Macroeconomic Theory: Assignment 6

Exercise 1. (AK Model) Consider the infinite horizon growth model. An infinitely-lived representative household values consumption in each period using the period utility is $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma > 0$, and the subjective discount factor $\beta \in (0,1)$. Production only requires the input of capital and takes the following functional form $y_t = Ak_t$ with $A > 0$. The accumulation of capital follows

$$k_{t+1} = Ak_t - c_t$$

and the initial capital stock is $k_0 > 0$. Then the sequential problem is

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \quad \text{subject to} \quad Ak_t = k_{t+1} + c_t.$$

$v(k) = \max_{k' \leq Ak} \{ u(Ak_t - k') + \beta v(k') \}$

1. (a) Write down the functional equation of the associated time t value function

   (b) Conjecture that the value function takes the form

   $$v(k) = E\frac{k^{1-\sigma}}{1-\sigma}, \tag{1}$$

   where $E > 0$ is an undetermined coefficient. Solve for E.

   (c) Solve for the policy function h such that $k_{t+1} = h(k_t)$.

Exercise 2. (Habit Persistence) Consider an infinite horizon growth model. The household's utility function takes the form

$$\ln c_t + \phi \ln c_{t-1},$$

where $\phi > 0$, and the capital accumulation follows

$$c_t + k_{t+1} = Ak_t^{\alpha}.$$

Then the sequential problem is

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\ln(c_t) + \gamma \ln(c_{t-1})\right] \quad \text{subject to} \quad Ak_t^{\alpha} = k_{t+1} + c_t.$$

Write down functional equation for the associated time t value function.

1. Write down the functional equation of the associated time t value function (Hint: the state variables include the current capital stock k_t and the previous consumption c_{t-1})

2. We guess that the time t value function takes the following form

$$v(k_t, c_{t-1}) = E + F \ln k_t + G \ln c_{t-1}$$

   Where E, F, G are undetermined coefficients. Use guess and verify to solve for the value function $v(k_t, c_{t-1})$ and the policy function $k_{t+1} = h(k_t, c_{t-1})$

E1

1. $v(k) = \max\limits_{k' \leq Ak} \left\{ \frac{(Ak-k')^{1-\sigma}}{1-\sigma} + \beta v(k') \right\}$ ※

2. Consider $Tv(k) = \max\limits_{k' \leq Ak} \left\{ \frac{(Ak-k')^{1-\sigma}}{1-\sigma} + \beta v(k') \right\}$, find fixed pt. for T.

Guess $v(k) = E \cdot \frac{k^{1-\sigma}}{1-\sigma}$

then $Tv(k) = \max\limits_{k' \leq Ak} \left\{ \frac{(Ak-k')^{1-\sigma}}{1-\sigma} + \beta E \cdot \frac{k'^{1-\sigma}}{1-\sigma} \right\}$

FOC: $\frac{1}{(Ak-k')^{\sigma}} = \beta \cdot E \cdot \frac{1}{k'^{\sigma}} \Rightarrow k'^{\sigma} = \beta E \cdot (Ak-k')^{\sigma}$

$\Rightarrow k' = (\beta E)^{\frac{1}{\sigma}} \cdot (Ak-k') \begin{cases} \Rightarrow (Ak-k') = k' \cdot (\beta E)^{-\frac{1}{\sigma}} \\ \Rightarrow k' = Ak \cdot [1+(\beta E)^{-\frac{1}{\sigma}}]^{-1} \end{cases}$

FOC完，反而會先用兩邊都有k'的等式化簡

So, $Tv(k) = \frac{1}{1-\sigma} \cdot \left(\frac{k'}{(\beta E)^{\frac{1}{\sigma}}} \right)^{1-\sigma} + \beta E \cdot \frac{k'^{1-\sigma}}{1-\sigma}$

$= \frac{k'^{1-\sigma}}{1-\sigma} \cdot \left[\frac{1}{(\beta E)^{\frac{1-\sigma}{\sigma}}} + \beta E \right] = \frac{k'^{1-\sigma}}{1-\sigma} \cdot \beta E \cdot [1+(\beta E)^{-\frac{1}{\sigma}}]$

$= \frac{k^{1-\sigma}}{1-\sigma} \cdot A^{1-\sigma} [1+(\beta E)^{-\frac{1}{\sigma}}]^{\sigma-1} \cdot \beta E \cdot [1+(\beta E)^{-\frac{1}{\sigma}}]$

$= \frac{k^{1-\sigma}}{1-\sigma} \cdot A^{1-\sigma} \cdot \beta E \cdot [1+(\beta E)^{-\frac{1}{\sigma}}]^{\sigma}$

$\because Tv(k) = v(k) \quad \therefore E = A^{1-\sigma} \cdot \beta E \cdot [1+(\beta E)^{-\frac{1}{\sigma}}]^{\sigma}$

$\Rightarrow 1 = A^{1-\sigma} \cdot \beta \cdot [1+(\beta E)^{-\frac{1}{\sigma}}]^{\sigma} \Rightarrow [A^{\sigma-1} \cdot \beta^{-1}]^{\frac{1}{\sigma}} = 1+(\beta E)^{-\frac{1}{\sigma}}$

$\Rightarrow A^{\frac{\sigma-1}{\sigma}} \cdot \beta^{-\frac{1}{\sigma}} - 1 = (\beta E)^{-\frac{1}{\sigma}} \Rightarrow A^{\frac{\sigma-1}{\sigma}} - \beta^{\frac{1}{\sigma}} = E^{-\frac{1}{\sigma}}$

$\Rightarrow E = [A^{\frac{\sigma-1}{\sigma}} - \beta^{\frac{1}{\sigma}}]^{-\sigma}$ ※

3. $\because (\beta E)^{-\frac{1}{\sigma}} = A^{\frac{\sigma-1}{\sigma}} \cdot \beta^{-\frac{1}{\sigma}} - 1$

$\therefore k_{t+1} = Ak_t \cdot [1+(\beta E)^{-\frac{1}{\sigma}}]^{-1} = Ak_t \cdot [1+A^{\frac{\sigma-1}{\sigma}} \cdot \beta^{-\frac{1}{\sigma}} - 1]^{-1}$

$= Ak_t \cdot A^{\frac{1-\sigma}{\sigma}} \cdot \beta^{\frac{1}{\sigma}} = A^{\frac{1}{\sigma}} \cdot \beta^{\frac{1}{\sigma}} \cdot k_t$

So, $h(k_t) = A^{\frac{1}{\sigma}} \cdot \beta^{\frac{1}{\sigma}} \cdot k_t$ ※

E2

1. $v(k_t, C_{t-1}) = \max\limits_{k_{t+1} \le A k_t^\alpha} \left\{ \ln(A k_t^\alpha - k_{t+1}) + \gamma \ln C_{t-1} + \beta\, v(k_{t+1}, C_t) \right\}$

2. Consider $Tv(k_t, C_{t-1}) = \max\limits_{k_{t+1} \le A k_t^\alpha} \left\{ \ln(A k_t^\alpha - k_{t+1}) + \gamma \ln C_{t-1} + \beta\, v(k_{t+1}, C_t) \right\}$

Guess $v(k_t, C_{t-1}) = E + F \ln k_t + G \ln C_{t-1}$

then
$$Tv(k_t, C_{t-1}) = \max_{k_{t+1} \le A k_t^\alpha} \Big\{ \ln(A k_t^\alpha - k_{t+1}) + \gamma \ln C_{t-1} + \beta \left[E + F \ln k_{t+1} + G \ln(A k_t^\alpha - k_{t+1}) \right] \Big\}$$

FOC: $\dfrac{1}{A k_t^\alpha - k_{t+1}} = \beta \cdot \left(\dfrac{F}{k_{t+1}} - \dfrac{G}{A k_t^\alpha - k_{t+1}} \right)$

$\Rightarrow 1 = \beta \cdot \dfrac{F(A k_t^\alpha - k_{t+1})}{k_{t+1}} - \beta G, \quad \Rightarrow k_{t+1} = \beta \cdot F \cdot (A k_t^\alpha - k_{t+1}) - \beta G k_{t+1}$

$\Rightarrow A k_t^\alpha - k_{t+1} = \dfrac{1+\beta G}{\beta F} k_{t+1}$

$\Rightarrow (1 + \beta F + \beta G) k_{t+1} = A k_t^\alpha \quad \Rightarrow k_{t+1} = \dfrac{\beta \cdot F \cdot A}{1 + \beta F + \beta G} k_t^\alpha$

So
$$\begin{aligned}
Tv(k_t, C_{t-1}) &= \ln\left(\frac{1+\beta G}{\beta F} k_{t+1}\right) + \gamma \ln C_{t-1} + \beta\left[E + F \ln k_{t+1} + G \ln\left(\frac{1+\beta G}{\beta F} k_{t+1}\right)\right] \\
&= \ln \frac{1+\beta G}{\beta F} + \gamma \ln C_{t-1} + \beta E + \beta G \cdot \ln \frac{1+\beta G}{\beta F} \\
&\quad + \ln k_{t+1} + \beta F \ln k_{t+1} + \beta G \cdot \ln k_{t+1} \\
&= (1+\beta G) \ln \frac{1+\beta G}{\beta F} + \gamma \ln C_{t-1} + \beta E + (1 + \beta F + \beta G) \cdot \ln k_{t+1} \\
&= (1+\beta G) \ln \frac{1+\beta G}{\beta F} + \gamma \ln C_{t-1} + \beta E \\
&\quad + (1 + \beta F + \beta G) \cdot \ln \frac{\beta \cdot F \cdot A}{1 + \beta F + \beta G} k_t^\alpha \\
&= \left[(1+\beta G) \ln \frac{(1+\beta G) A}{1 + \beta F + \beta G} + \beta F \ln \frac{\beta F A}{1 + \beta F + \beta G} + \beta E \right] \\
&\quad + \alpha (1 + \beta F + \beta G) \ln k_t \\
&\quad + \gamma \ln C_{t-1}
\end{aligned}$$

$\because Tv(k_t, C_{t-1}) = v(k_t, C_{t-1}) \quad \therefore F = \alpha(1 + \beta F + \beta G), \quad G = \gamma$

$\Rightarrow F = \dfrac{\alpha(1+\beta\gamma)}{1-\alpha\beta}$

$\frac{E}{\alpha} - \beta F$

Also, $E = \left(\dfrac{E}{\alpha} - \beta F\right) \cdot \ln \dfrac{A \cdot \left(\frac{\cancel{F}}{\alpha} - \beta \cancel{F}\right)}{\frac{\cancel{F}}{\alpha}} + \beta F \cdot \ln \dfrac{\beta \cancel{F} A}{\frac{\cancel{F}}{\alpha}} + \beta E$

$= \left(\dfrac{E}{\alpha} - \beta F\right) \cdot \ln(A - \alpha\beta \cancel{A}) + \beta F \cdot \ln \alpha\beta A + \beta E$

$\Rightarrow (1-\beta) E = \dfrac{1+\beta\gamma}{1-\alpha\beta} \ln(A - \alpha\beta) + \dfrac{\alpha\beta(1+\beta\gamma)}{1-\alpha\beta} \cdot \ln \dfrac{\alpha\beta A}{A - \alpha\beta}$

$$\Rightarrow \quad E = \frac{1}{1-\beta}\left[\frac{1+\beta\gamma}{1-\alpha\beta}\ln(A-\alpha\beta) + \frac{\alpha\beta(1+\beta\gamma)}{1-\alpha\beta}\cdot \ln\frac{\alpha\beta A}{A-\alpha\beta}\right]$$

Hence,
$$v(k_t, C_{t-1}) = \frac{1}{1-\beta}\left[\frac{1+\beta\gamma}{1-\alpha\beta}\ln(A-\alpha\beta) + \frac{\alpha\beta(1+\beta\gamma)}{1-\alpha\beta}\cdot \ln\frac{\alpha\beta A}{A-\alpha\beta}\right]$$
$$+ \frac{\alpha(1+\beta\gamma)}{1-\alpha\beta}\ln k_t$$
$$+ \gamma \ln C_{t-1}$$

Finally,
$$k_{t+1} = \frac{\beta\cdot F\cdot A}{1+\beta F+\beta G} k_t^\alpha = \frac{\frac{\alpha\beta(1+\beta\gamma)A}{1-\alpha\beta}}{1+\frac{\alpha\beta(1+\beta)\gamma}{1-\alpha\beta}+\beta\gamma}\cdot k_t^\alpha$$

So,
$$h(k_t, C_{t-1}) = \frac{\frac{\alpha\beta(1+\beta\gamma)A}{1-\alpha\beta}}{1+\frac{\alpha\beta(1+\beta\gamma)}{1-\alpha\beta}+\beta\gamma}\cdot k_t^\alpha$$
$$= \frac{\alpha\beta A/(1-\alpha\beta)}{1+\alpha\beta/(1-\alpha\beta)}\cdot k_t^\alpha$$
$$= \frac{\alpha\beta A}{(1-\alpha\beta)+\alpha\beta}\cdot k_t^\alpha = \alpha\beta A\cdot k_t^\alpha .$$

qed.