Problem Set 2

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To save time, I will omit the question description but only present the answers.

Answer 1.

a. No. Consider $X = (-\frac{\pi}{2}, \frac{\pi}{2}), U(x) = x, V(x) = \tan x$. If such f exists, then $f(\pi) > f(x)$ for all $x \in X$, which is impossible.

b. Yes. Let $X = \mathbb{R}$ and $\succsim = \geq$. Let $u(x) = \begin{cases} x+1 & \text{when } x \geq 0 \\ x & \text{when } x < 0 \end{cases}$

c. $2 \succ 1$, but $2 - \epsilon \sim 1 + \epsilon$ for all $\epsilon > 0$ small enough, which contradicts C1.

d. Notice that C3 $\iff$ C4 is trivial since $\{y|y \succsim x\}^c = \{y|x \succ y\}$ and $\{y|x \succsim y\}^c = \{y|y \succ x\}$.
C1 $\implies$ C4 is straight forward: Assume C1. If $y \succ x$, then there exists $r > 0$ such that $y' \in B(y, r), y' \succ x$. Hence, $\{y|y \succ x\}$ is open for all $x \in X$.
C4 $\implies$ C1 (the tricky part):Let $x \succ y$. We divide the question into two parts:
1. No element z satisfies $x \succ z \succ y$: There exists r_1, r_2, such that for all $x' \in B(x, r_1), y' \in B(y, r_2)$, $x' \succ y$ and $x \succ y'$. Since no such z exists, $x' \succsim x \succ y \succsim y'$. Hence, C1 holds.
2. There exists z that $x \succ z \succ y$: There exists r_1, r_2, such that for all $x' \in B(x, r_1), y' \in B(y, r_2)$, $x' \succ z \succ y'$. Hence, C1 holds.

Answer 2. Take $\mathbb{Q} \cap [0, 1]$ and $\geq$ on $\mathbb{Q} \cap [0, 1]$. If u is the utility function that returns only integer, then it at most contains $u(1) - u(0)$ values between 0 and 1 but there are infinitely many number that has different utility between 0 and 1.

Answer 3. (My version) Let $x_0 = x, z_0 = z$. We now define two sequence on the interval:

$$(x_n, z_n) = \begin{cases} (\frac{x_{n-1} + z_{n-1}}{2}, z_{n-1}) & \text{when } \frac{x_{n-1} + z_{n-1}}{2} \succsim y \\ (x_{n-1}, \frac{x_{n-1} + z_{n-1}}{2}) & \text{when } y \succ \frac{x_{n-1} + z_{n-1}}{2} \end{cases}$$

Notice that, by induction, we have $x_n \succsim y \succsim z_n$. Moreover, x_n, z_n are trivially convergent sequences with the same limit on the interval. Say the limit is m. Then by continuity, $y \sim m$.

Answer 3. (TA version) Notice that interval is a connected set, and $t|t \succ y$, $t|y \succ t$ are non-empty open. Hence, there must be an element which doesn't lies in their union, say m, hence $y \sim m$.

Answer 4. Say $(x_1, x_2) \succ^* (y_1, y_2)$,that is $\max\{x_1, x_2\} > \max\{y_1, y_2\}$, we must prove that $x_1^n + x_2^n > y_1^n + y_2^n$ for n big enough. W.L.O.G. let $x_1 = \max\{x_1, x_2\}$ and $y_1 = \max\{y_1, y_2\}$. $y_1^n + y_2^n \le 2y_1^n$ and $x_1^n + x_2^n \ge x_1^n$. Since $x_1 > y_1$, we have $\dfrac{x_1}{y_1} > 1$. Take N such that $\left(\dfrac{x_1}{y_1}\right)^N > 2$. For all $n > N$, we then have

$$x_1^n + x_2^n \ge x_1^n = y_1^n \left(\frac{x_1}{y_1}\right)^n > y_1^n \left(\frac{x_1}{y_1}\right)^N > 2y_1^n \ge y_1^n + y_2^n$$

Answer 5. We apply induction on $|X|$. If $|X| = 1$, then trivial. Consider $|X| = k > 1$.
Let $X = \{x_1, \cdots, x_k\}$ and $x_1 \succsim \cdots \succsim x_k$. By induction hypothesis, there exists u' on $X \setminus \{x_1\}$ representing $(\succsim, \succ\succ)$. We now define u in the following way:
1. $x_1 \sim x_2$: Define $u(x_1) = u'(x_2)$ and $u(x_i) = u'(x_i)$ for $i \neq 1$
2. $x_1 \succ x_2$:
(1). If for all $i > 1$, $x_1 \succ\succ x_i$: Define $u(x_1) = u'(x_2) + 2, u(x_i) = u'(x_i)$ for $i \neq 1$. It is an representation since $u(x_1) - u(x_i) \ge u(x_1) - u(x_2) > 1$.
(2). If there exists $i > 1$, x_1 not $\succ\succ$ x_i: Let α be the maximal that $x_2 \sim x_\alpha$. Define $u''(x_i) = u'(x_i)$ for $i \neq 2, \cdots, \alpha$ and $u''(x_2) = \cdots = u''(x_\alpha) = u'(x_2) - \dfrac{1}{n}$ for n big enough such that $u'(x_2) > u'(x_i) + 1 \implies u'(x_2) - \dfrac{1}{n} > u'(x_i) + 1$ and $u'(x_2) > u'(x_i) \implies u'(x_2) - \dfrac{1}{n} > u'(x_i)$ for all i. Notice that u'' still represents $(\succsim, \succ\succ)$. W.L.O.G, we can let u' be a function such that if x_2 not $\succ\succ$ x_i then $u'(x_2) < u(x_i) + 1$.
Define $u(x_1) = u'(x_k) + 1$ for which k is the maximal that satisfies x_1 not $\succ\succ$ x_k and $u(x_i) = u'(x_i)$ otherwise. This is an representation since for $i > k$, $u(x) = u'(x_k) + 1 > u'(x_i) + 1$ ($x_k \succ x_i$ by maximality of k); for $i = 2, \cdots k$, x_2 not $\succ\succ$ x_k, or $x_1 \succ\succ x_k$ and so we have the inequality: $u(x_k) \le u(i) \le u(x_2) < u(x_k) + 1 = u(x_1)$.

Answer 6.

a. Yes.
(S-1) One is approximately the same as itself.
(S-2) If one is approximately the same as another, then so is the converse.

(S-3) If x_n is approximately the same as y_n and $x_n \to x, y_n \to y$, then x, y should be approximately the same.
(S-4) If two numbers are between the two which are already approximately the same, then also should the two.
(S-5) For each element, if one can pick an element close enough to the given one, then the two should be approximately the same.
(S-6) If the given element change continuously, then should the range of "approximately the same as the given element".

b.

Only S-3 is not that trivial. If $(x_n, y_n) \in X \times X$ converges, then $(x_n, y_n) \to (x, y)$ for some x, y. We have $x_n \to x$ and $y_n \to y$ and so $x_n - y_n \to x - y$. Since taking absolute value is continuous $|x_n - y_n| \to |x - y|$. By $|x_n - y_n| \leq \epsilon$, we have $|x - y| < \epsilon \Rightarrow xS_\epsilon y$

c.

We first claim that M is continous:
(1) M conti at 0: For all sequence $\{x_n\} \searrow 0$, we have $M(x_n)$ decreasing and bounded below ($M(0)$ is a trivial lower bound), hence $\lim_{n\to\infty} M(x_n)$ exists. By $M(x_n)Sx_n$, we have, by continuity, $\lim_{n\to\infty} M(x_n)S0$. Thus, $\lim_{n\to\infty} M(x_n) \leq M(0)$. However, $\geq$ is trivial since $M(x_n) \geq M(0)$, we have M is continuous at 0.
(2) M conti at 1: $M(1-\epsilon) \geq 1 - \epsilon$, trivial.
(3) For all $t \in (0, 1)$, M conti at t: It suffices to prove that for all $x_n \searrow t$ and $y_n \nearrow t$, $\lim_{n\to\infty} M(x_n) = \lim_{n\to\infty} M(y_n) = M(t)$.
For all sequence $\{x_n\} \searrow t$, we have $M(x_n)$ decreasing and bounded below ($M(t)$ is a trivial lower bound), hence $\lim_{n\to\infty} M(x_n)$ exists. By $M(x_n)Sx_n$, we have, by continuity, $\lim_{n\to\infty} M(x_n)St$. Thus, $\lim_{n\to\infty} M(x_n) \leq M(t)$. However, $\geq$ is trivial since $M(x_n) \geq M(t)$.
Let $\{y_n\} \nearrow t$. If $t > \inf\{x|M(x) = 1\}$, then trivial. We suppose $t \leq \inf\{x|M(x) = 1\}$, that is, if $t' < t$, then $M(t') < 1$.
We have $M(y_n)$ increasing and bounded above ($M(t)$ is a trivial upper bound), hence $\lim_{n\to\infty} M(y_n)$ exists. $\lim_{n\to\infty} M(y_n) \leq M(t)$ is trivial since $M(y_n) \leq M(t)$.
We want to use m to help us argue the other side, so we here first observe m.
Claim: For all $s \leq t$, we have $m(M(s)) = s$
$s \geq m(M(s))$ is trivial. Notice that $M(m(M(s))) \geq M(s)$, and M is strictly increasing below s. Hence, $m(M(s)) \geq s$.
Thus, the claim is proven.

For the other side, since $\lim M(y_n) \geq M(y_n)$,

$$m(\lim_{n\to\infty} M(y_n)) \geq m(M(y_n)) \Rightarrow m(\lim_{n\to\infty} M(y_n)) \geq \lim_{n\to\infty} m(M(y_n)) = \lim_{n\to\infty} y_n = t = m(M(t))$$
$$\Rightarrow \lim_{n\to\infty} M(y_n) \geq M(t)$$

Remind that the last $\Rightarrow$ relies on $t \neq 0$.

Next, we claim that $\exists n \in N$ such that $M^n(0) = 1$, where we denote $M^2(0) = M(M(0))$, $M^n(0) = M(M^{n-1}(0))$ for $n > 2$.
Suppose not, then since M is continuous $M(\lim_{n\to\infty} M^n) = \lim_{n\to\infty} M^{n+1}(0) = \lim_{n\to\infty} M^n(0)$. However, since $\lim_{n\to\infty} M^n(0) \neq 1$, $M(\lim_{n\to\infty} M^n(0)) > \lim_{n\to\infty} M^n(0)$, which leads to a contradiction.

Last, we define H. W.L.O.G, let $\epsilon = M(0)$.
Let N be the number that $M^N(0) = 1$, and let t be the minimal that $M^{N-1}(t) = 1$. Let

$$H(x) = \begin{cases} x & \text{when } x \in [0, M(0)] \\ M(0)i + r & \text{when } x \in (M^i(0), M^{i+1}(0)] \text{ and } x = M^i(r) \text{ for } i = 1, \cdots, N-2 \\ M(0)(N-1) + r & \text{when } x \in (M^{N-1}(0), 1] \text{ and } x = M^{N-1}(r) \end{cases}$$

This can be check by $aSb \iff H(b) \leq H(a) + M(0)$ when $b > a$ straightforwardly.

Comment My thought of this proof is like the following. First, I want to construct H straightforwardly, so I write down

$$H(x) = \begin{cases} x & \text{when } x \in [0, M(0)] \\ M(0)i + r & \text{when } x \in (M^i(0), M^{i+1}(0)] \text{ and } x = M^i(r) \text{ for } i = 1, \cdots, N-2 \\ M(0)(N-1) + r & \text{when } x \in (M^{N-1}(0), 1] \text{ and } x = M^{N-1}(r) \end{cases}$$

in the very beginning. And I observed that if this function works, then the procedure of taking M repeatedly needs to attain 1. Notice that $M^n(0)$ is an increasing sequence, using "monotone+bounded $\Longrightarrow$ convergence", and try to apply M on its limit seems to be a natural way to say the limit must be 1. Hence, we need $M(\lim M^n) = \lim M(M^n)$, which means that we need M : continuous.