Macroeconomic Theory

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Constrained Optimization

Inequality Constraints

A general Optimization Problem with equality and inequality constraints is as follows

a vector

$$\max_{x \in X} \quad f(x)$$

$$\text{subject to} \quad \begin{cases} h_i(x) \geq 0,\ i = 1, \ldots, n \\ l_j(x) = 0,\ j = 1, \ldots, m \end{cases}$$

The corresponding Lagrangian is defined to be

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{n} \mu_i h_i(x) + \sum_{j=1}^{m} \lambda_j l_j(x)$$

Inequality Constraints

Thm 條件

$\begin{cases} \text{convex}: & f \text{ concave}, \ h_i \text{ concave}, \ l_j \text{ affine} \\ & (a+b_1x_1+b_2x_2+\cdots+b_nx_n=0) \\ \text{slaters' condition}: & \exists x \text{ s.t. } h_i(x)>0 \ \forall i. \end{cases}$

↳ 本堂課都會假設這些條件成立

Theorem

(Karush–Kuhn–Tucker Conditions) Consider a general optimization problem with zero duality gap. Suppose that x^* is a local optimum, and f, h_i, l_j are continuous differentiable at x^*. Then there exist λ^*, μ^* such that the following conditions are satisfied:

1. (First order condition)

$$\nabla L = \nabla f(x^*) + \sum_{i=1}^{n} \mu_i^* \nabla h_i(x^*) + \sum_{j=1}^{m} \lambda_j \nabla l_j(x^*) = 0$$

$$\nabla L = \begin{pmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \\ \vdots \\ \frac{\partial L}{\partial x_n} \end{pmatrix}$$

Inequality Constraints

2. (Feasibility)

$$
\begin{aligned}
h_i(x) &\geq 0 \text{ for all } i \\
l_j(x) &= 0 \text{ for all } j \\
\mu_i^* &\geq 0 \text{ for all } i
\end{aligned}
$$

3. (Complementary Slackness)

$$\mu_i^* h_i(x^*) = 0 \text{ for all } i$$

(See Convex Optimization/Boyd and Vandenberghe, Chapter 5)

Lagrange Multipliers

- The variables λ^*, μ^* are called Lagrange multiplers. What does a lagrange multiplier stand for?
- Consider the following consumer's problem

$$
\begin{aligned}
\max_{x_1, x_2} \quad & u(x_1, x_2) \\
\text{subject to} \quad & 3 \; m - p_1 x_1 - p_2 x_2 = 0
\end{aligned}
$$

- The Lagrangian:

$$L = u(x_1, x_2) + \lambda \left(m - p_1 x_1 - p_2 x_2\right)$$

- First order conditions:

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} &= 0 : 1 \; \frac{\partial u}{\partial x_1}(x_1^*, x_2^*) = \lambda^* p_1 \\
\frac{\partial L}{\partial x_2} &= 0 : 2 \; \frac{\partial u}{\partial x_2}(x_1^*, x_2^*) = \lambda^* p_2
\end{aligned}
$$

- Combining the budget constraint, we can solve for $(x_1^*, x_2^*, \lambda^*)$

$\hookrightarrow$ 1 2 3

Shadow Price

- Let $x_1^* = \hat{x}_1(m)$, and $x_2^* = \hat{x}_2(m)$. Then the optimal utility $\hat{u}(m) \equiv u(\hat{x}_1(m), \hat{x}_2(m))$. → 根據不同 m, 最適選擇會不同
- $\frac{d\hat{u}}{dm}(m)$ is called the shadow price, when represents the amount of utility you can obtain if you increase one unit of budget
- Or, the price (in terms of utility) one would like to pay for increasing one unit of the budget capacity

$\frac{d\hat{u}}{dm}(m)$: 多增加 1單位財富, 會多獲得的效用,

↳ 願意花多少成本去多增加 1單位財富

(shadow price)

Shadow Price

$\lambda^* p_1$ → 上2页

$\parallel$

$$
\begin{aligned}
\frac{d\hat{u}}{dm}(m) &= \frac{\partial u}{\partial x_1}(\hat{x}_1(m), \hat{x}_2(m))\frac{d\hat{x}_1}{dm}(m) \\
&\quad + \frac{\partial u}{\partial x_2}(\hat{x}_1(m), \hat{x}_2(m))\frac{d\hat{x}_2}{dm}(m) \\
&= \lambda^* \underline{\left[p_1 \frac{d\hat{x}_1}{dm}(m) + p_2 \frac{d\hat{x}_2}{dm}(m)\right]} = \lambda^* \cdot 1
\end{aligned}
$$

- Note that $p_1\hat{x}_1(m) + p_2\hat{x}_2(m) = m$. We take derivative with respect to m to both sides of the budget constraint

$$p_1 \frac{d\hat{x}_1}{dm}(m) + p_2 \frac{d\hat{x}_2}{dm}(m) = 1$$

So

- The Lagrange multiplier is equal to the shadow price:

$$\lambda^* = \frac{d\hat{u}}{dm}(m)$$

shadow price → 上一页

Complementary Slackness

- Suppose now the constraint is inequality, then we need to consider the complementary slackness condition

$\lambda = 0$ (shadow price is 0)

$\Uparrow$

不願去耗任何成本來增加財富

$\Uparrow$

$$\lambda \left(m - p_1 x_1 - p_2 x_2 \right) = 0$$

↱ 最佳選擇不會用完錢 ⇒ 本來就不需那麼多錢 ⇒ 再多增加錢也不會更爽

- If the buget constraint is nonbinding $m - p_1 x_1 - p_2 x_2 > 0$, it must be the case that the consumer does not need that much consumption, and in this case, an increase in the consumers money holding does not improve utility $(\lambda = 0)$
- If the shadow price is positive $\lambda > 0$. Meaning that increasing budget can improve utility, then the consumer must use up its budget, so $m - p_1 x_1 - p_2 x_2 = 0$ → $\lambda > 0$ ⇒ 需要更多錢 ⇒ 目前錢不夠會全用完
- Note that there is an additional constraint $\lambda \geq 0$ when the constraint is an inequality

Slater's Condition: Example

- $$\max_{x,y\in\mathbb{R}} \quad xy$$
  $$\text{subject to} \quad (x+y-2)^2 \leq 0$$
- The budget constraint is equivalent to $x+y=2$
- The solution is $x=y=1$

Slater's Condition: Example

- $$\max_{x,y\in\mathbb{R}} \quad xy$$
  $$\text{subject to} \quad (x+y-2)^2 \leq 0$$

- $$L = xy - \lambda(x+y-2)^2$$

- First order conditions
  $$\begin{aligned} y - 2\lambda(x+y-2) &= 0 \\ x - 2\lambda(x+y-2) &= 0 \end{aligned}$$

- Complementary slackness condition
  $$\lambda(x+y-2)^2 = 0$$

- The Lagrange multiplier $\lambda \geq 0$

Slater's Condition: Example

- If $\lambda^* = 0$, then $y^* = x^* = 0$
- If $\lambda^* > 0$, then $x + y - 2 = 0$, and this implies that $y^* = x^* = 0$
- In either cases, the feasibility is violated, meaning that there is no allocation satisfy the KKT conditions

Two-Period Model

Two-Period Model

- A household has an initial capital k_0, and capital is used to produce outputs
- Outputs can be consumed or invested or be just thrown away without using it

free disposal

$$c_t + x_t \leq f(k_t)$$

- Investment today is used to increase the future capital stock. The law of motion for capital is,

$$k_{t+1} = (1-\delta)k_t + x_t$$

where δ is the depreciation rate of capital

- We substitute investments as the net difference in capital, then

$$c_t + \underbrace{k_{t+1} - (1-\delta)k_t}_{x_t} \leq f(k_t)$$

Two-Period Model

- The household's optimization problem:

$$\max_{c_0, c_1, k_1, k_2} u(c_0) + \beta u(c_1) \tag{P.1}$$

$$\text{subject to} \quad \begin{cases} f(k_0) + (1-\delta)k_0 - c_0 - k_1 \geq 0 \\ f(k_1) + (1-\delta)k_1 - c_1 - k_2 \geq 0 \\ c_0, c_1, k_1, k_2 \geq 0 \end{cases}$$

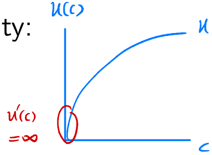
只要不可能消耗得超過當期有的全部東西就可以 $(n_0 < 0)$

Where the discount factor β is strictly between 0 and 1

- Outputs are free disposal

Two-Period Model

- Standard assumptions on consumption utility:
  - $u'(c) > 0, u''(c) < 0$
  - $\lim_{c \to 0} u'(c) = \infty$ (Inada condition) ← 無論如何一定會消費
  - $\lim_{c \to \infty} u'(c) = 0$



- Standard assumptions on production function
  - $f'(k) > 0,\ f''(k) < 0$
  - $\lim_{k \to 0} f'(k) = \infty$ (Inada condition)
  - $\lim_{k \to \infty} f'(k) = 0$
- The utility function is strictly concave, and the constraints are strictly concave $\Rightarrow$ A convex optimization problem

↳ 使用KKT thm 的條件

Two-Period Model

- The Lagragian

∵有它們≥0的假設
∴必須加此
保件

$$
\begin{aligned}
L \quad = \quad & u(c_0) + \beta u(c_1) + \lambda_0 \left[\, f(k_0) + (1-\delta)k_0 - c_0 - k_1 \right] \\
& + \lambda_1 \left[f(k_1) + (1-\delta)k_1 - c_1 - k_2 \right] \\
& \underline{+ \gamma_0 c_0 + \gamma_1 c_1 + \eta_1 k_1 + \eta_2 k_2}
\end{aligned}
$$

(t=0,1)
雖然是兩期模型,
但允許在第1期末投資,
∴會有第2期資本

- First order conditions

FOC = 0

$$
\begin{aligned}
c_0 \quad &: \quad u'(c_0^*) - \lambda_0^* + \gamma_0^* = 0 \\
c_1 \quad &: \quad \beta u'(c_1^*) - \lambda_1^* + \gamma_1^* = 0 \\
k_1 \quad &: \quad -\lambda_0^* + \lambda_1^* f \left['(k_1^*) + (1-\delta) \right] + \eta_1^* = 0 \\
k_2 \quad &: \quad -\lambda_1^* + \eta_2^* = 0
\end{aligned}
$$

Two-Period Model

CS = 0

- Complementary slackness conditions

$$
\begin{aligned}
\lambda_0^* [\, f(k_0) + (1-\delta)k_0 - c_0 - k_1] &= 0 \\
\lambda_1^* [f(k_1) + (1-\delta)k_1 - c_1 - k_2] &= 0 \\
\gamma_0^* c_0^* &= 0 \\
\gamma_1^* c_1^* &= 0 \\
\eta_1^* k_1^* &= 0 \\
\eta_2^* k_2 &= 0
\end{aligned}
$$

Two-Period Model

- $(c_0 \geq 0)$ The first order condition and complementary slackness condition w.r.t. c_0:

$\infty \to (\to\leftarrow)$

$$
\begin{aligned}
\text{FOC:} \quad u'(c_0^*) - \lambda_0^* + \gamma_0^* &= 0 \\
\text{C.S:} \quad \gamma_0^* c_0^* &= 0
\end{aligned}
$$

反證: $\infty - \lambda_0^* + \gamma_0^* = 0$, but $\lambda_0^* < \infty$ $(\to\leftarrow)$

  - Because $\lim_{c \to 0} u'(c^*) = \infty$, $c_0^* = 0$ cannot be the optimal solution $\Rightarrow c_0^* > 0$
  - By complementary slackness condition: $c_0^* > 0 \Rightarrow \gamma_0^* = 0$

- $(c_1 \geq 0)$ 同上 Similarly, by the first order condition and complementary slackness condition w.r.t. c_1 :

$$
\begin{aligned}
\beta u'(c_1^*) - \lambda_1^* + \gamma_1^* &= 0 \\
\gamma_1^* c_1^* &= 0
\end{aligned}
$$

  - We must have $c_1^* > 0$, and by complementary slackness conditionthus $\gamma_1^* = 0$

Two-Period Model

$(k_1 \geq 0)$

- The first order condition w.r.t. k_1:
  $-\lambda_0^* + \lambda_1^* \left[f'(k_1^*) + (1-\delta) \right] + \eta_1^* = 0$
  - Because $\lim_{k \to 0} f'(k) = \infty$, $k_1^* = 0$ cannot be a solution $\Rightarrow$ $k_1^* > 0$
- By complementary slackness condition: $\eta_1^* k_1^* = 0$
  - $k_1^* > 0 \Rightarrow \eta_1^* = 0$
- To summarize, the inada conditions rule out binding nonnegativity constraints for c_0, c_1, and k_1

↳ c_0, c_1, k_1 都不可能 $=0$,
毕竟只要增加一點,多獲得的效用就接近 ∞!
$\Rightarrow$ ∴ 一定要加 Inada condition.

Two-Period Model

直觉=0，因为目前模型根本不管第2期效用

($k_2 \geq 0$)
- Because $\gamma_0^* = 0, \gamma_1^* = 0, \eta_1^* = 0$, the first order conditions can be rewritten as (改写 p17)

FOC:

$$
\begin{aligned}
u'(c_0^*) &= \lambda_0^* \\
\beta u'(c_1^*) &= \lambda_1^* \\
\lambda_0^* &= \lambda_1^* \left[f'(k^*) + (1-\delta)\right] \\
\lambda_1^* &= \eta_2^*
\end{aligned}
$$

CS:
- The complementary slackness condition $\eta_2^* k_2 = 0$ can be rewritten as $\lambda_1^* k_2^* = 0$
- Note that $c_1^* > 0 \Rightarrow \lambda_1^* = \beta u'(c_1^*) > 0$, and thus, we must have $k_2^* = 0$

↳ $\lambda_1^* > 0$ means 我还想要更多钱!

直觉: $k_2^* > 0$ $\begin{cases} \text{utility} = 0 \\ \quad \wedge \\ \text{cost} = \text{少掉 c 能增加的效用} \end{cases}$ $\Rightarrow k_2^*$ should $= 0$.

Two-Period Model

- Question: are the budget constraints binding?
- Complementary slackness condition

$$\overset{\because >0}{\underbrace{\lambda_0^*}_{=u'(c_0^*)\ \text{by 上页}}}\underbrace{\left[f(k_0^*)+(1-\delta)k_0^*-c_0^*-k_1^*\right]}_{\therefore =0}=0$$

- Note that $c_0 > 0 \Rightarrow \lambda_0 = u'(c_0) > 0$, and thus, by complementary slackness condition, we must have $f(k_0^*)+(1-\delta)k_0^*-c_0^*-k_1^*=0$
- That is, since the marginal utility of consumption is greater than zero, the household should always use up the budget

$u'(c_0^*)>0 \Rightarrow [\ ,\ ] = 0$

↳钱全用完

↳也符合 $\lambda_0^*>0$ ∴应用完钱的直觉

Two-Period Model

第1期也一樣

- Similarly, by complementary slackness condition

$$\overset{\because > 0}{\lambda_1^*}\left[\underset{\therefore < 0}{\underline{f(k_1^*) + (1-\delta)k_1^* - c_1^* - k_2^*}}\right] = 0$$

- Note that $c_1 > 0 \Rightarrow \lambda_1 = \beta u'(c_1) > 0$, and thus, by complementary slackness condition, we must have $f(k_1^*) + (1-\delta)k_1^* - c_1^* - k_2^* = 0$

Two-Period Model

- To summarize, we have the following first order conditions

全部加總

$$\begin{cases} u'(c_0^*) & = & \lambda_0^* \\ \beta u'(c_1^*) & = & \lambda_1^* \\ \lambda_0^* & = & \lambda_1^* \left[f'(k_1^*) + (1-\delta) \right] \end{cases}$$

$\Downarrow$

- The Euler equation:

$$① \quad u'(c_0^*) = \beta \left[f'(k_1^*) + (1-\delta) \right] u'(c_1^*)$$

- We have the following budget constraint

$$\begin{aligned} ② \quad f(k_0^*) + (1-\delta)k_0^* - c_1^* - k_1^* &= 0 \\ ③ \quad f(k_1^*) + (1-\delta)k_1^* - c_2^* &= 0 \end{aligned}$$

?不確定

一定有 unique sol.

$\Rightarrow$ ∵ 1. u' is 單調

2. f(k)

- We can solve for the optimal (c_0^*, c_1^*, k_1^*)

①②③ 三條式子解 3 未知數

Two-Period Model

- Given the inada conditions and the complementary conditions, we know that the budget constraints must bind and we must have $c_0, c_1, k_1 > 0$, and $k_2 = 0$
- Thus, we can simplify the optimization problem as

$$\max_{c_0, c_1, k_1} \quad u(c_0) + \beta u(c_1) \tag{P.1}$$

$$\text{subject to} \quad \begin{cases} f(k_0) + (1-\delta)k_0 - c_0 - k_1 = 0 \\ f(k_1) + (1-\delta)k_1 - c_1 = 0 \end{cases}$$

↳ 之後就直接寫 =0，∵有 inada 最後就會推成這樣

Two-Period Model

- Let $u(c) = \ln c$, $f(k) = Ak^\alpha$
- Euler equation

$$\frac{1}{c_0} = \beta\alpha A k_1^{*\alpha-1}\frac{1}{c_1}$$

- Budget constraints

$$\begin{aligned} f(k_0) - c_0^* - k_1^* &= 0 \rightarrow c_0^* = f(k_0) - k_1^* \\ Ak_1^{*\alpha} - c_1^* &= 0 \rightarrow c_1^* = Ak_1^{*\alpha} \end{aligned}$$

- Substitute c_1^* and c_2^* as functions of k^*

$$Ak_1^{*\alpha} = \beta\alpha A k_1^{*\alpha-1}(f(k_0) - k_1^*)$$

$$\Rightarrow \quad k_1^* = \frac{\beta\alpha}{1+\beta\alpha} f(k_0)$$

$$c_0^* = \frac{1}{1+\beta\alpha} f(k_0)$$

$$c_1^* = Ak_1^{*\alpha}$$

(代入)
c_0^*, c_1^*