Chapter 1

Linear Equations in Linear Algebra

Exercise 1.1

1.

(i) $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & h \\ 0 & 0 & 7-2h \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

For consistency,

$7 - 2h = 0 \quad \Rightarrow h = 7/2$

(ii) $\begin{bmatrix} 1 & -2 & 3 \\ 3 & h & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 3 & h+6 & -11 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$

For consistency,

$h + 6 \neq 0 \quad \Rightarrow h \neq -6$

2.

(i) $\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$ $R_2 \to R_2 - 4R_1$

(a) For no. solution,

$8 - 4h = 0 \qquad k - 8 \neq 0$

$\Rightarrow h = 2 \qquad$ and $\qquad k \neq 8$

(b) For unique solution,

$8 - 4h \neq 0$

$\Rightarrow h \neq 2$

(c) For infinitely many solutions,

$8 - 4h = 0 \qquad$ and $\qquad k - 8 = 0$

$\Rightarrow h = 2 \qquad$ and $\qquad k = 8$

(ii) $\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$

(a) For No solution, $h - 9 = 0 \Rightarrow h = 9$ and $k - 6 \neq 0 \Rightarrow k \neq 6$.

(b) For unique solution,

$h - 9 \neq 0; \qquad k - 6 = 0$

$\Rightarrow h = 9, \qquad k = 6$

3.

(i) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$

$\sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ $R_2 \to -\frac{1}{5}R_2$

$\sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ $R_1 \to R_1 - 4R_2$

Corresponding system is

$x_1 + 3x_2 = -5$

x_2 is free variable.

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$x_3 = 3$

Using back substitution, we get

$x_1 = -5 - 3x_2$

x_2 is free variable.

(ii) $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -6 & 5 \\ 0 & -2 & 7 & -6 \end{bmatrix}$ $R_1 \to R_2$

$\sim \begin{bmatrix} 1 & 0 & -5 & 4 \\ 0 & 1 & -6 & 5 \end{bmatrix}$ $R_1 \to R_1 + 2R_2$

Corresponding system is

$x_1 - 5x_3 = 4$

$x_2 - 6x_3 = 5$

x_3 is free variable

$\therefore x_1 = 4 + 5x_3$

$x_2 = 5 + 6x_3$

x_3 is free variable.

(iii) $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_2 \to R_2 + 3R_1,\ R_3 \to R_3 + 2R_1$

Corresponding system is $3x_1 - 4x_2 + 2x_3 = 0$

x_2, x_3 are free variables

$\therefore x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3$

x_2, x_3 are free variables.

4.

(i) $\begin{bmatrix} 2 & 3 & 4 & 20 \\ 3 & 4 & 5 & 26 \\ 3 & 5 & 6 & 31 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 4 & 20 \\ 0 & -1/2 & -1 & -4 \\ 0 & 1/2 & 0 & 1 \end{bmatrix}$ $R_2 \to R_2 - \frac{3}{2}R_1,\ R_3 \to R_3 - \frac{3}{2}R_1$

$\sim \begin{bmatrix} 2 & 3 & 4 & 20 \\ 0 & -1/2 & -1 & -4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$ $R_3 \to R_3 + R_2$

$\sim \begin{bmatrix} 2 & 3 & 4 & 20 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -1 & -3 \end{bmatrix}$ $R_2 \to 2R_2$

Corresponding system is

$2x + 3y + 4z = 20$

$-y - 2z = -8$

$-z = -3$

Using back substitution,

$x = 1, y = 2, z = 3$

(ii) $\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$\sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{bmatrix}$ $R_2 \to R_3$

$$\sim\begin{bmatrix}1&0&-3&8\\0&1&5&-2\\0&0&5&-5\end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim\begin{bmatrix}1&0&-3&8\\0&1&5&-2\\0&0&0&-3\end{bmatrix} R_3 \to R_3 - R_2$$

Since last row is off the form $[0 \quad 0 \quad 0 \quad b]$, $b \neq 0$. So, the given system is inconsistent.

5.

(i) $$\begin{bmatrix}1&0&3&0&2\\0&1&0&-3&3\\0&-2&3&2&1\\3&0&0&7&-5\end{bmatrix} \sim \begin{bmatrix}1&0&3&0&2\\0&1&-3&3&3\\0&-2&-3&8&7\\0&0&-3&7&-8\end{bmatrix} R_4 \to R_4 - 3R_1$$

$$\sim\begin{bmatrix}1&0&3&0&2\\0&1&-3&3&3\\0&0&-3&8&7\\0&0&0&-1&-15\end{bmatrix} R_4 \to R_4 - R_3$$

Since, last column is not pivot column, so the system is consistent.

(ii) $$\begin{bmatrix}0&1&-8&8\\2&-3&2&1\\5&-8&7&1\end{bmatrix} \sim \begin{bmatrix}2&-3&2&1\\0&1&-8&8\\5&-8&7&1\end{bmatrix} R_1 \to R_2$$

$$\sim\begin{bmatrix}2&-3&2&1\\0&1&-8&8\\0&-1/2&2&-3/2\end{bmatrix} R_3 \to R_3 - \frac{5}{2}R_1$$

$$\sim\begin{bmatrix}2&-3&2&1\\0&1&-8&8\\0&0&-2&5/2\end{bmatrix} R_3 \to R_3 + \frac{1}{2}R_2$$

Since last column is not pivot column, so the system is consistent.

6: Let $A = \begin{bmatrix}1&3&4\\-4&2&-6\\-3&-2&-7\end{bmatrix}$ and $b = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible b_1, b_2, b_3?

Solution: The augmented matrix of $Ax = b$ is

$$\begin{bmatrix}1&3&4&b_1\\-4&2&-6&b_2\\-3&-2&-7&b_3\end{bmatrix}.$$

$$\sim\begin{bmatrix}1&3&4&b_1\\0&14&10&b_2+4b_1\\0&7&5&b_3+3b_1\end{bmatrix} \quad \because \text{ applying } \begin{matrix}R_2 \to R_2 + 4R_1\\R_3 \to R_3 + 3R_1\end{matrix}$$

$$\sim\begin{bmatrix}1&3&4&b_1\\0&14&10&b_2+4b_1\\0&0&0&b_1-(1/2)b_2+b_3\end{bmatrix} \quad [\text{applying } R_3 \to R_3 - (1/2)R_2]$$

This shows that the equation $Ax = b$ is inconsistent for every b, because some choice of b can make $b_1 - \frac{1}{2}b_2 + b_3$ is non-zero.

Exercise 1.2

1. (i) $$u + v = \begin{bmatrix}6\\-1\\5\end{bmatrix} + \begin{bmatrix}-3\\4\\6\end{bmatrix} = \begin{bmatrix}3\\3\\5\end{bmatrix}$$

and $$3u - 2v = 3\begin{bmatrix}6\\-1\\5\end{bmatrix} - 2\begin{bmatrix}-3\\4\\6\end{bmatrix} = \begin{bmatrix}18+6\\-3-8\\15-12\end{bmatrix} = \begin{bmatrix}24\\-11\\3\end{bmatrix}.$$

(ii) Similar to (i).

2. (i) $$[a_1 \; a_2 \; a_3 \; b] = \begin{bmatrix}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&5&2\\0&1&4&3\\0&2&8&6\end{bmatrix} R_2 \to R_2 + 2R_1$$

$$\begin{bmatrix}1&0&5&2\\0&1&4&3\\0&0&0&0\end{bmatrix} R_3 \to R_3 - 2R_2$$

Thus system $x_1 a_1 + x_2 a_2 + x_3 a_3 = b$ is consistent. Hence, b is linear combination of a_1, a_2, a_3.

(ii) Similar to (i).

3.

(i) $$\begin{bmatrix}1&-4&2&3\\0&3&5&-7\\-2&8&-4&-3\end{bmatrix} \sim \begin{bmatrix}1&-4&2&3\\0&3&5&-7\\0&0&0&3\end{bmatrix} R=3 \to R_3 + 2R_1$$

This system $Ax = b$ is inconsistent.

Hence, b is not linear combination of columns of A.

(ii) Similar to (i).

4.

(i) $$\begin{bmatrix}1&-2&4\\4&-3&1\\-2&7&h\end{bmatrix} \sim \begin{bmatrix}1&-2&4\\0&5&-15\\0&3&h+8\end{bmatrix} R_2 \to R_2 - 4R_1, R_3 \to R_3 + 2R_1$$

$$\sim\begin{bmatrix}1&-2&4\\0&1&-3\\0&3&h+8\end{bmatrix} R_2 \to \frac{1}{5}R_1$$

$$\sim\begin{bmatrix}1&-2&4\\0&1&-3\\0&0&h+17\end{bmatrix} R_3 \to R_3 - 3R_2$$

If b is in span $\{u, v\}$

Then $h + 17 = 0 \Rightarrow h = -17$.

(ii) Similar to (i).

5. Solved in book.

6. The augmented matrix of $Ax = b$ is

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 7 & 5 & b_3+3b_1 \end{bmatrix} \text{[applying } \begin{matrix} R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 0 & 0 & b_1-(1/2)b_2+b_3 \end{bmatrix} \quad \text{[applying } R_3 \to R_3 - (1/2) R_2]$$

This shows that the equation $Ax = b$ is inconsistent for every b, because some choice of b can make $b_1 - \frac{1}{2} b_2 + b_3$ is non-zero.

Exercise 1.3

A.

1. $\begin{bmatrix} 6 & 4 & 2 \\ 3 & -5 & -34 \end{bmatrix} \sim \begin{bmatrix} 6 & 4 & 2 \\ 0 & -7 & -35 \end{bmatrix} \; R_2 \to R_2 - \frac{1}{2} R_1$

$\sim \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix} \; R_1 \to \frac{1}{2} R_1, R_2 \to -\frac{1}{7} R_2$

Corresponding system is

$3x + 2y = 1$

$y = 5$

Using back substitution,

$x = -3, y = 5$

2. $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 5 \\ 3 & 1 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -1 \\ 0 & -2 & -2 & -10 \end{bmatrix} \; R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & -9 \end{bmatrix} \; R_3 \to R_3 - R_2$

Corresponding system is,

$x + y + z = 6$

$-2y = -1$

$-2z = -9$

Using back substituting

$x = 1, y = 1/2, z = 9/2$

3-6 → Similar to Q. 1.

B.

(i) $\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \; R_2 \to R_2 + R_1, R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; R_3 \to R_3 + 3 R_2$

Here x_3 is free variable so system has non trivial solution.

Corresponding system is

$3x_1 + 5x_2 - 4x_3 = 0$

$3x_2 = 0$

x_3 is free variable.

Using back substitution,

$x_1 = \frac{4}{3} x_3;\; x_2 = 0;\; x_3$ is favourable.

$\therefore$ Solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$

(ii) Similar to (i).

Exercise 1.4

1. Consider homogeneous system $x_1v_1 + x_2v_2 + x_3v_3 = 0$

(i) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

It is in echelon (reduced) form and x_1, x_2, x_3 are all basic variables, so homogeneous system has only trivial solution $x_1 = 0, x_2 = 0, x_3 = 0$. Hence, vectors are linearly independent.

(ii) Vectors are not multiples of each other. So, they are linearly independent.

(iii) Vectors are not multiples of each other. So, they are linearly independent.

(v) Vector the given vectors one vector is zero vector, so vectors are linearly dependent.

(vi) Vectors are not multiples of each other. So, they are linearly independent.

2.

(i) $\begin{bmatrix} 0 & -8 & 5 & 0 \\ 3 & -2 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 1 & -3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 3 & -2 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \; R_1 \leftrightarrow R_4$

$\sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \; R_2 \to R_2 - 3R_1, R_2 \to R_2 + R_1$

$\sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 0 & -10/7 & 0 \\ 0 & 0 & 51/7 & 0 \end{bmatrix} \; R_3 \to R_3 - \frac{2}{7} R_2, R_4 \to R_4 - \frac{8}{7} R_2$

$\sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 7 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \; R_3 \to -\frac{7}{10} R_3, R_4 \to \frac{7}{51} R_4$

$$\sim\begin{bmatrix}1 & -3 & 2 & 0\\ 0 & 7 & -2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} R_4 \to R_4 - R_3$$

No, free variables, hence columns of given matrix are linearly independent.

(ii) Similar to (i).

3.

(i) Consider homogeneous system

$x_1v_1 + x_2v_2 + x_3v_3 = 0$

It's augmented matrix is

$$\begin{bmatrix}1 & -3 & 5 & 0\\ -3 & 9 & -7 & 0\\ 2 & -6 & h & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -3 & 5 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & h-10 & 0\end{bmatrix} R_2 \to R_2 + 3R_1,\ R_3 \to R_3 - 2R_1$$

Here, x_2 is free variable for all values of h.

So, vectors are linearly independent for all values of h.

(ii) $$\begin{bmatrix}1 & 3 & -1 & 0\\ -1 & -5 & 5 & 0\\ 4 & 7 & h & 0\end{bmatrix} \sim \begin{bmatrix}1 & 3 & -1 & 0\\ 0 & -2 & 4 & 0\\ 0 & -5 & h+4 & 0\end{bmatrix} R_2 \to R_2 + R_1,\ R_3 \to R_3 - 4R_1$$

$$\sim\begin{bmatrix}1 & 3 & -1 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & h-6 & 0\end{bmatrix} R_2 \to -\frac{1}{2}R_2$$

For vectors to be linearly dependent. x_3 must be free variable i.e. if $h - 6 = 0 \Rightarrow h = 6$.

(iii) Similar to (ii).

Chapter 2

Transformation

Exercise 2

1. $T(u) = Au = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1\\ -2\end{bmatrix} = \begin{bmatrix}-1\\ -2\end{bmatrix}$

   $T(v) = Av = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}a\\ b\end{bmatrix}$

2. Hint: See questions no. 1.

3. Consider $Ax = b$, where $x = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

   It's augmented matrix is

   $\left[\begin{array}{ccc|c}1 & 0 & -2 & -1\\ -2 & 1 & 6 & 7\\ 3 & -2 & -5 & -3\end{array}\right]$

   $R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - 3R_1$

   $\sim \left[\begin{array}{ccc|c}1 & 0 & -2 & -1\\ 0 & 1 & 2 & 5\\ 0 & -2 & 1 & 0\end{array}\right]$

   $R_3 \to R_3 + 2R_2$

   $\sim \left[\begin{array}{ccc|c}1 & 0 & -2 & -1\\ 0 & 1 & 2 & 5\\ 0 & 0 & 5 & 10\end{array}\right]$

   $R_3 \to \frac{1}{5}R_3$

   $\sim \left[\begin{array}{ccc|c}1 & 0 & -2 & -1\\ 0 & 1 & 2 & 5\\ 0 & 0 & 1 & 2\end{array}\right]$

   Corresponding system is,

   $x_1 - 2x_3 = -1$

   $x_2 + 2x_3 = 5$

   $x_3 = 2$

   Using back substitution

   $x_1 = 3,\ x_2 = 1,\ x_3 = 2$

4. Hints: See Q. No. 3

5. Hints: Given

6. Consider $Ax = b$, where $x = \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$

   It's augmented matrix is,

   $\left[\begin{array}{cccc|c}1 & -4 & 7 & -5 & -1\\ 0 & 1 & -4 & 3 & 1\\ 2 & -6 & 6 & -4 & 0\end{array}\right]$

   $R_3 \to R_3 - 2R_1$

   $\sim \left[\begin{array}{cccc|c}1 & -4 & 7 & -5 & -1\\ 0 & 1 & -4 & 3 & 1\\ 0 & 2 & 8 & 6 & 2\end{array}\right]$

   $R_3 \to \frac{1}{2}R_3$

   $\sim \left[\begin{array}{cccc|c}1 & -4 & 7 & -5 & -1\\ 0 & 1 & -4 & 3 & 1\\ 0 & 1 & -4 & 3 & 1\end{array}\right]$

   $R_3 \to R_3 - R_2$

   $\sim \left[\begin{array}{cccc|c}1 & -4 & 7 & -5 & -1\\ 0 & 1 & -4 & 3 & 1\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$

   Since there is no row of the form $[0\ \ 0\ \ 0\ \ 0 \mid b]$, $b \neq 0$ so the system $Ax = b$ is consistent. Thus, b is in the range of linear transformation $x \to Ax$.

7.

(i) Here $T(x_1, x_2, x_3, x_4) = \begin{bmatrix}0\\ x_1 + x_2\\ x_2 + x_3\\ x_3 + x_4\end{bmatrix}$

$= \begin{bmatrix}0x_1 + 0x_2 + 0x_3 + 0x_4\\ 1x_1 + 1x_2 + 0x_3 + 0x_4\\ 0x_1 + 1x_2 + 1x_3 + 0x_4\\ 0x_1 + 0x_2 + 1x_3 + 1x_4\end{bmatrix}$

$= \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$

$= Ax$

Where, $A = \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$ is standard matrix for T and hence T is linear transformation.

(ii) Hint: see 7(i)

8.

(i) The standard matrix for T is

$A = [T(e_1)\ \ T(e_2)]$

$= \begin{bmatrix}3 & -5\\ 1 & 2\\ 3 & 0\\ 1 & 0\end{bmatrix}$

(ii) Hint: See Q. No. 8(i).

9.

$$T(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 5x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= Ax$$

So, $T(x) = (3, 8)$

$\Rightarrow$ $Ax = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

It's augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 4 & 5 & 8 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -4 \end{array}\right]$$

$\Rightarrow$ $x_1 + x_2 = 3$

$x_2 = -4$

Using back substitution,

$x_1 = 7$, $x_2 = -4$

$\therefore$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

10. Hint: Given in the question.
11. Hint: Given in the question.

12. See solution of Q. No. 7(i)

We have,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Using row operations, we get

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Showing that x_4 is free variable, so columns of A are linearly dependent. Hence, T is not one to one.

Also,

Since, $Ax = b$ is inconsistent for all $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \in \mathbb{R}^4$, so A is not onto.

13. Hint: See solution. Q. No. 12

Chapter 3

Matrix Algebra

Exercise 3.1

1. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$.

(i) Here,

$$BA = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 26 & -25 & 13 \\ 14 & -20 & 7 \end{bmatrix}$$

(ii) Here, A has more entries in a row than B has in a column. So, AB is not possible.

2. Let, $$A = \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}.$$

Then,

$$A - 5I = \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -14 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$.

Here,

$$AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

And $$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

Thus, $$AB = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} = AC.$$

4.

(i) $|A| = 2 \neq 0$, so A is nonsingular.

(iii) $|A| = 1 \neq 0$, so A is nonsingular.

5. (i) $\begin{vmatrix} 3 & 2 \\ 7 & 4 \end{vmatrix} = 12 - 14 = -2 \neq 0.$

So A is non singular.

6. Let, $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$

$|A| = 32 - 30 = 2$

Cofactor of $a_{11} = 4$ Cofactor of $a_{12} = -5$

Cofactor of $a_{21} = -6$ Cofactor of $a_{22} = 8$

$$\text{adj. }(A) = \begin{bmatrix} 4 & -5 \\ -6 & 8 \end{bmatrix}' = \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}}{2} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$$

$\therefore$ Since A^{-1} exists,

$X = A^{-1} C$

$\Rightarrow$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\Rightarrow$ $= \begin{bmatrix} 7 \\ -9 \end{bmatrix}$

7.

(i) $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$

$\sim \begin{bmatrix} 5 & 7 \\ 0 & -9/5 \end{bmatrix}$ $R_2 \to R_2 + \frac{3}{5}R_1$

$\therefore$ No. of pivot = 2

Size of A = 2

Hence, by investible matrix theorem, A is invertible ($\because$ no. of pivot = size of A).

(ii) Similar to (i).

(iii) $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix}$ $R_1 \to R_2$

$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix}$ $R_3 \to R_3 + 4R_1$

$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 + 3R_2$

No. of pivot = 2,

Size of A = 3

Hence, A is not invertible.

8. Note: An algorithm to fund A^{-1}, if A^{-1} exists

$[A : I] \sim [I : A^{-1}]$

$[A^{-1} : I] \sim [I : A]$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ -3 & 3 & -1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Using element row operations, we shall arrive

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 & 4 & 1 \\ 0 & 0 & 1 & 3 & 5 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

9. $A^T = \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -1 \\ -2 & -2 & 9 \end{bmatrix}$

$R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 4 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$\sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

Since, number of pivot in $A^T = 3 =$ size of A^T

$\therefore$ A^T is invertible.

10. See Q. No. 6.

Exercise 3.2

1. Not possible since $A_{11} B_{21}$ is not defined.

2. $Col_1(A)\ row_1(B) = \begin{bmatrix} -3 \\ 1 \end{bmatrix} [a\ \ b] = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix}$

$Col_2(A) . row_2(B) = \begin{bmatrix} 1 \\ -4 \end{bmatrix} [c\ \ d] = \begin{bmatrix} c & d \\ -4c & -4d \end{bmatrix}$

$Col_3(A) . row_3(B) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} [e\ \ f] = \begin{bmatrix} 2e & 2f \\ 5e & 5f \end{bmatrix}$

$\therefore$ $AB = \sum_{k=1}^{3} col_k(A) . row_k(B) = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} + \begin{bmatrix} c & d \\ -4c & -4d \end{bmatrix} + \begin{bmatrix} 2e & 2f \\ 5e & 5f \end{bmatrix}$

$= \begin{bmatrix} -3a + c + 2c & -3b + d + 2f \\ a - 4c + 5e & b - 4d + 5f \end{bmatrix}$

3. Similar to Q. No. 2

4. Here,

$A_{11}^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$; $A_{22}^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

and $\quad -A_{11}^{-1} A_{12} A_{22}^{-1} = \begin{bmatrix} 13 & 39 \\ 8 & -23 \end{bmatrix}$

Using formula for block upper triangular matrix for A^{-1},

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 13 & 39 \\ 1 & -2 & 8 & -23 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

Exercise 3.3

1. $I - C = \begin{bmatrix} 0.9 & -0.6 \\ -0.5 & 0.8 \end{bmatrix}$ Find $(I - C)^{-1}$ and use, $x = (I - C)^{-1} d$

2. $I - C = \begin{bmatrix} 0.8 & -0.2 & 0.0 \\ -0.3 & 0.9 & -0.3 \\ -0.1 & 0.0 & 0.8 \end{bmatrix}$

The augmented matrix of $(I - C)x = d$

$$\left[\begin{array}{ccc|c} 0.8 & -0.2 & 0.0 & 40 \\ -0.3 & 0.9 & -0.3 & 60 \\ -0.1 & 0.0 & 0.8 & 80 \end{array}\right]$$

using elementary row operations,

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 82.8 \\ 0 & 1 & 0 & 131 \\ 0 & 0 & 1 & 110.3 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 82.8 \\ 131 \\ 110 \end{bmatrix} \approx \begin{bmatrix} 83 \\ 131 \\ 110 \end{bmatrix}$$

3. Translation matrix $= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$(x, y) \to (3, 1)$

Rotation matrix $= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\xrightarrow{R} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Note: Translation transformation matrix $= \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$

and rotation transformation $= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. Similar

5, 6, 7 $\to$ See example

8. $$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\text{and } L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix}.$$

Chapter 4

Determinants

Exercise 4.1

1.

Here,

$$A = \begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = (3) \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} - 0 + (4) \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$$

$= (3)(-3-10) - 0 + (4)(10-0)$

$= -39 + 40$

$= 1.$

2.

Here,

$$A = \begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = (0) - (5) \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} - (1) \begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix}$$

$= 0 - (5)(4-0) + 1(16+6)$

$= -20 + 22$

$= 2.$

Q.3 - Q.8 Similar to Q.1.

9.

Here

$$A = \begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix}$$

[Expanded from a_{31}, so $(-1)^{3+1} = 1$]

$= (2)(5) \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}$

$= (2)(5)(7-6) = 10.$

10.

Here,

$$A = \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = (-3) \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

[Expanded from a_{23}, so $(-1)^{2+3} = -1$]

$= (-3)\left(5 \begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} - 0 + 4 \begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix}\right)$

$= (-3)(5(-10+12) - 0 + 4(-6+4))$

$= (-3)(10-8) = -6.$

Q.11 - Q.14 → Similar to Q.9.

Q.15 - Q.18 → Similar to Q.1.

19.

Here,

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

This shows if two rows are interchanged, the determinant changes its sign.

20. Here,

$\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = (18) - (20) = -2$

and $\begin{vmatrix} 3 & 4 \\ 5+3k & 6+4k \end{vmatrix} = (18+12k) - (20+12k) = -2$

$\Rightarrow \begin{vmatrix} 3 & 4 \\ 5+3k & 6+4k \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 3k & 4k \end{vmatrix}$

$= \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + k \begin{vmatrix} 3 & 4 \\ 3 & 4 \end{vmatrix}$

$= \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + k(0)$

$= \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$

One row times k (scalar) is added (or subtract) to another row, the determinant does not change.

Q.21 → Similar to Q.20

Q.22, 23 → Similar to Q.19

24.

Here, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{vmatrix} = (1)(1)(1) = 1$

The triangular determinant is same as the multiple of its leading diagonal entries.

Q.25 - Q.27 Similar to Q.24

28.

Here, $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 - (1) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0 = -(1)(1-0) = -1.$

29.

Here, $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 - 0 + (1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = (1)(0-1) = -1$

30. Since the elementary scaling matrix with k on its diagonal is an identity matrix with one non-zero entry is replaced by k. That is the elementary scaling matrix with k on its diagonal is:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Clearly the matrix is a triangular matrix, so its determinant is the product of its diagonal elements. So, the determinant of the matrix is k.

31.

Let,

$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then,

$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

Therefore,

$\det(EA) = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$

$\det(A) = \begin{vmatrix} a & d \\ c & b \end{vmatrix} = ad - bc,$

$\det(E) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

Thus,

det (E) det (A) = (–1) (ad – bc) = bc – ad = det (EA).

Q.32 - 34 → Similar to Q.31.

35. Let

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

Then

$$5A = 5\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 20 & 10 \end{bmatrix}$$

Therefore,

$$\det(A) = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 6 - 4 = 2$$

$$\det(5A) = \begin{vmatrix} 15 & 5 \\ 20 & 10 \end{vmatrix} = 150 - 100 = 50$$

And,

$$5\det(A) = 5(2) = 10 \neq \det(5A)$$

36. Let,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and let k is a scalar.

Then

$$kA = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Since,

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

And,

$\det(kA) = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2ad - k^2bc = k^2(ad - bc) = k^2 \det(A).$

Thus,

det (kA) = k^2 det (A) when k is 2×2 matrix.

Exercise 4.2

1. If two rows of a determinant A, are interchanged to produce a determinant B, then det (A) = – det (B).
In our problem, the first and second rows are interchanged.

2. If a row of a matrix A is multiplied by a scalar value k, produce a matrix B then

$$k\det(A) = \det(B)$$

In our problem, the first row of the determinant

$$\begin{vmatrix} 1 & -3 & 2 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix}$$

is multiplied by 2, produce another determinant which is given in right side.

3. If a multiple of one row of a matrix A is added to another row to produce a matrix B then det (B) = det (A).
In our problem, the first row is multiplied by –2 and add it with second row then

$$\det(A) = \begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 5 & -4 & 7 \end{vmatrix}$$

produce another matrix B where

$$\det(B) = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 5 & -4 & 7 \end{vmatrix}$$

Then det (A) = det (B).

4.

Similar to Q.3.
Here first row is multiplied by -3 and is added with third row.

5.

Here,

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - 3R_2]$$

$= (1)(1)(3)$ [Multiple leading diagonal entries]
$= 3$

Q.6 - Q.9 → Similar to Q.5.

10. Here

$$\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & -2 & 0 & 8 & -1 \\ 0 & -4 & 8 & 2 & 13 \end{vmatrix} \quad \begin{matrix} \text{Applying} \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - 3R_1 \\ R_5 \to R_5 - 3R_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & -4 & 7 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad \begin{matrix} \text{Applying} \\ R_4 \to R_4 + R_2 \\ R_5 \to R_5 + 2R_2 \end{matrix}$$

$$= - \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -4 & 7 & -3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad \text{Interchanging third and fourth rows.}$$

$= (-1)(1)(2)(-4)(3)(1)$
$= 24$

11.
Here,

$$\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 0 & 0 & 2 & 1 \end{vmatrix} \quad [\text{Applying } R_4 \to R_4 - 2R_1]$$

$$= (-5) \begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 9 \\ 0 & 2 & 1 \end{vmatrix} \quad \left[\text{Expanded from 2nd column.}\right]$$

$$= (-5) \begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & 3 \\ 0 & 2 & 1 \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 + 2R_1]$$

$$= (-5)(3) \begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix}$$

$= (-5)(3)(-2 - 6)$
$= 120.$

Q.12 - Q.14 → Similar to Q.11.

15. Let,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Now,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = (5) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \left[\text{Taking-out the common factor 5 from 3rd row.}\right]$$

$= (5)(7) = 35.$

16. Similar to Q.15.

17. Let,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Now,

$$\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = (-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \left[\text{Interchanging the 2nd and 3rd row.}\right]$$

$= (-1)(7) = -7$

18. Let,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Now,

$$\begin{vmatrix} a & b & c \\ 2d + a & 2e + b & 2f + c \\ g & h & i \end{vmatrix} = (2) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$$

$= (2)(7) + 0$ $\begin{bmatrix}\text{1st and 2nd rows have}\\ \text{same value in 2nd}\\ \text{determinant so its}\\ \text{value is 0.}\end{bmatrix}$

$= 14$

19. Similar to Q.18.

20.

Here,

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 \\ -3 & -5 & 0 \\ 1 & 2 & 1 \end{vmatrix} \begin{bmatrix}\text{Applying}\\ R_2 \to R_2 - 4R_3\end{bmatrix}$$

$$= (1)\begin{vmatrix} 2 & 3 \\ -3 & -5 \end{vmatrix}$$

$$= (1)(-10 + 9)$$

$$= -1 \neq 0$$

This means the given matrix is invertible.

Note: **The matrix A is invertible if det (A) ≠ 0.**

21. → Similar to Q.20.

22. Let

$$v_1 = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, v_2 = \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$$

Here,

$$\det[v_1 \quad v_2 \quad v_3] = \begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix}$$

$$= (-1)(-7)\begin{vmatrix} 6 & -5 \\ -7 & 6 \end{vmatrix} + 0 + (-1)(2)\begin{vmatrix} 4 & -3 \\ 6 & -5 \end{vmatrix}$$

$$= (7)(36 - 35) - 2(-20 + 18)$$

$$= 7 + 4$$

$$= 11 \neq 0$$

This means v_1, v_2, v_3 are linearly independent.

Note: **The vectors $v_1, v_2, \ldots, v_n$ are linearly independent if $\det[v_1, v_2, \ldots, v_n] \neq 0$ and linearly dependent for otherwise.**

Q. 23 - 24 → Similar to Q.22.

Exercise 4.3

1.

Let the given system is in the form Ax = b then

$$A = \begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here,

$$\det(A) = \begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix} = 20 - 14 = 6 \neq 0$$

So, the system has unique solution. Also,

$$\det(A_1(b)) = \begin{vmatrix} 3 & 7 \\ 1 & 4 \end{vmatrix} = 12 - 7 = 5$$

$$\det(A_2(b)) = \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = 5 - 6 = -1$$

Now, by Cramer's rule,

$$x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{5}{6}$$

and $$x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{-1}{6}$$

Thus, the solution of given system is $x_1 = \frac{5}{6}, x_2 = \frac{-1}{6}$.

Q.2 - Q.4 → Similar to Q.1.

5. Let the given system is in the form Ax = b then

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 7 \\ -8 \\ -3 \end{bmatrix}$$

Here, $$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (2)\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + 0$$

$$= 2(0 - 1) + 3(2 - 0)$$

$$= -2 + 6$$

$$= 4 \neq 0$$

So, the system has unique solution. Also,

$$\det(A_1(b)) = \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} = -(1)\begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix} + 0 - (1)\begin{vmatrix} 7 & 0 \\ -8 & 1 \end{vmatrix}$$

$$= -1(-16 + 3) + 0 - 1(7 - 0) = 13 - 7 = 6$$

$$\det(A_2(b)) = \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} -8 & 1 \\ -3 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 7 & 0 \\ -3 & 2 \end{vmatrix} + 0$$
$$= 2(-16+3) + 3(14-0) + 0$$
$$= -26 + 42$$
$$= 16$$

$$\det(A_3(b)) = \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} = 2\begin{vmatrix} 0 & -8 \\ 1 & -3 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 7 \\ 1 & -3 \end{vmatrix} + 0$$
$$= 2(0+8) + 3(-3-7) + 0$$
$$= 16 - 30$$
$$= -14$$

Now, by Cramer's rule,

$$x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{6}{4} = \frac{3}{2} = 1.5$$
$$x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{16}{4} = 4$$
$$x_3 = \frac{\det(A_3(b))}{\det(A)} = \frac{-14}{4} = \frac{-7}{2} = -3.5$$

Thus, the solution of given system is $x_1 = \frac{2}{3}$, $x_2 = 4$, $x_3 = \frac{-7}{2}$.

6. Similar to Q.5.

7. Let the given system is in $Ax = b$ then

$$A = \begin{bmatrix} 6s & 4 \\ 9 & 2s \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, b = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Here,

$$\det(A) = \begin{bmatrix} 6s & 4 \\ 9 & 2s \end{bmatrix} = 12s^2 - 36 = 12(s^2-3) \neq 0 \text{ for } s \neq \pm\sqrt{3}.$$

This means the has unique solution for $s \neq \pm\sqrt{3}$.

Also,
$$\det(A_1(b)) = \begin{vmatrix} 5 & 4 \\ -2 & 2s \end{vmatrix} = 10s + 8$$
$$\det(A_2(b)) = \begin{vmatrix} 6s & 5 \\ 9 & -2 \end{vmatrix} = -12s - 45$$

Now, by Cramer's rule

$$x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{10s+8}{12(s^2-3)} = \frac{5s+4}{6(s^2-3)}$$
$$x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{-12s-45}{12(s^2-3)} = \frac{-4s-15}{4(s^2-3)}$$

Thus, the solution of the given system is

$$x_1 = \frac{5s+4}{6(s^2-3)}, x_2 = \frac{-4s-15}{4(s^2-3)} \text{ for } s \neq \pm\sqrt{3}.$$

Q.8 - Q.10 → Similar to Q.7.

11.

Let $$A = \begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Here,

$$\det(A) = \begin{vmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{vmatrix} = -(3)\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} + 0 - 0$$
$$= -3(-2+1)$$
$$= 3 \neq 0$$

This means A is invertible and A^{-1} exists.

Here, the cofactors of A are,

$$c_{11} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \qquad c_{12} = -\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = -3$$
$$c_{13} = \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 3 \qquad c_{21} = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = -(-2+1) = 1$$
$$c_{22} = \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1 \qquad c_{23} = -\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = 2$$
$$c_{31} = \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} = 0 \qquad c_{32} = -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -3$$
$$c_{33} = \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} = 6$$

Therefore, the cofactor matrix of A is

$$\text{Cofactor matrix of A} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 0 & -3 & 3 \\ 1 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix}$$

So, the adjoint matrix of A is,

$$\text{Adj}(A) = (\text{Cofactor matrix of A})^T$$
$$= \begin{bmatrix} 0 & -3 & 3 \\ 1 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix}$$

Now, the inverse matrix of A i.e., A^{-1} is,

$$A^{-1} = \frac{1}{\det(A)}\text{Adj.}(A) = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 0 \\ -1 & -1/3 & -1 \\ 1 & 2/3 & 2 \end{bmatrix}$$

Q.12 - Q.16 → Similar to Q.11.

17. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Here,

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

This means A is invertible and A^{-1} exists for $ad - bc \neq 0$.

The cofactor matrix of A is

$$\text{cofactor matrix of } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

and the adjoint matrix of A is,

$$\text{adj.}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Now, the inverse matrix (A^{-1}) of A is,

$$A^{-1} = \frac{1}{\det(A)} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

18.

Let all the entries of a matrix A, are integers.

Given that $\det(A) = 1 \neq 0$. So, the inverse of A, exist. And,

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \text{adj}(A) \ldots \text{(i)}$$

Since the multiple, addition and subtraction between integers, is again an integer. So, all the entries of cofactor matrix, are integers. Therefore, all the entries of adjoint matrix, are integers.

So, by (i) all the entries of A^{-1}, are integers.

19.

Let the vertices of the parallelogram are A(0, 0), B(5, 2), C(6, 4), D(11, 6).

Then, $u = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5, 2) - (0, 0) = (5, 2)$

$v = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (6, 4) - (0, 0) = (6, 4)$

Since u and v are adjacent sides of the parallelogram ABCD.

Now, the area of the paralleogram ABCD is

$$\text{area} = \det \begin{bmatrix} u \\ v \end{bmatrix} = \begin{vmatrix} 5 & 2 \\ 6 & 4 \end{vmatrix} = 20 - 12 = 8$$

Thus, the area of the parallelogram ABCD is 8 sq. units.

Q.20 – Q.22 → Similar to Q. 19.

23.

Let A is invertible. So, A^{-1} exists. And, we have

$$AA^{-1} = I$$

So, $\det(AA^{-1}) = \det(I) = 1$

$\Rightarrow \quad \det(A)\det(A^{-1}) = 1$

$\Rightarrow \quad \det(A^{-1}) = \dfrac{1}{\det(A)}.$

24. Let the one vertex of the parallelopiped is at origin and the adjacent vertices are at (1, 0, –2), (1, 2, 4) and (7, 1, 4). Then,

$$\det(A) = \begin{vmatrix} 1 & 1 & 7 \\ 0 & 2 & 1 \\ -2 & 4 & 4 \end{vmatrix}.$$

$$= 1\begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} + 0 - 2\begin{vmatrix} 1 & 7 \\ 2 & 1 \end{vmatrix}.$$

$= 4 + 0 - 26.$

$= -22.$

Thus, the volume of the parallelepiped with $|-22| = 22$.

25.

(i) Given that S is the parallelogram determined by the vectors $b_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. So,

$$\det(S) = \begin{vmatrix} -2 & -2 \\ 3 & 5 \end{vmatrix} = -10 + 6 = -4.$$

Thus,

$$\text{Area of } S = |-4| = 4.$$

And given that

$$A = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}$$

Then,

$$\det(A) = \begin{vmatrix} 6 & -2 \\ -3 & 2 \end{vmatrix} = 12 - 6 = 6.$$

Therefore, the a area of S under the mapping $x \rightarrow Ax$ is,

area of image of S = Area of T(S)

= |det A| [Area of S]

= 6×4 = 24 sq. unit

(ii) Similar to (i).

26. The matrix represented to a square having vertices at O(0, 0, 1), A(1, 0, 1), B(0, 1, 1) and C(1, 1, 1) is

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and given matrix is

$$A = \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the effect of pre-multiplication of S by A is,

$$S' = A.S = \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a & 1+a & 2+a \\ b & 1+b & 1+b \\ 1 & 1 & 1 \end{bmatrix}$$

This means the vertices of the effect of the square A are O'(a, b, 1), A'(1 +a, 1 + b, 1), B(2 +a,1 + b, 1).

Chapter 5

Vector Space

Exercise 5.1

1. **Solution.** $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} x \ge 0, y \ge 0 \right\}$

a. If u, v $\in$ W then u and v lies in 1st quadrant i.e., u and v have non-negative entries, we know that the sum of non-negative numbers are non-negative so, u + v has non-negative entries. Thus, u + v $\in$ W

b. Take, $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in W$ and $c = -1$

Then, $cu = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin W$ so W is not a vector space.

2. **Solution.** Given, W be the union of the 1st and 3rd quadrants in the xy-plane

i.e., $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x\,y \ge 0 \right\}$

a. Let $u = \begin{bmatrix} x \\ y \end{bmatrix} \in W$ and c be any scalar then

$cu = c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W since $xy \ge 0$, so $(cx)(cy) = c^2 (xy) \ge 0$

b. Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in W$ and $v = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \in W$

but $u + v = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \notin W$ because u + v does not lies in 1st or 3rd quadrants.

c. W is not a vector space.

3. **Solution.** Given, $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 1 \right\}$

Let $u = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \in H$ and $c = 4$ then

$cu = 4 \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin H$

H is not closed under multiplication. So, it is not subspace of R^2.

4. **Solution.**

(a) Given, $P(t) = at^2$

So $P(t) = \text{Span}\,\{t^2\}$

Using theorem 1, the set span by $\{t^2\}$ is subspace of P_n.

(b) No, the polynomial of the form $P(t) = a + t^2$ is not subspace of P_n because, the set does not contains zero vectors.

(c) No, the set is not closed under multiplication by scalars which are not integers.

(d) Yes, it is subspace of P_n because the zero vector is in this set H. If p and q is H then $(p + q)(0) = p(0) + q(0) = 0 + 0 = 0$

So, $p + q \in H$

and $(cp)(0) = c.\,p(0) = c\,.\,0 = 0$ so $cp \in H$.

5. **Solution.** Given $H = \left\{ \begin{bmatrix} -2S \\ 5S \\ 3S \end{bmatrix} \right\} = \left\{ S \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \right\} = \{S\,V\}$ where, $v = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \in R^3$

$\therefore$ H = span {v} and H is subspace of R^3 because H is spaned by v and v is in R^3 which is vector space (by theorem 1).

6. **Solution.** Given $H = \left\{ \begin{bmatrix} 5t \\ 0 \\ -2t \end{bmatrix} \right\} = \{tv\}$ where, $v = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} \in R^3$

Since, the set H = Span {v} where $v \in R^3$ so using theorem 1 H is subspace of R^3.

7. **Solution.** Given, $W = \left\{ \begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix} \right\} = \left\{ b \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} = \{(bv_1 + cv_2)\}$

where $v_1 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Since, W = Span $\{v_1, v_2\}$ and $v_1, v_2 \in R^3$ so using theorem 1, W is subspace of R^3.

8. **Solution.**

Given, $W = \left\{ \begin{bmatrix} 2s + 4t \\ 3s \\ 2s - 3t \\ 3t \end{bmatrix} \right\} = \left\{ S \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -3 \\ 3 \end{bmatrix} \right\} = \{(sv_1 + tv_2)\}$

where $v_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 3 \end{bmatrix}$

Which shows that W = Span $\{v_1, v_2\}$ and since $v_1, v_2 \in R^4$. So by using theorem 1 W is subspace of R^4.

9. **Solution.**

a. The vector w is not in the set $\{v_1, v_2, v_3\}$. There are 3 vectors in the set $\{v_1, v_2, v_3\}$.

b. The set span $\{v_1, v_2, v_3\}$ contains infinitely many vectors.

c. The vector w is in subspace spanned by $\{v_1, v_2, v_3\}$. iff the equation $av_1 + bv_2 + cv_3 = w$ has a solution.

By using row reducing the augmented matrix for this system

$$[v_1\ v_2\ v_3\ w] = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ So the equation has a solution because there is free variables.

Thus, w is in subspace spanned by $\{v_1, v_2, v_3\}$.

10. **Solution.** If w in the subspace spanned by $\{v_1, v_2, v_3\}$, iff the equation $av_1 + bv_2 + cv_3 = w$ has a solution.

Since, $[v_1 \; v_2 \; v_3 \; w]$

$$= \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ -1 & 3 & 6 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which show that $av_1 + bv_2 + cv_3 = W$ has a solution.

$\therefore$ w is in the subspace spanned by $\{v_1, v_2, v_3\}$.

11. **Solution.**

a. Given $W = \left\{\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}\right\}$

This set does not contains the zero vector, so it is not vector space.

b. Similarly, it also does not contains zero vector.

c. Given, $W = \left\{\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} \; a, b, c, \in R\right\}$

$$= \left\{a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$= \{av_1 + bv_2 + cv_3, a, b, c \in R\}$$

where, $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

which shows that $W = \text{Span}\{v_1, v_2, v_3\}$ since, $v_1, v_2, v_3 \in R^4$

So W is subspace of R^4 i.e., W is vector space.

Here Set $S = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$

d. We have,

$$W = \left\{\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

$\therefore$ $S = \left\{\begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a set that spans W.

Exercise 5.2

1. **Solution.** $AW = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3-15+12 \\ 6-6+0 \\ -8+12-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$

$\therefore$ W is in Nul A.

Similarly, $Au = 0$ So, u is also in Nul A.

2. **Solution.**

(i) 1st we find the general solution of $Ax = 0$ in term of the free variabl

Since, $[A, 0] \sim \begin{bmatrix} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}$

Here, x_3 and x_4 are free variables so

$x_1 = 2x_3 - 4x_4$

$x_2 = -3x_3 + 2x_4$

$\therefore$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ A spanning set for Nul A is $\left\{\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}$

(ii) Similarly as (i) we get $\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

(iii) Similarly we get $\left\{\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}\right\}$

3. **Solution.**

(i) We have,

$$\begin{bmatrix} 2s+t \\ r-s+2t \\ 3r+s \\ 2r-s-t \end{bmatrix} = r\begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$\therefore$ $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} b-c \\ 2b+3d \\ b+3c-3d \\ c+d \end{bmatrix} = b\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ 1 & 3 & -3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} b \\ c \\ d \end{bmatrix}$

$\therefore$ $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ 1 & 3 & -3 \\ 0 & 1 & 1 \end{bmatrix}$

4. **Solution.**

(i) The matrix A is 4×2 order thus NulA is a subspace of R^2 and col A is a subspace of R^4.

$\therefore$ For NulA, $k = 2$, for ColA, $k = 4$

Similarly for

(ii) $k = 3$ for NulA and ColA

(iii) $k = 5$ for NulA and $k = 2$ for ColA

(iv) $k = 5$ for NulA and $k = 1$ for ColA

5. **Solution.** Consider the system with agumeted matrix

$[A \;\; w] = \begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

which shows that the system is consistent so w is in ColA.

Since, $Aw = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ w is in NulA.

6. **Solution.** Since,

$[A, W] \sim \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is consistent

$\therefore$ w is in ColA.

Also $Aw = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad\therefore$ w is in NulA

7. **Solution.**

For (i) we have,

$[A \;\; 0] = \begin{bmatrix} 6 & -4 & 0 \\ -3 & 2 & 0 \\ -9 & 6 & 0 \\ 9 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ The general solution, $x_1 = \frac{2}{3}x_2$

x_2 free variables

$\therefore$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (taking $x_2 = 3$)

$\therefore$ $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is in NulA

Any non zero vector of column of A is in ColA.

$\therefore$ $\begin{bmatrix} 6 \\ -3 \\ -9 \\ 9 \end{bmatrix}$ is in ColA.

(ii) We have

$[A \;\; 0] = \begin{bmatrix} 5 & -2 & 3 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ -5 & 7 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -2 & -2 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Which shows that the system is consistent so the general solution is

$x_1 = -x_3$

$x_2 = -x_3$

x_3 free

$\therefore$ $x = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ taking $x_3 = -1$

$\therefore$ $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is in NulA.

Any column of A is a nonzero vector is ColA.

8. **Solution.**

a. Since the augmented matrix

$[B \;\; a_3] \sim \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $[B \;\; a_5] \sim \begin{bmatrix} 1 & 0 & 0 & 10/3 \\ 0 & 1 & 0 & -26/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Which shows that both system is consistent

$\therefore$ a_3 and a_5 are in ColB.

b. We have using above

$A \sim \begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ The general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$\therefore$ The spanning set for Nul A is

$$\left\{ \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

c. The reduced row echelon form of A shows that the columns of A are linearly dependent and do not span R^4.

$\therefore$ T is neither one to one nor onto.

9. **Solution.** We have,

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= Ax$$

Where $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Since, kerT = Nul A so

$$[A \ \ 0] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$\therefore$ The general solution, $x_1 = x_3$, $x_2 = -x_3$, x_3 free

$\therefore$ $x = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ $\ker T = \left\{ x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ x_3 \in R \right\}$

<u>For ImT</u> we have,

$$\text{Col } A = \text{span } \{a_1, a_2, a_3\}$$

$$= \left\{ a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a+b \\ b+c \end{pmatrix} : a, b, c \in R \right\}$$

10. **Solution.** By rearranging the equations that describe the elements of H, we see that H is the set of all solution of the following system of homogeneous linear equations

$$a - 2b + 5c - d = 0$$
$$-a - b + c = 0$$

Thus by using theorem 2, H is a subspace of R^4.

Exercise 5.3

1. **Solution.**

(i) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

and $A = [v_1 \ \ v_2 \ \ v_3]$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Since there is no free variables, all columns of A are pivot columns so $\{v_1 \ v_2 \ v_3\}$ is a basis for R^3.

(ii), (iii), (iv) and (v) $\rightarrow$ Similarly to (i).

(vi) Let $v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$

$$A = [v_1 \ \ v_2 \ \ v_3]$$

$$= \begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & -3 & -15 \\ 0 & 1 & 13 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 13 \\ 0 & -3 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 13 \\ 0 & 0 & 24 \end{bmatrix}$$

Which shows that $\{v_1, v_2, v_3\}$ is basis for R^3 because each columns and row of A has pivot positions.

vii. Since there is more vectors then their entries. So the set is linearly dependent so it is not basis for R^3.

viii. Similar as vii.

2. **Solution.**

Given, $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$

Since $A = [v_1 \ \ v_2] = \begin{bmatrix} 1 & -2 \\ -2 & 7 \\ 3 & -9 \end{bmatrix}$ has at most two pivot positions which shows that $\{v_1, v_2\}$ can not span R^3.

The set $\{v_1, v_2\}$ is not basis for R^3.

No, $\{v_1, v_2\}$ is not basis for R^2 because v_1 and v_2 are not from R^2.

Solution.

Let $A = [v_1 \ \ v_2 \ \ v_3 \ \ v_4]$

$$= \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 1 & 1/5 & -1 \\ 0 & 1 & 1/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 1 & 1/5 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Since, 1st and 2nd columns of matrix A are pivot columns.

$\{v_1, v_2\}$ is basis for subspace W.

Solution.

Since,

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = sv_1 + sv_2$$

which shows that every vector in H is a linear combination of v_1 and v_2 so $\{v_1, v_2\}$ spans H.

Also since $\{v_1, v_2\}$ is linearly independent so it is basis for H.

Solution.

We find 1st general solution of $Ax = 0$ for

$$[A \ \ 0] = \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 3 & -2 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & -2 & 10 & -8 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $x_1 = 3x_3 - 2x_4$, $x_2 = 5x_3 - 4x_4$, x_3 free, x_4 free

$$x = \begin{bmatrix} 3x_3 - 2x_4 \\ 5x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

The set $\left\{\begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}\right\}$ is basis for Nul A.

6. **Solution.**

a. Given $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$

Now, if we reduced A as in echelon form then

$$A \sim \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that 1st and 2nd columns are pivot columns so

$\left\{\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}\right\}$ is basis for col A.

To find basis for Nul A, we find the general solution of

$Ax = 0$

Here, $x_1 = -6x_3 - 5x_4$

$$x_2 = -\frac{5}{2}x_3 - \frac{3}{2}x_4$$

x_3 free, x_4 free

$$\therefore \quad x = x_3\begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $\left\{\begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for Nul A.

b. Given $A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}$ doing same as above we get

$\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 9 \\ 9 \end{bmatrix}\right\}$ is basis for col A.

and $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for Nul A.

7. **Solution.**

Given, $y = -3x$

or $3x + y = 0$

or $[3 \ \ 1]\begin{bmatrix} x \\ y \end{bmatrix} = 0$

or $AX = 0$ where $A = [3 \;\; 1]$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

Since, $[A \;\; 0] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

which shows that x is basic and y is free variables.

So,

The general solution

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}y \\ y \end{bmatrix} = y\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} = -3y\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\therefore$ $\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$ is a basis for Nul A

8. **Solution.** Since $4v_1 + 5v_2 - 3v_3 = 0$ which shows that each of the vectors is a linear combination of the others.

So, the sets $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ all span H

Since, none of the three vectors is a multiple of any of the others, the set $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are linearly independent and thus each forms a basis for H.

Exercise 5.4

1. **Solution.**

(i) We have, $x = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ -25 \end{bmatrix} + \begin{bmatrix} -12 \\ 18 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

(ii) Similarly, we get $\begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 8 \\ -5 \\ 1 \end{bmatrix}$

2. **Solution.** (i) Let $[x]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ then,

$x = c_1 b_1 + c_2 b_2$

$$\begin{pmatrix} -1 \\ -6 \end{pmatrix} = \begin{pmatrix} c_1 \\ -4c_1 \end{pmatrix} + \begin{pmatrix} 2c_2 \\ -3c_2 \end{pmatrix}$$

$\therefore$ $c_1 + 2c_2 = -1$ .... (i)

$-4c_1 - 3c_2 = -6$ .... (ii)

Solving we get

$c_1 = 3$, $c_2 = -2$

$\therefore$ $[x]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(ii) We have,

$[b_1 \;\; b_2 \;\; b_3 \;\; x]$

$$= \begin{bmatrix} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore$ $[x]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

(iii) Similarly, we get $[x]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

3. **Solution.**

(i) $P_B = [b_1 \;\; b_2] = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$

(ii) $P_B = [b_1 \;\; b_2 \;\; b_3] = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ 6 & -4 & 3 \end{bmatrix}$

4. **Solution.**

(i) Here, $P_B = \begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

So $P_B^{-1} = \frac{1}{-1}\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}$

$\therefore$ $[x]_B = P_B^{-1} x = \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(ii) Similarly as above we get $[x]_B = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$

5. **Solution.** Given,

$1 + 2t^2, 4 + t + 5t^2, 3 + 2t$

So the coordinate vectors are (1, 0, 2), (4, 1, 5) and (3, 2, 0) respectively.

Now, using these vectors in columns in A, the augmented matrix

$$\begin{pmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that columns of A are linearly dependent. So the corresponding polynomials are linearly dependent.

6. **Solution.** Let (c_1, c_2, c_3) be the coordinate vector of p(t)

Then,

$c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + 2t + t^2) = 1 + 4t + 7t^2$

or $(c_1 + c_3) + (c_2 + 2c_3)t + (c_1 + c_2 + c_3)t^2 = 1 + 4t + 7t^2$

Comparing we get

$c_1 + c_3 = 1$ .... (i)

$c_2 + 2c_3 = 4$ .... (ii)

$c_1 + c_2 + c_3 = 7$ .... (iii)

Now, the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

So $[P]_B = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$

7. **Solution.** Solve as Q. No. 6 then we get

$$[P]_B = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

8. **Solution.**

(a) The coordinate vectors of given polynomials are (1, 0, 1), (0, 1, -3) and (1, 1, -3) respectively.

Let A be a matrix using them as in columns then

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

which shows that the matrix A is invertible. So the three columns of A form a basis for R^3. So the corresponding polynomials are form a basis for P_2. (Isomorphism between R^3 and P_2).

(b) Since, $[q]_B = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ so

$q = -1 . P_1 + 1 . P_2 + 2 . P_3$

$q = -(1 + t^2) + (t - 3t^2) + 2(1 + t - 3t^2)$

$q = 1 + 3t - 10t^2$

$\therefore$ $q(t) = 1 + 3t - 10t^2$

Chapter 6

Vector Space Continued

Exercise 6.1

1. **Solution.**

(i) Since $H = \left\{\begin{bmatrix} 2a \\ -4b \\ -2a \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + b\begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}\right\}$

which shows that $H = \text{span}\{v_1, v_2\}$ where $v_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}$.

Since, v_1 and v_2 are not multiple of each other so, $\{v_1, v_2\}$ is linearly independent. Thus, it is basis for H and dim H = 2.

(ii) Since, $H = \left\{\begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}\right\}$

$= \{(av_1 + bv_2 + cv_3): a, b, c \in R\}$

which shows that $H = \text{span}\{v_1, v_2, v_3\}$

also, since $v_1 \pm 0$, v_2 is not a multiple of v_1 and v_3 is not multiple of v_1 and v_2.

$\therefore$ $\{v_1, v_2, v_3\}$ is linearly independent (or using $[v_1\ v_2\ v_3\ 0]$ show for linearly independent)

Thus, $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}\right\}$ is basis for H and dim H = 3

Similarly do for others

(iii) $\left\{\begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$, dim H = 2 (iv) $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 2 \\ 6 \end{bmatrix}\right\}$, dim H = 3

(v) We have, $H = \{(a, b, c): a - 3b + c = 0, b - 2c = 0, 2b - c = 0\}$

which shows that H = Nul A where $A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -1 \end{bmatrix}$

Since, $[A\ \ 0] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Which shows that the all columns of A are pivot columns and there is no free variables.

Thus, H = Nul A = {0}, and dim H = 0

2. **Solution.** According to questions we have,

$H = \left\{\begin{bmatrix} a \\ b \\ a \end{bmatrix} : a, b \in R\right\} = \left\{a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : a, b \in R\right\}$

$= \{(av_1 + bv_2): a, b \in R\}$

$= \text{span}\{v_1, v_2\}$

Since, $\{v_1, v_2\}$ is linearly independent so, it is basis

$\therefore$ dim H = 2

3. **Solution.**

(i) Let A form a matrix using these vectors in columns

$A = \begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

There are three pivot columns so

dim = 3

(ii) Similar as (i) we get dim = 3.

4. **Solution.**

(i) Since these are three pivot columns and there are 2 not pivot columns

So, dim Col A = 3

dim Nul A = 2

(ii) $A = \begin{bmatrix} 3 & 2 \\ -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$

So dim Col A = 2

dim Nul A = 0

(iii) dim col A = 2

dim Nul A = 2

(iv) $A \sim \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

dim Col A = 2

dim Nul A = 2

(v) $A \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

dim Col A = 3

dim Nul A = 2

Exercise 6.2

1. **Solution.**

(i) Since there are two pivot position so

Rank A = 2, dim NulA = 2

For bases, 1st and 2nd columns are pivot columns so

$\left\{\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}\right\}$ is basis for ColA

{(1, 0, –1, 5), (0, –2, 5, –6)} is basis for Row A.

For NulA, the general solution of $Ax = 0$ is

$x_1 = x_3 - 5x_4$

$x_2 = \frac{5}{2}x_3 - 3x_4$

x_3 free

x_4 free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{\begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right\}$ is basis for NulA.

similarly do for (ii) and (iii)

(ii) Rank A = 3, dim NulA = 3

Basis for col A = $\left\{\begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix}\right\}$

Basis for row A = {(2, 6, –6, 6, 3, 6), (0, 3, 0, 3, 3, 0), (0, 0, 0, 0, 3, 0)}

Basis for NulA = $\left\{\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

(iii) Rank A = 5, dim NulA = 1

Basis for ColA = $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 6 \\ 0 \\ -1 \end{bmatrix}\right\}$

Basis for row A = {(1, 1,–2, 0, 1,–2), (0, 1,–1, 0,–3,–1), (0, 0, 1, 1,–13, –1), (0, 0, 0, 0, 1, –1), (0, 0, 0, 0, 0, 1)}

Basis for NulA = $\left\{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

2. **Solution.** Using row-operation we have

$$A \sim \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1st and 2nd columns are pivot columns so

Rank A = 2, dim NulA = 3

Basis for col A = $\left\{\begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix}\right\}$

Basis for row A = {(1, –2, –4, 3, –2), (0, 3, 9, –12, 12)}

<u>For Nul A</u>

$$A \sim \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -5 & 6 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the general solution of $Ax = 0$ is

$x = -2x_3 + 5x_4 - 6x_5$

$x_2 = -3x_3 + 4x_4 - 4x_5$

x_3 free

x_4 free

x_5 free

$\therefore$ $\left\{\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is basis for Nul A.

3. **Solution.** By rank theorem dim NulA = 7 - Rank A = 7 - 3 = 4

Since, dim Row A = Rank A = 3

Also, rank A^T = dim ColA^T = dim Row A

$\therefore$ Rank $A^T = 3$

$\therefore$ dim NulA = 4, dim Row A = 3, Rank $A^T = 3$

4. **Solution.** Yes ColA = R^4 since A has four pivot columns. dim ColA = 4. Thus ColA is a four dimensional subspace of R^4, and ColA = R^4

No, NulA ≠ R^3, it is true that dim Nul A = 3, but NulA is a subspace of R^7.

Exercise 6.3

1. **Solution.** (a) Here, $[b_1]_c = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $[b_2]_c = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$

Now, $\underset{C \leftarrow B}{P} = [[b_1]_C\ [b_2]_C] = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$

(b) Since, $[x]_B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$[x]_c = C \overset{P}{\leftarrow} B\,[x]_B = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

2. **Solution.** Similar as Q. No. 1
we get

$C \overset{P}{\leftarrow} B = \begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix}$

$[x]_e = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$

3. **Solution.**

a. Here, $[a_1]_B = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$, $[a_2]_B = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $[a_3]_B = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$\therefore$ $B \overset{P}{\leftarrow} A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

b. Here, $[x]_A = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$\therefore$ $[x]_B = B \overset{P}{\leftarrow} A[x]_A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$

4. **Solution.** Do as Q. No. 3 we get

$D \overset{P}{\leftarrow} F = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$[x]_D = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$

5. **Solution.**

a. We have,

$[c_1\ c_2\ b_1\ b_2] \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}$

$\therefore$ $C \overset{P}{\leftarrow} B = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$

$B \overset{P}{\leftarrow} C = \left(C \overset{P}{\leftarrow} B\right)^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$

Similarly do for b, c, d

b. $C \overset{P}{\leftarrow} B = \begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix}$, $B \overset{P}{\leftarrow} C = \begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix}$

c. $C \overset{P}{\leftarrow} B = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$, $B \overset{P}{\leftarrow} C = \begin{bmatrix} 1/2 & 3/2 \\ 0 & -1 \end{bmatrix}$

d. $C \overset{P}{\leftarrow} B = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$, $B \overset{P}{\leftarrow} C = \begin{bmatrix} 0 & -1/2 \\ 1 & 3/2 \end{bmatrix}$

6. **Solution.** Let $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$ then

$[b_1]_c = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $[b_2]_c = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$, $[b_3]_c = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

$\therefore$ $C \overset{P}{\leftarrow} B = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$

since, $x = -1 + 2t$, So $[x]_c = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ $[x]_c = C \overset{P}{\leftarrow} B\,[x]_B$

$\therefore$ The augmented matrix is

$\left[C \overset{P}{\leftarrow} B\,[x]_C\right] = \begin{bmatrix} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ $[x]_B = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

7. **Solution.**

(i) Let $y_k = 2^k$. Then,

$y_{k+2} + 2y_{k+1} - 8y_k = 2^{k+2} + 2 \,.\, 2^{k+1} - 8.2^k$
$= 2^k (2^2 + 2^2 - 8)$
$= 2^k \times 0 = 0$ for all k

Since the difference equation holds for all k, 2^k is a solution.

Let $y_k = (-4)^k$, then

$y_{k+2} + 2y_{k+1} - 8y_k = (-4)^{k+2} + 2\,(-4)^{k+1} - 8\,(-4)^k$
$= (-4)^k \{(-4)^2 + 2 \,.\, (-4) - 8\}$
$= (-4)^k \,.\, 0 = 0$ for all k

$\therefore$ For all k, $(-4)^k$ is a solution.

(ii) Similarly do for (ii)

8. **Solution.** (i) Compute and row reduce the casorati matrix for the signals 1, 2^k and $(-2)^k$, setting $k = 0$,

$\begin{bmatrix} 1^0 & 2^0 & (-2)^0 \\ 1^1 & 2^1 & (-2)^1 \\ 1^2 & 2^2 & (-2)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the signals $\{1^k, 2^k, (-2)^k\}$ is linearly independent.

i) Similarly as above linearly independent.

i) Similarly as above linearly independent.

Solution. The auxiliary equation for this difference equation is $r^2 - r + \frac{2}{9} = 0$.

$r = \frac{2}{3}, \frac{1}{3}$

So two solutions of the difference equations are $\left(\frac{2}{3}\right)^k$ and $\left(\frac{1}{3}\right)^k$.

0. **Solution.** (a) Let H stand for 'Healthy' and I stand for 'ill' then

From		To
H	I	
0.95	0.45	H
0.05	0.55	I

So the stochastic matrix is $P = \begin{bmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{bmatrix}$

b) Since 20% of the students are ill on Monday, the initial state vector is $x_0 = \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix}$,

For Tuesday, $x_1 = Px_0 = \begin{bmatrix} 0.85 \\ 0.15 \end{bmatrix}$

$$x_2 = Px_1 = \begin{bmatrix} 0.875 \\ 0.125 \end{bmatrix}$$

Thus 15% of the students are ill on Tuesday, and 12.5% on Wednesday.

c) Since the student is well today, the initial state vector is $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then

$x_1 = Px_0 = \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix}$ $\quad x_2 = Px_1 = \begin{bmatrix} 0.925 \\ 0.075 \end{bmatrix}$

Thus, the probability that the student is well two days from now is 0.925.

Chapter 7

Eigenvalue and Eigenvector

Exercise 7.1

1. Solution.

Given, $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$

Now, $Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6-30 \\ 30-10 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4u$

Hence, u is eigenvector of A.

2. Solution.

Let $A = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$Au = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3+4 \\ -3+32 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix} \neq \lambda u$

So, u is not eigenvector of A.

3. Let $A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Now, $Au = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6+4 \\ 4-4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \neq \lambda u$

So, u is not eigenvector of A.

4. Let $A = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$

Now, $Au = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12-21+9 \\ -16+15+1 \\ 8-12+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0u$

Hence, u is eigenvector of A and 0 is eigen value.

5. Let $A = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$, $\lambda = 1$

Since, $[A - \lambda I \quad 0] = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}$ Since there is no free variables.

which shows that $Ax = 1x$ has trivial solution.

So, 1 is not eigen value of $\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$

6. Given, $\lambda = 5$, $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

we have, $[A - 5I \quad 0]$

$= \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ There is free variables so

$Ax = 5x$ has non trivial solution. So 5 is eigen value of A. So,

$2x_1 - 4x_2 = 0$

x_2 free

$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is eigen vector of A.

7. Given, $\lambda = 3$, $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Here, $[A - 3I \quad 0]$

$= \begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$ $\quad R_1 \to -\frac{1}{2}R_1$

$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$ $\quad R_2 \to R_2 - 3R_1$

$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$ $\quad R_2 \to -\frac{1}{2}R_2$

$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which shows that x_3 is free variables so

$Ax = 3x$ has no trivial solution. So 3 is an eigen value of A

and $\quad x_1 - x_2 - x_3 = 0$

$x_2 - 2x_3 = 0$

x_3 free

$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is eigen vector of A.

Given, $\lambda = 2$, $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

Here, $[A - 2I \quad 0]$

$$= \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows that x_2 and x_3 are free variables so $Ax = 2x$ has non trivial solution.

So, 2 is eigen value of A.

Given, hint is book

Given, $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$, $\lambda = 1$

Now, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here x_2 is free variable so

$x_1 = 0$

x_2 free

So $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is basis for the eigen space.

Given, $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$, $\lambda = 4$

Now, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 6 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_2 \to R_2 - \frac{4}{5} R_1$$

$6x_1 - 9x_2 = 0$

x_2 free

$$x = \begin{bmatrix} \frac{9}{6} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} = \frac{x_2}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ is basis for the eigen space.

c. Given, $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$, $\lambda = 10$

Now, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} -6 & -2 & 0 \\ -3 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 0 \\ -3 & -1 & 0 \end{bmatrix} \qquad R_1 \to -\frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x = \begin{bmatrix} -1/3\, x_2 \\ x_2 \end{bmatrix} = \frac{1}{3} x_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\therefore \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$ is basis for eigen space.

d. Given, $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, $\lambda = 2$

Now, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2 and x_3 are free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for eigen space.

e. Given, $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, $\lambda = 1, \lambda = 2$

Now, for $\lambda = 1$.

$[A - \lambda I \quad 0]$

$$= \begin{bmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad x_3 \text{ free variables. So,}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$ is basis for eigen space for $\lambda = 1$.

For $\lambda = 2$,

$$[A - \lambda I \quad 0]$$

$$= \begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad R_1 \to -\frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

x_2 and x_3 is free variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ is basis for eigen space corresponding to $\lambda = 2$.

11.

a. Given, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

Here, $\lambda = 0, 2, -1$ are eigen value.

b. Given, $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

Here, $\lambda = 4, 0, 3$ are eigen value.

c. $\lambda = 3, 2, 1$

12. We have, $Ax = \lambda x$

So, $A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$

Similarly, $A^3x = \lambda^3 x$

13. We have, $A^3x = \lambda^3 x$

$$= (-4)^3 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -64 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = 64 \begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -384 \\ 320 \end{bmatrix}$$

Exercise 7.2

1.

a. Given, $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

the characteristic polynomial is

$|A - \lambda I|$

$$\begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix}$$

$= 4 + \lambda^2 - 4\lambda - 49$

$= \lambda^2 - 4\lambda - 45$

For eigen value, $|A - \lambda I| = 0$

$\lambda^2 - 4\lambda - 45 = 0$

$\lambda^2 - 9\lambda + 5\lambda - 45 = 0$

$\lambda(\lambda - 9) + 5(\lambda - 9) = 0$

$(\lambda - 9)(\lambda + 5) = 0$

$\therefore \quad \lambda = -5, 9$

(b), (c), (d) do similar as (a).

e. Given, $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

The characteristic polynomial is

$|A - \lambda I|$

$$\begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 2-\lambda & 2 & -1 \\ -1+\lambda & 3-\lambda & 0 \\ -1 & -2 & 2-\lambda \end{vmatrix} \qquad R_2 \to R_2 - R_1$$

$$= 1-\lambda \begin{vmatrix} 2-\lambda & 2 & -1 \\ -1 & 1 & 0 \\ -1 & -2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 4-\lambda & 2 & -1 \\ 0 & 1 & 0 \\ -3 & -2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \times 1 \times \begin{vmatrix} 4-\pi & -1 \\ -3 & 2-\lambda \end{vmatrix}$$

$= (1-\lambda)(8 + \lambda^2 - 6\lambda - 3)$

$= (1-\lambda)(\lambda^2 - 6\lambda + 5)$

$= (1-\lambda)(\lambda - 1)(\lambda - 5)$

$= (\lambda - 1)^2(\lambda - 5)$

For eigen value, $|A - \lambda I| = 0$

$(\lambda - 1)^2(\lambda - 5) = 0$

$\therefore \quad \lambda = 1, 5$

Given, $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

For characteristic polynomial

$|A - \lambda I|$

$= \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix}$

$= \begin{vmatrix} -1-\lambda & 4 & -2 \\ 0 & 3-\lambda & -3+\lambda \\ -3 & 1 & 3-\lambda \end{vmatrix}$ $\quad R_2 \rightarrow R_2 - R_3$

$= (3-\lambda) \begin{vmatrix} -1-\lambda & 4 & -2 \\ 0 & 1 & -1 \\ -3 & 1 & 3-\lambda \end{vmatrix}$

$= (3-\lambda) \begin{vmatrix} -1-\lambda & 4 & 2 \\ 0 & 1 & 0 \\ -3 & 1 & 4-\lambda \end{vmatrix}$ $\quad C_3 \rightarrow C_3 + C_2$

$= (3-\lambda) \times 1 \times \begin{vmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix}$

$= (3-\lambda)(\lambda^2 - 3\lambda - 4 + 6)$

$= (3-\lambda)(\lambda^2 - 3\lambda + 2)$

$= (3-\lambda)(\lambda - 1)(\lambda - 2)$

For eigen value: $(3-\lambda)(\lambda-1)(\lambda-2) = 0$

$\therefore \quad \lambda = 1, 2, 3$

Given, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

So, characteristic polynomial is

$|A - \lambda I|$

$= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{vmatrix}$

$= (2-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix}$

$= (2-\lambda)(-1-\lambda)(4-\lambda)$

For eigen value, $|A - \lambda I| = 0$

$\lambda = -1, 2, 4$

ilarly, for (i) and (j) they are triangular matrix.

See in book similar as examples.

Solution.

Statement: Every square matrix A satisfies its characteristic equation i.e.,

$|A - \lambda I| = 0$

Given,

$A = \begin{bmatrix} 6 & 2 & -1 \\ -6 & -1 & 2 \\ 7 & 2 & -2 \end{bmatrix}$

For characteristic polynomial is $|A - \lambda I|$

$= \begin{vmatrix} 6-\lambda & 2 & -1 \\ -6 & -1-\lambda & 2 \\ 7 & 2 & -2-\lambda \end{vmatrix}$

$= \begin{vmatrix} 6-\lambda & 2 & -1 \\ -6 & -1-\lambda & 2 \\ 1+\lambda & 0 & -1-\lambda \end{vmatrix}$ $\quad R_3 \rightarrow R_3 - R_1$

$= \begin{vmatrix} 5-\lambda & 2 & -1 \\ -4 & -1-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{vmatrix}$

$= -1-\lambda \begin{vmatrix} 5-\lambda & 2 \\ -4 & -1-\lambda \end{vmatrix}$

$= (-1-\lambda)(\lambda^2 - 4\lambda + 3)$

$= -\lambda^3 + 3\lambda^2 + \lambda - 3$

Now, replacing λ by A and show,

$-A^3 + 3A^2 + A - 3I = 0$

Exercise 7.3

1. Given, $P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

We have, $A^5 = P D^5 P^{-1}$

$= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^5 \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 96 & 4 \\ 32 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$

$= \begin{bmatrix} -96+4 & 384-12 \\ -32+1 & 128-3 \end{bmatrix}$

$= \begin{bmatrix} -92 & 372 \\ -31 & 125 \end{bmatrix}$

2. Given, $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$\begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = 0$

$\lambda^2 - 8\lambda + 7 + 8 = 0$

$\lambda^2 - 8\lambda + 15 = 0$

$\lambda^2 - 5\lambda - 3\lambda + 15 = 0$

$$\lambda(\lambda - 5) - 3(\lambda - 5) = 0$$
$$(\lambda - 5)(\lambda - 3) = 0$$
$$\therefore \quad \lambda = 3, 5$$

For $\lambda = 3$, corresponding to eigen vector $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda = 5$, corresponding to eigen vector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So, $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

So, $A^4 = P D^4 P^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^4 & 0 \\ 0 & 3^4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5^4 & 3^4 \\ -5^4 & -2 \times 3^4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 5^4 - 3^4 & 5^4 - 3^4 \\ -2 \times 5^4 + 2 \times 3^4 & -5^4 + 2 \times 3^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1169 & 544 \\ -1088 & -463 \end{bmatrix}$$

3. See in answer
4.

a. Given, $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = 0$$
$$(1 - \lambda)(-1 - \lambda) = 0$$
$$\therefore \quad \lambda = 1, -1$$

For $\lambda = 1$, the eigen vector is

$$[A - \lambda I \quad 0]$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 6 & -2 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here x_2 is free variables so

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} x_2 \\ x_2 \end{bmatrix} = \frac{1}{3} x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\therefore \quad v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

For $\lambda = -1$, $\begin{bmatrix} 2 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

So, $AP = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$

$PD = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$

Since, AP = PD. So A is diagonalizable.

b. Given, $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

For eigen values, $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$$
$$\lambda^2 - 3\lambda + 2 - 12 = 0$$
$$\lambda^2 - 3\lambda - 10 = 0$$
$$\lambda^2 - 5\lambda + 2\lambda - 10 = 0$$
$$\lambda(\lambda - 5) + 2(\lambda - 5) = 0$$
$$(\lambda - 5)(\lambda + 2) = 0$$
$$\therefore \quad \lambda = -2, 5$$

Since, A is 2×2 matrix and there are two distinct eigen values. So A is diagonalizable.

c. Given, $A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = 0$$
$$\lambda^2 - 8\lambda + 15 + 1 = 0$$
$$\lambda^2 - 8\lambda + 16 = 0$$
$$(\lambda - 4)^2 = 0$$
$$\therefore \quad \lambda = 4$$

For eigen vector, $[A - 4I \quad 0]$

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

ıce, A is 2×2 matrix so we need two linearly independent eigen vectors but re is only one eigen vector.

is not diagonalizable.

iven, $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

ır eigen value, $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(\lambda^2 - 3\lambda + 2) = 0$$

$$(2-\lambda)(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = 1, 2$$

ɔr $\lambda = 1$, the eigen vector,

$$[A - I \quad 0]$$

$$= \begin{bmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Iere x_3 is free variable so

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

'or $\lambda = 2$, the eigen vector

$$[A - 2I \quad 0]$$

$$= \begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ $P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and PD = AP. So A is diagonalizable.

e. Given, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

For eigen value,

Since, the matrix is triangular matrix. So

$\lambda = 1, 2, 2$

For eigen vector,

$\lambda = 1$, $$[A - I \quad 0]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 free, $8x_2 = -x_3$ $x_1 + x_2 = 0$

$$x_2 = -\frac{1}{8}x_3 \quad x_1 = -x_2, \quad x_1 = \frac{1}{8}x_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{8}x_3 \\ \frac{-1}{8}x_3 \\ x_3 \end{bmatrix} = \frac{1}{8}x_3 \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$$

$\therefore$ $v_1 = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$

For $\lambda = 2$, $$[A - 2I \quad 0]$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 is free variable,

$$x = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Since, A is 3×3 matrix and there are only two linearly independent eigen vector. So A is not diagonalizable.

f, g, h, i, j do similar as above.

5. Given, $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Here, $Av_1 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3v_1.$

$\therefore$ $\lambda = 3$ is eigen value corresponding to v_1.

Similarly, $Av_2 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\, v_2$

$\lambda = -1$ is eigen value corresponding to v_2.

$\therefore$ $P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$$AP = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & -1 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & -1 \end{bmatrix}$$

Since, AP = PD

$\therefore$ A is diagonalizable.

Exercise 7.4

1. Given, $T(b_1) = 3c_1 - 2c_2 + 5c_3$

$\therefore$ $[T(b_1)]_c = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$

Similarly, $[T(b_2)]_c = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$

$\therefore$ $[T]_{B.C} = [[T(b_1)]_c\ [T(b_2)]_c]$

$$= \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

2. Similar as 1.

3. Given, $T(x, y) = (y, -5x + 13y, -7x + 16y)$

Let $T(b_1) = a_1 c_1 + a_2c_2 + a_3 c_3$ where, $[T(b_1)]_c = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$T(31) = a_1 (1, 0, -1) + a_2 (-1, 2, 2) + a_3 (0, 1, 2)$

$\therefore$ $(1, -5 \times 3 + 13 \times 1, -7 \times 3 + 16 \times 1) = (a_1 - a_2, 2a_2 + a_3, -a_1 + 2a_2 + 2a_3)$

$\therefore$ $a_1 - a_2 = 1$ .... (1)

$2a_2 + a_3 = -2$ .... (2)

$-a_1 + 2a_2 + 2a_3 = -5$ .... (3)

Solving (1) and (3), we get

$a_2 + 2a_3 = -4$ .... (4)

Using equation (4) and (2)

$$\begin{array}{r} 2a_2 + a_3 = -2 \\ 2a_2 + 4a_3 = -8 \\ - \quad - \quad + \\ \hline -3a_3 = 6 \end{array}$$

$a_3 = -2$

$\therefore$ $2a_2 + a_3 = -2$

$2a_2 - 2 = -2$

$a_2 = 0$

$\therefore$ $a_1 = 1$

Hence, $[T(b_1)]_c = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Similarly we get $[T(b_2)]_c = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

$\therefore$ $[T]_{B.C} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$

4. Given, $T(x, y) = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$

Let $T(b_1) = c_1b_1 + c_2b_2$

$T(1, 1) = c_1 (1, 1) + c_2 (-1, 0)$

$(0, 2) = (c_1 - c_2, c_1)$

$\therefore$ $c_1 = 2,\ c_1 - c_2 = 0$

$c_2 = 2.$

So, $[T(b_1)]_B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Similarly, $[T(b_2)]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\therefore$ $[T]_B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$

5. Given, $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$

So $T(1) = T(1 + 0\,.\,t + 0\,.\,t^2) = 0$

$T(t) = T(0 + 1 . t + 0 . t^2) = 1 + 0 = 1$

$T(t^2) = T(0 + 0 . t + 1 . t^2) = 0 + 2 . 1 . t = 2t$

$[T]_B = [\,[T(1)]_B\ [T(t)]_B\ [T(t^2)]_B\,]$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Given, $T(P(t)) = P(t) + t^2 P(t)$

So, $T(2 - t + t^2) = 2 - t + t^2 + t^2(2 - t + t^2)$

$= 2 - t + t^2 + 2t^2 - t^3 + t^4$

$= 2 - t + 3t^2 - t^3 + t^4$

For next,

Let $B = \{1, t, t^2\}$, $C = \{1, t, t^2, t^3, t^4\}$

Then,

$T(1) = 1 + t^2 . 1 = 1 + t^2 = 1 . 1 + 0 . t + 1 . t^2 + 0 . t^3 + 0 . t^4$

$$[T(1)]_C = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$T(t) = t + t^2 . t = 0 . 1 + 1 . t + 0 . t^2 + 1 . t^3 + 0 . t^4$

$$[T(t)]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$T(t^2) = t^2 + t^2 . t^2$

$= t^2 + t^4$

$= 0 . 1 + 0 . t + 1 . t^2 = 0 . t^3 + 1 . t^4$

$$[T(t^2)]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$[T]_{B,C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given, $T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1) t + (4a_1 + a_2) t^2$

$T(1) = 3 . 1 + (5 . 1 - 2 . 0) t + (4 . 0 + 0) t^2$

$= 3 + 5t$

$= 3 + 5t + 0 t^2$

$$[T(1)]_B = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$

$T(t) = 3 . 0 + (5 . 0 - 2 . 1) t + (4 . 1 + 0) t^2$

$= -2t + 4t^2$

$= 0 - 2t + 4t^2$

$$\therefore\ [T(t)]_B = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}$$

$T(t^2) = 3 . 0 + (5 . 0 - 2 . 0) t + (4 . 0 + 1) t^2$

$= t^2$

$= 0 + 0 . t + 1 . t^2$

$$\therefore\ [T(t^2)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore\ [T]_B = [\,[T(1)]_B\ [T(t)]_B\ [T(t^2)]_B\,]$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

8.

i. Given, $A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$

We know, $D = [T]_B = P^{-1} AP$, where $B = [b_1\ b_2]$ is basis for R^2.

So $P = [b_1\ b_2]$

The characteristic equation

$|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & -3 \\ -7 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 6\lambda - 16 = 0$

$\lambda^2 - 8\lambda + 2\lambda - 16 = 0$

$\lambda(\lambda - 8) + 2(\lambda - 8) = 0$

$(\lambda - 8)(\lambda + 2) = 0$

$\therefore\ \lambda = -2, 8$

For $\lambda = -2$,

$[A + 2I\ \ 0]$

$$= \begin{bmatrix} 7 & -3 & 0 \\ -7 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 7 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ 7x_1 - 3x_2 = 0 \Rightarrow x_1 = \frac{3x_2}{7}$, x_2 free

$$\therefore\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{7}x_2 \\ x_2 \end{bmatrix} = \frac{x_2}{7}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

So, $b_1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Similarly, for $\lambda = 8$

$[A - 8I\ \ 0]$

$$= \begin{bmatrix} -3 & -3 & 0 \\ -7 & -7 & 0 \end{bmatrix}$$

$$\sim\begin{bmatrix}-3 & -3 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore$ $x_1 = -x_2$, x_2 free

$\therefore$ $x = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-x_2\\ x_2\end{bmatrix} = x_2\begin{bmatrix}-1\\ 1\end{bmatrix}$

$\therefore$ $B = \left\{\begin{bmatrix}3\\ 7\end{bmatrix}, \begin{bmatrix}-1\\ 1\end{bmatrix}\right\}$ is basis for R^2.

ii. Similar as (i).

9. Given, $A = \begin{bmatrix}3 & 4\\ -1 & -1\end{bmatrix}$, $b_1 = \begin{bmatrix}2\\ -1\end{bmatrix}$, $b_2 = \begin{bmatrix}1\\ 2\end{bmatrix}$

We have, $P = [b_1 \;\; b_2] = \begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}$

$\therefore$ We have,

B-matrix = $D = P^{-1}AP$

$$= \frac{1}{5}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}3 & 4\\ -1 & -1\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & 11\\ -1 & -3\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}5 & 25\\ 0 & 5\end{bmatrix}$$

$$= \begin{bmatrix}1 & 5\\ 0 & 1\end{bmatrix}$$

10. Similar as 9.

Exercise 7.5

1.

i. Given, $A = \begin{pmatrix}1 & -2\\ 1 & 3\end{pmatrix}$

$|A - \lambda I| = 0$

$$\begin{vmatrix}1-\lambda & -2\\ 1 & 3-\lambda\end{vmatrix} = 0$$

$\lambda^2 - 4\lambda + 3 + 2 = 0$

$\lambda^2 - 4\lambda + 5 = 0$

$\therefore$ $\lambda = 2 \pm i$

For $\lambda = 2 - i$

$(A - \lambda I)x = 0$

$$\begin{pmatrix}-1+i & -2\\ 1 & 1+i\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 0$$

$(-1+i)x_1 - 2x_2 = 0$ .... (i)

$x_1 + (1+i)x_2 = 0$ .... (ii)

Since both equations are identical. So from equation (ii)

$x_1 = -(1+i)x_2$

Let $x_2 = -1$, then $x_1 = 1 + i$

$\therefore$ $x = \begin{bmatrix}1+i\\ -1\end{bmatrix}$ similarly for $\lambda = 2 + i$, we get $x = \begin{bmatrix}1-i\\ -1\end{bmatrix}$

Hence, $\lambda = 2 \pm i$, $x = \begin{bmatrix}1 \pm i\\ -1\end{bmatrix}$

Similarly solve for (ii) and iii)

(ii) $\lambda = 2 \pm 3i$, $\begin{bmatrix}1 \pm 3i\\ 2\end{bmatrix}$

(iii) $\lambda = 2 \pm 2i$, $\begin{bmatrix}1\\ 2 \pm 2i\end{bmatrix}$

2.

i. Given, $A = \begin{pmatrix}5 & -2\\ 1 & 3\end{pmatrix}$

For eigen value,

$|A - \lambda I| = 0$

$$\begin{vmatrix}5-\lambda & -2\\ 1 & 3-\lambda\end{vmatrix} = 0$$

$\lambda^2 - 8\lambda + 15 + 2 = 0$

$\lambda^2 - 8\lambda + 17 = 0$

$\therefore$ $\lambda = \dfrac{8 \pm \sqrt{(-8)^2 - 4.1.17}}{2 \times 1}$

$\lambda = \dfrac{8 \pm \sqrt{-4}}{2}$

$\therefore$ $\lambda = 4 \pm i$

For basis, $\lambda = 4 + i$

$(A - \lambda I)x = 0$

$$\begin{bmatrix}1-i & -2\\ 1 & -1-i\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$(1-i)x_1 - 2x_2 = 0$ .... (i)

$x_1 - (1+i)x_2 = 0$ .... (ii)

both are identical equation so

$x_1 = (1+i)x_2$

Let $x_2 = 1$ then $x_1 = 1 + i$

$\therefore$ $\begin{bmatrix}1+i\\ 1\end{bmatrix}$ is basis corresponding to $\lambda = 4 + i$.

Similarly, we get $\begin{bmatrix}1-i\\ 1\end{bmatrix}$ fro $\lambda = 4 - i$.

Given, $A = \begin{pmatrix} 1.52 & -0.7 \\ 0.56 & 0.4 \end{pmatrix}$

For eigen value,

$|A - \lambda I| = 0$

$\begin{bmatrix} 1.52-\lambda & -0.7 \\ 0.56 & 0.4-\lambda \end{bmatrix} = 0$

$\lambda^2 - 1.92\lambda + 0.608 + 0.392 = 0$

$\lambda^2 - 1.92\lambda + 1 = 0$

$\therefore \quad \lambda = \dfrac{1.92 \pm \sqrt{(-1.92)^2 - 4.1.1}}{2.1}$

$\lambda = \dfrac{1.92 \pm 0.56\, i}{2}$

$\therefore \quad \lambda = 0.96 \pm 0.28\, i$

For $\lambda = 0.96 + 0.28\, i$, $(A - \lambda I)\, x = 0$

$\begin{pmatrix} 0.56 - 0.28\, i & -0.7 \\ 0.56 & -0.56 - 0.28\, i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$(0.56 - 0.28i)\, x_1 - 0.7\, x_2 = 0 \quad \text{.... (i)}$

$0.56x_1 - (0.56 + 0.28i)\, x_2 = 0 \quad \text{.... (ii)}$

Both are identical equation so

$0.56\, x_1 = (0.56 + 0.28i)\, x_2$

Let $x_2 = 2$ then $x_1 = 2 + i$

$\begin{bmatrix} 2+i \\ 2 \end{bmatrix}$ is basis for $\lambda = 0.96 + 0.28i$

Similarly $\begin{bmatrix} 2-i \\ 2 \end{bmatrix}$ is basis for $\lambda = 0.96 - 0.28i$

Similar as (ii)

$\lambda = -0.6 + 0.8i, \begin{bmatrix} 2+i \\ 5 \end{bmatrix}, \quad \lambda = -0.6 - 0.8\, i, \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$

Given, $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$\begin{vmatrix} 5-\lambda & -5 \\ 1 & 1-\lambda \end{vmatrix} = 0$

or $\quad \lambda^2 - 6\lambda + 5 + 5 = 0$

$\lambda^2 - 6\lambda + 10 = 0$

$\lambda = \dfrac{6 \pm \sqrt{(-6)^2 - 4.1.10}}{2}$

$\lambda = 3 \pm i$

For $\quad \lambda = 3 - i$

We have, $\quad a = 3, b = 1$

$C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

For eigen vector corresponding to $\lambda = 3 - i$.

$(A - \lambda I)\, x = 0$

$\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$(2 + i)\, x_1 - 5x_2 = 0$

$x_1 + (-2 + i)\, x_2 = 0$

both are identical equation so

$x_1 = -(-2 + i)\, x_2$

Let $x_2 = 1$ then $x_1 = 2 - i$

$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$\therefore \quad P = [\text{Re } x \quad \text{Im } x] = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

For check

$AP = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 3 & -1 \end{bmatrix}$

$PC = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 3 & -1 \end{bmatrix}$

ii. Given, $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$\begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^2 - 8\lambda + 15 + 2 = 0$

$\lambda^2 - 8\lambda + 17 = 0$

$\therefore \quad \lambda = 4 \pm i$

For $\lambda = 4 - i$, $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$

For $\lambda = 4 - i$ we get basis

$x = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$\therefore \quad P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

So $\quad AP = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & -1 \end{bmatrix}$

$PC = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & -1 \end{bmatrix}$

iii. Similar as above we get

$P = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}, C = \begin{bmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{bmatrix}$

iv. Similarly as above we get

$$P = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix}$$

Exercise 7.6

1. Given,

$$A = \begin{bmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{bmatrix}, \text{ where } p = 0.2$$

For eigen values, $|A - \lambda I| = 0$

$$\begin{vmatrix} 0.5-\lambda & 0.4 \\ -0.2 & 1.1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 - 1.6\lambda + 0.55 + 0.08 = 0$

$\Rightarrow \lambda^2 - 1.6\lambda + 0.63 = 0$

$\lambda^2 - 0.9\lambda - 0.7\lambda + 0.63 = 0$

$\Rightarrow (\lambda - 0.9)(\lambda - 0.7) = 0$

$\therefore \lambda = 0.9, 0.7$

For $\lambda_1 = 0.9$ $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} -0.4 & 0.4 & 0 \\ -0.2 & 0.2 & 0 \end{bmatrix} \sim \begin{bmatrix} -0.4 & 0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding $\lambda_1 = 0.9$.

For $\lambda_2 = 0.7$, we have, $[A - \lambda I \quad 0]$

$$= \begin{bmatrix} -0.2 & 0.4 & 0 \\ -0.2 & 0.4 & 0 \end{bmatrix} \sim \begin{bmatrix} -0.2 & 0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore x = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is eigen vector corresponding to $\lambda_2 = 0.7$.

Hence, we have

$x_k = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2$

$x_k = c_1 (0.9)^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 (0.7)^k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

as $k \to 0$ then $(0.9)^k \to 0$ and $(0.7)^k \to 0$. So,

$x_k \to 0$ so owl population decline.

ii. The rat population is also decline?

2. Similar as 1.

3. **Solution.** The initial $x_0 = c_1 v_1 + c_2 v_2 + c_3 v_3$

So,

$$\begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Comparing and solving we get $c_1 = 2$, $c_2 = 1$, $c_3 = 3$ and we have,

$x_k = c_1 (\lambda_1)^k v_1 + c_2 (\lambda_2)^k v_2 + c_3 (\lambda_3)^k v_3$

$$x_k = 2 \,.\, (1)^k \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + 1 \,.\, \left(\frac{2}{3}\right)^k \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3 \,.\, \left(\frac{1}{3}\right)^k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$\therefore$ The general solution is

$$x_k = 2 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + \left(\frac{2}{3}\right)^k \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3 \,.\, \left(\frac{1}{3}\right)^k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

as $k \to \infty$, then we get $\left(\frac{2}{3}\right)^k \to 0$, $\left(\frac{1}{3}\right)^k \to 0$. So $x_k = 2 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$.

4. **Solution.** Given, $A = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$

For eigen value, $|A - \lambda I| = 0$

$$= \begin{vmatrix} 4-\lambda & -5 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 5\lambda - 6 = 0 \quad \therefore \lambda = 6, -1$

The eigen vector corresponding to $\lambda_1 = 6$ is $v_1 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$

The eigen vector corresponding to $\lambda_2 = -1$ is $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $x_1(t) = v_1 e^{\lambda_1 t}$ and $x_2(t) = v_2 e^{\lambda_2 t}$ be eigen functions satisfy the differentia equations $x' = Ax$, so their linear combination.

$\therefore x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$$x(t) = c_1 \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Since, $x_0 = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix}$, i.e., $x(0) = x_0$. So,

$$\begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix} = c_1 \begin{pmatrix} -5 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$-5c_1 + c_2 = 2.9$ .... (1)

$2c_1 + c_2 = 2.6$ .... (2)

Solving (1) and (2) we get

$c_1 = \frac{-3}{70}, c_2 = \frac{188}{70}$

$$x(t) = \frac{-3}{70} \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t} + \frac{188}{70} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \frac{15}{70} e^{6t} + \frac{188}{70} e^{-t} \\ \frac{-6}{70} e^{6t} + \frac{188}{70} e^{-t} \end{pmatrix}$$

Chapter 8

Orthogonality and Least Squares

Exercise 8.1

1. (i) Here,

$$u = \begin{bmatrix}-1\\2\end{bmatrix}, v = \begin{bmatrix}4\\6\end{bmatrix}$$

Then,

$$u.u = \begin{bmatrix}-1\\2\end{bmatrix}.\begin{bmatrix}-1\\2\end{bmatrix} = 1 + 4 = 5$$

$$v.u = \begin{bmatrix}4\\6\end{bmatrix}.\begin{bmatrix}-1\\2\end{bmatrix} = -4 + 12 = 8$$

$$\frac{v.u}{u.u} = \frac{8}{5}.$$

(ii) Here,

$$w = \begin{bmatrix}3\\-1\\-5\end{bmatrix}, x = \begin{bmatrix}6\\-2\\3\end{bmatrix}$$

Then,

$$w.w = \begin{bmatrix}3\\-1\\-5\end{bmatrix}.\begin{bmatrix}3\\-1\\-5\end{bmatrix} = 9 + 1 + 25 = 35.$$

$$x.w = \begin{bmatrix}6\\-2\\3\end{bmatrix}.\begin{bmatrix}6\\-2\\3\end{bmatrix} = 18 + 2 - 15 = 5$$

$$\frac{x.w}{w.w} = \frac{5}{35} = 7.$$

(iii) Here,

$$w = \begin{bmatrix}3\\-1\\-5\end{bmatrix}$$

Then,

$$w.w = \begin{bmatrix}3\\-1\\-5\end{bmatrix}.\begin{bmatrix}3\\-1\\-5\end{bmatrix} = 9 + 1 + 25 = 35$$

So,

$$\left(\frac{1}{w.w}\right)w = \frac{1}{35}\begin{bmatrix}3\\-1\\-5\end{bmatrix} = \begin{bmatrix}3/35\\-1/35\\-5/35\end{bmatrix} = \begin{bmatrix}3/35\\-1/35\\-1/7\end{bmatrix}.$$

(iv) Here, From Q.1, u.u = 5.
Then,

$$\left(\frac{1}{u.u}\right)u = \frac{1}{5}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-1/5\\2/5\end{bmatrix}.$$

(v) Here,

$$u = \begin{bmatrix}-1\\2\end{bmatrix}, v = \begin{bmatrix}4\\6\end{bmatrix}$$

Then

$$u.v = \begin{bmatrix}-1\\2\end{bmatrix}.\begin{bmatrix}4\\6\end{bmatrix} = -4 + 12 = 8$$

$$v.v = \begin{bmatrix}4\\6\end{bmatrix}.\begin{bmatrix}4\\6\end{bmatrix} = 16 + 36 = 52$$

Now

$$\left(\frac{u.v}{v.v}\right)v = \left(\frac{8}{52}\right)\begin{bmatrix}4\\6\end{bmatrix} = \left(\frac{2}{13}\right)\begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}8/13\\12/13\end{bmatrix}.$$

(vi) Here,

$$x = \begin{bmatrix}6\\-2\\3\end{bmatrix}, w = \begin{bmatrix}3\\-1\\-5\end{bmatrix}$$

Then,

$$x.w = \begin{bmatrix}6\\-2\\3\end{bmatrix}.\begin{bmatrix}3\\-1\\-5\end{bmatrix} = 18 + 2 - 15 = 5$$

$$x.x = \begin{bmatrix}6\\-2\\3\end{bmatrix}.\begin{bmatrix}6\\-2\\3\end{bmatrix} = 36 + 4 + 9 = 49$$

Now,

$$\left(\frac{x.w}{x.x}\right)x = \left(\frac{5}{49}\right)\begin{bmatrix}6\\-2\\3\end{bmatrix} = \begin{bmatrix}30/49\\-10/49\\15/49\end{bmatrix}.$$

(vii) From Q.3, w.w. = 35
So,

$$\|w\| = \sqrt{w.w} = \sqrt{35}.$$

(viii) From Q.6, x.x = 49.
So,

$$\|x\| = \sqrt{x.x} = \sqrt{49} = 7.$$

2. (i) Let, $u = \begin{bmatrix}-30\\40\end{bmatrix}$

Then,

$$u.u = \begin{bmatrix}-30\\40\end{bmatrix}.\begin{bmatrix}-30\\40\end{bmatrix} = 900 + 1600 = 2500.$$

Now, the unit vector in the direction of u is,

$$\frac{u}{\|u\|} = \left(\frac{1}{\sqrt{u.u}}\right)u = \frac{1}{\sqrt{2500}}\begin{bmatrix}-30\\40\end{bmatrix} = \frac{1}{50}\begin{bmatrix}-30\\40\end{bmatrix} = \begin{bmatrix}-3/5\\4/5\end{bmatrix}.$$

Let, $u = \begin{bmatrix}-6\\4\\-3\end{bmatrix}$

Then,

$$u.u = \begin{bmatrix}-6\\4\\-3\end{bmatrix}.\begin{bmatrix}-6\\4\\-3\end{bmatrix} = 36 + 16 + 9 = 61$$

Now, the unit vector in the direction of u is

$$\frac{u}{\|u\|} = \left(\frac{1}{\sqrt{u.u}}\right)u = \frac{1}{\sqrt{61}}\begin{bmatrix}-6\\4\\-3\end{bmatrix} = \begin{bmatrix}-6/\sqrt{61}\\4/\sqrt{61}\\-3/\sqrt{61}\end{bmatrix}.$$

Let, $u = \begin{bmatrix}7/4\\1/2\\1\end{bmatrix}$

Then,

$$u.u = \begin{bmatrix}7/4\\1/2\\1\end{bmatrix}.\begin{bmatrix}7/4\\1/2\\1\end{bmatrix} = \frac{49}{16} + \frac{1}{4} + 1 = \frac{49 + 4 + 16}{16} = \frac{69}{16}$$

Now, the unit vector in the direction of u is,

$$\frac{u}{\|u\|} = \left(\frac{1}{\sqrt{u.u}}\right)u = \left(\frac{4}{\sqrt{69}}\right)\begin{bmatrix}7/4\\1/2\\1\end{bmatrix} = \begin{bmatrix}7/\sqrt{69}\\2/\sqrt{69}\\4/\sqrt{69}\end{bmatrix}.$$

Let, $u = \begin{bmatrix}8/3\\2\end{bmatrix}$

Then,

$$u.u = \begin{bmatrix}8/3\\2\end{bmatrix}.\begin{bmatrix}8/3\\2\end{bmatrix} = \frac{64}{9} + 4 = \frac{64 + 36}{9} = \frac{100}{9}$$

Now, the unit vector in the direction of u is,

$$\frac{u}{\|u\|} = \left(\frac{1}{\sqrt{u.u}}\right)u = \left(\frac{3}{10}\right)\begin{bmatrix}8/3\\2\end{bmatrix} = \begin{bmatrix}4/5\\3/5\end{bmatrix}.$$

Let, $x\begin{bmatrix}10\\-3\end{bmatrix}$ and $y = \begin{bmatrix}-1\\-5\end{bmatrix}$

Then,

$$x - y = \begin{bmatrix}10\\-3\end{bmatrix} - \begin{bmatrix}-1\\-5\end{bmatrix} = \begin{bmatrix}11\\2\end{bmatrix}$$

So,

$$(x - y).(x - y) = \begin{bmatrix}11\\2\end{bmatrix}.\begin{bmatrix}11\\2\end{bmatrix} = 121 + 4 = 125$$

Now the distance between x and y is

$$dis(x, y) = \|x - y\| = \sqrt{(x - y).(x - y)} = \sqrt{125} = 5\sqrt{5}.$$

4.(i) Here,

$$a.b = \begin{bmatrix}8\\-5\end{bmatrix}.\begin{bmatrix}-2\\-3\end{bmatrix} = -16 + 15 = -1 \neq 0$$

So, a and b are not orthogonal to each other.

(ii) Here,

$$u.v = \begin{bmatrix}12\\3\\-5\end{bmatrix}.\begin{bmatrix}2\\-3\\3\end{bmatrix} = 24 - 9 - 15 = 0$$

So, u and v are orthogonal to each other.

(iii) Here,

$$u.v = \begin{bmatrix}3\\2\\-5\\0\end{bmatrix}.\begin{bmatrix}-4\\1\\-2\\6\end{bmatrix} = -12 + 2 + 10 + 0 = 0$$

So, u and v are orthogonal to each other.

(iv) Here,

$$y.z = \begin{bmatrix}-3\\7\\4\\0\end{bmatrix}.\begin{bmatrix}1\\-8\\15\\-7\end{bmatrix} = -3 - 56 + 60 + 0 = 1 \neq 0.$$

So, y and z are not orthogonal to each other.

5.

(i) Let

$$v = \begin{bmatrix}-30\\40\end{bmatrix}$$

Then,

$$\|v\| = \sqrt{(-30)^2 + (40)^2} = \sqrt{2500} = 50.$$

Therefore, the unit vector of v is,

$$\frac{v}{\|v\|} = \frac{1}{50}\begin{bmatrix}-30\\40\end{bmatrix} = \begin{bmatrix}-3/5\\4/5\end{bmatrix}.$$

(ii) Similar to (i).

6.

(i) Let

$u = (1, -3)$ and $v = (2, 4)$.

Let θ be the angle between u and v.

Here,

$$u.v = (1, -3).(2, 4) = 2 - 12 = 10$$

And,

$$\|u\| = \sqrt{1 + 9} = \sqrt{10}$$
$$\|v\| = \sqrt{4 + 16} = \sqrt{20}$$

Now, angle between u and v is,

$$\theta = \cos^{-1}\left(\frac{u.v}{\|u\|\,\|v\|}\right)$$

$$= \cos^{-1}\left(\frac{10}{\sqrt{10}\sqrt{20}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

(ii), (iii) Similar to (i).

7.

Let $u, v \in R^n$, then

$$[dis(u, -v)]^2 = [dis(u, v)]^2$$
$$\Leftrightarrow \|u + v\|^2 = \|u - v\|^2$$
$$\Leftrightarrow \|u + v\|^2 = \|u - v\|^2$$
$$\Leftrightarrow (u + v) . (u + v) = (u - v) . (u - v)$$
$$\Leftrightarrow u.u + u.v + v.u + v.v = u.u - u.v - v.u + v.v$$
$$\Leftrightarrow u.v + v.u = - u.v - v.u$$
$$\Leftrightarrow 4\, u.v = 0$$
$$\Leftrightarrow u.v = 0$$

Exercise 8.2

1. (i) Let,

$$u = \begin{bmatrix}-1\\4\\-3\end{bmatrix}, v = \begin{bmatrix}5\\2\\1\end{bmatrix}, w = \begin{bmatrix}3\\-4\\-7\end{bmatrix}$$

Then,

$$u.v = \begin{bmatrix}-1\\4\\-3\end{bmatrix}.\begin{bmatrix}5\\2\\1\end{bmatrix} = (-1)(5) + (4)(2) + (-3)(1) = -5 + 8 - 3 = 0$$

$$v.w = \begin{bmatrix}5\\2\\1\end{bmatrix}.\begin{bmatrix}3\\-4\\-7\end{bmatrix} = (5)(3) + (2)(-4) + (1)(-7) = 15 - 8 - 7 = 0$$

and,

$$u.w = \begin{bmatrix}-1\\4\\-3\end{bmatrix}.\begin{bmatrix}3\\-4\\-7\end{bmatrix} = (-1)(3) + (4)(-4) + (-3)(-7)$$

$$= -3 - 16 + 21 \neq 0$$

This shows {u, v} and {v, w} are orthogonal pairs but {u, w} is not orthogonal to each other.

(ii) Let,

$$u = \begin{bmatrix}1\\-2\\1\end{bmatrix}, v = \begin{bmatrix}0\\1\\2\end{bmatrix}, w = \begin{bmatrix}-5\\-2\\1\end{bmatrix}$$

Then,

$$u.v = \begin{bmatrix}1\\-2\\1\end{bmatrix}.\begin{bmatrix}0\\1\\2\end{bmatrix} = (1)(0) + (-2)(1) + (1)(2) = 0 - 2 + 2 = 0$$

$$v.w = \begin{bmatrix}0\\1\\2\end{bmatrix}.\begin{bmatrix}-5\\-2\\1\end{bmatrix} = (0)(-5) + (1)(-2) + (2)(1) = 0 - 2 + 2 = 0$$

and

$$u.w = \begin{bmatrix}1\\-2\\1\end{bmatrix}.\begin{bmatrix}-5\\-2\\1\end{bmatrix} = (1)(-5) + (-2)(-2) + (1)(1) = -5 + 4 + 1 = 0$$

This shows u & v, v & w and u & w are orthogonal sets. This means u, v, w are orthogonal to each other.

(iii) Let,

$$u = \begin{bmatrix}2\\-7\\-1\end{bmatrix}, v = \begin{bmatrix}-6\\-3\\9\end{bmatrix}, w = \begin{bmatrix}3\\1\\-1\end{bmatrix}$$

Then,

$$u.v = \begin{bmatrix}2\\-7\\-1\end{bmatrix}.\begin{bmatrix}-6\\-3\\9\end{bmatrix} = (2)(-6) + (-7)(-3) + (-1)(9) = -12 + 21 \neq 0$$

$$v.w = \begin{bmatrix}-6\\-3\\9\end{bmatrix}.\begin{bmatrix}3\\1\\-1\end{bmatrix} = (-6)(3) + (-3)(1) + (9)(-1)$$

$$= -18 - 3 - 9 = -30 \neq 0$$

and

$$u.w = \begin{bmatrix}2\\-7\\-1\end{bmatrix}.\begin{bmatrix}3\\1\\-1\end{bmatrix} = (2)(3) + (-7)(1) + (-1)(-1) = 6 - 7 + 1 = 0$$

This shows u is orthogonal to w but not orthogonal to v. Also, v and w are not orthogonal to each other.

(iv) Let,

$$u = \begin{bmatrix}2\\-5\\-3\end{bmatrix}, v = \begin{bmatrix}0\\0\\0\end{bmatrix} = 0, w = \begin{bmatrix}4\\-2\\6\end{bmatrix}$$

Here,

$$u.w = \begin{bmatrix}2\\-5\\-3\end{bmatrix}.\begin{bmatrix}4\\-2\\6\end{bmatrix} = (2)(4) + (-5)(-2) + (-3)(6) = 8 + 10 - 18 = 0$$

And the vector v is zero vector, so it is orthogonal to every other vector(s). Therefore, u, v and w are orthogonal to each other.

Let,

$$u = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, w = \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$$

Here,

$$u.v = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix} . \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} = (3)(-1) + (-2)(3) + (1)(-3) + (3)(4)$$

$$= -3 - 6 - 3 + 12 = 0$$

$$u.w = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix} . \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = (3)(3) + (-2)(8) + (1)(7) + (3)(0)$$

$$= 9 - 16 + 7 + 0 = 0$$

and

$$v.w = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} . \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = (-1)(3) + (3)(8) + (-3)(7) + (4)(0)$$

$$= -3 + 24 - 21 + 0 = 0$$

This shows u, v, w are orthogonal to each other.

Let,

$$u = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, v = \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, w = \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}$$

Here,

$$u.v = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix} . \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} = (5)(-4) + (-4)(1) + (0)(-3) + (3)(8)$$

$$= -20 - 4 + 0 + 24 = 0$$

$$v.w = \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} . \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} = (-4)(3) + (1)(3) + (-3)(5) + (8)(-1)$$

$$= -12 + 3 - 15 - 8 = -32 \neq 0$$

and

$$u.w = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix} . \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} = (5)(3) + (-4)(3) + (0)(5) + (3)(-1)$$

$$= 15 - 12 + 0 - 3 = 0$$

This shows u is orthogonal to both v and w but v is not orthogonal to w.

2. (i) Here,

$$u_1.u_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} . \begin{bmatrix} 6 \\ 4 \end{bmatrix} = (2)(6) + (-3)(4) = 12 - 12 = 0$$

This shows u_1 and u_2 are orthogonal to each other.

And,

$$x = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$$

Then,

$$u_1.x = \begin{bmatrix} 2 \\ -3 \end{bmatrix} . \begin{bmatrix} 9 \\ -7 \end{bmatrix} = 18 + 21 = 39,\ u_1.u_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} . \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 4 + 9 = 13$$

$$u_2.x = \begin{bmatrix} 6 \\ 4 \end{bmatrix} . \begin{bmatrix} 9 \\ -7 \end{bmatrix} = 54 - 28 = 26,\ u_2.u_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix} . \begin{bmatrix} 6 \\ 4 \end{bmatrix} = 36 + 16 = 52$$

Now, the vector x as the linear combination of u is

$$x = \left(\frac{u_1.x}{u_1.u_1}\right) u_1 + \left(\frac{u_2.x}{u_2.u_2}\right) u_2$$

$$= \left(\frac{39}{13}\right) u_1 + \left(\frac{26}{52}\right) u_2$$

$$= 3u_1 + \frac{1}{2} u_2$$

(ii) Here,

$$u_1.\ u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} . \begin{bmatrix} -2 \\ 6 \end{bmatrix} = (3)(-2) + (1)(6) = -6 + 6 = 0$$

This shows u_1 and u_2 are orthogonal to each other.

And,

$$u_1.x = \begin{bmatrix} 3 \\ 1 \end{bmatrix} . \begin{bmatrix} -6 \\ 3 \end{bmatrix} = -18 + 3 = -15$$

$$u_2.x = \begin{bmatrix} -2 \\ 6 \end{bmatrix} . \begin{bmatrix} -6 \\ 3 \end{bmatrix} = 12 + 18 = 30$$

$$u_1.u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} . \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 9 + 1 = 10$$

$$u_2.u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix} . \begin{bmatrix} -2 \\ 6 \end{bmatrix} = 4 + 36 = 40$$

Now, the vector x as the linear combination of u is,

$$x = \left(\frac{u_1.x}{u_1.u_1}\right) u_1 + \left(\frac{u_2.x}{u_2.u_2}\right) u_2 = \left(\frac{-15}{10}\right) u_1 + \left(\frac{30}{40}\right) u_2 = -\frac{3u_1}{2} + \frac{3u_2}{4}$$

(iii) Here,

$$u_1.u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} . \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = (1)(-1) + (0)(4) + (1)(1) = -1 + 0 + 1 = 0$$

$$u_1.u_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}.\begin{bmatrix}2\\1\\-2\end{bmatrix} = (1)(2) + (0)(1) + (1)(-2) = 2 + 0 - 2 = 0$$

and

$$u_2.u_3 = \begin{bmatrix}-1\\4\\1\end{bmatrix}.\begin{bmatrix}2\\1\\-2\end{bmatrix} = (-1)(2) + (4)(1) + (1)(-2) = -2 + 4 - 2 = 0$$

This show $\{u_1, u_2, u_3\}$ is a set of orthogonal vectors.

Next,

$$u_1.u_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}.\begin{bmatrix}1\\0\\1\end{bmatrix} = 1 + 0 + 1 = 2$$

$$u_2.u_2 = \begin{bmatrix}-1\\4\\1\end{bmatrix}.\begin{bmatrix}-1\\4\\1\end{bmatrix} = 1 + 16 + 1 = 18$$

$$u_3.u_3 = \begin{bmatrix}2\\1\\-2\end{bmatrix}.\begin{bmatrix}2\\1\\-2\end{bmatrix} = 4 + 1 + 4 = 9$$

$$u_1.x = \begin{bmatrix}1\\0\\1\end{bmatrix}.\begin{bmatrix}8\\-4\\-3\end{bmatrix} = 8 + 0 - 3 = 5$$

$$u_2.x = \begin{bmatrix}-1\\4\\1\end{bmatrix}.\begin{bmatrix}8\\-4\\-3\end{bmatrix} = -8 - 16 - 3 = -27$$

$$u_3.x = \begin{bmatrix}2\\1\\-2\end{bmatrix}.\begin{bmatrix}8\\-4\\-3\end{bmatrix} = 16 - 4 + 6 = 18$$

Now, the vector x as a linear combination of u is

$$x = \left(\frac{u_1.x}{u_1.u_1}\right)u_1 + \left(\frac{u_2.x}{u_2.u_2}\right)u_2 + \left(\frac{u_3.x}{u_3.u_3}\right)u_3$$

$$= \left(\frac{5}{2}\right)u_1 + \left(\frac{-27}{18}\right)u_2 + \left(\frac{18}{9}\right)u_3$$

$$= \frac{5}{2}u_1 - \frac{3}{2}u_2 + 2u_3$$

(iv) Here,

$$u_1.u_2 = \begin{bmatrix}3\\-3\\0\end{bmatrix}.\begin{bmatrix}2\\2\\-1\end{bmatrix} = (3)(2) + (-3)(2) + (0)(-1) = 6 - 6 + 0 = 0$$

$$u_1.u_3 = \begin{bmatrix}3\\-3\\0\end{bmatrix}.\begin{bmatrix}1\\1\\4\end{bmatrix} = (3)(1) + (-3)(1) + (0)(4) = 3 - 3 = 0 = 0$$

and

$$u_2.u_3 = \begin{bmatrix}2\\2\\-1\end{bmatrix}.\begin{bmatrix}1\\1\\4\end{bmatrix} = (2)(1) + (2)(1) + (-1)(4) = 2 + 2 - 4 = 0$$

This shows $\{u_1, u_2, u_3\}$ is a set of orthogonal vectors.

Next,

$$u_1.u_1 = \|u_1\|^2 = (3)^2 + (-3)^2 + (0)^2 = 9 + 9 + 0 = 18$$
$$u_2.u_2 = \|u_2\|^2 = (2)^2 + (2)^2 + (-1)^2 = 4 + 4 + 1 = 9$$
$$u_3.u_3 = \|u_3\|^2 = (1)^2 + (1)^2 + (4)^2 = 1 + 1 + 16 = 18$$

$$u_1.x = \begin{bmatrix}3\\-3\\0\end{bmatrix}.\begin{bmatrix}5\\-3\\1\end{bmatrix} = 15 + 9 + 0 = 24$$

$$u_2.x = \begin{bmatrix}2\\2\\-1\end{bmatrix}.\begin{bmatrix}5\\-3\\1\end{bmatrix} = 10 - 6 - 1 = 3$$

$$u_3.x = \begin{bmatrix}1\\1\\4\end{bmatrix}.\begin{bmatrix}5\\-3\\1\end{bmatrix} = 5 - 3 + 4 = 6$$

Now, the vector x as a linear combination of u is

$$x = \left(\frac{u_1.x}{u_1.u_1}\right)u_1 + \left(\frac{u_2.x}{u_2.u_2}\right)u_2 + \left(\frac{u_3.x}{u_3.u_3}\right)u_3$$

$$= \left(\frac{24}{18}\right)u_1 + \left(\frac{3}{9}\right)u_2 + \left(\frac{6}{18}\right)u_3$$

$$= \left(\frac{4}{3}\right)u_1 + \left(\frac{1}{3}\right)u_2 + \left(\frac{1}{3}\right)u_3$$

$$= \frac{1}{3}(4u_1 + u_2 + u_3)$$

3. Let,

$$y = \begin{bmatrix}1\\7\end{bmatrix}, u = \begin{bmatrix}-4\\2\end{bmatrix}$$

Then

$$y.u = \begin{bmatrix}1\\7\end{bmatrix}.\begin{bmatrix}-4\\2\end{bmatrix} = (1)(-4) + (7)(2) = -4 + 14 = 10$$

$$u.u = \begin{bmatrix}-4\\2\end{bmatrix}.\begin{bmatrix}-4\\2\end{bmatrix} = (-4)(-4) + (2)(2) = 16 + 4 = 20$$

Now, the orthogonal projection $\hat{y}$ of y onto u is,

$$\hat{y} = \left(\frac{y.u}{u.u}\right)u = \left(\frac{10}{20}\right)\begin{bmatrix}-4\\2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$

4. (i) Here,

$$y.u = \begin{bmatrix}2\\3\end{bmatrix}.\begin{bmatrix}4\\-7\end{bmatrix} = (2)(4) + (3)(-7) = 8 - 21 = -13$$

$$u.u = \begin{bmatrix}4\\-7\end{bmatrix}.\begin{bmatrix}4\\-7\end{bmatrix} = (4)(4) + (-7)(-7) = 16 + 49 = 65$$

Now, the orthogonal projection $\hat{y}$ of y onto u is

$$\hat{y} = \left(\frac{y.u}{u.u}\right) u = \left(\frac{-13}{65}\right) u = \left(\frac{-1}{5}\right) \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix}$$

Here,

$$\hat{y} + (y - \hat{y}) = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$$

$$= \begin{bmatrix} 10/5 \\ 15/5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = y.$$

This shows that y is a sum of two vectors $\hat{y}$ and $(y - \hat{y})$.

And,

$$\hat{y} \cdot (y - \hat{y}) = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} \cdot \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix} = \frac{-56}{25} + \frac{56}{25} = 0$$

This means $\hat{y}$ and $(y - \hat{y})$ are orthogonal to each other.

Thus, y is a sum of two orthogonal vectors $\hat{y}$ and $(y - \hat{y})$.

Next,

$$\hat{y} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} = \left(\frac{-1}{5}\right) \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} = \alpha u$$

This means $\hat{y}$ = span {u}. And,

$$u.(y - \hat{y}) = \begin{bmatrix} 4 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix} = \frac{56}{5} - \frac{56}{5} = 0$$

This shows $\hat{y}$ is span {u} and another vector $(y - \hat{y})$ is orthogonal to u.

Here,

$$y.u = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} = (2)(7) + (6)(1) = 14 + 6 = 20$$

$$u.u = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} = (7)(7) + (1)(1) = 49 + 1 = 50$$

Now, the orthogonal projection $\hat{y}$ of y onto u is

$$\hat{y} = \left(\frac{y.u}{u.u}\right) u = \left(\frac{20}{50}\right) u = \frac{2}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

Here,

$$\hat{y} + (y - \hat{y}) = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \left(\begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

$$= \begin{bmatrix} 10/5 \\ 30/5 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = y$$

This shows that y is a sum of two vectors $\hat{y}$ and $(y - \hat{y})$.

And,

$$\hat{y}.(y - \hat{y}) = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} \cdot \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix} = \frac{-56}{25} + \frac{56}{25} = 0$$

Next,

$$\hat{y} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \left(\frac{2}{5}\right) \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \alpha u$$

This means $\hat{y}$ = span {u}. And,

$$v.(y - \hat{y}) = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix} = \frac{-28}{5} + \frac{28}{5} = 0$$

This shows $\hat{y}$ is span {u} and another vector $(y - \hat{y})$ is orthogonal to u.

5. (i) Here,

$$y.u = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 6 \end{bmatrix} = (3)(8) + (1)(6) = 24 + 6 = 30$$

$$u.u = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 6 \end{bmatrix} = (8)(8) + (6)(6) = 64 + 36 = 100$$

Then, the orthogonal projection of y onto u is,

$$\hat{y} = \left(\frac{y.u}{u.u}\right) u = \left(\frac{30}{100}\right) \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 12/5 \\ 9/5 \end{bmatrix}$$

So,

$$y - \hat{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 12/5 \\ 9/5 \end{bmatrix} = \begin{bmatrix} 3 - 12/5 \\ 1 - 9/5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$$

Now, the distance from y to the line through u and the origin is

$$\text{Distance} = \|y - \hat{y}\| = \sqrt{(y - \hat{y}) \cdot (y - \hat{y})}$$

$$= \sqrt{\begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}}$$

$$= \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{25}{25}}$$

$$= 1$$

(ii) Here,

$$y.u = \begin{bmatrix} -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-3)(1) + (9)(2) = -3 + 18 = 15$$

$$u.u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1)(1) + (2)(2) = 1 + 4 = 5$$

Then, the orthogonal projection of y onto u is,

$$\hat{y} = \left(\frac{y.u}{u.u}\right) u = \left(\frac{15}{5}\right)\begin{bmatrix}1\\2\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$$

So,

$$y - \hat{y} = \begin{bmatrix}-3\\9\end{bmatrix} - \begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}-3-3\\9-6\end{bmatrix} = \begin{bmatrix}-6\\3\end{bmatrix}$$

Now, the distance from y to the line through u and the origin is

$$\text{Distance} = \|y - \hat{y}\| = \sqrt{(y-\hat{y}).(y-\hat{y})}$$
$$= \sqrt{(-6)^2 + (3)^2} = \sqrt{36+9} = \sqrt{45}$$

6. (i) Let,

$$u = \begin{bmatrix}0\\1\\0\end{bmatrix} \text{ and } v = \begin{bmatrix}0\\-1\\0\end{bmatrix}$$

Then

$$u.v = \begin{bmatrix}0\\1\\0\end{bmatrix}.\begin{bmatrix}0\\-1\\0\end{bmatrix} = (0)(0) + (1)(-1) + (0)(0) = 0 - 1 + 0 = -1.$$

This means {u, v} is a set of non–orthogonal vectors.

(ii) Let,

$$u = \begin{bmatrix}-0.6\\0.8\end{bmatrix} \text{ and } v = \begin{bmatrix}0.8\\0.6\end{bmatrix}$$

Then,

$$u.v = \begin{bmatrix}-0.6\\0.8\end{bmatrix}.\begin{bmatrix}0.8\\0.6\end{bmatrix} = (-0.6)(0.8) + (0.8)(0.6)$$
$$= -0.48 + 0.48 = 0$$

This shows {u, v} is a set of orthogonal vectors.

And,

$$u.u = \begin{bmatrix}-0.6\\0.8\end{bmatrix}.\begin{bmatrix}-0.6\\0.8\end{bmatrix} = 0.36 + 0.64 = 1.00 = 1$$

$$v.v = \begin{bmatrix}0.8\\0.6\end{bmatrix}.\begin{bmatrix}0.8\\0.6\end{bmatrix} = 0.64 + 0.36 = 1.00 = 1.$$

This means {u, v} is a set of orthonormal vectors.

(iii) Let,

$$u = \begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix} \text{ and } v = \begin{bmatrix}1/3\\2/3\\0\end{bmatrix}.$$

Then,

$$u.v = \begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}.\begin{bmatrix}1/3\\2/3\\0\end{bmatrix} = \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)(0)$$
$$= \frac{-2}{9} + \frac{2}{9} + 0 = 0.$$

This shows {u, v} is a set of orthogonal vectors.

And,

$$u.u = \begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}.\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix} = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = \frac{9}{9} = 1$$

$$v.v = \begin{bmatrix}1/3\\2/3\\0\end{bmatrix}.\begin{bmatrix}1/3\\2/3\\0\end{bmatrix} = \frac{1}{9} + \frac{4}{9} + 0 = \frac{5}{9} \neq 1$$

This shows that {u, v} is not an orthonormal set.

Now, normalizing v as

$$v' = \frac{v}{\|v\|} = \left(\frac{1}{\sqrt{v.v}}\right) v = \sqrt{\frac{9}{5}}\begin{bmatrix}1/3\\2/3\\0\end{bmatrix} = \frac{3}{\sqrt{5}}\begin{bmatrix}1/3\\2/3\\0\end{bmatrix} = \frac{3}{\sqrt{5}} u_2.$$

Therefore {u, v'} is an orthonormal set.

(iv) Let,

$$u_1 = \begin{bmatrix}1/\sqrt{10}\\3/\sqrt{20}\\3/\sqrt{20}\end{bmatrix}, u_2 = \begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{20}\\-1/\sqrt{20}\end{bmatrix}, u_3 = \begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

Then

$$u_1.u_2 = \begin{bmatrix}1/\sqrt{10}\\3/\sqrt{20}\\3/\sqrt{20}\end{bmatrix}.\begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{20}\\-1/\sqrt{20}\end{bmatrix}$$
$$= \left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right) + \left(\frac{3}{\sqrt{20}}\right)\left(\frac{-1}{\sqrt{20}}\right) + \left(\frac{3}{\sqrt{20}}\right)\left(\frac{-1}{\sqrt{20}}\right)$$
$$= \frac{3}{10} - \frac{3}{20} - \frac{3}{20}$$
$$= 0.$$

$$u_1.u_3 = \begin{bmatrix}1/\sqrt{10}\\3/\sqrt{20}\\3/\sqrt{20}\end{bmatrix}.\begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$
$$= \left(\frac{1}{\sqrt{10}}\right)(0) + \left(\frac{3}{\sqrt{20}}\right)\left(\frac{-1}{\sqrt{20}}\right) + \left(\frac{3}{\sqrt{20}}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$= 0 - \frac{3}{\sqrt{40}} + \frac{3}{\sqrt{40}}$$
$$= 0$$

$$u_2.u_3 = \begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{20}\\-1/\sqrt{20}\end{bmatrix}.\begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$
$$= \left(\frac{3}{\sqrt{10}}\right)(0) + \left(\frac{-1}{\sqrt{20}}\right)\left(\frac{-1}{\sqrt{2}}\right) + \left(\frac{-1}{\sqrt{20}}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$= 0 + \frac{1}{\sqrt{40}} - \frac{1}{\sqrt{40}}$$

$= 0$

This means $\{u_1, u_2, u_3\}$ is an orthogonal set.

Also,

$$u_1.u_1 = \left(\frac{1}{\sqrt{10}}\right)^2 + \left(\frac{3}{\sqrt{20}}\right)^2 + \left(\frac{3}{\sqrt{20}}\right)^2 = \frac{1}{10} + \frac{9}{20} + \frac{9}{20} = 1$$

$$u_2.u_2 = \left(\frac{3}{\sqrt{10}}\right)^2 + \left(\frac{-1}{\sqrt{20}}\right)^2 + \left(\frac{-1}{\sqrt{20}}\right)^2 = \frac{9}{10} + \frac{1}{20} + \frac{1}{20} = 1$$

$$u_3.u_3 = (0)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

This shows $\{u_1, u_2, u_3\}$ is an orthonormal set.

Similar to (iv).

Exercise 8.3

Here,

$$\frac{x.u_1}{u_1.u_1} = \frac{\begin{bmatrix}10\\-8\\2\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\-4\\-1\end{bmatrix}}{\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix} \cdot \begin{bmatrix}0\\1\\-4\\-1\end{bmatrix}} = \frac{(10)(0) + (-8)(1) + (2)(-4) + (0)(-1)}{(0)(0) + (1)(1) + (-4)(-4) + (-1)(-1)}$$

$$= \frac{0 - 8 - 8 + 0}{0 + 1 + 16 + 1} = \frac{-16}{18} = -\frac{8}{9}$$

$$\frac{x.u_2}{u_2.u_2} = \frac{\begin{bmatrix}10\\-8\\2\\0\end{bmatrix} \cdot \begin{bmatrix}3\\5\\1\\1\end{bmatrix}}{\begin{bmatrix}3\\5\\1\\1\end{bmatrix} \cdot \begin{bmatrix}3\\5\\1\\1\end{bmatrix}} = \frac{(10)(3) + (-8)(5) + (2)(1) + (0)(1)}{(3)(3) + (5)(5) + (1)(1) + (1)(1)}$$

$$= \frac{30 - 40 + 2 + 0}{9 + 25 + 1 + 1} = \frac{-8}{36} = -\frac{2}{9}$$

$$\frac{x.u_3}{u_3.u_3} = \frac{\begin{bmatrix}10\\-8\\2\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\\-4\end{bmatrix}}{\begin{bmatrix}1\\0\\1\\-4\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\\-4\end{bmatrix}} = \frac{(10)(1) + (-8)(0) + (2)(1) + (0)(-4)}{(1)(1) + (0)(0) + (1)(1) + (-4)(-4)}$$

$$= \frac{10 + 0 + 2 + 0}{1 + 0 + 1 + 16} = \frac{12}{18} = \frac{2}{3}$$

and

$$\frac{x.u_4}{u_4.u_4} = \frac{\begin{bmatrix}10\\-8\\2\\0\end{bmatrix} \cdot \begin{bmatrix}5\\-3\\-1\\1\end{bmatrix}}{\begin{bmatrix}5\\-3\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}5\\-3\\-1\\1\end{bmatrix}} = \frac{(10)(5) + (-8)(-3) + (2)(-1) + (0)(1)}{(5)(5) + (-3)(-3) + (-1)(-1) + (1)(1)}$$

$$= \frac{50 + 24 - 2 + 0}{25 + 9 + 1 + 1} = \frac{72}{36} = 2$$

Let x_1 is span by $\{u_1, u_2, u_3\}$ and x_2 is span by $\{u_4\}$ then,

$$x_1 + x_2 = \text{span}\{u_1, u_2, u_3\} + \text{span}\{u_4\}$$

$$= \left[\left(\frac{x.u_1}{u_1.u_1}\right)u_1 + \left(\frac{x.u_2}{u_2.u_2}\right)u_2 + \left(\frac{x.u_3}{u_3.u_3}\right)u_3\right] + \left[\left(\frac{x.u_4}{u_4.u_4}\right)u_4\right]$$

$$= \left(\frac{-8}{9}\right)\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix} + \left(\frac{-2}{9}\right)\begin{bmatrix}3\\5\\1\\1\end{bmatrix} + \left(\frac{2}{3}\right)\begin{bmatrix}1\\0\\1\\-4\end{bmatrix} + 2\begin{bmatrix}5\\-3\\-1\\1\end{bmatrix}$$

$$= \left(\frac{1}{9}\right)\left[(-8)\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix} + (-2)\begin{bmatrix}3\\5\\1\\1\end{bmatrix} + (6)\begin{bmatrix}1\\0\\1\\-4\end{bmatrix}\right] + (2)\begin{bmatrix}5\\-3\\-1\\1\end{bmatrix}$$

$$= \left(\frac{1}{9}\right)\begin{bmatrix}0 - 6 + 6\\-8 - 10 + 0\\32 - 2 + 6\\8 - 2 - 24\end{bmatrix} + \begin{bmatrix}10\\-6\\-2\\2\end{bmatrix}$$

$$= \begin{bmatrix}0\\-2\\4\\-2\end{bmatrix} + \begin{bmatrix}10\\-6\\-2\\2\end{bmatrix} = \begin{bmatrix}10\\-8\\2\\0\end{bmatrix} = x$$

This shows x is a sum of x_1 and x_2 where x_1 = span $\{u_1, u_2, u_3\}$ and x_2 = span $\{u_4\}$.

2. (i) Here

$$u_1.u_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\0\end{bmatrix} = (1)(-1) + (1)(1) + (0)(0) = -1 + 1 + 0 = 0$$

This means $\{u_1, u_2\}$ is an orthogonal set.

And,

$$u_1.u_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} = (1)(1) + (1)(1) + (0)(0) = 1 + 1 + 0 = 2$$

$$u_2.u_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\0\end{bmatrix} = (-1)(-1) + (1)(1) + (0)(0) = 1 + 1 + 0 = 2$$

$$y.u_1 = \begin{bmatrix}-1\\4\\3\end{bmatrix}.\begin{bmatrix}1\\1\\0\end{bmatrix} = -1+4+0 = 3$$

$$y.u_2 = \begin{bmatrix}-1\\4\\3\end{bmatrix}.\begin{bmatrix}-1\\1\\0\end{bmatrix} = 1+4+0 = 5$$

Now, the orthogonal projection $\hat{y}$ of y onto Span$\{u_1, u_2\}$ is

$$\hat{y} = \left(\frac{y.u_1}{u_1.u_1}\right)u_1 + \left(\frac{y.u_2}{u_2.u_2}\right)u_2$$

$$= \left(\frac{3}{2}\right)\begin{bmatrix}1\\1\\0\end{bmatrix} + \left(\frac{5}{2}\right)\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}(3/2)-(5/2)\\(3/2)+(5/2)\\0+0\end{bmatrix} = \begin{bmatrix}-1\\4\\0\end{bmatrix}.$$

(ii) Here

$$u_1.u_2 = \begin{bmatrix}3\\-1\\2\end{bmatrix}.\begin{bmatrix}1\\-1\\-2\end{bmatrix} = (3)(1)+(-1)(-1)+(2)(-2) = 3+1-4 = 0$$

This means $\{u_1, u_2\}$ is an orthogonal set.

And,

$$u_1.u_1 = \begin{bmatrix}3\\-1\\2\end{bmatrix}.\begin{bmatrix}3\\-1\\2\end{bmatrix} = (3)(3)+(-1)(-1)+(2)(2)$$

$$= 9+1+4 = 14$$

$$u_2.u_2 = \begin{bmatrix}1\\-1\\-2\end{bmatrix}.\begin{bmatrix}1\\-1\\-2\end{bmatrix} = 1+1+4 = 6$$

$$y.u_1 = \begin{bmatrix}-1\\2\\6\end{bmatrix}.\begin{bmatrix}3\\-1\\2\end{bmatrix} = -3-2+12 = 7$$

$$y.u_2 = \begin{bmatrix}-1\\2\\6\end{bmatrix}.\begin{bmatrix}1\\-1\\-2\end{bmatrix} = -1-2-12 = -15$$

Now, the orthogonal projection $\hat{y}$ of y onto Span $\{u_1, u_2\}$ is,

$$\hat{y} = \left(\frac{y.u_1}{u_1.u_1}\right)u_1 + \left(\frac{y.u_2}{u_2.u_2}\right)u_2$$

$$= \left(\frac{7}{14}\right)\begin{bmatrix}3\\-1\\2\end{bmatrix} + \left(\frac{-15}{6}\right)\begin{bmatrix}1\\-1\\-2\end{bmatrix} = \begin{bmatrix}(3/2)+(-5/2)\\(-1/2)+(5/2)\\(1)+(5)\end{bmatrix} = \begin{bmatrix}-1\\2\\6\end{bmatrix}$$

(iii) Here,

$$u_1.u_2 = \begin{bmatrix}-4\\-1\\1\end{bmatrix}.\begin{bmatrix}0\\1\\1\end{bmatrix} = (-4)(0)+(-1)(1)+(1)(1) = 0-1+1 = 0$$

This means $\{u_1, u_2\}$ is an orthogonal set.

And,

$$u_1.u_1 = \begin{bmatrix}-4\\-1\\1\end{bmatrix}.\begin{bmatrix}-4\\-1\\1\end{bmatrix} = 16+1+1 = 18$$

$$u_2.u_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}.\begin{bmatrix}0\\1\\1\end{bmatrix} = 0+1+1 = 2$$

$$y.u_1 = \begin{bmatrix}6\\4\\1\end{bmatrix}.\begin{bmatrix}-4\\-1\\1\end{bmatrix} = -24-4+1 = -27$$

$$y.u_2 = \begin{bmatrix}6\\4\\1\end{bmatrix}.\begin{bmatrix}0\\1\\1\end{bmatrix} = 0+4+1 = 5$$

Now, the orthogonal projection $\hat{y}$ of y onto Span $\{u_1, u_2\}$ is

$$\hat{y} = \left(\frac{y.u_1}{u_1.u_1}\right)u_1 + \left(\frac{y.u_2}{u_2.u_2}\right)u_2$$

$$= \left(-\frac{27}{18}\right)\begin{bmatrix}-4\\-1\\1\end{bmatrix} + \left(\frac{5}{2}\right)\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}(6)+0\\(3/2)+(5/2)\\(-3/2)+5/2\end{bmatrix} = \begin{bmatrix}6\\4\\1\end{bmatrix}.$$

3. (i) Here,

$$u_1.u_1 = \begin{bmatrix}1\\3\\-2\end{bmatrix}.\begin{bmatrix}1\\3\\-2\end{bmatrix} = (1)(1)+(3)(3)+(-2)(-2) = 1+9+4 = 14$$

$$u_2.u_2 = \begin{bmatrix}5\\1\\4\end{bmatrix}.\begin{bmatrix}5\\1\\4\end{bmatrix} = 25+1+16 = 42$$

$$y.u_1 = \begin{bmatrix}1\\3\\5\end{bmatrix}.\begin{bmatrix}1\\3\\-2\end{bmatrix} = 1+9-10 = 0$$

$$y.u_2 = \begin{bmatrix}1\\3\\5\end{bmatrix}.\begin{bmatrix}5\\1\\4\end{bmatrix} = 5+3+20 = 28$$

Then, the orthogonal projection $\hat{y}$ of y onto Span $\{u_1, u_2\}$ is

$$\hat{y} = \left(\frac{y.u_1}{u_1.u_1}\right)u_1 + \left(\frac{y.u_2}{u_2.u_2}\right)u_2$$

$$= \left(\frac{0}{14}\right)\begin{bmatrix}1\\3\\-2\end{bmatrix} + \left(\frac{28}{42}\right)\begin{bmatrix}5\\1\\4\end{bmatrix}$$

$$= \begin{bmatrix}0+(10/3)\\0+(2/3)\\0+(8/3)\end{bmatrix} = \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix}$$

And,

$$y-\hat{y} = \begin{bmatrix}1\\3\\5\end{bmatrix} - \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix} = \begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}$$

Now,

$$(y - \hat{y}) + \hat{y} = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix} + \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = y$$

And,

$$(y - \hat{y}) \cdot \hat{y} = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix} \cdot \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} = \frac{-70}{9} + \frac{14}{9} + \frac{56}{9} = 0$$

This shows y is a sum of two orthogonal vectors $y - \hat{y}$ and y where $\hat{y}$ is in W when W be the subspace spanned by u_1 and u_2.

Here,

$$u_1.u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = (1)(1) + (1)(1) + (0)(0) + (1)(1) = 1 + 1 + 0 + 1 = 3$$

$$u_2.u_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} = 1 + 9 + 1 + 4 = 15$$

$$u_3.u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 1 + 0 + 1 + 1 = 3$$

$$y.u_1 = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 4 + 3 + 0 - 1 = 6$$

$$y.u_2 = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} = -4 + 9 + 3 + 2 = 10$$

$$y.u_3 = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = -4 + 0 + 3 - 1 = -2$$

Now, the orthogonal projection $\hat{y}$ of y onto Span $\{u_1, u_2, u_3\}$ is

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1\, u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2\, u_2}\right) u_2 + \left(\frac{y \cdot u_3}{u_3\, u_3}\right) u_3$$

$$= \left(\frac{6}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{10}{15}\right) \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix} + \left(\frac{-2}{3}\right) \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + \left(\frac{-2}{3}\right) + \left(\frac{2}{3}\right) \\ 2 + 2 + 0 \\ 0 + \left(\frac{2}{3}\right) + \left(\frac{-2}{3}\right) \\ 2 + \left(\frac{-4}{3}\right) + \left(\frac{-2}{3}\right) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

And,

$$y - \hat{y} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Now, $(y - \hat{y}) + \hat{y} = y$

and $(y - \hat{y}) \cdot \hat{y} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} = 4 - 4 + 0 + 0 = 0$

This shows y is a sum of $y - \hat{y}$ and $\hat{y}$ where $\hat{y}$ is in W when W be subspace spanned by u_1, u_2 and u_3 and $y - \hat{y}$ is orthogonal to $\hat{y}$.

(iii) Here,

$$u_1 \cdot u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = (1)(1) + (1)(1) + (0)(0) + (-1)(-1) = 1+1+0+1 = 3$$

$$u_2 \cdot u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 1 + 0 + 1 + 1 = 3$$

$$u_3 \cdot u_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0 + 1 + 1 + 1 = 3$$

$$y \cdot u_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3 + 4 + 0 - 6 = 1$$

$$y \cdot u_2 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 3 + 0 + 5 + 6 = 14$$

$$y \cdot u_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0 - 4 + 5 - 6 = -5$$

Now, the orthogonal projection $\hat{y}$ of y onto span $\{u_1, u_2, u_3\}$ is

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 + \left(\frac{y \cdot u_3}{u_3 \cdot u_3}\right) u_3$$

$$= \left(\frac{1}{3}\right)\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \left(\frac{14}{3}\right)\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \left(\frac{-5}{3}\right)\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}\left(\frac{1}{3}\right)+\left(\frac{14}{3}\right)+0\\ \left(\frac{1}{3}\right)+0+\left(\frac{5}{3}\right)\\ 0+\left(\frac{14}{3}\right)+\left(\frac{-5}{3}\right)\\ \left(\frac{-1}{3}\right)+\left(\frac{14}{3}\right)+\left(\frac{5}{3}\right)\end{bmatrix} = \begin{bmatrix}5\\2\\3\\6\end{bmatrix}$$

So, $$y - \hat{y} = \begin{bmatrix}3\\4\\5\\6\end{bmatrix} - \begin{bmatrix}5\\2\\3\\6\end{bmatrix} = \begin{bmatrix}-2\\2\\2\\0\end{bmatrix}$$

Now,

$$(y - \hat{y}) + \hat{y} = y$$

and $$(y - \hat{y}) \cdot \hat{y} = \begin{bmatrix}5\\2\\3\\6\end{bmatrix} \cdot \begin{bmatrix}-2\\2\\2\\0\end{bmatrix} = -10 + 4 + 6 + 0 = 0$$

This shows y is a sum of two orthogonal vectors $y - \hat{y}$ and $\hat{y}$ where $\hat{y}$ is in $W = \text{Span}\{u_1, u_2, u_3\}$.

4. (i) Let $W = \text{Span}\{v_1, v_2\}$. Then the closest point to y in W is the orthogonal projection $\hat{y}$ of y onto span $\{v_1, v_2\}$.

Here,

$$v_1 \cdot v_1 = \begin{bmatrix}3\\1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}3\\1\\-1\\1\end{bmatrix} = (3)(3) + (1)(1) + (-1)(-1) + (1)(1)$$

$$= 9 + 1 + 1 + 1 = 12$$

$$v_2 \cdot v_2 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = 1 + 1 + 1 + 1 = 4$$

$$y \cdot v_1 = \begin{bmatrix}3\\1\\5\\1\end{bmatrix} \cdot \begin{bmatrix}3\\1\\-1\\1\end{bmatrix} = 9 + 1 - 5 + 1 = 6$$

$$y \cdot v_2 = \begin{bmatrix}3\\1\\5\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = 3 - 1 + 5 - 1 = 6$$

Now, the orthogonal projection $\hat{y}$ of y onto span $\{v_1, v_2\}$ is,

$$\hat{y} = \left(\frac{y \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{y \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$= \left(\frac{6}{12}\right)\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \left(\frac{6}{4}\right)\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)\\ \left(\frac{1}{2}\right)+\left(\frac{-3}{2}\right)\\ \left(\frac{-1}{2}\right)+\left(\frac{3}{2}\right)\\ \left(\frac{1}{2}\right)+\left(\frac{-3}{2}\right)\end{bmatrix} = \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}$$

Thus, the closest point to y in the subspace W spanned by v_1 and v_2 is (3, -1, 1, -1).

(ii) Let $W = \text{Span}\{v_1, v_2\}$. Then the closest point to y in W is the orthogona projection of of y onto span $\{v_1, v_2\}$.

Here,

$$v_1 \cdot v_1 = \begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} = (1)(1) + (-2)(-2) + (-1)(-1) + (2)(2)$$

$$= 1 + 4 + 1 + 4 = 10$$

$$v_2 \cdot v_2 = \begin{bmatrix}-4\\1\\0\\3\end{bmatrix} \cdot \begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = 16 + 1 + 0 + 9 = 26$$

$$y \cdot v_1 = \begin{bmatrix}3\\-1\\1\\13\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} = 3 + 2 - 1 + 26 = 30$$

$$y \cdot v_2 = \begin{bmatrix}3\\-1\\1\\13\end{bmatrix} \cdot \begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = -12 - 1 + 0 + 39 = 26$$

Now, the orthogonal projection $\hat{y}$ of y onto $\{v_1, v_2\}$ is

$$\hat{y} = \left(\frac{y \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{y \cdot v_2}{v_2 \cdot v_2}\right) v_1$$

$$= \left(\frac{30}{10}\right)\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} + \left(\frac{26}{26}\right)\begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = \begin{bmatrix}3+(-4)\\(-6)+(-1)\\(-3)+0\\6+(3)\end{bmatrix} = \begin{bmatrix}-1\\-5\\-3\\-9\end{bmatrix}$$

Thus, the closest point to y in the subspace W spanned by v_1 and v_2 is $(-1, -5, -3, -9)$.

Here,

$$v_1 \cdot v_1 = \begin{bmatrix}2\\-1\\-3\\1\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\-3\\1\end{bmatrix} = (2)(2) + (-1)(-1) + (-3)(-3) + (1)(1) = 15$$

$$v_2 \cdot v_2 = \begin{bmatrix}1\\1\\0\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\-1\end{bmatrix} = 1 + 1 + 0 + 1 = 3$$

$$z \cdot v_1 = \begin{bmatrix}3\\-7\\2\\3\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\-3\\1\end{bmatrix} = 6 + 7 - 6 + 3 = 10$$

$$z \cdot v_2 = \begin{bmatrix}3\\-7\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\-1\end{bmatrix} = 3 - 7 + 0 - 3 = -7$$

Now, the orthogonal projection $\hat{z}$ of z onto span $\{v_1, v_2\}$ is,

$$\hat{z} = \left(\frac{z \cdot v_1}{v_1 \cdot v_2}\right) v_1 + \left(\frac{z \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$= \left(\frac{10}{15}\right)\begin{bmatrix}2\\-1\\-3\\1\end{bmatrix} + \left(\frac{-7}{3}\right)\begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}\left(\frac{4}{3}\right)+\left(\frac{-7}{3}\right)\\ \left(\frac{-2}{3}\right)+\left(\frac{-7}{3}\right)\\ (-2)+0\\ \left(\frac{2}{3}\right)+\left(\frac{7}{3}\right)\end{bmatrix} = \begin{bmatrix}-1\\-3\\-2\\3\end{bmatrix}$$

Therefore, the best approximation to z by vectors of the form $c_1v_1 + c_2 v_2$ is $\hat{z} = c_1v_1 + c_2 v_2 = (-1, -3, -2, 3)$.

Here,

$$u_1 \cdot u_1 = \begin{bmatrix}-3\\-5\\1\end{bmatrix} \cdot \begin{bmatrix}-3\\-5\\1\end{bmatrix} = (-3)(-3) + (-5)(-5) + (1)(1) = 35$$

$$u_2 \cdot u_2 = \begin{bmatrix}-3\\2\\1\end{bmatrix} \cdot \begin{bmatrix}-3\\2\\1\end{bmatrix} = 9 + 4 + 1 = 14$$

$$y \cdot u_1 = \begin{bmatrix}5\\-9\\5\end{bmatrix} \cdot \begin{bmatrix}-3\\-5\\1\end{bmatrix} = -15 + 45 + 5 = 35$$

$$y \cdot u_2 = \begin{bmatrix}5\\-9\\5\end{bmatrix} \cdot \begin{bmatrix}-3\\2\\1\end{bmatrix} = -15 - 18 + 5 = -28$$

Then, the orthogonal projection $\hat{y}$ of y onto spanned by u_1 and u_2 is

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2$$

$$= \left(\frac{35}{35}\right)\begin{bmatrix}-3\\-5\\1\end{bmatrix} + \left(\frac{-28}{14}\right)\begin{bmatrix}-3\\2\\1\end{bmatrix} = \begin{bmatrix}(-3)+(6)\\(-5)+(-4)\\1+(-2)\end{bmatrix} = \begin{bmatrix}3\\-9\\-1\end{bmatrix}$$

Then,

$$y - \hat{y} = \begin{bmatrix}5\\-9\\5\end{bmatrix} - \begin{bmatrix}3\\-9\\-1\end{bmatrix} = \begin{bmatrix}2\\0\\6\end{bmatrix}$$

Now, the distance from y to the plane in $\mathbb{R}^3$ spanned by u_1 and u_2 is $\|y - \hat{y}\|$. That is,

$$\|y - \hat{y}\| = \sqrt{(2)^2 + 0^2 + (6)^2} = \sqrt{40}.$$

7. Let $y = \begin{bmatrix}4\\8\\1\end{bmatrix}$, $u_1 = \begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}$, $u_2 = \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$, and $W = \text{Span}\{u_1, u_2\}$.

Let $U = [u_1 \ \ u_2]$.

(i) Here,

$$U = [u_1 \ \ u_2] = \begin{bmatrix}2/3 & -2/3\\1/3 & 2/3\\2/3 & 1/3\end{bmatrix}$$

Then,

$$U^T = \begin{bmatrix}2/3 & 1/3 & 2/3\\-2/3 & 2/3 & 1/3\end{bmatrix}$$

Now,

$$U^T U = \begin{bmatrix}2/3 & 1/3 & 2/3\\-2/3 & 2/3 & 1/3\end{bmatrix}\begin{bmatrix}2/3 & -2/3\\1/3 & 2/3\\2/3 & 1/3\end{bmatrix}$$

$$= \begin{bmatrix}(4/9)+(1/9)+(4/9) & (-4/9)+(2/9)+(2/9)\\(-4/9)+(2/9)+(2/9) & (4/9)+(4/9)+(1/9)\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

and

$$UU^T = \begin{bmatrix}2/3 & -2/3\\1/3 & 2/3\\2/3 & 1/3\end{bmatrix}\begin{bmatrix}2/3 & 1/3 & 2/3\\-2/3 & 2/3 & 1/3\end{bmatrix}$$

$$= \begin{bmatrix} (4/9)+(4/9) & (2/9)+(-4/9) & (4/9)+(-2/9) \\ (2/9)+(-4/9) & (1/9)+(4/9) & (2/9)+(2/9) \\ (4/9)+(-2/9) & (2/9)+(2/9) & (4/9)+(1/9) \end{bmatrix}$$

$$= \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$$

(ii) Here,

$$u_1 . u_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} . \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1.$$

$$u_2 . u_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} . \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$$

$$y.u_1 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} . \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \frac{8}{3} + \frac{8}{3} + \frac{2}{3} = 6$$

$$y.u_2 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} . \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = -\frac{8}{3} + \frac{16}{3} + \frac{1}{3} = 3$$

Let W = span $\{u_1, u_2\}$. Then

$$\text{Proj}_W (y) = \left(\frac{y.u_1}{u_1.u_1}\right) u_1 + \left(\frac{y.u_2}{u_2.u_2}\right) u_2$$

$$= \left(\frac{6}{1}\right)\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} + \left(\frac{3}{1}\right)\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 4-2 \\ 2+2 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

Since $(UU^T)\, y = \text{Proj}_W (y)$

So $(UU^T)\, y = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$.

Exercise 8.4

1. Let,

$$x_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$$

Let $\{x_1, x_2\}$ is a basis for a subspace W.

Define,

$$v_1 = x_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v_2 = x_2 - \left(\frac{x_2 . v_1}{v_1 . v_1}\right) v_1$$

Here,

$$x_2 . v_1 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 24 + 0 + 6 = 30$$

and, $$v_1 . v_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 9 + 0 + 1 = 10$$

Then,

$$v_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \left(\frac{30}{10}\right)\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 8-9 \\ 5-0 \\ -6+3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$$

Thus, by gram Schmidt process, the orthogonal basis for W is $\{v_1, v_2\}$ that is,

$$\left\{\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}\right\}.$$

2. Let,

$$x_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$$

Let $\{x_1, x_2\}$ is a basis for a subspace W.

Define,

$$v_1 = x_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \quad \text{and} \quad v_2 = x_2 - \left(\frac{x_2 . v_1}{v_1 . v_1}\right) v_1$$

Here,

$$x_2 . v_1 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} . \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = 0 + 24 - 14 = 10$$

$$v_1 . v_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} . \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = 0 + 16 + 4 = 20$$

Then,

$$v_2 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \left(\frac{10}{20}\right)\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5-0 \\ 6-2 \\ -7-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$$

Thus, by Gram-Schmidt process, the orthogonal basis for W is $\{v_1, v_2\}$ that is,

$$\left\{\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}\right\}.$$

3. Let,

$$x_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

Let $\{x_1, x_2\}$ is a basis for a subspace W.

Define,

$$v_1 = x_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = x_2 - \left(\frac{x_2 . v_1}{v_1 . v_1}\right) v_1$$

Here,

$$x_2 . v_1 = \begin{bmatrix}4\\-1\\2\end{bmatrix}\begin{bmatrix}2\\-5\\1\end{bmatrix} = 8 + 5 + 2 = 15$$

$$v_1 . v_1 = \begin{bmatrix}2\\-5\\1\end{bmatrix} . \begin{bmatrix}2\\-5\\1\end{bmatrix} = 4 + 25 + 1 = 30$$

Then,

$$v_2 = \begin{bmatrix}4\\-1\\2\end{bmatrix} - \left(\frac{15}{30}\right)\begin{bmatrix}2\\-5\\1\end{bmatrix} = \begin{bmatrix}4-1\\(-1)-(-5/2)\\2-(1/2)\end{bmatrix} = \begin{bmatrix}3\\3/2\\3/2\end{bmatrix}$$

Thus, by Gram-Schmidt process the orthogonal basis for W is $\{v_1, v_2\}$ that is

$$\left\{\begin{bmatrix}2\\-5\\1\end{bmatrix}, \begin{bmatrix}3\\3/2\\3/2\end{bmatrix}\right\}.$$

Let,

$$x_1 = \begin{bmatrix}3\\-4\\5\end{bmatrix} \text{ and } x_2 = \begin{bmatrix}-3\\14\\-7\end{bmatrix}$$

Let $\{x_1, x_2\}$ is a basis for a subspace of W.
Define,

$$v_1 = x_1 = \begin{bmatrix}3\\-4\\5\end{bmatrix} \quad \text{and} \quad v_2 = x_2 - \left(\frac{x_2 . v_1}{v_1 . v_1}\right) v_1$$

Here,

$$x_2 . v_1 = \begin{bmatrix}-3\\14\\-7\end{bmatrix} . \begin{bmatrix}3\\-4\\5\end{bmatrix} = -9 - 56 - 35 = -100$$

$$v_1 . v_1 = \begin{bmatrix}3\\-4\\5\end{bmatrix} . \begin{bmatrix}3\\-4\\5\end{bmatrix} = 9 + 16 + 25 = 50$$

Then,

$$v_2 = \begin{bmatrix}-3\\14\\-7\end{bmatrix} - \left(\frac{-100}{50}\right)\begin{bmatrix}3\\-4\\5\end{bmatrix} = \begin{bmatrix}-3+6\\14-8\\-7+10\end{bmatrix} = \begin{bmatrix}3\\6\\3\end{bmatrix}$$

Thus, by Gram-Schmidt process, the orthogonal basis for a subspace W is $\{v_1, v_2\}$ that is

$$\left\{\begin{bmatrix}3\\-4\\5\end{bmatrix}, \begin{bmatrix}3\\6\\3\end{bmatrix}\right\}.$$

Let,

$$x_1 = \begin{bmatrix}1\\-4\\0\\1\end{bmatrix} \text{ and } x_2 = \begin{bmatrix}7\\-7\\-4\\1\end{bmatrix}$$

Let $\{x_1, x_2\}$ is a basis for a subspace of W.
Define

$$v_1 = x_1 \quad \text{and} \quad v_2 = x_2 - \left(\frac{x_2 . v_1}{v_1 . v_1}\right) v_1$$

Here,

$$x_2 . v_1 = \begin{bmatrix}7\\-7\\-4\\1\end{bmatrix} . \begin{bmatrix}1\\-4\\0\\1\end{bmatrix} = (7)(1) + (-7)(-4) + (-4)(0) + (1)(1)$$

$$= 7 + 28 + 0 + 1 = 36$$

$$v_1 . v_1 = \begin{bmatrix}1\\-4\\0\\1\end{bmatrix} . \begin{bmatrix}1\\-4\\0\\1\end{bmatrix} = 1 + 16 + 0 + 1 = 18$$

Then,

$$v_2 = \begin{bmatrix}7\\-7\\-4\\1\end{bmatrix} - \left(\frac{36}{18}\right)\begin{bmatrix}1\\-4\\0\\1\end{bmatrix} = \begin{bmatrix}7-2\\-7+8\\-4+0\\1-2\end{bmatrix} = \begin{bmatrix}5\\1\\-4\\-1\end{bmatrix}$$

Thus, by Gram-Schmidt process, the orthogonal basis for a subspace W is $\{v_1, v_2\}$ that is

$$\left\{\begin{bmatrix}1\\-4\\0\\1\end{bmatrix}, \begin{bmatrix}6\\2\\-4\\-1\end{bmatrix}\right\}.$$

6. Let,

$$x_1 = \begin{bmatrix}3\\-1\\2\\-1\end{bmatrix} \text{ and } x_2 = \begin{bmatrix}-5\\9\\-9\\3\end{bmatrix}$$

Let $\{x_1, x_2\}$ is a basis for a subspace of W.
Define,

$$v_1 = x_1 \quad \text{and} \quad v_2 = x_2 - \left(\frac{x_2 . v_1}{v_1 . v_1}\right) v_1$$

Here,

$$x_2 . v_1 = \begin{bmatrix}-5\\9\\-9\\3\end{bmatrix} . \begin{bmatrix}3\\-1\\2\\-1\end{bmatrix} = (-5)(3) + (9)(-1) + (-9)(2) + (3)(-1)$$

$$= -15 - 9 - 18 - 3 = -45$$

$$v_1 . v_1 = \begin{bmatrix}3\\-1\\2\\-1\end{bmatrix} . \begin{bmatrix}3\\-1\\2\\-1\end{bmatrix} = 9 + 1 + 4 + 1 = 15$$

Then,

$$v_2 = \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix} - \left(\frac{-45}{15}\right)\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -5-9 \\ 9-3 \\ -9+6 \\ 3-3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$$

Thus by Gram-Schmidt process, the orthogonal basis for a subspace W is $\{v_1, v_2\}$ that is,

$$\left\{\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}\right\}.$$

7. From Q. No. 3 we have, the orthogonal basis of the subspace W spanned by x_1 and x_2 is, $\{v_1, v_2\}$ where,

$$v_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

Here,

$$\|v_1\| = \sqrt{(2)^2 + (-5)^2 + (1)^2} = \sqrt{4+25+1} = \sqrt{30}$$

$$\text{and } \|v_2\| = \sqrt{(3)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{9 + \left(\frac{9}{4}\right) + \left(\frac{9}{4}\right)} = \frac{3\sqrt{6}}{2}$$

Now, the orthonormal basis of W is $\{u_1, u_2\}$ where

$$u_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{30}} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{30}} \\ \frac{-5}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \end{bmatrix}$$

$$\text{and } u_2 = \frac{1}{\|v_2\|} v_2 = \frac{2}{3\sqrt{6}}\begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

8. **Similar to Q. No. 7.**

9. Let,

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$$

Let the columns of A are x_1, x_2, x_3. So,

$$x_1 = (5, 1, -3, 1),\ x_2 = (9, 7, -5, 5).$$

Let $\quad v_1 = x_1 = (5, 1, -3, 1)$

Take, $\quad v_2 = x_2 - \frac{x_2 . v_1}{v_1 . v_1} v_1$

$$= (9, 7, -5, 5) - \frac{(9, 7, -5, 5).(5, 1, -3, 1)}{(5, 1, -3, 1).(5, 1, -3, 1)}(5, 1, -3, 1)$$

$$= (9, 7, -5, 5) - \frac{72}{36}(5, 1, -3, 1)$$

$$= (-1, 5, 1, 3).$$

Thus, $\{v_1, v_2\}$ be an orthogonal basis. Then, let $\{u_1, u_2\}$ be normalize of the orthogonal basis. So,

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(5, 1, -3, 1)}{6}$$

$$u_2 = \frac{v_2}{\|v_2'\|} = \frac{(-1, 5, 1, 3)}{\sqrt{6}}$$

Let Q be a matrix whose columns are u_1, u_2. Then

$$Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -1/2 & 1/6 \\ -1/6 & 1/2 \end{bmatrix}.$$

Since we have A = QR, by QR-factorization theorem. Here,

$$Q^TA = Q^T(QR) = Q^TQR = I\,R = R.$$

Now,

$$R = Q^TA$$

$$\Rightarrow R = \begin{bmatrix} 5/6 & 1/6 & -1/2 & 1/6 \\ -1/6 & 5/6 & 1/6 & 1/2 \end{bmatrix}\begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}.$$

(ii) Similar to (i).

Exercise 8.5

1. (i) Let,

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Then,

$$A^T A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+1 & -2-6-3 \\ -2-6-3 & 4+9+9 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

and

$$A^T b = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4+2-2 \\ 8-3+6 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

Then, the equation $A^TAx = A^T b$ implies

$$\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix} \quad \text{.... (i)}$$

Here, the augmented matrix of (i) is

$$\begin{bmatrix} 6 & -11 & : & -4 \\ -11 & 22 & : & 11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -11/6 & : & -2/3 \\ -11 & 22 & : & 11 \end{bmatrix} \text{ applying } R_1 \to R_1/6$$

$$\sim \begin{bmatrix} 1 & -11/6 & : & -2/3 \\ 0 & 11/6 & : & 11/3 \end{bmatrix} \text{ applying } R_2 \to R_2 + 11 R_1$$

$$\sim \begin{bmatrix} 1 & 0 & : & 3 \\ 0 & 11/6 & : & 11/3 \end{bmatrix} \text{ applying } R_1 \to R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & : & 3 \\ 0 & 1 & : & 2 \end{bmatrix} \text{ applying } R_2 \to \frac{6}{11} R_2$$

This $x_1 = 3$, $x_2 = 2$.

Since, we know the least square solutions of $Ax = b$ coincides with the non-empty solution of normal equations $A^TAx = A^Tb$. So, the least square solution for given problem is,

$$\hat{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Let,

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

Then,

$$A^TA = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4+4 & 2+0+6 \\ 2+0+6 & 1+0+9 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$$

and

$$A^Tb = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -10-16+2 \\ -5+0+3 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

Then, the equation $A^TAx = A^Tb$ implies

$$\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix} \quad \text{.... (i)}$$

The augmented matrix of (i) is

$$\begin{bmatrix} 12 & 8 & : & -24 \\ 8 & 10 & : & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2/3 & : & -2 \\ 8 & 10 & : & -2 \end{bmatrix} \text{ applying } R \to R_1/12$$

$$\sim \begin{bmatrix} 1 & 2/3 & : & -2 \\ 0 & 14/3 & : & 14 \end{bmatrix} \text{ applying } R_2 \to R_2 - 8 R_1$$

$$\sim \begin{bmatrix} 1 & 2/3 & : & -2 \\ 0 & 1 & : & 3 \end{bmatrix} \text{ applying } R_2 \to (3/14) R_2$$

$$\sim \begin{bmatrix} 1 & 0 & : & -4 \\ 0 & 1 & : & 3 \end{bmatrix} \text{ applying } R_1 \to R_1 - (2/3) R_2$$

This means $x_1 = -4$ and $x_2 = 3$.

Since we know the least square solutions of $Ax = b$ coincides with the non-empty solution of normal equations $A^TAx = A^Tb$. So, the least square solution for given problem is,

$$\hat{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

(iii) Let,

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

Then,

$$A^TA = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0+4 & -2-2+0+10 \\ -2-2+0+10 & 4+4+9+25 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 4 & 42 \end{bmatrix}$$

and

$$A^Tb = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-1+0+4 \\ -6+2-12+10 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

Then, the equation $A^TAx = A^Tb$ implies

$$\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} \quad \text{.... (i)}$$

The augmented matrix of (i) is,

$$\begin{bmatrix} 6 & 6 & : & 6 \\ 6 & 42 & : & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & : & 1 \\ 6 & 42 & : & -6 \end{bmatrix} \text{ applying } R_1 \to R_1/6$$

$$\sim \begin{bmatrix} 1 & 1 & : & 1 \\ 0 & 36 & : & -12 \end{bmatrix} \text{ applying } R_2 \to R_2 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 1 & : & 1 \\ 0 & 1 & : & -1/3 \end{bmatrix} \text{ applying } R_2 \to R_2/3$$

$$\sim \begin{bmatrix} 1 & 0 & : & 4/3 \\ 0 & 1 & : & -1/3 \end{bmatrix} \text{ applying } R_1 \to R_1 - R_2$$

This implies $x_1 = 4/3$ and $x_2 = -1/3$

Since, we know that least square solutions of $Ax = b$ coincides with the non-empty solution of normal equations $A^TAx = A^Tb$. So, the least square solution for given problem is,

$$\hat{x} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}.$$

(iv) Let

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

Then,

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 3-1+1 \\ 3-1+1 & 9+1+1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}.$$

and
$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5+1+0 \\ 15-1+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

Then the equation $A^TAx = A^Tb$ implies

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix} \quad \text{.... (i)}$$

The augmented matrix of (i) is,

$$\begin{bmatrix} 3 & 3 & : & 6 \\ 3 & 11 & : & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & : & 2 \\ 3 & 11 & : & 14 \end{bmatrix} \text{ applying } R_1 \to R_1/3$$

$$\sim \begin{bmatrix} 1 & 1 & : & 2 \\ 0 & 8 & : & 8 \end{bmatrix} \text{ applying } R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & : & 2 \\ 0 & 1 & : & 1 \end{bmatrix} \text{ applying } R_1 \to R_2/8$$

$$\sim \begin{bmatrix} 1 & 0 & : & 1 \\ 0 & 1 & : & 1 \end{bmatrix} \text{ applying } R_1 \to R_1 - R_2$$

This implies $x_1 = 1$, $x_2 = 1$.

Since, we have the least square solutions of $Ax = b$ coincides with the non-empty solution of normal equations $A^TAx = A^Tb$. So, the least squares solutions for given problem is,

$$\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

2. (i) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

Then,

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

and
$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$$

Then the equation $A^TAx = A^Tb$ implies

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix} \quad \text{.... (i)}$$

The augmented matrix of (i) is,

$$\begin{bmatrix} 4 & 2 & 2 & : & 14 \\ 2 & 2 & 0 & : & 4 \\ 2 & 0 & 2 & : & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 1/2 & : & 7/2 \\ 1 & 1 & 0 & : & 2 \\ 1 & 0 & 1 & : & 5 \end{bmatrix} \text{ applying } R_1 \to R_1/4,\ R_2 \to R_2/2 \text{ and } R_3 \to R_3/2$$

$$\sim \begin{bmatrix} 1 & 1/2 & 1/2 & : & 7/2 \\ 0 & 1/2 & -1/2 & : & -3/2 \\ 0 & -1/2 & 1/2 & : & 3/2 \end{bmatrix} \text{ applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 2 \\ 0 & 1/2 & -1/2 & : & -3/2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \text{ applying } R_1 \to R_1 - R_2,\ R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 2 \\ 0 & 1 & -1 & : & -3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \text{ applying } R_2 \to 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & -1 & : & -3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \text{ applying } R_2 \to R_2 + R_1$$

This implies $x_1 = -1$, $x_2 - x_3 = -3 \Rightarrow x_1 = -1$, $x_2 = -3 + x_3$.

Since, we have the least square solution for $Ax = b$ concides with the non-empty solution of the normal equation $A^TAx = A^Tb$. So, the general least square solution for given problem is,

$$\hat{x} = \begin{bmatrix} -1 \\ -3 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(ii) **Similar to (i).**

3. Using the result from Q 1(iii),

$$b - A\hat{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

Then,

$$\|b - A\hat{x}\| = \sqrt{(1)^2 + (3)^2 + (-3)^2 + (1)^2} = \sqrt{1+9+9+1} = 2\sqrt{5}.$$

Thus, the least squares error is 6.

Note: Remember that the distance between b and for any x in $\mathbb{R}^4$ is at least 6 units.

Using the result from Q 1(iv).

$$b - A\hat{x} = \begin{bmatrix}5\\1\\0\end{bmatrix} - \begin{bmatrix}1 & 3\\1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}5/2\\-1/2\end{bmatrix} = \begin{bmatrix}5\\1\\0\end{bmatrix} - \begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}4\\-2\\-2\end{bmatrix}$$

Then, $\|b - A\hat{x}\| = \sqrt{(4)^2 + (-2)^2 + (-2)^2} = \sqrt{16+4+4} = 2\sqrt{6}$

Thus, the least squares error is $2\sqrt{6}$.

(i) Let,

$$A = [u_1 \;\; u_2] = \begin{bmatrix}1 & 5\\3 & 1\\-2 & 4\end{bmatrix} \text{ and } b = \begin{bmatrix}4\\-2\\-3\end{bmatrix}$$

Then,

$$u_1 . u_1 = \begin{bmatrix}1\\3\\-2\end{bmatrix} . \begin{bmatrix}1\\3\\-2\end{bmatrix} = 1 + 9 + 4 = 14$$

$$u_2 . u_2 = \begin{bmatrix}5\\1\\4\end{bmatrix} . \begin{bmatrix}5\\1\\4\end{bmatrix} = 25 + 1 + 16 = 42$$

$$b . u_1 = \begin{bmatrix}4\\-2\\-3\end{bmatrix} . \begin{bmatrix}1\\3\\-2\end{bmatrix} = 4 - 6 + 6 = 4$$

$$b . u_2 = \begin{bmatrix}4\\-2\\-3\end{bmatrix} . \begin{bmatrix}5\\1\\4\end{bmatrix} = 20 - 2 - 12 = 6$$

Now, the orthogonal projection of b onto Col A is,

$$\hat{b} = \left(\frac{b . u_1}{u_1 . u_1}\right)u_1 + \left(\frac{b . u_2}{u_2 . u_2}\right)u_2 \quad \text{where } A = [u_1 \;\; u_2]$$

$$= \left(\frac{4}{14}\right)\begin{bmatrix}1\\3\\-2\end{bmatrix} + \left(\frac{16}{42}\right)\begin{bmatrix}5\\1\\4\end{bmatrix} = \begin{bmatrix}(2/7)+(5/7)\\(6/7)+(1/7)\\(-4/7)+(4/7)\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

Since, the least square solution $\hat{x}$ of $Ax = b$ is the solution of $A\hat{x} = \hat{b}$. And we know,

$$\hat{x} = \begin{bmatrix}\dfrac{b . u_1}{u_1 . u_1}\\[2mm]\dfrac{b . u_2}{u_2 . u_2}\end{bmatrix} \quad \text{where } A = [u_1 \;\; u_2]$$

i.e. $\hat{x} = \begin{bmatrix}2/7\\1/7\end{bmatrix}$.

(ii) – (iv) → Similar to Q. No. 5(i).

6. Here,

$$Au = \begin{bmatrix}3 & 4\\-2 & 1\\3 & 4\end{bmatrix}\begin{bmatrix}5\\-1\end{bmatrix} = \begin{bmatrix}15-4\\-10-1\\15-4\end{bmatrix} = \begin{bmatrix}11\\-11\\11\end{bmatrix}$$

and

$$Av = \begin{bmatrix}3 & 4\\-2 & 1\\3 & 4\end{bmatrix}\begin{bmatrix}5\\-2\end{bmatrix} = \begin{bmatrix}15-8\\-10-2\\15-8\end{bmatrix} = \begin{bmatrix}7\\-12\\7\end{bmatrix}$$

Since,

$$\|Au\| = \sqrt{3(11)^2} = 11\sqrt{3}$$

$$\|Av\| = \sqrt{3(49)^2} = 49\sqrt{3}$$

This shows $\|Au\| < \|Av\|$. Therefore, u is the least squares solution of $Ax = b$.

Exercise 8.6

Example 30: Find the equation of least-squares line that best fits the data points (2, 1), (5, 2), (7, 3) and (8, 3).

Solution: Let the equation be

$$y = \beta_0 + \beta_1 x \qquad \ldots \text{(i)}$$

Then at the points (0, 1), (1, 1), (2, 2), (3, 2), the line (i) gives,

$$\left.\begin{aligned} 1 = y_1 &= \beta_0 + 0\beta_1\\ 1 = y_2 &= \beta_0 + 1\beta_2\\ 2 = y_3 &= \beta_0 + 2\beta_2\\ 2 = y_4 &= \beta_0 + 3\beta_2 \end{aligned}\right\} \qquad \ldots \text{(ii)}$$

In the matrix form of (ii),

$$y = X\beta \qquad \ldots \text{(iii)}$$

where,

$$y = \begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix} = \begin{bmatrix}1\\1\\2\\2\end{bmatrix}, \; X = \begin{bmatrix}1 & 0\\1 & 1\\1 & 2\\1 & 3\end{bmatrix} \text{ and } \beta = \begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}.$$

The normal equation to (iii) is,

$$X^T X\beta = X^T y \qquad \ldots \text{(iv)}$$

Here,

$$X^T X = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\\1 & 2\\1 & 3\end{bmatrix} = \begin{bmatrix}4 & 6\\6 & 14\end{bmatrix}.$$

and

$$X^T y = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\1\\2\\2\end{bmatrix} = \begin{bmatrix}6\\11\end{bmatrix}.$$

Therefore (iv) becomes,

$$\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 84 - 66 \\ -36 + 44 \end{bmatrix} = \begin{bmatrix} 9/10 \\ 2/5 \end{bmatrix}.$$

Thus, (i) becomes,

$$y = 0.9 + 0.4x.$$

This is the required least squares line.

(ii) –(iv) Similar to (i).

Exercise 8.7

1. Given that the inner product in $\mathbb{R}^2$ is defined as

$$<x, y> = 4x_1y_1 + 5x_2y_2$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$

(i) Let

$$x = (1, 1) \text{ and } y = (5, -1)$$

Then,

$$||x||^2 = <x, x> = 4\,(1)\,(1) + 5(1)\,(1) = 4 + 5 = 9$$
$$||y||^2 = <y, y> = 4\,(5)\,(5) + 5\,(-1)\,(-1) = 100 + 5 = 105$$
$$<x, y> = 4\,(1)\,(5) + 5\,(1)\,(-1) = 20 - 5 = 15$$

So, $|<x, y>|^2 = |15|^2 = 225$

Therefore,

$$||x|| = 3,\ ||y|| = \sqrt{105} \quad \text{and} \quad |<x, y>|^2 = 225.$$

(ii) Let, $z = (z_1, z_2)$ is orthogonal to y for any z_1, z_2. So,

$$<z, y> = 0$$

i.e. $4\,(z_1)\,(5) + 5\,(z_2)\,(-1) = 0$

$\Rightarrow \quad 20z_1 - 5z_2 = 0$

$\Rightarrow \quad z_2 = 4z_1$

Thus,

$$z = (z_1, z_2) = (z_1, 4z_1) = z_1\,(1, 4)$$

This means any vector which is multiple of (1, 4), is orthogonal to y.

2. Given that the inner product in $\mathbb{R}^2$ is defined as

$$<x, y> = 4x_1\,y_1 + 5x_2\,y_2$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Let

$$x = (3, -2) \text{ and } y = (-2, 1)$$

Then,

$$||x||^2 = <x, x> = 4\,(3)\,(3) + 5\,(-2)\,(-2) = 36 + 20 = 56$$
$$||y||^2 = <y, y> = 4\,(-2)\,(-2) + 5\,(1)\,(1) = 16 + 5 = 21$$
$$<x, y> = 4\,(3)\,(-2) + 5\,(-2)\,(1) = -24 - 10 = -34$$

Now, the Cauchy–Schwarz inequality for x, y is,

$$|<x, y>| \le ||x||\ ||y||$$

i.e. $|-34| \le \sqrt{56}\,\sqrt{21}$

$$\Rightarrow \quad 34 \le 7\sqrt{8}\,\sqrt{3} = 14\sqrt{2}\,\sqrt{3} = 14\sqrt{6} = 34.29$$

This means the Cauchy–Schwarz inequality holds by given x and y.

3. Given that for $p, q \in \mathbb{P}_2$ is defined as

$$<p, q> = p(t_0)\,q(t_0) + p(t_1)\,q(t_1) + p(t_2)\,q(t_2)$$

Also given that

$$t_0 = -1,\ t_1 = 0,\ t_2 = 1.$$

Here,

$$p(t) = 4 + t,\ q(t) = 5 - 4t^2$$

So,

$$p(t_0) = p(-1) = 3 \qquad q(t_0) = q(-1) = 1$$
$$p(t_1) = p(0) = 4 \qquad q(t_0) = q(0) = 5$$
$$p(t_2) = p(1) = 5 \qquad q(t_0) = q(1) = 1$$

Then,

$$<p, q> (3)\,(1) + (4)\,(5) + (5)\,(1) = 28$$

4. Given that for $p, q \in \mathbb{P}_2$ is defined as

$$<p, q> = p(t_0)\,q(t_0) + p(t_1)\,q(t_1) + p(t_2)\,q(t_2)$$

Also given that

$$t_0 = -1,\ t_1 = 0,\ t_2 = 1$$

Here,

$$p(t) = 3t - t^2,\ q(t) = 3 + 2t^2$$

So,

$$p(t_0) = p(-1) = -4 \qquad q(t_0) = q(-1) = 5$$
$$p(t_1) = p(0) = 0 \qquad q(t_1) = q(0) = 3$$
$$p(t_2) = p(1) = 2 \qquad q(t_2) = q(1) = 5$$

Then,

$$<p, q> = (-4)\,(5) + (0)\,(3) + (2)\,(5) = -20 + 0 + 10 = -10$$

5. From Q. No. 3

$$||p||^2 = <p, p> = [p(t_0)]^2 + [p(t_1)]^2 + p(t_2)]^2$$
$$= 9 + 16 + 25$$
$$= 50$$

$\Rightarrow \quad ||p|| = \sqrt{50} = 5\sqrt{2}$

and,

$$||q||^2 = <q, q> = [q(t_0)]^2 + [q(t_1)]^2 + q(t_2)]^2$$
$$= 1 + 25 + 1$$
$$= 27$$

$\Rightarrow \quad ||q|| = \sqrt{27} = 3\sqrt{3}$

6. From Q. No. 4

$$||p||^2 = <p, p> = [p(t_0)]^2 + [p(t_1)]^2 + p(t_2)]^2$$
$$= 16 + 0 + 4$$
$$= 20$$

$\Rightarrow \quad ||p|| = \sqrt{20} = 2\sqrt{5}$

and

$$||q|| = <q, q> = [q(t_0)]^2 + [q(t_1)]^2 + q(t_2)]^2$$
$$= 25 + 9 + 25$$
$$= 59$$

$\Rightarrow \quad \|q\| = \sqrt{59}$.

Here, $y = \beta_0 + \beta_1 x$

Let, $x = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, y = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{bmatrix}$

For weight matrix, choose W with diagonal entries 1, 2, 2, 2, 1.

Then, $W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

So, $Wx = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -2 \\ 2 & 0 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}$

and $Wy = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 8 \\ 4 \end{bmatrix}$

Then, $(Wx)^T (Wx) = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ -2 & -2 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & -2 \\ 2 & 0 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1+4+4+4+1 & -2-4+0+4+2 \\ -2-4+0+4+2 & 4+4+0+4+4 \end{bmatrix}$

$= \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix}$.

and $(Wx)^T (Wy) = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ -2 & -2 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 4 \\ 8 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 0+0+8+16+4 \\ 0+0+0+16+8 \end{bmatrix} = \begin{bmatrix} 28 \\ 24 \end{bmatrix}$

Now, the normal equation is,

$(Wx)^T (Wx)\, \beta = (Wx)^T (Wy)$

i.e. $\begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 28 \\ 24 \end{bmatrix}$

Solving we get $\beta_0 = \frac{28}{14} = 2$, $\beta_1 = \frac{24}{16} = 1.5$

Thus, the least-squares line that fits the given data is $y = 2 + 1.5x$.

Chapter 9

Group and Subgroups

Exercise 9.1

1. What three things must we check to determine whether a function $\phi : S \to S'$ is an isomorphism of a binary structure?

Solution. To determine whether a function $\phi: S \to S'$ $<S, *>$ with $<S', *'>$, is an isomorphism: we have to check the following three conditions

(i) Does ϕ is one-to-one function? That is, suppose that $\phi(x) = \phi(y)$ in S' and deduce from this $x = y$ in s.

(ii) Does ϕ is onto S'? That is suppose that $s' \in S'$ is given and show that there does exist $s \in S$ such that $\phi\binom{s}{x} = s'$.

(iii) Show that if $\phi(x * y) = \phi(x) * \phi(y)$ for all $x, y \in S$. This is just a question of computation. Compute both sides of the equation and see whether they are the same.

2

a. $<\mathbb{Z}, +>$ with $<\mathbb{Z}, +>$ where $\phi(x) = -n$ for $n \in \mathbb{Z}$.

Step 1: ϕ is one to one: for any $x, y \in \mathbb{Z}$ with $\phi(x) = \phi(y)$

$\Rightarrow -x = -y \Rightarrow x = y$: Hence ϕ is one to one.

Step 2: ϕ is on to: for any $y \in \mathbb{Z}$ such that $\phi(-y) = -(-y) = y$

$\therefore$ ϕ is on to [for any $y \in \mathbb{Z}$ there is $-y \in \mathbb{Z}$. $\phi(-y) = -((-y)) = y$.

Step 3: Now $\phi(x + y) = -(x + y) = (-x) + (-y) = \phi(x) + \phi(y)$

Then, ϕ is isomorphism.

b. $<\mathbb{Z}, +>$ with $<\mathbb{Z}, +>$ where $\phi(x) = 2x$ for $x \in \mathbb{Z}$.

i. ϕ is one-to-one: Let $x \in \mathbb{Z}$, then

$\phi(x) = 2x$ and $y \in \mathbb{Z}$. $\phi(y) = 2y$

Such that $\phi(x) = \phi(y)$

$\Rightarrow 2x = 2y$

$\Rightarrow x = y$

$\therefore$ $\phi(x) = \phi(y) \Rightarrow x = y$

Thus ϕ is one to one.

ii. ϕ is one to : $\forall x = n \in \mathbb{Z}$ the number $\frac{n}{2}$ may not be integer $\phi\left(\frac{n}{2}\right) = 2 \times \frac{n}{2} = n$

Thus ϕ is not on to and hence ϕ is not isomorphism.

c. $<\mathbb{Z}, +>$ and $<\mathbb{Z}, +>$ where $\phi(n) = n + 1$

i. $\phi(x) = \phi(y)$ for $x, y \in \mathbb{Z}$

$\Rightarrow x + 1 = y + 1$

$\Rightarrow x = y$ $\quad\therefore$ ϕ is one to one

ii. ϕ is onto: for every $x \in \mathbb{Z}$ there exists

$x - 1 \in \mathbb{Z}$ such that $\phi(x - 1) = x - 1 + 1 = x$

$\therefore$ ϕ is onto

iii. For $x, y \in \mathbb{Z}$ and $\phi(x + y) = x + y - 1 \neq x - 1 + y - 1 = \phi(x) + \phi(y)$

Then, $\phi(x + y) \neq \phi(x) + \phi(y)$

$\therefore$ ϕ is not homomorphism and ϕ is not isomorphism.

d. $<Q, +>$ when $<Q, +>$ when $\phi(x) = \frac{x}{2}$ for $x \in Q$.

i. For $x, y, \in Q$ with $\phi(x) = \phi(y)$

$\Rightarrow \frac{x}{2} = \frac{y}{2}$

$\Rightarrow x = y$

$\therefore$ ϕ is one to one.

ii. For $y \in Q$ there exists $2y \in Q$ such that $\phi(2y) = \frac{2y}{2} = y$. Then ϕ is on to.

iii. Let $\phi(x + y) = \frac{x + y}{2} = \frac{x}{2} + \frac{y}{2} = \phi(x) + \phi(y)$

Hence ϕ is isomorphism.

e. $<Q, .>$ and $<Q, .>$ where $\phi(x) = x^2$ for $x \in Q$.

i. For $x, y \in Q$ with $\phi(x) = \phi(y)$

$\Rightarrow$ $x^2 = y^2$ may not be $x = y$, Since $x^2 = y^2$ is true,

also for $-x = -y$, $-x = y$, $x = -y$.

Then ϕ is not one to one and is not isomorphism.

f. $<\mathbb{R}, .>$ with $<\mathbb{R}, .>$ where $\phi(x) = x^3$ for $x \in \mathbb{R}$.

i. For $x, y \in \mathbb{R}$ with $\phi(x) = \phi(y)$

$x^3 = y^3$ and $x, y \in \mathbb{R}$

$\Rightarrow$ $x = y$ or $x^2 - xy + y^2 = 0$

$\because$ ϕ is not isomorphism, but not possible in general.

Then ϕ is one to one.

$\forall$ real number $x \in \mathbb{R}$ there is $x^{\frac{1}{3}} \in \mathbb{R}$. Such that $\phi(x^{1/3}) = (x^{1/3})^3 = x$

Then ϕ is on to.

For $x, y \in \mathbb{R}$. Then $\phi(xy) = (xy)^3$

$$= x^3 y^3$$

$$= \phi(x)\,\phi(y)$$

Then ϕ is an isomorphism

$<M_2(\mathbb{R}), .>$ with $<\mathbb{R}, .>$ where $\phi(A)$ is the determinant of matrix A.

For $A, B \in M_2(R)$ with $\phi(A) = \phi(B) \Rightarrow |A| = |B|$

It is not necessary to the matrix are equal for equal determinant. We can find different matrices with same determinant. Hence ϕ is not one to one and ϕ is not isomorphism.

$<M_1(R), .>$ where $\phi(A)$ is the determinant of matrix A.

For one to one, let $A, B \in M_1(R)$ with $\phi(A) = \phi(B) \Rightarrow |A| = |B|$, $A = B$, if the determinant of $|X|$ matrices are equal then the matrices must be equal.

For even $x \in R$ there exists a matrix. $A = [x]$ such that $\phi(A) = |x| = x$. Then ϕ is on to.

For $A, B \in M_1(R)$ with $\phi(AB) = |AB| = AB = |A|\,|B| = \phi(A)\,\phi(B)$

Here, both A and B are 1×1 matrices and AB is also 1×1 matrix.

Then ϕ is isomorphism.

$<\mathbb{R}, +>$ with $<\mathbb{R}^+, .>$ where $\phi(r) = \left(\frac{1}{2}\right)^r$ for $r \in \mathbb{R}$

Let $x, y \in \mathbb{R}$

Then $\phi(x) = \phi(y)$

$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^y$

$x = y$ and $\phi(x) = \phi(y) \Rightarrow x = y$

For $y \in \mathbb{R}^+$ there is $x \in \mathbb{R}$ such that $\phi(x) = y$, $\left(\frac{1}{2}\right)^x = y$ is true only for $x = 0 \in R$, $y \in^+ \mathbb{R}$. Hence ϕ is not isomorphism

3. Determine whether the binary operation * gives a group structure on the given set. If no group results, give the first axioms in order G_0, G_1, G_2, G_3 from definition that does not hold.

a. Let $*$ be defined on $\mathbb{Z}$ by letting $a * b = ab$.

Closure G_0: For any integers a and b the product ab is again an integer hence $a * b \in G_0$.

Associative G_1: For any integers $a, b, c \in \mathbb{Z}$.

$(a * b) * c = a * (b * c)$ is true; since multiplication of integers are always associative.

Existence of identity G_2: For any integers $a \in \mathbb{Z}$ there exists unique integer $1 \in \mathbb{Z}$ such that $a * 1 = a = 1 * a$.

Existence of inverse G_3: For any integers are $a \in \mathbb{Z}$ the integer $\frac{1}{a}$ in some cases may not be integer or not defined particularly for $a = 0$. Hence G_3 is not hold. Hence $(G, *)$ is not group.

b. Let $*$ be defined on $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ by letting $a * b = a + b$.

Solution.

G_0: For any integers n the integer 2n is always even then for any $a = 2n_1$ and $b = 2n_2$: $a * b = 2n_1 + 2n_2 = 2(n_1 + n_2)$ is again even integer. Thus $a * b \in G$.

For G_1: Now for any three even integers $a = 2n_1$, $b = 2n_2$, $c = 2n_3$

Then, $(a * b) * c = (2n_1 * 2n_2) * 2n_3$

$$= (2n_1 + 2n_2) * 2n_3 = 2n_1 + 2n_2 + 2n_3 = 2(n_1 + n_2 + n_3)$$

Again, $a * (b * c) = (2n_1) * (2n_2 * 2n_3) = 2n_1 * (2n_2 + 2n_3)$

$$= 2n_1 + 2n_2 + 2n_3$$

$$= 2(n_1 + n_2 + n_3)$$

$\therefore$ $(a * b) * c = a * (b * c)$. Thus, G_1 is hold.

G_2 : For any integer $a = 2n_1 \in G$ there exists $e = 0 = 2.0 \in G$ such that $a * e = a + e = 2n_1 + 0 = 2n_1 = a$. Thus, G_2 hold.

G_3 : For any integer $a = 2n_1 \in G$ there exists unique integer $a' = -2n_1$ such that $a * a' = a + a' = 2n_1 - 2n_1 = 0$. Thus, G_3 is also hold. $\therefore$ G is group with the binary operation $a * b = a + b$.

c. Let $*$ be defined on $\mathbb{R}^+$ by letting $a * b = \sqrt{ab}$.

Solution.

G_0: For any positive real numbers $a, b \in \mathbb{R}^+$, $a*b \in \mathbb{R}^+$.

Now,

G_1: For any $a, b, c \in \mathbb{R}^+$ then

$$a*(b*c) = a*\sqrt{bc} = \sqrt{a\sqrt{bc}}$$

$$\text{and } (a*b)*c = \sqrt{ab}*c = \sqrt{\sqrt{ab}\,c}$$

Since, $\sqrt{a\sqrt{bc}} \neq \sqrt{\sqrt{ab}\,c}$. Thus, elements of $\mathbb{R}^+$ are not associative under the binary operation '$*$'.

Thus, $\mathbb{R}^+$ is not group with respect to binary operation $*$.

d. Let $*$ be defined on Q by letting $a*b = ab$.

Solution.

G_0: For any rational number a, b

$a*b = ab$ is also rational i.e. $a*b \in Q$.

G_1: For any rational numbers $a, b, c \in Q$. Then,

$(a*b)*c = (ab)*c = abc$

$a*(b*c) = a*(bc) = abc$

which shows that elements of Q are associative.

G_2: For any $a \in Q$ there exists unique rational numbers $1 \in Q$, such that, $a*1 = 1*a = a\,1 = a$

G_3: For any $a \in Q$ there exists unique element $b = \frac{1}{a} \in Q$ for $a \neq 0$, Such that

$$a*b = a*\frac{1}{a} = 1$$

But for $a = 0$, $b = \frac{1}{a}$ is not defined, so multiplicative inverse for zero is not defined.

Hence, G is not a group.

e. Let $*$ be defined on the set $\mathbb{R}^*$ of non-zero real numbers by letting $a*b = \frac{a}{b}$.

Solution.

G_0: For any two non-zero real numbers a, b : $a*b = \frac{a}{b}$ is also non zero positive real numbers.

$\therefore$ $a*b \in \mathbb{R}$

G_1: For any three non-zero positive real numbers $a, b, c \in \mathbb{R}$: $(a*b)*c = \left(\frac{a}{b}\right)*$

$$c = \frac{\frac{a}{b}}{c} = \frac{a}{bc}$$

Again, $a*(b*c) = a*\left(\frac{b}{c}\right) = \frac{a}{\frac{b}{c}} = \frac{ac}{b}$

$\therefore$ $(a*b)*c \neq a*(b*c)$ $\quad\therefore$ Elements of $\mathbb{R}^*$ are not associative under binary operation $*$.

$\therefore$ $\mathbb{R}^*$ is not a group with binary operation $*$.

f. Let $*$ be defined on $\mathbb{C}$ by letting $a*b = |ab|$

Solution. G_0: For any two complex number a, b. Then,

$a*b = |a\,b|$ $\quad$ $|ab|$ is also complex number. (Every real number is a complex number)

G_1: For any complex numbers a, b, c. Then,

$(a*b)*c = |ab|*c = ||ab|c| = |a|\,|b|\,|c|$

Again,

$a*(b*c) = a*|bc| = |a\,|bc|\,| = |a|\,|b|\,|c|$

Thus, elements of $\mathbb{C}$ are associative.

G_2: For any $a \in \mathbb{C}$ there is an element $1 \in \mathbb{C}$. Such that

$a*1 = |a| = a$ is not in general.

Hence existence of identity element is not true. Then G is a group.

Exercise 9.2

Determine whether the given subset of the complex numbers is a subgroup of the group $\mathbb{C}$ of complex numbers under addition.

1. $\mathbb{R}$: Here $\mathbb{R}$ is a non-empty subset of the group $\mathbb{C}$.
   Closure: Sum of any two real numbers is again a real number.
   Existence of Additive Identity: For any real number $x \in R$ there exists $0 \in R$ such that $x + 0 = 0 + x = x$.
   Existence of additive inverse: For any real number $x \in R$ there exists $-x \in R$ such that $x + (-x) = 0 = (-x) + x$.
   Hence R is subgroup of $\mathbb{C}$.

Q^+

Here Q^+ is set of all positive rational numbers is non-empty: sum of two positive rational number is positive rational number.

Existence of additive identity: For any $x \in Q^+$

$0 \notin Q^+$ such that $0 + x = x + 0 = x$

Q^+ is not subgroup of $\mathbb{C}$.

$7\mathbb{Z} = \{0, 7, 14, \ldots\} \cup \{-7, -14, \ldots\}$

$= \{\ldots, -14, -7, 0, 7, 14, \ldots\}$

Closure: For each elements $x = 7n_1$ and $y = 7n_2$, where $n_1, n_2 \in \mathbb{Z}$. The sum $x + y = 7n_1 + 7n_2 = 7(n_1 + n_2) \in 7\mathbb{Z}$

Existence of identity element: For each $x = 7n \in 7\mathbb{Z}$ there exists an element $0 = 7.0 \in 7\mathbb{Z}$ such that $0 + x = 0 + 7n = 7n = x + 0$

Existence of inverse: For each element $x = 7n \in 7\mathbb{Z}$ there exists an element $-7n = -x \in 7\mathbb{Z}$ such that $x + (-x) = 7n - 7n = 0$

$$-x + x = -7n + 7n = 0$$

Thus, $7\mathbb{Z}$ form a sub-group under addition.

The set $i\mathbb{R}$ of pure imaginary numbers including 0.

lution. $H = \{ix, x \in \mathbb{R}\} \cup \{0\}$

Closure: Sum of two pure imaginary numbers including 0 are either 0 or pure imaginary. Hence elements of $i\mathbb{R}$ are closed under addition.

Existence of additive identity: For each element $x \in H\ \{0\}$. There exists $0 \in H\ \{0\}$. Such that, $x + 0 = 0 + x = x$.

Existence of additive inverse: For each $x \in H$ there exists $-x \in H$. Such that $x + (-x) = (-x) + x = 0$. Thus, H is a subgroup of $\mathbb{C}$.

The set πQ of rational multiples of π.

lution. Let $H = \{\pi x : x \text{ is rational numbers}\}$

Since $0 \in Q$. Hence $0 \in H$.

Closure: For any elements $x, y \in H$

Then $x = \pi n_1$ and $y = \pi n_2$ where $n_1, n_2 \in Q$. Then $x + y = \pi(n_1 + n_2) \in \pi Q = H$.

Existence of additive identity: For each $x \in \pi Q$ there is $0 \in \pi Q$. Such that $x + 0 = 0 + x = x$.

Existence of additive inverse: For each $x \in \pi Q$ say $x = \pi n$, $n \in Q$ there exists $y = -\pi n$, $-n \in Q$ such that

$x + y = 0 = y + x$

πQ is a group under addition.

The set $G = \{\pi^n \mid n \in \mathbb{Z}\}$

Here for each $x = \pi^{n_1}$

$$y = \pi^{n_2} \qquad \text{where } n_1, n_2 \in \mathbb{Z}$$

$x + y = \pi^{n_1} + \pi^{n_2}$ may not in the form of π^n. Hence, G is not a sub group under addition.

Which of the sets from 1 to 6 are subgroup of $\mathbb{C}^*$ (of non-zero complex number) under multiplication.

lution. $\mathbb{Q}^+$ is subgroup of $\mathbb{C}^*$ under multiplication and all other are not.

Determine whether the given set of invertible n×n matrices with real entries is a subgroup of GL(n, R).

8. The n×n matrices with determinant 2.

Solution. Here $G = \{A : A \text{ is } n\times n \text{ invertible matrix with real entries}\}$

$H = \{B : B \text{ is } n\times n \text{ invertible matrix with real entries of determinant } 2\}$

Closure: For B_1 and $B_2 \in H$ then $|B_1| = 2$, $|B_2| = 2$

Then $|B_1 B_2| = |B_1|\,|B_2| = 2\times 2 = 4$

Thus, element of H are not closed under multiplication of matrices.

9. The diagonal n×n matrices with no zeros on the diagonal.

Solution. (i) Let n = 3 and $H = \{A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}; a \neq 0, b \neq 0, c \neq 0.$

Let $A, B, \in H$ then,

$$A + B = \begin{bmatrix} a+x & 0 & 0 \\ 0 & b+y & 0 \\ 0 & 0 & c+z \end{bmatrix}$$

Where $B = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ $x \neq 0, y \neq 0, z \neq 0$, here A + B may have zero in its main diagonal.

$a + x, b + y, c + z$ may be zero.

Hence, H is not closed under addition and hence is not subgroup of $G(n, \mathbb{R})$.

(ii) If we consider the binary operation is multiplication of matrices then

Closure, $AB = \begin{bmatrix} ax & 0 & 0 \\ 0 & by & 0 \\ 0 & 0 & cz \end{bmatrix} \in H$. Since $ax, by, cz \neq 0$.

Existence of multiplicative identity: There exists a matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Such that $AI = IA = A$.

Existence of multiplicative inverse: For any $A \in H$ there exists,

$$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix} \in H$$

Such that $AA^{-1} = A^{-1}A = I$

Thus H is subgroup under multiplication.

10. The upper-triangular n×n matrices with no zeros on the diagonal.

Solution. $H = \{A : A \text{ is upper triangular with no zeros in diagonal}\}$

Existence of identity: Here, the identity matrix of n×n order is an upper triangular with no zeros in it's diagonal is an element of H. Thus, H has identity element.

Closure: Let $A, B \in H$. Then both A and B are upper triangular with no zeros in their diagonal. Now, AB is being product of two upper triangular matrices is again upper triangular with no zero in it's diagonal.

$\therefore AB \in H$

Existence of Inverse: Since, the determinant of upper triangular matrix is just product of main diagonal elements; here each diagonal elements are non-zero. The matrix are invertible and hence each of them has multiplicative inverse in H.

Thus, H is subgroup of G.

11. The n×n matrices with determinant –1.

$$H = \left[A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = -1 \right]$$

we consider 2×2 matrices.

$$\text{Let } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} : eh - gf = -1$$

Then $|AB| = |A|\ |B| = 1$

The elements of H are not closed under multiplication. Thus H is not subgroup.

12. $H = \{A : |A| \text{ is } -1 \text{ or } 1\}$

Solution. Here, H is non empty. Since the identity matrix of order n×n is in H with determinant 1, which play the role of identity element in H.

Closure: For any $A, B \in H$ then both A and B are non singular with their determinant 1 or –1.

$$|AB| = |A|\,|B| = \begin{cases} -1 \times 1 = -1 \text{ if } |A| = -1 \text{ and } |B| = 1 \\ -1 \times -1 = 1 \text{ if } |A| = -1 \text{ and } |B| = -1 \\ 1 \times -1 = -1 \text{ if } |A| = 1 \text{ and } |B| = -1 \\ 1 \times 1 = 1 \text{ if } |A| = 1,\ |B| = 1 \end{cases}$$

In all cases $|AB| = 1$ or -1 hence $AB \in H$

Existence of Inverse: Since the determinant of all $A \in H$ is either –1 or 1 in any cases A is non singular and hence A^{-1} exists.

Now, $AA^{-1} = I \Rightarrow |AA^{-1}| = |I| \Rightarrow |A|\ |A^{-1}| = 1$. if $|A| = 1$ then $|A^{-1}| = 1$, if $|A| = -1$ then $|A^{-1}| = -1$

∴ $A^{-1} \in H$. Hence H is subgroup of G.

13–20. Let F be the set of all real valued functions with domain $\mathbb{R}$ and let $\tilde{F}$ be the subset of F consisting of those functions that have a non-zero value at every point in $\mathbb{R}$. In exercises 14 through 19, determine whether the given subset of F with the induced operations is (a) a subgroup of group F under addition, (b) a subgroup of the group $\tilde{F}$ under multiplication.

13. F

Solution. $F = \{f : f \text{ is real valued function whose domain is } \mathbb{R}\}$

$\tilde{F} = \{f : f \text{ is real valued} : f(x) \neq 0, x \in \mathbb{R}\}$

Does $\tilde{F}$ is subgroup of F.

Closure: Let $f_1, f_2 \in \tilde{F}$. Then $\forall x \in \mathbb{R} : f_1(x) \neq 0, f_2(x) \neq 0$

Now,

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$

Here if $f_1(x) = 2$ and $f_2(x) = -2$. Then

$f_1(x) + f_2(x) = 2 - 2 = 0$ i.e. $f_1 + f_2 \notin \tilde{F}$

∴ $\tilde{F}$ is not subgroup under addition.

14. The subset of all $f \in F$, such that $f(1) = 0$.

Solution. Let $H = \{f \in F : f(1) = 0\}$

Here H is not empty, since $0 \in H$ since $0(1) = 0$

Closure: Let $f_1, f_2 \in H$. Then $f_1(1) = 0, f_2(1) = 0$

Then $(f_1 + f_2)(1) = f_1(1) + f_2(1) = 0 + 0 = 0$

∴ $f_1 + f_2 \in H$

ii. Elements of H are associative under addition i.e.

$[(f_1 + f_2) + f_3](1)$

$= (f_1 + f_2) + f_3(1)$

$= (f_1 + f_2)(1) + f_3(1) = 0 + 0 + 0 = 0$

$= f_1(0) + f_2(0)$

Again, $(f_1 + (f_2 + f_3))(1)$

$= f_1(1) + (f_2 + f_3)(1)$

$= f_1(1) + 0 + 0$

$= 0 + 0 + 0$

$= 0$

iii. For every $f \in H$

There exists $0 \in H$. Such that

$(f + 0)(1) = f(1) + 0(1) = 0 + 0 = 0 = f(1)$

Again, $\forall f \in H$, $(0 + f)(1) = 0(1) + f(1) = 0 + 0 = 0 = f(1)$

∴ $0 + f = f$

∴ $0 \in H$ is an additive identity of H.

Existence of additive inverse: $\forall f \in H$. There exists $-f \in H$. Such that $[f + (-f)](1) = f(1) - f(1) = 0 + 0 = 0 = 0(1)$

∴ $-f$ is inverse of f.

∴ H is a subgroup of F.

15. The subset of all $f \in \tilde{F}$ such that $f(1) = 1$

Solution. Here, $H = \{f \in \tilde{F} : f(1) = 1\}$

Since the identity map is in H. So, H is non empty.

Closure: For $f_1, f_2 \in H : (f_1 . f_2)(1) = f_1(1) . f_2(1) = 1 . 1 = 2$

∴ Elements of H are closed under multiplication

ii. Existence of identity is obvious.

i.e. $\forall f \in H$, there exists $I \in H$ such that $(f . I)(1) = f(1)\, I(1) = 1 . 1 = 1 = f(1)$

Similarly, $(I . f)(1) = f(1)$

(iii) Existence of inverse: Here $0 \notin H$, every other $f \in H$ has their inverse themselves and form a subgroup under multiplication.

16. $H = \{f \in \tilde{F} : f(0) = 1\}$

Solution.

Closure: For every $f, g \in H$, $(f . g)(0) = f(0) . g(0) = 1 . 1 = 1$

∴ $f . g \in H$

Existence of Identity: For all $f \in H$ the identity map $I(x) = x \notin H$. Such that $(f . I)(0) = f(0) . I(0) = 1 . 0 = 0 \neq f(0)$

Thus, H is not subgroup of $\tilde{F}$ under multiplication.

17. The subset of all $f \in \tilde{F}$ such that $f(0) = -1$

Solution. Here for all $f, g \in H : (f . g)(0) = f(0) . g(0) = -1 . -1 = 1$

∴ $f.g \notin H$

Hence, H is not subgroup of $\tilde{F}$ under multiplication.

The subset of all constant function in F.

tion. $H = \{f \in \tilde{F} : f(x) = c \forall \in \mathbb{R}\}$

Here $0 \in H$. Since $0(x) = 0 \ \forall x \in \mathbb{R}$ but the inverse of 0 mapping does not exists since 0 mapping is not one to one. Thus, H is not a subgroup under multiplication.

Write at least 5 elements of each of the following cyclic groups.

a. $25\mathbb{Z}$ under addition

$25\mathbb{Z} = \{-100, -75, -25, 0, 25, 50, 75, \ldots.\}$

b. $\left\{\left(\frac{1}{2}\right)^n \mid n \in \mathbb{Z}\right\}$ under multiplication}

$= \left\{\frac{1}{2}, \frac{1}{4}, 2, 4, \frac{1}{8}, \ldots\right\}$

c. $\{\pi^n \mid n \in \mathbb{Z}\}$ under multiplication

$= \left\{\pi, \frac{1}{\pi}, \pi^2, \frac{1}{\pi^2}, \ldots\ldots\right\}$

Which of the following groups are cyclic? For each cyclic group, list all the generators of the group. $G_1 = <\mathbb{Z}, +>$, $G_2 = <Q, +>$, $G_3 = <Q^+, .>$, $G_4 = <6\mathbb{Z}, +>$

tion. $G_1 = <\mathbb{Z}, +>$

Find the order of the cyclic subgroup of $\mathbb{Z}_4$ generated by 3.

tion. Here $\mathbb{Z}_4$ is a group under addition. The cyclic subgroup generated by 3 is itself $\mathbb{Z}_4$. Since, $3 + 3 = 2$, $3 + 3 + 3 = 1$, $3 + 3 + 3 + 3 = 0$, $3 + 3 + 3 + 3 + 3 = 3$. Hence order of cyclic subgroup of $\mathbb{Z}_4$ generated by 3 is 4 and order of the generator 3 is 3.

Chapter 10

Ring and Field

Exercise 10

1.

(a) $(12)(6) = 22$

(b) Since, $-8 = 18$; so $(20)(-8) = (20)(18) = 22$

(c) Since In Z_4, $-3 = 1$ and In Z_{11}, $-4 = 7$

So, $(-3, 5)(2, -4) = (1, 5)(2, 7) = (2, 2)$

2.

(a) $nZ = \{nm : \forall m \in Z\}$(supple $n \neq 0, 1$), otherwise is ring)

The addition is defined as $nm_1 + nm_2 = n(m_1 + m_2)$ ..... (1) $\forall m, m_2 \in Z$

and multiplication is defined as $nm_1 . nm_2 = n^2 (m_1m_2)$ ..... (2)

R_1: Here nZ is abelian group under b.0 addition

Because it is closed by (1) and associative, since for nm_1, nm_2 and $nm_3 \in nZ$.

$nm_1 + (nm_2 + nm_3) = n[m_1 + m_2 + m_3]$ and

$(nm_1 + nm_2) + nm_3 = n[m_1 + m_2 + m_3]$

Here, $0 = n . 0 \in nZ$ act as additive inverse and for every $nm \in nZ \exists -n\,m \in n\,Z$ st $nm + (-nm) = 0$.

So existence of inverse. Also, $nm_1 + nm_2 = n(m_1 + m_2) = n(m_2 + m_1) = nm_2 + nm_1$. So Abellian.

R_2: From (2) it is closed under binary operation.

R_3: Since, $(nm_1 . nm_2)(nm_3) = n^3 (m_1 m_2 m_3)$ and

$(nm_1 (nm_2 nm_3) = n_3 (m_1 m_2 m_3)$, so it is associative under b . 0 Multiplication.

R_4: Here $nm_1 . (nm_2 + nm_3) = nm_1 [n(m_2 + m_3)]$

$= (nm_1) n(m_2 + m_3)$

$= n^2 m_1 (m_2 + m_3)$

$= n^2 (m_1m_2 + m_1m_3)$

(b) $Z\times Z = \{(n, m) : \forall n_1 m\ Z\}$

The + binary operation defined as

$(n_1, m_1) + (n_2, m_2) = (n_1 + n_2, m_1 + m_2)$ .... (1)

$(n_1, m_1) . (n_2, m_2) = (n_1n_2, m_1m_2)$ .... (2)

R1: Here $Z\times Z$ is abelian group under b . o t.

It is closed from (1). It is associative also, because

$[(n_1, m_1) + (n_2, m_2)] + (n_3, m_3) = (n_1 + n_2 + n_3, m_1 + m_2 + m_3)$ and

$(n_1m_1) + [(n_2, m_2) + (n_3 + m_3)] = (n_1 + n_2 + n_3, m_1 + m_2 + m_3)$

Here $(0, 0) \in Z \times Z$ act as identity element.

For all $(n_1m) \in Z\times Z \exists (-n, -m) (Z \times Z)$ is inverse element so that $(n_1m) + (-n_1 -m) = (0, 0)$

Gain, $(n_1, m_1) + (n_2, m_2) = (n_1 + n_2, m_1 + m_2)$

$= (n_2 + n_1, m_2 + m_1)$

$= (n_2, m_2) + (n_2, m_1)$

$\therefore$ $Z \times Z$ is abelian.

R2: From (2), it is closed under b . 0.

R3: Here $[(n_1, m_1)(n_2, m_2)](n_3, m_3)] = (n_1 n_2 n_3, m_1 m_2 m_3)$ and

$(n_1, m_1)[(n_2, m_2)(n_3, m_3)] = (n_1 n_2 n_3, m_1 m_2 m_3)$. So, it is associative.

R4: Since $(n_1, m_1)[(n_2, m_2) + (n_3, m_3)] = (n_1, m_1)[(n_2 + n_3, m_2 + m_3)]$

$= [n_1 (n_2 + n_3, m_1(m_2 + m_3)]$

$= (n_1n_2 + n_1n_3, m_1 m_2 + m_1m_3)$

and $[(n_1, m_1)(n_2, m_2) + (n_1, m_1)(n_3, m_3)] = (n_1n_2, m_1m_2) + (n_1n_3, m_1m_3)$

$= (n_1n_2 + n_1n_3, m_1m_2 + m_1m_3)$

Hence, left distributive hold similarly for right distributive also T.

This is abelian group with identity because $(n_1, m_1)(n_2, m_2) = (n_2, m_1), (n_1, m_1)$ and $(1, 1) \in Z\times Z$. But it is not field because for $(2, 3) \in Z \times Z$ has no multiplication inverse.

(c) Z^+ is not group because $5 \in Z^+ - 5 \notin Z^+$ i.e. has no additive inverse of 5 in Z^+. Even though it satisfy all properties of ring.

(d) $G = \{a + b\sqrt{2}, : a, b \in Q\}$

In example (3) of book prove $G = \{a + b\sqrt{2}, : a, b \in Z\}$ is commutative ring. It has identity element also because $1 + 0\sqrt{2} \in G$ act as multiplicative identity and is field also because for every non zero element $a + b\sqrt{2} \in G \exists \frac{1}{a + b\sqrt{Z}}$

$= \frac{a - b\sqrt{Z}}{a^2 - 2b^2}$

$= \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}$ in Z

Such that $(a + b\sqrt{Z}) . \left(\frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{Z}\right) = 1$

(e) $M_2(R) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R\right\}$

addition b . 0 is defined as for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2(R)$

$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$ .... (1)

and multiplication b . 0 is defined as

$AB = \begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} + b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{pmatrix}$

R1: Here $M_2(R)$ is abelian group under b. 0 +.

Because it is closed under b. 0 + from (1).

Again, matrix addition is associative also.

i.e. $A + (B + C) = (A + B) + C$.

Here $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$ act as additive inverse and for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R) \exists \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, So, existence of inverse also. And matrix addition is commutative.

Also, because $A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$

and

$$B + A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ b_{21} + a_{21} & b_{22} + a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

$$= A + B$$

Here $M_2(R)$ is closed under b . 0 multiplication, shown from (2).

Matrix multiplication is associative. Also,

Because $(AB)C = A(BC)$

Here, $(AB)C = \begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} + b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

Similarly, we find $A(BC)$ then we get $(AB)C = A(BC)$

Here, $A(B + C) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ b_{21} + a_{21} & b_{22} + a_{22} \end{pmatrix}$

and $AB + AC = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

For additive inverse of $(3, 2) \in Z_4 \times Z_7$

$(3, 2)^i = (3^i, 2^i)$

For $3 \in Z_4$, $3^i = 1$

and $2 \in Z_7$, $2^i = 5$

Thus, $(3, 2)^i = (3^i, 2^i) = (1, 5)$

For multiplicative inverse of $(3, 2) \in Z_4 Z_7$

For $3 \in Z_4$ $\quad 3^i = 3$

$2 \in Z_7$ $\quad 2^i = 4$

$(3, 2)^i = (3^i, 2^i) = (3, 4)$

Given equation is

$x^2 - 5x + 6 = 0$

$x^2 - 3x - 2x + 6 = 0$

$x (x - 3) - 2 (x - 3) = 0$

$(x - 3) (x - 2) = 0$

$x = 2, 3$ also solution in Z_{12}. For other solution.

When $x = 4$; $x^2 - 5x + 6 = (x - 2) (x - 3) = (2) (1) \neq 0$

$x = 5, \quad = (3) (2) \neq 0$

$x = 6, \quad = (4) (3) \neq 0$

$x = 7, \quad = (5) (4) \neq 0$

$x = 8, \quad = (6) (5) \neq 0$

$x = 9, \quad = (7) (6) \neq 0$

$x = 10, \quad = (8) (7) \neq 0$

$x = 11, \quad = (9) (8) \neq 0$

$x = 6$ and $x = 11$ also solution of given equation is Z_{12}

5. $3x = 2$, in Z_7

Here, $3x - 2 = 0$

When $x = 1$, $3x-2 \neq 0$

$x = 2$, $3x - 2 \neq 0$

$x = 3$, $3x - 2 = 0$

$x = 4$, $3x - 2 \neq 0$

$x = 5$, $3x - 2 \neq 0$

$x = 6$, $3x - 2 \neq 0$

$\therefore$ $x = 3$ is solution in Z_7.

6.

(a) Z_{20}

Here, 2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18 are number whose ged with 20 are not 1. Hence these are zero divisor of Z_{20} because (2) (10) = (4) (5) = (6) (5) = (12) (5) = (8) (5) = (18) (10) = (16) (5) = (14) (10) = 0

(b) Z_{16}

Here, 2, 4, 6, 8, 10, 12, 14 are number whose ged with 16 are not 1. So, these are zero divisor of Z_{16} because

(4) (4) = (2) (8) = (10) (8) = (6) (8) = 0

(c) Z_{10}

Here, 2, 4, 5, 6, and 8 are zero division because

(2) (5) = (4) (5) = (6) (5) = (8) (5) = 0

(d) In Z_{11} has no zero divisor i.e. no $a, b \in Z_{11}$ St $ab = 0$

7. Z_7 is integral domain.

Since, it has commutative with identity and has no zero divisor so it integral domain. Here no two $a_1b \in Z_7$ st $ab = 0$.

8. Z_{12} is not integral domain because it has zero divisor since, $(4)(3) = 0$.

9. Z_{11} is field because it is ring (given) and commutative with identity and every non-zero elements has multiplicative inverse. Here, inverse of 1 is 1. Inver of 2 is 6. Inverse of 3 is 4. Inverse of 5 is 9, Inverse of 7 is 8, inverse of 10 is itself and inverse of 11 also itself.