ECON 7219, Semester 110.1, Assignment 1, Solutions

1. (a) The event that there is a misunderstanding is $Y = \{\omega_{AB}, \omega_{BA}\}$. In line with the awareness axiom, we can formalize awareness of an event Y as:

   i. Player i is aware of Y if $K_i(Y) = Y$, i.e., if player i knows Y whenever Y obtains.

   ii. Player i is (completely) unaware of Y if $K_i(Y) = \emptyset$, i.e., if player i cannot know Y.

   The two statements are formalized as $K_S Y = \emptyset$ and $K_L Y = \emptyset$, which both hold.

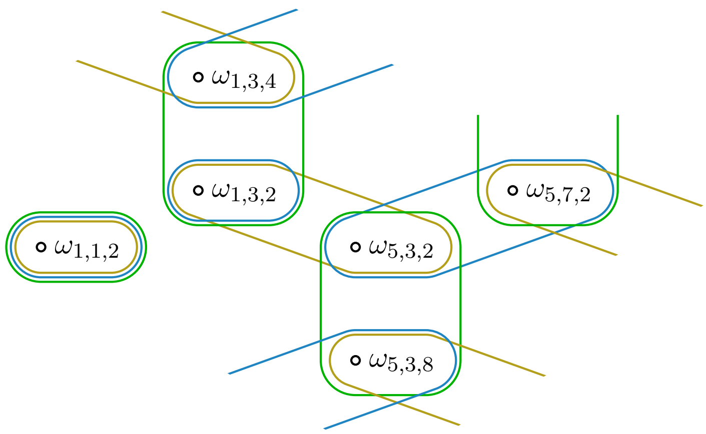
   (b) Not in the depicted model since, in any misunderstanding, both players are unaware of it. *Becoming aware of a misunderstanding would require a dynamic interaction, such as the Listener sharing what they understood or acting based on the information received.*

2. Consider a set Θ of possible states of nature, over which players $\mathcal{I}$ have belief hierarchies $\beta^* = (\beta_i^*)_{i\in\mathcal{I}}$. Proposition 2.9 shows that there is a one-to-one correspondence between the players' information sets and their belief hierarchies. Thus, we can write a belief hierarchy as a function $\beta(\omega)$ from the state of the world. The minimal belief space is thus the smallest belief space $(\mathcal{I}, \Omega, (P_i), (\mathcal{T}_i), \theta)$ that contains a state of the world ω_* with $\beta(\omega_*) = \beta^*$.

   Suppose that the minimal belief space contains a state of the world $\tilde{\omega} \notin C(\omega_*)$. We will show that the restriction of the minimal belief space to $C(\omega_*)$ also induces belief hierarchy $\beta|_{C(\omega_*)}(\omega_*) = \beta^*$. Since $C(\omega_*)$ is smaller than Ω, this is a contradiction to minimality of Ω.

   By definition, the event $C(\omega_*)$ is common knowledge in state ω_*. This implies that $C(\omega_*)$ is also common belief, i.e., $\omega_* \in B_{i_1} \dots B_{i_k} C(\omega_*)$ for any sequence $i_1, \dots, i_k$. This means that any k^{th} order beliefs of β^* are fully supported in $C(\omega_*)$. As a consequence, the restriction to $C(\omega_*)$ does not affect the generated belief hierarchy at all, i.e., $\beta|_{C(\omega_*)}(\omega_*) = \beta(\omega_*) = \beta^*$.

3. (a) Let us parametrize the states of the world as $\omega_{x,y,z}$, indicating that Anya, Bernadette, and Cheryl have the numbers x, y, and z, respectively, on their foreheads. The set Ω consists of all states $\omega_{x,y,x+y}$ and $\omega_{x,y,|x-y|}$ for any two distinct positive integers x and y.[1] In this model, Anya's information partition $\mathcal{T}_A$ contains singleton information sets $\{\omega_{x+y,x,y}\}$ and $\{\omega_{|x-y|,x,y}\}$ for all x and y with $|x-y| \in \{x, y\}$, and $\mathcal{T}_A$ contains $\{\omega_{x+y,x,y}, \omega_{|x-y|,x,y}\}$ otherwise. Information partitions of the other's are generated in the same way. Finally, the state of nature is $\theta(\omega_{x,y,z}) = (x, y, z)$. A partial illustration of the model is:



   (b) Let $\mathcal{T}_i^{(k)}$ denote the information partition of player i after the k^{th} announcement with the convention that Darren's announcement is announcement 0. Let $K_i^{(k)}$ indicate the associated knowledge operator. For each player i and each $k \geq 0$, let $Y_i^{(k)}$ denote the event that i knows her number after k announcements, i.e.,

$$Y_i^{(k)} = \bigcup_{x\in\mathbb{N}} K_i^{(k)}\{\theta_i = x\}.$$

[1] Observe that this implies Ω also contains $\omega_{x,x+y,y}$, $\omega_{x,|x-y|,y}$, $\omega_{x+y,x,y}$, and $\omega_{|x-y|,x,y}$ for any such x and y. There are other ways to describe Ω. For example, we could require $|x-y| \notin \{x, y\}$.

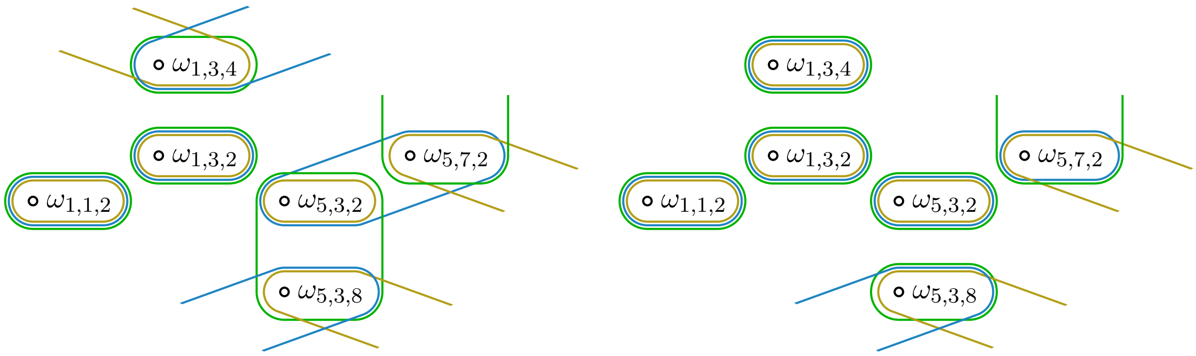


Figure 1: The left panel shows the information partitions after Bernadette's first-round announcement. The right panel shows he information partition after Cheryl's 1st-round and Anya's second-round announcement. No more information is revealed after that.

With announcement $k+1$, the non-announcing players learn to distinguish $\left\{Y_i^{(k)}, \left(Y_i^{(k)}\right)^c\right\}$ for $i = \mathrm{mod}(k+1, 3)$. For Anya's first announcement, Bernadette learns to distinguish $\omega_{x+y,x,y}$ from $\omega_{x+y,x+2y,y}$ for all x, y with $|x-y| \in \{x, y\}$. Similarly, Cheryl learns to distinguish $\omega_{x+y,x,y}$ from $\omega_{x+y,x,x+2y}$ for all x, y with $|x-y| \in \{x, y\}$. None of those states are in $C(\omega_{5,3,2})$, hence those are not visible in Figure 1. For ease of exposition, we restrict attention to $C(\omega_{5,3,2})$ for announcements $k > 1$, for which we obtain:

$$Y_2^{(1)} = \{\omega_{1,1,2}, \omega_{1,3,2}\}, \qquad Y_3^{(2)} = \{\omega_{1,1,2}, \omega_{1,3,2}, \omega_{1,3,4}\}$$
$$Y_1^{(3)} = \{\omega_{1,1,2}, \omega_{1,3,2}, \omega_{1,3,4}, \omega_{5,3,2}\}, \qquad \ldots$$

At that point, $\omega_{5,3,2}$ becomes common knowledge. Cheryl announces her number simply because of the turn order. She figures out her number at the same time as Bernadette.

(c) There are many such states. One easy way to find one is to look at a state $\omega_{x,y,z}$ that is connected to $\omega_{1,3,2}$ only through information sets of two players. Along such a sequence of states, the revealed information "travels" by two states in each round. For Anya and Bernadette, the first round can reveal only whether the true state is $\omega_{1,3,2}$. In round 2 it is revealed whether the true state is $\omega_{5,3,2}$ or $\omega_{5,7,2}$. In round k it is revealed whether the true state is $\omega_{4k-3,4k-5,2}$ or $\omega_{4k-3,4k-1,2}$. Thus, $\omega_{13,11,2}$ will do.

(d) Yes. Cheryl would be able to distinguish $\omega_{1,1,2}$ from $\omega_{1,1,0}$ because she knows that the number on her forehead is positive. This would delay the information revelation by one full round (three announcements).

4. (a) The belief hierarchies of Aaron and Blake are very similar to Andrew's and Flo's beliefs in the roadtrip example of the lecture notes. We start with two common belief states ω_1 and ω_2, add Aaron's beliefs using state ω_3, and use states ω_4 and ω_5 to add Blake's beliefs. To describe Cédric's belief hierarchy, we first need to add states that describe Cédric's beliefs of Aaron's and Blake's beliefs. Those are similar to Blake's belief hierarchy and each require two states ω_6–ω_9.[2] We can add Cédric's beliefs on top of that with states ω_{10} and ω_{11}. So far, the induced belief hierarchies are correct for Aaron in ω_3, for Blake in ω_4 and ω_5, and for Cédric in ω_{10} and ω_{11}, but no state induces the correct belief hierarchies for all players. We can achieve that by adding ω_{12}. See Table 1 for the belief table.

The players' information sets are given by those states of the world that induce the same first-order beliefs. If we enumerate the players' types / information sets, we obtain

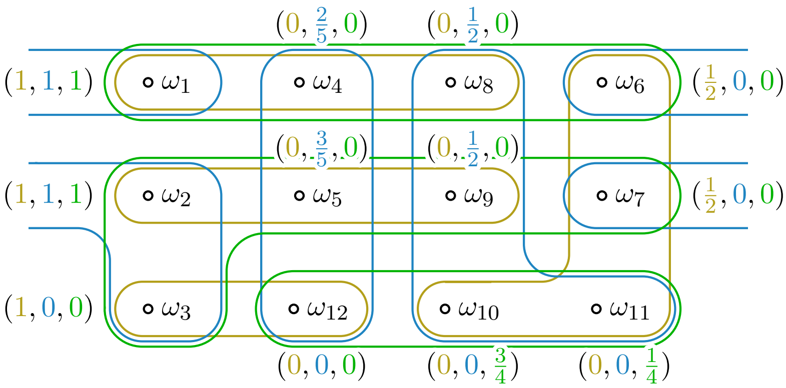
[2] Note that it matters how "this" in "this is common belief" in the last sentence is interpreted. In this solution, "this" refers to "the correct day". If "this" refers to "they both believe everybody else knows the correct day of the week," then Blake's and Cédric's beliefs in states ω_6 and ω_7 are instead concentrated on ω_6 and ω_7, respectively. Similarly, Aaron's and Cédric's beliefs in ω_8 and ω_9 are then concentrated on ω_8 and ω_9, respectively.

ω	$\theta(\omega)$	Posterior $P_{T_1(\omega)}$	Posterior $P_{T_2(\omega)}$	Posterior $P_{T_3(\omega)}$
ω_1	M	$[1\omega_1]$	$[1\omega_1]$	$[1\omega_1]$
ω_2	T	$[1\omega_2]$	$[1\omega_2]$	$[1\omega_2]$
ω_3	M	$[1\omega_3]$	$[1\omega_2]$	$[1\omega_2]$
ω_4	M	$[1\omega_1]$	$[\frac{2}{5}\omega_4, \frac{3}{5}\omega_5]$	$[1\omega_1]$
ω_5	T	$[1\omega_2]$	$[\frac{2}{5}\omega_4, \frac{3}{5}\omega_5]$	$[1\omega_2]$
ω_6	M	$[\frac{1}{2}\omega_6, \frac{1}{2}\omega_7]$	$[1\omega_1]$	$[1\omega_1]$
ω_7	T	$[\frac{1}{2}\omega_6, \frac{1}{2}\omega_7]$	$[1\omega_2]$	$[1\omega_2]$
ω_8	M	$[1\omega_1]$	$[\frac{1}{2}\omega_8, \frac{1}{2}\omega_9]$	$[1\omega_1]$
ω_9	T	$[1\omega_2]$	$[\frac{1}{2}\omega_8, \frac{1}{2}\omega_9]$	$[1\omega_2]$
ω_{10}	M	$[\frac{1}{2}\omega_6, \frac{1}{2}\omega_7]$	$[\frac{1}{2}\omega_8, \frac{1}{2}\omega_9]$	$[\frac{3}{4}\omega_{10}, \frac{1}{4}\omega_{11}]$
ω_{11}	T	$[\frac{1}{2}\omega_6, \frac{1}{2}\omega_7]$	$[\frac{1}{2}\omega_8, \frac{1}{2}\omega_9]$	$[\frac{3}{4}\omega_{10}, \frac{1}{4}\omega_{11}]$
ω_{12}	M/T	$[1\omega_3]$	$[\frac{2}{5}\omega_4, \frac{3}{5}\omega_5]$	$[\frac{3}{4}\omega_{10}, \frac{1}{4}\omega_{11}]$

Table 1: Belief table generated by the belief hierarchies.

i. $\tau_A^1 = \{\omega_1, \omega_4, \omega_8\}$, $\tau_A^2 = \{\omega_2, \omega_5, \omega_9\}$, $\tau_A^3 = \{\omega_3, \omega_{12}\}$, and $\tau_A^4 = \{\omega_6, \omega_7, \omega_{10}, \omega_{11}\}$.
ii. $\tau_B^1 = \{\omega_1, \omega_6\}$, $\tau_B^2 = \{\omega_2, \omega_3, \omega_7\}$, $\tau_B^3 = \{\omega_4, \omega_5, \omega_{12}\}$, and $\tau_B^4 = \{\omega_8, \omega_9, \omega_{10}, \omega_{11}\}$.
iii. $\tau_C^1 = \{\omega_1, \omega_4, \omega_6, \omega_8\}$, $\tau_C^2 = \{\omega_2, \omega_3, \omega_5, \omega_7, \omega_9\}$, $\tau_C^3 = \{\omega_{10}, \omega_{11}, \omega_{12}\}$.

Visually, we can depict the belief space as follows, where Blake's information sets "wrap around", i.e., connect left- and right-most states:



(b) Let us parametrize the player's first-order beliefs $\mu_i(\omega; M)$ through the probability the player assigns to Monday. Writing $\mu_i(\tau_i; M)$ as an abbreviation for $\mu_i(\omega; M)$ for all $\omega \in \tau_i$, we find that Aaron's and Blake's beliefsare:

i. $\mu_A(\tau_A^1; M) = 1$, $\mu_A(\tau_A^2; M) = 0$, $\mu_A(\tau_A^3; M) = 1$, and $\mu_A(\tau_A^4; M) = \frac{1}{2}$.
ii. $\mu_B(\tau_B^1; M) = 1$, $\mu_B(\tau_B^2; M) = 0$, $\mu_B(\tau_B^3; M) = \frac{2}{5}$, and $\mu_B(\tau_B^4; M) = \frac{1}{2}$.

A player accepts an event bet if they believe to be right at least half of the time, hence the events "Aaron accepts the bet" and "Blake accepts the bet" are

$$Y_A = \left\{\omega \in \Omega \,\middle|\, \mu_A(\omega; M) \geq \tfrac{1}{2}\right\} = \tau_A^1 \cup \tau_A^3 \cup \tau_A^4$$

$$Y_B = \left\{\omega \in \Omega \,\middle|\, \mu_B(\omega; M) \leq \tfrac{1}{2}\right\} = \tau_B^2 \cup \tau_B^3 \cup \tau_B^4.$$

The event where both players accept the bet is $Y = Y_A \cap Y_B = \{\omega_3, \omega_4, \omega_7, \omega_8, \omega_{10}, \omega_{11}, \omega_{12}\}$.

(c) We evaluate $P_{T_A(\omega_{12})}(Y) = 1$ and $P_{T_B(\omega_{12})}(Y) = \frac{2}{5}$.

(d) No, it does not. First, the belief space is one without a common prior. Second, the players' beliefs, with which they believe a bet is placed, is not common knowledge. The event "Aaron assigns belief 1 to Y" is τ_A^3 and the event "Blake assigns belief $\frac{2}{5}$ to Y" is τ_B^3, neither of which is common knowledge.