13. Social Learning

ECON 7219 – Games With Incomplete Information

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Motivating Example

Which Bubble Tea Shop Is Better?

Lining up for bubble tea:

- Do those people line up because they know this shop is the best?
- In that case, it is in my best interest to line up for the shop as well.
- Or did they make the same considerations as I did and simply line up, expecting others in the line must have information?

States and Payoffs

Unknown state:

- An unknown state θ can take one of two values in $\Theta = \{\vartheta_*, \vartheta_\times\}$.
- State ϑ_* will often be interpreted as the true state.
- A sequence of short-lived players have prior beliefs $\mu_0 \in \Delta(\Theta)$.

Actions and payoffs:

- Player t chooses an action $a_t \in \mathcal{A}$ and earns utility $u(a_t, \vartheta)$.
- Players observes the sequence $h_t = (a_0, \ldots, a_{t-1})$ of actions taken.
- We denote by $\mathcal{H}$ the set of all possible histories.

Available information:

- Player t observes a $\mathcal{Y}$-valued signal Y_t correlated with θ.
- Note that h_t is the public history at time t, whereas (h_t, y_t) is the private information of player t.

Probability Space and Beliefs

Probability Space:

- The outcome of the game is $(\theta, A_0, Y_0, A_1, Y_1, \ldots)$.
- We can choose $\omega = (\vartheta, a_0, y_0, a_1, y_1, \ldots)$ as states of the world.
- As usual, we set $H = (H_t)_{t \geq 0}$, defined by $H_t = (A_0, \ldots, A_{t-1})$.
- The filtration of public information is $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{F}_t$ is the σ-algebra generated by H_t, i.e., it is player t's information partition.

Beliefs:

- A strategy profile σ is a map $\sigma : \mathcal{H} \times \mathcal{Y} \to \Delta(\mathcal{A})$, where we denote by $\sigma(h_t, y_t; a_t)$ the probability with which a_t is chosen by player t.
- The belief process $\mu = (\mu_t)_{t \geq 0}$ is defined by $\mu_t := P_\sigma(\theta = \vartheta_* \mid \mathcal{F}_t)$.

Learning

Definition 13.1

Learning is complete in a strategy profile σ if for P_σ-a.e. $\omega \in \{\theta = \vartheta_*\}$, we have $\lim_{t\to\infty} \mu_t(\omega) = 1$. Learning is incomplete otherwise.

Public signals:

- If the signals were observable by every player, then learning would be complete in any strategy profile by the strong law of large numbers.

Private signals:

- Learning is complete if everybody follows their own signal.
- The problem is that doing so is not incentive compatible.
- Under what conditions is learning complete in equilibrium?

Which Bubble Tea Shop Is Better?

	ϑ_T	ϑ_C
T	1	0
C	0	1

Is Tiger Sugar or Chun Shui Tang better?

- The possible states are $\Theta = \{\vartheta_T, \vartheta_C\}$ with prior $\mu_0 \simeq \mu_0(\vartheta_T) = \frac{1}{2}$.
- Each player t receives a binary signal Y_t in $\mathcal{Y} = \{y_T, y_C\}$ with

$$\pi(y_T \mid \vartheta_T) = \pi(y_C \mid \vartheta_C) = p > \frac{1}{2}.$$

- Each player can go to (T)iger Sugar or to (C)hun Shui Tang.
- Suppose players mix 50–50 if they are indifferent.

First Two Players

First player:

- After receiving signal $y_0 = y_T$, action T is a best response since

$$\nu_0(y_T) = \frac{p\mu_0}{p\mu_0 + (1-p)(1-\mu_0)} = p > \frac{1}{2}.$$

- Similarly, action C is uniquely optimal after signal $y_0 = y_C$.

Second player:

- Player 0's action is perfectly revealing, hence player 1's prior is

$$\mu_1 = p1_{\{A_0=T\}} + (1-p)1_{\{A_0=G\}}.$$

- If player 1's signal coincides with player 0's, then player 1 follows suit.
- After a conflicting signal, player 1 is willing to mix since, for example,

$$\nu_1(T, y_C) = \frac{(1-p)\mu_1(T)}{(1-p)\mu_1(T) + p(1-\mu_1(T))} = \frac{1}{2}.$$

Third Player

Identical actions chosen by players 0 and 1:

- History $h_2 = (T, T)$ can arise in one of two ways:
  - Players 0 and 1 both observed signal y_T.
  - Player 0 observed signal y_T and player 1 observed signal y_C.
- Player 2's prior beliefs are

$$\mu_2(T,T) = \frac{\left(\frac{1}{2}(1-p)+p\right)p}{\left(\frac{1}{2}(1-p)+p\right)p + \left(\frac{1}{2}p + (1-p)\right)(1-p)} = \frac{p+p^2}{2(1-p+p^2)}.$$

- Suppose player 2's signal is y_C. The posterior is equal to

$$\nu_2(T,T,y_C) = \frac{(1-p)\mu_2(T,T)}{(1-p)\mu_2(T,T) + p(1-\mu_2(T,T))} = \frac{p+1}{3} > \frac{1}{2}.$$

- Player 2 follows suit after (T, T) or (C, C) regardless of his/her signal.

Third and Other Players

Different actions chosen by players 0 and 1:

- After history $h_2 = (T, C)$, player 2's prior beliefs are

$$\mu_2(T, C) = \frac{\frac{1}{2}(1-p)p}{\frac{1}{2}(1-p)p + \frac{1}{2}p(1-p)} = \frac{1}{2}.$$

- Therefore, player 2 follows her private signal after (T, C) and (C, T).

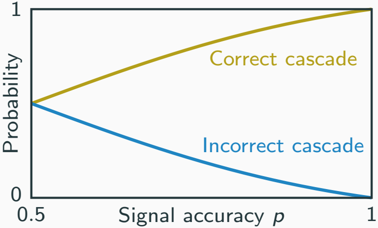
Forward induction:

- Even numbered players behave as player 2.
- Odd numbered players behave as player 1.

Informational cascades:

- An informational cascade occurs if players disregard their own signal.
- No more learning occurs after that and players herd on this action.

Which Bubble Tea Shop Is Better?



Which bubble tea shop is better?

- Because signals are of equal quality, no single signal realization can overturn a history with two more T than C or vice versa.
- Herding may occur as early as the third customer.
- An informational cascade occurs with probability 1.
- Learning is complete in equilibrium with probability $\frac{p+p^2}{2(1-p+p^2)}$.

Releasing Public Information

Do Ad Campaigns Help?

Advertising your bubble tea shop:

- After taking ECON 7219, you know how important early customers are and you run an ad campaign when you open your bubble tea shop.
- Because it is visible to anyone, the ad campaign is a public signal.
- Under what conditions does a public signal improve learning?

Releasing Public Information

Lemma 13.2

1. *The release of public information before the first player's decision can make some individuals worse off from an ex-ante perspective.*
2. *The release of public information after an informational cascade has occurred makes every player weakly better off.*

Proof:

- Statement 2. follows directly from Blackwell's theorem since, without the public signal, no more information is generated in the cascade.
- Statement 1. is shown with a counterexample.

Adopting a New Technology

	ϑ_I	ϑ_N
A	$1-c$	$-c$
N	0	0

Payoffs

Adopting a new technology:

- A new technology may be an improvement over the old technology (ϑ_I), in which case it yields a utility of 1, or it is not (ϑ_N).
- Players can either (A)dopt the new technology or (N)ot adopt it.
- Switching to the new technology incurs a cost $c < 1$.
- After observing others, should you make the transition?

Harmful Public Information

	y_G	y_M	y_B
ϑ_I	0.1	0.7	0.2
ϑ_N	0.05	0.55	0.4

Private signal

	s_G	s_B
ϑ_I	0.51	0.49
ϑ_N	0.49	0.51

Public signal

Setup:

- Everybody observes a private signal Y_t with values in $\{y_G, y_M, y_B\}$.
- A public signal S with values in $\{s_G, s_B\}$ is released in the beginning.
- While the public signal improves everybody's prior, it may make the adoption by the first player less informative.
- It turns out that for $c = 0.555$, the latter effect dominates and player 2's ex-ante expected value is lower with the public signal.

General Model

How to Interpret Deviations?

Going against the herd:

- Suppose the last 1 million individuals went to Tigar Sugar.
- Then one individual decides to go to Chun Shui Tang.
- Why has this deviation occurred?

Possible Explanations for Deviations

Possible explanations:

- That individual has more information than any of us and knows that despite the others' actions, the competition is better.
- It was an accident / an irrational decision.
- That individual simply has different preferences.

Incorporating these in the model:

- For any public beliefs μ_t, there has to be some signal realization that overturns the entire information before it.
- We need to ensure there is no uniformly most powerful signal.
- We model different preferences through players' types in $\mathcal{T}_p$.
- Irrationality can be expressed through commitment types in $\mathcal{T}_c$.

General Model

Players' types:

- Let $\mathcal{A}$ be a finite set of available actions and suppose that there exists a commitment type $\tau_a \in \mathcal{T}_c$ for every action $a \in \mathcal{A}$.
- There are finitely many payoff types $\tau \in \mathcal{T}_p$, earning utility $u(a, \vartheta, \tau)$.
- Types are independent and distributed according to a prior $\kappa \in \Delta(\mathcal{T})$, where we denote by $\mathcal{T} = \mathcal{T}_p \cup \mathcal{T}_c$ the set of all types.
- We suppose that types are drawn i.i.d. from κ.

Players' beliefs:

- The types' beliefs (before observing their private signal) coincide completely about θ as well as the other players' types.
- Thus, as with homogeneous preferences, we can work with the public belief process $\mu = (\mu_t)_{t \geq 0}$, which player t updates to $\nu_t(\mu_t, y_t)$.

Unboundedly Informative Signals

Definition 13.3

Suppose the private signals have conditional densities $f(y \mid \vartheta_*)$ and $f(y \mid \vartheta)$.

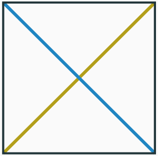
1. Player t's private beliefs solely from observing Y_t are

$$q(y_t) := \frac{f(y_t \mid \vartheta_*)}{f(y_t \mid \vartheta_*) + f(y_t \mid \vartheta)}.$$[1]

2. Signal Y_t is unboundedly informative if $\operatorname{conv}\big(\operatorname{supp} q(Y_t)\big) = [0, 1]$.

Example:

- A signal Y_t distributed on $(0, 1)$ with conditional densities $f(y \mid \vartheta_*) = 2y$ and $f(y \mid \vartheta) = 2 - 2y$.
- Then a signal arbitrarily close to 1 is arbitrarily more likely under ϑ_* than ϑ.



[1] This updating formula holds only because we have assumed $\mu_0(\vartheta_*) = \mu_0(\vartheta) = \frac{1}{2}$.

Updating Public Beliefs with Private Signals

Updating beliefs:

- Because we have assumed a uniform prior μ_0, we deduce

$$\nu_t(\mu_t, y_t) = \frac{f(y_t \mid \vartheta_*)\mu_t}{f(y_t \mid \vartheta_*)\mu_t + f(y_t \mid \vartheta)(1-\mu_t)}$$

$$= \frac{q_t \mu_t}{q_t \mu_t + (1-q_t)(1-\mu_t)} =: \nu(\mu_t, q_t),$$

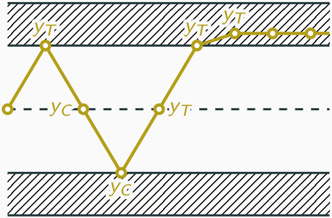
  where we denote by q_t the realized private beliefs at time t.

- Note that $\nu_t(\mu_t, q_t)$ is increasing in both μ_t and q_t.

Eliminating signals:

- We can directly work with private beliefs instead of signals.
- Let F_q denote the distribution function of q_t.

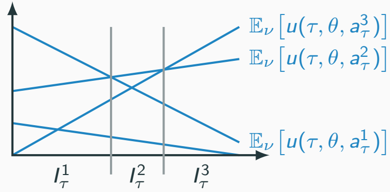
Information Cascades



Which bubble tea shop is better?

- An information cascade occurred if either two more positive signals were observed than negative signals or vice versa.
- Then, the same action was optimal after any private signal.
- Since public beliefs $\mu_t = p$ are updated to $\frac{1}{2}$ after a negative signal, public beliefs above p will always induce $a = T$ as best response.
- A cascade must start if $\mu_t > p$ or $\mu_t < 1 - p$.

Partition of Action Choices



Optimal choice of actions:

- The expected utility of type τ with posterior beliefs ν is

$$\mathbb{E}_\nu[u(\tau,\vartheta,a)] = \nu u(\tau,\vartheta_*,a) + (1-\nu)u(\tau,\vartheta,a).$$

- There exist actions $\mathcal{A}(\tau) = \left\{a_\tau^1,\ldots,a_\tau^{m_\tau}\right\}$ and an increasing sequence of intervals $I_\tau^1,\ldots,I_\tau^{m_\tau}$ that cover $[0,1]$ and touch at end points only, such that a_τ^k is a best response, given beliefs ν, if and only if $\nu \in I_\tau^k$.
- For any tie-braking convention, the equilibrium is unique.

Cascade Sets

Mapping private beliefs to actions:

- Since $\nu_t(\mu_t, q_t)$ is increasing, for any τ and μ_t there exist thresholds

$$0 = q_\tau^0(\mu_t) \leq q_\tau^1(\mu_t) \leq \ldots \leq q_\tau^{m_\tau}(\mu_t) = 1$$

such that a_τ^k maximizes τ's utility if and only if $q_t \in \left[q_\tau^{k-1}(\mu_t), q_\tau^k(\mu_t)\right]$.

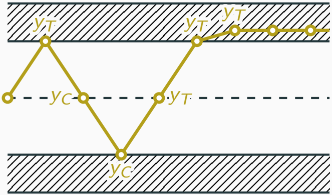
Cascade sets:

- If $q_t \in \left(q_\tau^{k-1}(\mu_t), q_\tau^k(\mu_t)\right]$ for any realization of q_t, then type τ will choose a_τ^k regardless of their private information.
- This happens if μ_t lies in type τ's cascade set of action a_τ^k

$$\mathcal{M}_\tau^k := \left\{\mu \in [0,1] \;\middle|\; \text{supp}(F_q) \subseteq \left[q_\tau^{k-1}(\mu_t), q_\tau^k(\mu_t)\right]\right\} \subseteq \text{int } I_\tau^k.$$

- Observing type τ reveals no new information if μ_t lies $\mathcal{M}_\tau = \bigcup \mathcal{M}_\tau^k$.
- Learning stops completely in $\mathcal{M} := \bigcap_{\tau \in \mathcal{T}_p} \mathcal{M}_\tau$.

Cascade Sets



Which bubble tea shop is better?

- Posteriors are partitioned into intervals $\mathcal{I}_\tau^1 = \left[0, \frac{1}{2}\right]$ and $\mathcal{I}_\tau^2 = \left[\frac{1}{2}, 1\right]$, on which actions $a_\tau^1 = C$ and $a_\tau^2 = T$, respectively, are optimal.
- Private signals are boundedly informative with $\text{supp}(F_q) = [1-p, p]$.
- Solving $\frac{q\mu}{q\mu+(1-q)(1-\mu)} = \frac{1}{2}$ for q we obtain $q_\tau^1(\mu_t) = 1 - \mu_t$.
- Therefore, $\text{supp}(F_q) \subseteq \left[q_\tau^1(\mu_t), 1\right]$ if and only if $1 - p \geq 1 - \mu_t$.
- In particular, $\mathcal{M}_\tau^2 = [p, 1]$ is the cascade set for action T.

Cascade Sets

Lemma 13.4

1. *If the signal is boundedly informative, then there exist* $0 < \underline{\mu} < \bar{\mu} < 1$ *such that* $\mathcal{M}_\tau^1 = [0, \underline{\mu}]$ *and* $\mathcal{M}_\tau^{m_\tau} = [\bar{\mu}, 1]$*.*
2. *If signals are unboundedly informative, then* $\mathcal{M}_\tau^1 = \{0\}$*,* $\mathcal{M}_\tau^{m_\tau} = \{1\}$*, and all other* $\mathcal{M}_\tau^k$ *are empty.*

Boundedly informative:

- If public beliefs are sufficiently certain of one outcome or the other, learning stops forever.

Unboundedly informative:

- All non-certain public beliefs can be overturned by a single signal.

Proof of Lemma 13.4

Boundedly informative:

- Let $\underline{q}$ and $\bar{q}$ be the lower and upper bound of supp F_q.
- Observe that $\text{supp}(F_q) \subseteq \left[q_\tau^{k-1}(\mu), q_\tau^k(\mu)\right]$ holds if and only if

$$\underline{q} \geq q_\tau^{k-1}(\mu) \qquad \text{and} \qquad \bar{q} \leq q_\tau^k(\mu). \tag{1}$$

- Therefore, $\bar{\mu}$ and $\underline{\mu}$ are given by $\underline{q} = q_\tau^{m_\tau - 1}(\bar{\mu})$ and $\bar{q} = q_\tau^1(\underline{\mu})$, i.e.,

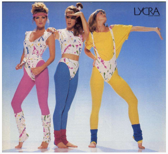
$$\frac{\underline{q}\bar{\mu}}{\underline{q}\bar{\mu} + (1-\underline{q})(1-\bar{\mu})} = \nu_\tau^{m_\tau - 1} \qquad \frac{\bar{q}\underline{\mu}}{\bar{q}\underline{\mu} + (1-\bar{q})(1-\underline{\mu})} = \nu_\tau^1. \tag{2}$$

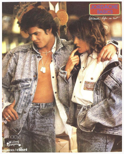
Unboundedly informative:

- If $\underline{q} = 0$ and $\bar{q} = 1$, then (1) implies $q_\tau^{k-1}(\mu) = 0$ and $q_\tau^k(\mu) = 1$.
- By (2), this is possible only if $k = 1$ and $\mu = 0$ or $k = m_\tau$ and $\mu = 1$.

Fashion Trends





Temporary cascades:

- Fashion trends persist for a while before they shift drastically.
- While there is a positive payoff externality for dressing like others, we can model changes of fashion trends as an information cascade.
- Consider an action a_τ^k and public beliefs μ_t with

$$[\varepsilon, 1-\varepsilon] \subseteq \left[q_\tau^{k-1}(\mu_t), q_\tau^k(\mu_t)\right] \subseteq [0,1] = \mathrm{supp}(F_q).$$

- Individuals with private beliefs in $[\varepsilon, 1-\varepsilon]$ best follow the trend.
- To against the herd, an individual needs a very informative signal.

Overturning Principle

Lemma 13.5

Suppose that preferences are homogeneous (= a single payoff type) and there is no noise (= no commitment types). Then after any deviation from a herd, uninformed individuals in any equilibrium follow suit.

Interpretation:

- Without heterogeneous preferences or noise, a deviation can only be attributed to the deviating individual having better information.
- With unboundedly informative signals, beliefs convergence to 0 or 1.

Proof:

- Because deviating from the herd to some action a_τ^k was an equilibrium decision, public beliefs shift to I_τ^k immediately.
- For an uninformed individual, the posterior equal the prior $\nu = \mu \in I_\tau^k$.

Check Your Understanding

True or false:

1. For any type τ, only the extremal actions a_τ^1 and $a_\tau^{m_\tau}$ can be played in a cascade set.
2. If there are more than two states in Θ, then the equilibrium is no longer essentially unique.
3. If signals are boundedly informative, an information cascade will occur with probability 1 in finite time.
4. If signals are boundedly informative, then a herd on the wrong action will arise with positive probability.

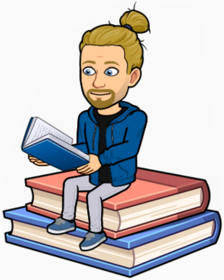
Scan me

Short-Answer Question:

5. What *is* the best bubble tea shop in Taiwan?

Literature

- S. Bikhchandani, D. Hirshleifer, and I. Welch: A Theory of Fads, Fashion, Custom, and Cultural Change as Information Cascades, Journal of Political Economy, **100** (1992), 992–1026
- A.V. Banerjee: A Simple Model of Herd Behavior, Quarterly Journal of Economics, **107** (1992), 797–817
- L. Smith and P. Sørensen: Pathological Outcomes of Observational Learning, Econometrica, **68** (2000), 371–398



Asymptotic Learning

General Result

General result?

- The overturning principle does not hold if there is noise or players attribute the deviation to somebody with different preferences.
- Can herding on the wrong action occur with informative signals?

Outline:

- Beliefs must be converge because they form a martingale.
- Belief convergence must imply convergence of actions in distribution.
- Because best responses are unique for almost every posterior, we must have action convergence for each type.
- For a single rational type, this will imply convergence to a cascade set.

Convergence of Beliefs

Lemma 13.6

For any strategy profile σ, let $\widehat{P}_\sigma := P_\sigma(\theta = \vartheta_*)$ denote the conditional probability measure that ϑ_* is the true state. For any strategy profile σ, the belief process μ converges $\widehat{P}_\sigma$-a.s. to a $(0, 1]$-valued random variable μ_∞.

Remarks:

- We already know from last week that the beliefs converge.
- The important addition is that beliefs cannot converge to 0 when ϑ_* is the true state, that is, beliefs cannot be completely wrong.

Proof

Convergence of likelihood ratio:

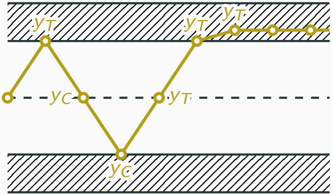
- Since $\mu_0 \in (0,1)$ and no signal is completely informative, we can define the likelihood ratio process $\lambda = (\lambda_t)_{t \geq 0}$ by setting $\lambda_t := \frac{1-\mu_t}{\mu_t}$.
- Analogously to the proof of Lemma 12.4, λ is a $\widehat{P}_\sigma$-martingale.
- By the martingale convergence theorem, λ converges $\widehat{P}_\sigma$-a.s. to a $[0,\infty]$-valued random variable λ_∞.[2]

Convergence to finite likelihood ratio:

- If $\widehat{P}_\sigma(\lambda_\infty = \infty) > 0$, it follows that $\widehat{\mathbb{E}}_\sigma[\lambda_\infty] = \infty$ because $\lambda_\infty \geq 0$.
- This is a contradiction to $\widehat{\mathbb{E}}_\sigma[\lambda_\infty] = \lambda_0 = \frac{1-\mu_0}{\mu_0} = 1$.
- We conclude that $\lambda_\infty < \infty$ and $\mu_\infty = \frac{1}{1+\lambda_\infty} > 0$ $\widehat{P}_\sigma$-a.s.

[2] For the martingale convergence theorem, it is sufficient that the martingale is bounded from one side, e.g., $\lambda \geq 0$.

Describing the Equilibrium Path



Which bubble tea shop is better?

- The choice of action dictates in which direction the public beliefs move.
- The amount the public beliefs shift depends on the current belief and the likelihood of the observed action, that is,

$$\mu_{t+1}(a_t; \vartheta) = \frac{P_{\mu_t,\sigma}(A_t = a_t \,|\, \theta = \vartheta)\mu_t(\vartheta)}{P_{\mu_t,\sigma}(A_t = a_t)} =: \varphi(a_t, \mu_t; \vartheta).$$

- The process $(a_{t-1}, \mu_t)_{t \geq 0}$ is a Markov chain!

Equilibrium Path as a Markov Chain

Choice of actions by payoff types:

- Given beliefs μ_t, in equilibrium type τ takes action a_τ^k with probability

$$\rho_\tau(a_\tau^k \,|\, \mu_t, \vartheta) := F_q(q_\tau^k(\mu_t) \,|\, \vartheta) - F_q(q_\tau^{k-1}(\mu_t) \,|\, \vartheta).$$

- Moreover, action $a \notin \mathcal{A}(\tau)$ is taken with probability $\rho_\tau(a \,|\, \mu_t, \vartheta) = 0$.

Markov chain:

- Recall that $\kappa \in \Delta(\mathcal{T})$ is the prior distribution over types.
- State (a_{t-1}, μ_t) transitions to state $(a_t, \varphi(a_t, \mu_t))$ with probability

$$P_{\mu_t,\sigma}(A_t = a_t \,|\, \theta = \vartheta) := \kappa(\tau_a) + \sum_{\tau \in \mathcal{T}_p} \kappa(\tau)\rho_\tau(a \,|\, \mu_t, \vartheta). \qquad (3)$$

- Note that $P_{\mu_t,\sigma}(A_t = a_t \,|\, \theta = \vartheta)$ and hence $\varphi(a_t, \mu_t)$ is continuous in μ_t if $F_q(\,\cdot \,|\, \vartheta)$ admits a density for each $\vartheta \in \Theta$.

Stationary Point of Markov Chain

Proposition 13.7

Suppose that $F_q(\cdot \mid \vartheta)$ admits a density for each $\vartheta \in \Theta$. In the equilibrium σ, for any $\mu \in \text{supp}(\mu_\infty)$ and every type τ, we must have

$$P_{\mu_t,\sigma}(A_t = a_t) = 0 \quad \text{or} \quad \varphi(a,\mu) = \mu \quad \text{for every } a \in \mathcal{A}. \tag{4}$$

Interpretation:

- In the limit, an action is either not chosen or it reveals nothing about the underlying state.

Idea of proof:

- Since the beliefs converge almost surely, $\varphi(a,\mu) = \mu$ must hold for almost every realization μ of μ_∞, i.e., for almost every $\mu \in \text{supp}(\mu_\infty)$.
- Continuity will give us the result for every $\mu \in \text{supp}(\mu_\infty)$.

Proof of Proposition 13.7

Setting up a contradiction:

- Suppose there exists $\mu_* \in \text{supp}(\mu_\infty)$ such that (4) is violated.
- By continuity, for every $\varepsilon > 0$ there exists a neighborhood U of μ_* with $P_{\mu,\sigma}(A_t = a) > \varepsilon$ and $|\varphi(a,\mu) - \mu| > \varepsilon$ for every $\mu \in U$. Define

$$U_\varepsilon := U \cap \left(\mu_* - \frac{\varepsilon}{2}, \mu_* + \frac{\varepsilon}{2}\right).$$

Beliefs cannot converge in U_ε:

- For any ω with $\mu_\infty(\omega) = \mu_*$, there exists $t_0(\omega)$ sufficiently large such that $\mu_t(\omega)$ lies in U_ε for any $t \geq t_0(\omega)$.
- If $\mu_t \in U_\varepsilon$, then $\mu_{t+1} \notin U$ with probability ε. Thus

$$P_\sigma(\mu_t \in U_\varepsilon \text{ for all } t \geq t_0 \,|\, \mu_\infty = \mu_*) = \lim_{t\to\infty} \varepsilon^t = 0.$$

Homogeneous Preferences

Theorem 13.8

Suppose that preferences are homogeneous, i.e., $\mathcal{T}_p = \{\tau\}$.

1. *In any equilibrium* σ*, the belief process* μ *converges* $\widehat{P}_\sigma$*-a.s. to a random variable* μ_∞ *with supp*$(\mu_\infty) \subseteq (0,1] \cap \mathcal{M}$.
2. *If the signals are unboundedly informative, then* $\mu_\infty = 1$ $\widehat{P}_\sigma$*-a.s.*

Interpretation:

- In equilibrium, the public beliefs must converge to a cascade set.
- If signals are unboundedly informative, learning is complete.
- Observe that the presence of noise has no impact asymptotically.

Proof of Statement 2.:

- This follows from Statement 1. and Lemma 13.6.

Proof of Theorem 13.8

Suppose the contrary is true:

- Fix any $\mu \in \operatorname{supp}(\mu_\infty)$ and suppose that it is not in $\mathcal{M}$.
- Then different actions must be played after different private signals.
- In, particular, there exists an action a_* such that either:
  - a_* is played only after $p \geq \frac{1}{2}$, hence $\rho_\tau(a_* \mid \mu, \vartheta_*) > \rho_\tau(a_* \mid \mu, \vartheta_\times)$.
  - a_* is played only after $p \leq \frac{1}{2}$, hence $\rho_\tau(a_* \mid \mu, \vartheta_\times) > \rho_\tau(a_* \mid \mu, \vartheta_*)$.

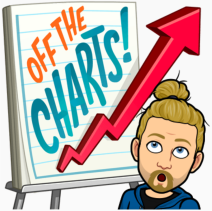
Derive a contradiction:

- By Proposition 13.7, we know $\varphi(a_*, \mu) = \mu$, hence

$$\mu(\vartheta) = \frac{P_{\mu,\sigma}(A_t = a_t \mid \theta = \vartheta)\mu(\vartheta)}{\sum_{\vartheta' \in \Theta} P_{\mu,\sigma}(A_t = a_t \mid \theta = \vartheta')\mu(\vartheta')} \qquad \text{for all } \vartheta \in \Theta.$$

- It follows that $P_{\mu,\sigma}(A_t = a_t \mid \theta = \vartheta_*) = P_{\mu,\sigma}(A_t = a_t \mid \theta = \vartheta_\times)$ and, hence, $\rho_\tau(a_* \mid \mu, \vartheta_*) = \rho_\tau(a_* \mid \mu, \vartheta_\times)$ by (3). This is a contradiction.

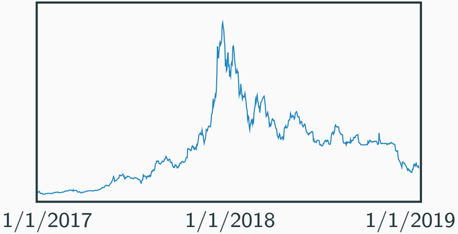
Asset Price Bubbles



An informational asset price model:

- States are $\Theta = \{\vartheta_U, \vartheta_O, \vartheta_I\}$: the asset is undervalued, overvalued, or it is valued at its intrinsic value.
- Investors decide whether to (B)uy, (N)ot buy, or (S)ell the asset.
- Slight model departure: state transitions based on underlying actions.
- Moreover, the available actions and utility depends on asset holdings and the utility of investing depends on other players' actions.

Bubbles and Crashes



Herds:

- Investors receive an (often very uninformative) private signal.
- A bubble is a herd on B, whereas a crash is a herd on S.
- Investor's preferences are homogeneous, hence we know that, eventually, the market corrects itself and the asset will be priced correctly.
- Nevertheless, riding the wrong herd can be very costly before that.

Source: https://www.coindesk.com/price/bitcoin

Bitcoin: All-Time High

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Tech

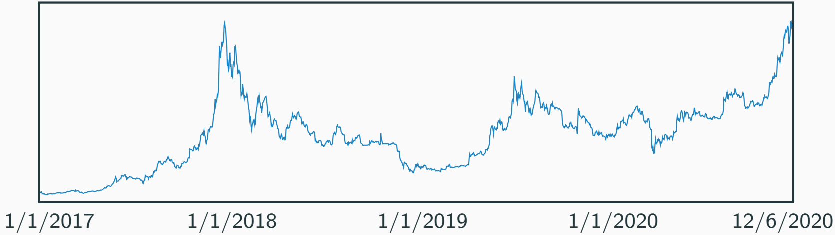
Bitcoin peaks at record high close to $20,000

4 days ago

Bitcoin has traded at its highest value to date, reaching $19,920.53 (£14,821) according to data-provider Coindesk.

Source: https://www.bbc.com/news/technology-55145444

Another Bitcoin Bubble?



Is this another bubble or is it for real?

- Bitcoin surpassed the peak of 2017 on December 2, 2020 at $19,920.53.
- Unfortunately, my signal is just as uninformative as yours.
- Nobel laureates Richard Thaler, Joseph Stiglitz, Robert Merton, Robert Shiller, and Paul Krugman are all outspoken against Bitcoin.
- Some additional analysis on Reuters and the Financial Times.

Source: https://www.coindesk.com/price/bitcoin

Heterogeneous Preferences

Why Did We Need Homogeneity of Preferences?

Heterogeneous preferences:

- For any $\mu \in \operatorname{supp}(\mu_\infty)$ and any action a played with positive probability, we still have $P_{\mu,\sigma}(A_t = a \,|\, \theta = \vartheta_*) = P_{\mu,\sigma}(A_t = a \,|\, \theta = \vartheta_\times)$.
- However, we can no longer deduce $\rho_\tau(a \,|\, \mu, \vartheta_*) = \rho_\tau(a \,|\, \mu, \vartheta_\times)$ since

$$P_{\mu,\sigma}(A_t = a_t \,|\, \theta = \vartheta) = \kappa(\tau_a) + \sum_{\tau \in \mathcal{T}_p} \kappa(\tau)\rho_\tau(a \,|\, \mu, \vartheta).$$

- Instead, we conclude that in aggregate, the payoff types play each action equally likely in any underlying state.

Confounding Outcome

Definition 13.9

Let $\rho_\tau(a \mid \mu, \vartheta)$ denote the probability with which payoff type τ plays a in equilibrium, given μ and ϑ. A confounding outcome is a public belief $\mu \notin \mathcal{M}$ such that for any two $\vartheta, \vartheta' \in \Theta$,

$$\sum_{\tau \in \mathcal{T}_p} \kappa(\tau)\rho_\tau(a \mid \mu, \vartheta) = \sum_{\tau \in \mathcal{T}_p} \kappa(\tau)\rho_\tau(a \mid \mu, \vartheta'). \qquad (5)$$

We denote by $\mathcal{K}$ the set of confounding outcomes.

Interpretation:

- Action choices depend on realizations of private signals.
- However, without knowing the active player's type, no inferences about the underlying state can be drawn.

Heterogeneous Preferences

Theorem 13.10

Suppose that preferences are heterogeneous, i.e., $|\mathcal{T}_p| \geq 2$.

1. *In any equilibrium σ, the belief process μ converges $\widehat{P}_\sigma$-a.s. to a random variable μ_∞ with supp$(\mu_\infty) \subseteq (0,1] \cap (\mathcal{M} \cup \mathcal{K})$.*

Moreover, for generic model parameters:

2. *At any $\mu \in \mathcal{K}$, at most two actions are played by each type.*
3. *If belief distributions are discrete or if signals are unboundedly informative with $|\mathcal{A}| > 2$, then $\mathcal{K} = \emptyset$.*

Interpretation:

- For confounded learning to exist robustly, it is necessary that either:
  - The signal is boundedly informative and continuous.
  - The signal is unboundedly informative and $|\mathcal{A}| = 2$.

Proof of Theorem 13.10

Proof of Statement 1.:

- In the same way as the proof of Theorem 13.8.

Proof of Statement 2.:

- Suppose that m actions are taken at a confounding outcome μ.
- The definition of a confounding outcome (5) is a system of $m-1$ linear equations in the single unknown μ.
- For generic parameters, this is not solvable unless $m = 2$.

Proof of Statement 3.:

- If signals are unboundedly informative, all actions are taken with positive probabilty at a confounding outcome, hence we need $|\mathcal{A}| = 2$ by 2.
- If the signal is discrete, then each side of (5) only assumes countably many values. For generic choices of $\kappa(\tau)$, it does not have a solution.

Example of a Confounding Outcome

	ϑ_T	ϑ_C
T	1	0
C	0	1

τ_C

	ϑ_T	ϑ_C
T	0	1
C	1	0

τ_N

Setup:

- There are two states $\Theta = \{\vartheta_T, \vartheta_C\}$, indicating whether Tiger Sugar or Chun Shui Tang makes the creamier bubble tea.
- Type τ_C likes creamy bubble teas, type τ_N does not.
- Each type can go to (T)iger Sugar or (C)hun Shui Tang.
- Suppose that signals are unboundedly informative with conditional densities given by $f(y \mid \vartheta_T) = 2y$ and $f(y \mid \vartheta_C) = 2 - 2y$.

Literature

- M.K. Brunnermeier: Asset Pricing under Asymmetric Information: Bubbles, Crashes, Technical Analysis, and Herding, Chapter 5, Oxford University Press, 2001
- H. Hong and J.C. Stein: A Unified Theory of Underreaction, Momentum Trading, and Overreaction in Asset Markets, Journal of Finance, **54** (1999), 2143–2184
- L. Smith and P. Sørensen: Pathological Outcomes of Observational Learning, Econometrica, **68** (2000), 371–398
- D. Acemoglu, M.A. Dahleh, I. Lobel, and A. Ozdaglar: Bayesian Learning in Social Networks, Review of Economic Studies, **78** (2011), 1201–1236