NTU Macroeconomic Theory I

Homework 3

- The deadline is 9:00AM of December 27, 2021.
- Please submit your own version to NTU COOL.

1 Durable goods and Calibration

Consider a neoclassical growth model with consumer durables. A representative agent has preferences defined over sequences of non-durables consumption, c_t, a stock of durables, d_t, and leisure, $1-h_t$. In particular, a representative's preferences are given by $\sum_{t=0}^{\infty} \beta^t [\alpha \log c_t + (1-\alpha) \log d_t + A \log(1-h_t)]$.

There is a single technology that is used to produce non-durable consumption, investment in capital, and consumer durables: $y_t = \gamma^t k_t^{\theta} h_t^{1-\theta}$, where $\gamma > 1$. Suppose that there are $N_0 = 1$ households in period 0 and that the population grows over time with the rate η, where $\eta > 1$. To be clear, we use capital letters to denote aggregate variables and small letters refer to per capita variables.

Suppose that investment in capital and durables become available in the following period. In addition, suppose that capital depreciates at the rate δ_k and consumer durables depreciate at the rate δ_d.

1. Describe a social planner's dynamic programming problem that gives equal weight to all households.
2. Derive a set of equations that characterize the balanced growth path for the economy. What is the growth rate of per capita output along the balanced growth path?
3. Suppose that you are given the following statistics computed from the U.S. data:

1. ~~max~~ $\sum \beta^t [\alpha \log C_t + (1-\alpha) \log D_t + A \log (1-H_t)]$ $\quad N\, u(c) \neq u(Nc)$

$$\text{s.t.} \begin{cases} C_t + K_{t+1} + D_{t+1} = \gamma^t K_t^\theta H_t^{1-\theta} + (1-\delta_k)K_t + (1-\delta_d)D_t. \\ N_{t+1} = \eta N_t, \quad N_0 = 1 \\ ?_t = N_t \cdot ?_t, \quad \text{where } ? = c, k, d, h \end{cases}$$

RC $\overset{\div N_t}{\Rightarrow}$ $c_t + \eta (k_{t+1} + d_{t+1}) = \gamma^t k_t^\theta h_t^{1-\theta} + (1-\delta_k) k_t + (1-\delta_d) d_t$

Let $x_t = g_x^t \cdot \hat{x}_t$

$$\Rightarrow g_c^t \cdot \hat{c}_t + \eta (g_k^{t+1} \hat{k}_{t+1} + g_d^{t+1} \hat{d}_{t+1}) = \gamma^t \cdot (g_k^t \cdot \hat{k}_t)^\theta \cdot \Big(\overset{h_t}{\cancel{g_h^t \cdot \hat{h}_t}}\Big)^{1-\theta} + (1-\delta_k)(g_k^t \cdot \hat{k}_t) + (1-\delta_d)(g_d^t \hat{d}_t)$$

$$\Rightarrow \hat{c}_t + \eta \Big[\Big(\frac{g_k}{g_c}\Big)^t \cdot g_k \cdot \hat{k}_{t+1} + \Big(\frac{g_d}{g_c}\Big)^t g_d \hat{d}_{t+1} \Big] = \Big(\frac{\gamma g_k^\theta}{g_c}\Big)^t \hat{k}_t^\theta \cdot \Big[\overset{h_t}{\cancel{\Big(\frac{g_h}{g_c}\Big)^t \hat{h}_t}}\Big]^{1-\theta} + (1-\delta_k)\Big[\Big(\frac{g_k}{g_c}\Big)^t \cdot \hat{k}_t\Big] + (1-\delta_d)\Big[\Big(\frac{g_d}{g_c}\Big)^t \hat{d}_t\Big]$$

$$\Rightarrow g_k = g_c = g_d, \quad \frac{\gamma g_k^\theta}{g_c} = 1 \quad \Rightarrow \quad \frac{\gamma g_c^\theta}{g_c} = 1 \quad \Rightarrow g_c = \Big(\frac{1}{\gamma}\Big)^{\frac{1}{\theta-1}} = \gamma^{\frac{1}{1-\theta}}.$$

Let $g \equiv g_k = g_c = g_d = \gamma^{\frac{1}{1-\theta}}$.

$$\begin{aligned} U &= \sum N_t \beta^t [\alpha \log g^t \hat{c} + (1-\alpha) \log g^t \hat{d} + A \log (1-h_t)] \\ &= \sum \eta^t \beta^t [\alpha \log \hat{c} + (1-\alpha) \log \hat{d} + A \log (1-h_t)] + \sum (\beta\eta)^t t \log g \end{aligned}$$

Note: $\sum (\beta\eta)^t t \log g = \Big(\frac{\beta\eta}{1-\beta\eta}\Big)^2 \log g$

$$V(\hat{d}, \hat{k}) = \max_{\{\hat{d}', \hat{k}', h\}} \{\alpha \log \hat{c} + (1-\alpha) \log \hat{d} + A \log (1-h_t) + (\eta\beta) V(\hat{d}', \hat{k}')\}$$
$$\text{s.t.} \quad \hat{c} + \eta \gamma^{\frac{1}{1-\theta}} (\hat{k}' + \hat{d}') = \hat{k}^\theta \cdot h^{1-\theta} + (1-\delta_k)\hat{k} + (1-\delta_d)\hat{d}$$

2. $$V(\hat{d}, \hat{k}) = \max_{\{\hat{d}', \hat{k}', h\}} \Big\{\alpha \log \Big[\hat{k}^\theta \cdot h^{1-\theta} + (1-\delta_k)\hat{k} + (1-\delta_d)\hat{d} - \big(\eta \gamma^{\frac{1}{1-\theta}} (\hat{k}' + \hat{d}')\big)\Big] + (1-\alpha) \log \hat{d} + A \log (1-h) + (\eta\beta) V(\hat{d}', \hat{k}')\Big\}$$

FOC to $[\hat{d}']$: $\alpha \cdot \dfrac{\eta \gamma^{\frac{1}{1-\theta}}}{\hat{c}} = \eta\beta V_d(\hat{d}', \hat{k}')$.

$[\hat{k}']$: $\alpha \cdot \dfrac{\eta \gamma^{\frac{1}{1-\theta}}}{\hat{c}} = \eta\beta V_k(\hat{d}', \hat{k}')$.

$[h]$: $\alpha \cdot \dfrac{(1-\theta)\hat{k}^\theta h^{-\theta}}{\hat{c}} = A \cdot \dfrac{1}{1-h}$

EC: $V_d(\hat{d}, \hat{k}) = \alpha \cdot \dfrac{(1-\delta_d)}{\hat{c}} + \dfrac{1-\alpha}{\hat{d}} \Rightarrow V_d(\hat{d}', \hat{k}') = \alpha \cdot \dfrac{(1-\delta_d)}{\hat{c}'} + \dfrac{1-\alpha}{\hat{d}'}$

$V_k(\hat{d}, \hat{k}) = \alpha \cdot \dfrac{\theta \hat{k}^{\theta-1} h^{1-\theta} + (1-\delta_k)}{\hat{c}} \Rightarrow V_k(\hat{d}', \hat{k}') = \alpha \cdot \dfrac{\theta \hat{k}'^{\theta-1} h'^{1-\theta} + (1-\delta_k)}{\hat{c}'}$

Combine :
$$\begin{cases} \dfrac{\alpha \gamma^{\frac{1}{1-\theta}}}{\beta \hat{c}} = \alpha \cdot \dfrac{(1-\delta_d)}{\hat{c}'} + \dfrac{1-\alpha}{\hat{d}'} & \dots \quad (\ast 1) \\ \dfrac{\alpha \gamma^{\frac{1}{1-\theta}}}{\beta \hat{c}} = \alpha \cdot \dfrac{\theta \hat{k}^{\theta-1} \hat{h}^{1-\theta} + (1-\delta_k)}{\hat{c}'} & \dots \quad (\ast 2) \end{cases}$$

And we have shown $g = \gamma^{\frac{1}{1-\theta}}$

3.
$$\begin{cases} 1-\theta = 0.6 \Rightarrow \theta = 0.4 \\ \dfrac{\bar{k}}{\bar{y}} = 3.5, \quad \dfrac{\bar{d}}{\bar{y}} = 0.9, \quad \dfrac{\bar{i}_k}{\bar{y}} = 0.2, \quad \dfrac{\bar{i}_d}{\bar{y}} = 0.05 \\ \gamma = 1.014, \quad \bar{h} = 0.31, \quad \eta = 1.015 \end{cases}$$

① $K_{t+1} = (1-\delta_k) K_t + I_t \Rightarrow \eta g \hat{k}_{t+1} = (1-\delta_k) \hat{k}_t + \hat{i}_k$

In S-S, $\hat{k}_{t+1} = \hat{k}_t \equiv \bar{k}$, $\hat{i}_k \equiv \bar{i}_k$

So, $\delta_k = \dfrac{\bar{i}_k}{\bar{k}} + 1 - \eta g = 0.0183$

Similarly, $\delta_d = \dfrac{\bar{i}_d}{\bar{d}} + 1 - \eta g = 0.0167$

② Check ①: In S-S, $\hat{d} = \hat{d}' \equiv \bar{d}$, $\hat{c} = \hat{c}' \equiv \bar{c}$, $\hat{y} = \hat{y}' \equiv \bar{y}$.

$$RC \Rightarrow \frac{\bar{c}}{\bar{y}} + \left(\eta \gamma^{\frac{1}{1-\theta}} + \delta_d - 1\right) \frac{\bar{d}}{\bar{y}} + \left(\eta \gamma^{\frac{1}{1-\theta}} + \delta_k - 1\right) \frac{\bar{k}}{\bar{y}} = 1$$

$$\Rightarrow \frac{\bar{c}}{\bar{y}} = 1 - \frac{\bar{i}_d}{\bar{y}} - \frac{\bar{i}_k}{\bar{y}} = 0.75.$$

③ Euler $\Rightarrow \alpha(1-\theta) \cdot \dfrac{\bar{k}^{\theta} \bar{h}^{1-\theta}}{\bar{c}} = A \cdot \dfrac{\bar{h}}{1-\bar{h}} \Rightarrow \dfrac{\alpha}{A} = 0.5616$

$$(\ast 1) \Rightarrow \alpha(1-\delta_d) + (1-\alpha)\frac{\bar{c}}{\bar{d}} = \frac{\alpha \gamma^{\frac{1}{1-\theta}}}{\beta} \Rightarrow \beta(1-\delta_d) = 1.0234 - \frac{1-\alpha}{\alpha} \cdot \frac{5}{6} \cdot \beta$$

$$(\ast 2) \Rightarrow \theta \cdot \frac{\bar{k}^{\theta} \cdot \bar{h}^{1-\theta}}{\bar{k}} + (1-\delta_k) = \frac{\gamma^{\frac{1}{1-\theta}}}{\beta} \Rightarrow \beta(1.1143 - \delta_k) = 1.0234$$

$\Rightarrow \beta = 0.93385$, $\alpha = 0.86087$, $A = 1.5685$

a. the average labor income share, 0.6;

b. the average capital to output ratio, 3.5;

c. the average consumer durable stock to output ratio, 0.9;

d. the average capital investment to output ratio, 0.2;

e. the average consumer durable purchases to output ratio, 0.05;

f. the average annual growth rate of per capita output, 1.014;

g. the average fraction of time that individuals spend on working in the market sector, 0.31;

h. the annual growth rate of population, 1.015.

Calibrate the economy so that the steady state matches the U.S. averages in these respects. Specifically, you need to find values for A, α, β, θ, γ, η, δ_k, and δ_d.

2 Life Expectancy

Consider a conditional probabilities of two states: life (z_1) and death (z_2). The probability of staying alive in a given period (such as a year) is p. The probability of dying is $1 - p$. If people die, they do not return to life. Therefore, the probability of transiting from z_2 to z_1 is zero.

A. Write down the transition matrix. What is the ergodic set(s)? Is there any transient state? If so, write down the transient state.

B. Solve the invariant distribution.

With the above information, we can compute the expected life duration (years). Denote the life duration by d.

C. If you are alive at period 0, what is the probability of living just one year? What is the probability of being alive for two years? What is the probability of being alive for t years?

D. Compute the life expectancy. If $p = 0.99$, How many years do you expect to be alive?

A.

$$\begin{bmatrix} p & 1-p \\ 0 & 1 \end{bmatrix}$$

The ergodic sets is $\{z_2\}$

There's a transient state z_1

B. Assume the invariant distribution is $\pi = [\pi_1 \quad 1-\pi_1]$.

By def, $\pi P = \pi$,

$\Rightarrow \pi_1 = p\pi_1$

For $p > 0$, $\pi_1 = 0$, implying that $\pi = [0 \quad 1]$.

C. $P(d=t) = p^t \cdot (1-p)$?!

D. $E[d] = \sum_{t=0}^{\infty} p^t \cdot (1-p) \cdot t$

$= (1-p) \sum p^t \cdot t$

$= (1-p) \cdot \frac{1}{(1-p)^2}$

$= (1-p)^{-1}$

With $p = 0.99$, $E[d] = 100$.