Learning Notes in Quantitative Methods

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Table 1: Contents between lectures and notes

Chu, 2021	Topic	Notes
Lecture 1	Number System	Section 1.1
	Set, Unions, Intersections	Section 1.2
	Negation	Section 1.3.2
	Necessary and Sufficient Condition	Section 1.3.3
	Basic Notations of Set Theory	Section 1.4
	Functions, Sequences, and Subsequences	Section 1.5
Homework 1	Question 1	Example 1.5.10
	Question 2	Theorem 1.5.12
	Question 3	Example 1.5.2
	Question 4	Example 1.5.9
	Question 5	Example 1.5.9
Lecture 2	Metric Spaces and Euclidean Space	Section 2.1
	Upper Bounds, Maximum Elements, Least Upper Bounds	Section 1.7
	Sequences and Limits	Section 3.1
Homework 2	Question 1	Example 3.2.2
	Question 2	Theorem 3.1.8
	Question 3	Theorem 3.1.8
	Question 4	Example **??**
	Question 5	Example **??**
	Question 6	Theorem **??**
	Question 7	Theorem **??**
	Question 8	Example 1.7.8
	Question 9	Example 3.1.4
Lecture 3	Lim Sup and Lim Inf	Section
Homework 3	Question 1	Theorem **??**
	Question 2	Theorem **??**
	Question 3	Example **??**

Contents

Chapter 1

Set Theory

1.1 The Real Number System

Definition 1.1.1 (real number system)**.**

1. The set of **natural numbers** is denoted by

$$\mathbb{N} = \{1, 2, 3, \ldots\}. \tag{1.1}$$

2. The set of **integers** is denoted by

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}. \tag{1.2}$$

3. The set of **rational numbers** is denoted by

$$\mathbb{Q} = \left\{ \frac{q}{p} \middle| p, q \in \mathbb{Z}, q \neq 0 \right\}. \tag{1.3}$$

4. The set of **real numbers** is denoted by $\mathbb{R}$.

[from Sundaram et al., 1996, p.2; Apostol, 1974, p.3,6-7; Chu, 2021, Lecture 1 p.1]

1. The real numbers are represented geometrically as points on a line, which is called the **real line** or the **real axis**.

2. Real numbers that are not rational are called **irrational numbers**.

Definition 1.1.2 (interval)**.**

1. By the **open interval** (a, b) we mean the set of all real numbers x such that $a < x < b$.
2. By the **closed interval** $[a, b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.
3. Occasionally we shall encounter **half-open intervals** $[a, b)$ and $(a, b]$; the first consists of x such that $a \leq x < b$, the second of all x such that $a < x \leq b$.

[from Rudin et al., 1964, p.31; Apostol, 1974, p.4; Sundaram et al., 1996, p.22]

1. The real line $\mathbb{R}$ is referred to as the open interval $(-\infty, \infty)$.
2. A single point is considered as a **degenerate** closed interval.

1.2 Sets, Unions, and Intersections

Definition 1.2.1 (set)**.** Let denote **sets** by capital letters such as A, B, X, and Y, and **elements** of these sets by lowercase letters such as a, b, x, and y.

1. An object a **belongs to** a set A is denoted by $a \in A$. If a is **not an element** of A, we write $a \notin A$.
2. If every element of a set B is also an element of a set A, we shall say that B is a **subset** of A and write $B \subset A$.
3. Two sets A and B are said to be **equal**, written $A = B$. If every element of A is also an element of B, and vice versa. That is, $A = B$ if we have both $A \subset B$ and $B \subset A$.
4. The **union** of two sets A and B, denoted $A \cup B = \{x | x \in A \text{ or } x \in B\}$, is the set which consists of all elements which are either in A or in B.
5. The **intersection** of two sets A and B, denoted $A \cap B = \{x | x \in A \text{ and } x \in B\}$, is the set which consists of all elements which belong to both A and B.
6. If $A \subset X$, the **complement** of A in X, denoted A^C, is defined as $A^C = \{x \in X | x \notin A\}$.

[from Sundaram et al., 1996, p.315-316; Rudin et al., 1964, p.3; Chu, 2020, Lecture 1 p.1; Chu, 2021, Lecture 1 p.1-3]

1.3 Logic

1.3.1 Propositions: Contrapositives and Converses

Definition 1.3.1 (imply). Given two propositions P and Q, the statement "If P, then Q" is interpreted as the statement that if the proposition P is true, then the statement Q is also true. We denote this by $P \Rightarrow Q$.

[from Sundaram et al., 1996, p.316-317]

1. Its **contrapositive** is the statement that "if Q is not true, then P is not true." We denote this by $\sim Q \Rightarrow \sim P$.
2. **A statement and its contrapositive are logically equivalent.** That is, if the statement is true, then the contrapositive is also true, while if the statement is false, so is the contrapositive.

Definition 1.3.2 (converse). The **converse** of the statement $P \Rightarrow Q$ is the statement that $Q \Rightarrow P$. That is, the statement that "if Q, then P."

[from Sundaram et al., 1996, p.318]

1. **There is no logical relationship between a statement and its converse.**
2. If a statement and its converse both hold, we express this by saying that "P **if and only if** Q," and denote this by $P \Leftrightarrow Q$.

1.3.2 Quantifiers and Negation

Definition 1.3.3 (logical quantifiers).

1. The **universal** or "**for all**" quantifier is used to denote that a property holds for every element a in some set A. We write $\forall$.
2. The **existential** or "**there exists**" quantifier denotes that the property holds for at least one element a in the set A. We write $\exists$.

[from Sundaram et al., 1996, p.318]

Definition 1.3.4 (negation). The **negation** of a proposition P is its denial $\sim P$.

[from Sundaram et al., 1996, p.318-319; Chu, 2020, Lecture 1 p.2; Chu, 2021, Lecture 1 p.3]

1. If the proposition P involves a universal quantifier, then its negation involves an existential quantifier: to deny the truth of a universal statement requires us to find just one case where the statement fails.
2. The negation of an existential quantifier involves a universal quantifier: to deny that there is at least one case where the proposition holds requires us to show that the proposition fails in every case.

Example 1.3.5. Negate the following statements.

1. A or B.
2. A and B.
3. If A, then B.
4. For all x, $A(x)$.
5. There exists x such that $A(x)$.
6. For every $x > 0$, there exists $y > 0$ such that $y^2 = x$.
7. For all $x \in S$, there is an $r > 0$ such that $B(x, r) \subset S$.

[from Chu, 2020, Lecture 1 p.2; Chu, 2021, Lecture 1 p.3-4]

Solution

1. Not A and not B.
2. Not A or not B.
3. Suppose A is true, but B is not true.
4. There exists x such that not $A(x)$.
5. For all x, not $A(x)$.
6. There exists an $x > 0$, $y^2 \neq x$ for all $y > 0$.

7. (a) There exists an $x \in S$, $B(x,r) \not\subset S$ for all $r > 0$.

   (b) There exists an $x \in S$, for all $r > 0$, there exists an $y \in B(x,r)$, but $y \notin S$.

□

Theorem 1.3.6 (Demorgan's laws)**.**

1. $\left(\bigcup_{i \in I} A_i\right)^C = \bigcap_{i \in I} A_i^C$
2. $\left(\bigcap_{i \in I} A_i\right)^C = \bigcup_{i \in I} A_i^C$

[from Wade, 2014, p.33; Rudin et al., 1964, p.33; Sundaram et al., 1996, p.25; Chu, 2020, Lecture 1 p.2; Chu, 2021, Lecture 1 p.3-4]

Proof.

1.

$$\begin{aligned} \text{LHS} &\Leftrightarrow \{x | \exists i, x \in A_i\}^C && (1.4)\\ &\Leftrightarrow \{x | \forall i, x \notin A_i\} && (1.5)\\ &\Leftrightarrow \{x | \forall i, x \in A_i^C\} && (1.6)\\ &\Leftrightarrow \text{RHS} && (1.7)\end{aligned}$$

2.

$$\begin{aligned} \text{LHS} &\Leftrightarrow \{x | \forall i, x \in A_i\}^C && (1.8)\\ &\Leftrightarrow \{x | \exists i, x \notin A_i\} && (1.9)\\ &\Leftrightarrow \{x | \exists i, x \in A_i^C\} && (1.10)\\ &\Leftrightarrow \text{RHS} && (1.11)\end{aligned}$$

□

1.3.3 Necessary and Sufficient Conditions

Definition 1.3.7 (necessary condition)**.** Suppose an implication of the form $P \Rightarrow Q$ is valid. Then, Q is said to be a **necessary condition** for P.

[from Sundaram et al., 1996, p.320; Chu, 2021, Lecture 1 p.4]

Definition 1.3.8 (sufficient condition)**.** Suppose an implication of the form $P \Rightarrow Q$ is valid. Then, P is said to be a **sufficient condition** for Q.

[from Sundaram et al., 1996, p.321; Chu, 2021, Lecture 1 p.4]

1.4 Ordered Pairs and Relations

Definition 1.4.1 (order pairs)**.** If a set of two elements a and b is **ordered**, we enclose the elements in parentheses (a, b). Then, a is called the first element, and b is called second element.

[from Apostol, 1974, p.33; Chu, 2021, Lecture 1 p.5]

Definition 1.4.2 (cartesian product)**.** Given two sets A and B, the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the **cartesian product** of A and B, and is denoted by $A \times B$.

[from Apostol, 1974, p.33; Chu, 2021, Lecture 1 p.5]

Definition 1.4.3 (relation)**.** Any set of ordered pairs is called a **relation**.

[from Apostol, 1974, p.34; Chu, 2021, Lecture 1 p.5]

1. If S is a relation, the set of all elements x that occur as first members of pairs (x, y) in S is called **domain** of S, denoted by $\mathfrak{D}(S)$.
2. The set of second members y is called the **range** of S, denoted by $\mathfrak{R}(S)$.

1.4.1 Binary Relation

Definition 1.4.4 (reflexive)**.** A binary relation R on the set X is **reflexive** if for all $x \in X$, xRx.

[from Osborne and Rubinstein, 2020, p.5]

Definition 1.4.5 (complete)**.** A binary relation R on the set X is **complete** if for all $x, y \in X$, xRy or yRx.

[from Osborne and Rubinstein, 2020, p.5]

Definition 1.4.6 (irreflexive). A binary relation R on the set X is **irreflexive** if for all $x, y \in X$, not xRx.

[from Osborne and Rubinstein, 2020, p.5]

Definition 1.4.7 (symmetric). A binary relation R on the set X is **symmetric** if for all $x, y \in X$, xRy implies yRx.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.8 (asymmetric). A binary relation R on the set X is **asymmetric** if for all $x, y \in X$, xRy implies not yRx.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.9 (antisymmetric). A binary relation R on the set X is **antisymmetric** if for all $x, y \in X$, xRy and yRx imply $x = y$.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.10 (transitive). A binary relation R on the set X is **transitive** if for all $x, y, z \in X$, xRy and yRz imply xRz.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.11 (quasi-transitive). A binary relation R on the set X is **transitive** if for all $x, y, z \in X$, xPy and yPz imply xPz.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.12 (acyclic). A binary relation R on the set X is **acyclic** if for all $x_1, x_2, \ldots, x_n \in X$, $x_1Px_2, x_2Px_3, \ldots, x_{n-1}Px_n$ imply x_1Rx_n.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.13 (negatively transitive). A binary relation R on the set X is **negatively transitive** if for all $x, y, z \in X$, not xRy and not yRz imply not xRz.

[from Osborne and Rubinstein, 2020, p.6]

1.5 Functions

Definition 1.5.1 (function)**.** A **function** F is a set of ordered pairs (x, y), no two of which have the same first member. That is, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.

[from Apostol, 1974, p.34-35; Rudin et al., 1964, p.24; Sundaram et al., 1996, p.41; Chu, 2021, Lecture 1 p.6]

1. The definition of function requires that for every x in the domain of F, there is exactly one y such that $(x, y) \in F$.
2. It is customary to call y the **value** of F at x and to write $y = F(x)$ instead of $(x, y) \in F$ to indicate that the pair (x, y) is in the set F.
3. When the domain $\mathfrak{D}(F)$ is a subset of $\mathbb{R}$, then F is called a **function of one real variable**.
4. If $\mathfrak{D}(F)$ is a subset of a cartesian product $A \times B$, then F is called a **function of two variable**.
5. If S is a subset of $\mathfrak{D}(F)$, we say that F is defined on S.
6. The set of $F(x)$ such that $x \in S$ is called the **image** of S under F, and is denoted by $F(S)$.
7. If T is any set which contain $F(S)$, then F is called a **mapping** from S into T. It is denoted by $F : S \to T$.
8. If $F(S) = T$, the mapping is said to be **onto** T.
9. A mapping of S into itself is called a **transformation**.

Example 1.5.2. Find the domain and range of following functions.

1. For all x in $[-3, 3)$,

$$F(x) = (x + 1)^2. \tag{1.12}$$

2. For all x in $\mathbb{R}$,

$$F(x) = 2^x. \tag{1.13}$$

[from Nagaoka and Miyaoka, 2007, p.11; Chu, 2021, Lecture 1 Homework Q3]

Solution

1.

$$\mathfrak{D}(F) = [-3, 3) \tag{1.14}$$

$$\mathfrak{R}(F) = [0, 16) \tag{1.15}$$

2.

$$\mathfrak{D}(F) = \mathbb{R} \tag{1.16}$$

$$\mathfrak{R}(F) = \mathbb{R}^{+} \tag{1.17}$$

Theorem 1.5.3. Two function F and G are **equal** if and only if

1. F and G have the same domain. That is,

$$\mathfrak{D}(F) = \mathfrak{D}(G). \tag{1.18}$$

2. for every x in $\mathfrak{D}(F)$,

$$F(x) = G(x). \tag{1.19}$$

[from Apostol, 1974, p.35]

Definition 1.5.4 (one-to-one)**.** Let F be a function defined on S. We say F is **one-to-one** on S if and only for every x and y in S, $F(x) = F(y)$ implies $x = y$.

[from Apostol, 1974, p.36; Chu, 2021, Lecture 1 p.7]

Definition 1.5.5 (converse)**.** Given a relation S, the new relation $\breve{S}$ defined by

$$\breve{S} = \{(a, b) | (b, a) \in S\} \tag{1.20}$$

is called the **converse** of S.

[from Apostol, 1974, p.36; Chu, 2021, Lecture 1 p.7]

Definition 1.5.6 (inverse)**.** Suppose that the relation F is a function. Consider the converse relation $\breve{F}$, which may or may not be a function. If $\breve{F}$ is a function, then $\breve{F}$ is called the **inverse** of F, and it denoted by F^{-1}.

[from Apostol, 1974, p.36; Chu, 2021, Lecture 1 p.7]

Theorem 1.5.7. If the function F is one-to-one on its domain, then $\check{F}$ is also a function.

[from Apostol, 1974, p.37; Chu, 2021, Lecture 1 p.7]

Proof. Let $(x,y) \in \check{F}$ and $(y,z) \in \check{F}$. By Definition 1.5.1 and Definition 1.5.5,

$$(x,y) \in \check{F} \Leftrightarrow (y,x) \in F \tag{1.21}$$

$$\Leftrightarrow F(y) = x \tag{1.22}$$

$$(x,z) \in \check{F} \Leftrightarrow (z,x) \in F \tag{1.23}$$

$$\Leftrightarrow F(z) = x. \tag{1.24}$$

We find

$$F(y) = F(z) = x. \tag{1.25}$$

By Definition 1.5.4,

$$y = z. \tag{1.26}$$

□

Definition 1.5.8 (composite function). Given two functions F and G such that $\mathfrak{R}(F) \subseteq \mathfrak{D}(G)$, we can form a new function, the **composite** $G \circ F$ of G and F, defined as follows: for every x in the domain of F,

$$(G \circ F)(x) = G\left[F(x)\right]. \tag{1.27}$$

[from Apostol, 1974, p.37; Chu, 2021, Lecture 1 p.8]

Example 1.5.9. Consider

$$F(x) = x^2 - 2, \quad x \in \mathbb{R} \tag{1.28}$$

$$G(x) = -x + 1, \quad x \in \mathbb{R}. \tag{1.29}$$

1. Check whether each of the two functions is a one-to-one functions.
2. Verify that $\mathfrak{R}(F) \subseteq \mathfrak{D}(G)$, and find the domain and range of $(G \circ F)(x)$.

[from Nagaoka and Miyaoka, 2007, p.11; Chu, 2021, Lecture 1 Homework Q4,Q5]

Solution

1. By Definition 1.5.4.

2. We know

$$\mathfrak{D}(F) = \mathbb{R} \tag{1.30}$$

$$\mathfrak{R}(F) = [-2, \infty) \tag{1.31}$$

$$\mathfrak{D}(G) = \mathbb{R} \tag{1.32}$$

$$\mathfrak{R}(G) = \mathbb{R}. \tag{1.33}$$

Because $\mathfrak{R}(F) \subseteq \mathfrak{D}(G)$, we can define a composite function

$$(G \circ F)(x) = -(x^2 - 2) + 1 = -x^2 + 3 \tag{1.34}$$

for all $x \in \mathbb{R}$. Moreover,

$$\mathfrak{D}(G \circ F) = \mathbb{R} \tag{1.35}$$

$$\mathfrak{R}(G \circ F) = (-\infty, 3]. \tag{1.36}$$

□

Example 1.5.10. Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. For $B \subset \mathbb{R}^m$, define $f^{-1}(B)$

$$f^{-1}(B) = \{x \in \mathbb{R}^n | f(x) \in B\}. \tag{1.37}$$

Show that for any subsets A_1, A_2 of $\mathbb{R}^n$ and B_1, B_2 of R^m:

1. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

4. $f^{-1}(B_1^C) = [f^{-1}(B_1)]^C$.

5. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

6. $A_1 \subseteq f^{-1}(f(A_1))$.

7. $f(f^{-1}(B_1)) \subseteq B_1$.

[from Sundaram et al., 1996, p.71; Nagaoka and Miyaoka, 2007, p.30; Chu, 2021, Lecture 1 p.8-9, Homework Q1]

Proof.

1. Skip.

2. By Definition 1.5.1,

$$f(A_1 \cup A_2) = \{b \in \mathbb{R}^m | \exists a \in A_1 \cup A_2 \text{ such that } f(a) = b\} \tag{1.38}$$

$$f(A_1) = \{b \in \mathbb{R}^m | \exists a_1 \in A_1 \text{ such that } f(a_1) = b\} \tag{1.39}$$

$$f(A_2) = \{b \in \mathbb{R}^m | \exists a_2 \in A_2 \text{ such that } f(a_2) = b\}. \tag{1.40}$$

   - ($\Rightarrow$)

     If $b \in f(A_1 \cup A_2)$, then there is a in either A_1 or A_2 such that $f(a) = b$.

     – If $a \in A_1$, then $b \in f(A_1)$.

     – If $a \in A_2$, then $b \in f(A_2)$.

     Thus, $b \in f(A_1) \cup f(A_2)$.

   - ($\Leftarrow$)

     If $b \in f(A_1) \cup f(A_2)$, then there is a in either A_1 or A_2 such that $f(a) = b$. Thus,

$$b \in f(A_1 \cup A_2). \tag{1.41}$$

3. Skip.

4. By Definition 1.5.1 and Definition 1.5.6,

$$f^{-1}(B_1^C) = \left\{a \in \mathbb{R}^n | f(a) \in B_1^C\right\} \tag{1.42}$$

$$[f^{-1}(B_1)]^C = \{a \in \mathbb{R}^n | f(a) \notin B_1\}. \tag{1.43}$$

   - ($\Rightarrow$)

     If $a \in f^{-1}(B_1^C)$, then $f(a) \in B_1^C$ and $f(a) \notin B_1$. Thus,

$$a \in [f^{-1}(B_1)]^C. \tag{1.44}$$

   - ($\Leftarrow$)

     If $a \in [f^{-1}(B_1)]^C$, then $f(a) \notin B_1$ and $f(a) \in B_1^C$. Thus,

$$a \in f^{-1}(B_1^C). \tag{1.45}$$

5. Skip.

6. Skip.

7. Skip.

$\square$

Example 1.5.11. The following functions F and G are defined for all real x by the equations given. In each case where the composite function $G \circ F$ can be formed, give the domain of $G \circ F$ and a formula for $(G \circ F)(x)$.

1.

$$F(x) = 1 - x \tag{1.46}$$

$$G(x) = x^2 + 2x. \tag{1.47}$$

2.

$$F(x) = x + 5 \tag{1.48}$$

$$G(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases} \tag{1.49}$$

3.

$$F(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 1, & \text{otherwise}, \end{cases} \tag{1.50}$$

$$G(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise}. \end{cases} \tag{1.51}$$

[from Apostol, 1974, p.43; Chu, 2021, Lecture 1 p.9-10]

Solution

1. Skip.
2. Skip.
3. We know

$$\mathfrak{D}(F) = \mathbb{R} \tag{1.52}$$

$$\mathfrak{R}(F) = [0, 2] \tag{1.53}$$

$$\mathfrak{D}(F) = \mathbb{R} \tag{1.54}$$

$$\mathfrak{R}(F) = [0, 1]. \tag{1.55}$$

Because $\mathfrak{R}(F) \subseteq \mathfrak{D}(G)$, we can define a composite function

$$(G \circ F)(x) = \begin{cases} 4x^2, & 0 \le x \le \frac{1}{2} \\ 0, & \frac{1}{2} < x \le 1 \\ 1 & \text{otherwise.} \end{cases} \tag{1.56}$$

Moreover,

$$\mathfrak{D}(G \circ F) = \mathbb{R} \tag{1.57}$$

$$\mathfrak{R}(G \circ F) = [0, 1]. \tag{1.58}$$

□

Theorem 1.5.12. Let $f : S \to T$ be a function. The following statements are equivalent.

1. f is one-to-one on S.
2. For all subsets A, B of S,
$$f(A \cap B) = f(A) \cap f(B). \tag{1.59}$$
3. For every subset A of S,
$$f^{-1}[f(A)] = A. \tag{1.60}$$
4. For all disjoint subsets A and B of S, the images $f(A)$ and $f(B)$ are disjoint.
5. For all subsets A and B of S with $B \subseteq A$, we have
$$f(A - B) = f(A) - f(B). \tag{1.61}$$

[from Apostol, 1974, p,44; Nagaoka and Miyaoka, 2007, p.30; Chu, 2021, Lecture 1 Homework Q2]

Proof. Skip. □

Definition 1.5.13 (finite sequence)**.** By a finite sequence of n terms, we shall understand a function F whose domain is the set of numbers $\{1, 2, \ldots, n\}$.

[from Apostol, 1974, p.37; Chu, 2021, Lecture 1 p.10]

Definition 1.5.14 (infinite sequence)**.** By an infinite sequence, we shall mean a function F whose domain is the set $\{1, 2, 3, \dots\}$ of all positive integers. The range of F is written

$$\{F_1, F_2, F_3, \dots, F_n\}, \tag{1.62}$$

and the function value F_n is called the nth **term** of the sequence.

[from Apostol, 1974, p.37; Chu, 2021, Lecture 1 p.10]

Definition 1.5.15 (similar)**.** Two sets A and B are called **similar**, or **equinumerous**, and we write $A \sim B$, if and only if there exists a one-to-one function F whose domain is the set A, and whose range is the set B.

[from Apostol, 1974, p.38]

Theorem 1.5.16.

1. Every set A is similar to itself.
2. If $A \sim B$, then $B \sim A$.
3. If $A \sim B$ and $B \sim C$, then $A \sim C$.

[from Apostol, 1974, p.38]

Proof.

1. Take F to be identity function. That is, for all x in A,

$$F(x) = x. \tag{1.63}$$

2. If F is a one-to-one function which makes $A \sim B$, then F^{-1} will make $B \sim A$.
3. Skip.

□

1.6 Finite, Infinite, Countable, and Uncountable Sets

Definition 1.6.1 (finite set)**.** A set S is called **finite**, and is said to contain n elements if

$$S \sim \{1, 2, \ldots, n\}. \tag{1.64}$$

[from Apostol, 1974, p.38]

1. The integer n is called the **cardinal number** of S.
2. The empty set is considered finite. Its cardinal number is defined to be 0.

Theorem 1.6.2. If

$$\{1, 2, \ldots, n\} \sim \{1, 2, \ldots, m\}, \tag{1.65}$$

then $m = n$.

[from Apostol, 1974, p.38]

Definition 1.6.3 (infinite set)**.** Sets which are not finite are called **infinite sets**.

[from Apostol, 1974, p.38]

Definition 1.6.4 (countably infinite set)**.** A set S is said to be **countably infinite** if it is equinumerous with the set of all positive integers. That is,

$$S \sim \{1, 2, 3, \ldots\}. \tag{1.66}$$

[from Apostol, 1974, p.39]

Definition 1.6.5 (countable, uncountable set)**.**

1. A set S is called **countable** if it is either finite or countably infinite.
2. A set which is not countable is called **uncountable**.

[from Apostol, 1974, p.39]

Theorem 1.6.6. Every subset of a countable set is countable.

[from Apostol, 1974, p.39]

Theorem 1.6.7. The set of all real numbers is uncountable.

[from Apostol, 1974, p.39]

1.7 Upper Bounds, Maximum Element, and Least Upper Bound

Definition 1.7.1 (order)**.** Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:

1. If $x \in S$ and $y \in S$, then one and only one of the statements

$$x < y \tag{1.67}$$

$$x = y \tag{1.68}$$

$$y < x \tag{1.69}$$

is true.

2. Let $x, y, z \in S$. If $x < y$ and $y < x$, then $x < z$.

[from Rudin et al., 1964, p.3]

1. The statement "$x < y$" may be read as "x is less than y" or "x is smaller than y" or "x preceeds y."

2. It is often convenient to write $y > x$ in place of $x < y$.

3. The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of these two is to hold. In other words, $x \leq y$ is the negation of $x > y$.

Definition 1.7.2 (order set)**.** An **order set** is a set S in which an order is defined.

[from Rudin et al., 1964, p.3]

Definition 1.7.3 (bound)**.** Suppose S is an ordered set, and $E \subset S$.

1. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is **bounded above**, and call β an **upper bound** of E.
2. If there exists a $\beta \in S$ such that $x \geq \beta$ for every $x \in E$, we say that E is **bounded below**, and call β an **lower bound** of E.

[from Rudin et al., 1964, p.3; Sundaram et al., 1996, p.14; Chu, 2020, Lecture 3 p.5; Chu, 2021, Lecture 2 p.5]

1. The set of **upper bounds** of E, denoted $U(E)$, is defined as

$$U(E) = \{\beta \in \mathbb{R} | \beta \geq x, \forall x \in E\}. \tag{1.70}$$

2. If

$$\beta \in E \cap U(E) \tag{1.71}$$

then β is called **maximum element** of E.

3. The set of **lower bounds** of E, denoted $L(E)$, is defined as

$$L(E) = \{\beta \in \mathbb{R} | \beta \leq x, \forall x \in E\}. \tag{1.72}$$

4. If

$$\beta \in E \cap L(E) \tag{1.73}$$

then β is called **minimum element** of E.

Definition 1.7.4 (supremum)**.** Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

1. α is an upper bound of E.
2. If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called the **least upper bound** of E or the **supremum** of E, and we write

$$\alpha = \sup E. \tag{1.74}$$

[from Rudin et al., 1964, p.4; Sundaram et al., 1996, p.14; Apostol, 1974, p.9; Chu, 2020, Lecture 3 p.5; Chu, 2021, Lecture 2 p.6]

Definition 1.7.5 (infimum)**.** Suppose S is an ordered set, $E \subset S$, and E is bounded below. Suppose there exists an $\alpha \in S$ with the following properties:

1. α is an lower bound of E.

2. If $\gamma > \alpha$, then γ is not an lower bound of E.

Then α is called the **greatest lower bound** of E or the **infimum** of E, and we write

$$\alpha = \inf E. \tag{1.75}$$

[from Rudin et al., 1964, p.4; Sundaram et al., 1996, p.14; Apostol, 1974, p.9; Chu, 2020, Lecture 3 p.5; Chu, 2021, Lecture 2 p.6]

Definition 1.7.6 (least-upper-bound property)**.** An ordered set S is said to have the **least-upper-bound property** if $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

[from Rudin et al., 1964, p.4]

1. If $\alpha = \sup E$ exists, then α may or may not be a member of E. For instance, let E_1 be the set of all $r \in Q$ with $r < 0$. Let E_2 be the set of all $r \in Q$ with $r \leq 0$. Then

$$\sup E_1 = \sup E_2 = 0, \tag{1.76}$$

and $0 \notin E_1, 0 \in E_2$.

2. We shall show that there is a close relation between greatest lower bounds and least upper bounds, and that every ordered set with the least-upper-bound property also has the greatest-lower-bound property.

Theorem 1.7.7. If the set of maximum elements exists, then

$$\max E = \sup E. \tag{1.77}$$

[from Chu, 2021, Lecture 2 p.6]

Proof. By Definition 1.7.3, (1.71), and Definition 1.7.4, the statement holds. □

Example 1.7.8. Find the maximum, supremum, minimum, and infimum of E.

1. $E = (0, 1)$
2. $E = (0, 1]$
3. $E = \left\{x \in \mathbb{R} | x = \frac{1}{n}, n \in \mathbb{R}\right\}$
4. $E = \left\{x \in \mathbb{R} | x = 1 - \frac{1}{n}, n \in \mathbb{R}\right\}$

[from Chu, 2021, Lecture 2 p.7, Homework Q8]

Solution

1. $\max E$ and $\min E$ don't exist.

$$\sup E = 1 \tag{1.78}$$

$$\inf E = 0 \tag{1.79}$$

2. $\min E$ doesn't exist.

$$\max E = 1 \tag{1.80}$$

$$\sup E = 1 \tag{1.81}$$

$$\inf E = 0 \tag{1.82}$$

3. $\min E$ doesn't exist.

$$\max E = 1 \tag{1.83}$$

$$\sup E = 1 \tag{1.84}$$

$$\inf E = 0 \tag{1.85}$$

4. $\max E$ doesn't exist.

$$\sup E = 1 \tag{1.86}$$

$$\min E = 0 \tag{1.87}$$

$$\inf E = 0 \tag{1.88}$$

□

Axiom 1.7.9 (completeness axiom)**.** Every nonempty set S of real numbers which is bounded above has a supremum. That is, there is a real number b such that $b = \sup S$.

[from Apostol, 1974, p.9; Chu, 2021, Lecture 2 p7]

Theorem 1.7.10. If $U(E)$ is nonempty, the supremum of E is well defined, i.e. there is a $x \in U(A)$ such that $x \leq u$ for all $u \in U(E)$. Similarly, if $L(E)$ is nonempty, the infimum of E is well defined, i.e. there is $y \in L(E)$ such that $y \geq l$ for all $l \in L(E)$.

[from Sundaram et al., 1996, p.14]

Theorem 1.7.11 (approximation property for suprema)**.** Suppose $\sup E$ is finite. Then, for all $\varepsilon > 0$, there is a $a(\varepsilon) \in E$ such that

$$\sup E \geq a(\varepsilon) > \sup E - \varepsilon. \tag{1.89}$$

[from Sundaram et al., 1996, p.16; Apostol, 1974, p.9-10; Chu, 2021, Lecture 2 p.7]

Proof. Suppose $\exists \varepsilon > 0$ such that $a \leq \sup E - \varepsilon$ for all $a \in E$. By Definion 1.7.3, $\sup E - \varepsilon$ would be an upper bound of E and $\sup E - \varepsilon < \sup E$, but it violates $\sup E$ is the least upper bound by Definition 1.7.4. This is a contradiction. □

Theorem 1.7.12. Suppose S is an ordered set with the least-upper-bounded property $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L \tag{1.90}$$

exists in S, and $\alpha = \inf B$. In particular, $\inf B$ exists in S.

[from Rudin et al., 1964, p.5]

Theorem 1.7.13 (comparison property)**.** Given nonempty subsets S and T of $\mathbb{R}$ such that $s \leq t$ for all $s \in S$ and $t \in T$. If T has a supremum, then S has a supremum and

$$\sup S \leq \sup T. \tag{1.91}$$

[from Apostol, 1974, p.10; Chu, 2021, Lecture 2 p.8]

Proof. (Contradiction) □

Theorem 1.7.14 (comparison property)**.** Given nonempty subsets S and T of $\mathbb{R}$ such that $s \leq t$ for all $s \in S$ and $t \in T$. If T has a supremum, then S has a supremum and

$$\sup S \leq \sup T. \tag{1.92}$$

[from Apostol, 1974, p.10; Chu, 2021, Lecture 2 p.8]

Chapter 2

Elements of Point Set Topology

Definition 2.0.1 (metric space)**.** A set M, whose elements we shall call **points**, is said to be a **metric space** if with any two points p and q of M there is associated a real number $d(p,q)$, call the **distance** from p to q, such that

1. (positivity) $d(p,q) \geq 0$;
2. (nondegeneracy) $d(p,q) = 0$ if and only if $p = q$;
3. (symmetry) $d(p,q) = d(q,p)$;
4. (triangle inequality) $d(p,q) \leq d(p,r) + d(r,q)$ for all $r \in M$.

[from Rudin et al., 1964, p.30; Marsden, Hoffman, et al., 1993, p.64; Chu, 2021, Lecture 2 p.1]

1. We denote a metric space by (M,d).
2. Any function with 4 properties in Definition 2.0.1 is called **distance function**, or a **metric**.

2.1 Euclidean Space

Definition 2.1.1 (Euclidean space)**.** **Euclidean space**, denoted $\mathbb{R}^n$, consists of all ordered n-tuples of real numbers. Symbolically, $\mathbb{R}^n = \{(x_1,\ldots,x_n) | x_1,\ldots,x_n \in \mathbb{R}\}$. Thus, $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) is the Cartesian product of $\mathbb{R}$ with itself n times.

[from Marsden, Hoffman, et al., 1993, p.57; Rudin et al., 1964, p.16; Apostol, 1974, p.47; Chu, 2021, Lecture 2 p.2]

1. Elements of $\mathbb{R}^n$ are generally denoted by (bold-face) single letters that stand for n-tuples such as

$$x = (x_1, \ldots, x_n), \tag{2.1}$$

and we speak of x as a **point** in $\mathbb{R}^n$.

2. **Addition** of n-tuples are defined by

$$x + y = (x_1 + y_1, \ldots, x_n + y_n). \tag{2.2}$$

3. **scalar multiplication** of n-tuples are defined by

$$\alpha x = (\alpha x_1, \ldots, \alpha x_n) \tag{2.3}$$

where $\alpha \in \mathbb{R}$.

Definition 2.1.2 (length, norm)**.** The **length** or **norm** of a vector x in $\mathbb{R}^n$ is defined by

$$\|x\| = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2}, \tag{2.4}$$

where $x = (x_1, \ldots, x_n)$.

[from Marsden, Hoffman, et al., 1993, p.59; Rudin et al., 1964, p.16; Apostol, 1974, p.48; Chu, 2020, Lecture 1 p.3-4; Chu, 2021, Lecture 2 p.4]

Definition 2.1.3 (distance, metric)**.** The **distance** or **metric** between two vectors x and y is the real number

$$d(x, y) = \|x - y\| = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2}. \tag{2.5}$$

[from Marsden, Hoffman, et al., 1993, p.59; Rudin et al., 1964, p.16; Apostol, 1974, p.48; Chu, 2020, Lecture 1 p.3-4; Chu, 2021, Lecture 2 p.2]

Definition 2.1.4 (inner product)**.** The **innder product** of x and y is defined by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i. \tag{2.6}$$

[from Marsden, Hoffman, et al., 1993, p.59; Rudin et al., 1964, p.16; Apostol, 1974, p.48; Chu, 2020, Lecture 1 p.3-4; Chu, 2021, Lecture 2 p.4]

Theorem 2.1.5 (properties of the inner product)**.**

1. (positivity) $x \cdot x \geq 0$.
2. (nondegeneracy) $x \cdot x = 0$ if and only if $x = 0$.
3. (distributivity) $x \cdot (y + w) = x \cdot y + x \cdot w$.
4. (multiplicativity) $\alpha x \cdot y = \alpha(x \cdot y)$ for $\alpha \in \mathbb{R}$.
5. (symmetry) $x \cdot y = y \cdot x$.

[from Marsden, Hoffman, et al., 1993, p.60-61; Rudin et al., 1964, p.16-17; Sundaram et al., 1996, p.4]

Theorem 2.1.6 (Cauchy-Schwarz inequality)**.** For vectors in $\mathbb{R}^n$, $|x \cdot y| \leq \|x\|\|y\|$.

[from Marsden, Hoffman, et al., 1993, p.61; Rudin et al., 1964, p.16-17; Sundaram et al., 1996, p.4-5; Chu, 2020, Lecture 1 p.4; Chu, 2021, Lecture 2 p.2-3]

Proof. The cases $x = 0$ or $y = 0$ are trivial, so we can assume $x \neq 0$ and $y \neq 0$. Let $\alpha = \frac{x \cdot y}{\|y\|^2}$, then

$$
\begin{aligned}
0 &\leq \|x - \alpha y\|^2 && (2.7)\\
&= (x - \alpha y) \cdot (x - \alpha y) && (2.8)\\
&= \|x\|^2 - 2\alpha x \cdot y + \alpha^2 \|y\|^2 && (2.9)\\
&= \|x\|^2 - 2\left(\frac{x \cdot y}{\|y\|^2}\right) x \cdot y + \left(\frac{x \cdot y}{\|y\|^2}\right)^2 \|y\|^2 && (2.10)\\
&= \|x\|^2 - 2\frac{(x \cdot y)^2}{\|y\|^2} + \frac{(x \cdot y)^2}{\|y\|^2} && (2.11)\\
&= \|x\|^2 - \frac{x \cdot y}{\|y\|^2} && (2.12)\\
\|x\|^2 &\geq \frac{(x \cdot y)^2}{\|y\|^2} && (2.13)\\
(x \cdot y)^2 &\leq \|x\|^2 \|y\|^2 && (2.14)\\
|x \cdot y| &\leq \|x\|\|y\| && (2.15)
\end{aligned}
$$

□

Theorem 2.1.7 (properties of the norm)**.**

1. (positivity) $\|x\| \geq 0$.
2. (nondegeneracy) $\|x\| = 0$ if and only if $x = 0$.
3. (multiplicativity) $\|\alpha x\| = |\alpha|\|x\|$ for $\alpha \in \mathbb{R}$.
4. (triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$.

[from Marsden, Hoffman, et al., 1993, p.61; Rudin et al., 1964, p.16-17; Sundaram et al., 1996, p.5-6; Apostol, 1974, p.48-49; Chu, 2020, Lecture 1 p.4]

Proof. (Triangle inequality)

$$\begin{aligned}
\|x + y\|^2 &= (x + y) \cdot (x + y) && (2.16)\\
&= x \cdot x + 2x \cdot y + y \cdot y && (2.17)\\
&\leq x \cdot x + 2|x \cdot y| + y \cdot y && (2.18)\\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 && (2.19)\\
&= (\|x\| + \|y\|)^2 && (2.20)\\
\|x + y\| &\leq \|x\| + \|y\| && (2.21)
\end{aligned}$$

□

Theorem 2.1.8 (properties of the distance)**.**

1. (positivity) $d(x, y) \geq 0$.
2. (nondegeneracy) $d(x, y) = 0$ if and only if $x = y$.
3. (symmetry) $d(x, y) = d(y, x)$.
4. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

[from Marsden, Hoffman, et al., 1993, p.61; Rudin et al., 1964, p.16-17; Sundaram et al., 1996, p.6; Chu, 2020, Lecture 1 p.4; Chu, 2021, Lecture 2 p.2]

Proof. (Triangle inequality)

$$d(x,z) = \|x-z\| = \|x-y+y-z\| \tag{2.22}$$

$$\leq \|x-y\| + \|y+z\| \tag{2.23}$$

□

Definition 2.1.9 (unit coordinate vector)**.** The **unit coordinate vector** u_k in $\mathbb{R}^n$ is the vector whose kth component is 1, and whose remaining components are 0.

[from Apostol, 1974, p.49; Chu, 2021, Lecture 2 p.4]

1. We can show that

$$u_1 = (1,0,\ldots,0) \tag{2.24}$$

$$u_2 = (0,1,\ldots,0) \tag{2.25}$$

$$\vdots \tag{2.26}$$

$$u_n = (0,0,\ldots,1). \tag{2.27}$$

2. If $x = (x_1,\ldots,x_n)$, then

$$x = x_1u_1 + \ldots + x_nu_n \tag{2.28}$$

and $x_i = x \cdot u_i$ for all $i = 1,\ldots,n$. The vectors u_i are called **basis vectors**.

Chapter 3

Limits and Continuity

3.1 Sequences and Limits

Definition 3.1.1 (sequence)**.** By a **sequence**, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \ldots$. The values of f, that is, the elements x_n, are called the **terms** of th sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a **sequence in** A, or a **sequence of elements of** A.

[from Rudin et al., 1964, p.26]

Definition 3.1.2 (converge sequence).

1. Let (M, d) be a metric space and x_k a sequence of points in M. We say that x_k **converges** to a point $x \in M$, written

$$\lim_{k \to \infty} x_k = x \tag{3.1}$$

or

$$x_k \to x \text{ as } k \to \infty, \tag{3.2}$$

provided that for every open set U containing x, there is an integer N such that $x_k \in U$ whenever $k \geq N$.

2. A sequence x_k in M **converges** to $x \in M$ if and only if for every $\varepsilon > 0$ there is an $N(\varepsilon)$ such that $k \geq N(\varepsilon)$ implies $d(x, x_k) < \varepsilon$.

[from Marsden, Hoffman, et al., 1993, p.120; Rudin et al., 1964, p.47; Sundaram et al., 1996, p.7; Chu, 2020, Lecture 1 p.4; Chu, 2021, Lecture 2 p.9]

Example 3.1.3. Show that $\{x_n\}_{n=1}^{\infty}$ with

$$x_n = \frac{1}{n} \tag{3.3}$$

converges to 0.

[from Chu, 2021, Lecture 2 p.9-10]

Proof. For all $\varepsilon > 0$, take

$$N(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 1 \tag{3.4}$$

such that

$$d(x_n, 0) = \frac{1}{n} < \varepsilon \tag{3.5}$$

if $n \geq N(\varepsilon)$. □

Example 3.1.4. Show that $\{x_n\}$ with $x_n = n$ does not converge in $\mathbb{R}$.

[from Chu, 2021, Lecture 2 Homework Q9]

Proof. Skip. □

Definition 3.1.5 (bounded, unbounded).

1. A sequence $\{x_k\}$ in $\mathbb{R}^n$ is called **bounded** sequence if there exists a real number M such that $\|x_k\| \leq M$ for all k.
2. A sequence $\{x_k\}$ in $\mathbb{R}^n$ is called **unbounded** sequence if for any $M \in \mathbb{R}$, there exists k such that $\|x_k\| > M$.

[from Sundaram et al., 1996, p.7; Chu, 2020, Lecture 1 p.2]

Theorem 3.1.6 (unique). Let $\{x_k\}$ be a sequence in a metric space M. If $x \in M$, $y \in M$, and if $\{x_k\}$ converges to x and to y, then $x = y$.

[from Rudin et al., 1964, p.48-49; Sundaram et al., 1996, p.7; Chu, 2020, Lecture 1 p.5; Chu, 2021, Lecture 2 p.10]

Proof. For all $\varepsilon > 0$, there is $N_x(\varepsilon), N_y(\varepsilon) \in \mathbb{N}$ such that

$$k \geq N_x(\varepsilon) \Rightarrow d(x_k, x) < \frac{\varepsilon}{2}, \tag{3.6}$$

$$k \geq N_y(\varepsilon) \Rightarrow d(x_k, y) < \frac{\varepsilon}{2}. \tag{3.7}$$

Taking

$$k \geq N = \max\left\{N_x(\varepsilon), N_y(\varepsilon)\right\}, \tag{3.8}$$

we have

$$d(x, y) \leq d(x, x_k) + d(x_k, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{3.9}$$

Because ε is arbitrary, we conclude that $d(x, y) = 0$. □

Theorem 3.1.7 (bounded). Let $\{x_k\}$ be a sequence in a metric space M. If $\{x_k\}$ is converges, then $\{x_k\}$ is bounded.

[from Rudin et al., 1964, p.48-49; Sundaram et al., 1996, p.7; Chu, 2020, Lecture 1 p.5; Chu, 2021, Lecture 2 p.11-12]

Proof. Suppose $x_k \to x$. That is, for all $\varepsilon > 0$, there is $N(\varepsilon) \in \mathbb{N}$ such that

$$k \geq N(\varepsilon) \Rightarrow d(x_k, x) < \varepsilon. \tag{3.10}$$

Let $\varepsilon = 1$ and $N(1)$ such that

$$k \geq N(1) \Rightarrow d(x_k, x) < 1. \tag{3.11}$$

By triangle inequality,

$$\|x_k\| = \|x_k - x + x\| \tag{3.12}$$

$$\leq \|x_k - x\| + \|x\| \tag{3.13}$$

$$\leq 1 + \|x\|. \tag{3.14}$$

Taking

$$M = \max\left\{\|x_1\|, \|x_2\|, \ldots, \|x_{N(1)-1}\|, 1 + \|x\|\right\}, \tag{3.15}$$

we find for all $k \in \mathbb{N}$

$$\|x_k\| \leq M. \tag{3.16}$$

□

Theorem 3.1.8. Suppose $\{s_k\}, \{t_k\}$ are complex sequences, and $\lim_{k\to\infty} s_k = s$, $\lim_{k\to\infty} t_k = t$. Then

1. $\lim_{k\to\infty} (s_k + t_k) = s + t$.
2. $\lim_{k\to\infty} (c + s_k) = c + s$ for any number c.
3. $\lim_{k\to\infty} cs_k = cs$ for any number c.
4. $\lim_{k\to\infty} s_k t_k = st$.
5. $\lim_{k\to\infty} \|s_k\| = \|s\|$.
6. If $\lim_{k\to\infty} s_k = s \neq 0$, then there is $N \in \mathbb{N}$ such that $\|s_k\| \geq \frac{\|s\|}{2} > 0$ if $k \geq N$.
7. $\lim_{k\to\infty} \frac{1}{s_k} = \frac{1}{s}$, provided $s_k \neq 0$ and $s \neq 0$.
8. $\lim_{k\to\infty} \frac{t_k}{s_k} = \frac{t}{s}$, provided $s_k \neq 0$ and $s \neq 0$.

[from Rudin et al., 1964, p.49; Tsai and Chen, 2016, p.87-90; Sundaram et al., 1996, p.67; Chu, 2020, Lecture 1 Homework Q3; Chu, 2021, Lecture 2 Homework Q2,Q3]

Proof. Skip. □

Theorem 3.1.9. Suppose $x_k \in \mathbb{R}^n, k \in \mathbb{N}$ and $x_k = (x_k^1, \ldots, x_k^n)$. Then $\{x_k\}$ converges to $x = (x^1, \ldots, x^n)$ if and only if $\lim_{n\to\infty} x_k^i = x^i$ $(1 \leq i \leq n)$.

[from Rudin et al., 1964, p.50; Sundaram et al., 1996, p.8-9; Chu, 2020, Lecture 1 p.6; Chu, 2021, Lecture 2 p.13-15]

Proof.

- $\Rightarrow$:

  If $x_k \to x$, then $\forall \varepsilon > 0, \exists N(\varepsilon)$ such that $d(x_k, x) < \varepsilon$ if $k \geq N(\varepsilon)$. The inequalities

$$d(x_k^i, x^i) = \left\{|x_k^i - x^i|^2\right\}^{1/2} \tag{3.17}$$

$$\leq \left\{\sum_{i=1}^{n} |x_k^i - x^i|^2\right\}^{1/2} \tag{3.18}$$

$$= d(x_k, x) < \varepsilon \tag{3.19}$$

  show that statement holds.

- $\Leftarrow$:

  If $\lim_{n\to\infty} x_k^i = x^i$, then $\forall \varepsilon > 0, \exists N_i(\varepsilon)$ such that $d\left(x_k^i, x^i\right) < \frac{\varepsilon}{\sqrt{n}}$ $(1 \leq k \leq n)$. Hence,

$$k \geq N(\varepsilon) \equiv \max\left\{N_1\left(\frac{\varepsilon}{\sqrt{n}}\right), N_2\left(\frac{\varepsilon}{\sqrt{n}}\right), \ldots, N_n\left(\frac{\varepsilon}{\sqrt{n}}\right)\right\} \tag{3.20}$$

  implies

$$d(x_k, x) = \left\{\sum_{i=1}^{n} |x_k^i - x^i|^2\right\}^{1/2} \tag{3.21}$$

$$< \left\{\sum_{i=1}^{n} \left(\frac{\varepsilon}{\sqrt{n}}\right)^2\right\}^{1/2} = \varepsilon. \tag{3.22}$$

□

Theorem 3.1.10. Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$ converging to a limit x. Suppose that for every k, we hae $a \leq x_k \leq b$, where $a = (a^1, \ldots, a^n)$ and $b = (b^1, \ldots, b^n)$ are some fixed vectors in $\mathbb{R}^n$. Then, it is also the case that $a \leq x \leq b$.

[from Sundaram et al., 1996, p.9; Chu, 2020, Lecture 1 p.7; Chu, 2021, Lecture 2 p.15-17]

Proof. (Contradiction.)

Since $x_k^i \to x^i$, $\forall \varepsilon > 0, \exists N(\varepsilon)$ such that $d\left(x_k^i, x^i\right) < \varepsilon$ if $k \geq N(\varepsilon)$. Let $\varepsilon = a^i - x^i$. By inequalities,

$$x_k^i - a^i = (x_k^i - x^i) + (x^i - a^i) \tag{3.23}$$

$$< (a^i - x^i) + (x^i - a^i) = 0. \tag{3.24}$$

We can find an integer N such that $x_k^i - x^i < a^i - x^i = \varepsilon$ if $k \geq N$. Therefore, $x_k^i - a^i < 0$ if $k > N$. A contradiction. □

Example 3.1.11. Show that $\left(\frac{1}{n}, 1 - \frac{1}{n}\right) \to (0, 1)$ as $n \to \infty$.

[from Chu, 2020, Lecture 1 Homework Q5]

Proof. Let $x_k = \left(\frac{1}{k}, 1 - \frac{1}{k}\right), x = (0, 1)$. If $k > N\left[\frac{\sqrt{2}}{k}\right]$, then

$$d(x_k, x) = d\left\{\left(\frac{1}{k}, 1 - \frac{1}{k}\right), (0, 1)\right\} \tag{3.25}$$

$$= \frac{\sqrt{2}}{k} < \varepsilon. \tag{3.26}$$

□

3.2 Subsequences and Limit Points

Definition 3.2.1 (subsequence)**.** Given a sequence $\{x_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\{x_{n_k}\}$ is called a **subsequence** of $\{x_n\}$.

[from Rudin et al., 1964, p.51; Sundaram et al., 1996 p.10]

1. If $\{x_{n_k}\}$ converges, its limit is called a **subsequential limit of** $\{x_n\}$**.**
2. It is clear that $\{x_n\}$ converges to x if and only if every subsequence of $\{x_n\}$ converge to x.
3. If a sequence contains a convergent subsequence, the limit of the convergent subsequence is called a **limit point** of the original sequence.

Example 3.2.2. Find converging subsequences of the following sequences.

1. $\{1, -1, 1, -1, 1, -1, \ldots\}$
2. $\{1, 0, 2, 0, 3, 0, 4, 0, \ldots\}$

[from Chu, 2020, Lecture 1 Homework Q4; Chu, 2021, Lecture 2 Homework Q1]

Solution

1. $\{x_{2n}\}_{n=1}^{\infty} \to -1$ and $\{x_{2n-1}\}_{n=1}^{\infty} \to 1$.

2. $\{x_{2n}\}_{n=1}^{\infty} \to 0$. □

Theorem 3.2.3. A point x is a limit point of the sequence $\{x_k\}$ if and only if for any $\varepsilon > 0$, there are infinitely many indices m for which $d(x, x_m) < \varepsilon$.

[from Sundaram et al., 1996, p.10]

3.3 Cauchy Sequences and Completeness

Definition 3.3.1 (Cauchy sequence). A sequence $\{x_k\}$ in a metric space M is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there is an integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ if $n \geq N(\varepsilon)$ and $m \geq N(\varepsilon)$.

[from Rudin et al., 1964, p.52; Marsden, Hoffman, et al., 1993, p.49; Sundaram et al., 1996, p.11]

Example 3.3.2. Prove that $\{x_k\}$ in $\mathbb{R}$ defined by $x_k = \frac{1}{k^2}$ for all k is a Cauchy sequence.

[from Sundaram et al., 1996, p.11]

Proof. For all $\varepsilon > 0$, put $N(\varepsilon) > \sqrt{\frac{2}{\varepsilon}}$. When $m, n \geq N(\varepsilon)$,

$$d(x_n, x_m) = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| \tag{3.27}$$

$$\leq \left|\frac{1}{n^2} + \frac{1}{m^2}\right| \tag{3.28}$$

$$\leq \left|\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right| = \varepsilon. \tag{3.29}$$

□

Theorem 3.3.3. A sequence $\{x_k\}$ in $\mathbb{R}^n$ is a Cauchy sequence if and only if each $i \in \{1, \ldots, n\}$, the sequence $\{x_k^i\}$ is a Cauchy sequence in $\mathbb{R}$.

[from Sundaram et al., 1996, p.11]

Proof.

- $\Rightarrow$: (If $\{x_k\}$ in $\mathbb{R}^n$ is a Cauchy sequence, the sequence $\{x_k^i\}$ is a Cauchy sequence in $\mathbb{R}$ for each $i \in \{1, \ldots, n\}$.)

  If $\{x_k\}$ is a Cauchy sequence, then $\forall \varepsilon > 0, \exists N(\varepsilon)$ such that $d(x_m, x_l) < \varepsilon$ if $m, l > N(\varepsilon)$. Therefore, for all $m, l \geq N(\varepsilon)$,

$$d(x_m^i, x_l^i) = \left\{|x_m^i - x_l^i|^2\right\}^{1/2} \tag{3.30}$$

$$\leq \left\{\sum_{i=1}^{n} |x_m^i - x_n^i|^2\right\}^{1/2} \tag{3.31}$$

$$= d(x_m, x_n) < \varepsilon. \tag{3.32}$$

  Put $N_i(\varepsilon) = N(\varepsilon)$, the statement holds.

- $\Leftarrow$: (If $\{x_k^i\}$ is a Cauchy sequence in $\mathbb{R}$ for each $i \in \{1, \ldots, n\}$, $\{x_k\}$ in $\mathbb{R}^n$ is a Cauchy sequence.)

  If $\{x_k^i\}$ is a Cauchy sequence, then $\forall \varepsilon > 0, \exists N_i\left(\frac{\varepsilon}{\sqrt{n}}\right)$ such that $d\left(x_m^i, x_l^i\right) < \frac{\varepsilon}{\sqrt{n}}$ if $m, l \geq N_i\left(\frac{\varepsilon}{\sqrt{n}}\right)$. Hence,

$$m, l \geq N(\varepsilon) \equiv \max\left\{N_1\left(\frac{\varepsilon}{\sqrt{n}}\right), N_2\left(\frac{\varepsilon}{\sqrt{n}}\right), \ldots, N_n\left(\frac{\varepsilon}{\sqrt{n}}\right)\right\} \tag{3.33}$$

  implies

$$d(x_m, x_l) = \left\{\sum_{i=1}^{n} |x_m^i - x_l^i|^2\right\}^{1/2} \tag{3.34}$$

$$< \left\{\sum_{i=1}^{n} \left(\frac{\varepsilon}{\sqrt{n}}\right)^2\right\}^{1/2} = \varepsilon. \tag{3.35}$$

□

Theorem 3.3.4.

1. In any metric space M, every convergent sequence is a Cauchy sequence.
2. Let X is a compact metric space. If $\{x_k\}$ is a Cauchy sequence in X, then $\{x_k\}$ converges to some point of X.
3. In $\mathbb{R}^n$, every Cauchy sequence converges.

[from Rudin et al., 1964, p.53 ; Sundaram et al., 1996, p.12]

Proof. (Only prove 1.)

If $x_k \to x$, then $\forall \varepsilon > 0, \exists N\left(\frac{\varepsilon}{2}\right)$ such that $d(x_k, x) < \frac{\varepsilon}{2}$ if $k \geq N\left(\frac{\varepsilon}{2}\right)$. Hence,

$$d(x_m, x_l) \leq d(x_m, x) + d(x, x_l) \tag{3.36}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{3.37}$$

if $m, l \geq N\left(\frac{\varepsilon}{2}\right)$. $\square$

1. A sequence $\{x_k\}$ in $\mathbb{R}^n$ is a Cauchy sequence if and only if it is a convergent sequence, i.e., if and only if there is $x \in \mathbb{R}^n$ such that $x_k \to x$.
2. The difference between the definition of convergence and the definition of a Cauchy sequence is that the limit is explicitly involved in the former, but not in the latter.
3. The fact that a sequence converges in $\mathbb{R}^n$ if and only if it is a Cauchy sequence is usually called the **Cauchy criterion** for convergence.

Definition 3.3.5 (completeness). A metric space in which every Cauchy sequence converges is said to be **complete**.

[from Rudin et al., 1964, p.54; Marsden, Hoffman, et al., 1993, p.123; Sundaram et al., 1996, p.12]

1. Theorem 3.3.4 says that all compact metric spaces and all Euclidean spaces are complete.
2. Theorem 3.3.4 implies that every closed subset E of a complete metric space X is complete.
3. An example of a metric space which is not complete is the space of all rational numubers, with $d(x, y) = |x - y|$.

Theorem 3.3.6. Let $\{x_k\}$ be a Cauchy sequence in $\mathbb{R}^n$. Then

1. $\{x_k\}$ is bounded.
2. $\{x_k\}$ has at most one limit point.

[from Sundaram et al., 1996, p.13; Marsden, Hoffman, et al., 1993, p.124]

Proof.

1. Suppose $\{x_k\}$ be a Cauchy sequence, $\forall \varepsilon > 0, \exists N(\varepsilon)$ such that $d(x_m, x_l) < \varepsilon$ if $m, l \geq N(\varepsilon)$. Now, take $\varepsilon = 1$ and $N(1)$ such that $d(x_m, x_n) < 1$ if $m, l \geq N(1)$. By triangle inequalty,

$$\begin{aligned} \|x_k\| &= \|x_k - x_{N(1)} + x_{N(1)}\| && (3.38) \\ &\leq \|x_k - x_{N(1)}\| + \|x_{N(1)}\| && (3.39) \\ &\leq 1 + \|x_{N(1)}\|. && (3.40) \end{aligned}$$

Put $M = \max\left\{\|x_1\|, \|x_2\|, \ldots, \|x_{N(1)-1}\|, 1 + \|x_{N(1)}\|\right\}$, then $\|x_k\| \leq M, \forall k \in \mathbb{N}$.

2. By Theorem 3.3.4 and Theorem 3.1.6, the statement holds.

□

Chapter 4

Old Exam

4.1 Midterm (Fall 2020)

Example 4.1.1. Consider the following utility maximization problem

$$\max_{x} \quad U(x) \tag{4.1}$$

$$\text{s.t.} \quad x \in \mathcal{D}(p, I) = \{x \in \mathbb{R}^n | p \cdot x \leq I, x_i \geq 0, \forall i\} \tag{4.2}$$

where $x = (x_1, x_2, \ldots, x_n)$, $U(x) = \sum_{i=1}^{n} u(x_i)$, $p \gg 0 \in \mathbb{R}^n$ and income $I > 0 \in \mathbb{R}$ are given. The function $u : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable everywhere on the domain,

$$u(\cdot) > 0, \quad u'(\cdot) > 0 > u''(\cdot), \quad \text{and} \quad \lim_{\Delta \to \infty} u'(\Delta) = \infty. \tag{4.3}$$

1. (2%) Define x^* as a maximizer to the utility maximization problem.
2. (5%) Explain why we need the assumption $p \gg 0$ to make the Weierstrass Theorem applicable.
3. (5%) Suppose $p_i = 0$ for some i. Because the Weierstrass Theorem does not hold, the solution to the utility maximization problem does not exist. State wherther th is statement is true or false. You need to explain your result.
4. (5%) Suppose $p_i = 0$ for some i, show that the solution to the utility maximization problem does not exist.
5. (8%) Show that the maximizer x^* must statisfy $p \cdot x^* = I$.
6. (20%) Show that the optimal solution $x^* \gg 0$.

[from Chu, 2020, Midterm Q1]

Example 4.1.2. Let

$$f(x,y) = 3x + 3xy - 3x^2 - y^3 \tag{4.4}$$

be a function defined on a set $\mathcal{D} = [0,1] \times [0,2]$; that is, $0 \le x \le 1$ and $0 \le y \le 2$. The function $f(x,y)$ is continuous on $\mathcal{D}$. Answer the following questions.

1. (10%) Show that the set $\mathcal{D}$ is closed in $\mathbb{R}^2$.
2. (5%) Is the set $\mathcal{D}$ bounded? Explain.
3. (5%) Is the set $\mathcal{D}$ convex? Explain.
4. (10%) Find the critical points that statisfy the first-order equation and classify these as local maximum, local minimum, or neither.
5. (10%) Find the global maximum and the global minimum values of $f(x,y)$ on $\mathcal{D}$.

[from Chu, 2020, Midterm Q2]

Example 4.1.3. Consider a set $A \subset \mathbb{R}$, and answer the following questions.

1. (2%) Define $\min A$.
2. (5%) Explain $\min A \in A \cap L(A)$, where $L(A)$ is the set of lower bounds of A.
3. (8%) Show that if $\min A$ exists, then $\inf A \in A$.

[from Chu, 2020, Midterm Q3]

4.2 Final Exam (Fall 2020)

Example 4.2.1. Please state whether each of the following statements is true or false. If your answer is true, give a brief proof or explaination to the statement. If your answer is false, you must provide a counter example, or explain why the statement is false.

1. (3%) If a set S is not closed, then S is open.
2. (3%) If $\sup A$ is finite, then $\max A$ exists and $\max A = \sup A$.
3. (3%) If $\sup A$ is infinite, then $\max A$ does not exist.

4. (3%) Suppose that a sequence is unbounded, then we cannot find any convergent subsequence from the sequence.

5. (3%) We find a sequence of points $\{x_k\}$ such that for each k, $x_k = \frac{1}{k} \in S$, and lim $\lim_{k\to\infty} x_k = 0 \in S$, so we've proved that the set S is compact.

6. (3%) Given a maximization problem with equality constraints, we check that the solution exists, solve all the critical points that meets the first-order equation of the Theorem of Lagrange, and verify the constraint qualification for each critical point. Then by comparing objective function at each critical points, we will find the global max.

7. (3%) Given a maximization problem with equality constraints, we find that the constraint qualification holds for every points in the constraint set, and we also find that there is no critical point satisfying the first-order equation of the Theorem of Largrange. Therefore, we can conclude that no solution exists.

[from Chu, 2020, Final Exam Q1]

Example 4.2.2. Consider the following coupled system of difference equations

$$x_{t+1} = 0.5x_t + 0.25y_t \tag{4.5}$$

$$y_{t+1} = 0.25x_t + 0.5y_t \tag{4.6}$$

with initial values x_0 and y_0.

1. (8%) Let

$$A = \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{pmatrix}. \tag{4.7}$$

   Find eigenvalues and eigenvectors of the matrix A.

2. (2%) Find the matrix P such that $P^{-1}AP \equiv D$, where D is a diagonal matrix with the form

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{4.8}$$

3. (2%) Characterize the solution to the coupled linear difference equation.

[from Chu, 2020, Final Exam Q2]

Example 4.2.3. (8%) Consider the following equation

$$y_t = c_t + i_t, \tag{4.9}$$

where (y^*, c^*, i^*) meets the above equation. Log-linearize the above equation around (y^*, c^*, i^*). Show that

$$\tilde{y}_t \approx \frac{c^*}{y^*}\tilde{c}_t + \frac{i^*}{y^*}\tilde{i}_t, \tag{4.10}$$

where

$$\tilde{c}_t \equiv \frac{c_t - c^*}{c^*}, \quad \tilde{i}_t \equiv \frac{i_t - i^*}{i^*}, \quad \text{and} \quad \tilde{y}_t \equiv \frac{y_t - y^*}{y^*}. \tag{4.11}$$

[from Chu, 2020, Final Exam Q3]

Example 4.2.4. Given the cost function

$$c(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2, \tag{4.12}$$

and consider the following competitive producer profit maximization problem

$$\max \quad f(x_1, x_2) \tag{4.13}$$

$$\text{s.t.} \quad \mathcal{D} = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0\}, \tag{4.14}$$

where $p_1 > 0$ and $p_2 > 0$. Please answer the following questions.

1. (3%) Show that the following condition (Inada condition) does not hold:

$$\lim_{x_1 \to 0} \frac{\partial f(x_1, x_2)}{\partial x_1} = \infty \quad \forall x_2 \quad \text{and} \quad \lim_{x_2 \to 0} \frac{\partial f(x_1, x_2)}{\partial x_2} = \infty \quad \forall x_1. \tag{4.15}$$

2. (3%) Show that $f(x_1, x_2)$ is a strictly concave function.

3. (2%) Set up the Lagrangian of the problem.

4. (8%) Write down all the Kuhn-Tucker first-order conditions.

5. (12%) Solve all critical points $(x_1, x_2, \lambda_1, \lambda_2)$ that meet the Kuhn-Tucker first-order conditions. Specify the parameter space such that each case exists. You also need to show why the case that $x_1 = 0$ and $x_2 = 0$ never meets the Kuhn-Tucker first-order conditions.

6. (2%) From the result you have in above question, is the local maximum (minimum) unique or are there multiple local maximum points?

7. (6%) We can actually conclude that the critical point that we find for each case is the global maximum, even without checking the existence of solution a prior or by comparing value at different critical points. Which Theorem(s) should we refer to? You need to write it (them) down and explain why the conditions in that theorem(s) are all met.

[from Chu, 2020, Final Exam Q4]

Example 4.2.5. Please read the following statement: Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$ converging to a limit x. Suppose that for every k, we have $x_k > a$, where $a = (a_1, a_2, \dots, a_n)$ is a fixed vector in $\mathbb{R}^n$. Then it is also the case that $x > a$. Now answer the following questions.

1. (4%) The above statement is not true. Please give a counter example. That is, find a sequence with $x_k > a$ for all k, but the limit is not greater than a.

2. (4%) Now I am going to "prove" the statement. Tell me why the below proof is incorrect.

   Suppose not. For every k, we have $x_k > a$, but there exists i such that $x^i \leq a^i$.

   $$x_k^i - a^i = (x_k^i - x^i) + (x^i - a^i). \tag{4.16}$$

   LHS is always positive. On the RHS, $x_k^i - x^i \to 0$ and $x^i - a^i \leq 0$. RHS will be less than or close to zero as k gets sufficiently large. We will find a contradiction.

[from Chu, 2020, Final Exam Q5]

Example 4.2.6. (15%) Compatify the constraint set for th expenditure minimization problem described in subsection 2.3.2 in Sundaram. Assume that the price vector p satisfies $p \gg 0$, and the utility function u is continuous on $\mathbb{R}^n_+$. You need to describe the new constraint set $\widehat{D(\overline{u})}$, and show that this set is compact.

[from Chu, 2020, Final Exam Q6]

Index

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