Balanced Growth Path

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Fall Semester

Basic Solow Growth Model

- Production function: $Y = F(K, L)$
- Output per worker: $y = f(k)$, where $k = \frac{K}{L}$ is capital per worker
  - Only k matters for y. Population size does not affect the relationship between y and k.
- At steady state:
  - $k = k^*$, $y^* = f(k^*)$, and $c^* = (1 - s)f(k^*)$, where s is saving rate
  - Capital, output, and consumption are **steady over time**.

Basic Solow Growth Model

- The basic Solow model implies **no growth** for an economy at steady state.
- However, this is not consistent with Macro data:
  - The website: Gapminder.org
  - Balanced growth path (BGP):
    -Kaldor, Nicholas, 1961, "Capital Accumulation and Economic Growth", in *The Theory of Capital*, edited by Friedrich A. Lutz and Douglas C. Hague. New York: St. Martin's Press.

Balanced Growth Path

- The stylized facts of **industrialized economies in the 20th century** reported in Kaldor Nicholas (1961).
- Kaldor facts:
  - The growth rate of GDP is roughly constant over time.
  - The capital labor ratio $\frac{K}{L}$ and the output-labor ratio $\frac{Y}{L}$ are growing at a roughly constant rate.
  - The real wage, $w = F_L$, is growing at a constant rate.
  - The capital-output ratio $\frac{K}{Y}$ is roughly constant over time. → doesn't grow
  - The investment-output ratio $\frac{I}{Y}$ is roughly constant over time.
  - The rental rate on capital, $r = F_K$, is constant over time.
  - The income share of capital, $\frac{rK}{Y}$ and the income share of labor $\frac{wL}{Y}$ are roughly constant over time.

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Solow Model and BGP

- **Balanced growth** is a situation in which C_t, Y_t, I_t, K_t, and w_t grow at constant, but possibly different rates.
- An economy that behaves according to the above facts is **along a balanced growth path**.
- The basic Solow model and the neoclassical growth model we introduced so far are inconsistent with the first three facts because these frameworks predict **no growth** in the long run.
- How to reconcile the neoclassical growth model with the Kaldor facts?

Macro focus on
1. Growth
2. Recycle
3. Money

$\longrightarrow$

1. Solow：只能解释 S-S & 外生变动
2. New classical："+ 多解释 Recycle
3. BGP：" + 终于解释了 S-S 的 Growth

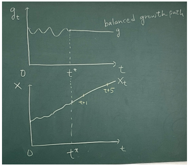
The Engine of Growth

- How to reconcile the neoclassical growth model with the Kaldor facts?
- Consider "the engine of growth" in the neoclassical growth model:
  1. - Exogenous growth: → model does not provide interpretation, just assumption.
       such as population growth, technological progress...
  2. - Endogenous growth: → optimal choice itself would grow
       such as human capital accumulation, endogenous population growth, spillover effects from production activity...

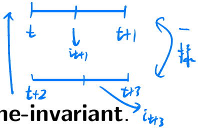
1. 單純假設不斷有外生的變動 → 造成成長

2. 解出的變數本身就是有成長

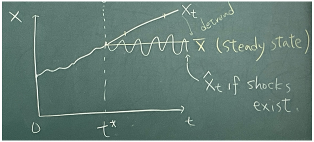
Dynamic Programming and BGP



各期的跨期决策应相同，
否则无法使用 recursive 来解



- Dynamic Programming
  - A dynamic programming problem should be **stationary**.
  - All functions in a dynamic programming problem are **time-invariant**.
- When "growth" is introduced into a dynamic programming problem,
  - The problem is non-stationary.
  - Some variables grow at some rates.
  - No steady state
- How to incorporate "growth" into our current framework? **Detrend**, removing the growth!!



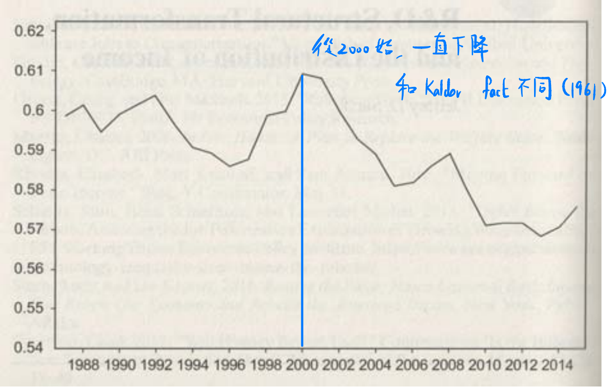
Why Detrend?

- We are more interested in steady-state properties.
標準化
- When we calibrate the model, some parameters are calibrated by some stationary relationships observed from data.
- We want to have a “manageable” model economy that is consistent with BGP.

The Decline in the Labor Income Share 1

The stylized facts have visibly broken down since around 2000.

Figure: Labor Income Share in the U.S.

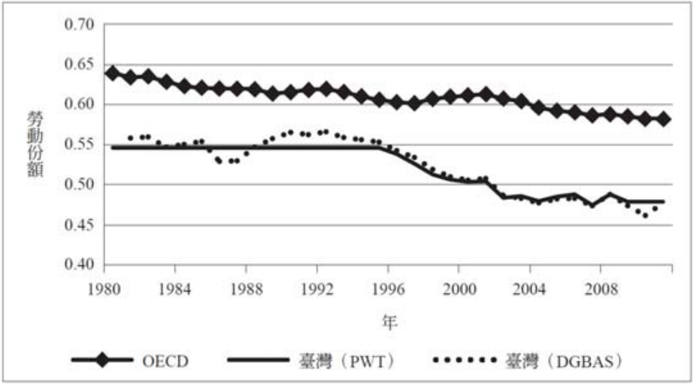


Source: Figure 1 in Sachs (2018).

$$Y \nearrow rK \Rightarrow \frac{rK}{Y} = \alpha \uparrow$$
$$Y \searrow wh \Rightarrow \frac{wh}{Y} = 1-\alpha \downarrow$$

The Decline in the Labor Income Share 2

Same pattern is observed in OECD and Taiwan.



Source: Figure 1 in Lin, Chang, and Lu (2017).

Necessary condition

The ~~necessary~~ sufficient conditions for the existence of BGP are

- Production function is constant returns to scale.
- Homothetic utility functions

Homogenous vs Homothetic Functions

- Homogenous of degree k:

$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, x_2, \dots, x_n), \forall t > 0.$$

- Homothetic function
  - A function is called homothetic if it is a monotone transformation of a homogenous function.
  - Example:
    $u(x, y) = xy$ is a homogenous function. we will call $v(x, y) = x^3 y^3 + xy$ and $w(x, y) = xy + 1$ as homothetic functions because $v(x, y) = [u(x, y)]^3 + u(x, y)$ and $w(x, y) = u(x, y) + 1$.

Examples for the Existence of BGP

- **Example 1**:
  The periodic utility function is $u(c_t, 1-h_t) = \log c_t - Bh_t$. When we consider growth and detrend the objective function,
  $\Rightarrow \max \sum_{t=0}^{\infty} (\beta\eta)^t [\log \hat{c}_t - Bh_t]$+constant.
- **Example 2**:
  $u(c_t, 1-h_t) = \log c_t + A\log(1-h_t)$,
  $\Rightarrow \max \sum_{t=0}^{\infty} (\beta\eta)^t [\log \hat{c}_t + A\log(1-h_t)]$+constant.
- **Example 3**:
  $u(c_t, 1-h_t) = \frac{(c_t^{\alpha}(1-h_t)^{1-\alpha})^{1-\sigma}}{1-\sigma}$,
  $\Rightarrow \max \sum_{t=0}^{\infty} \left(\beta\eta g^{\alpha(1-\sigma)}\right)^t \frac{(\hat{c_t}^{\alpha}(1-h_t)^{1-\alpha})^{1-\sigma}}{1-\sigma}$.

An Example for No BGP

Suppose the periodic utility function is
$u(c_t, 1-h_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + A\frac{(1-h_t)^{1-\sigma}}{1-\sigma}$.
With population growth rate η and exogenous technological progress g, the detrended lifetime utility function is:

$$\begin{aligned}
& \sum_{t=0}^{\infty} (\beta\eta)^t [\frac{c_t^{1-\sigma}}{1-\sigma} + A\frac{(1-h_t)^{1-\sigma}}{1-\sigma}] \\
= & \sum_{t=0}^{\infty} (\beta\eta)^t [\frac{(g^t \hat{c}_t)^{1-\sigma}}{1-\sigma} + A\frac{(1-h_t)^{1-\sigma}}{1-\sigma}] \\
= & \sum_{t=0}^{\infty} (\beta\eta g^{1-\sigma})^t \frac{\hat{c}_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} (\beta\eta)^t A\frac{(1-h_t)^{1-\sigma}}{1-\sigma}.
\end{aligned}$$

In this case, there is no balanced growth path.