + 2$ is 17.

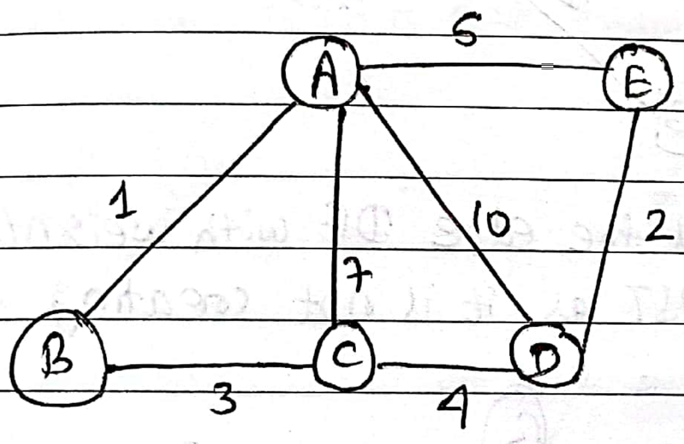
* Kruskal's Algorithm :-

It is an algorithm which finds a minimum spanning tree of a undirected edge-weighted graph. If a graph is connected, it finds minimum spanning tree.

It basically takes a connected weighted graph finds the subset of edges of that graph and forms a tree that includes every vertex which includes minimum sum of weights among all the spanning trees that can be formed from the graph.

Example:

The minimum spanning tree of following using Kruskal Algorithm:



The weight of the edges of the above graph is given in the below table:

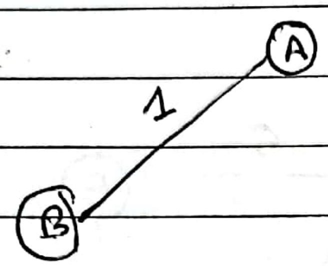
Edge	AB	AC	AD	AE	BC	CD	DE
weight	1	7	10	5	3	4	2

Now, sort the edges given above in the ascending order of their weights.

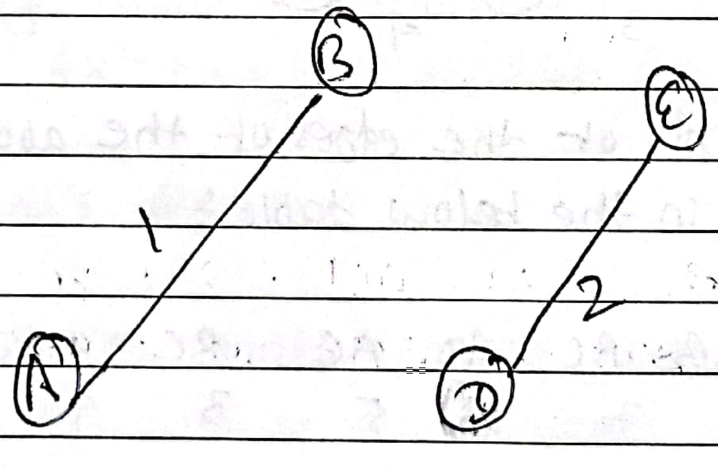
Edge	AB	DE	BC	CD	AE	AC	AD
weight	1	2	3	4	5	7	10

Now, let's start constructing the minimum spanning tree.

Step 1 - First, add the edge AB with weight 1 to the MST (minimum spanning tree).

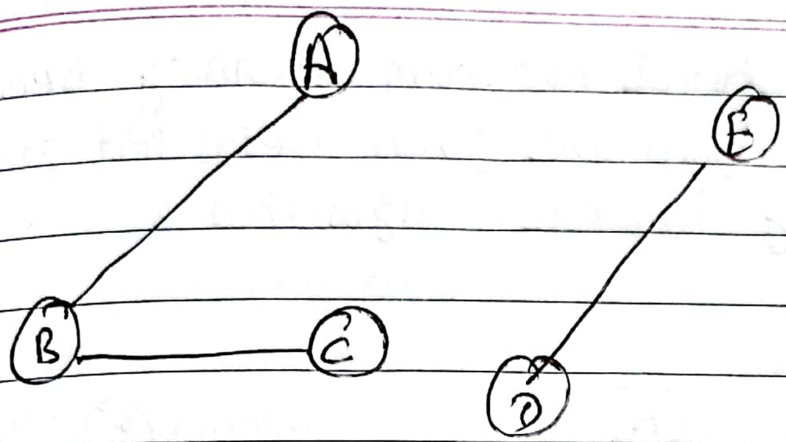


Step 2 - Add the edge DE with weight 2 to the MST as it is not creating the cycle.

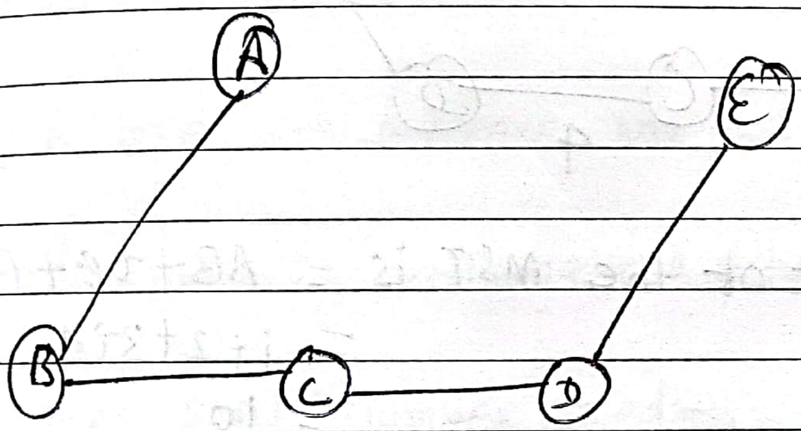


Step 3 :- Add the edge BC with weight 3 to the MST, as it is not creating the cycle or loop.

Date
Page



Step4 :- Now, pick the edge CD with weight 4 to the MST, as it is not forming the cycle.

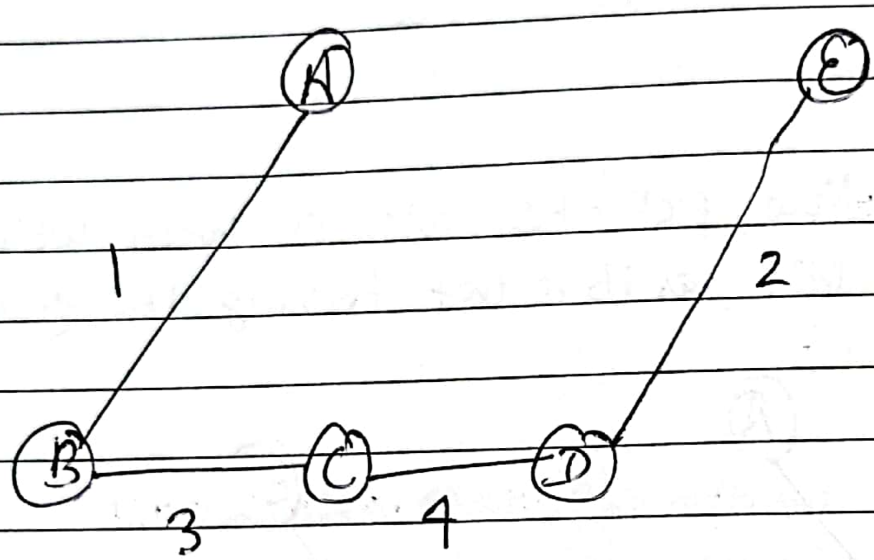


Step 5 :- After that, pick the edge AE with wt. 5. Including this edge will create the cycle. So discard it.

Step 6 :- Pick the edge AC with wt. 7, including this edge will create the cycle. So discard it.

Step 7 :- Pick the edge AD with wt. 10, Including this edge will also create the cycle, so discard it.

So, the final minimum spanning tree obtained from the given weighted graph by using Kruskal's algorithm is



The cost of the MST is = AB + DE + BC + CD

= 1 + 2 + 3 + 4

= 10.

Q) List any two applications of graph coloring theorem. Prove that "A tree with n vertices has n-1 edges".

→ The process of assigning a color to each vertex of a simple graph so that no two adjacent vertices are assigned to the same cover is called graph coloring.

Some applications of graph coloring is :-

(i) Making schedule on time table.
(ii) Mobile radio frequency assignment.
(iii) Sudoku
(iv) Register Allocation.
(v) Bipartitle Graph.
(vi) Map coloring.

Prove part :-

" A tree with n vertices has n-1 edge ".

Let's prove by mathematical induction

Assume p(n): Number of edges = n-1 for all tree with n vertices.

Basic step :-

P(1): for one vertex, there will be zero edges. So, it is true.

P(2): for two vertices, there will be one edge to connect them.

P(3): for three vertices, there should be two edge to connect them. So, it is true.

Inductive method :-

Let us assume for any arbitrary k, $p(k)$ is true i.e. for k no of vertices, no. of edges $= k-1$

Now, for $(k+1)$
the no. of edges will be $(k-1)$ + no. of edges required to add $(k+1)^{th}$ vertex.

Every vertex that is added to the tree contributes one edge to the tree.

Thus, no of edges required to add $(k+1)^{th}$ node $= 1$.

So, total no. of edges will be $= (k-1)+1$
$= k-1+1$
$= k$

Thus, $p(k+1)$ is true.

So, by using principle of mathematical induction it is proved that for n vertices, no. of edges $= n-1$.

classmate

Date

Page

Q) Define ceiling and floor function. Why do we need Inclusion - Exclusion principle? Make it clear with suitable example.

→ Let x be a real number. The floor function of x denoted by $\lfloor x \rfloor$ or floor (x), is defined to be the greatest integer that is less than or equal to x.

for example: $\lfloor 1.4 \rfloor = 1$ and $\lfloor 5 \rfloor = 5$

The ceiling function of x denoted by $\lceil x \rceil$ or ceil (x), is defined to be the greatest integer that is greater than or equal to x.

for example: $\lceil 5 \rceil = 5$ and $\lceil 1.5 \rceil = 2$

* Inclusive - Exclusive Principle :-

Next page see

It is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two or more finite steps.

Symbolically represented as

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

We need inclusion-exclusion Principle to distinguish the number of distinct elements which exist in a set.

An example to distinguish the number of integers in $\{1, 2, 3, 4 ----, 100\}$ that are divisible by 2, 3, or 5 is

Let $A = \{$an integer that is divisible by 2$\}$

$\therefore |A| = 50$

$B = \{$an integer that is divisible by 3$\}$

$\therefore |B| = 33$

$C = \{$an integer that is divisible by 5$\}$

$\therefore |C| = 20$

Now, $A \cap B = \{$divisible by 2 and 3$\} = 16$.

Similarly,

$|B \cap C| = 6$

$|A \cap C| = 10$

$|A \cap B \cap C| = 3$

By inclusion exclusion principle,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 50 + 33 + 20 - 16 - 10 - 6 + 3$

$= 74$

Hence, we distinguish the no. of integers that are divisible by 2, 3 or 5 using inclusion-exclusion principle.

Practice 2078 questions:-

QNo. 1 (DY).

QNo. 2 (skip if want

if time see youtube).

Q No 7 (same as above).

Q4) What is the coefficient of x^2 in $(1+x)^{11}$?

Describe how relation can be represented using matrix.

Solution:-

In short:

$$^{11}C_2 = \frac{11!}{2!(11-2)!} = 55.$$

Explanation:

In the example of $(a+b)^n$, the $(r+1)^{th}$ term denoted by t_{r+1} is given by:-

$$t_{r+1} = {}^nC_r \cdot a^{n-r} \cdot b^r$$

letting $a=1$, $b=x$ and $n=11$, we get,

$t_{r+1} = {}^{11}C_r \cdot 1^{11-r} \cdot x^r$

As, we need the coefficient of x^2, we have to take $r=2$.

$${}^{11}C_2 = \frac{11!}{2! \times (11-2)!} = 55$$

The coefficient of x^2 in $(1+x)^{11}$ is 55.

* Representation of Relation using Matrix:-

Let $A = \{a_1, a_2, ---, a_m\}$ and $B = \{b_1, b_2, --, b_n\}$ are finite sets containing m and n elements respectively. and set R be a relation from A to B then R can be represented by the mn matrix.

$$M_R = [m_{ij}] \text{ where } m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \ (a_i, b_j \notin R \end{cases}$$

The matrix M_R is called matrix of R.

Eg:-

Let $A = \{1, 3, 4\}$, $R = \{(1,1), (1,3), (3,3), (4,4)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

classmate
Date
Page

* Define zero - one matrix.

→ A matrix with entries that either 0 or 1 is called a zero - one matrix.

zero- one matrix are often used to represent discrete structures.

The boolean arithmetic is based on the boolean operations $\vee$ and $\wedge$, which operate on pairs of bits defined by:-

$$b_1 \wedge b_2 \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Q12) Represent any three set operations using venn-diagram. Give a recursive defined function to find the factorial of any given +ve integer

Part a)

Let take an example of three sets.

$X = \{1, 2, 5, 6, 7, 9\}$, $Y = \{1, 3, 4, 5, 6, 8\}$

and $Z = \{3, 5, 6, 7, 8, 10\}$.

Now, we will find required values such as $X \cap Y \cap Z$, $X \cap Y$ and $Y \cap Z$

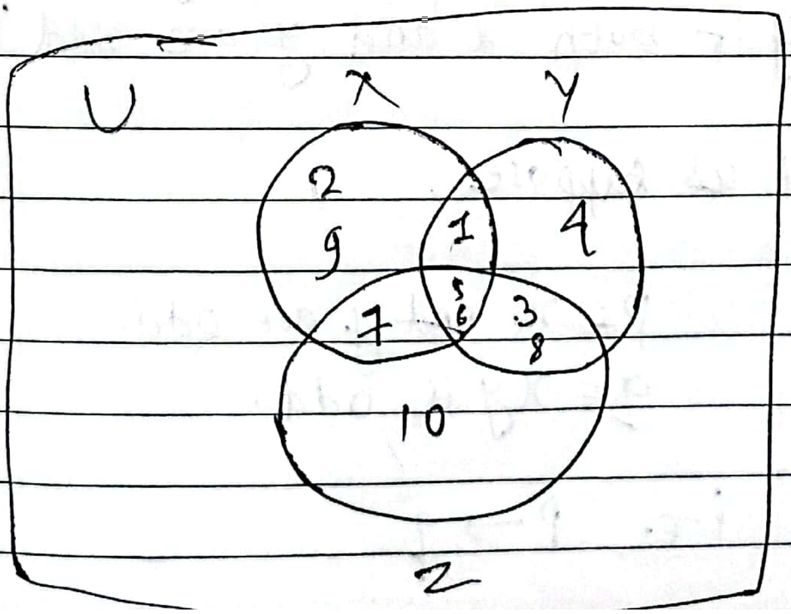
$X \cap Y \cap Z = \{5, 6\}$

$X \cap Y = \{1, 5, 6\}$

$Y \cap Z = \{3, 5, 6, 8\}$

$X \cap Z = \{5, 6, 7\}$

The venn-diagram of above set is :-



Part b)

If $n \in Z^+$ then factorial is defined as:-

$n! = n \cdot (n-1)(n-2) \cdot (n-3) \cdots 3 \cdot 2 \cdot 1.$

The recursive defined function to find factorial of any given +ve integer is

$$\begin{cases} f(n) = n \times f(n-1) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

(Also see book)

Q12 2075

Prove that the product xy is odd if and only if both x and y are odd integers.

→ Let us suppose,

p = x and y are odd.
q = xy is odd.

i.e. $p \rightarrow q$.

classmate

Date

Page

First case : $p \to q$ (direct proof)

If x and y are odd then xy is odd. By the definition of odd integers, $\exists$ integers a, b.

$x = 2a + 1$ and $y = 2b + 1$

$$
\begin{aligned}
xy &= (2a+1)(2b+1) \\
&= 4ab + 2a + 2b + 1 \\
&= 2(2ab + a + b) + 1 \\
&= 2c + 1, \text{ for some integer } c.
\end{aligned}
$$

Here, x is also odd by direct proof.

Second case : $\neg q \to \neg p$ (indirect proof)

The product xy is odd if and only if x and y are odd.

p = product xy is odd.
q = x and y are odd.

Let us suppose, x and y are even.
Now, we prove $\neg q \to \neg p$

By definition of even integers $\exists$ integers a, b

$x = 2a$ and $y = 2b$

$xy = 2a \times 2b$

$= 4ab$

$= 2(2ab)$

$= 2c$ for some integer c

So, xy is even.

Hence, by contraposition we can say xy is odd.

Therefore, By direct proof and contraposition, the product xy is odd if and only if x and y is odd.

Q) What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1?

Q) In how many ways can 4 girls and 4 boys be arranged alternatively on a round table?

Q) Show that if there are 30 students in a class, then at least two have same names that begin with the same letter.

Q) Also see model qsn circled one first and then others.

→ Here,

No. of students i.e. pigeons or objects $(N) = 30$

No. of pigeon holes (K) = letters of the alphabet $= 26$.

Now, According to pigeon hole principle,

If N objects are placed into K boxes, then there is at least one box containing at least

$$\left\lceil \frac{N}{K} \right\rceil \text{ objects}$$

so, $\left\lceil \frac{30}{26} \right\rceil = 2$

Hence, by the pigeonhole principle, there are at least two students who have a last name that begin with the same letter.

Q) Here,

Let E be the event that a bit string of length four contains at least two consecutive 0s and F be the event that the first bit is 1.

The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 1 equals $P(E/F)$.

Without restriction the number of ways of bit string of length four can be formed with digits 0 and 1 is $2^4 = 16$.

Hence, $n(S) = 16$.

Keeping 1 fixed at the beginning, there are 8 bit strings of length four that starts with 1. Thus, $n(F) = 8$.

$$p(F) = \frac{n(F)}{n(S)} = \frac{8}{16} = \frac{1}{2}$$

Again, $E \cap F = \{1000, 1100, 1001\}$, i.e. $n(E \cap F) = 3$

So $p(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{3}{16}$

Consequently,

$$P(E/F) = \frac{p(E \cap F)}{p(F)} = \frac{3/16}{1/2} = 3/8$$