Problem Set 4

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Answer 1.
Since a continuous preference relation has a continuous representation, say u. Notice that $[0,1]$ is compact, so it attains its maximum. Let the maximum be x^*.
Suppose that $a < b \leq x^*$, we have $x^* \succsim a$ by assumption on u and so $b \succ a$ by strong convexity. The other side is similar.

Answer 2.
Denote $\succsim$ be the preference on the space. Let $S = \{s > 0 | (0, s) \succsim (1, 0)\}$. We discuss the problem by the behavior of S.

1. $S = \emptyset$:
We have that $\forall s \in \mathbb{R}^+$, $(0, s) \prec (1, 0)$. Let $\succsim_1$ be the preference with respect to the first commodity and $\succsim_2$ be the preference with respect to the second commodity. We claim that the preference for this case is the lexicographic preference $\succsim'$ induced by $\succsim_1$ and $\succsim_2$.
Let $(x_1, y_1), (x_2, y_2)$ be two different bundle. W.L.O.G, we suppose that $(x_1, y_1) \succ' (x_2, y_2)$. It suffices to prove that $(x_1, y_1) \succ (x_2, y_2)$
(1) $x_1 > x_2$:
By quasi-linearity on the first commodity, $(x_1, y_1) \succ (x_2, y_2)$ iff $(x_1 - x_2, y_1) \succ (0, y_2)$. Next, by homotheticity, $(x_1 - x_2, y_1) \succ (0, y_2)$ iff $(1, \dfrac{y_1}{x_1 - x_2}) \succ (0, \dfrac{y_2}{x_1 - x_2})$. Moreover, $(1, \dfrac{y_1}{x_1 - x_2}) \succ (1, 0) \succ (0, \dfrac{y_2}{x_1 - x_2})$ is known by strict monotonicity and assumption that $S = \emptyset$.
(2) $x_1 = x_2, y_1 > y_2$ ::
By quasi-linearity on the first and the second commodity, $(x_1, y_1) \succ (x_2, y_2)$ iff $(0, y_1 - y_2) \succ (0, 0)$, which is straightforward by strict monotonicity.
2. $S = (0, \infty)$:
We have that $\forall s > 0$, $(0, s) \succ (0, \dfrac{s}{2}) \succsim (1, 0)$. Hence $\forall s > 0, (\dfrac{1}{s}, 0) \prec (0, 1)$.
Thus, $\forall t \in \mathbb{R}^+, (0, 1) \succ (t, 0)$. With a similar argument in $S = \emptyset$, we know that $\succsim = \succsim'$, where $\succsim'$ is the lexicographic preference induced by $\succsim_2$ and $\succsim_1$.

Notice that by strict monotonicity, if $S \neq \emptyset, (0,\infty)$, then $S = (m,\infty)$ or $[m,\infty)$ for some $m > 0$.

3. $S = (m, \infty)$:
We know that $(0,m) \prec (1,0)$ and for every $\epsilon > 0$, $(0, m+\epsilon) \succ (0, m + \frac{\epsilon}{2}) \succsim (1,0)$.
Rewrite tuple (x,y) as triple $(x, y, x + \frac{y}{m})$.
Let $\succsim_3$ be the preference represented by the utility function $u(x,y) = x + \frac{y}{m}$. We claim that $\succsim$ is the lexicographic preference induced by $\succsim_3$ and $\succsim_1$. Notice that the lexicographic preference satisfies strict monotonicity, quasi-linearity and homothecity.
By quasi-linearity on two commodities, it suffices to check the relation between $(a,b), (0,0)$,where $(a,b) \neq (0,0)$ and $(a,0), (0,b)$, where $a, b \neq 0$. The first case is trivial.
For the second case, by homothecity, it suffices to check $(1,0)$ and $(0,t)$. One is easy to check the lexicographic preference meets $\succsim$.

4. $S = [m,\infty)$ and $(0,m) \succ (1,0)$:
We have $(0,n) \prec (1,0)$ if $n < m$. Hence we have $(0,1) \succ (\frac{1}{m}, 0)$ and $(0.1) \prec (t, 0)$ if $t > \frac{1}{m}$. Notice that it just interchange the first and the second entry on the case $S = (m, \infty)$
Therefore, the preference is the lexicographic preference induced by $\succsim_3$ and $\succsim_1$.

5. $S = [m,\infty)$ and $(0,m) \sim (1,0)$:
We claim that $\succsim_3$ is the preference $\succsim$. To see this, notice that $\succsim_3$ is homothetic, strictly monotone and quasi-linear on two commodities.
With the same trick in 3., it suffices for us to check $(0,t)$ and $(1,0)$. Done!

Answer 3 One needs more benefit to give up same quantity of some commodity when one has less such commodity.
($1 \implies 4$) Let $(x_1, x_2) \sim (x_1 - \epsilon, x_2 + \delta_1) \sim (x_1 - 2\epsilon, x_2 + \delta_1 + \delta_2)$.
We have, by 1, $(x_1 - \epsilon, x_2 + \frac{\delta_1 + \delta_2}{2}) \succsim (x_1 - 2\epsilon, x_2 + \delta_1 + \delta_2)$.
By strong monotonicity, $\frac{\delta_1 + \delta_2}{2} \geq \delta_1 \Rightarrow \delta_2 \geq \delta_1$.
($4 \implies 1$) Suppose 4, by continuity, it suffices for us to prove that $x \succsim y \Rightarrow \lambda x + (1-\lambda) y \succsim y$ for all $\lambda \in \mathbb{Q}$.
Let $\lambda = \frac{p}{q}$ and $(x_1, x_2) \succsim (y_1, y_2)$ be different bundles. We want to show that $\lambda(x_1, x_2) + (1 - \lambda)(y_1, y_2) \succsim (y_1, y_2)$
(1) $x_1 \geq y_1, x_2, \geq y_2$: Staightforward by monotonicity.
(2) $x_1 > y_1, x_2 < y_2$: For simplicity, write $x = (x_1, x_2)$ and y in similar manner. For $\theta \in [0,1]$, there exists $\theta x \sim y$. For each $n = 1, \cdots, q-1$, there exists α_n such that $(\frac{q-n}{q}\theta x_1 + \frac{n}{q} y_1, \alpha_n) \sim \theta x \sim y$

since $(\frac{q-n}{q}\theta x_1 + \frac{n}{q}y_1, 0) \precsim y$ and $(\frac{q-n}{q}\theta x_1 + \frac{n}{q}y_1, y_2) \succsim x$. Notice that

$$\alpha_1 - \theta x_2 \le \alpha_2 - \alpha_1 \le \alpha_3 - \alpha_2 \le \cdots \le y_2 - \alpha_{q-1} \Rightarrow \alpha_n \le \frac{q-n}{q}(\theta x_2) + \frac{n}{q}y_2$$

Therefore, $\lambda x + (1-\lambda)y \succsim \lambda\theta x + (1-\lambda)y \succsim (\frac{q-n}{q}\theta x_1 + \frac{n}{q}y_1, \alpha_n) \sim y$.
(3) $x_1 < y_1, x_2 > y_2$: Similar to (2).

Answer 4
Notice that $(0,\cdots,0,x_i,0,\cdots,0) \sim (\alpha_i x_i, 0,\cdots,0)$ by reducing the problem to $K=2$ with commodity 1 and commodity k. We have

$$(0,x_2,0,\cdots,0) \sim (\alpha_2 x_2, 0,\cdots,0) \Rightarrow (x_1.x_2,\cdots,x_K) \sim (x_1+\alpha_2 x_2, 0, x_3,\cdots,x_k)$$

for some $\alpha_i > 0$ by quasi-linearity on all commodities. Similarly,

$$\begin{aligned}(x_1.x_2,\cdots,x_K) &\sim (x_1+\alpha_2 x_2, 0, x_3,\cdots,x_k)\\ &\sim (x_1+\alpha_2 x_2+\alpha_3 x_3, 0, 0, x_4,\cdots,x_K) \sim \cdots\\ &\sim (x_1+\alpha_2 x_2+\alpha_3 x_3+\cdots+\alpha_K x_k, 0,\cdots,0)\end{aligned}$$

By strong monotonicity, the preference can be represented by $\sum_{k=1}^{K}\alpha_k x_k$, where $\alpha_1 = 1$.

Answer 5
Let $T = \{t_2+\cdots+t_K | (0,t_2,\cdots,t_K) \succ (s,0,\cdots,0),\ \forall s>0\}$. Suppose that $T \neq \emptyset$, let $m = \inf T$. Depending on whether $m \in T$, we discuss it in two parts.
1. $m \in T$:
Let $m = \sum_{k=2}^{K} m_k$ and $(0,m_2,\cdots,m_K) \succ (s,0,\cdots,0) \forall s > 0$. Notice that $(1,0,0,\cdots,0) \prec (0,m_2,\cdots,m_K) \prec (1,m_2,\cdots,m_K)$. So there exists some $\lambda \in (0,1)$ such that $(1,\lambda m_2,\cdots,\lambda m_K) \sim (0,m_2,\cdots,m_K)$ by continuity.
Since $\sum_{k=2}^{K}\lambda m_k = \lambda m < m = \inf T$, there exists some $s^* > 0$ such that $(s^*,0,\cdots,0) \succsim (0,\lambda m_2,\cdots,\lambda m_K)$.
Hence $(s^*+1,0,\cdots,0) \succ (0,\lambda m_2,\cdots,\lambda m_K)$ by strict monotonicity in commodity 1 and

$$(s^*+2,0,\cdots,0) \succ (1,\lambda m_2,\cdots,\lambda m_K) \sim (0,m_2,\cdots,m_K)$$

by quasi-linearity in commodity 1. It leads to a contradiction.
2. $m \notin T$:
For all $n \in \mathbb{N}$, we have $y_n = (0,t_{n2},\cdots,t_{nK})$ with $\sum_{k=2}^{K} t_{nk} < m + \frac{1}{n}$ and $y_n \succ (s,0,\cdots,0)$ for all

$s > 0$.

Notice that $y_n \in [0, m+1]^K$ for all n. Hence, there exists a converge subsequence. W.L.O.G, let y_n converges to L. Let $L = (L_1, L_2, \cdots, L_K)$. Easy to see that $L_1 = 0$ and $L_2 + L_3 + \cdots + L_K = m$. We have some $(s^*, 0, \cdots, 0) \succsim L$ and so $(s^* + 1, 0, \cdots, 0) \succ L$. By continuity, there exists an neighborhood B of L such that $y \in B \Rightarrow y \prec (s^* + 1, 0, \cdots, 0)$, which contradicts to $y_n \to L$.

Conbining the two argument, we know that $T = \emptyset$, and there exists some $v((0, x_2, \cdots, x_K))$ for $x_2, \cdots, x_K$ such that

$$(0, x_2, \cdots, x_K) \sim (v((0, x_2, \cdots, x_K)), 0, \cdots, 0)$$

by continuity. Moreover, let $v((x_1, x_2, \cdots, x_K)) = x_1 + v((0, x_2, \cdots, x_K))$, then the statement is proven by quasilinearity in commodity 1.

Answer 6

a.

$$\begin{aligned}
(a_J, c_{-J}) \succ (b_J, c_{-J}) &\iff \sum_{i \in J} v_i(a_i) + \sum_{k \notin J} v_k(c_k) \geq \sum_{i \in J} v_i(b_i) + \sum_{k \notin J} v_k(c_k) \\
&\iff \sum_{i \in J} v_i(a_i) \geq \sum_{i \in J} v_i(b_i) \\
&\iff \sum_{i \in J} v_i(a_i) + \sum_{k \notin J} v_k(d_k) \geq \sum_{i \in J} v_i(b_i) + \sum_{k \notin J} v_k(d_k) \\
&\iff (a_J, d_{-J}) \succ (b_J, d_{-J})
\end{aligned}$$

b. We have $v_1(a) + v_2(b) \geq v_1(c) + v_2(d)$ and $v_1(c) + v_2(e) \geq v_1(f) + v_2(b)$.
Adding these two, we have

$$v_1(a) + v_2(b) + v_1(c) + v_2(e) \geq v_1(c) + v_2(d) + v_1(f) + v_2(b)$$

Hence, $v_1(a) + v_2(e) \geq v_1(f) + v_2(d)$ and thus, $(a, e) \succsim (f, d)$.
c. Consider the preference relation represented by $u(x, y) = x + xy^2 + y$. It satisfies condition S clearly since it only has two variables. However, it is not separated. This can be see by $(5, 2) \succsim (0, 27), (0, 7) \succsim (1, 2)$ but $(5, 7) \prec (1, 27)$, contradicting Hexagon-condition.

Answer 7

a. It is not differentiable since the improvement direction is not open, but it should be if it is differentiable.

b. The indifference curves are straight lines.

To see this, pick any bundle x and pick y which $y - x$ is perpendicular to $v(x)$. Notice that $y \precsim x$ since $y \notin D_{\succsim}(x)$. If $\delta \dot{v}(x) > 0$, then $((y-x)+\delta)\dot{v}(x) = \delta\dot{v}(x) > 0 \Rightarrow y+\delta \succsim x \succsim y \Rightarrow v(y) = v(x)$. Hence, by symmetry $y \succsim x$, which proves that the indifference curve is a straight line and moreover, perpendicular to $v(x)$ for x on the curve.

c. $\subseteq$: If not, say there exists some $d \in D(x)$ such that $x - d \succsim x$. Notice that, by convexity, $x - \lambda d \succsim x$ for $\lambda \in (0,1)$. Since $-d\dot{v}(x) < 0$, there exists δ with $\delta_k > (-d)_k$ on every entry k but $\delta\dot{v}(x) < 0$. We have $x + \lambda\delta \succ x - \lambda d \succsim x$, so δ is an improvement direction, which contradicts to differentiability.

For the rest of the problem, consider the preference represented by $u(x,y) = xy$, the direction that perpendicular to v is in E but not in $-D$.

d. It is not continuous since $(\frac{1}{n}, \frac{3}{2}) \sim (\frac{1}{2} + \frac{1}{n}, \frac{1}{2})$ but $(0, \frac{3}{2}) \succ (\frac{1}{2}, \frac{1}{2})$.

However, it is differentiable. It suffices to check the case that $x_1 + x_2 = 1$ and not hard to see that $\delta_1 + \delta_2 > 0$ iff $x + \delta \succ x$. Hence $v(x_1, x_2) = \frac{1}{\sqrt{2}}(1,1)$, done!