ECON 7219, Semester 110.1, Assignment 3, Solutions

1. (a) There are two approaches to finding the equilibrium. We assume its existence and then either derive the first-order necessary conditions or we equate its expected payments to the payments implied by the revenue equivalence theorem. Let us pursue the second approach here. Suppose that a symmetric equilibrium s in continuous and increasing strategies exists. Because s is increasing, bidder i wins the auction if and only if his/her valuation is higher than the valuation of all other bidders. Let $\theta_{-i}^{(1)} = \max_{j \neq i} \theta_j$ denote the highest valuation among i's opponents and let $\theta_{-i}^{(2)}$ denote the second-highest valuation among i's opponents. Bidder i's expected payments when he/she is of type ϑ_i are

$$\begin{aligned}\mathbb{E}_{\vartheta_i}\left[s\Big(\theta_{-i}^{(2)}\Big)1_{\left\{\theta_{-i}^{(1)}<\vartheta_i\right\}}\right] &= \int_0^1 \int_0^1 s(y)1_{\{x \leq \vartheta_i\}} f_{\theta_{-i}^{(1)},\theta_{-i}^{(2)}}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{\vartheta_i}\int_0^x s(y)(n-1)(n-2)y^{n-3}\,\mathrm{d}y\,\mathrm{d}x. \qquad (1)\end{aligned}$$

For the standard-uniform distribution, we obtain $P(\theta_{-i}^{(1)} \leq x) = (F(x))^{n-1} = x^{n-1}$ in the same way as in the lecture. We deduce that $\theta_{-i}^{(1)}$ has density function $(n-1)x^{n-2}$. By the revenue equivalence theorem, the expected payments in (1) have to be equal to

$$\mathbb{E}_{\vartheta_i}\left[\theta_{-i}^{(1)}1_{\left\{\theta_{-i}^{(1)}<\vartheta_i\right\}}\right] = \int_0^{\vartheta_i}(n-1)x^{n-1}\,\mathrm{d}x. \qquad (2)$$

Since both the functions in (1) and (2) are twice differentiable, equality of (1) and (2) implies that also their second derivatives are equal. This trick saves us from having to integrate. By virtue of the fundamental theorem of calculus, we obtain

$$s(\vartheta_i)(n-1)(n-2)\vartheta_i^{n-3} = (n-1)^2\vartheta_i^{n-2}.$$

If ϑ_i is different from 0, solving for $s(\vartheta_i)$ yields

$$s(\vartheta_i) = \frac{n-1}{n-2}\vartheta_i.$$

Since we have assumed s to be continuous, this also determines $s(0) = 0$. Observe that this strategy is indeed continuous and increasing. It remains to verify that s is an equilibrium strategy. Suppose that bidder i of type ϑ_i bids b but all of i's opponents follow strategy s. We verify that type ϑ_i's utility is maximized at $b = s(\vartheta_i)$ by verifying the *second-order sufficient condition.* With bid b, bidder i wins the auction in the event

$$\Big\{\max_{j\neq i} s(\theta_j) < b\Big\} = \bigcap_{j \neq i}\{s(\theta_j) < b\} = \bigcap_{j\neq i}\Big\{\theta_j < s^{-1}(b)\Big\} = \Big\{\theta_{-i}^{(1)} < s^{-1}(b)\Big\}.$$

Bidder i's expected utility from bidding b is thus

$$\begin{aligned}\mathbb{E}_{\vartheta_i}[u_i(\vartheta_i, b, s(\theta_{-i}))] &= \mathbb{E}_{\vartheta_i}\left[\Big(\vartheta_i - s\Big(\theta_{-i}^{(2)}\Big)\Big)1_{\left\{\theta_{-i}^{(1)}<s^{-1}(b)\right\}}\right] \\ &= \vartheta_i(s^{-1}(b))^{n-1} - \int_0^{s^{-1}(b)}\int_0^x (n-1)^2 y^{n-2}\,\mathrm{d}y\,\mathrm{d}x \\ &= \vartheta_i(s^{-1}(b))^{n-1} - \int_0^{s^{-1}(b)}(n-1)x^{n-1}\,\mathrm{d}x.\end{aligned}$$

Using that $s^{-1}(b) = \frac{n-2}{n-1}b$, taking the derivative of i's expected utility yields

$$\frac{\partial \mathbb{E}_{\vartheta_i}[u_i(\vartheta_i, b, s(\theta_{-i}))]}{\partial b} = (n-2)\Big(s^{-1}(b)\Big)^{n-2}\Big(\vartheta_i - s^{-1}(b)\Big).$$

Indeed, the first derivative is 0 for $b = s(\vartheta_i)$. So far so good. Using again the specific form of $s^{-1}(b)$, the product rule yields

$$\left.\frac{\partial^2 \mathbb{E}_{\vartheta_i}[u_i(\vartheta_i, b, s_{-i}(\theta_{-i}))]}{\partial b^2}\right|_{b=s(\vartheta_i)} = -\frac{(n-2)^3}{n-1}\vartheta_i^{n-3} < 0.$$

We conclude that $b = s(\vartheta_i)$ is a global best response to $s(\theta_{-i})$.

(b) Bidders make bids that are higher than their true valuation of the object. There is a benefit to overbidding your valuation to increase your chances of outbidding the second-highest bidder and hence increase your chances of winning the auction. This leads to a profit if the third-highest bid is below your true valuation and to a loss otherwise. At the equilibrium bid, the expected marginal gain from increasing your winning probability when you have to pay less than your valuation offsets the expected marginal loss from increasing your winning probability when you have to pay more than your valuation.

(c) The three conditions of the revenue equivalence theorem are satisfied: bidders' types are independent with a positive density on $[0, \bar{\vartheta}]$, the object is awarded to the highest bidder, and type 0 makes expected payments 0 because they never win the auction. By the revenue equivalence theorem, the expected revenue of the seller is $\mathbb{E}\Big[\theta^{(2)}\Big] = \frac{n-1}{n+1}$.

(d) Truthful reporting is not a dominant strategy. Suppose that the valuation of each bidder $i = 1, 2, 3$ is $\vartheta_i = i$. If bidder 2 reports 4 instead of his/her truthful valuation 2, then bidder 3 has a profitable deviation to over-report his/her valuation. Thus, truthful reporting is a best response only conditional on others reporting their valuations truthfully. Alternatively, one can verify that the preference reversal property fails to hold.

2. (a) The ex-post efficient social state is $q(\vartheta_L^a) = T$ and $q(\vartheta_L^b) = S$. Without Lisa, Mark would go to Switzerland, hence type ϑ_L^a imposes an externality of $v_M(S) - v_M(T) = 4$ on Mark, whereas type ϑ_L^b's report is not pivotal. Without Mark, Lisa would of type ϑ_L^a would go to Tanzania and Lisa of type ϑ_L^b would go to New Zealand. Mark thus imposes an externality of $v_L(N, \vartheta_L^b) - v_L(S, \vartheta_L^b) = 1$ on type ϑ_L^b, whereas his report is not pivotal if Lisa is of type ϑ_L^a. The pivot payments are thus

$$p_M^{\text{piv}}(\vartheta) = \begin{cases} 0 & \text{if } \vartheta_L = \vartheta_L^a, \\ 1 & \text{if } \vartheta_L = \vartheta_L^b, \end{cases} \qquad p_L^{\text{piv}}(\vartheta) = \begin{cases} 4 & \text{if } \vartheta_L = \vartheta_L^a, \\ 0 & \text{if } \vartheta_L = \vartheta_L^b. \end{cases}$$

The types' interim expected utilities in the pivot mechanisms are

$$\mathbb{E}\Big[u_M\Big(g^{\text{piv}}(\theta_L)\Big)\Big] = 0.5, \qquad u_L\Big(g^{\text{piv}}(\vartheta_L^a), \vartheta_L^a\Big) = 2, \qquad u_L\Big(g^{\text{piv}}(\vartheta_L^b), \vartheta_L^b\Big) = 3.$$

It follows that the participation subsidies are $\varphi_M = -0.5$ and $\varphi_L = 2$. In conclusion, the IR-VCG mechanism implements the ex-post efficient social state with payments

$$p_M^{\text{IR}}(\vartheta) = \begin{cases} 0.5 & \text{if } \vartheta_L = \vartheta_L^a, \\ 1.5 & \text{if } \vartheta_L = \vartheta_L^b, \end{cases} \qquad p_L^{\text{IR}}(\vartheta) = \begin{cases} 2 & \text{if } \vartheta_L = \vartheta_L^a, \\ -2 & \text{if } \vartheta_L = \vartheta_L^b. \end{cases}$$

Moreover, its ex-ante expected surplus is $\mathbb{E}[p_M(\theta_L)] + \mathbb{E}[p_L(\theta_L)] = 1$.

(b) The first problem is that the surplus is negative in state ϑ_L^b, hence Mark and Lisa cannot balance the budget by going to a dinner. The second issue is that this redistribution of surplus may distort incentives. Unfortunately, I made a mistake when designing the question, so this does not occur for the chosen numbers.

3. (a) In the indirect mechanism, every buyer reports their bids, every seller reports their asks, we construct supply and demand curves $S(p)$ and $D(p)$, and the market is cleared with a market clearing price p for which $D(p) = S(p)$. In particular, every buyer and seller in $D(p)$ and $S(p)$ will get to trade. A direct mechanism consists of the mapping $(p, q) : \Theta \to \mathcal{Q} \times \mathbb{R}^{2n}$, where we allow payments to be positive or negative. A social state indicates who holds the good after the transaction. The space of social states is thus $\mathcal{Q} = \left\{x \in \{0,1\}^{\{1,\dots,2n\}} \,\middle|\, \sum_i x_i = n\right\}$.

(b) Among all social states with $k \le n$ trades, social surplus is maximized if trade occurs between sellers and buyers $i \le k$. It remains to show that $k(\vartheta)$ trades are efficient. Indeed, since $\vartheta_S^i \le \vartheta_B^i$ for $i \le k(\vartheta)$, reducing the trade amount does not improve welfare. Moreover, since $\vartheta_S^i > \vartheta_B^i$ for $i > k(\vartheta)$, increasing the trade amount reduces welfare.

(c) We start by computing the pivot payments. We observe first that the absence of buyer or seller $i > k(\vartheta)$ does not affect the social state, hence their pivot payment is 0. Without buyer $i \le k(\vartheta)$, there are two possible scenarios:

i. If $\vartheta_S^{k(\vartheta)} \le \vartheta_B^{k(\vartheta)+1}$, then buyer $k(\vartheta)+1$ gets to buy the good instead of buyer i. Thus, i's presence imposes an externality of $\vartheta_B^{k(\vartheta)+1}$ on buyer $k(\vartheta)+1$.

ii. If $\vartheta_S^{k(\vartheta)} > \vartheta_B^{k(\vartheta)+1}$, then only $k(\vartheta)-1$ units of the good are traded and seller $k(\vartheta)$ gets to keep the item. Thus, i's presence imposes an externality of $\vartheta_S^{k(\vartheta)}$ on seller $k(\vartheta)$.

We conclude that $p_B^i(\vartheta) = \max\left\{\vartheta_B^{k(\vartheta)+1}, \vartheta_S^{k(\vartheta)}\right\}$ for any $i \le k(\vartheta)$, where $\vartheta_B^{k(\vartheta)+1}$ is understood to be 0 if $k(\vartheta) = n$. Similarly, the absence of seller $i \le k(\vartheta)$ may affect trading in two ways:

i. If $\vartheta_B^{k(\vartheta)} \ge \vartheta_S^{k(\vartheta)+1}$, then seller $k(\vartheta)+1$ gets to sell in i's stead. Thus, i's presence imposes an externality of $-\vartheta_S^{k(\vartheta)+1}$ on seller $k(\vartheta)+1$.

ii. If $\vartheta_B^{k(\vartheta)} < \vartheta_S^{k(\vartheta)+1}$, then only $k(\vartheta)-1$ units of the good are traded, leaving buyer $k(\vartheta)$ without a good. Thus, i's presence imposes an externality of $-\vartheta_B^{k(\vartheta)}$ on buyer $k(\vartheta)$.

We conclude that $p_S^i(\vartheta) = -\min\left\{\vartheta_S^{k(\vartheta)+1}, \vartheta_B^{k(\vartheta)}\right\}$ for any $i \le k(\vartheta)$, where $\vartheta_S^{k(\vartheta)+1}$ is understood to be ∞ if $k(\vartheta) = n$. Next, we compute the participation subsidy for the buyers. Buyers have outside option 0 and hence the participation subsidy is given by

$$\varphi_B = -\min_{\vartheta_B^i} \mathbb{E}_{\vartheta_B^i}\left[\vartheta_B^i 1_{\{i \le k(\theta)\}} - p_B^i(\theta)\right] = -\mathbb{E}_{\underline{\vartheta}_B}\left[\underline{\vartheta}_B 1_{\{k(\theta)=n\}} - \theta_S^n 1_{\{k(\theta)=n\}}\right],$$

where we have used the fact that the buyer's utility is minimized in the lowest type $\underline{\vartheta}_B$. Let F_S and f_S denote the distribution and density function, respectively, of $\max_j \theta_S^j$. Note that $P_{\underline{\vartheta}_B}(k(\theta) = n) = P\left(\max_j \theta_S^j \le \underline{\vartheta}_B\right) = F_S(\underline{\vartheta}_B)$ and hence

$$\varphi_B = -\underline{\vartheta}_B F_S(\underline{\vartheta}_B) + \int_{\underline{\vartheta}_S}^{\underline{\vartheta}_B} x f_S(x)\,\mathrm{d}x = -\int_{\underline{\vartheta}_S}^{\underline{\vartheta}_B} F_S(x)\,\mathrm{d}x.$$

Similarly, the highest type seller requires the largest participation subsidy and we obtain

$$\varphi_S = \bar{\vartheta}_S - \mathbb{E}_{\bar{\vartheta}_S}\left[\bar{\vartheta}_S 1_{\{k(\theta)<n\}} + \theta_B^n 1_{\{k(\theta)=n\}}\right] = \bar{\vartheta}_S P_{\bar{\vartheta}_S}(k(\theta) = n) - \mathbb{E}_{\bar{\vartheta}_S}\left[\theta_B^n 1_{\{k(\theta)=n\}}\right].$$

Let F_B and f_B be the distribution and density function, respectively, of $\min_j \theta_B^j$. Then

$$\varphi_S = \bar{\vartheta}_S\left(1 - F_B(\bar{\vartheta}_S)\right) - \int_{\bar{\vartheta}_S}^{\bar{\vartheta}_B} x f_B(x)\,\mathrm{d}x = \bar{\vartheta}_S - \bar{\vartheta}_B + \int_{\bar{\vartheta}_S}^{\bar{\vartheta}_B} F_B(x)\,\mathrm{d}x = -\int_{\bar{\vartheta}_S}^{\bar{\vartheta}_B}\left(1 - F_B(x)\right)\mathrm{d}x.$$

If the support of the buyers' and sellers' valuation is equal, the participation subsidies are 0. Since $p_B + p_S \leq 0$ and $p_B < p_S < 0$ with positive probability, the ex-ante expected revenue is negative. In particular, the IR-VCG mechanism is not budget-balanced.

(d) We verify that the conditions of Lemma 6.25 hold. The number of social states corresponds to the number of possible allocations of the good among buyers and sellers. In particular, it is finite. The ex-post efficient social state is unique unless there are several buyers or sellers who value the good equally. This is a measure-0 event, hence the ex-post efficient social state is unique for almost every $\vartheta \in \Theta$. By assumption, the types are one-dimensional with strictly positive density on their support. Finally, $v_i^B(q, \vartheta_i) = \vartheta_i 1_{\{q_i^B = 1\}}$ and $v_i^S(q, \vartheta_i) = \vartheta_i 1_{\{q_i^S = 1\}}$ are non-decreasing and absolutely continuous in ϑ_i for each q. Corollary 6.25 thus implies that there exists no such mechanism.