Contents

[04:30] This was ~~it~~ for signaling games, and I will continue with screening games.

1 Introduction

1.1 p.1 Job-Market Signaling

1.1.1 Separating behavior

[04:45] ~~So okay~~ the signaling games are all about the informed player moving first ~~and~~ because their strategy depends on the information that they have, [and] their actions sends some signal about the information that they have. ~~And~~ we can now think of the scenario ~~where~~ where the roles of the two players is inverted where first the uninformed player acts first, and tries to get separating behavior from the second player.

1.1.2 Screening

[05:15] ~~So~~ if we sort of invert our job market signaling application or ~~we say okay maybe~~ if the firm acts first, can they offer a wage level such that in response to this wage function the workers will have an incentive to separate? ~~And~~ this is then known as screening, and the idea is fairly similar. **The only difference is that the uninformed player moves first.**

1.2 p.2 Electing Government Officials

1.2.1 Uncertainty

[05:46] What are some examples of this? ~~So let's say~~ if we elect a government official, then at the beginning we're not sure whether that government official has our interests at heart or whether they simply abuse their office to make a personal benefit. ~~And~~ the idea is ~~okay~~ sure [that] they do send a signal during their campaign, but honestly it's a lot of cheap talk, so there is not a lot of information during the campaign. ~~So~~ **even after the campaign, we don't know for sure what a politician's intentions are.**

1.2.2 Electing government officials

[06:23] ~~Now, the question is~~ can we write a law that will incentivize only for the good types to even campaign for such an office? ~~And so those laws~~ there will be certain requirements for those laws. ~~So one~~ we want them to be sufficiently stringent such that people who want to abuse power don't even want to campaign. But ~~then~~ we want those laws to submission to be sufficiently lacked such that somebody who really wants to improve our society is not worried that some mistake that we make ~~hey makes~~ will bring him to jail for the rest of his life. So, those laws need to be sufficiently lacks to incentivize participation.

[07:05] ~~And the question is okay~~ how should we write such a law to screen out the different types of politicians?

1.3 p.3 A Naive Example: Air Fares

1.3.1 Airline tickets

[07:14] ~~And another example where~~ we have screening ~~would be~~ if ~~some retailer~~ some seller faces clients that are not sure what their willingness to pay is. ~~And so~~ an example would be ~~let's say~~ an airline which sells tickets

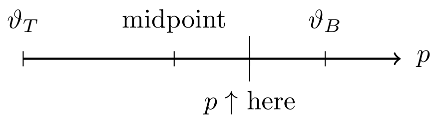
and it sells tickets to either ~~let's say~~ business travelers or tourists, and they value their tickets at fairly different prices. ~~And the question is~~ what price ~~should~~ should the airline set to ~~to~~ maximize the profit not knowing ~~what kind of~~ what type of buyer that they are facing?
[08:00] ~~So~~ we're going to build up here to the general screening model. ~~And so~~ let's start first with this naive example where we ask ~~okay~~ we simply sell a ticket to go from a to b. **We have two types of customers that could potentially want to fly, and we can simply charge one price for this destination.** ~~And so let's say that~~ the tourists value this trip at 10 thousand, the business travelers at 30 thousand, and the cost for the airline is 5 thousand. ~~And the question is~~ what should the optimal price be?

1.4 p.4 Extensive-Form Game

[08:40] ~~So~~ in this example, we can write this as a extensive-form game with where **the airline can choose an infinite number of ~~of~~ prices, and then after observing the price, the customer can decide whether they want to buy or not to buy the ticket.** ~~And~~ if we see the payoffs here, then it is a clear that the customers will only want to buy the ticket if it's cheaper than their evaluation of the trip.

1.4.1 Perfect Bayesian equilibrium

[09:11] [Now] what should the airline do as a best response to this? ~~So~~ **[at] first, if the airline posts any price that is not equal to exactly the willingness to pay of either the business traveler or the tourist, then this cannot be an equilibrium because they could always choose a slightly larger price and sell exactly to the same people.**
[09:34] ~~So let's say~~ if we have a set of prices and we have the willingness to pay, then choosing any intermediate point here cannot be in equilibrium because we could just increase the price and sell exactly to the same people.



[09:52] [Okay so we] **we conclude that ~~one of these~~ the only price in equilibrium can be one of these two** [ϑ_T, ϑ_B]**.** ~~Now question is what can~~ what can the response be ~~by the~~ by the travelers? ~~Well~~ suppose ~~first~~ that [if] they [type ϑ] choose to buy the price [ϑ] with probability smaller than 1, ~~and~~ then the airline has a profitable deviation to set the price slightly lower [$\vartheta - \varepsilon$] so that I can sell to all of these types.
[10:30] ~~So formally so formally~~ if we set ~~let's say~~ the price at the willingness to pay of the tourists,

$$p = \vartheta_T \tag{1}$$

and the tourists are willing to buy with a probability smaller than 1,

$$\sigma_2(\vartheta_T, B) = y < 1 \tag{2}$$

then ~~okay what's the~~ what's the expected payoff here of the firm? ~~Then will~~ they sell for sure to the business travelers ~~that's~~ μ_0, and ~~then~~ they sell with probability y to the tourist.

$$\mathbb{E}\left[u_1(\vartheta_T, \sigma(\theta))\right] = (p - c)\left[\mu_0 + y(1 - \mu_0)\right] \tag{3}$$

[11:29] ~~Now~~ in a deviation where they set the price slightly lower,

$$\text{[Deviation]}: \quad p = \vartheta_T - \varepsilon \tag{4}$$

~~so~~ deviation we set it slightly lower what do we get ask the utility, and then we sell to everybody.

$$\mathbb{E}\left[u_1(\vartheta_T - \varepsilon, \sigma(\theta))\right] = (p - \varepsilon - c) \cdot 1 \tag{5}$$

~~And so~~ if we solve this for ε,

$$\mathbb{E}\left[u_1(\vartheta_T - \varepsilon, \sigma(\theta))\right] \geq \mathbb{E}\left[u_1(\vartheta_T, \sigma(\theta))\right] \tag{6}$$

$$\Rightarrow \quad p - \varepsilon - c \geq (p - c)\left[\mu_0 + y(1 - \mu_0)\right] \tag{7}$$

then we see that ~~for any ε smaller or equal to~~

$$\varepsilon \leq (p - c)[1 - \mu_0 - y(1 - \mu_0)] = \underbrace{(p - c)}_{>0}(1 - \mu_0)\underbrace{(1 - y)}_{>0}. \tag{8}$$

This is profitable. ~~And we see indeed here~~ this is a positive quantity $[p - c > 0]$ ~~so yeah~~ because the price is larger than the cost, and there $[1 - y > 0]$ is a because y is smaller than 1.

[12:22] ~~All right so~~ we conclude that **if the equilibrium price is one of these two $[\vartheta_B, \vartheta_T]$, then those types of customers that are indifferent they have to buy it with certainty; otherwise, the airline would have an incentive to post a slightly lower price and instead sell to everybody.** So, there is only two options left ~~so~~ either it's **the price for the willingness to pay for the tourists**, in that case we sell to everybody,

$$\begin{aligned} \mathbb{E}\left[u_1(\vartheta_T, \sigma(\theta))\right] &= p - c = \vartheta_T - c && \quad (9)\\ &= 10 - 5 = 5 && \text{data from p.3} \quad (10) \end{aligned}$$

or it's **the willingness to pay of the business travelers**, in which case we only sell ~~to only sell~~ to the business travelers.

$$\begin{aligned} \mathbb{E}\left[u_1(\vartheta_B, \sigma(\theta))\right] &= (p - c)\mu_0 = (\vartheta_B - c)\,\mu_0 && \quad (11)\\ &= (30 - 5) \cdot 0.3 = 7.5 && \text{data from p.3} \quad (12) \end{aligned}$$

~~So what we get is that you expect the utility here? I guess the equilibrium is either this and we set so this was 5000, or and so here we only sell to a fraction μ_0 and it turns out that means we had a seven and a half thousand per ticket.~~

[13:50] ~~And~~ as a consequence, ~~the best or the equilibrium will be or~~ the equilibrium outcome will be that the airline charges exactly this price $[\vartheta_B]$, and the business travelers always buy. ~~Now~~ if we were to write down the equilibria that lead to this outcome, then ~~well~~ **for the airline we simply choose price p,**

$$\sigma_1 = p \tag{13}$$

the business travelers ~~they~~ buy if the price is low or equal to their willingness to pay,

$$\sigma_2(\vartheta_B) = \begin{cases} B, & \text{if } p \leq \vartheta_B \\ N, & \text{otherwise} \end{cases} \tag{14}$$

and **for the tourists,**

$$\sigma_2(\vartheta_T) = \begin{cases} B, & \text{if } p < \vartheta_T \\ \Delta(A_2^1), & \text{if } p = \vartheta_T \\ N, & \text{if } p > \vartheta_T \end{cases}. \tag{15}$$

~~so~~ if we look for the actual equilibria in that case, the tourist ~~does~~ is willing to mix at this price here [$p = \vartheta_T$], and this does not lead to a contradiction because this price is not proposed by the firm. ~~So only~~ only the type ~~the type~~ who is that their willingness to ~~to~~ pay in the price that is posted only that type has to accept with probability 1. It's possible that the other type mixes even ~~at their~~ at their willingness to pay.

1.5 p.5 Full Information Benchmark

1.5.1 Price discrimination

[15:50] Now, we see that **this equilibrium is inefficient. This equilibrium does not sell any tickets to tourists even though tourists are willing to travel, and they are willing to pay for it.** ~~And~~ one could think now ~~well~~ if price discrimination was legal. ~~So~~ if the airline was allowed ~~to say~~ to ask what are you willing to pay and then simply charge that price, then we would get a more efficient equilibrium because everybody who is willing to ~~to~~ travel will travel.
[16:24] ~~And as a result yeah so~~ as a result, **if price discrimination was legal, then the airline could offer the the high price ticket to the business traveler, and the low price took it to the tourists, and this would increase efficiency.**

1.6 p.6 Offering a Menu of Options

1.6.1 Screening idea

[16:50] ~~Now~~ this was sort of a naive way to look at a streaming model we post one price, and then as a result we screen out the types but we don't sell to the low type that we screened out. ~~Now~~ the idea is: can we do better than simply offering one type of ticket for one price? ~~And~~ this is in general the idea behind screening models that we choose ~~some other type~~ some other ticket that we can offer. ~~Let's say~~ **we can have a non-flexible ticket which has the willingness to pay as on the previous slide, or we can offer flexible tickets which are worth more to both types of buyers, but they're significantly worth more ~~to the business type~~ to the business travel.**

1.6.2 Questions

[17:38] ~~And we can now say okay~~ if we now post prices for both of these tickets, can we both post these prices such that ~~the~~ the types have an incentive to ~~self~~ select self separate. ~~So~~ it is possible that in equilibrium ~~then the high type will or~~ the business type will buy the flexible ticket, and the tourist will buy the non-flexible ticket. ~~so~~ that's the first question.
[18:08] The second question is that ~~we want to address then is okau in what in~~ what way ~~do~~ you have to do it so that the business type indeed buys the ticket that was designed for them, and does not buy the other ticket instead.

[18:19] ~~And then~~ lastly, ~~what~~ we're interested in ~~is~~ what ~~is~~ the optimal menu of tickets [are] that we should offer to their customers.

1.7 p.7 Screening / Principal-Agent Models / Contract Theory

[18:29] ~~And so~~ this is the idea behind the screening models, also known as principal agent models, or contract theory. ~~so~~ three different names for all [are] the same thing.

1.7.1 Contract theory

[18:42] The idea is that ~~yeah~~ **the player who moves first, also known as the principal, offers a menu of choices, so contracts to the agents, which are the players with information that move second, and then in response to these contracts the agents will choose which contract that they prefer.**
[19:03] ~~And~~ we assume ~~here~~ that contracts are legally enforceable, so the principal has full commitment power to follow their contract.

1.7.2 Incentive problems

[19:13] ~~Now~~ there are two possible incentive problems that we could face: **if the uninformed player moves first, and these are adverse selection problems**, ~~Like like here in this~~ in this example where we have seller and buyer, then the agent has private information about the true valuation and as a result the seller faces adverse selection.
[19:34] **A second possible type of incentive problem would be moral hazard where the principal cannot perfectly observe the agent's actions.** ~~So let's say~~ if we elect a politician, then sure we don't know what their type is but maybe also we cannot really observe everything that they do, so they have a lot of secrecy, and we can't really observe exactly what they do which creates incentive for the politician ~~to~~ to abuse their power. ~~And so~~ we will address both of these incentive questions.

1.7.3 Contractible variables

[20:04] ~~Formally, a contractible variable so~~ a contract can only depend on what is known as a contractible variable ~~which is~~ which has to be observable and verifiable.
[20:18] ~~so~~ you have to ~~go~~ be able to go to court and say this is exactly what happened and I can prove it, and then you can enforce ~~the~~ the contract. ~~So~~ specifically, it cannot depend on the player's types, and it cannot depend on the hidden actions.
[20:34] ~~And so~~ today we'll just look at the hidden types case because that nicely ties into the signaling games.

2 The Revelation Principle

2.1 p.8 Standard Setting

2.1.1 Contracting

[20:47] ~~So~~ if we write down this model more formally, then the principal and the agent ~~they~~ write a contract on some contractible variable q. ~~And~~ this could be the type of ticket [or] ~~this could be~~ some performance measure for the politicians, and that is verifiable. ~~And~~ we denounced by $\mathcal{Q}$ ~~here~~, the set of all values that this contractible variable can take.

[21:16] ~~And in these~~ in these settings, what we typically have in mind is that one of the two is typically responsible for this contractible variable, and is then compensated for ~~for~~ producing that output. ~~Yeah so in~~ in the selling example which is the standard example, the principal provides the ticket q and is compensated for doing that through a price. ~~And~~ this will be sort of our standard notation for ~~for~~ these sections we think of ~~the screening~~ the screening of a monopolist seller as sort of the standard example.

[21:53] ~~And~~ in that case, we would write **the utility of the principal as where they are paid minus the cost of providing** q,

$$u_1(q,p) = p - c(q) \tag{16}$$

and for the agent, **the value that they assign to** q **minus the price that they paid for it.**

$$u_2(q,p,\vartheta) = v(q,\vartheta) - p \tag{17}$$

~~Now if~~ it was a different example if ~~let's say~~ we hire an agent to run the company for us, then instead we would pay the agent that the patient would provide us with the company's returns. In that case, it would be flipped, and ~~so~~ then many of ~~the~~ the signs and inequalities would change throughout ~~throughout~~ the lecture. ~~So~~ to have it simpler, we'll just focus on the screening case ~~in~~ in these slides, ~~and then~~ and then we'll see learn all the tools how to derive the optimal contract, and then we're able to also do it if the signs are a little bit different.

2.1.2 Monopolistic seller

[22:47] ~~Yeah so~~ for example, q could be either the quality or the quantity of what we sell to the buyer, and then p is the price.

2.2 p.9 Contract

2.2.1 Definition 5.1

[22:58] ~~And~~ the contract is in general simply a price function of the set of ~~all possible that~~ all possible values that this contractible variable can take.

$$p : \mathcal{Q} \to \mathbb{R} \tag{18}$$

~~So~~ we had this could be ~~the~~ the number of items that are bought.

2.2.2 Different applications

[22:16] ~~Then~~ in different applications, this could be labor contracting, it could be the number of hours that are worked, and then p is the wage for the number of hours that you worked.

[23:25] In an insurance contract, we would have the deductible, and then the premium.

[23:32] In other examples, if you provide some kind of service, ~~and then~~ there will be some performance measure on how good the service has to be, and then p is the corresponding payment. ~~So~~ as an example think of the train service that is run by some trained company and the government has certain mandates ~~on~~ on what the trained service needs to provide, and for doing that, they get a subsidy for running it.

[23:58] if you hire an assistant professor, there are some performance measures in place mostly quantity and quality of publications, and then p the payment would be a step-function indicating whether you get promoted or not.
[24:11] A kickstarter would be an example. You have ~~Let's say~~ different pledge levels, and then you assign the rewards, and then people can self-select do I want to pledge this level because I want these rewards and so on.
[24:23] ~~Or~~ in general venture capital any startup, you would get a level of investment or I guess p ~~this should be no p~~ would be the investment level, and then the return [q] would accrue to the principal.
[24:42] So, these are all applications that we have in mind when we look at these screening models.

2.3 p.10 The Agent's Preferences

2.3.1 Finitely many states

[24:50] ~~Now so~~ we assume that the contractible variables and the types of the players are a subset of the real numbers.

$$\mathcal{Q}, \Theta \subseteq \mathbb{R} \tag{19}$$

~~And~~ if we think of only two types, ~~let's say~~ business travel or tourist, this did not necessarily have to be a subset of real numbers. What the advantage of doing that is ~~is that~~ there is some kind of natural order between the two with which we can rank it. ~~There is always so so I guess~~ in a formal language, the types are well ordered, ~~there is a~~ there is a strict reference relation on the set of types, and the same for all the possible states.
[25:28] ~~So~~ for example, ~~so~~ this allows us ~~then~~ to rank types and output in some very natural way. ~~So~~ in our screening example, **we could think that ~~e~~ the cost [c] is an increasing in quantity [q], and the value [v] is also increasing in quantity [q] and in the player's type [ϑ].**
[25:54] ~~Then~~ in addition to the monotonicity assumption, **we assume that the value function of the agent [v] has increasing differences.** ~~So~~ this means that if the state improves from q_1 to q_2, then the additional utility that an agent get is increasing with their type. ~~So~~ **an increase an improvement in the state is valid more by higher types for all possible improvements in the state.**

2.3.2 Example

[26:26] ~~So~~ in the screening example, where we have quantities ~~this~~ means additional quantity is always valid more by the higher type, or for the airfares, we can identify the different ticket classes somehow with their cost to the airline, and then any upgrade is worth more to the business travelers. ~~And so this assumption~~ the increasing differences assumption will be again what allows us to get this separating behavior, but whenever the state improves, then the higher types are more willing to pay more for this improvement.

2.4 p.11 Topkis' Theorem

2.4.1 Theorem 5.2

[27:05] ~~And~~ formally, this is known as Topkis' Theorem if we have a function that has increasing differences, and we have a low type [ϑ_1] and a high type [ϑ_2], then the high type will always choose weekly more than the low type if they maximize the function.

$$\underset{q}{\operatorname{argmax}}\, \varphi(q, \vartheta_i) \tag{20}$$

2.4.2 Interpretation

[27:23] ~~Now we've seen I guess the proof already in the image in the screening example, but let's also see the formal proof.~~

2.4.3 Proof

[27:30] ~~So~~ let's start with ~~with~~ any two quantities ~~and~~ because q_1 or any two maximal choices. ~~So~~ if q_1 is the best response by type one [ϑ_1], then it must be better than q_2 or weekly better at least, and the same goes for quantity two for type two [ϑ_2].

$$\varphi(q_1, \vartheta_1) \geq \varphi(q_2, \vartheta_1) \tag{21}$$

$$\varphi(q_2, \vartheta_2) \geq \varphi(q_1, \vartheta_2) \tag{22}$$

~~Then~~ if we add the two inequalities, bring all the terms of type two to the left side and all the terms of type one to the right side, then we get this expression.

$$\varphi(q_2, \vartheta_2) - \vartheta(q_1, \vartheta_2) \geq \varphi(q_2, \vartheta_1) - \vartheta(q_1, \vartheta_1) \tag{23}$$

~~And~~ because of increasing differences, we must have that q_2 is at least as large as q_1.

$$q_2 \geq q_1 \tag{24}$$

[28:09] How can we see that? ~~Well suppose~~ suppose q_2 is smaller.

$$q_2 < q_1 \tag{25}$$

~~Then okay then~~ because we know something about an increasing difference, but this would then be a decreasing difference. Let's flip the inequality.

$$\varphi(q_1, \vartheta_1) - \varphi(q_2, \vartheta_1) < \varphi(q_1, \vartheta_2) - \varphi(q_2, \vartheta_2) \leq \varphi(q_1, \vartheta_1) - \varphi(q_2, \vartheta_1) \tag{26}$$

~~So we would get where is it but~~ because we know that φ has increasing differences. We know that the increase from q_2 to q_1 has to be worth more to the higher type, then to the lower type, ~~and~~ this is a contradiction.
[29:20] ~~So~~ because of increasing differences, it must be that the higher type chooses the higher quantity, and this is the same that we've seen ~~in the~~ in the graph ~~in~~ in the job market signaling example.

2.5 p.12 Spence-Mirrlees Single-Crossing Property

2.5.1 Continuum of states

[29:36] ~~And then~~ if we have a continuum of states, ~~so~~ continuum of contractible variables, then we again assume that we have the Spence-Mirrlees single-crossing property. ~~So~~ in that case, we do assume that the cost function and the value function are differentiable, and that they satisfy these conditions.

$$v_q(q, \vartheta) \equiv \frac{\partial v(q, \vartheta)}{\partial q} > 0 \tag{27}$$

$$v_\vartheta(q, \vartheta) \equiv \frac{\partial v(q, \vartheta)}{\partial \vartheta} > 0 \tag{28}$$

2.5.2 SCP implies ID

[29:55] ~~And~~ we have already seen that **the Spence-Mirrlees single-crossing property implies increasing differences**, so I put it again on the slides just for the sake of completeness, but we've already seen the proof that ~~well~~ [the] difference we can always write it as the integral over the marginal change in q.

$$v(q_2, \vartheta_2) - v(q_1, \vartheta_2) = \int_{q_1}^{q_2} v_q(q, \vartheta_2) dq \tag{29}$$

$$> \int_{q_1}^{q_2} v_q(q, \vartheta_1) dq = v(q_2, \vartheta_1) - v(q_1, \vartheta_1), \quad \forall q_2 > q_1, \vartheta_2 > \vartheta_1 \tag{30}$$

~~And~~ because we have a single crossing property that this has to be larger or equal than what the increase is worth to the lower type.

[30:23] ~~And so~~ we still have Topkis' theorem, and we still have that ~~that~~ the higher types do select a higher quantity.

2.6 p.13 Full-Information Benchmark

2.6.1 Definition 5.3

[30:35] ~~Now~~ as a useful benchmark is to look at what happens if the principals or whoever designs the contract have full information. ~~So~~ if the principal had full information about the players types, then ~~what~~ what menu of options would they choose? ~~So~~ **in that case they would for each type separately, they would maximize the profit that they can get subject to the participation constraint of the agent.**

2.6.2 Interpretation

[31:07] ~~So~~ they must give each agent at least their reservation utility, which is what they would do if they don't get ~~any~~ any quantity from the seller, so they must get at least that much to participate in a contract.

$$\max_{(q,p) \in \mathcal{Q} \times \mathbb{R}} \quad p - c(q) \tag{31}$$

$$\text{s.t} \quad v(q, \vartheta) - p \geq v(0, \vartheta) \qquad \text{participation constraint} \tag{32}$$

~~Now what so~~ in the optimum, clearly the participation constraint has to be binding; otherwise, we could increase the price and extract the higher surplus ~~from~~ from the agent. ~~So~~ as a consequence, we can rewrite the objective function.

[31:58] ~~So and so~~ if we maximize this over q,

$$p - c(q) = v(q, \vartheta) - v(0, \vartheta) - c(q) \tag{33}$$

we see that ~~well this is in~~ this is the total surplus here [$v(q, \vartheta) - v(0, \vartheta)$], and the total surplus is maximized [when] the marginal utility is equal to the marginal cost.

$$v'(q, \vartheta) = c'(q) \tag{34}$$

~~So~~ we see that this is efficient.

[32:36] ~~So~~ **the consumption level here is socially efficient because the total surplus is maximized**, and then it's all given to the principal because **the principal has the first mover advantage**. ~~And so~~ **this is**

why we call this the first-best because this is the socially optimal outcome that could be achieved if there was no private information.
[32:57] ~~Now~~ in this case, solving this maximization problem is simple, but we see the same techniques that we'll use later in the incomplete information case. We find out which constraints are binding, and we plug them into the value function, and then we differentiate.

2.7 p.14 Air Fares: First-Best Contract

2.7.1 Setup

[33:17] ~~Okay so now~~ let's look at our example of the airline. ~~And let's see~~ if we can extract a higher surplus, if we offer two types of tickets, if we offer flexible tickets with the intent that the business traveler will buy it, and then not flexible ticket. ~~And~~ we can see here that this payoff function satisfies increasing differences because the upgrade from the not flexible ticket to the flexible ticket is worth more to the business traveler.
[33:50] ~~So let's say~~ the cost of the regular ticket is 5,000, then the cost of the flexible ticket is 8,000. ~~And so~~ we see that the marginal increase in cost is larger than the marginal increase in the willingness to pay of the tourists, so we should give the tourists the unflexible ticket. ~~So~~ our contract would be that we give the tourists the ~~let's say~~ not flexible ticket for ~~for~~ 10,000, and for the high type we give them the flexible ticket for 40,000.

$$(q_N, 10), \quad (q_F, 40) \text{ unit: thousand} \tag{35}$$

~~And what's our~~ what's our expected profit here?

$$\mu \cdot (40 - 8) + (1 - \mu) \cdot (10 - 5) = 5 + 27\mu \tag{36}$$

$$\overset{\mu=0.3}{=} 13.1 \qquad\qquad \text{unit: thousand} \tag{37}$$

~~And so~~ this is higher than what we had before because we're now also able to sell ~~to the~~ to the tourists.

2.8 p.15 Direct Revelation Contracts

2.8.1 Definition 5.4 and Direct revelation contract

[35:17] ~~Now~~ the first best what's nice about? It is that there's one bundle that is presented to each type, and so we can think of the contract samples the type of bundles that is offered to each type instead of as a function from quantities to prices. ~~And~~ this is what is known a direct revelation contract. ~~And~~ the idea is that it's a price

$$p : \Theta \to \mathbb{R} \tag{38}$$

and a quantity

$$q : \Theta \to \mathbb{R} \tag{39}$$

for each possible type ~~that the types~~ that we could face, and the idea is that we want the types to self-select the bundle that ~~we~~ we designed for them.
[35:56] ~~So~~ **an additional condition that we need is that the revelation contract is incentive compatible.** ~~So~~ every type has an incentive to choose the bundle that we designed for them, and ~~so~~ this will be the additional constraint that we have to satisfy when we design the optimal contract.

2.8.2 First-best contract

[36:14] ~~Now~~ **the first best contract is a direct revelation contract because we have one bundle for each type, but it may not be incentive compatible. It may be that the business type is not actually willing to ~~buy~~ pay 40 for a ticket if they can get the non-flexible ticket for a price of 10.**

2.9 p.16 Revelation Principle

2.9.1 Proposition 5.5

[36:38] ~~And~~ our first result here is the revelation principle which says that for any contract that we could write for any contract as a function of the observed quantities, and any best response ~~by the any best response~~ by the agent, we can find a direct revelation contract that is incentive compatible, and produces the same output. ~~And~~ this is known as the revelation principle because it means that we can always find ~~so~~ for any contract, that is incentive compatible we can find an incentive compatible direct revelation contract.

2.9.2 Proof

[37:19] ~~So the proof okay~~ incentive compatibility means that ~~that~~ no type has an incentive to pretend to be a different type. ~~And so~~ [at] first, we plug in the definition of the direct revelation contract. ~~and~~ because in the indirect contract, ~~because~~ q_* is the best response, this has to be lower or equal than ~~if the type reported or~~ if ~~the player~~ the agent reported the type truthfully.

$$\begin{aligned} v(q_*(\vartheta'),\vartheta) - p_*(\vartheta') &= v(q_*(\vartheta'),\vartheta) - p(q_*(\vartheta')) && \text{indirect} && (40)\\ &\leq v(q_*(\vartheta),\vartheta) - p(q_*(\vartheta)) && \text{truthful} && (41) \end{aligned}$$

[37:52] ~~Now~~ it's not too surprising ~~perhaps in the~~ in the framework of contract theory, but this is a very important result in mechanism design where we have an equivalent revelation principle.

2.10 p.17 Optimal Contract

2.10.1 Definition 5.6

[38:10] ~~and~~ then, we can write the optimal contract. ~~Okay~~ we maximize the expected profit, and subject to the individual rationality constraints and the incentive compatibility constraints for all the possible types.

$$\begin{aligned} \max_{q:\Theta\to\mathbb{R},p:\Theta\to\mathbb{R}} \quad & \mathbb{E}\left[p(\theta) - c(q(\theta))\right] && && (42)\\ \text{s.t.} \quad & v(q(\vartheta),\vartheta) - p(\vartheta) \geq v(0,\vartheta) && \mathrm{IR}_\vartheta && (43)\\ & v(q(\vartheta),\vartheta) - p(\vartheta) \geq v(q(\vartheta'),\vartheta') - p(\vartheta') \quad \forall\vartheta' \in \Theta && \mathrm{IC}_\vartheta && (44) \end{aligned}$$

2.10.2 Interpretation

[38:25] ~~And~~ we now call this the second-best outcome because typically the consumption will be lower than in the first-best. ~~So~~ we'll not be maximizing social welfare anymore. ~~So~~ it'll be second-best subject to the incentive constraints by the agents.

[38:43] ~~And~~ because we have the revelation principle, we can without loss of generality work with the simpler direct revelation contract where we just design a bundle for each type, and then check incentive compatibility.

3 Screening Two Types

3.1 p.19 Setup With Two Types

3.1.1 Notation

[39:00] ~~So~~ let's first look at the setting with two types. **With two types, any direct relation contract is simply a pair of a bundle**

$$\{(q_H, p_H), (q_L, p_L)\} \tag{45}$$

that we designed for each type so similar to the first-best contract that we had in the airlines example.

3.1.2 First-best contract

[39:17] ~~And~~ what is the first-best contract in that case? ~~Well so the reservation utility~~ ~~so~~ **in the first-best contract, because the principal extracts the full surplus, each type will get exactly their reservation utility.**

$$v(q_H^*, \vartheta_H) - p_H^* = v(0, \vartheta_H) \tag{46}$$

~~This should be p_H^* there not p_L^*. So what~~ they get in the contract if they select the bundle that we present them, then they simply get their reservation utility.
[39:54] ~~And now~~ we can see that doing so is not incentive compatible, so no first-best contract will be incentive compatible. ~~And~~ how can we see this? ~~Well if so~~ if the high type would instead select the bundle that we signed for the low type, what would they get? ~~Well~~ they would get ~~this~~ this value [$v(q_L^*, \vartheta_H)$] here. ~~So~~ how much is it above the reservation utility precisely? How much they value the low bundle minus the price they pay for it?

$$v(q_L^*, \vartheta_H) - v(0, \vartheta_H) > v(q_L^*, \vartheta_L) - v(0, \vartheta_L) \qquad \text{increasing difference} \tag{47}$$

$$\Rightarrow \quad v(q_L^*, \vartheta_H) - v(0, \vartheta_H^*) - p_L^* > v(q_L^*, \vartheta_L) - v(0, \vartheta_L) - p_L^* = 0 \qquad \text{IR}_L \tag{48}$$

~~And~~ because ~~so~~ this here is a difference right in quality we increase it from 0 to q_L^*, and ~~so~~ the high type has to value it strictly more than the low type. But this [$v(q_L^*, \vartheta_L) - v(0, \vartheta_L)$] is precisely the utility that makes the ~~low price~~ low type indifferent. ~~And~~ as a consequence, the high type gets strictly more by choosing the low bundle, then when they choose the bundle that we designed for them. So, **no first-best contract is incentive compatible.**

3.2 p.20 Maximization Problem

[40:59] ~~So~~ how can we fix this by solving this maximization problem with two types? We can write it out very explicitly. We have four constraints: 2 individual rationality constraints and 2 incentive compatibility constraints.

$$\max_{(q_H,p_H),(q_L,p_L)} \quad \mu(p_H - c(q_H)) + (1-\mu)(p_L - c(q_L)) \tag{49}$$
$$\text{s.t.} \quad v(q_H,\vartheta_H) - p_H \geq v(0,\vartheta_H) \qquad \text{IR}_H \tag{50}$$
$$v(q_L,\vartheta_L) - p_L \geq v(0,\vartheta_L) \qquad \text{IR}_L \tag{51}$$
$$v(q_H,\vartheta_H) - p_H \geq v(q_L,\vartheta_H) - p_L \qquad \text{IC}_H \tag{52}$$
$$v(q_L,\vartheta_L) - p_L \geq v(q_H,\vartheta_L) - p_H \qquad \text{IC}_L \tag{53}$$

~~Now~~ we could simply apply Kuhn-Tucker to this ~~to this~~ maximization problem, and find the maximum. But it's instructive and also easier to first think about which constraints are actually binding, and which ones can be relaxed, and then we can simplify the objective function and maximize an unconstrained function.

3.3 p.21 Binding Constraints

3.3.1 Lemma 5.7

[41:39] ~~So~~ the following lemma states precisely which constraints are binding, and which ones are redundant in the two type case. ~~So~~ we have that the individual rationality constraint, so the participation constraint for the low type binds and for the high type the incentive compatibility constraint binds whereas the other two are redundant constraints.

3.3.2 IR$_H$ is redundant

[42:07] ~~So~~ let's start with the individual rationality of the high type. ~~And~~ this one is implied by not wanting to pretend to be the low type and the individual rationality of the low type, and we get this again by increasing differences.

$$v(\hat{q}_H,\vartheta_H) - v(0,\vartheta_H) - \hat{p}_H \geq v(\hat{q}_L,\vartheta_H) - v(0,\vartheta_H) - \hat{p}_L > 0 \tag{54}$$

~~So, It has to be better~~ by incentive compatibility, it has to be better to choose the bundle that is designed for you than to choose the bundle that is designed for the low type, but from the previous slide, choosing the bundle for the low type gives us a utility strictly above the reservation utility. ~~And~~ so, **the individual rationality constraint for the high type [IR$_H$] is actually implied by not wanting to pretend to be the low type [IC$_H$] and the participation constraint of the low type [IR$_L$].**

3.3.3 IR$_L$ is binding

[42:55] ~~And then~~ for the low type, we do get that the participation constraint is binding, and the idea is that **there always has to be one type for which the participation constraint binds; otherwise, we can simply increase the price for every type. If we increase the price for every type, this does not change incentive constraints.** And if no participation constraint was binding, then we would increase our expected profit. So, we conclude that participation constraint for the low type has to be binding.

3.3.4 IC$_H$ is binding

[43:31] ~~And then knowing these two constraints binding or not binding, respectively, we can then move on to the incentive constraint of the high type. And~~ to show that it is binding, suppose that it is not. If it was not binding, then we can increase this time not all the payments but simply the payments by the high type. ~~Now~~ because the incentive constraint is not binding, then by increasing, ~~it then~~ the incentive constraint is still satisfied while it relaxes the incentive constraint of the low type because the low type has a smaller incentive to deviate to a bundle that is priced even higher. ~~And~~ it leaves the individual rationality constraint of the low type, and affect it. ~~So~~ **if he was not binding, we could improve our profit by simply charging more for the high bundle.** ~~And~~ as a consequence, the incentive compatibility constraint of the high type has to be binding.

3.4 p.22 Proof of Lemma 5.7

3.4.1 IC$_L$ is redundant

[44:36] ~~And then~~ we move on to the incentive compatibility of the low type. ~~Now~~ one very typical way of showing that a constraint is redundant is [that] we **look at the optimal solution without this constraint.** ~~And~~ then, we **show that this constraint is satisfied by this solution.**
[44:56] ~~And~~ the idea is that if we just drop the incentive constraint of the low type, then we have a maximization problem with fewer constraints. So, the maximum has to be better. ~~So~~ if the maximum also satisfies the constraint that we dropped, then the two maximization problems have to be the same. ~~So~~ let's denote here by this bundle here with r for the relaxed problem ~~the~~ solution if we drop the low types instead of constraint.

$$\{(q_H^r, p_H^r), (q_L^r, p_L^r)\} \tag{55}$$

[45:29] ~~Then~~ **if this constraint is not redundant, then it means that the low type has an incentive to deviate. This means that the low type ~~would get~~ would improve their utility by some positive amount, ~~let's say~~ ε, by choosing the bundle that was intended for the high type.**

$$u_2(q_H^r, p_H^r, \vartheta_L) - u_2(q_L^r, p_L^r, \vartheta_L) = \varepsilon > 0 \tag{56}$$

3.4.2 Offering only one bundle

[45:51] ~~And~~ the idea is that ~~now~~ we can construct a profitable deviation by the principle to exploit this ~~this~~ difference, and the idea is that we offer only one bundle to the two types in which case the incentive constraints are trivially satisfied because there is only one bundle to pick.
[46:12] ~~So let's say~~ if our profit is higher from the low bundle,

$$p_H^r - c(q_H^r) < p_L^r - c(q_L^r) \tag{57}$$

in that case we can just only offer the low bundle $[(q_L^r, p_L^r)]$. ~~and and yeah so~~ the incentive constraint for the high type is necessarily satisfied because we only offer one bundle, and the individual rationale constraint for the low type is satisfied ~~because~~ because it was already in the original contract here the low bundle was designed for the low type.
[46:53] ~~Now if~~ it's the other way around. If the high bundle is more profitable for the principal,

$$p_H^r - c(q_H^r) \geq p_L^r - c(q_L^r) \tag{58}$$

then we can simply offer the high bundle and charge an additional ε for the high bundle $[(q_H^r, p_H^r + \varepsilon)]$.

[47:05] This is again incentive compatible because there is only one constraint, and it's also satisfies the participation constraint of the low type because they got ε above their reservation utility from choosing the high bundle. ~~So~~ if we charge ε more for the high bundle, then they're still willing to buy it.

$$u_2(q_H^r, p_H^r + \varepsilon, \vartheta_L) = u_2(q_L^r, p_L^r, \vartheta_L) \geq v(0, \vartheta_L) \tag{59}$$

[47:30] ~~And so~~ this means that this is a contract that we have assumed to be the solution for the relaxed problem is not actually optimal if the incentive constraint for the low type is violated. So, this must imply is that the incentive compatibility of the low type ~~this constraint~~ is actually redundant.

3.5 p.23 Eliminating Payments from the Objective Function

[47:50] ~~So what~~ we're left with is two binding constraints and two constraints that we can draw. ~~Now~~ we can simplify our objective function by solving the binding constraints for two variables, and then replacing these variables in the objective function, and then maximizing a ~~not~~ unconstrained maximum.

3.5.1 Incentive-compatible payments

[48:10] ~~So let's say~~ we want to replace the player's payment. ~~Then okay~~ from the binding incentive constraint of the high type, we see that the payment of the high type has to exceed the payment of the low type precisely by the marginal utility that the high type gets from picking their own bundle over the bundle intended for the low type.

$$v\left(\hat{q}_H, \vartheta_H\right) - \hat{p}_H = v\left(\hat{q}_L, \vartheta_H\right) - \hat{p}_L \qquad \mathrm{IC}_H \tag{60}$$

$$\Rightarrow \quad \hat{p}_H = \hat{p}_L + v\left(\hat{q}_H, \vartheta_H\right) + v\left(\hat{q}_L, \vartheta_H\right) \tag{61}$$

[48:38] ~~And~~ we get the payment for the low type by the binding participation constraint.

$$v\left(\hat{q}_L, \vartheta_L\right) - \hat{p}_L = v\left(0, \vartheta_L\right) \qquad \mathrm{IR}_L \tag{62}$$

$$\Rightarrow \quad \hat{p}_L = v\left(\hat{q}_L, \vartheta_L\right) - v\left(0, \vartheta_L\right) \tag{63}$$

3.5.2 Information rent

[48:49] ~~Now~~ in order to ~~to~~ rewrite the objective function in the simplest way, let's now see what we can say already about ~~the~~ how much the high types utility exceeds the reservation utility. ~~So~~ what is this difference here in the contract?

$$v\left(\hat{q}_H, \vartheta_H\right) - \hat{p}_H - v(0, \vartheta_H) = v\left(\hat{q}_L, \vartheta_H\right) - v\left(\hat{q}_L, \vartheta_L\right) - \left(v(0, \vartheta_H) - v(0, \vartheta_L)\right) \geq 0 \tag{64}$$

~~And~~ using the payment of the high type and the low type, we see that it is equal to this difference here, the difference between choosing the low bundle type, and the difference between the reservation utilities. ~~And~~ because we have increasing differences, this inequality is strict if the low type gets a strictly positive quantity. [49:35] What this means is that the high type ~~now~~ gets a utility from the contract that it strictly exceeds the reservation utility, and we call this the **information rent**. So, we have to pay the high type something above the reservation utility for them to be willing to participate and to choose the bundle that we designed for them. [49:57] ~~And using~~ using this expression here for the high types information rent, we can now rewrite the objective function, and we can drop these two terms $[v(0, \vartheta_H) - v(0, \vartheta_L)]$ from the objective function because they don't depend on any terms that are chosen by the principle. ~~So~~ if we add or subtract this constant, it does not affect what the maximum will be.

3.6 p.24 Simplifying the Objective Function

3.6.1 Separable objective function

[50:21] ~~And then~~ we obtain this objective function here where we have replaced the payments of the two bundles using the two binding constraints that we had, and we see that the objective function has this ~~this~~ nice form where we have first the expected total surplus and minus the expected information rent that we pay to the high type.

$$V(q_H, q_L) \equiv \underbrace{\mu\left(v(q_H, \vartheta_H) - c(q_H)\right) + (1-\mu)\left(v(q_L, \vartheta_L) - c(q_L)\right)}_{\text{Total expected surplus}} \tag{65}$$

$$- \quad \underbrace{\mu\left(v(q_L, \vartheta_H) - v(q_L, \vartheta_L)\right)}_{\text{Expected information rent of } \vartheta_H} \tag{66}$$

3.6.2 Optimum

[00:00] ~~Good let's now~~ let's now see how this objective function is maximized. ~~So is the mic on can you yes okay so~~ the advantage of replacing or finding out which constraints are binding and which ones are not is that we can simply solve the binary constraints for one of these variables and eliminate those variables and then simply maximize this unconstrained optimization problem.
[00:31] ~~And~~ this optimization problem actually has a relatively simple structure because it's separable in the quantities that we give to the high type and the low type. ~~So,~~ we see here that **for the high type, the quantities only appear within this first term here** $[v(q_H, \vartheta_H) - c(q_H)]$**, and the information rent and the expected surplus of selling to the low type do not depend on the quantities that we designed for the high type at all.** ~~And so~~ as a consequence, **if we maximize this first expression** $[v(q_H, \vartheta_H) - c(q_H)]$**, this is precisely ~~where~~ where the derivative of cost is equal to the derivative of benefit. So, this is the same quantity as we sold them in the first-best.**
[01:13] ~~And~~ **then, the maximum with respect to the low type.** ~~So one thing that we know here is because~~ if we assume that v and c are differentiable, we know that the objective function is also differentiable. This means that the maximum is attained either for q_L equal to 0, or where the first order necessary condition is as satisfied. ~~And so~~ if we take the derivative with respect to q_L and solve for ~~say~~ the cost, then the marginal cost is equal to the marginal benefit plus this term here.

$$\frac{\partial V(q_H, q_L)}{\partial q_L} = (1-\mu)\left[v_q(q_L, \vartheta_L) - c'(q_L)\right] - \mu\left[v_q(q_L, \vartheta_H) - v_q(q_L, \vartheta_L)\right] = 0 \tag{67}$$

$$\Rightarrow \quad (1-\mu)c'(q_L) = (1-\mu)v_q(q_L, \vartheta_L) + \mu\left[v_q(q_L, \vartheta_L) - v_q(q_L, \vartheta_H)\right] \tag{68}$$

$$\Rightarrow \quad \underbrace{c'\left(\hat{q}_L\right)}_{=MC} = \underbrace{v_q\left(\hat{q}_L, \vartheta_L\right)}_{=MB} + \frac{\mu}{1-\mu}\left[v_q\left(\hat{q}_L, \vartheta_L\right) - v_q\left(\hat{q}_L, \vartheta_H\right)\right] \tag{69}$$

~~And~~ what is this term? ~~This term is~~ this term is the derivative of the information rent of that we need to give to the high type. ~~And~~ we see that ~~because we have~~ because it's increasing quantities [which] are increasing in type, [and] ~~what~~ we get is that this term here is negative. So, **the high type will always value the quantity**

designed for the low type more than the low type itself. ~~So this this term here~~ the second term here will be negative, and this means ~~that~~ that there will be a downward distortion ~~of~~ of the low type.

[02:35] The idea is that what we do here when we optimize this contract. Then, ~~okay~~ **for the high type**, the goal is the same as in first-best, but when we maximize with respect to the quantity for the low type, we have ~~two~~ two objectives: **one is maximize the expected surplus that we get from selling to the low type, and the other one is minimizing the expected information rent that we get to the high type.** ~~And so~~ to balance these two constraints ~~off~~, ~~we will~~ we are willing to accept a lower expected surplus from selling to the low type in order for the information rent that we need to give the high type for that to be lower.

3.7 p.25 Optimal Contract with Two Types

[03:16] ~~And so~~ if we summarize the optimal contract, then ~~yeah~~ the high type will get what they get in the first-best.

$$(1-\mu)\left(v(q_L,\vartheta_L)-c(q_L)\right) \geq \mu\left(v(q_L,\vartheta_H)-v(q_L,\vartheta_L)\right) \tag{70}$$

So, we say that there is no distortion at the top, and ~~the low type where the low type~~ the low type might not get anything.

$$\hat{q}_L = 0 \tag{71}$$

It might be possible that the optimal contract. That is, ~~just says~~ we don't sell to the low type at all. ~~And~~ this will happen precisely if the expected information rent would be more than the expected benefit from selling to them.

[03:49] ~~So if~~ if the expected benefit from selling to the low type is larger than the information rent, then we do sell to the low type, and ~~then~~ it's the solution to the first-order constraint. ~~And~~ then second once we know the quantities, we can solve for the payments of the two players through the two binding constraints that we had.

3.7.1 Shutdown

[04:11] ~~And to introduce some terminology,~~ we say that the low type is shut down if they don't get any quantities at all.

[04:18] We also say that the optimal contract involves shutdown in that case.

3.8 p.26 Comparison to the First-Best Contract

3.8.1 Optimal contract without shutdown

[04:25] ~~Yeah so I guess~~ the main ~~domain~~ intuition to take away from this result is that there is no distortion at the top the highest type gets what they would get in the first-best. And all the other types in this case, there is just one type but all the other types their quantities is distorted downwards. So, ~~that~~ we can save the information rent or some of it.

3.8.2 Optimal contract with shutdown

[04:53] ~~Yeah~~ if the optimal contract involves shutdown, ~~then~~ then the low types contract will simply be $(0,0)$. ~~And as a result~~ actually, the high type simply gets their reservation utility. ~~So~~ in that case, the high type does not get any information rent.

3.8.3 Different ordering of types

[05:14] ~~Now I should mention that if~~ we have a different parameterization [that] ~~let's say if~~: if we have a setting where the agent provides q to the principal, ~~so let's say~~ if the agent is a manager that manages a company. In that case, ~~um let's say~~ the order of types ~~would be~~ could be reversed if ϑ is sort of the cost of effort. ~~And so~~ if a numerical high type means a large cost, then we would get a reversal of the different types. In that case, the highest numerical type will be the type with the binding incentive constrained, and so on.
[05:50] So, whenever something of the model changes, then ~~we have to be careful to to um then~~ we can't just apply the theorem straight out. What we have to do in that case is we have to just use the same steps that we did figure out which ones are the binding our constraints, and then solve the maximization problem in the same way.
[06:10] ~~So~~ this is why I showed how we derive the results so that we can also use it when the model is slightly different.

3.9 p.27 Air Fares: Second-Best Contract

3.9.1 First-best contract

[06:18] ~~Good let's return to our example and~~ see what the second-best contract is here. ~~So~~ we've already found the first-best contract. ~~And~~ in this case, the theorem pretty much applies because we have exactly this setting that we are selling to a buyer who values the good at some type dependent way. So, what we get certainly in the contract here is that there is no distortion at the top. ~~So~~ the top gets the flexible ticket.

$$\hat{q}_H = q_H^* = \text{ flexible ticket} \tag{72}$$

3.9.2 Second-best contract

[07:11] ~~And~~ the decision where we should sell to the tourists ~~it~~ all depends on whether the expected information rent [from the high type] will be higher than the expected benefit from the low type. ~~So~~ for the expected benefit from the low type, ~~So if we sell them to~~ if we sell a positive quantity, no this should be I guess it'll be five, it will be the price of the ~~unflexible~~ not flexible.

$$\underbrace{(1-\mu)(10-5)}_{\text{expected benefit from selling to } \vartheta_L} \overset{\mu=0.3}{=} 3.5 < \underbrace{\mu(30-10)}_{\text{expected information rent of } \vartheta_H} \overset{\mu=0.3}{=} 6 \tag{73}$$

~~Ticket minus the cost so~~ this is ~~the~~ the expected benefit from selling. And if this is lower than the information rent, then we don't sell to the low type at all. What's the information rent? If we go back to the theorem, then the expected information rent is the expected difference in utilities ~~that~~ that the two types have in the low bundle. ~~And so for the low bundle, that is let's see what it was so pro bundle that's a 20. And~~ we can solve this for μ. ~~And what we get is. Yeah they should of course go the other direction.~~

$$5 < 25\mu \quad \Rightarrow \quad \mu > \frac{1}{5} \tag{74}$$

[09:44] ~~So~~ if the ratio the population ratio of business travelers is sufficiently high, then it's optimal to not sell to the tourists. But if the fraction of business travelers are small, then we must sell to the tourists as well. ~~And so in that case so~~ in this case, first q_L is 0, or is a no bundle at all. ~~So i guess~~ we could use 0 to denote that.

$$\mu > \frac{1}{5} \quad \Rightarrow \quad q_L = 0 \tag{75}$$

~~And~~ the optimal contract would be ~~so~~ if $\mu > \frac{1}{5}$, then we sell at the optimal price which was just 40 and 0.

$$\{(q_H^*, 40), (0, 0)\} \tag{76}$$

And if shutdown is not optimal [$\mu \leq \frac{1}{5}$], ~~then well~~ then the price for the low ticket ~~will have to bind with the insert~~ will have to be determined by the binding incentive constraint of high type. So, the low type gets the price. ~~So~~ in case [that] you do sell to the low type, it's the most flexible ticket.

$$q_L = \text{ not flexible} \tag{77}$$

$$p_L = 10 \tag{78}$$

$$p_H = p_L + v(q_H, \vartheta_H) - v(q_L, \vartheta_H) = 10 + 40 - 30 = 20 \qquad \text{IC}_H \tag{79}$$

We sell it at price 10 which makes the low type precise, and different we've been participating in not. ~~And we get that for the high type~~ for it to be incentive compatible to buy the high price ticket, a deviation has to not be profitable. [Then,] ~~we get so~~ we can only get ~~10~~ 10 more than the low price because this is precisely where the incentive comes straight fine.

[12:31] So, we see that if the optimal contract decides to both types, then the resulting price will be ~~much~~ much lower for the high type simply because the binding and standard constraint will force the price to be way lower.

3.10 p.28 Mobile Data

3.10.1 Mobile data plans

[12:48] ~~I already advanced the next example okay the next example is now now exactly what we have in the theorem because now~~ there is a quantity ~~that~~ that we provide to the customer. ~~So here~~ the quantity is ~~the~~ the bandwidth of the mobile data plan, and the utilities are given in in the following way here.

$$u_1(q, p) = p - \frac{3}{\log 2} q^2 \tag{80}$$

$$u_2(q, p, \vartheta) = \vartheta \frac{\log(q+1)}{\log 2} - p \tag{81}$$

~~So~~ the utilities have to be increasing. ~~And~~ in this case, the users have a logarithmic utility. ~~And~~ the reason why we divided by 2 is simply so that when q is 1, then the value is precisely equal to ϑ.

[13:36] ~~And~~ for ~~the~~ the carrier, ~~let's say~~ ChungHwa, what we get is that the cost. ~~Let's say~~ it's a quadratic cost of providing bandwidth at some point. They simply reach their bandwidth capacity.

[13:52] ~~And so let's see what in this case now let's solve this from from start to to finish~~ what is the first-best contract, [and] second-best contract? ~~And~~ let's compare the two.

3.10.2 First-best contract

[14:15] ~~All right for the first best contract okay so~~ **[for] the first best contract, we maximize the surplus for each type individually. ~~And~~ maximizing the surplus means we just add the utilities ~~of the~~ of**

the buyer and the seller.

$$
\begin{aligned}
u_1 + u_2 &= p - \frac{3}{\log 2}q^2 + \vartheta\frac{\log(q+1)}{\log 2} - p && (82)\\
&= \vartheta\frac{\log(q+1)}{\log 2} - \frac{3}{\log 2}q^2 && (83)\\
&\Rightarrow \vartheta\log(q+1) - 3q^2 && \text{get rid of constant} \quad (84)
\end{aligned}
$$

Maximizing this utility is the same as if we multiply the entire thing by constant. ~~So~~ let's get rid of the constant here. Simplify it and let's now take the derivative.

$$
\frac{\partial(u_1+u_2)}{\partial q} = \frac{\vartheta}{q+1} - 6q = 0 \tag{85}
$$

~~And~~ we will get a quadratic equation. Let's multiply everything with $q+1$, and then ~~say~~ divide by -6. ~~And~~ we use the quadratic formula to solve it.

$$
\begin{aligned}
& \vartheta - 6q(q+1) = 0 && (86)\\
\Rightarrow\quad & \vartheta - 6q^2 - 6q = 0 && (87)\\
\Rightarrow\quad & q^2 + q - \frac{\vartheta}{6} = 0 && (88)\\
\Rightarrow\quad & q = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\left(-\frac{1}{6}\right)}}{2\cdot 1} = \frac{-1 \pm \sqrt{1+\frac{2}{3}\vartheta}}{2} && (89)
\end{aligned}
$$

~~So that would be minus one now~~ the two solutions were only interested in the positive solution.

$$
q = \frac{-1+\sqrt{1+\frac{2}{3}\vartheta}}{2} \tag{90}
$$

~~So this is for any ϑ let's now see~~ for the low type [$\vartheta_L = 120$], ~~so if we plug in 120 we'll get the square root of 81 so four so~~ we'll give 4 GB to the low type.

$$
[\vartheta_L]: \quad q_L^* = \frac{-1+\sqrt{1+\frac{2}{3}\vartheta_L}}{2} = \frac{-1+\sqrt{1+\frac{2}{3}\cdot 120}}{2} = 4 \tag{91}
$$

~~And~~ to the high type [$\vartheta_H = 660$], ~~we get 660 to twenty four forty four forty one square root of four forty one this would be twenty~~ it's one 10 GB.

$$
[\vartheta_H]: \quad q_H^* = \frac{-1+\sqrt{1+\frac{2}{3}\vartheta_H}}{2} = \frac{-1+\sqrt{1+\frac{2}{3}\cdot 660}}{2} = 10 \tag{92}
$$

So, these are the data plans that if we can perfectly discriminate, this is ~~the~~ the amount that we would give them. ~~And~~ how much should we charge for those?

[18:12] ~~Also~~ **in the first-best, we don't need to pay information around. So, in the first-best, we can simply extract the entire surplus.** ~~Which means that okay~~ what's the reservation utility? The reservation utility here is 0, and ~~so~~ we simply the price isn't simply equal to

$$
p_\vartheta = v\left(q_\vartheta, \vartheta\right). \tag{93}
$$

[18:47] ~~So~~ for the high type, what we will get is

$$
p_H = \vartheta_H\frac{\log(q_H+1)}{\log 2} = 660\cdot\frac{\log(10+1)}{\log 2} \approx 2283.2249 \approx 2283 \tag{94}
$$

and for the low type,

$$p_L = \vartheta_L \frac{\log(q_L+1)}{\log 2} = 120 \cdot \frac{\log(4+1)}{\log 2} \approx 278.6314 \approx 279. \tag{95}$$

We see that **the price for the high type is very high because this is the first-best with price discrimination, where ChungHwa extracts the entire surplus.**

3.10.3 Second-best contract

[19:25] [However,] ~~and~~ in the second-best, we will see that the price will drop significantly because in this case the high type would have an incentive to deviate. ~~And~~ by the low type, even though ~~the high type~~ the high data users ~~would have~~ would like to use more data, ~~but~~ it's simply too highly priced if the buyers have private information.

[19:49] ~~Now,~~ for the second-best contract, ~~so~~ this utility is increasing. It satisfies the signal crossing property, so we can again apply the theorem straight away, and the quantity in the optimal contract for the high type is again 10 GB,

$$\hat{q}_H = q_H^* = 10 \tag{96}$$

but the quantity for the low type now ~~we~~ will ~~now~~ be lower because we trade off the expected payoff or the benefit from selling and the expected information rent.

[20:33] ~~So~~ let's first see what the solution is to this equation here [p.24 01:13]. ~~So c prime so~~ $c(q)$ was the quadratic function. ~~So~~ if we take the derivative, we get

$$\underbrace{c'(\hat{q}_L)}_{=MC} = \underbrace{v_q(\hat{q}_L, \vartheta_L)}_{=MB} + \frac{\mu}{1-\mu}\left[v_q(\hat{q}_L, \vartheta_L) - v_q(\hat{q}_L, \vartheta_H)\right] \tag{97}$$

$$\Rightarrow \quad \frac{3 \cdot 2}{\log 2} q_L = \frac{\vartheta_L}{\log 2} \cdot \frac{1}{q+1} + \frac{\mu}{1-\mu}\left(\frac{\vartheta_L}{\log 2} \cdot \frac{1}{q+1} - \frac{\vartheta_H}{\log 2} \cdot \frac{1}{q+1}\right) \tag{98}$$

~~So~~ to simplify the equation, we just multiply everything by logarithm of two times $q+1$. ~~So we get rid of all these terms. Okay what will we obtain~~ $\cancel{6q}$ ~~keep this one and~~ we see again that there will be some kind of downward distortion because of the information rent that we're trading off with the expected utility.

$$6q_L(q_L+1) = \vartheta_L - \frac{\mu}{1-\mu}(\vartheta_H - \vartheta_L) \tag{99}$$

~~So~~ let's divide everything by 6, and then we get a similar quadratic equation that we've solved before.

$$q_L(q_L+1) = \frac{1}{6}\left[\vartheta_L - \frac{\mu}{1-\mu}(\vartheta_H - \vartheta_L)\right] \tag{100}$$

$$0 = q_L^2 + q_L + \frac{1}{6}\left[\frac{\mu}{1-\mu}(\vartheta_H - \vartheta_L) - \vartheta_L\right] \tag{101}$$

~~So~~ let's plug in some numbers, and see what we obtain. ~~So the difference between how much they value the first GB is 540,~~ $\cancel{\vartheta_L}$ ~~is 120 and if we divide both by 60 a by 6.~~

$$q_L^2 + q_L + \frac{1}{6}\left[\frac{\mu}{1-\mu}(660-120) - 120\right] = 0 \tag{102}$$

$$q_L^2 + q_L + \underbrace{\left(\frac{90\mu}{1-\mu} - 20\right)}_{\equiv c} = 0 \tag{103}$$

~~So~~ let's ~~say~~ this is c all of this. ~~Then~~ the first thing that we can note is when will it be optimal to sell to the low type at all, when is this quantity strictly positive. ~~And~~ we see that is precisely the case if c is positive.

$$q_L = \frac{-1+\sqrt{1-4c}}{2} \tag{104}$$

~~So if c is positive or let's say~~ if c is 0 first, then this will exactly turn out to be 0,

$$q_L = \frac{-1+\sqrt{1-4\cdot 0}}{2} = 0 \tag{105}$$

and this is the cutoff layer [that] we don't provide any data plans ~~for~~ for casual users of data.
[24:51] ~~And so c is equal to zero where~~ let's ~~say~~ c is ~~larger~~ [less] than 0. ~~If well let's make common denominators~~

$$c \leq 0 \quad \Leftrightarrow \quad \frac{90\mu}{1-\mu} - 20 \tag{106}$$

$$\Leftrightarrow \quad \mu \leq \frac{2}{11} \tag{107}$$

[26:28] ~~And so again~~ we see that if the fraction of the population that doesn't value data as much is sufficiently low, then it's optimal not to sell to them at all.
[27:26] ~~Now~~ if it is optimal to sell to them, then we solve this one. Let's pick a specific value for μ. ~~So let's say~~ there is 10% of heavy data users in the in the population.

$$\hat{q}_L = \frac{-1+\sqrt{1-4c}}{2} \overset{\mu=0.1}{=} \frac{-1+\sqrt{41}}{2} \approx 2.7 \tag{108}$$

What will our optimal contract be? ~~So~~ if we plug in μ equals to 0.1, then this will turn out to be 2.7 GB. ~~And we see~~ there is a downward distortion of ~~of~~ the quantity for the bandwidth that the low type gets. ~~And so this is~~ 0.1 is indeed in the case where it's optimal to sell to the low type. What's the price?
[28:35] For the price, we again find the price ~~from the~~ from the binding participation constraint. ~~So~~ it's comparable to the price that we've had in the first-best.

$$\hat{p}_L = v\left(\hat{q}_L, \vartheta_L\right) - v(0, \vartheta_L) \tag{109}$$

$$= 120 \cdot \frac{\log(2.7+1)}{\log 2} - 120 \cdot \frac{\log(0+1)}{\log 2} \approx 226.5030 \approx 227 \tag{110}$$

~~But now~~ for the high type, we get a reduction in the price because they otherwise would choose the contract intended for the low type. ~~And~~ it will be 1264, almost half the price from it was before.

$$\hat{p}_H = \hat{p}_L + v\left(\hat{q}_H, \vartheta_H\right) - v(\hat{q}_L, \vartheta_H) \tag{111}$$

$$= 227 + 660 \cdot \frac{\log(10+1)}{\log 2} - 660 \cdot \frac{\log(2.7+1)}{\log 2} \approx 1264.4582 \approx 1264 \tag{112}$$

[29:52] ~~So~~ these are two examples where we have exactly this seller-buyer setting, and we can more or less directly apply the theorem. ~~So~~ in this case where q takes a continuum of possible states, we can directly apply it in the airline example — we only have two types of tickets, then we need to be a little bit careful how we do it.

3.11 p.29 Labor Contracting

[30:15] ~~But~~ then, there could be other applications where ~~where~~ a similar approach works, but the theorem itself does not apply straight away. So, one such application could be labor contracting where a union of workers negotiates with ~~with~~ a firm about the contract that they should be giving to their workers. ~~And~~ the union doesn't know what type of the firm, [and] ~~it is~~ the union does not know what is the efficiency of the firm ~~for~~ for the labor that they use. So, they don't know whether they should offer a high a contract with high pay for high amount of labor or a contract with low pay for lower-minded labor.

[30:58] ~~And so~~ the idea is now that ~~well~~ we just offer both possible contracts, then the company can pick which one they would like to choose. ~~Now~~ in this case, ~~the utility of the union let's say~~ [it] is essentially the utility of the representative worker in the union, and ~~in that case~~ it makes it makes sense to assume that the worker is not actually risk neutral. It makes sense to **assume that the worker has a strictly concave utility function in payments**, and ~~so~~ this already differs from the theorem that we have derived because we have assumed that the payments were linear there. Nevertheless, we can use ~~the same idea~~ the same approach to also solve this example. Then, the productivity of the firm is ~~is~~ scales linearly in their type. ~~And the question is~~ the question [is] that we always ask **what's the optimal contract that the union should now offer to the firm.**

3.11.1 First-best contract

[32:15] ~~Now for the first-best contract~~ for the first-best contract not too much has changed. We add the utilities of union, and ~~and the firms and~~ then we take the derivative to maximize the joint surplus.

$$u_1 + u_2 = v(p) - q + \vartheta f(q) - p \tag{113}$$

~~So~~ let's first maximize the the chosen quantity the produced labor. ~~So~~ if we maximize this with respect to q, then we get

$$\frac{\partial(u_1 + u_2)}{\partial q} = -1 + \vartheta f'(q) = 0 \tag{114}$$

and ~~so~~ the optimal output level is

$$q = (f')^{-1} \left(\frac{1}{\vartheta}\right). \tag{115}$$

~~So~~ because f is strictly concave, we the derivative is invertible, and so we can explicitly solve for the the first-best labor that a firm of type ϑ should ~~should~~ ask the workers to use.

[34:10] Now, what's the outside option here? ~~Well~~ I guess a strike would be the outside option for the union if the negotiations fail, ~~so in case of a strike~~ then the firm does not have to pay anything to their workers, but also they won't have any labor available. So again, for $(0, 0)$, we get our reservation utility which is again 0. ~~And so~~ the payments of the firm are then exactly equal to ~~to~~ the utility of the first-best. ~~So~~ the payment for type ϑ would then be

$$p_\vartheta = \vartheta \cdot f(q_\vartheta) \tag{116}$$

~~And~~ that's the first best contract already.

3.11.2 Second-best contract

[35:17] For the second-best contract, what should we do? ~~Well okay~~ in general, we have this maximization problem with the four constraints. ~~So~~ what do we maximize? We have four values that we need to choose [p.20].

$$V(p_L, q_L, p_H, q_H) = \mu\,(v(p_H) - q_H) + (1-\mu)\,(v(p_L) - q_L) \tag{117}$$

[37:31] ~~And~~ the participation constraint would be

$$\vartheta_H f(q_H) \geq p_H \qquad \mathrm{IR}_H \tag{118}$$

$$\vartheta_L f(q_L) \geq p_L \qquad \mathrm{IR}_R \tag{119}$$

~~So~~ these are the individual rationality constraints. ~~And~~ then, we have the incentive compatibility constraints

$$\vartheta_H f(q_H) - p_H \geq \vartheta_H f(q_L) - p_L \qquad \mathrm{IC}_H \tag{120}$$

$$\vartheta_L f(q_L) - p_L \geq \vartheta_L f(q_H) - p_H \qquad \mathrm{IC}_L \tag{121}$$

It's been helpful to see ~~which~~ which ~~are~~ constraints are redundant. ~~And~~ typically, only one of the individual rationality constraints is binding, and the other one is implied by the incentive compatibility constraint.
[39:10] ~~And so here let's see also since~~ since f is positive, if we multiply it with a higher value, this [$\vartheta_H f(q_H)$] is an increase. ~~So~~ this means that in this case ϑ_H will ~~also~~ be the high type. And we can show that the individual rationality constraint of the high type is indeed redundant.
[40:00] ~~So~~ for the high type, what do we need that the reservation you at least get the reservation utility which is 0? ~~And the idea is now to~~ [at] first, use the incentive constraint.

$$\vartheta_H f(q_H) - p_H \geq \vartheta_H f(q_L) - p_L \qquad \mathrm{IC}_H \tag{122}$$

$$\geq \vartheta_L f(q_L) - p_L \qquad \text{increasing differences} \tag{123}$$

$$\geq 0 \qquad \mathrm{IR}_R \tag{124}$$

[40:45] ~~and~~ so, indeed individual rationality for the high type is implied by incentive compatibility, increasing differences and individual rationality of the low type. ~~Then~~ for the other constraints, the argument is essentially the same to those that we had because in this example the order of the types is preserved, ~~so~~ ϑ_H is indeed the higher type. ~~And~~ in that case, we also get that the individual rationality constraint of the low type is binding because for at least one type, it's always binding, and then we get in the same fashion to before that the incentive constraint of the low type is redundant.
[41:35] ~~And so~~ in that case, we can again eliminate two variables for a form our optimization problem, and we can choose which two variables [are]. ~~This is and~~ if we look at our optimization function here, then it might actually be simpler if this time we replace the quantities with labor because ~~the~~ the value function is linear in quantities, and so we don't have to use the chain rule when we differentiate even though both will lead to the same solution. I would say this time it's simpler if we replace the quantities and not the prices. ~~So~~ let's see what we get.
[42:14] ~~So~~ for the quantity, the working hours of the low type,

$$q_L = (f)^{-1}\left(\frac{p_L}{\vartheta_L}\right) \tag{125}$$

~~We~~ this is given by the participation constraint. ~~And the quantity for the high type, is the quantity for the low type and then let's see the incentive constraint~~ ~~So~~ if we solve the incentive constraint for q_H, ~~first we move on~~

~~these to the other side so okay let's so first we have and we divide by ϑ_h and then we plug in what~~ we have found for the low type

$$f(q_H) = f(q_L) + \frac{p_H - p_L}{\vartheta_H} \qquad \text{IC}_H \tag{126}$$

$$= \frac{p_L}{\vartheta_L} + \frac{p_H - p_L}{\vartheta_H} \qquad \text{IR}_L \tag{127}$$

~~And so~~ if we solve it for q_h, get the inverse

$$q_H = f^{-1}\left(\frac{p_L}{\vartheta_L} + \frac{p_H - p_L}{\vartheta_H}\right). \tag{128}$$

[44:33] ~~And then so i guess~~ let's write out the function that we get at the end.

$$V(p_L, p_H) = \mu\left[v(p_H) - f^{-1}\left(\frac{p_L}{\vartheta_L} + \frac{p_H - p_L}{\vartheta_H}\right)\right] + (1-\mu)\left[v(p_L) - f^{-1}\left(\frac{p_L}{\vartheta_L}\right)\right] \tag{129}$$

[45:20] Now, two things are different from what we have had previously. ~~So~~ this is no longer separable in the ~~two~~ two quantities because it's connected here through this f^{-1}. ~~So~~ this is one difference ~~if players or~~ if one of the players is risk-averse. So, what we obtain is a maximization problem that is no longer separable. ~~So~~ it won't necessarily be true that ~~one~~ one player gets the first-best.
[45:51] Second, whether or not this will have a nice analytical solution what the result is will depend on precisely the assumptions on our v and f ~~and f~~ that we make. ~~And~~ but if we ~~let's say~~ choose ~~a~~ v as logarithm, and that is the square root of something, then we would be able to solve it. ~~And~~ we simply get a 2 by 2 system of equations in the two unknowns that we have whether how difficult it is to solve depends again on the specific functions that we deal with. ~~But~~ in general, this is the approach ~~we~~ we verify which type is ~~is~~ the lowest type. So to speak, ~~so which type~~ for which type the incentive, for which type the participation constraint finds, ~~and~~ sometimes it may be easier to solve it if we replace the quantities instead of the payments.

4 Ex-Ante Contracting

4.1 p.30 Funding Your Start-Up Idea

4.1.1 Venture capital

[00:11, point 1 and 2] ~~The idea behind excellent day~~ contracting is that **sometimes we write a contract before the agent has full information.** ~~And~~ such would be an example of a startup where you go see some venture capitalists, and ~~you~~ you want to have them invest in your startup. ~~And~~ in return, they will want to get ~~some~~ some return on their investment.

4.1.2 Difference

[00:34 point 1 and 2] ~~And~~ so, they can offer a menu of contracts that you have to accept ~~before you before you know your time, so before you know how good your business idea really is whether~~ before you know whether there is a market for your idea.
[00:48 point 2] ~~And so~~ Dragons' Den here ~~is one of these~~ is one of the British TV shows where ~~where~~ they decide on air whether they will offer to invest or not, and what we wonder now is how does this fact that the agent does not have full information at the time of contracting? How does that change contracting?

4.2 p.31 Ex-Ante Contracting

4.2.1 Participation and incentive constraints

[01:10, point 1] ~~Now the idea is that~~ the principal offers the menu of contracts at the beginning before the agent knows which type they are. ~~And so~~ **when the agent decides to participate or not, they don't know yet which type of they will be.** ~~**And**~~ **as a result, the participation constraint has to hold an expectation.**

$$[IR_0]: \quad \mu \underbrace{(v(q_H,\vartheta_H) - p_H - v(0,\vartheta_H))}_{\text{surplus of } \vartheta_H} + (1-\mu) \underbrace{(v(q_L,\vartheta_L) - p_L - v(0,\vartheta_L))}_{\text{information rent of } \vartheta_L} \tag{130}$$

~~So here~~ on the left hand side, we have the surplus of the high type, and [on the right hand side] here we have the surplus or the information rent of the low type. ~~And~~ then, we simply take the expectation over it.

[01:42, point 2] ~~And~~ **the incentive constraint is asked** ~~**before so**~~ **after you've accepted the contract, the agent will learn their type, and then they have to choose either this one** [(q_H, p_H)] **or this one** [(q_L, p_L)]**, but the outside option is not available anymore.** ~~And so~~ in the same way as before, we have the incentive constraint of the high type will be the binding constraint where the high type that is the utility that I get minus the payment that they make [$v(q_H,\vartheta_H) - p_H$].

$$[IC_H]: \quad v(q_H,\vartheta_H) - p_H \geq v(q_L,\vartheta_H) - p_L \tag{131}$$

4.2.2 Full surplus extraction

[02:09, point 1] Now, in the optimum, we've seen that a participation constraint always binds, and since here there is always only one participation constraint. This means that **the ex-ante participation constraint binds in the equilibrium, and** ~~**so this means that**~~ **no information rent is paid an expectation to the agent.**

[02:28, point 2] ~~And~~ as a result, the principal can extract the full expected surplus. ~~So~~ this means that they get to implement the first-best quantities q_H^* and q_L^*. ~~And~~ the only question that we now wonder is what are the payments that support these quantities in an incentive compatible contract.

[02:49, point 3] ~~So the first thing that we can see is well~~ if the ex-ante participation constraint binds, then one of these two information rents [$v(q_H,\vartheta_H) - p_H - v(0,\vartheta_H)$ and $v(q_L,\vartheta_L) - p_L - v(0,\vartheta_L)$] has to be negative. If we solve this equation here, ~~when it spins, for let's say the information rent of the low type~~ then we see that it's some negative expression times information rent of the positive type.

$$v(q_L^*,\vartheta_L) - p_L - v(0,\vartheta_L) = -\frac{\mu}{1-\mu}\left(v(q_H^*,\vartheta_H) - p_H - v(0,\vartheta_H)\right) \tag{132}$$

So, **only one of them can have a positive information rent, and the other type pays for the information rent of the one type.** Now, if we want to figure out what the payments are, then we can substitute from the binding incentive constraint. We can substitute ~~this~~ the utility of the high type [$v(q_H,\vartheta_H)$] with the utility of the high type if they chose the the different contracts [$v(q_L,\vartheta_H) - p_L$]. We plug it in here.

$$\mu\left(v(q_L,\vartheta_H) - p_L - v(0,\vartheta_H)\right) + (1-\mu)\left(v(q_L,\vartheta_L) - p_L - v(0,\vartheta_L)\right) = 0 \tag{133}$$

4.3 p.32 Ex-Ante Payments

4.3.1 Payments

[04:02, point 1] ~~and then~~ we obtain this identity here that tells us what the payments of the low type are.

$$p_L^0 = \mu \left(v(q_L^*, \vartheta_H) - v(0, \vartheta_H) \right) + (1 - \mu) \left(v(q_L^*, \vartheta_L) - v(0, \vartheta_L) \right) \tag{134}$$

~~And then~~ the payments of the high type are derived from the payments of the low type as well as the binding incentive constraint.

$$p_H^0 = p_L^0 + v(q_H^*, \vartheta_H) - v(q_L^*, \vartheta_H) \tag{135}$$

4.3.2 Information rent

[04:20, point 1] ~~Now~~ to see for which type the information rent is negative. Intuitively, we expect already that it will be negative for the low type, but we can verify it. ~~So~~ the information rent ~~it~~ takes this form — the utility in the contract minus the outside option, and if we plug in here p_L^0, the ex-ante payment then ~~we see that okay~~ we have

$$I_L^0 \equiv v(q_L^*, \vartheta_L) - p_L^0 - v(0, \vartheta_L) = -\mu \Delta v(q_L^*) < 0 \tag{136}$$

[05:15] and ~~so~~ we'll abbreviate that

$$\Delta v(q) \equiv v(q, \vartheta_H) - v(0, \vartheta_H) - \left(v(q, \vartheta_L) - v(0, \vartheta_L) \right). \tag{137}$$

From the identity that we've derived on the previous slide, we then have that information of the high type is

$$I_H^0 \equiv v(q_H^*, \vartheta_L) - p_H^0 - v(0, \vartheta_L) = (1 - \mu) \Delta v(q_L^*) > 0. \tag{138}$$

[05:35] ~~And~~ **because we have increasing differences this $\Delta v(q)$ is definitely positive, ~~and so indeed~~ the information rent of the low type will be negative, and the information rent of the high type will be positive.**

4.4 p.33 Limited Liability

4.4.1 Remark

[05:50, point 1] ~~Now okay so~~ in theory, this is a very good result for the principal because they can get the first-best outcome, the socially efficient outcome, and extract all the surplus just like as if they had full information. ~~And~~ the reason why they can do that is because at the time of contracting, they don't have an information disadvantage yet. They don't have to pay information rent because the agent does not have information yet. ~~And so~~ this reduces or eliminates the problem of adverse selection.

[06:23, point 2] ~~Now~~ put differently ~~this~~ means that the principal prices in the information that the agent will get after learning the contract, and they do that precisely by making the low type worse off, and having the low type pay for the information rent of the high type.

[06:43, point 3] ~~And~~ if we think of these contracts, then maybe this is a little bit surprising that the low type will pay at information of the high type in reality. Sometimes, there is a limit on how negative the information rent of one of the players can be.

4.4.2 Limited liability

[07:02, point 1] ~~Because~~ **in reality, there is limited liability.** ~~So~~ if you start a startup, then even if your idea goes wrong, ~~well~~ this was just a wasted investment of the venture capitalists, but you don't have to necessarily pay them had been fine even if you agreed to that before you can always declare bankruptcy. ~~So~~ there is always some limited liability.

[07:22, point 2] ~~So let's say~~ the liability is limited by some constant l. ~~And~~ in that case, the limited liability constraint looks very similar to the participation constraint that we had in the second-best contract.

$$[LL_L]: \quad v(q_L, \vartheta_L) - p_L \geq v(0, \vartheta_L) - l \tag{139}$$

~~So~~ without the minus cell here, this would just be the participation constraint of the second-best contract. ~~And now~~ the utility in the contract $[v(q_L, \vartheta_L) - p_L]$ has to be at least the outside utility $[v(0, \vartheta_L)]$ minus the bound on the liability $[l]$.

[07:54, point 3] ~~Now~~ because this the limited liability constraint and ~~individual~~ individual rationality constraint differ only by this constant l. **Exactly in the same way that we've seen in the second-best contract, the limited liability constraint for the high type ϑ_H is implied by the limited liability constraint of the low type $[LL_L]$ as well as the incentive constraint of the high type $[IC_H]$.** ~~So~~ what we get if we impose limited liability is we get ~~one additional~~ one additional constraint. ~~Now if the liability~~ if this bound on the liability is larger than than the payment or than the negative information rent that the low type would get in the ex-ante contract, then there is no difference. ~~Right then we can still~~ we can still design the optimal aside they contract, and as long as the information rent this is larger than $-l$, then ~~then~~ this also satisfies the limit liability constraint.

4.5 p.34 Participation Under Limited Liability

4.5.1 Binding liability constraint

[08:55, point 1] ~~And the more interesting case~~ if the opposite is true, so if the bound on the limited liability is smaller than minus the information rent of the low type, which was equal to this expression here.

$$l \leq -l_L^0 = \mu \Delta v(q_L^*) \tag{140}$$

[09:09, point 2] In that case, the limited liability constraint binds for the low type [LL$_L$], and ~~so~~ we have the limited liability constraint that binds we have the incentive compatibility constraint for the high type [IC$_H$] which binds.

[09:22, point 3] ~~And so~~ the last question is does the participation constraint the ex-ante participation constraint also bind? ~~And so~~ the standard approach here to ~~to~~ answer this question is that we first solve the problem under the relaxed constraints that we forget about the participation constraint, and simply solve the maximization problem under these two constraints. ~~And~~ as a result, if the solution to the relaxed maximization problem satisfies the original constraint, then this constraint here was redundant.

[09:55] ~~And~~ the reason for this is simply if we have a maximization problem with more constraints, then the maximum has to be or is at most as high as if we remove one constraint. ~~And so~~ as a consequence, if the solution to the relaxed problem satisfies all the constraints, then it has to be a maximum of the original maximization problem.

4.5.2 Binding participation constraint

[10:19, point 1] ~~Okay so~~ let's see under which conditions the participation constraint would bind. ~~So so without so~~ the relax problem looks exactly like the second-best constraint. The only difference is that the participation constraint of the low type is shifted by l. So, we can implement the same quantities, and the only differences that the payments are larger than or ~~the payments are~~ equal to the payments in the second-best contract plus l.

$$q_L^l = \hat{q}_L \tag{141}$$

$$q_H^l = \hat{q}_H^* \tag{142}$$

[10:56, point 2] ~~And~~ to see when the participation constraint binds, we simply plug in the binding limited liability constraints [LL$_L$], and the binding incentive compatibility constraint for the high type [IC$_H$] into the participation constraint. ~~Yes and yeah so~~ the participation constraint had the two terms here, so here the information of the low type which we know is a minus l if it binds. ~~And for the high type well~~ for the high type, it's the information of the life low type plus the amount that they get from ~~from~~ the binding incentive constraints. So, here this is ~~the difference between choosing the bundle that so~~ the difference between the valuation that the high type has for bundle (q, l) at that the low type.
[12:03, point 3] ~~And~~ if we solve this for ~~for~~ l, then we see that this binding if and only if l is larger or equal than this expression here. ~~And~~ the nice thing is this looks very similar to this expression up here except that this is the second-best quantity for the low type.

4.6 p.35 Contracting with Limited Liability

4.6.1 Theorem 5.9

[12:26] ~~So the quantity in the second-best contract and so~~ to summarize the results, we have here this theorem 5.9 that ~~distinguish the three~~ distinguishes the three cases under the limited liability constraint. ~~So~~ either limited liability is not binding in which case, we just have the optimal example contract

$$\{(q_H^*, q_H^0), (q_L^*, q_L^0)\} \tag{143}$$

if the limit of liability is so strong that it's below ~~this let's call it~~ the information rent in the second best contract

$$l \leq \mu \Delta v(\hat{q}_L) \tag{144}$$

then ~~we can~~ we can design the second-best contract. We can make it incentive compatible under limited liability, and we can even add l to the payments of the second-best contract. ~~And then~~ in between where all three constraints are binding, then what we do essentially is we solve this equation here

$$\Delta v(q_L^l) = \frac{l}{\mu} \tag{145}$$

for the payments or for the quantity of the low type in the optimal limited liability contract.
[13:32] ~~And so~~ if we were required to ~~let's say~~ find the limited liability contract what we would do is we first compute the first-best and the second-best contract. So, we can decide from these two equations [$l \geq \mu \Delta v(q_L^*)$ and $l \leq \mu \Delta v(\hat{q}_L)$] in which these three cases we are. Then, if we are either in this case [1] or in this case [3], then we already have found the optimal contract either in the first-best or second-best contract. ~~And then~~ only for in this case [2] here, we additionally have to solve this equation for the quantity of the low type.

4.7 p.36 Air Fares: Ex-Ante Ticket Sales

4.7.1 Ex-ante contracting

[14:23] In that case, we compute the optimal ex-ante contract which was to sell the flexible ticket to the business type,

$$q_B^* = q_F \quad \text{(flexible)} \tag{146}$$

the non-flexible ticket to the tourist.

$$q_T^* = q_N \text{ (not flexible)} \tag{147}$$

~~And~~ if we compute the payments using the format that we had we found that the payment for the low type was

$$p_L^0 = \mu(30-0) + (1-\mu)(10-0) = 10 + 20\mu \tag{148}$$

and the payment for the high type was

$$p_H^0 = (10 + 20\mu) + 40 - 30 = 20(1+\mu) \tag{149}$$

which left the low type an information rent of

$$v(q_L^*, \vartheta_L) - p_L^0 = 10 - (10 + 20\mu) = -20\mu. \tag{150}$$

and the high type an information rent of

$$v(q_H^*, \vartheta_H) - p_H = 40 - 20(1+\mu) = 20(1-\mu). \tag{151}$$

Then, ex-ante participation constraint of contract $\{(q_H^*, p_H^0), (q_L^*, p_L^0)\}$ is

$$\mu\left(v(q_H^*, \vartheta_H) - p_H - v(0, \vartheta_H)\right) + 1 - \mu\left(v(q_L^*, \vartheta_L) - p_L^0 - v(0, \vartheta_L)\right) = \mu \cdot 20(1-\mu) + (1-\mu)(-20)\mu = 0. \tag{152}$$

4.8 p.38 Check Your Understanding

4.8.1 Question 1

Q: An optimal contract is a perfect Bayesian equilibrium of a two-stage game, in whcih the principal moves first and the agent moves second.
[00:56] [True.] ~~Yes very good so~~ even though we introduce additional language, we say contract theory, agent [and] principal, this is still just a ~~two-stage~~ dynamic Bayesian game. It's a two-stage game with private information. ~~And so~~ all we're doing really is solving for the perfect Bayesian equilibrium. It's simply one example that has received so much attention because it's so widely applicable that it has their own name principal agent models, contract theory. ~~And so so yeah for the~~ there is different language that we use because it's so heavily studied, but it's still just a perfect basic equilibrium in a two-stage game.

4.8.2 Question 2

Q: Every agent with private information receives a positive information rent in the second-best contract.
[01:45] That is not true. Typically, the low type does not receive a positive information rent. ~~So~~ in the two type example, the low type doesn't receive one. ~~And~~ typically, the lowest type does not receive a positive information rent.

4.8.3 Question 3

Q: Ex-ante contracting is to the principal's advantage.
[02:08] That's correct because they don't face adverse selection. ~~So~~ if we contract at a stage in time where the agent does not have private information yet, then we don't have to pay anything to for the agent to reveal their information because they don't have any information. ~~And~~ the information that they learned later we can price it into the contract by punishing the low type and rewarding the high type.

4.8.4 Question 4

Q: In the language of contract theory, laws are contracts designed by the government (principal) for the citizen (agent).
[02:37] [True.] ~~Good yeah very good so~~ contract laws are contracts designed by the government for citizens to follow. ~~And~~ you don't have to follow a law, but there is just gonna be punishment if you don't. ~~And so~~ the government tries to think ~~well~~ if everybody maximizes their utility, we want them to follow the law. ~~And so~~ the punishments are chosen accordingly.

4.8.5 Question 5

Q: In every contract with private information by the agent, the principal must honor the agent's incentive and participation constraints.
[03:07] [False.] ~~So here~~ this depends on whether participation is voluntary or not. ~~So let's say~~ if we have the application of laws in mind, then our participation is not voluntary. If we live here, ~~we have to follow like~~ we are subject to the law. ~~And~~ if we break the law, ~~we~~ we face the consequences. ~~So it depends~~ if participation is voluntary like with selling mechanisms, then the incentive constraint and participation constraint we have to satisfy both. But in an example where participation is voluntary like if we design laws, [or] the optimal taxation example would be one where participation constraints do not have to be satisfied because you can't not pay taxes. ~~So~~ only if ~~so~~ true for the applications we've seen in class, but in general true only if participation is voluntary.

5 Screening a Continuum of Types

[00:01] ~~Okay welcome back to to our class. Last week we started the weight screening games. And~~ screening games are games where the uninformed party moves first, and then the formed party moves later. ~~And one example is~~ for example [p.4], if we screen customers depending on their type, ~~so in this case that we have the airline,~~ the airline does not know what kind of customer it faces, ~~and~~ you can ~~quote it can~~ quote a price, and then the customers decide whether to buy or not buy. ~~And so~~ we've seen that sort of the naive example was that we have only 1 ticket that goes from a to b. ~~And~~ by quoting 1 price, the airline can screen out customers, they can either choose a price where everybody will fly, but 2 [prices and tickets] is a price where only the more profitable clients will fly.
[01:01] ~~Now~~ **conceptually, these screening games are simpler to solve than the signaling games that we've seen because there is no basic updating going on at any point in time.** ~~So~~ the party with the information moves last, so there is no signaling effect anymore. ~~And so~~ because it is conceptually easier, we can

study it in almost full generality, and we've started with this in the case where there is two types last week. ~~And~~ today we will see what happens if there is a continuum of types. ~~So~~ if you're an airline and you're selling to customers, then it's not just two different valuations that the customers might have if you sell to the entire population more likely there is some distribution over the valuations that the players have. ~~And~~ this is what we will look at today.

[01:55, p.8] ~~Now~~ before we get into it, let's recall what we've done last week. ~~So~~ the basic setup is that we have some contractible variable q that the seller and the buyer write a contract on. ~~So~~ in general, ~~the this is a contractible variable~~ this contractible state q ~~this~~ could be many things. ~~So~~ depending on the application, ~~this could be and~~ we've seen quite a few applications. These could be the number of publications that assistant professors get, [and] this could be the quality of service; that you're providing to your customers, [and] this could be the number of units; that you're selling to them, [and] this could be the level of investment in a venture capital. ~~And so~~ this contractual variable is typically provided by one party to the other party. ~~And so~~ the signs of ~~of~~ the results that we derive will change depending on whether it's provided by the principal, or provided by the agent. ~~And so~~ here, we derive the theory in the case where the principle provides this contractual variable q to the agent, and then we've made this assumption that both principal and agent have a utility that is separable and linear in the payments. ~~And so~~ we have this form here where the principles utility is simply the payments that they receive from the agent minus the cost of producing q,

$$u_1(q,p) = p - c(q) \tag{153}$$

and for the agent it's the benefit of obtaining q minus the payment that they make for it.

$$u_2(q,p,\vartheta) = v(q,\vartheta) - p \tag{154}$$

~~And then~~ we've seen ~~several~~ several different contracts. ~~So~~ [at] first, we looked at the full information contract, also known as the first-best. ~~And~~ this is kind of the benchmark of what we can achieve. ~~So~~ the full information benchmark is where the seller in this case knows the buyer's types, and they're allowed to discriminate between buyers types. ~~And so~~ they can charge each buyer exactly their willingness to pay, and this leads to a contract with 2 properties. First, it's ~~the efficient~~ the socially efficient level of consumption, so ~~the~~ the socially efficient level of quantities bought. ~~And then~~ second, the entire surplus is extracted by the seller.

[04:29] Now, price discrimination is not legal. ~~So~~ this is typically not cannot be implemented this first-best contract, but it tells us what would the socially efficient consumption be. ~~And~~ then, we looked at the second-best contract which is sort of the optimal legal contract where the seller cannot discriminate among player types and what we obtained ~~then~~ was that the high type agent. In the two type setting, the high type of agent gets an information rent for the high type to choose the contract that was designed for them. ~~And this we're gonna~~ this is where we're gonna switch to a continuum of types.

5.1 p.40 Setup

5.1.1 Types and utilities

[05:13, point 1] ~~So now~~ there is a continuum of buyer evaluations for the product distributed on some interval with some density function $f(\vartheta)$ that is commonly known between agent and principle.

[05:28, point 2] ~~And~~ when we use a continuum of types, we need to make some additional assumptions ~~on the~~ on the utility function so that the problem is well-behaved. ~~And~~ one of them is absolute continuity, so we need

~~that~~ that this utility $v(q, \vartheta)$ is absolutely continuous. ~~And~~ this means that it's the integral of its derivative. ~~So~~ most continuous function that we know are absolutely continuous except ~~maybe if the~~ if the function oscillates very very fast ~~or something like that~~, then it might not be absolutely continuous.

5.1.2 First-best contract

[06:07, point 1] ~~And then the first best contract~~ with the first-best contract, ~~in also~~ when we have a continuum of types, ~~and~~ [we] simply maximizes the joint surplus, so we add up the utility of the agent and the principal. ~~And~~ what we obtain is simply the benefit of q minus ~~minus~~ the cost of providing it.

$$v(q, \vartheta) - c(q) \tag{155}$$

~~And~~ in the first-best, we know the type of the agent, and ~~so~~ we simply differentiate this ~~for~~ for every for every agent's type.

$$[q]: \quad v_q(q, \vartheta) - c'(q) = 0 \tag{156}$$

~~And~~ what we get is a relation that in the optimum.

$$v_q(q_\vartheta, \vartheta) = c'(q_\vartheta) \tag{157}$$

~~I guess~~ I should write v_q for the derivative with respect to q, and so this will be the first-best contract [that] will satisfy this equation.

5.1.3 Participation constraints

[07:06, point 1] ~~Now~~ if we move away from the first-best contract to the second-best contract, then the idea is again that we look at direct contract which is a contract that specifies for each type what they're supposed to buy as well as the price for that item. ~~And we have seen that~~ we can ~~then~~ write the maximization problem as finding the best direct contract subject to two types of constraints — **the incentive compatibility constraints which mean that every agent buys the bundle that is intended for them, and the individual rationality constraint that meant that every agent is better of ~~participating in the mechanism opposite~~ participating in the contract than not participating.**

[07:52, point 2] ~~Now in the~~ in the two type case, we have seen that the individual rationality constraint for one type binds and for the other one it was implied by the incentive compatibility constraint, and the individual rationality constrained for the other type. ~~this is again the case in~~ **if we have a continuum of types, ~~so~~ because we have increasing differences of this utility function v, then again in the same way that we had for two types, we see that the individual rationality constraint for every agent is implied by their incentive compatibility constraint as well as the individual rationality constraint of the lowest type.** ~~And~~ the proof works in exactly the same way because we didn't ~~we we didn't~~ require that there is only two types, so ~~all~~ all we needed is that for any two different types the higher types individual rationality constraint [IR$_\vartheta$] is implied by the lower type as well as the incentive compatibility constraint.

5.2 p.41 Incentive Compatibility

5.2.1 Lemma 5.10

[09:04, point 1] ~~And it turns out that in a~~ with a continuum of types, we have a fairly nice characterization of incentive compatible direct contracts, so they are instead of compatible if and only if the direct contract satisfies these two constraints. ~~So~~ the first constraint is that the quantity is non-decreasing in the buyer's type [ϑ]. ~~So~~ a higher type will always get ~~a higher a bundle~~ a bundle with higher quantity.
[9:34, point 2] ~~And then~~ the second one is an integral constraint that tells us that the payments in the contract are uniquely determined by the payment of the lowest type [$p(\underline{\vartheta})$], and the quantity that we provide to each type. ~~And so~~ we see that ~~that~~ everything else here is a constant, so ~~the only~~ the only variables that we have on the right hand side is the quantity for each type as well as the payment of the lowest type. ~~And and~~ this will greatly help us simplify the maximization problem because we can again get rid of all the payments, and then we're left with only the quantities over which we have to maximize.

5.3 p.46 Comparison to Two-Type Case

5.3.1 Comparison to Two-Type Case

[24:24, point 1] ~~Now~~ if we compare it to the two type case, then this in the two type case we also had that the payments are uniquely determined by the payment for the lowest type as well as the difference in the valuations between the bundle for the lowest type and ~~the high type~~ for the high type. ~~So~~ this was the binding incentive compatibility constraint. ~~And~~ we can get a very similar expression here ~~if the~~ if the contract is differentiable. ~~So~~ if the contract [q] is differentiable, we can rewrite this expression

$$p(\vartheta) = v\left(q(\vartheta), \vartheta\right) - U_2(\vartheta). \tag{158}$$

~~And we can~~ we can differentiate this quantity here, and obtain ~~a different~~ a different expression ~~for~~ for the payments.

$$\begin{aligned} p'(\vartheta) &= v_q\left(q(\vartheta), \vartheta\right) q'(x) + v_\vartheta\left(q(\vartheta), \vartheta\right) - U_2'(\vartheta) \qquad (159)\\ &= v_q\left(q(\vartheta), \vartheta\right) q'(x) \qquad (160) \end{aligned}$$

That is equivalent to the other one if q is differentiable, and this one looks exactly like we had in the two type case.
[25:24, point 2] ~~And so~~ we get the same idea, so we start with the payment for the lowest type and then we simply look at the difference in valuations for the different quantities that the higher types get.

$$p(\vartheta) = p(\underline{\vartheta}) + \int_{\underline{\vartheta}}^{\vartheta} v_q(q(x), x) q'(x) dx \tag{161}$$

5.4 p.47 Maximizing the Profit

5.4.1 Profit maximization

[25:40, point 1] ~~Okay so~~ this uniquely characterizes incentive compatible contracts. ~~And now~~ if we want to find ~~the best contract~~ the second-best contract, ~~so the~~ the best direct contract

$$V(p,q) = \mathbb{E}\left[p(\vartheta) - c(q(\vartheta))\right] \tag{162}$$
$$= \int_0^\infty \left[p(\vartheta) - c(q(\vartheta))\right] f(\vartheta) d\vartheta \tag{163}$$

subject to incentive compatibility

$$q(\vartheta) \geq 0 \tag{164}$$
$$p(\vartheta) = p(\underline{\vartheta}) + \int_{\underline{\vartheta}}^{\vartheta} v_q(q(x), x) q'(x) dx \tag{165}$$

and participation

$$p(\underline{\vartheta}) = v(0, \vartheta) \tag{166}$$

Then, we can rewrite it now in the following form, and the constraints that we had now is the participation constraint of the lowest type. Then, we have the incentive constraints which are one ethnicity constraint as well as this expression for the payments.

5.4.2 Typical Approach

[26:19, point 1] ~~And so~~ two of the constraints are very nice. The binding participation constraint is quite nice, and the payments is also binding constraint. What is a bit unusual is the constraint that q has to be increasing. ~~Now~~ the typical approach to how we solve these problems is that ~~we say well okay~~ let's look first at the relaxed problem where we don't enforce that the quantities are increasing. ~~And~~ let's see what the solution ~~how the solution~~ looks like. ~~Then~~ if the solution is a contract with increasing quantities, then we know that we have found the solution ~~to the~~ to the original problem because the relaxed problem has to give a better solution for the principle. Because we have fewer constraints, ~~and so~~ if the optimum of the relaxed problem satisfies the conditions of the original problem, then the two solutions agree. ~~And~~ that is where we'll we'll stop in today's class.

[27:20, point 2] There are some things that we can do if q is not ~~is not~~ monotonic to fix this problem later, but in this class we'll just look at a case where the relaxed problem is indeed the solution to the original problem.

5.5 p.48 Simplifying the Objective Function

5.5.1 Simplifying the objective function

[27:35] ~~So now~~ if we want to maximize this function and we drop the monotonicity constraint, then the typical way to do it is we plug in both of these expressions and to eliminate the payments completely from the objective function. ~~And then~~ we can maximize an unconstrained objective function.

[27:56, point 1] ~~And~~ if we do that, then we get a maximum that is very similar in form to what we've seen ~~in~~ in the two-type case. ~~That~~ it's the difference between the total surplus that is generated ~~from this~~ from this contract, and then minus the information rent that has to be paid ~~to the agent~~ for the agents to buy the contract that was intended for them. ~~and so~~ Here, ~~the information rent takes the form of~~ this information here comes ~~from the payments~~ from the expression for the payments.

$$V(q) \equiv \int_{\underline{\vartheta}}^{\overline{\vartheta}} \left(\underbrace{v\left(q(\vartheta), \vartheta\right) - c\left(q(\vartheta)\right)}_{\text{Total surplus}} - \underbrace{\int_{\underline{\vartheta}}^{\vartheta} v_{\vartheta}\left(q(x), x\right) dx}_{\text{Information rent}} \right) f(\vartheta) d\vartheta \tag{167}$$

[28:33, point 2] ~~Now~~ whenever we see an expression with a double integral, the first thing we do is we use Fubini's theorem to simplify the integral. ~~Maybe~~ as a reminder of Fubini's theorem, ~~so~~ we do here is we integrate over x and ϑ.

$$\int_{\underline{\vartheta}}^{\overline{\vartheta}} v_{\vartheta}\left(q(x), x\right) dx f(\vartheta) d\vartheta \tag{168}$$

The $\underline{\vartheta}$ is barely visible, but there is a $\underline{\vartheta}$ on both of these ~~on both of these~~ lower integral bounds. ~~And so~~ we maximize here, so x is always smaller than ϑ. ~~So~~ we're integrating over this area here. ~~All right so~~ for any point in here the x coordinate is smaller than ϑ. ~~And then~~ if we want to change the order of integration, so we'll first integrate our x and then for each x what we have to do is we have to then integrate over ϑ from x to the upper bound. ~~And so~~ if we do that, ~~so~~ we interchange the the order of the integrals.

$$\int_{\underline{\vartheta}}^{\overline{\vartheta}} v_{\vartheta}\left(q(x), x\right) \int_{x}^{\overline{\vartheta}} f(\vartheta) d\vartheta dx \tag{169}$$

~~And~~ then, the partial derivative here does not depend on ϑ. so here the subscript here this just means that it's the partial derivative with respect to ϑ, but it's not actually ~~it's not actually~~ a parameter that we integrate over. So, we can pull it out of the expectation what we're left. There is the integral over the density function which is just equal to to one minus the distribution of x.

$$1 - F(x) \tag{170}$$

~~And so~~ this entire expression would then be like

$$\int_{\underline{\vartheta}}^{\overline{\vartheta}} v_{\vartheta}\left(q(x), x\right) \left[1 - F(x)\right] dx \tag{171}$$

~~And so~~ we successfully simplify the double integral into a single integral.

[30:49, point 3] ~~Now~~ this integral is over x, the other one is over ϑ, but that's just the relabeling. ~~So~~ we can just rewrite these x's as as ϑ's, and then combine everything under one integral. ~~and so now~~ we here have still the total surplus, and then this is the the information rent that is paid to the buyer.

$$V(q) = \int_{\underline{\vartheta}}^{\overline{\vartheta}} \underbrace{\left(v\left(q(\vartheta), \vartheta\right) - c\left(q(\vartheta)\right) - v_{\vartheta}\left(q(\vartheta), \vartheta\right) \frac{1 - F(\vartheta)}{f(\vartheta)} \right)}_{\text{Virtual surplus}} f(\vartheta) d\vartheta \tag{172}$$

~~And so~~ **this total quantity here that we integrate over is called the virtual surplus because it is the surplus** ~~**that the**~~ **that the principal gets after paying the information rent to the agent.** ~~And now~~ if we want to maximize ~~this~~ the expected profit or ~~the extra revenue~~ the extra profit of the ~~of the~~ principle, we simply maximize this ~~this~~ integrate pointwise.

5.6 p.49 Maximizing the Virtual Surplus

5.6.1 Solving the relaxed problem

[31:47, point 1] ~~So~~ if for each $q(\vartheta)$ is whatever maximizes the virtual surplus, then at the end ~~the surplus or the~~ the profit of the principle is maximized. ~~And so~~ we have this function φ here that we want to maximize, and we can do that by taking the derivative.

$$\varphi(q,\vartheta) \equiv v(q,\vartheta) - c(q) - v_\vartheta(q,\vartheta)\frac{1-F(\vartheta)}{f(\vartheta)} \tag{173}$$

[32:00, point 2] ~~So~~ the first-order necessary constraint has to be satisfied which would be this one here.

$$v_q(q,\vartheta) - c'(q) - v_{\vartheta q}(q,\vartheta)\frac{1-F(\vartheta)}{f(\vartheta)} = 0 \tag{174}$$

~~So~~ this would be the first-order necessary concern to simply take the derivative with respect to q, and then there are several questions — ~~well okay when is this~~ when is this indeed a maximum and when is $\cancel{q}$ the solution to this one monotonic.

[32:41, point 3] ~~So well~~ if φ is concave, then this is definitely a maximum. ~~And~~ if it's strictly concave, then it's even unique.

[32:48, point 4] ~~And~~ so, ~~we can see that~~ we could guarantee that this is indeed a maximum if the second and third derivatives satisfy these inequalities.

$$v_{qq} \leq 0 \tag{175}$$

$$c'' \geq 0 \tag{176}$$

$$v_{\vartheta qq} \geq 0 \tag{177}$$

~~So~~ if v and c are sufficiently differentiable, then we can know ahead of time that this solution is indeed a maximum. ~~So yeah so here just if we take one more once more the derivative with respect to q, we just get the second derivative with respect to q for both v and c and then one more derivative of q for the last term.~~ ~~Okay so~~ in this class, we'll be satisfied with this result that if we have sufficiently differentiable cost and value functions that satisfy such that this maximization is concave, then we know ~~we'll get~~ a we'll get a maximum.

5.7 p.50 Monotonicity

5.7.1 Hazard rate

[33:44] ~~And now~~ the second question is: when is the solution to this maximization problem when is it a monotonic? ~~And here~~ this is a result that ~~we that~~ we had at the very beginning of when we looked at contract theory. This was Topkis' theorem. ~~And~~ **Topkis' theorem told us that if we have a function with increasing differences, ~~then the max if if~~ [and] different types choose the maximums then the higher type will always choose ~~the higher~~ the higher quantity.** ~~And so~~ this is precisely something that we have here. ~~So~~ if φ has increasing differences, ~~then~~ then we will know that the solution is monotonic. ~~So~~ **for increasing differences, one thing that is sufficient is if we have the signal crossing property.**

[34:35, point 1] ~~So~~ to rewrite this expression ~~let's rewrite this fraction~~ of 1 minus f of the distribution over density.

$$h(\vartheta) = \frac{f(\vartheta)}{1-F(\vartheta)} \tag{178}$$

[34:42, point 2] ~~Let's rewrite this as one over the hazard rate. So the hazard rate is is $f(\vartheta)$. So~~ this is the conditional density ~~that~~ that conditional value of ϑ given that ϑ is at least ϑ. ~~So I guess~~ the idea is if we have the conditional density given that ϑ is larger than some threshold, ~~and then in the limit as r goes to ϑ then~~ then we'll get the hazard rate. ~~And~~ it's known as the hazard rate because of actuarial sciences. ~~So~~ in actuarial sciences, ~~let's say~~ **if f is the distribution of your lifetime, then what is the hazard rate? The hazard rate is the likelihood of dying given that you have not died yet.** ~~And so~~ in the case of buyers, it's the likelihood that you're facing exactly type ϑ ~~if you haven't~~ if the buyer does not value the item anything less than ϑ.

5.7.2 Monotonicity

[36:09, point 2] ~~And so~~ by Topkis' theorem, we know that if the signal crossing property is satisfied ~~for~~ for the virtual surplus, then we know that the solution is indeed monotonic.
[36:22, point 3] ~~and again~~ we need to assume that v and c are sufficiently often differentiable, ~~but~~ then ~~again we can write it down~~ we can write down the condition ~~that~~ that would have the single crossing property.

$$\varphi_{q\vartheta} = v_{q\vartheta} - \frac{v_{q\vartheta\vartheta}}{h(\vartheta)} + v_{q\vartheta}\frac{h'(\vartheta)}{(h(\vartheta))^2} \tag{179}$$

~~Now~~ from the single crossing property for the valuation v, we already know that this [$v_{q\vartheta}$] is strictly positive. ~~So all~~ we need to make sure is that these ~~said~~ two terms here are non-negative.
[36:47, point 4] ~~And~~ this can be ensured by assuming that the hazard rate is not decreasing

$$h'(\vartheta) \geq 0 \tag{180}$$

as well as that this third derivative here of v is non-positive.

$$v_{q\vartheta\vartheta} \leq 0 \tag{181}$$

~~Now~~ these are a lot of constraints for the derivatives about our cost functions and value functions, but what we have found is a complete solution ~~for~~ of the second-best contract if c and v are sufficiently differentiable. ~~And~~ we even know that the solution to the relaxed problem is the solution to the original problem where the quantities will be increasing.

5.8 p.51 Second-Best Contract

5.8.1 Theorem 5.11

[37:28] ~~and~~ this then summarizes the second-best contract in this setting. ~~So the first thing~~ if we were to apply this theorem, the first thing that we do is **we check that all derivatives have to write the right signs, ~~and then~~ we can find the quantities from the first order constraint, and then lastly find the payments from the incentive compatibility constraint.**

5.8.2 Note

[38:00, point 1] ~~Now~~ I should mention here that **these conditions on the partial derivatives. There are sufficient conditions that guarantee that the solution to the relaxed problem is a solution to the original problem, but these are not necessary conditions.** ~~So if~~ if I ask you to find ~~some optimum~~ some optimal contract where one of these conditions is violated, then most likely it will still work. ~~You still have~~ you just have to find the solution to the relaxed problem first, and then verify manually that indeed the relax contract has increase in quantities. ~~And~~ typically if I give you some parameterization of some specific choice of c and d, this will be easy to verify.

5.9 p.52 Interpretation of the Optimal Contract

5.9.1 Indifference constraint

[38:55, point 1] ~~Any questions so far? Okay let's see if we can interpret the optimal contract, and then we'll see some examples of it. So first~~ the first-order condition that we have for the quantities, we can rewrite it in the following form

$$f(\vartheta)\left(v_q(q,\vartheta) - c'(q)\right) = (1 - F(\vartheta))\, v_{\vartheta q}(q,\vartheta) \tag{182}$$

[39:10, point 2] ~~So~~ on the left hand side of this expression ~~eight~~, what we have here is the marginal increase in the total surplus. ~~So~~ this is ~~the~~ the surplus that we maximize in the first-best contract. ~~And~~ this marginal increase in total surplus has to be equal in the optimum to the marginal increase in the information rent paid.
[39:21, point 3] ~~And so~~ what we see here on the right hand side is that ~~this is somehow the so~~ this is ~~the marginal~~ the marginal increase in benefit that a higher type gets. ~~And~~ it's not just for one type, but it's sort of multiplied by all the types that lie above type ϑ. ~~So~~ $1 - F(\vartheta)$ here is the probability that we have any type above it. ~~And if we remember this is similar to what we've had in the two type contract where we've had that. Well so~~ in the second-best contract, we faced this trade-off between ~~between~~ the revenue that we get from selling to low types, and this will be higher ~~if we~~ if we sell larger quantities, but this also ~~makes the incentive to deviate~~ for the higher types makes the incentives to deviate stronger. ~~And so~~ then here we have a condition precisely that ~~so~~ this is sort of an information when paid to all the higher types so that they don't deviate to the bundle that is in the intended for type ϑ.

5.9.2 Comparison to first-best

[40:35, point 1] ~~If we compare to the first best contract then we see okay~~ if we evaluate this at at the upper bound of ~~or of~~ the support of ϑ, then the distribution function is one at the upper bound.

$$q(\overline{\vartheta}) = q^*(\overline{\vartheta}) \tag{183}$$

~~And so~~ as a result, we get precisely the first-best quantity. ~~Ao again~~ similar to the two type contract, we have that at the top there is no distortion. **The quantity that is supplied to the highest type is exactly the same as in the first-best contract.**
[41:04, point 2] ~~And~~ this is because selling to the highest type is the most profitable, ~~so~~ the optimal contract sells the same quantity to the highest type, and then adjusts the quantity sold to the other type ~~to~~ to decrease the information rent paid to the higher types.

5.10 p.53 What Menu of Contracts Do You Offer?

5.10.1 Implementation

[41:24, point 1] ~~Now~~ this is the optimal direct contract, but typically what we would do is we don't say ~~okay~~ tell me precisely your utility function, and I'll tell you what bundle ~~you~~ you should get.
[41:36, point 2] What we do instead is we write a price function or a contract an indirect contract has a ~~function from the~~ price of a ~~so it's a price as a~~ function from the quantities that they can buy.

$$p : \mathcal{Q} \to \mathbb{R} \tag{184}$$

[41:52, point 3] ~~Then we simply say well~~ if you buy these many items, then this will be the price. Now, we can recover the indirect contract from the direct contract in the following way.

$$\Theta(q) \equiv \{\vartheta \in \Theta | \hat{q}(\vartheta) = q\} \tag{185}$$

~~So~~ the first thing to notice is that ~~so~~ **all the types that receive the same quantity, they also all have to make the same payments in the optimal contract by incentive compatibility.**

$$\hat{p}(\vartheta) = \hat{p}(\vartheta'), \quad \forall \vartheta, \vartheta' \in \Theta(q) \tag{186}$$

If one of the types made a lower payment in the contract, then every other type within that set would have an incentive to deviate because they could decrease their payments.
[42:32, point 4] ~~So~~ in the optimum, all the types that receive the same quantity have to make the same payments for it, and then we have essentially a one-to-one correspondence.

$$\hat{p}(\vartheta) = \hat{p}\left(\vartheta(q)\right), \quad \forall \vartheta(q) \in \Theta(q) \tag{187}$$

~~Then we can simply select one for each quantity~~ we can select one type from the set, and evaluate to indirect evaluate the direct contract at that type.
[42:54, point 5] ~~And~~ in the result, we indeed have that the optimal indirect contract provides exactly the same bundle to the optimal direct contract.

5.11 p.54 Different Parametrizations

5.11.1 Different parametrization

[43:16, point 1 and 2] ~~Now again a word of warning if we have different parameterizations of the model~~ if we instead have that the agent supplies q to the principal,

$$u_1(q, p) = v(q) - p \tag{188}$$

$$u_2(q, p, \vartheta) = p - c(q, \vartheta) \tag{189}$$

then the single crossing property would mean that ~~okay the cost of~~ the cost of producing quantity q is decreasing in the type.

$$c_{q\vartheta} < 0 \tag{190}$$

~~And~~ as a consequence, some of these signs that we have in the main theorem will change, but what does not change is the approach to proving it. ~~So~~ the same approach can be used for all of these contracts where we first solve incentive compatibility. This will typically mean some monotonicity constraint as well as a unique characterization of payments, and then maximizing the objective function by first dropping ~~the incentive~~ the monotonicity constraint, and then verifying that the monotonicity constraint is satisfied for the solution ~~of the~~ to the relaxed problem.

5.11.2 Lowest type

[44:19, point 1] ~~Now the lowest type so the lowest type here is let's say~~ **the lowest type is the type for which the participation constraint binds. ~~And~~ this is sort of the type that is worse off in the contract. That's why we call it the lowest type.** ~~Now~~ in the screening application where we have a monopolistic seller that ~~buys two different~~ sells to different buyers, the lowest type is indeed the type of values that get the least.

[44:47] Now in different applications, this ~~may be a~~ maybe ~~a~~ have a different numerical value. ~~So let's see if~~ if we have a setting where the cost or somehow ϑ is the cost, the unit cost of producing quantity q, then the person with the highest numerical value, the highest cost would be the lowest type because those are the ones that are worse off in the contract.

[45:16, point 2] ~~And so~~ depending on the application here, this could be the lowest type, the highest type, or in some application even an intermediate type might be the type that is worst off. ~~And so~~ it's always worth checking which type is the type that ~~for which some incentive~~ for which the participation constraint binds.

[45:38, point 3] ~~And then we do~~ we derive the payments and relative to the the lowest type which is the type for which the participation constraint binds.

5.11.3 Conclusion

[45:50] ~~Yeah so~~ whenever you have an application, that is not exactly like we have a theorem, then you need to be careful if you want to find the optimal contract.

5.12 p.55 Selling an Infinitely Divisible Good

5.12.1 Selling an infinitely divisible good

[03:21] ~~Good. So~~ we've developed the general result. Let's now see the application or two applications of it.

[03:34, point 1] ~~So~~ let's first look at an application where we sell an infinitely divisible good. ~~So I don't know~~ let's think ~~let's~~ maybe flour, you can just buy an arbitrary amount of flour that you want, ~~and~~ different types of baker's value flour differently, and to get a tractable scenario. Let's look at the example where the utility here is given by

$$v(q, \vartheta) = \vartheta\sqrt{q}. \tag{191}$$

~~And let's say the the buyers~~ the baker's types are uniformly distributed on $[0, 1]$.

[04:06, point 2] ~~And let's also say~~ for simplicity, ~~that~~ production cost is linear.

$$c(q) = q \tag{192}$$

~~Now~~ this is precisely the scenario that we have in the theorem. This is precisely the case we have a manipulate monopolistic seller who sells to a buyer of unknown type. ~~So~~ there is a good chance that we can apply the theorem.

5.12.2 Second-best contract

[04:28] ~~So let's see that that~~ the cost function and the value function here have the right qualitative properties. ~~So~~ in the theorem [p.51], we had the first derivatives of c and v that we were looking at as well as the Spence-Mirrless' single crossing property. ~~so~~ the first derivative of c is positive,

$$c'(q) = 1 > 0 \tag{193}$$

then the derivative of v is positive as well.

$$v_q(q, \vartheta) = \frac{\vartheta}{\sqrt{q}} > 0 \tag{194}$$

The derivative with respect to ϑ is positive, too,

$$v_\vartheta(q, \vartheta) = \sqrt{q} > 0 \tag{195}$$

and then the signal crossing property

$$v_{q\vartheta}(q, \vartheta) = \frac{1}{2\sqrt{q}} > 0 \tag{196}$$

~~And~~ we see that all those are positive, ~~so~~ so we are indeed in the setting of ~~theorem~~ the theorem that we had. ~~So~~ this means that the optimal quantity will satisfy this first-order constraint that we had here. ~~So the optimal quantity will satisfy this first order constraint. So let's see what this will imply.~~

[05:52] ~~So so the first derivative of respect to q, and for the uniform distribution,~~ we get

$$0 = \frac{\vartheta}{2\sqrt{q}} - 1 - \frac{1}{2\sqrt{q}} \cdot \frac{1-\vartheta}{1}. \tag{197}$$

~~Okay so~~ we can multiply everything through with the $\sqrt{q}$

$$\sqrt{q} = \frac{\vartheta}{2} - \frac{1-\vartheta}{2} \tag{198}$$

$$= \frac{2\vartheta - 1}{2}. \tag{199}$$

[07:02] ~~And so~~ we get that the quantity in the second-best contract should be

$$\hat{q}(\vartheta) = \left(\vartheta - \frac{1}{2}\right)^2. \tag{200}$$

~~Now we see that this is not so that for some~~ for some types, this will be negative. ~~And so~~ what this means is that for some types, we won't sell at all. ~~So and then~~ for those types, the first-order constraint does not have to be binding.

[07:54] ~~So yeah so here from the theorem, I forgot to mention this before the break~~ the first-order constraint binds only ~~for interior points~~ for interior solutions. ~~And~~ if we sell a positive quantity, this means that the maximum is either attained at 0 or at an interior point. ~~And so here~~ whenever this solution is an interior point, then it has to be given by the first-order constraint, but the optimum could also be 0. ~~And~~ so we see that the quantities in the second-best contract will be this form here if

$$\vartheta \geq \frac{1}{2} \tag{201}$$

and there will be 0, otherwise. ~~So we have again that~~ we have some low types that we don't sell to at all.

$$\hat{q}(\vartheta) = \begin{cases} \left(\vartheta - \frac{1}{2}\right)^2, & \text{if } \vartheta \geq \frac{1}{2} \\ 0, & \text{if } \vartheta < \frac{1}{2} \end{cases} \tag{202}$$

[09:21] ~~And now~~ knowing what the optimal quantity is, we can find the optimal payments as well. ~~So~~ how much does type ϑ value?

[10:03] Let's start first with the low types. ~~So~~ [for] the low types, they won't get anything supported with zero, and then we integrate over zero, so this will be zero.

$$\hat{p}(\vartheta) = 0, \quad \text{if } \vartheta < \frac{1}{2} \tag{203}$$

~~So~~ the low types don't pay anything.

[13:07] So now we know the ~~direct~~ direct contract.

$$\begin{aligned}
\hat{p}(\vartheta) &= \vartheta\sqrt{\left(\vartheta - \frac{1}{2}\right)^2} - 0 - \int_{\frac{1}{2}}^{\vartheta} \sqrt{\left(x - \frac{1}{2}\right)^2}\, dx & (204)\\
&= \vartheta\left(\vartheta - \frac{1}{2}\right) - \left[\frac{1}{2}x^2 - \frac{1}{2}x\right]_{x=\frac{1}{2}}^{\vartheta} & (205)\\
&= \frac{\vartheta^2}{2} - \frac{1}{8}, \quad \text{if } \vartheta \geq \frac{1}{2} & (206)
\end{aligned}$$

This quantity is intended for type ϑ supplied to type ϑ at this price here. ~~And~~ if we want to write it as a direct contract, ~~we~~ we invert this function [$q(\vartheta)$] here for ϑ,

$$\vartheta(q) = \sqrt{q} + \frac{1}{2} \tag{207}$$

[13:53] ~~And~~ we can plug this into the price to get the direct contract.

$$\begin{aligned}
p(q) &= \hat{p}\left(\vartheta(q)\right) & (208)\\
&= \frac{\left(\sqrt{q} + \frac{1}{2}\right)^2}{2} - \frac{1}{8} & (209)\\
&= \frac{q}{2} + \frac{\sqrt{q}}{2} & (210)
\end{aligned}$$

[15:02] ~~And~~ this is then the direct contract that we can quote to the bakers. ~~and so~~ we get ~~then~~ a strictly increasing function $\frac{q+\sqrt{q}}{2}$, and we see that the optimal contract here has a quantity discount. ~~So~~ the more you buy, the lower the price per kilo will be that you have to pay for flour. ~~And for types so~~ all the types between 0 and $\frac{1}{2}$, they'll just choose the lowest point, [and] they will not buy anything, so they can choose contract $(0, 0)$. ~~And~~ then for the highest typ, what's the highest quantity that can be bought? ~~So~~ we assume that ϑ is uniformly distributed on [0,1], so the highest quantity is bought by type 1 which is $\frac{1}{4}$. ~~And so~~ then, the contract will be a curve between ~~okay~~ you can buy anywhere between 0 and $\frac{1}{4}$ at this price that has a quantity discount.

5.13 p.56 Selling an Indivisible Good

5.13.1 Selling an indivisible good

[16:36, point 1] ~~A next example would now be~~ if we sell not a perfectly divisible good, but instead we sell an indivisible good. ~~So~~ we just sell a good where the quantity can either be 0 or 1. ~~And~~ let's say like in the supermarket. ~~So~~ you sell some item in the supermarket, and so what would the optimal pricing strategy be if you don't know how much your customers value, ~~let's say~~ [such as] an apple?

[17:05] ~~So so let's see~~ in general, this is still ~~quite~~ a quite close application of ~~of~~ the theorem that we've developed. ~~The only one~~ the only difference here is that the the quantity that we want in the optimal contract has to be either 0 or 1. So, we sell either no apple or we sell an apple. ~~And so~~ if we go back to the theorem that we have [p.50], ~~so~~ what does this mean? ~~Well okay~~ if the quantity is only 0 or 1, then we're not quite sure ~~what~~ what $\hat{q}$ here should be, what this solution should be. But, what we do know is that the solution to the optimal contract still has to be monotonic. For the incentive compatibility condition, we did not assume that q was differentiable or continuous, even so we still know that the quantity has to be monotonically increasing, and we still know that the payment has to be determined uniquely by the quantities that are supplied.

[18:20] Now, monotonicity of the quantity — ~~well if~~ the only way for a function to be monotonic if its values are in 0 and 1

$$\hat{q}(\vartheta) = \mathbb{1}_{\{\vartheta \geq \vartheta_0\}} \tag{211}$$

is that it's a step function. ~~So~~ there has to be some type above that type we sell it to, and below that type we don't sell the apple, too. ~~And so~~ that's the only way for us to have a monotonic function that takes value 0 or 1.

[19:02] ~~And~~ then, we can we can determine the payment. ~~So~~ for the payment, let's again distinguish two cases. If the type is lower than ϑ_0, then

$$\hat{p}(\vartheta) = 0 - 0 - \int_{\underline{\vartheta}}^{\vartheta} v_q\left(\hat{q}(x), x\right) dx \tag{212}$$

$$= - \int_{\underline{\vartheta}}^{\vartheta} \mathbb{1}_{\{x \geq \vartheta_0\}} dx \tag{213}$$

$$= 0, \quad \text{if } \vartheta < \vartheta_0 \tag{214}$$

[20:06] ~~Okay so here~~ in this example, because the quantity is 0 [or] 1, we can assume that ~~that~~ it's a the value is linear because you cannot have either zero or one, ~~and~~ so ϑ here is precisely the value that you attach ~~to the~~ to having exactly one apple. ~~And so the partial derivative of respect to ϑ is simply $\hat{q}$ but~~ we see that this is 0 on the entire interval that we integrate over because we've assumed that ϑ is smaller than ϑ_0. ~~And so as~~ one should expect in the optimal contract everybody who does not get an apple does not pay anything for the apple either.

[21:16] ~~Then for for higher types so~~ for higher types, ~~we now have I mean I have how much they value so okay here~~ the[y] value the apple exactly at ϑ, so this will be equal to ϑ [first two terms]. ~~And then this here we what~~ we integrate over a function is 1 precisely if $\vartheta \geq \vartheta_0$, so this integral here is the same as if we integrate from ϑ_0 to ϑ.

$$\hat{p}(\vartheta) = \underbrace{v\left(q(\vartheta), \vartheta\right) - v(0, \underline{\vartheta})}_{=\vartheta} - \underbrace{\int_{\underline{\vartheta}}^{\vartheta} \mathbb{1}_{\{x \geq \vartheta_0\}} dx}_{\int_{\vartheta_0}^{\vartheta} dx} \tag{215}$$

$$= \vartheta_0, \quad \text{if } \vartheta \geq \vartheta_0 \tag{216}$$

~~So~~ everybody who values the apple higher than this this cutoff type ϑ_0 will pay exactly ϑ_0. ~~And~~ this means that ~~the optimal~~ the optimal contract here to sell in a supermarket is a fixed price contract where you just quote a price. ~~You say the apple costs some fixed demand ϑ_0, and then everybody who values it more will buy it, and everybody who will not value this much won't buy it. So this is maybe you know an expected result, but~~ it's quite nice to think that ~~we've~~ what we've started and where we have gotten. So, we have started with very general a screen problem contract theory, and in general you know a contract could be something very complicated, and we've derived that if ~~if~~ it's a price for a single item that is not divisible, then what we see what is optimal is indeed what we see in our supermarkets.

5.14 p.60 Check Your Understanding

5.14.1 Question 1

Q: If the single-crossing property is satisfied, we obtain full separation in the equilibrium.
[00:19] [False.] ~~So~~ what does full separation in the equilibrium mean? It means that every type chooses a different bundle. ~~So~~ if every type chooses a different bundle, this means that the quantity for no two types is the same.

$$q(\vartheta) \neq q(\vartheta'), \quad \forall \vartheta, \vartheta' \in \Theta \tag{217}$$

~~And~~ this is particularly the case if q is strictly increasing. ~~So~~ **we get full separation precisely if q is strictly increasing, and we have seen that the signal crossing property is not sufficient for that.** ~~So~~ sometimes, it might happen that the q that we get from the relaxed problem is either only weakly increasing, or sometimes it's not even increasing at all. ~~And~~ then, we would ~~have to~~ have to take some additional steps afterwards. So, the [single-]crossing property is not sufficient here.

5.14.2 Question 2

Q: At most one type receives the same quantity in the first-best and the second-best contract.
[01:27] [False] This is again true if the quantity is strictly increasing, then there is only one type who will receive ~~the~~ the same quantity as in the first-best. If q is not strictly increasing, then there might be two types were grouped together, and both receive the quantity intended for the highest type. ~~And so~~ this in general would also be ~~be~~ false. It's only again the case if q is strictly increasing.

5.14.3 Question 3

Q: The quantity $\hat{q}(\vartheta)$ in the second-best contract is not monotonic if $v_{q\vartheta\vartheta} < 0$.
[02:09] [False.] ~~So~~ let's see the conditions here [p.50]. ~~Condition that we had so~~ what might happen is? ~~So~~ this is only a sufficient condition. It is not necessary. So, we see here that even if this [$v_{q\vartheta\vartheta}$] is ~~let's say~~ a little bit negative as long as the ratio [$v_{q\vartheta\vartheta/h(\vartheta)}$] here ~~let's say~~ is dominated by this term [$v_{q\vartheta}$] here, we can still apply Topkis' theorem ~~to~~ to get monotonicity of the solution. So, in general this is false because the opposite inequality is sufficient but not necessary.

5.14.4 Question 4

Q: Truthful reporting of one's type is a global best response if doing so is a local best response and $q(\vartheta)$ is monotonic.
[02:53] ~~And~~ this is true ~~correct~~. This is the whole magic of contract theory with continuum of types that if we have multiplicity of ~~of~~ the quantities and ~~a global best response~~ a local best response that is enough for a global best response.