Econ 7009: Midterm Exam

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You have **3 hours** to answer the following questions. The total number of points is **110**. You need to answer each questions in English. If you have stated a theorem in lecture notes (including homework), you may use it without proving it unless I explicitly ask you to, but you need to describe what the theorem is and why you can apply that theorem. For example, if there are some assumptions for that theorem to be applicable, you need to show that those assumptions are met in your problem.

1. (20 points) Consider a sequence $\{\boldsymbol{v_n}\}$ in $\boldsymbol{R^m}$. Show that $\|\boldsymbol{v_n}\|$ converges to 0 if and only if $\boldsymbol{v_n}$ converges to $\mathbf{0}$.

2. (14 points) Consider a sequence $\{\boldsymbol{x_n}\}_{n=1}^{\infty}$ in $\boldsymbol{R^m}$ converges to $\boldsymbol{x}$.

   (a) (8 points) Show that every subsequence of $\{\boldsymbol{x_n}\}_{n=1}^{\infty}$ converges.

   (b) (6 points) Given your answer from (a), why does a subsequence of $\{\boldsymbol{x_n}\}_{n=1}^{\infty}$ cannot converge to a point different from $\boldsymbol{x}$?

3. (6 points) Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in $\boldsymbol{R^1}$. Show that given any n

$$\inf\{x_n + y_n, x_{n+1} + y_{n+1}, ...\} \geq \inf\{x_n, x_{n+1}, ...\} + \inf\{y_n, y_{n+1}, ...\}$$

4. (8 points) Find the supremum and maximum for the following case: $X = \{x \in \boldsymbol{R} | x = \frac{n}{2n+1}, n = 1, 2, ...\}$

5. (8 points) The functions F and G are defined as $F(x) = 2x + 1, x > 0$ and

$$G(x) = \begin{cases} \frac{1}{x} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$$

   (a) Verify that $R(F) \subseteq D(G)$. Find the domain and range of $(G \circ F)(x)$ (Note: here I am asking the range that $(G \circ F)$ maps "on to")

   (b) Find the formula for $(G \circ F)(x)$

6. (8 points) Consider a function $f : S \to T$, with $A \subset S$ and $B \subset T$

   (a) Define the set of $f^{-1}(f(A))$

   (b) Define the set of $f(f^{-1}(B))$

7. (8 points) Suppose that a function is

$$f = \{(1,3), (2,5), (3,8), (4,10), (5,11), (6,4), (7,6), (8,8), (9,10), (10,12)\}$$

   (recall that a function is a set of ordered pairs), so this is a function from $\{1, 2, ..., 10\}$ into $\boldsymbol{R^1}$. Let $A = \{1, 2, 3\}$ and $B = \{6, 8, 10, 12, 15\}$.

   (a) Write down all the elements in $f^{-1}(f(A))$

   (b) Write down all the elements in $f(f^{-1}(B))$

8. (8 points) Suppose a and b are real numbers. Find a and b such that the following

vectors in $\mathbf{R}^4$ are linearly dependent.

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -1 \\ a \\ b \end{bmatrix}$$

9. (8 points)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

   (a) (5 points) Find the eigenvalues and eigenvectors of A

   (b) (3 points) There exists a unique solution to solve $Ax = b$ for every b. True or False?

10. (10 points) Discuss why the following proofs are incorrect.

   (a) Prove that if a sequence has only one limit point p, then this sequence converges to p.

   Let's prove by contradiction. Suppose that the sequence x_n does not converge to p. Then there exists an $\epsilon > 0$ and for all N such that $d(x_n, p) \geq \epsilon$ if $n \geq N$. Then we find an ϵ that there are finite indices n for which $d(x_n, p) < \epsilon$. Therefore, p cannot be a limit point.

   (b) Suppose that $x_n \to 0$. Show that $\lim_{n\to\infty} x_n \sin \frac{1}{x_n} = 0$

By $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$, we have

$$\lim_{n\to\infty} x_n \sin\frac{1}{x_n} = \lim_{n\to\infty} x_n \lim_{n\to\infty} \sin\frac{1}{x_n}.$$

Because the range of $\sin(\frac{1}{x_n})$ is bounded in $[-1, 1]$, and x_n tends to zero as n sufficiently large, the product of the two must eventually tend to zero.

11. (12 points) Please state whether each of the following statements is true or false. If your answer is true, give a brief proof or explanation to the statement. If your answer is false, you must provide a counter example, or explain why the statement is false.

   (a) If $\sup A$ exists, then $\sup A = \max A$

   (b) If $\sup A$ does not exist, then $\max A$ does not exist.

   (c) If $\max A$ exists, then $\sup A \in A$.

   (d) $\sup A$ has to be an element of A.