3. Bayesian Nash Equilibrium

ECON 7219 – Games With Incomplete Information

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Bayesian Game

Definition 2.19

A Bayesian game $\mathcal{G} = (\mathcal{I}, \Theta, \mathcal{T}, P, \mathcal{A}, u)$ consists of:

1. A finite set of players $\mathcal{I} = \{1, \ldots, n\}$,
2. A set of states of nature Θ.
3. A type space $\mathcal{T}_i$ for each player i, where each type $\tau_i \in \mathcal{T}_i$ determines:
   (i) the set $\mathcal{A}_i(\tau_i)$ of pure actions available to player i,
   (ii) player i's beliefs over the true state of nature θ and the type profile τ_{-i} of the other players via a posterior probability measure P_{τ_i} on $\Theta \times \mathcal{T}_{-i}$.
4. A payoff function $u_i : \mathcal{A} \times \Theta \to \mathbb{R}$ for each player $i \in \mathcal{I}$, where

$$\mathcal{A}_i := \bigcup_{\tau_i \in \mathcal{T}_i} \mathcal{A}_i(\tau_i), \qquad \text{and} \qquad \mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n.$$

Bayesian Nash Equilibrium

Definition 2.21

A Bayesian Nash equilibrium is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for every player i, every type τ_i, and every action $a_i \in \mathcal{A}_i(\tau_i)$,

$$\mathbb{E}_{\tau_i,\sigma}[u_i(A,\theta)] \geq \mathbb{E}_{\tau_i,(a_i,\sigma_{-i})}[u_i(A,\theta)].$$

Interpretation:

- No player has an incentive to deviate *after* learning their type.
- We say that $\sigma_i(\tau_i)$ maximizes player i's interim expected payoff.

Existence:

- If $\mathcal{T}_i$ and $\mathcal{A}_i$ is finite for every player i, then there exists at least one Bayesian Nash equilibrium in behavior strategies.

Examples of Bayesian Nash Eq.

Start-Up Problem: Motivation

Start-Up Problem: Model

	E	N
I	$0, -1$	$2, 0$
N	$2, 1$	$3, 0$

High Cost

	E	N
I	$1.5, -1$	$3.5, 0$
N	$2, 1$	$3, 0$

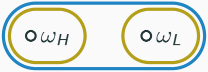
Low Cost

PChome currently holds a big market share in Taiwan's e-commerce. You have an idea how to make a better product:

- You decide whether or not to enter the market.
- PChome decides whether or not to innovate.
- You don't know whether PChome's innovation cost θ is high or low.

Entering the market is profitable if and only if PChome does not innovate.

Start-Up Problem: Types



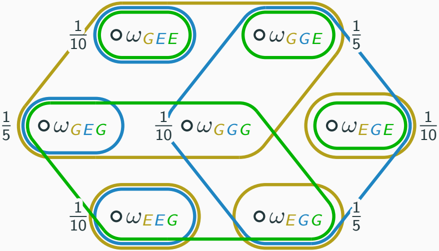
Independent payoff types:

- PChome has two types, the (H)igh-cost type and (L)ow-cost type.
- Since types are trivially independent, your beliefs are completely characterized by your prior $p = P_i(\theta = H)$.
- This is a very commonly studied case.

Belief hierarchies:

- It is common knowledge that PChome knows its type.
- It is common knowledge that You assign probability p to $\theta = H$.

Avalon: a Social Deduction Game



5-player game:

- There are three (G)ood characters and two (E)vil characters.
- Alignment (Good or Evil) is assigned randomly in the beginning. The distribution of this random assignment is the common prior P.
- For a subset of players $\{1, 2, 3\}$, the belief space is depicted above.

Avalon: Double Fails

	0 Fails	1 Fail	2 Fails
Good	1	−1	5
Evil	−1	1	−5

Players go on a quest:

- Every evil player on the quest can choose to (S)ucceed of (F)ail.
- Good characters cannot fail a quest, hence they must choose (S)ucceed.
- A double fail is particularly damning to the evil players because it unambiguously reveals that two members of the quest are evil.
- What is the Bayesian Nash equilibrium of this game?

Example with Heterogeneous Priors

	L	R
T	$2, 0$	$0, 1$
B	$0, 0$	$1, 0$

$\theta(\omega_1)$

	L	R
T	$0, 0$	$0, 0$
B	$1, 1$	$1, 0$

$\theta(\omega_2)$

	L	R
T	$0, 0$	$0, 0$
B	$1, 1$	$0, 0$

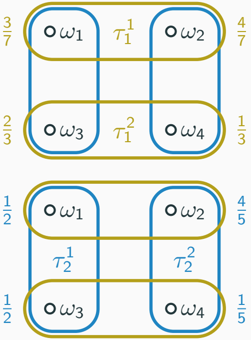
$\theta(\omega_3)$

	L	R
T	$0, 0$	$2, 1$
B	$0, 0$	$0, 2$

$\theta(\omega_4)$

Practice:

1. Find all Bayesian Nash equilibria.
   - Think first how to parametrize strategies.
   - Then use indifference principle or mutual best response functions.
2. Verify that this belief space indeed does not admit a common prior.
3. What are Player 1's second-order beliefs when he/she is of type τ_1^1?



Battle of the Bastards

Coordinating an attack:

- If Ramsay Bolton plans to fight in the field, a coordinated attack between Jon's forces and the Knights of the Vale would be successful.
- If, however, Ramsay remains within Whitestone castle, attacking and starting a siege would be futile because, as we know, winter is coming.

Image source: https://fineartamerica.com/featured/battle-of-the-bastards-a-game-of-thrones-joseph-oland.html

Battle of the Bastards: Payoffs

Open Battle

	A	N
A	1, 1	−1, 0
N	0, −1	0, 0

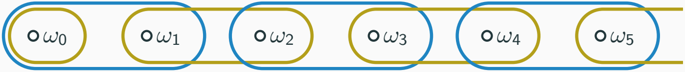
Siege

	A	N
A	−1, −1	−1, 0
N	0, −1	0, 0

Coordinating an attack:

- Suppose the common prior is $P(\theta = S) = \frac{2}{3}$.
- Jon Snow learns the type of battle θ from his spies and he sends a raven to the Knights of the Vale if $\theta = B$, informing them of θ.
- The Knights of the Vale and Jon Snow then send ravens back and forth, acknowledging the receipt of the previous letter.
- Each raven has a probability of being shot down with probability $\varepsilon > 0$.

Battle of the Bastards: Types



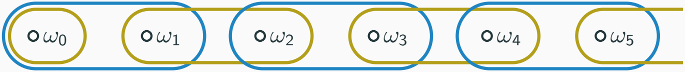
States of the world:

- In state ω_n, n ravens have been sent and $n-1$ ravens have arrived.
- The common prior is $P(\{\omega_0\}) = \frac{2}{3}$ and $P(\{\omega_n\}) = \frac{\varepsilon(1-\varepsilon)^{n-1}}{3}$.
- Commander of type $\tau_n = \{\omega_{n-1}, \omega_n\}$ for $n > 0$ does not know whether his own raven was shot down or the other commander's.

States of nature:

- The state of nature is $\theta(\omega_0) = S$ and $\theta(\omega_n) = B$ for $n > 1$.
- Jon of type τ_0 knows that $\theta = S$.
- Any commander of type τ_n with $n \geq 2$ knows that $\theta = B$.

Battle of the Bastards: Bayesian Nash Equilibrium



Subjective best responses:

- Jon Snow of type $\tau_0 = \{\omega_0\}$ knows that $\theta = S$, hence $\sigma_J(\tau_0) = N$.
- The posterior beliefs of the Knights of the Vale of type τ_1 are

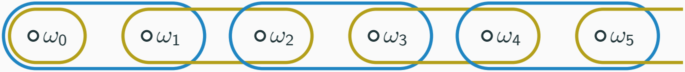
$$P_{\tau_1}(\{\omega_0\}) = \frac{P(\{\omega_0\})}{P(\tau_1)} = \frac{\frac{2}{3}}{\frac{2}{3} + \frac{\varepsilon}{3}} = \frac{2}{2+\varepsilon} \geq \frac{2}{3}.$$

- The Knights of the Vale thus believe Jon will not attack with

$$P_{\tau_1}(\sigma_J(T_J) = N) \geq P_{\tau_1}(T_J = \tau_0) \geq \frac{2}{3}.$$

- In particular, $\sigma_K(\tau_1) = N$ is the unique best response.

Battle of the Bastards: Bayesian Nash Equilibrium



Subjective best responses:

- Commander of type $\tau_n = \{\omega_n, \omega_{n-1}\}$ has posterior

$$P_{\tau_n}(\{\omega_{n-1}\}) = \frac{\varepsilon(1-\varepsilon)^{n-2}}{\varepsilon(1-\varepsilon)^{n-2} + \varepsilon(1-\varepsilon)^{n-1}} = \frac{1}{2(1-\varepsilon)} > \frac{1}{2}.$$

- This implies again that $\sigma_i(\tau_n) = N$ is the unique best response.

Bayesian Nash equilibria:

- For any $\varepsilon > 0$, the unique equilibrium outcome is (N, N).
- Due to lack of common knowledge, players take the less risky action.

Check Your Understanding

Finding a Bayesian Nash equilibrium:

In which order do you carry out the following steps?

1. Parametrize the players' strategies.
2. Identify the players' types.
3. Eliminate strictly dominated actions.
4. Find the players' best responses.
5. Compute the players' posterior beliefs.
6. Construct a belief table that represents a given belief hierarchy.

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Short-answer question:

7. Which of the above steps are optional?

Literature

- D. Fudenberg and J. Tirole: Game Theory, Chapter 6, MIT Press, 1991
- M. Maschler, E. Solan, and S. Zamir: Game Theory, Chapters 9.4, Cambridge University Press, 2013
- S. Tadelis: Game Theory: An Introduction, Chapter 12, Princeton University Press, 2013
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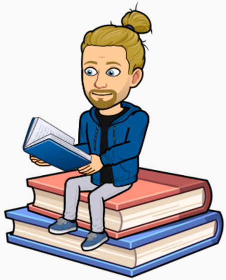
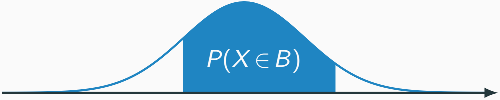


Infinite Belief Space

Finite Belief Spaces vs. Infinite Belief Spaces



Finite belief spaces:

- If $P_i(\tau_i) > 0$, player i of type τ_i updates his/her beliefs via Bayes' rule:

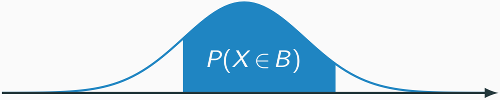
$$P_i(Y \mid \tau_i) = \frac{P_i(Y \cap \tau_i)}{P_i(\tau_i)}.$$

- If players have a common prior P, we can assume $P(\{\omega\}) > 0$ w.l.o.g.

Infinite belief spaces:

- If T_i has a continuous distribution, $P(T_i = \tau_i) = 0$ for any type τ_i.
- Even if Θ is finite, the universal belief space over Θ is infinite.

Continuous Random Variables



Continuous random variable:

- Suppose $X : \Omega \to \mathbb{R}$ is a random variable with density function f.
- The density function is related to the distribution $P \circ X^{-1}$ by

$$(P \circ X^{-1})(B) = P(X \in B) = \int_{x \in B} f(x)\,\mathrm{d}x.$$

- The cumulative distribution function F satisfies

$$F(x) = P(X \leq x) = (P \circ X^{-1})\big((-\infty, x]\big).$$

- Expectation of X under P is

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\,\mathrm{d}P(\omega) = \int_{\mathbb{R}} x\,\mathrm{d}(P \circ X^{-1})(x) = \int_{\mathbb{R}} x\,f(x)\,\mathrm{d}x.$$

General Types

Why show this?

- We do not want to restrict types to be discrete or admit a density.
- This is in line with the construction of the universal type space.

More general framework:

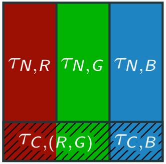
- Also allows us to update beliefs after probability-0 events.
- The distribution $P \circ X^{-1}$ is well defined for any probability measure P and any random variable X. Moreover, under P, X has expectation

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) \mathrm{d}P(\omega).$$

Random Variables Illustrated Graphically

$$\Omega = [0,1]^2$$

$$P = \text{area}$$



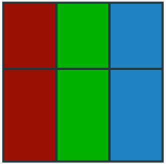
Examples:

- Roll of a 6-sided die takes values $\{1,2,3,4,5,6\}$.
- A random hat with equal probability of being red, green, and blue.
- About 8% of the male population are color blind.
- Player i's type is a random variable $T_i : \Omega \to \mathcal{T}_i$.
- Information set $\tau_i \in \mathcal{T}_i$ is the set, on which $T_i = \tau_i$ is realized.

Posterior Beliefs Illustrated Graphically

$\Omega = [0,1]^2$

$P_i =$ area



Posterior belief of type τ_i:

- Suppose player i has prior distribution P_i.
- The posterior beliefs of type $\tau_i \in \mathcal{T}_i$ about event Y are given by

$$P_{i,\tau_i}(Y) = P_i(Y \mid T_i = \tau_i) = \frac{P_i(Y \cap \{T_i = \tau_i\})}{P_i(T_i = \tau_i)}.$$

- Suppose a person's height is its type, which is normally distributed.
- Suppose Ben's type is 1.814159...m, which has prior probability 0.
- How do we compute Ben's posterior probability?

Regular Conditional Probability

Definition 3.1

Let $(\Omega, \mathcal{F}, P_i)$ be a probability space, let $(\mathcal{T}_i, \mathfrak{T}_i)$ be a measurable type space, and let $T_i : \Omega \to \mathcal{T}_i$ be a random variable, which indicates i's type.

A regular conditional probability $\pi_i : \mathcal{T}_i \times \mathcal{F} \to [0, 1]$ satisfies:

1. $\pi_i(\tau_i, \cdot\,)$ is a probability measure on $\mathcal{F}$ for all $\tau_i \in \mathcal{T}_i$,
2. $\pi_i(\,\cdot\,, Y)$ is a measurable function for every $Y \in \mathcal{F}$,
3. For all $Y \in \mathcal{F}$ and all $B \in \mathfrak{T}_i$,

$$P_i(Y \cap \{\omega \in \Omega \mid T_i(\omega) \in B\}) = \int_B \pi_i(\tau_i, Y) \overbrace{\mathrm{d}(P_i \circ T_i^{-1})(\tau_i)}^{\text{“}f(\tau_i)\,\mathrm{d}\tau_i\text{”}}$$

- $\pi_i(\tau_i, Y)$ is a measure-theoretic derivative with respect to $P_i(T_i = \tau_i)$.
- We define the posterior as $P_{i,\tau_i}(Y) := \pi_i(\tau_i, Y)$ so that updating of beliefs $P_i(Y) \mapsto P_{i,\tau_i}(Y)$ works even if $P_i(T_i = \tau_i) = 0$.

Infinite Belief Space

Definition 3.2

A belief space over a measure space (Θ, Ξ) of nature consists of:

- A finite set of players $\mathcal{I} = \{1, \ldots, n\}$.
- A probability space $(\Omega, \mathcal{F}, (P_i))$ with prior P_i for each $i \in \mathcal{I}$.
- A measurable type space $(\mathcal{T}_i, \mathfrak{T}_i)$, a random variable $T_i : \Omega \to \mathcal{T}_i$, and a regular conditional probability $\pi_i : \mathcal{T}_i \times \mathcal{F} \to [0, 1]$ for each player i.
- A random variable $\theta : \Omega \to \Theta$ indicating the true state of nature.

Note:

- The definition of a regular conditional probability implies that

$$\{\omega \in \Omega \mid \pi_i(\tau_i, \omega) > 0\} \subseteq \{T_i = \tau_i\}.$$

and that posterior beliefs are consistent with Bayes' rule if $P_i(\tau_i) > 0$.

First-Price Auction

(Sealed-Bid) First-Price Auction

Auction format:

- Players submit their bids simultaneously.
- Highest bidder obtains the auctioned object and pays his bid.
- Typically, a lot is drawn to break ties.

Independent private values:

- Each bidder $i = 1, \ldots, n$ values the auctioned object at θ_i.
- Bidders' valuations are independent across players: player i's value does not depend on the private values of others.
- Suitable for consumable items, for which bidders know their valuations.
- Independence implies that bidders' beliefs over opponents are determined completely be the common prior, hence $\tau_i = \theta_i$.

Suppose you are not Joey. How much should you bid?

Parametrizing the Auction

Types:

- Bidder i's type θ_i is distributed with density function $f_i > 0$ on $[0, \infty)$.
- Bidders' distribution functions are common knowledge among bidders.
- Together with independence of types, this defines the common prior

$$P(\theta \in B) := \int_{B_1} f_1(\vartheta_1)\,\mathrm{d}\vartheta_1 \cdots \int_{B_n} f_n(\vartheta_n)\,\mathrm{d}\vartheta_n.$$

Actions and payoffs:

- Each bidder i can submit a bid $b_i \in [0, \infty)$.
- Drawing lots: winner $i_*(b)$ is determined by uniform distribution on

$$\arg\max_{i \in \mathcal{I}} b_i = \left\{ i \in \mathcal{I} \;\middle|\; b_i = \max_{j \in \mathcal{I}} b_j \right\}.$$

- Bidder i's utility function is $u_i(\vartheta_i, b) = (\vartheta_i - b_i) 1_{\{i = i_*(b)\}}$.

Strategies and Equilibrium in First-Price Auction

Strategies:

- A pure strategy of bidder i is a map $s_i : \Theta_i \to [0, \infty)$.
- Bidder i's expected utility of strategy profile $s(\theta)$ is:

$$\mathbb{E}_{\vartheta_i}[u_i(\theta_i, s(\theta))] = (\vartheta_i - s_i(\vartheta_i))P\big(i = i_*(s(\theta))\big).$$

Bayesian Nash equilibrium:

- A pure-strategy Bayesian Nash equilibrium is a strategy profile s such that $s_i(\vartheta_i)$ maximizes $\mathbb{E}_{\vartheta_i}[u_i(\theta_i, b_i, s_{-i}(\theta_{-i}))]$ among all bids b_i.
- Is there a pure-strategy Bayesian Nash equilibrium s that is:
  - Symmetric, i.e., $s_i(\vartheta_i) = s_j(\vartheta_i)$ for any pair i, j.
  - Increasing, i.e., $s_i(\vartheta_i) > s_i(\vartheta_i')$ for all $\vartheta_i > \vartheta_i'$?

The Completely Unexpected

ω	$\theta_0(\omega)$	$P_{1,T_1(\omega)}$	$P_{2,T_2(\omega)}$
ω_1	ϑ_A	$\begin{bmatrix}1\omega_1\end{bmatrix}$	$\begin{bmatrix}1\omega_1\end{bmatrix}$
ω_2	ϑ_A	$\begin{bmatrix}1\omega_1\end{bmatrix}$	$\begin{bmatrix}1\omega_3\end{bmatrix}$
ω_3	ϑ_G	$\begin{bmatrix}1\omega_3\end{bmatrix}$	$\begin{bmatrix}1\omega_3\end{bmatrix}$

Additional states of nature:

- Suppose the state of nature has an additional dimension $\vartheta_0 \in \{\vartheta_A, \vartheta_G\}$, indicating whether this is an (A)uction or a (G)uessing game.
- The true state of the world is $(\omega_2, \vartheta_1, \vartheta_2)$, where Rachel believes ϑ_A is common belief and Joey believes ϑ_G is common belief.
- Each bidder i now has types $\tau_i = (\vartheta_i, \tau_i^0)$, where $\tau_i^0 \in \left\{\tau_i^A, \tau_i^G\right\}$.
- Rachel's prior assigns probability 0 to $\tau_i = \tau_i^G$.

The Completely Unexpected

Posterior after learning her type:

- Rachel learns her type $\tau_1 = (\vartheta_1, \tau_1^A)$.
- Her posterior over Joey's type is $P_{\tau_1}\left(\tau_2 \in \left\{\tau_2^G\right\} \times B_2\right) = 0$ and

$$P_{\tau_1}\left(\tau_2 \in \left\{\tau_2^A\right\} \times B_2\right) = \int_{B_2} f_2(\vartheta_2)\,\mathrm{d}\vartheta_2$$

for any set $B_2 \subseteq [0, \infty)$.

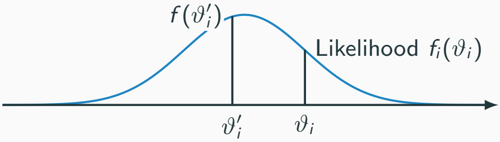
Posterior after talking to Joey:

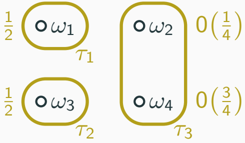
- If she learns that Joey believes it's a guessing game, we expect

$$P_{\tau_1}\left(\tau_2 \in \left\{\tau_2^G\right\} \times B_2\right) = \int_{B_2} f_2(\vartheta_2)\,\mathrm{d}\vartheta_2$$

and $P_{\tau_1}\left(\tau_2 \in \left\{\tau_2^A\right\} \times B_2\right) = 0$ for any $B_2 \subseteq [0, \infty)$.

Observing a Probability-Zero Event





Keeping track of relative likelihood:

- For continuous types, the density function f_i indicates the relative likelihood of probability-zero types $\vartheta_i, \vartheta_i'$.
- For discrete types, we specify the relative likelihood in parentheses.[1]
- In the example to the right, prior distribution assigns probability $\frac{1}{2}$ to each of ω_1 and ω_3. But upon learning τ_3, beliefs are $\left[\frac{1}{4}\omega_2, \frac{3}{4}\omega_4\right]$.
- In this notation, Rachel's posterior are

$$P_{\tau_1}\left(\tau_2 \in \{\tau_2^G\} \times B_2\right) = 0\left(\int_{B_2} f_2(\vartheta_2)\,\mathrm{d}\vartheta_2\right).$$

[1] $0(x)$ is a hyperreal number with standard part 0 and infinitesimal part x. Hyperreals are studied in the field of non-standard analysis.

Learning Probability-Zero Information

ω	$\theta(\omega)$	$P_{1,T_1(\omega)}$	$P_{2,T_2(\omega)}$
ω_1	ϑ_1	$[1\omega_1]$	$[1\omega_1]$
ω_2	ϑ_1	$[1\omega_1]$	$[\frac{1}{2}\omega_2, \frac{1}{2}\omega_3]$
ω_3	ϑ_2	$[1\omega_4]$	$[\frac{1}{2}\omega_2, \frac{1}{2}\omega_3]$
ω_4	ϑ_2	$[1\omega_4]$	$[1\omega_4]$

ϑ_1:

	L	R
T	0, 0	0, 1
B	−10, 0	1, 1

ϑ_2:

	L	R
T	1, 2	−10, 0
B	0, 2	0, 0

Practice:

- How many distinct types does each player have?
- What is the (non-standard) prior distribution on Ω?
- What does Player 2 believe about Player 1's knowledge of θ?
- Compute the Bayesian Nash equilibria in this game.

Summary

Belief spaces with probability-0 types:

- As long as we are careful with the model and use regular conditional probabilities, updating of beliefs is no problem.
- If players' types admit a density function, regular conditional probabilities are characterized by conditional density functions.

Independent types:

- In most applications with infinitely many types that we will see, players types are assumed to be independent.
- Then, player i's type reveals nothing about the other players' types, hence the posterior is equal to the prior.
- As a consequence, player's type coincides with his/her beliefs about θ.

Check Your Understanding

True or false:

1. If $\mathcal{A}_i$ is an interval, then any $a_i \in \mathcal{A}_i$ with

$$\frac{\partial \mathbb{E}_{\tau_i, a_i, \sigma_{-i}}[u_i(A, \theta)]}{\partial a_i} = 0, \quad \frac{\partial^2 \mathbb{E}_{\tau_i, a_i, \sigma_{-i}}[u_i(A, \theta)]}{\partial a_i^2} < 0$$

   is a best response by player i of type τ_i.

2. If $\mathcal{A}_i$ is an interval, then every best response $a_i \in \mathcal{A}_i$ of type τ_i must satisfy the above two conditions.

3. In an infinite belief space, we have $P_i(\tau_i) = 0$ for every type τ_i.

4. Each player's type in an infinite belief space admits a density function.

5. For finite belief spaces, a non-standard prior distribution consists of a standard prior distribution P_i and a posterior for each τ_i with $P_i(\tau_i) = 0$.

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Literature

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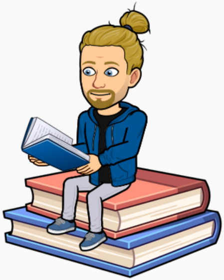


Dynamic Games

Selling Farmland

Two of Taiwan's most valuable crops are tea and rice.

Annual average yield:

- Tea: 5.35m NTD/km^2.
- Rice: 4.2m NTD/km^2.

A Rice Farmer considers selling his/her land to a Tea Farmer.

- Suppose high/low-quality soil yields 50% above/below average.
- The Rice Farmer knows the quality, but the Tea Farmer does not.
- Simultaneously, the Tea Farmer proposes a price $p \geq 0$ and the Rice Farmer quotes a set $\mathcal{P}$ of prices he/she will accept.

Source: https://www.visualcapitalist.com/the-worlds-most-valuable-cash-crop/

Selling Farmland: Bayesian Nash equilibria

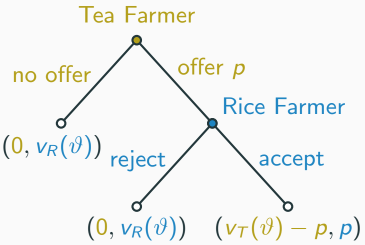
Bayesian Nash equilibria:

- Because of adverse selection, trade can occur only if the soil quality is low, for which the value of the land is $v_R(L) = 4.2$ and $v_T(L) = 5.4$.
- Any $p = \min \mathcal{P}(L) \leq 5.4$ constitutes a trade equilibrium.
- Any $p < 4.2$ and $\min \mathcal{P}(L) > 5.4$ is a no-trade equilibrium.

Understanding the equilibria:

- Trade may occur at any price in the interval $[4.2, 5.4]$.
- Which price is realized depends on the bargaining power of the farmers.
- We can model bargaining power through game dynamics.
- The no-trade equilibrium is inefficient. It is the result of a miscoordination that can be avoided if players do not act simultaneously.

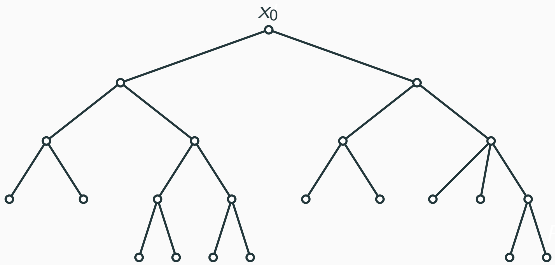
Selling Farmland: a Dynamic Setting



Adding dynamics:

- If the Tea Farmer gets to make a take-it-or-leave-it offer, then the Rice Farmer will agree to any sale at price $p \geq 4.2$.
- Anticipating this response, the Tea Farmer best offers $p = 4.2$.
- Conversely, if the Rice Farmer gets to make a take-it-or-leave-it offer, then the Tea Farmer will agree to any sale at price $p \leq 5.4$.
- Anticipating this response, the Rice Farmer best asks for $p = 5.4$.

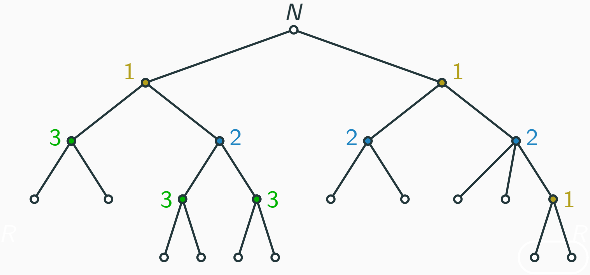
Extensive-Form Game



Game tree: Arborescence $(\mathcal{X}, \prec)$

- $\mathcal{X}$ is a finite set of nodes x, partially ordered by precedence relation $\prec$:
  - $\prec$ is asymmetric: $x \prec x'$ implies $x' \nprec x$ $\rightarrow$ prevents cycles.
  - $\prec$ is transitive: $x \prec x'$ and $x' \prec x''$ implies $x \prec x''$.
- There exists $x_0 \in \mathcal{X}$ with $x_0 \prec x$ for all $x \in \mathcal{X} \setminus \{x_0\}$ called initial node.
- Each node $x \neq x_0$ has exactly one immediate predecessor, that is, one $x' \prec x$ such that $x'' \prec x$ and $x'' \neq x'$ implies $x'' \prec x'$.

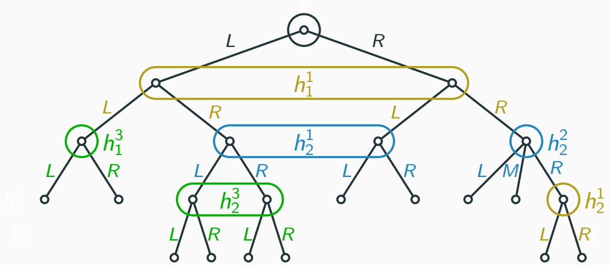
Extensive-Form Game



Players and payoffs:

- There is a finite set of strategic players $\mathcal{I} = \{1, \ldots, n\}$.
- There is a special non-strategic player N, called nature.
- Let $\mathcal{Z} := \{x \in \mathcal{X} \mid \nexists x' \in \mathcal{X} \text{ with } x \prec x'\}$ be the set of terminal nodes.
- Player $i \in \mathcal{I}$ receives a payoff $u^i : \mathcal{Z} \to \mathbb{R}$ at terminal node $z \in \mathcal{Z}$.
- A map $i : \mathcal{X} \setminus \mathcal{Z} \to \mathcal{I} \cup \{N\}$ indicates the active player at node x.

Extensive-Form Game



Information and actions:

- We denote the set of player i's nodes by $\mathcal{X}_i := \{x \in \mathcal{X} \setminus \mathcal{Z} \mid i(x) = i\}$.
- Let $\mathcal{H}_i$ be a partition of $\mathcal{X}_i$ into information sets $h_i \in \mathcal{H}_i$ such that each node $x \in h_i$ has the same number of successor nodes.
- Player i has actions $\mathcal{A}(h_i)$ available at information set $h_i \in \mathcal{H}_i$.
- For each $x \in h$, there is a bijection of $\mathcal{A}(h)$ to successors of x, indicating which node is reached from x when $a \in \mathcal{A}(h)$ is played.

Extensive-Form Game

Definition 3.3

An extensive-form game $\mathcal{G} = (\mathcal{X}, \prec, \mathcal{I}, i, (\mathcal{H}_i), (\mathcal{A}(h)), (u_i))$ consists of:

1. An arborescence $(\mathcal{X}, \prec)$ with terminal nodes $\mathcal{Z}$,
2. A set of players $\mathcal{I}$,
3. A map $i : \mathcal{X} \setminus \mathcal{Z} \to \mathcal{I} \cup \{N\}$ indicating the active player,
4. A partition $\mathcal{H}_i$ of $\mathcal{X}_i = \{x \in \mathcal{X} \setminus \mathcal{Z} \mid i(x) = i\}$ for each $i \in \mathcal{I}$,
5. A map $\mathcal{A}(h)$ indicating the available actions at h in $\mathcal{H} = \bigcup_{i \in \mathcal{I}} \mathcal{H}_i$,
6. A payoff function $u_i : \mathcal{Z} \mapsto \mathbb{R}$ for each player i.

Note: We refer to information sets in dynamic settings by h because those typically correspond to histories of observed actions.

Imperfect Information

Definition 3.4

Information is perfect if nature has no moves and each information set is a singleton. Information is imperfect otherwise.

Possible causes for imperfect information:

- Incomplete information: there is uncertainty about payoffs or players' types. Typically, this corresponds to unobserved moves by nature.
- Strategic uncertainty: simultaneous-move games.
- Imperfect monitoring: instead of observing actions directly, players observe a signal, whose distribution is affected by the chosen actions.

Strategies

Definition 3.5

Let $\mathcal{A}_i := \bigcup_{h_i \in \mathcal{H}_i} \mathcal{A}(h_i)$ denote all of player i's actions.

1. A pure strategy of player i is a map $s_i : \mathcal{H}_i \to \mathcal{A}_i$ such that $s_i(h_i) \in \mathcal{A}(h_i)$ for every $h_i \in \mathcal{H}_i$. Let $\mathcal{S}_i$ denote the set of i's pure strategies.
2. A mixed strategy of player i is a distribution $\sigma_i \in \Delta(\mathcal{S}_i)$.
3. A behavior strategy of player i is a map $\sigma_i : \mathcal{H}_i \to \Delta(\mathcal{A}_i)$ such that $\sigma_i(h_i) \in \Delta\mathcal{A}(h_i)$ for every $h_i \in \mathcal{H}_i$.

Remark:

- We denote by $\sigma_i(h_i; a_i)$ the probability that a_i is chosen under $\sigma_i(h_i)$.
- For $\mathcal{H}_i = \mathcal{T}_i$, these notions coincide with strategies in Bayesian games.

Outcome of an Extensive-Form Game

Outcome:

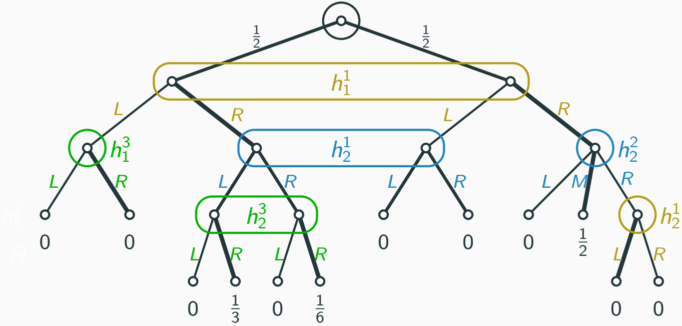
- Each terminal node $z \in \mathcal{Z}$ is reached by a unique sequence of actions.
- The outcome of an extensive-form game is a $\mathcal{Z}$-valued random variable Z or, equivalently, a sequence A of realized actions leading to Z.

Induced probability measure:

- For any node x, let $h(x)$ denote the information set at x and let a_x denote the taken at the predecessor of x that leads to x.
- For any $x \in \mathcal{X}$, let $(x_0, x_1, \dots, x_k)$ denote the sequence of nodes leading to $x = x_k$. Then x is reached under σ with probability

$$P_\sigma(\{x\}) = \prod_{j=0}^{k-1} \sigma_{i(x_j)}(h(x_j); a_{x_{j+1}}).$$

Probability Distribution over Outcomes



Example:

- Player 1 chooses $\sigma_1(h_1^1) = R$ and $\sigma_1(h_1^2) = L$,
- Player 2 chooses $\sigma_2(h_2^1) = \frac{2}{3}L + \frac{1}{3}R$ and $\sigma_2(h_2^2) = M$,
- Player 3 chooses $\sigma_3(h_3^1) = R$ and $\sigma_3(h_3^2) = R$.

Determining $\mathrm{P}_\sigma(\mathrm{z})$**:** multiply probabilities of edges between z and x_0.

Nash Equilibrium

Definition 3.6

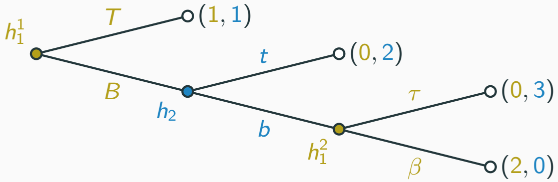
A Nash equilibrium in an extensive-form game is a strategy profile σ such that for every player i and every pure strategy s_i

$$\mathbb{E}_\sigma[u_i(Z)] \geq \mathbb{E}_{(s_i,\sigma_{-i})}[u_i(Z)].$$

Finding Nash equilibria:

- Note that this definition is identical to Nash equilibrium in a static game with $\mathcal{A} = \mathcal{S}$ and payoff functions $\widehat{u}_i(\sigma) = \mathbb{E}_\sigma[u_i(Z)]$.
- The transformed game is called the strategic-form game.
- We already know how to find Nash equilibria in static games.

Example of an Extensive-Form Game



Pure strategies:

- Player 2 has two pure strategies: $s_2(h_2) = t$ and $s_2(h_2) = b$.
- Player 1 has four pure strategies: any combinations of $s_1(h_1^1) \in \{T, B\}$ and $s_1(h_1^2) \in \{\tau, \beta\}$.
- Formally, a strategy involves a decision at every information set even if h_1^2 is never reached due to Player 1's own decision at h_1^1.

Reduced Strategic Form

	t	b
$T\tau$	1, 1	1, 1
$T\beta$	1, 1	1, 1
$B\tau$	0, 2	0, 3
$B\beta$	0, 2	2, 0

$\longrightarrow$

	t	b
T	1, 1	1, 1
$B\tau$	0, 2	0, 3
$B\beta$	0, 2	2, 0

Reduced-strategic form:

1. List all pure strategies of player i in i^{th} dimension of a payoff matrix.
2. Add payoff vector $\mathbb{E}_s[u(Z)]$ to cell corresponding to $s = (s^1, \ldots, s^n)$.
3. Combine payoff equivalent strategies (= duplicate rows/columns).

Nash equilibria are $(T, xt + (1-x)b)$ with $x \geq \frac{1}{2}$.

Harsanyi's Equivalence

Theorem 3.7

Let $\mathcal{G}$ be a Bayesian game with finitely many states of the world Ω, finitely many actions, and a common prior P with $P(\{\omega\}) > 0$ for every $\omega \in \Omega$.

- *The Bayesian game is equivalent to the extensive-form game $\mathcal{G}'$, in which nature chooses the players types in the first move.*
- *A Bayesian Nash equilibrium of $\mathcal{G}$ is a Nash equilibrium of the reduced strategic-form game associated with $\mathcal{G}'$.*

Two interpretations:

- A mixed Nash equilibrium of the strategic-form game corresponds to a mixed-strategy Nash equilibrium in the extensive-form game.
- Last week we have interpreted each type τ_i as a separate player, leading to a Nash equilibrium in behavior strategies.

Realization Equivalence

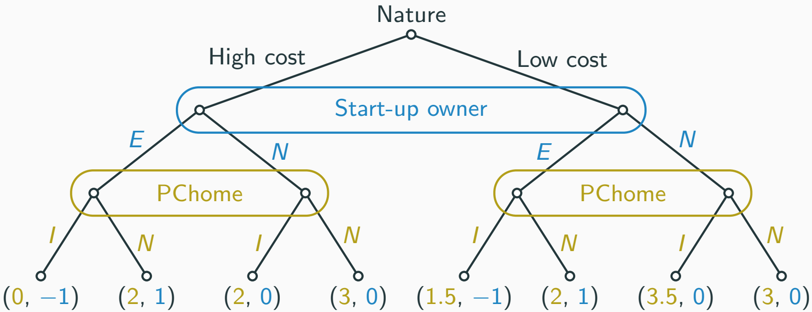
Definition 3.8

Two strategy profiles σ and $\widehat{\sigma}$ are realization equivalent if they induce the same distribution over outcomes, that is, $P_\sigma = P_{\widehat{\sigma}}$.

Mixed and behavior strategies:

- By Kuhn's theorem they are equivalent.
- Each pure strategy profile s leads to a unique terminal node z_s.
- For any behavior strategy profile σ, we can find a realization equivalent mixed strategy profile $\widehat{\sigma}$ by setting $\widehat{\sigma}(s) = P_\sigma(\{z_s\})$.
- For any mixed strategy profile $\widehat{\sigma}$, we can find a realization equivalent behavior strategy profile σ by setting $\sigma_i(h; a) = \sum_s \widehat{\sigma}_i(s) s_i(h; a)$.

Start-Up Problem as an Extensive-Form Game



	E	N
I	$0, -1$	$2, 0$
N	$2, 1$	$3, 0$

High cost

	E	N
I	$1.5, -1$	$3.5, 0$
N	$2, 1$	$3, 0$

Low cost

Summary

Extensive-form games:

- A very general framework for dynamic interactions that allows imperfect information for a variety of reasons.
- Game tree provides a good visual for small games.
- Drawbacks of extensive-form games: they are inherently finite and somewhat notationally cumbersome.

Harsanyi's equivalence:

- If players' have a common prior, a finite Bayesian game can be written as a game tree, where Nature selects players' types.
- However, this equivalence did not exploit the dynamics embedded in extensive-form games yet.

Check Your Understanding

True or false:

1. In a game tree, it is impossible that two different pure strategies lead to the same terminal node.
2. If information is imperfect, then it is incomplete.
3. Any single static game can be the reduced strategic-form game of many extensive-form games.
4. In a game of complete information, the probability measure P_σ defined here coincides with P_α defined by $P_\alpha(A = a) = \alpha(a)$.

Scan me

Short-answer question:

5. Suppose a pure strategy is a book with instructions on what to do in any given situation. Is going to a library and picking a book at random a mixed or a behavior strategy?

Literature

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