Algebra 2021 BMO A2

February 20, 2025

Problem. *Find all functions such that $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x^2 + y) \geq \left(\frac{1}{x} + 1\right) f(y)$$

Solution. *$f(x) = 0$ is the only solution.*

First of all we will prove that $f(x)$ is non-negative for all reals.

Insert $(x, y) = (0.5, y - 0.25)$ and $(x, y) = (-0.5, y - 0.25)$ in original equation we get

$$f(y) \geq max(3 * f(y - 0.25), -f(y - 0.25)) \geq 0$$

Hence f is a non negative function, now if $b > a$ then inserting $(x, y) = (\sqrt{b - a}, a)$ we get

$$f(b) \geq (\frac{1}{\sqrt{b - a}} + 1)f(a) \geq f(a)$$

hence f is non decreasing function. Now if $f(a)! = 0$ for any of the a then $f(a) > 0$ and $f(a + 1) > 0$, let $p = f(a + 1)/f(a)$,note that $p >= 1$, then insert $(x, y) = (1/2p, a)$ we get

$$f(\frac{1}{4p^2} + a) \geq (2p + 1)f(a) > pf(a) = f(a + 1)$$

This is a contradiction as f is non decreasing. Hence $f(x) = 0$ is the only solution

Exploration. *N/A*

Tags. *Functional Equation , FE , BMO , Shortlist , BMO-Shortlist , Algebra*