Algebra FE integer

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Problem. *Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer n we have*

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002$$

Solution. *$f(n) = n + 667$ is the only solution. Clearly it works , let's prove it's the only one.*

In general we will prove that if $c \in \mathbb{N}$,then $f(n) = n + c$ is the only solution to $2n + 3c \leq f(f(n)) + f(n) \leq 2n + 3c + 1$ so let's focus on this equation instead.

Let's say $f(n) \neq (n + c)$ for some n , then

if $f(n) = n + c + k$ where $k > 0$ then

$f(n) + c - 2k \leq f(f(n)) \leq f(n) + c - (2k - 1)$

if $f(n) = n + c - k$ where $k > 0$ then

$f(n) + c + 2k \leq f(f(n)) \leq f(n) + c + (2k + 1)$

This means if $f(a) > a + c$ then $f(f(a)) < f(a) + c$ and vice versa.

Let's say $f(n) \neq n + c$ for all n, choose a p such that $f(p) \neq p + c$, if $f(p) < p + c$ choose $p := f(p)$. Now we have $f(p) > p + c$. let's define the sequece

$$T_0 = p, T_n = f(T_{n-1})$$

Let's define another integer sequence $G_k = T_k - T_{k-1} - c$. Note that $G_1 > 0$ and G_k alternates sign hence G_k is positive for odd k and negative otherwise.

Note that for any number j

$$T_{2j} - T_0 - 2jc = \sum_{i=1}^{i=j} (G_{2i-1} + G_{2i})$$

Note that for even m we have $G_m \leq -(2G_{m-1} - 1)$ Now as $G_1 = k > 0$, this implies $G_2 \leq -(2k - 1)$ and $G_3 \geq 2(2k - 1) \geq 2k$. Hence for odd m we have $G_m \geq 2G_{m-2}$. Hence $G_{2t-1} \geq 2^{t-1}k$. Now note that $G_{2i-1} + G_{2i} \leq G_{2i-1} - (2G_{2i-1} - 1) \leq 1 - G_{2i-1} \leq 1 - 2^{i-1}k$.

Hence

$$T_{2j} - T_0 - 2jc = \sum_{i=1}^{i=j} (G_{2i-1} + G_{2i}) \leq \sum_{i=1}^{i=j} (1 - 2^{i-1}k) = (j - (2^j - 1)k).$$

So

$$T_{2j} \leq T_0 + (2c+1)j - (2^j - 1)k.$$

Now note that if j is big enough then right hand side will be negative hence T_{2j} will be negative which is a contradiction. Therefore we have $f(n) = n + c$. QED

Exploration. *N/A*

Tags. *Algebra , functional equation , FE , natural numbers , Number theory, inequality*