

10. Repeated Games and Reputations

ECON 7219 – Games With Incomplete Information

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Advantages and Disadvantages of Extensive-Form Games

Advantages of extensive-form games:

- It is a very general model, allowing imperfect information due to strategic uncertainty, payoff uncertainty, or imperfect observation.
- A game tree provides a good visual for small games.

Disadvantages of extensive-form games:

- The notion of a game tree inherently restricts the game to be finite.
- The notation is quite cumbersome: keeping track of the transition between nodes, information sets, the identity of the active player, etc.
- The generality makes it hard to characterize equilibria analytically.

Impose additional structure:

- In a **repeated game**, players repeatedly play the same stage game.

Beliefs in Extensive-Form Games

Off-path beliefs:

- Off-path beliefs are a nuisance already in 2-player, 2-type, 2-action signaling games. It becomes exponentially worse in longer games.

Payoff vs. information:

- Uninformed players face a trade-off between earning a higher payoff and learning more information at every non-terminal information set.

Improving tractability:

- There will be a single strategic type and a variety of **commitment types**, who invariably play a commonly known strategy.
- Therefore, off-path belief assign probability 1 to the strategic type.
- The uninformed player is **short-lived** and cares only about the payoff today. Such a player does not care about informational gain.

Repeated Games

Tipping in a Restaurant

	<i>E</i>	<i>S</i>
<i>E</i>	2, 2	-1, 3
<i>S</i>	3, -1	0, 0



Tipping:

- The **Waiter** can exert (*E*)ffort or he/she can (*S*)hirk.
- The **Customer** may choose to tip (*E*) or not to tip (*S*).
- Isolated interaction: *S* is the strictly dominant action for each player.

Incentives for tipping:

- The **Customer** is unsure whether he/she will go back in the future.
- Possibility for future retaliation by the **Waiter** may provide incentives.

Repeated Game

Repeating the stage game:

- Players repeat stage game $\mathcal{G} = (\mathcal{I}, \mathcal{A}, u)$ a total of $T \leq \infty$ times.
- Periods are labeled by $t = 0, \dots, T - 1$ for simplicity of notation.

Perfect monitoring:

- At the end of each period, players observe the chosen action profiles, that is, the history h^t at time t is of the form $h^t = (a^0, \dots, a^{t-1})$.
- Let \mathcal{H}^t denote the set of t -period histories and let $\mathcal{H}(T) = \bigcup_{t=0}^T \mathcal{H}^t$.
- We often omit the dependence on T if it is clear from context.

Strategies:

- A pure strategy of player i is a map $\sigma_i : \mathcal{H} \rightarrow \mathcal{A}_i$.
- A behavior strategy of player i is a map $\sigma_i : \mathcal{H} \rightarrow \Delta(\mathcal{A}_i)$.

Distribution over Outcomes

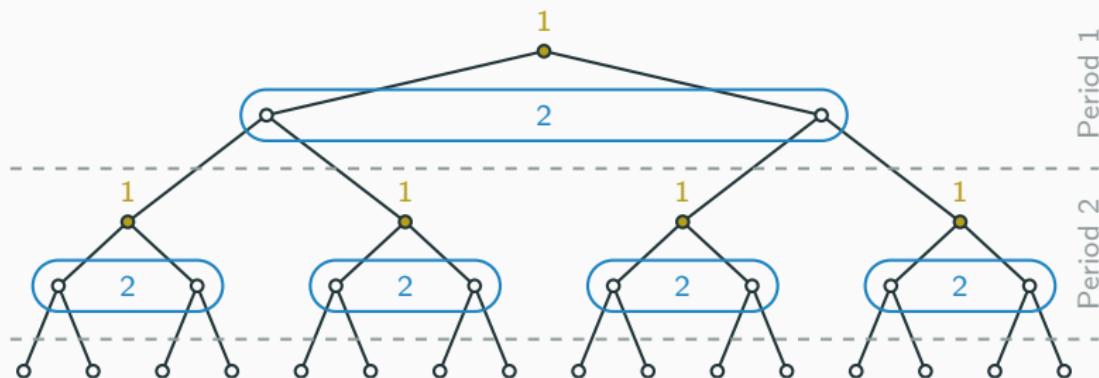
Outcome:

- The **outcome** of a repeated game is the random sequence $A = (A^t)_{t \geq 0}$.
- Conditional on $H^t := (A^0, \dots, A^{t-1})$, the random variables A_i^t and A_j^t are independent for any $j \neq i$ and any $t \geq 0$.
- We denote by $H = (H^t)_{t \geq 0}$ the sequence of (random) histories.

Distribution over outcomes:

- Play of strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ induces a probability measure P_σ , under which A_i^t is distributed according to $\sigma_i(H^t)$.
- For finitely repeated games, P_σ is identical to the probability measure we defined for extensive-form games.

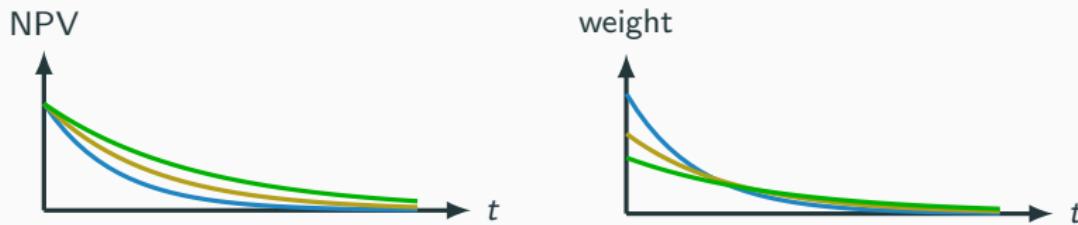
Game Tree



Continuation games:

- The only proper subgames start in some period t with history h^t .
- The continuation game is **identical** for each history $h^t \in \mathcal{H}^t$ and every player has the same information h^t in the continuation game.
- Because the game may be infinite, there may be no terminal nodes. Payoffs are earned at the end of each period instead.

Payoffs



Payoffs:

- Players discount future payoffs with a discount factor $\delta \in (0, 1)$.
- Players maximize their **average discounted payoff** is

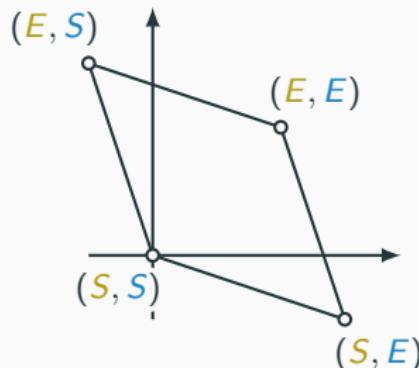
$$U(\sigma) := \frac{1 - \delta}{1 - \delta^T} \sum_{t=0}^{T-1} \delta^t \mathbb{E}_\sigma [u(A^t)]. \quad (1)$$

Benefits of considering average payoffs:

- Repeated-game payoffs are in the same units as stage-game payoffs.
- It allows a meaningful comparison between different δ and T .

Feasible Payoffs

	E	S
E	2, 2	-1, 3
S	3, -1	0, 0



Feasible payoffs:

- Repeated-game payoffs are a weighted sum of stage-game payoffs.
- Any strategy profile thus attains a payoff vector in $\mathcal{V} := \text{conv}(u(\mathcal{A}))$.
- Conversely, any payoff vector in \mathcal{V} can be attained by some strategy profile in the infinitely repeated game.
- In either case, we call \mathcal{V} the set of **feasible payoffs**.

How Reasonable Is an Infinitely Repeated Game?

Game with random termination:

- At the end of each period: continue the game with probability δ .
- Note that game ends in finite time with probability 1.
- The players' undiscounted expected payoff in this game is also (1).

Random termination and discounting:

- If, in addition, players discount future payoffs with discount factor ρ , then the discounted expected payoff is

$$U_i(\sigma) = (1 - \delta\rho) \sum_{t=0}^{\infty} (\delta\rho)^t \mathbb{E}_{\sigma} [u_i(A^t)].$$

- This is just a repeated game with discount factor $\delta' = \delta\rho$.

Repeated Play of Static Nash Equilibria

Lemma 10.1

Let $\mathcal{A}_{\text{Nash}} \subseteq \mathcal{A}$ denote the set of stage-game Nash equilibria, also called static Nash equilibria. Any strategy profile σ with $\sigma(h^t) = \alpha^t \in \mathcal{A}_{\text{Nash}}$ for any $h^t \in \mathcal{H}^t$ is a subgame-perfect equilibrium of the repeated game.

Intuition:

- Since α^s is chosen independently of history for any $s \geq t$, nothing any player i does in period t affects i 's continuation payoff.
- It is optimal to maximize the period t payoff by playing α_i^{t-1} .

Intertemporal incentives:

- In any subgame-perfect equilibrium with $\sigma(h^t) \notin \mathcal{A}_{\text{Nash}}$, players need to be rewarded for playing $\sigma(h^t)$ or punished for not playing $\sigma(h^t)$

Continuation Strategies and Payoffs

Definition 10.2

Consider a behavior strategy profile σ .

1. The **continuation strategy** $\sigma_i|_{h^t}$ of player i after history h^t is the map $\sigma_i|_{h^t} : \mathcal{H}(T - t) \rightarrow \Delta(\mathcal{A}_i)$ with $\sigma_i|_{h^t}(h^s) = \sigma_i(h^t h^s)$.
2. The **continuation value** after history h^t is

$$U(\sigma; h^t) := \frac{1 - \delta}{1 - \delta^{T-t}} \sum_{s=t}^{T-1} \delta^{s-t} \mathbb{E}_{\sigma} [u(A^s) \mid H^t = h^t]$$

or, equivalently, $U(\sigma|_{h^t})$ in the $(T - t)$ -repeated game.

Note:

- Infinitely repeated games are particularly tractable since every continuation game is identical to the entire game.

Decomposition of the Continuation Value

Evolution of the continuation value:

- For any strategy profile σ and any history h^t ,

$$\begin{aligned}
 U(\sigma; h^t) &= \frac{1 - \delta}{1 - \delta^{T-t}} \sum_{s=t}^{T-1} \delta^{(s-t)} \mathbb{E}_\sigma [u(A^s) \mid H^t = h^t] \\
 &= \frac{1 - \delta}{1 - \delta^{T-t}} \mathbb{E}_{\sigma(h^t)} [u(A^t)] + \frac{1 - \delta}{1 - \delta^{T-t}} \delta \sum_{s=t+1}^{T-1} \delta^{s-t-1} \mathbb{E}_\sigma [u(A^s) \mid H^t = h^t] \\
 &= \underbrace{\frac{1 - \delta}{1 - \delta^{T-t}}}_{=: 1 - \delta_{T-t}} \mathbb{E}_{\sigma(h^t)} [u(A^t)] + \underbrace{\frac{\delta(1 - \delta^{T-t-1})}{1 - \delta^{T-t}}}_{=: \delta_{T-t}} \mathbb{E}_{\sigma(h^t)} [U(\sigma|_{h^t A^t})].
 \end{aligned}$$

- The relative weight δ_{T-t} of the continuation value is increasing from $\delta_0 = 0$ to $\delta_\infty = \delta$ in the length of the continuation game.
- Idea: we can use continuation strategies after histories $h^t a^t$ to support non-static Nash play in period t .

Intertemporal Incentives

Intertemporal incentives:

- Players are willing to play $\sigma(h^t)$ after history h^t if and only if

$$(1 - \delta_{T-t})\mathbb{E}_{\sigma(h^t)}[u_i(A^t)] + \delta_{T-t}\mathbb{E}_{\sigma(h^t)}[U_i(\sigma|_{h^t A^t})] \\ \geq (1 - \delta_{T-t})\mathbb{E}_{a_i, \sigma_{-i}(h^t)}[u_i(A^t)] + \delta_{T-t}\mathbb{E}_{a_i, \sigma_{-i}(h^t)}[U_i(\sigma|_{h^t A^t})].$$

for every $a_i \in \mathcal{A}_i$ and every player i .

Credible punishments/rewards:

- In any SPE σ , we know that $\sigma|_{h^t a^t}$ is an SPE of the continuation game after $h^t a^t$, hence the rewards/punishments are credible.
- To construct an SPE that supports play of α , we need to find reward/punishment SPEs $\sigma|_{h^t a^t}$ that incentivize play of α .

Product-Choice Game

	<i>H</i>	<i>L</i>
<i>H</i>	2, 3	0, 2
<i>L</i>	3, 0	1, 1



Static game:

- The Firm can produce a (*H*)igh or a (*L*)ow-quality product.
- The Customers chooses a (*H*)igh- or a (*L*)ow-priced product.
- The unique static Nash equilibrium is (*L*, *L*).

Subgame-perfect equilibria:

- Repeated play of (*L*, *L*) is an SPE by Lemma 10.1.
- Is there an SPE, in which the high-quality product is produced?

Uniqueness of SPE

Lemma 10.3

In a finitely repeated game with a unique static Nash equilibrium α , the unique SPE is repeated play of α .

Intuition:

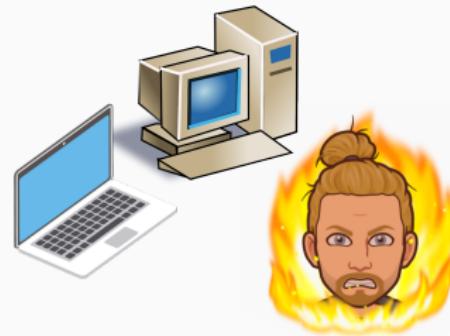
- In the last period, α must be played regardless of history, which destroys incentives in earlier periods by backward induction.

Intertemporal incentives require:

- At least two continuation equilibria with non-identical payoffs:
 - One “reward equilibrium” for following the equilibrium strategy,
 - One “punishment equilibrium” for deviating.
- Sufficiently high discount factor so that the difference in continuation payoffs outweighs potential profits in the current period.

Grim Trigger

	<i>H</i>	<i>L</i>
<i>H</i>	2, 3	0, 2
<i>L</i>	3, 0	1, 1



Candidate strategy profile:

- The grim-trigger strategy profile σ is defined by

$$\sigma_i(h) = \begin{cases} H & \text{if } h = (\textcolor{brown}{H}, \textcolor{blue}{H}), \dots, (\textcolor{brown}{H}, \textcolor{blue}{H}), \\ L & \text{otherwise.} \end{cases}$$

- Punishment continuation strategy profile is definitely subgame perfect.
- Is grim trigger an SPE in the infinitely repeated game?
- Potential problem: there are infinitely many deviations to consider.

The One-Shot Deviation Principle

Theorem 10.4 (One-shot deviation principle)

A strategy profile σ in a repeated game is subgame perfect if and only if no player i has a profitable one-shot deviation, i.e., a deviation $\tilde{\sigma}_i$ such that $\tilde{\sigma}_i(h) = \sigma_i(h)$ for all but a single history $h \in \mathcal{H}$.

Interpretation:

- The continuation profile after a one-shot deviation is subgame perfect, hence a two-shot deviation cannot improve upon a one-shot deviation.
- For the argument to work at infinity, discounting is required.

Importance:

- This immensely simplifies verifying subgame perfection.

Summary

Constructing equilibria:

- Time homogeneity and the one-shot deviation principle greatly simplify verifying that a given strategy profile is an SPE.
- We can construct SPEs by attaching punishments to unfavorable outcomes. This construction hinges on finding good punishment SPE.
- What do we do if the static Nash equilibrium is not a good punishment?
 - We take Econ 8008 – Microeconomic Theory II.

Reputation games:

- Our knowledge of complete-information repeated games suffices.
- Next, we simplify the players' beliefs by introducing commitment types.

Reputation Games

Reputations?

	<i>H</i>	<i>L</i>
<i>H</i>	2, 3	0, 2
<i>L</i>	3, 0	1, 1



Reputations?

- Grim trigger profile can be interpreted as maintaining a reputation for playing *H*: deviating destroys one's reputation forever.
- The set of continuation equilibria, however, is history-independent.
- Repeated play of *H* does not make the customer believe it is more likely that *H* will be played. There is no **reputation effect**.
- Consequently, no SPE supports *H* in a finitely repeated game.

Reputation Models

Reputation models:

- There is uncertainty about the quality/type of an institution.
- Individuals who interact with the institution report their experience and update social beliefs about the quality/type of the institution.
- Individuals are “short-lived”, i.e., they interact with institution once.

Reputation effect:

- By repeatedly acting in a certain way, individuals must come to expect this behavior. The institution builds a reputation for acting that way.
- Institutions with a good reputation enjoy a benefit.
- Building a reputation may be costly in the short-run.

Newly Opened Restaurant



Updating of beliefs:

- Customers are unsure about the quality of the restaurant.
- Social beliefs are updated via Google/Yelp reviews.

Reputation effect:

- Building a good reputation is costly and it may require skilled personnel, developing unique recipes, good interior design, etc.
- If the restaurant manages to establish a good reputation, it receives more customers, or it can increase prices.

Electing Government Officials



Politician's type:

- Citizen's are unsure whether a politician is true to their word.
- Politician's past performance is an imperfect reflection about the politician's intentions and an indication of their future behavior.

Building a reputation:

- Politician works hard even when he/she is only appointed to low offices.
- If the politician succeeds in building a good reputation, he/she may be elected to a higher office that earns him/her many benefits.

Commitment Types

Definition 10.5

1. A **commitment type** is a type ϑ_c that invariably plays strategy $\sigma_1(\vartheta_c)$.
 2. A commitment type ϑ_c is **simple** if there exists $\alpha_1 \in \Delta(\mathcal{A}_1)$ such that $\sigma_1(\vartheta_c, h^t) = \alpha_1$ for any history h^t . We also denote that type by ϑ_{α_1} .
-

Remark:

- Commitment types do not maximize a utility function.
- In the product-choice game, a firm that is committed to delivering a high-quality product is a simple commitment type ϑ_H .
- Commitment types do not falter: they never deviate from their strategy profile, hence any deviations are attributed to payoff types.

Reputation Game

Player 1's types:

- We denote by Θ_c the set of player 1's possible commitment types.
- Additionally, player 1 may be a **payoff type** ϑ_p who maximizes

$$U_1(\sigma) := (1 - \delta) \sum_{t=0}^{\infty} \delta^t \mathbb{E}_{\sigma} [u_1(A^t)],$$

where P_{σ} is defined from σ and prior beliefs $\mu_0 \in \Delta(\Theta)$ as usual.

- We denote by $\Theta = \Theta_c \cup \{\vartheta_p\}$ the set of all types.

Strategies:

- A strategy of Player 1 is a map $\sigma_1 : \Theta \times \mathcal{H} \rightarrow \Delta(\mathcal{A}_1)$.
- Type ϑ chooses a_1 after history h^t with probability $\sigma_1(\vartheta, h^t; a_1)$.
- It will be convenient to abbreviate $\sigma(\vartheta_p) = (\sigma_1(\vartheta_p), \sigma_2)$.

Updating of Beliefs

Beliefs:

- Let $\mu(h^t) \in \Delta(\Theta)$ denote player 2's beliefs about θ after history h^t , which assigns probability $\mu(h^t; \vartheta) = P_\sigma(\theta = \vartheta | H^t = h^t)$ to type ϑ .
- Player 2's belief process $(\mu_t)_{t \geq 0}$ is defined by $\mu_t = \mu(H^t)$.
- Observe that $(\mu_t)_{t \geq 0}$ is a sequence of random variables.

Updating of beliefs:

- After observing history $h^{t+1} = h^t a^t$ with $P_\sigma(h^{t+1}) > 0$, the beliefs are

$$\mu(h^{t+1}; \vartheta) = \frac{\sigma(\vartheta, h^t; a^t) \mu(h^t; \vartheta)}{\sum_{\vartheta' \in \Theta} \sigma(\vartheta', h^t; a^t) \mu(h^t; \vartheta')}.$$

- After any other history, the beliefs are updated to $\mu(h^{t+1}) = \delta_{\vartheta_p}$.
- Consequence: any strategy profile leads to well-defined off-path beliefs.

Short-Lived Players

Player 2 is short-lived:

- Each instance of player 2 “lives” for only one round and plays a stage-game best reply to the expected action by Player 1, given $\mu(h^t)$.
- For any action α_1 , denote player 2’s stage-game best responses by

$$\mathcal{B}_2(\alpha_1) := \left\{ \alpha_2 \in \Delta(\mathcal{A}_2) \mid u_2(\alpha_1, \alpha_2) = \max_{a_2 \in \mathcal{A}_2} u_2(\alpha_1, a_2) \right\}.$$

- Given beliefs $\mu(h^t)$, player 2 expects to see $\alpha_1^{\sigma, \mu}(h^t)$ defined by

$$\alpha_1^{\sigma, \mu}(h^t; a_1) = \sum_{\vartheta \in \Theta} \sigma_1(\vartheta, h^t; a_1) \mu(h^t; \vartheta).$$

- Player 2 plays an action α_2 from $\mathcal{B}_2(\alpha_1^{\sigma, \mu}(h^t))$, which maximizes

$$\mathbb{E}_{\sigma} [u_2(A_1^t, \alpha_2) \mid H^t = h^t] = \sum_{\vartheta \in \Theta} u_2(\sigma_1(\vartheta, h^t), \alpha_2) \mu(h^t; \vartheta).$$

Equilibria in Reputation Games

Definition 10.6

Strategy profile σ is a **Nash equilibrium of the reputation game** if for all $\hat{\sigma}_1$, $U_1(\sigma) \geq U_1(\hat{\sigma}_1, \sigma_2)$ and for all h^t with $P_\sigma(H^t = h^t) > 0$ and all $a_2 \in \mathcal{A}_2$,

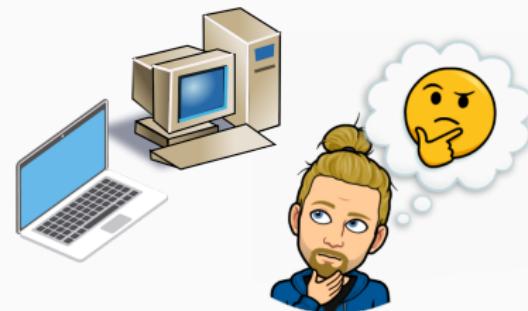
$$\mathbb{E}_\sigma[u_2(A^t) \mid H^t = h^t] \geq \mathbb{E}_\sigma[u_2(A_1^t, a_2) \mid H^t = h^t].$$

Refinements:

- Player 2 is short-lived and maximizes myopically, given his/her beliefs.
- Future threats or rewards have no impact on player 2's behavior, hence we need not worry whether threats by player 1 are credible or not.
- Nevertheless, refinements may help us find equilibria quicker.
- Since off-path beliefs are well-defined, subgame perfection, Bayesian perfection, and sequential equilibria all coincide.

Two-Stage Product-Choice Game

	<i>H</i>	<i>L</i>
<i>H</i>	2, 3	0, 2
<i>L</i>	3, 0	1, 1



Firm has two types:

- Commitment type ϑ_H is committed to playing *H* in both periods.
- Payoff type maximizes the sum of discounted payoffs.

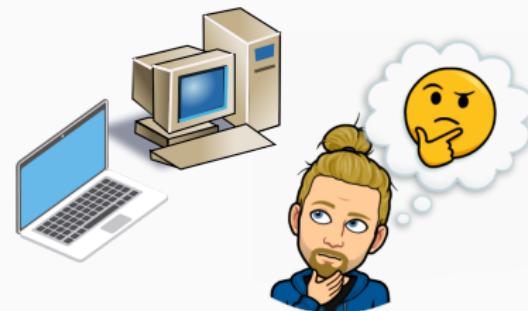
Equilibrium in second period:

- Payoff type must choose *L* in the second period because it is dominant.
- In period 2, *Customers* best responds with *H* if and only if

$$3\mu(h^1; \vartheta_H) \geq 2\mu(h^1; \vartheta_H) + 1 - \mu(h^1; \vartheta_H) \Leftrightarrow \mu(h^1; \vartheta_H) \geq \frac{1}{2}.$$

Two-Stage Product-Choice Game

	<i>H</i>	<i>L</i>
<i>H</i>	2, 3	0, 2
<i>L</i>	3, 0	1, 1



Optimistic customers:

- If $\mu_0(\vartheta_H) \geq \frac{1}{2}$, the *Customers* choose *H* in the first period.
- Mimicking the commitment type leads to $\mu(H; \vartheta_H) = \mu_0(\vartheta_H) \geq \frac{1}{2}$.
- Choosing *H* followed by *L* nets the *Firm* $2 + 3\delta$.
- Choosing *L* twice yields $3 + \delta$, which is preferable only if $\delta \leq \frac{1}{2}$.

Pessimistic customers:

- If $\mu_0(\vartheta_H) < \frac{1}{2}$, the *Customers* choose *L* in the first period.
- Is repeated play of (*L*, *L*) an equilibrium?

Two-Stage Product-Choice Game

Profitable deviation?

- Candidate profile $\sigma(h) = (\textcolor{blue}{L}, \textcolor{red}{L})$ for every history h .
- Consider a deviation by the **Firm** to H in the first period.
- After observing H , the **Customers**' beliefs in the second period are

$$\mu(\textcolor{blue}{H}; \vartheta_H) = \frac{\mu_0(\vartheta_H)}{\mu_0(\vartheta_H) + \sigma_1(\vartheta_p, \textcolor{blue}{H})\mu_0(\vartheta_p)} = 1.$$

- The **Customers**' best response in period 2 is to choose $\textcolor{red}{L}$.

Reputation effect:

- Type ϑ_p has an incentive to mimic type ϑ_H and play H in period 1.
- Building the reputation is costly: doing so nets the **Firm** 0 in period 1.
- The **Firm** can exploit the reputation and earn 3 in period 2.
- If $3\delta \geq 1 + \delta$, i.e., $\delta \geq \frac{1}{2}$, then doing so is profitable.

Impact of Reputations

Power of reputations:

- In a finitely repeated game, repeated play of (L, L) is the only SPE.
- If there is a small possibility $\mu_0 > 0$ that the firm is committed to producing a high-quality product, (L, L) is no longer an equilibrium.
- Playing L is suboptimal because it fails to exploit the reputation effect.

Strong variation of beliefs:

- A typical ingredient in reputation arguments is the strong variation of beliefs after a deviation by the payoff type.
- Choosing H with certainty may not be part of an equilibrium. Nevertheless, any equilibrium must be robust to this ϑ_H -imitating deviation.

Question: Can we find an equilibrium?

Parametrizing the Strategy

Parametrizing the Firm's strategy:

- In any Nash equilibrium σ , the commitment type must choose H after any history and the payoff type must choose L in the second period.
- The Firm's strategy σ_1 is entirely parametrized by $\sigma_1(\vartheta_p, \emptyset; H) = x$.

Updating of beliefs:

- Let us identify $\mu(h) \cong \mu(h; \vartheta_H)$ and write $\mu(h; \vartheta_p) = 1 - \mu$.
- $\mu_1 := \mu(A^1)$ is a random variable, taking values

$$\mu(H) = \frac{\mu_0}{\mu_0 + x(1 - \mu_0)}, \quad \mu(L) = 0$$

on the events $\{A_1^0 = H\}$ and $\{A_1^0 = L\}$, respectively.

- It follows that $P_\sigma(\mu_1 = \mu(H)) = x$ and $P_\sigma(\mu_1 = 0) = 1 - x$, i.e., the Firm's action affects magnitude and likelihood of the posterior.

Customers' Best Response

Parametrizing the **Customers'** strategy:

- Since $\mu(L) = 0$, the **Customers** best responds to $\sigma_2(L) = L$.
- Let z_0 and z_1 denote the probabilities of choosing H in the first period and after history H , respectively.

Best-response correspondence:

- In first period and after observing H , the **Customers** expect to see H with probability $P_\sigma(A_1^0 = H) = \mu_0 + x(1 - \mu_0)$ and $\mu(H)$, respectively.
- The **Customers** are willing to choose H in the first period if and only if

$$\mu_0 + x(1 - \mu_0) \geq \frac{1}{2} \iff x \geq \frac{1 - 2\mu_0}{2(1 - \mu_0)} =: x_0.$$

- The **Customers** are willing to choose H after observing H if and only if

$$\mu(H) \geq \frac{1}{2} \iff x \leq \frac{\mu_0}{1 - \mu_0} =: x_1.$$

Firm's Best Response

Firm's best response:

- The strategic type maximizes the discounted sum of payoffs

$$\begin{aligned} U_1(x) &= \mathbb{E}_\sigma[u_1(A^0)] + \delta \mathbb{E}_\sigma[u_1(A^1) | A_1^0 = H] P_\sigma(A_1^0 = H) \\ &\quad + \delta \mathbb{E}_\sigma[u_1(A^1) | A_1^0 = L] P_\sigma(A_1^0 = L) \\ &= 1 - x + 2z_0 + \delta(1 + 2xz_1). \end{aligned}$$

- The partial derivative with respect to x is $\frac{\partial U_1(x)}{\partial x} = -1 + 2\delta z_1$.
- The best-response correspondence is

$$\mathcal{B}_1(z_1) = \begin{cases} x = 1 & \text{if } z_1 > \frac{1}{2\delta}, \\ x \in [0, 1] & \text{if } z_1 = \frac{1}{2\delta}, \\ x = 0 & \text{if } z_1 < \frac{1}{2\delta}. \end{cases}$$

- Note that \mathcal{B}_1 does not depend on z_0 .

Consistency

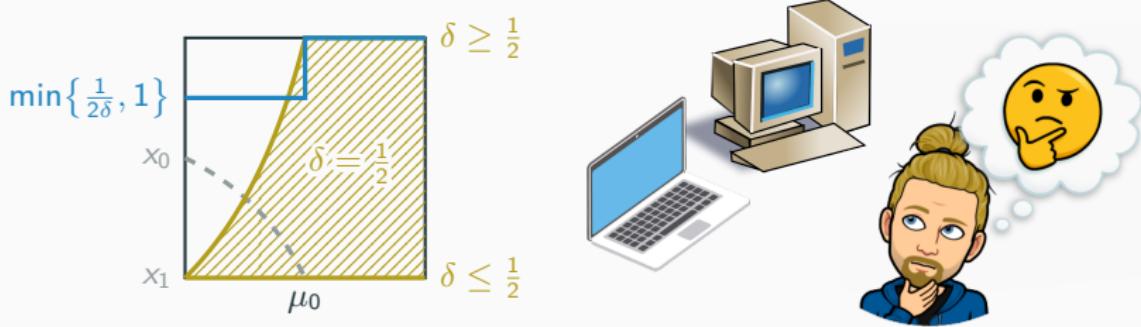
Optimistic Customers: $\mu_0 > \frac{1}{2}$

- Then $x_0 < 0$ and $x_1 > 1$, hence $x_0 < \textcolor{blue}{x} < x_1$ and $\textcolor{blue}{z}_0 = \textcolor{blue}{z}_1 = 1$.
- Thus, $\textcolor{blue}{x} = 1$ if $\delta > \frac{1}{2}$, $\textcolor{blue}{x} \in [0, 1]$ if $\delta = \frac{1}{2}$, and $\textcolor{blue}{x} = 0$ if $\delta < \frac{1}{2}$.

Pessimistic Customers: $\mu_0 < \frac{1}{2}$

- If $\textcolor{blue}{x} > x_1$, then \mathcal{B}_2 implies $\textcolor{blue}{z}_1 = 0$, hence \mathcal{B}_1 implies $\textcolor{blue}{x} = 0$. \times
- If $\textcolor{blue}{x} = x_1$, then \mathcal{B}_1 implies $\textcolor{blue}{z}_1 = \frac{1}{2\delta}$ and, hence, $\delta \geq \frac{1}{2}$. \checkmark
- If $\textcolor{blue}{x} < x_1$, then \mathcal{B}_2 implies $\textcolor{blue}{z}_1 = 1$, hence \mathcal{B}_1 implies $\delta \leq \frac{1}{2}$.
 - If $\delta < \frac{1}{2}$, then \mathcal{B}_1 implies $\textcolor{blue}{x} = 0$. \checkmark
 - If $\delta = \frac{1}{2}$, then any $\textcolor{blue}{x} < x_1$ is consistent. \checkmark
- In either case, $\textcolor{blue}{z}_0 = 1$ if $\textcolor{blue}{x} > x_0$, $\textcolor{blue}{z}_0 \in [0, 1]$ if $\textcolor{blue}{x} = x_0$, and $\textcolor{blue}{z}_0 = 0$ if $\textcolor{blue}{x} < x_0$.

Equilibria in Behavior Strategies



Loss of value:

- For $\mu_0 < \frac{1}{4}$, the **Customers** do not purchase the high-priced product in the first period since $x = x_1 < x_0$.
- The firm would be better off choosing $x = x_0$ since it can build a reputation while earning a higher payoff in the first period.
- However, the **Firm** cannot resist the temptation to deviate.
- If the **Firm** had the power to commit, its utility would increase.

Check Your Understanding

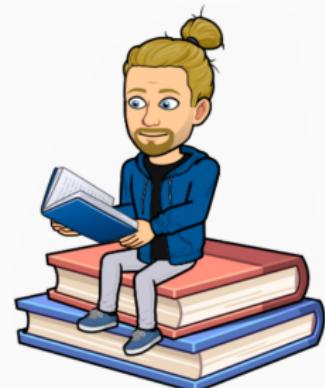
True or false:

1. The one-shot deviation principle does not apply to extensive-form games with perfect information.
2. The one-shot deviation principle does not apply to Nash equilibria.
3. If the long-lived player has multiple payoff types, then it is insufficient to consider Nash equilibria.
4. Short-lived players have a unique best response for any beliefs.
5. There is no reputation effect in standard repeated games because the continuation game is equivalent to the original game.



Literature

- G.J. Mailath and L. Samuelson: **Repeated Games and Reputations: Long-Run Relationships**, Chapter 15, Oxford University Press, 2006
- S. Tadelis: **Game Theory: An Introduction**, Chapter 17, Princeton University Press, 2013
- P. Milgrom, and J. Roberts: Predation, Reputation and Entry Deterrence, **Journal of Economic Theory**, 27 (1982), 280–312
- D. Kreps, and R. Wilson: Reputation and Imperfect Information, **Journal of Economic Theory**, 27 (1982), 253–279



Reputation Effect

Reputation Effect

Evolution of beliefs:

- We know how player 2 updates beliefs about player 1's type.
- However, player 2 does not care whether he/she faces a commitment type or a strategic type that mimics the commitment type.
- More relevant: what are player 2's beliefs about expected play?

Outcomes of the perfect-monitoring game:

- The outcome of the game (θ, A) is an Ω -valued random variable.
- Let Ω be the set of all possible outcomes $\omega = (\vartheta, a^0, a^1, \dots)$.
- We want to exploit the correlation of A and θ to express player 2's beliefs about expected continuation play.
- Player 2 cannot observe θ : A_2^t is h^t -conditionally independent of θ .

Expectation of Future Play

Reputation effect:

- In the product-choice game, it was beneficial for the strategic player to mimic the commitment type ϑ_H .
- Fix an action $\hat{a}_1 \in \mathcal{A}_1$ and a simple commitment type $\vartheta_{\hat{a}_1} \in \Theta_c$.
- Does player 2 come to expect \hat{a}_1 after repeated play of \hat{a}_1 ?

Expectation of future play:

- Given history h^t , player 2 expects to see \hat{a}_1 with a probability

$$q(h^t) := P_\sigma(A_1^t = \hat{a}_1 \mid H^t = h^t) = \sum_{\vartheta \in \Theta} \sigma_1(\vartheta, h^t; \hat{a}_1) \mu(h^t; \vartheta).$$

- The sequence $(Q_t)_{t \geq 0}$ defined by $Q_t := q(H^t)$ captures how player 2's expectation of seeing \hat{a}_1 evolves over time. Note that Q_t is a random variable: before time t , it is unknown which h^t will materialize.

Mimicking a Commitment Type

Mimicking a commitment type:

- Let $\Omega' := \{\omega \in \Omega \mid A_1^t(\omega) = \hat{a}_1 \text{ for all } t\}$ denote the set of outcomes, in which \hat{a}_1 is played in every period.
- Ω' contains outcomes $\omega = (\vartheta, a^0, a^1, \dots)$ with:
 - Player 1 is the commitment type $\vartheta_{\hat{a}_1}$,
 - Player 1 is the strategic type that imitates $\vartheta_{\hat{a}_1}$,
 - Player 1 is any other commitment type (or a strategic type) that plays \hat{a}_1 with positive probability in every period and \hat{a}_1 was always realized.

How does expectation about future play evolve on Ω' ?

- Intuition: q_t is increasing on Ω' .
- However, this intuition need not be correct.

Evolution of Q_t in the Product-Choice Game

Commitment types:

- Simple commitment type ϑ_H .
- Commitment type ϑ_t plays H in period $s < t$ and L after.
- Let $\hat{a}_1 = H$, that is, $\Omega' = \{\omega \in \Omega \mid A_1^t(\omega) = H\}$.

Evolution of Q_t :

- Consider the strategy profile $\sigma(\vartheta_p, h^t) = (H, H)$ for any h^t .
- Then $q_0 = 1 - \mu_0(\vartheta_0)$ and

$$q_1(H, H) = \frac{1 - \mu_0(\vartheta_0) - \mu_0(\vartheta_1)}{1 - \mu_0(\vartheta_0)}.$$

- Note that $q_1(H, H) < q_0$ if $\mu_0(\vartheta_1) > \mu_0(\vartheta_0)(1 - \mu_0(\vartheta_0))$.

Whether Q_t increases or decreases depends not only on σ , but also on prior beliefs and on commitment types that are being ruled out.

Reputation Effect

Lemma 10.7

For any $\zeta \in [0, 1)$, let $N_\zeta := |\{t \mid Q_t \leq \zeta\}|$. Fix $\hat{a}_1 \in \mathcal{A}_1$ with $\vartheta_{\hat{a}_1} \in \Theta_c$ and suppose that $\mu_0(\vartheta_{\hat{a}_1}) \in [\mu_*, 1)$ for some $\mu_* > 0$. For any profile σ ,

$$P_\sigma \left(N_\zeta > \frac{\ln \mu_*}{\ln \zeta} \mid \Omega' \right) = 0.$$

Moreover, $\mu_t(\vartheta_{\hat{a}_1})$ is non-decreasing for P_σ -almost every $\omega \in \Omega'$.

- If payoff type mimics commitment type $\vartheta_{\hat{a}_1}$, player 2 must expect \hat{a}_1 with large probability in all but finitely many periods.
- Bound $\frac{\ln \mu^*}{\ln \zeta}$ on number of those periods does not depend on σ .
- $\mu_t(\vartheta_{\hat{a}_1})$ may not converge to 1: if $\sigma(\vartheta_p)$ mimics the commitment type, player 2 places positive beliefs on facing a $\vartheta_{\hat{a}_1}$ -imitating payoff type.

Proof: Step 1

Monotonicity of μ :

- Fix σ and $h^{t+1} = (h^t, a^t)$ with $a_1^t = \hat{a}_1$ and $P_\sigma(H^{t+1} = h^{t+1}) > 0$.
- Recall $q(h^t) = P_\sigma(A_1^t = \hat{a}_1 | H^t = h^t)$. Bayes' rule implies that

$$\begin{aligned} P_\sigma(\theta = \vartheta_{\hat{a}_1} | h^{t+1}) &= \frac{P_\sigma(A^t = a^t | \vartheta_{\hat{a}_1}, h^t)P_\sigma(\theta = \vartheta_{\hat{a}_1} | h^t)}{P_\sigma(A^t = a^t | h^t)} \\ &= \frac{P_\sigma(A_2^t = a_2^t | \vartheta_{\hat{a}_1}, h^t)\mu(h^t; \vartheta_{\hat{a}_1})}{q(h^t)\sigma_2(h^t; a_t^2)} = \frac{\mu(h^t; \vartheta_{\hat{a}_1})}{q(h^t)}. \end{aligned}$$

- Since $q(h^t) \leq 1$, this implies that $\mu(h^{t+1}; \vartheta_{\hat{a}_1}) \geq \mu(h^t; \vartheta_{\hat{a}_1})$.
- Moreover, for every $\omega = (\vartheta, a^0, a^1, \dots)$ in Ω' with $P_\sigma((\theta, A) = \omega) > 0$, history (a^0, \dots, a^t) of any length $t + 1$ is of the above form.
- This shows that $\mu_t(\vartheta_{\hat{a}_1})$ is non-decreasing for P_σ -a.e. $\omega \in \Omega'$.

Proof: Step 2

Finding a bound for ζ :

- Fix $\omega = (\vartheta, a^0, a^1, \dots) \in \Omega'$ with $P_\sigma((\theta, A) = \omega) > 0$ and denote by $h^t = (a^0, \dots, a^{t-1})$ the associated histories.
- From previous slide: $\mu(h^t; \vartheta_{\widehat{a}_1}) = q(h^t)\mu(h^{t+1}; \vartheta_{\widehat{a}_1})$ for all t .
- Since $\mu_0(\vartheta_{\widehat{a}_1}) \geq \mu^* > 0$ by assumption, we obtain

$$\mu^* \leq \mu_0(\vartheta_{\widehat{a}_1}) = q(h^0)\mu(h^1; \vartheta_{\widehat{a}_1}) = q(h^0)q(h^1)\mu(h_2; \vartheta_{\widehat{a}_1})$$

$$= \dots = \left(\prod_{s=0}^{t-1} q(h^s) \right) \mu(h^t; \vartheta_{\widehat{a}_1}) \leq \prod_{s=0}^{t-1} q(h^s).$$

- Taking the limit as $t \rightarrow \infty$ yields $\mu^* \leq \prod_{s=0}^{\infty} q(h^s)$, hence

$$\mu_* \leq \prod_{s=0}^{\infty} q(h^s) \leq \prod_{s: q(h^s) \leq \zeta} q(h^s) \prod_{s: q(h^s) > \zeta} q(h^s) \leq \prod_{s: q(h^s) \leq \zeta} q(h^s) \leq \zeta^{N_\zeta(\omega)}.$$

Proof: Conclusion

Conclusion:

- We have deduced $\mu_* \leq \zeta^{N_\zeta(\omega)}$ for P_σ -almost every $\omega \in \Omega'$, hence

$$P_\sigma(\mu_* \leq \zeta^{N_\zeta} \mid \Omega') = 1.$$

- Therefore,

$$P_\sigma\left(N_\zeta > \frac{\ln \mu^*}{\ln \zeta} \mid \Omega'\right) = P_\sigma(\ln(\zeta^{N_\zeta}) < \ln \mu^* \mid \Omega') = 0.$$

Exploiting the Reputation Effect

Reputation effect:

- Imitating a simple commitment type $\vartheta_{\hat{a}_1}$ leads player 2 to expect \hat{a}_1 with high probability ζ in all but finitely many periods.
- Choosing ζ sufficiently high will cause player 2 to best reply to \hat{a}_1 .

Which commitment type should player 1 imitate?

- If player 1 could commit to playing $a_1 \in \mathcal{A}_1$, he/she would get:

$$\underline{v}_1(a_1) = \min_{a_2 \in \mathcal{B}_2(a_1)} u_1(a_1, a_2).$$

- Player 1 best imitates commitment type $\vartheta_{\hat{a}_1}$ that maximizes $\underline{v}_1(\hat{a}_1)$.

Reputation Bound

Theorem 10.8

Let $\widehat{\mathcal{A}}_1 \subseteq \mathcal{A}_1$ denote the set of player 1's pure actions a_1 , for which there is a simple commitment type ϑ_{a_1} with $\mu_0(\vartheta_{a_1}) > 0$. Suppose \mathcal{A}_2 is finite and $\mu_0(\vartheta_p) > 0$. Then there exists $K(\mu_0)$ such that

$$\inf_{Eq. \sigma} U_1(\sigma) \geq \delta^K \max_{a_1 \in \widehat{\mathcal{A}}_1} \underline{v}_1(a_1) + (1 - \delta^K) \min_{a \in \mathcal{A}} u_1(a).$$

Interpretation:

- Player 1 builds a reputation to be his preferred commitment type \widehat{a}_1 .
- Reputation effect: in all but $K(\mu_0)$ periods, player 2 best replies to \widehat{a}_1 .
- Trade-off: building a reputation is costly, but yields player 1 his/her highest-possible commitment payoff in the long-run.

Note: Bound applies to *any* Nash equilibrium payoff.

Proof: Step 1

Choose preferred commitment type:

- Let \hat{a}_1 be player 1's preferred simple commitment type:

$$\hat{a}_1 \in \arg \max_{a_1 \in \hat{\mathcal{A}}_1} v_1(a_1).$$

- Let $\mu^* := \mu_0(\hat{a}_1)$. Note that by assumption $\mu^* > 0$.

Choose ζ such that player 2 best replies to \hat{a}_1 :

- Because \mathcal{A}_2 is finite, there exists $\zeta \in (0, 1)$ such that if $\alpha_1(\hat{a}_1) > \zeta$,

$$\mathcal{B}_2(\alpha_1) \subset \mathcal{B}_2(\hat{a}_1).$$

- Fix a Nash equilibrium σ . Then for all h^t , for which $q(h^t) > \zeta$,

$$\mathcal{B}_2(\mathbb{E}_\sigma[\sigma_1^t(h^t, \theta) \mid h^t]) \subset \mathcal{B}_2(\hat{a}_1).$$

Proof: Step 2

Apply reputation effect:

- Set $K = \frac{\ln \mu(\vartheta_{\hat{a}_1})}{\ln \zeta}$.
- Lemma 10.7 implies that, conditional on player 1 always playing \hat{a}_1 , $q_t \leq \zeta$ for no more than K periods t with P_σ -probability 1.
- Let $\hat{\sigma}_1$ be the deviation that always plays \hat{a}_1 . It follows that

$$P_{\hat{\sigma}_1, \sigma_2}(\{t \mid Q_t \leq \zeta\} < K) = P_\sigma(\{t \mid Q_t \leq \zeta\} < K \mid \Omega') = 1.$$

- Since σ_1 has no profitable deviations

$$U_1(\sigma) \geq U_1(\hat{\sigma}_1, \sigma_2) \geq \delta^K \max_{a_1 \in \hat{\mathcal{A}}_1} \underline{v}_1(a_1) + (1 - \delta^K) \min_{a \in \mathcal{A}} u_1(a).$$

Patient Long-Lived Player

Corollary 10.9

Let $\vartheta_{\hat{a}_1}$ be player 1's preferred simple commitment type with $\mu_0(\vartheta_{\hat{a}_1}) > 0$. Suppose that \mathcal{A}_2 is finite and $\mu_0(\vartheta_p) > 0$. For any $\varepsilon > 0$, there exists $\underline{\delta} \in (0, 1)$ such that for any $\delta > \underline{\delta}$,

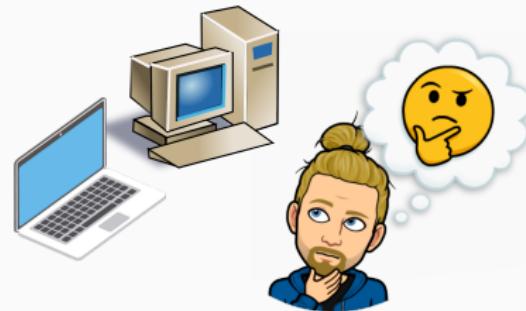
$$\inf_{Eq. \sigma} U_1(\sigma) \geq \max_{a_1 \in \mathcal{A}_1} v_1(a_1) - \varepsilon.$$

As long-lived player gets patient:

- Building a reputation becomes cheaper.
- As $\delta \rightarrow 1$ reputation effect dominates all other considerations and only a single equilibrium payoff remains.

Reputation Bound in Product-Choice Game

	<i>H</i>	<i>L</i>
<i>H</i>	2, 3	0, 2
<i>L</i>	3, 0	1, 1



Reputation bound:

- As long as there exists a commitment type ϑ_H with $\mu_0(\vartheta_H) > 0$, Theorem 10.8 establishes a lower bound for all Nash equilibria.
- Lower bound on equilibrium is increasing in δ because “building the reputation” is comparatively cheaper.
- By Corollary 10.9, payoff in **any** equilibrium converges to 2 as $\delta \rightarrow 1$.

Summary

Reputation effect:

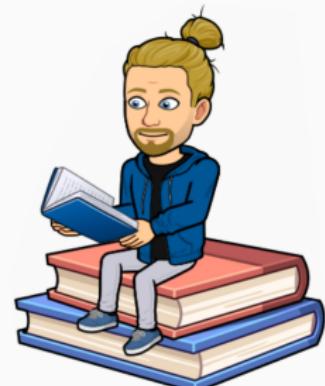
- If the strategic type plays a commitment action \hat{a}_1 repeatedly, the short-lived player must come to expect it eventually.
- Specifically, in all but finitely many periods, player 2 will best respond to the commitment action \hat{a}_1 .

Patient players:

- In every equilibrium, a reputation is built for the best commitment type.
- Building a reputation is costly in the short-run, but not doing so is suboptimal because it fails to exploit the reputation effect.
- Reputation effect is much stronger in an infinitely repeated game than in the twice repeated game.

Literature

- ❖ G.J. Mailath and L. Samuelson: **Repeated Games and Reputations: Long-Run Relationships**, Chapter 15, Oxford University Press, 2006
- ❖ S. Tadelis: **Game Theory: An Introduction**, Chapter 17, Princeton University Press, 2013
- ❖ D. Fudenberg, and D. K. Levine: Reputation and Equilibrium Selection in Games with a Patient Player, **Econometrica**, **57** (1989), 759–778



Reputations with Long-Lived Players

Motivation for Reputation Games

Short-lived players:

- Can a sequence of customers, citizens, etc. come to expect certain behavior because they have observed it repeatedly?
- How can we connect the continuation game in a meaningful way to players' observation without sacrificing tractability?

Original motivation:

- Finitely repeated games with a unique stage-game Nash equilibrium have a unique SPE, which is often deemed unintuitive.
- If players believed that there was a small chance of facing a cooperative type, would an outcome be observed that is closer to our expectation?

Partnership Game

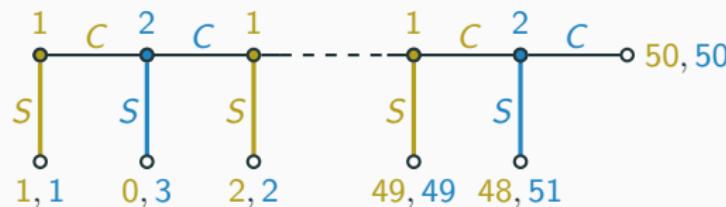
	E	S
E	2, 2	-1, 3
S	3, -1	0, 0



Group project:

- With complete information, the unique SPE predicts that there will be no effort, even if the horizon $T < \infty$ is potentially very long.
- Intuitively, we expect behavior closer to grim trigger if T is large.
- Does the set of equilibria change if players believe the other is a hard-working “grim-trigger type” with some small prior probability?

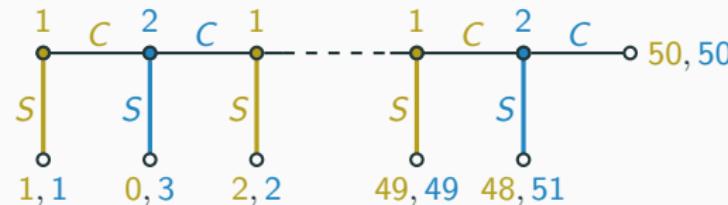
Failure to Replicate Subgame Perfection in the Lab



Centipede game:

- Suppose you are in Joseph's lab, participating in an experiment whether individuals act in a subgame perfect way.
- A typical such lab experiment is the **centipede game**, in which players iteratively choose to either (C)ontinue or (S)top the game.
- Lab experiments typically do not confirm the unique SPE.
- One possible explanation is that we believe there are commitment types who choose to continue at least once.

Modeling Strategic Uncertainty



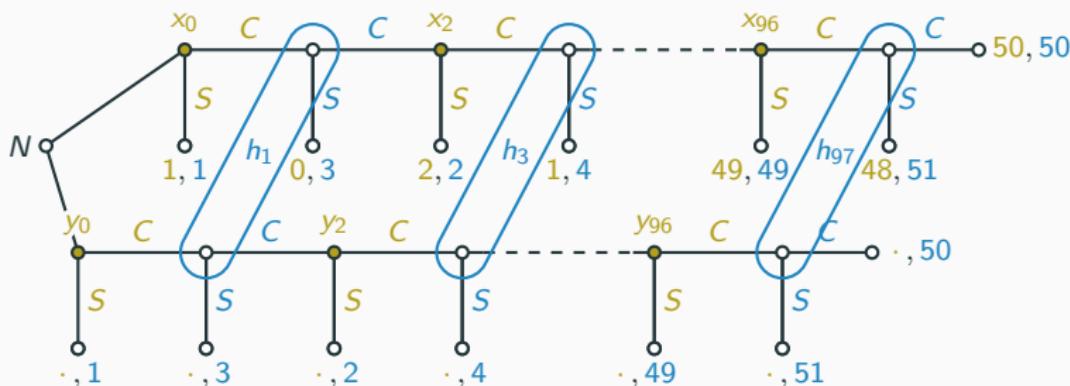
Commitment types:

- Consider a set of commitment types ϑ_t for each $t \geq 0$, who choose C at their first t decision nodes and S forever after.
- If we place sufficiently high beliefs on facing a commitment type, choosing C in the first period becomes uniquely optimal.

Reputations:

- For what priors will a reputation be built in this game?
- How long is the reputation maintained? → Let's find out!

Setting up the Reputation Game



Commitment types:

- Suppose that **Player 1** is either a payoff type or a commitment type ϑ_C who chooses C at every node.
- Player 2** has only one type and places prior $\mu_0 > 0$ on facing ϑ_C .
- For simplicity of notation, write $u_1(x_n, S)$ and $u_2(h_n, S)$ the utility for stopping at node x_n or information set h_n , respectively.

Strict Improvement Every Two Periods

Claim 1

Consider any perfect Bayesian equilibrium $\sigma = (\sigma_1, \sigma_2)$.

1. If $\sigma_2(h_n) = C$, then $\sigma_1(\vartheta_p, x_{n-1}) = C$.
 2. If $\sigma_2(h_n) = S$, then $\sigma_1(\vartheta_p, x_{n-1}) = S$.
-

Proof of statement 1:

- If $\sigma_2(h_n) = C$, then $\sigma_1(x_{n-1}) = C$ yields a utility of at least

$$u_1(\sigma) \geq u_1(x_{n+1}, S) > u_1(x_{n-1}, S).$$

Proof of statement 2:

- If $\sigma_2(h_n) = S$, then choosing $\sigma_1(x_{n-1}) = C$ yields a utility of

$$u_1(x_{n-1}, S) - 1 < u_1(x_{n-1}, S).$$

Strict Improvement Every Two Periods

Claim 2

If $\sigma_2(h_n) = C$ for some n in a PBE σ , then $\sigma_2(h_k) = C$ and $\sigma_1(\vartheta_p, x_k) = C$ for $k < n$. In particular, $\mu(h_n) = \mu_0$.

Proof by induction:

- If $\sigma_2(h_n) = C$ for some n , then both types of Player 1 choose C at x_{n-1} or y_{n-1} , respectively, by Claim 1.
- Since σ is a PBE, choosing $\sigma_2(h_{n-2}) = C$ yield a utility of at least

$$u_2(\sigma) \geq u_2(h_n, S) > u_2(h_{n-2}, S).$$

- By backward induction, $\sigma_2(h_k) = C$ for all $k < n$.
- By Claim 1, $\sigma_1(\vartheta_p, x_k) = C$ for all $k < n$.

Climbing the Centipede

Claim 3

In any perfect Bayesian equilibrium σ , we have $\sigma_2(h_n; C) > 0$ except for $n = 97$, where $\sigma_2(h_{97}) = S$.

Proof:

- Suppose towards a contradiction that $\sigma_2(h_n; C) = 0$ for $n < 97$.
- Then $\sigma_1(\vartheta_p, x_{n-1}) = S$ by Claim 1.
- Thus, conditional on reaching information set h_n , Player 2 must place beliefs 1 on facing the commitment type.
- Therefore, continuing at h_n is optimal with the intent to stop at h_{97} .

Climbing the Centipede

Claim 4

In any perfect Bayesian equilibrium σ , we have $\sigma_1(\vartheta_p, x_n; C) > 0$ except for $n = 96$, where $\sigma_1(\vartheta_p, x_{96}) = S$.

Proof:

- Suppose towards a contradiction that $\sigma_1(\vartheta_p, x_n; C) = 0$ for $n < 96$.
- Then, conditional on reaching information set h_{n+1} , Player 2 must place beliefs 1 on facing the commitment type.
- Therefore, continuing at h_n is optimal with the intent to stop at h_{97} .
- Then $\sigma_1(\vartheta_p, x_n) = C$ by Claim 1.

Three Equilibrium Phases



Indifference:

- At every information set h_n , in which Player 2 mixes, he/she must receive a utility of $u_2(\sigma) = u_2(h_n, S)$ by the indifference principle.
- Equating $u_2(h_n, S)$ to the expected utility of continuing yields

$$u_2(h_n, S) = (\mu(h_n) + \sigma_1(\vartheta_p, x_{n+1}; C)(1 - \mu(h_n))) u_2(h_{n+2}, S)$$

$$+ \sigma_1(\vartheta_p, x_{n+1}; S)(1 - \mu(h_n)) (u_2(h_n, S) - 1).$$

- Using $u_2(h_{n+2}, S) = u_2(h_n, S) + 1$, we can solve this for

$$\sigma_1(\vartheta_p, x_{n+1}; S) = \frac{1}{2(1 - \mu(h_n))}.$$

Three Equilibrium Phases



Indifference:

- The indifference principle for Player 1 yields

$$u_1(x_n, S) = \sigma_2(h_{n+1}; C) u_1(x_{n+2}, S) + \sigma_2(h_{n+1}; S) (u_1(x_n, S) - 1),$$

which we solve for $\sigma_2(h_{n+1}; S) = \frac{1}{2}$.

Beliefs:

- Bayesian updating implies that

$$\mu(h_{n+2}) = \frac{\mu(h_n)}{\mu(h_n) + (1 - \sigma_1(\vartheta_p, x_{n+1}; S))(1 - \mu(h_n))} = \frac{\mu(h_n)}{2}.$$

- Since $\mu(h_{97}) = 1$, we obtain $\mu(h_n) = (\frac{1}{2})^{\frac{97-n}{2}}$ during the mixing phase.

Three Equilibrium Phases

Lemma 10.10

Consider the depicted version of the centipede game with $\mu_0(\vartheta_C) \geq \left(\frac{1}{2}\right)^{48}$. Then the unique perfect Bayesian equilibrium is characterized by

1. Both players choosing C at x_n , h_n for $n < 97 + \frac{\log(\mu_0)}{\log(2)} =: n_0$.
2. Both players mixing at x_n , h_n for $n_0 \leq n < 96$ with probability

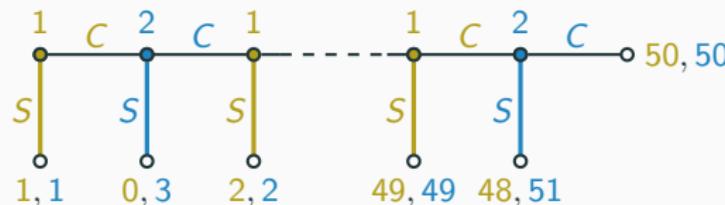
$$\sigma_1(\vartheta_p, x_n; S) = \frac{1}{2 - \left(\frac{1}{2}\right)^{\frac{96-n}{2}}}, \quad \sigma_2(h_n; S) = \frac{1}{2}.$$

3. Both players choosing S at x_{96} and h_{97} , respectively.
-

Note:

- It takes $K(\mu_0) = -\frac{\log(\mu_0)}{\log(2)}$ periods to build the reputation.
- Observe that $K(\mu_0)$ depends only on μ_0 , not on T .

Reputations and Expected Play



Reputations:

- “Reputations” are built at the end of the game.
- Long-lived players understand that, in equilibrium, they will be built. It is thus not necessary to build them in the beginning.

Expected play:

- At the beginning, both players expect to see C with certainty.
- As the game progresses, it becomes more likely that players stop.
- This is precisely what we expect to see in the lab!

Partnership Game

Proposition 10.11

Suppose that players assign beliefs $\mu_0 > 0$ that their opponent is the grim trigger commitment type in a T -repeated partnership game. Then there exists a constant $K(\mu_0)$, independent of T , such that in any PBE, both players exert effort with certainty in all but the last $K(\mu_0)$ periods.

Remarks:

- For large T , players exert effort in almost every period.
- The proof works along the same lines.
- Because ϑ_G is unforgiving, shirking is equivalent to stopping in the centipede.



Summary

Finitely repeated games:

- Players cooperate with decreasing probability as the game progresses.
- As a consequence, beliefs of facing the commitment type rise only at the end, i.e., reputations are built at the end.
- Nevertheless, expected play models observed behavior quite well.

Infinitely repeated games:

- Building of reputations cannot be deferred to the end.
- In infinitely repeated reputation games with perfect monitoring and a single commitment type, if a reputation is built in equilibrium, it is built in the first period.

Literature

- G.J. Mailath and L. Samuelson: **Repeated Games and Reputations: Long-Run Relationships**, Chapter 17, Oxford University Press, 2006
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