ECON 7011, Semester 110.1, Assignment 4, Solutions

1. (a) Let us parametrize a mixed strategy profile $\sigma = (\sigma_1, \sigma_2)$ by $\sigma_1 = xL + (1-x)R$ and $\sigma_2 = y\ell + (1-y)r$, respectively. Expected utilities are

$$u_1(\sigma) = x + 2y(1-x), \qquad u_2(\sigma) = 2x + (1-x)y.$$

Partial derivatives are $\frac{\partial u_1(\sigma)}{\partial x} = 1 - 2y$ and $\frac{\partial u_2(\sigma)}{\partial y} = 1 - x$, hence

$$\mathcal{B}_1(y) = \begin{cases} x = 1 & \text{if } y < \frac{1}{2}, \\ x \in [0,1] & \text{if } y = \frac{1}{2}, \\ x = 0 & \text{if } y > \frac{1}{2}, \end{cases} \qquad \mathcal{B}_2(x) = \begin{cases} y = 1 & \text{if } x < 1, \\ y \in [0,1] & \text{if } x = 1. \end{cases}$$

We verify consistency:

i. If $x < 1$, then $\mathcal{B}_2$ implies $y = 1$, hence $\mathcal{B}_1$ implies $x = 0$. This is consistent.
ii. If $x = 1$, then $\mathcal{B}_1$ implies $y \le \frac{1}{2}$, which is consistent.

We obtain the mixed-strategy Nash equilibria (R, ℓ) and $(L, y\ell + (1-y)r)$ for $y \le \frac{1}{2}$. Since this game has no proper subgames, these are all subgame perfect.

(b) Let us parametrize a behavior strategy profile $\sigma = (\sigma_1, \sigma_2)$ by $\sigma_1 = xL + (1-x)R$ and $\sigma_2 = y\ell + (1-y)r$, respectively. Expected utilities are

$$u_1(\sigma) = x + (1-x)y(2+2x) \qquad u_2(\sigma) = 2x + (1-x)(3x(1-y) + (1-x)y).$$

Partial derivatives are $\frac{\partial u_1(\sigma)}{\partial x} = 1 - 4xy$ and $\frac{\partial u_2(\sigma)}{\partial y} = (1-x)(1-4x)$, hence

$$\mathcal{B}_1(y) = \begin{cases} x = 1 & \text{if } y < \frac{1}{4}, \\ x = \frac{1}{4y} & \text{if } y \ge \frac{1}{4}, \end{cases} \qquad \mathcal{B}_2(x) = \begin{cases} y = 1 & \text{if } x < \frac{1}{4}, \\ y \in [0,1] & \text{if } x \in \left\{\frac{1}{4}, 1\right\}, \\ y = 0 & \text{if } x \in (\frac{1}{4}, 1). \end{cases}$$

We verify consistency:

i. If $x < \frac{1}{4}$, then $\mathcal{B}_2$ implies $y = 1$, hence $\mathcal{B}_1$ implies $x = \frac{1}{4}$. This is inconsistent.
ii. If $x = \frac{1}{4}$, then $\mathcal{B}_1$ implies $y = 1$, hence $\mathcal{B}_1$ implies $x = \frac{1}{4}$. This is consistent.
iii. If $x \in (\frac{1}{4}, 1)$, then $\mathcal{B}_2$ implies $y = 0$, hence $\mathcal{B}_1$ implies $x = 1$. This is inconsistent.
iv. If $x = 1$, then $\mathcal{B}_1$ implies $y \le \frac{1}{4}$, which is consistent.

We obtain the Nash equilibria $(\frac{1}{4}L + \frac{3}{4}R, \ell)$ and $(L, y\ell + (1-y)r)$ for $y \le \frac{1}{4}$ in behavior strategies, which are all subgame perfect.

(c) All subgame-perfect equilibria in mixed strategies admit a realization-equivalent behavior strategy profile since only Player 1 has imperfect recall and he/she plays pure strategies. However, the strategy profiles (R, ℓ) and $(L, y\ell + (1-y)r)$ for $y \in (\frac{1}{4}, \frac{1}{2}]$ are not subgame-perfect equilibria in behavior strategies because Player 1 has profitable deviations that are not available to him/her in mixed strategies. Specifically, Player 1 finds it profitable to deviate to a mixture between L and R that allows him/her to reach his/her preferred outcome $(4, 0)$ with positive probability.

(d) No. In the SPE $(\frac{1}{4}L + \frac{3}{4}R, \ell)$, Player 1 receives 1 when L is realized and 2 when R is realized. Both pure-strategy payoffs are below the expected utility $\frac{17}{8}$ in the SPE because the desirable outcome $(4, 0)$ cannot be reached in a pure strategy due to imperfect recall. With this kind of imperfect recall, where Player 1 forgets whether he/she has taken a move, Player 1's expected utility is no longer linear in his/her probability of mixing, hence he/she need not be indifferent. This does not contradict the indifference principle because this extensive-form game does not have a reduced-strategic form since there are nodes that cannot be reached by pure strategies.

2. (a) Let us parametrize a mixed action profile $\alpha = (\alpha_1, \alpha_2)$ by $\alpha_1 = xR + (1-x)C$ and $\alpha_2 = yW + (1-y)S$. The Management's expected utility and its first derivative are

$$u_1(\alpha) = 6xy + 7(1-x)y + 2x(1-y) = 2x + 7y - 3xy, \qquad \frac{\partial u_1(\alpha)}{\partial x} = 2 - 3y.$$

By symmetry, the best-response correspondences for the two players are

$$\mathcal{B}_1(y) = \begin{cases} x = 0 & \text{if } y > \frac{2}{3}, \\ x \in [0,1] & \text{if } y = \frac{2}{3}, \\ x = 1 & \text{if } y < \frac{2}{3}, \end{cases} \qquad \mathcal{B}_2(x) = \begin{cases} y = 0 & \text{if } x > \frac{2}{3}, \\ y \in [0,1] & \text{if } x = \frac{2}{3}, \\ y = 1 & \text{if } x < \frac{2}{3}. \end{cases}$$

We verify consistency:

i. If $y > \frac{2}{3}$, then $\mathcal{B}_1$ implies $x = 0$, hence $\mathcal{B}_2$ implies $y = 1$. This is consistent.
ii. If $y = \frac{2}{3}$, then $\mathcal{B}_2$ implies $x = \frac{2}{3}$, which is consistent.
iii. If $y < \frac{2}{3}$, then $\mathcal{B}_1$ implies $x = 1$, hence $\mathcal{B}_2$ implies $y = 0$. This is consistent.

The three static Nash equilibria are $\alpha^1 = (R, S)$, $\alpha^2 = (C, W)$, and $\alpha^3 = (\frac{2}{3}R + \frac{1}{3}S, \frac{2}{3}W + \frac{1}{3}S)$ with expected utilities $u(\alpha^1) = (2, 7)$, $u(\alpha^2) = (7, 2)$, and $u(\alpha^3) = \left(\frac{14}{3}, \frac{14}{3}\right)$.

(b) By subgame perfection, a static Nash equilibrium must be played after any history of length one. In order to support non-static Nash behavior in the first stage, play of (R, W) must lead to a higher continuation payoff than a unilateral deviation by either player. In order to be able to punish both players, the continuation profile must be $\sigma(R, W) = \alpha^3$, $\sigma(C, W) = \alpha^1$, and $\sigma(R, S) = \alpha^2$. Play after (C, S) can be any of the three static Nash equilibria. The resulting strategy profile is an SPE if and only if

$$6 + \frac{14}{3}\delta \geq 7 + 2\delta,$$

which is the case if and only if $\delta \geq \frac{3}{8}$.

(c) Again, since neither party plays a best response in (C, S), the continuation profile must be $\sigma(C, S) = \alpha^3$, $\sigma(R, S) = \alpha^1$, and $\sigma(C, W) = \alpha^2$ and (R, W) can be any of the three static Nash equilibria. This is an SPE if and only if

$$\frac{14}{3}\delta \geq 2 + 2\delta,$$

which is equivalent to $\delta \geq \frac{3}{4}$.

(d) Four conditions are necessary for a strike to arise in equilibrium. First, it is necessary that the strike is followed by cooperative behavior, in which Workers return to work and wages are raised with high probability. Second, the continuation $\sigma(C, W) = (C, W)$ implies that Workers believe that if they do not go on strike, wages remain low. Third, the continuation $\sigma(R, S) = (R, S)$ implies that Management believes Workers will keep making more demands if Management meets the demands too easily. Fourth, it is necessary that Workers and Management are both far-sighted enough ($\delta \geq \frac{3}{4}$).

3. (a) Given that e has already been chosen, u_1 is strictly decreasing in w. The unique best reply by the Shareholders to any e is thus $w = 0$. Anticipating this, the Manager's utility function is $u_2(w, e) = -c(e)$, which is maximized at $e = 0$.

(b) i. The strongest punishment available to the Manager is $w(e) = 0$ for $e \neq e_*$. Since cost of effort is increasing, the best response among $e \neq e_*$ is $e = 0$, at which the manager's utility is 0. To incentivize effort level e_*, it is thus necessary to offer $w(e_*) \geq c(e_*)$. Moreover, e_* is a unique best response if $w(e_*) > c(e_*)$.

ii. The total surplus of managing the company is $V(1+ae)-c(e)$. Since the Shareholders can extract the entire surplus, they want to incentivize effort level $e_* = Va$ at which the total surplus is maximized. For any $b > 0$, let w_b be the contract that offers $w_b(e) = 0$ for $e \neq e_*$ and $w_b(e_*) = c(e_*) + b$. For any $b > 0$, the Manager's unique best response is e_*. For $b = 0$, the Manager is indifferent among effort levels e_* and 0. For any $x \in [0, 1]$, let $e_x(w_b)$ denote the Manager's best response to w_b, in which he/she chooses effort 0 with probability x if $b = 0$. If $x > 0$, then the Shareholders have a profitable deviation to w_b for any $b < \frac{1}{2}xa^2V^2$ because

$$u_1(w_b, e_x(w_b)) = V + \frac{1}{2}a^2V^2 - b > V + (1-x)\frac{1}{2}a^2V^2 = u_1(w_0, e_x(w_0)).$$

Thus, $x = 0$ is the unique subgame-perfect equilibrium response to w_0. The unique SPE outcome is effort level $e_* = Va$ and a paid salary of $c(e_*) = \frac{1}{2}a^2V^2$.