1.A Static Games with Complete Information

ECON 7219 – Games with Incomplete Information

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From Individual Choice Theory to Game Theory

Individual choice:

- We know we can model preferences over outcomes as utility functions.
- A rational individual should maximize their expected utility.
- Often, the outcome also depends on the choices made by others, who also rationally maximize their own expected utilities.

Examples:

- Competing firms have to take into account the precise set by others.
- Political treaties must provide incentives for countries to abide by it.
- Your strategy in a board game depends on the other players.
- On a date you may order different items than you would by yourself.

Introduction to Game Theory

Static Game with Complete Information

Definition 1.1

A game with complete information $\mathcal{G} = \big(\mathcal{I}, (\mathcal{A}_i), (u_i)\big)$ consists of:

1. A finite set of players $\mathcal{I} = \{1, \dots, n\}$,
2. A set $\mathcal{A}_i$ of pure actions available to player i for each $i \in \mathcal{I}$,
3. A payoff function $u_i : \mathcal{A} \to \mathbb{R}$ for each player $i \in \mathcal{I}$, where

$$\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$$

is the set of pure action profiles $a = (a_1, \dots, a_n)$.

Remark:

- Crucially, each player's payoff can depend on every players' actions.
- $u : \mathcal{A} \to \mathbb{R}^n$ defined by $u = (u_1, \dots, u_n)$ is the players' payoff vector.

Assumptions About the Players

We impose:

1. Rationality: each player maximizes his/her expected utility.
2. Awareness: each player knows $\mathcal{G}$, that is, who is playing, the available actions of each player, and each player's preferences over outcomes.
3. It is common knowledge that players are rational and aware (CKR).

Common knowledge:

- Imposition 3 above is a very strong assumption, but we need it for almost everything in game theory.
- It allows us to analyze situations which require higher-order reasoning.
- In a game with complete information, the players' payoff functions are constant over all states of the world.

Rock, Paper, Scissors

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

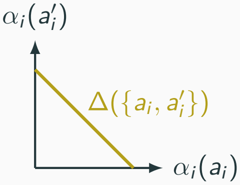
Rock paper scissors:

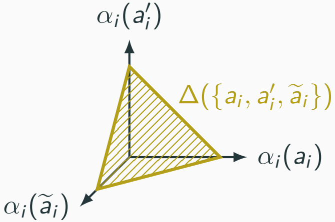
- The players $\mathcal{I} = \{1, 2\}$ are the literal players of the game.
- The available pure actions are $\mathcal{A}_1 = \mathcal{A}_2 = \{\text{Rock}, \text{Paper}, \text{and Scissors}\}$.
- Payoffs for player 1 and 2 are given in the payoff matrix.

Need to randomize:

- Players decide with which frequency to choose each of the pure actions.

Mixed Actions





Randomizing actions:

- A mixed action of player i is a distribution $\alpha_i \in \Delta(\mathcal{A}_i)$.
- If $\mathcal{A}_i$ is finite, a distribution over $\mathcal{A}_i$ is an $|\mathcal{A}_i|$-dimensional vector with

$$\alpha_i(a_i) \in [0,1] \quad \forall\ a_i \in \mathcal{A}_i, \qquad \textstyle\sum_{a_i \in \mathcal{A}_i} \alpha_i(a_i) = 1,$$

where $\alpha_i(a_i)$ indicates the probability with which $a_i \in \mathcal{A}_i$ is selected.[1]

[1] The set of n-dimensional vectors with non-negative entries that sum up to 1 is called the $(n-1)$-simplex.

Outcome

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

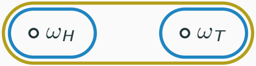
What do players observe?

- Suppose Fern chooses P and Ben chooses R and S with 50% each.
- The outcome of the game is (P, R) and (P, S) with 50% each.

The outcome, denoted $A = (A_1, \ldots, A_n)$, is a random variable such that:

1. each player i's realized action A_i is distributed according to α_i,
2. the players' actions are realized independently of each other.

Embedding Mixed Actions in an Aumann Model



Implementing mixed actions:

- Ben can implement the mixed action by flipping a coin and choosing R if heads comes up and S if tails comes up.
- The states of the world encode all uncertainty of the interaction.
- Fern does not see the coin flip and cannot anticipate the result.

Independence:

- If ω and ω' differ only through the outcome of player i's mixing, then ω and ω' lie in the same information set of every other player j.

Payoffs in a Mixed Action Profile

Ex-post payoff:

- In the action profile $\left(P, \frac{1}{2}R + \frac{1}{2}S\right)$, the players receive

$$u_1(A) = \begin{cases} 1 & \text{if } A_2 = R, \\ -1 & \text{if } A_2 = S, \end{cases} \qquad u_2(A) = \begin{cases} -1 & \text{if } A_2 = R, \\ 1 & \text{if } A_2 = S. \end{cases}$$

- Each player i's realized payoff $u_i(A)$ is a random variable as well.

Ex-ante expected payoff:

- Risk-neutral players maximize their expected payoff under the probability measure P_α defined by $P_\alpha(A_i = a_i) = \alpha_i(a_i)$.
- Expected payoff of mixed action profile $\alpha = (\alpha_1, \dots, \alpha_n)$ is

$$u_i(\alpha) := \mathbb{E}_\alpha[u_i(A)] = \sum_{a \in \mathcal{A}} u_i(a) P_\alpha(A = a) = \sum_{a \in \mathcal{A}} u_i(a) \prod_{j=1}^{n} \alpha_j(a_j).$$

Nash Equilibrium

Nash Equilibrium

Definition 1.2

A Nash equilibrium of a static game $\mathcal{G} = \big(\mathcal{I}, (\mathcal{A}_i), (u_i)\big)$ is a (possibly mixed) action profile α such that for every $i \in \mathcal{I}$ and every $a_i \in \mathcal{A}_i$,

$$\mathbb{E}_\alpha[u_i(A)] \geq \mathbb{E}_{(a_i,\alpha_{-i})}[u_i(A)],$$

where we denote $\alpha_{-i} = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$.

Interpretation:

1. Each player i predicts the behavior α_{-i} of his/her opponents correctly.
2. Each player i best replies to his/her conjecture with α_i.

Note: Requirement 1. is a rather strong requirement.

Nash Equilibrium as a Solution Concept

Correctly predicting behavior:

- A player with incorrect beliefs about opponent's play has ex-post regret.
- Correct predictions of opponents' play gives us a workable theory.
- As a theoretical solution concept, it is extremely useful.
- We later see justifications when the assumption is reasonably satisfied.

Existence of Nash equilibria:

- If $\mathcal{A}$ is finite, existence in mixed actions follows from Nash (1950).
- If $\mathcal{A}$ is convex and compact and u is continuous and quasi-concave, existence in pure actions follows from Glicksberg (1952).
- If $\mathcal{A}$ is closed and u is bounded and upper semi-continuous, existence in mixed actions follows from Dasgupta and Maskin (1986).

Tipping

	T	N
G	2, 2	−1, 3
B	3, −1	0, 0

Available actions:

- Client chooses to (T)ip or (N)ot tip.
- Waiter provides (G)ood or (B)ad service.

Strictly dominated actions:

- Regardless of the quality of service, the Client is better off not tipping.
- Regardless of tips, it is better for the Waiter to provide bad service.
- The unique Nash equilibrium is (B, N).

Strictly Dominated Strategies

Lemma 1.3

Action $a_i \in \mathcal{A}_i$ is strictly dominated by $\widetilde{a}_i \in \mathcal{A}_i$ if $u_i(a_i, a_{-i}) < u_i(\widetilde{a}_i, a_{-i})$ for all $a_{-i} \in \mathcal{A}_{-i}$. Strictly dominated actions cannot be played in any Nash equilibrium α, that is, $\alpha_i(a_i) = 0$ for any strictly dominated action a_i.

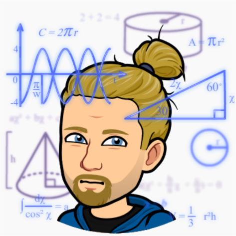
More generally:

- Nash equilibrium assigns positive weight only to actions a_i that survive iterated deletion of strictly dominated strategies.
- Remove in round k those strategies that are strictly dominated when opponents play strategies that survive round k of elimination.

Note:

- Nash equilibria can assign positive weight to weakly dominated actions.

Guessing Two Thirds of the Average

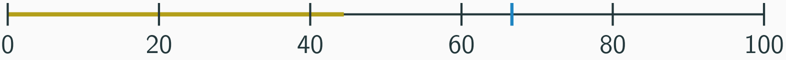


Setup of game:

- Guess a number between 0 and 100, that is, choose $a_i \in [0, 100]$.
- The goal is to guess two thirds of the average of all guesses, i.e.,

$$u_i(a) = -\left| a_i - \frac{2}{3n} \sum_{j=1}^{n} a_j \right|.$$

Iterated Elimination



Rationality:

- Guessing above 200/3 is strictly dominated by guessing 200/3.

Knowledge of rationality:

- I know nobody else will guess more than 200/3.
- Consequently, any guess above 400/9 is dominated by guessing 400/9.

Common knowledge of rationality:

- Everybody should guess 0. This is the unique Nash equilibrium.
- In any other outcome, some player's beliefs are incorrect.
- CKR can justify correct beliefs about opponents' play.

Indifference Principle

Lemma 1.4

In a mixed Nash equilibrium α, every player i is indifferent between the pure actions in the support of α_i, that is,

$$u_i(a_i, \alpha_{-i}) = u_i(\alpha)$$

for every $a_i \in \operatorname{supp}(\alpha_i)$, where $\operatorname{supp}(\alpha_i) := \{a_i \in \mathcal{A}_i \mid \alpha_i(a_i) > 0\}$.

Remark:

- The indifference principle is useful to find completely mixed equilibria, which are equilibria that assign positive weight to every pure action.
- The indifference principle is of no help if we do not know $\operatorname{supp}(\alpha_i)$.

Finding Completely Mixed Equilibria

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Need to randomize:

- Parametrize mixed action α_i for $i = 1, 2$ by $(r_i, p_i, 1 - r_i - p_i)$.
- Player 1's choice must make Player 2 indifferent between R, P, and S:

$$u_2(\alpha_1, P) = 2r_1 + p_1 - 1 \;=\; u_2(\alpha_1, S) = p_1 - r_1 \quad \text{yields} \quad r_1 = \tfrac{1}{3},$$

$$u_2(\alpha_1, R) = 1 - r_1 - 2p_1 \;=\; u_2(\alpha_1, S) = p_1 - r_1 \quad \text{yields} \quad p_1 = \tfrac{1}{3}.$$

- By symmetry, both players choose $\alpha^i_* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Best-Response Correspondences

Drawbacks of using the indifference principle:

- If there is a Nash equilibrium α with $\operatorname{supp}_i \alpha_i = \{R, P, S\}$, then it is α_*.
- It does not say anything about mixtures among, say, $\{P, R\}$ or $\{R, S\}$.

Best-response correspondence:

- Player i's best-response correspondence $\mathcal{B}_i$ is defined by

$$\mathcal{B}_i(\alpha_{-i}) = \left\{ \alpha_i \in \Delta(\mathcal{A}_i) \;\middle|\; u_i(\alpha_i, \alpha_{-i}) = \max_{a_i \in \mathcal{A}_i} u_i(a_i, \alpha_{-i}) \right\}.$$

- We can find Nash equilibria by:
  - computing $\mathcal{B}_i(\alpha_{-i})$ for each player i using differentiation,
  - verify consistency: find α such that $\alpha_i \in \mathcal{B}_i(\alpha_{-i})$ for each player i.

Crossing an Intersection

	C	W
C	$-4, -4$	$1, 0$
W	$0, 1$	$0, 0$

Crossing an intersection:

- Both Pedestrian and Driver can either (C)ross or (W)ait.
- Parametrize α by $\alpha_1(C) = x$ and $\alpha_2(C) = y$.
- The expected utilities of Pedestrian and Driver are

$$u_1(\alpha) = -4xy + x(1-y), \qquad u_2(\alpha) = -4xy + y(1-x).$$

- The partial derivatives are $\frac{u_1(\alpha)}{\partial x} = 1 - 5y$ and $\frac{\partial u_2(\alpha)}{\partial y} = 1 - 5x$.

Image source: https://www.taiwannews.com.tw/en/news/3506399

Verifying Consistency

Best-response correspondences:

$$\mathcal{B}_1(y) = \begin{cases} x = 1 & \text{if } y < \frac{1}{5}, \\ x \in [0,1] & \text{if } y = \frac{1}{5}, \\ x = 0 & \text{if } y > \frac{1}{5}, \end{cases} \qquad \mathcal{B}_2(x) = \begin{cases} y = 1 & \text{if } x < \frac{1}{5}, \\ y \in [0,1] & \text{if } x = \frac{1}{5}, \\ y = 0 & \text{if } x > \frac{1}{5}. \end{cases}$$

Verifying consistency:

- Go through the three cases:
  - If $y < \frac{1}{5}$, then $\mathcal{B}_1$ implies $x = 1$, hence $\mathcal{B}_2$ implies $y = 0$. ✓
  - If $y = \frac{1}{5}$, then $\mathcal{B}_2$ implies $x = \frac{1}{5}$, hence $\mathcal{B}_1$ implies $y = \frac{1}{5}$. ✓
  - If $y > \frac{1}{5}$, then $\mathcal{B}_1$ implies $x = 0$, hence $\mathcal{B}_2$ implies $y = 1$. ✓

$\Rightarrow$ The Nash equilibria are (C, W), (W, C), and $\left(\frac{1}{5}C + \frac{4}{5}W, \frac{1}{5}C + \frac{4}{5}W\right)$.

Correctly Predicting Behavior

Common knowledge of rationality:

- In some games, a unique action profile survives IESDS.

By design:

- There is explicit communication on which equilibrium will be played.
- Example: the law informs people how they are expected to behave.

Evolutionary process:

- Players continually observe opponents' strategies and they adapt by best responding iteratively until the process converges.
- Example: at a poker table, you may update your beliefs about the frequency, with which opponents bluff and adapt your strategy.
- Example: social norms are upheld because we are all following them.

What Is a Nash Equilibrium?

Consistent outcomes:

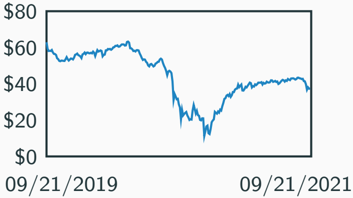
- Nash equilibria are the set of outcomes that are consistent with mutually rational behavior, in which nobody's beliefs are ex-post invalidated.
- We should view the set of Nash equilibria as the solution to a game.

Individual Nash equilibria as predictions:

- This is reasonable if it is a focal point through societal norms, laws, pre-game communication, or repeated interactions.
- Moreover, if the Nash equilibrium is unique, players can derive it.
- If neither of these situations apply, we should be cautious to use a single Nash equilibrium outcome as a prediction.

Continuous Action Sets

Cournot Duopoly



Cournot duopoly:

- Firms $i = 1, 2$ produce a quantity $q_i \in [0, \infty)$ at unit cost c_i.
- Suppose the market price is $p(q) = p_0 - q_1 - q_2$ so that i's payoff is

$$u_i(q) = (p(q) - c_i)q_i = (p_0 - c_i)q_i - q_i^2 - q_1 q_2.$$

- We still find Nash equilibria through best-response correspondences.

Source: https://www.macrotrends.net/2516/wti-crude-oil-prices-10-year-daily-chart

Concave Utilities: Unique Best Responses

Lemma 1.5

If u_i is strictly concave in a_i, then $\mathcal{B}_i(\alpha_{-i})$ is a singleton for every α_{-i}.

In this class:

- Whenever $\mathcal{A}_i$ is a continuum, we will impose that u_i is differentiable and strictly concave and that player i uses only pure strategies.
- In this setting, a pure-strategy Nash equilibrium exists by Debreu (1952), Glicksberg (1952), and Fan (1952).

Consequence: $\mathcal{B}_i(a_{-i})$ is either

- a boundary point of $\mathcal{A}_i$,
- an interior point a_i, at which $\frac{\partial u_i(a)}{\partial a_i} = 0$.

Cournot Duopoly: Best Responses

Best response correspondence:

- At the boundary point $q_i = 0$, firm i makes $u_i(q) = 0$.
- If the maximum is attained at an interior point $\widehat{q}_i$, then

$$0 = \left.\frac{\partial u_i(q)}{\partial q_i}\right|_{q_i = \widehat{q}_i} = (p_0 - c_i) - 2\widehat{q}_i - q_{-i}.$$

- By concavity of u_i, this is indeed a local maximum, not a minimum.
- The best response is thus to choose $\widehat{q}_i$ if $u_i(\widehat{q}_i, q_{-i}) \geq 0$ and to choose $q_i = 0$ otherwise. This can be simplified to

$$\mathcal{B}_i(q_{-i}) = \frac{(p_0 - c_i - q_{-i})^+}{2}.$$

- The result is incorrect without the positive part $(\,\cdot\,)^+ = \max(\,\cdot\,, 0)$.

Cournot Duopoly: Consistency

Best-response correspondences:

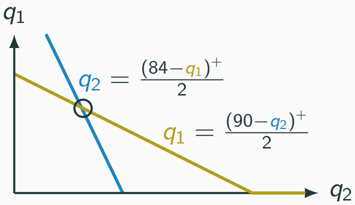
- Suppose that $p_0 = 100$, $c_1 = 10$, and $c_2 = 16$ so that

$$\mathcal{B}_1(q_2) = \frac{(90 - q_2)^+}{2}, \qquad \mathcal{B}_2(q_1) = \frac{(84 - q_1)^+}{2}.$$

Verifying consistency:

- Go through the three cases:
  - If $q_2 = 0$, then $\mathcal{B}_1$ implies $q_1 = 45$, hence $\mathcal{B}_2$ implies $q_2 = 19.5$. ✗
  - If $q_2 \geq 90$, then $\mathcal{B}_1$ implies $q_1 = 0$, hence $\mathcal{B}_2$ implies $q_2 = 42$. ✗
  - If $q_2 \in (0, 90)$, then $\mathcal{B}_1$ implies $q_1 = \frac{90 - q_2}{2}$ and $\mathcal{B}_2$ implies $q_2 = \frac{84 - q_1}{2}$. ✓
- The latter case is consistent, yielding the Nash equilibrium $(32, 26)$.

Cournot Duopoly: Linear Price Function



Mutual best response:

- Geometrically, mutual best responses are intersections of $\mathcal{B}_i(q_{-i})$.

Practice problem:

- Suppose the price and cost are as in this example, but:
  - Firm 1 chooses a pure quantity $q_1 \in [0, \infty)$.
  - Firm 2 can produce only $q_2 \in \{0, 20, 40\}$, but has the ability to mix.

Summary

Key points in the definition of a Nash equilibrium α:

- Each player i conjectures the behavior α_{-i} of opponents correctly,
- Each player i best replies to his/her conjecture with α_i.

How to find all Nash equilibria:

0. Eliminate strictly dominated strategies.
1. Parametrize the players' strategies.
2. Find the best-response correspondences.
3. Verify consistency by distinguishing cases.

Main concept:

- Finding mutual best responses.

Check Your Understanding

	L	M	R
T	2, 1	1, 2	2, 0
D	1, 3	0, 1	3, 1

	L	R
T	2, 1	0, 0
D	2, 1	1, 3

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Short-answer questions:

1. Find all the Nash equilibria in the above two games.
2. Find all the Nash equilibria in the practice problem on Slide 26.

True or false:

3. No strictly dominated actions are in the support of any Nash equilibrium.
4. In every outcome of a mixed Nash equilibrium, every player best responds to the other players' outcome.
5. We can find all Nash equilibria by applying the indifference principle to all possible combinations of the support.

Literature

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