6. Mechanism Design II

ECON 7219 – Games With Incomplete Information

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Recap: Mechanism Design

Mechanism design:

- Implementing a social choice $g : \Theta \to \mathcal{X}$ over alternatives $\mathcal{X}$ when players' preferences over alternatives are private information in Θ_i.
- By revelation principle: may restrict attention to direct mechanisms, in which we simply ask participants to reveal their types/preferences.

Implementation:

- Bayesian implementation: truth-telling is a Bayesian Nash equilibrium.
- Dominant-strategy implementation: truth-telling is weakly dominant.

Dominant-strategy implementation is preferable:

- Players need not think about other participant's preferences to realize that truth-telling is an equilibrium.
- Players need to report only their payoff types (i.e., preferences over alternatives) and not their infinite hierarchy of beliefs.

Recap: Selling Mechanism

Setting:

- The set of alternatives $\mathcal{X} = \{0, 1, \ldots, n\} \times \mathbb{R}^n$ consists of:
  - An allocation $q \in \Delta(\{0, 1, \ldots, n\})$ of the good to players.
  - Payment or transfer p_i from each player i to the mechanism designer.

Optimal selling mechanism:

- Buyers receive an information rent to reveal their valuation.
- The buyer with the highest virtual valuation obtains the good.
- If buyers are symmetric, it is a second-price auction with reserve price.

Applications of the optimal mechanism:

- Revenue equivalence theorem.
- Quick way to find the unique symmetric increasing BNE for any auction format, in which the highest bid wins the auction.

Recap: Time Line of Direct Mechanisms

Ex-ante stage:

- Mechanism designer and players know the joint distribution of types, but players' types have not been realized yet.
- Mechanism designer designs the mechanism.

Interim stage:

- Players observe their type and decide whether or not to participate.
- Players decide which type to report.

Ex-post stage:

- Players' reports are publicly revealed.

Revenue maximizers: care about ex-ante expected revenue.

Benevolent designers: prefer ex-post criteria over ex-ante criteria.

Dominant-Strategy Mechanisms

Benefit of Second-Price Auction

Solving the second-price auction:

- You want to win the auction if and only if $\vartheta_i \geq \max_{j \neq i} b_j$.
- Bid b_i wins if $b_i \geq \max_{j \neq i} b_j$, hence we should bid $s_i(\vartheta_i) = \vartheta_i$.

Weakly dominant strategy:

- Note that $s_i(\vartheta_i) = \vartheta_i$ is a best response to any profile of bids b_{-i} by the opponents, that is, without knowing opponents' strategy profile.
- $s(\vartheta) = \vartheta$ is a Bayesian Nash equilibrium in weakly dominant strategies.
- This is cognitively much simpler for participants than solving for the unique symmetric increasing BNE of a first-price auction.

Which social choice functions can we implement in dominant strategies?

Implementation in Dominant Strategies

Definition 6.1

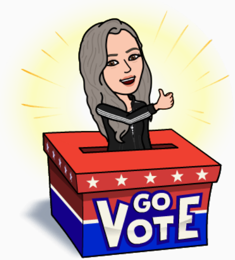
A mechanism $\Gamma = (\mathcal{S}_1, \ldots, \mathcal{S}_n, h)$ implements social choice function g in dominant strategies if there exists $s^* \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ such that:

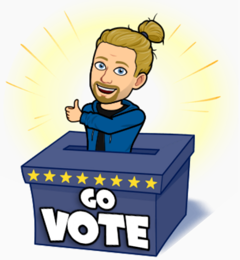
- $g(\vartheta(\tau)) = h(s^*(\tau))$ for every $\tau \in \mathcal{T}$,
- for each player i, every τ_i, every $s_{-i} \in \mathcal{S}_{-i}$ and every $s_i \in \mathcal{S}_i$,

$$u_i\big(h(s_i^*(\tau_i), s_{-i}), \vartheta_i(\tau_i)\big) \geq u_i\big(h(s_i, s_{-i}), \vartheta_i(\tau_i)\big).$$

- Beliefs about other players are not relevant.
- A player's utility depends on his type only through his payoff type ϑ_i.
- Revelation principle holds for dominant strategies: regardless of other player's reported preference, it is a best response to report truthfully.

Voting





Dominant-Strategy Voting:

- For mechanisms with many participants, it becomes increasingly demanding for players to figure out the Bayesian Nash equilibrium.
- In voting mechanisms, in particular, it would be desirable if there exists a welfare-maximizing dominant-strategy mechanism.

Voting Ballots

Voting ballots:

- There are finitely many alternatives $\mathcal{X} = \{x_1, \ldots, x_m\}$ to choose from.
- We can think of $\vartheta_i \in \Theta_i$ as a complete preference relation on $\mathcal{X}$.
- We can equivalently write $u_i(x_k, \vartheta_i) \geq u_i(x_\ell, \vartheta_i)$ or $x_k \succeq_{\vartheta_i} x_\ell$.

Strict preferences:

- A preference relation $\succ_{\vartheta_i}$ is strict if player i of type ϑ_i is not indifferent between any two alternatives x_k, x_ℓ, i.e., $x_k \succ_{\vartheta_i} x_\ell$ or $x_\ell \succ_{\vartheta_i} x_k$.
- In voting we are rarely indifferent between two candidates.

Vacation Destination

A	B	C	D	E
a	f	t	m	f
t	a	m	a	a
m	m	a	t	m
f	t	f	f	t

Vacation destination:

- Aaron, Blake, Cameron, Denise, and Eva are planning a vacation.
- Candidate destinations are Australia, France, Mexico, or Thailand.
- How should the friends aggregate their preferences?

Plurality: France

- Select the most frequently top-ranked option.
- In case of a tie, select from those the most frequently 2^{nd}-ranked, etc.

Vacation Destination

A	B	C	D	E
a	f	t	m	f
t	a	m	a	a
m	m	a	t	m
f	t	f	f	t

Ranked choice: Mexico

- Eliminate the option that is least frequently top-ranked.
- In case of a tie, eliminate from those the least frequently 2$^{\text{nd}}$-ranked, etc.
- Repeat the process until only one option remains.

Condorcet voting: Australia

- Compare all options bilaterally.
- Select the option that wins the most direct comparisons.

Implementability Through Preference Reversal

Lemma 6.2

A social choice function g is truthfully implementable in dominant strategies if and only if for any player i, any $\vartheta_{-i} \in \Theta_{-i}$, and any $\vartheta_i, \vartheta_i' \in \Theta_i$:

$$g(\vartheta_i, \vartheta_{-i}) \succeq_{\vartheta_i} g(\vartheta_i', \vartheta_{-i}) \quad \textit{and} \quad g(\vartheta_i', \vartheta_{-i}) \succeq_{\vartheta_i'} g(\vartheta_i, \vartheta_{-i})$$

- If player i's type changes from ϑ_i to ϑ_i', then his/her preference ranking over alternatives $g(\vartheta_i, \vartheta_{-i})$ and $g(\vartheta_i', \vartheta_{-i})$ must weakly reverse.

Proof:

- Fix a player i and preference ϑ_i. Preference reversal for any ϑ_i' and any ϑ_{-i} is equivalent to ϑ_i maximizing $u_i(g(\cdot, \vartheta_{-i}), \vartheta_i)$ for any ϑ_{-i}.
- g is implementable in dominant strategies, i.e., reporting ϑ_i is weakly dominant, if and only if ϑ_i maximizes $u_i(g(\cdot, \vartheta_{-i}), \vartheta_i)$ for any ϑ_{-i}.

Dictatorial Choice Functions

Definition 6.3

Consider a social choice function $g : \Theta \to \mathcal{X}$ is dictatorial on a subset $\mathcal{X}' \subseteq \mathcal{X}$ of alternatives if there is a player i such that for all $\vartheta \in \Theta$,

$$g(\vartheta) \in \{x \in \mathcal{X}' \mid x \succeq_{\vartheta_i} y \text{ for all } y \in \mathcal{X}'\}.$$

Interpretation:

- In a dictatorial choice function, there is one player (the dictator) whose favorite outcome is implemented for any report of preferences ϑ.
- Dictatorial choice functions are implementable in dominant strategies:
  - Truth-telling is weakly dominant for the dictator since his/her preferred choice from his/her report is implemented.
  - Truth-telling is weakly dominant for others since their report is ignored.

Gibbard Sattertwaithe Theorem

Theorem 6.4 (Gibbard-Sattertwaithe Theorem)

Suppose that $\mathcal{X}$ is finite, $g(\Theta)$ contains at least three elements, and each $\vartheta_i \in \Theta_i$ is a strict preference relation for every player i. Then g is truthfully implementable in dominant strategies if and only if it is dictatorial on $g(\Theta)$.

Interpretation:

- This is an impossibility result since dictatorial choice functions are trivial and quite often undesirable.
- To accomplish anything meaningful, we need to relax dominant-strategy implementation or allow indifference between alternatives.
- Monetary transfers (such as in selling mechanisms) are one way to break the Gibbard-Sattertwaithe theorem.

Step 1: Montonicity

Definition 6.5

1. Define the lower contour set $\mathcal{L}_i(x, \vartheta) := \{y \in \mathcal{X} \mid y \preceq_{\vartheta_i} x\}$.
2. Social choice function g is monotone if for any two profiles $\vartheta, \vartheta' \in \Theta$ with $g(\vartheta) = x$ and $\mathcal{L}_i(x, \vartheta) \subseteq \mathcal{L}_i(x, \vartheta')$ for each i, we have $g(\vartheta') = x$.

Lemma 6.6

Suppose that players have strict preferences on $\mathcal{X}$. If $g : \Theta \to \mathcal{X}$ is truthfully implementable in dominant strategies, then g is monotone.

Interpretation:

- If $g(\vartheta) = x$ incentivizes truthful reporting under ϑ and x is preferred to more alternatives under ϑ', then truthful reporting requires $g(\vartheta') = x$.

Monotonicity Visualized

1	2	3	$\cdots$	n
x	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\cdot$	y	x	$\cdots$	$\cdot$
$\cdot$	x	y	$\cdots$	x
y	$\cdot$	$\cdot$	$\cdots$	y

$g(\vartheta) = x$

1	2	3	$\cdots$	n
x	$\cdot$	x	$\cdots$	$\cdot$
$\cdot$	y	$\cdot$	$\cdots$	$\cdot$
$\cdot$	x	y	$\cdots$	x
y	$\cdot$	$\cdot$	$\cdots$	y

$g(\vartheta') = x$

1	2	3	$\cdots$	n
x	y	$\cdot$	$\cdots$	$\cdot$
$\cdot$	$\cdot$	x	$\cdots$	$\cdot$
y	x	$\cdot$	$\cdots$	x
$\cdot$	$\cdot$	y	$\cdots$	y

$g(\vartheta'') = x$

Fix a preference profile ϑ such that the social choice is $g(\vartheta) = x$.

Implications of monotonicity:

1. Let ϑ' be obtained from ϑ by moving x up in player i's preference order. Then $\mathcal{L}_i(x, \vartheta) \subseteq \mathcal{L}_i(x, \vartheta')$, hence monotonicity implies $g(\vartheta') = x$.
2. Let ϑ'' be obtained from ϑ by interchanging the order of i's preferences only above or below x. Then $\mathcal{L}_i(x, \vartheta) = \mathcal{L}_i(x, \vartheta'')$, hence $g(\vartheta'') = x$.

Proof of Lemma 6.6

Setup:

- Suppose g is truthfully implementable in dominant strategies.
- Fix $\vartheta, \vartheta' \in \Theta$ with $\mathcal{L}_i(g(\vartheta), \vartheta) \subseteq \mathcal{L}_i(g(\vartheta), \vartheta')$ for each player i.
- To show monotonicity we have to show $g(\vartheta') = g(\vartheta)$.

Proof:

- Truthful implementability implies that $g(\vartheta'_1, \vartheta_{-1}) \succeq_{\vartheta'_1} g(\vartheta)$ and

$$g(\vartheta'_1, \vartheta_{-1}) \in \mathcal{L}_1(g(\vartheta), \vartheta) \subseteq \mathcal{L}_1(g(\vartheta), \vartheta').$$

- The latter is equivalent to $g(\vartheta) \succeq_{\vartheta'_1} g(\vartheta'_1, \vartheta_{-1})$.
- Since preferences are strict, indifference implies $g(\vartheta) = g(\vartheta'_1, \vartheta_{-1})$.
- In the same way we get

$$g(\vartheta'_1, \vartheta_{-1}) = g(\vartheta'_1, \vartheta'_2, \vartheta_3, \ldots, \vartheta_n) = \ldots = g(\vartheta').$$

Step 2: Set Monotonicity

Definition 6.7

Social choice function g is set-monotone if for any set $\mathcal{X}' \subseteq g(\Theta)$ and any preference profiles $\vartheta, \vartheta' \in \Theta$ with $g(\vartheta) \in \mathcal{X}'$ and

$$x \succ_{\vartheta_i'} y \text{ and } y \succ_{\vartheta_i} x \text{ only if } x, y \in \mathcal{X}',$$

we must have $g(\vartheta') \in \mathcal{X}'$.

Interpretation:

- If preferences are reversed only among alternatives in $\mathcal{X}'$, then no alternative outside of $\mathcal{X}'$ can become better than $g(\vartheta)$.

Step 2: Set Monotonicity

Lemma 6.8

If a social choice function g is monotone, then it is set-monotone.

Setup of proof:

- Fix a set $\mathcal{X}' \subseteq g(\Theta)$, a preference profile ϑ with $g(\vartheta) \in \mathcal{X}'$, and a preference profile ϑ' that reverses preferences only $x, y \in \mathcal{X}'$.
- We have to show $g(\vartheta') \in \mathcal{X}'$.

Proof by contradiction:

- If $g(\vartheta') \notin \mathcal{X}'$, then $g(\vartheta') \succeq_{\vartheta_i} x$ if and only if $g(\vartheta') \succeq_{\vartheta_i'} x$.
- In particular, $\mathcal{L}_i(g(\vartheta'), \vartheta) = \mathcal{L}_i(g(\vartheta'), \vartheta')$ for each i.
- Monotonicity implies that $g(\vartheta') = g(\vartheta) \in \mathcal{X}'$, a contradiction.

Step 3: Unanimity

Definition 6.9

Social choice function g respects unanimity if for any $x, y \in g(\Theta)$, we have $g(\vartheta) \neq y$ for any ϑ with $x \succ_{\vartheta_i} y$ for every player i.

Lemma 6.10

Any monotone choice function g respects unanimity.

Interpretation:

- If everybody prefers x to y, then the social choice cannot be y.
- If everybody's first choice is x, the social choice must be x.

Step 3: Unanimity

1	2	3	⋯	n
x	·	x	⋯	·
·	y	·	⋯	·
·	x	y	⋯	x
y	·	·	⋯	y

$g(\vartheta_x) = x$

1	2	3	⋯	n
x	x	x	⋯	x
·	y	·	⋯	·
·	·	y	⋯	·
y	·	·	⋯	y

$g(\vartheta'_x) = x$

1	2	3	⋯	n
x	x	x	⋯	x
·	·	·	⋯	·
y	·	y	⋯	y
·	y	·	⋯	·

$g(\vartheta''_x) = x$

1	2	3	⋯	n
·	·	x	⋯	·
x	·	·	⋯	x
y	x	y	⋯	y
·	y	·	⋯	·

$g(\vartheta) \neq y$

Proof of Lemma 6.10:

- Fix ϑ with $x \succ_{\vartheta_i} y$ for every player i and fix $\vartheta_x \in \Theta$ with $g(\vartheta_x) = x$.
- Change ϑ_x to ϑ'_x by moving x to the top of everybody's preferences.
- Obtain ϑ''_x from ϑ'_x by rearranging choices below x to match ϑ.
- By monotonicity, we must have $g(\vartheta'_x) = g(\vartheta''_x) = x$.
- Finally, obtain ϑ by swapping x with $z \in \mathcal{X}'$ with $y \notin \mathcal{X}'$
- By set-monotonicity, $g(\vartheta) \in \{x\} \cup \mathcal{X}'$, hence $g(\vartheta) \neq y$.

Step 4: Existence of Local Dictator

Claim

For every alternative $x \in g(\Theta)$, there exists a player i_x such that $g(\vartheta) = x$ for any preference profile ϑ, for which $u_i(\cdot, \vartheta_{i_x})$ is maximized in x.

Proof setup:

- Fix any two alternatives $x, y \in g(\Theta)$.
- There must exist $\vartheta^x, \vartheta^y \in \Theta$ such that $x = g(\vartheta^x)$, $y = g(\vartheta^y)$.
- Denote by Θ_x, Θ_y the non-empty set of preference relations, under which every player ranks x and y at the top, respectively.
- Unanimity: $g(\vartheta) = x$ for any $\vartheta \in \Theta_x$ and $g(\vartheta) = y$ for any $\vartheta \in \Theta_y$.
- Let $\vartheta^0 \in \Theta_x$ be such that y is the least preferred choice of every player.

Step 4: Existence of Local Dictator

$$
\begin{array}{ccc}
1 & \cdots & n \\
\hline
x & \cdots & x \\
\cdot & \cdots & \cdot \\
\vdots & & \vdots \\
y & \cdots & y
\end{array}
\qquad
\begin{array}{ccccccc}
1 & \cdots & i_x - 1 & i_x & i_x + 1 & \cdots & n \\
\hline
y & \cdots & y & x & x & \cdots & x \\
x & \cdots & x & y & \cdot & \cdots & \cdot \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\cdot & \cdots & \cdot & \cdot & y & \cdots & y
\end{array}
\qquad
\begin{array}{ccccccc}
1 & \cdots & i_x - 1 & i_x & i_x + 1 & \cdots & n \\
\hline
y & \cdots & y & y & x & \cdots & x \\
x & \cdots & x & x & \cdot & \cdots & \cdot \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\cdot & \cdots & \cdot & \cdot & y & \cdots & y
\end{array}
$$

$$g(\vartheta^0) = x \qquad g(\vartheta^1) = x \qquad g(\vartheta^2) = y$$

Changing preferences:

- Let us change the preference profile in increasing order of players by moving up y in the preference order.
- As long as $x \succ y$, the social choice does not change by monotonicity.
- Once we interchange x and y, $g(\vartheta) \in \{x, y\}$ by set-monotonicity.
- Since $g(\vartheta) = y$ for $\vartheta \in \Theta_y$, there exists least player i_x, for which the social choice switches to y when $y \succ_{\vartheta^2} x$.

Step 4: Existence of Local Dictator

$$\begin{array}{ccccccc}
1 & \cdots & i_x - 1 & i_x & i_x + 1 & \cdots & n \\
\hline
y & \cdots & y & y & \cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot & x & \cdot & \cdots & \cdot \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\cdot & \cdots & \cdot & \cdot & x & \cdots & x \\
x & \cdots & x & \cdot & y & \cdots & y
\end{array}$$

$$g(\vartheta^3) = y$$

$$\begin{array}{ccccccc}
1 & \cdots & i_x - 1 & i_x & i_x + 1 & \cdots & n \\
\hline
y & \cdots & y & x & \cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot & y & \cdot & \cdots & \cdot \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\cdot & \cdots & \cdot & \cdot & x & \cdots & x \\
x & \cdots & x & \cdot & y & \cdots & y
\end{array}$$

$$g(\vartheta^4) = x$$

Changing preferences:

- Moving x all the way to the bottom for $i \neq i_x$ does not change the social choice by monotonicity, hence $g(\vartheta^3) = y$.
- Interchanging x and y for i_x implies $g(\vartheta^4) \in \{x, y\}$ by set-monotonicity.
- However, $g(\vartheta^4) = y$ would imply $g(\vartheta^1) = y$ by monotonicity.
- Therefore, we must have $g(\vartheta^4) = x$.

Step 4: Existence of Local Dictator

$$\begin{array}{ccccccc}
1 & \cdots & i_x - 1 & i_x & i_x + 1 & \cdots & n \\
\hline
\cdot & \cdots & \cdot & x & \cdot & \cdots & \cdot \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
z & \cdots & z & \cdot & z & \cdots & z \\
y & \cdots & y & z & x & \cdots & x \\
x & \cdots & x & y & y & \cdots & y
\end{array}$$

$$g(\vartheta^5) = x$$

$$\begin{array}{ccccccc}
1 & \cdots & i_x - 1 & i_x & i_x + 1 & \cdots & n \\
\hline
\cdot & \cdots & \cdot & x & \cdot & \cdots & \cdot \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
z & \cdots & z & \cdot & z & \cdots & z \\
y & \cdots & y & z & y & \cdots & y \\
x & \cdots & x & y & x & \cdots & x
\end{array}$$

$$g(\vartheta^6) = x$$

Changing preferences:

- By monotonicity, we must have $g(\vartheta^5) = x$.
- Since $g(\Theta)$ has at least three elements, there is $z \in g(\Theta) \setminus \{x, y\}$.
- Set-monotonicity implies $g(\vartheta^6) \in \{x, y\}$. However, $g(\vartheta^6) = y$ is impossible because g respects unanimity, hence $g(\vartheta^6) = x$.
- By monotonicity, $g(\vartheta) = x$ for any ϑ, for which i_x ranks x at the top.

Step 5: Existence of Supreme Dictator

Conclusion of proof:

- The claim shows that there exists a dictator i_x for any $x \in g(\Theta)$, that is, $g(\vartheta) = x$ for any ϑ such that $u_{i_x}(\,\cdot\,, \vartheta_{i_x})$ maximized in x.
- This includes ϑ for which $u_j(\,\cdot\,, \vartheta_j)$ is maximized in $y \in g(\Theta)$.
- Thus, no $j \neq i_x$ can be the dictator for $y \neq x$.
- Therefore, $i_x = i_y$ for any $y \in g(\Theta)$, hence i_x is the supreme dictator.

This concludes the proof of the Gibbard-Sattertwaithe Theorem.

Summary

Dominant-strategy implementation:

- Is preferable to Bayesian implementation because players to not have to take into account strategic considerations of others.
- Dominant-strategy implementation implies intuitive properties like monotonicity, set-monotonicity, and respecting unanimity.

Gibbard-Sattertwaithe theorem:

- Only dictatorial social choices can be implemented if players have strict preferences over at least 3 outcomes.
- The theorem does not apply to selling mechanisms (and other settings) because players may be indifferent between states.

Literature

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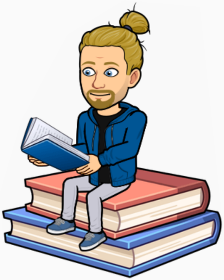


Quasi-Linear Preferences

Quasi-Linear Preferences

Adding monetary transfers:

- Set $\mathcal{X} = \mathcal{Q} \times \mathbb{R}^n$, where $\mathcal{Q}$ is a finite set of social states.
- Each alternative $x = (q, p_1, \ldots, p_n)$ consist of a social state q and transfer p_i from player i to the mechanism designer.
- Monetary transfers from i to j are incorporated via $p_i = -p_j$.
- Social choice $g = (q, p)$ consists of $q : \Theta \to \Delta(\mathcal{Q})$ and $p : \Theta \to \mathbb{R}^n$.

Quasi-linear utilities:

- Player i's utility function is quasi-linear if

$$u_i(x, \vartheta_i) = v_i(q, \vartheta_i) - p_i.$$

- Utilities are linear and additively separable in money.
- Function $v_i(q, \vartheta_i)$ is player i's money-equivalent of social state q.

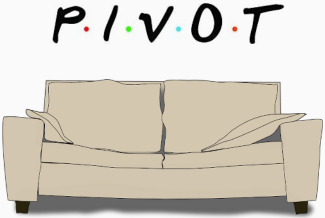
Vacation Destination

	A	B	C	D	E
Australia	3	2	2	3	1
France	0	4	2	3	1
Mexico	3	3	1	0	2
Thailand	2	4	2	1	1

Vacation destination:

- Aaron, Blake, Cameron, Denise, and Eva are planning a vacation.
- Candidate destinations are Australia, Mexico, France, or Thailand.
- Going on a vacation certainly has a money-equivalent: how much are you willing to spend on a vacation to destination X.

Roommate Problem



Buying a new couch:

- Cost of the new couch is \$15,000.
- Alan, Britt, Cedric, and Diane each value having the new couch at

$$\vartheta_A = \$6{,}000, \quad \vartheta_B = \$5{,}500, \quad \vartheta_C = \$5{,}000, \quad \vartheta_D = \$2{,}000,$$

  drawn independently and uniformly from $[\$1{,}000, \$7{,}000]$.
- How can we elicit truthful reporting of roommate's values?

Efficiency

Lemma 6.11

In any ex-post efficient alternative $x^* = (q^*, p_1^*, \dots, p_n^*)$, the social state q^* maximizes $\sum_{i=1}^n v_i(q, \vartheta_i)$. Such a social state is called ex-post efficient.

Proof by contradiction:

- Fix preferences ϑ and suppose that x^* is ex-post efficient but that there exists $\widetilde{q}$ with $\sum_{i=1}^n v_i(\widetilde{q}, \vartheta_i) > \sum_{i=1}^n v_i(q^*, \vartheta_i)$.
- Define the transfers

$$\widetilde{p}_i := p_i^* - \big(v_i(q^*, \vartheta_i) - v_i(\widetilde{q}, \vartheta_i)\big) - \frac{1}{n}\sum_{i=1}^n \big(v_i(\widetilde{q}, \vartheta_i) - v_i(q^*, \vartheta_i)\big).$$

- Then $(\widetilde{q}, \widetilde{p}_1, \dots, \widetilde{p}_n)$ is a Pareto improvement since

$$v_i(\widetilde{q}_i, \vartheta_i) - \widetilde{p}_i = v_i(q_i^*, \vartheta_i) - p_i^* + \frac{1}{n}\sum_{i=1}^n \big(v_i(\widetilde{q}, \vartheta_i) - v_i(q^*, \vartheta_i)\big).$$

Implementing Ex-Post Efficient States

Realizing Pareto improvements:

- Suppose we start with any social state $q(\vartheta)$.
- For any $\widetilde{q}(\vartheta)$ with higher social surplus than $q(\vartheta)$, some player must be willing to compensate the others for choosing $\widetilde{q}(\vartheta)$ instead of $q(\vartheta)$.
- We can iterate this procedure until we reach an ex-post efficient $q^*(\vartheta)$.

Top-down approach:

- Start directly with ex-post efficient $q(\vartheta)$.
- If player i's preferences were ignored, the others would implement social state $\widehat{q}_i(\vartheta_{-i})$ that maximizes $\sum_{j\neq i} v_j(q, \vartheta_j)$.
- Thus, player i is willing to make payments $v_i(q(\vartheta), \vartheta_i) - v_i(\widehat{q}_i(\vartheta_{-i}), \vartheta_i)$.
- This payment is positive only if $q(\vartheta) \neq \widehat{q}_i(\vartheta_i)$, that is, if player i is pivotal for the social choice.

Pivot Mechanism

Definition 6.12

A pivot mechanism is a direct mechanism $\Gamma = (\mathcal{T}_1, \ldots, \mathcal{T}_n, (q, p))$ such that $q(\vartheta(\tau))$ is ex-post efficient and

$$p_i^{\text{piv}}(\vartheta) := \sum_{j \neq i} v_j(\widehat{q}_i(\vartheta_{-i}), \vartheta_j) - \sum_{j \neq i} v_j(q(\vartheta), \vartheta_j),$$

where, for every player i, $\widehat{q}_i : \Theta_{-i} \to \mathcal{Q}$ is an ex-post efficient allocation in the society without i, i.e., it maximizes $\sum_{j \neq i} v_j(q, \vartheta_j)$ among all $q \in \mathcal{Q}$.

Idea behind payments:

- Every player i pays the externality he/she imposes on others.
- The payments align social preferences with individual preferences.
- Player i's payment is at most $v_i(q(\vartheta), \vartheta_i) - v_i(\widehat{q}_i(\vartheta_{-i}), \vartheta_i)$.

Truth-Telling in Pivot Mechanism

Proposition 6.13

A pivot mechanism is dominant-strategy incentive-compatible.

Proof:

- Suppose player i with type ϑ_i reports r_i and players $-i$ report ϑ_{-i}.
- Specific form of payments and ex-post efficiency of $g(\vartheta)$ imply that

$$\begin{aligned} v_i(q(r_i,\vartheta_{-i}),\vartheta_i) - p_i^{\text{piv}}(r_i,\vartheta_{-i}) &= \sum_{j=1}^{n} v_j(q(r_i,\vartheta_{-i}),\vartheta_j) - \sum_{j\neq i} v_j(\widehat{q}_i(\vartheta_{-i}),\vartheta_j) \\ &\leq \sum_{j=1}^{n} v_j(q(\vartheta_i,\vartheta_{-i}),\vartheta_j) - \sum_{j\neq i} v_j(\widehat{q}_i(\vartheta_{-i}),\vartheta_j). \end{aligned}$$

- Since this holds independently of whether ϑ_{-i} is a truthful report or not, reporting truthfully is weakly dominant for player i.

Payments in Pivot Mechanism

Pivots:

- Player i is pivotal for social state q at ϑ if $q(\vartheta) = q$ but $\widehat{q}_i(\vartheta_{-i}) \neq q$.
- If i is not pivotal for $q(\vartheta)$, then $p_i^{piv}(\vartheta) = 0$.

Payments:

- Payments satisfy $0 \leq p_i^{\text{piv}}(\vartheta) \leq v_i(q(\vartheta), \vartheta_i) - v_i(\widehat{q}_i(\vartheta_{-i}), \vartheta_i)$.
- Each pivotal player pays his externality and is happy to do so.
- If social states are costless, the mechanism designer never runs a deficit.

Individual rationality:

- If we view $\min_q v_i(q, \vartheta_i)$ as player i's outside option, then the pivot mechanism is ex-post individually rational because

$$u_i(q(\vartheta), \vartheta_i) = v_i(q(\vartheta), \vartheta_i) - p_i^{\text{piv}}(\vartheta) \geq v_i(\widehat{q}_i(\vartheta_{-i}), \vartheta_i) \geq \min_q v_i(q, \vartheta_i).$$

Second-Price Auction is a Pivot Mechanism

Symmetric second-price auction (without reserve price):

- Note that the social state is the allocation of the good.
- Bidder i with $\vartheta_i = \max_j \vartheta_j$ wins the auction.
- No bidder $j \neq i$ is pivotal at ϑ, hence $p_j(\vartheta) = 0$ for $j \neq i$.
- In absence of bidder i, the second-highest bidder j would win.
- Winner i imposes externality $v_j(\widehat{q}_i(\vartheta), \vartheta_j) = \vartheta_j$ on j, hence i pays ϑ_j.

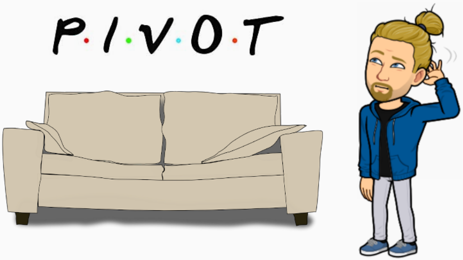
Vacation Destination

	A	B	C	D	E
Australia	3	2	2	3	1
France	0	4	2	3	1
Mexico	3	3	1	0	2
Thailand	2	4	2	1	1

Vacation destination:

- Aaron, Blake, Cameron, Denise, and Eva are planning a vacation.
- Candidate destinations are Australia, Mexico, France, or Thailand.
- Suppose that players' types are independent and that the common prior places a uniform value among $\{0, 1, 2, 3, 4\}$ for each destination.
- What are the pivot payments in the above setting?

Roommate Problem



Buying a new couch:

- Buying the couch imposes a social cost $c = \$15{,}000$.
- Alan, Britt, Cedric, and Diane each value having a new couch at

$$\vartheta_A = \$6{,}000, \quad \vartheta_B = \$5{,}500, \quad \vartheta_C = \$5{,}000, \quad \vartheta_D = \$2{,}000,$$

  drawn independently and uniformly from $[\$1{,}000, \$7{,}000]$.
- What are the payments in the pivot mechanism for above values?

Vickrey-Clarke-Groves Mechanism

Dealing with a Surplus or Deficit

What do we do with a surplus?

- Problem: returning money to players may distort incentives.
- Destroying the surplus is not efficient and it may be illegal.
- In the selling mechanism, the "surplus" goes to the seller. This causes no inefficiency because it is simply a transfer.

Can we charge players to overcome a deficit?

- Problem: additional charge may distort incentives.
- In particular, players may not be willing to participate.

Goal: take the redistribution into account from the beginning.

Budget Balance

Definition 6.14

A mechanism $\Gamma = (\mathcal{T}_1, \ldots, \mathcal{T}_n, (q, p))$ is

1. Ex-post budget balanced if $\sum_{i=1}^n p_i(\vartheta) = 0$ for every $\vartheta \in \Theta$.
2. Ex-ante budget balanced if $\sum_{i=1}^n \mathbb{E}[p_i(\theta)] = 0$.

Lemma 6.15

A mechanism $\Gamma = (\mathcal{T}_1, \ldots, \mathcal{T}_n, (q, p))$ is ex-post efficient if and only if it is ex-post budget-balanced and $q(\vartheta)$ is ex-post efficient

- Second-price auction is ex-post budget balanced if we add the seller as player 0 and set $p_0(\vartheta) = -\sum_{i=1}^n p_i(\vartheta)$.
- The second-price auction without reserve price is ex-post efficient.

Vickrey-Clarke-Groves Mechanism

Definition 6.16

A Vickrey-Clarke-Groves mechanism (or VCG mechanism) is a direct mechanism $\Gamma = (\mathcal{T}_1, \ldots, \mathcal{T}_n, (q, p))$ such that $q(\vartheta)$ is ex-post efficient and

$$p_i(\vartheta) = h_i(\vartheta_{-i}) - \sum_{j \neq i} v_j(q(\vartheta), \vartheta_j). \tag{1}$$

for every player i, where $h_i : \Theta_{-i} \to \mathbb{R}$ does not depend on i's valuation.

Remark:

- Pivot mechanism is the special case $h_i(\vartheta_{-i}) = \sum_{j \neq i} v_j(\widehat{q}_i(\vartheta_{-i}), \vartheta_j)$.
- Second term in (1) aligns social preferences with individual preferences.
- First term in (1) allows us to adjust payments and, hence, the surplus, without affecting incentives for truthful reporting.

Dominant-Strategy Implementability

Definition 6.17

A VCG mechanism is dominant-strategy incentive compatible.

Proof: since the term $h_i(\vartheta_{-i})$ does not affect player i's incentives, the proof is analogous to the pivot mechanism.

History:

- Vickrey (1961) derived the mechanism for auctions, which is why second-price auctions are also called Vickrey auctions.
- Clarke (1971) derived the pivot mechanism.
- Groves (1973) derived the general case.

The VCG-mechanism is an extension of the second-price auction.

What Makes VCG Mechanisms Special?

Remark

In many settings, VCG mechanisms are the only dominant-strategy incentive-compatible mechanisms with an ex-post efficient social state.

Examples:

- We show in Theorem 6.24 that this is true if Θ_i is one-dimensional.
- Green and Laffont (1979) show that this is true if the type space is sufficiently rich.[1]
- Krishna and Maenner (2001) show that this is true if Θ_i is a convex subset of Euclidean space and $v_i(q, \vartheta_i)$ is convex in ϑ_i.

[1] A type space is "rich" if for every utility function $\widehat{v}_i$ representing i's preferences over $\mathcal{Q}$, there exists $\vartheta_i \in \Theta_i$ with $\widehat{v}_i(q) = v_i(q, \vartheta_i)$.

Individual Rationality

Incentives in VCG mechanism:

- Adjusting $h(\vartheta_{-i})$ does not affect incentives for truthful reporting, but it may affect incentives to participate in the mechanism.
- Recall that a mechanism is interim individually rational with outside options $IR_i : \mathcal{T}_i \to \mathbb{R}$ if for every player i and every $\tau_i \in \mathcal{T}_i$,

$$\mathbb{E}_{\tau_i}[u_i(g(\theta), \vartheta_i(\tau_i)] \geq IR(\tau_i).$$

Participation subsidy:

- With quasi-linear utilities, giving a participation subsidy

$$\varphi_i = \max_{\tau_i \in \mathcal{T}_i} \big(IR_i(\tau_i) - \mathbb{E}_{\tau_i}[u_i(g(\theta), \vartheta_i(\tau_i))] \big)$$

  guarantees that i has incentive to participate for any type $\tau_i \in \mathcal{T}_i$.
- Note that the participation subsidy could be negative.

IR-VCG Mechanism

Definition 6.18

The individually rational VCG mechanism (or IR-VCG mechanism) with ex-post efficient social state $q(\vartheta)$ and outside options IR_i has payments

$$p_i^{\mathsf{IR}}(\vartheta) = p_i^{\mathsf{piv}}(\vartheta) - \varphi_i^{\mathsf{piv}},$$

with $\varphi_i^{\mathsf{piv}} = \max_{\tau_i \in \mathcal{T}_i}\big(IR_i(\tau_i) - \mathbb{E}_{\tau_i}\big[u_i(g^{\mathsf{piv}}(\theta), \vartheta_i(\tau_i))\big]\big)$.

Remark:

- The IR-VCG mechanism is dominant-strategy incentive-compatible because the participation subsidy does not depend on the reported type.
- If $IR_i(\vartheta_i) = v_i(\widehat{q}_i(\vartheta_{-i}), \vartheta_i)$, then $\varphi_i^{\mathsf{piv}} \leq 0$ as we have seen on slide 34.

Vacation Destination

	A	B	C	D	E
Australia	3	2	2	3	1
France	0	4	2	3	1
Mexico	3	3	1	0	2
Thailand	2	4	2	1	1

Vacation destination:

- Aaron, Blake, Cameron, Denise, and Eva are planning a vacation.
- Suppose that players' types are independent and that the common prior places a uniform value among $\{0, 1, 2, 3, 4\}$ for each destination.
- Suppose that Blake is currently very busy with work so that his outside option is $IR_B(\vartheta) = 2$ for any ϑ. What is the IR-VCG mechanism?

Property Rights

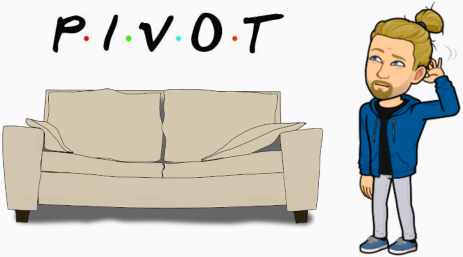
Definition 6.19

Player i has property rights over social state $q_* \in \mathcal{Q}$ if q_* is not available without i's participation. This can be incorporated by imposing an ex-post individual rationality constraint with $IR_i(\vartheta)$ for all ϑ with $q(\vartheta) = q_*$.

Examples:

- In a selling mechanism, the seller has property rights over the good.
- In a procurement auction, sellers need to be paid for their services.
- The roommate who owns the old couch needs to agree to get rid of it.

Roommate Problem



Buying a new couch:

- Buying the couch imposes a social cost $c = \$15{,}000$.
- Suppose that Alan, Britt, and Cedric each value having the new couch at \$2,000, \$4,000, and \$6,000 with equal probability.
- Suppose that the old couch belongs to Cedric, who values it at \$3,000.
- What is the IR-VCG mechanism with Cedric's property rights?

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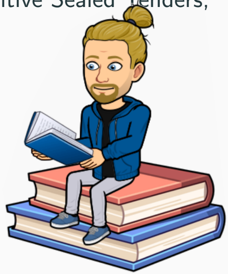


One-Dimensional Types

Players' Types in Selling Mechanisms

Incentive compatibility in selling mechanisms:

- The expected probability $\bar{q}_i(\vartheta_i)$ of receiving the good must be non-decreasing in player i's valuation ϑ_i in any direct selling mechanism.
- This gave rise to a very nice revenue equivalence result.
- Mathematically, this characterization requires one-dimensional types.

What does one-dimensionality mean?

- There is an ordering of types such that "higher types" attach a strictly higher utility to the state q_i of receiving the good.
- If the order is complete (any two types are comparable), then we can re-organize the types according to their utility.

General social states: with respect to which state do we order the types?

One-Dimensional Types

Definition 6.20

Suppose player i has a complete, transitive preference relation $\succeq_i$ over social states $\mathcal{Q}$. For any two $\vartheta_i, \vartheta_i' \in \Theta_i$, we say ϑ_i is a higher type than ϑ_i' with respect to $\succeq_i$ if for any $q, q' \in \mathcal{Q}$ with $q \succ_i q'$, we have[1]

$$v_i(q, \vartheta_i) - v_i(q', \vartheta_i) > v_i(q, \vartheta_i') - v_i(q', \vartheta_i'), \tag{2}$$

and for any $q, q' \in \mathcal{Q}$ with $q \approx_i q'$, we have[2]

$$v_i(q, \vartheta_i) - v_i(q', \vartheta_i) = v_i(q, \vartheta_i') - v_i(q', \vartheta_i') = 0.$$

We also write $\vartheta_i \succ_i \vartheta_i'$ if ϑ_i is a higher type than ϑ_i'.

Interpretation:

- The marginal gain of a higher social state is larger for higher types.
- For types ranked according to $\succ_i$, v_i has increasing differences.

[2] $\approx_i$ and $\succ_i$ are derived from $\succeq_i$ by $q \approx_i q'$ if $q \succeq_i q'$ and $q' \succeq_i q$ and $q \succ_i q'$ if $q \succeq_i q'$ and $q' \not\succeq_i q$.

One-Dimensional Types

Definition 6.21

Player i's type space Θ_i is one-dimensional if there exists a complete, transitive preference relation $\succeq_i$ over social states $\mathcal{Q}$ such that the induced order on Θ_i is complete.[3]

Interpretation:

- Fix any two alternatives $q, q' \in \mathcal{Q}$ with $q \succ_i q'$.
- We can assign to any type a real number $r_i(\vartheta_i) := v_i(q, \vartheta_i) - v_i(q', \vartheta_i)$, indicating ϑ_i's marginal utility of a change from q' to q.
- Because $\succ_i$ is a strict order on Θ_i, the map $r_i : \Theta_i \to \mathbb{R}$ is injective.
- The map r_i is an embedding of Θ_i into $\mathbb{R}$.

[3] Note that, in general, the order $\succ_i$ on Θ_i is typically incomplete.

Comparison with Auctions

Preference relation on $\mathcal{Q}$:

- Buyer i strictly prefers social state q_i, in which i receives the good, to any other social state q, that is, $q_i \succ_i q$.
- Buyer i is indifferent between $q, q' \in \mathcal{Q} \setminus \{q_i\}$, that is, $q \approx q'$.

Induced preference relation on Θ_i:

- For any $q \in \mathcal{Q} \setminus \{q_i\}$ and $\vartheta_i \in \Theta_i$, we have $v_i(q_i, \vartheta_i) - v_i(q, \vartheta_i) = \vartheta_i$.
- For any two $q, q' \in \mathcal{Q} \setminus \{q_i\}$, we have $v_i(q, \vartheta_i) = v_i(q', \vartheta_i) = 0$.
- Therefore, $\vartheta_i \succ_i \vartheta_i'$ if and only if $\vartheta_i > \vartheta_i'$.

Embedding into $\mathbb{R}$:

- Any such embedding assigns value $r_i(\vartheta_i) = v_i(q_i, \vartheta_i) = \vartheta_i$.

Dominant-Strategy Implementability

Lemma 6.22

Suppose that for each player i, there exists a preference relation $\succeq_i$ over $\mathcal{Q}$, with respect to which Θ_i one-dimensional. Then there exist payments $p : \Theta \to \mathbb{R}^n$ such that (q, p) is dominant-strategy implementable if and only if for any $\vartheta, \vartheta' \in \Theta$ with $\vartheta_i \succ_i \vartheta'_i$, we have $q(\vartheta) \succeq_i q(\vartheta')$.

Interpretation:

- We say that such a choice q is monotone with respect to $\succeq_i$.
- This is the equivalent of statement (i) of Lemma 6.11 for arbitrary social states and one-dimensional type spaces.

Proof of Necessity

Similarly to the proof of Lemma 6.11:

- Suppose that (q, p) is dominant-strategy implementable.
- This implies that for any $\vartheta \in \Theta$ and $r_i \in \Theta_i$, we have

$$\begin{aligned} u_i(r_i, \vartheta_i) \leq u_i(\vartheta_i, \vartheta_i) &= u_i(\vartheta_i, r_i) + v_i(q(\vartheta), \vartheta_i) - v_i(q(\vartheta), r_i) \\ &\leq u_i(r_i, r_i) + v_i(q(\vartheta), \vartheta_i) - v_i(q(\vartheta), r_i). \end{aligned}$$

- Subtracting $u_i(r_i, \vartheta_i)$ on both sides yields

$$v_i(q(\vartheta), \vartheta_i) - v_i(q(r_i, \vartheta_{-i}), \vartheta_i) \geq v_i(q(\vartheta), r_i) - v_i(q(r_i, \vartheta_{-i}), r_i). \quad (3)$$

- Suppose towards a contradiction that $r_i \succ_i \vartheta_i$, but $q(\vartheta) \succ_i q(r_i, \vartheta_{-i})$.
- Then (3) contradicts the increasing difference property (2).
- Multiplying (3) with -1 and repeating this step for the case $\vartheta_i \succ_i r_i$ shows that q is monotone with respect to $\succeq_i$.

Proof of Sufficiency

Setup and trivial case:

- Fix a player i, a preference relation $\succeq_i$ over $\mathcal{Q}$, and a report ϑ_{-i}.
- Let $\mathcal{Q}^1, \ldots, \mathcal{Q}^m$ be a partition of $\mathcal{Q}$ such that $q \approx_i q'$ for any $q, q' \in \mathcal{Q}^k$ as well as $q \succ_i q'$ for any $q \in \mathcal{Q}^k, q' \in \mathcal{Q}^\ell$ with $k > \ell$.
- If $m = 1$, then i's report does not affect i's preference over social states, hence truthful reporting is weakly dominant.

Partition of Player i's types:

- If $m \geq 2$, define $\Theta_i^k := \left\{\vartheta_i \in \Theta_i \;\middle|\; q(\vartheta_i, \vartheta_{-i}) \in \mathcal{Q}^k\right\}$.
- One-dimensionality and monotonicity with respect to $\succeq_i$ imply that $\vartheta_i \succ_i \vartheta_i'$ for any $\vartheta_i \in \Theta_i^k$, $\vartheta_i' \in \Theta_i^\ell$ with $k > \ell$.
- Thus, types in Θ_i^k are higher than types in Θ_i^ℓ for $k > \ell$.

Proof of Sufficiency

Separating payments:

- For each k and $\ell < k$, choose any $q \in \mathcal{Q}^k$ and $q' \in \mathcal{Q}^{k-1}$ and set

$$p_i^k := \inf_{\vartheta_i \in \Theta_i^k} \big(v_i(q, \vartheta_i) - v_i(q', \vartheta_i)\big) \geq \sup_{\vartheta_i \in \Theta_i^\ell} \big(v_i(q, \vartheta_i) - v_i(q', \vartheta_i)\big) \geq 0.$$

- Note that p_i^k is non-negative since alternative q is higher than q'.
- Any type in Θ_i^k is willing to pay p_i^k for an outcome in $\mathcal{Q}^k$ over $\mathcal{Q}^{k-1}$.
- The ranking of types implies that for any $q \in \mathcal{Q}^\ell$, $q' \in \mathcal{Q}^{\ell-1}$, and $k > \ell$,

$$\inf_{\vartheta_i \in \Theta_i^k} \big(v_i(q, \vartheta_i) - v_i(q', \vartheta_i)\big) \geq \inf_{\vartheta_i \in \Theta_i^\ell} \big(v_i(q, \vartheta_i) - v_i(q', \vartheta_i)\big) = p_i^\ell.$$

- Thus, a type in Θ_i^k is willing to pay $p_k + p_{k-1}$ for an outcome in $\mathcal{Q}^k$ over $\mathcal{Q}^{k-2}$ or to pay $\sum_{j=\ell+1}^k p_j$ for an outcome in $\mathcal{Q}^k$ over $\mathcal{Q}^\ell$

Proof of Sufficiency

Payments:

- For any $\vartheta_i \in \Theta_i^k$, define the transfers $p_i(\vartheta) := \sum_{k=2}^{\ell} p_i^k$.
- The argument on the previous slide shows that a type $\vartheta_i \in \Theta_i^k$ has no incentive to report a lower type.
- Suppose that type ϑ_i reports a higher type $\vartheta_i' \in \Theta_i^\ell$ with $\ell > k$.
- For $j = k, \ldots, \ell$, let q^j be any element of $\mathcal{Q}^j$. Then

$$
\begin{aligned}
u_i(q(\vartheta_i', \vartheta_{-i}), \vartheta_i) - u_i(q(\vartheta), \vartheta_i) &= v_i(q^\ell, \vartheta_i) - v_i(q^k, \vartheta_i) - \sum_{j=k+1}^{\ell} p_i^j \\
&= \sum_{j=k+1}^{\ell} \big(\underbrace{v_i(q^j, \vartheta_i) - v_i(q^{j-1}, \vartheta_i) - p_i^j}_{\leq 0}\big).
\end{aligned}
$$

- Reporting a different type in Θ_i^k has no impact on the social choice, hence truthful reporting is weakly dominant.

Revenue Equivalence

Lemma 6.23

Suppose that the following conditions hold:

1. *Set $\mathcal{Q}$ of social states is finite.*
2. *Θ_i is one-dimensional and convex for each player i, i.e., $\Theta_i = [\underline{\vartheta}_i, \bar{\vartheta}_i]$ and $v_i(q, \vartheta_i)$ is non-decreasing in ϑ_i for each $q \in \mathcal{Q}$.*
3. *$v_i(q, \vartheta_i)$ is absolutely continuous in ϑ_i for each $q \in \mathcal{Q}$.*

For any dominant-strategy mechanism Γ, let Q denote the random variable implementing $q : \Theta \to \Delta(\mathcal{Q})$. Then payments p in Γ are equal to

$$p_i(\vartheta_i, \vartheta_{-i}) = p_i(\underline{\vartheta}_i, \vartheta_{-i}) + \mathbb{E}_{\vartheta_i}[v_i(Q, \vartheta_i)] - \mathbb{E}_{\underline{\vartheta}_i}[v_i(Q, \underline{\vartheta}_i)] - \sum_{q \in \mathcal{Q}} \int_{\underline{\vartheta}_i}^{\vartheta_i} \frac{\partial v_i(q, x)}{\partial x} P_x(Q = q)\, \mathrm{d}x. \tag{4}$$

Discussion of Lemma 6.23

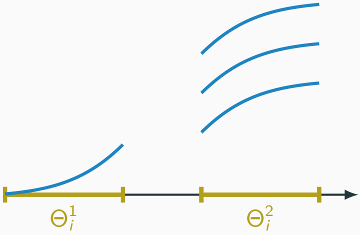
Comparison to Lemma 6.11:

- Lemma 6.22 and Lemma 6.23 are generalizations of statements (i) and (ii) of Lemma 6.11, respectively.
- Indeed, payments are determined by q and payment of the lowest type.

Dominant-strategy implementability:

- Imposing dominant-strategy implementability means that truthful reporting holds for all reports ϑ_{-i}, hence (4) has to hold for all ϑ_{-i}.
- If we replace dominant-strategy implementation with Bayesian implementation, $p_i(\vartheta_i, \vartheta_i)$ is replaced with $\bar{p}_i(\vartheta_i) := \mathbb{E}_{\vartheta_i}[p_i(\theta)]$ in (4).

Discussion of Lemma 6.23



Necessity of assumptions:

- The revenue equivalence was established by integrating over the player's marginal utility, hence we need absolute continuity of v_i.
- If Θ_i was not interval, then payments would be unique only up to payments of the lowest type in each connected component of Θ_i.
- While the payment of the lowest type is determined by the participation constraint, payments in other connected components are not.

Proof of Lemma 6.23

Proof setup:

- Fix any dominant-strategy incentive compatible mechanism Γ.
- Let Q denote the random variable realizing the choice $q : \Theta \to \Delta(\mathcal{Q})$.
- Since v_i is non-decreasing, it has a weak derivative. Since v_i is absolutely continuous, v_i is the antiderivative of its weak derivative.

Integration by parts:

- Analogous to the proof statement (ii) of Lemma 6.11, it follows that

$$p_i(\vartheta_i, \vartheta_{-i}) = p_i(\underline{\vartheta}_i, \vartheta_{-i}) + \mathbb{E}_{\vartheta_i}[v_i(Q, \vartheta_i)] - \mathbb{E}_{\underline{\vartheta}_i}[v_i(Q, \underline{\vartheta}_i)] - \sum_{q \in \mathcal{Q}} \int_{\underline{\vartheta}_i}^{\vartheta_i} \frac{\partial v_i(q, x)}{\partial x} P_x(Q = q) \, \mathrm{d}x.$$

VCG Mechanisms

Theorem 6.24

Suppose that the conditions of Lemma 6.23 are satisfied. Then:

1. *Any dominant-strategy incentive-compatible mechanism implementing an ex-post efficient social state is a VCG mechanism.*
2. *The IR-VCG mechanism implementing ex-post efficient q maximizes the ex-ante expected surplus among all incentive-compatible and individually rational mechanisms that implement q.*

Interpretation:

- If we insist on implementing an ex-post efficient social state (and types are one-dimensional), then IR-VCG mechanisms are optimal:
  - They maximize the expected surplus.
  - They are dominant-strategy implementable.

Proof of Theorem 6.24

Statement 1:

- Fix such a mechanism implementing (q, p). Lemma 6.23 implies that payments are determined uniquely up to $p_i(\underline{\vartheta}_i, \vartheta_{-i})$.
- Any VCG mechanism implementing $(q, \widetilde{p})$ satisfies (4), hence

$$\begin{aligned} p_i(\vartheta) &= \widetilde{p}_i(\vartheta) - \widetilde{p}_i(\underline{\vartheta}_i, \vartheta_{-i}) + p_i(\underline{\vartheta}_i, \vartheta_{-i}) \\ &= \widetilde{h}_i(\vartheta_{-i}) - \widetilde{p}_i(\underline{\vartheta}_i, \vartheta_{-i}) + p_i(\underline{\vartheta}_i, \vartheta_{-i}) - \sum_{j \neq i} v_j(q(\vartheta), \vartheta_j). \end{aligned}$$

- Therefore, p_i is a VCG payment with

$$h_i(\vartheta_{-i}) = \widetilde{h}_i(\vartheta_{-i}) - \widetilde{p}_i(\underline{\vartheta}_i, \vartheta_{-i}) + p_i(\underline{\vartheta}_i, \vartheta_{-i}).$$

Proof of Theorem 6.24

Statement 2:

- Fix q and let p_i^{IR} denote the payments of the IR-VCG mechanism.
- p_i^{IR} satisfies (4) pointwise, hence also in expectation.
- For any incentive compatible mechanism implementing (q, p), Lemma 6.23 implies that $\bar{p}_i(\vartheta_i) = \bar{p}_i^{\mathrm{IR}}(\vartheta_i) + c_i = \bar{p}_i^{\mathrm{piv}}(\vartheta_i) - \varphi_i^{\mathrm{piv}} + c_i$.
- Since φ_i^{piv} is the smallest participation subsidy that makes pivot payments individually rational, we get $c_i \leq 0$ and $\bar{p}_i(\vartheta_i) \leq \bar{p}_i^{\mathrm{IR}}(\vartheta_i)$.
- The ex-ante expected surplus

$$S = \sum_{i=1}^{n} \int_{\underline{\vartheta}_i}^{\bar{\vartheta}_i} \bar{p}_i(\vartheta_i) f_i(\vartheta_i)\, \mathrm{d}\vartheta_i$$

is thus maximized in the IR-VCG mechanism.

Uniqueness

Corollary 6.25

Suppose that the conditions of Lemma 6.23 are satisfied, as well as:

1. *Each player's type θ_i admits a positive density $f_i(\vartheta_i)$ on $[\underline{\vartheta}_i, \bar{\vartheta}_i]$.*
2. *The ex-post efficient social state is unique for almost every $\vartheta \in \Theta$.*

Then the expected surplus of any IR-VCG mechanism is unique.

Implication:

- These conditions are fairly often satisfied in applied work.
- Corollary 6.25 thus gives us a very quick way to establish whether there exists an IC, IR, budget balanced mechanism.

Example:

- In a selling mechanism, the ex-post efficient social state is unique up to preferences ϑ, in which $\max_i \vartheta_i$ is attained by more than one buyer.

Provision of a Public Good

Public goods mechanism:

- Social state $q \in \{0, 1\}$ indicates whether the agreement is signed.
- Enforcing the agreement comes at a social cost c, which signatories contribute through reduced GHG emissions.
- Suppose countries' valuations θ_i of the climate agreement are independent and distributed on $[\underline{\vartheta}, \bar{\vartheta}]$ with density $f_i(\vartheta_i) > 0$.
- Country i's utility is $u_i(q, p, \vartheta_i) = v_i(q, \vartheta_i) - p_i = q\vartheta_i - p_i$.
- What is the expected surplus of the IR-VCG mechanism?

Find the IR-VCG Mechanism

Step 1: Find ex-post efficient social state

- If the good is produced, there is a social cost $c > 0$.
- Social state q is ex-post efficient if it maximizes non-monetary utility

$$\sum_{i=1}^{n} v_i(q, \vartheta_i) - cq = q\left(\sum_{i=1}^{n} \vartheta_i - c\right).$$

- It follows that the ex-post efficient social state is

$$q(\vartheta) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \vartheta_i \geq c, \\ 0 & \text{otherwise.} \end{cases}$$

Find the IR-VCG Mechanism

Step 2: Determine the pivot payments

- What should the social state $\widehat{q}_i$ be in country i's absence? By definition, a public good is non-excludable, hence i cannot really be absent.
- If country i did not participate in the mechanism, it would value the agreement at least at $\underline{\vartheta}$, hence $\widehat{q}_i(\vartheta_{-i}) = q(\underline{\vartheta}, \vartheta_{-i})$.
- Since country i's report is pivotal only if $q(\vartheta) = 1$ and $q(\underline{\vartheta}, \vartheta_{-i}) = 0$, we can write the pivot payments as:

$$p_i^{\text{piv}}(\vartheta) = \big(q(\vartheta) - q(\underline{\vartheta}, \vartheta_{-i})\big)\left(c - \sum_{j \neq i} \vartheta_j\right) \geq 0.$$

Find the IR-VCG Mechanism

3. Determine minimum participation subsidy:

- Suppose there is no outside option, that is, $IR(\vartheta_i, \vartheta_{-i}) = 0$. Then

$$\varphi_i^{\text{pivot}} = -\min_{\vartheta_i \in \Theta_i} u_i(g^{\text{pivot}}(\vartheta), \vartheta_i)$$

- We minimize

$$u_i(g^{\text{pivot}}(\vartheta), \vartheta_i) = \vartheta_i q(\vartheta) - \left(q(\vartheta) - q(\underline{\vartheta}, \vartheta_{-i})\right)\left(c - \sum_{j \neq i} \vartheta_j\right)$$

$$= \left(\sum_{j=1}^{n} \vartheta_j - c\right) q(\vartheta) + q(\underline{\vartheta}, \vartheta_{-i})\left(\sum_{j \neq i} \vartheta_j - c\right).$$

- This is minimized for $\vartheta_i = \underline{\vartheta}$, hence $\varphi_i^{\text{pivot}} = -\underline{\vartheta} q(\underline{\vartheta}, \vartheta_{-i})$.

Find the IR-VCG Mechanism

4. **Conclusion:** IR-VCG mechanism (q, p^{IR}) is determined by

$$q(\vartheta) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \vartheta_i \geq c, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$p_i^{\mathrm{IR}}(\vartheta) = \underline{\vartheta} q(\underline{\vartheta}, \vartheta_{-i}) + \big(q(\vartheta) - q(\underline{\vartheta}, \vartheta_{-i})\big)\left(c - \sum_{j \neq i} \vartheta_j\right).$$

Payments:

- If $q(\vartheta) = 0$, then $q(\underline{\vartheta}, \vartheta_{-i}) = 0$ and hence $p_i^{\mathrm{IR}}(\vartheta) = 0$ for each i.
- If $q(\underline{\vartheta}, \vartheta_{-i}) = 1$, then $q(\vartheta) = 1$ and hence $p_i^{\mathrm{IR}}(\vartheta) = \underline{\vartheta}$.
- If country i is pivotal, then $p_i^{\mathrm{IR}}(\vartheta) = c - \sum_{j \neq i} \vartheta_j$.

Question: Does it run an expected surplus?

Expected Surplus of IR-VCG Mechanism

Trivial cases:

- If $n\underline{\vartheta} \geq c$, then $q(\vartheta) = q(\underline{\vartheta}, \vartheta_{-i}) = 1$ for every country i.
- Thus, each country contributes $\underline{\vartheta}$, hence the surplus is $n\underline{\vartheta} - c \geq 0$.
- If $n\bar{\vartheta} \leq c$, then $q(\vartheta) = 0$, hence the surplus is 0.

Non-trivial cases:

- Suppose that $n\underline{\vartheta} < c < n\bar{\vartheta}$.
- Case 1: $q(\vartheta) = 0$. Then payments and cost are 0, i.e., the surplus is 0.
- Case 2: $q(\vartheta) = q(\underline{\vartheta}, \vartheta_{-i}) = 1$ for every country i. Then no country is pivotal and everybody pays $\underline{\vartheta}$. This leads to a deficit since $n\underline{\vartheta} < c$.

Expected Surplus of IR-VCG Mechanism

Non-trivial cases:

- Case 3: $q(\vartheta) = 1$ and players in $\mathcal{P}$ are pivotal. Then

$$\begin{aligned}
\sum_{i=1}^{n} p_i^{\mathrm{IR}}(\vartheta) &= \sum_{i\in\mathcal{P}}\left(c - \sum_{j\neq i}\vartheta_j\right) + \sum_{i\notin\mathcal{P}}\underline{\vartheta} \\
&= |\mathcal{P}|c - |\mathcal{P}|\sum_{j\notin\mathcal{P}}\vartheta_j - (|\mathcal{P}|-1)\sum_{j\in\mathcal{P}}\vartheta_j + \sum_{i\notin\mathcal{P}}\underline{\vartheta} \\
&= |\mathcal{P}|c - (|\mathcal{P}|-1)\sum_{j=1}^{n}\vartheta_j - \sum_{i\notin\mathcal{P}}(\vartheta_i - \underline{\vartheta}) \\
&\leq |\mathcal{P}|c - (|\mathcal{P}|-1)c - \sum_{i\notin\mathcal{P}}(\vartheta_i - \underline{\vartheta}) = c - \sum_{i\notin\mathcal{P}}(\vartheta_i - \underline{\vartheta}) \leq c.
\end{aligned}$$

- Note that the last inequality holds strictly almost-surely.

Ex-ante expected surplus: Since Cases 2 & 3 occur with positive probability, the ex-ante expected surplus is negative in the non-trivial case.

Impossibility Result

Proposition 6.26

An incentive-compatible individually rational ex-post efficient public goods mechanism exists if and only if either $n\underline{\vartheta} \geq c$ *or* $n\bar{\vartheta} \leq c$.

Remark:

- Private information prevents ex-post efficiency except in trivial cases.
- Note: the pivot mechanism runs a deficit because social state $q = 1$ has a social cost c associated with it.

What do we do next?

- We have to accept that either some payments are wasted for some ϑ or that the social state is sometimes inefficient.
- Next week we will see how to find the Bayesian-optimal mechanism.

Proof of Proposition 6.26

Proof of sufficiency:

- If $n\overline{\vartheta} \leq c$, then the public good is never produced.
- Payments of 0 are incentive-compatible and individually rational.
- If $n\underline{\vartheta} \geq c$, then the public good is always produced.
- Payments of $\frac{c}{n} \leq \underline{\vartheta}$ are incentive-compatible and individually rational.

Proof of necessity:

- IR-VCG runs an expected deficit if $n\underline{\vartheta} < c < n\overline{\vartheta}$,
- By Corollary 6.25, there exists no incentive-compatible, individually rational, and ex-post efficient mechanism.

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