NTU Macroeconomic Theory I

Homework 2

- The deadline is 9:00AM of December 13, 2021.
- Please submit your own version to NTU COOL.

1. Human Capital with Externality

Consider a model economy populated by a continuum of measure one of consumers (i.e. the sum of all consumers is one) with preferences:

$$\sum_{t=0}^{\infty} \beta^t \ln c_t$$

The technology for producing the consumption/investment good is operated by competitive firms using the technology:

$$y_t = (h_{1t})^\alpha \bar{h}_t^{\eta}$$

where $\bar{h}_t$ is the average level of human capital in the economy, which is taken as given by the firms. Human capital is produced using the production function:

$$h_{t+1} = B h_{2t}$$

Households have to allocate total human capital between its two uses (one for labor supply, h_1, and the other for human capital investment, h_2) subject to:

$$h_t = h_{1t} + h_{2t}$$

In equilibrium, individual human capital equals average human capital: $h_t = \bar{h}_t$.

(1) Write down a firm's profit maximization problem and derive a set of equations that characterize firm's optimal decision. Use w_t to denote the wage rate for labor supply at time t.

(2) Write down a consumer's utility maximization problem and derive a set of equations that characterize the consumer's optimal decision.

(3) Find the growth rates along the balanced growth path. Hint: consumption and human capital both grow at constant, but not necessarily identical rates.

1) $\max (h_{1t})^{\alpha} \bar{h}_t^{\eta} - w_t \cdot h_{1t}$

FOC to $[h_{1t}]$: $\alpha (h_{1t})^{\alpha-1} \cdot \bar{h}_t^{\eta} = w_t$

(2) $\max\limits_{\{c_t, h_{1t}, h_{2t}\}} \sum\limits_{t=0}^{\infty} \beta^t \ln c_t$ s.t. $\begin{cases} c_t = w_t h_{1t} \\ h_{t+1} = B \cdot h_{2t} \\ h_t = h_{1t} + h_{2t} \end{cases}$

$\Rightarrow \max\limits_{\{h_{1t}, h_{2t}\}} \sum\limits_{t=0}^{\infty} \beta^t \ln (w_t h_{1t})$ s.t. $\begin{cases} h_{t+1} = B \cdot h_{2t} \\ h_t = h_{1t} + h_{2t} \end{cases}$

$\Rightarrow \mathcal{L} = \sum\limits_{t=0}^{\infty} \left[\beta^t \ln (w_t h_{1t})\right] + \left[\lambda_t (h_{1\,t+1} + h_{2\,t+1} - B h_{2t})\right]$

FOC to $[h_{1t}]$: $\frac{\cancel{w_t}}{\cancel{w_t} h_{1t}} \cdot \beta^t = \lambda_{t-1}$

to $[h_{2t}]$: $\lambda_t B = \lambda_{t-1}$

$\Rightarrow \beta B h_{1t} = h_{1\,t+1}$

$\Rightarrow \frac{c_{t+1}}{c_t} = \frac{w_{t+1} h_{1\,t+1}}{w_t h_{1t}} = \frac{w_{t+1} \beta B}{w_t}$

(3) ?

(1) Write down a firm's profit maximization problem and derive a set of equations that characterize firm's optimal decision. Use w_t to denote the wage rate for labor supply at time t.

(2) Write down a consumer's utility maximization problem and derive a set of equations that characterize the consumer's optimal decision.

(3) Find the growth rates along the balanced growth path. Hint: consumption and human capital both grow at constant, but not necessarily identical rates.

2. Two Alternative Technologies

Consider an economy with a large number of infinitely lived identical households with preferences given by,

$$\sum_{t=0}^{\infty} \beta^t N_t \log c_t$$

Each household is endowed with k_0 units of capital in period 0 and 1 unit of labor each period. The number of households in period t is N_t, where $N_{t+1} = \eta N_t$, $\eta > 1$. The technology is given by:

$$Y_t = \gamma^t K_t^{\mu} N_t^{\phi} L_t^{1-\mu-\phi}$$

In these technologies, $\gamma > 1$ is the rate of exogenous total factor productivity growth, K_t is total (not *per capita*) capital, Y_t is total output, and L_t is the total stock of land. Land is assumed to be a fixed factor, it cannot be produced and does not depreciate. To simplify with out loss of generality, assume that $L_t = 1$ for all t. The resource constraint, assuming 100 percent depreciation of capital each period, is given by

$$N_t c_t + K_{t+1} \leq Y_t$$

(1) Formulate, as a dynamic programming problem, the social planner's problem that weights all individuals utility equally. That is, the planner weights utility in period t by the number of identical agents alive in that period.

(2) Characterize the balanced growth path of this economy. Solve explicitly for the growth rate of per capita consumption (c_t) along this path.

(3) Repeat (1) and (2) using the technology instead: $Y_t = \gamma^t K_t^{\theta} N_t^{1-\theta}$.

(4) Compare how the rate of population growth η affects the rate of per capita growth in the two cases and provide the intuition.

(1) $N_t c_t + K_{t+1} = r^t K_t^u N_t^{\phi} L_t^{1-u-\phi}$

$\Rightarrow c_t + \frac{\eta}{\eta} \cdot \frac{K_{t+1}}{N_t} = r^t k_t^u \left(\frac{L_t}{N_t}\right)^{1-u-\phi}$ where $k_t = \frac{K_t}{N_t}$

$\Rightarrow c_t + \eta k_{t+1} = r^t \cdot k_t^u \cdot \left(\frac{1}{\eta^t N_0}\right)^{1-u-\phi}$

WLOG, let $N_0 = 1$.

Let $\hat{x}_t$ be stationary variable, s.t. $\hat{x}_t = \frac{x_t}{(g_x)^t}$, where g_t is the growth rate along BGP.

$\Rightarrow g_c^t \hat{c}_t + \eta (g_k)^{t+1} \cdot \hat{k}_{t+1} = \left(\frac{r}{\eta^{1-u-\phi}}\right)^t \cdot (g_k^t)^u \cdot \hat{k}_t^u$

$\Rightarrow \hat{c}_t + \eta g_k \cdot \left(\frac{g_k}{g_c}\right)^t \cdot \hat{k}_{t+1} = \left(\frac{g_k^u \cdot r}{g_c \cdot \eta^{1-u-\phi}}\right)^t \cdot \hat{k}_t^u$

$\because$ both side should be stationary

$\therefore$ ① $g_k = g_c \equiv g$

② $\frac{g_k^u \cdot r}{g_c \cdot \eta^{1-u-\phi}} = 1 \Rightarrow g^{u-1} \cdot \frac{r}{\eta^{1-u-\phi}} = 1 \Rightarrow \frac{r}{\eta^{1-u-\phi}} = g^{1-u}$

$\Rightarrow \left(\frac{r}{\eta^{1-u-\phi}}\right)^{\frac{1}{1-u}} = g$

$\Rightarrow \hat{c}_t + \eta \cdot \left(\frac{r}{\eta^{1-u-\phi}}\right)^{\frac{1}{1-u}} \cdot \hat{k}_{t+1} = \hat{k}_t^u$

$$\sum_{t=0}^{\infty} \beta^t N_t \log c_t = \sum_{t=0}^{\infty} (\beta\eta)^t \cdot \log(\hat{c}_t \cdot g^t)$$
$$= \sum_{t=0}^{\infty} (\beta\eta)^t \cdot \log(\hat{c}_t) + \sum_{t=0}^{\infty} (\beta\eta)^t \cdot t \log g.$$
$$= \sum_{t=0}^{\infty} (\beta\eta)^t \cdot \log(\hat{c}_t) + \underline{(\beta\eta) \log g}, \quad \text{where } \beta\eta < 1$$

?

P.S. $\sum_{t=0}^{\infty} (\beta\eta)^t \cdot t = \frac{1-\beta\eta}{1-\beta\eta} \sum_{t=0}^{\infty} (\beta\eta)^t \cdot t$

$= \frac{1-\beta\eta}{1-\beta\eta} [\beta\eta + (\beta\eta)^2 \cdot 2 + (\beta\eta)^3 \cdot 3 + \cdots]$

$= \frac{\beta\eta}{1-\beta\eta} [1 + \beta\eta \cdot 2 + (\beta\eta)^2 \cdot 3 + \cdots - (\beta\eta) - (\beta\eta)^2 \cdot 2 - \cdots]$

$= \frac{\beta\eta}{1-\beta\eta} [1 + \beta\eta + (\beta\eta)^2 + \cdots]$

$= \frac{\beta\eta}{1-\beta\eta} \cdot \frac{1}{1-\beta\eta} = \frac{\beta\eta}{(1-\beta\eta)^2}$?

$\Rightarrow$ Social planner's DPP: $V(\hat{k}) = \max_{\hat{c}, \hat{k}'} \log(\hat{c}) + (\beta\eta) V(\hat{k}')$

s.t. $\hat{c} + (r\eta^{\phi})^{\frac{1}{1-u}} \cdot \hat{k}' = (\hat{k})^u$, k_0 is given

(2) $$V(\hat{k}) = \max_{\hat{k}'} \log\left[(\hat{k})^{\alpha} - (\gamma\eta\phi)^{\frac{1}{1-\alpha}}\cdot\hat{k}'\right] + (\beta\eta)\,V(\hat{k}')$$

FOC to $[\hat{k}']$: $$\frac{-(\gamma\eta\phi)^{\frac{1}{1-\alpha}}}{(\hat{k})^{\alpha} - (\gamma\eta\phi)^{\frac{1}{1-\alpha}}\cdot\hat{k}'} + (\beta\eta)V'(\hat{k}') = 0$$

EC : $$V'(\hat{k}) = \frac{\alpha(\hat{k})^{\alpha-1}}{(\hat{k})^{\alpha} - (\gamma\eta\phi)^{\frac{1}{1-\alpha}}\cdot\hat{k}'}$$

$$\Rightarrow V'(\hat{k}') = \frac{\alpha(\hat{k}')^{\alpha-1}}{(\hat{k}')^{\alpha} - (\gamma\eta\phi)^{\frac{1}{1-\alpha}}\cdot\hat{k}''}$$

Euler equation : $$\beta\eta\cdot\frac{\alpha(\hat{k}')^{\alpha-1}}{(\hat{k}')^{\alpha} - (\gamma\eta\phi)^{\frac{1}{1-\alpha}}\cdot\hat{k}''} = \frac{(\gamma\eta\phi)^{\frac{1}{1-\alpha}}}{(\hat{k})^{\alpha} - (\gamma\eta\phi)^{\frac{1}{1-\alpha}}\cdot\hat{k}'}$$

In S-S, let $\hat{k}' = \hat{k}'' = \hat{k} = \hat{k}^*$, $\hat{c}' = \hat{c}'' = \hat{c} = \hat{c}^*$

then $$\beta\eta\cdot\frac{\alpha(\hat{k}^*)^{\alpha-1}}{(\hat{k}^*)^{\alpha} - (\gamma\eta\phi)^{\frac{1}{1-\alpha}}\cdot\hat{k}^*} = \frac{(\gamma\eta\phi)^{\frac{1}{1-\alpha}}}{(\hat{k}^*)^{\alpha} - (\gamma\eta\phi)^{\frac{1}{1-\alpha}}\cdot\hat{k}^*}$$

$$\Rightarrow \beta\eta\cdot\alpha(\hat{k}^*)^{\alpha-1} = (\gamma\eta\phi)^{\frac{1}{1-\alpha}}$$

$$\Rightarrow \hat{k}^* = \left[\frac{1}{\beta\eta\alpha}\cdot(\gamma\eta\phi)^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\alpha-1}}$$

$$\hat{c}^* = (\hat{k}^*)^{\alpha} - (\gamma\eta\phi)^{\frac{1}{1-\alpha}}\cdot\hat{k}^*$$

(3) $$N_t c_t + K_{t+1} = \gamma^t K_t^{\theta} N_t^{1-\theta}$$

$$\Rightarrow c_t + \frac{\eta}{\eta}\cdot\frac{K_{t+1}}{N_t} = \gamma^t\cdot k_t^{\theta},$$ where $k_t \equiv \frac{K_t}{N_t}$

$$\Rightarrow c_t + \eta\, k_{t+1} = \gamma^t\cdot k_t^{\theta}$$

Let $\tilde{x}_t$ be stationary variable, s.t. $\tilde{x}_t = \frac{x_t}{b_x{}^t}$, where b_x is the growth rate along BGP.

then $$b_c{}^t\cdot\tilde{c}_t + \eta\cdot b_k^{t+1}\cdot\tilde{k}_{t+1} = \gamma^t\cdot(b_k^t\cdot\tilde{k}_t)^{\theta}$$

$$\Rightarrow \tilde{c}_t + \eta\cdot b_k\cdot\left(\frac{b_k}{b_c}\right)^t\tilde{k}_{t+1} = \left(\frac{\gamma(b_k)^{\theta}}{b_c}\right)^t\cdot\tilde{k}_t^{\theta}$$

$\Rightarrow$ $\because$ both side should be stationary

$\therefore$ ① $b_k = b_c \equiv b$

② $$\frac{\gamma(b_k)^{\theta}}{b_c} = \gamma\cdot b^{\theta-1} = 1 \Rightarrow b = \left(\frac{1}{\gamma}\right)^{\frac{1}{\theta-1}} = \gamma^{\frac{1}{1-\theta}}$$

$$\Rightarrow \tilde{c}_t + \eta\cdot\gamma^{\frac{1}{1-\theta}}\cdot\tilde{k}_{t+1} = \tilde{k}_t^{\theta}$$

$$\sum_{t=0}^{\infty} \beta^t N_t \log c_t = \sum_{t=0}^{\infty} (\beta\eta)^t \cdot \log(\tilde{c}_t \cdot b^t)$$
$$= \sum_{t=0}^{\infty} (\beta\eta)^t \cdot \log(\tilde{c}_t) + \sum_{t=0}^{\infty} (\beta\eta)^t \cdot t \log b.$$
$$= \sum_{t=0}^{\infty} (\beta\eta)^t \cdot \log(\tilde{c}_t) + (\beta\eta) \log b, \quad \text{where } \beta\eta < 1$$

$\Rightarrow$ Social planner's DPP: $V(\tilde{k}) = \max_{\tilde{c}, \tilde{k}'} \left\{ \log \tilde{c} + \beta\eta\, V(\tilde{k}') \right\}$

$$\text{s.t. } \tilde{c}_t + \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}_{t+1} = \tilde{k}_t^{\theta}$$

$$\Rightarrow V(\tilde{k}) = \max_{\tilde{k}'} \left\{ \log\left(\tilde{k}^{\theta} - \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}'\right) + \beta\eta\, V(\tilde{k}') \right\}$$

FOC to $[\tilde{k}']$: $\dfrac{-\eta \cdot \gamma^{\frac{1}{1-\theta}}}{\tilde{k}^{\theta} - \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}'} + \beta\eta\, V'(\tilde{k}') = 0$

EC: $V'(\tilde{k}) = \dfrac{\theta \tilde{k}^{\theta-1}}{\tilde{k}^{\theta} - \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}'} \Rightarrow V'(\tilde{k}') = \dfrac{\theta \tilde{k}'^{\theta-1}}{\tilde{k}'^{\theta} - \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}''}$

Euler eq: $\beta\eta \dfrac{\theta \tilde{k}'^{\theta-1}}{\tilde{k}'^{\theta} - \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}''} = \dfrac{\eta \cdot \gamma^{\frac{1}{1-\theta}}}{\tilde{k}^{\theta} - \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}'}$

In S-S, $\beta\eta \dfrac{\theta \tilde{k}^{*\theta-1}}{\tilde{k}^{*\theta} - \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}^{*}} = \dfrac{\eta \cdot \gamma^{\frac{1}{1-\theta}}}{\tilde{k}^{*\theta} - \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}^{*}}$

$$\Rightarrow \beta\cancel{\eta} \cdot \theta \tilde{k}^{*\theta-1} = \cancel{\eta} \cdot \gamma^{\frac{1}{1-\theta}}$$
$$\Rightarrow \tilde{k}^{*} = \left[\frac{1}{\beta\theta} \cdot \gamma^{\frac{1}{1-\theta}}\right]^{\frac{1}{\theta-1}}$$
$$\Rightarrow \tilde{c}^{*} = \tilde{k}^{*\theta} - \eta \cdot \gamma^{\frac{1}{1-\theta}} \cdot \tilde{k}^{*}$$

(4) In (2), $g = \left(\dfrac{\gamma}{\eta^{1-\alpha-\phi}}\right)^{\frac{1}{1-\alpha}}$

In (3), $b = \gamma^{\frac{1}{1-\theta}}$

① The rate of population growth η does not affect b.

② When η increases, g would decrease. It's intuitive since the production is bounded by the land in the first case, i.e. $\eta \uparrow \Rightarrow$ land per capita $\downarrow \Rightarrow g \downarrow$