**ECON 7011, Semester 110.1, Assignment 3, Solutions**

1. (a) In the game to the left, let $x_i^k$ denote player $i$'s $k^{\text{th}}$ decision node for $i, k = 1, 2$. By backward induction $\sigma_2(x_2^2) = r$ in any SPE $\sigma$. This choice makes Player 1 indifferent at $x_1^2$, hence we parametrize $\sigma_1(x_1^2) = y\ell + (1-y)r$. Depending on $y$, Player 2 might be willing to choose $L$ or $R$, hence we parametrize $\sigma_2(x_2^1) = zL + (1-z)R$. Regardless of the decisions that follow, Player 1 will choose $R$ at $x_1^1$. Expected utilities of Player 2 in $\sigma$ are $u_2(\sigma) = z + 3(1-z)(1-y)$. Player 2's best response is, therefore,

$$\mathcal{B}_2(y) = \begin{cases} z = 1 & \text{if } y > \frac{2}{3}, \\ z \in [0,1] & \text{if } y = \frac{2}{3}, \\ z = 0 & \text{if } y < \frac{2}{3}. \end{cases}$$

All SPE thus satisfy either $y > \frac{2}{3}$ and $z = 1$, $y = \frac{2}{3}$ and $z \in [0,1]$, or $y < \frac{2}{3}$ and $z = 0$. In the game to the right, let $x_0$ denote the root, let $x_L$ and $x_R$ denote the left and right node, respectively, that start a proper subgame, and let $h_1$ denote Player 1's non-singleton information set. By backward induction, Player 2 must play $\sigma_2(x_R) = L$. For the subgame starting at $x_L$, we can parametrize $\sigma_1(h_1) = x\ell + (1-x)r$ and $\sigma_2(x_L) = yA + (1-y-z)B + zC$. The expected utilities in the subgame are

$$u_1(\sigma) = 3y + (1-y-z)(1+2x) + zx, \quad u_2(\sigma) = 2y + (1-y-z)(2-x) + z(1+2x)$$

with partial derivatives

$$\frac{\partial u_1(\sigma)}{\partial x} = 2 - 2y - z, \qquad \frac{\partial u_2(\sigma)}{\partial y} = x, \qquad \frac{\partial u_2(\sigma)}{\partial z} = 3x - 1.$$

Since $y + z \leq 1$, the derivative $\frac{\partial u_1(\sigma)}{\partial x}$ is positive unless $y = 1$, at which point Player 1 is indifferent. The resulting best-response correspondences are

$$\mathcal{B}_1(y) = \begin{cases} x = 1 & \text{if } y < 1, \\ x \in [0,1] & \text{if } y = 1, \end{cases} \qquad \mathcal{B}_2(x) = \begin{cases} z = 1 & \text{if } x > \frac{1}{2}, \\ y + z = 1 & \text{if } x = \frac{1}{2}, \\ y = 1 & \text{if } x < \frac{1}{2}. \end{cases}$$

We verify consistency:

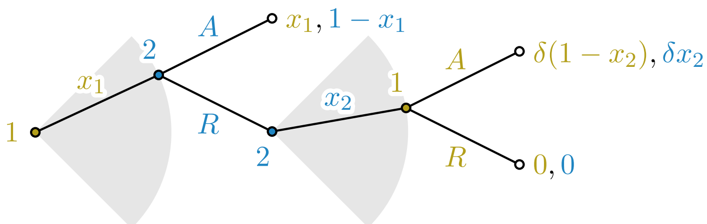
i. If $y < 1$, then $\mathcal{B}_1$ implies $x = 1$, hence $\mathcal{B}_2$ implies $z = 1$. This is consistent.

ii. If $y = 1$, then $\mathcal{B}_2$ implies $x \leq \frac{1}{2}$. This is consistent.

The Nash equilibrium outcomes in the two cases are $(1, 3)$ and $(3, 2)$, respectively. Thus, if $(\ell, C)$ is played in $\mathcal{G}(x_L)$, then Player 1 will choose $R$ at $x_0$. If $(A, x\ell + (1-x)r)$ is played in $\mathcal{G}(x_L)$ for $x \leq \frac{1}{2}$, then Player 1 will choose $L$ at $x_0$.

(b) Because the SPE are not unique in either game, common knowledge of rationality is insufficient. Both games require that players are rational and have correct conjectures.

2. (a) We can draw the extensive-form game as follows:



(b) We solve the game backwards through the tree. We first show that in any SPE, the responder in the last period must accept any proposal with probability 1. This follows immediately from rationality if he/she receives a positive share in expectation. Suppose towards a contradiction that $x_i^T = 1$, but the responder accepts only with probability $y < 1$. Then the proposer has a profitable deviation to $x_i^T > y$, a contradiction. Given that the responder accepts any share, the proposer must propose $x_i^T = 1$.

Anticipating this unique continuation equilibrium, the responder in period $T-1$ knows that he/she will get $\delta$ if he/she rejects. For the same reason as above, the responder must accept any proposal $x_i^{T-1} \leq 1-\delta$ that grants him/her at least a share $\delta$. The proposer will thus optimally offer a share $x_i^{T-1} = 1-\delta$. Iterating this argument, the responder in period $T-2$ must get $\delta x_i^{T-1} = \delta(1-\delta)$, hence

$$x_i^{T-2} = 1 - \delta x_i^{T-1} = 1 - \delta(1-\delta) = 1 - \delta + \delta^2 = \sum_{k=0}^{2}(-\delta)^k.$$

This is a partial sum of a geometric series $\sum_{k=0}^{\infty}(-\delta)^k = \frac{1}{1+\delta}$. We can thus rewrite

$$x_i^t = \sum_{k=0}^{T-t}(-\delta)^k = \sum_{k=0}^{\infty}(-\delta)^k - (-\delta)^{T-t+1}\sum_{k=0}^{\infty}(-\delta)^k = \frac{1-(-\delta)^{T-t+1}}{1+\delta}. \tag{1}$$

We now show by mathematical induction that in any SPE, the proposer in period $t$ must propose $x_i^t$ as given in (1) and that the responder must accept with probability 1. We have already established the base case $t = T$, for which (1) reduces to 1. The inductive step is analogous as the proof in the last period. Suppose the inductive hypothesis is true for all $t' > t$, i.e., $x_i^{t+1}$ is given as in (1). Then the responder in period $t$ can ensure $\delta x_i^{t+1}$ by rejecting the proposal, hence he accepts any proposal $x_i^t \leq 1 - \delta x_i^{t+1}$. Anticipating this behavior, the proposer must propose

$$x_i^t = 1 - \delta x_i^{t+1} = \frac{1+\delta-\delta(1-(-\delta)^{T-t})}{1+\delta} = \frac{1-(-\delta)^{T-t+1}}{1+\delta}.$$

The unique SPE outcome is, therefore, the split $(x_1^1, 1-x_1^1)$.

(c) Let us denote by $x_T^{\text{even}}$ and $x_T^{\text{odd}}$ the equilibrium share of the first proposer when $T$ is even or odd, respectively. Of course, $T$ cannot be both even and odd, but the difference

$$x_T^{\text{odd}} - x_T^{\text{even}} = \frac{1+\delta^T}{1+\delta} - \frac{1-\delta^T}{1+\delta} = \frac{2\delta^T}{1+\delta}$$

gives us a sense of how valuable the last-mover advantage is for player 1. We can capture the value of the last-mover advantage for player 2 by

$$(1-x_T^{\text{even}}) - (1-x_T^{\text{odd}}) = x_T^{\text{odd}} - x_T^{\text{even}} = \frac{2\delta^T}{1+\delta}.$$

The value of the first move for the last-proposer and last-responder, respectively, are

$$x_T^{\text{odd}} - (1-x_T^{\text{even}}) = \frac{1-\delta}{1+\delta}, \qquad x_T^{\text{even}} - (1-x_T^{\text{odd}}) = \frac{1-\delta}{1+\delta}.$$

Having the last-mover advantage is thus more valuable if and only if $2\delta^T > 1-\delta$.

(d) Rationality alone is not sufficient to pin down the behavior in the last period since any response is a best response to $x_i^T = 1$. Therefore, we need that the players are rational and that their conjectures are correct.

3. (a) We first solve for Player $i$'s best response, given $q_{-i}$. Since $u_i$ is not differentiable at $q_{-i}$, the maximum may be attained at a boundary point, at $q_{-i}$, or at an interior point with derivative 0. The partial derivative at $q_i \neq q_{-i}$ is

$$\frac{\partial u_i(q)}{\partial q_i} = \begin{cases} 1 - 2cq_i & \text{if } q_i < q_{-i}, \\ -1 - 2cq_i & \text{if } q_i > q_{-i}. \end{cases}$$

Strict concavity implies that $\mathcal{B}_i(q_{-i}) = \min\left\{\frac{1}{2c}, q_{-i}\right\}$. We verify consistency:

i. If $q_1 > \frac{1}{2c}$, then $\mathcal{B}_2$ implies $q_2 = \frac{1}{2c}$, hence $\mathcal{B}_1$ implies $q_1 = \frac{1}{2}$. This is inconsistent.

ii. If $q_1 \leq \frac{1}{2c}$, then $\mathcal{B}_2$ implies $q_2 = q_1 \leq \frac{1}{2c}$, hence $\mathcal{B}_1$ implies $q_1 = q_2$. This is consistent.

In conclusion, there are continuum of Nash equilibria, in which both players give each other gifts of identical quality $q_i \leq \frac{1}{2c}$.

(b) Drew's best response, given $q_1$, is $q_2^*(q_1) = \min\left\{\frac{1}{2c}, q_1\right\}$ identically as above. Anticipating this response, Cameron's utility of giving gift $q_1$ is

$$\begin{aligned} u_1^*(q_1) := u_1(q_1, q_2^*(q_1)) &= \min\left\{\tfrac{1}{2c}, q_1\right\} - \left|\min\left\{\tfrac{1}{2c}, q_1\right\} - q_1\right| - cq_1^2 \\ &= 2\min\left\{\tfrac{1}{2c}, q_1\right\} - q_1 - cq_1^2. \end{aligned}$$

For $q_1 < \frac{1}{2c}$, we have $u_1(q_1) = q_1 - cq_1^2$ with derivative $u_1'(q_1) = 1 - 2cq_1 > 0$. For $q_1 > \frac{1}{2c}$, we have $u_1(q_1) = \frac{1}{c} - q_1 - cq_1^2$ with derivative $u_1'(q_1) = -1 - 2cq_1 < 0$. We conclude that Cameron's unique best-response is $q_1 = \frac{1}{2c}$. Thus, the unique subgame-perfect equilibrium is $q_1 = \frac{1}{2c}$ and $q_2(q_1) = \min\left\{\frac{1}{2c}, q_1\right\}$. Note that $q_2$ is a function, mapping any $q_1$ to a gift quality. In particular, $q_2 \equiv \frac{1}{2c}$ is not Drew's equilibrium strategy.

(c) Observe that regardless of Cameron's choice of gift quality $q_1 \leq \frac{1}{2c}$, he/she will receive a gift of equal quality in return, similarly to the Nash equilibria. The sequential nature of the second game, however, means that Cameron essentially gets to select among Nash equilibrium outcomes, from which he/she prefers the highest-utility equilibrium. In the simultaneous-move game, no player has the power to unilaterally select among equilibria.