USA TSTST/4

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Problem. *Suppose that n and k are positive integers such that*

$$1 = \underbrace{\varphi(\varphi(\ldots\varphi(}_{k \ \ times} n)\ldots)).$$

Prove that $n \leq 3^k$.

Here $\varphi(n)$ denotes Euler's totient function, i.e. $\varphi(n)$ denotes the number of elements of $\{1, \ldots, n\}$ which are relatively prime to n. In particular, $\varphi(1) = 1$.

Proposed by Linus Hamilton

Solution.

Lemma.

$$\forall n \in \mathbb{N}, \frac{n}{\varphi(n)} \leq 3 \cdot \left(\frac{3}{2}\right)^{v_2(\varphi(n)) - v_2(n)}$$

Proof. Note that if $n = 2^k$ for some $k \geq 0$ then this result holds trivially. Now let's assume that some odd prime divides n. Let S denote the set of all primes dividing n and G denotes set of all odd primes dividing n. Then note that $v_2(\varphi(n)) \geq v_2(n) + |G| - 1$ as each odd prime will add at least one power of 2 and only one power of 2 can be taken out from the even part. Hence $|G| \leq v_2(\varphi(n)) - v_2(n) + 1$ Now note that.

$$\frac{n}{\varphi(n)} = \prod_{p \in S}\left(1 + \frac{1}{p-1}\right) \leq 2 \cdot \prod_{p \in G}\left(1 + \frac{1}{p-1}\right) \leq 2 \cdot \left(\frac{3}{2}\right)^{|G|} \leq 2 \cdot \left(\frac{3}{2}\right)^{v_2(\varphi(n)) - v_2(n) + 1}$$

which proves our original claim. □

Now let $f_t(m) = \frac{\varphi^{t-1}(m)}{\varphi^t(m)}$ where $t, m \in \mathbb{N}$ and $g^t(m)$ is the function g applied to m total t times (if $t = 0$ then $g^t(m) = m$). Then note that

$$\prod_{i=1}^{i=k} f_t(n) \leq \prod_{i=1}^{i=k} 3 \cdot \left(\frac{3}{2}\right)^{v_2(\varphi^t(n)) - v_2(\varphi^{t-1}(n))}$$

Hence

$$n \leq 3^k \cdot \left(\frac{3}{2}\right)^{-v_2(n)} \leq 3^k$$

QED

Exploration. *Basically the solution, looking into powers of 2 was the key. We can improve bound to* $2 \cdot (3^{k-1})$ *but this is clean*

Tags. *NT , Euler totient , phi*