USAMO 2018/4

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Problem. *Let p be a prime and $a_1, \ldots, a_p$ be integers.*

Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + k, \ldots a_p + pk$$

produce at least $\frac{p}{2}$ distinct remainders upon division by p

Solution. *Note that if $p = 2$ the claim is trivially true, so let's assume $p \geq 3$ Let $N(G)$ denote the number of edges in a simple graph G.*

Consider p different simple graphs $G_0, G_2, \ldots G_{p-1}$. Each of these graphs have p nodes labelled from 1 to p. Graph G_k has an edge between i and j iff

$a_i + ik \equiv a_j + jk \mod p$.

note that edge between node i and node j is only present in graph G_k where $k \equiv (a_i - a_j)(j - i)^{-1} \mod p$ .Hence all of the edges of each graph are disjoint and every possible edge is present in at least one graph.

this implies

$$\sum_{i=0}^{i=p-1} N(G_i) = p(p-1)/2$$

hence by pigeonhole there exists a graph G_k such that $N(G_k) \leq (p-1)/2$. As number of connected components is at least (nodes - edges) therefore we get number of connected components at least $(p+1)/2$ which proves our original claim

Exploration. *Lots of thinking on choosing the right k , but then graph idea struck me*

Tags. *NT , Combinatorics , Graph*