Macroeconomic Theory: Assignment 5: Value Function Iterations

Exercise 1. [**Value Function Iterations**] Consider an economy where the household chooses an infinite sequence of consumption and next period's capital stock $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to solve the following sequential problem

$$\begin{array}{ll} \max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{subject to} & \left\{ \begin{array}{l} c_t + k_{t+1} = g(k_t) \\ c_t, k_{t+1} \geq 0 \end{array} \right. , \text{ for } t = 0 \ldots \infty \end{array}$$

The utility function and the production function take the following form:

$$\begin{aligned} u(c) &= \frac{c^{1-\theta} - 1}{1-\theta}, \\ g(k_t) &= Ak_t^{\alpha} + (1-\delta)k_t \end{aligned}$$

The associated contraction mapping is defined as

$$Tv(k) = \max_{k' \in X, k' \leq g(k)} u(g(k) - k') + \beta v(k')$$

Let $A = 1$, $\delta = 0.5$, $\theta = 0.8$, $\beta = 0.9$, $\alpha = 0.5$. Follow the instructions below to find the fixed point of the contraction mapping by using value function iterations:

1. Follow instructions below to construct the fixed point of the contraction mapping T:

Step 1. [**Define the domain and image of the value function**]**:** Discretize the domain of state variable by defining a vector

$$\bar{X} = \left\{ 0 = k_0, k_1, k_2, \ldots, k_n = \bar{k} \right\}.$$

(Hint: the suggested grid size is 0.005.). Construct a vector $v = \{v_0, v_1, \ldots, v_n\}$ to represent the initial value function $v(k)$; that is, $v_i = v(k_i)$, for $i = 0, 1, \ldots, n$.

Step 2. **[Given $v(k)$, solve for $Tv(k)$]:** construct a vector

$$Tv = \{Tv_0, Tv_1, \ldots, Tv_n\},$$

where $Tv_i = Tv(k_i)$ such that

$$Tv(k_i) = \max_{k' \in X, k' \leq g(k_i)} u(g(k_i) - k') + \beta v(k') \qquad (\star)$$

Moreover, construct a policy vector $pol = \{pol_0, pol_1, \ldots, pol_n\}$, where pol_i is the **index** of k' that solves $(\star)$.

Step 3. **[Repeat Step 2. Conduct value function iterations]:** Let $\epsilon \equiv \max_i \{|Tv_i - v_i|\}$ denote the sup norm. After computing ϵ, replace v by Tv, and replace the old policy vector by the new policy vector. Repeat Step 2 for 100 times. What is the ϵ you obtain after 100 times of iterations?

2. Apply the policy vector *pol* to find the dynamics of the economy. Let $A = 1$, $\delta = 0.5$, $\theta = 0.8$, $\beta = 0.95$, $\alpha = 0.5$. Suppose that the economy was initially at the steady state (For $t = -5, -4, -3, -2, -1$). At time $t = 0$, the time discount factor β decreases from 0.95 to 0.9 unexpectedly and permanently. Characterize the dynamic path of capital, consumption, output, and investment before and after the TFP shock.

3. Compare the dynamic paths you drew in Question 1.2 (by using **value function iteration**) with the dynamic paths you drew in Assignment 4, Part 2 (by using **steady state linearization)**. To compare, simply put the dynamic paths under these two approaches in the same figure and check whether they are close enough.

Question 2: Suppose that T is a contraction mapping with modulus β on a space (V, ρ). Let $v_n = T^n v_0$ and $\rho(v_1, v_0) = 1$, and the fixed point of T is denoted by v^*. Given $\epsilon > 0$, what is the smallest number of iterations that guarantees that $\rho(v_n, v^*) < \epsilon$.