# Algebra 2021 BMO A2

February 20, 2025

**Problem.** *Find all functions such that* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x^2 + y) \geq \left(\frac{1}{x} + 1\right) f(y)$$

**Solution.** $f(x) = 0$ *is the only solution.*

*First of all we will prove that* $f(x)$ *is non-negative for all reals.*

*Insert* $(x, y) = (0.5, y-0.25)$ *and* $(x, y) = (-0.5, y-0.25)$ *in original equation we get*

$$f(y) \geq max(3 * f(y - 0.25), -f(y - 0.25)) \geq 0$$

*Hence* $f$ *is a non negative function, now if* $b >= a$ *then inserting* $(x, y) = (\sqrt{b-a}, a)$ *we get*

$$f(b) \geq (\frac{1}{\sqrt{b-a}} + 1)f(a) \geq f(a)$$

*hence* $f$ *is non decreasing function. Now if* $f(a)! = 0$ *for any of the* $a$ *then* $f(a) > 0$ *and* $f(a+1) > 0$ *, let* $p = f(a+1)/f(a)$ *,note that* $p >= 1$ *, then insert* $(x, y) = (1/2p, a)$ *we get*

$$f(\frac{1}{4p^2} + a) \geq (4p^2 + 1)f(a) > pf(a) = f(a+1)$$

*This is a contradiction as* $f$ *is non decreasing. Hence* $f(x) = 0$ *is the only solution*

**Exploration.** *N/A*

**Tags.** *Functional Equation , FE , BMO , Shortlist , BMO-Shortlist , Algebra*