4. Dynamic Games

ECON 7219 – Games With Incomplete Information

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Extensive-Form Games

Selling Farmland

Two of Taiwan's most valuable crops are tea and rice.

Annual average yield:

- Tea: 5.35m NTD/km^2.
- Rice: 4.2m NTD/km^2.

A Rice Farmer considers selling his/her land to a Tea Farmer.

- Suppose high/low-quality soil yields 50% above/below average.
- The Rice Farmer knows the quality, but the Tea Farmer does not.
- Simultaneously, the Tea Farmer proposes a price $p \geq 0$ and the Rice Farmer quotes a set $\mathcal{P}$ of prices he/she will accept.

Source: https://www.visualcapitalist.com/the-worlds-most-valuable-cash-crop/

Selling Farmland: Bayesian Nash equilibria

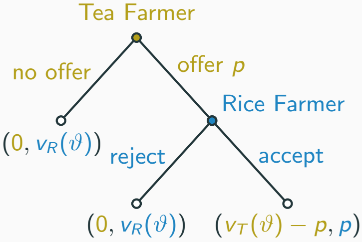
Bayesian Nash equilibria:

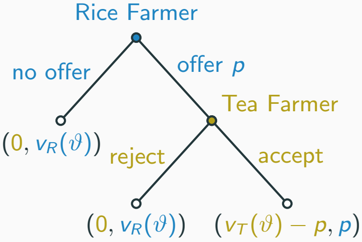
- Because of adverse selection, trade can occur only if the soil quality is low, for which the value of the land is $v_R(L) = 4.2$ and $v_T(L) = 5.4$.
- Any $p = \min \mathcal{P}(L) \leq 5.4$ constitutes a trade equilibrium.
- Any $p < 4.2$ and $\min \mathcal{P}(L) > 5.4$ is a no-trade equilibrium.

Understanding the equilibria:

- Trade may occur at any price in the interval $[4.2, 5.4]$.
- Which price is realized depends on the bargaining power of the farmers.
- We can model bargaining power through game dynamics.
- The no-trade equilibrium is inefficient. It is the result of a miscoordination that can be avoided if players do not act simultaneously.

Selling Farmland: a Dynamic Setting

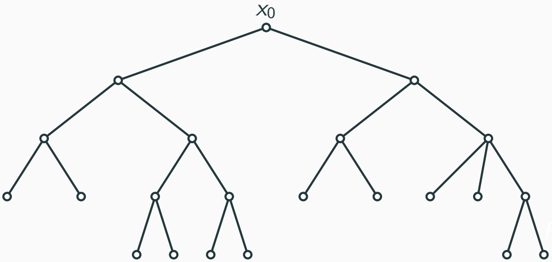




Adding dynamics:

- If the Tea Farmer gets to make a take-it-or-leave-it offer, then the Rice Farmer will agree to any sale at price $p \geq 4.2$.
- Anticipating this response, the Tea Farmer best offers $p = 4.2$.
- Conversely, if the Rice Farmer gets to make a take-it-or-leave-it offer, then the Tea Farmer will agree to any sale at price $p \leq 5.4$.
- Anticipating this response, the Rice Farmer best asks for $p = 5.4$.

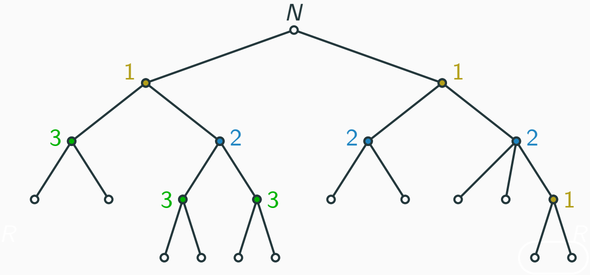
Extensive-Form Game



Game tree: Arborescence $(\mathcal{X}, \prec)$

- $\mathcal{X}$ is a finite set of nodes x, partially ordered by precedence relation $\prec$:
  - $\prec$ is asymmetric: $x \prec x'$ implies $x' \not\prec x$ $\rightarrow$ prevents cycles.
  - $\prec$ is transitive: $x \prec x'$ and $x' \prec x''$ implies $x \prec x''$.
- There exists $x_0 \in \mathcal{X}$ with $x_0 \prec x$ for all $x \in \mathcal{X} \setminus \{x_0\}$ called initial node.
- Each node $x \neq x_0$ has exactly one immediate predecessor, that is, one $x' \prec x$ such that $x'' \prec x$ and $x'' \neq x'$ implies $x'' \prec x'$.

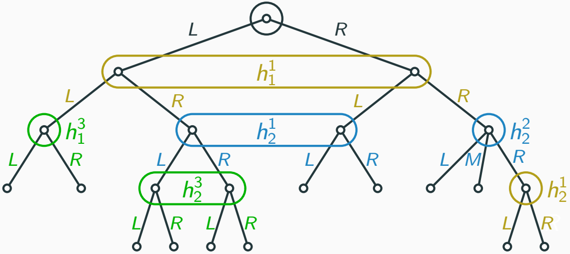
Extensive-Form Game



Players and payoffs:

- There is a finite set of strategic players $\mathcal{I} = \{1, \ldots, n\}$.
- There is a special non-strategic player N, called nature.
- Let $\mathcal{Z} := \{x \in \mathcal{X} \mid \nexists x' \in \mathcal{X} \text{ with } x \prec x'\}$ be the set of terminal nodes.
- Player $i \in \mathcal{I}$ receives a payoff $u^i : \mathcal{Z} \to \mathbb{R}$ at terminal node $z \in \mathcal{Z}$.
- A map $i : \mathcal{X} \setminus \mathcal{Z} \to \mathcal{I} \cup \{N\}$ indicates the active player at node x.

Extensive-Form Game



Information and actions:

- We denote the set of player i's nodes by $\mathcal{X}_i := \{x \in \mathcal{X} \setminus \mathcal{Z} \mid i(x) = i\}$.
- Let $\mathcal{H}_i$ be a partition of $\mathcal{X}_i$ into information sets $h_i \in \mathcal{H}_i$ such that each node $x \in h_i$ has the same number of successor nodes.
- Player i has actions $\mathcal{A}(h_i)$ available at information set $h_i \in \mathcal{H}_i$.
- For each $x \in h$, there is a bijection of $\mathcal{A}(h)$ to successors of x, indicating which node is reached from x when $a \in \mathcal{A}(h)$ is played.

Extensive-Form Game

Definition 4.1

An extensive-form game $\mathcal{G} = (\mathcal{X}, \prec, \mathcal{I}, i, (\mathcal{H}_i), (\mathcal{A}(h)), (u_i))$ consists of:

1. An arborescence $(\mathcal{X}, \prec)$ with terminal nodes $\mathcal{Z}$,
2. A set of players $\mathcal{I}$,
3. A map $i : \mathcal{X} \setminus \mathcal{Z} \to \mathcal{I} \cup \{N\}$ indicating the active player,
4. A partition $\mathcal{H}_i$ of $\mathcal{X}_i = \{x \in \mathcal{X} \setminus \mathcal{Z} \mid i(x) = i\}$ for each $i \in \mathcal{I}$,
5. A map $\mathcal{A}(h)$ indicating the available actions at h in $\mathcal{H} = \bigcup_{i \in \mathcal{I}} \mathcal{H}_i$,
6. A payoff function $u_i : \mathcal{Z} \mapsto \mathbb{R}$ for each player i.

Note: We refer to information sets in dynamic settings by h because those typically correspond to histories of observed actions.

Imperfect Information

Definition 4.2

Information is perfect if nature has no moves and each information set is a singleton. Information is imperfect otherwise.

Possible causes for imperfect information:

- Incomplete information: there is uncertainty about payoffs or players' types. Typically, this corresponds to unobserved moves by nature.
- Strategic uncertainty: simultaneous-move games.
- Imperfect monitoring: instead of observing actions directly, players observe a signal, whose distribution is affected by the chosen actions.

Strategies

Definition 4.3

Let $\mathcal{A}_i := \bigcup_{h_i \in \mathcal{H}_i} \mathcal{A}(h_i)$ denote all of player i's actions.

1. A pure strategy of player i is a map $s_i : \mathcal{H}_i \to \mathcal{A}_i$ such that $s_i(h_i) \in \mathcal{A}(h_i)$ for every $h_i \in \mathcal{H}_i$. Let $\mathcal{S}_i$ denote the set of i's pure strategies.
2. A mixed strategy of player i is a distribution $\sigma_i \in \Delta(\mathcal{S}_i)$.
3. A behavior strategy of player i is a map $\sigma_i : \mathcal{H}_i \to \Delta(\mathcal{A}_i)$ such that $\sigma_i(h_i) \in \Delta\mathcal{A}(h_i)$ for every $h_i \in \mathcal{H}_i$.

Remark:

- We denote by $\sigma_i(h_i; a_i)$ the probability that a_i is chosen under $\sigma_i(h_i)$.
- For $\mathcal{H}_i = \mathcal{T}_i$, these notions coincide with strategies in Bayesian games.

Outcome of an Extensive-Form Game

Outcome:

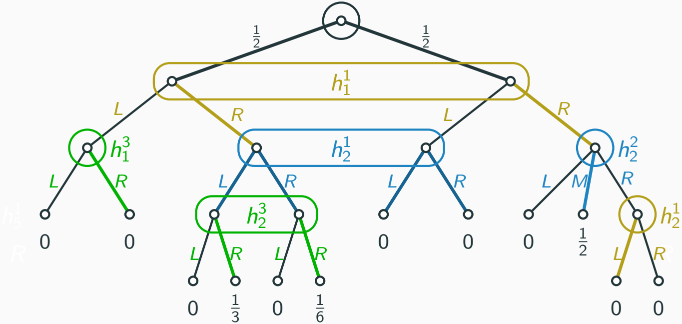
- Each terminal node $z \in \mathcal{Z}$ is reached by a unique sequence of actions.
- The outcome of an extensive-form game is a $\mathcal{Z}$-valued random variable Z or, equivalently, a sequence A of realized actions leading to Z.

Induced probability measure:

- For any node x, let $h(x)$ denote the information set at x and let a_x denote the taken at the predecessor of x that leads to x.
- For any $x \in \mathcal{X}$, let $(x_0, x_1, \ldots, x_k)$ denote the sequence of nodes leading to $x = x_k$. Then x is reached under σ with probability

$$P_\sigma(\{x\}) = \prod_{j=0}^{k-1} \sigma_{i(x_j)}(h(x_j); a_{x_{j+1}}).$$

Probability Distribution over Outcomes



Example:

- Player 1 chooses $\sigma_1(h_1^1) = R$ and $\sigma_1(h_1^2) = L$,
- Player 2 chooses $\sigma_2(h_2^1) = \frac{2}{3}L + \frac{1}{3}R$ and $\sigma_2(h_2^2) = M$,
- Player 3 chooses $\sigma_3(h_3^1) = R$ and $\sigma_3(h_3^2) = R$.

Determining $\mathrm{P}_\sigma(\mathrm{z})$**:** multiply probabilities of edges between z and x_0.

Nash Equilibrium

Definition 4.4

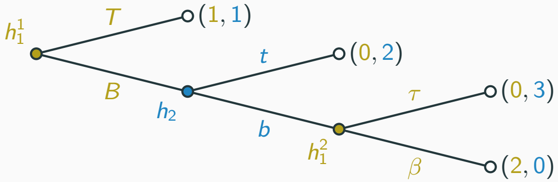
A Nash equilibrium in an extensive-form game is a strategy profile σ such that for every player i and every pure strategy s_i

$$\mathbb{E}_{\sigma}[u_i(Z)] \geq \mathbb{E}_{(s_i,\sigma_{-i})}[u_i(Z)].$$

Finding Nash equilibria:

- Note that this definition is identical to Nash equilibrium in a static game with $\mathcal{A} = \mathcal{S}$ and payoff functions $\widehat{u}_i(\sigma) = \mathbb{E}_{\sigma}[u_i(Z)]$.
- The transformed game is called the strategic-form game.
- We already know how to find Nash equilibria in static games.

Example of an Extensive-Form Game



Pure strategies:

- Player 2 has two pure strategies: $s_2(h_2) = t$ and $s_2(h_2) = b$.
- Player 1 has four pure strategies: any combinations of $s_1(h_1^1) \in \{T, B\}$ and $s_1(h_1^2) \in \{\tau, \beta\}$.
- Formally, a strategy involves a decision at every information set even if h_1^2 is never reached due to Player 1's own decision at h_1^1.

Reduced Strategic Form

	t	b
$T\tau$	$1, 1$	$1, 1$
$T\beta$	$1, 1$	$1, 1$
$B\tau$	$0, 2$	$0, 3$
$B\beta$	$0, 2$	$2, 0$

$\longrightarrow$

	t	b
T	$1, 1$	$1, 1$
$B\tau$	$0, 2$	$0, 3$
$B\beta$	$0, 2$	$2, 0$

Reduced-strategic form:

1. List all pure strategies of player i in i^{th} dimension of a payoff matrix.
2. Add payoff vector $\mathbb{E}_s[u(Z)]$ to cell corresponding to $s = (s^1, \ldots, s^n)$.
3. Combine payoff equivalent strategies (= duplicate rows/columns).

Nash equilibria are $(T, yt + (1-y)b)$ with $y \geq \frac{1}{2}$.

Harsanyi's Equivalence

Theorem 4.5

Let $\mathcal{G}$ be a Bayesian game with finitely many actions, finite type spaces $\mathcal{T}_i$, and a common prior P with $P(\tau_i) > 0$ for every $\tau_i \in \mathcal{T}_i$.

- *The Bayesian game is equivalent to the extensive-form game $\mathcal{G}'$, in which nature chooses the players types in the first move.*
- *A Bayesian Nash equilibrium of $\mathcal{G}$ is a Nash equilibrium of the reduced strategic-form game associated with $\mathcal{G}'$.*

Two interpretations:

- A mixed Nash equilibrium of the strategic-form game corresponds to a mixed-strategy Bayesian Nash equilibrium in the extensive-form game.
- Last week we have interpreted each type τ_i as a separate player, leading to a Bayesian Nash equilibrium in behavior strategies.

Realization Equivalence

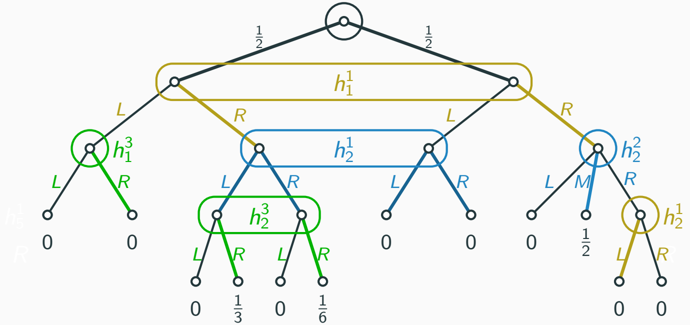
Definition 4.6

Two strategy profiles σ and $\widehat{\sigma}$ are realization equivalent if they induce the same distribution over outcomes, that is, $P_\sigma = P_{\widehat{\sigma}}$.

Kuhn's theorem:

- Mixed and Behavior strategies are realization equivalent.
- Each pure strategy profile s leads to a unique terminal node z_s.
- For any behavior strategy profile σ, we can find a realization equivalent mixed strategy profile $\widehat{\sigma}$ by setting $\widehat{\sigma}(s) = P_\sigma(\{z_s\})$.
- For any mixed strategy profile $\widehat{\sigma}$, we can find a realization equivalent behavior strategy profile σ by setting $\sigma_i(h; a) = \sum_s \widehat{\sigma}_i(s) s_i(h; a)$.

Probability Distribution over Outcomes

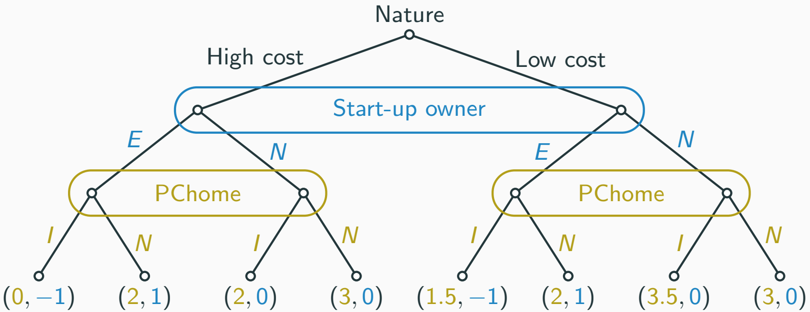


Behavior strategy:

- Player 1 chooses $\sigma_1(h_1^1) = R$ and $\sigma_1(h_1^2) = L$,
- Player 2 chooses $\sigma_2(h_2^1) = \frac{2}{3}L + \frac{1}{3}R$ and $\sigma_2(h_2^2) = M$,
- Player 3 chooses $\sigma_3(h_3^1) = R$ and $\sigma_3(h_3^2) = R$.

Mixed strategy: Choose LM with probability $\frac{2}{3}$ and RM with probability $\frac{1}{3}$.

Start-Up Problem as an Extensive-Form Game



	E	N
I	0, −1	2, 0
N	2, 1	3, 0

High cost

	E	N
I	1.5, −1	3.5, 0
N	2, 1	3, 0

Low cost

Summary

Extensive-form games:

- A very general framework for dynamic interactions that allows imperfect information for a variety of reasons.
- Game tree provides a good visual for small games.
- Drawbacks of extensive-form games: they are inherently finite and somewhat notationally cumbersome.

Harsanyi's equivalence:

- If players' have a common prior, a finite Bayesian game can be written as a game tree, where Nature selects players' types.
- However, this equivalence did not exploit the dynamics embedded in extensive-form games yet.

Check Your Understanding

True or false:

1. In a game tree, it is impossible that two different pure strategies lead to the same terminal node.
2. If information is imperfect, then it is incomplete.
3. Any single static game can be the reduced strategic-form game of many extensive-form games.
4. In a game of complete information, the probability measure P_σ defined here coincides with P_α defined by $P_\alpha(A = a) = \alpha(a)$.

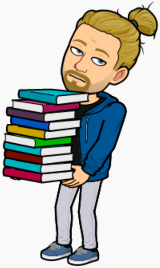
Scan me

Short-answer question:

5. Suppose a pure strategy is a book with instructions on what to do in any given situation. Is going to a library and picking a book at random a mixed or a behavior strategy?

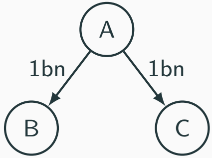
Literature

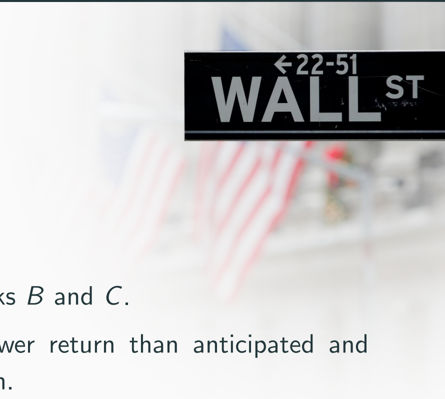
- D. Fudenberg and J. Tirole: Game Theory, Chapter 3, MIT Press, 1991
- M. Maschler, E. Solan, and S. Zamir: Game Theory, Chapter 3, Cambridge University Press, 2013
- S. Tadelis: Game Theory: An Introduction, Chapter 7, Princeton University Press, 2013
- H. Kuhn: Extensive games and the problem of information, in: Contributions to the Theory of Games, Vol. II, Annals of Mathematics Studies No. 28, Princeton University Press, 1953, 193–216.
- J.Harsanyi: Games with Incomplete Information Played by 'Bayesian' Players, Part I-III, Management Science, **14** (1967–68), 159–182, 320–334, 486–502.



Subgame Perfection

Bail-ins and Bailouts

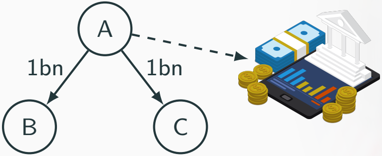




Consider a 3-bank network:

- Bank A has liabilities of 1bn to banks B and C.
- Bank A's investments yielded a lower return than anticipated and bank A's assets are worth only 1.6bn.
- If Bank A declares bankruptcy, losses of 200m are incurred (asset liquidation, legal fees, etc.) before A repays 0.7bn each to B and C.
- Creditors B and C have an incentive to bail in A and write down their claims on A by 200m each to save the bankruptcy costs.

Bail-ins and Bailouts



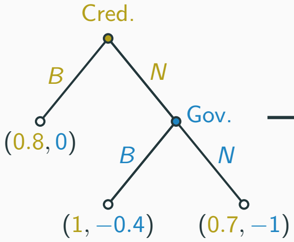
Presence of government:

- Bankruptcy of bank A may cause severe losses to the economy.
- Government also has an incentive to rescue bank A.

Bail-in vs. bailout:

- Banks B and C will coordinate a bail-in only if they believe the government will not step in with a bailout.
- Government would like to "threaten" that it will not bail out A.
- Bail-in will be organized if B and C deem the threat credible.

Bail-in / Bailout as Extensive-Form Game

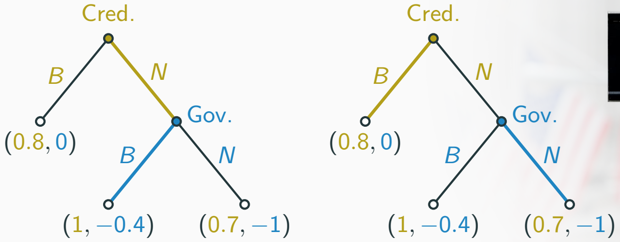


	B	N
B	0.8, 0	0.8, 0
N	1, −0.4	0.7, −1

Bail-in / bailout as an extensive-form game:

- Creditors choose whether or not to bail in bank A.
- Government then chooses whether or not to bail out bank A.
- Government is the "lender of last resort" and moves last.
- To find Nash equilibria: transform the game into the strategic form.
- Pure-strategy Nash equilibria are (B, N) and (N, B).

Comparing the two Equilibria



Equilibrium (B, N):

- Government threatens to not bail out bank A if Creditors choose N.
- Given that Creditors bail in A, Government's action does not matter.

Problem:

- Given that the Government gets to act, it is suboptimal to choose N.
- We say that the Government's no-bailout threat is not credible.

Subgame

Definition 4.7

Consider an extensive-form game $(\mathcal{X}, \prec, \mathcal{I}, i, \mathcal{H}, \mathcal{A}, u)$. For any $x \in \mathcal{X}$, let $\mathcal{X}(x) := \{x\} \cup \{x' \in \mathcal{X} \mid x \prec x'\}$ denote the sub-tree starting at x. Then $\mathcal{G}(x) = (\mathcal{X}(x), \prec, \mathcal{I}, i|_x, \mathcal{H}|_x, \mathcal{A}|_x, u|_x)$ is a subgame of $\mathcal{G}$ if:

1. For any $h \in \mathcal{H}$: if $x' \in \mathcal{X}(x)$ for some $x' \in h$, then $h \subseteq \mathcal{X}(x)$,
2. $i|_x, \mathcal{H}|_x, \mathcal{A}|_x$, and $u|_x$ are the restrictions of $i, \mathcal{H}, \mathcal{A}$, and u to $\mathcal{X}(x)$.

Two points worth noticing:

- Point 1. says that an information set is either completely contained in the subgame or the intersection with the subgame is empty.
- This applies to the node x starting subgame $\mathcal{G}(x)$, i.e., $h(x) = \{x\}$.

At any $x' \in \mathcal{X}(x)$, it is common knowledge that $\mathcal{I}$ are playing the subgame.

Subgame Perfection

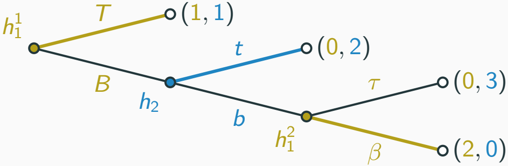
Definition 4.8

A behavior strategy profile σ is a subgame perfect equilibrium if its restriction to $\mathcal{G}(x)$ is a Nash equilibrium for every proper subgame $\mathcal{G}(x)$.

Properties:

- Every finite extensive-form game has an SPE in behavior strategies.
- Refinement: every subgame perfect equilibrium is a Nash equilibrium.
- Subgame perfection eliminates non-credible threats.
- If all information sets are singletons, then subgame perfect equilibria coincide with the solution found by backward induction.

Finding Nash Equilibria vs. Finding SPE



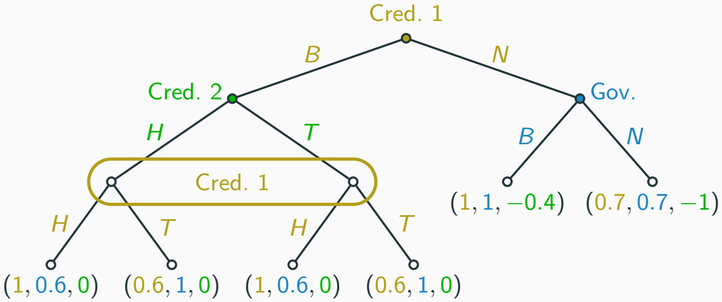
Nash equilibria:

- Bring the game into the reduced strategic form.
- Eliminate all strictly dominated strategies.
- Find the intersection of the players' best response correspondences.
- Nash equilibria are $(T\alpha, yt+(1-y)b)$ with $y \geq \frac{1}{2}$ and $\alpha \in \Delta(\mathcal{A}(h_1^2))$.

Subgame-perfect equilibria:

- Find the backward-induction solution is often extremely simple.
- The unique subgame-perfect equilibrium in this game is $(T\beta, t)$.

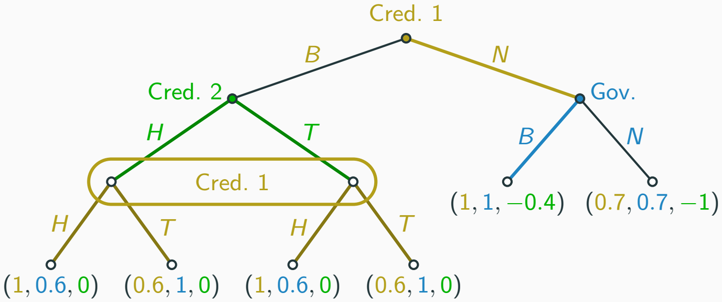
Bail-In Subgame



Modification of the game:

- Creditor 1 gets to decide whether the creditors organize a bail-in.
- If they do, contributions are decided by a matching pennies game.
- Not solvable by backward induction.

Dynamic Programming Principle



Dynamic programming:

- If subgame $\mathcal{G}$ has a unique equilibrium $\sigma_{\mathcal{G}}$, then replacing $\mathcal{G}$ with $\mathbb{E}_{\sigma_{\mathcal{G}}}[u(Z)]$ retains subgame perfect equilibria in the rest of the tree.
- If subgame $\mathcal{G}$ has multiple equilibria, doing this for all subgame perfect equilibrium outcomes of $\mathcal{G}$ yields all subgame perfect equilibria.

Subgame Perfection on the Equilibrium Path

Theorem 4.9

Let σ be a Nash equilibrium in an extensive-form game $\mathcal{G}$. For every node x, for which $\mathcal{G}(x)$ is a proper subgame with $P_\sigma(\{x\}) > 0$, the restriction $\sigma|_{\mathcal{G}(x)}$ is a Nash equilibrium of the subgame $\mathcal{G}(x)$.

Equilibrium path:

- Any node x with $P_\sigma(\{x\}) > 0$ is said to lie on the path of strategy profile σ because it is reached with positive probability.
- Any other node lies off the path.

Consequence:

- Any Nash equilibrium is subgame perfect on the equilibrium path.
- Subgame perfection refines only behavior off the path.

Proof of Theorem 4.9

Suppose the theorem fails:

- Suppose there is a Nash equilibrium σ, in which some player i has a profitable deviation $\widehat{\sigma}_i$ in the subgame $\mathcal{G}(x)$ with $P_\sigma(\{x\}) > 0$.

Derive a contradiction:

- Let $\widetilde{\sigma}_i$ be the strategy that plays σ_i outside of $\mathcal{G}(x)$ and $\widehat{\sigma}_i$ in $\mathcal{G}(x)$.
- Since $\widetilde{\sigma}_i$ and σ_i agree outside of $\mathcal{G}(x)$, we have $P_{\widetilde{\sigma}_i,\sigma_{-i}}(\{x\}) = P_\sigma(\{x\})$ and $P_{\widetilde{\sigma}_i,\sigma_{-i}}(\{z\}) = P_\sigma(\{z\})$ for any terminal node $z \notin \mathcal{G}(x)$. Thus,

$$\begin{aligned}\mathbb{E}_{\widetilde{\sigma}_i,\sigma_{-i}}[u_i(Z)] = {} & \mathbb{E}_{\widetilde{\sigma}_i,\sigma_{-i}}[u_i(Z) \mid Z \notin \mathcal{G}(x)]\big(1 - P_{\widetilde{\sigma}_i,\sigma_{-i}}(\{x\})\big) \\ & + \mathbb{E}_{\widetilde{\sigma}_i,\sigma_{-i}}[u_i(Z) \mid Z \in \mathcal{G}(x)] P_{\widetilde{\sigma}_i,\sigma_{-i}}(\{x\}) \\ > {} & \mathbb{E}_\sigma[u_i(Z)].\end{aligned}$$

- This stands in contradiction to σ being a Nash equilibrium.

Completely Mixed Strategies

Complete mixture:

- A mixed strategy σ_i is completely mixed if it assigns positive weight $\sigma_i(s_i)$ to every pure strategy s_i.
- A behavior strategy σ_i is completely mixed if $\sigma_i(h_i; a_i) > 0$ for every $a_i \in \mathcal{A}(h_i)$ and every $h_i \in \mathcal{H}_i$.

Theorem 4.10

A Nash equilibrium in completely mixed strategies is subgame perfect.

Proof:

- If σ is completely mixed, then $P_\sigma(\{x\}) > 0$ for every node x.
- By previous result, $\sigma|_{\mathcal{G}(x)}$ is a Nash equilibrium of $\mathcal{G}(x)$ for every x.

Uniqueness

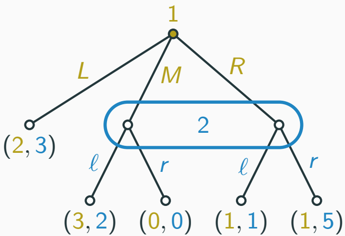
Theorem 4.11

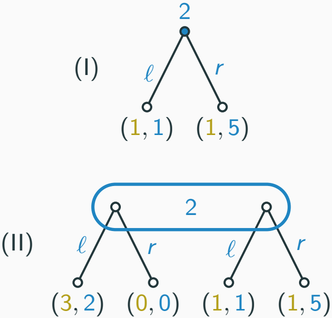
An extensive form game of perfect information has a unique SPE if no two terminal nodes award the same payoffs to any player.

Proof:

- Consider an immediate predecessor node x of $\mathcal{Z}$.
- Since no player is indifferent between any two terminal nodes, player $i(x)$ will choose a pure action.
- Using backward induction, no player will ever be indifferent between two actions, hence all choices except one are dominated.

Limitations of Subgame Perfection

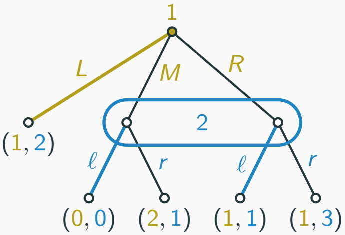


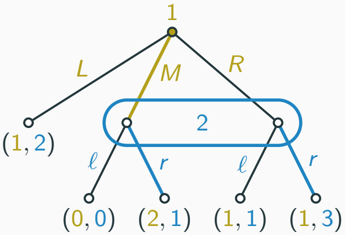


No proper subgames:

- Game (I) is no proper subgame because player 2 knows that player 1 played R, which is information he does not have in $\mathcal{G}$.
- Game (II) is no proper subgame because the relative probabilities player 2 assigns to the two nodes depend on player 1's action.

Seemingly Dominated Strategies





Two subgame-perfect equilibria:

- Given that Player 2 gets to act, it is strictly dominant to choose r.
- Nevertheless, (L, ℓ) is a subgame-perfect equilibrium.
- By definition, a subgame starts at a singleton information set, hence the only subgame is the entire game itself.
- As a consequence, all Nash equilibria are trivially subgame perfect.

Summary

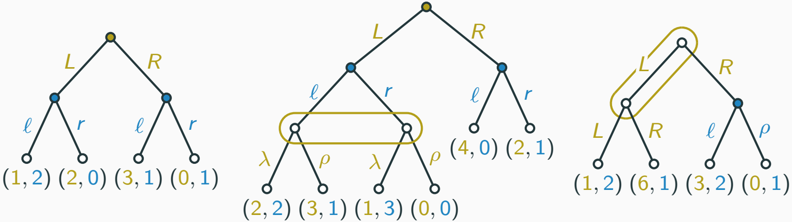
Refinement off the path:

- Any Nash equilibrium is already optimal on the equilibrium path.
- Subgame perfection refines Nash equilibria off the equilibrium path.

Eliminating non-credible threats:

- Subgame perfection eliminates threatened continuation strategies that are suboptimal if carried out.
- Subgame perfection cannot "cut through" information sets.

Check Your Understanding



Short-answer questions:

1. Find the SPE in the three games.
2. How many SPE are there in each game?
3. How many outcomes are consistent with SPE?

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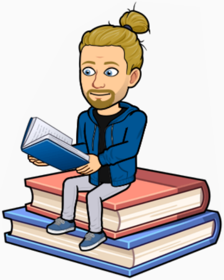
True or false:

4. The restriction $\sigma|_{\mathcal{G}(x)}$ of an SPE σ to subgame $\mathcal{G}(x)$ is an SPE of $\mathcal{G}(x)$.

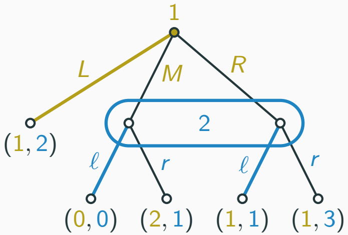
Literature

- D. Fudenberg and J. Tirole: Game Theory, Chapter 3, MIT Press, 1991
- M. Maschler, E. Solan, and S. Zamir: Game Theory, Chapter 7.1, Cambridge University Press, 2013
- S. Tadelis: Game Theory: An Introduction, Chapter 8, Princeton University Press, 2013
- R. Selten: Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit, Zeitschrift für die gesamte Staatswissenschaft, **121** (1965), 301–324



Perfect Bayesian Equilibrium

Extending Subgame Perfection



Basic idea:

- If a player has beliefs μ_h over an information set h, he/she can aggregate expected payoffs from subgames $\mathcal{G}(x)$ that start at $x \in h$.
- We want players to act optimally, given their beliefs.
- Moreover, beliefs should be derived from Bayes' rule.

Beliefs on the Equilibrium Path

Beliefs over continuation games:

- For any information set h that is reached with positive probability under σ, Bayesian updating induces beliefs over h given by

$$\mu_h(x) := P_\sigma(\{x\} \mid h) = \frac{P_\sigma(\{x\})}{P_\sigma(h)}.$$

- Moreover, any node x' that can be reached from h has conditional probability of being reached, given that h is reached, of

$$P_\sigma(\{x'\} \mid h) = \frac{P_\sigma(\{x'\})}{P_\sigma(h)}.$$

- This allows us to aggregate expected payoffs on h by maximizing the conditional expected value $\mathbb{E}_\sigma[u_i(Z) \mid h]$.
- However, we want strategies to be optimal also off the path.

Beliefs off the Equilibrium Path

Beliefs over continuation games:

- Suppose a player has some beliefs μ_h over information set h.
- Given beliefs μ_h, any node x that is reachable from $x_0 \in h$ via sequence $(x_0, x_1, \ldots, x_k)$ with $x = x_k$ is reached with probability

$$P_{\mu_h,\sigma}(\{x\}) = \mu_h(\{x_0\}) \prod_{j=0}^{k-1} \sigma_{i(x_j)}(h(x_j); a_{x_{j+1}})$$

  when continuation profile σ is played, where $h(x_j)$ denotes the information set at x_j and $a_{x_{j+1}}$ denotes the taken at x_j that leads to x_{j+1}.

Remark:

- Probability $P_{\mu_h,\sigma}(\{x\})$ multiplies $\mu(\{x_0\})$ with the probability of all edges in the tree between x_0 and x.
- On the path, $P_{\mu_h,\sigma}$ coincides with $P_\sigma(\cdot \mid h)$ if $\mu_h = P_\sigma(\cdot \mid h)$.

Sequential Rationality

Definition 4.12

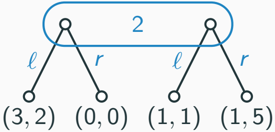
1. A belief system $\mu = (\mu_h)_{h \in \mathcal{H}}$ assigns beliefs to each information set.
2. A strategy profile σ is sequentially rational, given belief system μ, if for every player i, every information set $h \in \mathcal{H}_i$, and every deviation s_i,

$$\mathbb{E}_{\mu_h,\sigma}[u_i(Z)] \geq \mathbb{E}_{\mu_h,s_i,\sigma_{-i}}[u_i(Z)].$$

3. A pair (μ, σ) is called an assessment.

Sequential rationality:

- Allows us to "cut through" information sets.
- Allows us to analyze improper subgames.



Perfect Bayesian Equilibrium

Definition 4.13

An assessment (μ, σ) is a perfect Bayesian equilibrium for prior P if

1. σ is sequentially rational given μ,
2. μ is derived from P via Bayes rule "wherever possible."
3. μ can be anything where Bayes rule "is impossible."

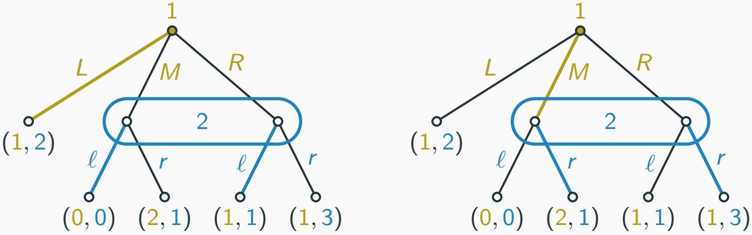
Wherever possible means:

- On the equilibrium path, we must have $\mu_h = P_\sigma(\,\cdot \mid h\,)$.
- At any singleton information set $h = \{x\}$, we must have $\mu_h = \delta_x$.[1]
- For any proper subgame $\mathcal{G}(x)$, we must have $\mu_h = P_{\delta_x, \sigma_x}(\,\cdot \mid h\,)$ on the path of the continuation equilibrium $\sigma|_x$.

Implication: Any perfect Bayesian equilibrium is subgame perfect.

[1] δ_x is called the Dirac measure and places probability 1 on x.

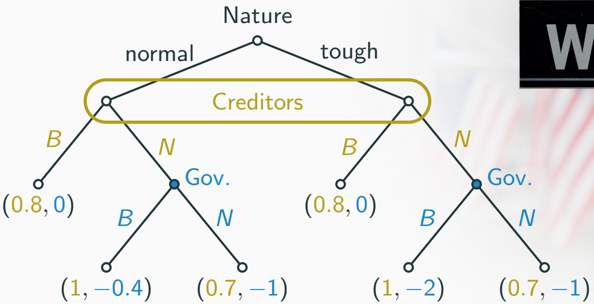
Seemingly Dominated Strategies



Proper refinement off the path:

- (L, ℓ) is not a perfect Bayesian equilibrium.
- Regardless of Player 2's belief at his/her information set, it cannot be sequentially rational to play a dominated strategy.
- (M, r) is a perfect Bayesian equilibrium.

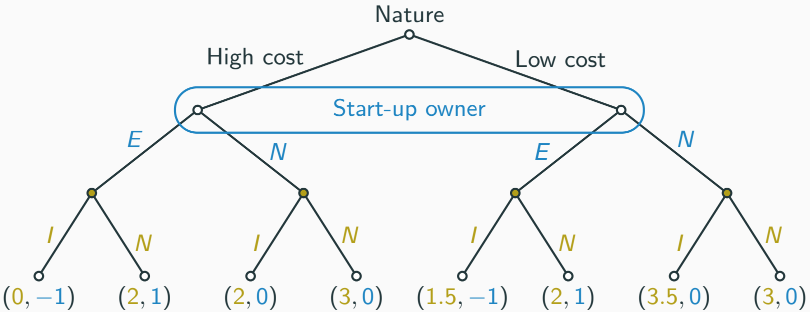
Two Types of Government



Tough government:

- Instead of being the welfare-maximizing normal type, the government may be tough on banks because elections are coming up.
- What is the perfect Bayesian Nash equilibrium for prior $P(\text{tough}) = p$?

Start-Up Problem as an Extensive-Form Game



	E	N
I	0, −1	2, 0
N	2, 1	3, 0

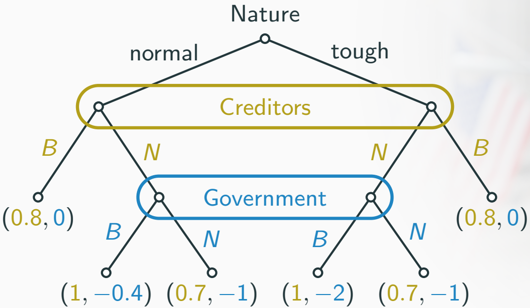
High cost

	E	N
I	1.5, −1	3.5, 0
N	2, 1	3, 0

Low cost

Sequential Equilibrium

Unknown Type of Government



Unknown voter preferences:

- The government is not aware whether voters prefer a government that is tough on banks or not.
- What are the perfect Bayesian Nash equilibrium with $P(\text{tough}) = 0.3$?

Forgetful Government

In equilibrium (B, N):

- The Government's information set lies off the equilibrium path of any proper subgame, hence beliefs are unrestricted in a PBE.
- The choice N is sequentially rational for strong beliefs that voters prefer a tough government.
- Perfect Bayesian equilibrium allows Government to "forget" its prior over types simply because the Creditors have acted unexpectedly.

This issue is resolved by:

- Sequential equilibria in general extensive-form games.
- Keeping track of relative beliefs in dynamic Bayesian games.

Consistent Belief Systems

Definition 4.14

A belief system $\mu = (\mu_h)_{h \in \mathcal{H}}$ is consistent with strategy profile σ if there exists a sequence of completely mixed strategies $(\sigma_k)_{k \geq 0}$ such that

$$\lim_{k \to \infty} \sigma_k = \sigma, \qquad \mu_h = \lim_{k \to \infty} P_{\sigma_k}(\,\cdot\,|h).$$

Consistency imposes:

- Along the approximating sequence, $P_{\sigma_k}(\,\cdot\,|h)$ is well defined for any h.
- As a consequence, $\mu = \lim_{k \to \infty} P_{\sigma_k}$ is a belief system and beliefs in μ are given by Bayesian updating wherever possible.
- If players are called upon to act unexpectedly, players have an explanation how they ended up there (via "trembles" σ_k).

Sequential Equilibrium

Definition 4.15

An assessment (σ, μ) is a sequential equilibrium if

1. σ is sequentially rational, given μ,
2. μ is consistent with σ.

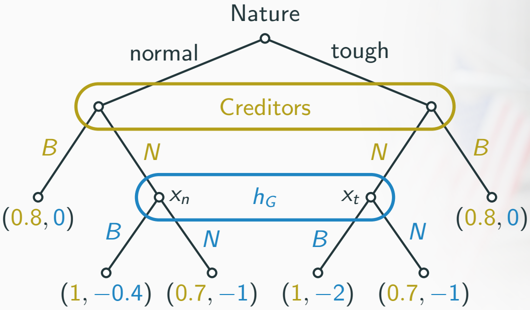
Refinements:

- Every sequential equilibrium is a perfect Bayesian equilibrium and hence also a subgame-perfect equilibrium.

Advantages and disadvantages:

- Restricts off-path beliefs to reasonable beliefs.
- It is tedious to verify whether an assessment is a sequential equilibrium.

Unknown Type of Government



Unknown voter preferences:

- Any completely mixed approximating sequence σ^k must place positive probability on N, hence $P_{\sigma^k}(\{x_t\} \mid h_G) = 0.3$.
- The unique beliefs in the limit as $k \to \infty$ is $\mu_{h_G}(\{x_t\}) = 0.3$.
- In the unique sequential equilibrium, (N, B) is played.

Summary

Solution concepts:

Nash $\supsetneq$ Subgame perfect $\supsetneq$ Perfect Bayesian $\supsetneq$ Sequential

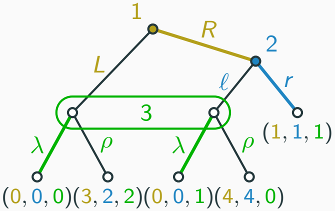
Advantages of extensive-form games:

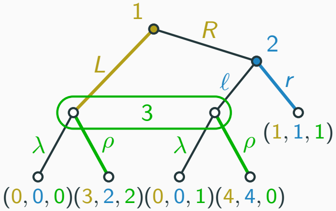
- Very general with various possible reasons for imperfect information.
- Drawing the game tree helps us understand small examples.

Limitations of extensive-form games:

- Restriction to finitely many types and actions.
- Restriction to games with common prior over types.
- While easily interpreted visually, they are unwieldy notationally.

Check Your Understanding





True or false:

1. Subgame-perfect equilibria are sequentially rational.
2. Subgame perfection and perfect Bayesian equilibria both eliminate non-credible threats.
3. We cannot refine sequential equilibria any further.

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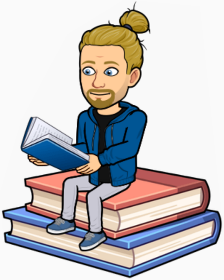
Short-answer question:

4. Which of the two subgame-perfect equilibria are sequential equilibria?

Literature

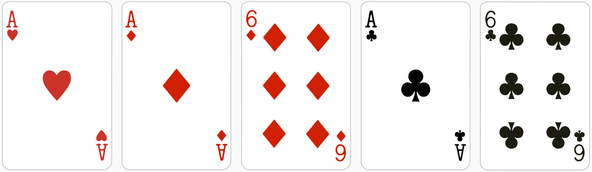
- D. Fudenberg and J. Tirole: Game Theory, Chapter 8, MIT Press, 1991
- M. Maschler, E. Solan, and S. Zamir: Game Theory, Chapter 7.4, Cambridge University Press, 2013
- S. Tadelis: Game Theory: An Introduction, Chapter 15, Princeton University Press, 2013
- D.M. Kreps and R. Wilson: Sequential Equilibria, Econometrica, **50** (1982), 863–894
- D. Fudenberg and J. Tirole: Perfect Bayesian Equilibrium and Sequential Equilibrium, Journal of Economic Theory, **53** (1991), 236–260



Poker

Good Fold? Bad Fold?

Bluffing on the River



Model the situation:

- John Juanda holds 6♡ 6♠.
- He loses to A♠, but wins against anything else.
- What are the types? actions? strategies?
- What is Juanda's best reply here?
- What are the perfect Bayesian equilibria of the last round?

Bluffing on the River: Types and Actions

Types:

- Juanda needs to distinguish only two of Lebedev types: $\theta_L \in \{A, B\}$, where type A holds A♠ and type B does not.
- Suppose, for simplicity, that Juanda has only one type, i.e., his hand is common knowledge.

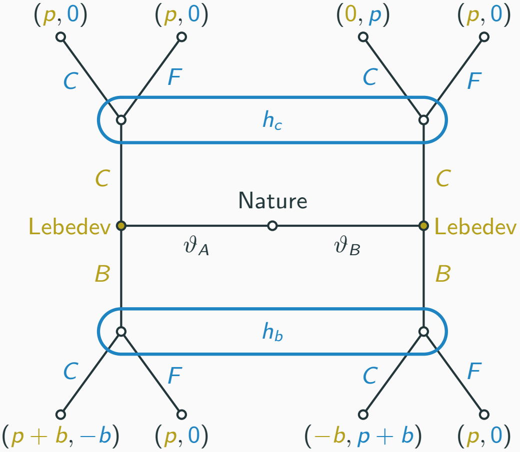
Actions:

- Lebedev can either bet an amount b or check ($=$ bet 0).
- Suppose, for simplicity, that Juanda can only call or fold (no raise).

Are the simplifying assumptions reasonable?

- Juanda can only call or fold if Lebedev goes all in.
- Hand ranges in poker are often "polarized" in the sense that a person who raises a large amount is either bluffing or has a monster hand.

Bluffing on the River: Extensive-Form Game



Bluffing on the River: Strategies and Beliefs

Parametrize strategies:

- Lebedev's strategy specifies $\sigma_L(\vartheta_A)$ and $\sigma_L(\vartheta_B)$ and Juanda's strategy specifies $\sigma_J(h_C)$ and $\sigma_J(h_B)$. We parametrize them by

$$\alpha = \sigma_L(\vartheta_A; B), \quad \beta = \sigma_L(\vartheta_B; B), \quad \gamma = \sigma_J(h_B; C), \quad \delta = \sigma_J(h_C; C).$$

Parametrize beliefs:

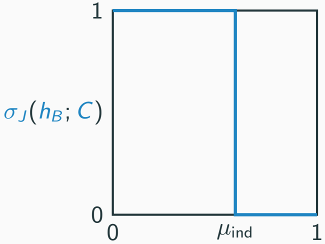
- Suppose $\mu_0 \in [0,1]$ indicates Jaunda's prior that Lebedev is of type ϑ_A.
- Under strategy profile $\sigma = (\sigma_L, \sigma_J)$, those beliefs are updated to

$$\mu(h_B) = P_\sigma(\vartheta_A \mid h_B) = \frac{P_\sigma(\vartheta_A \cap h_B)}{P_\sigma(h_B)}$$

$$= \frac{P_\sigma(h_B \mid \vartheta_A) P_\sigma(\vartheta_A)}{P_\sigma(h_B \mid \vartheta_A) P_\sigma(\vartheta_A) + P_\sigma(h_B \mid \vartheta_B) P_\sigma(\vartheta_B)} = \frac{\alpha\mu_0}{\alpha\mu_0 + \beta(1-\mu_0)}.$$

- Updated beliefs after observing C are given analogously.

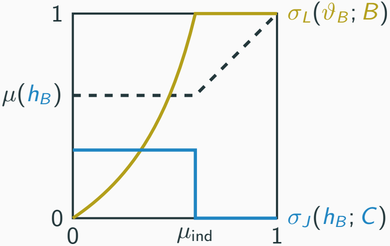
Bluffing on the River: Posterior Beliefs



Posterior beliefs:

- The posterior beliefs increase if $\sigma_L(\vartheta_A; B) > \sigma_L(\vartheta_B; B)$, they decrease if $\sigma_L(\vartheta_A; B) < \sigma_L(\vartheta_B; B)$, and they are μ_0 if $\sigma_L(\vartheta_A; B) = \sigma_L(\vartheta_B; B)$.
- There exists a cutoff μ_{ind} of posterior beliefs, for which Juanda is indifferent between calling a bet of size b and folding.
- If $\mu(h_B) < \mu_{\text{ind}}$, the unique best response is to call.
- If $\mu(h_B) > \mu_{\text{ind}}$, the unique best response is to fold.

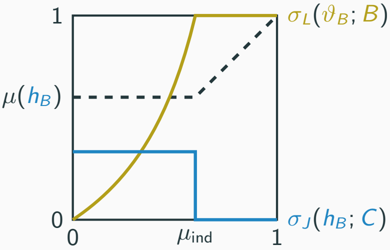
Bluffing on the River: Undominated PBE



Optimistic prior: $\mu_0 < \mu_{\text{ind}}$

- If type ϑ_B never bluffs, Juanda will never call type ϑ_A because he would be certain that he is facing type ϑ_A after seeing a bet.
- To keep calls in Juanda's best responses, type ϑ_B bluffs with a probability that makes Juanda indifferent between calling and folding.
- In equilibrium, Juanda must call a bet with a probability that makes the bluffing type ϑ_B indifferent between betting and checking.

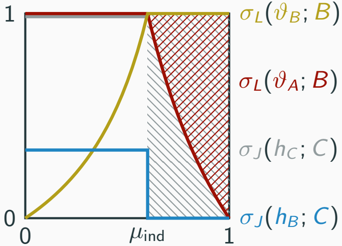
Bluffing on the River: Undominated PBE



Pessimistic prior: $\mu_0 > \mu_{\text{ind}}$

- If Lebedev wishes to keep calls in Juanda's range, he must decrease Juanda's posterior by not betting with certainty when he is of type ϑ_A.
- However, if Juanda calls with positive probability, type ϑ_A must bet.
- Thus, there is no equilibrium that keeps calls in Juanda's range.
- Since type ϑ_A cannot elicit a call, the only thing Lebedev can do is to protect the pot as type ϑ_B by betting with certainty.

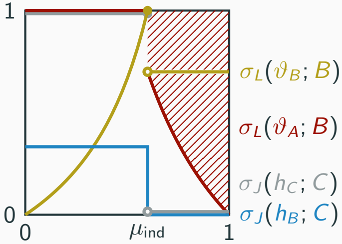
Bluffing on the River: Other PBE



Pessimistic prior: $\mu_0 > \mu_{\text{ind}}$

- Since type ϑ_A cannot elicit a call, he is indifferent between betting and checking as long as folding remains a best response for Juanda.
- This requires ϑ_A to bet with a probability so that $\mu(h_B) \geq \mu_{\text{ind}}$.
- Since type ϑ_B bets with certainty, Juanda is willing to fold to a bet as long as betting with certainty remains the unique best response for ϑ_B.
- This simply requires that Juanda calls with positive probability.

Bluffing on the River: Other PBE



Folding with certainty:

- If Juanda folds with certainty to a check when $\mu(h_C) \geq \mu_{\text{ind}}$, then ϑ_B does not need to protect the pot by betting with certainty.
- Type ϑ_B can bet with any probability as long the betting ratio of type ϑ_B to ϑ_A is sufficiently high so that $\mu(h_C) \geq \mu_{\text{ind}}$.

Question: Which PBE do we expect to see in reality?

Bluffing on the River: Comparing PBE

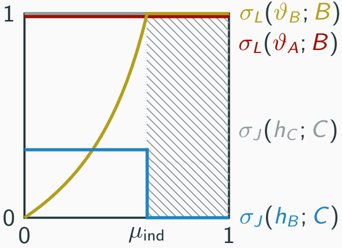
Dominated PBE:

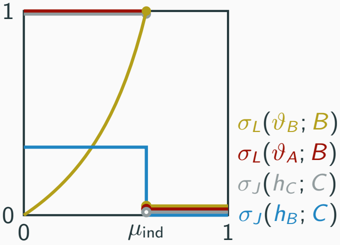
- Require players to be very certain of opponent's beliefs and strategies.
- To not bluff with certainty as type ϑ_B when $\mu_0 > \mu_{\text{ind}}$, Lebedev must know Juanda will fold with certainty because Juanda knows Lebedev folds the ace with a sufficiently high ratio to not make a call profitable.

Undominated PBE:

- Are much more robust to misspecification of players beliefs or changes in higher-order knowledge about players' strategies.
- Does sequential equilibrium rule out the dominated PBE?

Bluffing on the River: Off-Path Beliefs





Off-path beliefs:

- Off-path beliefs arise when an action is observed that no type is supposed to take under the conjectured strategy profile.
- Here, this happens if $\mu_0 \geq \mu_{\text{ind}}$ and $\sigma_L(\vartheta_B; B) = \sigma_L(\vartheta_A; B) \in \{0, 1\}$.
- If both types bet, $\sigma_J(h_C; C) < 1$ is supported by $\mu(h_C) = 1$.
- If both types check, $\sigma_J(h_B; C) = 0$ is supported by $\mu(h_B) \geq \mu_{\text{ind}}$.

Bluffing on the River: Off-Path Beliefs

Both types bet:

- σ_L is approximated by $\sigma_L^k(\vartheta_A; B) = 1 - \frac{1}{k}$ and $\sigma_L^k(\vartheta_B; B) = 1 - \frac{1}{k^2}$.
- The induced posterior after observing C is

$$\mu_k(h_C) = \frac{\frac{1}{k}\mu_0}{\frac{1}{k}\mu_0 + \frac{1}{k^2}(1-\mu_0)} = \frac{k\mu_0}{k\mu_0 + (1-\mu_0)} \xrightarrow{k\to\infty} 1.$$

- Those perfect Bayesian equilibria are also sequential equilibria.

Both types check:

- σ_L is approximated by $\sigma_L^k(\vartheta_A; B) = \frac{1}{k}$ and $\sigma_L^k(\vartheta_B; B) = \frac{1}{k^2}$.
- The induced posterior after observing B is

$$\mu_k(h_B) = \frac{\frac{1}{k}\mu_0}{\frac{1}{k}\mu_0 + \frac{1}{k^2}(1-\mu_0)} = \frac{k\mu_0}{k\mu_0 + (1-\mu_0)} \xrightarrow{k\to\infty} 1.$$

Conclusion: All perfect Bayesian equilibria are also sequential equilibria.

Refinements

Eliminating "unreasonable" off-path beliefs:

- Sequential equilibrium only eliminates irrational off-path beliefs.
- The intuitive criterion and uniform divinity are the most used refinements off-path beliefs to "reasonable" beliefs.

Eliminating weakly dominated strategies:

- Trembling-hand perfect equilibrium is a refinement of sequential equilibrium that eliminates weakly dominated strategies.
- Trembles in the approximating sequence σ^k form an equilibrium of a perturbed game where players are forced to make errors.
- Stable equilibria are those that are robust to misspecification of beliefs and often rule out weakly dominated strategies.
- Also, simply ruling out weakly dominated strategies is a well-motivated and perfectly valid equilibrium selection procedure.

Good Fold? Bad Fold?

What is the prior:

- It is a short-deck tournament, meaning there are no 2–5 in the deck.
- Juanda knows 7 out of 36 cards, 1 of the remaining 29 is the ace.
- There are five other players left in the tournament. If one of them had the ace, they would have stuck around, hence

$$\mu_0 = P(\theta_L = \vartheta_A \,|\, 7 \text{ cards}) = \frac{\binom{29}{9}}{\binom{29}{10}} = 50\%.$$

What have we observed:

- Juanda has folded, hence $\mu(h_B) \geq \mu_{\text{ind}} = \frac{p+b}{p+2b} = 84.3\%$.
- From $\mu(h_B) \geq \mu_{\text{ind}}$, we deduce that

$$\sigma_L(\vartheta_B; B) \leq \frac{\mu_0(1-\mu_{\text{ind}})}{\mu_{\text{ind}}(1-\mu_0)} = 18.6\%.$$

- Either Lebedev is known to be a very tight player or it was a bad fold.

Posterior Beliefs as a Random Variable

Posterior beliefs:

- Juanda's posterior μ is a function of Lebedev's realized action A_L

$$\mu = \begin{cases} \mu(h_B) & \text{if } A_L = B, \\ \mu(h_C) & \text{if } A_L = C. \end{cases}$$

- Lebedev's strategy affects the two values the posterior can take via

$$\mu(h_B) = \frac{\alpha\mu_0}{\alpha\mu_0 + \beta(1-\mu_0)}, \quad \mu(h_C) = \frac{(1-\alpha)\mu_0}{(1-\alpha)\mu_0 + (1-\beta)(1-\mu_0)}.$$

- Lebedev's strategy affects the probabilities of the two outcomes via

$$P_\sigma(\mu = \mu(h_B)) = P_\sigma(A_L = B) = \alpha\mu_0 + \beta(1-\mu_0),$$

$$P_\sigma(\mu = \mu(h_C)) = P_\sigma(A_L = C) = (1-\alpha)\mu_0 + (1-\beta)(1-\mu_0).$$

- The expectation of the posterior is equal to the prior: $\mathbb{E}_\sigma[\mu] = \mu_0$.

Summary

How to find PBE and SE:

1. Parametrize strategies efficiently (= no strictly dominated strategies).
2. Write down expected utilities and Bayesian updating of beliefs.
3. Find a consistent set of parameters from the best-response functions.
   - Typically, we distinguish corner solutions (like $\alpha = \delta = 1$) and interior solutions (like $\alpha \in (0, 1)$) and verify consistency with the conditions imposed by Bayesian updating and the best response functions.
   - Signaling games: pooling and separating equilibria are corner solutions.
4. Specify required off-path beliefs if those occur in equilibrium.
5. For SE: verify which off-path beliefs are consistent.

Hint: Drawing a graph helps. If you realize you do not know an equilibrium for some prior μ_0, then you are missing equilibria.

Literature

- D. Fudenberg and J. Tirole: Game Theory, Chapter 8.2, MIT Press, 1991
- S. Tadelis: Game Theory: An Introduction, Chapter 16, Princeton University Press, 2013
- M. Acevedo: Modern Poker Theory, D&B Publishing, 2019
- M. Spence: Job Market Signaling, Quarterly Journal of Economics, **87** (1973), 355–374