IMO 1998/3

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Problem. *Determine all positive integers k such that*

$$\frac{d(n^2)}{d(n)} = k$$

for some $n \in \mathbb{N}$.

Solution. *Note that $d(n^2)$ is odd so k can never be even. We claim that all odd numbers are possible.*

Let set $S := \{\frac{d(n^2)}{d(n)} : n \in \mathbb{N}\}$. Then let prime factorization of n be $\prod_{i=1}^{i=k} p_i^{a_i}$ where $a_i \in \mathbb{N}_0$. Then $\frac{d(n^2)}{d(n)} = \prod_{i=1}^{i=k}(2 - \frac{1}{a_i+1})$. Of course all numbers in S are of form $\prod_{i=1}^{i=k}(2-\frac{1}{a_i})$ where $a_i \in \mathbb{N}$, note that any number of form $\prod_{i=1}^{i=k}(2-\frac{1}{a_i})$ is also in S by choosing appropriate prime factors. So we only have to consider rationals of these forms.

First of all note that $1 \in S$ by choosing $n = 1$ (or choosing $k = 1, a_1 = 1$). Now if $a \in S$ and $b \in S$,then $ab \in S$. Now note that $\forall n, t \in \mathbb{N}$ we have $\frac{2^t(n-1)+1}{n} \in S$. This can be proved by induction on t. For $t = 1$ consider the sequence $k = 1, a_1 = n$. Now if for $t = m$, if we have $\frac{2^m(n-1)+1}{n} \in S$, then note that by choosing the sequence $k = 1, a_1 = (2^m(n-1)+1)$ we get $\frac{2^{m+1}(n-1)+1}{2^m(n-1)+1} \in S$. Hence their product $\frac{2^m(n-1)+1}{n} \cdot \frac{2^{m+1}(n-1)+1}{2^m(n-1)+1} \in S$. This completes the induction step. As we have $\frac{2^t(n-1)+1}{n} = 2^t - \frac{2^t-1}{n} \in S$. Letting $n = a \cdot (2^t - 1)$ we get $2^t - \frac{1}{a} \in S$ for all $a \in \mathbb{N}$.

Now let's prove any number of form $2t - 1 \in S$ where $t \in \mathbb{N}$. For $t = 1$, it is already established that $1 \in S$. Now let's suppose for the sake of induction that the given relation holds for all $t < m$ where $t > 1$. Then note that $2m - 1 = 2^k \cdot q - 1$ for some positive k where q is an odd number. Note that $q < (2m - 1)$ hence $q \in S$. Now as establised earlier that $2^k - \frac{1}{q} \in S$. We get $(2^k - \frac{1}{q}) \cdot q = 2^k \cdot q - 1 = 2m - 1 \in S$. This finishes the induction step. Hence all odd numbers k satisfy the given equation for some $n \in \mathbb{N}$.

Exploration. *Basically same as solution but with lots of messing around. using a computer and looking at construction of 7 helped to realised the $2^k - (2^k - 1)/n$ property*

Tags. *NT , Algebra*