Problem Set 7

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Answer 1.

a.

"The size of support" does not satisfy I. Let $Z = \{z_1, z_2, z_3\}$. We have $\frac{1}{2}[z_1] \oplus \frac{1}{2}[z_2] \succ [z_3]$. However,

$$\frac{1}{2}\left[\frac{1}{2}[z_1] \oplus \frac{1}{2}[z_2]\right] \oplus \frac{1}{2}\left[\frac{1}{2}[z_1] \oplus \frac{1}{2}[z_2]\right] = \frac{1}{2}[z_1] \oplus \frac{1}{2}[z_2] \prec \frac{1}{4}[z_1] \oplus \frac{1}{4}[z_2] \oplus \frac{1}{2}[z_3] = \frac{1}{2}\left[\frac{1}{2}[z_1] \oplus \frac{1}{2}[z_2]\right] \oplus \frac{1}{2}[z_3]$$

It does not satisfy C, either. Consider $\frac{1}{2}[z_1] \oplus \frac{1}{2}[z_2] \succ [z_3]$. Given any neighborhood of $[z_3]$, there is a lottery with support 3, which is more preferred than $\frac{1}{2}[z_1] \oplus \frac{1}{2}[z_2]$.

"Comparing the most likely prize" also does not satisfy C and I.

Let $Z = \{z_1, z_2, z_3\}$ with $z_1 \succ z_2 \succ z_3$. Suppose one let $[z_1] \sim \frac{1}{2}[z_1] \oplus \frac{1}{2}[z_2] \succ [z_2]$. However we cannot find any open ball centered in $\frac{1}{2}[z_1] \oplus \frac{1}{2}[z_2]$ which has all its element $\succ [z_2]$. For I, observe that $\frac{2}{3}[z_1] \oplus \frac{1}{3}[z_3] \succ [z_2]$. However,

$$\frac{1}{2}[\frac{2}{3}[z_1] \oplus \frac{1}{3}[z_3]] \oplus \frac{1}{2}[\frac{1}{2}[z_2] \oplus \frac{1}{2}[z_3] \prec \frac{1}{2}[z_2] \oplus \frac{1}{2}[\frac{1}{2}[z_2] \oplus \frac{1}{2}[z_3]$$

b. Thinking about picking a ball in a bag, and different color of ball may represent same prize.

"The size of support": One would prefer a bag with three different colors more than the one with only one color. However, we can let all those color represent exactly the same prize.

"Comparing the most likely prize": Let blue and red represent the prize [10] and black represent [20]. One would prefer the bag with 4 black balls ,3 blue balls and 3 red balls than the bag with 4 black balls and 6 blue balls.

Answer 2.

a. $u([1]) = 1, u(0.5[0] \oplus 0.5[4]) = 2 - \frac{1}{4} \times 4 = 1, u(0.5[0] \oplus 0.5[2]) = \frac{3}{4}$. We have $[1] \sim 0.5[0] \oplus 0.5[4] \succ 0.5[0] \oplus 0.5[2]$. However, $u(\frac{1}{2}[1] \oplus \frac{1}{2}[0.5[0] \oplus 0.5[4]]) = \frac{-3}{4} \Rightarrow \frac{1}{2}[1] \oplus \frac{1}{2}[0.5[0] \oplus 0.5[4]] \prec 0.5[0] \oplus 0.5[2]$, which contradicts to vNM assumption.

b. It clearly satisfies C. For I, we calculate that $u(L) = Ex(L) - (Ex(L^2))$. Hence $u(\alpha L_1 \oplus (1-\alpha)L_2) = \alpha u(L_1) + (1-\alpha)u(L_2)$. It tells us that u is not only satisfies vNM assumption but also induced by a vNM utility representation.

Answer 3.
a. Let he choose L_1 between L_1 and L_2 and L_3 between L_3 and L_4, then the one is taking the lottery $\alpha L_1 \oplus (1-\alpha)L_3$. Similar when he or she choose another lottery. Therefore, we can formulate it as one is choosing from 4 lotteries, $\alpha L_1 \oplus (1-\alpha)L_3$,$\alpha L_1 \oplus (1-\alpha)L_4$,$\alpha L_2 \oplus (1-\alpha)L_3$,$\alpha L_2 \oplus (1-\alpha)L_4$.
b. Let $L_1 \succsim L_2$ and $L_3 \succsim L_4$ without loss of generality. Then $\alpha L_1 \oplus (1-\alpha)L_3 \succsim \alpha L_1 \oplus (1-\alpha)L_4$ and $\alpha L_1 \oplus (1-\alpha)L_3 \succsim \alpha L_2 \oplus (1-\alpha)L_3$ by independence. Moreover, by transitivity and $\alpha L_1 \oplus (1-\alpha)L_4 \succsim \alpha L_2 \oplus (1-\alpha)L_4$, we then have

$$\alpha L_1 \oplus (1-\alpha)L_3 \succsim \alpha L_2 \oplus (1-\alpha)L_4,$$

done!

Answer 4.
a. $\alpha[z] \oplus (1-\alpha)L_1 \succ \alpha[z] \oplus (1-\alpha)L_2$ iff $L_1 \oplus L_2$ is known by vNM. By the Bayesian updating rule, we know that the probability distribution of $\alpha[z] \oplus (1-\alpha)L_i$ after removing $[z]$ is L_i, done!
b. Suppose one is using "the most likely prize" to compare two lotteries. Let $z_1 \succ z_2 \succ z_3 \succ z_4 \succ z_5$ in Z. One prefers $\frac{1}{3}[z_1] \oplus \frac{2}{9}[z_3] \oplus \frac{2}{9}[z_4] \oplus \frac{2}{9}[z_5]$ then $\frac{1}{3}[z_1] \oplus \frac{2}{3}[z_2]$. However, one also prefers $[z_2]$ to $\frac{1}{3}[z_3] \oplus \frac{1}{3}[z_4] \oplus \frac{1}{3}[z_5]$.
c. Suppose removing $[z_1]$ from $\frac{1}{2}[z_1] \oplus \frac{1}{3}[z_2]\frac{1}{6}[z_3]$ would become $\frac{1}{2}[z_2] \oplus \frac{1}{2}[z_3]$ and removing $[z_1]$ from $\frac{1}{2}[z_1] \oplus \frac{1}{4}[z_2]\frac{1}{4}[z_3]$ would become $\frac{2}{3}[z_2] \oplus \frac{1}{3}[z_3]$. Then we can easily find a violation.

Answer 5.
a. Suppose the higher the number is, the higher the income is.
Intuitively, we have $[2] \sim [1]$. However, $\frac{1}{2}[2] \oplus \frac{1}{2}[1] \prec [1]$ since we only care about egalitarianism.
b. $[z] \succsim L$ for all lottery L and sim iff L is also degenerate.
Two examples: 1. by $-var(L)$ 2. by minimum income-maximum income.

Answer 6.
a. Trivial.
b. Pick a quality q_0. Notice that for $t > 1$, we have $\frac{1}{t}[(q,t)] \oplus (1-\frac{1}{t})[(q,0)]$ and similar trick can be done with $t < 1$ (interchange the role of (q,t) and $(q,1)$). We define $u(q_0,t)$ as $v(q_0)t$ and it is vNM representation for the lottery with quality restricted to q_0. We claim that this utility can extend to any q.
By continuity, there exists δ_1, δ_2 such that $[(q_0,1)] \succ \delta_1[(q,1)] \oplus (1-\delta_1)[(q,0)] \sim [(q,\delta_1)]$ and

$[(q,1)] \succ [(q_0,\delta_2)]$.
Notice that $[(q_0,1)] \sim \delta_1[(q_0,\dfrac{1}{\delta_1})]\oplus(1-\delta_1)[q_0,0] \sim \delta_1[(q_0,\dfrac{1}{\delta_1})]\oplus(1-\delta_1)[q,0] \Rightarrow [(q_0,\dfrac{1}{\delta_1})] \succ [q,1] \succ [(q_0,\delta_2)]$. Hence, there exists $k \in (0,1)$ such that $[(q,1)] \sim k[(q_0,\dfrac{1}{\delta_1}] \oplus (1-k)[(q_0,\delta_2] \sim [(q_0,t_q)]$ for some $t_q > 0$. What remains to prove is $[(q,t)] \sim [(q_0,t_q t)]$. For $t > 1$, we have

$$\frac{1}{t}[(q,t)]\oplus(1-\frac{1}{t})[(q,0)] \sim [q,1] \sim [(q_0,t_q t)] \sim \frac{1}{t}[(q_0,t_q t)]\oplus(1-\frac{1}{t})[(q_0,0)] \sim \frac{1}{t}[(q_0,t_q t)]\oplus(1-\frac{1}{t})[(q,0)]$$

By I, $[(q,t)] \sim [(q_0,t_q t)]$. For $t < 1$, similar, just interchange the role of (q,t) and $(q,1)$ in the last part.

Answer 8.

a. A schedule is a lottery with prizes as activities.

b. 1. The combination of activities is homogenous, that is, it does not affect the rule of aggregation.

2. The time point of activity is irrelevant to the preference.

c. To sleep 20 minutes in every hour is not the same as to sleep 8 hours at night.