ECON 7219, Semester 110.1, Assignment 2, Solutions

1. (a) Let $\mu(\tau_i)$ denote type τ_i's belief that the defendant is guilty. Bayesian updating implies

$$\mu(\tau_i) = \frac{f(\tau_i \mid \vartheta_G)\mu_0}{f(\tau_i \mid \vartheta_G)\mu_0 + f(\tau_i \mid \vartheta_I)(1-\mu_0)} = \frac{\tau_i\mu_0}{1-\tau_i-\mu_0+2\tau_i\mu_0}. \tag{1}$$

We can parametrize a cut-off strategy with cutoff τ_* by

$$\sigma_i(\tau_i) = \begin{cases} C & \text{if } \tau_i \geq \tau_*, \\ A & \text{if } \tau_i < \tau_*.^1 \end{cases}$$

A type τ_i who votes to acquit the defendant gains a positive utility if and only if the defendant is innocent, hence his/her expected utility is

$$\mathbb{E}_{\sigma,\tau_i}[u_i(\theta, A, A_{-i})] = P_{\sigma,\tau_i}(\theta = \vartheta_I) = (1-\mu(\tau_i)). \tag{2}$$

A type τ_i who votes to convict the defendant can gain a positive utility both for convicting and acquitting the defendant. It follows from the law of total probability that

$$\begin{aligned}\mathbb{E}_{\sigma,\tau_i}[u_i(\theta, C, A_{-i})] &= \mathbb{E}_{\sigma,\tau_i}\left[1_{\{A_{-i}=C\}}1_{\{\theta=\vartheta_G\}} + 1_{\{A_{-i}=A\}}1_{\{\theta=\vartheta_I\}}\right] \\ &= P_{\sigma,\tau_i}(A_{-i}=C \mid \theta=\vartheta_G)\mu(\tau_i) + P_{\sigma,\tau_i}(A_{-i}=A \mid \theta=\vartheta_I)(1-\mu(\tau_i)).\end{aligned}$$

Since T_{-i} is conditionally independent of T_i, given θ, we can simplify this expression via

$$P_{\sigma,\tau_i}(A_{-i}=C \mid \theta=\vartheta_G) = P(T_{-i} \geq \tau_* \mid \theta=\vartheta_G) = \int_{\tau_*}^1 2\tau_{-i}\,\mathrm{d}\tau_{-i} = 1-\tau_*^2,$$

$$P_{\sigma,\tau_i}(A_{-i}=A \mid \theta=\vartheta_I) = P(T_{-i} < \tau_* \mid \theta=\vartheta_I) = \int_0^{\tau_*} 2-2\tau_{-i}\,\mathrm{d}\tau_{-i} = 2\tau_*-\tau_*^2.$$

We conclude that voting to convict nets an expected utility of

$$\mathbb{E}_{\sigma,\tau_i}[u_i(\theta, C, A_{-i})] = (1-\tau_*^2)\mu(\tau_i) + (2\tau_*-\tau_*^2)(1-\mu(\tau_i)) = 2\tau_*-\tau_*^2+(1-2\tau_*)\mu(\tau_i). \tag{3}$$

Because the expected utilities for voting A and C are continuous in τ_i, the cutoff type must be indifferent between the two. Equating (2) and (3) for $\tau_i = \tau_*$ yields

$$(1-\tau_*)(1-2\mu(\tau_*)-\tau_*) = 0. \tag{4}$$

Since $\tau_* = 1$ does not yield a cutoff strategy, we equate $\mu(\tau_*) = (1-\tau_*)/2$ with (1) for $\tau_i = \tau_*$ to obtain the quadratic equation

$$(1-2\mu_0)\tau_*^2 + (\mu_0-2)\tau_* + 1 - \mu_0 = 0. \tag{5}$$

If $\mu_0 = \frac{1}{2}$, then the equation is linear with the solution $\tau_* = \frac{1}{3}$. If $\mu_0 \neq \frac{1}{2}$, then the quadratic formula implies that (5) has two solutions

$$\tau_*^{\pm} := \frac{2-\mu_0 \pm \sqrt{\mu_0 + 7\mu_0(1-\mu_0)}}{2(1-2\mu_0)}.$$

Note that τ_*^+ is negative if $\mu_0 < \frac{1}{2}$ and it is larger than 1 if $\mu_0 > \frac{1}{2}$ since the numerator is larger than the denominator. Thus, only τ_*^- can be the cutoff. It remains to show

[1] At the cutoff, the judges are indifferent, hence we could also set $\sigma_i(\tau_*) = xA + (1-x)C$.

that $\tau_*^- \in (0,1)$. To that end, we note that $\mu_0^2 < \mu_0 < \mu_0 + 7\mu_0(1-\mu_0)$. Taking the square-root and adding $2 - 2\mu_0$ on both sides yields

$$2 - \mu_0 < \sqrt{\mu_0 + 7\mu_0(1-\mu_0)} + 2(1-\mu_0).$$

Subtracting the square-root term and dividing by $2(1-\mu_0)$ yields $\tau_*^- < 1$. To observe that $\tau_*^- > 0$, note that the numerator is positive if and only if

$$4 - 4\mu_0 + \mu_0^2 = (2-\mu_0)^2 > 8\mu_0 - 7\mu_0^2. \tag{6}$$

Bringing all terms to the left hand side and factoring (6) yields $4(1-2\mu_0)(1-\mu_0) > 0$. Therefore, the numerator of τ_*^- is positive if and only if the denominator is positive, hence $\tau_*^- > 0$. We conclude that the cutoff must be

$$\tau_* = \begin{cases} \frac{1}{3} & \text{if } \mu_0 = \frac{1}{2}, \\ \tau_*^- & \text{otherwise.} \end{cases}$$

So far, we have only shown that if there is a BNE in symmetric cutoff strategies, it must have cutoff τ_*. It remains to verify that no type has an incentive to deviate to conclude that σ as above is indeed a BNE (equivalent to justifying why C and A are played above and below the cutoff, respectively). To that end, note that

$$\frac{\partial \mu(\tau_i)}{\partial \tau_i} = \frac{\mu_0(1-\tau_i-\mu_0+2\tau_i\mu_0) + \tau_i\mu_0(1-2\mu_0)}{(1-\tau_i-\mu_0+2\tau_i\mu_0)^2} = \frac{\mu_0(1-\mu_0)}{(1-\tau_i-\mu_0+2\tau_i\mu_0)^2} > 0.$$

Together with (2) and (3), this implies that

$$\mathbb{E}_{\sigma,\tau_i}[u_i(\theta, C, A_{-i})] - \mathbb{E}_{\sigma,\tau_i}[u_i(\theta, A, A_{-i})] = 2\tau_* - \tau_*^2 + 2(1-\tau_*)\mu(\tau_i)$$

is increasing in τ_i, showing that no type has an incentive to deviate.

(b) We evaluate τ_*^- for $\mu_0 = 0.2$ and obtain $\tau_*^- \simeq 0.543$. It follows that beliefs at the cutoff are $\mu(\tau_*) = 22.87\%$ and the defendant is convicted with probability

$$P_\sigma(A_1 = C, A_2 = C) = (1-\tau_*^2)^2\mu_0 + (1-\tau_*)^4(1-\mu_0) \simeq 13.46\%.$$

To understand why the rate of conviction is so low, note that a judge's vote matters only if the other judge votes to convict. In order to overturn the other judge's signal in equilibrium, a judge must be quite certain that the defendant is innocent. Because the signals are correlated (through θ), the judges overcompensate in equilibrium.

2. (a) Let us parametrize a pure strategy profile by $\sigma_1(\vartheta_L) = q_L^*$, $\sigma_1(\vartheta_H) = q_H^*$, and $\sigma_2 = q_2^*$. Firm 1 knows the demand, hence each type ϑ maximizes $u_1(\vartheta, q) = (\vartheta - q_1 - q_2 - c)q_1$. It will be convenient to note that $f(x) = (a-x)x$ is a parabola with zeroes at 0 and a, hence its global maximum on $\mathbb{R}$ is attained at $\frac{a}{2}$ and its global maximum on $\mathbb{R}^+$ at $\frac{a^+}{2}$. Therefore, the best response correspondence by Firm 1 of type ϑ to q_2 is

$$\mathcal{B}_1(q_2) = \frac{(\vartheta - q_2 - c)^+}{2}. \tag{7}$$

Firm 2 does not know θ and maximizes its expected utility

$$\mathbb{E}_{\mu,\sigma}[u_2(\vartheta, q)] = \Big(\mu(\vartheta_H - q_H^*) + (1-\mu)(\vartheta_L - q_L^*)\Big) - q_2 - c)q_2.$$

For the parameters in the question, Firm 2's best response is

$$\mathcal{B}_2(q_L^*, q_H^*) = \Big(35 - \tfrac{1}{4}q_H^* - \tfrac{1}{4}q_L^*\Big)^+. \tag{8}$$

Note that this implies $q_2 \leq 35 \leq \vartheta_L - c \leq \vartheta_H - c$, hence $\mathcal{B}_1$ implies $q_L^*, q_H^* > 0$. In particular, (7) also holds without the positive part. We verify consistency:

- If $q_2 = 0$, then $\mathcal{B}_1$ implies $q_L^* = 30$ and $q_H^* = 40$. Therefore, $\mathcal{B}_2$ implies $q_2 = 17.5$, a contradiction.
- If $q_2 > 0$, then (8) holds without positive part, hence we can plug (8) into (7) for each type to get a system of two linear equations in two unknowns

$$2q_H^* = \vartheta_H - 45 + \frac{1}{4}q_H^* + \frac{1}{4}q_L^*, \qquad 2q_L^* = \vartheta_L - 45 + \frac{1}{4}q_H^* + \frac{1}{4}q_L^*.$$

  We can solve this for $q_L^* = \frac{55}{3}$ and $q_H^* = \frac{85}{3}$. Therefore, $\mathcal{B}_2$ implies $q_2^* = \frac{70}{3}$.

(b) We parametrize a pure strategy profile by $\sigma_1(\vartheta_L) = q_L^*$, $\sigma_1(\vartheta_H) = q_H^*$, and $\sigma_2(q_1) = q_2^*(q_1)$ and denote by $\mu(q_1) = P_\sigma(\theta = \vartheta_H \mid q_1)$ Firm 2's beliefs that demand is high after observing q_1. We begin with those parts of the equilibrium characterization that are common to pooling and separating equilibria. Given arbitrary beliefs μ, Firm 2's expected utility is

$$\mathbb{E}_{\mu,\sigma}[u_2(\theta, q) \mid q_1] = \Big(\mathbb{E}_\mu[\theta] - q_1 - q_2 - c\Big)q_2.$$

Therefore, Firm 2's global best response is uniquely given by

$$\hat{q}_2(q_1, \mu) = \frac{\Big(\mathbb{E}_\mu[\theta] - q_1 - c\Big)^+}{2}.$$

Next, we determine the off-path beliefs that make off-path deviations of Firm 1 the least attractive. Firm 1's utility in an off-path deviation q_1 is

$$u_1\Big(\vartheta, q_1, \hat{q}_2(q_1, \mu(q_1))\Big) = \Big(\vartheta - q_1 - \hat{q}_2(q_1, \mu(q_1)) - c\Big)q_1.$$

Since $\hat{q}_2$ is non-decreasing in μ, off-path deviations are least attractive if $\mu(q_1) = 1$. We thus set off-path beliefs to $\mu(q_1) = 1$, hence $q_2^*(q_1) = \hat{q}_2(q_1, 1)$ for $q_1 \notin \{q_L^*, q_H^*\}$.

i. In a separating pure-strategy equilibrium, we have $q_L^* \neq q_H^*$ and, hence, $\mu(q_L^*) = \vartheta_L$ and $\mu(q_H^*) = \vartheta_H$. Given the above off-path beliefs, Firm 2's best response is

$$q_2^*(q_L^*) = \frac{(\vartheta_L - q_L^* - c)^+}{2}, \qquad q_2^*(q_1) = \frac{(\vartheta_H - q_1 - c)^+}{2} \text{ if } q_1 \neq q_L^*. \tag{9}$$

In equilibrium, Firm 1 cannot have any profitable on-path or off-path deviations. Since off-path beliefs are identical to on-path beliefs for type ϑ_H, q_H^* must be the global maximum of $u_1(\vartheta_H, q_1, q_2^*(q_1))$. Equation (9) implies that $q_2^*(q_1) = 0$ holds only if Firm 1 chooses $q_1 \geq \vartheta - c$. Such a choice yields a negative utility, which cannot be optimal. Consequently, we can omit the positive part in (9) and deduce $q_H^* = \frac{1}{2}(\vartheta_H - c) = 40$. Firm 2 responds with $\frac{1}{4}(\vartheta_H - c) = 20$, which yields a utility of

$$u_1(\vartheta_H, q_H^*, q_2^*(q_H^*))) = \frac{1}{8}(\vartheta_H - c)^2 = 800 \tag{10}$$

to Firm 1. Since q_H^* is the global maximum, off-path deviations cannot be profitable for type ϑ_H. Moreover, on-path deviations are deterred if

$$800 \geq u_1(\vartheta_H, q_L^*, q_2^*(q_L^*)) = \frac{1}{2}(2\vartheta_H - \vartheta_L - c - q_L^*)q_L^*, \tag{11}$$

where we have used that on the path, the positive part in (9) can be omitted. Let us abbreviate $a = 2\vartheta_H - \vartheta_L - c = 100$. Inequality (11) holds with equality if

$$\frac{a \pm \sqrt{a^2 - 6400}}{2} = \frac{100 \pm 60}{2} = \{20, 80\}.$$

We conclude that (11) holds if and only if $q_L^* \notin (20, 80)$. For type ϑ_L, any deviation to $q_1 \geq \vartheta - c = 80$ yields a negative utility. Any deviation to $q_1 < \vartheta - c = 80$ prompts response $q_2^*(q_1) > 0$ by Firm 2. Therefore, Firm 1's utility in such a deviation is

$$u_1(\vartheta_L, q_1, q_2^*(q_1)) = \frac{1}{2}(2\vartheta_L - \vartheta_H - c - q_1)q_1.$$

Consequently, the most profitable deviation is a deviation to $\hat{q}_1 = 20$. This prompts reply $q_2^*(\hat{q}_1) = 30$ by Firm 2, netting Firm 1 a utility of

$$u_1(\vartheta_L, \hat{q}_1, q_2^*(\hat{q}_1)) = (70 - 20 - 30 - 10) \cdot 20 = 200. \tag{12}$$

This yields the incentive constraint for the low type

$$200 \leq u_1(\vartheta_L, q_H^*, q_2^*(q_H^*)) = \frac{1}{2}(\vartheta_L - c - q_L^*)q_L^*. \tag{13}$$

This is a quadratic inequality that holds with equality at

$$\frac{\vartheta_L - c \pm \sqrt{(\vartheta_L - c)^2 - 1600}}{2} = \frac{60 \pm 20\sqrt{5}}{2} = \left\{30 - 10\sqrt{5}, 30 + \sqrt{5}\right\}.$$

Therefore (13) is satisfied for any $q_L^* \in \left[30 - 10\sqrt{5}, 30 + \sqrt{5}\right]$. We conclude that in any separating equilibrium, we have $q_H^* = 40$, $q_L^* \in \left[30 - 10\sqrt{5}, 20\right]$, and

$$q_2^*(q_1) = \begin{cases} \frac{(\vartheta_L - q_L^* - c)^+}{2} & \text{if } q_1 = q_L^*, \\ \frac{(\vartheta_H - q_1 - c)^+}{2} & \text{if } q_1 \neq q_L^*. \end{cases}$$

ii. In a pooling equilibrium, we have $q_L^* = q_H^* =: q_1^*$ and, hence $\mu(q_L^*) = \mu(q_H^*) = \mu_0$. Firm 2's off-path responses are as in (9) and the on-path response is

$$q_2^*(q_1^*) = \frac{(\frac{1}{2}\vartheta_L + \frac{1}{2}\vartheta_H - q_1^* - c)^+}{2}.$$

Again, we can drop the positive part since type ϑ_L does not have an incentive to choose $q_1^* > \frac{1}{2}(\vartheta_L + \vartheta_H) - c$. As in (10) and (12), off-path deviations of types ϑ_H and ϑ_L, yield at most 800 and 200, respectively. Thus, the only conditions required for a pooling equilibrium are

$$800 \leq u_1(\vartheta_H, q_1^*, q_2^*(q_1^*)) = \tfrac{1}{2}\left(\tfrac{3}{2}\vartheta_H - \tfrac{1}{2}\vartheta_L - c - q_1^*\right)q_1^*$$

$$200 \leq u_1(\vartheta_L, q_1^*, q_2^*(q_1^*)) = \tfrac{1}{2}\left(\tfrac{3}{2}\vartheta_L - \tfrac{1}{2}\vartheta_H - c - q_1^*\right)q_1^*$$

Those are quadratic inequalities that are satisfied in the regions

$$[45 - 5\sqrt{17}, 45 + 5\sqrt{17}] \qquad \text{and} \qquad [10, 40],$$

respectively. Both inequalities hold at the intersection of those regions, i.e., at $q_1^* \in [45 - 5\sqrt{17}, 40]$. Firm 2's reply is

$$q_2^*(q_1) = \begin{cases} \frac{(70 - q_1)^+}{2} & \text{if } q_1 = q_1^*, \\ \frac{(\vartheta_H - q_1 - c)^+}{2} & \text{if } q_1 \neq q_1^*. \end{cases}$$

(c) The similarity is that Firm 2 has the same information when it gets to act since no information is reveled in a pooling equilibrium. Two differences are that in the sequential-move game, Firm 1 has the first-mover advantage, and in the simultaneous-move game, Firm 1 need not worry about the information it reveals, hence both types maximize their respective expected utility individually.

(d) Firm 2's expected utility in the simultaneous-move game is 544.44, whereas the maximal ex-ante expected utility in a separating equilibrium is 542.71, attained for $q_L^* = 30 - 10\sqrt{5}$. In the simultaneous-move game, Firm 2 has an informational disadvantage, but in the separating equilibrium it has a last-mover disadvantage. In this setting, the information gained in a separating equilibrium is not worth having to move last.

3. (a) Observe first that it is a strict best response for the hackers to attack after history h_N. Given that $\sigma_2(h_N) = A$ is played in any PBE, I is a strict best response for type ϑ_C. We can thus parametrize a strategy profile $\sigma = (\sigma_1, \sigma_2)$ by

$$\sigma_1(\vartheta_I; I) = x, \qquad \sigma_2(h_I; A) = y.$$

Let us parametrize the Hackers' beliefs by the probability that they assign to type ϑ_C. The hackers' beliefs after observing I and N, respectively, are

$$\mu(h_I) = \frac{\mu_0}{\mu_0 + x(1-\mu_0)}$$

and $\mu(h_N) = 0$ unless $x = 1$, in which case beliefs are unrestricted. By the law of total probability, type ϑ_I's expected payoff is

$$\begin{aligned}\mathbb{E}_\sigma[u_1(\vartheta_I, A)] &= \mathbb{E}_\sigma[u_1(\vartheta_I, A) \,|\, A_1 = I]x + \mathbb{E}_\sigma[u_1(\vartheta_I, A) \,|\, A_1 = N](1-x) \\ &= x(-1-3y) - 3(1-x) = x(2-3y) - 3.\end{aligned}$$

After observing h_I, the hackers' expected utility is

$$\mathbb{E}_\sigma[u_2(\theta, A) \,|\, h_I] = y(-\mu(h_I) + 2(1-\mu(h_I)) = y(2 - 3\mu(h_I)).$$

The best response correspondences are

$$\mathcal{B}_1(y) = \begin{cases} x = 1 & \text{if } y < \frac{2}{3}, \\ x \in [0,1] & \text{if } y = \frac{2}{3}, \\ x = 0 & \text{if } y > \frac{2}{3}. \end{cases} \qquad \mathcal{B}_2(h_I) = \begin{cases} y = 1 & \text{if } \mu(h_I) < \frac{2}{3}, \\ y \in [0,1] & \text{if } \mu(h_I) = \frac{2}{3}, \\ y = 0 & \text{if } \mu(h_I) > \frac{2}{3}. \end{cases}$$

We verify consistency by analyzing pooling, separating, and semi-separating equilibria.

i. In a pooling equilibrium we have $x = 1$ and $\mu(h_I) = \mu_0$. Therefore, $\mathcal{B}_1$ implies $y \leq \frac{2}{3}$. If $y > 0$, $\mathcal{B}_2$ requires $\mu(h_I) = \mu_0 = \frac{2}{3}$. If $y = 0$, then $\mu(h_I) = \mu_0 \geq \frac{2}{3}$. Off-path beliefs $\mu(h_N)$ are unrestricted since $\sigma_2(h_N) = A$ is a strictly dominant response.

ii. In a separating equilibrium we have $x = 0$ and, hence, $\mu(h_I) = 1$ and $\mu(h_N) = 1$. Therefore, $\mathcal{B}_2$ implies $y = 0$, which requires $x = 1$, a contradiction. We conclude that there are no separating equilibria.

iii. Suppose now that $x \in (0,1)$, i.e., we have a semi-separating equilibrium. This requires $y = \frac{2}{3}$ by $\mathcal{B}_1$, hence $\mu(h_I) = \frac{2}{3}$ by $\mathcal{B}_2$. The latter equality we can solve for $x = \frac{\mu_0}{2(1-\mu_0)}$. This is consistent with $x \in (0,1)$ if $\mu_0 \in (0, \frac{2}{3})$.

(b) This is a signaling game because investing into cyberdefense is not costless.