2. Belief Spaces and Belief Hierarchies

ECON 7219 – Games With Incomplete Information

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Absence of Knowledge

Aumann model of incomplete information:

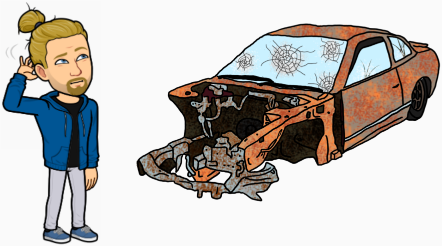
- Allows us to describe higher-order knowledge between players and how players update their higher-order knowledge given new information.

In absence of knowledge:

- We don't just say "whelp, there's nothing I can do." We form beliefs.

Belief Spaces

Is the Mechanic Honest?



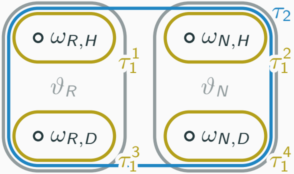
Visiting the mechanic:

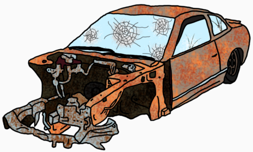
- You experience car troubles and go see the mechanic.
- The mechanic provides a much longer list with items that need fixing.

Incomplete information:

- Is the mechanic honest?
- Does the car actually need those expensive repairs?

Aumann Model of Incomplete Information

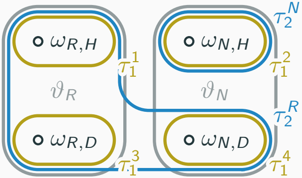


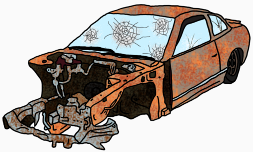


States:

- Payoff relevant is only whether your car needs expensive repairs or not, that is, the possible states of nature are $\Theta = \{\vartheta_R, \vartheta_N\}$.
- We represent the Mechanic's (H)onesty or (D)ishonesty by incorporating this uncertainty into the states of the world $\omega_{R,H}$, $\omega_{R,D}$, $\omega_{N,H}$, $\omega_{N,D}$.
- The Mechanic knows both, hence has singleton information sets.
- We do not know either, hence we have a single information set.

The Mechanic's Recommendation





Recommendation:

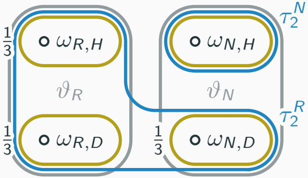
- The Mechanic can recommend (E)xtensive repairs or (M)inor repairs.
- Suppose his/her decision rule is $\sigma_1(\tau_1^2) = M$ and $\sigma_1(\tau_1) = E$ for $\tau_1 \neq \tau_1^2$.
- The recommendation allows us to distinguish the events

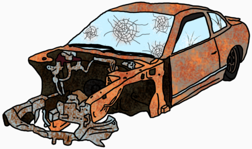
$$\{A_1 = M\} = \{\omega \in \Omega \mid A_1(\omega) = M\} = \{\omega_{N,H}\},$$

$$\{A_1 = E\} = \{\omega \in \Omega \mid A_1(\omega) = E\} = \{\omega_{R,H}, \omega_{R,D}, \omega_{N,D}\}.$$

- Our information sets are $\tau_2^N = \{\omega_{N,H}\}$ and $\tau_2^R = \{\omega_{R,H}, \omega_{R,D}, \omega_{N,D}\}$.

Knowledge and Beliefs





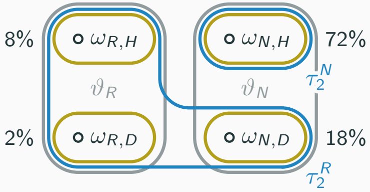
Knowledge:

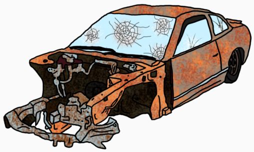
- Upon hearing M, we know $\theta = \vartheta_N$ since $T_2(\omega_{N,H}) \subseteq \{\theta = \vartheta_N\}$.
- We do not know θ after hearing E.

Beliefs:

- Upon learning that the true state is in τ_2^R, we form beliefs about its states, i.e., we assign probabilities to the three states in τ_2^R.
- In the above example, our beliefs are $P_{\tau_2^R}(\theta = \vartheta_R) = \frac{2}{3}$.

How Do We Form Beliefs?





Updating beliefs:

- Suppose that 20% of mechanics are dishonest and that, independently, our car needs serious repair with 10% probability.
- These are our prior beliefs P, held before we learn the information set.
- After learning the information set τ_2^R, we update our beliefs to the posterior beliefs $P_{\tau_2^R}(\theta = \vartheta_R) = P\left(\theta = \vartheta_R \,\middle|\, \tau_2^R\right)$ via Bayes' rule.
- In general, players may have differing priors $P_i \neq P_j$.

Probability Measure

Definition 2.1

A probability measure P is a map that assigns to each observable event $Y \subseteq \Omega$ a probability $P(Y) \in [0,1]$ such that $P(\Omega) = 1$ and

$$P\left(\bigcup_{n\in\mathbb{N}} Y_n\right) = \sum_{n\in\mathbb{N}} P(Y_n) \tag{1}$$

for any sequence $(Y_n)_{n\in\mathbb{N}}$ of pairwise disjoint observable events.[1]

Interpretation:

- Eq. (1) is known as σ-additivity: the probability that one of two disjoint events occurs is the sum of the probabilities with which either occurs.
- Eq. (1) also implies $P(Y^c) = 1 - P(Y)$.
- A probability measure is essentially a "distribution" on Ω.

[1] If Ω is finite, any event is observable.

Posterior Beliefs via Bayes' Rule

Lemma 2.2 (Bayes' rule)

For any two events $X, Y \subseteq \Omega$ with $P(Y) > 0$,

$$P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)}.$$

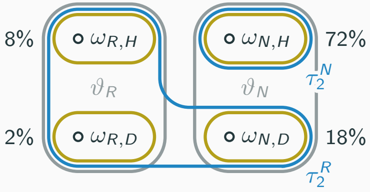
Updating beliefs:

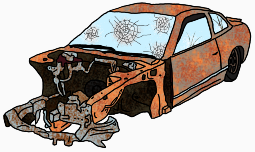
- Player i with prior P_i receives information that the true state is in τ_i.
- If $P_i(\tau_i) > 0$, then for any $\omega \in \Omega$ and any event $Y \subseteq \Omega$, we have

$$P_i(\{\omega\} \mid \tau_i) = \frac{P_i(\{\omega\} \cap \tau_i)}{P_i(\tau_i)}, \qquad P_i(Y \mid \tau_i) = \frac{P_i(Y \cap \tau_i)}{P_i(\tau_i)}.$$

- Player i's posterior beliefs $P_{\tau_i}(Y) := P_i(Y \mid \tau_i)$ is a probability measure concentrated on τ_i: any event outside of τ_i is known to be impossible.

From Prior to Posterior Beliefs





Updating beliefs:

- Suppose our prior P is that 20% of mechanics are dishonest and that, independently, our car needs serious repair with 10% probability.
- After a recommendation to do costly repairs, we update our beliefs to

$$P_{\tau_2^R}(\theta = \vartheta_R) = \frac{P\left(\{\theta = \vartheta_R\} \cap \tau_2^R\right)}{P\left(\tau_2^R\right)} = \frac{10}{28}.$$

- One can also calculate posteriors of individual states of the world

$$P_{\tau_2^R}(\{\omega_{R,H}\}) = \frac{8}{28}, \qquad P_{\tau_2^R}(\{\omega_{R,D}\}) = \frac{2}{28}, \quad P_{\tau_2^R}(\{\omega_{N,D}\}) = \frac{18}{28}.$$

Belief Space

Definition 2.3

A finite belief space $\left(\mathcal{I}, \Omega, (P_i), (\mathcal{T}_i), \theta\right)$ over states of nature Θ consists of:

1. A finite set of players $\mathcal{I} = \{1, \ldots, n\}$.
2. A finite set Ω of possible states of the world ω.
3. A probability measure P_i over Ω for each player i.
4. A partition $\mathcal{T}_i = \left\{\tau_i^1, \ldots, \tau_i^{m_i}\right\}$ of Ω for each player $i \in \mathcal{I}$.
5. A function $\theta : \Omega \to \Theta$, indicating the state of nature $\theta(\omega)$ in each ω.

Remarks:

- P_i is player i's prior belief (before learning his/her information).
- If $P_i = P$ for every player i, we say P is the common prior.
- In settings with common prior, we impose $P(\{\omega\}) > 0$ for every $\omega \in \Omega$.

Common vs. Heterogeneous Prior

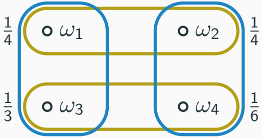
Common prior vs. heterogeneous prior:

- Common prior: it is common knowledge how θ is generated.
  - Example: cards in poker are dealt after uniformly shuffling the deck.
  - Example: everybody knows the population ratio of dishonest mechanics.
- Heterogeneous prior: players disagree on how θ is generated.
  - Example: one player believes the dealer in poker is cheating.
  - Example: some people mistrust mechanics more than others.

In research:

- In applied research, we often make the assumption that players have a common prior as, otherwise, we could prove anything.
- In theoretical research, we use heterogeneous priors for generality.

Simple Example of a Belief Space



Suppose the true state is ω_1:

- Player 1 knows $T_1(\omega_1) = \{\omega_1, \omega_2\}$. Thus, he believes

$$P_{T_1(\omega_1)}(\{\omega_1\}) = P(\{\omega_1\} \mid T_1(\omega_1)) = \frac{P(\{\omega_1\})}{P(\{\omega_1,\omega_2\})} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}} = \frac{1}{2}.$$

- Player 2 knows $T_2(\omega_1) = \{\omega_1, \omega_3\}$. Thus, he believes

$$P_{T_2(\omega_1)}(\{\omega_1\}) = P(\{\omega_1\} \mid T_2(\omega_1)) = \frac{P(\{\omega_1\})}{P(\{\omega_1,\omega_3\})} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{3}} = \frac{3}{7}.$$

- Similarly, $P(\{\omega_2\} \mid T_1(\omega_1)) = \frac{1}{2}$ and $P(\{\omega_3\} \mid T_2(\omega_1)) = \frac{4}{7}$.

Beliefs About the States of Nature

Definition 2.4

Player i's beliefs $\mu_i(\omega) \in \Delta(\Theta)$ over the state of nature in state ω are

$$\mu_i(\omega; \vartheta) := P_{T_i(\omega)}(\theta = \vartheta) = P_{T_i(\omega)}(\{\widetilde{\omega} \in \Omega \mid \theta(\widetilde{\omega}) = \vartheta\}).$$

Remark:

- $\mu_i(\omega)$ denotes the entire distribution over Θ, whereas $\mu_i(\omega; \vartheta)$ is the probability that $\mu_i(\omega)$ assigns to $\vartheta \in \Theta$.
- Because player i's information set changes with ω, so do the beliefs.
- For any ω, ω' in the same information set τ_i, the beliefs coincide.

Visiting the mechanic:

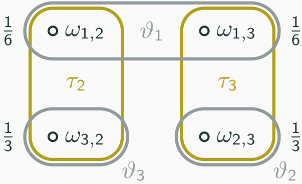
- Our beliefs are $\mu_2(\omega_{N,H}; \vartheta_R) = 0$ and $\mu_2(\omega; \vartheta_R) = \frac{5}{14}$ for $\omega \neq \omega_{N,H}$.

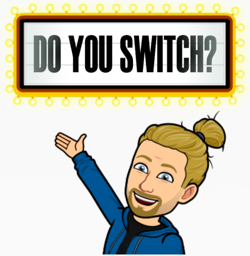
Monty Hall Problem

You are in a game show:

- There are 3 doors. Behind one is a car, behind the others a goat.
- After you pick a door, the game show host opens one of the other doors, behind which is a goat. He then asks you if you would like to switch or stick with your original guess.
- With what probability do you get the car if you switch?

Monty Hall Problem





Modeling the problem:

- Any initial choice has probability $\frac{1}{3}$ of being right. Suppose we choose 1.
- There are 4 states $\omega_{C,M}$: car is behind door C, Monty opens door M.
- If Monty opens door 3, we compute our posterior given τ_3:

$$P(\theta = \vartheta_1 \,|\, \tau_3) = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3}, \qquad P(\theta = \vartheta_2 \,|\, \tau_3) = \frac{\frac{1}{3}}{\frac{1}{6} + \frac{1}{3}} = \frac{2}{3}.$$

- This makes the underlying assumption that Monty opens each door that he is allowed to open with equal probability.

Summary

Belief spaces:

- An extension of knowledge models, in which players form beliefs rationally about any quantities they do not know.

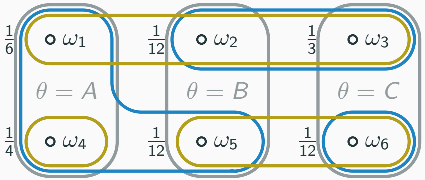
Updating of beliefs:

- Each player i starts with his/her prior beliefs P_i.
- Player i receives information τ_i / learns his/her type τ_i.
- Player i updates his/her beliefs to the posterior beliefs P_{τ_i}.

Beliefs:

- Players' beliefs about some event are computed as the posterior probability of the event, given their information set.
- Beliefs are correlated through the states of the world.

Check Your Understanding



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Short-answer questions:

1. Find the player's posterior beliefs in the above belief space.
2. Find the players' beliefs μ_1 and μ_2 about the state of nature.

True or False:

3. If players have no common prior, then θ is not randomly generated.
4. Suppose player i first receives information in $\mathcal{T}_i$, then information in $\mathcal{T}_*$. Applying Bayes' rule to $\mathcal{T}_i \cap \mathcal{T}_*$ yields the same as applying Bayes' rule twice: first to $\mathcal{T}_i$, then to $\mathcal{T}_*$.

Beliefs and Knowledge

Strong Beliefs

Proposition 2.5

Consider a finite beliefs space with common prior. Then player i knows event Y in information set τ_i if and only if $P_{\tau_i}(Y) = 1$.

Remarks:

- If players have heterogeneous priors or there are infinitely many states of the world, then the equivalence no longer holds.
- Knowledge, of course, still implies beliefs with probability 1.

Proof of necessity:

- Suppose i knows Y in τ_i, which is defined as $\tau_i \subseteq Y$.
- Monotonicity of probability measures implies that $P_{\tau_i}(Y) \geq P_{\tau_i}(\tau_i) = 1$.

Proof of Proposition 2.5

Proof of sufficiency:

- Suppose that $P(Y \mid \tau_i) = 1$ holds.
- In a finite belief space with common prior, $P(\{\omega\}) > 0$ for every ω.
- In particular, $P(\tau_i) > 0$, hence Bayes rule implies

$$1 = P_{\tau_i}(Y) = P(Y \mid \tau_i) = \frac{P(Y \cap \tau_i)}{P(\tau_i)}.$$

- Therefore, $P(Y \cap \tau_i) = P(\tau_i)$, which implies that $Y \cap \tau_i$ can differ from τ_i only by a probability-0 event.
- Since $P(\{\omega\}) > 0$ for every $\omega \in \Omega$, the only probability-0 event is the empty set. This implies that $Y \cap \tau_i = \tau_i$ and hence $\tau_i \subseteq Y$.

Aumann's Agreement Theorem

Theorem 2.6

Let $(\{1,2\},\Omega,\mathcal{T}_i,\theta)$ be a finite belief space with common prior P. The event "Player i ascribes probability p_i to Y" is $\left\{\omega \in \Omega \;\middle|\; P_{T_i(\omega)}(Y) = p_i\right\}$. Suppose there exists a state $\omega \in \Omega$, in which the two events

"Player 1 ascribes probability p_1 to Y"

"Player 2 ascribes probability p_2 to Y"

are both common knowledge. Then $p_1 = p_2$.

If people "agree to disagree":

- Either they have heterogeneous prior beliefs,
- Or they are not fully rational (e.g., incorrectly process new information).

Proof of Theorem 2.6

Proof setup:

- For $i = 1, 2$, denote $Y_i = \left\{\omega \in \Omega \mid P_{T_i(\omega)}(Y) = p_i\right\}$.
- Fix $\omega_* \in \Omega$, in which $Y_1 \cap Y_2$ is common knowledge.

Step 1: Reducing the problem to a self-evident set

- By definition of the common knowledge component $C(\omega_*)$, it satisfies $\omega_* \in C(\omega_*)$, $C(\omega_*) \subseteq Y_1 \cap Y_2$, and $K_i C(\omega_*) = C(\omega_*)$ for $i = 1, 2$.
- By Lemma 1.4, $K_i C(\omega_*) = C(\omega_*)$ implies that $C(\omega_*)$ is the disjoint union of information sets $C(\omega_*) = \bigcup_{\tau_i \in \mathcal{T}_i^*} \tau_i$ for some $\mathcal{T}_i^* \subseteq \mathcal{T}_i$.
- The common prior assumption implies that $P(\{\omega\}) > 0$ for every $\omega \in \Omega$ and hence $P(\tau_i) > 0$ for every $\tau_i \in \mathcal{T}_i^*$ and $P(C(\omega_*)) > 0$.

Proof of Theorem 2.6

Step 2: Use Bayes' rule

- Since any $\tau_i \in \mathcal{T}_i^*$ is a subset of Y_i, the definition of Y_i implies

$$p_i = P_{\tau_i}(Y) = \frac{P(Y \cap \tau_i)}{P(\tau_i)}. \tag{2}$$

- Because $C(\omega_*)$ is the disjoint union of information sets in $\mathcal{T}_i^*$, equation (2) and the law of total probability imply that

$$P(Y \cap C(\omega_*)) = \sum_{\tau_i \in \mathcal{T}_i^*} P(Y \cap \tau_i) = p_i \sum_{\tau_i \in \mathcal{T}_i^*} P(\tau_i) = p_i P(C(\omega_*)). \tag{3}$$

- Equating (3) for $i = 1, 2$ yields

$$p_1 P(C(\omega_*)) = P(Y \cap C(\omega_*)) = p_2 P(C(\omega_*)).$$

- Because $P(C(\omega_*)) > 0$, we deduce that $p_1 = p_2$.

No-Trade Theorem

Theorem 2.7

Suppose two rational, risk-neutral players with common prior P place a bet of size x on the outcome of an event Y, that is, player 1 pays x to player 2 if Y obtains and player 2 pays x to player 1 otherwise. Then

$$P\big(\text{“both players ascribe probability } \tfrac{1}{2} \text{ to } Y\text{”}\big) = 1.$$

Intuition:

- If player 1 accepts the bet, $P_{\tau_1}(Y) \leq \frac{1}{2}$ becomes common knowledge.
- If player 2 accepts the bet, $P_{\tau_2}(Y) \geq \frac{1}{2}$ becomes common knowledge.
- Similarly to the proof of the agreement theorem, it cannot be common knowledge that $P_{\tau_1}(Y) < \frac{1}{2}$ or $P_{\tau_2}(Y) > \frac{1}{2}$ on a set X with $P(X) > 0$.

Implementing the No-Trade Theorem

Communication protocol:

P1: I want to make a bet. Do you want to bet?

P2: Yes, I want to make this bet. Do you still want to bet?

P1: Yes, I want to make this bet. Do you still want to bet?

...

Repeating this procedure indefinitely, one player will eventually back down.

Why do bets happen?

- People have differing prior probabilities.
- People agree to a bet too quickly.

Literature

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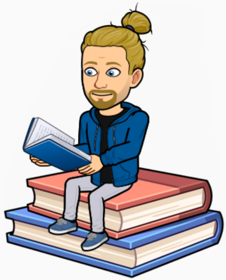


Belief Hierarchies

Avalon: a Social Deduction Game

Rules for a 5-player game:

- Each player i is randomly assigned an alignment $\theta_i \in \{\text{Good, Evil}\}$.
- The two evil players know each other.
- The good players don't know anybody.

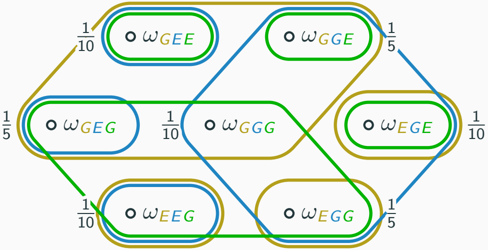
Going on a quest:

- Players choose teams to go on 5 progressively harder quests.
- Each player on a quest secretly chooses to (S)ucceed or (F)ail the quest.
- After the quest, only the aggregate successes and fails are revealed.
- Players update their beliefs via Bayes' rule and go on to the next quest.

Win condition:

- Good players win if 3 quests succeed, evil players win if 3 quests fail.

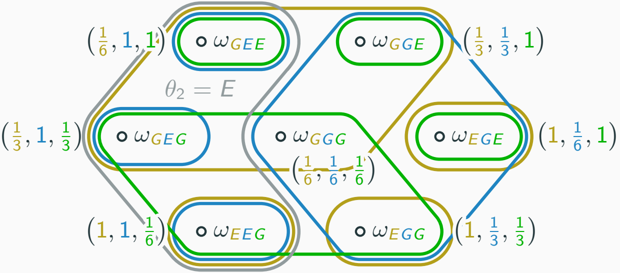
Avalon: a Social Deduction Game



3-player belief subspace:

- The state of nature is $\theta = (\theta_1, \theta_2, \theta_3)$, where θ_i is player i's alignment.
- In this belief subspace, there are 7 states of nature $\omega_{\vartheta_1\vartheta_2\vartheta_3}$.
- Evil players know their partner and, by exclusion, the good players.
- Good players don't know anybody's alignment except their own.
- The common prior P is the uniform distribution of roles after shuffling.

Avalon: Posterior Beliefs



Posterior beliefs:

- Triplet (x, y, z) in state ω indicates $P_{T_i(\omega)}(\{\omega\})$ for players $i = 1, 2, 3$.
- In state ω, each player i assigns posterior 0 to every $\omega' \notin T_i(\omega)$.
- Player 1's beliefs about θ_2 in state ω_{GGE} are

$$\mu_1(\omega_{GGE}; E) = P_{T_1(\omega_{GGE})}(\theta_2 = E) = P_{T_1(\omega_{GGE})}(\{\omega_{GEE}, \omega_{GEG}, \omega_{EEG}\}) = \tfrac{1}{2}.$$

- Player 1's beliefs about θ_2 in state ω_{EEG} are $P_{T_1(\omega_{EEG})}(\theta_2 = E) = 1$.

Strategic Considerations

Good players:

- Need to figure out everybody's alignment.
- Once they figure that out, they win by majority.
- Relevant are (mostly) the first-order beliefs.
- Correlation of beliefs matter for picking a team.

Evil players:

- Failing a quest brings them closer to the goal but reveals information.
- At which point should an evil player optimally fail a quest?
- Specifically, under what conditions can an evil player fail a quest but trick others into believing someone else did?
- We must consider higher-order beliefs.

Beliefs About Other Players Beliefs

Construction of beliefs:

- Player i's beliefs $\mu_i(\omega;\vartheta)$ measure the probability of the event

$$\{\theta = \vartheta\} = \{\widetilde{\omega} \in \Omega \mid \theta(\widetilde{\omega}) = \vartheta\} =: \theta^{-i}(\vartheta)$$

under player i's posterior beliefs in information set $T_i(\omega)$.

Beliefs over other players' beliefs:

- For any $\pi \in \Delta(\Theta)$, player i believes that j believes π with probability

$$\mu_{i,j}(\omega;\pi) := P_{T_i(\omega)}(\{\widetilde{\omega} \in \Omega \mid \mu_j(\widetilde{\omega}) = \pi\}).$$

- For a specific $\vartheta \in \Theta$, i expects j to believe ϑ with probability

$$\mu_{i,j}(\omega;\vartheta) := \mathbb{E}_{T_i(\omega)}[\mu_j(\vartheta)] = \sum_{\widetilde{\omega}\in\Omega} P_{T_i(\omega)}(\{\widetilde{\omega}\})\mu_j(\widetilde{\omega};\vartheta).$$

- Which one is more appropriate?

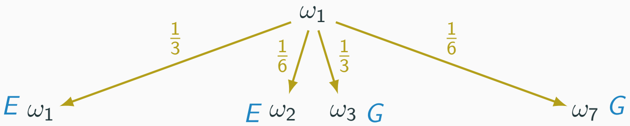
Belief Table: an Alternative Representation

ω	$\theta_2(\omega)$	Posterior $P_{T_1(\omega)}$	Posterior $P_{T_2(\omega)}$	Posterior $P_{T_3(\omega)}$
ω_1	E	$\left[\frac{1}{3}\omega_1, \frac{1}{6}\omega_2, \frac{1}{3}\omega_3, \frac{1}{6}\omega_7\right]$	$\left[1\omega_1\right]$	$\left[\frac{1}{3}\omega_1, \frac{1}{3}\omega_5, \frac{1}{6}\omega_6, \frac{1}{6}\omega_7\right]$
ω_2	E	$\left[\frac{1}{3}\omega_1, \frac{1}{6}\omega_2, \frac{1}{3}\omega_3, \frac{1}{6}\omega_7\right]$	$\left[1\omega_2\right]$	$\left[1\omega_2\right]$
ω_3	G	$\left[\frac{1}{3}\omega_1, \frac{1}{6}\omega_2, \frac{1}{3}\omega_3, \frac{1}{6}\omega_7\right]$	$\left[\frac{1}{3}\omega_3, \frac{1}{6}\omega_4, \frac{1}{3}\omega_5, \frac{1}{6}\omega_7\right]$	$\left[1\omega_3\right]$
ω_4	G	$\left[1\omega_4\right]$	$\left[\frac{1}{3}\omega_3, \frac{1}{6}\omega_4, \frac{1}{3}\omega_5, \frac{1}{6}\omega_7\right]$	$\left[1\omega_4\right]$
ω_5	G	$\left[1\omega_5\right]$	$\left[\frac{1}{3}\omega_3, \frac{1}{6}\omega_4, \frac{1}{3}\omega_5, \frac{1}{6}\omega_7\right]$	$\left[\frac{1}{3}\omega_1, \frac{1}{3}\omega_5, \frac{1}{6}\omega_6, \frac{1}{6}\omega_7\right]$
ω_6	E	$\left[1\omega_6\right]$	$\left[1\omega_6\right]$	$\left[\frac{1}{3}\omega_1, \frac{1}{3}\omega_5, \frac{1}{6}\omega_6, \frac{1}{6}\omega_7\right]$
ω_7	G	$\left[\frac{1}{3}\omega_1, \frac{1}{6}\omega_2, \frac{1}{3}\omega_3, \frac{1}{6}\omega_7\right]$	$\left[\frac{1}{3}\omega_3, \frac{1}{6}\omega_4, \frac{1}{3}\omega_5, \frac{1}{6}\omega_7\right]$	$\left[\frac{1}{3}\omega_1, \frac{1}{3}\omega_5, \frac{1}{6}\omega_6, \frac{1}{6}\omega_7\right]$

How to read the table:

- Player 2 is evil in states ω_1, ω_2, and ω_6, i.e., $\{\theta_2 = E\} = \{\omega_1, \omega_2, \omega_6\}$.
- In state ω_1 (and ω_2, ω_3, and ω_7), Player 1's posterior assigns probability $\frac{1}{3}$ to states ω_1 and ω_3 and probability $\frac{1}{6}$ to states ω_2 and ω_7.
- This notation makes it somewhat easier to read off higher-order beliefs.

Reading Beliefs from the Table

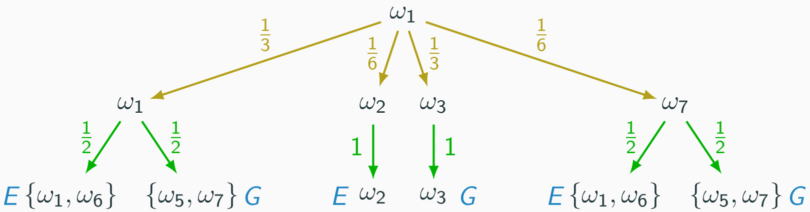


Steps to find $\mu_i(\omega_*; \vartheta)$:

- Evaluate player i's posterior at information set $T_i(\omega_*)$.
- For all states $\omega \in \operatorname{supp} P_{T_i(\omega_*)}$, evaluate $\theta(\omega)$.
- Sum up probabilities of all paths ending in $\{\theta = \vartheta\}$.

As before: Player 1 believes Player 2 is evil with probability $\frac{1}{2}$.

Reading Expected Beliefs from the Table

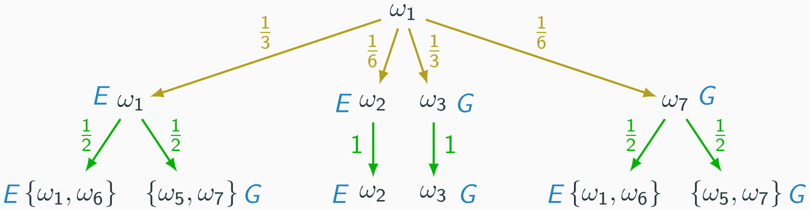


Steps to find $\mu_{i,j}(\omega_*; \vartheta)$ visually:

- Evaluate player i's posterior at information set $T_i(\omega_*)$.
- For all states $\omega' \in \operatorname{supp} P_{T_i(\omega_*)}$, evaluate player j's posterior $P_{T_j(\omega')}$.
- For all states $\omega \in \operatorname{supp} P_{T_j(\omega')}$, evaluate $\theta(\omega)$.
- Sum up probabilities of all paths ending in $\{\theta = \vartheta\}$.

Thus: Player 1 expects that Player 3 believes $\theta_2 = E$ with probability $\frac{5}{12}$.

Expected Beliefs Are Insufficient



Correlation among beliefs:

- With $P_{T_1(\omega)}$-probability $\frac{1}{3}$, Player 1 and Player 3 both believe $\theta_2 = E$.
- With $P_{T_1(\omega)}$-probability $\frac{5}{12}$, Player 1 and Player 3 both believe $\theta_2 = G$.
- With $P_{T_1(\omega)}$-probability $\frac{1}{4}$, Player 1 and Player 3 disagree about θ_2.

In general:

- Need to evaluate $\theta(\omega)$ in every step. Expected beliefs are insufficient.

Belief Hierarchy

Definition 2.8

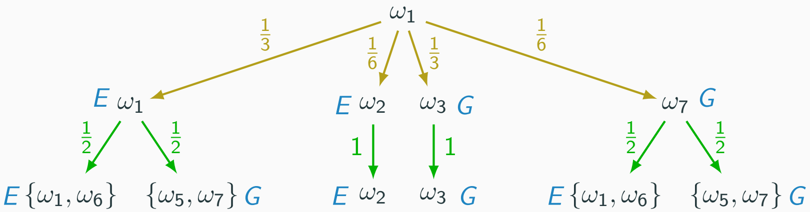
Set $\mathcal{Z}_1 := \Theta$ and define $\mathcal{Z}_k := \mathcal{Z}_{k-1} \times \Delta(\mathcal{Z}_{k-1})^{(n-1)}$ for $k \geq 2$.

1. A belief of order k is an element $\mu^k \in \Delta(\mathcal{Z}_k)$.
2. A belief hierarchy is an element $\beta \in \bigtimes_{k=1}^{\infty} \Delta(\mathcal{Z}_k)$.

Note:

- A belief hierarchy of player i is an element $\beta_i = (\mu_i^1, \mu_i^2, \ldots)$.
- Player i's beliefs of order k correlate his/her beliefs about the state with the opponents' beliefs of orders $1, \ldots, k-1$:
    - 1st-order beliefs are in $\Delta(\Theta)$.
    - 2nd-order beliefs are in $\Delta(\Theta \times \Delta(\Theta)^{n-1})$.
    - 3rd-order beliefs are in $\Delta\big(\Theta \times \Delta(\Theta)^{n-1} \times \Delta(\Theta \times \Delta(\Theta)^{n-1})^{n-1}\big)$.

Avalon: Second-Order Beliefs



Player 1's second-order beliefs about θ_2:

- With probability $\frac{1}{3}$: the state θ_2 is E, Player 2 knows E, and Player 3 believes E and G are equally likely.
- With probability $\frac{1}{6}$: the state θ_2 is E and Players 2 and 3 both know E.
- With probability $\frac{1}{3}$: the state θ_2 is G and Players 2 and 3 both know G.
- With probability $\frac{1}{6}$: the state θ_2 is G, Player 2 knows G, and Player 3 believes E and G are equally likely.

Type Space

Proposition 2.9

A belief space $(\mathcal{I}, \Omega, P, \mathcal{T}, \theta)$ uniquely determines a belief hierarchy $\beta_i(\tau_i)$ in each information set $\tau_i \in \mathcal{T}_i$ for each $i \in \mathcal{I}$. Moreover, β_i is injective.

Proof:

- The procedure on Slide 36 can be extended to arbitrarily high order.
- The generated tree for any $\omega, \omega' \in \tau_i$ coincides.
- If $\tau_i \neq \tilde{\tau}_i$, then first-order beliefs must differ, hence $\beta(\tau_i) \neq \beta_i(\tilde{\tau}_i)$.

Consequence:

- Player i has exactly $|\mathcal{T}_i|$ different possible outlooks on the world.
- Each $\tau_i \in \mathcal{T}_i$ describes one type that player i could be.
- We refer to $\mathcal{T}_i$ as player i's type space.

Types

Types:

- Recall that $T_i(\omega)$ is the information set $\tau_i \in \mathcal{T}_i$ with $\omega \in \tau_i$, that is, it is player i's type when the true state of the world is ω.
- Player i's type $T_i : \Omega \to \mathcal{T}_i$ is thus a $\mathcal{T}_i$-valued random variable.

Interpretation:

- Before they type is drawn (e.g., cards are dealt in poker), nobody knows what i's type will be.
- Player i will learn his/her type when receiving information τ_i since

$$\{T_i = \tau_i\} = \{\omega \in \Omega \mid T_i(\omega) = \tau_i\} = \tau_i$$

  is in player i's knowledge partition by construction.
- Other players will typically have incomplete information about i's type.

Check Your Understanding

True or false:

1. In the Avalon example, the belief table on Slide 31 generates only up to third-order beliefs.
2. At the beginning of a poker round, a players' belief hierarchy is completely specified by his/her hole cards.
3. Using the formalism of knowledge operators, the event Player 1 knows Player 2's alignment is $K_1(\theta_2)$.

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Short-answer questions:

4. In the Avalon example, what probability does Player 1 assign to Player 3 knowing Player 2's alignment in state ω_{GEG}?
5. Three players each wear a blue or a white hat. They can see the others' hats but not their own. How many types does each player have?

Minimal Belief Spaces

Starting with Belief Hierarchies

Construction of belief hierarchies from belief space:

- Players are aware of all the states of the world, everybody's information sets, and everybody's prior probability measure.
- These appear to be rather strong assumptions on players' knowledge.

Inverse construction:

- Players know only (some of) the states of nature and the other players.
- The primitives of such a model are the players' belief hierarchies.
- Can we construct a belief space that is consistent with $\beta = (\beta_1, \ldots, \beta_n)$?

Answer:

- We first construct the minimal belief space that contains a given β.
- Then we construct the universal belief space that contains all β.

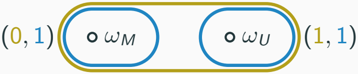
Oedipus: Belief Hierarchies

ω	$\theta(\omega)$	μ_1	μ_2
ω_U	ϑ_U	$[1\omega_U]$	$[1\omega_U]$
ω_M	ϑ_M	$[1\omega_U]$	$[1\omega_M]$

Belief hierarchies:

- There are two states of nature: in state ϑ_U Iocaste is unrelated to Oedipus, whereas as in state ϑ_M Iocaste is Oedipus' mother.
- Oedipus is not even aware of state ϑ_M. Therefore, his entire belief hierarchy is concentrated on ϑ_U: He believes the state us ϑ_U, he believes everybody else believes the state is ϑ_U, etc.
- The Oracle knows Iocaste is Oedipus' mother, but believes that Oedipus believes it is common belief that she is unrelated.

Oedipus: Belief Space



Belief space:

- Oedipus is not aware of ϑ_M. This implies that he has only a single information set / type $\tau_1 = \{\omega_M, \omega_U\}$ with prior $P_1(\omega_M) = 0$.
- Since Oedipus has only one type, his posterior is equal to his prior.
- The Oracle can distinguish the two states, hence $\tau_2^U = \{\omega_U\}$ and $\tau_2^M = \{\omega_M\}$. The oracle's prior is $P_2(\omega_M) = 1$.
- Oracle's posterior is $P_{T_2(\omega)}(\{\omega\}) = 1$ for both states ω.

Belief Operator

Implication:

- Proposition 2.5 indeed fails for heterogeneous priors. Belief with probability 1 does not imply knowledge.
- In fact, if some players are not aware of all states of the world, there is no chance they could know them.
- Can we define a concept analogous to the knowledge operator?

Definition 2.10

For any event $Y \subseteq \Omega$, let $B_i(Y) := \{\omega \in \Omega \mid \mu_i(\omega; Y) = 1\}$ denote the set of states, in which player i believes that event Y obtains.

Note:

- For finite belief spaces with common prior, $B_i = K_i$ by Proposition 2.5.

Belief Axioms

Proposition 2.11

Belief operator B_i satisfies the following properties:

(B1) *Axiom of awareness:* $B_i\Omega = \Omega$,

(B2) *Distribution axiom:* $B_i(X \cap Y) = B_iX \cap B_iY$ *for any events* X, Y,

(B3) *Axiom of introspection:* $B_i\big(B_iY\big) = B_iY$ *for any event* Y,

(B4) *Axiom of negative introspection:* $(B_iY)^c = B_i\big((B_iY)^c\big)$ *for any* Y.

Note:

- As we have seen, the axiom of knowledge cannot hold.
- Belief operator retains all the other properties of a knowledge operator.

Common Belief

Definition 2.12

An event $Y \subseteq \Omega$ is common belief in state ω if for every finite sequence of players $i_1, \dots, i_k$, we have $\omega \in B_{i_1} B_{i_2} \dots B_{i_k} Y$.

Proposition 2.13

An event Y is common belief in ω if and only if there exists $X \subseteq Y$ with:

1. $\mu_i(\omega; X) = 1$ *for every player i,*
2. $\mu_i(\omega'; X) = 1$ *for every player i and every $\omega' \in X$.*

Note:

- Property 2 is similar to the existence of a self-evident set in Lemma 1.9: once beliefs reach X, they are absorbed there.
- The crucial difference is property 1, as it allows $\omega \notin X$.

Minimal Belief Space

Constructing beliefs space from belief hierarchy:

1. Starting with common-belief states, we can construct the smallest belief table containing (Ω, θ, μ) such that players' first order beliefs μ_i for $i \in \mathcal{I}$ generate the desired belief hierarchy β.
2. We can now define the players' information sets as the set of states $\omega \in \Omega$, on which their belief hierarchies coincide:
$$T_i(\omega) := \{\widetilde{\omega} \in \Omega \mid \mu_i(\widetilde{\omega}) = \mu_i(\omega)\}.$$
3. The first-order beliefs thus define a family of posteriors $(P_{\tau_i})_{\tau_i \in \mathcal{T}_i}$ via $P_{\tau_i}(Y) = \mu_i(\omega; Y)$ for any $\omega \in \tau_i$.
4. By Lemma 2.14, there exists a prior probability P_i on Ω that is consistent with the family of posteriors $(P_{\tau_i})_{\tau_i \in \mathcal{T}_i}$.
5. The resulting belief space $(\mathcal{I}, \Omega, P, \mathcal{T}, \theta)$ contains belief hierarchy β.

Any Family of Posteriors Is Induced by a Prior

Lemma 2.14

For any partition $\mathcal{T}_i$ and any family of posteriors $(P_{\tau_i})_{\tau_i \in \mathcal{T}_i}$ with $P_{\tau_i}(\tau_i) = 1$, there exists a probability measure P_i on Ω with $P_{\tau_i}(Y) = P_i(Y \mid \tau_i)$.

Consequence:

- With the procedure on Slide 46 and Lemma 2.14, Belief spaces as in Definition 2.3 are able to capture any belief hierarchies.
- Belief spaces are a "complete" framework for analyzing incomplete information and higher-order beliefs about the uncertainty.[2]
- This is important: we have found the most general framework.

[2] Complete, here, is understood in the mathematical sense that there are no belief hierarchies we cannot capture.

Proof of Lemma 2.14

Step 1: defining a prior P_i

- The family of posteriors $(P_{\tau_i})_{\tau_i \in \mathcal{T}_i}$ determines the beliefs within each information set, but not across information sets.
- We have the freedom to choose the probabilities $c_{\tau_i} > 0$ of each $\tau_i \in \mathcal{T}_i$ under player i's prior P_i such that $\sum_{\tau_i \in \mathcal{T}_i} c_{\tau_i} = 1$ and set

$$P_i(Y) := \sum_{\tau_i \in \mathcal{T}_i} c_{\tau_i} P_{\tau_i}(Y).$$

- Since each P_{τ_i} is a probability measure, P_i is σ-additive and

$$P_i(\Omega) = \sum_{\tau_i \in \mathcal{T}_i} c_{\tau_i} P_{\tau_i}(\Omega) = \sum_{\tau_i \in \mathcal{T}_i} c_{\tau_i} = 1.$$

Proof of Lemma 2.14

Step 2: verifying posteriors using Bayes' rule

- It follows from Bayes' rule implies and the definition of P_i that

$$P_i(Y \mid \tau_i) = \frac{P_i(Y \cap \tau_i)}{P_i(\tau_i)} = \frac{\sum_{\tau_i' \in \mathcal{T}_i} c_{\tau_i'} P_{\tau_i'}(Y \cap \tau_i)}{\sum_{\tau_i' \in \mathcal{T}_i} c_{\tau_i'} P_{\tau_i'}(\tau_i)}. \qquad (4)$$

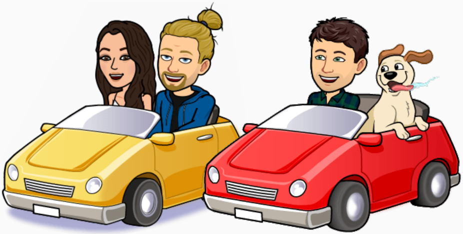
- Since $P_{\tau_i'}(\tau_i') = 1$ for any τ_i', it follows that for any $\tau_i \neq \tau_i'$,

$$0 \leq P_{\tau_i'}(Y \cap \tau_i) \leq P_{\tau_i'}(\tau_i) = 0.$$

- Therefore, (4) reduces to

$$P_i(Y \mid \tau_i) = \frac{c_{\tau_i} P_{\tau_i}(Y \cap \tau_i)}{c_{\tau_i}} = P_{\tau_i}(Y \cap \tau_i) + \underbrace{P_{\tau_i}(Y \cap \tau_i^c)}_{=0} = P_{\tau_i}(Y).$$

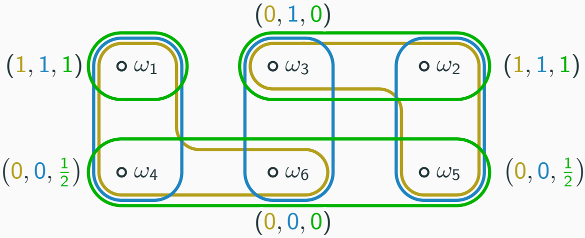
Turning Left or Right?



Andrew, Ben, and Flo are on a road trip. Arriving at a T-intersection, Andrew believes that they should turn right and that it is common belief among the three to turn right. Ben believes that they should turn right, that the others both believe they should turn left and that it is common belief to turn left. Flo believes that they should turn left or right with equal likelihood, that the others know the correct way to turn, and that the others believe it is common belief to turn that way. Find the minimal belief space that contains this belief hierarchy.

Turning Left or Right?

ω	$\theta(\omega)$	$P_{T_1(\omega)}$	$P_{T_2(\omega)}$	$P_{T_3(\omega)}$
ω_1	R	$[1(\omega_1)]$	$[1(\omega_1)]$	$[1(\omega_1)]$
ω_2	L	$[1(\omega_2)]$	$[1(\omega_2)]$	$[1(\omega_2)]$
ω_3	R	$[1(\omega_2)]$	$[1(\omega_3)]$	$[1(\omega_2)]$
ω_4	R	$[1(\omega_1)]$	$[1(\omega_1)]$	$\left[\frac{1}{2}(\omega_4), \frac{1}{2}(\omega_5)\right]$
ω_5	L	$[1(\omega_2)]$	$[1(\omega_2)]$	$\left[\frac{1}{2}(\omega_4), \frac{1}{2}(\omega_5)\right]$
ω_6	L/R	$[1(\omega_1)]$	$[1(\omega_3)]$	$\left[\frac{1}{2}(\omega_4), \frac{1}{2}(\omega_5)\right]$



Summary

Finding the minimal belief space:

1. Build each player's belief hierarchy "from the bottom up":
    - Start with the common belief states.
    - Progress one belief order at a time.
2. If after step 1. no state induces all belief hierarchies at the same time, include one last state to correlate belief hierarchies.
3. Any information set by player i contains all states with identical posterior beliefs by Proposition 2.9.
4. Find prior beliefs as in the proof of Lemma 2.14.

In an experiment:

- It may be easier to elicit belief hierarchies than belief spaces.
- We can carry out our analysis in the minimal belief space.

Universal Belief Space

Universal Belief Space

Universal belief space:

- We do not need to do this for each specific belief hierarchy.
- The universal belief space contains all possible belief hierarchies.

Recall:

- In a finite belief space, we called τ_i player i's type because there is a one-to-one correspondence to player i's belief hierarchies.
- Now we start with belief hierarchies and simply call those types.

Types:

- Let $\mathcal{B}_0 := \bigtimes_{k=1}^{\infty} \Delta(\mathcal{Z}_k)$ denote the space of all types/belief hierarchies.
- Do we need to consider belief hierarchies over types?

Coherent Belief Hierarchies

Definition 2.15

A belief hierarchy $\beta_i = (\mu_i^1, \mu_i^2, \ldots)$ is coherent if

1. the marginal distribution of μ_i^2 on $\Delta(\Theta)$ is μ^1,
2. the marginal distribution of μ_i^k on $\Delta(\mathcal{Z}_{k-2})$ coincides with marginal distribution of μ_i^{k-1} onto $\Delta(\mathcal{Z}_{k-2})$.

Let $\mathcal{B}_1$ denote the space of all coherent types/belief hierarchies.

Interpretation:

- Higher-order beliefs do not contradict lower-order beliefs.
- Ensures that questions like “what probability does player i ascribe to event Y” has an unequivocal answer.

Coherent Belief Hierarchies

Proposition 2.16

There exists a homeomorphism $\varphi : \mathcal{B}_1 \to \Delta(\Theta \times (\mathcal{B}_0)^{n-1})$.[3]

Illustration:

- A belief hierarchy is an element of

$$\beta \in \overbrace{\Delta(\Theta)}^{\Delta(\mathcal{Z}_1)} \times \overbrace{\Delta(\Theta \times \Delta(\mathcal{Z}_1)^{n-1})}^{\Delta(\mathcal{Z}_2)} \times \overbrace{\Delta(\Theta \times \Delta(\mathcal{Z}_1)^{n-1} \times \Delta(\mathcal{Z}_2)^{n-1})}^{\Delta(\mathcal{Z}_3)} \times \dots$$

$$\Delta\big(\Theta \times (\Delta(\mathcal{Z}_1)^{n-1} \times \Delta(\mathcal{Z}_2)^{n-1} \times \dots)\big) = \Delta(\Theta \times (\mathcal{B}_0)^{n-1})$$

- Coherency implies that beliefs over Θ, $\Delta(\mathcal{Z}_1)^{n-1}$, etc. are well defined.

[3] A homeomorphism is a continuous bijection with a continuous inverse: $\mathcal{B}_1$ and $\Delta(\Theta \times (\mathcal{B}_0)^{n-1})$ are identical in a topological sense.

Common Belief of Coherency

Implication of Proposition 2.16:

- A coherent type has well-defined beliefs over opponents' types.
- But: it does not determine i's beliefs over j's beliefs over i's type if believes it is possible that j believes i's type is not coherent.

Impose common belief of coherency:

- Define $\mathcal{B}_2 := \left\{\beta \in \mathcal{B}_1 \;\middle|\; \varphi(\beta)(\Theta \times (\mathcal{B}_1)^{n-1}) = 1\right\}$ denote the set
  - of all coherent types $\beta \in \mathcal{B}_1$ such that
  - they believe with probability 1 that opponents have coherent types.
- Inductively, let $\mathcal{B}_k := \left\{\beta \in \mathcal{B}_1 \;\middle|\; \varphi(\beta)(\Theta \times (\mathcal{B}_{k-1})^{n-1}) = 1\right\}$ and set

$$\mathcal{T} := \bigcap_{k=1}^{\infty} \mathcal{B}_k.$$

- $\mathcal{T}$ is the set of coherent types that satisfy common belief in coherency.

Universal Type Space

Theorem 2.17

1. *The universal type space* $\mathcal{T}$ *is homeomorphic to* $\Delta(\Theta \times \mathcal{T}^{n-1})$.
2. *Any belief space is a subspace of the universal belief space* $\Theta \times \mathcal{T}^n$.

Consequences:

- A type in $\mathcal{T}$ determines all higher-order beliefs over opponents' types.
- Can set $\Omega = \Theta \times \mathcal{T}^n$ without loss of generality.
- In particular, $\omega = (\theta(\omega), T_1(\omega), \ldots, T_n(\omega))$.

Summary

Belief hierarchies and types:

- Because a belief hierarchy completely determines a player's outlook on the game, we can characterize players through their belief hierarchies.
- A player's information set uniquely determines his/her belief hierarchy.
- Consequently, we refer to players' information sets as the players' types.

Finite/minimal belief spaces:

- They are tractable, but do not allow arbitrary beliefs hierarchies.
- If the belief hierarchy is fixed, we work with the minimal belief space.

Universal belief space / type space:

- The universal belief space allows all possible belief hierarchies over Θ.
- Most applied research will use the universal belief space.

Check Your Understanding

True or false:

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1. In a finite belief space with a common prior, common belief is identical to common knowledge.
2. In a belief space with n states of the world and a common prior, player i's 2^{nd} order beliefs put positive weight on at most n 1^{st}-order beliefs held by player j.
3. Every family of posteriors defines a unique belief hierarchy, but not every belief hierarchy defines a unique family of posteriors.
4. If one were to implement a software that predicts the likelihood of enemy roles in Avalon, one should use the universal type space.

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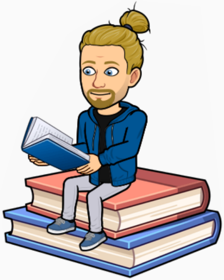


Infinite Belief Spaces

Banach-Tarski Paradox

Theorem 2.18

There is no measure P on a continuum Ω with the following 3 properties:

1. *P is invariant under rotations and translations.*
2. *$P(X \cup Y) = P(X) + P(Y)$ for X and Y disjoint.*
3. *$P(Y)$ is well-defined for every $Y \subseteq \Omega$.*

Consequences for belief spaces:

- Giving up 1. would imply that the parametrization affects the result.
- We must restrict attention to a subset of events, called observable.
- The set of observable events must be well-behaved: a σ-algebra.

Sigma-Algebras

Definition 2.19

A σ-algebra $\mathcal{F}$ is a collection of subsets $Y \subseteq \Omega$ such that:

1. Both $\emptyset$ and Ω are in $\mathcal{F}$.
2. If Y is in $\mathcal{F}$, then Y^c is in $\mathcal{F}$.
3. If $(Y_n)_{n \in \mathbb{N}}$ is in $\mathcal{F}$, then $\bigcup_{n \in \mathbb{N}}$ is in $\mathcal{F}$.

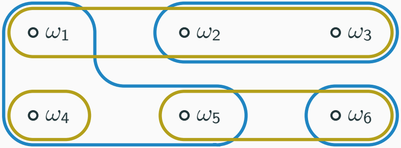
Why do we choose a σ-algebra:

1. This is consistency with the awareness axiom.
2. We want to be able to deduce that $P(Y^c) = 1 - P(Y)$.
3. This is the minimal condition for σ-additivity (1).

Interpretation:

- σ-algebras provide the minimal consistency for probability measures.

Examples of Sigma-Algebras



Information partitions:

- The σ-algebra generated by $\mathcal{T}_i$ is the smallest σ-algebra that contains every $\tau_i \in \mathcal{T}_i$. It consists of $\emptyset$ and all possible unions of $\tau_i \in \mathcal{T}_i$.
- The σ-algebra generated by $\mathcal{T}_1$ is

$$\{\emptyset, \tau_1^1, \tau_1^2, \tau_1^3, \tau_1^1 \cup \tau_1^2, \tau_1^1 \cup \tau_1^3, \tau_1^2 \cup \tau_1^3, \Omega\}.$$

Other examples:

- If Ω is countable, then the power set 2^Ω is a σ-algebra.
- The Borel σ-algebra in $\mathbb{R}^d$ is generated by all open (or closed) sets.

Rolling Dice

$\Omega = [0, 1]^2$

$P =$ area

1	2	3
4	5	6

Rolling dice:

- The roll of a 6-sided die is a random variable $X : \Omega \to \{1, \ldots, 6\}$.
- Since not every subset of Ω is measurable, we must be careful in how we choose the map $X : \Omega \to \{1, \ldots, 6\}$.
- If $\{\omega \in \Omega \mid X(\omega) = 6\}$ is not measurable, we cannot quantify $P(X = 6)$.

Measurability:

- A function $X : \Omega \to \mathcal{X}$ is measurable if for any measurable set $B \subseteq \mathcal{X}$, its pre-image $X^{-1}(B) := \{\omega \in \Omega \mid X(\omega) \in B\}$ lies in $\mathcal{F}$.

Probability Space

Definition 2.20

A probability space $(\Omega, \mathcal{F}, P)$ consists of:

1. A set Ω of states of the world.
2. A σ-algebra $\mathcal{F}$ of observable events $Y \subseteq \Omega$.
3. A probability measure P on $\mathcal{F}$.

Definition 2.21

A measurable space $(\mathcal{X}, \Xi)$ is a set $\mathcal{X}$ and a σ-algebra Ξ on $\mathcal{X}$.

Definition 2.22

An $\mathcal{X}$-valued random variable is a measurable function $X : \Omega \to \mathcal{X}$.

Representing Information

Type space:

- If player i's type space $\mathcal{T}_i$ is uncountable, we embed it with a σ-algebra $\mathfrak{T}_i$ of measurable subsets of types: $(\mathcal{T}_i, \mathfrak{T}_i)$ is a measurable space.
- Player i's type is a random variable $T_i : \Omega \to \mathcal{T}_i$.

Information partition:

- The family $\{T_i = \tau_i\}_{\tau_i \in \mathcal{T}_i}$ partitions Ω.
- Problem: not every combination of types is measurable.
- Player i's information is the σ-algebra $\mathcal{F}_i$ generated by T_i, which contains the sets $\{\omega \in \Omega \mid T_i(\omega) \in B\}$ for every $B \in \mathfrak{T}_i$.

Note:

- If $\mathfrak{T}_i$ contains $\{\tau_i\}$ (which it typically does), then $\mathcal{F}_i$ contains $\{T_i = \tau_i\}$.
- $\mathcal{F}_i$ is a sub-σ-algebra of $\mathcal{F}$ since T_i is measurable.

Decision Rules as Measurable Maps

Lemma 2.23

The following two are equivalent:

1. *A decision rule is a map $\sigma_i : \mathcal{T}_i \to \mathcal{A}_i$.*
2. *It's outcome $A_i = \sigma_i \circ T_i$ is an $\mathcal{F}_i$-measurable map $A_i : \Omega \to \mathcal{A}_i$.*

Illustration:

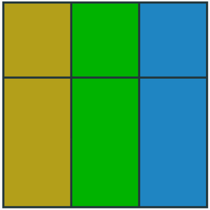
- Suppose $\mathcal{T}_i$ is finite. Measurability implies that

$$A_i^{-1}(a_i) := \{\omega \in \Omega \mid \sigma(T_i(\omega)) = a_i\}$$

  is either empty (in which case a_i is not played) or a union of $\tau_i \in \mathcal{T}_i$.
- Player i cannot play different actions on the same information set.
- General $\mathcal{T}_i$: $A_i^{-1}(a_i)$ is a measurable union of $\{T_i = \tau_i\}$.

Bayes' Rule Visualized

$$\Omega = [0,1]^2$$
$$\mathcal{F} = \text{Borel}$$
$$P = \text{area}$$



Predicting the color of a hat:

- The color of the hat is a random variable $\theta : \Omega \to \{Y, G, B\}$.
- Player i's type is a random variable $T_i : \Omega \to \mathcal{T}_i$.
- Information set $\tau_i \in \mathcal{T}_i$ is the set, on which $T_i = \tau_i$ is realized.
- The beliefs of type $\tau_i \in \mathcal{T}_i$ are given by

$$P_{\tau_i}(Y) = P_i(Y \mid T_i = \tau_i) = \frac{P_i(Y \cap \{T_i = \tau_i\})}{P_i(T_i = \tau_i)}.$$

- Suppose Player i's height $1.8141592653589793\ldots$m is his type.

Regular Conditional Probability

Definition 2.24

Let $(\Omega, \mathcal{F}, P_i)$ be a probability space, let $(\mathcal{T}_i, \mathfrak{T}_i)$ be a measurable space and $T_i : \Omega \to \mathcal{T}_i$ be a random variables. A map $\pi_i : \mathcal{T}_i \times \mathcal{F} \to [0,1]$ is a regular conditional probability if it satisfies:

1. $\pi_i(\tau_i, \cdot)$ is a probability measure on $\mathcal{F}$ for each $\tau_i \in \mathcal{T}_i$,
2. $\pi_i(\cdot, Y)$ is a measurable function for every $Y \in \mathcal{F}$,
3. For every $Y \in \mathcal{F}$ and every $B \in \mathfrak{T}_i$,

$$P_i(Y \cap \{\omega \in \Omega \mid T_i(\omega) \in B\}) = \int_B \pi_i(\tau_i, Y) \underbrace{\mathrm{d}(P_i \circ T_i^{-1})(\tau_i)}_{\text{“}f(\tau_i)\,\mathrm{d}\tau_i\text{”}} \quad (5)$$

Interpretation of Regular Conditional Probability

Interpretation:

- $P_{\tau_i}(Y) := \pi_i(\tau_i, Y)$ is type τ_i's posterior.
- Updating of beliefs $P_i(Y) \mapsto P_{\tau_i}(Y)$ works even if $P_i(T_i = \tau_i) = 0$.
- It follows from (5) for $B = \{\tau_i\}$ that P_{τ_i} is supported on $\{T_i = \tau_i\}$.

Why using regular conditional probabilities?

- Points 1. and 3. is simply a continuous statement of Bayes' rule.
- Point 2. guarantees that the event "player i assigns probability p to Y"

$$\left\{\omega \in \Omega \;\middle|\; P_{T_i(\omega)}(Y) = p\right\}$$

is observable. This allows higher-order considerations by others.

Infinite Belief Spaces

Definition 2.25

A belief space $(\mathcal{I}, \Omega, \mathcal{F}, (\mathcal{T}_i, \mathfrak{T}_i), (P_i), \theta)$ over a measurable space (Θ, Ξ) of states of nature consists of:

1. A finite set of players $\mathcal{I} = \{1, \ldots, n\}$.
2. A measurable space $(\Omega, \mathcal{F})$ of states of the world.
3. A measurable type space $(\mathcal{T}_i, \mathfrak{T}_i)$ for each player i and a random variable $T_i : \Omega \to \mathcal{T}_i$ indicating player i's type.
4. A prior probability measure P_i on $\mathcal{F}$ and a regular conditional probability $\pi_i : \mathcal{T}_i \times \mathcal{F} \to [0, 1]$ of posterior beliefs for each player i.
5. A random variable $\theta : \Omega \to \Theta$, indicating the state of nature.

Note: This definition reduces to Definition 2.3 if Θ, Ω, and $\mathcal{T}_i$ are all finite.

Knowledge in a Continuum

Knowledge:

- Recall the definition of player i's knowledge operator

$$K_i(Y) := \{\omega \in \Omega \mid T_i(\omega) \subseteq Y\} = \bigcup_{\tau_i \in \mathcal{T}_i : \{T_i = \tau_i\} \subseteq Y} \{T_i = \tau_i\}.$$

- Problem: the union to the right-hand side might not be measurable.
- Consequence: $K_i(Y)$ might not be an observable event, hence we cannot describe knowledge hierarchies.
- The Banach-Tarski paradox has a flavor similar to Heisenberg's uncertainty principle in physics.

Key takeaways:

- Concept of knowledge is too strong if there are a continuum of states.
- For belief spaces, everything works also with a continuum of types.

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