# 3. Bayesian Games

ECON 7011 – Micro I

Benjamin Bernard

# Selling Farmland

A Rice Farmer considers selling his/her land to a Tea Farmer.

**Annual average yield:**

- Tea: 5.35m NTD/km$^2$.
- Rice: 4.2m NTD/km$^2$.

**Incomplete information:**

- Suppose high/low-quality soil yields 50% above/below average.
- The Rice Farmer knows the quality, but the Tea Farmer does not.
- Can trade between the two farmers occur? If so, at which price?

Source: https://www.visualcapitalist.com/the-worlds-most-valuable-cash-crop/

# Modeling This Interaction

**Uncertainty:**

- An Aumann model of incomplete information describes the players' information/knowledge of the payoff-relevant state of nature $\theta$.
- Each player $i$'s information is described by a partition $\mathcal{T}_i$ of $\Omega$ into information sets $\tau_i$, also known as player $i$'s type.

**Decisions and utilities:**

- A decision by player $i$ is a map $\sigma_i : \mathcal{T}_i \to \mathcal{A}_i$.
- Each player $i$ earns a utility $u_i : \Theta \times \mathcal{A} \to \mathbb{R}$.

**Equilibrium:**

- If player $i$ of type $\tau_i$ does not know $\theta$, he/she forms beliefs about $\theta$.
- All players simultaneously aims to maximize $\mathbb{E}_{\tau_i}[u_i(\theta, \sigma(T))]$.

# Bayesian Nash Equilibrium

# Bayesian Game

## Definition 3.1

A Bayesian game $\mathcal{G} = \big(\mathcal{I}, \Theta, (\mathcal{T}_i), (P_i), (\mathcal{A}_i), (u_i)\big)$ consists of:

1. A finite set of players $\mathcal{I} = \{1, \ldots, n\}$,
2. A set of states of nature $\Theta$.
3. A type space $\mathcal{T}_i$ for each player $i$, where each type $\tau_i \in \mathcal{T}_i$ determines:
   (i) the set $\mathcal{A}_i(\tau_i)$ of pure actions available to player $i$,
   (ii) player $i$'s beliefs over the true state of nature $\theta$ and the type profile $\tau_{-i}$ of the other players via a posterior probability measure $P_{\tau_i}$ on $\Theta \times \mathcal{T}_{-i}$.
4. A payoff function $u_i : \Theta \times \mathcal{A} \to \mathbb{R}$ for each player $i \in \mathcal{I}$, where

$$\mathcal{A}_i := \bigcup_{\tau_i \in \mathcal{T}_i} \mathcal{A}_i(\tau_i), \qquad \text{and} \qquad \mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n.$$

# Start-up: an Improved e-Commerce Platform


## Incomplete information:

- Is PChome already innovating, which will make a start-up futile?
- More generally: what is PChome's cost to innovate?

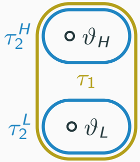
# Start-Up Problem: Model

| | $I$ | $N$ |
|---|---|---|
| $E$ | $-1, 0$ | $1, 2$ |
| $N$ | $0, 2$ | $0, 3$ |

$\vartheta_H$

| | $I$ | $N$ |
|---|---|---|
| $E$ | $-1, 1.5$ | $1, 2$ |
| $N$ | $0, 3.5$ | $0, 3$ |

$\vartheta_L$




**Types and prior beliefs:**

- PChome's innovation cost $\theta$ takes values in $\Theta = \{\vartheta_H, \vartheta_L\}$.
- PChome knows its innovation cost, hence PChome has two types.
- We do not know $\theta$ and we have prior beliefs $\mu = P_1(\theta = \vartheta_H)$.

**Actions and payoffs:**

- We decide whether we (E)nter the market or (N)ot.
- PChome decide whether they (I)nnovate or (N)ot.
- Entering the market is profitable only if PChome does not innovate.

# Strategies in a Bayesian Game

**Definition 3.2**

1. A pure strategy of player $i$ is a map $s_i : \mathcal{T}_i \to \mathcal{A}_i$ with $s_i(\tau_i) \in \mathcal{A}_i(\tau_i)$.
2. A mixed strategy $\sigma_i$ of player $i$ is a distribution over pure strategies.
3. A behavior strategy is a map $\sigma_i : \mathcal{T}_i \to \Delta(\mathcal{A}_i)$ with $\sigma_i(\tau_i) \in \Delta(\mathcal{A}_i(\tau_i))$.

**Interpretation:**

- Before learning $\tau_i$, player $i$ decides what he/she will do in any situation.
- As before, it maps player $i$'s information to his/her available actions.

**Mixed vs. behavior strategy:**

- In a mixed strategy, a player randomizes before learning his/her type.
- In a behavior strategy, a player randomizes after learning his/her type.

# Outcome and Expected Payoffs

**Outcome:**

- The outcome of the game is the tuple $(\theta, T, A)$.
- The realized action profile $A = (A_1, \ldots, A_n)$ satisfies for each player $i$:
  - Conditional on $\mathcal{T}_i = \tau_i$, $A_i$ is independent from $\theta$, $T_j$, and $A_j$ for $j \neq i$.
  - Conditional on $\mathcal{T}_i = \tau_i$, the distribution of $A_i$ is $\sigma_i(\tau_i) \in \Delta(\mathcal{A}_i)$.
- We denote by $\sigma_i(\tau_i; a_i)$ the probability that $\sigma_i(\tau_i)$ assigns to $a_i \in \mathcal{A}_i$.

**Induced probability measure:**

- Player $i$ of type $\tau_i$'s subjective distribution over $(\theta, A)$ is

$$P_{\tau_i,\sigma}(\theta = \vartheta, A = a) := \sum_{\tau_{-i} \in \mathcal{T}_{-i}} P_{\tau_i,\sigma}(\theta = \vartheta, A = a \,|\, T = \tau) P_{\tau_i}(T = \tau)$$

$$= \sum_{\tau_{-i} \in \mathcal{T}_{-i}} \sigma_i(\tau_i; a_i) P_{\tau_i}(\theta = \vartheta, T = \tau).$$

# Bayesian Nash Equilibrium

## Definition 3.3

A Bayesian Nash equilibrium (BNE) is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for every player $i$, every type $\tau_i$, and every action $a_i \in \mathcal{A}_i(\tau_i)$,

$$\mathbb{E}_{\tau_i,\sigma}[u_i(\theta, A)] \geq \mathbb{E}_{\tau_i,(a_i,\sigma_{-i})}[u_i(\theta, A)].$$

**Interpretation:**

- Each player best replies to his/her opponents' strategy profile, given their beliefs implied by information $\tau_i$ and Bayesian updating.
- We say that $\sigma_i(\tau_i)$ maximizes player $i$'s interim expected payoff.
- Similarly to the Nash equilibrium, this imposes that every player correctly predicts their opponents' strategies.
- No player has an incentive to deviate *after* learning their type.

# Start-Up Problem: Model

| | $I$ | $N$ |
|---|---|---|
| $E$ | $-1, 0$ | $1, 2$ |
| $N$ | $0, 2$ | $0, 3$ |

$\vartheta_H$

| | $I$ | $N$ |
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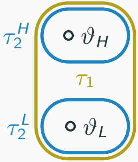


## Strategies:

- A behavior strategy is a map $\sigma_i : \mathcal{T}_i \to \Delta(\mathcal{A}_i)$.
- Our strategy is simply the choice of an action in $\Delta(\{E, N\})$.
- PChome's strategy is the choice of an action for each type.

## Finding a BNE:

- Observe that not innovating is dominant for type $\tau_2^H$.
- We parametrize a strategy profile $\sigma = (\sigma_1, \sigma_2)$ by $\sigma_1(\tau_1; E) = x$ and $\sigma_2(\tau_2^L; I) = y$ and find the best-response correspondences.

# Finding Bayesian Nash Equilibria

**Finding BNE with finitely many types:**

- Parametrize the strategy profile $\sigma : \mathcal{T} \to \mathcal{A}$.
- For each type $\tau_i$ of each player $i$:
  - Compute the subjective expected utility $\mathbb{E}_{\tau_i,\sigma}[u_i(\theta, A)]$.
  - Find $\tau_i$'s best response by maximizing his/her expected utility with respect to his/her decision variables.
- Verify consistency by going through all possible cases.

**Comparison to finding Nash equilbiria:**

- The approach is exactly the same with the two exceptions:
  - We have to find best-response correspondences for each type.
  - Expectation is taken over mixing, state of nature, and types.

# Existence

**Separate-players interpretation:**

- If $\mathcal{T}_i$ is finite, we can interpret each type $\tau_i$ as a separate player.
- Player $\tau_i$'s available pure actions are $\mathcal{A}_{\tau_i} := \mathcal{A}_i(\tau_i)$.
- Player $\tau_i$'s payoff in action profile $a = (a_{\tau_1^1}, a_{\tau_1^2}, a_{\tau_1^3}, \ldots, a_{\tau_n^{m_n}})$ is

$$u_{\tau_i}(a) := \sum_{\tau_{-i} \in \mathcal{T}_{-i}} u_i(\vartheta, a_{\tau_1}, \ldots, a_{\tau_n}) P_{\tau_i}(\theta = \vartheta, T_{-i} = \tau_{-i}).$$

- The Bayesian game is thus equivalent to the static game among players $\bigcup_{i \in \mathcal{I}} \mathcal{T}_i$ with available pure actions $\mathcal{A}_{\tau_i}$ and payoff functions $u_{\tau_i}$.

**Existence:**

- A BNE of a Bayesian game is a NE of the equivalent static game.
- If $\mathcal{T}_i$ and $\mathcal{A}_i$ is finite for each player $i$, existence of a BNE in behavior strategies thus follows from existence of mixed NE in static games.

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# Modeling the Interaction as a Game

**States of Nature:**

- Quality of soil $\theta$ can be low, medium, or high, that is, $\Theta = \{\vartheta_L, \vartheta_M, \vartheta_H\}$.
- Suppose the common prior $P$ assigns equal probability to all three.

**Types of players:**

- The Rice Farmer knows $\theta$: he has 3 information sets and hence 3 types.
- The Tea Farmer has a single type, hence his beliefs are equal to $P$.

**Actions and strategies:**

- The Tea Farmer can offer a price $p \geq 0$.
- The Rice Farmer of any type chooses set of prices $\mathcal{P}$ he/she accepts.
- A pure strategy of the Rice Farmer is thus a map

$$s_R : \left\{\tau_2^L, \tau_2^M, \tau_2^H\right\} \to \{\text{subsets of } \mathbb{R}_+\}.$$

# Utilities

**Value of the land:**

- The annual yield of tea and rice per km$^2$ is

$$y_T(\vartheta_L) = 2.7, \qquad y_T(\vartheta_M) = 5.35, \qquad y_T(\vartheta_H) = 8,$$
$$y_R(\vartheta_L) = 2.1, \qquad y_R(\vartheta_M) = 4.2, \qquad y_R(\vartheta_H) = 6.3.$$

- Suppose the value of the land is the yield in perpetuity, i.e., if players discount future payoffs with discount factor $\delta = \frac{1}{2}$, the value is

$$v_i(\vartheta) = \sum_{t=0}^{\infty} \delta^t y_i(\vartheta) = \frac{y_i(\vartheta)}{1-\delta} = 2y_i(\vartheta).$$

**Utilities:**

- Tea Farmer's utility is $u_T(\vartheta, p, \mathcal{P}) = (v_T(\vartheta) - p)1_{\{p \in \mathcal{P}\}}$,
- Rice Farmer's utility is $u_T(\vartheta, p, \mathcal{P}) = p \cdot 1_{\{p \in \mathcal{P}\}} + v_R(\vartheta)1_{\{p \notin \mathcal{P}\}}$.

# Dominated Strategies

**To sell or not to sell:**

- It cannot be optimal to sell at a price $p < v_R(\vartheta)$.
- Formally, $\mathcal{P}(\vartheta) \cap [0, v_R(\vartheta)) = \emptyset$ in a Bayesian Nash equilibrium.

**No trades with the high type:**

- For any price $p < 12.6$, type $\vartheta_H$ is not willing to sell.
- The expected value of the land to the Tea Farmer is

$$\mathbb{E}[v_T(\theta)] = \frac{1}{3}(5.4 + 10.7 + 16) = 10.7.$$

- Offering $p \geq 12.3 > 10.7$ is thus strictly dominated.
- No trades with type $\vartheta_H$ are incentive compatible for both farmers.

# Conditionally Dominated Strategies

**No trades with the medium type:**

- Knowing he/she won't trade with type $\vartheta_H$, the conditional expected value of the land to the Tea Farmer is

$$\mathbb{E}[v_T(\theta) \,|\, \theta \neq \vartheta_H] = \frac{1}{2}(5.4 + 10.7) = 8.05.$$

- For any price $p < 8.4$, however, type $M$ is not willing to sell.
- Offering $p \geq 8.4 > 8.05$ is thus strictly dominated.

**Trading only with the low type:**

- Tea Farmer will not offer more than $v_T(\vartheta_L) = 5.4$.
- Rice Farmer will not accept any less than $v_R(\vartheta_L) = 4.2$.

# Equilibrium Trades

**Equilibrium outcomes:**

- Trade at any price $p \in [4.2, 5.4]$ can be supported in equilibrium by

$$\mathcal{P}(L) \cap [4.2, 5.4] \neq \emptyset, \qquad p = \min \mathcal{P}(L).$$

- No trade is also an equilibrium outcome, supported by strategies

$$\mathcal{P}(L) \cap [4.2, 5.4] = \emptyset, \qquad p < 4.2.$$

**Inefficiency:**

- Trade can occur only with the low type, even though the Tea Farmer values the land more than the Rice Farmer in any state of nature.
- Private information can be a great source of inefficiency.
- Players use weakly dominated strategies in the no-trade equilibrium.

# Adverse Selection

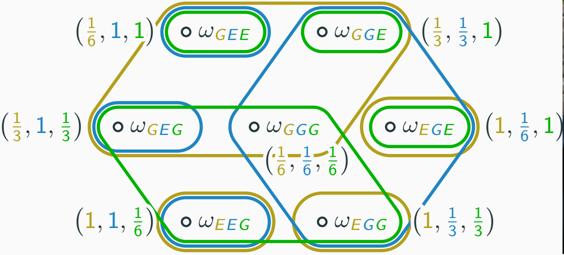
**Adverse selection:**

- Adverse selection arises in a market situation with asymmetric information between buyers and sellers.
- The informed side chooses to participate selectively in trades which benefit them the most, at the expense of the other trader.
- The uninformed side experiences an adverse selection of market participants, leaving it with worse-than-average trading partners.

**Equilibrium unraveling:**

- The equilibrium unravels if trade occurs only with the lowest type.
- The equilibrium need not unravel completely. If high- or low-quality soil affected yield by only 33%, trade could occur with type $\vartheta_M$.

# Avalon: a Social Deduction Game




**5-player game:**

- There are three (G)ood characters and two (E)vil characters.
- Alignment (Good or Evil) is assigned randomly in the beginning. The distribution of this random assignment is the common prior $P$.
- For a subset of players $\{1, 2, 3\}$, the belief space and the induced family of posterior beliefs is depicted above.

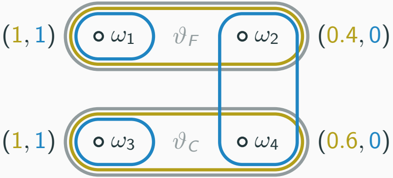
# Avalon: Double Fails

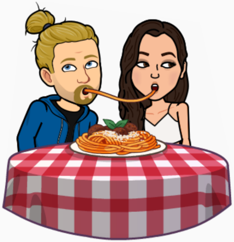
| | 0 Fails | 1 Fail | 2 Fails |
|---|---|---|---|
| Good | 1 | $-1$ | 5 |
| Evil | $-1$ | 1 | $-5$ |

**2 Players go on a quest:**

- Every evil player on the quest can choose to (S)ucceed of (F)ail.
- Good characters cannot fail a quest, hence they must choose (S)ucceed.
- A double fail is particularly damning to the evil players because it unambiguously reveals that two members of the quest are evil.
- What is the Bayesian Nash equilibrium of this game?

## Going on a Date



**Going out for dinner:**

- Maria and Tony can dress (F)ormally or (C)asually to go to a restaurant, whose dress code is either formal or casual, i.e., $\Theta = \{\vartheta_F, \vartheta_C\}$.
- Maria knows $\theta$, and she believes $\theta$ is common knowledge.
- Tony believes the dress code is casual with 60%, that Maria knows $\theta$, and that Maria believes $\theta$ is common knowledge.
- This belief hierarchy gives rise to the above belief space.

# Going on a Date

$\vartheta_F$

| | F | C |
|---|---|---|
| F | 2, 2 | 1, −2 |
| C | −2, 1 | 0, 0 |

$\vartheta_C$

| | F | C |
|---|---|---|
| F | 0, 0 | −2, 1 |
| C | 1, −2 | 2, 2 |

**Going out for dinner:**

- Suppose utilities are as above. What is the Bayesian Nash equilibrium?
- If the restaurant is indeed formal, how will the players dress?

# Check Your Understanding

**True or false:**

1. A strategy of an Avalon player involves a plan of action for every role he/she might have.
2. An equivalent definition of a Bayesian game $\mathcal{G}$ is: $\mathcal{G}$ is a belief space, in which each state of nature is a game of complete information.
3. In a finite Bayesian game with a common prior $P(\tau) > 0$ for every type profile $\tau$, any Nash equilibrium is a Bayesian Nash equilibrium.

Scan me

**Short-answer question:**

4. Suppose a pure strategy is a book with instructions on what to do in any given situation. Is going to a library and picking a book at random a mixed or a behavior strategy?

# Continuum of Types

# Number of Types and Actions

Number of types

| Number of actions | 1 | 2 | continuum |
|---|---|---|---|
| 2 | Works like finding | Nash equilbiria | cutoff strategies |
| cont. | | | monotonic strategies |

**Continuum of types:**

- In many applications, types are meaningfully ordered: a higher type values a certain good more, has lower production costs, etc.
- With two actions, this often leads to equilibria in cutoff strategies.
- With a continuum of actions, we often restrict attention to equilibria that are monotonic in the type because they are intuitive.
- Monotonicity often implies differentiability in equilibrium.

# Swapping Houses

**Swapping houses:**

- Two players $i = 1, 2$ own a house, whose values $v_i$ are drawn independently from a uniform distribution on $[0, 1]$.
- Because the grass is always greener on the other side, each player $i$ values player $j$'s house at $\frac{3}{2}v_j$.
- Suppose the players announce simultaneously whether they want to swap the house with the other player and that the house is swapped only if they both announce they want to swap.
- What are the Bayesian Nash equilibria of this game?

# First-Price Auction

# (Sealed-Bid) First-Price Auction

**Auction format:**

- Players submit their bids simultaneously.
- Highest bidder obtains the auctioned object and pays his bid.
- Typically, a lot is drawn to break ties.

**Independent private values:**

- Each bidder $i = 1, \dots, n$ values the auctioned object at $\theta_i$.
- Bidders' valuations are independent across players: player $i$'s value does not depend on the private values of others.
- Suitable for consumable items, for which bidders know their valuations.
- Independence implies that bidders' beliefs over opponents are determined completely be the common prior, hence $\tau_i = \theta_i$.

Suppose you are not Joey. How much should you bid?

# Parametrizing the Auction

**Types:**

- Bidder $i$'s type $\theta_i$ is distributed with density function $f_i > 0$ on $[0, \infty)$.
- Bidders' distribution functions are common knowledge among bidders.
- Together with independence of types, this defines the common prior

$$P(\theta \in B) := \int_{B_1} f_1(\vartheta_1)\, \mathrm{d}\vartheta_1 \cdots \int_{B_n} f_n(\vartheta_n)\, \mathrm{d}\vartheta_n.$$

**Actions and payoffs:**

- Each bidder $i$ can submit a bid $b_i \in [0, \infty)$.
- Drawing lots: winner $i_*(b)$ is determined by uniform distribution on

$$\arg\max_{i \in \mathcal{I}} b_i = \left\{ i \in \mathcal{I} \;\middle|\; b_i = \max_{j \in \mathcal{I}} b_j \right\}.$$

- Bidder $i$'s utility function is $u_i(\vartheta_i, b) = (\vartheta_i - b_i) 1_{\{i = i_*(b)\}}$.

# Strategies and Equilibrium in First-Price Auction

**Strategies:**

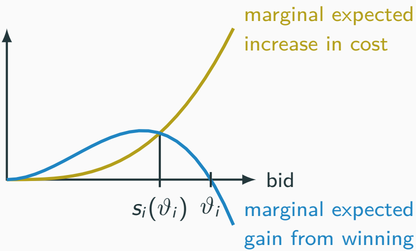
- A pure strategy of bidder $i$ is a map $s_i : \Theta_i \to [0, \infty)$.
- Bidder $i$'s expected utility of strategy profile $s(\theta)$ is:

$$\mathbb{E}_{\vartheta_i}[u_i(\theta_i, s(\theta))] = (\vartheta_i - s_i(\vartheta_i))P\big(i = i_*(s(\theta))\big).$$

**Bayesian Nash equilibrium:**

- A pure-strategy Bayesian Nash equilibrium is a strategy profile $s$ such that $s_i(\vartheta_i)$ maximizes $\mathbb{E}_{\vartheta_i}[u_i(\theta_i, b_i, s_{-i}(\theta_{-i}))]$ among all bids $b_i$.
- Is there a pure-strategy Bayesian Nash equilibrium $s$ that is:
  - Symmetric, i.e., $s_i(\vartheta_i) = s_j(\vartheta_i)$ for any pair $i, j$.
  - Increasing, i.e., $s_i(\vartheta_i) > s_i(\vartheta_i')$ for all $\vartheta_i > \vartheta_i'$,
  - And differentiable?

# Bidding Incentives in the First Price Auction




**First-price auction:**

- Increasing the bid increases the probability of winning, but also increases the price you pay for the object.
- In equilibrium, the marginal gain of winning the auction precisely offsets marginal increase in the price you pay.

Image generated for a 4-player auction with independent standard-uniformly distributed types and $\vartheta_i = 0.9$.

# Summary

**Bayesian games:**

- There is uncertainty about a payoff-relevant state of nature $\theta$.
- A player's type determines the information he/she has about $\theta$ as well as which actions the player has available.
- Players' belief hierarchies are either determined via a belief space or specified explicitly through a belief table.
- A strategy in a Bayesian games specifies a decision for each type.

**Bayesian Nash equilibrium:**

- Players maximize their interim expected value, given their beliefs.
- In a Bayesian Nash equilibrium, no player has an incentive to deviate after learning their type.

# Literature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- M. Maschler, E. Solan, and S. Zamir: Game Theory, Chapters 9.4 and 10.5, Cambridge University Press, 2013
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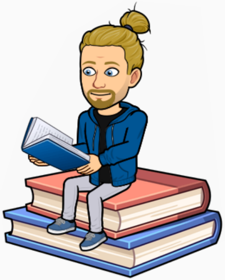

# Dynamic Games

# Selling Farmland

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**Annual average yield:**

- Tea: 5.35m NTD/km$^2$.
- Rice: 4.2m NTD/km$^2$.

**Bayesian Nash equilibrium:**

- Suppose high/low-quality soil yields 50% above/below average.
- The Rice Farmer knows the quality, but the Tea Farmer does not.
- Simultaneously, the Tea Farmer proposes a price $p \geq 0$ and the Rice Farmer quotes a set $\mathcal{P}$ of prices he/she will accept.

Source: https://www.visualcapitalist.com/the-worlds-most-valuable-cash-crop/

# Selling Farmland: Bayesian Nash equilibria

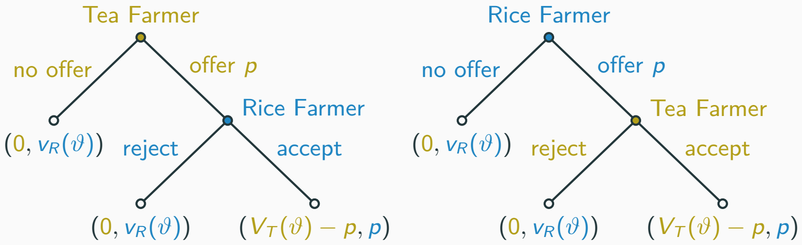
**Bayesian Nash equilibria:**

- Because of adverse selection, trade can occur only if the soil quality is low, for which the value of the land is $v_R(L) = 4.2$ and $v_T(L) = 5.4$.
- Any $p = \min \mathcal{P}(L) \leq 5.4$ constitutes a trade equilibrium.
- Any $p < 4.2$ and $\min \mathcal{P}(L) > 5.4$ is a no-trade equilibrium.

**Understanding the equilibria:**

- Trade may occur at any price in the interval $[4.2, 5.4]$.
- Which price is realized depends on the bargaining power of the farmers.
- We can model bargaining power through game dynamics.
- The no-trade equilibrium is inefficient. It is the result of a miscoordination that can be avoided if players do not act simultaneously.

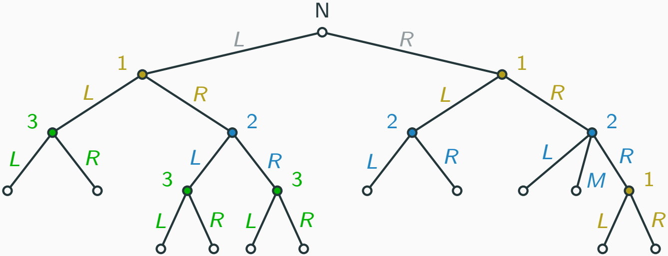
# Selling Farmland: a Dynamic Setting




**Adding dynamics:**

- If the Tea Farmer gets to make a take-it-or-leave-it offer, then the Rice Farmer will agree to any sale at price $p \geq 4.2$.
- Anticipating this response, the Tea Farmer best offers $p = 4.2$.
- Conversely, if the Rice Farmer gets to make a take-it-or-leave-it offer, then the Tea Farmer will agree to any sale at price $p \leq 5.4$.
- Anticipating this response, the Rice Farmer best asks for $p = 5.4$.

# The Mechanics of the Game

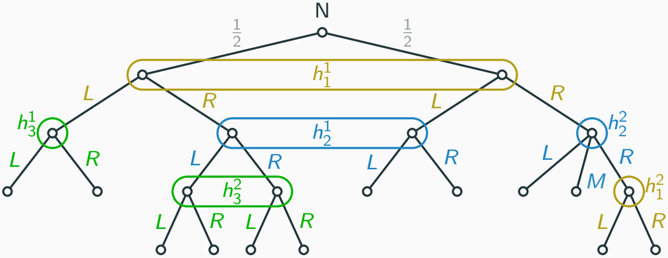



**Game tree:**

- A game tree $\mathcal{X}$ consists of nodes $x$ that describe the state of play:[1]
  - The game starts at the root of the tree $x_0$.
  - At each node $x$, the active player $i(x)$ chooses an action $a$ from $\mathcal{A}_i(x)$, which advances the game to the successor node $x_a$ of $x$.
  - The game ends with payoff vector $u(z)$ when terminal node $z$ is reached.

[1] This is not a complete mathematical description of a game tree. See Chapter 3 in Fudenberg and Tirole (1991).

# Available Information




**Information sets:**

- An information set $h_i$ contains node that player $i$ cannot distinguish.

  *In $h_2^1$, Player 2 cannot tell whether $(L, R)$ or $(R, L)$ has been played.*

- We denote by $\mathcal{H}_i$ the collection of all of $i$'s information sets.
- We require that $\mathcal{A}_i(x) = \mathcal{A}_i(x')$ for any two $x, x' \in h_i$ as, otherwise, player $i$ can distinguish $x$ and $x'$ via the set of available actions.

# Extensive-Form Game

**Definition 3.4**

An extensive-form game $\mathcal{G} = (\mathcal{X}, \prec, \mathcal{I}, i, (\mathcal{H}_i), (\mathcal{A}(h)), (u_i))$ consists of:

1. A tree $\mathcal{X}$ with root $x_0$ and set of terminal nodes $\mathcal{Z}$,
2. A set of players $\mathcal{I}$,
3. A map $i : \mathcal{X} \setminus \mathcal{Z} \to \mathcal{I} \cup \{N\}$ indicating the active player,
4. A partition $\mathcal{H}_i$ of $\mathcal{X}_i = \{x \in \mathcal{X} \setminus \mathcal{Z} \mid i(x) = i\}$ for each $i \in \mathcal{I}$,
5. A map $\mathcal{A}(h)$ indicating the available actions at $h$ in $\mathcal{H} = \bigcup_{i \in \mathcal{I}} \mathcal{H}_i$,
6. A payoff function $u_i : \mathcal{Z} \mapsto \mathbb{R}$ for each player $i$.

**Note:** We refer to information sets in dynamic settings by $h$ because those typically correspond to histories of observed actions.

# Perfect vs. Imperfect Information

**Definition 3.5**

Information is perfect if nature has no moves and each information set is a singleton. Information is imperfect otherwise.

**Possible causes for imperfect information:**

- Strategic uncertainty: simultaneous-move games.
- Imperfect monitoring: instead of observing actions directly, players observe a signal, whose distribution is affected by the chosen actions.
- Incomplete information: there is uncertainty about payoffs or players' types. Typically, this corresponds to unobserved moves by nature.

# Strategies

### Definition 3.6

Let $\mathcal{A}_i := \bigcup_{h_i \in \mathcal{H}_i} \mathcal{A}_i(h_i)$ denote all of player $i$'s actions.

1. A pure strategy of player $i$ is a map $s_i : \mathcal{H}_i \to \mathcal{A}_i$ such that $s_i(h_i) \in \mathcal{A}_i(h_i)$ for every $h_i \in \mathcal{H}_i$. Let $\mathcal{S}_i$ denote the set of $i$'s pure strategies.
2. A mixed strategy of player $i$ is a distribution $\sigma_i \in \Delta(\mathcal{S}_i)$.
3. A behavior strategy of player $i$ is a map $\sigma_i : \mathcal{H}_i \to \Delta(\mathcal{A}_i)$ such that $\sigma_i(h_i) \in \Delta\mathcal{A}_i(h_i)$ for every $h_i \in \mathcal{H}_i$.

**Remark:**

- We denote by $\sigma_i(h_i; a_i)$ the probability that $a_i$ is chosen under $\sigma_i(h_i)$.
- For $\mathcal{H}_i = \mathcal{T}_i$, these notions coincide with strategies in Bayesian games.

# Outcomes

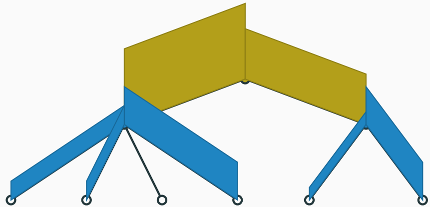
**The states of the game:**

- Since immediate predecessor nodes are unique, for each node $x$ there is precisely one path $(x_0, x_1, \ldots, x_k)$ that leads from $x_0$ to $x = x_k$.
- Node $x$ is thus reached if and only if $(a^{x_1}, \ldots, a^{x_k})$ has been played.

**Outcome:**

- We set the outcome $A$ to be the sequence of all realized actions.
- Note: if player used mixed actions, $A$ is a random variable.
- The player's ex-post payoff in outcome $A$ is $u(A) := u(z(A))$, where $z(A)$ denotes the terminal payoff reached when $A$ is realized.

# Probability Distribution over Outcomes




**Distribution under $\sigma$:**

- $P_\sigma(x) =$ multiply the probability of all edges leading to $x$.
- Formally, this is given by $P_\sigma(x_0) = 1$ and for $x \neq x_0$:

$$P_\sigma(x) := \prod_{j=1}^{k} \sigma_{i(x_{j-1})}(h_i(x_{j-1}); a_{x_j}),$$

  where $x = x_k$ is reached by play of $a_{x_1}, \ldots, a_{x_k}$ at $x_0, \ldots, x_{k-1}$.
- Each player $i$ maximizes his/her expected utility $\mathbb{E}_\sigma[u_i(A)]$.

# Harsanyi's Equivalence

**Theorem 3.7**

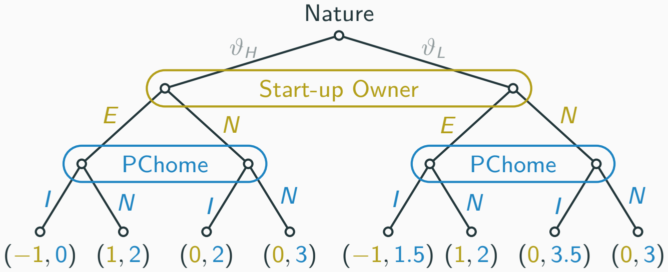
*Let $\mathcal{G}$ be a Bayesian game with finitely many actions, finite type spaces $\mathcal{T}_i$, and a common prior $P$ with $P(\tau_i) > 0$ for every $\tau_i \in \mathcal{T}_i$.*

- *The Bayesian game is equivalent to the extensive-form game $\mathcal{G}'$, in which nature chooses the players types in the first move.*
- *A Bayesian Nash equilibrium of $\mathcal{G}$ is a Nash equilibrium of the reduced strategic-form game associated with $\mathcal{G}'$.*

**A different interpretation:**

- It may be helpful to visualize to flow of a game with a game tree.
- Finding BNE, however, does not necessarily get easier.
- A mixed Nash equilibrium of the strategic-form game corresponds to a mixed-strategy Bayesian Nash equilibrium in the extensive-form game.

# Start-Up Problem as an Extensive-Form Game




**Finding BNE using Harsanyi's equivalence:**

- Transform the game into its strategic form. Expectations over moves by nature correspond to expectations with respect to the common prior.
- Find the Nash equilibrium of the strategic-form game.
- The equivalence illustrates that information sets from unobserved actions and incomplete information are identical.

# Summary

**Extensive-form games:**

- A very general framework for dynamic interactions that allows imperfect information for a variety of reasons.
- Game tree provides a good visual for small games.
- Drawbacks of extensive-form games: they are inherently finite and somewhat notationally cumbersome.

**Harsanyi's equivalence:**

- If players' have a common prior, a finite Bayesian game can be written as a game tree, where Nature selects players' types.
- However, this equivalence did not exploit the dynamics embedded in extensive-form games yet.

# Check Your Understanding

**True or false:**

1. In a game tree, it is impossible that two different pure strategies lead to the same terminal node.
2. If information is imperfect, then it is incomplete.
3. The probability measure $P_\sigma$ defined here coincides with $P_\sigma$ defined in a non-dynamic Bayesian game.

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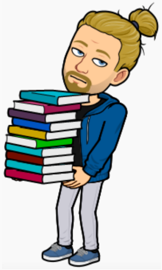
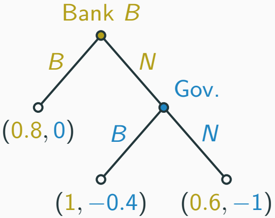
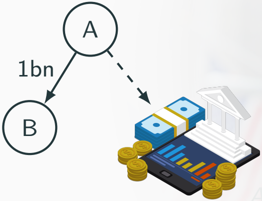


# Subgame Perfection

## Bail-ins and Bailouts





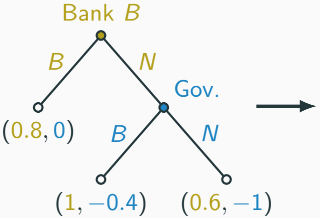
**Bailing in a defaulting bank:**

- Bank $A$ has liabilities of \$1bn to Bank $B$, but only \$800m in assets.
- If bank $A$ declares bankruptcy, additional losses of \$200m are incurred (asset liquidation, legal fees, etc.) before $A$ repays \$600m to bank $B$.
- Bank $B$ has an incentive to (B)ail in bank $A$ and accept \$800m.

**Presence of government:**

- Bankruptcy of bank $A$ may cause severe losses to the economy.
- Government has an incentive to rescue bank $A$ if Bank $B$ does not.

# Bail-ins and Bailouts

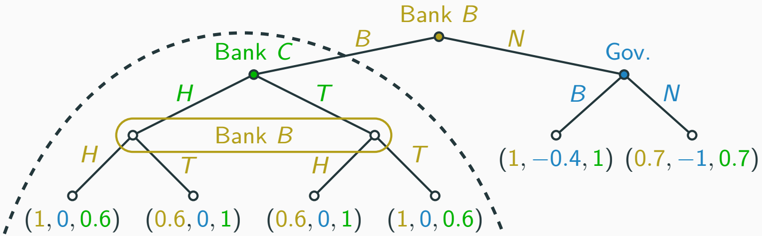



| | B | N |
|---|---|---|
| B | $0.8, \underline{0}$ | $0.8, \underline{0}$ |
| N | $\underline{1}, \underline{-0.4}$ | $0.6, -1$ |


**Bail-in vs. bailout:**

- Pure-strategy Nash equilibria are $(B, N)$ and $(N, B)$.
- In $(B, N)$, the Government "threatens" that it will not bail out $A$:
  - Because of this threat, Bank $B$ is willing to bail in bank $A$.
  - Given that Bank $B$ bails in $A$, the Government's action does not matter.
- By backward induction, this no-bailout threat is not credible: given that the Government gets to act, choosing $N$ is suboptimal.

# Bail-In Subgame




**Variant with multiple creditors:**

- Bank $A$ has \$1.6bn assets and \$1bn liabilities each to banks $B$ and $C$.
- If Bank $A$ defaults, bankruptcy losses of \$200m are incurred before $A$ repays \$700m to each $B$ and $C$.
- Bank $B$ gets to decide whether the creditors organize a bail-in and, if they do, the contributions are decided by a matching pennies game.
- Not solvable by backward induction because information is imperfect.

# Subgame

**Definition 3.8**

Consider an extensive-form game $(\mathcal{X}, \prec, \mathcal{I}, i, \mathcal{H}, \mathcal{A}, u)$. For any $x \in \mathcal{X}$, let $\mathcal{X}_x := \{x\} \cup \{x' \in \mathcal{X} \mid x \prec x'\}$ denote the sub-tree starting at $x$. Then $\mathcal{G}(x) = (\mathcal{X}_x, \prec, \mathcal{I}, i|_x, \mathcal{H}|_x, \mathcal{A}|_x, u|_x)$ is a (proper) subgame of $\mathcal{G}$ if:

1. For any $h_i \in \mathcal{H}_i$: if $x' \in \mathcal{X}(x)$ for some $x' \in h_i$, then $h_i \subseteq \mathcal{X}_x$,
2. $i|_x, \mathcal{H}|_x, \mathcal{A}|_x$, and $u|_x$ are the restrictions of $i, \mathcal{H}, \mathcal{A}$, and $u$ to $\mathcal{X}_x$.

**Two points worth noticing:**

- Point 1. says that an information set is either completely contained in the subgame or the intersection with the subgame is empty.
- This applies to the node $x$ starting subgame $\mathcal{G}(x)$, i.e., $h_i(x) = \{x\}$.

At any $x' \in \mathcal{X}_x$, it is common knowledge that $\mathcal{I}$ are playing the subgame.

# Subgame Perfection

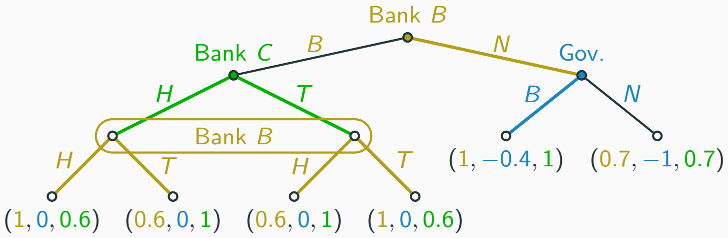
**Definition 3.9**

A behavior strategy profile $\sigma$ is a subgame perfect equilibrium (SPE) if its restriction to $\mathcal{G}(x)$ is a Nash equilibrium for every proper subgame $\mathcal{G}(x)$.

**Properties:**

- Since the entire game is a proper subgame, any subgame-perfect equilibrium is also a Nash equilibrium: SPE is a refinement.
- Subgame perfection eliminates non-credible threats off the equilibrium path that are suboptimal to carry out.
- If all information sets are singletons, then subgame perfect equilibria coincide with the solution found by backward induction.
- Subgame perfection is applicable to games with imperfect information.

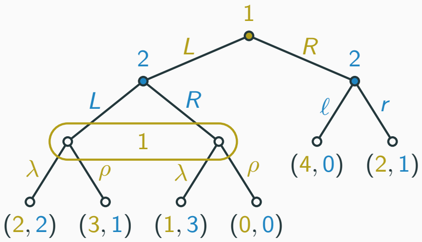
# Dynamic Programming Principle




**Dynamic programming:**

- Bail-in subgame has a unique Nash equilibrium $\left(\frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T\right)$
- If subgame $\mathcal{G}$ has a unique equilibrium $\sigma_{\mathcal{G}}$, then replacing $\mathcal{G}$ with $\mathbb{E}_{\sigma_{\mathcal{G}}}[u(Z)]$ retains subgame perfect equilibria in the rest of the tree.
- If subgame $\mathcal{G}$ has multiple equilibria, doing this for all subgame perfect equilibrium outcomes of $\mathcal{G}$ yields all subgame perfect equilibria.

# Example




**Goal:** Find all the subgame-perfect equilibria.

# Finding subgame-perfect equilibria

**Finding subgame-perfect equilibria:**

- Use dynamic programming to proceed backwards through the tree.
- For each proper subgame $\mathcal{G}(x)$:
  - Let $\widehat{\mathcal{X}}(x)$ denote the nodes $x' \in \mathcal{X}_x$, for which $\mathcal{G}(x') \subsetneq \mathcal{G}(x)$ is a proper subgame of $\mathcal{G}(x)$ that is not contained in any other subgame of $\mathcal{G}(x)$.
  - For every combination of continuation equilibria $(\sigma|_{x'})_{x' \in \widehat{\mathcal{X}}(x)}$, replace $\mathcal{G}(x')$ with terminal node that yields payoffs $u(\sigma|_{x'})$.
  - Transform the "pruned" version of $\mathcal{G}(x)$ into its reduced-strategic form.
  - Find the Nash equilibria as we did in static games.
  - Transform the Nash equilibria into equivalent behavior strategies.

**Note:** If there are multiple subgames with multiple subgame-perfect equilibria, this can be quite a tedious procedure.

# Subgame Perfection on the Equilibrium Path

**Proposition 3.10**

*Let $\sigma$ be a Nash equilibrium in an extensive-form game $\mathcal{G}$. For every node $x$, for which $\mathcal{G}(x)$ is a proper subgame with $P_\sigma(\{x\}) > 0$, the restriction $\sigma|_{\mathcal{G}(x)}$ is a Nash equilibrium of the subgame $\mathcal{G}(x)$.*

**Equilibrium path:**

- Any node $x$ with $P_\sigma(\{x\}) > 0$ is said to lie on the path of strategy profile $\sigma$ because it is reached with positive probability.
- Any other node lies off the path.

**Consequence:**

- Any Nash equilibrium is subgame perfect on the equilibrium path.
- Subgame perfection refines only behavior off the path.

# Proof of Proposition 3.10

**Suppose the proposition fails:**

- Suppose there is a Nash equilibrium $\sigma$, in which some player $i$ has a profitable deviation $\widehat{\sigma}_i$ in the subgame $\mathcal{G}(x)$ with $P_\sigma(\{x\}) > 0$.

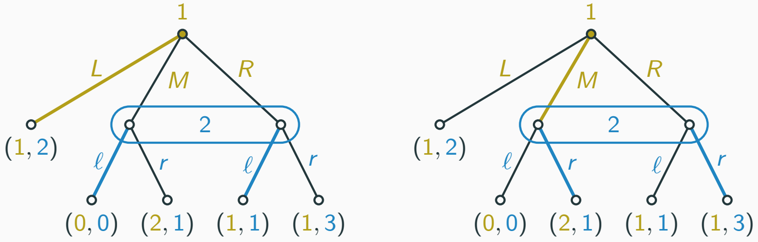
**Derive a contradiction:**

- Let $\widetilde{\sigma}_i$ be the strategy that plays $\sigma_i$ outside of $\mathcal{G}(x)$ and $\widehat{\sigma}_i$ in $\mathcal{G}(x)$.
- Since $\widetilde{\sigma}_i$ and $\sigma_i$ agree outside of $\mathcal{G}(x)$, we have $P_{\widetilde{\sigma}_i,\sigma_{-i}}(\{x\}) = P_\sigma(\{x\})$ and $P_{\widetilde{\sigma}_i,\sigma_{-i}}(\{z\}) = P_\sigma(\{z\})$ for any terminal node $z \notin \mathcal{G}(x)$. Thus,

$$\begin{aligned}\mathbb{E}_{\widetilde{\sigma}_i,\sigma_{-i}}[u_i(Z)] = \mathbb{E}_{\widetilde{\sigma}_i,\sigma_{-i}}[u_i(Z)\,|\,Z \notin \mathcal{G}(x)]\big(1 - P_{\widetilde{\sigma}_i,\sigma_{-i}}(\{x\})\big)& \\ + \mathbb{E}_{\widetilde{\sigma}_i,\sigma_{-i}}[u_i(Z)\,|\,Z \in \mathcal{G}(x)]P_{\widetilde{\sigma}_i,\sigma_{-i}}(\{x\})& \\ > \mathbb{E}_\sigma[u_i(Z)].&\end{aligned}$$

- This stands in contradiction to $\sigma$ being a Nash equilibrium.

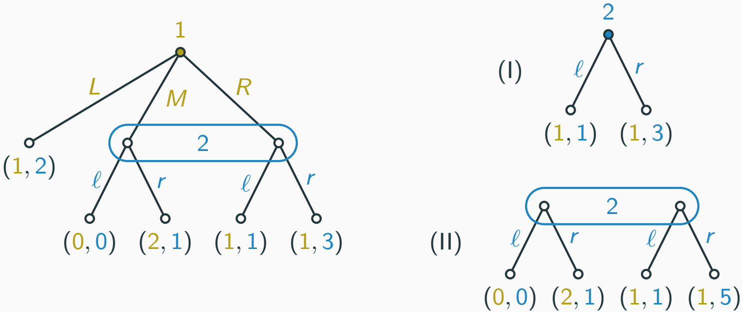
# Seemingly Dominated Strategies




**Two subgame-perfect equilibria:**

- Given that Player 2 gets to act, it is strictly dominant to choose $r$.
- Nevertheless, $(L, \ell)$ is a subgame-perfect equilibrium.
- By definition, a subgame starts at a singleton information set, hence the only subgame is the entire game itself.
- As a consequence, all Nash equilibria are trivially subgame perfect.

# Limitations of Subgame Perfection




**No proper subgames:**

- Game (I) is no proper subgame because Player 2 knows that Player 1 played $R$, which is information he does not have in $\mathcal{G}$.
- Game (II) is no proper subgame because the relative probabilities Player 2 assigns to the two nodes depend on Player 1's action.

# Summary

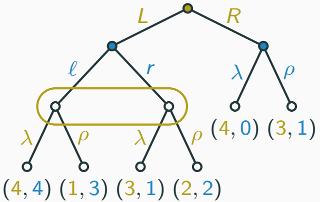
**Refinement off the path:**

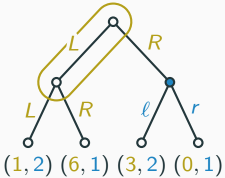
- Any Nash equilibrium is already optimal on the equilibrium path.
- Subgame perfection refines Nash equilibria off the equilibrium path.

**Eliminating non-credible threats:**

- Similarly to backward induction, subgame perfection eliminates threatened continuation strategies that are suboptimal if carried out.
- Contrary to backward induction, existence of a subgame perfect equilibrium is guaranteed in any extensive-form game.
- Sometimes, subgame perfect equilibria are unintuitive since subgame perfection cannot “cut through” information sets.

# Check Your Understanding

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**Short-answer questions:**

1. How many SPE are there in each game?
2. How many outcomes can arise from an SPE in each game?

**True or false:**

3. A Nash equilibrium $\sigma$ differs from a subgame-perfect equilibrium only on a set of nodes with $P_\sigma$-measure 0.
4. In an extensive-form game with a continuum of actions, an SPE exists if the action set at each information set is closed and bounded.

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