ECON 7011, Semester 110.1, Assignment 5, Solutions

1. (a) Denote the players' types by $\tau_1^1 = \{\omega_1\}$, $\tau_1^2 = \{\omega_2, \omega_3\}$, $\tau_2^1 = \{\omega_1, \omega_2\}$, and $\tau_2^1 = \{\omega_3\}$. For the singleton information sets, the posteriors assign probability 1 to that state. The non-singleton information sets τ_1^2 and τ_2^1 assign belief $\frac{2}{3}$ and $\frac{1}{2}$, respectively, to state ω_2.

   (b) Observe that the players have a strictly dominant action in state ϑ_1. Since type τ_1^1 knows $\theta = \vartheta_1$, he/she must choose D, i.e., $\sigma_1(\tau_1^1) = D$. For the remaining types, let us parametrize the strategy profiles by $\sigma_1(\tau_1^2) = xT + (1-x)D$, $\sigma_2(\tau_2^1) = yL + (1-y)R$, and $\sigma_2(\tau_2^2) = zL + (1-z)R$. Type τ_1^2 knows $\theta = \vartheta_2$, hence his/her expected utility is

$$\mathbb{E}_\sigma\Big[u_1(\theta, A)\,\Big|\,\tau_1^2\Big] = 2x\Big(1 - \frac{2}{3}y - \frac{1}{3}z\Big) + 3(1-x)\Big(\frac{2}{3}y + \frac{1}{3}\Big).$$

The partial derivative with respect to x is

$$\frac{\partial \mathbb{E}_\sigma[u_1(\theta, A)\,|\,\tau_1^2]}{\partial x} = 2 - \frac{10}{3}y - \frac{5}{3}z = \frac{5}{3}\Big(\frac{6}{5} - 2y - z\Big).$$

Type τ_2^2 also knows $\theta = \vartheta_2$. His/her expected utility is, therefore,

$$\mathbb{E}_\sigma\Big[u_2(\theta, A)\,\Big|\,\tau_2^2\Big] = (3zx + z(1-x) + 2(1-z)(1-x)).$$

The partial derivative with respect to z is

$$\frac{\partial \mathbb{E}_\sigma[u_2(\theta, A)\,|\,\tau_2^2]}{\partial z} = 3x + (1-x) - 2(1-x) = 4x - 1.$$

Type τ_2^1 knows that if $\theta = \vartheta_1$, then he/she will face D. His/her expected utility is

$$\mathbb{E}_\sigma\Big[u_2(\theta, A)\,\Big|\,\tau_2^1\Big] = -\frac{1}{2}y + \frac{1}{2}(3yx + y(1-x) + 2(1-y)(1-x)).$$

The partial derivative with respect to y is $2x - 1$. The best-response correspondences are

$$\mathcal{B}_1(y, z) = \begin{cases} x = 1 & \text{if } 2y + z < \frac{6}{5}, \\ x \in [0,1] & \text{if } 2y + z = \frac{6}{5}, \\ x = 0 & \text{if } 2y + z > \frac{6}{5}, \end{cases}$$

$$\mathcal{B}_{\tau_2^1}(x) = \begin{cases} y = 1 & \text{if } x > \frac{1}{2}, \\ y \in [0,1] & \text{if } x = \frac{1}{2}, \\ y = 0 & \text{if } x < \frac{1}{2}, \end{cases} \qquad \mathcal{B}_{\tau_2^2}(x) = \begin{cases} z = 1 & \text{if } x > \frac{1}{4}, \\ z \in [0,1] & \text{if } x = \frac{1}{4}, \\ z = 0 & \text{if } x < \frac{1}{4}. \end{cases}$$

We verify consistency:

- If $x > \frac{1}{2}$, then $y = z = 1$, hence $x = 0$, a contradiction.
- If $x = \frac{1}{2}$, then $z = 1$ and $y = \frac{1}{10}$. This is a BNE.
- If $x < \frac{1}{2}$, then $y = 0$, hence $x = 1$, a contradiction.

   (c) The types will be τ_1^2 and τ_2^1, hence both players will mix.

2. Let b_ϑ denote the campaign budgets of the Incumbent. Since the Incumbent knows ϑ, we simply differentiate the interim expected utility function to obtain

$$\frac{\partial u_I(b, \vartheta)}{\partial b_\vartheta} = \frac{b_C \vartheta}{(b_\vartheta + b_C)^2} - 1, \qquad \frac{\partial^2 u_I(b, \vartheta)}{\partial {b_\vartheta}^2} = -\frac{b_C \vartheta}{(b_\vartheta + b_C)^3} < 0 \tag{1}$$

for $\vartheta = 1, 2$. The latter shows that the utility function is strictly concave for positive campaign contributions, hence any solution $b_\vartheta^* \geq \frac{1}{10}$ to the first-order necessary condition is a global maximum, i.e., a best response. Thus, we can obtain an explicit best-response function by solving the first-order condition for b_ϑ:

$$b_\vartheta = \max\left\{\frac{1}{10}, \sqrt{b_C \vartheta} - b_C\right\},$$

where we have used that only the positive solution of the square root can exceed the minimum campaign contributions. The ex-ante expected utility for the Challenger is

$$\mathbb{E}[u_C(b, \vartheta)] = \frac{1}{2}\frac{b_C}{b_1 + b_C} + \frac{b_C}{b_2 + b_C} - b_C.$$

The first- and second-order derivatives are

$$\frac{\partial u_C(b, \vartheta)}{\partial b_C} = \frac{1}{2}\frac{b_1}{(b_1 + b_C)^2} + \frac{b_2}{(b_2 + b_C)^2} - 1, \tag{2}$$

$$\frac{\partial^2 u_C(b, \vartheta)}{\partial b_C^2} = -\frac{1}{2}\frac{b_1}{(b_1 + b_C)^3} - \frac{b_2}{(b_2 + b_C)^3} < 0,$$

again showing that any solution $b_2^* \geq \frac{1}{10}$ is a best response. The first-order condition in (1) implies that $\frac{1}{(b_\vartheta + b_C)^2} = \frac{1}{b_C \vartheta}$. Substituting these expressions into (2) yields $b_1 + b_2 = 2b_C$. Substituting b_1 and b_2 with their explicit best-response correspondences, this yields

$$0 = (1 + \sqrt{2})\sqrt{b_C} - 4b_C = \sqrt{b_C}\Big((1 + \sqrt{2}) - 4\sqrt{b_C}\Big)$$

with unique positive solution $b_C^* = \frac{3+2\sqrt{2}}{16}$. We deduce that $b_1^* = \frac{1+2\sqrt{2}}{16}$ and $b_2^* = \frac{5+2\sqrt{2}}{16}$. Since all three budgets exceed the minimum budget $\frac{1}{10}$, we deduce that $\sigma = (\sigma_I, \sigma_C)$ with $\sigma_I(\vartheta) = b_\vartheta^*$ and $\sigma_C = b_C^*$ is the unique Bayesian Nash equilibrium.

3. (a) Let us parametrize $\sigma_i(\vartheta_H)$ and $\sigma_i(\vartheta_L)$ by distribution functions F and G, respectively. Suppose towards a contradiction that G assigns positive weight to $b > \vartheta_L$, i.e., $G(\vartheta_L) < 1$. Then we can find a profitable deviation $\tilde{\sigma}_i(\vartheta_L)$ parametrized by $\tilde{G}$ that assigns all weight above ϑ_L to ϑ_L, that is, $\tilde{G}(b) = G(b)$ for $b < \vartheta_L$ and $\tilde{G}(\vartheta_L) = 1$. Indeed,

$$\mathbb{E}_{\tilde{\sigma}_i, \sigma_{-i}}[u_1(B, \vartheta_L)] = \mathbb{E}_\sigma[u_1(B, \vartheta_L) \,|\, B \leq \vartheta_L] P_\sigma(B \leq L) > \mathbb{E}_\sigma[u_1(B, \vartheta_L) \,|\, B].$$

Suppose now that G assigns positive weight to $b < \vartheta_L$. Then we can find a profitable deviation $\tilde{G}$ that bids a slightly larger amount. For a specific parametrization, we can set $\tilde{G}\left(\frac{b+\vartheta_L}{2}\right) = G(b)$ for all $b < \vartheta_L$, which shrinks the distribution G towards ϑ_L by a factor of 2. The probability of winning with bid b when facing strategy σ_{-i} is

$$\begin{aligned} H(b) &= \frac{1}{2}\Big(G(b-) + F(b-)\Big) + \frac{1}{4}\Big(\Delta G(b) + \Delta F(b)\Big) \\ &= \frac{1}{4}\Big(G(b) + F(b) + G(b-) + F(b-)\Big). \end{aligned} \tag{3}$$

Using the substitution $b' = 2b - \vartheta_L$, the expected utility of bidding according to $\tilde{G}$ is

$$\mathbb{E}_{\tilde{\sigma}_i, \sigma_{-i}}[u_1(B, \vartheta_L)] = \int_0^{\vartheta_L} (\vartheta_L - b) H(b) \, \mathrm{d}\tilde{G}(b) = \int_{-\vartheta_L}^{\vartheta_L} \frac{1}{2}(\vartheta_L - b') H\left(\frac{b' + \vartheta_L}{2}\right) 2 \, \mathrm{d}G(b').$$

Since F and G are non-decreasing, (3) implies that H is non-decreasing. Thus,

$$\mathbb{E}_{\tilde{\sigma}_i, \sigma_{-i}}[u_1(B, \vartheta_L)] > \int_{-\vartheta_L}^{\vartheta_L} (\vartheta_L - b') H(b') \, \mathrm{d}G(b') = \mathbb{E}_\sigma[u_1(B, \vartheta_L)], \tag{4}$$

where the inequality is strict because G (and hence H) assigns strictly positive weight below ϑ_L. The last equality in (4) follows since $H(b') = 0$ for $b' < 0$.

(b) Suppose that $\sigma_i(\vartheta_H) = b < \vartheta_H$ for $i = 1, 2$. Following σ yields

$$\mathbb{E}_\sigma[u_i(B, \vartheta_H)] = \frac{1}{2}(\vartheta_H - b) + \frac{1}{4}(\vartheta_H - b) = \frac{3}{4}(\vartheta_H - b)$$

for type ϑ_H. A deviation to $b' = b + \varepsilon$ yields expected utility

$$\mathbb{E}_{b',\sigma_{-i}}[u_i(B, \vartheta_H)] = \vartheta_H - b - \varepsilon.$$

Thus, such a deviation is profitable if $\varepsilon < \frac{1}{4}(\vartheta - b)$. Suppose that $\sigma_i(\vartheta_H) = b \geq \vartheta_H$. Then type ϑ_H has a non-positive utility, whereas deviation $b' = \vartheta_L + \varepsilon$ yields

$$\mathbb{E}[u_i(b', \sigma_{-i}(\theta_{-i}), \vartheta_H)] = \frac{1}{2}(\vartheta_H - \vartheta_L - \varepsilon),$$

which is profitable for any $0 < \varepsilon < \vartheta_H - \vartheta_L$.

(c) The same argument as in (a) shows that $\underline{b} \geq \vartheta_L$. To see why $\underline{b} = \vartheta_L$, note that type ϑ_H's expected utility of a bid $b > \vartheta_L$ is

$$\mathbb{E}_{b,\sigma_{-i}}[u_i(B, \vartheta_H)] = \frac{1}{2}(\vartheta_H - b) + \frac{1}{2}(\vartheta_H - b)F(b) = \frac{1}{2}(\vartheta_H - b)\big(1 + F(b)\big). \tag{5}$$

The expected utility of bid $b \searrow \underline{b}$ is, therefore,

$$\lim_{b \searrow \underline{b}} \mathbb{E}_{b,\sigma_{-i}}[u_i(B, \vartheta_H)] = \frac{1}{2}(\vartheta_H - \underline{b}). \tag{6}$$

If $\underline{b} > \vartheta_L$, then a deviation to $b' \in (\vartheta_L, \underline{b})$ yields expected utility

$$\mathbb{E}_{b',\sigma_{-i}}[u_i(B, \vartheta_H)] = \frac{1}{2}(\vartheta_H - b') > \frac{1}{2}(\vartheta_H - \underline{b}),$$

contradicting the fact that any $b \in (\underline{b}, \bar{b}]$ is a best response.

(d) It follows from $\underline{b} = \vartheta_L$, Equation (6), and the indifference principle that any b in the support of F yields the same expected utility $\frac{1}{2}(\vartheta_H - \vartheta_L)$. Because ties are broken 50–50, bidding $b = \vartheta_L$ yields only $\frac{1}{4}(\vartheta_H - \vartheta_L)$ and, hence, is not a best response.

(e) The indifference principle requires that bid $\bar{b}$ yields $\frac{1}{2}(\vartheta_H - \vartheta_L)$ as well. This implies

$$\frac{1}{2}(\vartheta_H - \vartheta_L) = \mathbb{E}_{\bar{b},\sigma_{-i}}[u_i(B, \vartheta_H)] = (\vartheta_H - \bar{b}),$$

which we solve for $\bar{b} = \frac{\vartheta_L + \vartheta_H}{2}$. Using (5), we now deduce that $F(b) = \frac{b - \vartheta_L}{\vartheta_H - b}$.

(f) The arguments in (c)–(e) show that $\mathbb{E}_\sigma[u_i(B, \vartheta_H)] = \frac{1}{2}(\vartheta_H - \vartheta_L)$ and that a deviation to $b = \vartheta_L$ is not profitable. Any deviation to $b < \vartheta_L$ yields utility 0 and any deviation to $b > \vartheta_H$ yields a negative expected utility, hence such deviations are not profitable either. Finally, a deviation to $b \in (\bar{b}, \vartheta_H)$ yields utility

$$\mathbb{E}_{b,\sigma_{-i}}[u_i(B, \vartheta_H)] = (\vartheta_H - b) < (\vartheta_H - \bar{b}) = \frac{1}{2}(\vartheta_H - \vartheta_L).$$

Since no deviations are profitable, we conclude that σ is a Bayesian Nash equilibrium.