Midterm Exam
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- You may use theorems and lemmas used and derived in the lecture notes and videos.
- Unless replaced by a different definition, the definitions from the lecture notes apply.
- Do not write your *name* or *student ID* on the exam or we will *null* your result. Instead, put your solutions into the envelope that has been provided to you.
- Per room, only one student can leave to go to the bathroom at any given time. Put your exam and notes into the envelope before leaving the room.

Problem 1 (easy)

In the lecture, we have seen two definitions of preference according to the questionnaires Q and R.

Definition 1. Preferences on a set $\mathcal{X}$ are a function f that assigns to any pair (x, y) of distinct elements of $\mathcal{X}$ one of the three "values" $x \succ y$, $y \succ x$, or I so that for any three different elements x, y, and z in $\mathcal{X}$, the following two conditions hold:

- No order effect: $f(x, y) = f(y, x)$.
- Transitivity: if $f(x, y) = x \succ y$ and $f(y, z) = y \succ z$, then $f(x, z) = x \succ z$, and if $f(x, y) = I$ and $f(y, z) = I$, then $f(x, z) = I$.

Definition 2. Preferences on a set $\mathcal{X}$ are a binary relation $\succsim$ on $\mathcal{X}$ satisfying:

- Reflexivity: For any $x \in \mathcal{X} : x \succsim x$.

- Completeness: For any distinct $x, y \in \mathcal{X}$, $x \succsim y$ or $y \succsim x$.
- Transitivity: For any $x, y, z \in \mathcal{X}$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

A researcher came up with the idea of a new questionnaire L that would drastically reduce the number of questions a subject needs to answer. In fact, the questionnaire consists of a single task of the following form:

$\mathbf{L}(\mathcal{X})$: List the alternatives in $\mathcal{X}$ in descending order of how much you like them. Write only one alternative in each row.

1.
2.
3. ...

Questions

1. Define preferences based on the questionnaire L in a sensible manner. (10 Points)
2. Show that this definition is not equivalent to the definitions according to the questionnaires Q and R. (15 Points)

Problem 2 (standard)

A researcher tries to elicit expected utility preferences by offering budget allocations over lotteries to an individual.

Let $L(\mathbb{R}_+)$ be the set of finite support lotteries on the nonnegative real numbers. Let $\mathcal{Y}$ be a finite subset $\{l_1, \dots, l_n\}$ of $L(\mathbb{R}_+)$ that contains at least two distinct lotteries.

An allocation is a function $a : \mathcal{Y} \to \mathbb{R}_{\geq 0}$. A budget set with prices p and budget w is the set $\mathcal{B}(p, w) = \{a \in \mathcal{A} : \sum_{l \in y} a(l)p(l) \leq w\}$. The decision maker has preferences over all allocations $\mathcal{A}$. Denote by $\pi = \prod_{i=1}^n \mathit{support}(l_i)$ the possible combinations of payoffs of lotteries. Assume that the lotteries are independent, i.e. the payoffs $(x_1, \dots, x_n) \in \pi$ have a probability $\prod_{i=1}^n l_i(x_i)$.

Definition 3. $\succsim$ has an expected utility representation if it can be represented by $U(a) = \sum_{(x_i)_{i=1}^n \in \pi} \left(\prod_{i=1}^n l_i(x_i)\right) u\left(\sum_{i=1}^n x_i a(l_i)\right)$ where $u : \mathbb{R}_+ \to \mathbb{R}$ is the Bernoulli utility over payoffs.

Questions

1. Provide an example of a monotone and continuous preference on allocations that does not have an expected utility representation. (standard, 5 Points)
2. Suppose we can only observe a demand correspondence $d_{\succsim}(\mathcal{B}(p, w)) = \{a \in \mathcal{B}(p, w) : a \succsim b, \forall b \in \mathcal{B}(p, w)\}$ induced by the preference $\succsim$ on budget sets. Can we identify the preference relation $\succsim$ of the decision maker? How do we find out if $a \succsim b$? (standard, 5 Points)

3. Suppose $\succsim$ has expected utility representations U_1 and U_2. Is U_1 an affine transformation of U_2? (7 Points)

4. Suppose the lotteries are not necessarily independent, i.e., the probability of payoffs $p(x_1, \ldots, x_n)$ is arbitrary. If $\succsim$ has expected utility representations U_1 and U_2 with Bernoulli utilities u_1 and u_2, respectively, is u_1 an affine transformation of u_2? (8 Points)

Problem 3 (moderately difficult)

It has been an important conundrum in research why decision makers in Taiwan at the same time love to purchase insurances but gamble on Chinese New Year. One possible explanation is that decision makers exhibit "skewness" in their preferences in that they highly value large gains that occur with small probability.

Let $L(Z)$ be the set of lotteries on a finite set Z.

Definition 4. A preference $\succsim_1$ is more gain-skewed than $\succsim_2$ if for all $p, q, r \in L(Z)$ such that $p \succsim_1 q \succsim_1 r$ and all $0 < \alpha \leq \beta \leq 1/2$ we have that if $\alpha p \oplus (1-\alpha) r \succsim_2 \beta q \oplus (1-\beta) r$, then $\alpha p \oplus (1-\alpha) r \succsim_1 \beta q \oplus (1-\beta) r$.

Definition 5. A preference $\succsim_1$ is more risk averse than $\succsim_2$ if for any lottery p and degenerate lottery $[x]$, $p \succsim_1 [x]$ implies $p \succsim_2 [x]$.

Questions

1. Show that if $\succsim_1$ and $\succsim_2$ satisfy the vNM axioms, then $\succsim_2$ is more risk averse than $\succsim_1$ if $\succsim_1$ is more gain-skewed than $\succsim_2$. (10 points)

2. Show that if $\succsim_1$ and $\succsim_2$ satisfy the vNM axioms, then $\succsim_1$ is more gain-skewed than $\succsim_2$ if $\succsim_2$ is more risk averse than $\succsim_1$. (10 points)

3. How would you weaken the definition to allow for an expected utility maximizing decision maker to be more gain skewed than a risk neutral decision maker and purchase both lottery tickets and insurances? (5 Points)

Problem 4 (difficult)

Let $\mathcal{X} = \mathbb{R}_+^K$ be a set of consumption bundles.

For any two relations $\succsim_1$ and $\succsim_2$ on $\mathcal{X}$, denote by $\succsim_3 = \succsim_1 \cap \succsim_2$ the relation such that for all $x, y \in \mathcal{X}$, $x \succsim_3 y$ if and only if $x \succsim_1 y$ and $x \succsim_2 y$.

For any two relations $\succsim_1$ and $\succsim_2$ on $\mathcal{X}$, denote by $\succsim_1 \subseteq \succsim_2$ that for all $x, y \in \mathcal{X}$, if $x \succsim_1 y$, then $x \succsim_2 y$.

Definition 6. A preference relation $\succsim$ is between preference relations $\succsim'$ and $\succsim''$ if $\succsim' \cap \succsim'' \subseteq \succsim$.

Questions

1. Suppose preference relation $\succsim$ is between monotone preferences $\succsim'$ and $\succsim''$. Show that $\succsim$ need not be monotone. (standard, 5 points)

2. Change the definition of "between" to "truly between" to ensure that any preference relation $\succsim$ is monotone if it is truly between two monotone preferences. (10 points)

3. Prove or disprove that a preference relation $\succsim$ is continuous if it is truly between monotone continuous preferences $\succsim'$ and $\succsim''$. (10 points)