Macroeconomic Theory - ~~Recursive~~ Guess and verify Methods

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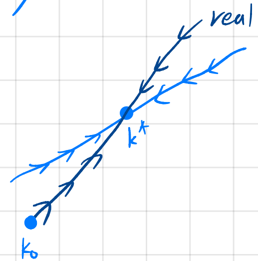
value func. iteration

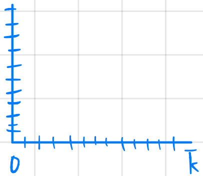
steady state linearization



Good
1. 不用怕k_0離太遠的問題
2. 一樣不會遇到此問題

Bad
1. if k_0 is far away from k^*, the accuracy ↓
2. constraint case：在不同 k's 下可能會有 or 沒 constraint
有可能k_0和k^*太遠，導致k_0有，k^*沒有
此時取線性逼近結果會很有問題
∵ 有沒有 constraint 下的結果差很多

Bad 3. Curse of dimension



solve $V(k) = \max_{k'} L(k, k')$

→ 必須算 n×n 個值 (only 1 S-S variable)

if # of S-S variable ↑太多，
電腦就算不出來
(e.g. $1 \to 2 \Rightarrow n^2 \to n^4$)

Good 3. 變數增加，只是矩陣變大，仍能用matlab算

$$\begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_t \\ c_t \end{bmatrix}$$

⇓

$$\begin{bmatrix} k_{t+1} \\ m_{t+1} \\ l_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_t \\ m_t \\ l_t \\ c_t \end{bmatrix}$$

↓
∴很多總經model都用這個算

Guess and verify

- We want to find a $v(k)$ satisfying

$$v(k) = \max_{k' \in [0, g(k)]} u(g(k) - k') + \beta v(k')$$

- We can consider the following mapping of functions $T : V \rightarrow V$:

$$Tv(k) = \max_{k' \in [0, g(k)]} u(g(k) - k') + \beta v(k')$$

- The solution of the functional equation is a fixed point of T

$$Tv = v$$

- To solve for $v(x)$, we basically look for the fixed point of the operation T

Guess and verify

而不只是前面解出的均衡

We always want the analytical sol., ∵ 背後有很多經濟直覺,

But very hard

∴ Now 教你如何 Guess and verify!

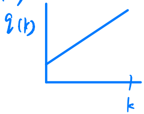
Suppose 知道 the shape of func.

- Suppose that $u(c) = \ln(c)$, $g(k) = Ak^{\alpha}$, then

$$Tv(k) = \max_{k' \leq [0, Ak^{\alpha}]} \ln(Ak^{\alpha} - k') + \beta v(k')$$

- Our goal is to find $v(k)$

e.g. linear func.



⇓

一定是 $ak+b$ 形式!

⇓

只需猜 a, b (係數)

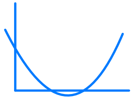
※ the form of value func.

is highly correlated with ① utility func.

② production func.

③ capital rule

e.g. quadratic func.



⇒ 一定 ax^2+bk+c

Guess and verify

- We guess that the value function has the form 假定 form

$$v(k) = \underbrace{E}_{} + \underbrace{F}_{} \ln k$$

↓ 只要求係數

- Where E and F are parameters we want to solve
- Step 1: Given $v(k)$, solve for $(Tv)(k)$ assume $u=\ln x$, $f(k)=Ak^{\alpha}$

$$v(k) = \max_{k' \le Ak^{\alpha}} \left\{ \ln(Ak^{\alpha} - k') + \beta \left[E + F \ln k' \right] \right\}$$

solve k'^{*}!

直接就把 k'^{*}代回去最快 但有點麻煩

- The first order condition:

$$\frac{1}{Ak^{\alpha} - k'^{*}} = \beta \frac{F}{k'^{*}}$$

這條比較好用！ ∵直接求到k程 ← (★)

$$\Rightarrow \quad Ak^{\alpha} - k'^{*} = \frac{1}{\beta F} k'^{*}$$

$$\Rightarrow \underline{k'^{*}} = \frac{\beta F}{1 + \beta F} \underline{Ak^{\alpha}} = h(k)$$

已知今日和明日k的關係
只剩F不知 → Goal!

Guess and verify

Step 2: Substitute the results of the first order condition into the functional equation

$$v(k) = \max_{k' \leq Ak^{\alpha}} \ln(Ak^{\alpha} - k') + \beta \left[E + F \ln k' \right]$$

$$\begin{aligned}
RHS &= \ln(Ak^{\alpha} - k'^{*}) + \beta \left[E + F \ln k'^{*} \right] \\
&\overset{\text{代入}}{=} \ln(\frac{1}{\beta F} k'^{*}) + \beta \left[E + F \ln k'^{*} \right] \\
&= \ln(k'^{*}) - \ln \beta F + \beta \left[E + F \ln k'^{*} \right] \\
&= (1 + \beta F) \ln k'^{*} - \ln \beta F + \beta E
\end{aligned}$$

by 上页(*)

先不要代入k^*
先整理

Guess and verify

- Substitute $k'^{*} = \frac{\beta F}{1+\beta F} A k^{\alpha}$ into the equation

$$
\begin{aligned}
&= (1+\beta F)\ln \frac{\beta F}{1+\beta F} A k^{\alpha} - \ln \beta F + \beta E \\
&= (1+\beta F)\left[\ln \frac{\beta F}{1+\beta F} A + \alpha \ln k\right] - \ln \beta F + \beta E \\
&= \underbrace{(1+\beta F)\left[\ln \frac{\beta F}{1+\beta F} A\right] - \ln \beta F + \beta E}_{E} + \underbrace{(1+\beta F)\alpha}_{F} \ln k
\end{aligned}
$$

- Step 3: Compare the coefficients

Coeffient!

$$LHS = E + F \ln k$$

Guess and verify

Then

$$
\begin{aligned}
& F = \alpha + \beta F \alpha \\
\Rightarrow \quad & F = \frac{\alpha}{1 - \beta\alpha}
\end{aligned}
$$

$$
\begin{aligned}
E \quad &= \quad (1 + \beta F)\left[\ln \frac{\beta F}{1 + \beta F} A\right] - \ln \beta F + \beta E \\
&\Rightarrow \quad E = \frac{1}{1 - \beta}\left[\beta F \ln \beta F + (1 + \beta F)\ln \frac{A}{1 + \beta F}\right]
\end{aligned}
$$

Guess and verify

啥意思？ k' 的func？

Policy function: 解出F后，我能解出 k'

$$
\begin{aligned}
k' &= \frac{\beta F}{1+\beta F} A k^{\alpha} \\
\Rightarrow k' &= \frac{\frac{\beta\alpha}{1-\alpha\beta}}{1+\frac{\beta\alpha}{1-\alpha\beta}} A k^{\alpha} \\
&= \beta\alpha A k^{\alpha}
\end{aligned}
$$

Example 1: CRRA Utility

- Utility function: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$
- Capital accumalation $k' = Ak - c$ → $c = Ak - k'$
- Assume that $\beta A^{1-\sigma} < 1$

$$V(k) = \max_{k' \leq Ak} \frac{(Ak - k')^{1-\sigma}}{1-\sigma} + \beta V(A')$$

打错了?

- Conjecture $V(k) = E\frac{k^{1-\sigma}}{1-\sigma}$

猜

→ FOC

→ 整理RHS

→ LHS = RHS → 找E → 得 V(k)

Example 2: Habit Persistence

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \ln(c_t) + \gamma \ln(c_{t-1}) \\ \text{subject to} \quad & Ak_t^{\alpha} = k_{t+1} + c_t \\ & c_{-1} = 0 \end{aligned}$$

Guess:

$$v(k_t, c_{t-1}) = E + F \ln k_t + G \ln c_{t-1}$$

where E, F, G are unknown variables