Learning Notes in Quantitative Methods

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Chapter 1

Set Theory

1.1 The Real Number System

Definition 1.1.1 (real number system)**.**

1. The set of **natural numbers** is denoted by

$$\mathbb{N} = \{1, 2, 3, \dots\}. \tag{1.1}$$

2. The set of **integers** is denoted by

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}. \tag{1.2}$$

3. The set of **rational numbers** is denoted by

$$\mathbb{Q} = \left\{ \frac{q}{p} \middle| p, q \in \mathbb{Z}, q \neq 0 \right\}. \tag{1.3}$$

4. The set of **real numbers** is denoted by $\mathbb{R}$.

[from Sundaram et al., 1996, p.2; Apostol, 1974, p.3,6-7; Chu, 2021, Lecture 1 p.1]

1. The real numbers are represented geometrically as points on a line, which is called the **real line** or the **real axis**.

2. Real numbers that are not rational are called **irrational numbers**.

Definition 1.1.2 (interval)**.**

1. By the **open interval** (a, b) we mean the set of all real numbers x such that $a < x < b$.
2. By the **closed interval** $[a, b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.
3. Occasionally we shall encounter **half-open intervals** $[a, b)$ and $(a, b]$; the first consists of x such that $a \leq x < b$, the second of all x such that $a < x \leq b$.

[from Rudin et al., 1964, p.31; Apostol, 1974, p.4; Sundaram et al., 1996, p.22]

1. The real line $\mathbb{R}$ is referred to as the open interval $(-\infty, \infty)$.
2. A single point is considered as a **degenerate** closed interval.

1.2 Sets, Unions, and Intersections

Definition 1.2.1 (set)**.** Let denote **sets** by capital letters such as A, B, X, and Y, and **elements** of these sets by lowercase letters such as a, b, x, and y.

1. An object a **belongs to** a set A is denoted by $a \in A$. If a is **not an element** of A, we write $a \notin A$.
2. If every element of a set B is also an element of a set A, we shall say that B is a **subset** of A and write $B \subset A$.
3. Two sets A and B are said to be **equal**, written $A = B$. If every element of A is also an element of B, and vice versa. That is, $A = B$ if we have both $A \subset B$ and $B \subset A$.
4. The **union** of two sets A and B, denoted $A \cup B = \{x | x \in A \text{ or } x \in B\}$, is the set which consists of all elements which are either in A or in B.
5. The **intersection** of two sets A and B, denoted $A \cap B = \{x | x \in A \text{ and } x \in B\}$, is the set which consists of all elements which belong to both A and B.
6. If $A \subset X$, the **complement** of A in X, denoted A^C, is defined as $A^C = \{x \in X | x \notin A\}$.

[from Sundaram et al., 1996, p.315-316; Rudin et al., 1964, p.3; Chu, 2020, Lecture 1 p.1; Chu, 2021, Lecture 1 p.1-3]

1.3 Logic

1.3.1 Propositions: Contrapositives and Converses

Definition 1.3.1 (imply)**.** Given two propositions P and Q, the statement "If P, then Q" is interpreted as the statement that if the proposition P is true, then the statement Q is also true. We denote this by $P \Rightarrow Q$.

[from Sundaram et al., 1996, p.316-317]

1. Its **contrapositive** is the statement that "if Q is not true, then P is not true." We denote this by $Q \Rightarrow \ P$.

2. **A statement and its contrapositive are logically equivalent.** That is, if the statement is true, then the contrapositive is also true, while if the statement is false, so is the contrapositive.

Definition 1.3.2 (converse)**.** The **converse** of the statement $P \Rightarrow Q$ is the statement that $Q \Rightarrow P$. That is, the statement that "if Q, then P."

[from Sundaram et al., 1996, p.318]

1. **There is no logical relationship between a statement and its converse.**

2. If a statement and its converse both hold, we express this by saying that "P **if and only if** Q," and denote this by $P \Leftrightarrow Q$.

1.3.2 Quantifiers and Negation

Definition 1.3.3 (logical quantifiers)**.**

1. The **universal** or "**for all**" quantifier is used to denote that a property holds for every element a in some set A. We write $\forall$.

2. The **existential** or "**there exists**" quantifier denotes that the property holds for at least one element a in the set A. We write $\exists$.

[from Sundaram et al., 1996, p.318]

Definition 1.3.4 (negation)**.** The **negation** of a proposition P is its denial $\sim P$.

[from Sundaram et al., 1996, p.318-319; Chu, 2020, Lecture 1 p.2; Chu, 2021, Lecture 1 p.3]

1. If the proposition P involves a universal quantifier, then its negation involves an existential quantifier: to deny the truth of a universal statement requires us to find just one case where the statement fails.
2. The negation of an existential quantifier involves a universal quantifier: to deny that there is at least one case where the proposition holds requires us to show that the proposition fails in every case.

Example 1.3.5. Negate the following statements.

1. A or B.
2. A and B.
3. If A, then B.
4. For all x, $A(x)$.
5. There exists x such that $A(x)$.
6. For every $x > 0$, there exists $y > 0$ such that $y^2 = x$.
7. For all $x \in S$, there is an $r > 0$ such that $B(x, r) \subset S$.

[from Chu, 2020, Lecture 1 p.2; Chu, 2021, Lecture 1 p.3-4]

Solution

1. Not A and not B.
2. Not A or not B.
3. Suppose A is true, but B is not true.
4. There exists x such that not $A(x)$.
5. For all x, not $A(x)$.
6. There exists an $x > 0$, $y^2 \neq x$ for all $y > 0$.

7. (a) There exists an $x \in S$, $B(x,r) \not\subset S$ for all $r > 0$.

   (b) There exists an $x \in S$, for all $r > 0$, there exists an $y \in B(x,r)$, but $y \notin S$.

□

Theorem 1.3.6 (Demorgan's laws)**.**

1. $\left(\bigcup_{i \in I} A_i\right)^C = \bigcap_{i \in I} A_i^C$

2. $\left(\bigcap_{i \in I} A_i\right)^C = \bigcup_{i \in I} A_i^C$

[from Wade, 2014, p.33; Rudin et al., 1964, p.33; Sundaram et al., 1996, p.25; Chu, 2020, Lecture 1 p.2; Chu, 2021, Lecture 1 p.3-4]

Proof.

1.

$$\begin{aligned} \text{LHS} &\Leftrightarrow \{x | \exists i, x \in A_i\}^C && (1.4)\\ &\Leftrightarrow \{x | \forall i, x \notin A_i\} && (1.5)\\ &\Leftrightarrow \{x | \forall i, x \in A_i^C\} && (1.6)\\ &\Leftrightarrow \text{RHS} && (1.7) \end{aligned}$$

2.

$$\begin{aligned} \text{LHS} &\Leftrightarrow \{x | \forall i, x \in A_i\}^C && (1.8)\\ &\Leftrightarrow \{x | \exists i, x \notin A_i\} && (1.9)\\ &\Leftrightarrow \{x | \exists i, x \in A_i^C\} && (1.10)\\ &\Leftrightarrow \text{RHS} && (1.11) \end{aligned}$$

□

1.3.3 Necessary and Sufficient Conditions

Definition 1.3.7 (necessary condition)**.** Suppose an implication of the form $P \Rightarrow Q$ is valid. Then, Q is said to be a **necessary condition** for P.

[from Sundaram et al., 1996, p.320; Chu, 2021, Lecture 1 p.4]

Definition 1.3.8 (sufficient condition)**.** Suppose an implication of the form $P \Rightarrow Q$ is valid. Then, P is said to be a **sufficient condition** for Q.

[from Sundaram et al., 1996, p.321; Chu, 2021, Lecture 1 p.4]

1.4 Ordered Pairs and Relations

Definition 1.4.1 (order pairs)**.** If a set of two elements a and b is **ordered**, we enclose the elements in parentheses (a, b). Then, a is called the first element, and b is called second element.

[from Apostol, 1974, p.33; Chu, 2021, Lecture 1 p.5]

Definition 1.4.2 (cartesian product)**.** Given two sets A and B, the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the **cartesian product** of A and B, and is denoted by $A \times B$.

[from Apostol, 1974, p.33; Chu, 2021, Lecture 1 p.5]

Definition 1.4.3 (relation)**.** Any set of ordered pairs is called a **relation**.

[from Apostol, 1974, p.34; Chu, 2021, Lecture 1 p.5]

1. If S is a relation, the set of all elements x that occur as first members of pairs (x, y) in S is called **domain** of S, denoted by $\mathfrak{D}(S)$.
2. The set of second members y is called the **range** of S, denoted by $\mathfrak{R}(S)$.

1.4.1 Binary Relation

Definition 1.4.4 (reflexive)**.** A binary relation R on the set X is **reflexive** if for all $x \in X$, xRx.

[from Osborne and Rubinstein, 2020, p.5]

Definition 1.4.5 (complete)**.** A binary relation R on the set X is **complete** if for all $x, y \in X$, xRy or yRx.

[from Osborne and Rubinstein, 2020, p.5]

Definition 1.4.6 (irreflexive). A binary relation R on the set X is **irreflexive** if for all $x, y \in X$, not xRx.

[from Osborne and Rubinstein, 2020, p.5]

Definition 1.4.7 (symmetric). A binary relation R on the set X is **symmetric** if for all $x, y \in X$, xRy implies yRx.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.8 (asymmetric). A binary relation R on the set X is **asymmetric** if for all $x, y \in X$, xRy implies not yRx.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.9 (antisymmetric). A binary relation R on the set X is **antisymmetric** if for all $x, y \in X$, xRy and yRx imply $x = y$.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.10 (transitive). A binary relation R on the set X is **transitive** if for all $x, y, z \in X$, xRy and yRz imply xRz.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.11 (quasi-transitive). A binary relation R on the set X is **transitive** if for all $x, y, z \in X$, xPy and yPz imply xPz.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.12 (acyclic). A binary relation R on the set X is **acyclic** if for all $x_1, x_2, \ldots, x_n \in X$, $x_1Px_2, x_2Px_3, \ldots, x_{n-1}Px_n$ imply x_1Rx_n.

[from Osborne and Rubinstein, 2020, p.6]

Definition 1.4.13 (negatively transitive). A binary relation R on the set X is **negatively transitive** if for all $x, y, z \in X$, not xRy and not yRz imply not xRz.

[from Osborne and Rubinstein, 2020, p.6]

1.5 Functions

Definition 1.5.1 (function)**.** A **function** F is a set of ordered pairs (x, y), no two of which have the same first member. That is, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.

[from Apostol, 1974, p.34-35; Rudin et al., 1964, p.24; Sundaram et al., 1996, p.41; Chu, 2021, Lecture 1 p.6]

1. The definition of function requires that for every x in the domain of F, there is exactly one y such that $(x, y) \in F$.
2. It is customary to call y the **value** of F at x and to write $y = F(x)$ instead of $(x, y) \in F$ to indicate that the pair (x, y) is in the set F.
3. When the domain $\mathfrak{D}(F)$ is a subset of $\mathbb{R}$, then F is called a **function of one real variable**.
4. If $\mathfrak{D}(F)$ is a subset of a cartesian product $A \times B$, then F is called a **function of two variable**.
5. If S is a subset of $\mathfrak{D}(F)$, we say that F is defined on S.
6. The set of $F(x)$ such that $x \in S$ is called the **image** of S under F, and is denoted by $F(S)$.
7. If T is any set which contain $F(S)$, then F is called a **mapping** from S into T. It is denoted by $F : S \to T$.
8. If $F(S) = T$, the mapping is said to be **onto** T.
9. A mapping of S into itself is called a **transformation**.

Example 1.5.2. Find the domain and range of following functions.

1. For all x in $[-3, 3)$,

$$F(x) = (x + 1)^2. \tag{1.12}$$

2. For all x in $\mathbb{R}$,

$$F(x) = 2^x. \tag{1.13}$$

[from Nagaoka and Miyaoka, 2007, p.11; Chu, 2021, Lecture 1 Homework Q3]

Solution

1.

$$\mathfrak{D}(F) = [-3, 3) \tag{1.14}$$

$$\mathfrak{R}(F) = [0, 16) \tag{1.15}$$

2.

$$\mathfrak{D}(F) = \mathbb{R} \tag{1.16}$$

$$\mathfrak{R}(F) = \mathbb{R}^{+} \tag{1.17}$$

Theorem 1.5.3. Two function F and G are **equal** if and only if

1. F and G have the same domain. That is,

$$\mathfrak{D}(F) = \mathfrak{D}(G). \tag{1.18}$$

2. for every x in $\mathfrak{D}(F)$,

$$F(x) = G(x). \tag{1.19}$$

[from Apostol, 1974, p.35]

Definition 1.5.4 (one-to-one)**.** Let F be a function defined on S. We say F is **one-to-one** on S if and only for every x and y in S, $F(x) = F(y)$ implies $x = y$.

[from Apostol, 1974, p.36; Chu, 2021, Lecture 1 p.7]

Definition 1.5.5 (converse)**.** Given a relation S, the new relation $\breve{S}$ defined by

$$\breve{S} = \{(a, b) | (b, a) \in S\} \tag{1.20}$$

is called the **converse** of S.

[from Apostol, 1974, p.36; Chu, 2021, Lecture 1 p.7]

Definition 1.5.6 (inverse)**.** Suppose that the relation F is a function. Consider the converse relation $\breve{F}$, which may or may not be a function. If $\breve{F}$ is a function, then $\breve{F}$ is called the **inverse** of F, and it denoted by F^{-1}.

[from Apostol, 1974, p.36; Chu, 2021, Lecture 1 p.7]

Theorem 1.5.7. If the function F is one-to-one on its domain, then $\check{F}$ is also a function.

[from Apostol, 1974, p.37; Chu, 2021, Lecture 1 p.7]

Proof. Let $(x, y) \in \check{F}$ and $(y, z) \in \check{F}$. By Definition 1.5.1 and Definition 1.5.5,

$$\begin{aligned} (x, y) \in \check{F} &\Leftrightarrow (y, x) \in F && (1.21) \\ &\Leftrightarrow F(y) = x && (1.22) \\ (x, z) \in \check{F} &\Leftrightarrow (z, x) \in F && (1.23) \\ &\Leftrightarrow F(z) = x. && (1.24) \end{aligned}$$

We find

$$F(y) = F(z) = x. \tag{1.25}$$

By Definition 1.5.4,

$$y = z. \tag{1.26}$$

□

Definition 1.5.8 (composite function). Given two functions F and G such that $\mathfrak{R}(F) \subseteq \mathfrak{D}(G)$, we can form a new function, the **composite** $G \circ F$ of G and F, defined as follows: for every x in the domain of F,

$$(G \circ F)(x) = G[F(x)]. \tag{1.27}$$

[from Apostol, 1974, p.37; Chu, 2021, Lecture 1 p.8]

Example 1.5.9. Consider

$$\begin{aligned} F(x) &= x^2 - 2, \quad x \in \mathbb{R} && (1.28) \\ G(x) &= -x + 1, \quad x \in \mathbb{R}. && (1.29) \end{aligned}$$

1. Check whether each of the two functions is a one-to-one functions.
2. Verify that $\mathfrak{R}(F) \subseteq \mathfrak{D}(G)$, and find the domain and range of $(G \circ F)(x)$.

[from Nagaoka and Miyaoka, 2007, p.11; Chu, 2021, Lecture 1 Homework Q4,Q5]

Solution

1. By Definition 1.5.4.

2. We know

$$\mathfrak{D}(F) = \mathbb{R} \tag{1.30}$$

$$\mathfrak{R}(F) = [-2, \infty) \tag{1.31}$$

$$\mathfrak{D}(G) = \mathbb{R} \tag{1.32}$$

$$\mathfrak{R}(G) = \mathbb{R}. \tag{1.33}$$

Because $\mathfrak{R}(F) \subseteq \mathfrak{D}(G)$, we can define a composite function

$$(G \circ F)(x) = -(x^2 - 2) + 1 = -x^2 + 3 \tag{1.34}$$

for all $x \in \mathbb{R}$. Moreover,

$$\mathfrak{D}(G \circ F) = \mathbb{R} \tag{1.35}$$

$$\mathfrak{R}(G \circ F) = (-\infty, 3]. \tag{1.36}$$

□

Example 1.5.10. Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. For $B \subset \mathbb{R}^m$, define $f^{-1}(B)$

$$f^{-1}(B) = \{x \in \mathbb{R}^n | f(x) \in B\}. \tag{1.37}$$

Show that for any subsets A_1, A_2 of $\mathbb{R}^n$ and B_1, B_2 of R^m:

1. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.
2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
4. $f^{-1}(B_1^C) = [f^{-1}(B_1)]^C$.
5. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
6. $A_1 \subseteq f^{-1}(f(A_1))$.
7. $f(f^{-1}(B_1)) \subseteq B_1$.

[from Sundaram et al., 1996, p.71; Nagaoka and Miyaoka, 2007, p.34; Chu, 2021, Lecture 1 p.8-9, Homework Q1]

Proof.

1. Skip.

2. By Definition 1.5.1,

$$f(A_1 \cup A_2) = \{b \in \mathbb{R}^m | \exists a \in A_1 \cup A_2 \text{ such that } f(a) = b\} \tag{1.38}$$

$$f(A_1) = \{b \in \mathbb{R}^m | \exists a_1 \in A_1 \text{ such that } f(a_1) = b\} \tag{1.39}$$

$$f(A_2) = \{b \in \mathbb{R}^m | \exists a_2 \in A_2 \text{ such that } f(a_2) = b\} . \tag{1.40}$$

   - ($\Rightarrow$)

     If $b \in f(A_1 \cup A_2)$, then there is a in either A_1 or A_2 such that $f(a) = b$.

     – If $a \in A_1$, then $b \in f(A_1)$.

     – If $a \in A_2$, then $b \in f(A_2)$.

     Thus, $b \in f(A_1) \cup f(A_2)$.

   - ($\Leftarrow$)

     If $b \in f(A_1) \cup f(A_2)$, then there is a in either A_1 or A_2 such that $f(a) = b$. Thus,

$$b \in f(A_1 \cup A_2). \tag{1.41}$$

3. Skip.

4. By Definition 1.5.1 and Definition 1.5.6,

$$f^{-1}(B_1^C) = \left\{a \in \mathbb{R}^n | f(a) \in B_1^C\right\} \tag{1.42}$$

$$\left[f^{-1}(B_1)\right]^C = \{a \in \mathbb{R}^n | f(a) \notin B_1\} . \tag{1.43}$$

   - ($\Rightarrow$)

     If $a \in f^{-1}(B_1^C)$, then $f(a) \in B_1^C$ and $f(a) \notin B_1$. Thus,

$$a \in \left[f^{-1}(B_1)\right]^C . \tag{1.44}$$

   - ($\Leftarrow$)

     If $a \in \left[f^{-1}(B_1)\right]^C$, then $f(a) \notin B_1$ and $f(a) \in B_1^C$. Thus,

$$a \in f^{-1}(B_1^C). \tag{1.45}$$

5. Skip.

6. Skip.

7. Skip.

$\square$

Example 1.5.11. The following functions F and G are defined for all real x by the equations given. In each case where the composite function $G \circ F$ can be formed, give the domain of $G \circ F$ and a formula for $(G \circ F)(x)$.

1.

$$F(x) = 1 - x \tag{1.46}$$

$$G(x) = x^2 + 2x. \tag{1.47}$$

2.

$$F(x) = x + 5 \tag{1.48}$$

$$G(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases} \tag{1.49}$$

3.

$$F(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 1, & \text{otherwise}, \end{cases} \tag{1.50}$$

$$G(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise}. \end{cases} \tag{1.51}$$

[from Apostol, 1974, p.43; Chu, 2021, Lecture 1 p.9-10]

Solution

1. Skip.

2. Skip.

3. We know

$$\mathfrak{D}(F) = \mathbb{R} \tag{1.52}$$

$$\mathfrak{R}(F) = [0, 2] \tag{1.53}$$

$$\mathfrak{D}(F) = \mathbb{R} \tag{1.54}$$

$$\mathfrak{R}(F) = [0, 1]. \tag{1.55}$$

Because $\mathfrak{R}(F) \subseteq \mathfrak{D}(G)$, we can define a composite function

$$(G \circ F)(x) = \begin{cases} 4x^2, & 0 \leq x \leq \frac{1}{2} \\ 0, & \frac{1}{2} < x \leq 1 \\ 1 & \text{otherwise.} \end{cases} \tag{1.56}$$

Moreover,

$$\mathfrak{D}(G \circ F) = \mathbb{R} \tag{1.57}$$

$$\mathfrak{R}(G \circ F) = [0, 1]. \tag{1.58}$$

□

Theorem 1.5.12. Let $f : S \to T$ be a function. The following statements are equivalent.

1. f is one-to-one on S.
2. For all subsets A, B of S,

$$f(A \cap B) = f(A) \cap f(B). \tag{1.59}$$

3. For every subset A of S,

$$f^{-1}[f(A)] = A. \tag{1.60}$$

4. For all disjoint subsets A and B of S, the images $f(A)$ AND $f(B)$ are disjoint.
5. For all subsets A and B of S with $B \subseteq A$, we have

$$f(A - B) = f(A) - f(B). \tag{1.61}$$

[from Apostol, 1974, p,44; Chu, 2021, Lecture 1 Homework Q2]

Proof. Skip. □

Definition 1.5.13 (finite sequence)**.** By a finite sequence of n terms, we shall understand a function F whose domain is the set of numbers $\{1, 2, \ldots, n\}$.

[from Apostol, 1974, p.37; Chu, 2021, Lecture 1 p.10]

Definition 1.5.14 (infinite sequence)**.** By an infinite sequence, we shall mean a function F whose domain is the set $\{1, 2, 3, \ldots\}$ of all positive integers. The range of F is written

$$\{F_1, F_2, F_3, \ldots, F_n\}, \tag{1.62}$$

and the function value F_n is called the nth **term** of the sequence.

[from Apostol, 1974, p.37; Chu, 2021, Lecture 1 p.10]

Definition 1.5.15 (similar)**.** Two sets A and B are called **similar**, or **equinumerous**, and we write $A \sim B$, if and only if there exists a one-to-one function F whose domain is the set A, and whose range is the set B.

[from Apostol, 1974, p.38]

Theorem 1.5.16.

1. Every set A is similar to itself.
2. If $A \sim B$, then $B \sim A$.
3. If $A \sim B$ and $B \sim C$, then $A \sim C$.

[from Apostol, 1974, p.38]

Proof.

1. Take F to be identity function. That is, for all x in A,

$$F(x) = x. \tag{1.63}$$

2. If F is a one-to-one function which makes $A \sim B$, then F^{-1} will make $B \sim A$.

3. Skip.

□

1.6 Finite, Infinite, Countable, and Uncountable Sets

Definition 1.6.1 (finite set). A set S is called **finite**, and is said to contain n elements if

$$S \sim \{1, 2, \ldots, n\}. \tag{1.64}$$

[from Apostol, 1974, p.38]

1. The integer n is called the **cardinal number** of S.
2. The empty set is considered finite. Its cardinal number is defined to be 0.

Theorem 1.6.2. If

$$\{1, 2, \ldots, n\} \sim \{1, 2, \ldots, m\}, \tag{1.65}$$

then $m = n$.

[from Apostol, 1974, p.38]

Definition 1.6.3 (infinite set). Sets which are not finite are called **infinite sets**.

[from Apostol, 1974, p.38]

Definition 1.6.4 (countably infinite set). A set S is said to be **countably infinite** if it is equinumerous with the set of all positive integers. That is,

$$S \sim \{1, 2, 3, \ldots\}. \tag{1.66}$$

[from Apostol, 1974, p.39]

Definition 1.6.5 (countable, uncountable set).

1. A set S is called **countable** if it is either finite or countably infinite.
2. A set which is not countable is called **uncountable**.

[from Apostol, 1974, p.39]

Theorem 1.6.6. Every subset of a countable set is countable.

[from Apostol, 1974, p.39]

Theorem 1.6.7. The set of all real numbers is uncountable.

[from Apostol, 1974, p.39]

1.7 Upper Bounds, Maximum Element, and Least Upper Bound

Definition 1.7.1 (order)**.** Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:

1. If $x \in S$ and $y \in S$, then one and only one of the statements

$$x < y \tag{1.67}$$

$$x = y \tag{1.68}$$

$$y < x \tag{1.69}$$

   is true.

2. Let $x, y, z \in S$. If $x < y$ and $y < x$, then $x < z$.

[from Rudin et al., 1964, p.3]

1. The statement "$x < y$" may be read as "x is less than y" or "x is smaller than y" or "x preceeds y."

2. It is often convenient to write $y > x$ in place of $x < y$.

3. The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of these two is to hold. In other words, $x \leq y$ is the negation of $x > y$.

Definition 1.7.2 (order!order set)**.** An **order set** is a set S in which an order is defined.

[from Rudin et al., 1964, p.3]

Definition 1.7.3 (bound)**.** Suppose S is an ordered set, and $E \subset S$.

1. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is **bounded above**, and call β an **upper bound** of E.
2. If there exists a $\beta \in S$ such that $x \geq \beta$ for every $x \in E$, we say that E is **bounded below**, and call β an **lower bound** of E.

[from Rudin et al., 1964, p.3; Sundaram et al., 1996, p.14; Chu, 2020, Lecture 3 p.5]

1. The set of **upper bounds** of E, denoted $U(E)$, is defined as

$$U(E) = \{\beta \in \mathbb{R} | \beta \geq x, \forall x \in E\}. \tag{1.70}$$

2. The set of **lower bounds** of E, denoted $L(E)$, is defined as

$$L(E) = \{\beta \in \mathbb{R} | \beta \leq x, \forall x \in E\}. \tag{1.71}$$

Definition 1.7.4. Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

1. α is an upper bound of E.
2. If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called the **least upper bound** of E (that there is at most one such α is clear from 2.) or the **supremum** of E, and we write

$$\alpha = \sup E. \tag{1.72}$$

The **greatest lower bound**, or **infimum**, of a set E which is bounded below is defined in the same manner: the statement

$$\alpha = \inf E \tag{1.73}$$

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E.

[from Rudin et al., 1964, p.4; Sundaram et al., 1996, p.14; Chu, 2020, Lecture 3 p.5]

Definition 1.7.5 (least-upper-bound property)**.** An ordered set S is said to have the **least-upper-bound property** if $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

[from Rudin et al., 1964, p.4]

1. If $\alpha = \sup E$ exists, then α may or may not be a member of E. For instance, let E_1 be the set of all $r \in Q$ with $r < 0$. Let E_2 be the set of all $r \in Q$ with $r \leq 0$. Then

$$\sup E_1 = \sup E_2 = 0, \tag{1.74}$$

and $0 \notin E_1, 0 \in E_2$.

2. We shall show that there is a close relation between greatest lower bounds and least upper bounds, and that every ordered set with the least-upper-bound property also has the greast-lower-bound property.

Theorem 1.7.6. If $U(E)$ is nonempty, the supremum of E is well defined, i.e. there is a $x \in U(A)$ such that $x \leq u$ for all $u \in U(E)$. Similarly, if $L(E)$ is nonempty, the infimum of E is well defined, i.e. there is $y \in L(E)$ such that $y \geq l$ for all $l \in L(E)$.

[from Sundaram et al., 1996, p.14]

Theorem 1.7.7 (approximation property for suprema)**.** Suppose $\sup E$ is finite. Then, for all $\varepsilon > 0$, there is a $a(\varepsilon) \in E$ such that

$$\sup E \geq a(\varepsilon) > \sup E - \varepsilon. \tag{1.75}$$

[from Sundaram et al., 1996, p.16]

Proof. Suppose $\exists \varepsilon > 0$ such that $a \leq \sup E - \varepsilon$ for all $a \in E$. By Definion 1.7.3, $\sup E - \varepsilon$ would be an upper bound of E and $\sup E - \varepsilon < \sup E$, but it violates $\sup E$ is the least upper bound by Definition 1.7.4. A contradiction. □

Theorem 1.7.8. Suppose S is an ordered set with the least-upper-bounded property $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L \tag{1.76}$$

exists in S, and $\alpha = \inf B$. In particular, $\inf B$ exists in S.

[from Rudin et al., 1964, p.5]

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