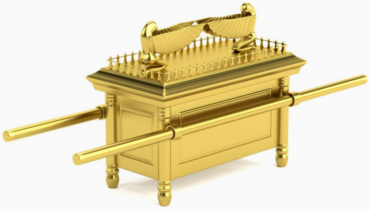
12. Dynamic Mechanism Design

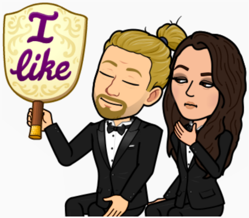
ECON 7219 – Games With Incomplete Information

Benjamin Bernard

Dynamic Revelation Principle

Dynamic Selling Mechanism





Before the auction:

- Auctioneer advertises the item he has for sale.
- Potential bidders form beliefs about their true valuation of the object.
- Potential bidders decide whether or not to attend the auction.

At the auction:

- Bidders examine the item more closely and learn their true valuation.
- Bidders bid for the item and the highest bidder wins.

Types and Utilities

Type space:

- Player i's utility depends only on their ex-post type $\vartheta_i \in \Theta_i$.
- At the time of entering the mechanism, player i has some information about ϑ_i available, reflected in their ex-ante type $\tau_i \in \mathcal{T}_i$.
- We suppose types are one-dimensional: $\Theta_i = [\underline{\vartheta}_i, \bar{\vartheta}_i]$ and $\mathcal{T}_i = [\underline{\tau}_i, \bar{\tau}_i]$.

Quasi-linear utilities:

- The set of alternatives $\mathcal{X}$ is $\mathcal{Q} \times \mathbb{R}^n$, where $q \in \mathcal{Q}$ is the social state.
- We assume $u_i(p, q, \vartheta_i) = v_i(q, \vartheta_i) - p_i$ such that $0 \leq \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \leq K$.
- Types are one-dimensional: there exists an order $\succ_i$ of elements in $\mathcal{Q}$ such that v_i has increasing differences, i.e., $q_H \succ q_L$ and $\vartheta_i > \vartheta_i'$ imply

$$v_i(q_H, \vartheta_i) - v_i(q_L, \vartheta_i) \geq v_i(q_H, \vartheta_i') - v_i(q_L, \vartheta_i').$$

Information of the Ex-Ante Type

Independent types:

- Types (T_i, θ_i) and (T_j, θ_j) are independent for any i, j.
- Joint distribution F_i of (T_i, θ_i) for any i is common knowledge.

Information:

- Player i's ex-ante information are reflected by his/her beliefs $F_i(\vartheta_i \,|\, \tau_i)$ about his/her true valuation.
- We impose that $F_i(\vartheta_i \,|\, \tau_i)$ is decreasing in τ_i for any $\vartheta_i \in (\underline{\vartheta}_i, \bar{\vartheta}_i)$: higher values of τ_i make high values of ϑ_i more likely.
- The support of $F_i(\vartheta_i \,|\, \tau_i)$ is independent of τ_i: the ex-ante type does not provide any certainty about the payoff type.
- The partial derivative $\partial F_i(\vartheta_i \,|\, \tau_i)/\partial \tau_i$ is bounded.

Timing in Dynamic Mechanism Design

Ex ante:

- Joint distribution of ex-ante and ex-post types is commonly known.
- Mechanism designer designs the mechanism.
- Players learn their ex-ante type and report an ex-ante type.

Interim:

- Players observe their ex-post type (payoff type).
- Players decide what ex-post type to report.

Ex post:

- Players' reports are publicly revealed.

Individual rationality and incentive compatibility:

- Individual rationality has to be satisfied only at the ex-ante stage.
- Incentive compatibility has to hold at the ex-ante and interim stage.

Direct Dynamic Mechanism

Direct dynamic mechanism:

- A direct dynamic mechanism is a pair (p, q) with

$$q : \mathcal{T} \times \Theta \to \Delta(\mathcal{Q}), \qquad p : \mathcal{T} \times \Theta \to \mathbb{R}^n.$$

- Denote by $\alpha_q(\tau, \vartheta)$ the probability that q is chosen.

Players' strategies:

- Players do not observe the ex-ante report of other players.
- We can write a strategy σ_i of player i using two maps

$$\sigma_i^0 : \mathcal{T}_i \to \mathcal{T}_i, \qquad \sigma_i^1 : \mathcal{T}_i \times \Theta_i \times \mathcal{T}_i \to \Theta_i.$$

Dynamic Revelation Principle

Proposition 12.1

For any indirect mechanism Γ and any PBE σ of that mechanism, there exists a direct dynamic mechanism Γ' and PBE $\widehat{\sigma}$ with

$$\widehat{\sigma}_i^0(\tau_i) = \tau_i, \qquad \widehat{\sigma}_i^1(\tau_i, \vartheta_i, \tau_i) = \vartheta_i,$$

that induces the same distribution over outcomes in Γ' as σ does in Γ.

Remark:

- Dynamic revelation principle specifies truth-telling only on the path.
- Truth-telling off the path will follow from payoff independence of τ_i.

Proof: Set $\Gamma' = \Gamma \circ \sigma$. Any deviation from $\widehat{\sigma}$ in Γ' has a corresponding deviation from σ in Γ and cannot be profitable.

Interim and Ex-Ante Utilities

Interim expected utility:

- Player i's interim expected utility of reporting $r_i \in \Theta_i$ is

$$u_i(R_i, r_i \mid \tau_i, \vartheta_i) := \mathbb{E}_{\tau_i, \vartheta_i}[v_i(q(R_i, T_{-i}, r_i, \theta_{-i}), \vartheta_i) - p_i(R_i, T_{-i}, r_i, \theta_{-i})]$$

  conditional on ex-ante report R_i and truthful reporting by others.

Ex-ante expected utility:

- Player i's ex-ante utility for reporting $R_i \in \mathcal{T}_i$ is

$$U_i(R_i, \sigma_i^1 \mid \tau_i) = \int_{\Theta_i} u_i(R_i, \sigma_i^1(\tau_i, \vartheta_i, R_i) \mid \tau_i, \vartheta_i) f_i(\vartheta_i \mid \tau_i)\, \mathrm{d}\vartheta_i,$$

  conditional on future report σ_i^1 and truthful reporting by others.
- Denote by $\widehat{U}_i(R_i \mid \tau_i) = U_i(R_i, \widehat{\sigma}_i^1 \mid \tau_i)$ the ex-ante utility of reporting R_i, conditional on truthful future report $\widehat{\sigma}_i^1(\tau_i, \vartheta_i, R_i) = \vartheta_i$.

Payoff Independence of the Ex-Ante Type

Payoff independence:

- Conditional on truthful reporting by others, player i's interim utility is

$$
\begin{aligned}
u_i(R_i, r_i \mid \tau_i, \vartheta_i) &= \mathbb{E}_{\tau_i,\vartheta_i}[v_i(q(R_i, T_{-i}, r_i, \theta_{-i}), \vartheta_i) - p_i(R_i, T_{-i}, r_i, \theta_{-i})] \\
&= \sum_{q\in Q} v_i(q, \vartheta_i) \underbrace{\mathbb{E}_{\tau_i,\vartheta_i}[\alpha_q(R_i, T_{-i}, r_i, \theta_{-i})]}_{=:\,\bar{\alpha}_q(R_i,r_i)} - \underbrace{\mathbb{E}_{\tau_i,\vartheta_i}[p_i(R_i, T_{-i}, r_i, \theta_{-i})]}_{=:\,\bar{p}_i(R_i,r_i)} \\
&=: u_i(r_i \mid R_i, \vartheta_i)
\end{aligned}
$$

- Knowing the ex-ante type is no longer valuable because:
  - At the interim stage, player i knows ϑ_i already.
  - Types are independent, hence it does not help predict the others' types.
- It will be convenient to introduce the notation

$$
\bar{v}_i(r_i \mid R_i, \vartheta_i) := \mathbb{E}_{R_i,\vartheta_i}[v_i(q(R_i, T_{-i}, r_i, \theta_{-i}), \vartheta_i)] = \sum_{q\in Q} v_i(q, \vartheta_i)\bar{\alpha}_q(R_i, r_i).
$$

Incentive Compatibility

Definition 12.2

A direct mechanism is incentive compatible if:

1. It is incentive compatible with respect to the ex-post type, i.e., for every type (τ_i, ϑ_i), and every ex-post report $r_i \in \Theta_i$,

$$u_i(\vartheta_i \mid \tau_i, \vartheta_i) \geq u_i(r_i \mid \tau_i, \vartheta_i).$$

2. It is incentive-compatible with respect to the ex-ante type, i.e., for every τ_i, every future report σ_i^1, and every ex-ante report $R_i \in \mathcal{T}_i$

$$\widehat{U}_i(\tau_i \mid \tau_i) \geq U_i(R_i, \sigma_i^1 \mid \tau_i).$$

Incentive Compatibility

Lemma 12.3

A direct mechanism is incentive-compatible if and only if it satisfies

1. *For every type* (τ_i, ϑ_i)*, and every ex-post report* $r_i \in \Theta_i$*,*
$$u_i(\vartheta_i \,|\, \tau_i, \vartheta_i) \geq u_i(r_i \,|\, \tau_i, \vartheta_i).$$
2. *For every ex-ante type* τ_i *and every ex-ante report* $R_i \in \mathcal{T}_i$
$$\widehat{U}_i(\tau_i \,|\, \tau_i) \geq \widehat{U}_i(R_i, \,|\, \tau_i).$$

Importance:

- Since ex-ante type τ_i does not affect interim utilities, truth-telling on the path is sufficient to prevent deviations off the path as well.
- Thus, the revelation principle implies truth-telling also off the path.

Proof of Lemma 12.3

Proof of necessity:

- $\widehat{U}_i(\tau_i \mid \tau_i) \geq U_i(R_i, \sigma_i^1 \mid \tau_i)$ for any σ_i^1 implies $\widehat{U}_i(\tau_i \mid \tau_i) \geq \widehat{U}_i(R_i \mid \tau_i)$.

Proof of sufficiency:

- Incentive-compatibility with respect to the ex-post type implies

$$\begin{aligned} U_i(R_i, \sigma_i^1 \mid \tau_i) &= \int_{\Theta_i} u_i(\sigma_i^1(\tau_i, \vartheta_i, R_i) \mid R_i, \vartheta_i) f_i(\vartheta_i \mid \tau_i) \, \mathrm{d}\vartheta_i \\ &\leq \int_{\Theta_i} u_i(\vartheta_i \mid R_i, \vartheta_i) f_i(\vartheta_i \mid \tau_i) \, \mathrm{d}\vartheta_i \\ &= \widehat{U}_i(R_i \mid \tau_i) \leq \widehat{U}_i(\tau_i \mid \tau_i). \end{aligned}$$

Revenue Equivalence

Characterizing Incentive Compatibility

Static mechanism with one-dimensional types:

- Incentive compatibility = monotonicity + revenue equivalence.
- Selling mechanism:
  - Higher type has to receive the object with a higher likelihood.
  - Marginal increase in expected payments are equal to the marginal benefit of the increase in likelihood to obtain the item.
- Does a similar result hold for dynamic mechanisms?

Interim incentive compatibility:

- For a given ex-ante type τ_i, incentive compatibility with respect to the ex-post type is identical to the static case.

Incentive Compatibility With Respect to the Ex-Post Type

Lemma 12.4

A direct dynamic mechanism is incentive compatible with respect to the ex-post type if and only if for every player i and every ex-ante type τ_i:

1. *"Monotonic" social state: $h(\tau_i, \vartheta_i)$ is non-decreasing in ϑ_i, where*

$$h(\tau_i, \vartheta_i) := \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(\tau_i, \vartheta_i).$$

2. *"Revenue equivalence" determines expected payments:*

$$\bar{p}_i(\tau_i, \vartheta_i) = \bar{p}_i(\tau_i, \underline{\vartheta}) + \sum_{q \in Q} \big(v_i(q, \vartheta_i) \bar{\alpha}_q(\tau_i, \vartheta_i) - v_i(q, \underline{\vartheta}) \bar{\alpha}_q(\tau_i, \underline{\vartheta}) \big)$$

$$- \sum_{q \in Q} \int_{\underline{\vartheta}}^{\vartheta_i} \frac{\partial v_i(q, x)}{\partial \vartheta_i} \bar{\alpha}_q(\tau_i, x) \, \mathrm{d}x.$$

Incentive Compatibility With Respect to the Ex-Ante Type

Revenue equivalence:

- It turns out that a revenue-equivalence result holds.
- Step 1: show absolute continuity and differentiability of $\widehat{U}_i(\tau_i \mid \tau_i)$.
- Step 2: use incentive compatibility to show the integral condition.

Monotonicity:

- Monotonicity of the allocation function with respect to the ex-ante type is sufficient, but not necessary for incentive compatibility.
- Step 1: show a simple counterexample to necessity.
- Step 2: show sufficiency.

Absolute Continuity

Lemma 12.5

Suppose that there exist K_F and $(K_q)_{q\in Q}$ such that

$$\left|\frac{\partial v_i(q,\vartheta_i)}{\partial \vartheta_i}\right| \le K_q, \qquad \left|\frac{\partial F_i(\vartheta_i \mid \tau_i)}{\partial \tau_i}\right| \le K_F.$$

Then for any incentive-compatible mechanism, $\widehat{U}_i(\tau_i \mid \tau_i)$ is increasing and absolutely continuous in τ_i.

Interpretation:

- Ex-ante expected utility is increasing in τ_i under truth-telling.

Selling mechanism:

- $K_q = 1$ since $\frac{\partial v_i(q,\vartheta_i)}{\partial \vartheta_i} = 1_{\{q=q_i\}}$.

Proof of Monotonicity

Proof of monotonicity:

- Integration by parts yields

$$\widehat{U}_i(R_i \mid \tau_i) = \int_{\Theta_i} \underbrace{u_i(\vartheta_i \mid R_i, \vartheta_i)}_{\downarrow} \underbrace{f_i(\vartheta_i \mid \tau_i)}_{\uparrow} \, \mathrm{d}\vartheta_i$$

$$= u_i(\bar{\vartheta} \mid R_i, \bar{\vartheta}) - \int_{\Theta_i} \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(R_i, \vartheta_i) F_i(\vartheta_i \mid \tau_i) \, \mathrm{d}\vartheta_i.$$

- For $\tau_i^2 > \tau_i^1$, ex-ante incentive compatibility implies

$$\widehat{U}_i(\tau_i^2 \mid \tau_i^2) - \widehat{U}_i(\tau_i^1 \mid \tau_i^1) \geq \widehat{U}_i(\tau_i^1 \mid \tau_i^2) - \widehat{U}_i(\tau_i^1 \mid \tau_i^1)$$

$$= \int_{\Theta_i} \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(\tau_i^1, \vartheta_i) \underbrace{\left(F_i(\vartheta_i \mid \tau_i^1) - F_i(\vartheta_i \mid \tau_i^2)\right)}_{\geq 0 \text{ by FOSD}} \, \mathrm{d}\vartheta_i \geq 0.$$

- This shows that $\widehat{U}(\tau_i \mid \tau_i)$ is non-decreasing.

Proof of Absolute Continuity

Bounding the difference:

- Ex-ante incentive compatibility implies that for any τ_i^2, τ_i^1,

$$\begin{aligned}\widehat{U}_i(\tau_i^2 \mid \tau_i^2) - \widehat{U}_i(\tau_i^1 \mid \tau_i^1) &\le \widehat{U}_i(\tau_i^2 \mid \tau_i^2) - \widehat{U}_i(\tau_i^2 \mid \tau_i^1) \\ &\le \sup_{R_i \in \mathcal{T}_i} \widehat{U}_i(R_i \mid \tau_i^2) - \widehat{U}_i(R_i \mid \tau_i^1).\end{aligned}$$

- Along the same lines, we obtain

$$\widehat{U}_i(\tau_i^2 \mid \tau_i^2) - \widehat{U}_i(\tau_i^1 \mid \tau_i^1) \ge \inf_{R_i \in \mathcal{T}_i} \widehat{U}_i(R_i \mid \tau_i^2) - \widehat{U}_i(R_i \mid \tau_i^1).$$

- Together, the two conditions yield

$$\left|\widehat{U}_i(\tau_i^2 \mid \tau_i^2) - \widehat{U}_i(\tau_i^1 \mid \tau_i^1)\right| \le \sup_{R_i \in \mathcal{T}_i} \left|\widehat{U}_i(R_i \mid \tau_i^2) - \widehat{U}_i(R_i \mid \tau_i^1)\right|.$$

Proof of Absolute Continuity

Lipschitz continuity:

- For $\tau_i^2 > \tau_i^1$, we have $F_i(\vartheta_i \mid \tau_i^1) - F_i(\vartheta_i \mid \tau_i^2) > 0$, hence

$$\begin{aligned}
\left|\widehat{U}_i(\tau_i^2 \mid \tau_i^2) - \widehat{U}_i(\tau_i^1 \mid \tau_i^1)\right| &\leq \sup_{R_i \in \mathcal{T}_i} \left|\widehat{U}_i(R_i \mid \tau_i^2) - \widehat{U}_i(R_i \mid \tau_i^1)\right| \\
&\leq \int_{\Theta_i} \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(R_i, \vartheta_i) \big(F_i(\vartheta_i \mid \tau_i^1) - F_i(\vartheta_i \mid \tau_i^2)\big) \, \mathrm{d}\vartheta_i \\
&\leq \int_{\Theta_i} \max_q K_q \left| \frac{\partial F_i(\vartheta_i \mid \tau_i')}{\partial \tau_i} \right| \left|\tau_i^2 - \tau_i^1\right| \, \mathrm{d}\vartheta_i \\
&\leq \max_q K_q K_F (\bar{\vartheta} - \underline{\vartheta}) \left|\tau_i^2 - \tau_i^1\right|,
\end{aligned}$$

  where we have used the mean-value theorem.

Thus, $\widehat{U}_i(\tau_i \mid \tau_i)$ is Lipschitz continuous and hence absolutely continuous.

Revenue Equivalence for Ex-Ante Utility

Proposition 12.6

Let $U_i(\tau_i) := \widehat{U}_i(\tau_i, \tau_i)$. For any incentive-compatible direct mechanism:

1. *U_i is differentiable everywhere except at most countably many points. At any point of differentiability τ_i, we have*

$$U_i'(\tau_i) = -\int_{\Theta_i} \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(\tau_i, \vartheta_i) \frac{\partial F_i(\vartheta_i \mid \tau_i)}{\partial \tau_i} \, \mathrm{d}\vartheta_i.$$

2. *For every ex-ante type τ_i, we have*

$$U_i(\tau_i) = U_i(\underline{\tau}) - \int_{\underline{\tau}}^{\tau_i} \int_{\Theta_i} \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(t, \vartheta_i) \frac{\partial F_i(\vartheta_i \mid t)}{\partial \tau_i} \, \mathrm{d}\vartheta_i \, \mathrm{d}t.$$

Corollary: An incentive-compatible mechanism is ex-ante individually rational if and only if $U_i(\underline{\tau}) \geq 0$.

Proof of Proposition 12.6

Derivative with respect to ex-ante type:

- Since $\widehat{U}_i(\tau_i \mid \tau_i)$ is monotonic, it is differentiable almost everywhere.
- Since $F_i(\vartheta_i \mid \tau_i)$ is differentiable with respect to τ_i,

$$\widehat{U}_i(R_i \mid \tau_i) = u_i(\bar{\vartheta} \mid R_i, \bar{\vartheta}) - \int_{\Theta_i} \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(R_i, \vartheta_i) F_i(\vartheta_i \mid \tau_i) \, \mathrm{d}\vartheta_i$$

is differentiable as well, with derivative

$$\frac{\partial \widehat{U}_i(R_i \mid \tau_i)}{\partial \tau_i} = - \int_{\Theta_i} \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(R_i, \vartheta_i) \frac{\partial F_i(\vartheta_i \mid \tau_i)}{\partial \tau_i} \, \mathrm{d}\vartheta_i .$$

Proof of Proposition 12.6

Revenue equivalence:

- Incentive-compatibility implies:

$$\frac{\widehat{U}_i(\tau_i \,|\, \tau_i) - \widehat{U}(\tau_i + \delta \,|\, \tau_i + \delta)}{\delta} \leq \frac{\widehat{U}_i(\tau_i \,|\, \tau_i) - \widehat{U}(\tau_i \,|\, \tau_i + \delta)}{\delta},$$

$$\frac{\widehat{U}_i(\tau_i - \delta \,|\, \tau_i - \delta) - \widehat{U}(\tau_i \,|\, \tau_i)}{\delta} \geq \frac{\widehat{U}_i(\tau_i \,|\, \tau_i - \delta) - \widehat{U}(\tau_i \,|\, \tau_i)}{\delta}.$$

- At any differentiability point of $U_i(\tau_i) := \widehat{U}_i(\tau_i \,|\, \tau_i)$, we obtain

$$\left.\frac{\partial \widehat{U}_i(R_i \,|\, \tau_i)}{\partial \tau_i}\right|_{R_i = \tau_i} \leq U_i'(\tau_i) \leq \left.\frac{\partial \widehat{U}_i(R_i \,|\, \tau_i)}{\partial \tau_i}\right|_{R_i = \tau_i}.$$

- Therefore,

$$U_i'(\tau_i) = -\int_{\Theta_i} \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(R_i, \vartheta_i) \frac{\partial F_i(\vartheta_i \,|\, \tau_i)}{\partial \tau_i} \, \mathrm{d}\vartheta_i.$$

Revenue Equivalence for Payments

Proposition 12.7

If a direct dynamic mechanism is incentive-compatible, then:

$$\bar{p}_i(\tau_i, \vartheta_i) = \bar{p}_{i,\underline{\vartheta}_i}(\tau_i) + \sum_{q \in Q} v_i(q, \vartheta_i)\bar{\alpha}_q(\tau_i, \vartheta_i) - \int_{\underline{\vartheta}}^{\vartheta_i} h_i(\tau_i, x)\,\mathrm{d}x,$$

where $h_i(\tau_i, \vartheta_i) := \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(\tau_i, \vartheta_i)$ *and*

$$\bar{p}_{i,\underline{\vartheta}_i}(\tau_i) := \bar{p}_i(\underline{\tau}, \underline{\vartheta}) - \sum_{q \in Q} v_i(q, \underline{\vartheta})\bar{\alpha}_q(\underline{\tau}, \underline{\vartheta}) + \int_{\underline{\tau}}^{\tau_i} \int_{\underline{\vartheta}}^{\bar{\vartheta}} h_i(t, \vartheta) \frac{\partial F_i(\vartheta_i \mid t)}{\partial \tau_i}\,\mathrm{d}\vartheta_i\,\mathrm{d}t$$
$$+ \int_{\underline{\vartheta}}^{\bar{\vartheta}} \int_{\underline{\vartheta}}^{\vartheta_i} \big(h_i(\tau_i, x) f_i(\vartheta_i \mid \tau_i) - h_i(\underline{\tau}, x) f_i(\vartheta_i \mid \underline{\tau})\big)\,\mathrm{d}x\,\mathrm{d}\vartheta_i$$

Note: Expected payments are determined uniquely up to $\bar{p}_i(\underline{\tau}, \underline{\vartheta})$.

Proof of Proposition 12.7

Use revenue equivalence for ex-ante utility:

- Recall the definition of the ex-ante utility

$$U_i(\tau_i) = \int_{\Theta_i} \left(\sum_{q \in Q} v_i(q, \vartheta_i) \bar{\alpha}_q(\tau_i, \vartheta_i) - \bar{p}_i(\tau_i, \vartheta_i) \right) f_i(\vartheta_i \,|\, \tau_i) \,\mathrm{d}\vartheta_i,$$

- From revenue equivalence for ex-ante utilities, we obtain

$$\begin{aligned}\int_{\Theta_i} \bar{p}_i(\tau_i, \vartheta_i) f(\vartheta_i | \tau_i) \,\mathrm{d}\vartheta_i = & \sum_{q \in Q} \int_{\Theta_i} v_i(q, \vartheta_i) \bar{\alpha}_q(\tau_i, \vartheta_i) f_i(\vartheta_i \,|\, \tau_i) \,\mathrm{d}\vartheta_i \\ & + \int_{\Theta_i} \left(\bar{p}_i(\underline{\tau}, \vartheta_i) - \sum_{q \in Q} v_i(q, \vartheta_i) \bar{\alpha}_q(\underline{\tau}, \vartheta_i) \right) f_i(\vartheta_i \,|\, \underline{\tau}) \,\mathrm{d}\vartheta_i \\ & + \int_{\underline{\tau}}^{\tau} \int_{\Theta_i} h_i(t, \vartheta_i) \frac{\partial F_i(\vartheta_i \,|\, t)}{\partial \tau_i} \,\mathrm{d}\vartheta_i \,\mathrm{d}t\end{aligned}$$

Proof of Proposition 12.7

Use revenue equivalence for ex-post type:

- Replacing $\bar{p}_i(\tau_i, \vartheta_i)$ and $\bar{p}_i(\underline{\tau}, \vartheta_i)$ with the expression from revenue equivalence for the ex-post type yields

$$\bar{p}_i(\tau_i, \underline{\vartheta}) = \bar{p}_i(\underline{\tau}, \underline{\vartheta}) + \sum_{q \in Q} v_i(q, \underline{\vartheta})\big(\bar{\alpha}_q(\tau_i, \underline{\vartheta}) - \bar{\alpha}_q(\underline{\tau}, \underline{\vartheta})\big)$$

$$+ \int_{\underline{\tau}}^{\tau_i} \int_{\underline{\vartheta}}^{\bar{\vartheta}} h_i(t, \vartheta_i) \frac{\partial F(\vartheta_i \mid t)}{\partial \tau_i} \, \mathrm{d}\vartheta_i \, \mathrm{d}t$$

$$+ \int_{\underline{\vartheta}}^{\bar{\vartheta}} \int_{\underline{\vartheta}}^{\vartheta_i} \big(h_i(\tau_i, x) f(\vartheta_i | \tau_i) - h_i(\underline{\tau}, x) f(\vartheta_i \mid \underline{\tau})\big) \, \mathrm{d}x \, \mathrm{d}\vartheta_i.$$

- Result now follows from revenue equivalence with respect to the ex-post type with the above expression for $\bar{p}_i(\tau_i, \underline{\vartheta})$.

Monotonicity with Respect to Ex-Ante Type May Fail

Selling to a single buyer:

- There are two states: buyer obtains the good ($q = 1$) or not ($q = 0$).
- Let $q_i(R_i, \vartheta_i) = \bar{\alpha}_q(R_i, \vartheta_i)$ denote the probability of selling the good.
- Let $\bar{p}_i(\tau_i, \vartheta_i) = p_i(\tau_i, \vartheta_i)$, and suppose q_i and p_i are twice differentiable.

Ex-ante utility:

- Since $\frac{\partial v_i(q,\vartheta_i)}{\partial \vartheta_i} = 1_{\{q=1\}}$, we obtain

$$\widehat{U}_i(R_i \mid \tau_i) = u_i(\bar{\vartheta} \mid R_i, \bar{\vartheta}) - \int_{\Theta_i} \underbrace{\sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(R_i, \vartheta_i)}_{q_i(R_i, \vartheta_i)} F_i(\vartheta_i \mid \tau_i) \, \mathrm{d}\vartheta_i$$

$$= \bar{\vartheta} q_i(R_i, \bar{\vartheta}) - p_i(R_i, \bar{\vartheta}) - \int_{\Theta_i} q_i(R_i, \vartheta_i) F_i(\vartheta_i \mid \tau_i) \, \mathrm{d}\vartheta_i.$$

Monotonicity with Respect to Ex-Ante Type May Fail

First- and second-order constraints:

$$\left.\frac{\partial \widehat{U}_i(R_i \mid \tau_i)}{\partial R_i}\right|_{R_i=\tau_i} = \bar{\vartheta} q_i'(\tau_i, \bar{\vartheta}) - p_i'(\tau_i, \bar{\vartheta}) - \int_{\Theta_i} q_i'(\tau_i, \vartheta_i) F_i(\vartheta_i \mid \tau_i)\, \mathrm{d}\vartheta_i = 0,$$

$$\left.\frac{\partial^2 \widehat{U}_i(R_i \mid \tau_i)}{\partial R_i^2}\right|_{R_i=\tau_i} = \bar{\vartheta} q_i''(\tau_i, \bar{\vartheta}) - p_i''(\tau_i, \bar{\vartheta}) - \int_{\Theta_i} q_i''(\tau_i, \vartheta_i) F_i(\vartheta_i \mid \tau_i)\, \mathrm{d}\vartheta_i \leq 0.$$

Differentiate first-order constraint with respect to τ_i:

$$0 = \underbrace{\bar{\vartheta} q_i''(\tau_i, \bar{\vartheta}) - p_i''(\tau_i, \bar{\vartheta}) - \int_{\Theta_i} q_i''(\tau_i, \vartheta_i) F_i(\vartheta_i \mid \tau_i)\, \mathrm{d}\vartheta_i}_{\leq 0 \text{ by SOSC}} - \int_{\Theta_i} q_i'(\tau_i, \vartheta_i) \frac{\partial F_i(\vartheta_i \mid \tau_i)}{\partial \tau_i}\, \mathrm{d}\vartheta_i.$$

Therefore, $q_i(\tau_i, \vartheta_i)$ is increasing in τ_i “on average”:

$$\int_{\Theta_i} q_i'(\tau_i, \vartheta_i) \frac{\partial F_i(\vartheta_i \mid \tau_i)}{\partial \tau_i}\, \mathrm{d}\vartheta_i \leq 0.$$

Monotonicity

Lemma 12.8

Suppose $q(\tau, \vartheta)$ is such that $h_i(\tau_i, \vartheta_i) = \sum_{q \in Q} \frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} \bar{\alpha}_q(\tau_i, \vartheta_i)$ is non-decreasing in τ_i and ϑ_i for every player i. Then there exist payments $p(\tau, \vartheta)$ such that the direct mechanism (q, p) is incentive-compatible.

Interpretation:

- The allocation of an incentive-compatible dynamic mechanism may not be monotonic in the ex-ante type.
- Monotonicity, however, is sufficient for incentive-compatibility.

Proof of Lemma 12.8

Incentive compatibility of ex-post type:

- Suppose that $q(\tau, \vartheta)$ satisfies the monotonicity constraint.
- By Proposition 12.7, we must define $\bar{p}_i(\tau_i, \vartheta_i)$ via the revenue equivalence for payments, choosing $p_i(\underline{\tau}, \underline{\vartheta})$ such that $U_i(\underline{\tau}) = 0$.
- The mechanism (q, p) is incentive compatible with respect to ex-post type by Lemma 12.4
- By Lemma 12.3, it remains to show $U_i(\tau_i) \geq \widehat{U}_i(R_i \mid \tau_i)$.

Proof of Lemma 12.8

Incentive compatibility of ex-post type:

- Recall that

$$\frac{\partial \widehat{U}_i(R_i \mid \tau_i)}{\partial \tau_i} = -\int_{\Theta_i} h_i(R_i, \vartheta_i) \frac{\partial F_i(\vartheta_i \mid \tau_i)}{\partial \tau_i} \, \mathrm{d}\vartheta_i.$$

- The revenue equivalence for ex-ante utilities yields

$$\begin{aligned} U_i(\tau_i) - \widehat{U}_i(R_i \mid \tau_i) &= \widehat{U}_i(\tau_i \mid \tau_i) - \widehat{U}_i(R_i \mid R_i) + \widehat{U}_i(R_i \mid R_i) - \widehat{U}_i(R_i \mid \tau_i) \\ &= \int_{R_i}^{\tau_i} U'(t) - \frac{\partial \widehat{U}_i(R_i \mid t)}{\partial \tau_i} \, \mathrm{d}t \\ &= \int_{R_i}^{\tau_i} \int_{\Theta_i} \underbrace{\big(h_i(R_i, \vartheta_i) - h_i(t, \vartheta_i)\big)}_{\leq 0} \underbrace{\frac{\partial F_i(\vartheta_i \mid t)}{\partial \tau_i}}_{\leq 0 \text{ by FOSD}} \, \mathrm{d}\vartheta_i \, \mathrm{d}t \end{aligned}$$

- This shows that (q, p) is incentive-compatible.

Dynamic vs. Static Mechanisms

Revelation principle:

- The dynamic revelation principle gives us truth-telling only on the equilbirium path.
- If the ex-ante information is not directly payoff relevant, then truth-telling on the path is sufficient for off-path truth-telling.

Revenue equivalence:

- Revenue equivalence for ex-post type holds as in the static case.
- If $\frac{\partial F_i(\vartheta_i \mid \tau_i)}{\partial \tau_i}$ is bounded, then it also holds for the ex-ante type.

Monotonicity:

- Monotonicity for the ex-post type holds as in the static case.
- Monotonicity for the ex-ante type is sufficient for incentive-compatibility, but it is not be necessary.

Optimal Selling Mechanism

Optimal Selling Mechanism

Setup:

- There are $i = 1, \dots, n$ potential buyers.
- There are $n+1$ social states q_i: i obtains the good and q_0: the seller keeps the good. Suppose the seller places no value on the item.
- Buyer i obtains the item with subjective probability

$$\bar{q}_i(\tau_i, \vartheta_i) := P_{\tau_i, \vartheta_i}(q(\tau, \vartheta) = q_i) = \bar{\alpha}_q(\tau_i, \vartheta_i) = h_i(\tau_i, \vartheta_i),$$

where we have used that $\frac{\partial v_i(q, \vartheta_i)}{\partial \vartheta_i} = 1_{\{q = q_i\}}$.

Marginal distribution of ex-ante type:

- Let $g_i(\tau_i) := \int_{\Theta_i} f_i(\tau_i, \vartheta_i)\,\mathrm{d}\vartheta_i$ denote the marginal density function.
- Let $G_i(\tau_i) := \int_{\underline{\tau}}^{\tau_i} g_i(t)\,\mathrm{d}t$ denote the marginal distribution function.

Optimal Selling Mechanism

Expected revenue from a single buyer:

- By revenue equivalence, payments are determined by allocation rule.
- Seller's expected revenue from buyer i is

$$Rev_i = \int_{\mathcal{T}_i} \int_{\Theta_i} \vartheta_i \bar{q}_i(\tau_i, \vartheta_i) f_i(\vartheta_i \mid \tau_i) g_i(\tau_i) \, \mathrm{d}\vartheta_i \, \mathrm{d}\tau_i - \int_{\mathcal{T}_i} U_i(\tau_i) g(\tau_i) \, \mathrm{d}\tau_i.$$

- Revenue is maximized if $U_i(\underline{\tau}) = 0$.
- Revenue equivalence for ex-ante utilities thus yields

$$\begin{aligned}\int_{\mathcal{T}_i} \underbrace{U_i(\tau_i)}_{\downarrow} \underbrace{(-g(\tau_i))}_{\uparrow} \, \mathrm{d}\tau_i &= -\int_{\mathcal{T}_i} (1 - G_i(\tau_i)) U_i'(\tau_i) \, \mathrm{d}\tau_i \\ &= \int_{\mathcal{T}_i} \int_{\Theta_i} (1 - G_i(\tau_i)) \bar{q}_i(\tau_i, \vartheta_i) \frac{\partial F_i(\vartheta_i \mid \tau_i)}{\partial \tau_i} \, \mathrm{d}\vartheta_i \, \mathrm{d}\tau_i.\end{aligned}$$

Optimal Selling Mechanism

Expected revenue from a single buyer:

- Define i's virtual valuation as

$$\psi_i(\tau_i, \vartheta_i) := \vartheta_i + \frac{1 - G_i(\tau_i)}{g_i(\tau_i)} \frac{\partial F_i(\vartheta_i \mid \tau_i)/\partial \tau_i}{f_i(\vartheta_i \mid \tau_i)}.$$

- Then the seller's expected revenue from buyer i is

$$Rev_i = \int_{\mathcal{T}_i} \int_{\Theta_i} \psi_i(\tau_i, \vartheta_i) \bar{q}_i(\tau_i, \vartheta_i) f_i(\vartheta_i \mid \tau_i) g_i(\tau_i) \, \mathrm{d}\vartheta_i \, \mathrm{d}\tau_i.$$

- Since $\bar{q}_i(\tau_i, \vartheta_i) = \int_{\mathcal{T}_{-i}} \int_{\Theta_{-i}} q_i(\tau, \vartheta) f_{-i}(\vartheta_{-i} \mid \tau_{-i}) g_{-i}(\tau_{-i})$, we obtain

$$Rev_i = \int_{\mathcal{T}} \int_{\Theta} \psi_i(\tau_i, \vartheta_i) q_i(\tau, \vartheta) f(\vartheta \mid \tau) g(\tau) \, \mathrm{d}\vartheta \, \mathrm{d}\tau.$$

Optimal Selling Mechanism

Seller's expected revenue:

- Total revenue equals

$$Rev = \int_{\mathcal{T}} \int_{\Theta} \sum_{i=1}^{n} \psi_i(\tau_i, \vartheta_i) q_i(\tau, \vartheta) f(\vartheta \mid \tau) g(\tau) \, \mathrm{d}\vartheta \, \mathrm{d}\tau.$$

- Revenue is maximized for

$$q_i(\tau, \vartheta) = \begin{cases} 1 & \text{if } \psi_i(\tau_i, \vartheta_i) \geq \max\{\max_j \psi_j(\tau_j, \vartheta_j), 0\} \\ 0 & \text{otherwise.} \end{cases}$$

- If q_i is non-decreasing, then payments p given by revenue equivalence make the mechanism incentive-compatible.

Optimal Selling Mechanism

Proposition 12.9

Suppose $\psi_i(\tau_i, \vartheta_i)$ is non-decreasing in τ_i and ϑ_i. Then an incentive-compatible and individually rational direct dynamic mechanism (q, p) maximizes the seller's expected revenue if and only if

$$q_i(\tau, \vartheta) = \begin{cases} 1 & \text{if } \psi_i(\tau_i, \vartheta_i) \geq \max\{\max_j \psi_j(\tau_j, \vartheta_j), 0\} \\ 0 & \text{otherwise,} \end{cases}$$

and payments $p_i(\tau_i, \vartheta_i)$ are determined from revenue equivalence such that $U_i(\underline{\tau}) = 0$ for every buyer i.

Information Rent

Information rent consists of two components:

- The hazard rate $\frac{g_i(\tau_i)}{1-G_i(\tau_i)}$ of the ex-ante type.
- Informativeness measure $\frac{\partial F_i(\vartheta_i \mid \tau_i)/\partial \tau_i}{f_i(\vartheta_i \mid \tau_i)}$ that captures how the buyers' knowledge of their valuation changes with ex-ante type.

Reserve price:

- Since ψ_i is non-decreasing, there exists a cutoff

$$\widehat{p}_i(\tau_i) := \min\{\vartheta_i \in \Theta_i \mid \psi_i(\vartheta_i, \tau_i) \geq 0\}$$

  such that $\psi(\vartheta_i, \tau_i) < 0$ if and only if $\vartheta_i < \widehat{p}_i(\tau_i)$.
- Thus, the good is sold if and only if $\vartheta_i \geq \widehat{p}_i(\tau_i)$.
- Since ψ_i is non-decreasing, the reserve price $\widehat{p}_i(\tau_i)$ is non-increasing in the announced ex-ante type τ_i.

Optimal Selling Mechanism With a Single Buyer

Payments:

- For a pair (τ_i, ϑ_i) with $\vartheta_i < \widehat{p}_i(\tau_i)$:

$$p_i(\tau_i, \vartheta_i) = p_{i,\underline{\vartheta}_i}(\tau_i) + \vartheta_i \underbrace{q_i(\tau_i, \vartheta_i)}_{=0} - \int_{\underline{\vartheta}_i}^{\vartheta_i} \underbrace{q_i(\tau_i, x)}_{=0} \, \mathrm{d}x = p_{i,\underline{\vartheta}_i}(\tau_i).$$

- For a pair (τ_i, ϑ_i) with $\vartheta_i \geq \widehat{p}_i(\tau_i)$:

$$p_i(\tau_i, \vartheta_i) = p_{i,\underline{\vartheta}_i}(\tau_i) + \vartheta_i \underbrace{q_i(\tau_i, \vartheta_i)}_{=1} - \int_{\widehat{p}_i(\tau_i)}^{\vartheta_i} \underbrace{q_i(\tau_i, x)}_{=1} \, \mathrm{d}x$$

$$= p_{i,\underline{\vartheta}_i}(\tau_i) + \widehat{p}_i(\tau_i).$$

- $p_{i,\underline{\vartheta}_i}(\tau_i)$ is determined by revenue equivalence and $U_i(\underline{\tau}) = 0$.

Optimal Selling Mechanism With a Single Buyer

Proposition 12.10

Suppose $\psi_i(\tau_i, \vartheta_i)$ is non-decreasing in τ_i and ϑ_i. There exist prices $p_{i,\underline{\vartheta}_i}(\tau_i)$ and $\widehat{p}_i(\tau_i)$ such that the optimal selling mechanism takes the form

$$q_i(\tau_i, \vartheta_i) = \begin{cases} 1 & \text{if } \vartheta_i \geq \widehat{p}_i(\tau_i), \\ 0 & \text{otherwise,} \end{cases}$$

$$p_i(\tau_i, \vartheta_i) = \begin{cases} p_{i,\underline{\vartheta}_i}(\tau_i) + \widehat{p}_i(\tau_i) & \text{if } \vartheta_i \geq \widehat{p}_i(\tau_i), \\ p_{i,\underline{\vartheta}_i}(\tau_i) & \text{otherwise,} \end{cases}$$

Implementation:

- Offer a menu of option contracts $\left(p_{i,\underline{\vartheta}_i}(\tau_i), \widehat{p}_i(\tau_i)\right)_{\tau_i}$ to the buyer.
- In contract $\left(p_{i,\underline{\vartheta}_i}(\tau_i), \widehat{p}_i(\tau_i)\right)$, buyer buys a call option with exercise price $\widehat{p}_i(\tau_i)$ ($=$ right to buy item at price $\widehat{p}_i(\tau_i)$) for the price $p_{i,\underline{\vartheta}_i}(\tau_i)$.

Optimal Selling Mechanism With Multiple Buyers

If the informativeness measure is a function of only the ex-ante type

$$\frac{\partial F_i(\vartheta_i \mid \tau_i)/\partial \tau_i}{f_i(\vartheta_i \mid \tau_i)} = \phi_i(\tau_i),$$

then the optimal selling mechanism allows a similar interpretation. Set

$$\widehat{p}_i(\tau_i) := -\frac{1 - G_i(\tau_i)}{g_i(\tau_i)}\phi_i(\tau_i).$$

Implementation:

- A the beginning, buyers can buy a premium $\widehat{p}_i(\tau_i)$ for price $p_{i,\underline{\vartheta}_i}(\tau_i)$.
- After buyers learn their valuation, buyers bid in a second-price auction without reserve price, but in addition to the second-highest bid they also have to pay the acquired premium $\widehat{p}_i(\tau_i)$.
- It is thus a weakly dominant strategy to bid their true valuation minus the premium, i.e., to bid their virtual valuation.

Value of Private Information

Decomposition of Information

Decomposition into initial and additional information:

- First, every participant i observes realization τ_i of T_i.
- Then, participant observes $A_i = F(\theta_i \mid \tau_i)$.
- Note that A_i is a transformation of the random variable θ_i via $F(\,\cdot \mid \tau_i)$.
- It is distributed on $[0, 1]$ and its conditional distribution is

$$P(A_i \leq \alpha_i \mid \tau_i) = P(F(\theta_i \mid \tau_i) \leq \alpha_i \mid \tau_i) = P\big(\theta_i \leq F_i^{-1}(\alpha_i \mid \tau_i) \big| \tau_i\big)$$

$$= F_i(F_i^{-1}(\alpha_i \mid \tau_i) \mid \tau_i) = \alpha_i.$$

- A_i is uniformly distributed and stochastically independent of T_i.

Interpretation:

- T_i is a noisy signal of θ_i, where A_i is the noise.
- Upon learning the noise $A_i = \alpha_i$, buyer i learns $\vartheta_i = F^{-1}(\alpha_i \mid \tau_i)$.

Value of Additional Information

Consider two mechanisms:

- A_i is learned privately by the buyer.
- A_i is observed publicly, hence the seller does not need to elicit α_i.
- Difference between seller's revenue is information rent for α_i.

$\mathbf{A_i}$ is learned privately:

- This is ismorphic to the case we have studied.
- Define the virtual valuation

$$\psi_i(\tau_i, \alpha_i) = F^{-1}(\alpha_i \,|\, \tau_i) + \frac{1 - G_i(\tau_i)}{g_i(\tau_i)} \frac{\partial F_i(F_i^{-1}(\alpha_i \,|\, \tau_i) \,|\, \tau_i)/\partial \tau_i}{f_i(F_i^{-1}(\alpha_i \,|\, \tau_i) \,|\, \tau_i)}$$

$$= F^{-1}(\alpha_i \,|\, \tau_i) + \frac{1 - G_i(\tau_i)}{g_i(\tau_i)} \frac{\partial F_i^{-1}(\alpha_i \,|\, \tau_i)}{\partial \tau_i}.$$

- Sell the item to buyer with the highest non-negative virtual valuation.

Optimal Selling Mechanism for Publicly Observed Information

Proposition 12.11

Suppose $\psi_i(\tau_i, \alpha_i)$ is non-decreasing in τ_i and α_i for every player i. Then the optimal selling mechanism when α is privately observed is also optimal when α is publicly observed.

Interpretation:

- Additional information can be elicited from the seller at no cost.
- Private information before entering the mechanism is more powerful than private information learned afterwards.
- The seller would like to contract early to minimize adverse selection.

Note:

- The result does not hold if ψ_i is not monotonic.

Revenue Equivalence for Publicly Observed Information

Lemma 12.12

For any incentive-compatible direct selling mechanism $(q(\tau,\alpha), p(\tau,\alpha))$ *with publicly observable* α, *we have*

$$U_i(\tau_i) = U_i(\underline{\tau}) - \int_{\underline{\tau}}^{\tau_i} \int_0^1 \frac{\partial F_i^{-1}(\alpha_i \mid t)}{\partial \tau_i} \bar{q}_i(t, \alpha_i) \, \mathrm{d}\alpha_i \, \mathrm{d}t.$$

Proof:

- Since $U_i(\tau_i + \delta) \geq \widehat{U}_i(\tau_i \mid \tau_i + \delta)$, we obtain

$$\frac{U_i(\tau_i + \delta) - U_i(\tau_i)}{\delta} \geq \int_0^1 \frac{F_i^{-1}(\alpha_i \mid \tau_i + \delta) - F_i^{-1}(\alpha_i \mid \tau_i)}{\delta} \bar{q}_i(\tau_i, \alpha_i) \, \mathrm{d}\alpha_i.$$

- Do the same for $\tau_i - \delta$, take limits, integrate.

Proof of Proposition 12.11

Expected revenue from a single buyer:

- The seller's expected revenue from buyer i is

$$Rev_i = \int_{\mathcal{T}_i} \int_0^1 F_i^{-1}(\alpha_i \mid \tau_i)\bar{q}_i(\tau_i, \alpha_i) g_i(\tau_i) \, \mathrm{d}\alpha_i \, \mathrm{d}\tau_i - \int_{\mathcal{T}_i} U_i(\tau_i) g(\tau_i) \, \mathrm{d}\tau_i.$$

- Revenue equivalence for publicly observed α thus yields

$$\int_{\mathcal{T}_i} \underbrace{U_i(\tau_i)}_{\downarrow} \underbrace{(-g(\tau_i))}_{\uparrow} \, \mathrm{d}\tau_i = -\int_{\mathcal{T}_i} (1 - G_i(\tau_i)) U_i'(\tau_i) \, \mathrm{d}\tau_i$$

$$= \int_{\mathcal{T}_i} \int_0^1 (1 - G_i(\tau_i)) \frac{\partial F_i^{-1}(\alpha_i \mid \tau_i)}{\partial \tau_i} \bar{q}_i(\tau_i, \vartheta_i) \, \mathrm{d}\alpha_i \, \mathrm{d}\tau_i.$$

- Therefore, we obtain $Rev_i = \int_{\mathcal{T}} \int_0^1 \psi_i(\tau_i, \alpha_i) q_i(\tau_i, \alpha_i) g(\tau_i) \, \mathrm{d}\alpha_i \, \mathrm{d}\tau_i$ for the same ψ_i as for privately observed α.
- The remainder of proof works as before.

Literature

T. Börgers: An Introduction to the Theory of Mechanism Design, Chapter 11, Oxford University Press, 2015