Macro Theory I Part 2 - Assignment 2

Solution suggested by Shang-Chieh Huang

December 13, 2021

Question 1. Human Capital with Externality

(1) Write down a firm's profit maximization problem and derive a set of equations that characterize firm's optimal decision.

$$\max_{h_{1t}^f} \quad (h_{1t}^f)^\alpha (\bar{h}_t)^\eta - w_t h_{1t}^f$$

F.O.C.:

$$[h_{1t}^f]: \; w_t = \alpha (h_{1t}^f)^{\alpha-1} (\bar{h}_t)^\eta \tag{1}$$

Eq.(1) characterizes firm's optimal decision.

(2) Write down a consumer's utility maximization problem and derive a set of equations that characterize consumer's optimal decision.

$$\begin{aligned} \max_{\{c_t, h_{1t}, h_{2t}, h_{t+1}\}_{t=0}^\infty} & \sum_{t=0}^\infty \beta^t \ln c_t \\ \text{s.t.} \qquad & c_t = w_t h_{1t} \\ & h_{t+1} = B h_{2t} \\ & h_t = h_{1t} + h_{2t} \end{aligned}$$

Since

$$\begin{aligned} c_t &= w_t h_{1t} \\ &= w_t (h_t - h_{2t}) \\ &= w_t \left(h_t - \frac{h_{t+1}}{B} \right), \end{aligned}$$

the consumer's utility maximization problem can be written as

$$\begin{aligned} \max_{\{c_t, h_{t+1}\}_{t=0}^\infty} & \sum_{t=0}^\infty \beta^t \ln c_t \\ \text{s.t.} \qquad & c_t = w_t \left(h_t - \frac{h_{t+1}}{B} \right) \end{aligned} \tag{2}$$

F.O.C.:

$$[c_t]: \frac{1}{c_t} = \lambda_t$$
$$[h_{t+1}]: \frac{\lambda_t w_t}{B} = \beta \lambda_{t+1} w_{t+1}$$

Combining these two first order conditions gives us

$$\frac{c_{t+1}}{c_t} = \beta B \frac{w_{t+1}}{w_t} \tag{3}$$

Eq.(2) and Eq.(3) characterize the consumer's optimal decision.

(3) Find the growth rates along the balanced growth path.

At the equilibrium, $h_t = \bar{h}_t$ and $h_{1t} = h_{1t}^f$. Thus, from Eq.(3),

$$\begin{aligned}
\frac{c_{t+1}}{c_t} &= \beta B \frac{w_{t+1}}{w_t} \\
&= \beta B \frac{\alpha (h_{1t+1}^f)^{\alpha-1} (\bar{h}_{t+1})^{\eta}}{\alpha (h_{1t}^f)^{\alpha-1} (\bar{h}_t)^{\eta}} \\
&= \beta B \frac{\alpha (h_{1t+1})^{\alpha-1} (h_{t+1})^{\eta}}{\alpha (h_{1t})^{\alpha-1} (h_t)^{\eta}} \\
&= \beta B \left(\frac{h_{1t+1}}{h_{1t}}\right)^{\alpha-1} \left(\frac{h_{t+1}}{h_t}\right)^{\eta} \qquad (4)
\end{aligned}$$

Look for the growth rate along the balanced growth path. Suppose that $c_t = g_c^t c^*$, $h_t = g_h^t h^*$, $h_{1t} = g_{h1}^t h_1^*$ and $h_{2t} = g_{h2}^t h_2^*$.

1. From $h_t = h_{1t} + h_{2t}$, we have

$$\begin{aligned}
& g_h^t h^* = g_{h1}^t h_1^* + g_{h2}^t h_2^* \\
\Longrightarrow \qquad & h^* = \left(\frac{g_{h1}}{g_h}\right)^t h_1^* + \left(\frac{g_{h2}}{g_h}\right)^t h_2^*
\end{aligned}$$

   This equation is stationary if

$$g_h = g_{h1} = g_{h2}. \tag{5}$$

2. From Eq.(2),

$$
\begin{aligned}
c_t &= w_t \left(h_t - \frac{h_{t+1}}{B} \right) \\
&= \alpha (h_{1t}^f)^{\alpha-1} (\bar{h}_t)^{\eta} \left(h_t - \frac{h_{t+1}}{B} \right) \\
&= \alpha (h_{1t})^{\alpha-1} (h_t)^{\eta} \left(h_t - \frac{h_{t+1}}{B} \right) \\
\implies \quad g_c^t c^* &= \alpha (g_{h1}^t h_1^*)^{\alpha-1} (g_h^t h^*)^{\eta} \left(g_h^t h^* - \frac{g_h^{t+1} h^*}{B} \right) \\
&= \alpha h_1^{*\alpha-1} h^{*\eta} \left(g_{h1}^{\alpha-1} g_h^{\eta} \right)^t \left(g_h^t h^* - \frac{g_h^{t+1} h^*}{B} \right)
\end{aligned}
$$

By Eq.(5), $g_h = g_{h1} = g_{h2}$ implies

$$
\begin{aligned}
g_c^t c^* &= \alpha h_1^{*\alpha-1} h^{*\eta} g_h^{(\alpha+\eta-1)t} \left(g_h^t h^* - \frac{g_h^{t+1} h^*}{B} \right) \\
&= \alpha h_1^{*\alpha-1} h^{*\eta+1} g_h^{(\alpha+\eta)t} \left(1 - \frac{g_h}{B} \right) \\
\implies \quad c^* &= \alpha h_1^{*\alpha-1} h^{*\eta+1} \left(\frac{g_h^{\alpha+\eta}}{g_c} \right)^t \left(1 - \frac{g_h}{B} \right).
\end{aligned}
$$

The equation above is stationary if

$$
g_c = g_h^{\alpha+\eta}. \tag{6}
$$

3. From Eq.(4) and Eq.(5),

$$
\begin{aligned}
\frac{c_{t+1}}{c_t} &= \beta B \left(\frac{h_{1t+1}}{h_{1t}} \right)^{\alpha-1} \left(\frac{h_{t+1}}{h_t} \right)^{\eta} \\
\implies \quad \frac{g_c^{t+1} \hat{c}}{g_c^t \hat{c}} &= \beta B \left(\frac{g_{h1}^{t+1} \hat{h}_1}{g_{h1}^t \hat{h}_1} \right)^{\alpha-1} \left(\frac{g_h^{t+1} \hat{h}}{g_h^t \hat{h}} \right)^{\eta} \\
\implies \quad g_c &= \beta B g_{h1}^{\alpha-1} g_h^{\eta} \\
&= \beta B g_h^{\alpha+\eta-1}.
\end{aligned}
$$

Plugging g_c in Eq.(6) into the above equation yields

$$
g_h^{\alpha+\eta} = \beta B g_h^{\alpha+\eta-1}.
$$

Thus,

$$
g_h = g_{h1} = g_{h2} = \beta B
$$

and

$$
g_c = (\beta B)^{\alpha+\eta}
$$

Question 2. Two Alternative Technologies

(1) Formulate the social planner's dynamic programming problem.

The standard procedure is to transform the economy into a **stationary** one, starting from the resource constraint:

$$N_t c_t + K_{t+1} = \gamma^t K_t^\mu N_t^\phi L_t^{1-\mu-\phi}$$

Remove population growth:

Divide it by the **growth component** N_t, then we have

$$c_t + \frac{N_{t+1}}{N_t}\frac{K_{t+1}}{N_{t+1}} = \gamma^t \left(\frac{K_t}{N_t}\right)^\mu \left(\frac{N_t}{N_t}\right)^\phi \left(\frac{L_t}{N_t}\right)^{1-\mu-\phi}$$

Let $k_t \equiv \frac{K_t}{N_t}$ and $L_t = 1$ for all t, the above equation can be transformed as

$$c_t + \eta k_{t+1} = \gamma^t k_t^\mu \left(\frac{1}{\eta^t N_0}\right)^{1-\mu-\phi}$$

All variables are transformed to be per capita terms. Without loss of generality, I normalize N_0 to unity.

Remove technology growth:

For some variable x, define $\hat{x}_t = \frac{x_t}{g_x^t}$, where g_x is the growth rate of x along the balanced growth path. The above resource constraint can be rewritten as

$$g_c^t \hat{c}_t + \eta g_k^{t+1}\hat{k}_{t+1} = \left(\frac{\gamma}{\eta^{1-\mu-\phi}}\right)^t \left(g_k^t \hat{k}_t\right)^\mu$$

Divide it by g_c^t:

$$\hat{c}_t + \eta g_k \left(\frac{g_k}{g_c}\right)^t \hat{k}_{t+1} = \left[\left(\frac{\gamma}{\eta^{1-\mu-\phi}}\right)\left(\frac{g_k^\mu}{g_c}\right)\right]^t \hat{k}_t^\mu \tag{7}$$

Eq.(7) is stationary if

$$g_k = g_c$$

and

$$\gamma g_k^\mu = \eta^{1-\mu-\phi} g_c$$

These two equations imply

$$g_c = g_k = \left(\frac{\gamma}{\eta^{1-\mu-\phi}}\right)^{\frac{1}{1-\mu}} \equiv g,$$

where g is common growth rate. Thus, resource constraint, Eq.(7), can be rewritten as

$$\hat{c}_t + \eta g_k \left(\frac{g_k}{g_c}\right)^t \hat{k}_{t+1} = \left[\left(\frac{\gamma}{\eta^{1-\mu-\phi}}\right)\left(\frac{g_k^\mu}{g_c}\right)\right]^t \hat{k}_t^\mu$$

$$\Longrightarrow \qquad \hat{c}_t + \eta g_k \hat{k}_{t+1} = \hat{k}_t^\mu$$

$$\Longrightarrow \qquad \hat{c}_t + (\gamma\eta^\phi)^{\frac{1}{1-\mu}}\hat{k}_{t+1} = \hat{k}_t^\mu$$

We now move to the utility function. Since $N_t = \eta^t N_0 = \eta^t$ and $c_t = \hat{c}_t g^t$, the utility function can be rewritten as

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t N_t \log c_t &= \sum_{t=0}^{\infty} (\beta\eta)^t \log(\hat{c}_t g^t) \\
&= \sum_{t=0}^{\infty} (\beta\eta)^t \log \hat{c}_t + \sum_{t=0}^{\infty} (\beta\eta)^t t \log g \\
&= \sum_{t=0}^{\infty} (\beta\eta)^t \log \hat{c}_t + \frac{\beta\eta}{(1-\beta\eta)^2} \log g,
\end{aligned}$$

where $\beta\eta < 1$ and

$$\begin{aligned}
\sum_{t=0}^{\infty} (\beta\eta)^t t \log g &= [\beta\eta + 2(\beta\eta)^2 + 3(\beta\eta)^2 + ...] \log g \\
&= \beta\eta[1 + 2\beta\eta + 3(\beta\eta)^2 + ...] \log g \\
&= \frac{\beta\eta}{1-\beta\eta}[(1-\beta\eta) + 2\beta\eta(1-\beta\eta) + 3(\beta\eta)^2(1-\beta\eta) + ...] \log g \\
&= \frac{\beta\eta}{1-\beta\eta}[1 - \beta\eta + 2\beta\eta - 2(\beta\eta)^2 + 3(\beta\eta)^2 - 3(\beta\eta)^3 + ...] \log g \\
&= \frac{\beta\eta}{1-\beta\eta}[1 + \beta\eta + (\beta\eta)^2 + (\beta\eta)^3 + ...] \log g \\
&= \frac{\beta\eta}{(1-\beta\eta)^2} \log g.
\end{aligned}$$

Finally, the social planner's dynamic programming problem is

$$\begin{aligned}
V(\hat{k}) = &\max_{\hat{c}, \hat{k}'} \ \log \hat{c} + \beta\eta V(\hat{k}') \\
&\text{s.t.} \quad \hat{c} + (\gamma\eta^{\phi})^{\frac{1}{1-\mu}} \hat{k}' = \hat{k}^{\mu}
\end{aligned}$$

(2) Characterize the balanced growth path of this economy. Solve explicitly for the growth rate of per capita consumption (c_t) along this path.

Replacing $\hat{c} = \hat{k}^{\mu} - (\gamma\eta^{\phi})^{\frac{1}{1-\mu}} \hat{k}'$, the social planner's dynamic programming problem becomes

$$V(\hat{k}) = \max_{\hat{k}'} \ \log \left[\hat{k}^{\mu} - (\gamma\eta^{\phi})^{\frac{1}{1-\mu}} \hat{k}'\right] + \beta\eta V(\hat{k}')$$

F.O.C.:

$$[\hat{k}'] : \ \beta\eta V'(\hat{k}') = \frac{(\gamma\eta^{\phi})^{\frac{1}{1-\mu}}}{\hat{k}^{\mu} - (\gamma\eta^{\phi})^{\frac{1}{1-\mu}} \hat{k}'}. \tag{8}$$

Envelope condition:

$$\begin{aligned}
V'(\hat{k}) &= \frac{\mu \hat{k}^{\mu-1}}{\hat{k}^{\mu} - (\gamma\eta^{\phi})^{\frac{1}{1-\mu}} \hat{k}'} \\
\implies \quad V'(\hat{k}') &= \frac{\mu \hat{k}'^{\mu-1}}{\hat{k}'^{\mu} - (\gamma\eta^{\phi})^{\frac{1}{1-\mu}} \hat{k}''}.
\end{aligned} \tag{9}$$

Combining Eq.(8) and Eq.(9) gives us Euler equation:

$$\beta\eta\frac{\mu\hat{k}'^{\mu-1}}{\hat{k}'^{\mu}-\left(\gamma\eta^{\phi}\right)^{\frac{1}{1-\mu}}\hat{k}''}=\frac{\left(\gamma\eta^{\phi}\right)^{\frac{1}{1-\mu}}}{\hat{k}^{\mu}-\left(\gamma\eta^{\phi}\right)^{\frac{1}{1-\mu}}\hat{k}}.$$

At the steady state, $\hat{k}=\hat{k}'=\hat{k}''\equiv\hat{k}^*$ and $\hat{c}=\hat{c}'=\hat{c}''\equiv\hat{c}^*$. The Euler equation at the steady state is

$$\beta\eta\mu\hat{k}^{*\mu-1}=\left(\gamma\eta^{\phi}\right)^{\frac{1}{1-\mu}}.$$

It implies that

$$\hat{k}^*=\left[\frac{\beta\mu}{\left(\gamma\eta^{\phi+\mu-1}\right)^{\frac{1}{1-\mu}}}\right]^{\frac{1}{1-\mu}},$$

and

$$\hat{c}^*=\hat{k}^{*\mu}-\left(\gamma\eta^{\phi}\right)^{\frac{1}{1-\mu}}\hat{k}^*.$$

(3) Repeat (1) and (2) using the technology instead: $Y_t=\gamma^tK_t^{\theta}N_t^{1-\theta}$.

The standard procedure is to transform the economy into a **stationary** one, starting from the resource constraint:

$$N_tc_t+K_{t+1}=\gamma^tK_t^{\theta}N_t^{1-\theta}$$

Divide it by the **growth component** N_t, then we have

$$c_t+\frac{N_{t+1}}{N_t}\frac{K_{t+1}}{N_{t+1}}=\gamma^t\left(\frac{K_t}{N_t}\right)^{\theta}\left(\frac{N_t}{N_t}\right)^{1-\theta}$$

Let $k_t\equiv\frac{K_t}{N_t}$ for all t, the above equation can be transformed as

$$c_t+\eta k_{t+1}=\gamma^tk_t^{\theta}$$

All variables are transformed to be per capita terms. Without loss of generality, I normalize N_0 to unity. For some variable x, define $\tilde{x}_t=\frac{x_t}{r_x^t}$, where r_x is the growth rate of x along the balanced growth path. The above resource constraint can be rewritten as

$$r_c^t\tilde{c}_t+\eta r_k^{t+1}\tilde{k}_{t+1}=\gamma^t\left(r_k^t\tilde{k}_t\right)^{\theta}$$

Divide it by r_c^t:

$$\tilde{c}_t+\eta r_k\left(\frac{r_k}{r_c}\right)^t\tilde{k}_{t+1}=\left[\frac{\gamma r_k^{\theta}}{r_c}\right]^t\tilde{k}_t^{\theta}\tag{10}$$

Eq.(10) is stationary if

$$r_k=r_c$$

and

$$\gamma r_k^{\theta} = r_c$$

These two equations imply

$$r_c = r_k = \gamma^{\frac{1}{1-\theta}} \equiv r,$$

where g is common growth rate. Thus, resource constraint, Eq.(10), can be rewritten as

$$\begin{aligned} & \tilde{c}_t + \eta r_k \left(\frac{r_k}{r_c}\right)^t \tilde{k}_{t+1} = \left[\frac{\gamma r_k^{\theta}}{r_c}\right]^t \tilde{k}_t^{\theta} \\ \Longrightarrow \quad & \tilde{c}_t + \eta r_k \tilde{k}_{t+1} = \tilde{k}_t^{\theta} \\ \Longrightarrow \quad & \tilde{c}_t + \eta \gamma^{\frac{1}{1-\theta}} \tilde{k}_{t+1} = \tilde{k}_t^{\theta} \end{aligned}$$

We now move to the utility function. Since $N_t = \eta^t N_0 = \eta^t$ and $c_t = \tilde{c}_t r^t$, the utility function can be rewritten as

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t N_t \log c_t &= \sum_{t=0}^{\infty} (\beta\eta)^t \log(\tilde{c}_t r^t) \\ &= \sum_{t=0}^{\infty} (\beta\eta)^t \log \tilde{c}_t + \sum_{t=0}^{\infty} (\beta\eta)^t t \log r \\ &= \sum_{t=0}^{\infty} (\beta\eta)^t \log \tilde{c}_t + \frac{\beta\eta}{(1-\beta\eta)^2} \log r, \end{aligned}$$

where $\beta\eta < 1$.

Finally, the social planner's dynamic programming problem is

$$V(\tilde{k}) = \max_{\tilde{k}'} \ \log\left[\tilde{k}^{\theta} - \eta\gamma^{\frac{1}{1-\theta}} \tilde{k}'\right] + \beta\eta V(\tilde{k}')$$

F.O.C.:

$$[\tilde{k}']: \ \beta\eta V(\tilde{k}') = \frac{\eta\gamma^{\frac{1}{1-\theta}}}{\tilde{k}^{\theta} - \eta\gamma^{\frac{1}{1-\theta}} \tilde{k}'}. \tag{11}$$

Envelope condition:

$$\begin{aligned} & V'(\tilde{k}) = \frac{\theta \tilde{k}^{\theta-1}}{\tilde{k}^{\theta} - \eta\gamma^{\frac{1}{1-\theta}} \tilde{k}'} \\ \Longrightarrow \quad & V'(\tilde{k}') = \frac{\theta \tilde{k}'^{\theta-1}}{\tilde{k}'^{\theta} - \eta\gamma^{\frac{1}{1-\theta}} \tilde{k}''}. \end{aligned} \tag{12}$$

Combining Eq.(11) and Eq.(12) gives us Euler equation:

$$\beta\eta \frac{\theta \tilde{k}'^{\theta-1}}{\tilde{k}'^{\theta} - \eta\gamma^{\frac{1}{1-\theta}} \tilde{k}''} = \frac{\eta\gamma^{\frac{1}{1-\theta}}}{\tilde{k}^{\theta} - \eta\gamma^{\frac{1}{1-\theta}} \tilde{k}'}.$$

Evaluating Euler equation at steady state yields the result:

$$\tilde{k}^* = \left(\frac{\beta\theta}{\gamma^{\frac{1}{1-\theta}}} \right)^{\frac{1}{1-\theta}},$$

and

$$\tilde{c}^* = \tilde{k}^{*\theta} - \eta\gamma^{\frac{1}{1-\theta}}\tilde{k}^*.$$

(4) Compare how the rate of population growth η affects the rate of per capita growth in the two cases and provide the intuition.

The growth rate in the first case:

$$g = \left(\frac{\gamma}{\eta^{1-\mu-\phi}} \right)^{\frac{1}{1-\mu}}.$$

The growth rate in the second case:

$$r = \gamma^{\frac{1}{1-\theta}}.$$

The population growth rate affects only the growth rate in the first case, g. It is because the production is bounded by a fixed factor of production, land. While the population grows land per capita decreases, slowing down the growth rate.