Algebra FE Inequality

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Problem. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x+y) + y \leq f(f(f(x)))$$

holds for all $x, y \in \mathbb{R}$

Solution. *All functions such that $f(x) = c - x$ where $c \in \mathbb{R}$ are the only solution to the following equation. It's clear all such function satisfy the following equation, lets prove they are the only one.*

Let $P(x, y)$ be the assertion that $f(x+y) + y \leq f(f(f(x)))$.

Then $P(x, f(f(x)) - x) \implies f(f(x)) \leq x$. Hence ,

$$f(x+y) + y \leq f(f(f(x))) \leq f(x)$$

which implies $f(x+y) + x + y \leq f(x) + x$. Let $g(x) = f(x) + x$, then $g(x+y) \leq g(x)$.

This implies $g(x)$ is contant, hence $f(x) = c - x$, and all such functions satisfy the assertion.

Exploration. *N/A*

Tags. *Algebra , FE , Functional Equation , Functional Inequality , Inequality*