Macroeconomic Theory: Assignment 1

In the following questions, utility functions and production functions are assumed to satisfy the standard assumptions we discussed in class.

Question 1. (30%) (Capital Adjustment) Consider an economy populated by a large number of identical households. Each household has a preference given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),\ 0 < \beta < 1,$$

where c_t denotes the consumption. Households use capital, k_t, to produce consumption goods and investment goods:

$$c_t + x_t \leq f(k_t).$$

The capital available in the next period, k_{t+1}, is a function of the investment, x_t, and the current capital, k_t:

$$k_{t+1} = \phi k_t \left(\frac{x_t}{k_t}\right)^{\alpha}, 0 < \alpha < 1.$$

The technology captures the idea that the yield of an additional unit of investment is decreasing in the size of the existing capital stock.

1. (5%) Write down the optimization problem of the economy.
2. (5%) Write down the first order conditions of the problem.
3. (10%) Discuss the existence and uniqueness of the steady state.
4. (10%) Describe the impact of an increase in the productivity parameter ϕ on the steady state level of capital and consumption.

Question 2. (30%) (Habit Persistence) Consider an economy populated by a large number of households with utility functions given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t - \phi z_t).$$

The variable c_t is individual consumption, and z_t is a measure of lagged consumption. To be precise,

$$z_t = \sum_{j=0}^{t-1} (1-\eta)^j c_{t-1-j}.$$

It follows that, alternatively, it is possible to describe the law of motion for z_t as

$$\begin{aligned} z_{t+1} &= (1-\eta) z_t + c_t, \\ z_0 &= 0. \end{aligned}$$

In this setting, z_t is a measure of "habit persistence," as it implies that the utility of any given level of consumption is smaller if the level of past consumption is higher. Finally, the production and investment is given by

$$\begin{aligned} c_t + x_t &= f(k_t), \\ k_{t+1} &\leq (1-\delta) k_t + x_t. \end{aligned}$$

1. (5%) Write down the optimization problem of the economy.
2. (5%) Find the condition that (ϕ, η) is required to satisfy in order to maintain the holding of steady state nonnegative condition: $c^* - \phi z^* \geq 0$.
3. (10%) Write down the first order conditions of the problem.
4. (10%) How does the model say about the impact of changes in ϕ on the steady state level of capital and consumption.

1. $\max \sum \beta^t u(c_t - \phi z_t)$ s.t. $\begin{cases} c_t + x_t = f(k_t) \\ k_{t+1} \leq (1-\delta)k_t + x_t \\ z_{t+1} = (1-\eta)z_t + c_t \end{cases}$

$$\Rightarrow \max_{z_{t+1},\, k_{t+1}} \sum \beta^t u\left(z_{t+1} + (\eta-\phi-1)z_t\right) \quad \text{s.t. } [z_{t+1} - (1-\eta)z_t] + [k_{t+1} - (1-\delta)k_t] \leq f(k_t)$$

↳ 是希望我這樣做嗎?

2. At S-S, $z^* + (\eta-\phi-1)z^* \geq 0 \Rightarrow (\eta-\phi)z^* \geq 0 \Rightarrow \eta \geq \phi$.

3. $$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left\{ u\left(z_{t+1} + (\eta-\phi-1)z_t\right) + \lambda_t \left(f(k_t) - [z_{t+1} - (1-\eta)z_t] - [k_{t+1} - (1-\delta)k_t] \right) \right\}$$

FOC: $[z_{t+1}]$: $$\beta^t\left[u'(z_{t+1} + (\eta-\phi-1)z_t) - \lambda_t\right] + \beta^{\cancel{t+1}}\left[u'(z_{t+2} + (\eta-\phi-1)z_{t+1})\cdot(\eta-\phi-1) + \lambda_{t+1}(1-\eta)\right] = 0$$

這要怎麼整理?

$[k_{t+1}]$: $$\lambda_t = \beta \cdot \lambda_{t+1}\left[f'(k_{t+1}) + (1-\delta)\right].$$

3. $$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left\{ u(c_t - \phi z_t) + \lambda_t\left[f(k_t) + (1-\delta)k_t - k_{t+1} - c_t\right] + \gamma_t\left[z_{t+1} - (1-\eta)z_t - c_t\right] \right\}$$

FOC: $[c_t]$: $u_c(t) + \lambda_t + \gamma_t = 0$

$[z_{t+1}]$: $\gamma_t + \beta\left[u_z(t+1)\cdot(-\phi) + \gamma_{t+1}(1-\eta)\right] = 0$

$[k_{t+1}]$: $\lambda_t = \beta \lambda_{t+1}\left[f'(k_{t+1}) + (1-\delta)\right]$

4. How ϕ affect c^*, k^*?

① k^* is determined by $\cancel{\lambda^*} = \beta \cancel{\lambda^*}\left[f'(k^*) + (1-\delta)\right]$

$$\Rightarrow 1 = \beta\left[f'(k^*) + (1-\delta)\right]$$

So, k^* would not be affected by ϕ.

② BC: $c^* + \delta k^* = f(k^*)$

c^* is determined by $f(k^*) - \delta k^*$.

Since k^* would not be affected, c^* also remains constant.

Question 3. **(40%) (Productivity in Capital Goods Sector)** Consider an economy populated by a large number of households with utility functions given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), 0 < \beta < 1.$$

There are two sectors, 1 and 2. Sector 1 produces consumption goods, c_t; sector 2 produces investment goods, x_t. In each period, a unit of capital k_t can be allocated to sector 1 or to sector 2. Let k_{it} denote the quantity of capital allocated to sector i, then

$$k_t = k_{1t} + k_{2t}.$$

The production of consumption goods and investment goods follows

$$\begin{aligned} c_t &= z^1 f^1(k_{1t}), \\ x_t &= z^2 f^2(k_{2t}), \end{aligned}$$

where $f^1(k_{1t})$, and $f^2(k_{1t})$ are the production function of sector 1 and sector 2, and z^1 and z^2 are their corresponding technology parameters. Note that capital are fully malleable and can be costlessly reallocated across sectors in each period, so it is sufficient to apply the aggregate capital accumulation process:

$$k_{t+1} = (1 - \delta)k_t + x_t.$$

1. (10%) Write down the optimization problem of the economy.
2. (10%) Write down the first order conditions of the problem.
3. (10%) Discuss the existence and uniqueness of the steady state.
4. (10%) Someone argues that an increase in z^2 increases k_2 and crowds out the capital used to produce consumption goods, k_1, and thus, decreases the consumption in the steady state. Do you agree or disagree this statement? Prove your argument by showing how an increase in z^2 influences the steady state consumption.

1. $\max_{c_t, k_{1t+1}, k_{2t+1}} \sum \beta^t u(c_t)$ s.t. $\begin{cases} c_t = z^1 f^1(k_{1t}) \\ x_t = z^2 f^2(k_{2t}) \\ k_t = k_{1t} + k_{2t} \\ k_{t+1} = (1-\delta) k_t + x_t \end{cases}$

$\max_{k_{1t+1}, k_{2t+1}} \sum \beta^t u[z^1 f^1(k_{1t})]$ s.t. $(k_{1t+1} + k_{2t+1}) = (1-\delta)(k_{1t} + k_{2t}) + z^2 f^2(k_{2t})$

2. $\mathcal{L} = \max \sum \beta^t \{ u[z^1 f^1(k_{1t})] + \lambda_t [(1-\delta)(k_{1t} + k_{2t}) + z^2 f^2(k_{2t}) - (k_{1t+1} + k_{2t+1})]$

FOC: $[k_{1t+1}]$: $\lambda_t = \beta [u'[z^1 f^1(k_{1t+1})] \cdot z^1 f^{1\prime}(k_{1t+1}) + \lambda_{t+1} (1-\delta)]$

$[k_{2t+1}]$: $\lambda_t = \beta \cdot \lambda_{t+1} \cdot [(1-\delta) + z^2 f^{2\prime}(k_{2t+1}) - 1]$

$[c_t]$: $u'(c_t) = \lambda_t^1$

$[k_{1t}]$: $\lambda_t^1 \cdot z^1 F_k^1(k_{1t}) = \mu_t$

$[k_{2t}]$: $\lambda_t^2 \cdot z^2 \cdot F_k^2(k_{2t}) = \mu_t$

$[k_{t+1}]$: $\lambda_t^2 = \beta(1-\delta) \lambda_{t+1}^2 + \mu_{t+1} \beta$

$\Rightarrow u'(c^*) \cdot z^1 F_k^1(k^*) = [\beta(1-\delta)\lambda^* + \mu^* \beta] z^2 \cdot F_k^2(k^*)$