Lobbying with a continuum of policies

The politician's utility $\mathbb{E}_\nu[u(\theta, a)] = -a^2 + 2a\mathbb{E}_\nu[\theta] - \mathbb{E}_\nu[\theta^2]$ is maximized at $\hat{a}(\nu) = \mathbb{E}_\nu[\theta]$. Note that $\mathbb{E}_\nu[\theta] = \int_0^1 x \,\mathrm{d}\nu(x)$ does not have expectation ν since ν is a distribution on $[0,1]$. Substituting $a_*(\theta) = \lambda\theta + (1-\lambda)\vartheta$ into the sender's interim expected utility yields

$$\begin{aligned}
\hat{v}(\nu) = \mathbb{E}_\nu[v(\hat{a}(\nu), \theta)] &= -\mathbb{E}_\nu\left[\mathbb{E}_\nu[\theta]^2 + \lambda^2\theta^2 + (1-\lambda)^2\vartheta_*^2\right] \\
&\quad + 2\mathbb{E}_\nu[\lambda\theta\mathbb{E}_\nu[\theta] + (1-\lambda)\vartheta_*\mathbb{E}_\nu[\theta] - \lambda(1-\lambda)\vartheta_*\mathbb{E}_\nu[\theta]] \\
&= (2\lambda - 1)\mathbb{E}_\nu[\theta]^2 + (1-\lambda)^2\vartheta_*\mathbb{E}_\nu[\theta] - (1-\lambda)^2\vartheta_*^2 + \lambda^2\mathbb{E}_\nu\left[\theta^2\right].
\end{aligned}$$

By Lemma 8.9, without loss of generality, we can restrict attention to finitely supported Bayes-plausible distributions ψ. For any any function f, linearity of the integral implies

$$\mathbb{E}_\psi[\mathbb{E}_\nu[f(\theta)]] = \sum_{\nu\in\mathrm{supp}(\psi)} \psi(\nu) \int_0^1 f(x)\,\mathrm{d}\nu(x) = \int_0^1 f(x)\,\mathrm{d}\left(\sum_{\nu\in\mathrm{supp}(\psi)} \psi(\nu)\nu(x)\right) = \int_0^1 f(x)\,\mathrm{d}\mu(x).$$

In particular, $c(\mu) := (1-\lambda)^2\vartheta_*\mathbb{E}_\nu[\theta] - (1-\lambda)^2\vartheta_*^2 + \lambda^2\mathbb{E}_\nu[\theta^2]$ does not depend on ψ other than $\mathbb{E}_\psi[\nu] = \mu$. Therefore, the sender's maximization problem becomes

$$\max_{\psi\in\Delta(\Delta(\Theta))\,:\,\mathbb{E}_\psi[\nu]=\mu} \mathbb{E}_\psi[\hat{v}(\nu)] = c(\mu) + \max_{\psi\in\Delta(\Delta(\Theta))\,:\,\mathbb{E}_\psi[\nu]=\mu} (2\lambda-1)\mathbb{E}_\nu[\theta]^2.$$

Let us next show that $\mathbb{E}_\nu[\theta]^2$ is convex in ν. To do so, we set $\nu_t = t\nu_1 + (1-t)\nu_0$ for $\nu_0, \nu_1 \in \Delta(\Theta)$ and any $t \in [0,1]$. Then

$$\begin{aligned}
\mathbb{E}_{\nu_t}[\theta]^2 &= t^2\mathbb{E}_{\nu_1}[\theta]^2 + 2t(1-t)\mathbb{E}_{\nu_1}[\theta]\mathbb{E}_{\nu_0}[\theta] + (1-t)^2\mathbb{E}_{\nu_0}[\theta]^2 \\
&\leq t^2\mathbb{E}_{\nu_1}[\theta]^2 + t(1-t)\left(\mathbb{E}_{\nu_0}[\theta]^2 + \mathbb{E}_{\nu_1}[\theta]^2\right) + (1-t)^2\mathbb{E}_{\nu_0}[\theta]^2 \\
&= t\mathbb{E}_{\nu_1}[\theta]^2 + (1-t)\mathbb{E}_{\nu_0}[\theta]^2,
\end{aligned}$$

where we have used that $0 \leq \left(\mathbb{E}_{\nu_0}[\theta] - \mathbb{E}_{\nu_1}[\theta]\right)^2 = \mathbb{E}_{\nu_0}[\theta]^2 + \mathbb{E}_{\nu_1}[\theta]^2 - 2\mathbb{E}_{\nu_0}[\theta]\mathbb{E}_{\nu_1}[\theta]$. Convexity of $\mathbb{E}_\nu[\theta]^2$ implies that:

- If $\lambda < \frac{1}{2}$, then $\hat{v}$ is concave, hence $V = \hat{v}$. The completely uninformative signal is optimal.
- If $\lambda > \frac{1}{2}$, then $\hat{v}$ is convex. The completely informative signal is optimal.
- Finally, if $\lambda = \frac{1}{2}$, then any signal is optimal.