This sample solution is an attempt to be as comprehensive as possible. To achieve full score on the exam, much less was required.

Problem 1

Answers:

1. (Method 1)
   Preferences on a set $\mathcal{X}$ is a finite sequence (a_i) such that for all $x \in \mathcal{X}$, x appears in (a_i) once and only once.

   (Method 2)
   Define xPy if and only if x precedes y in the list. Then preferences on a set $\mathcal{X}$ is a binary relation P on $\mathcal{X}$ satisfying:

   (a) Completeness: For any $x, y \in \mathcal{X}$, either xPy or yPx.
   (b) Transitivity: For any $x, y, z \in \mathcal{X}$, if xPy and yPz, then xPz.
   (c) Asymmetry: If xPy, then not yPx.

   Please note that as economists, we take the data that we receive from a survey as given. However, we may exclude data that is inconsistent with the assumed format of the data. For example, Min-Feng, who is your subject and a mischievous child, may submit a list $(xyzyx)$, a list $(xxxxx)$, or a list (Min-Feng $bxyz$) when $X = \{a, b, x, y, z\}$. You may then drop Min-Feng's list from your data.

   You may feel inclined to come up with a definition of preferences according to the list that allows you to account for an agent to feel indifferent. This is of course impossible without changing the questionnaire and therefore false. After all, when defining preferences according to the questionnaire R we also did not try to come up with a preference definition that includes "x is twice as good as y".

2. Let $T : \mathcal{R} \to \mathcal{L}$. To show that the definition is not equivalent to R, we can try to create a translation "table" of T and show that this must necessarily fail. For simplicity, let $\mathcal{X} = \{x, y, z\}$.

   Suppose we say x is strictly preferred to y if and only if x precedes y in the list. Then $T(x \sim y \succ z)$ can not be represented by any list and hence T is not well-defined, i.e., $\mathcal{L}$ does not allow indifference.

   Suppose we say x is weakly preferred to y if x precedes y in the list. Then we have

   (a) $T(x \succ y \succ z) = xyz = T(x \sim y \succ z)$ and

(b) $T(x \sim y \succ z) = xyz$ and yxz.

That is, either T is not well-defined as a function or T is not one-to-one even if we somehow manage to define T by adding some tie-breaking rules.

Another way is to argue that for $n \geq 2$ many possible alternatives the number of distinct preference relations $\succsim$ according to Definition 2 is strictly greater than the number of distinct preference relations based on the new definition. For two alternatives, $\succsim = \{(x,x),(y,y),(x,y),(y,x)\}$, $\succsim = \{(x,x),(y,y),(y,x)\}$, or $\succsim = \{(x,x),(y,y),(x,y)\}$ are possible preferences according to Definition 2. According to the new definition, only $\succ = \{(x,y)\}$ and $\succ = \{(y,x)\}$ are possible preferences.

Note also that after showing that the definition according to L is not equivalent to the definition according to R, it follows from the equivalence of the definitions proved in the lecture notes that the definition according to L can also not be equivalent to the definition according to Q.

Problem 2

Answers:

1. For any $a, b \in \mathcal{A}$, define $a \succsim b$ if and only if $\sum_i a(l_i) \geq \sum_i b(l_i)$. Hence, $\succsim$ is monotone and continuous. However, consider the following example.

   Let $\mathcal{Y} = \{l_1, l_2, l_3\}$. Let

$$\begin{aligned} l_1 &= 0.5[10] \oplus 0.5[20], \\ l_2 &= 0.5[20] \oplus 0.5[30], \text{ and} \\ l_3 &= 0.25[30] \oplus 0.5[40] \oplus 0.25[50]. \end{aligned}$$

   Let $a(l_1) = a(l_2) = 1$ and $a(l_3) = 0$. Let $b(l_1) = b(l_2) = 0$ and $b(l_3) = 1$. Then $a \succ b$ but

$$U(a) = 0.25u(30) + 0.5u(40) + 0.25u(50) = U(b).$$

   Another simple example is given by Leontief preferences with utility representation $U(a) = \min_i a(l_i)$. A similar argument as above applies which is left as an exercise.

2. If we let each l_i be a commidity indexed by i, then each $a \in \mathcal{A}$ is a consumption bundle. Hence, the results in Lecture 5: Consumer Choice should apply here. A general way to find out whether $a \succsim b$ is to find a budget set $\mathcal{B}(p, w)$ such that a is chosen while b is not. If furthermore $\sum_i b(l_i)p(l_i) < w$, then we say $a \succ b$. It may be the case that such a budget set does not exist and we perhaps need to rely on indirectly revealed preferences. That is, we try to find $x^1, \ldots, x^n$ such that $a \succsim x^1$, $x^i \succsim x^{i+1}$ for all i, and $x^n \succsim b$ in the above sense. Naturally, we want to impose some restrictions on the demand correspondence, such as GARP so that this method does not yield a contradiction. Still, it is possible that indirectly revealed preferences do not work either. In this case, we may want to extend an acyclic but not complete relation to a weak order.

   Further details (not needed for a correct answer): Suppose we have a finite set of data $\{(a^j, p^j, w^j)\}_{j=1}^n$, where $a^j \in d_{\succsim}(\mathcal{B}(p^j, w^j))$. By Afriat's theorem, the data satisfy the Generalized Axiom of Revealed Preference (GARP) if and only if there exists a continuous, strictly increasing and concave function that rationalizes it. Still, this may not be the decision-maker's true preferences. To make matters worse, it may be the case that two functions U_1 and U_2 rationalizes the data, but $U_1(a) \geq U_1(b)$ while $U_2(a) < U_2(b)$ for some $a, b \in \mathcal{A}$.

3. A simple counterexample is the case in which $\mathcal{Y}$ only contains constant lotteries. Suppose $l_1 = [70]$ and $l_2 = [11]$. Then

$$U_1(a) = u_1\Big(70a(l_1) + 11a(l_2)\Big).$$

where the product $\prod_{i=1}^{2} l(x_i) = (1 \cdot 1)$ is trivial as the lotteries are constant. Now any strictly increasing transformation $U_2 = T \circ U_1$ represents the same preference and has expected utility form over these lotteries:

$$U_2(a) = T\Big(u_1\Big(70a(l_1) + 11a(l_2)\Big)\Big).$$

We also need to exclude that there are too many lotteries with zero payoff. For example, suppose $l_1 = \frac{1}{2}[1] \oplus \frac{1}{2}[0]$ and $l_2 = [0]$. Then $\pi = \{(1,0),(0,0)\}$ and

$$U_1(a) = \frac{1}{2}u_1\big(a(l_1)\big) + \frac{1}{2}u_1(0).$$

Note that the second term is always fixed as a constant and does not influence the decision-maker's preference. Thus, any stictly increasing transformation of u_1 represents the same preference.

Answers that showed that under fairly general conditions under which U_1 must be an affine transformation of U_2 also counted as correct. However, it is much more difficult to show this.

A fairly straightforward way to establish uniqueness of the representation up to affine transformations is to assume that $l_1 = [c]$ for some strictly positive real number c and that there is at least one other nontrivial lottery. We can then define certainty equivalents as $(CE(a), 0) \sim (a(l_1), a(l_2))$. Notice that certainty equivalents always exist for expected utility representations of the given form if preferences are monotone (why?).

For simplicity, assume that another lottery fulfills $l_2 = 1/2[1] \oplus 1/2[0]$. Notice that we can thus generate any binary lottery $1/2[x] \oplus 1/2[y]$ with $x \geq y$ via allocation $(y, x - y)$. Suppose now that U_1 and U_2 are not affine transformations of another. Then,

$$\frac{1-q_1}{q_1} \equiv \frac{U_1(x) - U_1(y)}{U_1(a) - U_1(b)} \neq \frac{U_2(x) - U_2(y)}{U_2(a) - U_2(b)} \equiv \frac{1-q_2}{q_2}$$

for some $a, b, c, d \in \mathcal{A}$ which is equivalent to

$$\frac{1-q_1}{q_1} = \frac{u_1(CE(a)) - u_1(CE(b))}{u_1(CE(c)) - u_1(CE(d))} \neq \frac{u_2(CE(a)) - u_2(CE(b))}{u_2(CE(c)) - u_2(CE(d))} = \frac{1-q_2}{q_2}$$

Without loss of generality (why?), assume that $1 \geq \frac{1-q_1}{q_1} > \frac{1-q_2}{q_2}$. Now consider a binary dyadic number, $q_1 < n/2^k < q_2$, where n and k are natural numbers.

Without loss of generality, n is odd. Notice that

$$\frac{n}{2^k}u_1(CE(a)) + \frac{2^k-n}{2^k}u_1(CE(d)) > \frac{n}{2^k}u_1(CE(b)) + \frac{2^k-n}{2^k}u_1(CE(c))$$

but

$$\frac{n}{2^k}u_2(CE(a)) + \frac{2^k-n}{2^k}u_2(CE(d)) < \frac{n}{2^k}u_2(CE(b)) + \frac{2^k-n}{2^k}u_2(CE(c)).$$

While there does not necessarily exist a lottery of the form $\frac{n}{2^k}[CE(a)] \oplus \frac{2^k-n}{2^k}[CE[d]]$, we know that

$$\frac{n}{2^k}u_1(CE(a)) + \frac{2^k-n}{2^k}u_1(CE(d)) = u_1(CE(1/2[CE(1/2[\ldots] \oplus 1/2[\ldots])] \oplus 1/2[\ldots]))$$

where the iteration of certainty equivalents is k layers deep and $CE(a)$ appears n times. It follows that the preferences represented by U_1 and U_2 are not identical, a contradiction. (How would you need to change the argument to allow for l_2 to be of the form $\alpha[x] \oplus (1-\alpha)[y]$? Difficult: can you generalize this argument to the case in which l_2 has more than two outcomes?).

4. This question is similar to the previous question but now even if all lotteries are non-trivial we can construct a counterexample quite easily:

   Suppose $l_1 = \frac{1}{2}[70] \oplus \frac{1}{2}[0]$ and $l_2 = \frac{1}{2}[11] \oplus \frac{1}{2}[0]$. However, assume that they are perfectly correlated such that the high payoff is paid either in both lotteries, or in neither. Then,

$$U_1(a) = \frac{1}{2}u_1\Big(70a(l_1) + 11a(l_2)\Big) + \frac{1}{2}u_1(0).$$

   and any monotone transformation of U_1 can also be written in this form.

Problem 3

Answers:

Before doing this problem, we extend a result in Lecture Notes. The proof is very similar.

Lemma 1 (Monotonicity)**.** Let $\succsim$ be a preference relation over $L(Z)$ satisfying I. Let $p, q \in L(Z)$ and $1 \geq \alpha \geq \beta \geq 0$. Then

$$p \succsim q \Leftrightarrow \alpha p \oplus (1-\alpha)q \succsim \beta p \oplus (1-\beta)q.$$

Proof. If either $\alpha = 1$ or $\beta = 0$, the claim is implied by I. Suppose $1 > \alpha$ and $\beta > 0$. Then

$$\begin{aligned}
& p \succsim q \\
\Leftrightarrow \quad & \alpha p \oplus (1-\alpha)q \succsim q \\
\Leftrightarrow \quad & \alpha p \oplus (1-\alpha)q \succsim \frac{\beta}{\alpha}(\alpha p \oplus (1-\alpha)q) \oplus (1-\frac{\beta}{\alpha})q = \beta p \oplus (1-\beta)q.
\end{aligned}$$

$\square$

1. Assume by contradiction that there exist p and $[x]$ such that $p \succsim_2 [x]$ but $[x] \succ_1 p$. Then $p \succsim_2 [x] \succsim_1 p$. By Lemma: Monotonicity,

   $$p \succsim_2 [x] \Leftrightarrow (1-\alpha)p \oplus \alpha[x] \succsim_2 (1-\beta)p \oplus \beta[x].$$

   if $\alpha \leq \beta$ That is, $\alpha[x] \oplus (1-\alpha)p \succsim_2 \beta[x] \oplus (1-\beta)p$. Since $\succsim_1$ is more gain-skewed than $\succsim_2$, we have

   $$\alpha[x] \oplus (1-\alpha)p \succsim_1 \beta[x] \oplus (1-\beta)p.$$

   By Lemma: Monotonicity, we have $p \succsim_1 [x]$, a contradiction.

2. The statement is unfortunately false without further qualification. Despite the fact that the statement is false, one can still attempt to prove it to see how to construct a counterexample.

   First, we show the outcomes (degenerate lotteries) are ranked in the same way. In particular, for the best ouctome, denoted by M_1 for $\succsim_1$ and M_2 for $\succsim_2$, since $M_2 \succsim_2 [x]$ implies $M_2 \succsim_1 [x]$, we have $M_1 = M_2 \equiv M$. Similarly, for the worst outcome, $m_1 = m_2 \equiv m$.

   Second, since vNM utility function is unique up to positive affine transformation, we may without loss of generality assume $u_1(M) = u_2(M)$ and $u_1(m) = u_2(m)$. Since $\succsim_2$ is more risk averse than $\succsim_1$, we have u_2 over outcomes must be a concave transformation of u_1 over outcomes.

Finally, we know that any lottery is as good as some combination of the best and the worst outcome. Hence, we can rewrite p, q, and r into different combinations of $[M]$ and $[m]$. When doing this, we can realize what the problem is. If p, q, and r are not adjacent binary lotteries, then we may have preference reversal. That is, $p \succsim_1 q \succsim_1 r$, but not $p \succsim_2 q \succsim_2 \succsim_r$.

Our counterexample builds on the above observation. We can set $u_1(M) = u_2(M)$ and $u_1(m) = u_2(m)$. For simplicity, we let $\succsim_1$ be risk neutral and hence $\succsim_2$ can be any concave function. Again, for simplicity, I pick a quadratic function. It would be great if we can have $r \succsim_2 p \succsim_2 q$. We move r to the best position while keep the relative position of p and q the same. By doing so, we can guarantee $\alpha p \oplus (1-\alpha)r \succsim_2 \beta q \oplus (1-\beta)r$ for any $\alpha \leq \beta$.

Consider the following counterexmple. Let $Z = \{0, 10, 40, 100\}$. Let $\succsim_1$ and $\succsim_2$ be respectively represented by:

$$
\begin{aligned}
U_1(p) &= \sum_z p(z)u_1(z) = \sum_z p(z)z; \\
U_2(p) &= \sum_z p(z)u_2(z) = \sum_z p(z)\left[-\frac{(z-100)^2}{100} + 100\right].
\end{aligned}
$$

Since $u_1(z)$ is linear and $u_2(z)$ is concave, we have $\succsim_2$ is more risk averse than $\succsim_1$. Consider the following three lotteries:

$$
\begin{aligned}
p &= 0.5[10] \oplus 0.5[100] \Rightarrow U_1(p) = 55 \text{ and } U_2(p) = 59.5. \\
q &= 0.5\ \ [0] \oplus 0.5[100] \Rightarrow U_1(p) = 50 \text{ and } U_2(p) = 50. \\
r &= \qquad [40] \qquad\qquad \Rightarrow U_1(p) = 40 \text{ and } U_2(p) = 64.
\end{aligned}
$$

Thus, $p \succsim_1 q \succsim_1 r$ ($r \succsim_2 p \succsim_2 q$). Take some α and β with $\beta > \frac{3}{2}\alpha$. For example, $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{2}$. Then,

$$
U_2\Big(\alpha p \oplus (1-\alpha)r\Big) = \frac{503}{8} \geq 57 = U_2\Big(\beta q \oplus (1-\beta)r\Big).
$$

However,

$$
U_1\Big(\alpha p \oplus (1-\alpha)r\Big) = \frac{175}{4} < \frac{95}{2} = U_1\Big(\beta q \oplus (1-\beta)r\Big).
$$

The statement is only correct if we weaken the definition of more gain-skewed as follows:

Definition 1. *A preference $\succsim_1$ is more gain-skewed than $\succsim_2$ if for all $p, q, r \in Z$ such that $p \succsim_1 q \succsim_1 r$ and all $0 < \alpha \leq \beta \leq 1/2$ we have that if $\alpha p \oplus (1-\alpha)r \succsim_2 \beta q \oplus (1-\beta)r$, then $\alpha p \oplus (1-\alpha)r \succsim_1 \beta q \oplus (1-\beta)r$.*

Thus, we need to restrict p, q, r to be outcomes. (In fact, p, q, r can also be binary lotteries of outcomes as long as the supports of .)

3. Let us first find a condition that is equivalent to the comparison of risk aversion and that keeps the spirit of more gain-skewed. Let $L_2(Z)$ be the set of binary lotteries, i.e., lotteries with two elements in the support. Let an outcome $y \in Z$ be *between* the support of $p \in L_2(Z)$ if $x, z \in support(p)$ and $x \prec y \prec z$. A binary lottery is *adjacent* if there are no outcomes between its support.

   Definition 2 (Comparative Gain-Skewedness)**.** *A preference $\succsim_1$ is more gain-skewed than $\succsim_2$ if for all binary adjacent lotteries $p, q, r \in L(Z)$ such that $p \succsim_1 q \succsim_1 r$ and all $0 \leq \alpha \leq \beta \leq 1/2$ we have that if $\alpha p \oplus (1-\alpha)r \succsim_2 \beta q \oplus (1-\beta)r$ then $\alpha p \oplus (1-\alpha)r \succsim_1 \beta q \oplus (1-\beta)r$.*

   Notice that the comparative nature of the assumption on preferences allows us to work with a "risk neutral" decision maker without actually knowing monetary payoffs associated with the outcomes! Let's say we have found a decision maker who is "risk neutral". Since in vNM preferences the expected utility of a risk-neutral decision maker is an affine transformation of the expected value, we can determine (an affine transformation of) the expected value of every lottery from the preferences of the risk-neutral decision maker.

   Therefore, for the decision maker to buy insurances, we need to assume that the decision maker is more risk averse than the risk-neutral decision maker for a subset of $L(Z)$. In order for the decision maker to purchase lottery tickets, we need the decision maker to be less risk averse than the risk-neutral decision maker for another subset of $L(Z)$. It is for example sensible to assume that for some z^*, we have that $\succsim_1$ is less risk averse / more gain-skewed than $\succsim_2$ on $L(\overline{Z})$ where $\overline{Z} = \{z : z \succsim_1 z'\}$ while also assuming that $\succsim_1$ is more risk averse / less gain-skewed than $\succsim_2$ on $L(\underline{Z})$ where $\underline{Z} = \{z : z' \succsim_1 z\}$.

Problem 4

Answers:

A smart way to do this problem is by viewing $\succsim$ as a subset of $\mathcal{X}^2$.

1. Let $\succsim$ be total indifference, i.e., $x \succsim y$ for any $x, y \in \mathcal{X}$ or equivalently, $\succsim = \mathcal{X}^2$. Then $\succsim$ is not monotone but $\succsim' \cap \succsim'' \subseteq \succsim$ for any $\succsim'$ and $\succsim''$.

2. The total indifference $\succsim$ in our answer to the first question is too "large", i.e., contains too many pairs $(x, y) \in \succsim$, i.e. $x \succsim y$. This gives us a hint how to change the "truly between" condition. We add a condition to restrict its "size" as a subset of the cartesian product. We say a preference relation is truly between preference relations $\succsim'$ and $\succsim''$ if
$$\succsim' \cap \succsim'' \subseteq \succsim \subseteq \succsim' \cup \succsim'',$$
where $\succsim' \cup \succsim''$ is the relation $\succsim^*$ such that for any x, $y \in \mathcal{X}$, $x \succsim^* y$ if and only if $x \succsim' y$ or $x \succsim'' y$.

   Note that $\succsim \subseteq \succsim' \cup \succsim''$ is equivalent to $\succ' \cap \succ'' \subseteq \succ$ by taking complements on both sides and reversing the subset relation. The former notation demonstrates the idea of the name "betweenness" while the latter shows that this fixes the issue in Q1.

   It remains to show that $\succsim$ is monotone if it is truly between two monotone preferences. First suppose $x_k > y_k$ for all k but $y \succ x$. Then $y \succsim x$ and thus either $y \succsim' x$ or $y \succsim'' x$, a contradiction. Thus, we must have $x \succ y$. Suppose now $x_k \geq y_k$ for all k. Then $x \succsim' y$ and $x \succsim'' y$. Thus $x \succsim y$.

3. Continuity of $\succsim$ is equivalent to closedness of $\succsim$ as a subset of $\mathcal{X}^2$. Although $\succsim' \cap \succsim''$ and $\succsim' \cup \succsim''$ are closed, it can be the case that $\succsim$ is not closed. We shall use a counterexample to disprove the claim in the question. Let $\mathcal{X} = \mathbb{R}^2_+$. Let $\succsim_1$ be represented by $u(\mathbf{x}) = x_1$ and $\succsim_2$ be represented by $v(\mathbf{x}) = x_2$. Let $\succsim$ be lexicographic preference. Note that
$$\succsim' \cap \succsim'' = \{(\mathbf{x}, \mathbf{y}) : x_1 \geq y_1 \text{ and } x_2 \geq y_2\}$$
$$\succsim' \cup \succsim'' = \{(\mathbf{x}, \mathbf{y}) : x_1 \geq y_1 \text{ or } x_2 \geq y_2\}$$
Hence, $\succsim = \{(\mathbf{x}, \mathbf{y}) : [x_1 > y_1] \text{ or } [x_1 = y_1 \text{ and } x_2 \geq y_2]\}$ is truly between $\succsim_1$ and $\succsim_2$. Both $\succsim_1$ and $\succsim_2$ are monotone and continuous but $\succsim$ is not continuous.