2. Belief Spaces and Bayesian Games

ECON 7219 – Games With Incomplete Information

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Last Week

Definition 1.7

Let Θ be a set of finitely many states of nature. An Aumann model of incomplete information over Θ consists of:

1. A finite set of players $\mathcal{I} = \{1, \ldots, n\}$.
2. A finite set Ω of possible states of the world ω.
3. A partition $\mathcal{T}_i = \left\{\tau_i^1, \ldots, \tau_i^{m_i}\right\}$ of Ω into information sets τ_i for each i.
4. A function $\theta : \Omega \to \Theta$, indicating the state of nature $\theta(\omega)$ in each ω.

Knowledge:

- Denote by $T_i(\omega)$ the information set $\tau_i \in \mathcal{T}_i$ with $\omega \in \tau_i$.
- A player i knows event $Y \subseteq \Omega$ in state ω if $T_i(\omega) \subseteq Y$.

Absence of Knowledge

Aumann model of incomplete information:

- Allows us to describe higher-order knowledge between players and how players update their higher-order knowledge given new information.

In absence of knowledge:

- We don't just say "whelp, there's nothing I can do." We form beliefs.

Belief Spaces

Belief Space

Definition 2.1

A finite belief space $(\mathcal{I}, \Omega, (P_i), (\mathcal{T}_i), \theta)$ over states of nature Θ consists of:

1. A finite set of players $\mathcal{I} = \{1, \ldots, n\}$.
2. A finite set Ω of possible states of the world ω.
3. A probability measure P_i over Ω for each player i.
4. A partition $\mathcal{T}_i = \{\tau_i^1, \ldots, \tau_i^{m_i}\}$ of Ω for each player $i \in \mathcal{I}$.
5. A function $\theta : \Omega \to \Theta$, indicating the state of nature $\theta(\omega)$ in each ω.

Remarks:

- P_i is player i's prior belief (before learning his/her information).
- If $P_i = P$ for every player i, we say P is the common prior.
- In settings with common prior, we impose $P(\{\omega\}) > 0$ for every $\omega \in \Omega$.

Relation to Probability Theory

Definition 2.2

A probability space $(\Omega, \mathcal{F}, P)$ consists of

1. A set Ω of states of the world.
2. A collection $\mathcal{F}$ of observable events $Y \subseteq \Omega$.
3. A probability measure P:
   (i) Assigns to each $Y \in \mathcal{F}$ a probability $P(Y) \in [0, 1]$ with $P(\Omega) = 1$.
   (ii) P is σ-additive, that is, for any sequence $(Y_n)_{n\in\mathbb{N}}$ of observable events with $Y_n \cap Y_m = \emptyset$ for any $n \neq m$, we have $P\left(\bigcup_{n\in\mathbb{N}} Y_n\right) = \sum_{n\in\mathbb{N}} P(Y_n)$.

Observable events:

- If Ω is countable, any event is measurable and we choose $\mathcal{F} = 2^{\Omega}$.
- If Ω is uncountable, not every event is measurable by the Banach-Tarski paradox. We then select a well-behaved set $\mathcal{F}$ of observable events.[1]

[1] Formally, well-behaved means that $\mathcal{F}$ is a σ-algebra.

Random Variable

Definition 2.3

An $\mathcal{X}$-valued random variable X is a measurable function $X : \Omega \to \mathcal{X}$.

- Measurability of X means that for any measurable set $B \subseteq \mathcal{X}$, the pre-image $X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) = x\}$ lies in $\mathcal{F}$.
- As a result, we can measure the probability of the event $\{X \in B\}$ by:

$$P(X \in B) = P(\{\omega \in \Omega \mid X(\omega) \in B\}) = P\big(X^{-i}(B)\big).$$

- $P \circ X^{-1}$ is a probability measure on $\mathcal{X}$, called the distribution of X, assigning to $B \subseteq \mathcal{X}$ the probability $(P \circ X^{-1})(B) = P\big(X^{-i}(B)\big)$.
- Note that Ω serves the same role as it does in knowledge models: to allow for correlation among random variables.

Discrete Random Variable

$\Omega = [0,1]^2$

$P =$ area

1	2	3
4	5	6

Roll of a 6-sided die:

- The outcome space is $\mathcal{X} = \{1,2,3,4,5,6\}$.
- The roll is a random variable $X : \Omega \to \mathcal{X}$ such that for each $x \in \mathcal{X}$,

$$P(X = x) = P(\{\omega \in \Omega \mid X(\omega) = x\}) = P\big(X^{-1}(\{x\})\big) = \frac{1}{6}.$$

- The distribution $P \circ X^{-1}$ assigns to each $x \in \mathcal{X}$ probability $\frac{1}{6}$.
- The expectation of X corresponds to integrating out the randomness

$$\mathbb{E}[X] = \int_\Omega X(\omega)\mathrm{d}P(\omega) = \sum_{x\in\mathcal{X}} x\,(P \circ X^{-1})(x) = \sum_{x\in\mathcal{X}} x\,P(X = x).$$

Uncertainty as Random Variables

> *"While in theory randomness is an intrinsic property, in practice, randomness is incomplete information."*
>
> *– Nassim Nicholas Taleb*

State of nature:

- The true state of nature $\theta : \Omega \to \Theta$ is a Θ-valued random variable, whose outcome is simply not observed by (some of) the players.

Common prior vs. heterogeneous prior:

- Common prior: it is common knowledge how θ is generated.
  - Example: cards in poker are dealt after uniformly shuffling the deck.
- Heterogeneous prior: players disagree on how θ is generated.
  - Example: one player believes the dealer is cheating.
- In either case, player *i* updates his/her beliefs based on information τ_i.

Posterior Beliefs via Bayes' Rule

Lemma 2.4 (Bayes' rule)

For any two events $X, Y \subseteq \Omega$ with $P(Y) > 0$,

$$P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)}.$$

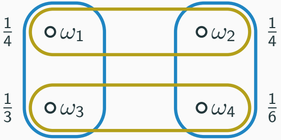
Updating beliefs:

- Player i with prior P_i receives information that the true state is in τ_i.
- If $P_i(\tau_i) > 0$, player i updates his/her beliefs using Bayes rule:
  - For any state $\omega \in \Omega$ or, more generally, for any event $Y \subseteq \Omega$

$$P_i(\{\omega\} \mid \tau_i) = \frac{P_i(\{\omega\} \cap \tau_i)}{P_i(\tau_i)}, \qquad P_i(Y \mid \tau_i) = \frac{P_i(Y \cap \tau_i)}{P_i(\tau_i)}.$$

  - Player i's posterior beliefs $P_{i,\tau_i}(Y) := P_i(Y \mid \tau_i)$ is a probability measure concentrated on τ_i: any event outside of τ_i is known to be impossible.

Computing Posterior Beliefs



Suppose the true state is ω_1:

- Player 1 knows $T_1(\omega_1) = \{\omega_1, \omega_2\}$. Thus, he believes

$$P(\{\omega_1\} \mid T_1(\omega_1)) = \frac{P(\{\omega_1\})}{P(\{\omega_1,\omega_2\})} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}} = \frac{1}{2}.$$

- Player 2 knows $T_2(\omega_1) = \{\omega_1, \omega_3\}$. Thus, he believes

$$P(\{\omega_1\} \mid T_2(\omega_1)) = \frac{P(\{\omega_1\})}{P(\{\omega_1,\omega_3\})} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{3}} = \frac{3}{7}.$$

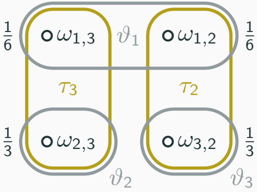
- Similarly, $P(\{\omega_2\} \mid T_1(\omega_1)) = \frac{1}{2}$ and $P(\{\omega_3\} \mid T_2(\omega_1)) = \frac{4}{7}$.

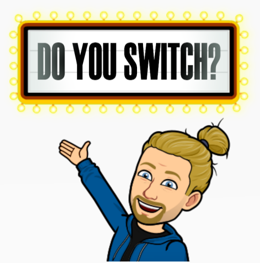
Monty Hall Problem

You are in a game show:

- There are 3 doors. Behind one is a car, behind the others a goat.
- After you pick a door, the game show host opens one of the other doors, behind which is a goat. He then asks you if you would like to switch or stick with your original guess.
- With what probability do you get the car if you switch?

Monty Hall Problem





Modeling the problem:

- Any initial choice has probability $\frac{1}{3}$ of being right. Suppose we choose 1.
- There are 4 states $\omega_{C,M}$: car is behind door C, Monty opens door M.
- If Monty opens door 3, we compute our posterior given τ_3:

$$P(\vartheta_1 \,|\, \tau_3) = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3}, \qquad P(\vartheta_2 \,|\, \tau_3) = \frac{\frac{1}{3}}{\frac{1}{6} + \frac{1}{3}} = \frac{2}{3}.$$

- This makes the underlying assumption that Monty opens each door that he is allowed to open with equal probability.

Beliefs About the States of Nature:

Definition 2.5

Player i's beliefs $\mu_i(\omega) \in \Delta(\Theta)$ over the state of nature in state ω are

$$\mu_i(\omega; \vartheta) := P_{i,T_i(\omega)}(\theta = \vartheta) = P_{i,T_i(\omega)}(\{\widetilde{\omega} \in \Omega \mid \theta(\widetilde{\omega}) = \vartheta\}).$$

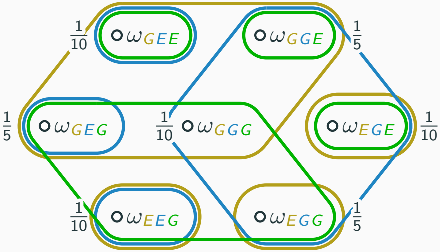
Remark:

- $\mu_i(\omega)$ denotes the entire distribution over Θ, whereas $\mu_i(\omega; \vartheta)$ is the probability that $\mu_i(\omega)$ assigns to $\vartheta \in \Theta$.
- For any ω, ω' in the same information set τ_i, the beliefs coincide.

Monty Hall problem:

- Relevant for our decision are our beliefs over the state of nature θ, indicating behind which door the car is.
- States of the world are only a tool to update our beliefs correctly.

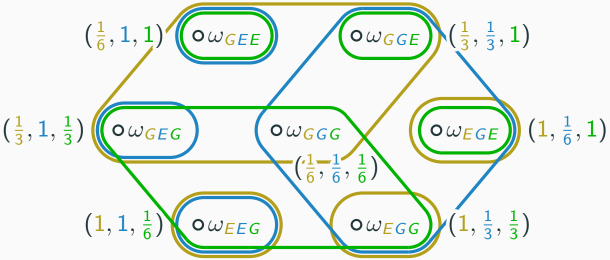
Avalon: a Social Deduction Game



5-player game:

- There are three (G)ood characters and two (E)vil characters.
- Alignment (Good or Evil) is assigned randomly in the beginning. The distribution of this random assignment is the common prior P.
- For a subset of players $\{1, 2, 3\}$, the belief space is depicted above.

Avalon: Posterior Beliefs



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Family of posterior beliefs:

- In each state $\omega_{\vartheta_1\vartheta_2\vartheta_3}$, player i computes their posterior beliefs via Bayes' rule, given the information of their character's alignment ϑ_i.
- Triplet (x, y, z) in state ω indicates $P_{T_i(\omega)}(\{\omega\})$ for players $i = 1, 2, 3$.

What are Player 1's beliefs about θ_2 in states ω_{GGE} and ω_{EEG}?

Beliefs and Knowledge

Strong Beliefs

Proposition 2.6

Consider a finite beliefs space with common prior. Then player i knows event Y in information set τ_i if and only if $P_{\tau_i}(Y) = 1$.

Remarks:

- If players have heterogeneous priors or there are infinitely many states of the world, then the equivalence no longer holds.
- Knowledge, of course, still implies beliefs with probability 1.

Proof of necessity:

- Suppose i knows Y in τ_i, which is defined as $\tau_i \subseteq Y$.
- σ-additivity of probability measures implies that $P_{\tau_i}(Y) \geq P_{\tau_i}(\tau_i) = 1$.

Proof of Proposition 2.6

Proof of sufficiency:

- Suppose that $P(Y \mid \tau_i) = 1$ holds.
- In a finite belief space with common prior, $P(\{\omega\}) > 0$ for every ω.
- In particular, $P(\tau_i) > 0$, hence Bayes rule implies

$$1 = P_{\tau_i}(Y) = P(Y \mid \tau_i) = \frac{P(Y \cap \tau_i)}{P(\tau_i)}.$$

- Therefore, $P(Y \cap \tau_i) = P(\tau_i)$, which implies that $Y \cap \tau_i$ can differ from τ_i only by a probability-0 event.
- Since $P(\{\omega\}) > 0$ for every $\omega \in \Omega$, the only probability-0 event is the empty set. This implies that $Y \cap \tau_i = \tau_i$ and hence $\tau_i \subseteq Y$.

Aumann's Agreement Theorem

Theorem 2.7

Let $(\{1,2\},\Omega,\mathcal{T}_i,\theta)$ be a finite belief space with common prior P. The event "Player i ascribes probability p_i to Y" is $\{\omega \in \Omega \mid P_{T_i(\omega)}(Y) = p_i\}$. Suppose there exists a state $\omega \in \Omega$, in which the two events

"Player 1 ascribes probability p_1 to Y"

"Player 2 ascribes probability p_2 to Y"

are both common knowledge. Then $p_1 = p_2$.

If people "agree to disagree":

- either they have heterogeneous prior distributions,
- or they are not fully rational (e.g., incorrectly process new information).

Proof of Theorem 2.7

Proof setup:

- For $i = 1, 2$, denote $Y_i = \left\{\omega \in \Omega \;\middle|\; P_{T_i(\omega)}(Y) = p_i\right\}$.
- Fix $\omega_* \in \Omega$, in which $Y_1 \cap Y_2$ is common knowledge.

Step 1: Reducing the problem to a self-evident set

- By definition of the common knowledge component $C(\omega_*)$, it satisfies $\omega_* \in C(\omega_*)$, $C(\omega_*) \subseteq Y_1 \cap Y_2$, and $K_i C(\omega_*) = C(\omega_*)$ for $i = 1, 2$.
- By Lemma 1.10, $K_i C(\omega_*) = C(\omega_*)$ implies that $C(\omega_*)$ is the disjoint union of information sets $C(\omega_*) = \bigcup_{\tau_i \in \mathcal{T}_i^*} \tau_i$ for some $\mathcal{T}_i^* \subseteq \mathcal{T}_i$.
- The common prior assumption implies that $P(\{\omega\}) > 0$ for every $\omega \in \Omega$ and hence $P(\tau_i) > 0$ for every $\tau_i \in \mathcal{T}_i^*$ and $P(C(\omega_*)) > 0$.

Proof of Theorem 2.7

Step 2: Use Bayes' rule

- Since any $\tau_i \in \mathcal{T}_i^*$ is a subset of Y_i, the definition of Y_i implies

$$p_i = P_{\tau_i}(Y) = \frac{P(Y \cap \tau_i)}{P(\tau_i)}. \tag{1}$$

- Because $C(\omega_*)$ is the disjoint union of information sets in $\mathcal{T}_i^*$, equation (1) and the law of total probability imply that

$$P(Y \cap C(\omega_*)) = \sum_{\tau_i \in \mathcal{T}_i^*} P(Y \cap \tau_i) = p_i \sum_{\tau_i \in \mathcal{T}_i^*} P(\tau_i) = p_i P(C(\omega_*)). \tag{2}$$

- Equating (2) for $i = 1, 2$ yields

$$p_1 P(C(\omega_*)) = P(Y \cap C(\omega_*)) = p_2 P(C(\omega_*)).$$

- Because $P(C(\omega_*)) > 0$, we deduce that $p_1 = p_2$.

No-Trade Theorem

Theorem 2.8

Suppose two rational, risk-neutral players with common prior P place a bet of size x on the outcome of an event Y, that is, player 1 pays x to player 2 if Y obtains and player 2 pays x to player 1 otherwise. Then

$$P\big(\text{“both players ascribe probability } \tfrac{1}{2} \text{ to } Y\text{”}\big) = 1.$$

Intuition:

- If player 1 accepts the bet, $P_{\tau_1}(Y) \leq \frac{1}{2}$ becomes common knowledge.
- If player 2 accepts the bet, $P_{\tau_2}(Y) \geq \frac{1}{2}$ becomes common knowledge.
- Similarly to the proof of the agreement theorem, it cannot be common knowledge that $P_{\tau_1}(Y) < \frac{1}{2}$ or $P_{\tau_2}(Y) > \frac{1}{2}$ on a set X with $P(X) > 0$.

Implementing the No-Trade Theorem

Communication protocol:

P1: I want to make a bet. Do you want to bet?

P2: Yes, I want to make this bet. Do you still want to bet?

P1: Yes, I want to make this bet. Do you still want to bet?

...

Repeating this procedure indefinitely, one player will eventually back down.

Why do bets happen?

- People have differing prior probabilities.
- People agree to a bet too quickly.

Summary

Belief spaces:

- An extension of knowledge models, in which players form beliefs rationally about any quantities they do not know.

Updating of beliefs:

- Each player i starts with his/her prior beliefs P_i.
- Player i receives information τ_i / learns his/her type τ_i.
- Player i updates his/her beliefs to the posterior beliefs P_{τ_i}.

Beliefs:

- Players' beliefs about the state of nature or any event are computed as the posterior probability of the corresponding pre-image.
- Beliefs about the states of the world serve to correlate beliefs.

Check Your Understanding

True or false:

1. If players receive information twice, they update their beliefs via Bayes' rule twice.
2. If players have no common prior, then the state of nature θ is not randomly generated.
3. Using the formalism of knowledge operators, the event Player 1 knows Player 2's alignment is $K_1(\theta_2)$.
4. By Aumann's agreement theorem it is not possible that two rational poker players have differing beliefs on the hole cards of a third player.

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Short-answer questions:

5. Three players each wear a blue or a white hat. They can see the others' hats but not their own. How many types does each player have?

Literature

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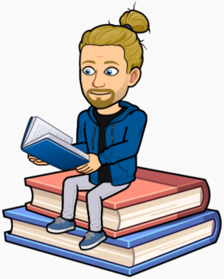


Belief Hierarchies

Beliefs About Other Players Beliefs:

Construction of beliefs:

- Player i's beliefs $\mu_i(\omega; \vartheta)$ measure the probability of the event

$$\{\theta = \vartheta\} = \{\widetilde{\omega} \in \Omega \mid \theta(\widetilde{\omega}) = \vartheta\} =: \theta^{-i}(\vartheta)$$

  under player i's posterior beliefs in information set $T_i(\omega)$.
- In the same way, we can measure $f^{-1}(x)$ for any function $f : \Omega \to \mathcal{X}$.

Beliefs over other players' beliefs:

- For any $\pi \in \Delta(\Theta)$, player i believes that j believes π with probability

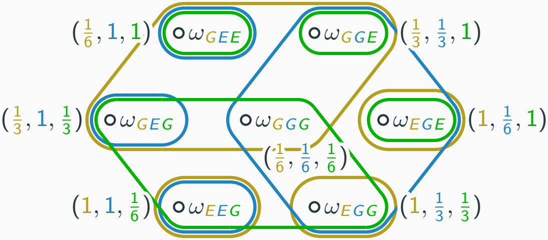
$$\mu_{i,j}(\omega; \pi) := P_{i,T_i(\omega)}(\{\widetilde{\omega} \in \Omega \mid \mu_j(\widetilde{\omega}) = \pi\}).$$

- For a specific $\vartheta \in \Theta$, i expects j to believe ϑ with probability

$$\mu_{i,j}(\omega; \vartheta) := \mathbb{E}_{i,T_i(\omega)}[\mu_j(\vartheta)] = \sum_{\widetilde{\omega} \in \Omega} P_{i,T_i(\widetilde{\omega})}(\{\widetilde{\omega}\})\mu_j(\widetilde{\omega}, \vartheta).$$

- Which one is more appropriate?

Avalon: Posterior Beliefs



Family of posterior beliefs:

- In each state $\omega_{\vartheta_1\vartheta_2\vartheta_3}$, player i computes their posterior beliefs via Bayes' rule, given the information of their character's alignment ϑ_i.
- Triplet (x, y, z) in state ω indicates $P_{T_i(\omega)}(\{\omega\})$ for players $i = 1, 2, 3$.

What are Player 1's higher-order beliefs about θ_2?

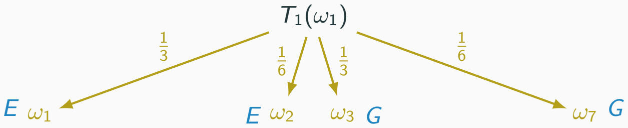
Belief Table

ω	$\theta_2(\omega)$	Posterior $P_{\tau_1(\omega)}$	Posterior $P_{\tau_2(\omega)}$	Posterior $P_{\tau_3(\omega)}$
ω_1	E	$\left[\frac{1}{3}\omega_1, \frac{1}{6}\omega_2, \frac{1}{3}\omega_3, \frac{1}{6}\omega_7\right]$	$[1\omega_1]$	$\left[\frac{1}{3}\omega_1, \frac{1}{3}\omega_5, \frac{1}{6}\omega_6, \frac{1}{6}\omega_7\right]$
ω_2	E	$\left[\frac{1}{3}\omega_1, \frac{1}{6}\omega_2, \frac{1}{3}\omega_3, \frac{1}{6}\omega_7\right]$	$[1\omega_2]$	$[1\omega_2]$
ω_3	G	$\left[\frac{1}{3}\omega_1, \frac{1}{6}\omega_2, \frac{1}{3}\omega_3, \frac{1}{6}\omega_7\right]$	$\left[\frac{1}{3}\omega_3, \frac{1}{6}\omega_4, \frac{1}{3}\omega_5, \frac{1}{6}\omega_7\right]$	$[1\omega_3]$
ω_4	G	$[1\omega_4]$	$\left[\frac{1}{3}\omega_3, \frac{1}{6}\omega_4, \frac{1}{3}\omega_5, \frac{1}{6}\omega_7\right]$	$[1\omega_4]$
ω_5	G	$[1\omega_5]$	$\left[\frac{1}{3}\omega_3, \frac{1}{6}\omega_4, \frac{1}{3}\omega_5, \frac{1}{6}\omega_7\right]$	$\left[\frac{1}{3}\omega_1, \frac{1}{3}\omega_5, \frac{1}{6}\omega_6, \frac{1}{6}\omega_7\right]$
ω_6	E	$[1\omega_6]$	$[1\omega_6]$	$\left[\frac{1}{3}\omega_1, \frac{1}{3}\omega_5, \frac{1}{6}\omega_6, \frac{1}{6}\omega_7\right]$
ω_7	G	$\left[\frac{1}{3}\omega_1, \frac{1}{6}\omega_2, \frac{1}{3}\omega_3, \frac{1}{6}\omega_7\right]$	$\left[\frac{1}{3}\omega_3, \frac{1}{6}\omega_4, \frac{1}{3}\omega_5, \frac{1}{6}\omega_7\right]$	$\left[\frac{1}{3}\omega_1, \frac{1}{3}\omega_5, \frac{1}{6}\omega_6, \frac{1}{6}\omega_7\right]$

How to read the table:

- Player 2 is evil in states ω_1, ω_2, and ω_6, i.e., $\{\theta_2 = E\} = \{\omega_1, \omega_2, \omega_6\}$.
- In state ω_1, Player 2 knows that the state is ω_1, hence $\mu_2(\omega_1; E) = 1$.
- In state ω_1, Player 1's posterior assigns probability $\frac{1}{2}$ to $\{\theta_2 = E\}$, which implies that his/her beliefs over θ_2 are hence $\mu_1(\omega_1; E) = \frac{1}{2}$.

Find Beliefs Visually

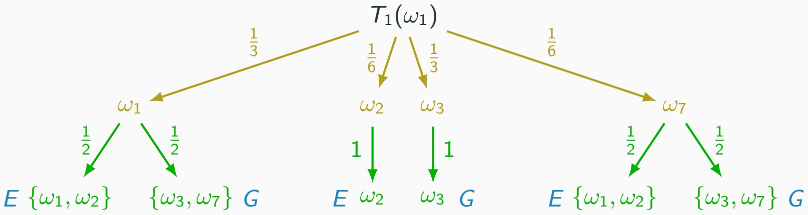


Steps to find $\mu_i(\omega_*; \vartheta)$ visually:

- Evaluate player i's posterior at information set $T_i(\omega_*)$.
- For all states $\omega \in \operatorname{supp} P_{i,T_i(\omega_*)}$, evaluate $\theta(\omega)$.
- Sum up probabilities of all paths ending in $\{\theta = \vartheta\}$.

As before: Player 1 believes Player 2 is evil with probability $\frac{1}{2}$.

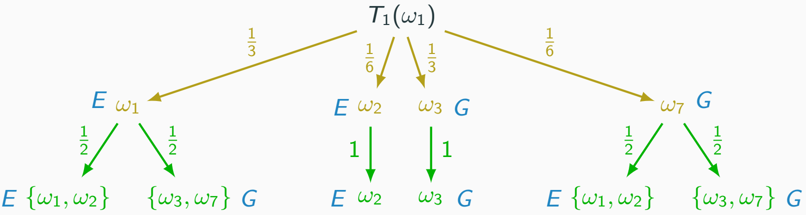
Find Expected Beliefs Visually



Steps to find $\mu_{i,j}(\omega_*; \vartheta)$ visually:

- Evaluate player i's posterior at information set $T_i(\omega_*)$.
- For all states $\omega' \in \operatorname{supp} P_{i,T_i(\omega_*)}$, evaluate player j's posterior $P_{j,T_j(\omega')}$.
- For all states $\omega \in \operatorname{supp} P_{j,T_j(\omega')}$, evaluate $\theta(\omega)$.
- Sum up probabilities of all paths ending in $\{\theta = \vartheta\}$.

Expected Beliefs Are Insufficient



We deduce:

- Player 1 expects that Player 3 believes $\theta_2 = E$ with probability $\frac{5}{12}$.
- However, expected beliefs cannot capture correlation among beliefs:
  - With $P_{\tau_1(\omega_1)}$-probability $\frac{1}{3}$, Player 1 and Player 3 both believe $\theta_2 = E$.
  - With $P_{\tau_1(\omega_1)}$-probability $\frac{5}{12}$, Player 1 and Player 3 both believe $\theta_2 = G$.
  - With $P_{\tau_1(\omega)}$-probability $\frac{1}{4}$, Player 1 and Player 3 disagree about θ_2.

Belief Hierarchy

Definition 2.9

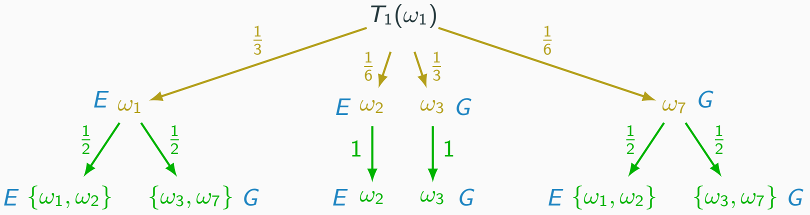
Set $\mathcal{Z}_1 := \Theta$ and define $\mathcal{Z}_k := \mathcal{Z}_{k-1} \times \Delta(\mathcal{Z}_{k-1})^{(n-1)}$ for $k \geq 2$.

1. A belief of order k is an element $\mu^k \in \Delta(\mathcal{Z}_k)$.
2. A belief hierarchy is an element $\beta \in \bigtimes_{k=1}^{\infty} \Delta(\mathcal{Z}_k)$.

Remark:

- A belief hierarchy of player i is an element $\beta_i = (\mu_i^1, \mu_i^2, \ldots)$.
- $\mathcal{Z}_k$ is the set of possible values k^{th}-order beliefs can take: 1^{st}-order beliefs take values in Θ, 2^{nd}-order beliefs take values in $\Theta \times \Delta(\Theta)^{n-1}$, etc.
- Player i's belief of order k correlates the values $\mathcal{Z}_{k-1}$ that player i's $k-1^{\text{st}}$-order beliefs take with the other players' beliefs of order $k-1$.

Avalon: Second-Order Beliefs



Player 1's second-order beliefs about θ_2:

- With probability $\frac{1}{3}$: the state θ_2 is E, Players 2 knows E, and Player 3 believes E and G are equally likely.
- With probability $\frac{1}{6}$: the state θ_2 is E and Players 2 and 3 both know E.
- With probability $\frac{1}{3}$: the state θ_2 is G and Players 2 and 3 both know G.
- With probability $\frac{1}{6}$: the state θ_2 is G, Players 2 knows G, and Player 3 believes E and G are equally likely.

Type Space

Proposition 2.10

A belief space $(\mathcal{I}, \Omega, P, \mathcal{T}, \theta)$ uniquely determines a belief hierarchy $\beta_i(\tau_i)$ in each information set $\tau_i \in \mathcal{T}_i$ for each $i \in \mathcal{I}$. Moreover, β_i is injective.

Proof:

- The procedure on Slide 29 can be extended to arbitrarily high order.
- The generated tree for any $\omega, \omega' \in \tau_i$ coincides.
- If $\tau_i \neq \tilde{\tau}_i$, then first-order beliefs must differ, hence $\beta(\tau_i) \neq \beta_i(\tilde{\tau}_i)$.

Consequence:

- Player i has exactly $|\mathcal{T}_i|$ different possible outlooks on the world.
- Each $\tau_i \in \mathcal{T}_i$ describes one type that player i could be.
- We refer to $\mathcal{T}_i$ as player i's type space.

Types

Types:

- Recall that $T_i(\omega)$ is the information set $\tau_i \in \mathcal{T}_i$ with $\omega \in \tau_i$, that is, it is player i's type when the true state of the world is ω.
- Player i's type $T_i : \Omega \to \mathcal{T}_i$ is thus a $\mathcal{T}_i$-valued random variable.

Interpretation:

- Before they type is drawn (e.g., cards are dealt in poker), nobody knows what i's type will be.
- Player i will learn his/her type when receiving information τ_i since

$$\{T_i = \tau_i\} = \{\omega \in \Omega \mid T_i(\omega) = \tau_i\} = \tau_i$$

  is in player i's knowledge partition by construction.
- Other players will typically have incomplete information about i's type.

Minimal Belief Spaces

Starting with Belief Hierarchies

Construction of belief hierarchies from belief space:

- Players are aware of all the states of the world, everybody's information sets, and everybody's prior probability measure.
- These appear to be rather strong assumptions on players' knowledge.

Inverse construction:

- Players know only (some of) the states of nature and the other players.
- The primitives of such a model are the players' belief hierarchies.
- Can we construct a belief space that is consistent with $\beta = (\beta_1, \ldots, \beta_n)$?

Answer:

- We first construct the minimal belief space that contains a given β.
- Then we construct the universal belief space that contains all β.

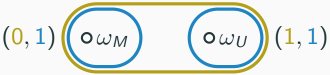
Oedipus: Belief Hierarchies

ω	$\theta(\omega)$	μ_1	μ_2
ω_U	ϑ_U	$[1\omega_1]$	$[1\omega_1]$
ω_M	ϑ_M	$[1\omega_1]$	$[1\omega_2]$

Belief hierarchies:

- There are two states of nature: in state ϑ_U Iocaste is unrelated to Oedipus, whereas as in state ϑ_M Iocaste is Oedipus' mother.
- Oedipus is not even aware of state ϑ_M. Therefore, his entire belief hierarchy is concentrated on ϑ_U: He believes the state us ϑ_U, he believes everybody else believes the state is ϑ_U, etc.
- The Oracle knows Iocaste is Oedipus' mother, but believes that Oedipus believes it is common belief that she is unrelated.

Oedipus: Belief Space



Belief space:

- Oedipus not being aware of ϑ_M implies that he has only a single information set / type $\tau_1 = \{\omega_M, \omega_U\}$ with prior $P_1(\omega_M) = 0$.
- The Oracle can distinguish the two states, hence $\tau_2^U = \{\omega_U\}$ and $\tau_2^M = \{\omega_M\}$. The oracle's prior is $P_2(\omega_M) = 1$.

Posterior beliefs:

- Since Oedipus has only one type, his posterior is equal to his prior.
- Oracle's posterior is $P_{2,\tau_2(\omega)}(\{\omega\}) = 1$ for both states ω.

Belief Operator

Implication:

- Proposition 2.6 indeed fails for heterogeneous priors. Belief with probability 1 does not imply knowledge.
- In fact, if some players are not aware of all states of the world, there is no chance they could know them.
- Can we define a concept analogous to the knowledge operator?

Definition 2.11

For any event $Y \subseteq \Omega$, let $B_i(Y) := \{\omega \in \Omega \mid \mu_i(\omega; Y) = 1\}$ denote the set of states, in which player i believes that event Y obtains.

Note:

- For finite belief spaces with common prior, $B_i = K_i$ by Proposition 2.6.

Belief Axioms

Proposition 2.12

Belief operator B_i satisfies the following properties:

(B1) *Axiom of awareness:* $B_i\Omega = \Omega$,

(B2) *Distribution axiom:* $B_i(X \cap Y) = B_iX \cap B_iY$ *for any events* X, Y,

(B3) *Axiom of introspection:* $B_i\big(B_iY\big) = B_iY$ *for any event* Y,

(B4) *Axiom of negative introspection:* $(B_iY)^c = B_i\big((B_iY)^c\big)$ *for any* Y.

Note:

- As we have seen, the axiom of knowledge cannot hold.
- Belief operator retains all the other properties of a knowledge operator.

Common Belief

Definition 2.13

An event $Y \subseteq \Omega$ is common belief in state ω if for every finite sequence of players $i_1, \ldots, i_k$, we have $\omega \in B_{i_1} B_{i_2} \ldots B_{i_k} Y$.

Proposition 2.14

An event Y is common belief in ω if and only if:

1. $\mu_i(\omega; Y) = 1$ *for every player i,*
2. $\mu_i(\omega'; Y) = 1$ *for every player i and every $\omega' \in Y$.*

Note:

- Property 2 is similar to the existence of a self-evident set in Lemma 1.14: once beliefs reach Y, they are absorbed there.
- The crucial difference is property 1, as it allows $\omega \notin Y$.

Minimal Belief Space

Constructing beliefs space from belief hierarchy:

1. Starting with common-belief states, we can construct the smallest belief table containing (Ω, θ, μ) such that players' first order beliefs μ_i for $i \in \mathcal{I}$ generate the desired belief hierarchy β.
2. We can now define the players' information sets as the set of states $\omega \in \Omega$, on which their belief hierarchies coincide:
$$T_i(\omega) := \{\widetilde{\omega} \in \Omega \mid \mu_i(\widetilde{\omega}) = \mu_i(\omega)\}.$$
3. The fist-order beliefs thus define a family of posteriors $(P_{i,\tau_i})_{\tau_i \in \mathcal{T}_i}$ via $P_{i,\tau_i}(Y) = \mu_i(\omega; Y)$ for any $\omega \in \tau_i$.
4. By Lemma 2.15, there exists a prior probability P_i on Ω that is consistent with the family of posteriors $(P_{i,\tau_i})_{\tau_i \in \mathcal{T}_i}$.
5. The resulting belief space $(\mathcal{I}, \Omega, P, \mathcal{T}, \theta)$ contains belief hierarchy β.

Any Family of Posteriors Is Induced by a Prior

Lemma 2.15

For any partition $\mathcal{T}_i$ and any family of posteriors $(P_{\tau_i})_{\tau_i \in \mathcal{T}_i}$ with $P_{\tau_i}(\tau_i) = 1$, there exists a probability measure P_i on Ω with $P_{\tau_i}(Y) = P_i(Y \mid \tau_i)$.

Proof, step 1:

- Choose any $c_{\tau_i} > 0$ such that $\sum_{\tau_i \in \mathcal{T}_i} c_{\tau_i} = 1$ and set

$$P_i(Y) := \sum_{\tau_i \in \mathcal{T}_i} c_{\tau_i} P_{\tau_i}(Y).$$

- Since each P_{τ_i} is a probability measure, P_i is σ-additive and

$$P_i(\Omega) = \sum_{\tau_i \in \mathcal{T}_i} c_{\tau_i} = 1.$$

Proof of Lemma 2.15

Proof, step 2:

- It follows from Bayes' rule implies and the definition of P_i that

$$P_i(Y \mid \tau_i) = \frac{P_i(Y \cap \tau_i)}{P_i(\tau_i)} = \frac{\sum_{\tau_i' \in \mathcal{T}_i} c_{\tau_i'} P_{\tau_i'}(Y \cap \tau_i)}{\sum_{\tau_i' \in \mathcal{T}_i} c_{\tau_i'} P_{\tau_i'}(\tau_i)}. \tag{3}$$

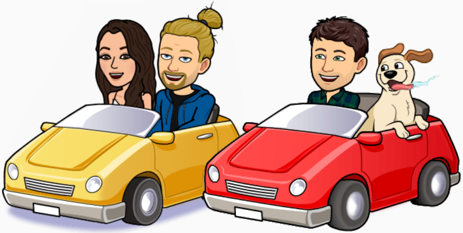
- Since $P_{\tau_i'}(\tau_i') = 1$ for any τ_i', it follows that for any $\tau_i \neq \tau_i'$,

$$0 \leq P_{\tau_i'}(Y \cap \tau_i) \leq P_{\tau_i'}(\tau_i) = 0.$$

- Therefore, (3) reduces to

$$P_i(Y \mid \tau_i) = \frac{c_{\tau_i} P_{\tau_i}(Y \cap \tau_i)}{c_{\tau_i}} = P_{\tau_i}(Y \cap \tau_i) + P_{\tau_i}(Y \cap \tau_i^c) = P_{\tau_i}(Y).$$

Turning Left or Right?



Andrew, Ben, and Flo are on a road trip. At an intersection, Andrew believes that they should turn right and that this is common belief among the three. Ben believes that they should turn right, that the others both believe they should turn left, that the others believe everyone else believes that they should turn left, and so on. Flo believes that they should turn left or right with equal likelihood, that the others know the correct way to turn, and that the others believe it is common belief to turn that way. Find the minimal belief space that contains this belief hierarchy.

Universal Type Space

Universal Belief Space

Universal belief space:

- We do not need to do this for each specific belief hierarchy.
- The universal belief space contains all possible belief hierarchies.

Recall:

- In a finite belief space, we called τ_i player i's type because there is a one-to-one correspondence to player i's belief hierarchies.
- Now we start with belief hierarchies and simply call those types.

Types:

- Let $\mathcal{B}_0 := \bigtimes_{k=1}^{\infty} \Delta(\mathcal{Z}_k)$ denote the space of all types/belief hierarchies.
- Do we need to consider belief hierarchies over types?

Coherent Belief Hierarchies

Definition 2.16

A belief hierarchy $\beta_i = (\mu_i^1, \mu_i^2, \dots)$ is coherent if

- the marginal distribution of μ_i^2 on $\Delta(\Theta)$ is μ^1,
- the marginal distribution of μ_i^k on $\Delta(\mathcal{Z}_{k-2})$ coincides with marginal distribution of μ_i^{k-1} onto $\Delta(\mathcal{Z}_{k-2})$.

Let $\mathcal{B}_1$ denote the space of all coherent types/belief hierarchies.

Interpretation:

- Higher-order beliefs do not contradict lower-order beliefs.
- Ensures that questions like "what probability does player i ascribe to event Y" has an unequivocal answer.

Coherent Belief Hierarchies

Proposition 2.17

There exists a homeomorphism $\varphi : \mathcal{B}_1 \to \Delta(\Theta \times (\mathcal{B}_0)^{n-1})$.[2]

Illustration:

- A belief hierarchy is an element of

$$\beta \in \overbrace{\Delta(\Theta)}^{\Delta(\mathcal{Z}_1)} \times \overbrace{\Delta(\Theta \times \Delta(\mathcal{Z}_1)^{n-1})}^{\Delta(\mathcal{Z}_2)} \times \overbrace{\Delta(\Theta \times \Delta(\mathcal{Z}_1)^{n-1} \times \Delta(\mathcal{Z}_2)^{n-1})}^{\Delta(\mathcal{Z}_3)} \times \dots$$

$$\Delta\big(\Theta \times (\Delta(\mathcal{Z}_1)^{n-1} \times \Delta(\mathcal{Z}_2)^{n-1} \times \dots)\big) = \Delta(\Theta \times (\mathcal{B}_0)^{n-1})$$

- Coherency implies that beliefs over Θ, $\Delta(\mathcal{Z}_1)^{n-1}$, etc. are well defined.

[2] A homeomorphism is a continuous bijection with a continuous inverse: $\mathcal{B}_1$ and $\Delta(\Theta \times (\mathcal{B}_0)^{n-1})$ are identical in a topological sense.

Common Belief of Coherency

Implication of Proposition 2.17:

- A coherent type has well-defined beliefs over opponents' types.
- But: it does not determine i's beliefs over j's beliefs over i's type if believes it is possible that j believes i's type is not coherent.

Impose common belief of coherency:

- Define $\mathcal{B}_2 := \left\{\beta \in \mathcal{B}_1 \mid \varphi(\beta)(\Theta \times (\mathcal{B}_1)^{n-1}) = 1\right\}$ denote the set
    - of all coherent types $\beta \in \mathcal{B}_1$ such that
    - they believe with probability 1 that opponents have coherent types.
- Inductively, let $\mathcal{B}_k := \left\{\beta \in \mathcal{B}_1 \mid \varphi(\beta)(\Theta \times (\mathcal{B}_{k-1})^{n-1}) = 1\right\}$ and set

$$\mathcal{T} := \bigcap_{k=1}^{\infty} \mathcal{B}_k.$$

- $\mathcal{T}$ is the set of coherent types that satisfy common belief in coherency.

Universal Type Space

Theorem 2.18

1. *The universal type space $\mathcal{T}$ is homeomorphic to $\Delta(\Theta \times \mathcal{T}^{n-1})$.*
2. *Any belief space is a subspace of the universal belief space $\Theta \times \mathcal{T}^n$.*

Consequences:

- A type in $\mathcal{T}$ determines all higher-order beliefs over opponents' types.
- Can set $\Omega = \Theta \times \mathcal{T}^n$ without loss of generality.
- In particular, $\omega = (\theta(\omega), T_1(\omega), \ldots, T_n(\omega))$.

Summary

Belief hierarchies and types:

- Because a belief hierarchy completely determines a player's outlook on the game, we can characterize players through their belief hierarchies.
- A player's information set uniquely determines his/her belief hierarchy.
- Consequently, we refer to players' information sets as the players' types.

Finite/minimal belief spaces:

- They are tractable, but do not allow arbitrary beliefs hierarchies.
- If the belief hierarchy is fixed, we work with the minimal belief space.

Universal belief space / type space:

- The universal belief space allows all possible belief hierarchies over Θ.
- Any research should be carried out with the universal type space.

Check Your Understanding

True or false:

1. In the Avalon example, the belief table on Slide 27 generates only up to third-order beliefs.
2. In poker, a players' belief hierarchy at the beginning of the round is completely specified by his hole cards.
3. In a finite belief space with a common prior, common belief is identical to common knowledge.
4. Every family of posteriors defines a unique belief hierarchy, but not every belief hierarchy defines a unique family of posteriors.

Scan me

Short-answer question:

5. In the Avalon example, what probability does Player 1 assign to Player 3 knowing Player 2's alignment in state ω_{GEG}?

Literature

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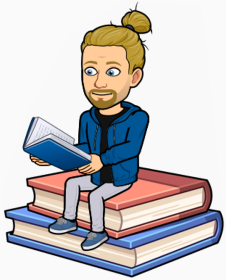

Bayesian Nash Equilibrium

Bayesian Game

Definition 2.19

A Bayesian game $\mathcal{G} = \big(\mathcal{I}, \Theta, (\mathcal{T}_i), (P_i), (\mathcal{A}_i), (u_i)\big)$ consists of:

1. A finite set of players $\mathcal{I} = \{1, \dots, n\}$,
2. A set of states of nature Θ.
3. A type space $\mathcal{T}_i$ for each player i, where each type $\tau_i \in \mathcal{T}_i$ determines:
   (i) the set $\mathcal{A}_i(\tau_i)$ of pure actions available to player i,
   (ii) player i's beliefs over the true state of nature θ and the type profile τ_{-i} of the other players via a posterior probability measure P_{i,τ_i} on $\Theta \times \mathcal{T}_{-i}$.
4. A payoff function $u_i : \mathcal{A} \times \Theta \to \mathbb{R}$ for each player $i \in \mathcal{I}$, where

$$\mathcal{A}_i := \bigcup_{\tau_i \in \mathcal{T}_i} \mathcal{A}_i(\tau_i), \qquad \text{and} \qquad \mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n.$$

Bayesian Game: Remarks About the Definition

Definition using type spaces:

- Convenient for applications because we do not need to specify Ω.
- Even though the probability space is less visible, the true state θ and the players' types $T_1, \ldots, T_n$ are still random variables.
- Player i's type τ_i determines his/her beliefs over $\Theta \times \mathcal{T}_{-i}$, which we denote again by P_{i,τ_i}.

What type space to choose?

- In real-word applications, one often uses the universal type space.
- In this class, we often work with specific types spaces that are particularly tractable (either finite or types are independent).

Bayesian Game: Remarks About the Definition

Independent types with common prior P:

- If T_i is independent of T_j for $j \neq i$, then player i's type reveals nothing about the other players' types.
- Higher-order beliefs are thus common knowledge and determined by P.
- As a consequence, player's type coincides with his/her beliefs about θ.

States of nature:

- From a theoretical standpoint, it is not necessary to include Θ in the definition of a Bayesian game.
- Instead, payoffs could depend directly on type profile $\tau = (\tau_1, \ldots, \tau_n)$.
- Since Θ is often observable in applications, we choose this approach.

Strategies in a Bayesian Game

Definition 2.20

1. A pure strategy of player i is a map $s_i : \mathcal{T}_i \to \mathcal{A}_i$ with $s_i(\tau_i) \in \mathcal{A}_i(\tau_i)$.
2. A mixed strategy σ_i of player i is a distribution over pure strategies.
3. A behavior strategy is a map $\sigma_i : \mathcal{T}_i \to \Delta(\mathcal{A}_i)$ with $\sigma_i(\tau_i) \in \Delta(\mathcal{A}_i(\tau_i))$.

Interpretation:

- By Proposition 2.10, Player i's belief hierarchy is determined by a prior probability P_i and information τ_i that player i learns.
- Before learning τ_i, player i decides what he/she will do in any situation.

Mixed vs. behavior strategy:

- In a mixed strategy, a player randomizes before learning his/her type.
- In a behavior strategy, a player randomizes after learning his/her type.

Outcome and Expected Payoffs

Outcome:

- The outcome of the game is the tuple (θ, T, A).
- The realized action profile $A = (A_1, \ldots, A_n)$ satisfies for each player i:
  - Conditional on $\mathcal{T}_i = \tau_i$, A_i is independent from θ, T_j, and A_j for $j \neq i$.
  - Conditional on $\mathcal{T}_i = \tau_i$, the distribution of A_i is $\sigma_i(\tau_i) \in \Delta(\mathcal{A}_i)$.
- We denote by $\sigma_i(\tau_i; a_i)$ the probability that $\sigma_i(\tau_i)$ assigns to $a_i \in \mathcal{A}_i$.

Induced probability measure:

- Player i of type τ_i's subjective distribution over (θ, A) is

$$\begin{aligned} P_{\tau_i,\sigma}(\theta = \vartheta, A = a) &:= \sum_{\tau_{-i} \in \mathcal{T}_{-i}} P_{\tau_i,\sigma}(\theta = \vartheta, A = a \,|\, T = \tau) P_{i,\tau_i}(T = \tau) \\ &= \sum_{\tau_{-i} \in \mathcal{T}_{-i}} \sigma_i(\tau_i; a_i) P_{i,\tau_i}(\theta = \vartheta, T = \tau), \end{aligned}$$

where P_{i,τ_i} is player i's posterior belief.

Bayesian Nash Equilibrium

Definition 2.21

A Bayesian Nash equilibrium is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for every player i, every type τ_i, and every action $a_i \in \mathcal{A}_i(\tau_i)$,

$$\mathbb{E}_{\tau_i,\sigma}[u_i(\theta, A)] \geq \mathbb{E}_{\tau_i,(a_i,\sigma_{-i})}[u_i(\theta, A)].$$

Interpretation:

- No player has an incentive to deviate *after* learning their type.
- We say that $\sigma_i(\tau_i)$ maximizes player i's interim expected payoff.
- Player i's ex-ante expected payoff is the expected payoff before player i learns his/her type, i.e., the expectation given the prior distribution

$$\mathbb{E}_{i,\sigma}[u_i(\theta, A)] = \sum_{\tau_i \in \mathcal{T}_i} \mathbb{E}_{\tau_i,\sigma}[u_i(\theta, A)] P_i(T_i = \tau_i).$$

Existence

Separate-players interpretation:

- If $\mathcal{T}_i$ is finite, we can interpret each type τ_i as a separate player.
- Player τ_i's available pure actions are $\mathcal{A}_{\tau_i} := \mathcal{A}_i(\tau_i)$.
- Player τ_i's payoff in action profile $a = (a_{\tau_1^1}, a_{\tau_1^2}, a_{\tau_1^3}, \ldots, a_{\tau_n^{m_n}})$ is

$$u_{\tau_i}(a) := \sum_{\tau_{-i} \in \mathcal{T}_{-i}} u_i(\vartheta, a_{\tau_1}, \ldots, a_{\tau_n}) P_{i,\tau_i}(\theta = \vartheta, T_{-i} = \tau_{-i}).$$

- The Bayesian game is thus equivalent to the static game among players $\bigcup_{i \in \mathcal{I}} \mathcal{T}_i$ with available pure actions $\mathcal{A}_{\tau_i}$ and payoff functions u_{τ_i}.

Existence:

- A BNE of a Bayesian game is a NE of the equivalent static game.
- If $\mathcal{T}_i$ and $\mathcal{A}_i$ is finite for each player i, existence of a BNE in behavior strategies thus follows from existence of mixed NE in static games.

Selling Farmland

Two of Taiwan's most valuable crops are tea and rice.

Annual average yield:

- Tea: 5.35m NTD/km^2.
- Rice: 4.2m NTD/km^2.

A Rice Farmer considers selling his/her land to a Tea Farmer.

- Suppose high/low-quality soil yields 50% above/below average.
- The Rice Farmer knows the quality, but the Tea Farmer does not.
- Can trade between the two farmers occur? If so, at which price?

Source: https://www.visualcapitalist.com/the-worlds-most-valuable-cash-crop/

Modeling the Interaction as a Game

States of Nature:

- Quality of soil θ can be low, medium, or high, that is, $\Theta = \{L, M, H\}$.
- Suppose the common prior P assigns equal probability to all three.

Types of players:

- The Rice Farmer knows θ: he has 3 information sets and hence 3 types.
- The Tea Farmer has a single type, hence his beliefs are equal to P.

Actions and strategies:

- The Tea Farmer can offer a price $p \geq 0$.
- The Rice Farmer of any type chooses set of prices $\mathcal{P}$ he/she accepts.
- A pure strategy of the Rice Farmer is thus a map

$$s_R : \{L, M, H\} \to \{\text{subsets of } \mathbb{R}_+\}.$$

Utilities

Value of the land:

- The annual yield of tea and rice per km^2 is

$$y_T(L) = 2.7, \qquad y_T(M) = 5.35, \qquad y_T(H) = 8,$$
$$y_R(L) = 2.1, \qquad y_R(M) = 4.2, \qquad y_R(H) = 6.3.$$

- Suppose the value of the land is the yield in perpetuity, i.e., if players discount future payoffs with discount factor $\delta = \frac{1}{2}$, the value is

$$v_i(\vartheta) = \sum_{t=0}^{\infty} \delta^t y_i(\vartheta) = \frac{y_i(\vartheta)}{1-\delta} = 2y_i(\vartheta).$$

Utilities:

- Tea Farmer's utility is $u_T(p, \mathcal{P}, \vartheta) = (v_T(\vartheta) - p)1_{\{p \in \mathcal{P}\}}$,
- Rice Farmer's utility is $u_R(p, \mathcal{P}, \vartheta) = p \cdot 1_{\{p \in \mathcal{P}\}} + v_R(\vartheta)1_{\{p \notin \mathcal{P}\}}$.

Dominated Strategies

To sell or not to sell:

- It cannot be optimal to sell at a price $p < v_R(\vartheta)$.
- Formally, $\mathcal{P}(\vartheta) \cap [0, v_R(\vartheta)) = \emptyset$ in a Bayesian Nash equilibrium.

No trades with type H:

- For any price $p < 12.6$, type H is not willing to sell.
- The expected value of the land to the Tea Farmer is

$$\mathbb{E}[v_T(\theta)] = \frac{1}{3}(5.4 + 10.7 + 16) = 10.7.$$

- Offering $p \geq 12.3 > 10.7$ is thus strictly dominated.
- No trades with type H are incentive compatible for both farmers.

Conditionally Dominated Strategies

No trades with type M:

- Knowing he/she won't trade with type H, the conditional expected value of the land to the Tea Farmer is

$$\mathbb{E}[v_T(\theta) \,|\, \theta \neq H] = \frac{1}{2}(5.4 + 10.7) = 8.05.$$

- For any price $p < 8.4$, however, type M is not willing to sell.
- Offering $p \geq 8.4 > 8.05$ is thus strictly dominated.

Trading only with type L:

- Tea Farmer will not offer more than $y_T(L) = 5.4$.
- Rice Farmer will not accept any less than $y_R(L) = 4.2$.

Equilibrium Trades

Equilibrium outcomes:

- Trade at any price $p \in [4.2, 5.4]$ can be supported in equilibrium by

$$\mathcal{P}(L) \cap [4.2, 5.4] \neq \emptyset, \qquad p = \min \mathcal{P}(L).$$

- No trade is also an equilibrium outcome, supported by strategies

$$\mathcal{P}(L) \cap [4.2, 5.4] = \emptyset, \qquad p < 4.2.$$

Inefficiency:

- Trade can occur only with the low type, even though the Tea Farmer values the land more than the Rice Farmer in any state of nature.
- Private information can be a great source of inefficiency.
- Players use weakly dominated strategies in the no-trade equilibrium.

Adverse Selection

Adverse selection:

- Adverse selection arises in a market situation with asymmetric information between buyers and sellers.
- The informed side chooses to participate selectively in trades which benefit them the most, at the expense of the other trader.
- The uninformed side experiences an adverse selection of market participants, leaving it with worse-than-average trading partners.

Equilibrium unraveling:

- The equilibrium unravels if trade occurs only with the lowest type.
- The equilibrium need not unravel completely. If high- or low-quality soil affected yield by only 33%, trade could occur with the M-type.

Adverse Selection in Insurance

Facing many buyers:

- Suppose an insurance company offers one universal insurance.
- The insurance policy is priced such that the premiums are equal to the expected claims from the population.
- Since the insurance policy is too expensive for healthy individuals, those will opt out, leaving the insurance with clients of below average health.

Summary

Bayesian games:

- There is uncertainty about a payoff-relevant state of nature θ.
- A player's type determines the information he/she has about θ as well as which actions the player has available.
- Players' belief hierarchies are either determined via a belief space or specified explicitly through a belief table.
- A strategy in a Bayesian games specifies a decision for each type.

Bayesian Nash equilibrium:

- Players maximize their interim expected value, given their beliefs.
- In a Bayesian Nash equilibrium, no player has an incentive to deviate after learning their type.

Check Your Understanding

True or false:

1. A strategy of an Avalon player involves a plan of action for every role he/she might have.
2. The outcome of a game is a random variable.
3. For a finite Bayesian game with a positive a prior, any Nash equilibrium is a Bayesian Nash equilibrium.
4. An equivalent definition of a Bayesian game $\mathcal{G}$ is: $\mathcal{G}$ is a belief space, in which each state of nature is a game of complete information.

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Short-Answer Question:

5. What is the belief hierarchy in the farming example if $\theta = M$?

Literature

- D. Fudenberg and J. Tirole: Game Theory, Chapter 6, MIT Press, 1991
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- S. Tadelis: Game Theory: An Introduction, Chapter 12, Princeton University Press, 2013
- J.Harsanyi: Games with Incomplete Information Played by 'Bayesian' Players, Part I-III, Management Science, **14** (1967, 1968), 159–182, 320–334, 486–502.