SOLUTIONS MANUAL TO MATHEMATICAL ANALYSIS BY TOM APOSTOL - SECOND EDITION-CHAPTER ONE

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CHAPTER 1

THE REAL AND COMPLEX NUMBER SYSTEMS

1.1 INTRODUCTION

Mathematical analysis studies concepts related in some way to real numbers, so we begin our study of analysis with a discussion of the real-number system.

Several methods are used to introduce real numbers. One method starts with the positive integers 1, 2, 3, ... as undefined concepts and uses them to build a larger system, the positive *rational numbers* (quotients of positive integers), their negatives, and zero. The rational numbers, in turn, are then used to construct the *irrational numbers*, real numbers like $\sqrt{2}$ and π which are not rational. The rational and irrational numbers together constitute the real-number system.

Although these matters are an important part of the foundations of mathematics, they will not be described in detail here. As a matter of fact, in most phases of analysis it is only the *properties* of real numbers that concern us, rather than the methods used to construct them. Therefore, we shall take the real numbers themselves as undefined objects satisfying certain axioms from which further properties will be derived. Since the reader is probably familiar with most of the properties of real numbers discussed in the next few pages, the presentation will be rather brief. Its purpose is to review the important features and persuade the reader that, if it were necessary to do so, all the properties could be traced back to the axioms. More detailed treatments can be found in the references at the end of this chapter.

For convenience we use some elementary set notation and terminology. Let S denote a set (a collection of objects). The notation $x \in S$ means that the object x is in the set S, and we write $x \notin S$ to indicate that x is not in S.

A set S is said to be a *subset* of T, and we write $S \subseteq T$, if every object in S is also in T. A set is called *nonempty* if it contains at least one object.

We assume there exists a nonempty set $\mathbf{R}$ of objects, called real numbers, which satisfy the ten axioms listed below. The axioms fall in a natural way into three groups which we refer to as the *field axioms*, the *order axioms*, and the *completeness axiom* (also called the *least-upper-bound axiom* or the *axiom of continuity*).

1.2 THE FIELD AXIOMS

Along with the set $\mathbf{R}$ of real numbers we assume the existence of two operations, called *addition* and *multiplication*, such that for every pair of real numbers x and y

the *sum* $x + y$ and the *product* xy are real numbers uniquely determined by x and y satisfying the following axioms. (In the axioms that appear below, x, y, z represent arbitrary real numbers unless something is said to the contrary.)

Axiom 1. $x + y = y + x$, $xy = yx$ (*commutative laws*).

Axiom 2. $x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$ (*associative laws*).

Axiom 3. $x(y + z) = xy + xz$ (*distributive law*).

Axiom 4. *Given any two real numbers x and y, there exists a real number z such that $x + z = y$. This z is denoted by $y - x$; the number $x - x$ is denoted by* 0. *(It can be proved that* 0 *is independent of x.) We write $-x$ for* $0 - x$ *and call $-x$ the negative of x.*

Axiom 5. *There exists at least one real number $x \neq 0$. If x and y are two real numbers with $x \neq 0$, then there exists a real number z such that $xz = y$. This z is denoted by y/x; the number x/x is denoted by* 1 *and can be shown to be independent of x. We write x^{-1} for $1/x$ if $x \neq 0$ and call x^{-1} the reciprocal of x.*

From these axioms all the usual laws of arithmetic can be derived; for example, $-(-x) = x$, $(x^{-1})^{-1} = x$, $-(x - y) = y - x$, $x - y = x + (-y)$, etc. (For a more detailed explanation, see Reference 1.1.)

1.3 THE ORDER AXIOMS

We also assume the existence of a relation $<$ which establishes an ordering among the real numbers and which satisfies the following axioms:

Axiom 6. *Exactly one of the relations $x = y$, $x < y$, $x > y$ holds.*

NOTE. $x > y$ means the same as $y < x$.

Axiom 7. *If $x < y$, then for every z we have $x + z < y + z$.*

Axiom 8. *If $x > 0$ and $y > 0$, then $xy > 0$.*

Axiom 9. *If $x > y$ and $y > z$, then $x > z$.*

NOTE. A real number x is called *positive* if $x > 0$, and *negative* if $x < 0$. We denote by $\mathbf{R}^+$ the set of all positive real numbers, and by $\mathbf{R}^-$ the set of all negative real numbers.

From these axioms we can derive the usual rules for operating with inequalities. For example, if we have $x < y$, then $xz < yz$ if z is positive, whereas $xz > yz$ if z is negative. Also, if $x > y$ and $z > w$ where both y and w are positive, then $xz > yw$. (For a complete discussion of these rules see Reference 1.1.)

NOTE. The symbolism $x \leq y$ is used as an abbreviation for the statement:

$$\text{"}x < y \quad \text{or} \quad x = y.\text{"}$$

Thus we have $2 \leq 3$ since $2 < 3$; and $2 \leq 2$ since $2 = 2$. The symbol $\geq$ is similarly used. A real number x is called *nonnegative* if $x \geq 0$. A pair of simultaneous inequalities such as $x < y$, $y < z$ is usually written more briefly as $x < y < z$.

The following theorem, which is a simple consequence of the foregoing axioms, is often used in proofs in analysis.

Theorem 1.1. *Given real numbers a and b such that*

$$a \leq b + \varepsilon \qquad \textit{for every } \varepsilon > 0. \tag{1}$$

Then $a \leq b$.

Proof. If $b < a$, then inequality (1) is violated for $\varepsilon = (a - b)/2$ because

$$b + \varepsilon = b + \frac{a - b}{2} = \frac{a + b}{2} < \frac{a + a}{2} = a.$$

Therefore, by Axiom 6 we must have $a \leq b$.

Axiom 10, the completeness axiom, will be described in Section 1.11.

1.4 GEOMETRIC REPRESENTATION OF REAL NUMBERS

The real numbers are often represented geometrically as points on a line (called the *real line* or the *real axis*). A point is selected to represent 0 and another to represent 1, as shown in Fig. 1.1. This choice determines the scale. Under an appropriate set of axioms for Euclidean geometry, each point on the real line corresponds to one and only one real number and, conversely, each real number is represented by one and only one point on the line. It is customary to refer to the *point x* rather than the point representing the real number x.

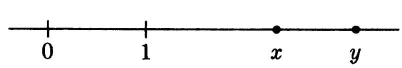

Figure 1.1

The order relation has a simple geometric interpretation. If $x < y$, the point x lies to the left of the point y, as shown in Fig. 1.1. Positive numbers lie to the right of 0, and negative numbers to the left of 0. If $a < b$, a point x satisfies the inequalities $a < x < b$ if and only if x is *between* a and b.

1.5 INTERVALS

The set of all points between a and b is called an *interval*. Sometimes it is important to distinguish between intervals which include their endpoints and intervals which do not.

NOTATION. The notation $\{x : x \text{ satisfies } P\}$ will be used to designate the set of all real numbers x which satisfy property P.

Definition 1.2. *Assume $a < b$. The open interval (a, b) is defined to be the set*

$$(a, b) = \{x : a < x < b\}.$$

The closed interval $[a, b]$ is the set $\{x : a \le x \le b\}$. The half-open intervals $(a, b]$ and $[a, b)$ are similarly defined, using the inequalities $a < x \le b$ and $a \le x < b$, respectively. Infinite intervals are defined as follows:

$$(a, +\infty) = \{x : x > a\}, \qquad [a, +\infty) = \{x : x \ge a\},$$

$$(-\infty, a) = \{x : x < a\}, \qquad (-\infty, a] = \{x : x \le a\}.$$

The real line **R** is sometimes referred to as the open interval $(-\infty, +\infty)$. A single point is considered as a "degenerate" closed interval.

NOTE. The symbols $+\infty$ and $-\infty$ are used here purely for convenience in notation and are not to be considered as being real numbers. Later we shall extend the real-number system to include these two symbols, but until this is done, the reader should understand that all real numbers are "finite."

1.6 INTEGERS

This section describes the *integers*, a special subset of **R**. Before we define the integers it is convenient to introduce first the notion of an *inductive set.*

Definition 1.3. *A set of real numbers is called an inductive set if it has the following two properties:*

a) *The number* 1 *is in the set.*

b) *For every x in the set, the number $x + 1$ is also in the set.*

For example, **R** is an inductive set. So is the set $\mathbf{R}^+$. Now we shall define the positive integers to be those real numbers which belong to every inductive set.

Definition 1.4. *A real number is called a positive integer if it belongs to every inductive set. The set of positive integers is denoted by $\mathbf{Z}^+$.*

The set $\mathbf{Z}^+$ is itself an inductive set. It contains the number 1, the number $1 + 1$ (denoted by 2), the number $2 + 1$ (denoted by 3), and so on. Since $\mathbf{Z}^+$ is a subset of every inductive set, we refer to $\mathbf{Z}^+$ as the *smallest* inductive set. This property of $\mathbf{Z}^+$ is sometimes called the *principle of induction.* We assume the reader is familiar with proofs by induction which are based on this principle. (See Reference 1.1.) Examples of such proofs are given in the next section.

The negatiyes of the positive integers are called the *negative integers.* The positive integers, together with the negative integers and 0 (zero), form a set **Z** which we call simply the *set of integers.*

1.7 THE UNIQUE FACTORIZATION THEOREM FOR INTEGERS

If n and d are integers and if $n = cd$ for some integer c, we say d is a *divisor* of n, or n is a *multiple* of d, and we write $d|n$ (read: d divides n). An integer n is called a *prime* if $n > 1$ and if the only positive divisors of n are 1 and n. If $n > 1$ and n is not prime, then n is called *composite*. The integer 1 is neither prime nor composite.

This section derives some elementary results on factorization of integers, culminating in the *unique factorization theorem*, also called *the fundamental theorem of arithmetic*.

The fundamental theorem states that (1) every integer $n > 1$ can be represented as a product of prime factors, and (2) this factorization can be done in only one way, apart from the order of the factors. It is easy to prove part (1).

Theorem 1.5. *Every integer $n > 1$ is either a prime or a product of primes.*

Proof. We use induction on n. The theorem holds trivially for $n = 2$. Assume it is true for every integer k with $1 < k < n$. If n is not prime it has a positive divisor d with $1 < d < n$. Hence $n = cd$, where $1 < c < n$. Since both c and d are $<n$, each is a prime or a product of primes; hence n is a product of primes.

Before proving part (2), uniqueness of the factorization, we introduce some further concepts.

If $d|a$ and $d|b$ we say d is a *common divisor* of a and b. The next theorem shows that every pair of integers a and b has a common divisor which is a linear combination of a and b.

Theorem 1.6. *Every pair of integers a and b has a common divisor d of the form*

$$d = ax + by$$

where x and y are integers. Moreover, every common divisor of a and b divides this d.

Proof. First assume that $a \geq 0$, $b \geq 0$ and use induction on $n = a + b$. If $n = 0$ then $a = b = 0$, and we can take $d = 0$ with $x = y = 0$. Assume, then, that the theorem has been proved for $0, 1, 2, \ldots, n - 1$. By symmetry, we can assume $a \geq b$. If $b = 0$ take $d = a$, $x = 1$, $y = 0$. If $b \geq 1$ we can apply the induction hypothesis to $a - b$ and b, since their sum is $a = n - b \leq n - 1$. Hence there is a common divisor d of $a - b$ and b of the form $d = (a - b)x + by$. This d also divides $(a - b) + b = a$, so d is a common divisor of a and b and we have $d = ax + (y - x)b$, a linear combination of a and b. To complete the proof we need to show that every common divisor divides d. Since a common divisor divides a and b, it also divides the linear combination $ax + (y - x)b = d$. This completes the proof if $a \geq 0$ and $b \geq 0$. If one or both of a and b is negative, apply the result just proved to $|a|$ and $|b|$.

NOTE. If d is a common divisor of a and b of the form $d = ax + by$, then $-d$ is also a divisor of the same form, $-d = a(-x) + b(-y)$. Of these two common divisors, the nonnegative one is called the *greatest common divisor* of a and b, and is denoted by $\gcd(a, b)$ or, simply by (a, b). If $(a, b) = 1$ then a and b are said to be *relatively prime*.

Theorem 1.7 (Euclid's Lemma). *If $a|bc$ and $(a, b) = 1$, then $a|c$.*

Proof. Since $(a, b) = 1$ we can write $1 = ax + by$. Therefore $c = acx + bcy$. But $a|acx$ and $a|bcy$, so $a|c$.

Theorem 1.8. *If a prime p divides ab, then $p|a$ or $p|b$. More generally, if a prime p divides a product $a_1 \cdots a_k$, then p divides at least one of the factors.*

Proof. Assume $p|ab$ and that p does not divide a. If we prove that $(p, a) = 1$, then Euclid's Lemma implies $p|b$. Let $d = (p, a)$. Then $d|p$ so $d = 1$ or $d = p$. We cannot have $d = p$ because $d|a$ but p does not divide a. Hence $d = 1$. To prove the more general statement we use induction on k, the number of factors. Details are left to the reader.

Theorem 1.9 (Unique factorization theorem). *Every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.*

Proof. We use induction on n. The theorem is true for $n = 2$. Assume, then, that it is true for all integers greater than 1 and less than n. If n is prime there is nothing more to prove. Therefore assume that n is composite and that n has two factorizations into prime factors, say

$$n = p_1 p_2 \cdots p_s = q_1 q_2 \cdots q_t. \tag{2}$$

We wish to show that $s = t$ and that each p equals some q. Since p_1 divides the product $q_1 q_2 \cdots q_t$, it divides at least one factor. Relabel the q's if necessary so that $p_1 | q_1$. Then $p_1 = q_1$ since both p_1 and q_1 are primes. In (2) we cancel p_1 on both sides to obtain

$$\frac{n}{p_1} = p_2 \cdots p_s = q_2 \cdots q_t.$$

Since n is composite, $1 < n/p_1 < n$; so by the induction hypothesis the two factorizations of n/p_1 are identical, apart from the order of the factors. Therefore the same is true in (2) and the proof is complete.

1.8 RATIONAL NUMBERS

Quotients of integers a/b (where $b \neq 0$) are called *rational numbers.* For example, $1/2$, $-7/5$, and 6 are rational numbers. The set of rational numbers, which we denote by $\mathbf{Q}$, contains $\mathbf{Z}$ as a subset. The reader should note that all the field axioms and the order axioms are satisfied by $\mathbf{Q}$.

We assume that the reader is familiar with certain elementary properties of rational numbers. For example, if a and b are rational, their average $(a + b)/2$ is also rational and lies between a and b. Therefore between any two rational numbers there are infinitely many rational numbers, which implies that if we are given a certain rational number we cannot speak of the "next largest" rational number.

1.9 IRRATIONAL NUMBERS

Real numbers that are not rational are called *irrational*. For example, the numbers $\sqrt{2}$, e, π and e^{π} are irrational.

Ordinarily it is not too easy to prove that some particular number is irrational. There is no simple proof, for example, of the irrationality of e^{π}. However, the irrationality of certain numbers such as $\sqrt{2}$ and $\sqrt{3}$ is not too difficult to establish and, in fact, we easily prove the following:

Theorem 1.10. *If n is a positive integer which is not a perfect square, then $\sqrt{n}$ is irrational.*

Proof. Suppose first that n contains no square factor >1. We assume that $\sqrt{n}$ is rational and obtain a contradiction. Let $\sqrt{n} = a/b$, where a and b are integers having no factor in common. Then $nb^2 = a^2$ and, since the left side of this equation is a multiple of n, so too is a^2. However, if a^2 is a multiple of n, a itself must be a multiple of n, since n has no square factors >1. (This is easily seen by examining the factorization of a into its prime factors.) This means that $a = cn$, where c is some integer. Then the equation $nb^2 = a^2$ becomes $nb^2 = c^2n^2$, or $b^2 = nc^2$. The same argument shows that b must also be a multiple of n. Thus a and b are both multiples of n, which contradicts the fact that they have no factor in common. This completes the proof if n has no square factor >1.

If n has a square factor, we can write $n = m^2k$, where $k > 1$ and k has no square factor >1. Then $\sqrt{n} = m\sqrt{k}$; and if $\sqrt{n}$ were rational, the number $\sqrt{k}$ would also be rational, contradicting that which was just proved.

A different type of argument is needed to prove that the number e is irrational. (We assume familiarity with the exponential e^x from elementary calculus and its representation as an infinite series.)

Theorem 1.11. *If $e^x = 1 + x + x^2/2! + x^3/3! + \cdots + x^n/n! + \cdots$, then the number e is irrational.*

Proof. We shall prove that e^{-1} is irrational. The series for e^{-1} is an alternating series with terms which decrease steadily in absolute value. In such an alternating series the error made by stopping at the nth term has the algebraic sign of the first neglected term and is less in absolute value than the first neglected term. Hence, if $s_n = \sum_{k=0}^{n} (-1)^k/k!$, we have the inequality

$$0 < e^{-1} - s_{2k-1} < \frac{1}{(2k)!},$$

from which we obtain

$$0 < (2k - 1)!\,(e^{-1} - s_{2k-1}) < \frac{1}{2k} \le \frac{1}{2}, \tag{3}$$

for any integer $k \ge 1$. Now $(2k - 1)!\, s_{2k-1}$ is always an integer. If e^{-1} were rational, then we could choose k so large that $(2k - 1)!\, e^{-1}$ would also be an integer. Because of (3) the difference of these two integers would be a number between 0 and $\frac{1}{2}$, which is impossible. Thus e^{-1} cannot be rational, and hence e cannot be rational.

NOTE. For a proof that π is irrational, see Exercise 7.33.

The ancient Greeks were aware of the existence of irrational numbers as early as 500 B.C. However, a satisfactory theory of such numbers was not developed until late in the nineteenth century, at which time three different theories were introduced by Cantor, Dedekind, and Weierstrass. For an account of the theories of Dedekind and Cantor and their equivalence, see Reference 1.6.

1.10 UPPER BOUNDS, MAXIMUM ELEMENT, LEAST UPPER BOUND (SUPREMUM)

Irrational numbers arise in algebra when we try to solve certain quadratic equations. For example, it is desirable to have a real number x such that $x^2 = 2$. From the nine axioms listed above we cannot prove that such an x exists in $\mathbf{R}$ because these nine axioms are also satisfied by $\mathbf{Q}$ and we have shown that there is no rational number whose square is 2. The completeness axiom allows us to introduce irrational numbers in the real-number system, and it gives the real-number system a property of continuity that is fundamental to many theorems in analysis.

Before we describe the completeness axiom, it is convenient to introduce additional terminology and notation.

Definition 1.12. *Let S be a set of real numbers. If there is a real number b such that $x \le b$ for every x in S, then b is called an upper bound for S and we say that S is bounded above by b.*

We say *an* upper bound because every number greater than b will also be an upper bound. If an upper bound b is also a member of S, then b is called the *largest member* or the *maximum element* of S. There can be at most one such b. If it exists, we write

$$b = \max S.$$

A set with no upper bound is said to be *unbounded above*.

Definitions of the terms *lower bound, bounded below, smallest member* (or *minimum element*) can be similarly formulated. If S has a minimum element we denote it by $\min S$.

Examples

1. The set $\mathbf{R}^+ = (0, +\infty)$ is unbounded above. It has no upper bounds and no maximum element. It is bounded below by 0 but has no minimum element.
2. The closed interval $S = [0, 1]$ is bounded above by 1 and is bounded below by 0. In fact, $\max S = 1$ and $\min S = 0$.
3. The half-open interval $S = [0, 1)$ is bounded above by 1 but it has no maximum element. Its minimum element is 0.

For sets like the one in Example 3, which are bounded above but have no maximum element, there is a concept which takes the place of the maximum element. It is called the *least upper bound* or *supremum* of the set and is defined as follows:

Definition 1.13. *Let S be a set of real numbers bounded above. A real number b is called a least upper bound for S if it has the following two properties:*

a) *b is an upper bound for S.*

b) *No number less than b is an upper bound for S.*

Examples. If $S = [0, 1]$ the maximum element 1 is also a least upper bound for S. If $S = [0, 1)$ the number 1 is a least upper bound for S, even though S has no maximum element.

It is an easy exercise to prove that a set cannot have two different least upper bounds. Therefore, if there is a least upper bound for S, there is *only* one and we can speak of *the* least upper bound.

It is common practice to refer to the least upper bound of a set by the more concise term *supremum*, abbreviated *sup*. We shall adopt this convention and write

$$b = \sup S$$

to indicate that b is the supremum of S. If S has a maximum element, then $\max S = \sup S$.

The *greatest lower bound*, or *infimum* of S, denoted by $\inf S$, is defined in an analogous fashion.

1.11 THE COMPLETENESS AXIOM

Our final axiom for the real number system involves the notion of supremum.

Axiom 10. *Every nonempty set S of real numbers which is bounded above has a supremum; that is, there is a real number b such that $b = \sup S$.*

As a consequence of this axiom it follows that every nonempty set of real numbers which is bounded below has an infimum.

1.12 SOME PROPERTIES OF THE SUPREMUM

This section discusses some fundamental properties of the supremum that will be useful in this text. There is a corresponding set of properties of the infimum that the reader should formulate for himself.

The first property shows that a set with a supremum contains numbers arbitrarily close to its supremum.

Theorem 1.14 (Approximation property). *Let S be a nonempty set of real numbers with a supremum, say $b = \sup S$. Then for every $a < b$ there is some x in S such that*

$$a < x \leq b.$$

Proof. First of all, $x \leq b$ for all x in S. If we had $x \leq a$ for every x in S, then a would be an upper bound for S smaller than the least upper bound. Therefore $x > a$ for at least one x in S.

Theorem 1.15 (Additive property). *Given nonempty subsets A and B of $\mathbf{R}$, let C denote the set*

$$C = \{x + y : x \in A, \quad y \in B\}.$$

If each of A and B has a supremum, then C has a supremum and

$$\sup C = \sup A + \sup B.$$

Proof. Let $a = \sup A$, $b = \sup B$. If $z \in C$ then $z = x + y$, where $x \in A$, $y \in B$, so $z = x + y \leq a + b$. Hence $a + b$ is an upper bound for C, so C has a supremum, say $c = \sup C$, and $c \leq a + b$. We show next that $a + b \leq c$. Choose any $\varepsilon > 0$. By Theorem 1.14 there is an x in A and a y in B such that

$$a - \varepsilon < x \quad \text{and} \quad b - \varepsilon < y.$$

Adding these inequalities we find

$$a + b - 2\varepsilon < x + y \leq c.$$

Thus, $a + b < c + 2\varepsilon$ for every $\varepsilon > 0$ so, by Theorem 1.1, $a + b \leq c$.

The proof of the next theorem is left as an exercise for the reader.

Theorem 1.16 (Comparison property). *Given nonempty subsets S and T of $\mathbf{R}$ such that $s \leq t$ for every s in S and t in T. If T has a supremum then S has a supremum and*

$$\sup S \leq \sup T.$$

1.13 PROPERTIES OF THE INTEGERS DEDUCED FROM THE COMPLETENESS AXIOM

Theorem 1.17. *The set $\mathbf{Z}^+$ of positive integers* 1, 2, 3, ... *is unbounded above.*

Proof. If $\mathbf{Z}^+$ were bounded above then $\mathbf{Z}^+$ would have a supremum, say $a = \sup \mathbf{Z}^+$. By Theorem 1.14 we would have $a - 1 < n$ for some n in $\mathbf{Z}^+$. Then $n + 1 > a$ for this n. Since $n + 1 \in \mathbf{Z}^+$ this contradicts the fact that $a = \sup \mathbf{Z}^+$.

Theorem 1.18. *For every real x there is a positive integer n such that $n > x$.*

Proof. If this were not true, some x would be an upper bound for $\mathbf{Z}^+$, contradicting Theorem 1.17.

1.14 THE ARCHIMEDEAN PROPERTY OF THE REAL NUMBER SYSTEM

The next theorem describes the Archimedean property of the real number system. Geometrically, it tells us that any line segment, no matter how long, can be covered by a finite number of line segments of a given positive length, no matter how small.

Theorem 1.19. *If $x > 0$ and if y is an arbitrary real number, there is a positive integer n such that $nx > y$.*

Proof. Apply Theorem 1.18 with x replaced by y/x.

1.15 RATIONAL NUMBERS WITH FINITE DECIMAL REPRESENTATION

A real number of the form

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n},$$

where a_0 is a nonnegative integer and $a_1, \ldots, a_n$ are integers satisfying $0 \le a_i \le 9$, is usually written more briefly as follows:

$$r = a_0 . a_1 a_2 \cdots a_n.$$

This is said to be a *finite decimal representation* of r. For example,

$$\frac{1}{2} = \frac{5}{10} = 0.5, \qquad \frac{1}{50} = \frac{2}{10^2} = 0.02, \qquad \frac{29}{4} = 7 + \frac{2}{10} + \frac{5}{10^2} = 7.25.$$

Real numbers like these are necessarily rational and, in fact, they all have the form $r = a/10^n$, where a is an integer. However, not all rational numbers can be expressed with finite decimal representations. For example, if $\frac{1}{3}$ could be so expressed, then we would have $\frac{1}{3} = a/10^n$ or $3a = 10^n$ for some integer a. But this is impossible since 3 does not divide any power of 10.

1.16 FINITE DECIMAL APPROXIMATIONS TO REAL NUMBERS

This section uses the completeness axiom to show that real numbers can be approximated to any desired degree of accuracy by rational numbers with finite decimal representations.

Theorem 1.20. *Assume $x \ge 0$. Then for every integer $n \ge 1$ there is a finite decimal $r_n = a_0 . a_1 a_2 \cdots a_n$ such that*

$$r_n \le x < r_n + \frac{1}{10^n}.$$

Proof. Let S be the set of all nonnegative integers $\le x$. Then S is nonempty, since $0 \in S$, and S is bounded above by x. Therefore S has a supremum, say $a_0 = \sup S$. It is easily verified that $a_0 \in S$, so a_0 is a nonnegative integer. We call a_0 the *greatest integer* in x, and we write $a_0 = [x]$. Clearly, we have

$$a_0 \le x < a_0 + 1.$$

Now let $a_1 = [10x - 10a_0]$, the greatest integer in $10x - 10a_0$. Since $0 \le 10x - 10a_0 = 10(x - a_0) < 10$, we have $0 \le a_1 \le 9$ and

$$a_1 \le 10x - 10a_0 < a_1 + 1.$$

In other words, a_1 is the largest integer satisfying the inequalities

$$a_0 + \frac{a_1}{10} \le x < a_0 + \frac{a_1 + 1}{10}.$$

More generally, having chosen $a_1, \ldots, a_{n-1}$ with $0 \le a_i \le 9$, let a_n be the largest integer satisfying the inequalities

$$a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} \le x < a_0 + \frac{a_1}{10} + \cdots + \frac{a_n + 1}{10^n}. \tag{4}$$

Then $0 \le a_n \le 9$ and we have

$$r_n \le x < r_n + \frac{1}{10^n},$$

where $r_n = a_0 . a_1 a_2 \cdots a_n$. This completes the proof. It is easy to verify that x is actually the supremum of the set of rational numbers $r_1, r_2, \ldots$.

1.17 INFINITE DECIMAL REPRESENTATIONS OF REAL NUMBERS

The integers $a_0, a_1, a_2, \ldots$ obtained in the proof of Theorem 1.20 can be used to define an infinite decimal representation of x. We write

$$x = a_0 . a_1 a_2 \cdots$$

to mean that a_n is the largest integer satisfying (4). For example, if $x = \frac{1}{8}$ we find $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and $a_n = 0$ for all $n \ge 4$. Therefore we can write

$$\tfrac{1}{8} = 0.125000 \cdots$$

If we interchange the inequality signs $\le$ and $<$ in (4), we obtain a slightly different definition of decimal expansions. The finite decimals r_n satisfy $r_n < x \le r_n + 10^{-n}$ although the digits $a_0, a_1, a_2, \ldots$ need not be the same as those in (4). For example, if we apply this second definition to $x = \frac{1}{8}$ we find the infinite decimal representation

$$\tfrac{1}{8} = 0.124999 \cdots$$

The fact that a real number might have two different decimal representations is merely a reflection of the fact that two different sets of real numbers can have the same supremum.

1.18 ABSOLUTE VALUES AND THE TRIANGLE INEQUALITY

Calculations with inequalities arise quite frequently in analysis. They are of particular importance in dealing with the notion of absolute value. If x is any real number, the absolute value of x, denoted by $|x|$, is defined as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

A fundamental inequality concerning absolute values is given in the following:

Theorem 1.21. *If $a \geq 0$, then we have the inequality $|x| \leq a$ if, and only if, $-a \leq x \leq a$.*

Proof. From the definition of $|x|$, we have the inequality $-|x| \leq x \leq |x|$, since $x = |x|$ or $x = -|x|$. If we assume that $|x| \leq a$, then we can write $-a \leq -|x| \leq x \leq |x| \leq a$ and thus half of the theorem is proved. Conversely, let us assume $-a \leq x \leq a$. Then if $x \geq 0$, we have $|x| = x \leq a$, whereas if $x < 0$, we have $|x| = -x \leq a$. In either case we have $|x| \leq a$ and the theorem is proved.

We can use this theorem to prove the *triangle inequality.*

Theorem 1.22. *For arbitrary real x and y we have*

$$|x + y| \leq |x| + |y| \qquad \textit{(the triangle inequality).}$$

Proof. We have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Addition gives us $-(|x| + |y|) \leq x + y \leq |x| + |y|$, and from Theorem 1.21 we conclude that $|x + y| \leq |x| + |y|$. This proves the theorem.

The triangle inequality is often used in other forms. For example, if we take $x = a - c$ and $y = c - b$ in Theorem 1.22 we find

$$|a - b| \leq |a - c| + |c - b|.$$

Also, from Theorem 1.22 we have $|x| \geq |x + y| - |y|$. Taking $x = a + b$, $y = -b$, we obtain

$$|a + b| \geq |a| - |b|.$$

Interchanging a and b we also find $|a + b| \geq |b| - |a| = -(|a| - |b|)$, and hence

$$|a + b| \geq \big||a| - |b|\big|.$$

By induction we can also prove the generalizations

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

and

$$|x_1 + x_2 + \cdots + x_n| \geq |x_1| - |x_2| - \cdots - |x_n|.$$

1.19 THE CAUCHY–SCHWARZ INEQUALITY

We shall now derive another inequality which is often used in analysis.

Theorem 1.23 (Cauchy–Schwarz inequality). *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are arbitrary real numbers, we have*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

Moreover, if some $a_i \neq 0$ equality holds if and only if there is a real x such that $a_k x + b_k = 0$ for each $k = 1, 2, \ldots, n$.

Proof. A sum of squares can never be negative. Hence we have

$$\sum_{k=1}^{n} (a_k x + b_k)^2 \ge 0$$

for every real x, with equality if and only if each term is zero. This inequality can be written in the form

$$Ax^2 + 2Bx + C \ge 0,$$

where

$$A = \sum_{k=1}^{n} a_k^2, \qquad B = \sum_{k=1}^{n} a_k b_k, \qquad C = \sum_{k=1}^{n} b_k^2.$$

If $A > 0$, put $x = -B/A$ to obtain $B^2 - AC \le 0$, which is the desired inequality. If $A = 0$, the proof is trivial.

NOTE. In vector notation the Cauchy–Schwarz inequality takes the form

$$(\mathbf{a} \cdot \mathbf{b})^2 \le \|\mathbf{a}\|^2 \|\mathbf{b}\|^2,$$

where $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ are two n-dimensional vectors,

$$\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^{n} a_k b_k,$$

is their dot product, and $\|\mathbf{a}\| = (\mathbf{a} \cdot \mathbf{a})^{1/2}$ is the length of $\mathbf{a}$.

1.20 PLUS AND MINUS INFINITY AND THE EXTENDED REAL NUMBER SYSTEM R*

Next we extend the real number system by adjoining two "ideal points" denoted by the symbols $+\infty$ and $-\infty$ ("plus infinity" and "minus infinity").

Definition 1.24. *By the extended real number system* $\mathbf{R}^*$ *we shall mean the set of real numbers* $\mathbf{R}$ *together with two symbols $+\infty$ and $-\infty$ which satisfy the following properties:*

a) *If $x \in \mathbf{R}$, then we have*

$$x + (+\infty) = +\infty, \qquad x + (-\infty) = -\infty,$$
$$x - (+\infty) = -\infty, \qquad x - (-\infty) = +\infty,$$
$$x/(+\infty) = x/(-\infty) = 0.$$

b) *If* $x > 0$, *then we have*

$$x(+\infty) = +\infty, \qquad x(-\infty) = -\infty.$$

c) *If* $x < 0$, *then we have*

$$x(+\infty) = -\infty, \qquad x(-\infty) = +\infty.$$

d) $(+\infty) + (+\infty) = (+\infty)(+\infty) = (-\infty)(-\infty) = +\infty,$
$(-\infty) + (-\infty) = (+\infty)(-\infty) = -\infty.$

e) *If* $x \in \mathbf{R}$, *then we have* $-\infty < x < +\infty$.

NOTATION. We denote $\mathbf{R}$ by $(-\infty, +\infty)$ and $\mathbf{R}^*$ by $[-\infty, +\infty]$. The points in $\mathbf{R}$ are called "finite" to distinguish them from the "infinite" points $+\infty$ and $-\infty$.

The principal reason for introducing the symbols $+\infty$ and $-\infty$ is one of convenience. For example, if we define $+\infty$ to be the sup of a set of real numbers which is not bounded above, then every nonempty subset of $\mathbf{R}$ has a supremum in $\mathbf{R}^*$. The sup is finite if the set is bounded above and infinite if it is not bounded above. Similarly, we define $-\infty$ to be the inf of any set of real numbers which is not bounded below. Then every nonempty subset of $\mathbf{R}$ has an inf in $\mathbf{R}^*$.

For some of the later work concerned with limits, it is also convenient to introduce the following terminology.

Definition 1.25. *Every open interval* $(a, +\infty)$ *is called a neighborhood of* $+\infty$ *or a ball with center* $+\infty$. *Every open interval* $(-\infty, a)$ *is called a neighborhood of* $-\infty$ *or a ball with center* $-\infty$.

1.21 COMPLEX NUMBERS

It follows from the axioms governing the relation $<$ that the square of a real number is never negative. Thus, for example, the elementary quadratic equation $x^2 = -1$ has no solution among the real numbers. New types of numbers, called *complex numbers*, have been introduced to provide solutions to such equations. It turns out that the introduction of complex numbers provides, at the same time, solutions to general algebraic equations of the form

$$a_0 + a_1x + \cdots + a_nx^n = 0,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are arbitrary real numbers. (This fact is known as the *Fundamental Theorem of Algebra.*)

We shall now define complex numbers and discuss them in further detail.

Definition 1.26. *By a complex number we shall mean an ordered pair of real numbers which we denote by* (x_1, x_2). *The first member,* x_1, *is called the real part of the complex number; the second member,* x_2, *is called the imaginary part. Two complex numbers* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *are called equal, and we write* $x = y$, *if,* *and only if,* $x_1 = y_1$ *and* $x_2 = y_2$. *We define the sum* $x + y$ *and the product* xy *by the equations*

$$x + y = (x_1 + y_1, x_2 + y_2), \qquad xy = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1).$$

NOTE. The set of all complex numbers will be denoted by **C**.

Theorem 1.27. *The operations of addition and multiplication just defined satisfy the commutative, associative, and distributive laws.*

Proof. We prove only the distributive law; proofs of the others are simpler. If $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$, then we have

$$\begin{aligned} x(y + z) &= (x_1, x_2)(y_1 + z_1, y_2 + z_2) \\ &= (x_1 y_1 + x_1 z_1 - x_2 y_2 - x_2 z_2, x_1 y_2 + x_1 z_2 + x_2 y_1 + x_2 z_1) \\ &= (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1) + (x_1 z_1 - x_2 z_2, x_1 z_2 + x_2 z_1) \\ &= xy + xz. \end{aligned}$$

Theorem 1.28.

$$(x_1, x_2) + (0, 0) = (x_1, x_2), \qquad (x_1, x_2)(0, 0) = (0, 0),$$

$$(x_1, x_2)(1, 0) = (x_1, x_2), \qquad (x_1, x_2) + (-x_1, -x_2) = (0, 0).$$

Proof. The proofs here are immediate from the definition, as are the proofs of Theorems 1.29, 1.30, 1.32, and 1.33.

Theorem 1.29. *Given two complex numbers* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$, *there exists a complex number* z *such that* $x + z = y$. *In fact,* $z = (y_1 - x_1, y_2 - x_2)$. *This* z *is denoted by* $y - x$. *The complex number* $(-x_1, -x_2)$ *is denoted by* $-x$.

Theorem 1.30. *For any two complex numbers* x *and* y, *we have*

$$(-x)y = x(-y) = -(xy) = (-1, 0)(xy).$$

Definition 1.31. *If* $x = (x_1, x_2) \neq (0, 0)$ *and* y *are complex numbers, we define* $x^{-1} = [x_1/(x_1^2 + x_2^2), -x_2/(x_1^2 + x_2^2)]$, *and* $y/x = yx^{-1}$.

Theorem 1.32. *If* x *and* y *are complex numbers with* $x \neq (0, 0)$, *there exists a complex number* z *such that* $xz = y$, *namely,* $z = yx^{-1}$.

Of special interest are operations with complex numbers whose imaginary part is 0.

Theorem 1.33.

$$\begin{aligned} (x_1, 0) + (y_1, 0) &= (x_1 + y_1, 0), \\ (x_1, 0)(y_1, 0) &= (x_1 y_1, 0), \\ (x_1, 0)/(y_1, 0) &= (x_1/y_1, 0), \qquad \textit{if } y_1 \neq 0. \end{aligned}$$

NOTE. It is evident from Theorem 1.33 that we can perform arithmetic operations on complex numbers with zero imaginary part by performing the usual real-number operations on the real parts alone. Hence the complex numbers of the form $(x, 0)$ have the same arithmetic properties as the real numbers. For this reason it is convenient to think of the real number system as being a special case of the complex number system, and we agree to identify the complex number $(x, 0)$ and the real number x. Therefore, we write $x = (x, 0)$. In particular, $0 = (0, 0)$ and $1 = (1, 0)$.

1.22 GEOMETRIC REPRESENTATION OF COMPLEX NUMBERS

Just as real numbers are represented geometrically by points on a line, so complex numbers are represented by points in a plane. The complex number $x = (x_1, x_2)$ can be thought of as the "point" with coordinates (x_1, x_2). When this is done, the definition of addition amounts to addition by the *parallelogram law*. (See Fig. 1.2.)

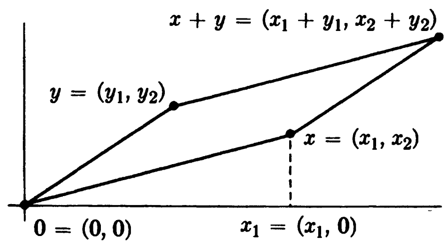

Figure 1.2

The idea of expressing complex numbers geometrically as points on a plane was formulated by Gauss in his dissertation in 1799 and, independently, by Argand in 1806. Gauss later coined the somewhat unfortunate phrase "complex number." Other geometric interpretations of complex numbers are possible. Instead of using points on a plane, we can use points on other surfaces. Riemann found the sphere particularly convenient for this purpose. Points of the sphere are projected from the North Pole onto the tangent plane at the South Pole and thus there corresponds to each point of the plane a definite point of the sphere. With the exception of the North Pole itself, each point of the sphere corresponds to exactly one point of the plane. This correspondence is called a *stereographic projection*. (See Fig. 1.3.)

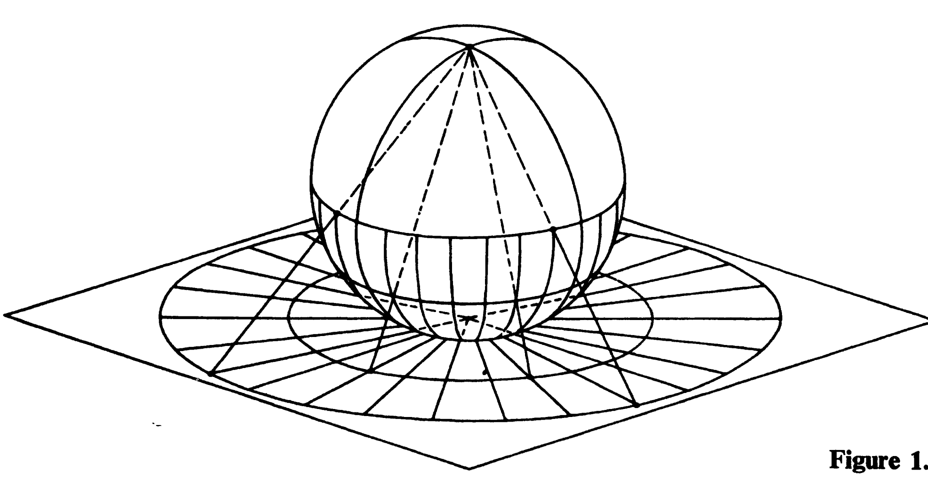

Figure 1.3

1.23 THE IMAGINARY UNIT

It is often convenient to think of the complex number (x_1, x_2) as a two-dimensional vector with components x_1 and x_2. Adding two complex numbers by means of Definition 1.26 is then the same as adding two vectors component by component. The complex number $1 = (1, 0)$ plays the same role as a unit vector in the horizontal direction. The analog of a unit vector in the vertical direction will now be introduced.

Definition 1.34. *The complex number* $(0, 1)$ *is denoted by i and is called the imaginary unit.*

Theorem 1.35. *Every complex number* $x = (x_1, x_2)$ *can be represented in the form* $x = x_1 + ix_2$.

Proof. $x_1 = (x_1, 0), \quad ix_2 = (0, 1)(x_2, 0) = (0, x_2),$
$x_1 + ix_2 = (x_1, 0) + (0, x_2) = (x_1, x_2).$

The next theorem tells us that the complex number i provides us with a solution to the equation $x^2 = -1$.

Theorem 1.36. $i^2 = -1$.

Proof. $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$.

1.24 ABSOLUTE VALUE OF A COMPLEX NUMBER

We now extend the concept of absolute value to the complex number system.

Definition 1.37. *If* $x = (x_1, x_2)$, *we define the modulus, or absolute value, of x to be the nonnegative real number $|x|$ given by*

$$|x| = \sqrt{x_1^2 + x_2^2}\,.$$

Theorem 1.38.

i) $|(0, 0)| = 0$, *and* $|x| > 0$ *if* $x \neq 0$.
ii) $|xy| = |x|\,|y|$.
iii) $|x/y| = |x|/|y|$, *if* $y \neq 0$.
iv) $|(x_1, 0)| = |x_1|$.

Proof. Statements (i) and (iv) are immediate. To prove (ii), we write $x = x_1 + ix_2$, $y = y_1 + iy_2$, so that $xy = x_1y_1 - x_2y_2 + i(x_1y_2 + x_2y_1)$. Statement (ii) follows from the relation

$$|xy|^2 = x_1^2y_1^2 + x_2^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2) = |x|^2|y|^2.$$

Equation (iii) can be derived from (ii) by writing it in the form $|x| = |y|\,|x/y|$.

Geometrically, $|x|$ represents the length of the segment joining the origin to the point x. More generally, $|x - y|$ is the distance between the points x and y. Using this geometric interpretation, the following theorem states that one side of a triangle is less than the sum of the other two sides.

Theorem 1.39. *If x and y are complex numbers, then we have*

$$|x + y| \le |x| + |y| \qquad (\textit{triangle inequality}).$$

The proof is left as an exercise for the reader.

1.25 IMPOSSIBILITY OF ORDERING THE COMPLEX NUMBERS

As yet we have not defined a relation of the form $x < y$ if x and y are arbitrary complex numbers, for the reason that it is impossible to give a definition of $<$ for complex numbers which will have all the properties in Axioms 6 through 8. To illustrate, suppose we were able to define an order relation $<$ satisfying Axioms 6, 7, and 8. Then, since $i \ne 0$, we must have either $i > 0$ or $i < 0$, by Axiom 6. Let us assume $i > 0$. Then taking, $x = y = i$ in Axiom 8, we get $i^2 > 0$, or $-1 > 0$. Adding 1 to both sides (Axiom 7), we get $0 > 1$. On the other hand, applying Axiom 8 to $-1 > 0$ we find $1 > 0$. Thus we have both $0 > 1$ and $1 > 0$, which, by Axiom 6, is impossible. Hence the assumption $i > 0$ leads us to a contradiction. [Why was the inequality $-1 > 0$ not already a contradiction?] A similar argument shows that we cannot have $i < 0$. Hence the complex numbers cannot be ordered in such a way that Axioms 6, 7, and 8 will be satisfied.

1.26 COMPLEX EXPONENTIALS

The exponential e^x (x real) was mentioned earlier. We now wish to define e^z when z is a complex number in such a way that the principal properties of the real exponential function will be preserved. The main properties of e^x for x real are the law of exponents, $e^{x_1}e^{x_2} = e^{x_1+x_2}$, and the equation $e^0 = 1$. We shall give a definition of e^z for complex z which preserves these properties and reduces to the ordinary exponential when z is real.

If we write $z = x + iy$ (x, y real), then for the law of exponents to hold we want $e^{x+iy} = e^x e^{iy}$. It remains, therefore, to define what we shall mean by e^{iy}.

Definition 1.40. *If $z = x + iy$, we define $e^z = e^{x+iy}$ to be the complex number* $e^z = e^x (\cos y + i \sin y)$.

This definition* agrees with the real exponential function when z is real (that is, $y = 0$). We prove next that the law of exponents still holds.

---

* Several arguments can be given to motivate the equation $e^{iy} = \cos y + i \sin y$. For example, let us write $e^{iy} = f(y) + ig(y)$ and try to determine the real-valued functions f and g so that the usual rules of operating with real exponentials will also apply to complex exponentials. Formal differentiation yields $e^{iy} = g'(y) - if'(y)$, if we assume that $(e^{iy})' = ie^{iy}$. Comparing the two expressions for e^{iy}, we see that f and g must satisfy the equations $f(y) = g'(y)$, $f'(y) = -g(y)$. Elimination of g yields $f(y) = -f''(y)$. Since we want $e^0 = 1$, we must have $f(0) = 1$ and $f'(0) = 0$. It follows that $f(y) = \cos y$ and $g(y) = -f'(y) = \sin y$. Of course, this argument *proves* nothing, but it strongly suggests that the definition $e^{iy} = \cos y + i \sin y$ is reasonable.

Theorem 1.41. *If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then we have*

$$e^{z_1}e^{z_2} = e^{z_1+z_2}.$$

Proof.

$$e^{z_1} = e^{x_1}(\cos y_1 + i \sin y_1), \qquad e^{z_2} = e^{x_2}(\cos y_2 + i \sin y_2),$$

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}e^{x_2}[\cos y_1 \cos y_2 - \sin y_1 \sin y_2 \\ &\quad + i(\cos y_1 \sin y_2 + \sin y_1 \cos y_2)]. \end{aligned}$$

Now $e^{x_1}e^{x_2} = e^{x_1+x_2}$, since x_1 and x_2 are both real. Also,

$$\cos y_1 \cos y_2 - \sin y_1 \sin y_2 = \cos (y_1 + y_2)$$

and

$$\cos y_1 \sin y_2 + \sin y_1 \cos y_2 = \sin (y_1 + y_2),$$

and hence

$$e^{z_1}e^{z_2} = e^{x_1+x_2}[\cos (y_1 + y_2) + i \sin (y_1 + y_2)] = e^{z_1+z_2}.$$

1.27 FURTHER PROPERTIES OF COMPLEX EXPONENTIALS

In the following theorems, z, z_1, z_2 denote complex numbers.

Theorem 1.42. *e^z is never zero.*

Proof. $e^z e^{-z} = e^0 = 1$. Hence e^z cannot be zero.

Theorem 1.43. *If x is real, then $|e^{ix}| = 1$.*

Proof. $|e^{ix}|^2 = \cos^2 x + \sin^2 x = 1$, and $|e^{ix}| > 0$.

Theorem 1.44. *$e^z = 1$ if, and only if, z is an integral multiple of $2\pi i$.*

Proof. If $z = 2\pi i n$, where n is an integer, then

$$e^z = \cos (2\pi n) + i \sin (2\pi n) = 1.$$

Conversely, suppose that $e^z = 1$. This means that $e^x \cos y = 1$ and $e^x \sin y = 0$. Since $e^x \neq 0$, we must have $\sin y = 0$, $y = k\pi$, where k is an integer. But $\cos (k\pi) = (-1)^k$. Hence $e^x = (-1)^k$, since $e^x \cos (k\pi) = 1$. Since $e^x > 0$, k must be even. Therefore $e^x = 1$ and hence $x = 0$. This proves the theorem.

Theorem 1.45. *$e^{z_1} = e^{z_2}$ if, and only if, $z_1 - z_2 = 2\pi i n$ (where n is an integer).*

Proof. $e^{z_1} = e^{z_2}$ if, and only if, $e^{z_1 - z_2} = 1$.

1.28 THE ARGUMENT OF A COMPLEX NUMBER

If the point $z = (x, y) = x + iy$ is represented by polar coordinates r and θ, we can write $x = r \cos \theta$ and $y = r \sin \theta$, so that $z = r \cos \theta + ir \sin \theta = re^{i\theta}$.

The two numbers r and θ uniquely determine z. Conversely, the positive number r is uniquely determined by z; in fact, $r = |z|$. However, z determines the angle θ only up to multiples of 2π. There are infinitely many values of θ which satisfy the equations $x = |z| \cos \theta$, $y = |z| \sin \theta$ but, of course, any two of them differ by some multiple of 2π. Each such θ is called an *argument* of z but *one* of these values is singled out and is called the *principal argument* of z.

Definition 1.46. *Let $z = x + iy$ be a nonzero complex number. The unique real number θ which satisfies the conditions*

$$x = |z| \cos \theta, \qquad y = |z| \sin \theta, \qquad -\pi < \theta \leq +\pi$$

is called the principal argument of z, denoted by $\theta = \arg(z)$.

The above discussion immediately yields the following theorem:

Theorem 1.47. *Every complex number $z \neq 0$ can be represented in the form $z = re^{i\theta}$, where $r = |z|$ and $\theta = \arg(z) + 2\pi n$, n being any integer.*

NOTE. This method of representing complex numbers is particularly useful in connection with multiplication and division, since we have

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad \text{and} \qquad \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

Theorem 1.48. *If $z_1 z_2 \neq 0$, then $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2\pi n(z_1, z_2)$, where*

$$n(z_1, z_2) = \begin{cases} 0, & \text{if } -\pi < \arg(z_1) + \arg(z_2) \leq +\pi, \\ +1, & \text{if } -2\pi < \arg(z_1) + \arg(z_2) \leq -\pi, \\ -1, & \text{if } \quad \pi < \arg(z_1) + \arg(z_2) \leq 2\pi. \end{cases}$$

Proof. Write $z_1 = |z_1| e^{i\theta_1}$, $z_2 = |z_2| e^{i\theta_2}$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$. Then $z_1 z_2 = |z_1 z_2| e^{i(\theta_1 + \theta_2)}$. Since $-\pi < \theta_1 \leq +\pi$ and $-\pi < \theta_2 \leq +\pi$, we have $-2\pi < \theta_1 + \theta_2 \leq 2\pi$. Hence there is an integer n such that $-\pi < \theta_1 + \theta_2 + 2n\pi \leq \pi$. This n is the same as the integer $n(z_1, z_2)$ given in the theorem, and for this n we have $\arg(z_1 z_2) = \theta_1 + \theta_2 + 2\pi n$. This proves the theorem.

1.29 INTEGRAL POWERS AND ROOTS OF COMPLEX NUMBERS

Definition 1.49. *Given a complex number z and an integer n, we define the* nth *power of z as follows:*

$$z^0 = 1, \qquad z^{n+1} = z^n z, \qquad \text{if } n \geq 0,$$

$$z^{-n} = (z^{-1})^n, \qquad \text{if } z \neq 0 \text{ and } n > 0.$$

Theorem 1.50, which states that the usual laws of exponents hold, can be proved by mathematical induction. The proof is left as an exercise.

Theorem 1.50. *Given two integers m and n, we have, for $z \neq 0$,*

$$z^n z^m = z^{n+m} \quad \text{and} \quad (z_1 z_2)^n = z_1^n z_2^n.$$

Theorem 1.51. *If $z \neq 0$, and if n is a positive integer, then there are exactly n distinct complex numbers $z_0, z_1, \ldots, z_{n-1}$ (called the* n*th* roots of *z), such that*

$$z_k^n = z, \qquad \text{for each } k = 0, 1, 2, \ldots, n-1.$$

Furthermore, these roots are given by the formulas

$$z_k = Re^{i\phi_k}, \qquad \text{where} \qquad R = |z|^{1/n},$$

and

$$\phi_k = \frac{\arg(z)}{n} + \frac{2\pi k}{n} \qquad (k = 0, 1, 2, \ldots, n-1).$$

NOTE. The n nth roots of z are equally spaced on the circle of radius $R = |z|^{1/n}$, center at the origin.

Proof. The n complex numbers $Re^{i\phi_k}$, $0 \le k \le n-1$, are distinct and each is an nth root of z, since

$$(Re^{i\phi_k})^n = R^n e^{in\phi_k} = |z| e^{i[\arg(z) + 2\pi k]} = z.$$

We must now show that there are no other nth roots of z. Suppose $w = Ae^{i\alpha}$ is a complex number such that $w^n = z$. Then $|w|^n = |z|$, and hence $A^n = |z|$, $A = |z|^{1/n}$. Therefore, $w^n = z$ can be written $e^{in\alpha} = e^{i[\arg(z)]}$, which implies

$$n\alpha - \arg(z) = 2\pi k \qquad \text{for some integer } k.$$

Hence $\alpha = [\arg(z) + 2\pi k]/n$. But when k runs through all integral values, w takes only the distinct values $z_0, \ldots, z_{n-1}$. (See Fig. 1.4.)

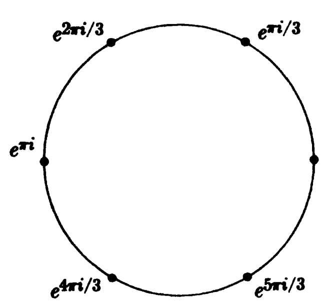

Figure 1.4

1.30 COMPLEX LOGARITHMS

By Theorem 1.42, e^z is never zero. It is natural to ask if there are other values that e^z cannot assume. The next theorem shows that zero is the only exceptional value.

Theorem 1.52. *If z is a complex number $\neq 0$, then there exist complex numbers w such that $e^w = z$. One such w is the complex number*

$$\log|z| + i\arg(z),$$

and any other such w must have the form

$$\log|z| + i\arg(z) + 2n\pi i,$$

where n is an integer.

Proof. Since $e^{\log|z|+i\arg(z)} = e^{\log|z|}e^{i\arg(z)} = |z|e^{i\arg(z)} = z$, we see that $w = \log|z| + i\arg(z)$ is a solution of the equation $e^w = z$. But if w_1 is any other solution, then $e^w = e^{w_1}$ and hence $w - w_1 = 2n\pi i$.

Definition 1.53. *Let $z \neq 0$ be a given complex number. If w is a complex number such that $e^w = z$, then w is called a logarithm of z. The particular value of w given by*

$$w = \log|z| + i\arg(z)$$

is called the principal logarithm of z, and for this w we write

$$w = \operatorname{Log} z.$$

Examples

1. Since $|i| = 1$ and $\arg(i) = \pi/2$, $\operatorname{Log}(i) = i\pi/2$.
2. Since $|-i| = 1$ and $\arg(-i) = -\pi/2$, $\operatorname{Log}(-i) = -i\pi/2$.
3. Since $|-1| = 1$ and $\arg(-1) = \pi$, $\operatorname{Log}(-1) = \pi i$.
4. If $x > 0$, $\operatorname{Log}(x) = \log x$, since $|x| = x$ and $\arg(x) = 0$.
5. Since $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \pi/4$, $\operatorname{Log}(1 + i) = \log\sqrt{2} + i\pi/4$.

Theorem 1.54. *If $z_1z_2 \neq 0$, then*

$$\operatorname{Log}(z_1z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2\pi i n(z_1, z_2),$$

where $n(z_1, z_2)$ is the integer defined in Theorem 1.48.

Proof.

$$\begin{aligned}\operatorname{Log}(z_1z_2) &= \log|z_1z_2| + i\arg(z_1z_2)\\ &= \log|z_1| + \log|z_2| + i[\arg(z_1) + \arg(z_2) + 2\pi n(z_1, z_2)].\end{aligned}$$

1.31 COMPLEX POWERS

Using complex logarithms, we can now give a definition of complex powers of complex numbers.

Definition 1.55. *If $z \neq 0$ and if w is any complex number, we define*

$$z^w = e^{w \operatorname{Log} z}.$$

Examples

1. $i^i = e^{i \operatorname{Log} i} = e^{i(i\pi/2)} = e^{-\pi/2}$.
2. $(-1)^i = e^{i \operatorname{Log}(-1)} = e^{i(i\pi)} = e^{-\pi}$.
3. If n is an integer, then $z^{n+1} = e^{(n+1)\operatorname{Log} z} = e^{n \operatorname{Log} z} e^{\operatorname{Log} z} = z^n z$, so Definition 1.55 does not conflict with Definition 1.49.

The next two theorems give rules for calculating with complex powers:

Theorem 1.56. *$z^{w_1} z^{w_2} = z^{w_1+w_2}$ if $z \neq 0$.*

Proof. $z^{w_1+w_2} = e^{(w_1+w_2)\operatorname{Log} z} = e^{w_1 \operatorname{Log} z} e^{w_2 \operatorname{Log} z} = z^{w_1} z^{w_2}$.

Theorem 1.57. *If $z_1 z_2 \neq 0$, then*

$$(z_1 z_2)^w = z_1^w z_2^w e^{2\pi i w\, n(z_1, z_2)},$$

where $n(z_1, z_2)$ is the integer defined in Theorem 1.48.

Proof. $(z_1 z_2)^w = e^{w \operatorname{Log}(z_1 z_2)} = e^{w[\operatorname{Log} z_1 + \operatorname{Log} z_2 + 2\pi i\, n(z_1, z_2)]}$.

1.32 COMPLEX SINES AND COSINES

Definition 1.58. *Given a complex number z, we define*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

NOTE. When z is real, these equations agree with Definition 1.40.

Theorem 1.59. *If $z = x + iy$, then we have*

$$\cos z = \cos x \cosh y - i \sin x \sinh y,$$
$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

Proof.

$$\begin{aligned} 2 \cos z &= e^{iz} + e^{-iz} \\ &= e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x) \\ &= \cos x(e^y + e^{-y}) - i \sin x(e^y - e^{-y}) \\ &= 2 \cos x \cosh y - 2i \sin x \sinh y. \end{aligned}$$

The proof for $\sin z$ is similar.

Further properties of sines and cosines are given in the exercises.

1.33 INFINITY AND THE EXTENDED COMPLEX PLANE C*

Next we extend the complex number system by adjoining an ideal point denoted by the symbol ∞.

Definition 1.60. *By the extended complex number system* $\mathbf{C}^*$ *we shall mean the complex plane* $\mathbf{C}$ *along with a symbol ∞ which satisfies the following properties:*

a) *If $z \in \mathbf{C}$, then we have $z + \infty = z - \infty = \infty$, $z/\infty = 0$.*

b) *If* $z \in \mathbf{C}$, *but* $z \neq 0$, *then* $z(\infty) = \infty$ *and* $z/0 = \infty$.

c) $\infty + \infty = (\infty)(\infty) = \infty$.

Definition 1.61. *Every set in* $\mathbf{C}$ *of the form* $\{z : |z| > r \geq 0\}$ *is called a neighborhood of* ∞, *or a ball with center at* ∞.

The reader may wonder why two symbols, $+\infty$ and $-\infty$, are adjoined to $\mathbf{R}$ but only one symbol, ∞, is adjoined to $\mathbf{C}$. The answer lies in the fact that there is an ordering relation $<$ among the real numbers, but no such relation occurs among the complex numbers. In order that certain properties of real numbers involving the relation $<$ hold without exception, we need two symbols, $+\infty$ and $-\infty$, as defined above. We have already mentioned that in $\mathbf{R}^*$ every nonempty set has a sup, for example.

In $\mathbf{C}$ it turns out to be more convenient to have just one ideal point. By way of illustration, let us recall the stereographic projection which establishes a one-to-one correspondence between the points of the complex plane and those points on the surface of the sphere distinct from the North Pole. The apparent exception at the North Pole can be removed by regarding it as the geometric representative of the ideal point ∞. We then get a one-to-one correspondence between the extended complex plane $\mathbf{C}^*$ and the total surface of the sphere. It is geometrically evident that if the South Pole is placed on the origin of the complex plane, the exterior of a "large" circle in the plane will correspond, by stereographic projection, to a "small" spherical cap about the North Pole. This illustrates vividly why we have defined a neighborhood of ∞ by an inequality of the form $|z| > r$.

EXERCISES

Integers

1.1 Prove that there is no largest prime. (A proof was known to Euclid.)

1.2 If n is a positive integer, prove the algebraic identity

$$a^n - b^n = (a - b)\sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

1.3 If $2^n - 1$ is prime, prove that n is prime. A prime of the form $2^p - 1$, where p is prime, is called a *Mersenne prime*.

1.4 If $2^n + 1$ is prime, prove that n is a power of 2. A prime of the form $2^{2^m} + 1$ is called a *Fermat prime*. *Hint.* Use Exercise 1.2.

1.5 The *Fibonacci numbers* 1, 1, 2, 3, 5, 8, 13, ... are defined by the recursion formula $x_{n+1} = x_n + x_{n-1}$, with $x_1 = x_2 = 1$. Prove that $(x_n, x_{n+1}) = 1$ and that $x_n = (a^n - b^n)/(a - b)$, where a and b are the roots of the quadratic equation $x^2 - x - 1 = 0$.

1.6 Prove that every nonempty set of positive integers contains a smallest member. This is called the *well-ordering principle*.

Rational and irrational numbers

1.7 Find the rational number whose decimal expansion is $0.3344444\ldots$

1.8 Prove that the decimal expansion of x will end in zeros (or in nines) if, and only if, x is a rational number whose denominator is of the form 2^n5^m, where m and n are nonnegative integers.

1.9 Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

1.10 If a, b, c, d are rational and if x is irrational, prove that $(ax + b)/(cx + d)$ is usually irrational. When do exceptions occur?

1.11 Given any real $x > 0$, prove that there is an irrational number between 0 and x.

1.12 If $a/b < c/d$ with $b > 0$, $d > 0$, prove that $(a + c)/(b + d)$ lies between a/b and c/d.

1.13 Let a and b be positive integers. Prove that $\sqrt{2}$ always lies between the two fractions a/b and $(a + 2b)/(a + b)$. Which fraction is closer to $\sqrt{2}$?

1.14 Prove that $\sqrt{n - 1} + \sqrt{n + 1}$ is irrational for every integer $n \geq 1$.

1.15 Given a real x and an integer $N > 1$, prove that there exist integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$. *Hint.* Consider the $N + 1$ numbers $tx - [tx]$ for $t = 0, 1, 2, \ldots, N$ and show that some pair differs by at most $1/N$.

1.16 If x is irrational prove that there are infinitely many rational numbers h/k with $k > 0$ such that $|x - h/k| < 1/k^2$. *Hint.* Assume there are only a finite number $h_1/k_1, \ldots, h_r/k_r$ and obtain a contradiction by applying Exercise 1.15 with $N > 1/\delta$, where δ is the smallest of the numbers $|x - h_i/k_i|$.

1.17 Let x be a positive rational number of the form

$$x = \sum_{k=1}^{n} \frac{a_k}{k!},$$

where each a_k is a nonnegative integer with $a_k \leq k - 1$ for $k \geq 2$ and $a_n > 0$. Let $[x]$ denote the greatest integer in x. Prove that $a_1 = [x]$, that $a_k = [k!\,x] - k[(k - 1)!\,x]$ for $k = 2, \ldots, n$, and that n is the smallest integer such that $n!\,x$ is an integer. Conversely, show that every positive rational number x can be expressed in this form in one and only one way.

Upper bounds

1.18 Show that the sup and inf of a set are uniquely determined whenever they exist.

1.19 Find the sup and inf of each of the following sets of real numbers:

a) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$, where p, q, and r take on all positive integer values.

b) $S = \{x : 3x^2 - 10x + 3 < 0\}$.

c) $S = \{x : (x - a)(x - b)(x - c)(x - d) < 0\}$, where $a < b < c < d$.

1.20 Prove the comparison property for suprema (Theorem 1.16).

1.21 Let A and B be two sets of positive numbers bounded above, and let $a = \sup A$, $b = \sup B$. Let C be the set of all products of the form xy, where $x \in A$ and $y \in B$. Prove that $ab = \sup C$.

1.22 Given $x > 0$ and an integer $k \geq 2$. Let a_0 denote the largest integer $\leq x$ and, assuming that $a_0, a_1, \ldots, a_{n-1}$ have been defined, let a_n denote the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \ldots + \frac{a_n}{k^n} \leq x.$$

a) Prove that $0 \leq a_i \leq k - 1$ for each $i = 1, 2, \ldots$

b) Let $r_n = a_0 + a_1 k^{-1} + a_2 k^{-2} + \cdots + a_n k^{-n}$ and show that x is the sup of the set of rational numbers $r_1, r_2, \ldots$

NOTE. When $k = 10$ the integers $a_0, a_1, a_2, \ldots$ are the digits in a decimal representation of x. For general k they provide a representation in the scale of k.

Inequalities

1.23 Prove Lagrange's identity for real numbers:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2.$$

Note that this identity implies the Cauchy–Schwarz inequality.

1.24 Prove that for arbitrary real a_k, b_k, c_k we have

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \leq \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^2\right)^2\left(\sum_{k=1}^{n} c_k^4\right).$$

1.25 Prove Minkowski's inequality:

$$\left(\sum_{k=1}^{n} (a_k + b_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}.$$

This is the triangle inequality $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$ for n-dimensional vectors, where $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ and

$$\|\mathbf{a}\| = \left(\sum_{k=1}^{n} a_k^2\right)^{1/2}.$$

1.26 If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, prove that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leq n \sum_{k=1}^{n} a_k b_k.$$

Hint. $\sum_{1 \leq j \leq k \leq n} (a_k - a_j)(b_k - b_j) \geq 0.$

Complex numbers

1.27 Express the following complex numbers in the form $a + bi$.

a) $(1 + i)^3$,
b) $(2 + 3i)/(3 - 4i)$,
c) $i^5 + i^{16}$,
d) $\frac{1}{2}(1 + i)(1 + i^{-8})$.

1.28 In each case, determine all real x and y which satisfy the given relation.

a) $x + iy = |x - iy|$, b) $x + iy = (x - iy)^2$, c) $\sum_{k=0}^{100} i^k = x + iy$.

1.29 If $z = x + iy$, x and y real, the complex conjugate of z is the complex number $\bar{z} = x - iy$. Prove that:

a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, b) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, c) $z\bar{z} = |z|^2$,

d) $z + \bar{z} =$ twice the real part of z,

e) $(z - \bar{z})/i =$ twice the imaginary part of z.

1.30 Describe geometrically the set of complex numbers z which satisfies each of the following conditions:

a) $|z| = 1$, b) $|z| < 1$, c) $|z| \le 1$,

d) $z + \bar{z} = 1$, e) $z - \bar{z} = i$, f) $\bar{z} + z = |z|^2$.

1.31 Given three complex numbers z_1, z_2, z_3 such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$. Show that these numbers are vertices of an equilateral triangle inscribed in the unit circle with center at the origin.

1.32 If a and b are complex numbers, prove that:

a) $|a - b|^2 \le (1 + |a|^2)(1 + |b|^2)$.

b) If $a \ne 0$, then $|a + b| = |a| + |b|$ if, and only if, b/a is real and nonnegative.

1.33 If a and b are complex numbers, prove that

$$|a - b| = |1 - \bar{a}b|$$

if, and only if, $|a| = 1$ or $|b| = 1$. For which a and b is the inequality $|a - b| < |1 - \bar{a}b|$ valid?

1.34 If a and c are real constants, b complex, show that the equation

$$az\bar{z} + b\bar{z} + \bar{b}z + c = 0 \qquad (a \ne 0,\ z = x + iy)$$

represents a circle in the xy-plane.

1.35 Recall the definition of the inverse tangent: given a real number t, $\tan^{-1}(t)$ is the unique real number θ which satisfies the two conditions

$$-\frac{\pi}{2} < \theta < +\frac{\pi}{2}, \qquad \tan\theta = t.$$

If $z = x + iy$, show that

a) $\arg(z) = \tan^{-1}\left(\dfrac{y}{x}\right)$, if $x > 0$,

b) $\arg(z) = \tan^{-1}\left(\dfrac{y}{x}\right) + \pi$, if $x < 0$, $y \ge 0$,

c) $\arg(z) = \tan^{-1}\left(\dfrac{y}{x}\right) - \pi$, if $x < 0$, $y < 0$,

d) $\arg(z) = \dfrac{\pi}{2}$ if $x = 0$, $y > 0$; $\arg(z) = -\dfrac{\pi}{2}$ if $x = 0$, $y < 0$.

1.36 Define the following "pseudo-ordering" of the complex numbers: we say $z_1 < z_2$ if we have either

i) $|z_1| < |z_2|$ or ii) $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$.

Which of Axioms 6, 7, 8, 9 are satisfied by this relation?

1.37 Which of Axioms 6, 7, 8, 9 are satisfied if the pseudo-ordering is defined as follows? We say $(x_1, y_1) < (x_2, y_2)$ if we have either

i) $x_1 < x_2$ or ii) $x_1 = x_2$ and $y_1 < y_2$.

1.38 State and prove a theorem analogous to Theorem 1.48, expressing $\arg(z_1/z_2)$ in terms of $\arg(z_1)$ and $\arg(z_2)$.

1.39 State and prove a theorem analogous to Theorem 1.54, expressing $\operatorname{Log}(z_1/z_2)$ in terms of $\operatorname{Log}(z_1)$ and $\operatorname{Log}(z_2)$.

1.40 Prove that the nth roots of 1 (also called the nth roots of unity) are given by α, $\alpha^2, \ldots, \alpha^n$, where $\alpha = e^{2\pi i/n}$, and show that the roots $\neq 1$ satisfy the equation

$$1 + x + x^2 + \cdots + x^{n-1} = 0.$$

1.41 a) Prove that $|z^i| < e^{\pi}$ for all complex $z \neq 0$.

b) Prove that there is no constant $M > 0$ such that $|\cos z| < M$ for all complex z.

1.42 If $w = u + iv$ (u, v real), show that

$$z^w = e^{u \log|z| - v \arg(z)} e^{i[v \log|z| + u \arg(z)]}.$$

1.43 a) Prove that $\operatorname{Log}(z^w) = w \operatorname{Log} z + 2\pi i n$, where n is an integer.

b) Prove that $(z^w)^\alpha = z^{w\alpha} e^{2\pi i n\alpha}$, where n is an integer.

1.44 i) If θ and a are real numbers, $-\pi < \theta \leq +\pi$, prove that

$$(\cos\theta + i\sin\theta)^a = \cos(a\theta) + i\sin(a\theta).$$

ii) Show that, in general, the restriction $-\pi < \theta \leq +\pi$ is necessary in (i) by taking $\theta = -\pi$, $a = \frac{1}{2}$.

iii) If a is an integer, show that the formula in (i) holds without any restriction on θ. In this case it is known as DeMoivre's theorem.

1.45 Use DeMoivre's theorem (Exercise 1.44) to derive the trigonometric identities

$$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta,$$
$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

valid for real θ. Are these valid when θ is complex?

1.46 Define $\tan z = (\sin z)/(\cos z)$ and show that for $z = x + iy$, we have

$$\tan z = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}.$$

1.47 Let w be a given complex number. If $w \neq \pm 1$, show that there exist two values of $z = x + iy$ satisfying the conditions $\cos z = w$ and $-\pi < x \leq +\pi$. Find these values when $w = i$ and when $w = 2$.

1.48 Prove Lagrange's identity for complex numbers:

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 = \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \sum_{1 \le k < j \le n} |a_k \bar{b}_j - a_j \bar{b}_k|^2.$$

Use this to deduce a Cauchy–Schwarz inequality for complex numbers.

1.49 a) By equating imaginary parts in DeMoivre's formula prove that

$$\sin n\theta = \sin^n \theta \left\{ \binom{n}{1} \cot^{n-1} \theta - \binom{n}{3} \cot^{n-3} \theta + \binom{n}{5} \cot^{n-5} \theta - + \cdots \right\}.$$

b) If $0 < \theta < \pi/2$, prove that

$$\sin (2m + 1)\theta = \sin^{2m+1}\theta \, P_m(\cot^2 \theta)$$

where P_m is the polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1} x^m - \binom{2m+1}{3} x^{m-1} + \binom{2m+1}{5} x^{m-2} - + \cdots$$

Use this to show that P_m has zeros at the m distinct points $x_k = \cot^2 \{\pi k/(2m + 1)\}$ for $k = 1, 2, \ldots, m$.

c) Show that the sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m + 1} = \frac{m(2m - 1)}{3},$$

and that the sum of their squares is given by

$$\sum_{k=1}^{m} \cot^4 \frac{\pi k}{2m + 1} = \frac{m(2m - 1)(4m^2 + 10m - 9)}{45}.$$

NOTE. These identities can be used to prove that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ and $\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$. (See Exercises 8.46 and 8.47.)

1.50 Prove that $z^n - 1 = \prod_{k=1}^{n} (z - e^{2\pi ik/n})$ for all complex z. Use this to derive the formula

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}} \qquad \text{for } n \ge 2.$$

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The Real And Complex Number Systems

Integers

1.1 Prove that there is no largest prime.

Proof: Suppose p is the largest prime. Then $p! + 1$ is **NOT** a prime. So, there exists a prime q such that

$$q \,|p! + 1 \Rightarrow q \,|1$$

which is impossible. So, there is no largest prime.

Remark: There are many and many proofs about it. The proof that we give comes from A**rchimedes 287-212 B. C.** In addition, **Euler Leonhard (1707-1783)** find another method to show it. The method is important since it develops to study the theory of numbers by analytic method. The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 91-93. (Chinese Version)**

1.2 If n is a positive integer, prove the algebraic identity

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k}$$

Proof: It suffices to show that

$$x^n - 1 = (x - 1) \sum_{k=0}^{n-1} x^k.$$

Consider the right hand side, we have

$$\begin{aligned}(x-1)\sum_{k=0}^{n-1}x^k &= \sum_{k=0}^{n-1}x^{k+1}-\sum_{k=0}^{n-1}x^k \\ &= \sum_{k=1}^{n}x^k-\sum_{k=0}^{n-1}x^k \\ &= x^n-1.\end{aligned}$$

1.3 If $2^n - 1$ is a prime, prove that n is prime. A prime of the form $2^p - 1$, where p is prime, is called a Mersenne prime.

Proof: If n is not a prime, then say $n = ab$, where $a > 1$ and $b > 1$. So, we have

$$2^{ab}-1=(2^a-1)\sum_{k=0}^{b-1}(2^a)^k$$

which is not a prime by **Exercise 1.2**. So, n must be a prime.

Remark: The study of **Mersenne prime** is important; it is related with so called **Perfect number**. In addition, there are some **OPEN** problem about it. For example, **is there infinitely many Mersenne nembers?** The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 13-15. (Chinese Version)**

1.4 If $2^n + 1$ is a prime, prove that n is a power of 2. A prime of the form $2^{2^m} + 1$ is called a **Fermat prime.** Hint. Use exercise 1.2.

Proof: If n is a not a power of 2, say $n = ab$, where b is an odd integer. So,

$$2^a+1 \,\big|\, 2^{ab}+1$$

and $2^a + 1 < 2^{ab} + 1$. It implies that $2^n + 1$ is not a prime. So, n must be a power of 2.

Remark: (1) In the proof, we use the identity

$$x^{2n-1}+1=(x+1)\sum_{k=0}^{2n-2}(-1)^k x^k.$$

Proof: Consider

$$\begin{aligned}(x+1)\sum_{k=0}^{2n-2}(-1)^k x^k &= \sum_{k=0}^{2n-2}(-1)^k x^{k+1} + \sum_{k=0}^{2n-2}(-1)^k x^k \\ &= \sum_{k=1}^{2n-1}(-1)^{k+1} x^k + \sum_{k=0}^{2n-2}(-1)^k x^k \\ &= x^{2n+1}+1.\end{aligned}$$

(2) The study of **Fermat number** is important; for the details the reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 15. (Chinese Version)**

1.5 The Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, ...$ are defined by the recursion formula $x_{n+1} = x_n + x_{n-1}$, with $x_1 = x_2 = 1$. Prove that $(x_n, x_{n+1}) = 1$ and that $x_n = (a^n - b^n) / (a - b)$, where a and b are the roots of the quadratic equation $x^2 - x - 1 = 0$.

Proof: Let $d = g.c.d.\,(x_n, x_{n+1})$, then

$$d\,|x_n \text{ and } d\,|x_{n+1} = x_n + x_{n-1}\ .$$

So,

$$d\,|x_{n-1}\ .$$

Continue the process, we finally have

$$d\,|1\ .$$

So, $d = 1$ since d is positive.

Observe that

$$x_{n+1} = x_n + x_{n-1},$$

and thus we consider

$$x^{n+1} = x^n + x^{n-1},$$

i.e., consider

$$x^2 = x + 1 \text{ with two roots, } a \text{ and } b.$$

If we let

$$F_n = (a^n - b^n) / (a - b)\,,$$

then it is clear that

$$F_1 = 1,\ F_2 = 1,\ \text{and } F_{n+1} = F_n + F_{n-1} \text{ for } n > 1.$$

So, $F_n = x_n$ for all n.

Remark: The study of the **Fibonacci numbers** is important; the reader can see the book, **Fibonacci and Lucas Numbers with Applications by Koshy and Thomas.**

1.6 Prove that every nonempty set of positive integers contains a smallest member. This is called the **well–ordering Principle.**

Proof: Given $(\phi \neq) S (\subseteq N)$, we prove that if S contains an integer k, then S contains the smallest member. We prove it by **Mathematical Induction of second form** as follows.

As $k = 1$, it trivially holds. Assume that as $k = 1, 2, ..., m$ holds, consider as $k = m + 1$ as follows. In order to show it, we consider two cases.

(1) If there is a member $s \in S$ such that $s < m + 1$, then by Induction hypothesis, we have proved it.

(2) If every $s \in S$, $s \geq m + 1$, then $m + 1$ is the smallest member.

Hence, by **Mathematical Induction**, we complete it.

Remark: We give a fundamental result to help the reader get more. We will prove the followings are equivalent:

(A. Well–ordering Principle) every nonempty set of positive integers contains a smallest member.

(B. Mathematical Induction of first form) Suppose that $S (\subseteq N)$, if S satisfies that

$$\begin{gathered}(1).\ 1 \text{ in } S\\(2).\ \text{As } k \in S,\ \text{then } k+1 \in S.\end{gathered}$$

Then $S = N$.

(C. Mathematical Induction of second form) Suppose that $S (\subseteq N)$, if S satisfies that

$$\begin{gathered}(1).\ 1 \text{ in } S\\(2).\ \text{As } 1, ..., k \in S,\ \text{then } k+1 \in S.\end{gathered}$$

Then $S = N$.

Proof: $(A \Rightarrow B)$: If $S \neq N$, then $N - S \neq \phi$. So, by (**A**), there exists the smallest integer w such that $w \in N - S$. Note that $w > 1$ by (1), so we consider $w - 1$ as follows.

Since $w - 1 \notin N - S$, we know that $w - 1 \in S$. By (2), we know that $w \in S$ which contadicts to $w \in N - S$. Hence, $S = N$.

$(B \Rightarrow C)$: It is obvious.

$(C \Rightarrow A)$: We have proved it by this exercise.

Rational and irrational numbers

1.7 Find the rational number whose decimal expansion is 0.3344444444....

Proof: Let $x = 0.3344444444...$, then

$$\begin{aligned} x &= \frac{3}{10} + \frac{3}{10^2} + \frac{4}{10^3} + ... + \frac{4}{10^n} + .., \text{ where } n \geq 3 \\ &= \frac{33}{10^2} + \frac{4}{10^3}\left(1 + \frac{1}{10} + ... + \frac{1}{10^n} + ..\right) \\ &= \frac{33}{10^2} + \frac{4}{10^3}\left(\frac{1}{1 - \frac{1}{10}}\right) \\ &= \frac{33}{10^2} + \frac{4}{900} \\ &= \frac{301}{900}. \end{aligned}$$

1.8 Prove that the decimal expansion of x will end in zeros (or in nines) if, and only if, x is a rational number whose denominator is of the form $2^n 5^m$, where m and n are nonnegative integers.

Proof: $(\Leftarrow)$Suppose that $x = \frac{k}{2^n 5^m}$, if $n \geq m$, we have

$$\frac{k5^{n-m}}{2^n 5^n} = \frac{5^{n-m}k}{10^n}.$$

So, the decimal expansion of x will end in zeros. Similarly for $m \geq n$.

$(\Rightarrow)$Suppose that the decimal expansion of x will end in zeros (or in nines).

For case $x = a_0.a_1a_2\cdots a_n$. Then

$$x = \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{10^n} = \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n}.$$

For case $x = a_0.a_1a_2\cdots a_n999999\cdots$. Then

$$\begin{aligned} x &= \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n} + \frac{9}{10^{n+1}} + ... + \frac{9}{10^{n+m}} + ... \\ &= \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n} + \frac{9}{10^{n+1}}\sum_{j=0}^{\infty} 10^{-j} \\ &= \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n} + \frac{1}{10^n} \\ &= \frac{1+\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n}. \end{aligned}$$

So, in both case, we prove that x is a rational number whose denominator is of the form 2^n5^m, where m and n are nonnegative integers.

1.9 Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

Proof: If $\sqrt{2}+\sqrt{3}$ is rational, then consider

$$\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right) = 1$$

which implies that $\sqrt{3}-\sqrt{2}$ is rational. Hence, $\sqrt{3}$ would be rational. It is impossible. So, $\sqrt{2}+\sqrt{3}$ is irrational.

Remark: (1)$\sqrt{p}$ is an irrational if p is a prime.

Proof: If $\sqrt{p} \in Q$, write $\sqrt{p} = \frac{a}{b}$, where $g.c.d.\,(a,b) = 1$. Then

$$b^2p = a^2 \Rightarrow p\left|a^2\right. \Rightarrow p\,|a \qquad (*)$$

Write $a = pq$. So,

$$b^2p = p^2q^2 \Rightarrow b^2 = pq^2 \Rightarrow p\left|b^2\right. \Rightarrow p\,|b\,. \qquad (*')$$

By (*) and (*'), we get

$$p\,|g.c.d.\,(a,b) = 1$$

which implies that $p = 1$, a contradiction. So, $\sqrt{p}$ is an irrational if p is a prime.

Note: There are many and many methods to prove it. For example, the reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 19-21. (Chinese Version)**

(2) Suppose $a, b \in N$. Prove that $\sqrt{a} + \sqrt{b}$ is rational if and only if, $a = k^2$ and $b = h^2$ for some $h, k \in N$.

Proof: ($\Leftarrow$) It is clear.
($\Rightarrow$) Consider

$$\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a^2 - b^2,$$

then $\sqrt{a} \in Q$ and $\sqrt{b} \in Q$. Then it is clear that $a = h^2$ and $b = h^2$ for some $h, k \in N$.

1.10 If a, b, c, d are rational and if x is irrational, prove that $(ax + b)/(cx + d)$ is usually irrational. When do exceptions occur?

Proof: We claim that $(ax + b)/(cx + d)$ is rational if and only if $ad = bc$.
($\Rightarrow$)If $(ax + b)/(cx + d)$ is rational, say $(ax + b)/(cx + d) = q/p$. We consider two cases as follows.
(i) If $q = 0$, then $ax + b = 0$. If $a \neq 0$, then x would be rational. So, $a = 0$ and $b = 0$. Hence, we have

$$ad = 0 = bc.$$

(ii) If $q \neq 0$, then $(pa - qc)x + (pb - qd) = 0$. If $pa - qc \neq 0$, then x would be rational. So, $pa - qc = 0$ and $pb - qd = 0$. It implies that

$$qcb = qad \Rightarrow ad = bc.$$

($\Leftarrow$)Suppose $ad = bc$. If $a = 0$, then $b = 0$ or $c = 0$. So,

$$\frac{ax + b}{cx + d} = \begin{cases} 0 \text{ if } a = 0 \text{ and } b = 0 \\ \frac{b}{d} \text{ if } a = 0 \text{ and } c = 0 \end{cases}.$$

If $a \neq 0$, then $d = bc/a$. So,

$$\frac{ax + b}{cx + d} = \frac{ax + b}{cx + bc/a} = \frac{a(ax + b)}{c(ax + b)} = \frac{a}{c}.$$

Hence, we proved that if $ad = bc$, then $(ax + b)/(cx + d)$ is rational.

1.11 Given any real $x > 0$, prove that there is an irrational number between 0 and x.

Proof: If $x \in Q^c$, we choose $y = x/2 \in Q^c$. Then $0 < y < x$. If $x \in Q$, we choose $y = x/\sqrt{2} \in Q$, then $0 < y < x$.

Remark: (1) There are many and many proofs about it. We may prove it by the concept of **Perfect set**. The reader can see the book, **Principles of Mathematical Analysis written by Walter Rudin, Theorem 2.43, pp 41.** Also see the textbook, **Exercise 3.25**.

(2) Given a and $b \in R$ with $a < b$, there exists $r \in Q^c$, and $q \in Q$ such that $a < r < b$ and $a < q < b$.

Proof: We show it by considering four cases. (i) $a \in Q$, $b \in Q$. (ii) $a \in Q$, $b \in Q^c$. (iii) $a \in Q^c$, $b \in Q$. (iv) $a \in Q^c$, $b \in Q^c$.

(i) ($a \in Q$, $b \in Q$) Choose $q = \frac{a+b}{2}$ and $r = \frac{1}{\sqrt{2}}a + \left(1 - \frac{1}{\sqrt{2}}\right) b$.

(ii) ($a \in Q$, $b \in Q^c$) Choose $r = \frac{a+b}{2}$ and let $c = \frac{1}{2^n} < b - a$, then $a + c := q$.

(iii) ($a \in Q^c$, $b \in Q$) Similarly for (iii).

(iv) ($a \in Q^c$, $b \in Q^c$) It suffices to show that there exists a rational number $q \in (a, b)$ by (ii). Write

$$b = b_0.b_1b_2 \cdots b_n \cdots$$

Choose n large enough so that

$$a < q = b_0.b_1b_2 \cdots b_n < b.$$

(It works since $b - q = 0.000..000b_{n+1}... \leq \frac{1}{10^n}$)

1.12 If $a/b < c/d$ with $b > 0$, $d > 0$, prove that $(a + c) / (b + d)$ lies bwtween the two fractions a/b and c/d

Proof: It only needs to conisder the substraction. So, we omit it.

Remark: The result of this exercise is often used, so we suggest the reader keep it in mind.

1.13 Let a and b be positive integers. Prove that $\sqrt{2}$ always lies between the two fractions a/b and $(a + 2b) / (a + b)$. Which fraction is closer to $\sqrt{2}$?

Proof: Suppose $a/b \leq \sqrt{2}$, then $a \leq \sqrt{2}b$. So,

$$\frac{a + 2b}{a + b} - \sqrt{2} = \frac{\left(\sqrt{2} - 1\right)\left(\sqrt{2}b - a\right)}{a + b} \geq 0.$$

In addition,

$$\begin{aligned}
\left(\sqrt{2}-\frac{a}{b}\right)-\left(\frac{a+2b}{a+b}-\sqrt{2}\right) &= 2\sqrt{2}-\left(\frac{a}{b}+\frac{a+2b}{a+b}\right) \\
&= 2\sqrt{2}-\frac{a^2+2ab+2b^2}{ab+b^2} \\
&= \frac{1}{ab+b^2}\left[\left(2\sqrt{2}-2\right)ab+\left(2\sqrt{2}-2\right)b^2-a^2\right] \\
&\geq \frac{1}{ab+b^2}\left[\left(2\sqrt{2}-2\right)a\frac{a}{\sqrt{2}}+\left(2\sqrt{2}-2\right)\left(\frac{a}{\sqrt{2}}\right)^2 - \right. \\
&= 0.
\end{aligned}$$

So, $\frac{a+2b}{a+b}$ is closer to $\sqrt{2}$.

Similarly, we also have if $a/b > \sqrt{2}$, then $\frac{a+2b}{a+b} < \sqrt{2}$. Also, $\frac{a+2b}{a+b}$ is closer to $\sqrt{2}$ in this case.

Remark: Note that

$$\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b} < \frac{2b}{a} \text{ by Exercise 12 and 13.}$$

And we know that $\frac{a+2b}{a+b}$ is closer to $\sqrt{2}$. We can use it to approximate $\sqrt{2}$. Similarly for the case

$$\frac{2b}{a} < \frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}.$$

1.14 Prove that $\sqrt{n-1}+\sqrt{n+1}$ is irrational for every integer $n \geq 1$.

Proof: Suppose that $\sqrt{n-1}+\sqrt{n+1}$ is rational, and thus consider

$$\left(\sqrt{n+1}+\sqrt{n-1}\right)\left(\sqrt{n+1}-\sqrt{n-1}\right) = 2$$

which implies that $\sqrt{n+1}-\sqrt{n-1}$ is rational. Hence, $\sqrt{n+1}$ and $\sqrt{n-1}$ are rational. So, $n-1=k^2$ and $n+1=h^2$, where k and h are positive integer. It implies that

$$h = \frac{3}{2} \text{ and } k = \frac{1}{2}$$

which is absurb. So, $\sqrt{n-1}+\sqrt{n+1}$ is irrational for every integer $n \geq 1$.

1.15 Given a real x and an integer $N > 1$, prove that there exist integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$. Hint. Consider the $N+1$ numbers $tx - [tx]$ for $t = 0, 1, 2, ..., N$ and show that some pair differs by at most $1/N$.

Proof: Given $N > 1$, and thus consider $tx - [tx]$ for $t = 0, 1, 2, ..., N$ as follows. Since

$$0 \leq tx - [tx] := a_t < 1,$$

so there exists two numbers a_i and a_j where $i \neq j$ such that

$$|a_i - a_j| < \frac{1}{N} \Rightarrow |(i - j)\, x - p| < \frac{1}{N}, \text{ where } p = [jx] - [ix]\,.$$

Hence, there exist integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$.

1.16 If x is irrational prove that there are infinitely many rational numbers h/k with $k > 0$ such that $|x - h/k| < 1/k^2$. Hint. Assume there are only a finite number $h_1/k_1, ..., h_r/k_r$ and obtain a contradiction by applying Exercise 1.15 with $N > 1/\delta$, where δ is the smallest of the numbers $|x - h_i/k_i|$.

Proof: Assume there are only a finite number $h_1/k_1, ..., h_r/k_r$ and let $\delta = \min_{i=1}^{r} |x - h_i/k_i| > 0$ since x is irrational. Choose $N > 1/\delta$, then by **Exercise 1.15**, we have

$$\frac{1}{N} < \delta \leq \left| x - \frac{h}{k} \right| < \frac{1}{kN}$$

which implies that

$$\frac{1}{N} < \frac{1}{kN}$$

which is impossible. So, there are infinitely many rational numbers h/k with $k > 0$ such that $|x - h/k| < 1/k^2$.

Remark: (1) There is another proof by **continued fractions.** The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 270. (Chinese Version)**

(2) The exercise is useful to help us show the following lemma. $\{ar + b : a \in Z, b$ where $r \in Q^c$ is dense in R. It is equivalent to $\{ar : a \in Z\}$, where $r \in Q^c$ is dense in $[0, 1]$ modulus 1.

Proof: Say $\{ar+b : a \in Z, b \in Z\} = S$, and since $r \in Q^c$, then by **Exercise 1.16**, there are infinitely many rational numbers h/k with $k > 0$ such that $|kr - h| < \frac{1}{k}$. Consider $(x-\delta, x+\delta) := I$, where $\delta > 0$, and thus choosing k_0 large enough so that $1/k_0 < \delta$. Define $L = |k_0 r - h_0|$, then we have $sL \in I$ for some $s \in Z$. So, $sL = (\pm)[(sk_0)r - (sh_0)] \in S$. That is, we have proved that S is dense in R.

1.17 Let x be a positive rational number of the form

$$x = \sum_{k=1}^{n} \frac{a_k}{k!},$$

where each a_k is nonnegative integer with $a_k \leq k-1$ for $k \geq 2$ and $a_n > 0$. Let $[x]$ denote the largest integer in x. Prove that $a_1 = [x]$, that $a_k = [k!x] - k[(k-1)!x]$ for $k = 2, ..., n$, and that n is the smallest integer such that $n!x$ is an integer. Conversely, show that every positive rational number x can be expressed in this form in one and only one way.

Proof: $(\Rightarrow)$First,

$$\begin{aligned}
[x] &= \left[a_1 + \sum_{k=2}^{n} \frac{a_k}{k!}\right] \\
&= a_1 + \left[\sum_{k=2}^{n} \frac{a_k}{k!}\right] \text{ since } a_1 \in N \\
&= a_1 \text{ since } \sum_{k=2}^{n} \frac{a_k}{k!} \leq \sum_{k=2}^{n} \frac{k-1}{k!} = \sum_{k=2}^{n} \frac{1}{(k-1)!} - \frac{1}{k!} = 1 - \frac{1}{n!} < 1.
\end{aligned}$$

Second, fixed k and consider

$$k!x = k! \sum_{j=1}^{n} \frac{a_j}{j!} = k! \sum_{j=1}^{k-1} \frac{a_j}{j!} + a_k + k! \sum_{j=k+1}^{n} \frac{a_j}{j!}$$

and

$$(k-1)!x = (k-1)! \sum_{j=1}^{n} \frac{a_j}{j!} = (k-1)! \sum_{j=1}^{k-1} \frac{a_j}{j!} + (k-1)! \sum_{j=k}^{n} \frac{a_j}{j!}.$$

So,

$$
\begin{aligned}
[k!x] &= \left[k!\sum_{j=1}^{k-1}\frac{a_j}{j!} + a_k + k!\sum_{j=k+1}^{n}\frac{a_j}{j!}\right] \\
&= k!\sum_{j=1}^{k-1}\frac{a_j}{j!} + a_k \text{ since } k!\sum_{j=k+1}^{n}\frac{a_j}{j!} < 1
\end{aligned}
$$

and

$$
\begin{aligned}
k\left[(k-1)!x\right] &= k\left[(k-1)!\sum_{j=1}^{k-1}\frac{a_j}{j!} + (k-1)!\sum_{j=k}^{n}\frac{a_j}{j!}.\right] \\
&= k\,(k-1)!\sum_{j=1}^{k-1}\frac{a_j}{j!} \text{ since } \ (k-1)!\sum_{j=k}^{n}\frac{a_j}{j!} < 1 \\
&= k!\sum_{j=1}^{k-1}\frac{a_j}{j!}
\end{aligned}
$$

which implies that

$$a_k = [k!x] - k\left[(k-1)!x\right] \ \text{ for } k = 2, ..., n.$$

Last, in order to show that n is the smallest integer such that $n!x$ is an integer. It is clear that

$$n!x = n!\sum_{k=1}^{n}\frac{a_k}{k!} \in Z.$$

In addition,

$$
\begin{aligned}
(n-1)!x &= (n-1)!\sum_{k=1}^{n}\frac{a_k}{k!} \\
&= (n-1)!\sum_{k=1}^{n-1}\frac{a_k}{k!} + \frac{a_n}{n} \\
&\notin Z \text{ since } \frac{a_n}{n} \notin Z.
\end{aligned}
$$

So, we have proved it.

($\Leftarrow$)It is clear since every a_n is uniquely deermined.

Upper bounds

1.18 Show that the sup and the inf of a set are uniquely determined whenever they exists.

Proof: Given a nonempty set $S\,(\subseteq R)$, and assume $\sup S = a$ and $\sup S = b$, we show $a = b$ as follows. Suppose that $a > b$, and thus choose $\varepsilon = \frac{a-b}{2}$, then there exists a $x \in S$ such that

$$b < \frac{a+b}{2} = a - \varepsilon < x < a$$

which implies that

$$b < x$$

which contradicts to $b = \sup S$. Similarly for $a < b$. Hence, $a = b$.

1.19 Find the sup and inf of each of the following sets of real numbers:

(a) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$, where p, q, and r take on all positive integer values.

Proof: Define $S = \{2^{-p} + 3^{-q} + 5^{-r} : p, q, r \in N\}$. Then it is clear that $\sup S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$, and $\inf S = 0$.

(b) $S = \{x : 3x^2 - 10x + 3 < 0\}$

Proof: Since $3x^2 - 10x + 3 = (x-3)(3x-1)$, we know that $S = \left(\frac{1}{3}, 3\right)$. Hence, $\sup S = 3$ and $\inf S = \frac{1}{3}$.

(c) $S = \{x : (x-a)(x-b)(x-c)(x-d) < 0\}$, where $a < b < c < d$.

Proof: It is clear that $S = (a, b) \cup (c, d)$. Hence, $\sup S = d$ and $\inf S = a$.

1.20 Prove the comparison property for suprema (**Theorem 1.16**)

Proof: Since $s \leq t$ for every $s \in S$ and $t \in T$, fixed $t_0 \in T$, then $s \leq t_0$ for all $s \in S$. Hence, by **Axiom 10**, we know that $\sup S$ exists. In addition, it is clear $\sup S \leq \sup T$.

Remark: There is a useful result, we write it as a reference. Let S and T be two nonempty subsets of R. If $S \subseteq T$ and $\sup T$ exists, then $\sup S$ exists and $\sup S \leq \sup T$.

Proof: Since $\sup T$ exists and $S \subseteq T$, we know that for every $s \in S$, we have

$$s \leq \sup T.$$

Hence, by **Axiom 10**, we have proved the existence of $\sup S$. In addition, $\sup S \leq \sup T$ is trivial.

1.21 Let A and B be two sets of positive numbers bounded above, and let $a = \sup A$, $b = \sup B$. Let C be the set of all products of the form xy, where $x \in A$ and $y \in B$. Prove that $ab = \sup C$.

Proof: Given $\varepsilon > 0$, we want to find an element $c \in C$ such that $ab - \varepsilon < c$. If we can show this, we have proved that $\sup C$ exists and equals ab.

Since $\sup A = a > 0$ and $\sup B = b > 0$, we can choose n large enough such that $a - \varepsilon/n > 0$, $b - \varepsilon/n > 0$, and $n > a + b$. So, for this $\varepsilon' = \varepsilon/n$, there exists $a' \in A$ and $b' \in B$ such that

$$a - \varepsilon' < a' \text{ and } b - \varepsilon' < b'$$

which implies that

$$ab - \varepsilon' \left(a + b - \varepsilon'\right) < a'b' \text{ since } a - \varepsilon' > 0 \text{ and } b - \varepsilon' > 0$$

which implies that

$$ab - \frac{\varepsilon}{n}(a + b) < a'b' := c$$

which implies that

$$ab - \varepsilon < c.$$

1.22 Given $x > 0$, and an integer $k \geq 2$. Let a_0 denote the largest integer $\leq x$ and, assumeing that $a_0, a_1, ..., a_{n-1}$ have been defined, let a_n denote the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} \leq x.$$

Note: When $k = 10$ the integers $a_0, a_1, ...$ are the digits in a decimal representation of x. For general k they provide a representation in the scale of k.

(a) Prove that $0 \leq a_i \leq k - 1$ for each $i = 1, 2, ...$

Proof: Choose $a_0 = [x]$, and thus consider

$$[kx - ka_0] := a_1$$

then

$$0 \leq k(x - a_0) < k \Rightarrow 0 \leq a_1 \leq k - 1$$

and

$$a_0 + \frac{a_1}{k} \leq x \leq a_0 + \frac{a_1}{k} + \frac{1}{k}.$$

Continue the process, we then have

$$0 \leq a_i \leq k - 1 \text{ for each } i = 1, 2, ...$$

and

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} \leq x < a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} + \frac{1}{k^n}. \qquad (*)$$

(b) Let $r_n = a_0 + a_1 k^{-1} + a_2 k^{-2} + ... + a_n k^{-n}$ and show that x is the sup of the set of rational numbers $r_1, r_2, ...$

Proof: It is clear by (a)-(*).

Inequality

1.23 Prove **Lagrange's identity** for real numbers:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2.$$

Note that this identity implies that **Cauchy-Schwarz inequality**.

Proof: Consider

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \sum_{1 \leq k, j \leq n} a_k^2 b_j^2 = \sum_{k=j} a_k^2 b_j^2 + \sum_{k \neq j} a_k^2 b_j^2 = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k \neq j} a_k^2 b_j^2$$

and

$$\left(\sum_{k=1}^{n} a_k b_k\right)\left(\sum_{k=1}^{n} a_k b_k\right) = \sum_{1\le k,j\le n} a_k b_k a_j b_j = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k\neq j} a_k b_k a_j b_j$$

So,

$$\begin{aligned}
\left(\sum_{k=1}^{n} a_k b_k\right)^2 &= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) + \sum_{k\neq j} a_k b_k a_j b_j - \sum_{k\neq j} a_k^2 b_j^2 \\
&= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) + 2\sum_{1\le k<j\le n} a_k b_k a_j b_j - \sum_{1\le k<j\le n} a_k^2 b_j^2 + a_j^2 b_k^2 \\
&= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\le k<j\le n} \left(a_k b_j - a_j b_k\right)^2 .
\end{aligned}$$

Remark: (1) The reader may recall the relation with **Cross Product** and **Inner Product**, we then have a fancy formula:

$$\|x \times y\|^2 + |< x, y >|^2 = \|x\|^2 \|y\|^2 ,$$

where $x, y \in R^3$.

(2) We often write

$$< a, b >:= \sum_{k=1}^{n} a_k b_k,$$

and the Cauchy-Schwarz inequality becomes

$$|< x, y >| \le \|x\| \|y\| \text{ by } \textbf{Remark } (1).$$

1.24 Prove that for arbitrary real a_k, b_k, c_k we have

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \le \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^2\right)^2\left(\sum_{k=1}^{n} c_k^4\right).$$

Proof: Use **Cauchy-Schwarz inequality** twice, we then have

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 &= \left[\left(\sum_{k=1}^{n} a_k b_k c_k\right)^2\right]^2 \\ &\leq \left(\sum_{k=1}^{n} a_k^2 c_k^2\right)^2 \left(\sum_{k=1}^{n} b_k^2\right)^2 \\ &\leq \left(\sum_{k=1}^{n} a_k^4\right)^2 \left(\sum_{k=1}^{n} c_k^4\right) \left(\sum_{k=1}^{n} b_k^2\right)^2 \\ &= \left(\sum_{k=1}^{n} a_k^4\right) \left(\sum_{k=1}^{n} b_k^2\right)^2 \left(\sum_{k=1}^{n} c_k^4\right).\end{aligned}$$

1.25 Prove that **Minkowski's inequality**:

$$\left(\sum_{k=1}^{n} (a_k + b_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}.$$

This is the triangle inequality $\|a+b\| \leq \|a\| + \|b\|$ for $n-$dimensional vectors, where $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and

$$\|a\| = \left(\sum_{k=1}^{n} a_k^2\right)^{1/2}.$$

Proof: Consider

$$\begin{aligned}\sum_{k=1}^{n} (a_k + b_k)^2 &= \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\sum_{k=1}^{n} a_k b_k \\ &\leq \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2} \quad \text{by Cauchy-Schwarz i} \\ &= \left[\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}\right]^2.\end{aligned}$$

So,

$$\left(\sum_{k=1}^{n}\left(a_k+b_k\right)^2\right)^{1/2} \leq \left(\sum_{k=1}^{n}a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n}b_k^2\right)^{1/2}.$$

1.26 If $a_1 \geq ... \geq a_n$ and $b_1 \geq ... \geq b_n$, prove that

$$\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right) \leq n\left(\sum_{k=1}^{n}a_kb_k\right).$$

Hint. $\sum_{1\leq j\leq k\leq n}\left(a_k-a_j\right)\left(b_k-b_j\right) \geq 0.$

Proof: Consider

$$0 \leq \sum_{1\leq j\leq k\leq n}\left(a_k-a_j\right)\left(b_k-b_j\right) = \sum_{1\leq j\leq k\leq n}a_kb_k+a_jb_j - \sum_{1\leq j\leq k\leq n}a_kb_j+a_jb_k$$

which implies that

$$\sum_{1\leq j\leq k\leq n}a_kb_j+a_jb_k \leq \sum_{1\leq j\leq k\leq n}a_kb_k+a_jb_j. \tag{*}$$

Since

$$\begin{aligned}\sum_{1\leq j\leq k\leq n}a_kb_j+a_jb_k &= \sum_{1\leq j<k\leq n}a_kb_j+a_jb_k+2\sum_{k=1}^{n}a_kb_k \\ &= \left(\sum_{1\leq j<k\leq n}a_kb_j+a_jb_k+\sum_{k=1}^{n}a_kb_k\right)+\sum_{k=1}^{n}a_kb_k \\ &= \left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right)+\sum_{k=1}^{n}a_kb_k,\end{aligned}$$

we then have, by (*)

$$\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right)+\sum_{k=1}^{n}a_kb_k \leq \sum_{1\leq j\leq k\leq n}a_kb_k+a_jb_j. \tag{**}$$

In addition,

$$\sum_{1\leq j\leq k\leq n} a_kb_k + a_jb_j$$

$$= \sum_{k=1}^{n} a_kb_k + na_1b_1 + \sum_{k=2}^{n} a_kb_k + (n-1)\,a_2b_2 + ... + \sum_{k=n-1}^{n} a_kb_k + 2a_{n-1}b_{n-1} + \sum_{k=n} a$$

$$= n\sum_{k=1}^{n} a_kb_k + a_1b_1 + a_2b_2 + ... + a_nb_n$$

$$= (n+1)\sum_{k=1}^{n} a_kb_k$$

which implies that, by (**),

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leq n\left(\sum_{k=1}^{n} a_kb_k\right).$$

Complex numbers

1.27 Express the following complex numbers in the form $a + bi$.

(a) $(1+i)^3$

Solution: $(1+i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i$.

(b) $(2+3i)\,/\,(3-4i)$

Solution: $\frac{2+3i}{3-4i} = \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)} = \frac{-6+17i}{25} = \frac{-6}{25} + \frac{17}{25}i$.

(c) $i^5 + i^{16}$

Solution: $i^5 + i^{16} = i + 1$.

(d) $\frac{1}{2}\,(1+i)\,(1+i^{-8})$

Solution: $\frac{1}{2}\,(1+i)\,(1+i^{-8}) = 1 + i$.

1.28 In each case, determine all real x and y which satisfy the given relation.

(a) $x + iy = |x - iy|$

Proof: Since $|x - iy| \geq 0$, we have

$$x \geq 0 \text{ and } y = 0.$$

(b) $x + iy = (x - iy)^2$

Proof: Since $(x - iy)^2 = x^2 - (2xy)\,i - y^2$, we have

$$x = x^2 - y^2 \text{ and } y = -2xy.$$

We consider tow cases: (i) $y = 0$ and (ii) $y \neq 0$.

(i) As $y = 0 : x = 0$ or 1.

(ii) As $y \neq 0 : x = -1/2$, and $y = \pm\frac{\sqrt{3}}{2}$.

(c) $\sum_{k=0}^{100} i^k = x + iy$

Proof: Since $\sum_{k=0}^{100} i^k = \frac{1-i^{101}}{1-i} = \frac{1-i}{1-i} = 1$, we have $x = 1$ and $y = 0$.

1.29 If $z = x + iy$, x and y real, the complex conjugate of z is the complex number $\bar{z} = x - iy$. Prove that:

(a) Conjugate of $(z_1 + z_2) = \bar{z}_1 + \bar{z}_2$

Proof: Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{(x_1 + x_2) + i\,(y_1 + y_2)} \\
&= (x_1 + x_2) - i\,(y_1 + y_2) \\
&= (x_1 - iy_1) + (x_2 - iy_2) \\
&= \bar{z}_1 + \bar{z}_2.
\end{aligned}$$

(b) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Proof: Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}
\overline{z_1 z_2} &= \overline{(x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)} \\
&= (x_1 x_2 - y_1 y_2) - i\,(x_1 y_2 + x_2 y_1)
\end{aligned}$$

and

$$\begin{aligned}
\bar{z}_1 \bar{z}_2 &= (x_1 - iy_1)\,(x_2 - iy_2) \\
&= (x_1 x_2 - y_1 y_2) - i\,(x_1 y_2 + x_2 y_1)\,.
\end{aligned}$$

So, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(c) $z\bar{z} = |z|^2$

Proof: Write $z = x + iy$ and thus

$$z\bar{z} = x^2 + y^2 = |z|^2 .$$

(d) $z + \bar{z}$ =twice the real part of z

Proof: Write $z = x + iy$, then

$$z + \bar{z} = 2x,$$

twice the real part of z.

(e) $(z - \bar{z})/i$ =twice the imaginary part of z

Proof: Write $z = x + iy$, then

$$\frac{z - \bar{z}}{i} = 2y,$$

twice the imaginary part of z.

1.30 Describe geometrically the set of complex numbers z which satisfies each of the following conditions:

(a) $|z| = 1$

Solution: The unit circle centered at zero.

(b) $|z| < 1$

Solution: The open unit disk centered at zero.

(c) $|z| \leq 1$

Solution: The closed unit disk centered at zero.

(d) $z + \bar{z} = 1$

Solution: Write $z = x + iy$, then $z + \bar{z} = 1$ means that $x = 1/2$. So, the set is the line $x = 1/2$.

(e) $z - \bar{z} = i$

Proof: Write $z = x + iy$, then $z - \bar{z} = i$ means that $y = 1/2$. So, the set is the line $y = 1/2$.

(f) $z + \bar{z} = |z|^2$

Proof: Write $z = x + iy$, then $2x = x^2 + y^2 \Leftrightarrow (x-1)^2 + y^2 = 1$. So, the set is the unit circle centered at $(1, 0)$.

1.31 Given three complex numbers z_1, z_2, z_3 such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$. Show that these numbers are vertices of an equilateral triangle inscribed in the unit circle with center at the origin.

Proof: It is clear that three numbers are vertices of triangle inscribed in the unit circle with center at the origin. It remains to show that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$. In addition, it suffices to show that

$$|z_1 - z_2| = |z_2 - z_3|.$$

Note that

$$|2z_1 + z_3| = |2z_3 + z_1| \text{ by } z_1 + z_2 + z_3 = 0$$

which is equivalent to

$$|2z_1 + z_3|^2 = |2z_3 + z_1|^2$$

which is equivalent to

$$(2z_1 + z_3)(2\bar{z}_1 + \bar{z}_3) = (2z_3 + z_1)(2\bar{z}_3 + \bar{z}_1)$$

which is equivalent to

$$|z_1| = |z_3|.$$

1.32 If a and b are complex numbers, prove that:

(a) $|a - b|^2 \leq \left(1 + |a|^2\right)\left(1 + |b|^2\right)$

Proof: Consider

$$\begin{aligned}\left(1 + |a|^2\right)\left(1 + |b|^2\right) - |a - b|^2 &= (1 + \bar{a}a)\left(1 + \bar{b}b\right) - (a - b)\left(\bar{a} - \bar{b}\right) \\ &= (1 + \bar{a}b)\left(1 + a\bar{b}\right) \\ &= |1 + \bar{a}b|^2 \geq 0,\end{aligned}$$

so, $|a-b|^2 \leq \left(1+|a|^2\right)\left(1+|b|^2\right)$

(b) If $a \neq 0$, then $|a+b| = |a| + |b|$ if, and only if, b/a is real and nonnegative.

Proof: ($\Rightarrow$)Since $|a+b| = |a| + |b|$, we have

$$|a+b|^2 = (|a| + |b|)^2$$

which implies that

$$\operatorname{Re}(\bar{a}b) = |a| |b| = |\bar{a}| |b|$$

which implies that

$$\bar{a}b = |\bar{a}| |b|$$

which implies that

$$\frac{b}{a} = \frac{\bar{a}b}{\bar{a}a} = \frac{|\bar{a}| |b|}{|a|^2} \geq 0.$$

($\Leftarrow$) Suppose that

$$\frac{b}{a} = k, \text{ where } k \geq 0.$$

Then

$$|a+b| = |a+ka| = (1+k)|a| = |a| + k|a| = |a| + |b|.$$

1.33 If a and b are complex numbers, prove that

$$|a-b| = |1-\bar{a}b|$$

if, and only if, $|a| = 1$ or $|b| = 1$. For which a and b is the inequality $|a-b| < |1-\bar{a}b|$ valid?

Proof: ($\Leftrightarrow$) Since

$$\begin{aligned}
|a-b| &= |1-\bar{a}b| \\
&\Leftrightarrow \left(\bar{a}-\bar{b}\right)(a-b) = (1-\bar{a}b)\left(1-a\bar{b}\right) \\
&\Leftrightarrow |a|^2 + |b|^2 = 1 + |a|^2 |b|^2 \\
&\Leftrightarrow \left(|a|^2 - 1\right)\left(|b|^2 - 1\right) = 0 \\
&\Leftrightarrow |a|^2 = 1 \text{ or } |b|^2 = 1.
\end{aligned}$$

By the preceding, it is easy to know that

$$|a-b| < |1-\bar{a}b| \Leftrightarrow 0 < \left(|a|^2-1\right)\left(|b|^2-1\right).$$

So, $|a-b| < |1-\bar{a}b|$ if, and only if, $|a| > 1$ and $|b| > 1$. (Or $|a| < 1$ and $|b| < 1$).

1.34 If a and c are real constant, b complex, show that the equation

$$az\bar{z} + b\bar{z} + \bar{b}z + c = 0 \ (a \neq 0, z = x+iy)$$

represents a circle in the $x-y$ plane.

Proof: Consider

$$z\bar{z} - \frac{b}{-a}\bar{z} - \frac{\bar{b}}{-a}z + \frac{b}{-a}\left[\overline{\left(\frac{b}{-a}\right)}\right] = \frac{-ac+|b|^2}{a^2},$$

so, we have

$$\left|z - \left(\frac{b}{-a}\right)\right|^2 = \frac{-ac+|b|^2}{a^2}.$$

Hence, as $|b|^2 - ac > 0$, it is a circle. As $\frac{-ac+|b|^2}{a^2} = 0$, it is a point. As $\frac{-ac+|b|^2}{a^2} < 0$, it is not a circle.

Remark: The idea is easy from the fact

$$|z-q| = r.$$

We square both sides and thus

$$z\bar{z} - q\bar{z} - \bar{q}z + \bar{q}q = r^2.$$

1.35 Recall the definition of the inverse tangent: given a real number t, $\tan^{-1}(t)$ is the unique real number θ which satisfies the two conditions

$$-\frac{\pi}{2} < \theta < +\frac{\pi}{2}, \ \ \tan\theta = t.$$

If $z = x+iy$, show that

(a) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$, if $x > 0$

Proof: Note that in this text book, we say $\arg(z)$ is the principal argument of z, denoted by $\theta = \arg z$, where $-\pi < \theta \leq \pi$.
So, as $x > 0$, $\arg z = \tan^{-1}\left(\frac{y}{x}\right)$.

(b) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$, if $x < 0$, $y \geq 0$

Proof: As $x < 0$, and $y \geq 0$. The point (x, y) is lying on $S = \{(x, y) : x < 0,\ y \geq$
Note that $-\pi < \arg z \leq \pi$, so we have $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$.

(c) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) - \pi$, if $x < 0$, $y < 0$

Proof: Similarly for (b). So, we omit it.

(d) $\arg(z) = \frac{\pi}{2}$ if $x = 0$, $y > 0$; $\arg(z) = -\frac{\pi}{2}$ if $x = 0$, $y < 0$.

Proof: It is obvious.

1.36 Define the folowing "**pseudo-ordering**" of the complex numbers: we say $z_1 < z_2$ if we have either
(i) $|z_1| < |z_2|$ or (ii) $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$.
Which of Axioms 6,7,8,9 are satisfied by this relation?

Proof: (1) For axiom 6, we prove that it holds as follows. Given $z_1 = r_1 e^{i\arg(z_1)}$, and $r_2 e^{i\arg(z_2)}$, then if $z_1 = z_2$, there is nothing to prove it. If $z_1 \neq z_2$, there are two possibilities: (a) $r_1 \neq r_2$, or (b) $r_1 = r_2$ and $\arg(z_1) \neq \arg(z_2)$. So, it is clear that axiom 6 holds.

(2) For axiom 7, we prove that it does not hold as follows. Given $z_1 = 1$ and $z_2 = -1$, then it is clear that $z_1 < z_2$ since $|z_1| = |z_2| = 1$ and $\arg(z_1) = 0 < \arg(z_2) = \pi$. However, let $z_3 = -i$, we have

$$z_1 + z_3 = 1 - i > z_2 + z_3 = -1 - i$$

since

$$|z_1 + z_3| = |z_2 + z_3| = \sqrt{2}$$

and

$$\arg(z_1 + z_3) = -\frac{\pi}{4} > -\frac{3\pi}{4} = \arg(z_2 + z_3).$$

(3) For axiom 8, we prove that it holds as follows. If $z_1 > 0$ and $z_2 > 0$, then $|z_1| > 0$ and $|z_2| > 0$. Hence, $z_1 z_2 > 0$ by $|z_1 z_2| = |z_1||z_2| > 0$.

(4) For axiom 9, we prove that it holds as follows. If $z_1 > z_2$ and $z_2 > z_3$, we consider the following cases. Since $z_1 > z_2$, we may have (a) $|z_1| > |z_2|$ or (b) $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$.
As $|z_1| > |z_2|$, it is clear that $|z_1| > |z_3|$. So, $z_1 > z_3$.

As $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$, we have $\arg(z_1) > \arg(z_3)$. So, $z_1 > z_3$.

1.37 Which of Axioms 6,7,8,9 are satisfied if the **pseudo-ordering** is defined as follows? We say $(x_1, y_1) < (x_2, y_2)$ if we have either (i) $x_1 < x_2$ or (ii) $x_1 = x_2$ and $y_1 < y_2$.

Proof: (1) For axiom 6, we prove that it holds as follows. Given $x = (x_1, y_1)$ and $y = (x_2, y_2)$. If $x = y$, there is nothing to prove it. We consider $x \neq y$: As $x \neq y$, we have $x_1 \neq x_2$ or $y_1 \neq y_2$. Both cases imply $x < y$ or $y < x$.

(2) For axiom 7, we prove that it holds as follows. Given $x = (x_1, y_1)$, $y = (x_2, y_2)$ and $z = (z_1, z_3)$. If $x < y$, then there are two possibilities: (a) $x_1 < x_2$ or (b) $x_1 = x_2$ and $y_1 < y_2$.

For case (a), it is clear that $x_1 + z_1 < y_1 + z_1$. So, $x + z < y + z$.

For case (b), it is clear that $x_1 + z_1 = y_1 + z_1$ and $x_2 + z_2 < y_2 + z_2$. So, $x + z < y + z$.

(3) For axiom 8, we prove that it does not hold as follows. Consider $x = (1, 0)$ and $y = (0, 1)$, then it is clear that $x > 0$ and $y > 0$. However, $xy = (0, 0) = 0$.

(4) For axiom 9, we prove that it holds as follows. Given $x = (x_1, y_1)$, $y = (x_2, y_2)$ and $z = (z_1, z_3)$. If $x > y$ and $y > z$, then we consider the following cases. (a) $x_1 > y_1$, or (b) $x_1 = y_1$.

For case (a), it is clear that $x_1 > z_1$. So, $x > z$.

For case (b), it is clear that $x_2 > y_2$. So, $x > z$.

1.38 State and prove a theorem analogous to Theorem 1.48, expressing $\arg(z_1/z_2)$ in terms of $\arg(z_1)$ and $\arg(z_2)$.

Proof: Write $z_1 = r_1 e^{i\arg(z_1)}$ and $z_2 = r_2 e^{i\arg(z_2)}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i[\arg(z_1) - \arg(z_2)]}.$$

Hence,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2),$$

where

$$n(z_1, z_2) = \begin{cases} 0 \text{ if } -\pi < \arg(z_1) - \arg(z_2) \leq \pi \\ 1 \text{ if } -2\pi < \arg(z_1) - \arg(z_2) \leq -\pi \\ -1 \text{ if } \pi < \arg(z_1) - \arg(z_2) < 2\pi \end{cases}.$$

1.39 State and prove a theorem analogous to Theorem 1.54, expressing $Log\,(z_1/z_2)$ in terms of $Log\,(z_1)$ and $Log\,(z_2)$.

Proof: Write $z_1 = r_1 e^{i\arg(z_1)}$ and $z_2 = r_2 e^{i\arg(z_2)}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i[\arg(z_1) - \arg(z_2)]}.$$

Hence,

$$\begin{aligned} Log\,(z_1/z_2) &= \log\left|\frac{z_1}{z_2}\right| + i\arg\left(\frac{z_1}{z_2}\right) \\ &= \log|z_1| - \log|z_2| + i\left[\arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2)\right] \text{ by xercise 1.38} \\ &= Log\,(z_1) - Log\,(z_2) + i2\pi n(z_1, z_2). \end{aligned}$$

1.40 Prove that the nth roots of 1 (also called the nth roots of unity) are given by $\alpha, \alpha^2, ..., \alpha^n$, where $\alpha = e^{2\pi i/n}$, and show that the roots $\neq 1$ satisfy the equation

$$1 + x + x^2 + ... + x^{n-1} = 0.$$

Proof: By **Theorem 1.51**, we know that the roots of 1 are given by $\alpha, \alpha^2, ..., \alpha^n$, where $\alpha = e^{2\pi i/n}$. In addition, since

$$x^n = 1 \Rightarrow (x - 1)\left(1 + x + x^2 + ... + x^{n-1}\right) = 0$$

which implies that

$$1 + x + x^2 + ... + x^{n-1} = 0 \text{ if } x \neq 1.$$

So, all roots except 1 satisfy the equation

$$1 + x + x^2 + ... + x^{n-1} = 0.$$

1.41 (a) Prove that $|z^i| < e^\pi$ for all complex $z \neq 0$.

Proof: Since

$$z^i = e^{iLog(z)} = e^{-\arg(z) + i\log|z|},$$

we have

$$\left|z^i\right| = e^{-\arg(z)} < e^{\pi}$$

by $-\pi < \arg(z) \leq \pi$.

(b) Prove that there is no constant $M > 0$ such that $|\cos z| < M$ for all complex z.

Proof: Write $z = x + iy$ and thus,

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

which implies that

$$|\cos x \cosh y| \leq |\cos z|.$$

Let $x = 0$ and y be real, then

$$\frac{e^y}{2} \leq \frac{1}{2}\left|e^y + e^{-y}\right| \leq |\cos z|.$$

So, there is no constant $M > 0$ such that $|\cos z| < M$ for all complex z.

Remark: There is an important theorem related with this exercise. We state it as a reference. (**Liouville's Theorem)** A bounded entire function is constant. The reader can see the book, **Complex Analysis by Joseph Bak, and Donald J. Newman, pp 62-63. Liouville's Theorem** can be used to prove the much important theorem, **Fundamental Theorem of Algebra.**

1.42 If $w = u + iv$ $(u, v$ real), show that

$$z^w = e^{u \log|z| - v \arg(z)} e^{i[v \log|z| + u \arg(z)]}.$$

Proof: Write $z^w = e^{wLog(z)}$, and thus

$$\begin{aligned} wLog(z) &= (u + iv)(\log|z| + i \arg(z)) \\ &= [u \log|z| - v \arg(z)] + i[v \log|z| + u \arg(z)]. \end{aligned}$$

So,

$$z^w = e^{u \log|z| - v \arg(z)} e^{i[v \log|z| + u \arg(z)]}.$$

1.43 (a) Prove that $Log\left(z^{w}\right) = wLog\ z + 2\pi in$.

Proof: Write $w = u + iv$, where u and v are real. Then

$$\begin{aligned} Log\left(z^{w}\right) &= \log\left|z^{w}\right| + i\arg\left(z^{w}\right) \\ &= \log\left[e^{u\log|z| - v\arg(z)}\right] + i\left[v\log|z| + u\arg(z)\right] + 2\pi in \text{ by Exercise1.42} \\ &= u\log|z| - v\arg(z) + i\left[v\log|z| + u\arg(z)\right] + 2\pi in. \end{aligned}$$

On the other hand,

$$\begin{aligned} wLogz + 2\pi in &= (u + iv)\left(\log|z| + i\arg(z)\right) + 2\pi in \\ &= u\log|z| - v\arg(z) + i\left[v\log|z| + u\arg(z)\right] + 2\pi in. \end{aligned}$$

Hence, $Log\left(z^{w}\right) = wLog\ z + 2\pi in$.

Remark: There is another proof by considering

$$e^{Log(z^{w})} = z^{w} = e^{wLog(z)}$$

which implies that

$$Log\left(z^{w}\right) = wLogz + 2\pi in$$

for some $n \in Z$.

(b) Prove that $\left(z^{w}\right)^{\alpha} = z^{w\alpha}e^{2\pi in\alpha}$, where n is an integer.

Proof: By (a), we have

$$\left(z^{w}\right)^{\alpha} = e^{\alpha Log(z^{w})} = e^{\alpha(wLogz + 2\pi in)} = e^{\alpha wLogz}e^{2\pi in\alpha} = z^{\alpha w}e^{2\pi in\alpha},$$

where n is an integer.

1.44 (i) If θ and a are real numbers, $-\pi < \theta \le \pi$, prove that

$$(\cos\theta + i\sin\theta)^{a} = \cos(a\theta) + i\sin(a\theta).$$

Proof: Write $\cos\theta + i\sin\theta = z$, we then have

$$\begin{aligned} (\cos\theta + i\sin\theta)^{a} = z^{a} = e^{aLogz} &= e^{a\left[\log\left|e^{i\theta}\right| + i\arg\left(e^{i\theta}\right)\right]} = e^{ia\theta} \\ &= \cos(a\theta) + i\sin(a\theta). \end{aligned}$$

Remark: Compare with the **Exercise 1.43-(b)**.

(ii) Show that, in general, the restriction $-\pi < \theta \leq \pi$ is necessary in (i) by taking $\theta = -\pi$, $a = \frac{1}{2}$.

Proof: As $\theta = -\pi$, and $a = \frac{1}{2}$, we have

$$(-1)^{\frac{1}{2}} = e^{\frac{1}{2}Log(-1)} = e^{\frac{\pi}{2}i} = i \neq -i = \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right).$$

(iii) If a is an integer, show that the formula in (i) holds without any restriction on θ. In this case it is known as **DeMorvre's theorem**.

Proof: By **Exercise 1.43**, as a is an integer we have

$$(z^w)^a = z^{wa},$$

where $z^w = e^{i\theta}$. Then

$$\left(e^{i\theta}\right)^a = e^{i\theta a} = \cos(a\theta) + i\sin(a\theta).$$

1.45 Use **DeMorvre's theorem** (Exercise 1.44) to derive the trigino-metric identities

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

valid for real θ. Are these valid when θ is complex?

Proof: By **Exercise 1.44-(iii)**, we have for any real θ,

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta).$$

By **Binomial Theorem**, we have

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

and

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

For complex θ, we show that it holds as follows. Note that $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, we have

$$\begin{aligned}
3\cos^2 z \sin z - \sin^3 z &= 3\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2\left(\frac{e^{iz}-e^{-iz}}{2i}\right) - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^3 \\
&= 3\left(\frac{e^{2zi}+e^{-2zi}+2}{4}\right)\left(\frac{e^{iz}-e^{-iz}}{2i}\right) + \frac{e^{3zi}-3e^{iz}+3e^{-iz}-e}{8i} \\
&= \frac{1}{8i}\left[3\left(e^{2zi}+e^{-2zi}+2\right)\left(e^{zi}-e^{-zi}\right) + \left(e^{3zi}-3e^{iz}+3e^{-iz}-e\right.\right. \\
&= \frac{1}{8i}\left[\left(3e^{3zi}+3e^{iz}-3e^{-iz}-3e^{-3zi}\right) + \left(e^{3zi}-3e^{iz}+3e^{-iz}-e\right.\right. \\
&= \frac{4}{8i}\left(e^{3zi}-e^{-3zi}\right) \\
&= \frac{1}{2i}\left(e^{3zi}-e^{-3zi}\right) \\
&= \sin 3z.
\end{aligned}$$

Similarly, we also have

$$\cos^3 z - 3\cos z \sin^2 z = \cos 3z.$$

1.46 Define $\tan z = \sin z/\cos z$ and show that for $z = x + iy$, we have

$$\tan z = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}.$$

Proof: Since

$$\begin{aligned}
\tan z &= \frac{\sin z}{\cos z} = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{\sin x\cosh y + i\cos x\sinh y}{\cos x\cosh y - i\sin x\sinh y} \\
&= \frac{(\sin x\cosh y + i\cos x\sinh y)(\cos x\cosh y + i\sin x\sinh y)}{(\cos x\cosh y - i\sin x\sinh y)(\cos x\cosh y + i\sin x\sinh y)} \\
&= \frac{\left(\sin x\cos x\cosh^2 y - \sin x\cos x\sinh^2 y\right) + i\left(\sin^2 x\cosh y\sinh y + \cos^2 x\cosh\right.}{(\cos x\cosh y)^2 - (i\sin x\sinh y)^2} \\
&= \frac{\sin x\cos x\left(\cosh^2 y - \sinh^2 y\right) + i(\cosh y\sinh y)}{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} \text{ since } \sin^2 x + \cos^2 x = 1 \\
&= \frac{(\sin x\cos x) + i(\cosh y\sinh y)}{\cos^2 x + \sinh^2 y} \text{ since } \cosh^2 y = 1 + \sinh^2 y \\
&= \frac{\frac{1}{2}\sin 2x + \frac{i}{2}\sinh 2y}{\cos^2 x + \sinh^2 y} \text{ since } 2\cosh y\sinh y = \sinh 2y \text{ and } 2\sin x\cos x = \sin 2x \\
&= \frac{\sin 2x + i\sinh 2y}{2\cos^2 x + 2\sinh^2 y} \\
&= \frac{\sin 2x + i\sinh 2y}{2\cos^2 x - 1 + 2\sinh^2 y + 1} \\
&= \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} \text{ since } \cos 2x = 2\cos^2 x - 1 \text{ and } 2\sinh^2 y + 1 = \cosh 2y.
\end{aligned}$$

1.47 Let w be a given complex number. If $w \neq \pm 1$, show that there exists two values of $z = x + iy$ satisfying the conditions $\cos z = w$ and $-\pi < x \leq \pi$. Find these values when $w = i$ and when $w = 2$.

Proof: Since $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, if we let $e^{iz} = u$, then $\cos z = w$ implies that

$$w = \frac{u^2+1}{2u} \Rightarrow u^2 - 2wu + 1 = 0$$

which implies that

$$(u-w)^2 = w^2 - 1 \neq 0 \text{ since } w \neq \pm 1.$$

So, by **Theorem 1.51**,

$$\begin{aligned}
e^{iz} &= u = w + \left|w^2-1\right|^{1/2} e^{i\phi_k}, \text{ where } \phi_k = \frac{\arg\left(w^2-1\right)}{2} + \frac{2\pi k}{2}, \ k = 0, 1. \\
&= w \pm \left|w^2-1\right|^{1/2} e^{i\left(\frac{\arg\left(w^2-1\right)}{2}\right)}
\end{aligned}$$

So,

$$ix-y = i\,(x+iy) = iz = \log\left|w \pm \left|w^2-1\right|^{1/2} e^{i\frac{\arg\left(w^2-1\right)}{2}}\right| + i \arg\left(w \pm \left|w^2-1\right|^{1/2} e^{i(\ldots}\right.$$

Hence, there exists two values of $z = x+iy$ satisfying the conditions $\cos z = w$ and

$$-\pi < x = \arg\left(w \pm \left|w^2-1\right|^{1/2} e^{i\left(\frac{\arg\left(w^2-1\right)}{2}\right)}\right) \le \pi.$$

For $w = i$, we have

$$iz = \log\left|\left(1 \pm \sqrt{2}\right) i\right| + i \arg\left(\left(1 \pm \sqrt{2}\right) i\right)$$

which implies that

$$z = \arg\left(\left(1 \pm \sqrt{2}\right) i\right) - i \log\left|\left(1 \pm \sqrt{2}\right) i\right|.$$

For $w = 2$, we have

$$iz = \log\left|2 \pm \sqrt{3}\right| + i \arg\left(2 \pm \sqrt{3}\right)$$

which implies that

$$z = \arg\left(2 \pm \sqrt{3}\right) - i \log\left|2 \pm \sqrt{3}\right|.$$

1.48 Prove Lagrange's identity for complex numbers:

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 = \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \sum_{1 \le k < j \le n} \left(a_k \bar{b}_j - \bar{a}_j b_k\right)^2.$$

Use this to deduce a Cauchy-Schwarz ineqality for complex numbers.

Proof: It is the same as the **Exercise 1.23**; we omit the details.

1.49 (a) By eqating imaginary parts in DeMoivre's formula prove that

$$\sin n\theta = \sin^n \theta \left\{ \tbinom{n}{1} \cot^{n-1}\theta - \tbinom{n}{3} \cot^{n-3}\theta + \tbinom{n}{5} \cot^{n-5}\theta - + \ldots \right\}$$

Proof: By **Exercise 1.44 (i)**, we have

$$\begin{aligned}
\sin n\theta &= \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} \sin^{2k-1}\theta \cos^{n-(2k-1)}\theta \\
&= \sin^n\theta \left\{ \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} \cot^{n-(2k-1)}\theta \right\} \\
&= \sin^n\theta \left\{ \binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - +... \right\}.
\end{aligned}$$

(b) If $0 < \theta < \pi/2$, prove that

$$\sin(2m+1)\theta = \sin^{2m+1}\theta P_m\left(\cot^2\theta\right)$$

where P_m is the polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - +...$$

Use this to show that P_m has zeros at the m distinct points $x_k = \cot^2\{\pi k/(2m+1)\}$ for $k = 1, 2, ..., m$.

Proof: By (a),

$$\begin{aligned}
&\sin(2m+1)\theta \\
&= \sin^{2m+1}\theta\left\{\binom{2m+1}{1}\left(\cot^2\theta\right)^m - \binom{2m+1}{3}\left(\cot^2\theta\right)^{m-1} + \binom{2m+1}{5}\left(\cot^2\theta\right)^{m-2} - +...\right\} \\
&= \sin^{2m+1}\theta P_m\left(\cot^2\theta\right), \text{ where } P_m(x) = \sum_{k=1}^{m+1}\binom{2m+1}{2k-1}x^{m+1-k}. \qquad (*)
\end{aligned}$$

In addition, by (*), $\sin(2m+1)\theta = 0$ if, and only if, $P_m(\cot^2\theta) = 0$. Hence, P_m has zeros at the m distinct points $x_k = \cot^2\{\pi k/(2m+1)\}$ for $k = 1, 2, ..., m$.

(c) Show that the sum of the zeros of P_m is given by

$$\sum_{k=1}^{m}\cot^2\frac{\pi k}{2m+1} = \frac{m(2m-1)}{3},$$

and the sum of their squares is given by

$$\sum_{k=1}^{m} \cot^4 \frac{\pi k}{2m+1} = \frac{m(2m-1)(4m^2+10m-9)}{45}.$$

Note. There identities can be used to prove that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ and $\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$. (See Exercises 8.46 and 8.47.)

Proof: By (b), we know that sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} x_k = \sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m+1} = -\left(\frac{-\binom{2m+1}{3}}{\binom{2m+1}{1}}\right) = \frac{m(2m-1)}{3}.$$

And the sum of their squares is given by

$$\begin{aligned}\sum_{k=1}^{m} x_k^2 &= \sum_{k=1}^{m} \cot^4 \frac{\pi k}{2m+1} \\ &= \left(\sum_{k=1}^{m} x_k\right)^2 - 2\left(\sum_{1\le i<j\le n} x_i x_j\right) \\ &= \left(\frac{m(2m-1)}{3}\right)^2 - 2\left(\frac{\binom{2m+1}{5}}{\binom{2m+1}{1}}\right) \\ &= \frac{m(2m-1)(4m^2+10m-9)}{45}.\end{aligned}$$

1.50 Prove that $z^n - 1 = \prod_{k=1}^{n}\left(z - e^{2\pi i k/n}\right)$ for all complex z. Use this to derive the formula

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

Proof: Since $z^n = 1$ has exactly n distinct roots $e^{2\pi i k/n}$, where $k = 0, ..., n-1$ by **Theorem 1.51.** Hence, $z^n - 1 = \prod_{k=1}^{n}\left(z - e^{2\pi i k/n}\right)$. It implies that

$$z^{n-1} + ... + 1 = \prod_{k=1}^{n-1}\left(z - e^{2\pi i k/n}\right).$$