NT 2022 JBMO N1

February 20, 2025

Problem. *Determine all pairs* (k, n) *of positive integers that satisfy*

$$1! + 2! + \cdots + k! = 1 + 2 + \cdots + n$$

Solution. *Note that of* $k < 7$ *only 3 solutions exist namely* $(1, 1)$ *,* $(2, 2)$ *and* $(5, 17)$ *. If* $k \geq 7$ *then in this case we have*

$$2 * (1! + 2! + \cdots + k!) = n(n + 1)$$

Taking modulo 7 *both sides we get* $3 \equiv n(n+1) \mod 7$ *, this is a contraditction as* $(n(n + 1))$ *can never have a residue* 3 *mod* 7 *. Hence these are the only solutions*

Exploration. *This equation appears to be solvable by modular arithmetic. Just a matter of choosing right mod, tried with a computer and 7 works just fine. Another intuition on why modular arithmetic should work is if* $k \geq p$ *for some* p *then left side is always constant modulo* p *while left side is*

$$\frac{1}{2}\left(\left(n + \frac{1}{2}\right)^2 - \frac{1}{4}\right)$$

This should only take half the values as quadratic residues can only take half the values modulo p*. Since left and right side feels independent, each value of* p *has a* 50% *chance of working. Hence a modular search for contradiction feels natural.*

Tags. *Number Theory , Diophantine equation , Factorial , JBMO , Shortlist , Triangular Numbers*