8. Mechanism Design III

ECON 7219 – Games With Incomplete Information

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Vickrey-Clarke-Groves Mechanism

Definition 7.16

A Vickrey-Clarke-Groves mechanism (or VCG mechanism) is a direct mechanism $\Gamma = (\mathcal{T}_1, \dots, \mathcal{T}_n, (q, p))$ such that $q(\vartheta)$ is ex-post efficient and

$$p_i(\vartheta) = h_i(\vartheta_{-i}) - \sum_{j \neq i} v_j(q(\vartheta), \vartheta_j). \tag{1}$$

for every player i, where $h_i : \Theta_{-i} \to \mathbb{R}$ does not depend on i's valuation.

Remark:

- Second term in (1) aligns social preferences with individual preferences.
- First term in (1) allows us to adjust payments and, hence, the surplus, without affecting incentives for truthful reporting.
- IR-VCG mechanism maximizes $h_i(\vartheta_{-i})$ subject to individual rationality.

IR-VCG Mechanism

Optimality of IR-VCG Mechanism:

- In many settings, the IR-VCG mechanism is the optimal mechanism that implements an ex-post efficient social state in the sense that:
    - It is dominant-strategy implementable.
    - It maximizes the ex-ante expected revenue among all such mechanisms.
- We have shown this for one-dimensional types.

Attaining a balance budget:

- If the IR-VCG mechanism runs a deficit, we have to allow either that:
    - Payments are burned for some ϑ.
    - Sometimes the implemented social state q is inefficient.
- If the IR-VCG mechanism runs an expected surplus, we can balance the budget, but we have to give up dominant-strategy implementability.

Budget Balance

Achieving a Balanced Budget

Proposition 8.1

Suppose types are independent and admit a common prior. If a direct incentive-compatible mechanism $\Gamma : (\mathcal{T}_1, \ldots, \mathcal{T}_n, h)$ with $h = (q, p)$ runs an ex-ante expected surplus, then $\Gamma' = (\mathcal{T}_1, \ldots, \mathcal{T}_n, (q, p^B))$ with

$$p_i^B(\tau) = \mathbb{E}_{\tau_i}[p_i(T)] - \mathbb{E}_{\tau_{mod(i,n)+1}}\left[p_{mod(i,n)+1}(T)\right] + \mathbb{E}\left[p_{mod(i,n)+1}(T)\right] - \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[p_j(T)].$$

is an ex-post budget balanced direct mechanism. Moreover:

1. Γ' *is Bayesian incentive-compatible,*
2. Γ' *is weakly preferred to* Γ *by every individual.*

Interpretation of Payments

Redistributing surplus:

- The expected surplus is distributed to the n individuals via the term

$$-\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[p_j(T)].$$

- However, doing so only balances the budget ex ante.

Ex-post budget balance:

- Together with $\mathbb{E}_{\tau_i}[p_i(T)]$, the second term in

$$\mathbb{E}\left[p_{\mathrm{mod}(i,n)+1}(T)\right] - \mathbb{E}_{\tau_{\mathrm{mod}(i,n)+1}}\left[p_{\mathrm{mod}(i,n)+1}(T)\right] \qquad (2)$$

guarantees that the budget is balanced for any report τ.

- Because types are independent with common prior, the terms in (2) have the same expected value under player i's posterior beliefs P_{τ_i}.

Ex-Ante Budget Balance vs. Ex-Post Budget Balance

Definition 8.2

Two mechanisms (q, p) and (q', p') are equivalent if if $q = q'$ and every type τ_i's interim expected payments are identical for every reported type r_i:

$$\mathbb{E}_{\tau_i}[p_i(r_i, T_{-i})] = \mathbb{E}_{\tau_i}[p_i'(r_i, T_{-i})].$$

Corollary 8.3

Suppose types are independent and admit a common prior. For every ex-ante budget-balanced mechanism, there exists an equivalent ex-post budget-balanced mechanism.

Proof: Apply Proposition 8.1 to an ex-ante budget-balanced mechanism.

Proof of Proposition 8.1

Incentive-compatibility:

- Suppose i reports type r_i and everybody else reports truthfully.
- Player i's interim expected utility is

$$\begin{aligned} U_i^B(r_i,\tau_i) &= \mathbb{E}_{\tau_i}[v_i(q(r_i, T_{-i}),\vartheta_i(\tau_i))] - \mathbb{E}_{\tau_i}\left[p_i^B(r_i, T_{-i})\right] \\ &= \mathbb{E}_{\tau_i}[v_i(q(r_i, T_{-i}),\vartheta_i(\tau_i))] - \mathbb{E}_{\tau_i}[p_i(r_i, T_{-i})] + \sum_{j=1}^{n} \frac{\mathbb{E}[p_j(T)]}{n} \\ &\leq \mathbb{E}_{\tau_i}[u_i(q(\tau_i, T_{-i}),\vartheta_i(\tau_i))] + \sum_{j=1}^{n} \frac{\mathbb{E}[p_j(T)]}{n} = U_i^B(\tau_i,\tau_i). \end{aligned}$$

- Therefore, truthful reporting is a Bayesian Nash equilibrium.
- Finally, $U_i^B(\tau_i,\tau_i) \geq \mathbb{E}_{\tau_i}[u_i(g(\tau_i, T_{-i}),\tau_i)]$ shows that i prefers Γ'.

Home Improvement

Home improvement:

- Alexa and Siri have enough savings to either build a (D)ance studio or a (S)wimming pool. The set of social states is $\mathcal{Q} = \{D, S\}$.
- Suppose payoff types θ_i are independent and uniformly distributed on $\Theta_i = \{1, \ldots, 9\}$ with utilities $v_i(S, \vartheta_i) = \vartheta_i + 5$ and $v_i(D, \vartheta_i) = 2\vartheta_i$.
- Let us find an IC, IR, ex-post efficient budget balanced mechanism.

Home Improvement

Home improvement:

- Suppose now that only Alexa is able to build either the swimming pool or the dance studio, that is, she has property rights over D and S.
- Suppose that $IR_A(\vartheta) = 10$ if $q(\vartheta) \in \{D, S\}$.
- We need to add a third state N, in which nothing is built.
- Does an IC, IR, ex-post efficient budget balanced mechanism exist?

Shortcuts

Finding Ex-Post Efficient Budget-Balanced Mechanisms

Current approach:

1. Find the IR-VCG mechanism.
2. Verify whether it runs an expected surplus.
3. Balance the budget via Proposition 8.1.

Expected externality mechanism:

- If nobody has property rights and no social state incurs a social cost, then the pivot payments can be redistributed in a simpler way.
- Expected externality mechanism is a shortcut to 3.

Lemma 8.5:

- Provides a shortcut to 2. if the answer is negative.
- This is particularly useful if we anticipate the answer to be negative.

Expected-Externality Mechanism

Definition 8.4

For an ex-post efficient choice of social state $q : \Theta \to \mathcal{Q}$, the payments in the expected-externality mechanism implementing (q, p^{EE}) are

$$p_i^{\text{EE}}(\tau) = \mathbb{E}_{\tau_i}\left[p_i^{\text{piv}}(\theta)\right] - \mathbb{E}_{\tau_{\text{mod}(i,n)+1}}\left[p_{\text{mod}(i,n)+1}^{\text{piv}}(\theta)\right]. \tag{3}$$

Interpretation:

- If nobody has property rights and no social state incurs a social cost, then $p_i^{\text{piv}}(\vartheta) \geq 0$, hence redistribution in (3) preserves IC and IR.
- In the expected externality mechanism, player i pays the interim expected externality that he/she imposes to player $i-1$ (modulo n).
- Since i receives the expected externality imposed by $i+1$, the net payments are given by (3).

Home Improvement

Home improvement:

- Recall that θ_i for $i = A, S$ is uniformly distributed on $\Theta_i = \{1, \ldots, 9\}$ with utilities $v_i(S, \vartheta_i) = \vartheta_i + 5$ and $v_i(D, \vartheta_i) = 2\vartheta_i$.
- Recall that the pivot payments are

$$p_i(\vartheta) = (5 - \vartheta_{-i})1_{\{10-\vartheta_i<\vartheta_{-i}\leq 5\}} + (\vartheta_{-i} - 5)1_{\{5<\vartheta_{-i}\leq 10-\vartheta_1\}}.$$

- Let us find the expected-externality mechanism.

Cheat Code for Dominant-Strategy Implementability

Lemma 8.5

An incentive-compatible, ex-post budget-balanced VCG mechanism implementing ex-post efficient social state $q : \Theta \to \mathcal{Q}$ exists if and only if there exist functions $H_i : \Theta_{-i} \to \mathbb{R}$ for $i = 1, \ldots, n$ such that for every $\vartheta \in \Theta$,

$$\sum_{i=1}^{n} v_i(q(\vartheta), \vartheta_i) = \sum_{i=1}^{n} H_i(\vartheta_{-i}).$$

Remark:

- Does not make a statement about individual rationality.
- Nevertheless, if the condition is violated, then no incentive-compatible, individually rational, ex-post budget-balanced VCG mechanism exists.

Proof of Lemma 8.5

Proof of necessity:

- Let $(q(\vartheta), p(\vartheta))$ be an ex-post budget balanced VCG mechanism.
- Recall that payments in a VCG mechanism are of the form

$$p_i(\vartheta) = h_i(\vartheta_{-i}) - \sum_{j \neq i} v_j(q(\vartheta), \vartheta_j).$$

- Ex-post budget balance implies that

$$0 = \sum_{i=1}^{n} p_i(\vartheta) = \sum_{i=1}^{n} h_i(\vartheta_{-i}) - (n-1) \sum_{i=1}^{n} v_i(q(\vartheta), \vartheta_i).$$

- Therefore,

$$\sum_{i=1}^{n} v_i(q(\vartheta), \vartheta_i) = \frac{1}{n-1} \sum_{i=1}^{n} h_i(\vartheta_{-i})$$

  is of the desired form.

Proof of Lemma 8.5

Proof of sufficiency:

- Suppose $q : \Theta \to \mathcal{Q}$ is ex-post efficient and there exist $H_i : \Theta_{-i} \to \mathbb{R}$ such that for every $\vartheta \in \Theta$,

$$\sum_{i \in \mathcal{I}} v_i(q(\vartheta), \vartheta_i) = \sum_{i \in \mathcal{I}} H_i(\vartheta_{-i}).$$

- Set $h_i(\vartheta_{-i}) = (n-1)H_i(\vartheta_{-i})$ and define payments

$$p_i(\vartheta) = h_i(\vartheta_{-i}) - \sum_{j \neq i} v_j(q(\vartheta), \vartheta_j).$$

- By definition, $\big(q(\vartheta), p(\vartheta)\big)$ is a VCG mechanism.
- We verify that it is ex-post budget-balanced:

$$\sum_{i=1}^{n} p_i(\vartheta) = (n-1)\sum_{i=1}^{n} v_i(q(\vartheta), \vartheta_i) - \sum_{i=1}^{n}\sum_{j \neq i} v_j(q(\vartheta), \vartheta_j) = 0.$$

Bilateral Trade

Bilateral Trade:

- Seller S values the good at θ_S with density $f_S(\vartheta_S) > 0$ on $[\underline{\vartheta}_S, \bar{\vartheta}_S]$.
- Buyer B values the good at θ_B with density $f_B(\vartheta_B) > 0$ on $[\underline{\vartheta}_B, \bar{\vartheta}_B]$.
- Social state $q \in \{0, 1\}$ indicates whether trade occurs and utilities are

$$u_S(q, p, \vartheta_S) = (1 - q)\vartheta_S - p_S, \qquad u_B(q, p, \vartheta_B) = q\vartheta_B - p_B.$$

Myerson-Sattertwaithe Theorem

Theorem 8.6 (Myerson-Sattertwaithe Theorem)

An incentive-compatible, individually rational, ex-post efficient mechanism exists if and only if $\underline{\vartheta}_B \geq \bar{\vartheta}_S$ *or* $\underline{\vartheta}_S \geq \bar{\vartheta}_B$.

Interpretation:

- An ex-post efficient mechanism exists only in trivial cases.
- Incomplete information imposes some inefficiency on trade, i.e., there are always some states, in which trade does not occur despite $\vartheta_B > \vartheta_S$.
- Regardless of extensive-form game, some types of buyers are unwilling to trade because they are adversely selected against.

Proof of Theorem 8.6

Step 1: Find the ex-post efficient social state

- Social welfare

$$v_S(q,\vartheta_S) + v_B(q,\vartheta_B) = (1-q)\vartheta_S + q\vartheta_B = \vartheta_S + q(\vartheta_B - \vartheta_S)$$

is maximized in $q(\vartheta) = 1_{\{\vartheta_B \geq \vartheta_S\}}$.

Step 2: Trivial cases

- If $\underline{\vartheta}_S \geq \bar{\vartheta}_B$, no trade is ex-post efficient, hence we need no mechanism.
- If $\underline{\vartheta}_B \geq \bar{\vartheta}_S$, then (q,p) for any $p_B = -p_S \in [\bar{\vartheta}_S, \underline{\vartheta}_B]$ is incentive-compatible, ex-post individually rational, and ex-post budget balanced.

Proof of Theorem 8.6

Step 3: Non-trivial case

- Suppose $[\underline{\vartheta}_B, \bar{\vartheta}_B] \cap [\underline{\vartheta}_S, \bar{\vartheta}_S]$ has non-empty interior and that there exists an ex-post budget balanced VCG mechanism that implements q.
- By Lemma 8.5, there exist $H_B(\vartheta_S)$ and $H_S(\vartheta_B)$ such that:

$$H_B(\vartheta_S) + H_S(\vartheta_B) = v_S(q(\vartheta), \vartheta_S) + v_B(q(\vartheta), \vartheta_B) = \max(\vartheta_B, \vartheta_S).$$

- For any $\vartheta, \vartheta' \in [\underline{\vartheta}_B, \bar{\vartheta}_B] \cap [\underline{\vartheta}_S, \bar{\vartheta}_S]$ with $\vartheta < \vartheta'$, this imposes

$$\vartheta = \max(\vartheta, \vartheta) = H_B(\vartheta) + H_S(\vartheta), \qquad \vartheta' = \max\big(\vartheta, \vartheta'\big) = H_B(\vartheta) + H_S(\vartheta'),$$

$$\vartheta' = \max\big(\vartheta', \vartheta\big) = H_B(\vartheta') + H_S(\vartheta), \quad \vartheta' = \max\big(\vartheta', \vartheta'\big) = H_B(\vartheta') + H_S(\vartheta').$$

- Adding equations $(1) + (4)$ and $(2) + (3)$ shows a contradiction.

Proof of Theorem 8.6

Conclusion of proof:

- Lemma 8.5 implies that no incentive-compatible, ex-post budget-balanced VCG mechanism exists.
- In particular, no IR-VCG mechanism runs an expected surplus.
- By statement 2. of Theorem 7.25, there exists no Bayesian incentive-compatible, individually rational, and ex-post efficient mechanism.

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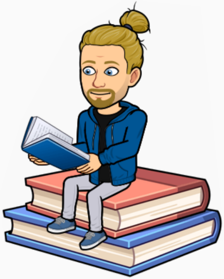

Bayesian-Optimal Mechanism

Bayesian-Optimal Mechanism

Definition 8.7

The Bayesian-optimal mechanism is the mechanism that maximizes the designer's objective function (welfare or revenue) among all incentive-compatible, individually rational, and ex-post budget-balanced mechanisms.

Remark:

- Note that we do not require dominant-strategy incentive compatibility.
- As a consequence, we can impose ex-ante budget balance instead and then achieve ex-post budget balance by Corollary 8.3.
- For one-dimensional independent types, incentive compatibility and individual rationality are characterized similarly to Lemmas 7.23 and 7.24:
  - Incentive-compatibility conditions give rise to revenue equivalence.
  - Individual rationality determine expected payments of lowest types.

Provision of a Public Good

Public goods mechanism:

- Social state $q \in \{0, 1\}$ indicates whether the agreement is signed.
- Enforcing the agreement comes at a social cost c, which signatories contribute through reduced GHG emissions.
- Suppose countries' valuations θ_i of the climate agreement are independent and distributed on $[\underline{\vartheta}, \bar{\vartheta}]$ with density $f_i(\vartheta_i) > 0$.
- Country i's utility is $u_i(q, p, \vartheta_i) = v_i(q, \vartheta_i) - p_i = q\vartheta_i - p_i$.

Impossibility Result

Proposition 7.27

An incentive-compatible individually rational ex-post efficient mechanism exists if and only if either $n\underline{\vartheta} \geq c$ or $n\bar{\vartheta} \leq c$.

What do we do next?

- We have to accept that either some payments are wasted for some ϑ or that the social state is sometimes inefficient.
- Let us find the Bayesian-optimal mechanism:
  - Simply by treating as a constrained maximization problem.
  - The characterization works similarly to the selling mechanism.

Objective Function

Constrained maximization problem:

- The objective function is the joint utility of the social state

$$V(q,p) = \int_{\Theta} \left(q(\vartheta) \sum_{i=1}^{n} \vartheta_i - c \right) f(\vartheta) \, \mathrm{d}\vartheta.$$

- Maximize $V(p,q)$ subject to the incentive compatibility, individual rationality, and ex-post budget balance constraints.

Simplifying the problem:

- Characterize incentive constraints first through a monotonicity and a "revenue equivalence" constraint.
- It is sufficient to impose ex-ante budget balance by Corollary 8.3.

Incentive Compatibility Constraints

Monotonicity:

- As usual, let us abbreviate $\bar{q}_i(\vartheta_i) = \mathbb{E}_{\vartheta_i}[q(\theta)]$.
- Individual i has no incentive to misrepresent his type as r_i:

$$\begin{aligned} u_i(r_i, \vartheta_i) \leq u_i(\vartheta_i, \vartheta_i) &= u_i(\vartheta_i, r_i) + \bar{q}_i(\vartheta_i)(\vartheta_i - r_i) \\ &\leq u_i(r_i, r_i) + \bar{q}_i(\vartheta_i)(\vartheta_i - r_i). \end{aligned} \tag{4}$$

- Subtracting $u_i(r_i, \vartheta_i)$ shows that (4) is equivalent to

$$\big(\bar{q}_i(\vartheta_i) - \bar{q}_i(r_i)\big)(\vartheta_i - r_i) \geq 0.$$

- Therefore, $\bar{q}_i$ is non-decreasing.

Incentive Compatibility Constraints

Revenue-equivalence condition:

- Let us abbreviate $\bar{p}_i(\vartheta_i) := \mathbb{E}_{\vartheta_i}[p_i(\theta)]$. As in the selling mechanism,

$$U_i(\vartheta_i) := \mathbb{E}_{\vartheta_i}[u_i(q(\theta), p(\theta), \vartheta_i)] = \bar{q}_i(\vartheta_i)\vartheta_i - \bar{p}_i(\vartheta_i).$$

  is differentiable almost everywhere with derivative $\bar{q}_i(\vartheta_i)$.

- Integrating U_i from $\underline{\vartheta}$ to ϑ_i yields

$$\bar{p}_i(\vartheta_i) = -U_i(\underline{\vartheta}) + \bar{q}_i(\vartheta_i)\vartheta_i - \int_{\underline{\vartheta}}^{\vartheta_i} \bar{q}_i(x)\,\mathrm{d}x. \tag{5}$$

- Similarly to the selling mechanism, monotonicity of $\bar{q}_i$ and (5) are also sufficient for incentive compatibility.

Individual rationality:

- IC mechanism is individually rational if and only if $U_i(\underline{\vartheta}) \geq 0$.

Budget Balance

Ex-ante budget balance:

- Using (5) and solving the double integral by Fubini's theorem yields

$$
\begin{aligned}
S &= \sum_{i=1}^{n} \int_{\underline{\vartheta}}^{\bar{\vartheta}} \big(\bar{p}_i(\vartheta_i) - cq(\vartheta)\big) f_i(\vartheta_i)\,\mathrm{d}\vartheta_i \\
&= \int_{\Theta} q(\vartheta)\left[\sum_{i=1}^{n}\left(\vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}\right) - c\right] f(\vartheta)\,\mathrm{d}\vartheta - \sum_{i=1}^{n} U_i(\underline{\vartheta}).
\end{aligned}
$$

- Ex-ante budget balance imposes $S = 0$.

Combined constraint:

- Budget balance and individual rationality combined yield

$$
\int_{\Theta} q(\vartheta)\left[\sum_{i=1}^{n}\left(\vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}\right) - c\right] f(\vartheta)\,\mathrm{d}\vartheta \geq 0.
$$

Objective Function with Constraints

Simplified maximization problem:

- Maximize the objective function

$$V(p, q) = \int_{\Theta} q(\vartheta) \left(\sum_{i=1}^{n} \vartheta_i - c \right) f(\vartheta) \, \mathrm{d}\vartheta$$

subject to the constraints that $\bar{q}_i$ is non-decreasing and

$$\int_{\Theta} q(\vartheta) \left[\sum_{i=1}^{n} \left(\vartheta_i - \frac{1 - F_i(\vartheta_i)}{f_i(\vartheta_i)} \right) - c \right] f(\vartheta) \, \mathrm{d}\vartheta \geq 0.$$

Typical approach:

1. Forget about incentive-compatibility constraint.
2. Write the relaxed problem using Karush-Kuhn-Tucker conditions.
3. Impose conditions on distribution such that $\bar{q}$ is increasing.

Karush-Kuhn-Tucker Conditions

KKT Conditions: Choice $q(\vartheta)$ solves the relaxed maximization problem if and only if there exists $\lambda \geq 0$ such that q maximizes

$$\int_\Theta q(\vartheta)\left(\sum_{i=1}^n \vartheta_i - c\right) f(\vartheta)\,\mathrm{d}\vartheta + \lambda \int_\Theta q(\vartheta)\left[\sum_{i=1}^n \left(\vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}\right) - c\right] f(\vartheta)\,\mathrm{d}\vartheta$$

$$= \int_\Theta q(\vartheta)(1+\lambda)\left[\sum_{i=1}^n \left(\vartheta_i - \frac{\lambda}{1+\lambda}\frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}\right) - c\right] f(\vartheta)\,\mathrm{d}\vartheta$$

and, moreover, $\lambda = 0$ only if

$$\int_\Theta q(\vartheta)\left[\sum_{i=1}^n \left(\vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}\right) - c\right] f(\vartheta)\,\mathrm{d}\vartheta > 0.$$

Optimal Provision of Public Good

Pointwise maximization:

- The integrand is maximized if $q(\vartheta) = 1$ if and only if

$$\sum_{i=1}^{n} \vartheta_i > c + \sum_{i=1}^{n} \frac{\lambda}{1+\lambda} \frac{1 - F_i(\vartheta_i)}{f_i(\theta)}.$$

- If $\lambda = 0$, then $q(\vartheta)$ is ex-post efficient.
- We know from Proposition 7.27 that no IC, IR, ex-post efficient and budget balanced mechanism exists.
- We conclude that $\lambda > 0$ is necessary.

Incentive compatibility:

- If $\psi_i(\vartheta_i) = \vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta)}$ is increasing, then the problem is solved.
- If ψ_i is not increasing, use Myerson's ironing.

Bayesian-Optimal Mechanism

Proposition 8.8

Suppose that $n\underline{\vartheta} < c < n\overline{\vartheta}$ and that each ψ_i is increasing. A mechanism is incentive compatible, individually rational, ex-ante budget balanced, and it maximizes expected welfare among all such mechanisms if and only if:

1. *There is $\lambda > 0$, such that for all $\vartheta \in \Theta$:*

$$q(\theta) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \vartheta_i > c + \sum_{i=1}^{n} \frac{\lambda}{1+\lambda} \frac{1-F_i(\vartheta_i)}{f_i(\theta)}, \\ 0 & \text{otherwise.} \end{cases}$$

2. $\bar{p}_i(\vartheta_i) = \bar{q}_i(\vartheta_i)\vartheta_i - \int_{\underline{\vartheta}}^{\vartheta_i} \bar{q}_i(x)\,\mathrm{d}x.$
3. $\int_{\Theta} q(\vartheta)\left[\sum_{i=1}^{n} \psi_i(\vartheta_i) - c\right] f(\vartheta)\,\mathrm{d}\vartheta = 0.$

Revenue-Maximizing Mechanism

Proposition 8.9

Suppose that ψ_i is increasing for every player $i = 1, \ldots, n$. A mechanism is incentive compatible, individually rational, and maximizes expected revenue among all such mechanisms if and only if:

1. *For all $\vartheta \in \Theta$:*

$$q(\theta) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \psi_i(\vartheta_i) > c, \\ 0 & \text{otherwise.} \end{cases}$$

2. $\bar{p}_i(\vartheta_i) = \bar{q}_i(\vartheta_i)\vartheta_i - \int_{\underline{\vartheta}}^{\vartheta_i} \bar{q}_i(x)\,\mathrm{d}x.$

Proof: analogous to optimal selling mechanism.

Revenue-Maximizing vs. Welfare-Maximizing Mechanisms

Revenue-maximizing:

$$q(\theta) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \vartheta_i > c + \sum_{i=1}^{n} \frac{1-F_i(\vartheta_i)}{f_i(\theta)}, \\ 0 & \text{otherwise.} \end{cases}$$

Welfare-maximizing: there exists $\lambda > 0$,

$$q(\theta) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \vartheta_i > c + \sum_{i=1}^{n} \frac{\lambda}{1+\lambda} \frac{1-F_i(\vartheta_i)}{f_i(\theta)}, \\ 0 & \text{otherwise.} \end{cases}$$

- There are inefficiencies in both due to information rent.
- $\frac{\lambda}{1+\lambda} < 1$: lower quantity is supplied by a revenue-maximizing designer.

Selection of Mechanisms

Allocation of Goods

Social planner allocates $m < n$ goods:

- Each individual has a unit demand for the good,
- Individuals' valuation is distributed on $[\underline{\vartheta}, \bar{\vartheta}]$,
- Social planner places value 0 on the good.
- Mechanism specifies allocation of goods and payments.

Optimal Taxation

Economy of a continuum of consumers/producers:

- Type is the individuals' skill level, distributed according to density f.
- Mechanism $g(\vartheta) = \big(q(\vartheta), p(\vartheta)\big)$ assigns
  - Production level $q(\vartheta)$ (labor),
  - Consumption $p(\vartheta) = q(\vartheta) - z(q(\vartheta))$ for tax rate z.
- Suppose everyone has the same quasi-linear utility

$$u(g(\vartheta)) = p(\vartheta) - v(q(\vartheta)),$$

i.e., people like consuming, but dislike effort.

Vacation Destination

Mark and Lisa are a Dutch couple, planning a vacation:

- Possible destinations are {"New Zealand", "Switzerland", "Tanzania"}.
- Mark is an open book and his preferences are commonly known:

$$v_M(NZ) = 7, \qquad v_M(S) = 9, \qquad v_M(T) = -1.$$

- Lisa preferences are not known to Mark. He believes they are

$$v_M(NZ, \vartheta_L^a) = 5, \qquad v_M(S, \vartheta_L^a) = 2, \qquad v_M(T, \vartheta_L^a) = 7,$$

$$v_M(NZ, \vartheta_L^b) = 8, \qquad v_M(S, \vartheta_L^b) = 6, \qquad v_M(T, \vartheta_L^b) = 1,$$

  with equal likelihood.
- Lisa is very busy with work, her outside option is $v_L(W) = 4$.

Double Auction

There are n sellers and n buyers:

- Sellers have types distributed on $[\underline{\vartheta}_S, \bar{\vartheta}_S]$.
- Buyers have types distributed on $[\underline{\vartheta}_B, \bar{\vartheta}_B]$.
- Mechanism specifies allocation of goods and payments.

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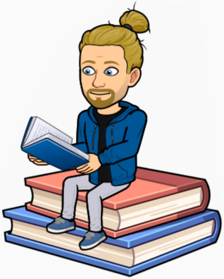

Interdependent Types

Decomposition of Types

Decomposition of types:

- Each type τ_i assigns positive probability only to one $\vartheta_i(\tau_i) \in \Theta_i$:

$$\tau_i \simeq \delta_{\vartheta_i(t_i)} \otimes \tau_i|_{\Theta_{-i} \times \mathcal{T}^{n-1}},$$

  where $\delta_{\vartheta_i(\tau_i)}$ is the Dirac measure at $\vartheta_i(\tau_i)$.

- We can decompose a player's type $\tau_i \simeq \big(\vartheta_i(\tau_i), \beta_i(\tau_i)\big)$ into his/her payoff type $\vartheta_i(\tau_i)$ and his/her belief type $\beta_i(\tau_i) := \tau_i|_{\Theta_{-i} \times \mathcal{T}^{n-1}}$.

Interdependence types:

- If players' preferences are independent, then $\beta_i(\tau_i) = P_i$ for any type τ_i, hence types are uniquely determined by ϑ_i and P_i.
- What changes if types are no longer independent?

Failure of Revenue Equivalence

Failure of revenue equivalence:

- Let $U_i(r_i, \tau_i)$ be τ_i's interim expected utility of reporting type r_i.
- If types are independent, then

$$U_i(r_i, \tau_i) = \mathbb{E}_{\tau_i}\left[v_i\big(q(r_i, T_{-i}), \vartheta_i(\tau_i)\big)\right] - \mathbb{E}_{\tau_i}[p_i(r_i, T_{-i})]$$

$$= \mathbb{E}\left[v_i\big(q(r_i, T_{-i}), \vartheta_i(\tau_i)\big)\right] - \bar{p}_i(r_i).$$

- Expected payments depend only on report, but not on type. Thus

$$0 = \frac{\partial U_i(r_i, \tau_i)}{\partial r_i}\bigg|_{r_i=\tau_i} = \frac{\partial \mathbb{E}\left[v_i\big(q(r_i, T_{-i}), \vartheta_i(\tau_i)\big)\right]}{\partial r_i}\bigg|_{r_i=\tau_i} - \bar{p}_i'(\tau_i)$$

  implies that q determines expected payments up to a constant.
- This is no longer possible when types are interdependent.

Interdependent Types

If the type space is finite:

- Let π denote the joint probability mass function of T.
- Suppose for simplicity that $\pi(\tau) > 0$ for every $\tau \in \mathcal{T}$.
- Player i of type τ_i has beliefs on T_{-i} with probability mass function

$$\pi_{T_{-i}|\tau_i}(\tau_{-i} \mid \tau_i) = \frac{\pi(\tau_i, \tau_{-i})}{\sum_{\tau'_{-i} \in \mathcal{T}_{-i}} \pi(\tau_i, \tau'_{-i})}.$$

If the type vector T admits a density:

- Let $f_i(\tau_i) = \int_{\mathcal{T}_{-i}} f(\tau_i, \tau_{-i})\, \mathrm{d}\tau_{-i}$ denote type T_i's marginal density.
- Player i of type τ_i has beliefs on T_{-i} with density

$$f_{T_{-i}|\tau_i}(\tau_{-i} \mid \tau_i) = \frac{f(\tau_i, \tau_{-i})}{f_i(\tau_i)}.$$

Crémer-McLean Condition

Definition 8.10

The distribution π satisfies the Crémer-McLean condition if for no player i of type $\tau_i \in \mathcal{T}_i$, there are weights $\lambda_{\tau_i'}$ that satisfy

$$\pi(\cdot \mid \tau_i) = \sum_{\tau_i' \in \mathcal{T}_i \setminus \{\tau_i\}} \lambda_{\tau_i'} \, \pi(\cdot \mid \tau_i'),$$

i.e., posterior beliefs of individual i's types are linearly dependent.

Crémer-McLean condition is violated if:

- Two types of player i have the same posterior beliefs,
- Player i has redundant types,
- Players' types are independent.

Anything is Possible

Proposition 8.11

Suppose that the distribution π satisfies the Crémer-McLean condition. Consider any direct mechanism $\big(q(\tau), p(\tau)\big)$. Then there is an equivalent direct mechanism $\big(q(\tau), p'(\tau)\big)$ that is Bayesian incentive-compatible.

Recall: two mechanisms are equivalent if

- They have the same decision rule q,
- They lead to the same interim expected payments:

$$\mathbb{E}_{\tau_i}[p_i(\tau_i, T_{-i})] = \mathbb{E}_{\tau_i}[p_i'(\tau_i, T_{-i})].$$

Consequence:

- If $\big(q(\tau), p(\tau)\big)$ is interim individually rational, so is $\big(q(\tau), p'(\tau)\big)$.

Auction with Interdependent Types

Consider the mechanism:

- Let q allocate the good to i if $\vartheta_i(\tau_i) = \max_j \vartheta_j(\tau_j)$.
- Demand payment $p(\tau) = \vartheta_i(\tau_i) 1_{\{q(\tau)=i\}}$.
- This mechanism is individually rational.

If Crémer-McLean condition is satisfied:

- Adjust payments to $p'(\tau)$ to make truthful reporting incentive-compatible so that interim expected payments remain unchanged.
- Therefore, ex-ante revenue is unaffected by this change.
- Auctioneer gains the same expected revenue as if he knew the types.
- Buyers earn no information rent.

What is going on?

What Is Going On?

Idea of proof:

- Make payments dependent on reported belief type $\beta_i(\tau_i)$.
- Add incentives to report belief type truthfully.
- Due to Crémer-McLean condition, no two types have the same beliefs. Truthfully reporting belief types reports payoff types truthfully as well.
- Make punishments for untruthful reporting of beliefs arbitrarily high.
- Reporting of belief types outweighs reporting of payoff types.

Farkas' Lemma

Lemma 8.12

Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^m$. Exactly one of the following two statements holds true:

(i) *There exists $x \in \mathbb{R}^n$ with $Ax = b$ and $x \geq 0$.*

(ii) *There exists $y \in \mathbb{R}^m$ with $A^\top y \geq 0$ and $b^\top y < 0$, where the vector inequalities hold element-wise.*

Apply the lemma:

- Fix individual i of type $\tau_i \in \mathcal{T}_i$.
- Let $b = \pi(\cdot \mid \tau_i)$, hence $m = |\mathcal{T}_{-i}|$.
- Let A be the matrix of column vectors $\pi(\cdot \mid \tau_i')$ for $\tau_i' \in \mathcal{T}_i \setminus \{\tau_i\}$.
- By Crémer-McLean condition, (i) does not hold, hence (ii) holds.

Proof of Proposition 8.11

By Farkas' lemma:

- There exists $y \in \mathbb{R}^m$ for $m = |\mathcal{T}_{-i}|$, such that

$$\pi(\cdot \mid \tau_i)^\top y < 0, \qquad \pi(\cdot \mid \tau_i')^\top y \geq 0 \quad \forall \tau_i' \in \mathcal{T}_i \setminus \{\tau_i\}.$$

- Index elements of y by τ_{-i} such that for any $\tau_i' \in \mathcal{T}_i$,

$$\sum_{\tau_{-i} \in \mathcal{T}_{-i}} y(\tau_{-i}) \pi(\tau_{-i} \mid \tau_i') = \mathbb{E}_{\tau_i'}[y(T_{-i})].$$

- Farkas' lemma guarantees existence of a function $y : \mathcal{T}_{-i} \to \mathbb{R}$ with

$$\mathbb{E}_{\tau_i}[y(T_{-i})] < 0, \qquad \mathbb{E}_{\tau_i'}[y(T_{-i})] \geq 0 \quad \forall \tau_i' \in \mathcal{T}_i.$$

- Define payments $p_i'(\tau) = p_i(\tau) + c\big(y_{\tau_i}(\tau_{-i}) - \mathbb{E}_{\tau_i}[y_{\tau_i}(T_{-i})]\big)$.
- Incentives to reveal the truth are strict.
- Conditional on truthtelling, interim expected payments are the same.

Auction of an Indivisible Good

	ϑ_1	ϑ_2	ϑ_3
ϑ_1	0.2	0.1	0.05
ϑ_2	0.1	0.1	0.1
ϑ_3	0.05	0.1	0.2

Auction with interdependent types:

- There are two buyers with three possible valuations $\Theta_i = \{\vartheta_1, \vartheta_2, \vartheta_3\}$ such that payoff type ϑ_k values the good at k.
- Valuations are not independent, but instead drawn from π.
- How does the revenue-maximizing auction look like?

Auction of an Indivisible Good

	ϑ_1	ϑ_2	ϑ_3
ϑ_1	$\frac{1}{2}, \frac{1}{2}$	$0, 1$	$0, 1$
ϑ_2	$1, 0$	$\frac{1}{2}, \frac{1}{2}$	$0, 1$
ϑ_3	$1, 0$	$1, 0$	$\frac{1}{2}, \frac{1}{2}$

Allocation

	ϑ_1	ϑ_2	ϑ_3
ϑ_1	$\frac{1}{2}, \frac{1}{2}$	$0, 2$	$0, 3$
ϑ_2	$2, 0$	$1, 1$	$0, 3$
ϑ_3	$3, 0$	$3, 0$	$\frac{3}{2}, \frac{3}{2}$

IR Payments

Applying Crémer-McLean construction:

- Start with full-information auction as indicated above.
- Find separating payments $y_i(\vartheta)$ for player i such that for $r_i \neq \vartheta_i$,

$$\mathbb{E}_{\vartheta_i}[y_i(\vartheta_i, \theta_{-i})] = 0, \qquad \mathbb{E}_{\vartheta_i}[y_i(r_i, \theta_{-i})] < 0.$$

- Add sufficiently large multiple to IR payments.

Auction of an Indivisible Good

	ϑ_1	ϑ_2	ϑ_3
ϑ_1	$-3, -3$	$3, 3$	$6, 6$
ϑ_2	$3, 3$	$-6, -6$	$3, 3$
ϑ_3	$6, 6$	$3, 3$	$-3, -3$

Belief Elicitation

	ϑ_1	ϑ_2	ϑ_3
ϑ_1	$-\frac{5}{2}, -\frac{5}{2}$	$3, 5$	$6, 9$
ϑ_2	$5, 3$	$-5, -5$	$3, 6$
ϑ_3	$9, 6$	$6, 3$	$-\frac{3}{2}, -\frac{3}{2}$

Optimal Payments

Problems for implementing in practice:

- Not dominant-strategy implementable.
- Mechanism designer has to be extremely certain of prior distribution.

Welfare-maximizing mechanism:

- Can we carry out a similar construction?
- How does the construction interfere with budget balance?

Identifiable Distributions

Definition 8.13

Distribution π satisfies the identifiability condition if, for any other distribution $\mu \neq \pi$ with $\mu(\tau) > 0$ for all $t \in \Theta$, there exists i and $\tau_i \in \Theta_i$, such that for any collection of non-negative weights $(\lambda_{\tau_i}(\tau_i'))_{\tau_i' \in \Theta_i}$, we have

$$\mu(\,\cdot\,|\,\tau_i) \neq \sum_{\tau_i' \in \Theta_i} \lambda_{\tau_i}(\tau_i')\,\pi(\,\cdot\,|\,\tau_i').$$

Remark:

- The Crémer-McLean condition says that the posterior of no type is a linear combination of the same individual's posteriors of other types.
- The identifiability condition says no other distribution is replicated for all agents of all types by randomizing over π.

Achieving Budget Balance

Proposition 8.14

Suppose that π satisfies the Crémer-McLean and the identifiability conditions. For any ex-ante budget balanced direct mechanism $\big(q(\tau), p(\tau)\big)$, there exists an equivalent Bayesian incentive-compatible and ex-post budget balanced mechanism $\big(q(\tau), p'(\tau)\big)$.

Idea of proof: (see Kosenok and Severinov (2008) for full proof)

- Using the Crémer-McLean condition, we can construct an equivalent Bayesian incentive-compatible mechanism.
- Interim expected payments are the same, but ex-post budget balance may be violated by belief elicitation scheme.
- Adjustment of payments that do not violate truthful revelation for any beliefs the opponent might hold requires identifiability.

Yet Another Auction

	ϑ_1	ϑ_4	ϑ_5
ϑ_0	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{2}{11}$
ϑ_2	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$

Joint distribution of valuations

Does not fall into one of previously studied cases:

- Types are not independent, hence second-price auction is not optimal.
- Types do not satisfy the Crémer-McLean condition.
- The following lemma shows that types with the same belief types are conditionally independent, given the belief profile β.

Common Prior

Proposition 8.15

Suppose the type space $\mathcal{T}$ is finite with common prior π such that $\pi(\tau) > 0$ for all $\tau \in \mathcal{T}$. For any belief type profile β that can arise in $\mathcal{T}$, we have

$$\pi(\vartheta(\tau) \,|\, \beta) = \pi(\vartheta_1(\tau_1) \,|\, \beta) \ldots \pi(\vartheta_n(\tau_n) \,|\, \beta),$$

that is, conditional on belief types, the payoff types are independent.

Proof:

- Recall that $\mathcal{T}_i \cong \Delta(\Theta \times \mathcal{T}_{-i})$ and $\tau_i \cong (\vartheta_i(\tau_i), \beta_i(\tau_i))$, where $\vartheta_i(\tau_i)$ is the marginal on Θ_i and $\beta_i(\tau_i)$ is the marginal on $\Theta_{-i} \times \mathcal{T}_{-i}$.
- For any $\tau_1, \tau_1' \in \mathcal{T}_1$ with $\beta_1(\tau_1) = \beta_1(\tau_1') = \beta_1$, we have

$$\pi(\vartheta_{-1}(\tau_{-1}) \,|\, \beta, \vartheta_1(\tau_1)) = \beta_1(\vartheta_{-1}(\tau_{-1})) = \pi(\vartheta_{-1}(\tau_{-1}) \,|\, \beta, \vartheta_1(\tau_1')).$$

- This shows that $\pi(\vartheta(\tau) \,|\, \beta) = \pi(\vartheta_1(\tau_1) \,|\, \beta)\pi(\vartheta_{-1}(\tau_{-1}) \,|\, \beta)$.

Mechanisms in Common-Prior Setting

Proposition 8.16

Suppose the type space $\mathcal{T}$ is finite with common prior π such that

1. $\pi(\tau) > 0$ *for all* $\tau \in \mathcal{T}$,
2. *Posteriors* $\beta_i(\tau_i)$ *for* $\tau_i \in \mathcal{T}_i$ *are linearly independent.*

Consider a direct mechanism $\big(q(\tau), p(\tau)\big)$ *such that for any player* i *and any* $\tau_i, \tau_i' \in \mathcal{T}_i$ *with* $\beta_i(\tau_i) = \beta_i(\tau_i')$, *type* τ_i *has no incentive to report* τ_i'. *Then there exists an equivalent Bayesian incentive-compatible direct mechanism.*

Proof:

- Use Crémer-McLean construction to elicit beliefs truthfully.
- Conditional on reported beliefs, truth-telling is incentive compatible.

Significance of the Result

If types admit a common prior:

- Use Crémer-McLean construction to elicit beliefs truthfully.
- For each reported belief type β, payoff types are independent.

Given report β:

- Player i can report only types τ_i with $\beta_i(\tau_i) = \beta_i$.
- Reporting type r_i when the true type is τ_i yields interim utility

$$U_i(r_i, \tau_i, \beta) = \mathbb{E}_\beta[v_i(q(r_i, T_{-i}), \vartheta_i(\tau_i))] - \mathbb{E}_\beta[p(r_i, T_{-i})].$$

- All techniques developed earlier apply.
- See Farinha Luz (2013) for a general auction setting.

Yet Another Auction, Continued

	ϑ_1	ϑ_4	ϑ_5
ϑ_0	0, 1	0, 1	0, 1
ϑ_2	1, 0	0, 1	0, 1

Allocation

	ϑ_1	ϑ_4	ϑ_5
ϑ_0	0, 1	0, 4	0, 4
ϑ_2	2, 0	0, 4	0, 4

IR and IC(ϑ_4, ϑ_5) Payments

Construction of optimal auction:

- Crémer-McLean: only need to analyze incentives for $\vartheta \in \{\vartheta_4, \vartheta_5\}$.
- Optimal allocation and optimal IR and IC(ϑ_4, ϑ_5) is indicated above.
- Note that ϑ_4 and ϑ_5 have an incentive to pretend being of type ϑ_1. However, those incentives can be thwarted with belief elicitation.

Yet Another Auction, Continued

	ϑ_1	ϑ_4	ϑ_5
ϑ_0	$0, 2$	$0, -1$	$0, -1$
ϑ_2	$0, -1$	$0, 1$	$0, 1$

Belief Elicitation

	ϑ_1	ϑ_4	ϑ_5
ϑ_0	$0, 9$	$0, 0$	$0, 0$
ϑ_2	$2, -4$	$0, 8$	$0, 8$

Optimal Payments

No Common Prior

No Common Prior

In a setting with a common prior:

- Mechanism designer shares the prior of the participants.
- Crémer-McLean construction leaves expected revenue unaffected.
- Revenue maximizer can extract the full surplus.

Without common prior:

- We typically do not assume a prior for the mechanism designer.
- Mechanism designer's uncertainty is simply given by the type space $\mathcal{T}$.
- We instead search for undominated mechanisms.

Undominated Mechanisms

Definition 8.17

1. A performance measure $w(\mathcal{T}, g, t) \in \mathbb{R}^m$ evaluates mechanism g when players truthfully report type t from type space $\mathcal{T}$.
2. A mechanism g is undominated for performance measure $w(\mathcal{T}, g, t)$ if there exists no other mechanism g' such that for every $t \in \mathcal{T}$ and every $k = 1, \ldots, m$, we have $w^k(\mathcal{T}, g', t) \geq w^k(\mathcal{T}, g, t)$ and there exist t_0, k_0 with $w^{k_0}(\mathcal{T}, g', t) > w^{k_0}(\mathcal{T}, g, t)$.

Performance measures:

- Ex-post Pareto welfare $\big(u_i(g(\tau), \tau_i)\big)_{i \in \mathcal{I}}$.
- Interim Pareto welfare $\big(\mathbb{E}_{\beta_i(\tau_i)}[u_i(g(\tau_i, T_i), \tau_i)]\big)_{i \in \mathcal{I}}$.
- Revenue $\sum_{i \in \mathcal{I}} p_i(\tau)$ for mechanism $g(\tau) = \big(q(\tau), p(\tau)\big)$.

No Undominated Mechanisms

Proposition 8.18

Suppose utilities are quasi-linear. For generic type spaces without common prior, there is no undominated mechanism with respect to the interim Pareto welfare criterion or the revenue criterion.

Idea of Proof:

- Recall from the no-trade theorem (Theorem 2.8), that rational players are not willing to bet if they share a common prior.
- Players with differing priors are willing to enter bets because at the interim stage, they may both believe they are better off in expectation.
- Quasi-linear utilities allow us to price in bets of arbitrarily large size into any mechanism $\rightarrow$ interim Pareto welfare is unbounded.
- Charge players to enter bets $\rightarrow$ revenue is unbounded.

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