TRIBHUVAN UNIVERSITY

Institution of Science and Technology

Bachelor Level/First Year/Second Semester/Science
Computer Science and Information Technology [MTH. 163]
(Mathematics II)

Full Marks: 80
Pass Marks: 32
Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

TU QUESTIONS-ANSWERS 2075

Group 'A'

Attempt any three questions: (3×10 = 30)

1. **When a system of linear equations is consistent and inconsistent? Give an example for each. Test the consistency and solve: x + y + z = 4, x + 2y + 2z = 2, 2x + 2y + z = 5.** [2+1+7]

Solution:

Definition (Consistent and Inconsistent System)

A system of linear equations is called consistent if it has solution (that may be one solution or infinitely many solutions) and called inconsistent if it has no solution.

Example: Consider a system

$$x_1 - 2x_2 = -1$$
$$x_1 - 3x_2 = -3$$

This system has solution (3, 2). So, the system is consistent and has unique solution.

Consider a system

$$x_1 - x_2 = 1$$
$$-3x_1 + 3x_2 = -3$$

Here second equation is the thrice time multiple of first. So, the system has infinite solutions and is consistent.

Consider a system

$$x_1 + 2x_2 = -1$$
$$x_1 + 2x_2 = 2$$

Here the equations represent parallel lines. So, the system has no solution. So, the system is inconsistent.

Problem Part:

Solution: Given system is;

$$x + y + z = 4$$
$$x + 2y + 2z = 2$$
$$2x + 2y + z = 5$$

The matrix notation of the system is,

$$\begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 1 & 2 & 2 & : & 2 \\ 2 & 2 & 1 & : & 5 \end{bmatrix}$$

Apply $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$ then the above matrix reduces to

$$\sim\begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 1 & 1 & : & -2 \\ 0 & 0 & -1 & : & -3 \end{bmatrix}$$

Apply $R_3 \to -R_3$ then the above matrix reduces to

$$\sim\begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 1 & 1 & : & -2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$ is triangular form.

The equation form of the matrix notation is,

$$x + y + z = 4 \quad \ldots \text{(i)}$$
$$y + z = -2 \quad \ldots \text{(ii)}$$
$$z = 3 \quad \ldots \text{(iii)}$$

From (iii), we get $z = 3$.

then (ii) gives, $y = -5$

And, (i) gives, $x = 6$

Thus, the solution of the given linear system is $(x, y, z) = (3, -5, 6)$.

2. What is the condition of a matrix to have an inverse? Find the inverse of the matrix, $\begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}$. [2+8]

Solution: If A is an invertible matrix then there is a matrix C such that

$$AC = I = CA.$$

In such case, C is called inverse of A and write as $C = A^{-1}$.

Condition for existence of inverse of a matrix.

Let A be a given matrix then the inverse of A i.e. A^{-1} exists if det (A) is non-zero.

Problem Part:

Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$

$$\text{Here, det (A)} = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{vmatrix}$$
$$= 1(25 + 24) + 2(-5 - 30) - 1(4 - 25)$$
$$= 49 - 70 + 21$$
$$= 0$$

So, the inverse of A does not exist.

3. Define linearly independent set of vectors with an example. Show that the vectors (1, – 4, 3), (0, 3, 1) and (3, –5, 4) are linearly independent. Do they form a basis? Justify. [2+5+3]

Solution: Definition (Linearly independent and dependent vectors)

An indexed set of vectors $\{v_1, v_2, \ldots v_p\}$ in V is said to be linearly independent if the vector equation.

$$c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0$$

has only the trivial solution, i.e. $c_1 = 0, c_2 = 0, \ldots c_p = 0$.

For otherwise, the vectors are linearly dependent.

Example: The set {(1, 0, 0), (0, 1, 0), (0, 0, 1)} is linearly independent vectors.

Problem Part:

Given vectors are (1, –4, 3), (0, 3, 1) and (3, –5, 4).

Here we have to show that v_1, v_2, v_3 are linearly independent and they span $\mathbb{R}^3$.

For linearly independent, $Ax = 0$

$$\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 3 & -5 & 4 & 0 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 7 & -5 & 0 \end{bmatrix} \text{[Applying } R_3 \to R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -22 & 0 \end{bmatrix} \text{[Applying } R_2 \to 3R_3 - 7R_2]$$

No basic variable so having a trivial solution. Thus v_1, v_2, v_3 are linearly independent.

For v_1, v_2, v_3 span R^3

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 0 & 3 & 1 \\ 3 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & -22 \end{bmatrix}$$

Each row has pivot so, column of A span R^3. So, $\{v_1, v_2, v_3\}$ span R^3.
Thus $\{v_1, v_2, v_3\}$ is basic for R^3.

4. **Find a least square solution of $Ax = b$ for $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix}$.**

Solution: Let,

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix}.$$

Here,

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{bmatrix}$$

and,

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 30 \\ 38 \end{bmatrix}.$$

Since we have the set of least squares solutions of $Ax = b$ coincides with the non-empty set of solutions of $A^TAx = A^Tb$.

Therefore,

$$\hat{x} = (A^TA)^{-1} (A^Tb) \qquad \text{...... (i)}$$

Here,

$$|A^TA| = \begin{vmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{vmatrix} = 84 \neq 0.$$

Then,

$$(A^TA)^{-1} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}.$$

Therefore (i) becomes,

$$\hat{x} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Group 'B'

Attempt any ten questions: (10×5 = 50)

5. Change into reduced echelon form of the matrix:

$$\begin{bmatrix} 0 & 3 & -6 \\ 3 & -7 & 8 \\ 3 & -9 & 12 \end{bmatrix}.$$

Solution: Here,

$$\begin{bmatrix} 0 & 3 & -6 \\ 3 & -7 & 8 \\ 3 & -9 & 12 \end{bmatrix}$$

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\sim \begin{bmatrix} 3 & -9 & 12 \\ 3 & -7 & 8 \\ 0 & 3 & -6 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & -4 \\ 0 & 3 & -6 \end{bmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1]$$

$$\sim \begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\text{Applying } R_3 \to R_3 - R_2]$$

$$\sim \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \text{Applying } R_1 \to \frac{1}{3} R_1 \\ R_2 \to \frac{1}{2} R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\text{Applying } R_1 \to R_1 + 2R_2]$$

This is the reduced echelon form of the original matrix.

6. Define linear transformation with an example. Is a transformation T: $\mathcal{R}^2 \to \mathcal{R}^3$ defined by T(x, y) = (3x + y, 5x + 7y, x + 3y) linear? Justify. [2 + 3]

Solution:

Definition (linear transformation)

A transformation $T : U \to V$ is linear if for $u, v \in U$ and for any scalar a

(i) $T(u + v) = T(u) + T(v)$

(ii) $T(au) = aT(u)$

Alternatively,

A transformation $T : U \to V$ is called linear if for any $u, v \in U$ and for any two scalars a, b,

$$T(au + bv) = aT(u) + bT(v)$$

Problem Part:

Let $T : R^2 \to R^3$ is defined by

$$T(x, y) = (3x + y, 5x + 7y, x + 3y)$$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ then $u, v \in R^2$

Also, let a, b are two scalars,

$$\begin{aligned} \text{Here, } T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= (3(ax_1 + bx_2) + (ay_1 + by_2), 5(ax_1 + bx_2) \\ &\qquad + 7(ay_1 + by_2), (ax_1 + bx_2) + 3(ay_1 + by_2)) \\ &= a(3x_1 + y_1, 5x_1 + 7y_1, x_1 + 3y_1) + b(3x_2 + y_2, 5x_2 \\ &\qquad + 7y_2, x_2 + 3y_2) \\ &= aT(x_1, y_1) + bT(x_2, y_2) \\ &= aT(u) + bT(v) \end{aligned}$$

This means T is linear.

7. **Let $A = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix}$. What value(s) of k, if any, will make AB = BA?**

Solution: Let $A = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix}$.

Here,

$$AB = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix} = \begin{bmatrix} -9-2k & 0 \\ 45+9k & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k-5 & -2k-9 \end{bmatrix}$$

And, suppose AB = BA. That is,

$$\begin{bmatrix} -9-2k & 0 \\ 45+9k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k-5 & -2k-9 \end{bmatrix}.$$

This, implies,

$-9-2k = 1 \Rightarrow k = -5.$

$45 + 9k = -k - 5 \Rightarrow k = -5$ (which is same as above).

Thus, at $k = -5$, we get AB = BA.

8. **Define determinant. Evaluate without expanding $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$.**

Solution: Definition (Determinant)

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ of n terms of the form

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

Problem Part:

Here,

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - 3R_2]$$

$= (1)(1)(3)$ [Multiple leading diagonal entries]

$= 3$

9. **Define subspace of a vector space. Let $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, s, t \in R \right\}$. Show that H is a subspace of $\mathcal{R}^3$.**

Solution: Definition (Vector Subspace)

Let V be a vector space over the field K. Then a non-empty subset W of V is called a subspace of V if W satisfies the conditions:

(i) $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$.

(ii) $aw \in W$ for all $w \in W$, $a \in K$.

(iii) $0 \in W$.

Problem Part:

Let $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, s, t \in R \right\}$.

(i) Taking, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, is an zero element in W.

(ii) For all $\alpha, \beta \in R$ and $w_1 = \begin{bmatrix} s_1 \\ t_1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} s_2 \\ t_2 \\ 0 \end{bmatrix} \in W$ then

$$\alpha w_1 + \beta w_2 = \begin{bmatrix} \alpha s_1 + \beta s_2 \\ \alpha t_1 + \beta t_2 \\ 0 \end{bmatrix} \in W.$$

Hence, W is a subspace of V.

10. Find the dimension of the null space and column space of

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Solution: Let,

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 13 & 26 & -26 \end{pmatrix} \quad \begin{pmatrix} \text{Applying} \\ R_2 \to 3R_2 + R_1 \\ R_3 \to 3R_3 + 2R_1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{pmatrix} \quad \begin{pmatrix} \text{Applying} \\ R_2 \to \frac{1}{5} R_2 \text{ and } R_3 \to \frac{1}{13} R_3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \text{Applying} \\ R_3 \to R_3 - R_2 \end{pmatrix}$$

In the echelon form of A, there are three free variables x_2, x_4 and x_5. So, the dimension of Nul A is 3. Also, it has two pivot columns that is first and third column, so dim Col A is 2.

11. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

Let λ be a scalar value such that

$$\det(A - \lambda I) = 0$$

i.e. $\begin{vmatrix} 6-\lambda & 3 & -8 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (6-\lambda)(2-\lambda)(-3-\lambda) + 1[3(-3-\lambda)] = 0$

$\Rightarrow \quad (3+\lambda)[(6-\lambda)(2-\lambda) + 3] = 0$

$\Rightarrow \quad (3 + \lambda)(\lambda^2 - 8\lambda + 15) = 0$

$\Rightarrow \quad (3 + \lambda)(\lambda - 5)(\lambda - 3) = 0$

This gives, $\lambda = 3, 5, -3$

So, the eigen values corresponding to A are $\lambda = 3, 5, -3$.

And the eigen vector X corresponding to A at the eigen values λ is,

$Ax = \lambda x$

$$\Rightarrow \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \ldots\ldots\ldots\ldots (1)$$

At $\lambda = 3$,

$6x_1 + 3x_2 - 8x_3 = 3x_1$

$-2x_2 = 3x_2$

$x_1 - 3x_3 = 3x_3$

These implies $\quad \ldots\ldots\ldots\ldots (2)$

$3x_1 + 3x_2 - 8x_3 = 0$

$5x_2 = 0 \quad \ldots\ldots\ldots (3) \quad \ldots\ldots\ldots (4)$

$x_1 - 6x_3 = 0$

From (3), $x_2 = 0$

From (4) $x_1 = 6x_3$

From (2), $x_3 = \frac{1}{8}(3x_1 + 3x_2) = \frac{18x_3}{8}$

$\Rightarrow \quad 10x_3 = 0$

$\Rightarrow \quad x_3 = 0$

Then, $x_1 = 0$

Thus, $x = (x_1, x_2, x_3) = (0, 0, 0)$ be eigen vector of at $\lambda = 3$.

Next at $\lambda = 5$,

$6x_1 + 3x_2 - 8x_3 = 5x_1$

$-2x_2 = 5x_2$

$x_1 - 3x_3 = 5x_3$

These implies

$x_1 + 3x_2 - 8x_3 = 0 \quad \ldots\ldots\ldots (5)$

$7x_2 = 0 \quad \ldots\ldots\ldots (6)$

$x_1 - 8x_3 = 0 \quad \ldots\ldots\ldots (7)$

From (6), $x_2 = 0$

From (7), $x_1 = 8x_3$

From (5), $8x_3 = x_1 + 3x_2 = 8x_3$

$\Rightarrow \quad 0 = 0$

That is x_3 is free.

Therefore, $x = (8x_3, 0, x_3)$ be the eigen vector of A at $\lambda = 5$.

Next $\lambda = -3$

$6x_1 + 3x_2 - 8x_3 = -3x_1$

$-2x_2 = -3x_2$

$-3x_3 = -3x_3$

This implies,

$9x_1 + 3x_2 - 8x_3 = 0 \quad \ldots\ldots\ldots (8)$

$x_2 = 0$

$x_1 = 0$

And (i) gives $x_3 = 0$

Therefore, $x = (x_1, x_2, x_3) = (0, 0, 0)$ be eigen vector of A at $\lambda = -3$.

12. Find the LU factorization of the matrix $\begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix}$.

Solution: Let,

$$A = \begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix} \sim \begin{pmatrix} 2 & 5 \\ 0 & -22 \end{pmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 - 3R_1 \end{bmatrix}$$
$$= U.$$

Here, U has pivot values 2 in first column, –22 in second column.
The entries of column of pivot value and to be reduced value which are determine the row reduction of A to U are,

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} \quad [-22]$$
$$\div 2 \quad \div(-22)$$
$$\downarrow \quad \downarrow$$
$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad [1]$$

Therefore L is,

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

Thus,

$$A = \begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 0 & -22 \end{pmatrix} = LU.$$

13. Define group. Show that the set of all integers Z forms group under addition operation.

Solution: Definition (Group)

Let $(G, *)$ be a binary structure then G is said to be a group with the binary operation $*$ if the following conditions are satisfied.

1. Closure: For all $a, b \in G$ then $a*b \in G$.
2. Associativity: For all $a, b, c \in G$ then $(a * b) * c = a * (b * c)$.
3. Existence of identity element: For any $a \in G$ there exist an element $e \in G$ such that, $a* e = e * a = a$.
4. Existence of inverse element: For any $a \in G$ there exists $a' \in G$ such that $a * a' = e = a' * a$.

where * is additive or multiplicative operation.

Problem Part:

Let $\mathbb{Z}$ be a set of all integers.

1. Closure: For all $a, b \in \mathbb{Z}$ then $a + b$ is again an integer, so $(a + b) \in \mathbb{Z}$.
2. Associativity: For all $a, b, c \in \mathbb{Z}$ then $(a + b) + c = a + (b + c)$.
3. Existence of identity element: For any $a \in \mathbb{Z}$ there exist $0 \in \mathbb{Z}$ such that, $a + 0 = 0 + a = a$.
4. Existence of inverse element: For any $a \in \mathbb{Z}$ there exist $(-a) \in \mathbb{Z}$ such that $a + (-a) = 0 = (-a) + a$.

This means $\mathbb{Z}$ is group under addition.

15. Define ring with an example. Compute the product in the given ring (–3, 5) (2, –4) in $Z_4 \times Z_{11}$.

Solution: Definition of Ring:

A non–empty set R together with two binary operator + and • denoted by <R, +, •> is called ring if the following conditions are satisfied:

(i) Closure: For all a, b $\in$ R then $(a + b) \in R$.
(ii) Commutative: For all a, b $\in$ R then $a + b = b + a$.
(iii) Associativity: For all a, b, c $\in$ R then $(a + b) + c = a + (b + c)$.
(iv) Existence of identity element: For all $a \in R$ there exists an element $0 \in R$ such that $a + 0 = 0 + a = a$.
(v) Existence of inverse element: For all $a \in R$ there exists $a' \in R$ such that $a + a' = 0 = a' + a$.
(vi) Closure: For all a, b $\in$ R then $ab \in R$.
(vii) Associativity: For all a, b, c $\in$ R then $(ab)c = a(bc)$.
(viii) Distributive: For all a, b, c $\in$ R,

$$a(b + c) = ab + ac \quad \text{(left distributive)}$$
$$(a + b)c = ac + bc \quad \text{(right distributive).}$$

Example: A set of real number R is a ring.

Problem Part:

Since in Z_4, $-3 = 1$ and in Z_{11}, $-4 = 7$.

So, $(-3, 5)(2, -4) = (1, 5)(2, 7) = (2, 2)$.

15. State and prove the Pythagorean Theorem of two vectors and verify this for u = (1, –1) and v = (1, 1).

Solution: The Pythagorean Theorem

Two vectors u and v are orthogonal if and only if

$$||u + v||^2 = ||u||^2 + ||v||^2.$$

Proof:

First suppose that u and v are orthogonal. Therefore,

$$u \cdot v = 0 \quad \text{....(i)}$$

Since $||u||^2 = u \cdot u$. So,

$$\begin{aligned} ||u + v||^2 &= (u + v)\cdot(u + v) \\ &= u\cdot(u + v) + v\cdot(u + v) \\ &= u\cdot u + u\cdot v + v\cdot u + v\cdot v \\ &= ||u||^2 + 0 + 0 + ||v||^2 \quad \text{(using (i)]} \\ &= ||u||^2 + ||v||^2 \end{aligned}$$

Conversely, suppose that

$$||u + v||^2 = ||u||^2 + ||v||^2$$
$$\Rightarrow (u + v)\cdot(u + v) = ||u||^2 + ||v||^2$$
$$\Rightarrow u\cdot u + u\cdot v + v\cdot u + v\cdot v = ||u||^2 + ||v||^2$$
$$\Rightarrow ||u||^2 + u\cdot v + v\cdot u + ||v||^2 = ||u||^2 + ||v||^2$$
$$\Rightarrow u\cdot v + v\cdot u = 0$$
$$\Rightarrow 2u\cdot v = 0$$
$$\Rightarrow u\cdot v = 0.$$

This means the vectors u and v are orthogonal.

Problem Part:

Let u = (1, –1) and v = (1, 1). Then

$$||u + v||^2 = ||(1, -1) + (1, 1)||^2 = ||(2, 0)||^2 = \left(\sqrt{4 + 0}\right)^2 = 4.$$

And,

$$||u||^2 = ||(1, -1)||^2 = \left(\sqrt{1 + 1}\right)^2 = 2.$$
$$||v||^2 = ||(1, 1)||^2 = \left(\sqrt{1 + 1}\right)^2 = 2.$$

Thus, $||u + v||^2 = ||u||^2 + ||v||^2$.

This means u and v verifies the Pythagorean Theorem.

TU QUESTIONS-ANSWERS 2076

Group 'A'

Attempt any three questions: **(3×10 = 30)**

1. **When a system of linear equation is consistent and inconsistent? Give an example for each. Test the consistency and solve the system of equations: $x - 2y = 5, -x + y + 5z = 2, y + z = 0$.** **[2 + 2 + 6]**

Solution:

A system of linear equations is called consistent if it has solution (that may be one solution or infinitely many solutions) and called inconsistent if it has no solution.

Problem Part:

Solution: Given system is,

$$x - 2y = 5$$
$$-x + y + 5z = 2$$
$$y + z = 0$$

The matrix notation of the system is,

$$\begin{bmatrix} 1 & -2 & 0 & : & 5 \\ -1 & 1 & 5 & : & 2 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

Apply $R_2 \to R_2 + R_1$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & -2 & 0 & : & 5 \\ 0 & -1 & 5 & : & 7 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$; then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & -2 & 0 & : & 5 \\ 0 & -1 & 5 & : & 7 \\ 0 & 0 & 6 & : & 7 \end{bmatrix} \text{ is triangular form.}$$

The equation form of the matrix notation is,

$$x - 2y = 5 \quad \ldots (i)$$
$$-y + 5z = 7 \quad \ldots (ii)$$
$$6z = 7 \quad \ldots (iii)$$

From (iii), we get $z = \frac{7}{6}$.

then (ii) gives, $y = -\frac{7}{6}$

And, (i) gives, $x = -\frac{5}{6}$

Thus, the solution of the given linear system is $(x, y, z) = \left(\frac{-5}{6}, \frac{7}{6}, \frac{7}{6}\right)$.

2. **What is the condition of a matrix to have an inverse? Find the inverse of the matrix, $A = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.** **[2+8]**

Solution: Let A be a given matrix then the inverse of A i.e. A^{-1} exists if det(A) is non-zero.

Problem Part:

Let $A = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

Here, det (A) $= \begin{vmatrix} 5 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{vmatrix} = 5(0+9) - 1(8-12) + 2(-3-0)$

$= 45 + 4 - 6 = 43 \neq 0$

So, the inverse of A exists.

By algorithm,

$$[A \quad I] = \begin{bmatrix} 5 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & 4 & 5 & -1 & 0 \\ 0 & -19 & 32 & 0 & -4 & 1 \end{bmatrix} \quad \begin{matrix}\text{[Applying} \\ R_2 \to 5R_2 - R_1 \\ R_3 \to 5R_3 - 4R_1]\end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \quad \begin{matrix}\text{[Applying} \\ R_3 \to R_3 + 3R_2]\end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}.$$

$$\sim [I \quad A^{-1}].$$

Thus, A^{-1} exists and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

3. Find a least square solution of Ax = b for $A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$. [10]

Solution: Let,

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}.$$

Here,

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 90 \end{bmatrix}$$

and,

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 45 \end{bmatrix}.$$

Since we have the set of least squares solutions of Ax = b coincides with the non-empty set of solutions of $A^TAx = A^Tb$.

Therefore,

$$[A^TA : A^Tb] = \begin{bmatrix} 4 & 0 & : & 8 \\ 0 & 90 & : & 45 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & : & 2 \\ 0 & 1 & : & 0.5 \end{bmatrix} \quad \begin{bmatrix}\text{Performing} \\ R_1 \to \frac{1}{4}R_1 \\ R_3 \to \frac{1}{90}R_3\end{bmatrix}$$

From the last matrix,

$$x_1 = 2 \text{ and } x_2 = 0.5$$

Then, the least squares solution of $Ax = b$ is

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

4. Let T is a linear transformation. Find the standard matrix of T such that

i. T: $R^2 \rightarrow R^4$ by $T(e_1) = (3, 1, 3, 1)$ and $T(e_2) = (-5,2,0,0)$ where $e_1 = (1, 0)$ and $e_2) = (0, 1)$.

ii. T: $R^2 \rightarrow R^2$ rotates point as the origin through $\frac{3\pi}{2}$ radians counter clockwise.

iii. T: $R^2 \rightarrow R^2$ is a vertical shear transformation that maps e_1 into $e_1 - 2e_2$ but leaves vector e_2 unchanged. **[4 + 3 + 3]**

Solution:

(i) Let T is a linear transformation T: $\mathbb{R}^2 \rightarrow \mathbb{R}^4$ is defined by $T(e_1) = (3, 1, 3, 1)$ and $T(e_2) = (-5, 2, 0, 0)$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.
The standard matrix for T is

$$A = [T(e_1) \quad T(e_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

(ii) Solution: Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let the vector rotates through an angle $\frac{3\pi}{2}$ in counterclockwise direction. Then,

$$T(e_1) = \begin{bmatrix} \cos(3\pi/2) \\ \sin(3\pi/2) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and $$T(e_2) = \begin{bmatrix} -\sin(3\pi/2) \\ \cos(3\pi/2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now, the standard matrix A for T is,

$$A = [T(e_1) \quad T(e_2)] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(iii) Let a linear transformation T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a vertical shear transformation that maps e_1 into $e_1 - 2e_2$ but leaves vector e_2 unchanged. Therefore,

$$T(e_2) = e_2 = (0, 1)$$

and $$T(e_1) = e_1 - 2e_2 = (1, 0) - 2(0, 1) = (1, -2)$$

Thus,

$$A = [T(e_1) \quad T(e_2)] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

Group 'B'

Attempt any ten questions: (10×5 = 50)

5. For what value of h will y be in span $\{v_1, v_2, v_3\}$ where $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$?

Solution:

Let $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$.

We have if $x_1v_1 + x_2v_2 + x_3v_3 = y$ has solution, then b is in the plane spanned by v_1, v_2 and v_3.

Let y is in the plane spanned by v_1, v_2 and v_3. So, the equation

$$Ax = y$$

has solution for $A = [v_1 \quad v_2 \quad v_3]$

Here the augmented matrix of $Ax = b$ is,

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix}$$

Apply $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + 2R_1$ then

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix}$$

Again apply $R_3 \to R_3 - 3R_2$. Then

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

Clearly, the matrix gives solution only if

$$h - 5 = 0.$$

$$\Rightarrow \quad h = 5.$$

Thus, for h = 5, the vector y is in the plane spanned by v_1, v_2 and v_3.

6. **Let us define a linear transformation $T: R^2 \to R^2$ by $T(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$. Find the image under T of $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $u + v = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.**

Solution: Let

$$T(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, u + v = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Therefore,

$$Tu = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$Tv = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$T(u+v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

7. **Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. Determine the value (s) of k if any wil make AB = BA.**

Solution: Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$.

Here,

$$AB = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 8+15 & -10+5k \\ -12+3 & 15+k \end{bmatrix} = \begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 8+15 & 20-5 \\ 6-3k & 15+k \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$$

And, suppose AB = BA. That is,

$$\begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$$

This, implies,

$$-10 + 5k = 15 \Rightarrow k = 5.$$

$$-9 = 6 - 3k \Rightarrow k = 5 \text{ (which is same as above).}$$

Thus, at $k = 5$, we get $AB = BA$.

8. Define determinant. Compute the determinant without expanding
$\begin{bmatrix} -2 & 8 & -9 \\ -1 & 7 & 0 \\ 1 & -4 & 2 \end{bmatrix}$. **[1 + 4]**

Solution: For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ of n terms of the form

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

Problem Part:

Here,

$$\begin{bmatrix} -2 & 8 & -9 \\ -1 & 7 & 0 \\ 1 & -4 & 2 \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & -9 \\ 0 & 6 & 9 \\ 0 & 0 & -5 \end{bmatrix} \quad \begin{bmatrix} \text{Performing} \\ R_2 \to 2R_2 - R_1 \\ R_3 \to 2R_3 + R_1 \end{bmatrix}$$

$$= (-2) \times 6 \times (-5)$$

$$= 60$$

9. Define null space. Find the basis for the null space of the matrix
$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$. **[1 + 4]**

Solution: Let A be a $m \times n$ matrix then null-space of the matrix A is denoted by Nul A and defined by

$$\text{Nul } A = \{x : x \in \mathbb{R}^n : Ax = 0\}.$$

Problem Part:

The augmented matrix of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ is

$$\begin{pmatrix} 1 & 2 & 3 & : & 0 \\ 0 & -1 & -2 & : & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & : & 0 \\ 0 & -1 & -2 & : & 0 \end{pmatrix}$$

Pivot is in 1st and 2nd column so x_1 and x_2 are basic variables and x_3 is free variable. Hence $Ax = 0$ has non-trivial solution. For solution

From 1st row, $\quad x_1 - x_3 = 0$

$\Rightarrow \quad x_1 = x_3$

From 2nd row, $\quad -x_2 - 2x_3 = 0$

$\Rightarrow \quad x_2 = -2x_3$

and x_3 is free variable

Thus, solution, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Thus, Nul A $= \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} : a \in R \right\}$ and here $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \neq P$ and span Nul A

Hence, $\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}$ is basis for Nul A.

10. Let B = {b_1, b_2} and C = {c_1, c_2} be bases for a vector space V, and suppose $b_1 = -c_1 + 4c_2$ and $b_2 = 5c_1 - 3c_2$. Find the change of coordinate matrix for a vector space and find $[x]_C$ for $x = 5b_1 + 3b_2$. [2.5 + 2.5]

Solution: We know that

$$[x]_C = \underset{C\leftarrow B}{P}[x]_B \quad \text{......(1)}$$

and $$\underset{C\leftarrow B}{P} = [[b_1]_C \quad [b_2]_C]$$

Given that,

$$b_1 = -c_1 + 4c_2 \quad \text{i.e. } [b_1]_C = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$b_2 = 5c_1 - 3c_2 \quad \text{i.e. } [b_2]_C = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

and $x = 5b_1 + 3b_2$ i.e. $[x]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Thus, $\underset{C\leftarrow B}{P} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$ is called change of coordinate matrix from B to C.

From (1)

$$[x]_C = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

11. Find the eigenvalues of the matrix $\begin{bmatrix} 6 & 5 \\ -8 & -6 \end{bmatrix}$.

Solution: Given,

$$A = \begin{bmatrix} 6 & 5 \\ -8 & -6 \end{bmatrix}$$

So, the characteristic equation of A is

$$|A - \lambda I| = 0.$$

Here,

$$A - \lambda I = \begin{bmatrix} 6 & 5 \\ -8 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 6-\lambda & 5 \\ -8 & -6-\lambda \end{bmatrix}$$

Thus, characteristic equation of A is,

$$|A - \lambda I| = 0.$$

or, $\begin{vmatrix} 6-\lambda & 5 \\ -8 & -6-\lambda \end{vmatrix} = 0.$

or, $(6-\lambda)(-6-\lambda) + 40 = 0.$

or, $-(6-\lambda)(6+\lambda) + 40 = 0.$

or, $\lambda^2 - 36 + 40 = 0.$

or, $\lambda^2 = -4$

This gives,

$$\lambda = \pm 2i$$

Therefore, the eigenvalues of A are $\lambda = 2i$ and $\lambda = -2i$.

12. Find the QR factorization of the matrix $\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$. Let the columns of A are x_1, x_2. So,

$$x_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = (2, 3) \text{ and } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (1, -1)$$

Then,

$$v_1 = x_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = (2, 3)$$

Take,
$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1 = (1, -1) - \left(\frac{(1, -1)\cdot(2, 3)}{(2, 3)\cdot(2, 3)}\right)(2, 3)$$
$$= (1, -1) - \left(\frac{-1}{13}\right)(2, 3)$$
$$= \frac{1}{13}[(13, -13) + (2, 3)]$$
$$= \frac{1}{13}(15, 10)$$
$$= \frac{5}{13}(3, 2)$$

Set, v' = (3, 2)

Thus, $\{v_1, v_2'\}$ be an orthogonal basis. Let $\{u_1, u_2\}$ be normalize of the orthogonal basis. So,

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(2, 3)}{\sqrt{13}}$$
$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{(3, 2)}{\sqrt{13}}$$

Let Q be a matrix whose columns are u_1, u_2. Then

$$Q = \begin{bmatrix} 2/\sqrt{13} & 3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Since we have A = QR, by QR–factorization theorem. Here,

$$Q^T A = Q^T (QR) = Q^T QR = IR = R.$$

Now,

$$R = Q^T A$$
$$\Rightarrow R = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 13 & -1 \\ 12 & 1 \end{bmatrix}$$

13. Define binary operation. Determine whether the binary operation * is associative or commutative or both where * is defined on Q by letting $x * y = \frac{x+y}{3}$. **[5]**

Solution: Definition of Binary Operations:

A binary operation * on a set S is a function mapping S × S into S. Given, for each (a, b)∈ S × S, we will denote the element * ((a, b)) of S by a * b.

Problem Part:

Given,

$$x * y = \frac{x+y}{3} \text{ for all } x, y \in Q$$

Associative: for any x, y, z ∈ Q we need to show

$$x * (y * z) = (x * y) * z$$

Here,

$$x * (y * z) = x * \left(\frac{y+z}{3}\right) = \frac{x + \frac{y+z}{3}}{3} = \frac{3x+y+z}{9}$$

$$(x * y) * z = \frac{x+y}{3} * z = \frac{\frac{x+y}{3} + z}{3} = \frac{x+y+3z}{9}$$

Since $x * (y * z) \neq (x * y) * z$ so it is not associative.

Cumulative: For any $x, y \in Q$ we need to show $x * y = y * x$

Since,

$$x * y = \frac{x+y}{3} = \frac{y+x}{3} = y * x$$

$\therefore$ It is commutative.

Hence, the binary operation $*$ on Q is commutative but not associative.

14. Show that the ring $(Z_4, +_4, \bullet_4)$ is not an integral domain. [5]

Solution: Since $Z_4 = \{0, 1, 2, 3\}$ is ring under binary operation $+_4$ and $\bullet_4$. It is commutative ring and is with identity, also. Here,

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\bullet_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Here, Z_4 has zero divisor 2 because $2 \bullet 2 = 0$. Hence, Z_4 is not integral domain.

15. Find the vector x determined by the coordinate vector $[x]_\beta = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}$ where

$$\beta = \left\{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}\right\}$$ [5]

Solution: Given that

$$[x]_\beta = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix} \text{ and } \beta = \left\{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}\right\}$$

Since

$$\beta = \{b_1, b_2, b_3\} = \left\{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}\right\}$$

So,

$$b_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, b_3 = \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}$$

Here,

$$x = (-4) b_1 + 8b_2 + (-7)b_3 = -4\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} - 7\begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$

TU QUESTIONS-ANSWERS 2078

Group 'A'

Attempt any three questions: **(3×10 = 30)**

1. **Define system of linear equations. When a system of equations is consistent? Determine if the system**

$$-2x_1 - 3x_2 + 4x_3 = 5$$
$$x_2 - 2x_1 = 4$$
$$x_1 + 3x_2 - x_3 = 2$$

Solution: A system of linear equations is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. That is, a linear system has solution if and only if an echelon form of the augmented matrix has no row of the form $[0 \;\; 0 \;\; \dots \;\; 0 \;\; b]$ with $b \neq 0$.

If the system is not consistent then it is inconsistent.

Problem Part:

Given a system of linear equations is

$$-2x_1 - 3x_2 + 4x_3 = 5$$
$$x_2 - 2x_1 = 4$$
$$x_1 + 3z_2 - z_3 = 2$$

The augmented matrix of the given system is,

$$\begin{bmatrix} -2 & -2 & 0 & : & 5 \\ 1 & 1 & 5 & : & 2 \\ 1 & 1 & 1 & : & 0 \end{bmatrix}$$

Applying $R_2 \to R_2 + R_1$ then

$$\begin{bmatrix} 1 & -2 & 0 & : & 5 \\ 0 & -1 & 5 & : & 7 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$ then

$$\begin{bmatrix} 1 & -2 & 0 & : & 5 \\ 0 & -1 & 5 & : & 7 \\ 0 & 0 & 6 & : & 7 \end{bmatrix}$$

Here, no last row is in form of $[0 \;\; 0 \;\; 0 \;\; b]$, $b \neq 0$. So, system is consistent. And, from the last matrix, we get

$$x - 2y = 5 \quad \dots \text{(i)}$$
$$-y + 5z = 7 \quad \dots \text{(ii)}$$
$$6z = 7 \quad \dots \text{(iii)}$$

From (iii) we get, $z = \dfrac{7}{6}$

From (ii) we get, $y = 5z - 7 = \dfrac{35}{6} - 7 = \dfrac{-7}{6}$

From (i) we get, $x = 5 + 2y = 5 - \dfrac{14}{6} = 5 - \dfrac{7}{3} = \dfrac{8}{3}$

Thus, the solution of given system is $(x, y, z) = \left(\dfrac{8}{3}, \dfrac{-7}{6}, \dfrac{7}{6}\right)$.

2. **Define linear transformation with an example. Let** $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$b = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **and define a transformation** $T : R^2 \to R^2$ **by** $T(x) = Ax$ **then**

(a) Find T(v)

(b) Find $z \in R^2$ whose image under T is b.

Solution: (Definition: Linear Transformation)

A transformation (or mapping) T is linear if for all $\alpha, \beta \in F$, u, v domain of T, is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta t(v).$$

Problem Part:

Let, $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$.

Given that the transformation $T : R^2 \to R^2$ by $T(x) = Ax$.

Now,

(a) $T(v) = Av = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$.

(b) Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Suppose x in R^2 whose image under T is b. Then

$$T(x) = b \Rightarrow Ax = b$$

$$\Rightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

The augmented matrix of $Ax = b$ is,

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \qquad \text{[Apply } R_2 \to R_2 - 3R_1,\ R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 4 & -2 \end{bmatrix} \qquad \text{[Apply } R_2 \to R_2/14$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{[Apply } R_3 \to R_3 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{[Apply } R_1 \to R_1 - 3R_2$$

This implies $x_1 = 1.5$ and $x_2 = -0.5$

Thus, $x = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$ in R^2 whose image under T is b.

3. **Find the LU factorization of** $\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$

Solution: Let,

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + 3R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 4R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_4 \to R_4 - 2R_3 \end{bmatrix}$$

$$= U.$$

Here, U has pivot values 2 in first column, 3 in second column, third column has no pivot value, 2 in fourth column and 5 in fifth column. The entries of column of pivot value and to be reduced value which are determine the row reduction of A to U are,

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad [5]$$

$$\div 2 \qquad \div 3 \qquad \div 2 \qquad \div 5$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad [1]$$

Therefore L is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Thus,

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= LU.$$

4. **Find a least square solution of the inconsistent system Ax = b for**

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

Solution: Given that

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

Here,

$$A^T A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

and,

$$A^T b = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}.$$

Since we have the set of least squares solutions of $Ax = b$ coincides with the non-empty set of solutions of $A^T Ax = A^T b$.

Therefore,

$$\hat{x} = (A^T A)^{-1} (A^T b) \qquad \dots (i)$$

Here,

$$|A^T A| = \begin{vmatrix} 6 & -11 \\ -11 & 22 \end{vmatrix} = 11 \neq 0.$$

Then,

$$(A^T A)^{-1} = \frac{1}{11} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix}.$$

Therefore (i) becomes,

$$\hat{x} = \frac{1}{11} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 33 \\ 19 \end{bmatrix}.$$

Thus, the least square solution is $\begin{bmatrix} 3 \\ 19/11 \end{bmatrix}$.

Group 'B'

Attempt any ten questions: (10×5 = 50)

5. **Determination the column of the matrix A are linearly independent, where $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$.**

Solution: The augmented matrix of $Ax = 0$ be,

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix}$$ [Interchanging 1st and 2nd row and then apply $R_3 \rightarrow R_3 - 5R_1$]

$$\sim \begin{bmatrix} \boxed{1} & 2 & -1 & 0 \\ 0 & \boxed{1} & 4 & 0 \\ 0 & 0 & \boxed{13} & 0 \end{bmatrix}$$ [Apply $R_3 \rightarrow R_3 + 2R_2$]

Here, x_1, x_2 and x_3 are basic variable and has no free variable, so there exist trivial solution. This means the columns of A are linearly independent.

6. **When two column vectors in R^2 are equal? Give an example. Compute $u + 3v$, $u - 2v$, where $u = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.** 1+4

Solution: Definition (Equal Vectors)

Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal. Otherwise, the vectors are **not equal**.

Example (i) $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, (ii) $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Here in (i), $u = v$ if $u_1 = v_1$ and $u_2 = v_2$.

But in (ii), u and v are not equal because $3 \neq 2$ however $2 = 2$.

Problem Part: Let,

$$u = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Now, $$u + 3v = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 11 \end{bmatrix}$$

and $$u - 2v = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}.$$

7. **Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and define $T : R^2 \to$ by $T(x) = Ax$, find the image under T of $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.**

Solution: Let

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, u = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ and } v = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Then

$$Tu = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$Tv = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

8. **Find the eigenvalues of $\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.** **5**

Solution:

Let λ be the eigenvalues of the given matrix. Then characteristics equation is

$$\begin{vmatrix} 3-\lambda & 6 & -8 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(3-\lambda)\begin{vmatrix} -\lambda & 6 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow$ $(3-\lambda)(-\lambda)(2-\lambda) = 0$

$\Rightarrow$ $\lambda = 3, \lambda = 0, \lambda = 2$

Thus the eigenvalues of A are $\lambda = 3, \lambda = 0, \lambda = 2$.

9. **Define null space of a matrix A. Let $A = \begin{bmatrix} -1 & -3 & 2 \\ -5 & -9 & 1 \end{bmatrix}$, and $v = \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix}$. Then show that v is the null A.** **1+4**

Solution: Let A be a m×n matrix then null-space of the matrix A is denoted by Nul A and defined by

$$\text{Nul } A = \{x: x \in R^n: Ax = 0\}.$$

Problem Part:

Given that

$$A = \begin{bmatrix} -1 & -3 & 2 \\ -5 & -9 & 1 \end{bmatrix}, \text{ and } v = \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix}$$

Now,

$$Av = \begin{bmatrix} -1 & -3 & 2 \\ -5 & -9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

This means v is the null space of A. That is, v is the null A.

10. **Verify that 1^k, $(-2)^k$, 3^k are linearly independent signals.** 5

Solution:

We have, $u_k = 1^k, v_k = (-2)^k, w_k = 3^k$

By Casorati matrix we have

$$\begin{bmatrix} u_k & v_k & w_k \\ u_{k+1} & v_{k+1} & w_{k+1} \\ u_{k+2} & v_{k+2} & w_{k+2} \end{bmatrix}$$

$$= \begin{bmatrix} 1^k & (-2)^k & 3^k \\ 1^{k+1} & (-2)^{k+1} & 3^{k+1} \\ 1^{k+2} & (-2)^{k+2} & 3^{k+2} \end{bmatrix}$$

taking k = 0, we have,

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 3 & 8 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 10 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_3 \to R_3 + R_2 \end{bmatrix}$$

Since, each columns are pivot columns which shows that the Casorati matrix is invertible for k = 0

Thus, 1^k, $(-2)^k$ and 3^k are linearly independent.

11. **If $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^n, where $A = PDP^{-1}$ and $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.** 5

Solution: Let, $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

Given that $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

Since $A = PDP^{-1}$. So,

$$A^2 = (PDP^{-1})^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}$$

Again,

$$A^3 = A.A^2 = (PDP^{-1})(PD^2P^{-1}) = PD(P^{-1}P)D^2P^{-1} = PDID^2P^{-1}$$
$$= PD^3P^{-1}$$

So, in general, for $n \geq 1$,
$A^n = PD^nP^{-1}$.

Now,

$$A^n = P\,D^n\,P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 5^n & 3^n \\ -5^n & -2 \times 3^n \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \times 5^n - 3^n & 5^n - 3^n \\ -2 \times 5^4 + 2 \times 3^n & -5^n + 2 \times 3^n \end{bmatrix}$$

12. Find a unit vector v of u = (1, –2, 2, 3) I the direction of u. 5

Solution: Let

$$u = (1, -2, 2, 3)$$

Then,

$$\|u\| = \sqrt{(1)^2 + (-2)^2 + (2)^2 + (3)^2} = 3.$$

Therefore, the unit vector of u is,

$$\frac{u}{\|u\|} = \frac{(1, -2, 2, 3)}{3} = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}, 1\right).$$

13. Prove that the two vectors u and v are perpendicular to each other if and only if the line through u is perpendicular bisector of the line segment from – u to v. 5

Solution: The two vectors u and v are perpendicular to each other. Then

$$u \bullet v = 0$$

Since the line segment from –u to v is v + u. And, its bisector is $\frac{v+u}{2}$.

Now, $u \bullet \frac{v+u}{2} = \frac{u \bullet u}{2}$

14. Let an operation * be defined on Q* by $a * b = \frac{ab}{2}$. Then show that Q* forms a group. 5

Solution: Given that * is defined on $\mathbb{Q}^+$ by $a * b = \frac{ab}{2}$.

Closure: For all a, b $\in \mathbb{Q}^+$, $a*b = \frac{ab}{2} \in \mathbb{Q}^+$.

That is, the elements of $\mathbb{Q}^+$ are closed under '*'.

Associativity: For all a, b, c $\in \mathbb{Q}^+$,

$$(a * b) * c = \frac{ab}{2} * c = \frac{abc}{4}.$$

Again, $a * (b * c) = a * \frac{bc}{2} = \frac{abc}{4}$.

Thus, $*$ is associative

Existence of Identity: For all, $a \in Q^+$;

$$a * 2 = \frac{a \cdot 2}{2} = a \quad \text{and} \quad 2 * a = \frac{2a}{2} = a$$

Hence 2 is the identity element for $*$.

Existence of Inverse:

Finally $a * b = 2 = b * a$, where b is inverse of a under $*$.

$$\frac{ab}{2} = 2 = \frac{ba}{2} \Rightarrow b = \frac{4}{a} \in Q^+$$

Therefore, $\frac{4}{a}$ is an inverse for a. Hence Q^+ is a group under the binary operation $*$.

Thus, $*$ satisfies all conditions for group under addition, so Q^+ is a group under addition.

15. Define ring and show that set of real numbers with respect to addition and multiplication operation is a ring. 2+

Solution: Definition of Ring:

A non-empty set R together with two binary operator + and • denoted by <R, +, •> is called ring if the following conditions are satisfied:

R1: < R, +> is abelian group.

R2 : Closed under multiplication:

for all a, b $\in$ R, ab $\in$ R

R3: Associative under multiplication:

for all a, b $\in$ R, (ab)c = a(bc).

R4: Distributive:

for all a, b, c $\in$ R, a(b + c) = ab + ac (left distributive)

for all a, b, c $\in$ R, (a + b)c = ac + bc (right distributive)

Problem Part:

Let R be the set all real numbers.

1. For abelian group under addition
   - i. closure: $\forall$ a, b $\in$R we have a + b $\in$ R
   - ii. Associative: $\forall$ a, b, c $\in$R we have (a + b) + c = a + (b + c)

     e.g. if a = – 2, b = 3, c = 1 then

     $(a + b) + c = (-2 + 3) + 1 = 1 + 1 = 2$

     $a + (b + c) = -2 + (3 + 1) = -2 + 4 = 2$

     So it is associative
   - iii. Existence of identity

     Since there is an element 0 $\in$ R s.t. $\forall$ a $\in$ R we have

     $0 + a = a + 0 = a,$
   - iv. Existence of inverse

     $\forall$a $\in$ R there exist –a $\in$ R such that

     $a + (-a) = (-a) + a = 0$

     e.g. if 2, 3, . . . are in Z then – 2, – 3, . . . also are in R.
   - v. Commutative

     $\forall$ a, b, $\in$ R we have a + b = b + a

     e.g. if a = – 3, b = 2 then

     $a + b = -3 + 2 = -1$

     $b + a = 2 - 3 = -1$

2. Closed under multiplication

   $\forall$ a, b $\in$R we have ab $\in$ R

   eg. if a = – 3, b = 4 then ab = (– 3) • 4 = – 12 $\in$R

3. Associative under multiplication

   $\forall$ a, b, c $\in$R we have a • (b • c) = (a • b) • c

   e.g. if a = 2, b = – 3, c = 5 then

   $a \bullet (b \bullet c) = 2 \bullet (-3 \bullet 5) = 2 \bullet (-15) = -30$

   $(a \bullet b) \bullet c = (2 \bullet (-3)) \bullet 5 = -6 \bullet 5 = -30$

4. Distributive law

   $\forall$ a, b, c, $\in$ R we have a • (b + c) = a • b + a • c and

   $(a + b) \bullet c = a \bullet c + b \bullet c$

   eg. if a = 5 , b = 2, c = – 3 we have

   $a \bullet (b + c) = 5 \bullet (2 - 3) = 5 \bullet (-1) = -5$

   $a \bullet b + a \bullet c = 5 . 2 + 5 \bullet (-3) = 10 - 15 = -5$

   Similarly,

   $(a + b) \bullet c = (5 + 2) \bullet (-3) = 7 \bullet (-3) = -21$

   $a \bullet c + b \bullet c = 5 . (-3) + 2 \bullet (-3) = -15 - 6 = -21$

   Hence, (R, + , •) forms a ring.