Macro Theory I Part 2 - Final (2020)

Solution suggested by Shang-Chieh Huang

January 3, 2022

A. Habit Persistent Model

A-1: Write down a representative household's problem as a dynamic programming.

$$
\begin{aligned}
V(k, c_{-1}, K, C_{-1}) = \max_{c,k',h} \; & u(c, c_{-1}, 1-h) + \beta V(k', c, K', C) \\
\text{s.t.} \quad & c + k' = r(K, C_{-1})k + w(K, C_{-1})h + (1-\delta)k \\
& K' = G_K(K, C_{-1}) \\
& C = G_C(K, C_{-1})
\end{aligned}
$$

A-2: Write down the firm's problem.

$$
\max_{K^f, H^f} \; F(K^f, H^f) - r(K, C_{-1})K^f + w(K, C_{-1})H^f
$$

A-3: Define a ***recursive competitive equilibrium*** for the economy.

A *Recursive Competitive Equilibrium* is a set of equations

1. a value function: $V(k, c_{-1}, K, C_{-1})$,
2. individual's decision rules: $c(k, c_{-1}, K, C_{-1})$, $h(k, c_{-1}, K, C_{-1})$ and $k'(k, c_{-1}, K, C_{-1})$,
3. decision rules for the firm: $h^f(K, C_{-1})$ and $k^f(K, C_{-1})$,
4. pricing functions: $w(K, C_{-1})$ and $r(K, C_{-1})$,
5. a low of motion for aggregate state variable $K' = G_K(K, C_{-1})$ and $C = G_C(K, C_{-1})$

such that

1) given $r(K, C_{-1})$, $w(K, C_{-1})$, $G_K(K, C_{-1})$ and $G_C(K, C_{-1})$,

   $V(k, c_{-1}, K, C_{-1})$, $c(k, c_{-1}, K, C_{-1})$, $h(k, c_{-1}, K, C_{-1})$ and $k'(k, c_{-1}, K, C_{-1})$ solve the household's dynamic programming problem;

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2) given $r(K, C_{-1})$ and $w(K, C_{-1})$,

   the decision rules $h^f(K, C_{-1})$ and $k^f(K, C_{-1})$ solve the firm's problem;

3) market clearing condition:

   – labor:

$$h^f(K, C_{-1}) = h(K, C_{-1}, K, C_{-1});$$

   – capital:

$$k^f(K, C_{-1}) = K;$$

   – consumption goods: clear by Walras' Law;

4) perceptions are correct:

$$K' = k'$$

$$\Longrightarrow G_K(K, C_{-1}) = k'(K, C_{-1}, K, C_{-1});$$

$$C = c$$

$$\Longrightarrow G_C(K, C_{-1}) = c(K, C_{-1}, K, C_{-1}).$$

B. Transitional Probability

B-1: Compute an invariant distribution for this Markov chain. Is this invariant distribution unique? Explain.

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 1 *(Stokey and Lucas P.321) An set $E \subset S$ is called an **ergodic set** if*

$$Pr(z_{t+1} \in E | z_t \in E) = 1,$$

for $z_i \in E$, and if no proper subset of E has this property.

The sets satisfy that $Pr(z_{t+1} \in E | z_t \in E) = 1$ are $\{z_1\}$, $\{z_2\}$ and $\{z_1, z_2\}$. But $\{z_1, z_2\}$ has subsets $\{z_1\}$ and $\{z_2\}$ that have same property. Also, we can know that $\{z_1\}$ and $\{z_2\}$ do not have a subset that have same property. That is, the ergodic sets include $\{z_1\}$ and $\{z_2\}$, which obviously are not unique. Thus, we cannot have an unique invariant distribution in this case.

Assume that the invariant distribution is $\pi = [\pi_1 \quad 1 - \pi_1]$. By definition, we have

$$\pi P = \pi.$$

Thus, we have

$$\begin{aligned} \pi_1 &= \pi_1 \\ 1 - \pi_1 &= 1 - \pi_1. \end{aligned}$$

Therefore, the invariant distribution is $[x \quad 1 - x]$, where $x \in [0, 1]$.

B-2: Compute an invariant distribution for this Markov chain. Is this invariant distribution unique? Explain.

$$p = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.7 & 0.3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Assume that the invariant distribution is $\pi = [\pi_1 \quad \pi_2 \quad 1 - \pi_1 - \pi_2]$. By definition, we have

$$\pi P = \pi.$$

Thus, we have

$$\begin{aligned} 0.3\pi_1 + 0.7\pi_2 + (1 - \pi_1 - \pi_2) &= \pi_1 \\ 0.7\pi_1 + 0.3\pi_2 &= \pi_2 \\ 0 &= 1 - \pi_1 - \pi_2. \end{aligned}$$

Therefore, we can solve for $\pi_1 = 0.5$ and $\pi_2 = 0.5$. Hence, the invariant distribution is $\pi = [0.5 \quad 0.5 \quad 0]$.

The sets satisfy that $Pr(z_{t+1} \in E | z_t \in E) = 1$ are $\{z_1, z_2\}$ and $\{z_1, z_2, z_3\}$. But $\{z_1, z_2, z_3\}$ has a subset $\{z_1, z_2\}$ that have same property; that is, the ergodic set is $\{z_1, z_2\}$, which obviously is unique. Thus, we can have an unique invariant distribution in this case.

C. Dynamic Programming

C-1: Derive a set of first order and envelope conditions for this for this problem.

The DPP can be rewritten as

要代入算!!

$$v(s) = \max_{d} \left\{ R(s,d) + \beta v(B(s,d)) \right\}.$$

F.O.C.: 求极

chain rule!

$$R_d(s,d) + \beta v'(s') B_d(s,d) = 0,$$

where $R_d(s,d) \equiv \partial R(s,d)/\partial d$ and $B_d(s,d) \equiv \partial B(s,d)/\partial d$.

Envelope Condition:

对 state variable 偏微分

等于 2 回

就是对 v 的参数偏微分

$$\boxed{v'(s)} = R_s(s,d) + \beta v'(s') B_s(s,d),$$

where $R_s(s,d) \equiv \partial R(s,d)/\partial s$ and $B_s(s,d) \equiv \partial B(s,d)/\partial s$.

C-2: Derive a set of equations that determine a steady state for this problem.

Rearranging the envelope condition, we have

$$v'(s') = \frac{v'(s) - R_s(s,d)}{\beta B_s(s,d)}.$$

Plugging it into the first order condition yields

$$R_d(s,d) + \left[\frac{v'(s) - R_s(s,d)}{B_s(s,d)} \right] B_d(s,d) = 0.$$

再做一次??

After some rearrangement, we have

$$v'(s) = R_s(s,d) - R_d(s,d) \frac{B_s(s,d)}{B_d(s,d)}$$

$$\Longrightarrow v'(s') = R_s(s',d') - R_d(s',d') \frac{B_s(s',d')}{B_d(s',d')}.$$

Substitute $v'(s')$ in the first order condition, we have the Euler equation

$$R_d(s,d) + \beta \left[R_s(s',d') - R_d(s',d') \frac{B_s(s',d')}{B_d(s',d')} \right] B_{\cancel{s}\,d}(s,d) = 0.$$

Evaluating the Euler equation and the law of motion, $s' = B(s,d)$, at the steady state, i.e. $d = d^*$ and $s = s^*$, we have

$$0 = R_d(s^*,d^*) + \beta \left[R_s(s^*,d^*) - R_d(s^*,d^*) \frac{B_s(s^*,d^*)}{B_d(s^*,d^*)} \right] B_{\cancel{s}\,d}(s^*,d^*)$$

$$s^* = B(s^*,d^*).$$

These two equations can solve for the two unknowns, d^* and s^*.

D.Two-period OLG Model

D-1: Write down the maximization problem for an agent born at $t \geq 0$. What is his/her optimal decision on savings?

$$\begin{aligned} \max_{c_t^y, c_{t+1}^o, s_t} \quad & \log c_t^y + \beta \log c_{t+1}^o \\ \text{s.t.} \quad & c_t^y + s_t = w_t \\ & c_{t+1}^o = R_{t+1} s_t \end{aligned} \tag{1}$$

Combining two constraints gives us

$$c_t^y + \frac{c_{t+1}^o}{R_{t+1}} = w_t. \tag{2}$$

F.O.C.:

$$\begin{aligned} [c_t^y]: \quad & \frac{1}{c_t^y} = \lambda \\ [c_{t+1}^o]: \quad & \frac{\beta}{c_{t+1}^o} = \frac{\lambda}{R_{t+1}}. \end{aligned}$$

Combining the above equations, we have

$$\frac{1}{c_t^y} = \beta R_{t+1} \frac{1}{c_{t+1}^o}. \tag{3}$$

Combining Eq.(2) and Eq.(3) yields

$$\begin{aligned} c_t^y &= \frac{w_t}{1+\beta} \\ c_{t+1}^o &= \frac{\beta R_{t+1} w_t}{1+\beta}. \end{aligned}$$

From Eq.(1),

$$s_t = \frac{\beta w_t}{1+\beta}.$$

D-2: Write down the maximization problem for initial old. What is his/her optimal decision?

$$\begin{aligned} \max_{c_0^o} \quad & \log c_0^o \\ \text{s.t. } & c_0^o = R_0 s_{-1} \end{aligned}$$

The optimal decision is simply $c_0^o = R_0 s_{-1}$.

D-3: Write down the firm's maximization problem. What are the firm's optimal decision?

$$\max_{K_t^f, N_t^f} (1+g)^t A (K_t^f)^\alpha (N_t^f)^{1-\alpha} - w_t N_t^f - r_t K_t^f$$

F.O.C.:

$$\begin{aligned}
[K_t^f]: \quad r_t &= \alpha(1+g)^t A (K_t^f)^{\alpha-1}(N_t^f)^{1-\alpha} \\
&= \alpha(1+g)^t A \left(k_t^f\right)^{\alpha-1} \\
[N_t^f]: \quad w_t &= (1-\alpha)(1+g)^t A (K_t^f)^\alpha (N_t^f)^{-\alpha} \\
&= (1-\alpha)(1+g)^t A \left(k_t^f\right)^\alpha .
\end{aligned}$$

D-4: Derive a set of equations that characterize the detrended equilibrium for the economy

The set of equations that characterize the equilibrium are

$$\frac{1}{c_t^y} = \beta R_{t+1} \frac{1}{c_{t+1}^o} \tag{4}$$

$$c_t^y + s_t = w_t \tag{5}$$

$$c_{t+1}^o = R_{t+1} s_t \tag{6}$$

RC

+

Law of motion!

$$r_t = \alpha(1+g)^t A \left(k_t^f\right)^{\alpha-1} \tag{7}$$

$$w_t = (1-\alpha)(1+g)^t A \left(k_t^f\right)^\alpha \tag{8}$$

$$R_t = r_t + 1 - \delta \tag{9}$$

$$c_t^y + \frac{c_t^o}{1+n} + (1+n)k_{t+1} = (1+g)^t A k_t^\alpha + (1-\delta)k_t \tag{10}$$

?

mkt clear

$$s_t = (1+n)k_{t+1} \tag{11}$$

Define $x_t = g_x^t \hat{x}_t$. From Eq.(10), we have

$$g_{cy}^t \hat{c}_t^y + g_{co}^t \frac{\hat{c}_t^o}{1+n} + (1+n) g_k^{t+1} \hat{k}_{t+1} = (1+g)^t A \left(g_k^t \hat{k}_t\right)^\alpha + (1-\delta) g_k^t \hat{k}_t .$$

Divide the equation above by g_k^t, we have

$$\left(\frac{g_{cy}}{g_k}\right)^t \hat{c}_t^y + \left(\frac{g_{co}}{g_k}\right)^t \frac{\hat{c}_t^o}{1+n} + (1+n) g_k \hat{k}_{t+1} = \left(\frac{1+g}{g_k^{1-\alpha}}\right)^t A \hat{k}_t^\alpha + (1-\delta)\hat{k}_t .$$

This equation is stationary if

$$g_{cy} = g_{co} = g_k = (1+g)^{\frac{1}{1-\alpha}} . \tag{12}$$

From Eq(5), we have

$$\hat{c}_t^y + \left(\frac{g_s}{g_{cy}}\right)^t \hat{s} = \left(\frac{g_w}{g_{cy}}\right)^t \hat{w}_t .$$

This equation is stationary if

$$g_{cy} = g_s = g_w. \tag{13}$$

From Eq.(7), we have

$$r_t = \alpha \left(\frac{1+g}{g_{kf}^{\alpha-1}} \right)^t A \left(\hat{k}_t^f \right)^{\alpha-1}.$$

This equation is stationary if

$$g_{kf} = (1+g)^{\frac{1}{1-\alpha}}. \tag{14}$$

From Eq.(12), Eq.(13) and Eq.(14),

$$g_{cy} = g_{co} = g_k = g_s = g_w = g_{kf} = (1+g)^{\frac{1}{1-\alpha}}.$$

Note that r_t and R_t would not growth.

Thus, the set of equations that characterize the detrended equilibrium for the economy is

$$\begin{aligned}
\frac{(1+g)^{\frac{1}{1-\alpha}} \hat{c}_{t+1}^o}{\hat{c}_t^y} &= \beta R_{t+1} \\
\hat{c}_t^y + \hat{s}_t &= \hat{w}_t \\
(1+g)^{\frac{1}{1-\alpha}} \hat{c}_{t+1}^o &= R_{t+1} \hat{s}_t \\
r_t &= \alpha A \left(\hat{k}_t^f \right)^{\alpha-1} \\
\hat{w}_t &= (1-\alpha) A \left(\hat{k}_t^f \right)^{\alpha} \\
R_t &= r_t + 1 - \delta \\
\hat{c}_t^y + \frac{\hat{c}_t^o}{1+n} + (1+n)(1+g)^{\frac{1}{1-\alpha}} \hat{k}_{t+1} &= A \hat{k}_t^\alpha + (1-\delta)\hat{k}_t \\
\hat{s}_t &= (1+n)(1+g)^{\frac{1}{1-\alpha}} \hat{k}_{t+1}
\end{aligned}$$

D-5: Derive the condition for dynamic inefficiency for the economy.

Dynamic inefficiency exists when the saving rate implied by OLG model does not follow golden rule.

In steady state, the resource constraint becomes

$$c^{*y} + \frac{c^{*o}}{1+n} + (1+n)(1+g)^{\frac{1}{1-\alpha}} k^* = A k^{*\alpha} + (1-\delta) k^*.$$

Rearrange the equation, we have

$$c^{*y} + \frac{c^{*o}}{1+n} = A k^{*\alpha} - \left[(1+n)(1+g)^{\frac{1}{1-\alpha}} - (1-\delta) \right] k^*.$$

Define aggregate consumption at the steady state $\bar{c} = c^{*y} + \frac{c^{*o}}{1+n}$.

The golden rule saving rate maximizes $\bar{c}$ satisfying the following first order condition:

$$\frac{\partial \bar{c}}{\partial s} = \left\{ \alpha A k^{*\alpha-1} - \left[(1+n)(1+g)^{\frac{1}{1-\alpha}} - (1-\delta) \right] \right\} \frac{\partial k^*}{\partial s} = 0.$$

if < 0 <0 → inefficiency!

$f'(k_{ss})$

Thus when $\alpha A k^{*\alpha-1} < (1+n)(1+g)^{\frac{1}{1-\alpha}} - (1-\delta)$, the marginal gain of aggregate consumption from saving is negative. In other words, the economy overaccumulates capital, resulting in dynamic inefficiency.