8. Information Design

ECON 7219 – Games With Incomplete Information

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Informativeness of Experiments

Mixing Sodium With Water

Does sodium explode in water?

- State of nature $\Theta = \{\vartheta_E, \vartheta_N\}$ indicates whether it explodes in water.
- To learn more about the state, we can design an experiment.
- Throw a block of sodium into a bucket of water and observe:
  - If the block explodes, we conclude $\theta = \vartheta_E$.
  - If the block does not explode, we conclude $\theta = \vartheta_N$.

Conducting an Experiment

Definition 8.1

An experiment about an unknown state θ is the observation of a signal S with known conditional distribution $\pi(s \mid \vartheta)$ over some signal space $\mathcal{S}$.

Learning from the experiment:

- If S and θ are correlated, we can learn from the experiment via

$$\nu(\vartheta \mid s) = \frac{\pi(s \mid \vartheta)\mu(\vartheta)}{\sum_{\vartheta'} \pi(s \mid \vartheta')\mu(\vartheta')}.$$

Mixing sodium with water:

- The experiment is perfectly informative since

$$\pi(\text{Explosion} \mid \vartheta_E) = 1, \qquad \pi(\text{No explosion} \mid \vartheta_N) = 1.$$

Distribution of Posteriors

Posteriors as random variables:

- Before conducting the experiment, the outcome S is unknown.
- The posterior $\nu(\vartheta) := \nu(\vartheta \,|\, S)$ for any $\vartheta \in \Theta$ is a $[0,1]$-valued random variable, taking value $\nu(\vartheta \,|\, s)$ for any $s \in \mathcal{S}$ with prior probability

$$P(S = s) = \sum_{\vartheta' \in \Theta} \pi(s \,|\, \vartheta')\mu(\vartheta').$$

- $\nu = \big(\nu(\vartheta_1), \ldots, \nu(\vartheta_n)\big)$ is thus a $\Delta(\Theta)$-valued random variable.
- The distribution ψ of ν is an element from $\Delta(\Delta(\Theta))$.

Mixing sodium with water:

- If our prior is $\mu \in \Delta(\Theta)$, our posterior is $\delta_{\vartheta_E} 1_{\{\theta = \vartheta_E\}} + \delta_{\vartheta_N} 1_{\{\theta = \vartheta_N\}}$ with distribution $\mu\delta_{\vartheta_E} + (1-\mu)\delta_{\vartheta_N}$, where δ is the Dirac measure.

Bayes-Plausible Posteriors

Definition 8.2

A distribution of posteriors $\psi \in \Delta(\Delta(\Theta))$ is Bayes plausible for a prior distribution $\mu \in \Delta(\Theta)$ if $\mathbb{E}_\psi[\nu] = \mu$.

Posteriors induced by experiment:

- Given prior μ, any experiment $\pi(s \mid \vartheta)$ must induce a distribution of posteriors ψ that is Bayes-plausible for μ.
- Indeed, since ψ is supported on $\big(\nu(s)\big)_{s \in \mathcal{S}}$, for each $\vartheta \in \Theta$, we obtain

$$\mathbb{E}_\psi[\nu(\vartheta)] = \sum_{\nu \,\in\, \mathrm{supp}\,\psi} \psi(\nu)\nu(\vartheta) = \sum_{s \in \mathcal{S}} P(S = s)\nu(\vartheta \mid s)$$

$$= \sum_{s \in \mathcal{S}} \pi(s \mid \vartheta)\mu(\vartheta) = \mu(\vartheta).$$

- This implies that $\mathbb{E}_\psi[\nu] = \mu$, hence ψ is Bayes plausible for μ.

Going on a Date

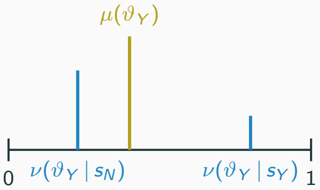
	ϑ_Y	ϑ_N
s_Y	0.6	0.1
s_N	0.4	0.9

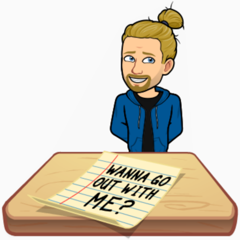
Does she like me?

- The state of nature $\Theta = \{\vartheta_Y, \vartheta_N\}$ indicates whether she does.
- My experiment could be to ask her on a date with possible outcomes $s_n =$ "She says no" or $s_y =$ "She says yes."
- If $\mu_0 = 0.4$, the posterior beliefs after observing s_Y and s_N are

$$\nu(\vartheta_Y \mid s_Y) = \frac{\frac{3}{5}\cdot\frac{2}{5}}{\frac{3}{5}\cdot\frac{2}{5} + \frac{1}{10}\cdot\frac{3}{5}} = \frac{4}{5}, \qquad \nu(\vartheta_Y \mid s_N) = \frac{\frac{2}{5}\cdot\frac{2}{5}}{\frac{2}{5}\cdot\frac{2}{5} + \frac{9}{10}\cdot\frac{3}{5}} = \frac{8}{35}.$$

Going on a Date





Posteriors as a random variable:

- The posterior of state ϑ_Y is $\nu(\vartheta_Y) = \frac{4}{5}1_{\{S=s_Y\}} + \frac{8}{35}1_{\{S=s_N\}}$.
- Since $P(S = s_Y) = \frac{3}{5} \cdot \frac{2}{5} + \frac{1}{10} \cdot \frac{3}{5} = \frac{3}{10}$, the distribution of posteriors is

$$\psi = \frac{3}{10}\delta_{\frac{4}{5}} + \frac{7}{10}\delta_{\frac{8}{35}}.$$

- The expectation of $\nu(\vartheta_Y)$ is $\frac{3}{10} \cdot \frac{4}{5} + \frac{7}{10} \cdot \frac{8}{35} = \frac{2}{5} = \mu(\vartheta_Y)$.
- The distribution ψ is called a mean-preserving spread of μ.

Mean-Preserving Spreads

Definition 8.3

A distribution ν is a mean-preserving spread (MPS) of a distribution μ if there exist random variables $N \sim \nu$, $M \sim \mu$, and ε such that $N \stackrel{\mathrm{d}}{=} M + \varepsilon$ and $\mathbb{E}[\varepsilon \mid M = m] = 0$ for every $m \in \operatorname{supp} \mu$.

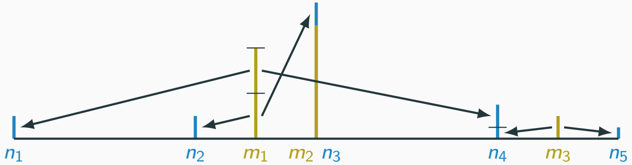
Interpretation:

- Each realization m of M is spread by a mean-zero random variable $\varepsilon \mid M = m$ to $\operatorname{supp}(\nu)$ such that, overall, the distribution ν is attained.
- Probability weights $\mu(m)$ are spread to $\operatorname{supp}(\nu)$ in a mean-preserving way.

Uses of mean-preserving spreads:

- Generalizes Bayes-plausibility to non-trivial priors.
- Allows us to compare informativeness of two experiments.

Visualization of a Mean-Preserving Spread



Mean-preserving spread:

- Suppose that $\Theta = \{\vartheta_1, \vartheta_2\}$ so that $\Delta(\Theta) = [0, 1]$.
- If $M \sim \mu$ is a $\Delta(\Theta)$-valued random variable, then the weight $\mu(m)$ of each realization m is spread to $\operatorname{supp} \nu$ in a mean-preserving way.
- The spread of m_i to $\operatorname{supp} \nu = \{n_1, \ldots, n_5\}$ corresponds to $\varepsilon \mid M = m_i$.

Increase in Variance:

- Note that N has the same mean as M, but higher variance.
- Does higher variance not typically imply higher uncertainty?

Perfectly Informative and Uninformative Signals

Perfectly informative signal:

- For finite Θ, the posterior ν after a perfectly informative signal satisfies

$$\nu = \sum_{\vartheta \in \Theta} \delta_\vartheta 1_{\{\theta = \vartheta\}} = \delta_\theta.$$

- The distribution of ν is thus "equal" to the distribution of θ:

$$P(\nu = \delta_\vartheta) = P(\theta = \vartheta).$$

- Nevertheless, the distinction between θ and δ_θ is an important one:
  - More variance in θ means more noise $\Rightarrow$ less information.
  - More variance in ν means that information is partitioned into smaller information sets $\Rightarrow$ more information.

Perfectly uninformative signal:

- Posterior $\nu = \delta_\mu$ is constant and equal to the prior distribution μ.

Mean-Preserving Spreads and Bayes-Plausibility

Bayes plausibility and MPS:

- The notion of mean-preserving spreads is an extension of Bayes plausibility to conducting experiments with a non-trivial prior.

Experiments:

- Bayes plausibility implies that the distribution of posteriors induced by an experiment $\pi(s \mid \vartheta)$ is a mean-preserving spread of δ_μ.
- Indeed, we can set $\varepsilon = \nu(\cdot \mid s)1_{\{S=s\}} - \mu$.
- Clearly, $\mathbb{E}[\varepsilon \mid M = \mu] = \mathbb{E}[\varepsilon] = 0$ since $\operatorname{supp} M = \{\mu\}$.

Additional uses of MPS:

- Allows us to compare distribution of posteriors after two experiments.
- Does a mean-preserving spread always signify better information?

Comparisons of Experiments

Definition 8.4

Suppose that Θ is finite.

1. A matrix R is row stochastic or column stochastic if $R_{ij} \in [0, 1]$ and the sum over each row or column, respectively, equals 1.
2. Experiment S_1 with signals in $\mathcal{S}_1$ is (Blackwell) more informative than experiment S_2 with signals in $\mathcal{S}_2$ if there exists a column-stochastic $|\mathcal{S}_2| \times |\mathcal{S}_1|$-matrix R with $\pi_2(\,\cdot \mid \vartheta) = R\pi_1(\,\cdot \mid \vartheta)$ for any state ϑ.

Interpretation:

- For any true state ϑ, the distribution $\pi_2(\,\cdot \mid \vartheta)$ is a garbling of $\pi_1(\,\cdot \mid \vartheta)$.
- Experiment 1 is statistically sufficient for experiment 2 because we can recover experiment 2 by garbling the outcome of experiment 1 with R.

Laughing at a Bad Joke

	ϑ_Y	ϑ_N
s_L	0.6	0.4
s_F	0.4	0.6

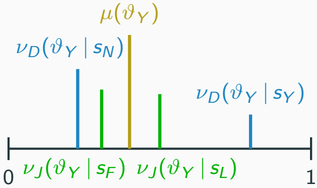
A less informative experiment:

- Suppose instead of asking her on a date, you make a bad joke and see if she laughs (s_L) or frowns (s_F). For $\mu_0 = 0.4$, the posteriors are

$$\nu(\vartheta_Y \mid s_Y) = \frac{\frac{3}{5} \cdot \frac{2}{5}}{\frac{3}{5} \cdot \frac{2}{5} + \frac{2}{5} \cdot \frac{3}{5}} = \frac{1}{2}, \quad \nu(\vartheta_Y \mid s_N) = \frac{\frac{2}{5} \cdot \frac{2}{5}}{\frac{2}{5} \cdot \frac{2}{5} + \frac{3}{5} \cdot \frac{3}{5}} = \frac{4}{13}.$$

- Signal s_L arises with prior probability $P(S = s_L) = \frac{3}{5} \cdot \frac{2}{5} + \frac{2}{5} \cdot \frac{3}{5} = \frac{12}{25}$.
- The distribution of posteriors is $\frac{12}{25}\delta_{\frac{1}{2}} + \frac{13}{25}\delta_{\frac{4}{13}}$.

Laughing at a Bad Joke



A less informative experiment:

- Let π_J and π_D indicate the experiments of making a joke and asking her out on a date, respectively. Is π_J a garbling of π_D?
- Indeed, there exists a column-stochastic matrix R such that

$$\begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix} = R \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \quad \Longrightarrow \quad R = \begin{bmatrix} 0.76 & 0.36 \\ 0.24 & 0.64 \end{bmatrix}.$$

- We can recover π_J from π_D by interpreting s_Y as s_L with 0.76 probability and interpreting s_N as s_L with 0.36 probability.

Equivalent Comparisons of Experiments

Proposition 8.5

Suppose Θ is finite. For any prior $\mu \in \Delta(\Theta)$ and any experiments S_i for $i = 1, 2$ with distribution of posteriors ψ_i, the following are equivalent:

1. *S_1 is more informative than experiment S_2.*
2. *ψ_1 is a mean-preserving spread of ψ_2.*

Remark:

- The two notions are equivalent comparisons of the two experiments.
- Relation through garblings is easily interpreted, but in information design it is often easier to work with mean-preserving spreads.
- Many other notions are equivalent as well; see Blackwell (1953).
- The results also holds if Θ is any measure space.

Mean-Preserving Spread Associated with Experiments

Claim

Fix a prior μ and two experiments S_i for $i = 1, 2$ on $\mathcal{S}_i = \left\{s_i^1, \ldots, s_i^{n_i}\right\}$ such that the induced distribution of posteriors ψ_i is $\psi_i(\nu_i(\cdot \mid s_i)) = P(S_i = s_i)$. Then ψ_1 is a mean-preserving spread of ψ_2 if and only if there exists a row-stochastic $|\mathcal{S}_2| \times |\mathcal{S}_1|$-matrix E with

$$\nu_2(\cdot \mid s_2^k) = \sum_{j=1}^{|\mathcal{S}_1|} E_{kj} \nu_1(\cdot \mid s_1^j), \qquad P(S_1 = s_1^j) = \sum_{k=1}^{|\mathcal{S}_2|} E_{kj} P\left(S_2 = s_2^k\right).$$

Proof of claim:

- If ψ_1 is an MPS of ψ_2, there exists a $\Delta(\Theta)$-valued random variable ε:
  - $\mathbb{E}\left[\varepsilon \,\middle|\, \nu_2(\cdot \mid S_2) = \nu_2(\cdot \mid s_2^k)\right] = 0$ for every $s_2^k \in \mathcal{S}_2$.
  - $\nu_1(\cdot \mid S_1)$ is distributed identically to $\nu_2(\cdot \mid S_2) + \varepsilon$.

Proof of Claim

Proof of necessity:

- Note that $\varepsilon|_{S_2=s_2^k}$ has to spread $\nu(\cdot \mid s_2^k)$ to $\operatorname{supp}\psi_1$ in a mean-preserving way, hence ε_k can only take values in $\nu(\cdot \mid s_1^j) - \nu(\cdot \mid s_2^k)$.
- Set $E_{kj} := P\big(\varepsilon = \nu(\cdot \mid s_1^j) - \nu(\cdot \mid s_2^k) \,\big|\, S_2 = s_2^k\big)$.
- Conditional mean 0 implies that

$$0 = \mathbb{E}\big[\varepsilon \,\big|\, S_2 = s_2^k\big] = \sum_{j=1}^{|\mathcal{S}_1|} \big(\nu(\cdot \mid s_1^j) - \nu(\cdot \mid s_2^k)\big) E_{kj}.$$

- Equality in distribution implies that

$$\begin{aligned} P(S_1 = s_1) &= P\big(\nu_2(\cdot \mid S_2) + \varepsilon = \nu_1(\cdot \mid s_1)\big) \\ &= \sum_{k=1}^{|\mathcal{S}_2|} \underbrace{P\big(\nu_2(\cdot \mid S_2) + \varepsilon = \nu_1(\cdot \mid s_1) \,\big|\, S_2 = s_2^k\big)}_{=E_{kj}} P\big(S_2 = s_2^k\big). \end{aligned}$$

Proof of Claim

Proof of sufficiency:

- Suppose that such a matrix E exist.
- For any $k = 1, \ldots, |\mathcal{S}_2|$, let ε_k be a random variable independent of S_1 and S_2 such that

$$P\big(\varepsilon_k = \nu(\cdot \mid s_1^j) - \nu(\cdot \mid s_2^k)\big) = E_{kj}.$$

- Define random variable ε by

$$\varepsilon = \sum_{k=1}^{|\mathcal{S}_2|} \varepsilon_k 1_{\{S_2 = s_2^k\}}.$$

- It follows as on the previous slide that $\mathbb{E}\big[\varepsilon \,\big|\, S_2 = s_2^k\big] = 0$ and that $\nu_1(\cdot \mid S_1)$ is distributed identically to $\nu_2(\cdot \mid S_2) + \varepsilon$.

Proof of Proposition 8.5

Statement 2. implies statement 1.:

- Suppose ψ_1 is a mean-preserving spread of ψ_2.
- By the claim, there exists a row-stochastic matrix E such that

$$\pi_2(s_2^k \mid \vartheta) = \frac{\nu_2(\vartheta \mid s_2^k) P\big(S_2 = s_2^k\big)}{\mu(\vartheta)} = \sum_{j=1}^{|\mathcal{S}_1|} \frac{E_{kj}\, \nu_1(\vartheta \mid s_1^j) P\big(S_2 = s_2^k\big)}{\mu(\vartheta)}$$

$$= \sum_{j=1}^{|\mathcal{S}_1|} \underbrace{\frac{E_{kj}\, P\big(S_2 = s_2^k\big)}{P\big(S_1 = s_1^j\big)}}_{=:\, R_{kj}} \pi_1(s_1^j \mid \vartheta).$$

- R is column-stochastic since $\sum_{k=1}^{|\mathcal{S}_2|} E_{kj}\, P\big(S_2 = s_2^k\big) = P\big(S_1 = s_1^j\big)$.
- We conclude that S_1 is more informative than S_2.

Proof of Proposition 8.5

Statement 1. implies statement 2.:

- Suppose that S_1 is more informative that S_2, that is, there exists a column-stochastic matrix R such that $\pi_2(\,\cdot\,|\,\vartheta) = R\pi_1(\,\cdot\,|\,\vartheta)$.
- The transformation $E_{kj} = \frac{R_{kj} P\left(S_1 = s_1^j\right)}{P\left(S_2 = s_2^k\right)}$ from the previous slide yields

$$\sum_{j=1}^{|\mathcal{S}_1|} E_{kj}\, \nu_1(\vartheta \,|\, s_1^j) = \sum_{j=1}^{|\mathcal{S}_1|} \frac{R_{kj}\, \pi_1(s_1^j \,|\, \vartheta)\mu(\vartheta)}{P\left(S_2 = s_2^k\right)}$$

$$= \frac{\pi_2(s_2^k \,|\, \vartheta)\mu(\vartheta)}{P\left(S_2 = s_2^k\right)} = \nu_1(\vartheta \,|\, s_2^k).$$

- Moreover, column-stochasticity of R implies that

$$\sum_{k=1}^{|\mathcal{S}_2|} E_{kj}\, P\left(S_2 = s_2^k\right) = P\left(S_1 = s_1^j\right) \sum_{k=1}^{|\mathcal{S}_2|} R_{kj} = P\left(S_1 = s_1^j\right).$$

Is More Information Always Better?

Decision problem:

- Suppose a single decision-maker must choose an action based on the outcome of an experiment S, that is, choose a decision rule $\sigma : \mathcal{S} \to \mathcal{A}$.
- Let $u(a, \vartheta)$ denote the utility of action $a \in \mathcal{A}$ in state ϑ and denote by $u(a) := \big(u(a, \vartheta_1), \dots, u(a, \vartheta_n)\big)$ the vector of state-contingent payoffs.
- For decision rule σ, let $u(\sigma, \vartheta) = \mathbb{E}[u(\sigma(S), \vartheta) \,|\, \theta = \vartheta]$ denote the conditional expected value. Define the risk vector as

$$u(\sigma) := \big(u(\sigma, \vartheta_1), \dots, u(\sigma, \vartheta_n)\big).$$

- Agents with different risk preferences value $u(\sigma)$ differently.
- Let $\mathcal{U}(S) := \{u(\sigma) \mid \sigma : \mathcal{S} \to \mathcal{A}\}$ denote the set of all feasible risk vectors when the agent observes experiment S.

Blackwell's Theorem

Theorem 8.6 (Blackwell, 1953)

Suppose that Θ is finite and that $u(\mathcal{A})$ is compact and convex. For two experiments S_1, S_2, the following two statements are equivalent:

1. $\mathcal{U}(S_1) \supseteq \mathcal{U}(S_2)$.
2. S_1 *is more informative than* S_2.

Interpretation:

- A single decision maker can attain a larger set of outcomes if and only if he is better informed.
- Consequently, the optimal decision rule under $\mathcal{U}(S_1)$ must be at least as good as under $\mathcal{U}(S_2)$ for any utility function the agent may have.
- Information is always valuable.

Summary

Experiments:

- Experiments allow us to gain additional information about the state.
- Additional information is always valuable.
- The distribution of posteriors after an experiment is a mean-preserving spread of the distribution of priors.

Importance for information design:

- Information designer provides information to the players through an appropriately chosen "experiment."
- Rationality of the players restricts the information designer to induce distributions over posteriors that are mean-preserving spreads of prior.

Second-Order Stochastic Dominance

Definition 8.7

Let X and Y be real-valued random variables with distribution functions F_X and F_Y, respectively. X second-order stochastically dominates Y if

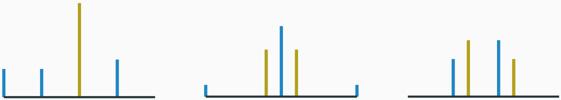
$$\int_{-\infty}^{x} F_Y(t)\,\mathrm{d}t \geq \int_{-\infty}^{x} F_X(t)\,\mathrm{d}t$$

for all x with strict inequality at some x.

Relation to MPS:

- F_Y is a mean-preserving spread of F_X if and only if $\mathbb{E}[X] = \mathbb{E}[Y]$ and X second-order stochastically dominates Y.
- This is very helpful if Θ consists of two states because $\Delta(\Theta) \simeq [0, 1]$, hence $\psi \in \Delta(\Delta(\Theta))$ is described by a distribution function on $[0, 1]$.

Check Your Understanding



Scan Me

True or false:

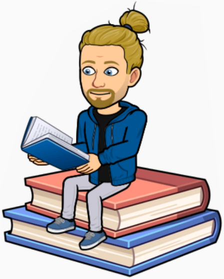
1. The above distributions of posteriors are mean-preserving spreads of the distribution of priors.
2. Suppose $|\Theta| = 2$ and $\mu = \delta_p$. The design of an experiment with 3 possible outcomes has 3 degrees of freedom.
3. If the distribution of posteriors has a higher mean than the prior, the distribution of posteriors is preferred by agents with any utility function.

Short-answer question:

4. Suppose $|\Theta| = 2$ and a player's prior is the distribution function F. What is the most informative Bayes-plausible distribution of posteriors?

Literature

D. Blackwell: Equivalent Comparisons of Experiments, Annals of Mathematical Statistics, **24** (1953), 265–272



Bayesian Persuasion

Bayesian Persuasion

Model:

- There are 2 players, called Sender and Receiver, who share a common prior $\mu \in \Delta(\Theta)$ about a state of nature $\vartheta \in \Theta$.
- The Receiver takes an action $a \in \mathcal{A}$ that determines the payoffs $v(a, \vartheta)$ and $u(a, \vartheta)$ of the Sender and Receiver, respectively.

Persuasion:

- Sender designs an experiment S with conditional distribution $\pi(s \mid \vartheta)$.
- After observing the signal $s \in \mathcal{S}$:
  - Both players update their beliefs via Bayes' rule to posterior $\nu(s)$.
  - The Receiver takes an action a that maximizes $\mathbb{E}_{\nu(s)}[u(a, \theta)]$.
- Equilibrium selection: if multiple actions maximize the Receiver's utility, we select the Sender-preferred action $\widehat{a}(\nu)$ among them.

Relation to Single-Agent Decision Problems

Receiver's problem:

- The Receiver precisely faces a single-agent decision problem.
- By Blackwell's theorem, any information is beneficial to the Receiver.

Sender's problem:

- The Sender designs the Receiver's information environment.
- Can the Sender benefit from persuasion even if the Receiver is perfectly rational and is aware with what intent the signal was created?

Bayesian persuasion vs. information design:

- An information design problem is considered to be Bayesian persuasion if the designer (Sender) is one of the players.
- Typically in Bayesian persuasion models there is only one Receiver.

Relation to Cheap Talk

Sender's information:

- The Sender may not know θ but decides how θ is investigated.
- The Sender may learn θ but can credibly commit to an information revelation policy before learning θ.

Bayesian persuasion vs. cheap talk:

- Signals in cheap talk models are free and non-verifiable.
- Signals in Bayesian persuasion models are free and verifiable.
- One can view Bayesian persuasion also as cheap talk, in which the Sender can commit to an information policy before learning the state.

Persuasion of a Judge

	ϑ_G	ϑ_I
C	$1, 1$	$1, 0$
A	$0, 0$	$0, 1$

Trial in court:

- A defendant stands in court for trial. The states are $\Theta = \{\vartheta_G, \vartheta_I\}$, indicating whether the defendant is guilty or innocent.
- Suppose the common prior μ is that 30% of defendants are guilty.
- The Judge wants to choose the just action: (C)onvict if the defendant is guilty and (A)cquit otherwise.
- The Prosecutor wants to convince the Judge to convict the defendant.

Persuasion of a Judge

	ϑ_G	ϑ_I
C	$1, 1$	$1, 0$
A	$0, 0$	$0, 1$

Choosing an experiment:

- The Prosecutor's investigation is an experiment that results in evidence more frequently if the defendant is guilty.
- While the Prosecutor can be selective about what to investigate, he/she is bound by law to reveal exculpatory evidence.
- Examples: the Prosecutor can choose to look for forensic evidence at the crime scene or summon eye witnesses or expert witnesses.

Choice of Signal

Perfectly revealing investigation:

- This is good for the Judge and leads to a 30% conviction rate.

Completely uninformative investigation:

- The Judge has not enough evidence to convict anybody and acquits every defendant. This is the worst-possible outcome for the Prosecutor.

Optimal signal:

- Sender chooses signal distribution $\pi(s \mid \vartheta)$ that maximizes

$$\mathbb{E}_\mu\left[v\left(\widehat{a}(\nu(S)), \theta\right)\right],$$

where $\widehat{a}$ is the Prosecutor-preferred best response of the Judge.

- What should the signal space $\mathcal{S}$ be?

Revelation Principle

Proposition 8.8

Suppose that $\widehat{a} : \Delta(\Theta) \to \mathcal{A}$ is the equilibrium response to signal S. Then there exists an action recommendation S' with $\mathcal{S}' \subseteq \mathcal{A}$ such that the equilibrium response a_* to S' satisfies $a_*(\nu(s')) = s'$ and $a_*(\nu) = \widehat{a}(\nu)$.

Pooling signals:

- Let $\mathcal{S}_a := \{s \in \mathcal{S} \mid \widehat{a}(\nu(s)) = a\}$ the signals, after which a is played.
- Pool signals in $\mathcal{S}_a$ and call the pooled signal a, that is, define signal S' with $\mathcal{S}' \subseteq \mathcal{A}$ via conditional distribution

$$\pi'(a \mid \vartheta) = \sum_{s \in \mathcal{S}_a} \pi(s \mid \vartheta),$$

where $\pi(s \mid \vartheta)$ is the conditional distribution of S.

Proof of Proposition 8.8

Posterior beliefs:

- Posterior beliefs after observing $s' = a$ are

$$\nu(\vartheta \mid a) = \frac{\pi'(a \mid \vartheta)\mu(\vartheta)}{P(S' = a)} = \sum_{s \in \mathcal{S}_a} \frac{\pi(s \mid \vartheta)\mu(\vartheta)}{P(S \in \mathcal{S}_a)} = \sum_{s \in \mathcal{S}_a} \nu(\vartheta \mid s) \frac{P(S = s)}{P(S \in \mathcal{S}_a)}.$$

Obedience:

- By linearity of the expectation, for any $a' \in \mathcal{A}$, we obtain

$$\begin{aligned} \mathbb{E}_{\nu(a)}[u(a,\theta)] &= \sum_{s \in \mathcal{S}_a} \mathbb{E}_{\nu(s)}[u(a,\theta)] P(S = s \mid S \in \mathcal{S}_a) \\ &\geq \sum_{s \in \mathcal{S}_a} \mathbb{E}_{\nu(s)}\left[u(a',\theta)\right] P(S = s \mid S \in \mathcal{S}_a) = \mathbb{E}_{\nu(a)}\left[u(a',\theta)\right]. \end{aligned}$$

- Moreover, a must be the Sender-preferred maximizer otherwise it would not be the Sender-preferred action for one signal $s \in S_a$.

What Distributions Can the Sender Induce?

Simplifying the maximization problem:

- Sender sends an action recommendation without loss of generality.
- The distribution of posteriors is Bayes plausible for any signal.
- Is the converse true: can any Bayes-plausible distribution of posteriors be induced by some action recommendation?

Bayes plausibility:

- Suppose that ν is the common posterior after observing the signal.
- The interim expected utility of the sender is $\widehat{v}(\nu) := \mathbb{E}_\nu\left[v\left(\widehat{a}(\nu), \theta\right)\right]$.
- The ex-ante expected utility under Bayes-plausible ψ is thus $\mathbb{E}_\psi[\widehat{v}(\nu)]$.

Any Bayes-Plausible Distribution Is Attainable

Lemma 8.9

Suppose that Θ is finite. Then the following statements are equivalent:

1. *There exists a signal S with expected value $v_* = \mathbb{E}_\mu[v(\widehat{a}(\nu(S)), \theta)]$.*
2. *There exists a Bayes-plausible distribution ψ with $v_* = \mathbb{E}_\psi[\widehat{v}(\nu)]$.*
3. *There exists a distribution ψ as in 2. with finite support.*

Importance:

- We know 1. $\Rightarrow$ 2., hence 2. $\Rightarrow$ 3. $\Rightarrow$ 1. is the important direction.
- Instead of finding the optimal signal, the Sender can instead solve:

$$\max_{\psi \in \Delta(\Delta(\Theta)) \,:\, \mathbb{E}_\psi[\nu] = \mu} \mathbb{E}_\psi[\widehat{v}(\nu)].$$

- Since the signal matters only insofar as it affects the posteriors, this eliminates one level of indirection.

Proof of Lemma 8.9

2. implies 3.:

- Suppose there exists Bayes-plausible ψ with $v_* = \mathbb{E}_\psi[\widehat{v}(\nu)]$.
- Show that there exists a Bayes-plausible distribution ψ_* with expected value v_* that is finitely supported, i.e., $\operatorname{supp} \psi_* = \{\nu_1, \ldots, \nu_m\}$ with

$$\mathbb{E}_{\psi_*}[\widehat{v}(\nu)] = \sum_{k=1}^{m} \psi_*(\nu_k)\widehat{v}(\nu_k) = v_*.$$

- The is achieved by applying Caratheodory's theorem.

Theorem 8.10 (Caratheodory's Theorem)

For any set $\mathcal{X} \subseteq \mathbb{R}^d$ and any $x \in \operatorname{conv} \mathcal{X}$, there exist $x_1, \ldots, x_{d+1}$ in $\mathcal{X}$ and $\lambda_1, \ldots, \lambda_{d+1}$ in $[0,1]$ with $\sum_{i=1}^{d+1} \lambda_i = 1$ such that $x = \sum_{i=1}^{d+1} \lambda_i x_i$.

Proof of Lemma 8.9

2. implies 3.:

- Since Θ is finite, the space $\Delta(\Theta)$ corresponds to $[0,1]^{|\Theta|-1}$. The graph $\Gamma(\widehat{v}) := \{(\nu, \widehat{v}(\nu)) \mid \nu \in \Delta(\Theta)\}$ of $\widehat{v}$ is thus a subset of $\mathbb{R}^{|\Theta|}$.
- Bayes-plausibility implies that $(\mu, v_*) = \mathbb{E}_\psi[(\nu, \widehat{v}(\nu))]$, which must lie in $\operatorname{conv}\Gamma(\widehat{v})$ because the expectation is a contraction.
- By Theorem 8.10, there exists ψ_* with $\operatorname{supp}\psi_* = \left\{\nu_1, \ldots, \nu_{|\Theta|+1}\right\}$ and

$$\mu = \sum_{k=1}^{|\Theta|+1} \psi_*(\nu_k)\nu_k, \qquad v_* = \sum_{k=1}^{|\Theta|+1} \psi_*(\nu_k)\widehat{v}(\nu_k).$$

- We conclude that ψ_* is Bayes-plausible attaining v_*.

Proof of Lemma 8.9

Step 2, construct a suitable signal:

- Choose a signal with values in $\mathcal{S} = \left\{s_1, \ldots, s_{|\Theta|+1}\right\}$ such that $\nu(s_k) = \nu_k$.
- We can attain this by setting

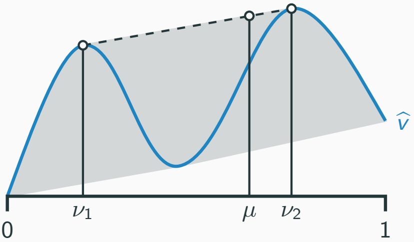
$$\pi(s_k \mid \vartheta) = \frac{\nu_k(\vartheta)\psi_*(\nu_k)}{\mu(\vartheta)}.$$

- We verify that this indeed induces the right posteriors:

$$\nu(\vartheta \mid s_k) = \frac{\pi(s_k \mid \vartheta)\mu(\vartheta)}{\sum_{\vartheta'} \pi(s_k \mid \vartheta')\mu(\vartheta')} = \frac{\nu_k(\vartheta)\psi_*(\nu_k)}{\sum_{\vartheta'} \nu_k(\vartheta')\psi_*(\nu_k)} = \nu_k(\vartheta).$$

- This concludes the proof of Lemma 8.9.

Visualization of Proof



Visualization:

- For any prior $\mu \in \Delta(\Theta)$ and any Bayes-plausible distribution of posteriors ψ, the pair $(\mu, \mathbb{E}_\psi[\widehat{v}(\nu)])$ lies in the convex hull of $\Gamma(\widehat{v})$.
- Conversely, any $(\mu, v_*) \in \Gamma(\widehat{v})$ can be attained through persuasion.
- For fixed μ, persuasion leads to expected payoff $\widehat{v}(\mu)$.
- It is optimal for the Sender to attain $V(\mu) = \sup\{v_* \mid (\mu, v_*) \in \Gamma(\widehat{v})\}$.

Bayesian Persuasion

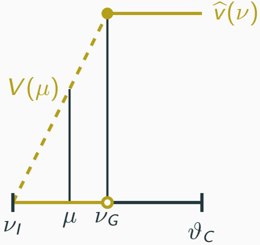
Theorem 8.11

Let $\widehat{a}(\nu)$ denote the equilibrium response and let $V(\nu)$ be the concavification of $\widehat{v}(\nu) = \mathbb{E}_\nu[v(\widehat{a}(\nu), \theta)]$. For prior any $\mu \in \Delta(\Theta)$:

1. *Optimal persuasion yields an expected payoff of $V(\mu)$ to the sender.*
2. *The sender strictly benefits from persuasion if and only if $V(\mu) > \widehat{v}(\mu)$.*
3. *The optimal distribution of posteriors ψ is supported on a finite subset $\{\nu_1, \ldots, \nu_m\}$ of $\{\nu \mid V(\nu) = \widehat{v}(\nu)\}$.*
4. *The optimal signal is supported on $\{s_1, \ldots, s_m\}$ with distribution*

$$\pi(s_k \mid \vartheta) = \frac{\nu_k(\vartheta)\psi_*(\nu_k)}{\mu(\vartheta)}.$$

Persuasion of a Judge



Induced posteriors:

- The Judge's equilibrium response is $\widehat{a}(\nu) = C \cdot 1_{\{\nu \geq 0.5\}} + A \cdot 1_{\{\nu < 0.5\}}$.
- The Prosecutor's interim expected payoff is $\widehat{v}(\nu) = 1_{\{\nu \geq 0.5\}}$.
- The Prosecutor strictly benefits from persuasion if $\mu \in (0, 0.5)$.
- The optimal distribution of posteriors is supported on $\{\nu_A, \nu_C\}$ with

$$\nu_A(\vartheta_G) = 0, \qquad \nu_C(\vartheta_G) = 0.5.$$

Persuasion of a Judge

	ϑ_G	ϑ_I
s_C	1	$\frac{3}{7}$
s_A	0	$\frac{4}{7}$

Optimal investigation:

- Bayes-plausibility implies $\psi(\nu_C)\nu_C(\vartheta_G) = \mu(\vartheta_G)$, hence $\psi(\nu_G) = 0.6$.
- The optimal signal takes values $\{s_A, s_C\}$ with distribution

$$\pi(s_A \mid \vartheta_I) = \frac{1 \cdot 0.4}{0.7} = \frac{4}{7}, \qquad \pi(s_C \mid \vartheta_G) = \frac{0.5 \cdot 0.6}{0.3} = 1.$$

- The optimal investigation obfuscates the truth in state ϑ_I by pooling it with ϑ_G as much as it is allowed by Bayes plausibility.

Lobbying

	ϑ_G	ϑ_B
P	$-2, 1$	$-2, -2$
R	$1, -2$	$1, 1$
A	$0, 0$	$0, 0$

Persuading a politician:

- A benevolent Politician is voting on a bill, whose net effect is either positive ϑ_G or negative ϑ_B with prior $\mu = P(\theta = \vartheta_G) = 0.6$.
- The Politician can (P)ass or (R)eject the bill or he/she can (A)bstain.
- Voting incorrectly is more hurtful than it is beneficial to vote correctly because politicians are afraid to be "on the wrong side of history."
- Suppose you represent an Interest Group who would like to see the bill rejected and you corroborate your lobbying efforts with a study.

Lobbying

Interest group aims to persuade a politician:

- Tobacco industry funds studies about the health effects of smoking.
- Pharmaceutical companies perform clinical trials to prove the effectiveness of their medication.

Concavification Approach

Finitely many states:

- Gives us extremely quick solutions to two-state persuasion problems.
- It is still useful with three states since $\Gamma(\widehat{v})$ can still be visualized.
- For more states, the concavification approach is difficult to apply.

Infinitely many states:

- Suppose that $\Omega = [0, 1]$ and that the Interest Group's payoff depends only on the mean $\mathbb{E}_\mu[\theta]$ of the Politician's posterior.
- This problem is similar to the case $|\Theta| = 2$ via the transformation $\widehat{\mu} := \mathbb{E}_\mu[\theta]$ and the concavification approach yields $V(\mu)$.
- While the optimal signal distribution cannot be deduced from it, we can see whether persuasion is necessary or not.

Lobbying With a Continuum of Policies

Benevolent politician:

- A benevolent Politician chooses a policy $a \in [0, 1]$.
- The optimal policy $\theta \in [0, 1]$ is distribution according to $\mu \in \Delta([0, 1])$.
- The Politician's utility is $u(\vartheta, a) = -(a - \vartheta)^2$, that is, he/she aims to minimize the quadratic distance $\mathbb{E}_\mu\left[(a - \theta)^2\right]$ of the true policy.

Interest group:

- The Interest Group has a preferred policy $a_*(\theta) = \lambda\theta + (1 - \lambda)\vartheta_*$, which depends on θ, but is biased towards ϑ_*.
- The utility function of the Interest Group is $v(a, \vartheta) = -(a - a_*(\vartheta))^2$.
- Which study should the Interest Group optimally commission?

Check Your Understanding

Finding the optimal signal:

In which order do you carry out the following steps?

1. Find the receiver's best response.
2. Find the optimal signal distribution.
3. Concavify the sender's payoff function.
4. Find the sender's interim expected payoff function.
5. Determine the support of the distribution of posteriors.

Scan Me

True or false:

6. We cannot use Bayesian persuasion if the state is known to the sender.
7. Designing state propaganda is a problem of Bayesian persuasion.
8. Creating a dating profile is a problem of Bayesian persuasion.

Literature

- R. Aumann and M.B. Maschler: Repeated Games with Incomplete Information, Chapter 6, MIT Press, 1995
- E. Kamenica and M. Gentzkow: Bayesian Persuasion, American Economic Review, **101** (2011), 2590–2615
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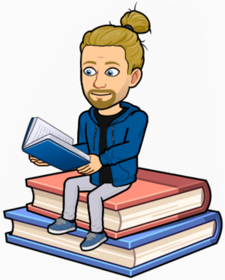


Information Design

Decomposition of Bayesian Games

Bayesian game:

- Consider a Bayesian game $\mathcal{G} = (\mathcal{I}, \Theta, \mathcal{T}, P, \mathcal{A}, u)$ over a finite set Θ of states of nature with a common prior P over $\Theta \times \mathcal{T}$.
- Recall that $\vartheta \in \Theta$ are the payoff-relevant states and the type spaces $\mathcal{T}_i$ capture the players' belief hierarchies over Θ.

Decomposition:

- A Bayesian game $\mathcal{G}$ can be decomposed into:
  - the basic game $\mathcal{G}_0 = (\mathcal{I}, \Theta, \mu, \mathcal{A}, u)$ describes the game's mechanics as a function of state θ, where $\mu \in \Delta(\Theta)$ is the marginal of P.
  - the information structure $\mathcal{J} = (\mathcal{T}, \pi)$ describes the players' information / belief hierarchies about θ, where $\pi(\tau \mid \vartheta) = P(\tau \mid \vartheta)$.

Note: an information structure $(\mathcal{T}, \pi)$ is a set of correlated experiments $T_1, \ldots, T_n$ whose signals are the players' types.

Mechanism Design vs. Information Design

Mechanism design vs. information design:

- Mechanism design: given an information structure $\mathcal{I}$, what is the optimal basic game $\mathcal{G}_0$ that the mechanism designer can choose?
- Information design: given a basic game $\mathcal{G}_0$, what is the optimal information structure $\mathcal{I}$ that the information designer can choose?

Providing information:

- Bayesian persuasion: the receiver is typically uninformed, hence the sender can induce any Bayes-plausible distribution of posteriors.
- If the receiver starts with some non-trivial initial information, the information designer can induce any mean-preserving spread.
- Multi-player information design with non-trivial initial information: we need to define a multi-player analogue to mean-preserving spreads.

Combined Information Structure

Definition 8.12

1. An information structure $(\mathcal{T}^*, \pi^*)$ is a combination of $(\mathcal{T}^1, \pi^1)$ and $(\mathcal{T}^2, \pi^2)$ if $\mathcal{T}_i^* = \mathcal{T}_i^1 \times \mathcal{T}_i^2$ and $\mathrm{marg}_{\mathcal{T}^i} \pi^* = \pi^i$ for $i = 1, 2$, that is,

$$\pi^i(\tau_i \mid \vartheta) = \sum_{\tau_{-i} \in \mathcal{T}_{-i}} \pi^*(\tau_i, \tau_{-i} \mid \vartheta).$$

2. An information structure $\mathcal{J}^*$ is an expansion of $\mathcal{J}^1$ if it is a combination of $\mathcal{J}^1$ and some other information structure $\mathcal{J}^2$.

Omniscient information designer:

- Note that T^1 and T^2 may be correlated under π^*.
- Information designer can design experiments that depend not only on the underlying state, but also on the information players already have.

Individual Sufficiency

Definition 8.13

Information structure $(\mathcal{T}^1, \pi^1)$ is individually sufficient for information structure $(\mathcal{T}^2, \pi^2)$ if there exists a combination $(\mathcal{T}^*, \pi^*)$ such that

$$\text{marg}_{\mathcal{T}^1 \times \mathcal{T}_i^2} \pi^*(\tau_i^2 \mid \tau^1, \vartheta) = \frac{\sum_{\tau_{-i}^2} \pi^*(\tau^1, \tau_i^2, \tau_{-i}^2 \mid \vartheta)}{\sum_{\tilde{\tau}_i^2 \tau_{-i}^2} \pi^*(\tau^1, \tilde{\tau}_i^2, \tau_{-i}^2 \mid \vartheta)}$$

is independent of τ_{-i}^1 and ϑ for every player i.

Interpretation:

- Independence means that neither does T_i^2 provide any new information about T_{-i}^1, nor does it provide new information about θ, given T_i^1.
- Similarly to the single-player case, one can show that $\mathcal{J}^1$ is individually sufficient for $\mathcal{J}^2$ if and only if $\mathcal{J}^1$ is an expansion of $\mathcal{J}^2$,

Relation to Single-Player Case

Blackwell informativeness:

- If the conditional distribution of T_i^2 given T_i^1 is independent of θ, we can write it as a column-stochastic matrix R with:

$$\pi^2(\cdot \mid \vartheta) = R\pi^1(\cdot \mid \vartheta).$$

- Thus, R is the garbling matrix in Definition 8.4.

Mean-preserving spread:

- Each type τ_i is the posterior belief in $\Delta(\Theta)$ after one specific signal.
- Information structure $(\mathcal{T}^*, \pi^*)$ is an expansion of $(\mathcal{T}^1, \pi^1)$ if it is the combination of $(\mathcal{T}^1, \pi^1)$ and some $(\mathcal{T}^2, \pi^2)$.
- Information structure $(\mathcal{T}^2, \pi^2)$ is the map $\varepsilon | T^1 = \tau^1$ that spreads each posterior τ^1 as visualized on Slide 8.

Bayes-Correlated Equilibrium

Definition 8.14

1. A decision rule $\rho : \Theta \times \mathcal{T} \to \Delta(\mathcal{A})$ is a distribution of action recommendations, which may be the result of an experiment on (θ, τ).
2. A decision rule ρ is a Bayes-correlated equilibrium of $(\mathcal{G}_0, (\mathcal{T}, \pi))$ if for every player i, every type $\tau_i \in \mathcal{T}_i$, and every deviation $a_i \in \mathcal{A}_i$,

$$\mathbb{E}_{\rho,\tau_i}[u_i(R, \theta) \,|\, R_i = r_i] \geq \mathbb{E}_{\rho,\tau_i}[u_i(a_i, R_{-i}, \theta) \,|\, R_i = r_i],$$

where R is a random variable with conditional distribution $\rho(T, \theta)$ and

$$P_{\rho,\tau_i}(R_{-i} = r_{-i}, \theta = \vartheta \,|\, R_i = r_i) = \frac{\sum_{\tau_{-i}} \rho(r_i, r_{-i} \,|\, \tau, \vartheta)\pi(\tau \,|\, \vartheta)\mu(\vartheta)}{\sum_{\tau_{-i}, r'_{-i}, \vartheta'} \rho(r_i, r'_{-i} \,|\, \tau, \vartheta')\pi(\tau \,|\, \vartheta)\mu(\vartheta)}.$$

- Players are aware of the conditional distribution of action recommendations R and obedience is incentive compatible, given r_i.

Theorem

Theorem 8.15

A decision rule ρ is a Bayes-correlated equilibrium of $(\mathcal{G}, \mathcal{I})$ if and only if there exist an expansion $\mathcal{I}_*$ of $\mathcal{I}$ and a Bayesian Nash equilibrium σ in $(\mathcal{G}, \mathcal{I}_*)$ such that ρ is the ex-ante distribution over outcomes induced by σ.

Implication:

- Any obedient decision rule can be implemented by:
  - Providing additional information to the players.
  - The players choosing a BNE of the information-designer's choice.[1]
- If the information designer has an objective function $v : \Theta \times \mathcal{A} \to \mathbb{R}$, the goal is to maximize $\mathbb{E}_\rho[v(R, \theta)]$ over all Bayes-correlated equilibria.

[1] This is identical to restricting attention to sender-preferred equilibria in problems of Bayesian persuasion.

Comparison to Mechanism Design

Two-step approach:

- In a first step, we characterize the set of all incentive-compatible mechanisms / obedient decision rules.
- In a second step, we maximize the designer's objective function.
- This approach is also feasible in situations where concavification is difficult to apply because $\mathbb{E}_\rho[v(R,\theta)]$ is linear in ρ.

Partial implementation:

- The players choosing the information designer's preferred BNE is similar to partial implementation in mechanism design.
- More recent literature has analyzed adversarial implementation, where players choose the information designer's least-preferred BNE.

Proof of Theorem 8.15: Sufficiency

Finding a suitable expansion:

- Fix a Bayes-correlated equilibrium ρ in an information structure $(\mathcal{T}, \pi)$.
- Let $(\mathcal{T}^*, \pi^*)$ be a combination of $(\mathcal{T}, \pi)$ and $(\mathcal{A}, \pi')$ such that

$$\pi^*(\tau, r \mid \vartheta) = \rho(r \mid \tau, \vartheta)\pi(\tau \mid \vartheta).$$

- By definition $(\mathcal{T}^*, \pi^*)$ is an expansion of $(\mathcal{T}, \pi)$.

Verifying Bayesian Nash equilibrium:

- Define the obedient strategy profile σ by setting $\sigma_i(\tau_i, r_i; a_i) = 1_{\{a_i = r_i\}}$.
- Obedience of ρ implies that σ is a BNE.

Proof of Theorem 8.15: Necessity

Finding a suitable decision rule:

- Fix a BNE σ in some expansion $(\mathcal{T}^*, \pi^*)$ of $(\mathcal{T}, \pi)$, that is, $(\mathcal{T}^*, \pi^*)$ is the combination of $(\mathcal{T}, \pi)$ and some $(\mathcal{T}', \pi')$.
- Let ρ be the ex-ante distribution over outcomes induced by σ.
- The probability assigned to each elementary outcome (a, τ, ϑ) is

$$\rho(a \mid \tau, \vartheta)\pi(\tau \mid \vartheta)\mu(\vartheta) = \sum_{\tau' \in \mathcal{T}'} \sigma(\tau, \tau'; a)\pi^*(\tau, \tau' \mid \vartheta)\mu(\vartheta).$$

Proof of Theorem 8.15: Necessity

Verifying obedience:

- For any recommendation r_i and any action a_i,

$$\mathbb{E}_{\rho,\tau_i}[u_i(a_i, R_{-i}, \theta) \mid R_i = r_i] = \sum_{r_{-i},\tau_{-i},\vartheta} u_i(a_i, r_{-i})\rho(r \mid \tau, \vartheta)\pi(\tau \mid \vartheta)\mu(\vartheta)$$

$$= \sum_{r_{-i},\tau_{-i},\vartheta} u_i(a_i, r_{-i}) \sum_{\tau' \in \mathcal{T}'} \sigma(\tau, \tau'; r)\pi^*(\tau, \tau' \mid \vartheta)\mu(\vartheta)$$

$$= \sum_{\tau_i' \in \mathcal{T}_i'} \sigma_i(\tau_i, \tau_i'; r_i)\mathbb{E}_{\tau_i,\tau_i',\sigma}[u_i(a_i, A_{-i}, \vartheta)]$$

- Since σ is a BNE for information structure $(\mathcal{T}^*, \pi^*)$, this expression is smaller than or equal to

$$\sum_{\tau_i' \in \mathcal{T}_i'} \sigma_i(\tau_i, \tau_i'; r_i)\mathbb{E}_{\tau_i,\tau_i',\sigma}[u_i(A, \vartheta)] = \mathbb{E}_{\rho,\tau_i}[u_i(R, \theta) \mid R_i = r_i].$$

Investment Game

	ϑ_G	ϑ_B
I	x	-1
N	0	0

Payoffs $u(a, \vartheta)$

Investment game:

- An Investor chooses to (I)nvest or (N)ot to invest. Investment yields a payoff $x \in (0, 1)$ in the good state ϑ_G and -1 in the bad state ϑ_B.
- Suppose that μ is uniform on $\Theta = \{\vartheta_G, \vartheta_B\}$ and that the Government wants to maximize the probability of investment, i.e., $v(a, \vartheta) = 1_{\{a=I\}}$.
- Let us compare the one-player setting with and without prior information, as well as the one- vs. two-player setting.

Investment Game: One Player Without Prior Information

Bayes-correlated equilibrium:

- A decision rule $\rho : \Theta \to \Delta(\mathcal{A})$ is a stochastic, state-contingent recommendation of investment. Denote $p_k = \rho(\vartheta_k; I)$.
- After seeing the recommendation to invest, the investor's posterior is

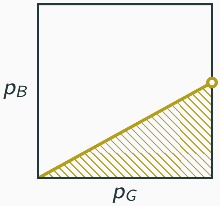
$$\nu(h_I) = \frac{\frac{1}{2} p_G}{\frac{1}{2}(p_G + p_B)}.$$

- The obedience constraint is thus $x\nu(h_I) - (1 - \nu(h_I)) \geq 0$.
- After seeing the recommendation not to invest, the posterior is

$$\nu(h_N) = \frac{\frac{1}{2}(1 - p_G)}{\frac{1}{2}(2 - p_G - p_B)}.$$

- The obedience constraint is thus $x\nu(h_N) - (1 - \nu(h_N)) \leq 0$.

Investment Game: One Player Without Prior Information



Bayes-correlated equilibria

	ϑ_G	ϑ_B
I	$\frac{1}{2}$	$\frac{1}{2}x$
N	0	$\frac{1}{2}(1-x)$

Ex-ante distribution $\rho(a \mid \vartheta)\mu(\vartheta)$ of optimal BCE

Maximizing objective:

- We maximize $V(p_G, p_B) = \mathbb{E}_\rho[v(A,\theta)] = \frac{1}{2}(p_G + p_B)$ subject to

$$p_G x \geq p_B, \qquad 1 - p_B \geq (1 - p_G)x.$$

- Note that the first constraint implies the second constraint, hence the government's objective is maximized at $p_G = 1$ and $p_B = x$.

Investment Game: One Player with Prior Information

	ϑ_G	ϑ_B
τ_g	x	-1
τ_b	0	0

Prior signal $\pi(\tau \mid \vartheta)$

Prior information:

- The investor observes a prior signal about the state, which is correct with probability $q > \frac{1}{2}$. The types are $\mathcal{T} = \{\tau_g, \tau_b\}$.
- A decision rule $\rho : \Theta \times \mathcal{T} \to \Delta(\mathcal{A})$ is now a recommendation that may depend on the true state and the realization of the prior signal.
- We parametrize it with a quadruple $(p_{Gg}, p_{Gb}, p_{Bg}, p_{Bb})$.

Investment Game: One Player Without Prior Information

Bayes-correlated equilibrium:

- After seeing the recommendation to invest, the investor's posterior is

$$\nu(\tau_g, h_I) = \frac{qp_{Gg}}{qp_{Gg} + (1-q)p_{Bg}}, \quad \nu(\tau_b, h_I) = \frac{(1-q)p_{Gb}}{qp_{Bb} + (1-q)p_{Gb}}.$$

- Obedience constraints now have to be satisfied for each type, that is,

$$qp_{Gg}x \geq (1-q)p_{Bg}, \qquad (1-q)p_{Gb}x \geq qp_{Bb}.$$

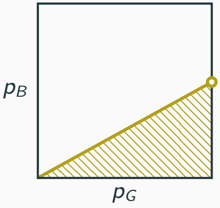
- After seeing the recommendation not to invest, the posterior is

$$\nu(\tau_g, h_N) = \frac{q(1-p_{Gg})}{q(1-p_{Gg}) + (1-q)(1-p_{Bg})}.$$

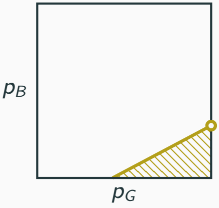
- The obedience constraint are, therefore, given by

$$q(1-p_{Gg})x \leq (1-q)(1-p_{Bg}), \quad (1-q)(1-p_{Gb})x \leq q(1-p_{Bb}).$$

Investment Game: One Player with Prior Information



BCE for $q = 0.6$ and $x = 0.55$



BCE for $q = 0.8$ and $x = 0.55$

Maximizing objective:

- We maximize $V(p_G, p_B) = \mathbb{E}_\rho[v(A,\theta)] = \frac{1}{2}(p_G + p_B)$ subject to

$$p_G x \geq p_B, \qquad p_G \geq q - \frac{1-q}{x},$$

where we denote $p_G = q p_{Gg} + (1-q) p_{Gb}$ and $p_B = (1-q) p_{Bg} + q p_{Bb}$.
- The obedience constraint for no-investment becomes binding for q sufficiently large that investment becomes the default action.

Impact of Additional Information

Definition 8.16

Let $BCE(\mathcal{G}, \mathcal{I})$ denote the set of all Bayes-correlated equilibrium outcomes in $(\mathcal{G}, \mathcal{I})$. We say that information structure $\mathcal{I}$ is more incentive-constrained than $\mathcal{I}'$ if $BCE(\mathcal{G}, \mathcal{I}) \subseteq BCE(\mathcal{G}, \mathcal{I}')$ for every basic game $\mathcal{G}$.

Theorem 8.17

Information structure $\mathcal{I}$ is individually sufficient for $\mathcal{I}'$ if and only if $\mathcal{I}$ is more incentive-constrained than $\mathcal{I}'$.

Interpretation:

- More prior information of the players means that the information designer can choose among a smaller set of information structures.

Investment Game: Two Players without Prior Information

ϑ_G	I	N
I	$x+\varepsilon, x+\varepsilon$	$x, 0$
N	$0, x$	$0, 0$

ϑ_B	I	N
I	$\varepsilon-1, \varepsilon-1$	$-1, 0$
N	$0, -1$	$0, 0$

Two investors:

- Investors get an extra utility of ε if they both invest.
- A decision rule $\rho : \Theta \times \mathcal{T} \to \Delta(\mathcal{A})$ is now a recommendation that may depend on the true state and the realization of the prior signal.
- Without loss of generality, we can restrict attention to symmetric decision rules and parametrize it by (p_G, r_G, p_B, r_B), where r_ϑ is the probability that both receive a recommendation to invest.

Investment Game: Two Players without Prior Information

	I	N
I	r_G	$p_G - r_G$
N	$p_G - r_G$	$1 + r_G - 2p_G$

ϑ_G

	I	N
I	r_B	$p_B - r_B$
N	$p_B - r_B$	$1 + r_B - 2p_B$

ϑ_B

Investment recommendations:

- If $\max\{0, 2p_\vartheta - 1\} \leq r_\vartheta \leq p_\vartheta$, then ρ is indeed a distribution.
- Since not investing is the default action without prior information, the obedience constraint after an investment recommendation is binding:

$$-p_B + p_G x + (r_B + r_G)\varepsilon \geq 0. \tag{1}$$

- The government aims to find the Bayes-correlated equilibrium that maximizes $p_B + p_G$. From (1) we see that we must have $p_G = r_G = 1$.

Investment Game: Two Players without Prior Information

Strategic complements:

- If $\varepsilon > 0$, then r_B relaxes the obedience constraint.
- It is optimal to set $r_B = p_B$ and solve (1) to obtain

$$p_B = r_B = \frac{x + \varepsilon}{1 - \varepsilon}.$$

Strategic substitutes:

- If $\varepsilon < 0$, then r_B tightens the obedience constraint.
- It is optimal to set $r_B = 0$ and solve (1) to obtain

$$p_B = \max\{x + \varepsilon, 0\} =: (x + \varepsilon)^+.$$

Investment Game: Two Players without Prior Information

ϑ_G:

	I	N
I	1	0
N	0	0

$\vartheta_B, \varepsilon > 0$:

	I	N
I	$\frac{x+\varepsilon}{1-\varepsilon}$	0
N	0	$\frac{1-x-2\varepsilon}{1-\varepsilon}$

$\vartheta_B, \varepsilon < 0$:

	I	N
I	0	$(x+\varepsilon)^+$
N	$(x+\varepsilon)^+$	$1-2(x+\varepsilon)^+$

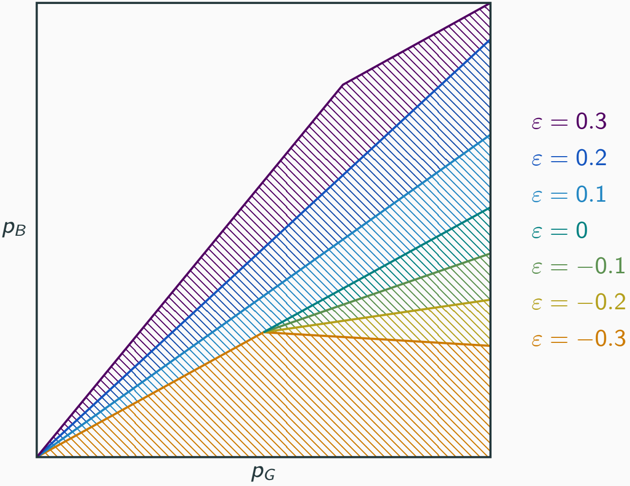
Strategic complements:

- If $\varepsilon > 0$, the optimal information structure is a public.
- It is commonly known among firms that they observe the same signal.

Strategic substitutes:

- If $\varepsilon < 0$, the optimal information structure is a private.
- The correlation among signals is minimized ($=$ maximally negative).

Investment Game: Two Players without Prior Information



BCE for $x = 0.55$

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