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Asymptotics (Large-Sample Theory)

Typically, in stats or econometrics, we derive the properties of estimators by **taking expectations** and **taking sample size goes to infinity**. For example, given *i.i.d.* data y_1, \dots, y_n and the corresponding expectation $\mathbb{E}[y_i] = \mu$, we are able to estimate

$$\hat{\mu} \equiv \frac{1}{n} \sum_{i=1}^n y_i \quad \text{and} \quad \mathbb{E}[\hat{\mu}] = \mu.$$

Another example yields

$$\begin{aligned} \hat{\beta}_{OLS} &= \left(\frac{1}{n} \sum_{i=1}^n x_i x_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n x_i y_i \right) \\ &\xrightarrow{p} \left(\mathbb{E}[x_i x_i'] \right)^{-1} \mathbb{E}[x_i y_i] \end{aligned}$$

Law of Large Numbers

Given z_1, z_2, \dots, z_n are *i.i.d.* (not necessary), we have

$$\bar{z}_n \equiv \frac{1}{n} \sum_{i=1}^n z_i \quad \text{and} \quad \bar{z}_n \xrightarrow{p} \mathbb{E}[z_i]$$

Note that it is Weak Law of Large Number (WLLN) and almost-sure convergence here.

Central Limit Theorem

Given z_1, z_2, \dots, z_n are *i.i.d.* (not necessary) and $\mathbb{E}[z_i] \equiv \mu$, where z_i are $k \times 1$ vectors, we have

$$\sqrt{n}(\bar{z}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \mathbb{E}[(z_i - \mu)(z_i - \mu)']),$$

where $\mathbb{E}[(z_i - \mu)(z_i - \mu)'] \equiv \text{Var}(z_i)$.

Least Square

Given the data

$$\begin{array}{ll} y_1, & y_2, \dots, y_n \quad \text{dependent variables} \\ 1 \times 1 & \\ x_1, & x_2, \dots, x_n \quad \text{independent variables,} \\ k \times 1 & \end{array}$$

we define

$$Y \equiv \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad X \equiv \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}.$$

Theorem. Suppose $g(x_i)$ is some function of x_i . Then, the conditional mean of y_i , $\mathbb{E}[y_i | x_i] \equiv \mu_i$, minimize $\mathbb{E}[y_i - g(x_i)]^2$. That is, $g(x_i) = \mu_i$ is the minimizer. \square

Denote the predicted y_i as \hat{y}_i and define $\hat{\mathbb{E}}[\cdot] \equiv \frac{1}{n} \sum_{i=1}^n (\cdot)$, we want to minimize

$$Q_\infty(\beta) \equiv \mathbb{E}[y_i - \hat{y}_i]^2 \quad \text{and} \quad Q_n(\beta) \equiv \hat{\mathbb{E}}[y_i - \hat{y}_i]^2$$

by using linear curve $\hat{y}_i = x'_i \beta$, where x'_i and β are $1 \times k$ and $k \times 1$ vectors, respectively. Note that econometrisians call Q as the objective function, and statistisians call it as the criterion function.

Theorem. The minimizer of $\mathbb{E}[y_i - x'_i \beta]^2$ is

$$\beta_\infty = \left(\mathbb{E}[x_i x'_i] \right)^{-1} (\mathbb{E}[x_i y_i]).$$

The minimizer of $\hat{\mathbb{E}}[y_i - \hat{y}_i]^2$ is

$$\hat{\beta} = \left(\hat{\mathbb{E}}[x_i x'_i] \right)^{-1} (\hat{\mathbb{E}}[x_i y_i]).$$

\square

Here, if we define

$$\begin{aligned} e_i &\equiv y_i - x'_i \beta_\infty \quad \text{and} \quad \hat{e}_i \equiv y_i - x'_i \hat{\beta} \\ E &\equiv \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \quad \quad \quad \hat{E} \equiv \begin{pmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_n \end{pmatrix}. \end{aligned}$$

then

$$\mathbb{E}[x_i e_i] = 0 \quad \text{and} \quad \hat{\mathbb{E}}[x_i \hat{e}_i] = 0.$$

Assume observations are *i.i.d.* since

$$\hat{\mathbb{E}}[x_i x'_i] \xrightarrow{p} \mathbb{E}[x_i x'_i] \quad \text{and} \quad \hat{\mathbb{E}}[x_i y_i] \xrightarrow{p} \mathbb{E}[x_i y_i]$$

therefore, we obtain $\hat{\beta} \xrightarrow{p} \beta_\infty$.

Remark. $x'_i \beta_\infty$ may not to be the **true** μ_i but we know $\hat{\beta}$ converges to β_∞ . \square

Remark.

$$\begin{aligned} Q_n(\beta) &= \frac{1}{n} \sum_{i=1}^n (y_i - x'_i \beta)^2 \xrightarrow{p} Q_\infty(\beta) = \mathbb{E}[y_i - x'_i \beta]^2 \\ \hat{\beta} &\equiv \arg \min_{\beta} Q_n(\beta) \xrightarrow{p} \beta_\infty \equiv \arg \min_{\beta} Q_\infty(\beta). \end{aligned}$$

Typically, in econometrics textbook, β_∞ is the true parameters. That is, **consistency means that estimators converge to true parameters in probability.** \square

Finite sample properties

Given the model $Y = X\beta_\infty + E$, we have

$$\hat{\beta} = (X'X)^{-1}(X'Y) \quad \text{and} \quad \hat{\beta} = \beta_\infty + (X'X)^{-1}X'E.$$

Note that X and Y are $n \times k$ and $n \times 1$ matrix and vector.

- We say the parameter as unbiasedness if $\mathbb{E}[\hat{\beta} | X] = \beta_\infty$ by assuming $\mathbb{E}[E | X] = 0$.
- We obtain

$$\mathbb{E}[(\hat{\beta} - \beta_\infty)(\hat{\beta} - \beta_\infty)' | X] = \sigma^2(X'X)^{-1}$$

by assuming $\mathbb{E}[EE' | X] = \sigma^2 I_n$.

- If $E \sim \mathcal{N}(0, \sigma^2 I_n)$, we obtain

$$\hat{\beta} | X \sim \mathcal{N}(\beta_\infty, \sigma^2(X'X)^{-1}).$$

Asymptotic properties (Large-Sample properties)

Given the model $y_i = x_i'\beta_\infty + e_i$, we have

$$\hat{\beta} = \left(\frac{1}{n} \sum_{i=1}^n x_i x_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n x_i y_i \right) = \beta_\infty + \left(\frac{1}{n} \sum_{i=1}^n x_i x_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n x_i e_i \right).$$

The last part is sometimes called the **sampling error**. Note that since $\frac{1}{n} \sum_{i=1}^n x_i e_i \xrightarrow{p} \mathbb{E}[x_i e_i] = 0$, we have the consistency property

$$\hat{\beta} \xrightarrow{p} \beta_\infty.$$

Next, by re-scaling and the subtraction, the estimators turns to

$$\sqrt{n}(\hat{\beta} - \beta_\infty) = \left(\frac{1}{n} \sum_{i=1}^n x_i x_i' \right)^{-1} \left(\sqrt{n} \frac{1}{n} \sum_{i=1}^n x_i e_i \right).$$

By CLT,

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n x_i e_i - \mathbb{E}[x_i e_i] \right) \xrightarrow{d} \mathcal{N}(0, \mathbb{E}[x_i x_i' e_i^2])$$

since $\mathbb{E}[x_i e_i] = 0$, therefore, it alters to

$$\begin{aligned} \sqrt{n}(\hat{\beta} - \beta_\infty) &= \left(\frac{1}{n} \sum_{i=1}^n x_i x_i' \right)^{-1} \left(\sqrt{n} \frac{1}{n} \sum_{i=1}^n x_i e_i \right) \\ &\xrightarrow{d} \left(\mathbb{E}[x_i x_i'] \right)^{-1} \mathcal{N}(0, \mathbb{E}[x_i x_i' e_i^2]) \\ &\rightarrow \mathcal{N}\left(0, \left(\mathbb{E}[x_i x_i'] \right)^{-1} \mathbb{E}[x_i x_i' e_i^2] \left(\mathbb{E}[x_i x_i'] \right)^{-1}\right). \end{aligned}$$

Assume that $\mathbb{E}[x_i x_i' e_i^2] = \mathbb{E}[x_i x_i'] \sigma^2$, where $\sigma^2 \equiv \mathbb{E}[e_i^2]$, the asymptotic covariance matrix is $\sigma^2 \left(\mathbb{E}[x_i x_i'] \right)^{-1}$.

Remark. The existence of inverse $(X'X)^{-1}$ and $\left(\mathbb{E}[x_i x_i'] \right)^{-1}$ means that there is no perfect multi-collinearity. \square

Theorem. $(X'X)^{-1}$ exists if and only if the columns of X are linearly independent. To elaborate, the eigenvalues of $X'X$ are not equal to 0. \square

Note that

$$\text{plim}_{n \rightarrow \infty} \frac{1}{n} X'X = \mathbb{E}[x_i x_i'].$$

The existence issues mentioned in the remark and the theorem above reveals the identification; that is, we can identify the **unknown** parameters.

Identification

These equations are identical:

$$\begin{aligned} \mathbb{E}[x_i x_i'] \beta &= \mathbb{E}[x_i y_i] \\ \left(\frac{1}{n} \sum_{i=1}^n x_i x_i' \right) \beta &= \frac{1}{n} \sum_{i=1}^n x_i y_i \\ (X'X) \beta &= X'Y \end{aligned}$$

Here if the inverse of $X'X$ exists; that is, $X'X$ has k equations and we have k unknown β .

Note: chi-square are the square of normal distribution.

Projection and residual

The projection matrix is defined as $P \equiv X(X'X)^{-1}X'$. To represent the projection matrix mathematically, it projects vectors into the subspace spanned by columns of X . To elaborate, for any vector V , PV is the linear combination of columns of X .

To be more econometrics, it comes from

$$\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y \equiv PY.$$

Now, we define another matrix $M \equiv I_n - P$, where M is $n \times n$. We have the following properties for M and P :

- P, M are symmetric.
- $PP = P$.
- $MM = M$.
- $PM = 0$ (important in the calculation of prediction errors).
- $\text{trace}(P) = k$ and $\text{trace}(M) = n - k$. For any square matrix A , $\text{trace}(A)$ is the sum of diagonal entries of A . Moreover, $\text{trace}(AB) = \text{trace}(BA)$.

Prediction error

Suppose we have the true **in-sample** data and model $y_i^{in} = \mu_i + e_i^{in}$ generated and estimated by in-sample data x_i , and there exists an **out-sample** data y_i^{out} generated by **same in-sample** x_i , i.e., $y_i^{out} = \mu_i + e_i^{out}$. Note that $y_i = \mu_i + e_i$ is really the **true model** given $\mathbb{E}[y_i | x_i] \equiv \mu_i \iff e_i \equiv y_i - \mu_i$. Now, we want to calculate the expected square errors

$$\mathbb{E}[(Y^{in} - \hat{Y})'(Y^{in} - \hat{Y}) | X] \quad \text{and} \quad \mathbb{E}[(Y^{out} - \hat{Y})'(Y^{out} - \hat{Y}) | X]$$

to specify the prediction power of the model. Observe that

$$\begin{aligned}
 Y^{in} - X\hat{\beta} &= \mu + E^{in} - X\hat{\beta} \\
 &= \mu + E^{in} - X(X'X)^{-1}X'(\mu + E^{in}) \\
 &= (I - P)\mu + (I - P)E^{in} \\
 Y^{out} - X\hat{\beta} &= \mu + E^{out} - X\hat{\beta} \\
 &= \mu + E^{out} - X(X'X)^{-1}X'(\mu + E^{in}) \\
 &= (I - P)\mu + E^{out} - PE^{in}.
 \end{aligned}$$

Hence, we can take the expectation of the square error

$$\begin{aligned}
 \mathbb{E}[(Y^{in} - \hat{Y})'(Y^{in} - \hat{Y}) | X] &= \mathbb{E}[\mu'M'M\mu + E^{in'}M'ME^{in} + \mu'M'ME^{in} + E^{in'}M'M\mu] \\
 &= (n - k)\sigma^2 + \mu'(I - P)\mu \\
 &\quad (\text{Since } \mu'M'ME^{in} = E^{in'}M'M\mu = 0) \\
 \mathbb{E}[(Y^{out} - \hat{Y})'(Y^{out} - \hat{Y}) | X] &= \mathbb{E}[\mu'M'M\mu + \mu'M'E^{out} - \mu'M'PE^{in} + E^{out'}M\mu + E^{out'}E^{out} \\
 &\quad - E^{out'}PE^{in} + E^{in'}PM\mu - E^{in'}P'E^{out} + E^{in'}P'PE^{in} | X] \\
 &= (n + k)\sigma^2 + \mu'(I - P)\mu.
 \end{aligned}$$

Only the **highlighted terms** remain, and others go to 0 after taking the expectation. The reasons for being 0 include

Independence: μ and E^{out} ; E^{in} and E^{out} are independent. Therefore, the expectation term goes to 0.

PM Matrix: $PM = 0$ by definition.

By dividing into n , prediction errors alter to

$$\begin{aligned}
 \frac{1}{n} \mathbb{E}[(Y^{in} - \hat{Y})'(Y^{in} - \hat{Y}) | X] &= \sigma^2 - \frac{k}{n}\sigma^2 + \frac{1}{n}\mu'(I - P)\mu \\
 \frac{1}{n} \mathbb{E}[(Y^{out} - \hat{Y})'(Y^{out} - \hat{Y}) | X] &= \sigma^2 + \frac{k}{n}\sigma^2 + \frac{1}{n}\mu'(I - P)\mu.
 \end{aligned}$$

Note that when $k > n$ (k is the number of variables), $X'X$ is not invertible. Consequently, it is not a case.

Now, comparing in-sample and out-sample prediction errors yields

$$\mathbb{E}[(Y^{out} - \hat{Y})'(Y^{out} - \hat{Y}) | X] - \mathbb{E}[(Y^{in} - \hat{Y})'(Y^{in} - \hat{Y}) | X] = 2k\sigma^2 \text{ (treated as a penalty)}.$$

There are many choices of the variable sets. Conventionally, people use the biggest approximating linear model to estimate σ^2 .

Remark.

- The in-sample prediction error always suggests to use more complex models.
- However, the out-sample prediction error exhibits a trade-off between **bias** and **variance**. It penalizes too much variables.

□

Remark. For any matrix A ,

$$\begin{aligned}
 \mathbb{E}[E'AE] &= \text{trace}(\mathbb{E}[E'AE]) \\
 &= \mathbb{E}[\text{trace}(E'AE)] \\
 &= \mathbb{E}[\text{trace}(AEE')] \\
 &= \text{trace}(A \mathbb{E}[EE']) \\
 &= \text{trace}(A\sigma^2 I_n) \\
 &= \sigma^2 \text{trace}(A).
 \end{aligned}$$

□

Model Selection Theory

Mallows CP

Mallows CP calculates

$$\mathbb{E}[(\mu - \hat{\mu})'(\mu - \hat{\mu}) | X] = k\sigma^2 + \mu'(I - P)\mu,$$

where $\hat{\mu} = \hat{Y} = \hat{X}\hat{\beta}$. The result is similar to the out-sample prediction error.

Nonlinear Least Square (NLS)

Given *i.i.d.* data

$$\begin{array}{ll}
 y_1, & y_2, \dots, y_n \text{ dependent variables} \\
 1 \times 1 & \\
 x_1, & x_2, \dots, x_n \text{ independent variables,} \\
 k \times 1 &
 \end{array}$$

and the model $y_i = f(x_i; \beta) + e_i$. The objective function is

$$Q_n(\beta) \equiv \frac{1}{n} \sum_{i=1}^n (y_i - f(x_i; \beta))^2 \xrightarrow{p} Q_\infty(\beta) \equiv \mathbb{E}[y_i - f(x_i; \beta)]^2.$$

Here, econometrisians impose some restrictions:

Identification assumption: $\beta_\infty \equiv \arg \min_{\beta} Q_\infty(\beta)$ uniquely exists.

Probability convergence assumption: $Q_n(\beta) \xrightarrow{p} Q_\infty(\beta)$ uniformly.

Then, we have

$$\hat{\beta} \equiv \arg \min_{\beta} Q_n(\beta) \xrightarrow{p} \beta_\infty \equiv \arg \min_{\beta} Q_\infty(\beta),$$

i.e., a consistent estimator.

Statistical properties

FOC results in the estimated parameter (here $\hat{\beta}$). Clearly,

$$0 = \frac{\partial Q_n(\hat{\beta})}{\partial \beta} = \frac{-2}{n} \sum_{i=1}^n (y_i - f(x_i; \hat{\beta})) \frac{\partial f(x_i; \hat{\beta})}{\partial \beta}.$$

Therefore, we need to use numerical methods to solve the nonlinear problems.

Mean value theorem

However, we can still estimate the nonlinear function by using the **mean value theorem**.

$$0 = \frac{\partial Q_n(\hat{\beta})}{\partial \beta} = \frac{\partial Q_n(\beta)}{\partial \beta} + \frac{\partial^2 Q_n(\beta_m)}{\partial \beta \partial \beta'} (\hat{\beta} - \beta),$$

where $\beta_m \in [\hat{\beta}, \beta]$. Since $\hat{\beta} \xrightarrow{p} \beta$, therefore, it gives $\beta_m \xrightarrow{p} \beta$. Moreover, rewriting the NLS problem as an asymptotic form gives

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{\partial^2 Q_n(\beta_m)}{\partial \beta \partial \beta'} \right)^{-1} \left(-\sqrt{n} \frac{\partial Q_n(\beta)}{\partial \beta} \right).$$

If

$$-\sqrt{n} \frac{\partial Q_n(\beta)}{\partial \beta} \xrightarrow{d} \mathcal{N} \left(0, \text{plim}_{n \rightarrow \infty} \left(n \frac{\partial Q_n(\beta)}{\partial \beta} \frac{\partial Q_n(\beta)'}{\partial \beta} \right) \right),$$

then the distribution asymptotes to

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} \mathcal{N} \left(0, \left(\text{plim}_{n \rightarrow \infty} \frac{\partial^2 Q_n(\beta)}{\partial \beta \partial \beta'} \right)^{-1} \left(\text{plim}_{n \rightarrow \infty} n \frac{\partial Q_n(\beta)}{\partial \beta} \frac{\partial Q_n(\beta)'}{\partial \beta} \right) \left(\text{plim}_{n \rightarrow \infty} \frac{\partial^2 Q_n(\beta)}{\partial \beta \partial \beta'} \right)^{-1} \right).$$

In addition,

$$\frac{\partial Q_n(\beta)}{\partial \beta} = \frac{1}{n} \sum_{i=1}^n \frac{\partial f_i(\beta)}{\partial \beta} e_i,$$

which leads to

$$\text{plim}_{n \rightarrow \infty} \left(n \frac{\partial Q_n(\beta)}{\partial \beta} \frac{\partial Q_n(\beta)'}{\partial \beta} \right) = \mathbb{E} \left[\frac{\partial f_i(\beta)}{\partial \beta} \frac{\partial f_i(\beta)'}{\partial \beta} \right] \quad \text{and} \quad \text{plim}_{n \rightarrow \infty} \frac{\partial^2 Q_n(\beta)}{\partial \beta \partial \beta'} = \mathbb{E} \left[\frac{\partial f_i(\beta)}{\partial \beta} \frac{\partial f_i(\beta)'}{\partial \beta} \right].$$

As a result, the asymptotic covariance of the NLS is

$$\sigma^2 \left(\mathbb{E} \left[\frac{\partial f_i(\beta)}{\partial \beta} \frac{\partial f_i(\beta)'}{\partial \beta} \right] \right)^{-1}.$$