5. Mechanism Design I

ECON 7219 – Games With Incomplete Information

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Motivation

Which Auction Should You Run?

(Sealed-bid) first-price auction:

- Auction participants have independent private values $\theta_i \sim F$ on $[0, \infty)$.
- Highest bidder obtains the auctioned object and pays his/her bid.
- We have derived a symmetric pure-strategy BNE s with

$$s_i(\vartheta_i) = \vartheta_i - \int_0^{\vartheta_i} \frac{F^{n-1}(x)}{F^{n-1}(\vartheta_i)} \, \mathrm{d}x = \mathbb{E}\left[\max_{j \neq i} \theta_j \,\middle|\, \max_{j \neq i} \theta_j \leq \vartheta_i\right]$$

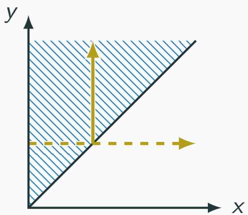
Fubini's Theorem

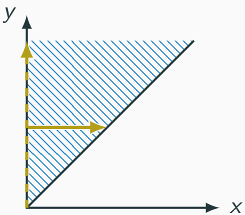
Theorem 5.1 (Fubini)

Consider two σ-finite measure spaces $\mathcal{X}$ and $\mathcal{Y}$. If a function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is $(\mathcal{X} \times \mathcal{Y})$-integrable, that is, if $\int_{\mathcal{X}\times\mathcal{Y}} |f(x, y)| \, \mathrm{d}(x, y) < \infty$, then

$$\int_{\mathcal{X}\times\mathcal{Y}} f(x, y) \, \mathrm{d}(x, y) = \int_{\mathcal{X}} \int_{\mathcal{Y}} f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_{\mathcal{Y}} \int_{\mathcal{X}} f(x, y) \, \mathrm{d}x \, \mathrm{d}y,$$

i.e., we can interchange the order of integration





Expectation of Non-Negative Random Variable

Lemma 5.2

Let X be a non-negative random variable with distribution function $F(x)$ and density $f(x)$. Then $\mathbb{E}[X] = \int_0^\infty (1 - F(x))\,\mathrm{d}x$.

Proof:

- By the definition of a density function

$$\int_0^\infty (1 - F(x))\,\mathrm{d}x = \int_0^\infty P(X > x)\,\mathrm{d}x = \int_0^\infty \int_x^\infty f(y)\,\mathrm{d}y\,\mathrm{d}x = \dots$$

- By Fubini's theorem

$$\dots = \int_0^\infty \int_0^y f(y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty y f(y)\,\mathrm{d}y = \mathbb{E}[X].$$

Expected Revenue of First-Price Auction

Re-evaluating strategy of first-price auction:

- Symmetric equilibrium strategy is

$$s_i(\vartheta_i) = \vartheta_i - \int_0^{\vartheta_i} \frac{F^{n-1}(x)}{F^{n-1}(\vartheta_i)}\,\mathrm{d}x = \int_0^{\vartheta_i} 1 - \frac{F^{n-1}(x)}{F^{n-1}(\vartheta_i)}\,\mathrm{d}x$$

$$= \mathbb{E}[\max_{j\neq i}\theta_j \,|\, \max_{j\neq i}\theta_j \leq \vartheta_i].$$

- Bidder i bid the expected highest valuation among i's opponents, conditional on wanting to win the auction at that price.

Expected revenue:

- Let $\theta^{(2)}$ denote the second-highest valuation among $(\theta_1, \dots, \theta_n)$.
- Expected revenue of first-price and second-price auctions are

$$\mathbb{E}\left[\theta^{(2)} \,\middle|\, \theta^{(2)} \leq \max_j \theta_j\right] = \mathbb{E}\left[\theta^{(2)}\right].$$

Are There Better Auctions to Run?

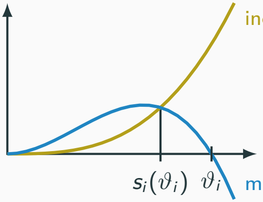
Second-price auction:

- Auction participants have independent private values $\theta_i \sim F_i$ on $[0, \infty)$.
- Players submit their bids simultaneously.
- Highest bidder obtains the object and pays the second-highest bid.

Which auction should you run?

- It may seem like a first-price auction must yield a higher revenue since the seller receives the highest rather than the second-highest bid.
- However, changing the auction rules may change the bidding behavior.
- This is a crucial aspect of any economic design: people react to changes in rules/regulations, i.e., players' actions must be endogenous.

Comparing Bidding Incentives



First-price auction:

- Increasing the bid increases the probability of winning, but also increases the price you pay for the object.
- In equilibrium, the marginal benefit of winning the auction precisely offsets marginal increase in price you pay.

Second-price auction:

- Increasing the bid only increases the probability of winning.

Image generated for a 4-player auction with independent standard-uniformly distributed types and $\vartheta_i = 0.9$.

(Sealed-Bid) Second-Price Auction

Model of the auction:

- Each bidder i can submit a bid $b_i \in [0, \infty)$.
- Drawing lots: winner $i_*(b)$ is determined by uniform distribution on

$$\arg\max_{i \in \mathcal{I}} b_i = \left\{ i \in \mathcal{I} \;\middle|\; b_i = \max_{j \in \mathcal{I}} b_j \right\}.$$

- Bidder i's utility function is $u_i(\vartheta_i, b) = (\vartheta_i - \max_{j \neq i} b_j) 1_{\{i = i_*(b)\}}$.

Solving the second-price auction:

- Bid affects your payoff only through the probability of winning.
- You want to win the auction if and only if $\vartheta_i \geq \max_{j \neq i} b_j$.
- Bid b_i wins if $b_i \geq \max_{j \neq i} b_j$, hence we should bid $s_i(\vartheta_i) = \vartheta_i$.
- Expected revenue $\mathbb{E}\left[\theta^{(2)}\right]$ is identical to first-price auction!

English Auction

Optimal Selling Mechanism?

Game theory:

- PBE tell us what to expect for a given auction model (mechanism).
- We can't keep changing the auction mechanism, solving it, and comparing the seller's expected revenue.
- We need a more universal approach.

Mechanism design:

- A selling mechanism is an extensive-form game that determines:
  - Allocation of the good to one (or none) of the buyers.
  - Payments from potential buyers to seller.
- What is the revenue-maximizing selling mechanism?
- Is there an efficient way to analyze such a problem?

Mechanisms and the Revelation Principle

Looking at Auctions Through a Different Lens

Informational problem:

- If the seller knew the buyers' valuations, he/she could sell it to the highest-valuation buyer and extract the full surplus.

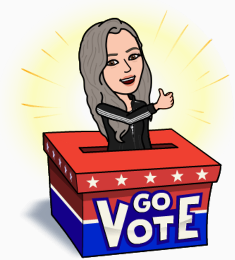
Eliciting private information:

- The first-price auction reveals the buyers' valuations in equilibrium due to monotonicity: bid b_i was made by type $s^{-1}(b_i)$.
- Buyer of type ϑ_i receives an information rent $\vartheta_i - s_i(\vartheta_i)$.
- First-price auction is mechanism that incentivizes bidders to reveal their private information in exchange for an information rent.

Mechanism design:

- Solving an allocation problem in a setting with asymmetric information.

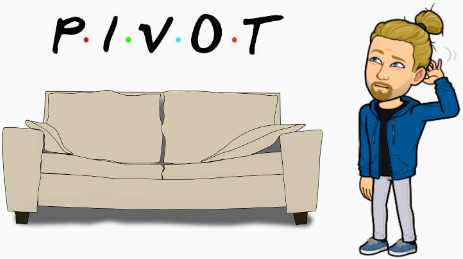
Voting



Voting mechanism:

- People have private information about their preference over candidates.
- If there are more than two alternatives, voters may have an incentive to not vote for their favorite candidate.
- How do we design an election that elicits preferences truthfully and represents the population most fairly?

Roommate Problem



Public goods mechanism:

- Should you and your roommate get that new couch, new X-box, etc?
- Roommates have private information about their willingness to contribute, with an incentive to underreport their willingness to pay.
- It may be that the sum of reported values is lower than the price of the couch even though the sum of valuations is high enough.
- How should you ask your roommates to prevent this inefficiency?

Mechanism Design Problem

These scenarios have in common:

- There is a set $\mathcal{X}$ of mutually exclusive alternatives.
- Each player i's preference θ_i over alternatives are private information.
- We try to "optimally" decide among alternatives in $\mathcal{X}$.

Problems:

- Criterion of optimality may depend on private information.
- Players may have an incentive to misrepresent their preferences.

Can we design a clever mechanism that:

- Elicits the private information truthfully?
- Implements the optimal choice from the designer's perspective?

Preferences

States of nature:

- Θ_i is the set of i's possible preferences over $\mathcal{X}$:
  - Preferences could be captured by utility function $u_i : \mathcal{X} \times \Theta_i \to \mathbb{R}$, that is, $u_i(x, \vartheta_i)$ is type ϑ_i's utility in alternative $x \in \mathcal{X}$.
  - Or Θ_i could be the set of preference relations over $\mathcal{X}$, that is, player i with preference ϑ_i prefers $x \in \mathcal{X}$ to $y \in X$ if $x \succ_{\vartheta_i} y$.
- The states of nature $\Theta = \Theta_1 \times \cdots \times \Theta_n$ are the players' preference profiles $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ over $\mathcal{X}$.

Players' types:

- Players know their own preferences but not the preferences of others.
- A player's type corresponds to beliefs over Θ and opponents' types.
- The type space is the universal type space $\mathcal{T} \simeq \Delta(\Theta \times \mathcal{T}^{n-1})$.

Decomposition of Types

Players know their preferences:

- Each type τ_i assigns positive probability only to one $\vartheta_i(\tau_i) \in \Theta_i$:

$$\tau_i \simeq \delta_{\vartheta_i(t_i)} \otimes \tau_i|_{\Theta_{-i} \times \mathcal{T}^{n-1}},$$

  where $\delta_{\vartheta_i(\tau_i)}$ is the Dirac measure at $\vartheta_i(\tau_i)$.
- We can decompose a player's type $\tau_i \simeq \big(\vartheta_i(\tau_i), \beta_i(\tau_i)\big)$ into his/her payoff type $\vartheta_i(\tau_i)$ and his/her belief type $\beta_i(\tau_i) := \tau_i|_{\Theta_{-i} \times \mathcal{T}^{n-1}}$.

Independent types:

- If players' preferences are independent, then $\beta_i(\tau_i) = P_i$ for any type τ_i, hence types are uniquely determined by ϑ_i and P_i.
- Most of mechanism design deals with this case.

Examples

Election:

- Players' payoff type ϑ_i is a preference ranking over candidates.
- Types are not independent since we are influenced by our social circle.

Auction:

- Players' payoff type ϑ_i is their valuation of the good.
- Independence of types is usually violated for investment goods.
- For consumption goods, independence of types appears reasonable.

Roommate problem:

- Players' payoff type ϑ_i is their valuation of having a new couch.
- Independence of types appears reasonable.

Social Choice Function

Definition 5.3

A social choice function is a map $g : \Theta \to \mathcal{X}$.

Mechanism designer's goal:

- Choosing the optimal alternative $x \in \mathcal{X}$ corresponds to implementing an optimal social choice function.
- Note that optimality may depend on the players' preferences, but not on the players beliefs over other players' types.

Criterion of optimality:

- Selling mechanism: optimal typically means revenue-maximizing.
- Voting: optimal typically means welfare-maximizing.

Static Mechanisms

Definition 5.4

A (static) mechanism $\Gamma = (\mathcal{S}_1, \ldots, \mathcal{S}_n, h)$ consists of:

1. A set of available pure strategies $\mathcal{S}_i$ for each player $i \in \mathcal{I}$,
2. A map $h : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{X}$ that assigns outcomes to alternatives.

What do we expect players do?

- In strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, player i's expected utility is

$$\mathbb{E}_{\tau_i,\sigma}[u_i(h(S), \vartheta_i(\tau_i))],$$

where S is the realization of σ.

- We expect players to play a Bayesian Nash equilibrium of Γ.

Implementation

Definition 5.5

1. A mechanism $\Gamma = (\mathcal{S}_1, \ldots, \mathcal{S}_n, h)$ (partially) implements social choice function g if there exists a Bayesian Nash equilibrium σ such that

$$g(\vartheta(\tau)) = h(s), \qquad \forall\, s \in \operatorname{supp} \sigma(\tau),\ \forall\, \tau \in \mathcal{T}. \tag{1}$$

2. A mechanism Γ fully implements g if (1) holds for the unique BNE.
3. g is implementable if there exists a mechanism Γ that implements g.

Natural questions to ask:

- Which social choice functions g are implementable?
- What are the "simplest" mechanisms that implement g?
- How problematic is partial implementation?

Multiplicity of Equilibria

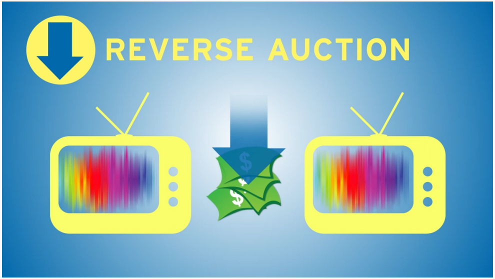
Multiplicity of equilibria:

- Recall that an equilibrium is an outcome, in which:
  - (i) Players best reply to their conjecture about opponents' strategies,
  - (ii) Players conjectures about opponents' strategies are correct.
- In absence of communication between players, (ii) can be hard to justify unless the equilibrium is unique.
- If there are several equilibria, players may miscoordinate.

Full vs. partial implementation:

- Full: mechanism designer can be fairly confident rational players will behave the way he/she wants them to.
- Partial: mechanism designer may need to tell players what to do.

Weak Implementation



Weak implementation:

- Weak implementation is often justified if there is a significant imbalance between mechanism designer and players.
- Example: Government is the designer of spectrum auctions, in which mobile carriers, TV stations, etc. participate.
- Other examples: Government vs. citizens, companies vs. users.

Direct Mechanisms

Definition 5.6

A direct mechanism is a mechanism, in which $\mathcal{S}_i = \mathcal{T}_i$ for each $i \in \mathcal{I}$.

Direct mechanism:

- Mechanism designer simply asks players to report their type.
- A social choice function g is truthfully implementable by a direct mechanism if $s_i(\tau_i) = \tau_i$ is a Bayesian Nash equilibrium that satisfies (1).

Incentive compatibility:

- A direct mechanism is incentive compatible if $s_i(\tau_i) = \tau_i$ is a BNE.
- It is a best response to report one's type truthfully in an incentive-compatible direct mechanism, conditional on others reporting truthfully.

Revelation Principle

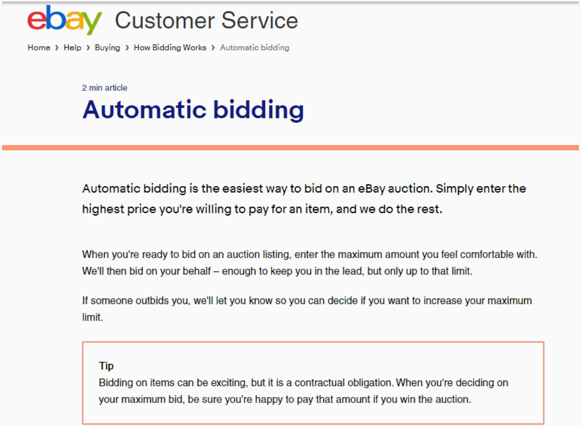
Proposition 5.7 (Revelation Principle)

A social choice function is implementable by a mechanism if and only if it is truthfully implementable by a direct mechanism.

Proof:

- Suppose $\Gamma := (\mathcal{S}_1, \ldots, \mathcal{S}_n, h)$ implements social choice function g.
- For BNE σ with $h(s) = g(\vartheta(\tau))$ for every $s \in \operatorname{supp} \sigma(\tau)$ and every $\tau \in \mathcal{T}$, define the direct mechanism $\Gamma' := (\mathcal{T}_1, \ldots, \mathcal{T}_n, h \circ \sigma)$.
- Clearly, Γ' implements social choice function g.
- Truthfully reporting one's type is an equilibrium:
  - Strategy $\sigma_i(\tau_i')$ is a unilateral deviation in the original mechanism,
  - No unilateral deviation in the original mechanism is profitable.

Interpretation of Revelation Principle



ebay Customer Service

Home › Help › Buying › How Bidding Works › Automatic bidding

2 min article

Automatic bidding

Automatic bidding is the easiest way to bid on an eBay auction. Simply enter the highest price you're willing to pay for an item, and we do the rest.

When you're ready to bid on an auction listing, enter the maximum amount you feel comfortable with. We'll then bid on your behalf – enough to keep you in the lead, but only up to that limit.

If someone outbids you, we'll let you know so you can decide if you want to increase your maximum limit.

Tip
Bidding on items can be exciting, but it is a contractual obligation. When you're deciding on your maximum bid, be sure you're happy to pay that amount if you win the auction.

Indirect mechanism: players compute BNE and act according to it.

Direct mechanism: mechanism designer computes BNE for players.

Simple Mechanisms

Revelation principle:

- In theory, it allows us to restrict attention to direct mechanisms.
- In practice, noticing that truthful reporting is a BNE may not be cognitively simpler than participating in an indirect mechanism.
- Direct mechanisms may require reporting of infinite belief hierarchies.

Cognitively simpler mechanisms:

- Restriction to Bayesian Nash equilibria in weakly dominant strategies.
- Restriction to strategies that depend only on first-order beliefs.
- Assuming independence of players' types and a common prior.

Time Line of Direct Mechanisms

Ex-ante stage:

- Mechanism designer and players know the joint distribution of types, but players' types have not been realized yet.
- Mechanism designer designs the mechanism.

Interim stage:

- Players observe their type.
- Players decide which type to report.

Ex-post stage:

- Players' reports are publicly revealed.

Voluntary Participation

Definition 5.8

Fix a direct mechanism $\Gamma = (\mathcal{T}_1, \ldots, \mathcal{T}_n, g)$ and suppose that player i has outside option $IR_i : \mathcal{T}_i \to \mathbb{R}$ if he/she doesn't participate in Γ.

1. Γ is ex-ante individually rational if $\mathbb{E}_i[u_i(g(T), \vartheta_i(T_i))] \geq \mathbb{E}_i[IR_i(T_i)]$.
2. Γ is interim individually rational if for every $\tau_i \in \mathcal{T}_i$,

$$\mathbb{E}_{\tau_i}[u_i(g(\theta), \vartheta_i(\tau_i))] \geq IR(\tau_i).$$

3. Γ is ex-post individually rational if $u_i(g(\vartheta), \vartheta_i(\tau_i)) \geq IR_i(\tau_i)$ for every τ_i.

- If participation is voluntary, Γ is x-individually rational if and only if choosing to participate at stage x is a best response by player i.
- Ex-post IR is more of a fairness than a participation constraint.

Deciding the Election in Court

Rigging the election:

- Before the election, republicans were rushing to fill the vacant seat on the Supreme Court so they will have a majority of Justices.
- If the Supreme Court had decided that enough mail-in ballots were invalid to declare Trump the winner, voters may feel worse off than if they did not vote. Such an election is not ex-post individually rational.

Evaluating a Mechanism

Definition 5.9

Fix a social choice function $g : \Theta \to \mathcal{X}$.

1. g is ex-post efficient if $g(\vartheta)$ is not Pareto dominated by another alternative $x \in \mathcal{X}$ for any preference profile $\vartheta \in \Theta$.
2. g maximizes ex-post (utilitarian) welfare if for every $\vartheta \in \Theta$,

$$g(\vartheta) \in \arg\max_{x \in \mathcal{X}} \sum_{i=1}^{n} u_i(x, \vartheta_i).$$

Examples:

- In the roommate example, ex-post efficiency may be most desirable.
- In elections, we wish to maximize ex-post welfare.

Summary

Mechanism design:

- Designing the rules/regulations of the game to implement a desirable social choice function.
- The players' behavior is endogenous and reactive to rule changes.
- The theory is very general with many potential applications.
- Without loss of generality, we can work with direct mechanisms.

Criteria to consider in applications:

- Any mechanism must be incentive compatible.
- Additional criteria are welfare or revenue maximization, efficiency, voluntary participation, fairness, or strategic simplicity.

Check Your Understanding

True or false:

1. In a mechanism, the players jointly decide which social choice function to implement.
2. Truthful reporting in Avalon does not require reporting of higher-order beliefs.
3. A welfare-maximizing mechanism designer prefers to maximize ex-post welfare over ex-ante welfare.
4. Direct mechanisms are cognitively the simplest mechanisms because players only have to report their type.
5. In the definition of full implementation, we could equivalently require that g is implemented by all BNE instead of a unique BNE.

Scan me

Literature

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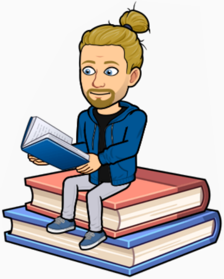


Optimal Selling Mechanism

Selling Mechanism: Setup

Setup:

- One indivisible good is for sale to n potential buyers.
- Each alternative $x = (y, p_1, \ldots, p_n)$ consists of:
  - An allocation y of the good to one (or none) of the buyers.
  - A payment p_i from each buyer $i = 1, \ldots, n$ to the seller.
- The set of alternative is $\mathcal{X} = \{0, 1, \ldots, n\} \times \mathbb{R}^n$, that is, payments p_i can be negative to allow transfers between players.

Buyers' types:

- Suppose buyers' valuations $\theta_1, \ldots, \theta_n$ are independently drawn from a common prior such that each $\theta_i \geq 0$ admits density function $f_i(\vartheta_i)$.
- Each buyer i's type is completely determined by his/her valuation ϑ_i.

Direct Selling Mechanism

Definition 5.10

A direct selling mechanism is a pair $(q, p) : \Theta \to \Delta(\{0, 1, \ldots, n\}) \times \mathbb{R}^n$:

1. $q_i(\vartheta)$ determines the probability that i obtains the good.
2. $p_i(\vartheta)$ is buyer i's deterministic payment to the seller.

Utilities:

- Buyer i's utility function is $u_i(q(\vartheta), p(\vartheta), \vartheta_i) = q_i(\vartheta)\vartheta_i - p_i(\vartheta)$.
- Buyer i's interim expected utility is

$$\mathbb{E}_{\vartheta_i}[u_i(q(\theta), p(\theta), \vartheta_i)] = \underbrace{\mathbb{E}_{\vartheta_i}[q_i(\theta)]}_{\bar{q}_i(\vartheta_i)}\vartheta_i - \underbrace{\mathbb{E}_{\vartheta_i}[p_i(\theta)]}_{\bar{q}_i(\vartheta_i)}.$$

- Buyer i's incentives depend on θ only through $\bar{q}_i(\vartheta_i)$ and $\bar{p}_i(\vartheta_i)$.

Time Line of Direct Selling Mechanism

Ex-ante stage:

- Seller knows distribution of types and designs the mechanism.

Interim stage:

- Players observe their type and decide whether or not to participate.
- Players decide which type to report.

Ex-post stage:

- Players' reports are publicly revealed.

Expected revenue:

- The ex-post revenue is $R(\vartheta) = \sum_{i=1}^{n} p_i(\vartheta)$.
- The seller maximizes the ex-ante expected revenue $\mathbb{E}[R(\theta)]$.

Incentive Compatibility

Lemma 5.11

A direct selling mechanism (q, p) is incentive compatible if and only if

(i) $\bar{q}_i(\vartheta_i)$ *is non-decreasing in* ϑ_i,

(ii) $\bar{p}_i(\vartheta_i) = \bar{p}_i(0) + \bar{q}_i(\vartheta_i)\vartheta_i - \int_0^{\vartheta_i} \bar{q}_i(x)\,\mathrm{d}x$.

Interpretation:

- Reporting a higher type cannot decrease the expected probability of obtaining the item.
- Expected payments are fully determined by:
  - Expected probability of receiving the object,
  - Expected payments of the lowest type.

Proof of Necessity

Setup:

- Denote $u_i(r_i, \vartheta_i) := \bar{q}_i(r_i)\vartheta_i - \bar{p}_i(r_i)$ denote i's utility of reporting valuation r_i when his true valuation is ϑ_i.
- Incentive compatibility: $u_i(\vartheta_i, \vartheta_i) \geq u_i(r_i, \vartheta_i)$ for every $\vartheta_i, r_i \in \Theta_i$.

Monotonicity:

- Suppose that (q, p) is incentive compatible. Then

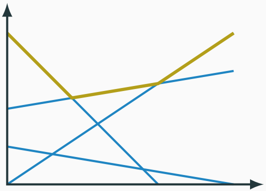
$$\begin{aligned} u_i(r_i, \vartheta_i) \leq u_i(\vartheta_i, \vartheta_i) &= u_i(\vartheta_i, r_i) + \bar{q}_i(\vartheta_i)(\vartheta_i - r_i) \\ &\leq u_i(r_i, r_i) + \bar{q}_i(\vartheta_i)(\vartheta_i - r_i). \end{aligned} \tag{2}$$

- Subtracting $u_i(r_i, \vartheta_i)$ shows that (2) is equivalent to

$$\big(\bar{q}_i(\vartheta_i) - \bar{q}_i(r_i)\big)(\vartheta_i - r_i) \geq 0. \tag{3}$$

- Inequality (3) holds if and only if $\bar{q}_i$ is non-decreasing.

Proof of Necessity



Differentiability:

- Incentive compatibility implies that

$$u_i(\vartheta_i, \vartheta_i) = \max_{r_i \in [\underline{\vartheta}_i, \bar{\vartheta}_i]} u_i(r_i, \vartheta_i)$$

- Since each $u_i(r_i, \vartheta_i) = \bar{q}_i(r_i)\vartheta_i - \bar{p}_i(r_i)$ is convex in ϑ_i, so is $u_i(\vartheta_i, \vartheta_i)$.
- Any convex function is differentiable almost everywhere.
- Convexity also implies that $u_i(\vartheta_i, \vartheta_i)$ absolutely continuous on $(0, \infty)$, i.e., it is the integral of its weak derivative.

Proof of Necessity

Characterization of expected payments:

- Inequality (2) shows that

$$u_i(r_i, r_i) - u_i(\vartheta_i, \vartheta_i) \geq \bar{q}_i(\vartheta_i)(r_i - \vartheta_i). \tag{4}$$

- Inverting the roles of ϑ_i and r_i in (4) and using (3) yields

$$u_i(\vartheta_i, \vartheta_i) - u_i(r_i, r_i) \geq \bar{q}_i(r_i)(\vartheta_i - r_i) \geq \bar{q}_i(\vartheta_i)(\vartheta_i - r_i). \tag{5}$$

- At a differentiability point of $u_i(\vartheta_i, \vartheta_i)$, (4) and (5) imply that

$$\bar{q}_i(\vartheta_i) \leq \lim_{r_i \to \vartheta_i} \frac{u_i(r_i, r_i) - u_i(\vartheta_i, \vartheta_i)}{r_i - \vartheta_i} \leq \bar{q}_i(\vartheta_i).$$

- Absolute continuity implies (ii) via

$$\bar{p}_i(\vartheta_i) = \bar{q}_i(\vartheta_i)\vartheta_i - u_i(\vartheta_i, \vartheta_i) = \bar{q}_i(\vartheta_i)\vartheta_i - u_i(0,0) - \int_0^{\vartheta_i} \bar{q}_i(x)\,\mathrm{d}x.$$

Proof of Sufficiency:

Proof of sufficiency:

- Suppose that (q, p) satisfies (i) and (ii). Then

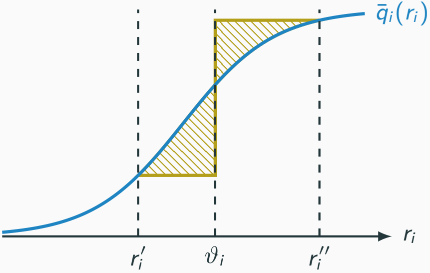
$$u_i(r_i, \vartheta_i) = \bar{q}_i(r_i)\vartheta_i - \bar{p}_i(r_i) \overset{\text{(ii)}}{=} \underbrace{\bar{q}_i(r_i)(\vartheta_i - r_i)}_{=\int_{r_i}^{\vartheta_i} \bar{q}_i(r_i)\, \mathrm{d}x} - \bar{p}_i(0) + \int_0^{r_i} \bar{q}_i(x)\, \mathrm{d}x$$

$$= -\bar{p}_i(0) + \int_0^{\vartheta_i} \bar{q}_i(x)\, \mathrm{d}x + \int_{\vartheta_i}^{r_i} \underbrace{\bar{q}_i(x) - \bar{q}_i(r_i)}_{\leq 0 \text{ by (i)}}\, \mathrm{d}x$$

$$\overset{\text{(ii)}}{\leq} \bar{q}_i(\vartheta_i)\vartheta_i - \bar{p}_i(\vartheta_i) = u_i(\vartheta_i, \vartheta_i).$$

- This shows that (q, p) is incentive compatible.

Incentive Compatibility Illustrated:



Misrepresenting types:

- The derivation on the previous slide shows that

$$u_i(\vartheta_i, \vartheta_i) - u_i(r_i, \vartheta_i) = \int_{\vartheta_i}^{r_i} \bar{q}_i(r_i) - \bar{q}_i(x)\, \mathrm{d}x \geq 0.$$

- Reporting type r_i instead of ϑ_i leads to a loss equal to the shaded area.

Individual Rationality

Lemma 5.12

An incentive-compatible direct selling mechanism (q, p) is interim individually rational with outside option 0 if and only if $\bar{p}_i(0) \leq 0$.

Proof:

- Statement (ii) of Lemma 5.11 implies that (q, p) is interim individually rational if and only if for every $\vartheta_i \in \Theta_i$,

$$0 \leq \mathbb{E}_{\vartheta_i}[u_i(q_i(\theta), p_i(\theta), \vartheta_i)] = \bar{q}_i(\vartheta_i)\vartheta_i - \bar{p}_i(\vartheta_i)$$

$$= -\bar{p}_i(0) + \int_0^{\vartheta_i} \bar{q}_i(x)\,\mathrm{d}x.$$

Maximizing the Revenue

Revenue maximization:

- The seller wishes to maximize the ex-ante expected revenue

$$\mathbb{E}[R(\theta)] = \sum_{i=1}^{n} \int_0^\infty \bar{p}_i(\vartheta_i) f_i(\vartheta_i)\, \mathrm{d}\vartheta_i.$$

  subject to the incentive compatibility and participation constraints.

- By Lemmas 5.11 and 5.12, this is equivalent to maximizing

$$V(p,q) = \sum_{i=1}^{n} \int_0^\infty \left(\bar{q}_i(\vartheta_i)\vartheta_i - \int_0^{\vartheta_i} \bar{q}_i(x)\, \mathrm{d}x \right) f_i(\vartheta_i)\, \mathrm{d}\vartheta_i + \sum_{i=1}^{n} \bar{p}_i(0).$$

  subject to $\bar{q}_i(\vartheta_i)$ being non-decreasing in ϑ_i and $\bar{p}_i(0) \leq 0$.

- It is clear that $\bar{p}_i(0) = 0$ is optimal. What should $\bar{q}_i$ be?

Maximizing the Revenue

Simplifying the objective function:

- Solve the double integral using Fubini's theorem

$$\int_0^\infty \int_0^{\vartheta_i} \bar{q}_i(x)\,\mathrm{d}x\, f_i(\vartheta_i)\,\mathrm{d}\vartheta_i = \int_0^\infty \bar{q}_i(x) \underbrace{\int_x^\infty f_i(\vartheta_i)\,\mathrm{d}\vartheta_i}_{1-F_i(x)}\,\mathrm{d}x$$

- Using $\bar{q}_i(\vartheta_i) = \mathbb{E}_{\vartheta_i}[q_i(\vartheta_i, \theta_{-i})]$ and Fubini's theorem again yields

$$\begin{aligned} V(p,q) &= \sum_{i=1}^n \int_0^\infty \bar{q}_i(\vartheta_i)\left(\vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}\right) f_i(\vartheta_i)\,\mathrm{d}\vartheta_i \\ &= \int_0^\infty \dots \int_0^\infty \sum_{i=1}^n q_i(\vartheta_i)\left(\vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}\right) f(\vartheta)\,\mathrm{d}\vartheta_1 \dots \mathrm{d}\vartheta_n, \end{aligned}$$

where we have used that $f(\vartheta) = f_1(\vartheta_1)\dots f_n(\vartheta_n)$ by independence.

Revenue-Maximizing Selling Mechanism

Theorem 5.13

For each potential buyer i, suppose that $\psi_i(\vartheta_i) := \vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}$ is strictly increasing. Then any revenue-maximizing incentive-compatible individually rational direct selling mechanism (q^*, p^*) satisfies:

1. $q_i^*(\vartheta) = \begin{cases} 1 & \text{if } \psi_i(\vartheta_i) > \max\{0, \max_{j \neq i} \psi_j(\vartheta_j)\} \\ 0 & \text{if } 0 > \max_j \psi_j(\vartheta_j). \end{cases}$
2. $\bar{p}_i^*(\vartheta_i) = \bar{q}_i^*(\vartheta_i)\vartheta_i - \int_0^{\vartheta_i} \bar{q}_i^*(x)\,\mathrm{d}x.$

- The allocation q is uniquely determined almost everywhere.
- Payments p have to satisfy 2. only in expectation.
- The simplest way is to satisfy 2. pointwise, i.e., for every ϑ_{-i}:

$$p_i^*(\vartheta) = q_i^*(\vartheta)\vartheta_i - \int_0^{\vartheta_i} q_i^*(x, \vartheta_{-i})\,\mathrm{d}x.$$

Interpretation of Optimal Allocation

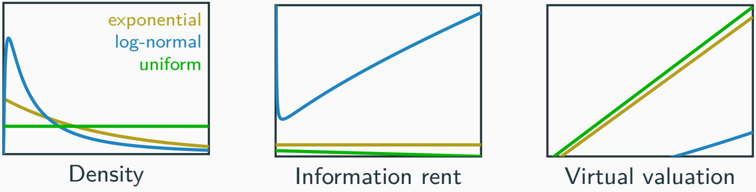
Interpretation of $\psi_i(\vartheta_i) = \vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}$:

- The seller's marginal revenue in buyers' types is $\max\{\max_i \psi_i(\vartheta_i), 0\}$.
- With complete information, the marginal revenue would be $\max_i \vartheta_i$.
- The difference $\frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}$ for each buyer i is the information rent buyer i receives for reporting his/her type truthfully.
- $\psi_i(\vartheta_i)$ is also called buyer i's virtual valuation.

Optimal allocation of the good:

- If the virtual valuation after accounting for information rent is negative for all potential buyers, the seller keeps the good.
- Otherwise, the buyer with the highest virtual valuation gets it.
- This does not have to be the buyer with the highest valuation.

Information Rent and Virtual Valuation



Information rent:

- Information rent can be written as

$$\psi_i(\vartheta_i) = \frac{1 - F_i(\vartheta_i)}{f_i(\vartheta_i)} = \int_{\vartheta_i}^{\infty} \frac{f_i(x)}{f_i(\vartheta_i)} \, \mathrm{d}x.$$

- Information rent is high if:
  - the likelihood $f_i(\vartheta_i)$ is low: type ϑ_i is unlikely.
  - the density does not decrease too fast above ϑ_i: type ϑ_i is well obscured.
- In particular, heavy-tailed distributions lead to high information rents.

Increasing Virtual Valuations

Necessity:

- If $\psi_i(\vartheta_i) = \vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}$ is not increasing, then q_i^* is not monotone.
- Lemma 5.11 implies that, then, (q_i^*, p^*) is not incentive compatible.

Question: for which distribution functions is ψ_i strictly increasing?

- It turns out that ψ_i is increasing for the most commonly used parametric distributions on subsets of $[0, \infty)$, such as Uniform, Exponential, Chi-squared, Gamma, Log-normal, and Pareto with finite mean.
- It is not satisfied for Pareto with infinite mean, but then the revenue is unbounded, hence there is no revenue-maximizing mechanism.

Later today: We will learn what to do when ψ_i is not increasing.

Interpretation of Optimal Payments

Actual buyer:

- Player i receives the item if and only if $\psi_i(\vartheta_i) \geq \max\{0, \max_{j \neq i} \psi_j(\vartheta_j)\}$.
- Since ψ_i is increasing, there exists a lowest report

$$r_i^*(\vartheta_{-i}) := \psi_i^{-1}\big(\max\{0, \max_{j \neq i} \psi_j(\vartheta_j)\}\big),$$

  given ϑ_{-i}, with which i receives the item.
- This implies that $q_i^*(\vartheta) = 1_{\{\vartheta_i > r_i^*(\vartheta_{-i})\}}$ and, hence,

$$p_i^*(\vartheta) = q_i^*(\vartheta)\vartheta_i - \int_0^{\vartheta_i} q_i^*(x, \vartheta_{-i})\,\mathrm{d}x = \vartheta_i - \int_{r_i^*(\vartheta_{-i})}^{\vartheta_i} \mathrm{d}x = r_i^*(\vartheta_{-i}).$$

Everybody else:

- Player j with $\psi_j(\vartheta_j) < \max\{0, \max_i \psi_i(\vartheta_i)\}$ satisfies $\vartheta_j < r_j^*(\vartheta_{-j})$.
- Thus, $q_j^*(r_j, \vartheta_{-j}) = 0$ for $r_j \leq \vartheta_j$, hence $p_j^*(\vartheta) = 0$.

Optimal Selling Mechanism Under Symmetry

Theorem 5.14

Suppose $F_i = F$ for every potential buyer i such that $\psi(x) := x - \frac{1-F(x)}{f(x)}$ is strictly increasing. Then the revenue-maximizing direct selling mechanism is a second-price auction with reserve price $\psi^{-1}(0)$.

Remark:

- Symmetry guarantees that the highest-valuation buyer also has the highest virtual valuation due to monotonicity of ψ.
- Buyer i with the highest valuation receives the good if ϑ_i exceeds the reserve price, and he/she pays $r_i^*(\vartheta_{-i}) = \max\{\max_{j\neq i} \vartheta_j, \psi^{-1}(0)\}$.
- The seller keeps the good if $\max_j \vartheta_j < \psi^{-1}(0)$.
- No other static mechanism could yield a higher expected revenue.

Efficiency of the Optimal Mechanism

Efficiency:

- By Lemma 7.9, a selling mechanism is ex-post efficient if and only if the buyer with the highest valuation gets the good.

Under symmetry:

- The only source of inefficiency is if the buyer keeps the good, that is, if everybody's valuation is below the reserve price.
- This is the well-known feature that a monopolist makes the good artificially scarce under incomplete information.

Without symmetry:

- The buyer with the highest virtual valuation receives the good, which may not be the buyer with the highest valuation.
- Incomplete information creates additional sources of inefficiency.

Optimality of First-Price Auction

First-price auction:

- Without reserve price, we have seen that the first-price auction yields the same expected revenue as the second-price auction.
- Even though it is not as obvious to interpret Theorem 5.13 as a first-price auction, one can show that a first-price auction with appropriately chosen reserve price also maximize the expected revenue.

Question: what other auction formats yield the same expected revenue?

Revenue Equivalence Theorem

Theorem 5.15

Consider any auction $\mathcal{G}$ with independent private types $\theta_1, \ldots, \theta_n$ such that

1. *Each type θ_i admits a positive density $f_i(\vartheta_i)$ on $[0, \infty)$.*
2. *The object is awarded to the highest bidder.*

In any symmetric, increasing pure-strategy BNE s, in which type 0 makes expected payments 0, the expected payments of type ϑ_i are

$$\bar{p}_i(\vartheta_i) = \mathbb{E}\left[\max_{j \neq i} \theta_j 1_{\{\max_{j \neq i} \theta_j \leq \vartheta_i\}}\right].$$

Moreover, the seller's ex-ante expected revenue is $\mathbb{E}\left[\theta^{(2)}\right]$.

Remark:

- This result allows us to characterize symmetric, increasing pure-strategy BNE in auctions without reserve price extremely quickly.
- Typically, the symmetric, increasing pure-strategy BNE is unique.

Proof of Theorem 5.15

Step 1: Expected payments

- Asking players to report their valuations and playing s for the players in auction $\mathcal{G}$ is an incentive-compatible direct selling mechanism.
- Since s is symmetric and increasing, $s(\vartheta_i) > \max_{j \neq i} s(\vartheta_j)$ if and only if $\vartheta_i = \max_j \vartheta_j$, hence $q_i(\vartheta) = 1_{\{\vartheta_i \geq \max_{j \neq i} \vartheta_j\}}$.
- Abbreviate $X_i = \max_{j \neq i} \theta_j$. Since $\bar{p}_i(0) = 0$, Lemma 5.11 implies that

$$
\begin{aligned}
\bar{p}_i(\vartheta_i) &= \vartheta_i P(X_i \leq \vartheta_i) - \int_0^{\vartheta_i} P(X_i \leq x)\,\mathrm{d}x \\
&= \vartheta_i P(X_i \leq \vartheta_i) - \mathbb{E}\left[\int_0^{\vartheta_i} 1_{\{X_i \leq x\}}\,\mathrm{d}x\right] \\
&= \mathbb{E}\big[\vartheta_i 1_{\{X_i \leq \vartheta_i\}} - (\vartheta_i - X_i) 1_{\{X_i \leq \vartheta_i\}}\big] = \mathbb{E}\big[X_i 1_{\{X_i \leq \vartheta_i\}}\big].
\end{aligned}
$$

Proof of Theorem 5.15

Step 2: Expected revenue

- Highest bidder wins a second-price auction without reserve price.
- Moreover, truthful reporting $s_i(\vartheta_i) = \vartheta_i$ is symmetric, increasing, with expected payment of 0 for type 0.
- The second-price auction satisfies the conditions of Theorem 5.15.
- Therefore, the ex-ante expected revenue of any such auction is

$$\mathbb{E}[R(\theta)] = \sum_{i=1}^{n} \mathbb{E}[\bar{p}_i(\theta_i)] = \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[X_i 1_{\{X_i \le \theta_i\}}\right]\right] = \mathbb{E}\left[\theta^{(2)}\right].$$

Finding Bidding Strategies

Corollary 5.16

Consider a sealed-bid first-price auction with independent types distributed on $[0, \infty)$ *with positive density* $f_i(\vartheta_i)$*. The unique symmetric, increasing pure-strategy BNE is* $s_i(\vartheta_i) = \mathbb{E}[\max_{j \neq i} \theta_j \,|\, \max_{j \neq i} \theta_j \leq \vartheta_i]$.

Proof:

- Fix any symmetric, increasing pure-strategy BNE s.
- If $s(0) > 0$, then $s(\vartheta_i) > \vartheta_i$ for ϑ_i sufficiently small. Bidding above ϑ_i is strictly dominated, hence we must have $s(0) = 0$.
- It follows that Theorem 5.15 applies to s. Since bidder i of type ϑ_i pays $s(\vartheta_i)$ if he/she wins, ϑ_i's expected payment is

$$s(\vartheta_i)P(\vartheta_i \geq X_i) = \mathbb{E}\big[X_i 1_{\{X_i \leq \theta_i\}}\big].$$

- Dividing by $P(\vartheta_i \geq X_i)$ yields the desired result.

Summary

Optimal selling mechanism:

- Incentive compatibility has a strikingly simple characterization:
  - Monotonicity of allocation function.
  - Payments are uniquely determined by $\bar{p}(0)$ and allocation function.
- If types are symmetric and ψ_i is increasing, the optimal selling mechanism is a second-price auction with reserve price.

Revenue equivalence:

- Any auction format, in which the highest-value bidder obtains the item, must yield the same ex-ante expected revenue.
- We can use the revenue equivalence result to prove uniqueness of symmetric increasing pure-strategy Bayesian Nash equilibria.

Check Your Understanding

True or false:

1. Every auction format has a unique BNE.
2. Every type receives a positive information rent in the optimal selling mechanism.
3. If the bidder with the highest valuation has the highest virtual valuation, the optimal selling mechanism is a second-price auction with reserve price.
4. A third-price auction yields the same ex-ante expected revenue as an auction that awards the good randomly to one of the two highest bidders, who each pay their bid regardless of who obtains the item.

Scan me

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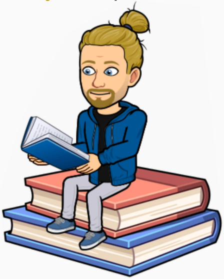
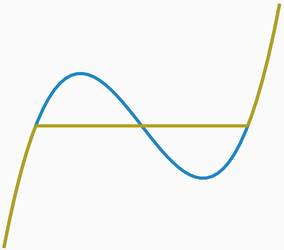
- G.A. Jehle and P.J. Reny: Advanced Microeconomic Theory, Chapter 9.4, Prentice Hall, 2011
- W. Vickrey: Counterspeculation, Auctions, and Competitive Sealed Tenders, Journal of Finance, **16** (1961), 8–37
- R.B. Myerson: Optimal Auction Design, Mathematics of Operations Research, **6** (1981), 58–73



Myerson's Ironing

Myerson's Ironing Technique



Myerson's ironing:

- For many distribution functions, $\psi_i(\vartheta_i) = \vartheta_i - \frac{1-F_i(\vartheta_i)}{f_i(\vartheta_i)}$ is increasing, hence the constructed mechanisms are incentive compatible.
- If ψ_i is not increasing, let us "iron" ψ_i by making the function constant on any interval, over which ψ_i is non-monotonic.
- The resulting function $\bar{\psi}_i$ is called an ironed virtual value function.

Distribution of Item

Breaking ties:

- After ironing, buyers have the same ironed virtual valuation with positive probability, hence we need to specify how ties are broken.
- Define $\mathcal{I}_{\bar{\psi}}(\vartheta) := \left\{ i \in \mathcal{I} \;\middle|\; \bar{\psi}_i(\vartheta_i) \geq \max\left\{\max_j \bar{\psi}_j(\vartheta_j), 0\right\}\right\}$ and set

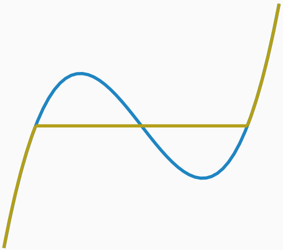
$$\widehat{q}_i(\bar{\psi}, \vartheta) = \begin{cases} \frac{1}{\left|\mathcal{I}_{\bar{\psi}}(\vartheta)\right|} & \text{if } i \in \mathcal{I}_{\bar{\psi}}(\vartheta), \\ 0 & \text{otherwise.} \end{cases}$$

Incentives:

- $\widehat{q}_i(\vartheta_i, \vartheta_{-i})$ is non-decreasing in ϑ_i for each ϑ_{-i}, hence so is $\bar{q}_i(\vartheta_i)$.
- Lemma 6.11 implies that $(\widehat{q}, p)$ is incentive compatible if we set

$$\bar{p}_i(\vartheta_i) = \bar{q}_i(\vartheta_i)\vartheta_i - \int_0^{\vartheta_i} \bar{q}_i(x)\, \mathrm{d}x.$$

Incentives



Intuitively:

- Because q_i is constant on an "ironed" interval, so is $\bar{p}_i$.
- Therefore, pooled types have no incentives to misrepresent as another type from the same "ironed" interval.
- Incentives of types at the edge of the "ironed" interval to misreport as a type outside the "ironed" interval remain unaffected.
- Therefore, truth telling is incentive compatible.

Which Ironing Point Should We Choose?

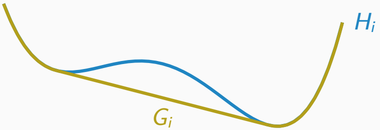
Individual rationality:

- Fix any $(\bar{\psi}_i)_i$ and consider an ironed interval $[a_i, b_i]$ of $\bar{\psi}_i$.
- If we want to maintain $\bar{p}_i(0) = 0$, then, at the optimum, $\bar{\psi}_i(\vartheta_i)$ for any $\vartheta_i \in [a_i, b_i]$ is equal to the expected virtual valuation over $[a_i, b_i]$.
- If $\bar{\psi}_i(\vartheta_i)$ is higher, the seller is paying less information rent on average, which requires a participation subsidy $\bar{p}_i(0) < 0$.

Maximizing revenue:

- If $\bar{\psi}_i(\vartheta_i)$ is lower than the expected virtual valuation over $[a_i, b_i]$, the seller is paying excessive information rents, which cannot be optimal.
- Moreover, the ironing interval must be minimal so that we do not pool types unnecessarily (cf. screening of customers).

Transforming the Problem



Transformation to quantile space:

- Define $h_i := \psi_i \circ F_i^{-1}$ so that $\psi_i(\vartheta_i) = h_i(F_i(\vartheta_i))$ and, hence,

$$\int_{a_i}^{b_i} \psi_i(\vartheta_i) f_i(\vartheta_i)\, \mathrm{d}\vartheta_i = H_i(F_i(b_i)) - H_i(F_i(a_i)).$$

  where H_i is the anti-derivative of h_i.
- Let G_i denote the convex hull of H_i, $g_i = G_i'$, and $\bar{\psi}_i(\vartheta_i) = g_i(F_i(\vartheta_i))$.
- Note that $G_i(F_i(a_i)) = H_i(F_i(a_i))$ and $G_i(F_i(b_i)) = H_i(F_i(b_i))$, hence

$$\int_{a_i}^{b_i} \psi_i(\vartheta_i) f_i(\vartheta_i)\, \mathrm{d}\vartheta_i = \int_{a_i}^{b_i} \bar{\psi}_i(\vartheta_i) f_i(\vartheta_i)\, \mathrm{d}\vartheta_i.$$

Myerson's Ironing Technique

Theorem 5.17

Let g_i be defined as before and set $\bar{\psi}_i(\vartheta_i) := g_i(F_i^{-1}(\vartheta_i))$. Then $(\widehat{q}\,(\bar{\psi}), \widehat{p})$ for $\widehat{p}$ defined from $\widehat{q}$ and (ii) in Lemma 6.11 is the revenue-maximizing incentive-compatible individually rational direct selling mechanism.

This is indeed a generalization:

- If ψ_i is increasing, then $h_i = \psi_i \circ F_i^{-1}$ is increasing, hence H_i is convex and equal to G_i. Therefore, $g_i = h_i$ and, hence, $\bar{\psi}_i = \psi_i$.

Interpretation:

- Buyer types, for which ψ_i is non-monotonic, are pooled together by ironing the buyers' virtual valuation functions.
- Buyer with highest positive ironed virtual valuation receives the good.

Proof of Theorem 5.17

Revenue maximization:

- For any probability assignment function q and p defined by (ii) of Lemma 6.11, the seller's expected revenue is

$$V(q) = \int_0^\infty \sum_{i=1}^n q_i(\vartheta_i)\bar{\psi}_i(\vartheta_i) f(\vartheta)\,\mathrm{d}\vartheta$$

$$+ \sum_{i=1}^n \int_0^\infty \bar{q}_i(\vartheta_i)(\psi_i(\vartheta_i) - \bar{\psi}_i(\vartheta_i)) f_i(\vartheta_i)\,\mathrm{d}\vartheta_i$$

- For any choice of $(\bar{\psi}_i)_i$, the first term is maximized in $\widehat{q}(\bar{\psi})$.
- It remains to show that the second term is maximized for $\bar{\psi}_i = g_i \circ F_i^{-1}$.

Proof of Theorem 5.17

Second term:

- Integration by parts yields

$$\int_0^\infty \underbrace{\bar{q}_i(\vartheta_i)}_{\downarrow} \underbrace{(h_i(F_i(\vartheta_i)) - g_i(F_i(\vartheta_i)))f_i(\vartheta_i)}_{\uparrow} \, \mathrm{d}\vartheta_i$$

$$= \bar{q}_i(\vartheta_i)\big(H_i(F_i(\vartheta_i)) - G_i(F_i(\vartheta_i))\big)\Big|_{\vartheta_i=0}^{\infty}$$

$$- \int_0^\infty \big(H_i(F_i(\vartheta_i)) - G_i(F_i(\vartheta_i))\big) \underbrace{\mathrm{d}\bar{q}_i(\vartheta_i)}_{\text{“}= \bar{q}_i'(\vartheta_i)\,\mathrm{d}\vartheta_i\text{”}} \leq 0.$$

- Since $\bar{q}_i$ is constant on ironed intervals and $H_i = G_i$ everywhere else, $(\bar{\psi}_i)_i$ maximizes the expected revenue.

Summary

Myerson's ironing:

- If the allocation function is decreasing over an interval $[\underline{\vartheta}_i, \bar{\vartheta}_i]$, we pool those types together by ironing the virtual valuation function.
- Truthful reporting is incentive compatible:
  - Within the interval because types are treated identically.
  - Towards outside the interval due to the incentives of boundary types.

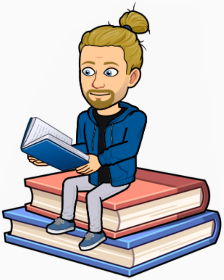
Individual rationality:

- Interim individual rationality remains unaffected if we set the ironed virtual valuation equal to $\mathbb{E}\left[\psi_i(\theta_i) \,\middle|\, \theta_i \in [\underline{\vartheta}_i, \bar{\vartheta}_i]\right]$.
- Optimally, we iron over an interval with

$$\psi_i(\underline{\vartheta}_i) \leq \mathbb{E}\left[\psi_i(\theta_i) \,\middle|\, \theta_i \in [\underline{\vartheta}_i, \bar{\vartheta}_i]\right] \leq \psi_i(\bar{\vartheta}_i).$$

Literature

- R.B. Myerson: Optimal Auction Design, Mathematics of Operations Research, **6** (1981), 58–73



Mechanisms in Large Economies

Selling an Item to the Entire Population?

Selling to a large population:

- What can we learn from mechanism design about selling items in a supermarket or a department store?

Modeling large economies:

- Setting $n = 23.78$ million is hardly feasible.
- One could take the limit as $n \to \infty$, but what guarantees that the model is continuous at infinity?
- Often more promising: model the economy as a continuum of agents.

Exact Law of Large Numbers

Theorem 5.18

Consider a continuum of independent random variables $(X_t)_{t\in[0,1]} \sim F$ on a Fubini extension of a typical probability space. Then the empirical distribution function is almost-surely equal to F.

Interpretation:

- The strong law of large numbers states that the empirical distribution of a sample of size n converges to F almost-surely as $n \to \infty$.
- The larger sample size reduces the variance of the estimation.
- If we have a continuous sample, the estimation error is 0.
- The proof requires non-standard analysis; see Sun (2006).

Selling an Item to the Entire Population

Corollary 5.19

The revenue-maximizing mechanism for selling an indivisible good to a continuum of buyers with density $f > 0$ is to sell the item at the fixed price $p := \psi^{-1}(0)$, where ψ is defined as usual.

Large economies:

- By the exact law of large numbers, a mechanism for a continuum of individuals is equivalent to a mechanism for a single individual.
- This is what we used in contract theory / screening in Econ 7011.

Proof:

- The second-price auction with reserve price for a single buyer means that the buyer obtains the good at price p if and only if $\vartheta_i \geq p$.

Optimal Taxation

Economy of a continuum of consumers/producers:

- Type is the individuals' skill level, distributed according to density f.
- Mechanism $g(\vartheta_i) = \big(q(\vartheta_i), p(\vartheta_i)\big)$ assigns
  - Production level $q(\vartheta_i)$ (labor),
  - Consumption $p(\vartheta_i) = q(\vartheta_i) - z(q(\vartheta_i))$ for tax rate z.
- Suppose everyone has the same quasi-linear utility function

$$u(g(\vartheta), \vartheta_i) = p(\vartheta_i) - c(q(\vartheta_i), \vartheta_i),$$

  where the cost of labor $c(q(\vartheta_i), \vartheta_i)$ depends on individual i's skill level.
- Because there is no aggregate uncertainty, p and q in an incentive-compatible mechanism do not depend on reported types of others.
- Question: what is the optimal tax schedule?

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