Lecture 3

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1 Problems in Text

p.33 The Weak Axiom of Revealed Preference
Condition α and β to gether are equivalent to WA for any choice correspondence with a domain satisfying that if a and B are included i the domain, then so is their intersection.

Answer
(WA $\implies$ α and β)
α: Take $b \in C(A)$. We have $a, b \in A \cap B = A$, $a \in C(B), b \in C(A)$. Hence, $a \in C(A)$.
β: $a, b \in A \cap B = A$, $a \in C(A)$, $b \in C(B)$, so $a \in C(B)$. (α and β $\implies$ WA)
Suppose $x, y \in A \cap B$, $x \in C(A)$ and $y \in C(B)$. By assumption, $A \cap B$ is also in domain, and $x, y \in C(A \cap B)$ by Condition α.
Notice that $x, y \in C(A \cap B)$ and $y \in C(B)$. Hence $x \in C(B)$ by Condition β.

Comment Notice that WA $\implies$ α and β generally holds while α and β $\implies$ WA needs some assumption about domain.

2 Problems Set

Answer 1
a. Yes, let $a \succ b$ if a makes more suffering than b.
b. Yes, let $a \succ b$ if a gets lower average score than b.
c. Yes, let $a \succ b$ if a is closer to the ideal point.
d.(not quite sure) No. It is more like a framing issue. Since any element appears only once in a set ($\{a\} = \{a, a\}$ in the structure of set). Hence, it is better to see as a choice problem as a tuple and one can find that $a = c((a, a, b)) \neq c((a, b, b)) = b$ although $c(\{a, a, b\}) = c(\{a, b\}) = c(\{a, b, b\})$.
e. $D \subset P(X)$ consists of all subsets A with $|A|$ be odd. No. Let $X = \{1, 2, 3, 4, 5\}, D = P(X)$. $c(\{1, 2, 3, 4, 5\}) = 3$ while $c(\{1, 2, 3\}) = 1$, violating condition α.

Answer 2

a. Let C can be rationalized. First, notice that if $a \in C(A)$, then it means that a is $\succsim$-maximal in A, and so is it in $C(A_1) \cup C(A_2) \subseteq A$.

Let's do the other side. Let $a \in C(C(A_1) \cup C(A_2))$. W.L.O.G, let $a \in C(A_1)$. Let $b \in C(A_2)$ and $z_1 \in A_1$, $z_2 \in A_2$. We have $a \succsim z_1$ by $a \in C(A_1)$. And by $a \in C(C(A_1) \cup C(A_2))$ and $b \in C(A_2)$, we have $a \succsim b \succsim z_2$. Thus, a is a $\succsim$-maximal in A.

b. Consider $X = \{\text{Princeton, Chicago, LSE}\}$ as page 37.

We have $C(\{\text{Princeton, Chicago, LSE}\}) = \text{LSE}$ but

$$C(\{\text{Princeton, LSE}\}) \cup C(\{\text{Chicago}\}) = \{\text{Princeton, Chicago}\}$$

and so $C(C(\{\text{Princeton, LSE}\}) \cup C(\text{Chicago})) \neq \text{LSE}$.

c. Let $a \in A \subset B$, and $a \in C(B)$. $C(B) = C(C(A) \cup C(B \setminus A)) \Rightarrow a \in C(A) \cup C(B \setminus A) \Rightarrow a \in C(A)$.

d. Best two. If $|A| = 1$, just take the one.

Answer 3

$C(A)$: Yes. Suppose that $a, b \in A \cap B$, $a \in C(A), b \in C(B)$.

If the number of $y \in X$ for which $V(a) \geq V(y)$ is at least $\frac{|X|}{2}$, then

$$C(A) = \{x \in A| \text{ the number of } y \in X \text{ for which } V(a) \geq V(y) \text{ is at least } \frac{|X|}{2}\ \}$$

Since, $a \in B$, the set $\{x \in B|$ the number of $y \in X$ for which $V(a) \geq V(y)$ is at least $\frac{|X|}{2}\ \}$ is not empty, hence

$$C(B) = \{x \in B| \text{ the number of } y \in X \text{ for which } V(a) \geq V(y) \text{ is at least } \frac{|X|}{2}\ \}$$

and so $a \in C(B)$.

If the number of $y \in X$ for which $V(a) \geq V(y)$ is less than $\frac{|X|}{2}$, then $C(A) = A$ and the number of $y \in X$ for which $V(b) \geq V(y)$ is also less than $\frac{|X|}{2}$. It tells us that $C(B) = B \Rightarrow a \in C(B)$.

$D(A)$: No. Let $X = \{1, 2, 3, 4, 5\}$ and $V(x) = x$. Let $A = \{1, 2, 3, 4\}, B = X$. We have $3 \in C(A)$, $4 \in C(B)$, while $3 \notin C(B)$.

$E(A)$: No. It fails condition β. Let a, b, c with $c \succ_1 a \succ_1 b$ and $b \succ_2 a \succ_2 c$. We have $C(\{a, b\}) = \{a, b\}$ but $C(\{a, b, c\}) = \{b, c\}$.

Comment

For $C(A)$, the choice correspondence is actually rational.

To see this, let $S_1 = \{x \in X | \text{the number of } y \text{ such that } V(x) \geq V(y) \text{ is at least } \frac{|X|}{2}\}$ and S_2

Answer 4

a. Complete, Asymmetric: Trivial

Transitive: Say xPy and yPz. Thus, $C(\{x,y\}) = x$ and $C(\{y,z\}) = y$. Notice that $class(x) \geq class(y) \geq class(z)$.

Hence, $class(x) \geq class(z)$. If $class(x) > class(z)$, done! If not, it means that $class(x) = class(y) = class(z)$ and x is a better element than y and y is a better element than z. Hence, x is a better element in z, done!

b. No. Consider $X = \{a,b,c\}$. Suppose that $class(a) = 2$, $class(b) = class(c) = 1$ and b is a better element than c.

We have $C(\{a,b\}) = a$ but $C(\{a,b,c\}) = b$, violating condition α.

Answer 5 (I want to discuss about this problem, please let me know if you have good examples)

Consider the choice set is the set of 升大學準備方式

a. Consider v as 學測分數, v^* as 繁星上想去大學需要的學測分數. u as 繁星上想去大學的機會. If the way that maximizes u does not let $v(a^*) \geq v^*$, then the one will go preparing for 學測.

It is not consistent with rational man paradigm. Let $u(a) > u(b) > u(c)$ and $v(c) > v(b) > v^* > v(a)$. Then $C(\{a,b,c\}) = c, C(\{b,c\}) = b$. b. Consider v as 學測分數 again, u as 特殊選材能上的學校, u^* as 最低能接受的特殊選材學校. Notice that 特殊選材 result does not depend on 學測分數. If one cannot go to the ideal school by 特殊選材, then the one will go preparing for 學測. It is consistent with rational man paradigm. Details are omitted.

Answer 6

a.

Satisfy I: 1. first in the list, 2. highest utility

Do not satisfy I: 1. middle in the list, 2. elements appears the most

b.

Order Invariance: $C(\langle a_1, \cdots, a_N \rangle) = C(\langle a_{\sigma(1)}, \cdots, a_{\sigma(N)} \rangle)$, where σ is a permutation on $\{1, \cdots, N\}$.

Duplication Invariance: $C(\langle a_1, \cdots, a_N \rangle) = C(\langle a_{k_1}, \cdots, a_{k_{N-1}} \rangle)$, where $\{k_i\}$ is a subsequence of $1, 2, \cdots, N$ and $\{a_i | 1 \leq i \leq N\} = \{a_{k_j} | 1 \leq j \leq N-1\}$.

Duplication Invariance$\Rightarrow$ Order Invariance:

We know that any permutation on $\{1, \cdots, N\}$ can be generated by $(1\ 2)$ and $(1\ 2\ \cdots\ N)$. We only do these two.

$(1\ 2)$: $C(\langle a_1, a_2, a_3, \cdots, a_N \rangle) = C(\langle a_1, a_2, a_1, a_3, \cdots, a_N \rangle) = C(\langle a_2, a_1, a_3, \cdots, a_N \rangle)$

$(1\ 2\ \cdots\ N)$: $C(\langle a_1, a_2, \cdots, a_N \rangle) = C(\langle a_1, a_2, a_3, \cdots, a_N, a_1 \rangle) = C(\langle a_2, \cdots, a_N, a_1 \rangle)$

b.(TA version) Let L' be L after permutation. Notice that $C(L) = C(\langle L, L' \rangle) = C(L')$

c. Suppose that the choice function satisfies Duplication Invariance and property I.
For a list L, $C(L)$ does not change after deleting the repeating elements in L. Hence, W.L.O.G, we may assume the lists do not have repeating elements. Notice the result of b., it tells us that as long as the set of list element remains unchanged, then the choice is unchanged. Therefore, C is actually a function only regards to the set of the list, but not the order of it. We then can well-define $C(\{a_1, \cdots, a_N\}) = C(\langle a_1, \cdots, a_N \rangle)$.
Next, by property I, we can see that C satisfies Condition α. And since C is defined on all list, it is defined on all subset of X. Hence, C is rational.
d. 1. maximizer of u, 2. first element in the list (not interesting)
e. 2. minimizer of u, 2. last element in the list (not interesting, either)

Answer 7

a.
Lexicographically rational $\implies$ reference point property: Take a be the $O-$maximal, done!
Reference point property $\implies$ Lexicographically rational: We say that $a \in P(A)$ if a is one of the element in A satisfies the description in the problem.
For $A = X$, pick $a_1 \in P(X)$. Then pick $a_i \in P(X \setminus \{a_1, \cdots, a_{i-1}\})$ recursively. We now define $a_i O a_j$ if $i > j$. We define $x \succ_{a_i} y$ if there exists $A \supseteq \{x, y\}$ which a_i is the O-maximal element in A and $C(A) = x$.
We shall prove that $\succ_{a_i}$ is asymmetric and transitive, so it can be extended to a preference relation and the choice in A can be seen as choosing $\succ_{a_i}$-maximal element in A when a_i is O-element in A.

Asymmetric: Let $x \succ_{a_i} y$ and $y \succ_{a_i} x$. It means that there exists sets $A_1, A_2 \supseteq \{x, y\}$ containing a_i as their O-maximal element such that $C(A_1) = x, C(A_2) = y$. Notice that a_i is O-maximal in $A_1 \cup A_2$. Without loss of generality, say $C(A_1 \cup A_2) \in A_1$, then by reference point property, $C(A_1 \cup A_2) = C(A_1) = x$. Since $x \in A_2$, so by reference point property, $C(A_2) = C(A_1 \cup A_2) = x \neq y$, which leads to an contradiction.

Transitivity: Let $x \succ_{a_i} y$ and $y \succ_{a_i} z$. It means that there exists $A_1 \supseteq \{x, y\}, A_2 \supseteq \{y, z\}$ containing a_i as their O-maximal in element such that $C(A_1) = x, C(A_2) = y$. Notice that a_i is O-maximal in $A_1 \cup A_2$.
If $C(A_1 \cup A_2) \in A_2$, then by reference point property, $C(A_1 \cup A_2) = C(A_2) = y \Rightarrow C(A_1) = y \neq x$, which leads to a contradiction.
Hence $C(A_1 \cup A_2) \in A_1$, and by reference point property $C(A_1 \cup A_2) = C(A_1) = x \Rightarrow x \succ_{a_i} z$.

b. Second best.

Answer 8

a.

Rationalized $\Rightarrow$ WWARP:

Let $\{x,y\} \subseteq B_1 \subseteq B_2$ and $c(\{x,y\}) = c(B_2) = x$. From $c(\{x,y\}) = x$, we knows that either $x \succ y$ or y is not a $\succ_k$-maximal for any k. If the latter, then $c(B_1) \neq y$. Say the former. Notice that $c(B_2) = x$ says that x is some $\succ_k$-maximal in B_2, and so is in B_1. By $x \succ y$, we knows that no matter y is some $\succ_k$-maximal or not, $c(B_1) \neq y$.

WWARP$\Rightarrow$ rationalized:

For every $A \subseteq X$, define $\succ_A = \{(c(A), b) | b \in A \setminus \{c(A)\}\}$ and $a \succ b$ if $c(\{a,b\}) = a$.

We now prove that a choice function which satisfies WWARP is rationalized with $\succ_A$ and $\succ$ we defined above.

In a set A, if x is some $\succ_S$-maximal in A, then by definition, we must have $x = c(S)$ and $S \supseteq A$. We shall prove that $c(A) \succ c(S)$ for all $S \supseteq A$ if $c(S) \neq c(A)$. Assume not, we then have $c(S) \succ c(A)$ and so $\{c(A), c(S)\} \subseteq A \subsetneq S$ with $c(S) = c(\{c(A), c(S)\})$, which forces $c(A) = c(S)$ by WWARP.

b.

Let $c(\{x,y\}) = c(B_2) = x$. If x is the most moral alternative in B_2, then so is it in $\{x,y\}$ and either y is not more selfish than x or $x \succsim_S l(y)$. Neither of the case is y possible to be $c(B_1)$. If x is the most selfish alternative in B_2, then so is it in $\{x,y\}$ and either y is not more moral than x or $l(x) \succ_S y$. If the former, then y is clearly impossible to be $c(B_1)$. If the latter, then the most selfish element z in B_1 satisfies $l(z) \succsim_S l(x)$ by $z \succsim_S x$. By transitivity $l(z) \succ_S y$ and hence $c(B_1) \neq y$.

c. Take the set of rationales as $\{\succsim_M, \succsim_S\}$. Moreover, we let $x \succ y$ if one of the following three happens: 1. $x \succ_M y$ and $x \succ_S y$. 2. $x \succ_M y$, $y \succ_S x$and $x \succsim_S l(y)$. 3. $x \succ_S y$, $y \succ_M x$ and $l(x) \succ_S y$.

Answer 9

a. The probability to choose a from A is the probability to choose a from B times the probability to choose something in B from A. Specifying B does not change relative probability of the member in B.

Examples:

1. Problem 4 but modified a little bit, after the class is chosen, then the elements in such class share the same probability. Consider $X = \{a,b,c\}$, where $class(a) > class(b) = class(c)$. Hence, $C(a|X) = 0$, $C(\{a,b\}|X) = C(a|X) + C(b|X) = 0 + \frac{1}{2}$, $C(a|\{a,b\}) = 1$.

2. Pick poker cards probabilistically in the following manner. If a set of poker cards A is given,

first sort them by suits. Assign the probability of each card with respect to their suit to satisfy the following two conditions.

(1) Every card in a given suit has the same probability.

(2) The probability to choose any suit is equal.

For example, $A = \{\clubsuit 3, \clubsuit 5, \clubsuit 7, \spadesuit A, \spadesuit J\}$, then the probability is $\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}$ respectively.

b. Let $v(x) = C(x|X)$. We then have

$$C(a|A) = \frac{C(a|X)}{C(A|X)} = \frac{C(a|X)}{\sum_{x \in A} C(x|X)} = \frac{v(a)}{\sum_{x \in A} v(x)}$$