5. Signaling and Communication

ECON 7219 – Games With Incomplete Information

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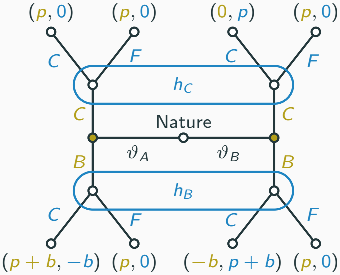
Signaling Games

Bluffing on the River: Information Setting



Information setting:

- Juanda perfectly observes Lebedev's actions.
- As a result, Juanda has one information set for each history of play.
- Only uncertainty is about the player's type, hence each of Juanda's information sets has two nodes, corresponding to the two types.
- This setting is special in several ways, which we will explore more.

Information Setting in Poker

Multi-stage game with observed actions:

- Players face each other in several stages or periods.
- Only imperfect information arises from uncertainty over types.
- Since actions are observed, updating of beliefs is relatively easy.

Signaling game:

- A two-stage game, in which a player with private information moves first, followed by a player without private information.
- Since a player's strategy depends on his/her types, the first player's action sends a signal about his/her type to the second player.
- In poker, players try to conceal information.
- If different types correspond to different individuals of a population, some individuals may want to reveal their type.

Signaling Game: Types and Strategies

Types:

- Player 1's type θ is distributed according to a prior $\mu_0 \in \Delta(\Theta)$.
- Player 2's type is common knowledge.

Strategies and Beliefs:

- As usual, a strategy of player 1 is a map $\sigma_1 : \Theta \to \Delta(\mathcal{A}_1)$.
- Since player 2 observes A_1, his/her strategy is a map $\sigma_2 : \mathcal{A}_1 \to \Delta(\mathcal{A}_2)$.
- Given $A_1 = a_1$, player 2 updates his/her beliefs about each $\vartheta \in \Theta$ to

$$\mu(a_1; \vartheta) = \frac{\sigma_1(\vartheta; a_1)\mu_0(\vartheta)}{\sum_{\vartheta' \in \Theta} \sigma_1(\vartheta'; a_1)\mu_0(\vartheta')}.$$

- Note that $\mu(a_1)$ is the element in $\Delta(\Theta)$ that assigns probability $\mu(a_1; \vartheta)$ to each $\vartheta \in \Theta$. Finally $\mu := \mu(A_1)$ is player 2's posterior.

Pooling and Separating Equilibria

Definition 5.1

1. A pooling equilibrium is a perfect Bayesian equilibrium, in which every type of the informed player chooses the same pure action.
2. A (fully) separating equilibrium is a perfect Bayesian equilibrium, in which no two types choose mixed actions with overlapping support.

Remark:

- Separating equilibria are fully revealing: the uninformed player will know exactly which type he/she is facing.
- Pooling equilibria reveal no information to the uninformed player.
- Because only one action is chosen on the path, we always have to specify off-path beliefs for pooling equilibria.
- Mixed-strategy PBE are also called semi-separating equilibria.

Job-Market Signaling

Job application:

- There are high- and low-productivity Workers, i.e., $\Theta = \{\vartheta_H, \vartheta_L\} \subseteq \mathbb{R}$.
- It is commonly known that a fraction $\mu_0 \in [0,1]$ of Workers are type ϑ_H.
- Workers choose an education level $e \geq 0$ and Firm offers a wage $w \geq 0$.
- Suppose utilities are $u_1(a, \vartheta) = w - \frac{e}{\vartheta}$ and $u_2(a, \vartheta) = -(w - \vartheta)^2$:
  - Cost of education is decreasing in the Workers' type.
  - Firm cares only about the Workers' type, not the education level.

Firm's Best Response

Firm's best response:

- Firm maximizes its conditional expected utility, given $A_1 = e$,

$$\begin{aligned}\mathbb{E}_\sigma[u_2(a,\theta) \mid e] &= -\mathbb{E}_\sigma\left[(w-\theta)^2 \mid e\right] \\ &= -\mu(e;\vartheta_H)(w-\vartheta_H)^2 - \mu(e;\vartheta_L)(w-\vartheta_L)^2.\end{aligned}$$

- Since $\mathbb{E}_\sigma[u_2(a,\theta) \mid e]$ is differentiable and strictly concave in w, the maximum is attained where the derivative is 0.
- The derivative with respect to w

$$\frac{\partial \mathbb{E}_\sigma[u_2(a,\theta) \mid e]}{\partial w} = -2\mu(e;\vartheta_H)(w-\vartheta_H) - 2\mu(e;\vartheta_L)(w-\vartheta_L) \overset{!}{=} 0.$$

- We deduce $w(e) = \mu(e;\vartheta_H)\vartheta_H + \mu(e;\vartheta_L)\vartheta_L = \mathbb{E}_\sigma[\theta \mid e]$.

Pooling Equilibria

On the equilibrium path:

- By definition of a pooling equilibrium, $\sigma_1(\vartheta_L) = \sigma_1(\vartheta_H) = e_*$.
- After observing e_*, posterior is μ_0, hence $w(e_*) = \mu_0\vartheta_H + (1-\mu_0)\vartheta_L$.

Off the equilibrium path:

- Easiest way to prevent deviations: assign pessimistic beliefs $\mu(e;\vartheta_L) = 1$ after observing $e \neq e_*$.
- This results in a wage of $w(e) = \vartheta_L$ being offered.

Deviations:

- Incentives to deviate are higher for the low type.
- Low type has no profitable deviation if $\vartheta_L \leq \mu_0\vartheta_H + (1-\mu_0)\vartheta_L - \frac{e_*}{\vartheta_L}$.
- Solving for e_* yields $e_* \leq (1-\mu_0)\vartheta_L(\vartheta_H - \vartheta_L)$.

Separating Equilibria

No mixed actions:

- Since the low type will be identified, he/she gets ϑ_L regardless of education level. It is thus optimal to choose $\sigma_1(\vartheta_L) = 0$.
- If the high type mixes among education levels, the lowest of those already fully reveals his/her type by definition.
- The high type must choose a pure action, that is, $\sigma_1(\vartheta_H) = e_*$.

No deviations:

- High type does not want to deviate: $\vartheta_L \leq \vartheta_H - \frac{e_*}{\vartheta_H}$.
- Low types does not want to deviate: $\vartheta_L \geq \vartheta_H - \frac{e_*}{\vartheta_L}$.
- Together: $\vartheta_L(\vartheta_H - \vartheta_L) \leq e_* \leq \vartheta_H(\vartheta_H - \vartheta_L)$.

Pessimistic beliefs:

- Any such e_* can be supported given $\mu(\vartheta_L \mid e) = 1$ for $e \neq e_*$.

Least-Separating Equilibrium

Least-Separating equilibrium:

- It seems reasonable that type ϑ_H would select the cheapest education level $\underline{e} = \vartheta_L(\vartheta_H - \vartheta_L)$ that separate him/herself from type ϑ_L.
- However, if $e_* \neq \underline{e}$, then pessimistic off-path beliefs $\mu(\vartheta_L \mid e) = 1$ imply he/she will get the low wage for doing that.

Type ϑ_H could argue:

- It would not make sense for type ϑ_L to play $\underline{e}$ as he/she would pay for additional education and get the same wage.
- It can only make sense for type ϑ_H to choose $\underline{e}$.
- The pessimistic off-path beliefs $\mu(\vartheta_L \mid e) = 1$ are thus counter-intuitive.
- This is the idea behind the intuitive criterion, refining off-path beliefs.

Best-Response Correspondences

Definition 5.2

If player 1 can convince player 2 that he/she is a type in the set $\Theta' \subseteq \Theta$ by playing action $a_1 \in \mathcal{A}_1$, then player 2's best response-correspondence is

$$\mathcal{B}_2(\Theta', a_1) = \bigcup_{\mu \in \Delta(\Theta')} \arg\max_{a_2 \in \mathcal{A}_2} \mathbb{E}_\mu[u_2(a_1, a_2, \theta)].$$

Interpretation:

- $\arg\max_{a_2 \in \mathcal{A}_2} \mathbb{E}_\mu[u_2(a_1, a_2, \theta)]$ is player 2's best-response correspondence if player 2 had specific beliefs μ over Θ.
- We want to say if player 2 revised his/her beliefs to *any* beliefs over types in Θ', player 1 would benefit.
- Therefore, we take the union over all such beliefs $\mu \in \Delta(\Theta')$.

Intuitive Criterion

Definition 5.3

Suppose Θ is finite. A perfect Bayesian equilibrium σ *fails* the intuitive criterion if there exists $\vartheta \in \Theta$, a set $\Theta' \subseteq \Theta \setminus \{\vartheta\}$, and $a_1 \in \mathcal{A}_1$ such that:

1. Type ϑ strictly benefits from playing a_1 if doing so separates him/herself from all the types in Θ', that is,
$$\min_{a_2 \in \mathcal{B}_2(\Theta \setminus \Theta', a_1)} u_1(a_1, a_2, \vartheta) > \mathbb{E}_{\vartheta,\sigma}[u_1(A, \theta)].$$
2. Any type $\vartheta' \in \Theta'$ strictly loses by playing a_1, that is,
$$\max_{a_2 \in \mathcal{B}_2(\Theta, a_1)} u_1(a_1, a_2, \vartheta') < \mathbb{E}_{\vartheta',\sigma}[u_1(A, \theta)].$$

Interpretation:

- The intuitive criterion refines off-path beliefs such that ϑ is separated from Θ' after play of a_1 without disadvantage to player 2 in equilibrium.

Intuitive Job-Market Signaling

Separating equilibria:

- All but the least separating fail the intuitive criterion because type ϑ_H could benefit by deviating to $\underline{e}$ without distorting incentives.

Incentives:

- The Firm pays wage $w = \vartheta_H$ to type ϑ_H in any separating equilibrium.
- Indeed, even if off-path beliefs assign probability 1 that the Worker is type ϑ_H for every $e \geq \underline{e}$, type ϑ_L cannot gain by deviating.

Job-Market Signaling

Pooling equilibria:

- For each $e_* \leq (1-\mu_0)\vartheta_L(\vartheta_H - \vartheta_L)$, there exists a pooling equilibrium, in which both types choose e_*.
- Pooling equilibria in this application cannot be refined by the intuitive criterion because both types benefit from getting cheaper education.

Semi-separating equilibria: to complete the analysis, we would also have to check for semi-separating equilibria.

Separating Equilibria in Poker

No separating equilibria:

- In a fully separating strategy, your opponents can deduce your hand.
- All worse hands will fold and only better hands will call.

Varying bid size:

- You can (and should) vary your bid size depending on all other factors.
- These include stack sizes, the size of the blinds, your position in the hand played, the progression of the tournament, etc.

Perfect Bayesian Equilibrium in Poker

Suitability:

- Equilibrium concepts assume that players know each others' strategies.
- In a two-player setting, poker is a zero-sum game:
  - If the opponent does not play the equilibrium strategy, he/she is playing a deviation from it, which cannot be profitable by definition.
  - Playing an equilibrium strategy guarantees a certain payoff, but yields more if the opponent does not play his/her part.
- In a multi-player setting, you may learn opponent's strategy over time.

Applying perfect Bayesian equilibria:

- Works very well for bluff-catcher scenarios on the river.
- Can we solve for the equilibrium of the entire hand?
- Yes, do it numerically with PioSOLVER.

Bet Sizing in Poker

SPR	Check	Bet 75%	Bet 100%	Bet 125%	Bet 150%	Bet 200%
10	70%	16.8%			13.2%	
5	70%	16.6%			13.5%	
3	70%	12.3%		17.7%		
2	69.3%		24.2%			6.5%
1.5	69%		15%		16%	
1	70%		30%			
0.75	66.6%	33.4%				

Bet sizing:

- Optimal bet sizing depends on your stack-to-pot ratio (SPR).
- Pokers solver pools on at most two bet sizes (all-ins in yellow).

Source: Table 90 in Acevedo (2019). Bet sizes used by solver in a unit-interval approximation of poker meant to simulate post-flop play.

Betting Hands: 30% Range

AA 1	AKs 1	AQs 1	AJs 1	ATs 1	A9s 1	A8s 1	A7s 1	A6s 1	A5s 1	A4s 1	A3s 1	A2s 1
AKo 1	KK 1	KQs 1	KJs 1	KTs 1	K9s 1	K8s 1	K7s 1	K6s 1	K5s 1	K4s 1	K3s 0	K2s 0
AQo 1	KQo 1	QQ 1	QJs 1	QTs 1	Q9s 1	Q8s 1	Q7s 1	Q6s 0	Q5s 0	Q4s 0	Q3s 0	Q2s 0
AJo 1	KJo 1	QJo 1	JJ 1	JTs 1	J9s 1	J8s 1	J7s 0.95	J6s 0	J5s 0	J4s 0	J3s 0	J2s 0
ATo 1	KTo 1	QTo 1	JTo 1	TT 1	T9s 1	T8s 1	T7s 1	T6s 0	T5s 0	T4s 0	T3s 0	T2s 0
A9o 1	K9o 0	Q9o 0	J9o 0	T9o 0	99 1	98s 1	97s 1	96s 1	95s 0	94s 0	93s 0	92s 0
A8o 1	K8o 0	Q8o 0	J8o 0	T8o 0	98o 0	88 1	87s 1	86s 1	85s 0	84s 0	83s 0	82s 0
A7o 1	K7o 0	Q7o 0	J7o 0	T7o 0	97o 0	87o 0	77 1	76s 1	75s 1	74s 0	73s 0	72s 0
A6o 0	K6o 0	Q6o 0	J6o 0	T6o 0	96o 0	86o 0	76o 0	66 1	65s 1	64s 0	63s 0	62s 0
A5o 0	K5o 0	Q5o 0	J5o 0	T5o 0	95o 0	85o 0	75o 0	65o 0	55 1	54s 0	53s 0	52s 0
A4o 0	K4o 0	Q4o 0	J4o 0	T4o 0	94o 0	84o 0	74o 0	64o 0	54o 0	44 1	43s 0	42s 0
A3o 0	K3o 0	Q3o 0	J3o 0	T3o 0	93o 0	83o 0	73o 0	63o 0	53o 0	43o 0	33 1	32s 0
A2o 0	K2o 0	Q2o 0	J2o 0	T2o 0	92o 0	82o 0	72o 0	62o 0	52o 0	42o 0	32o 0	22 1

Additional Ranges

AA 1	AKs 1	AQs 1	AJs 1	ATs 1	A9s 0	A8s 0	A7s 0	A6s 0	A5s 0	A4s 0	A3s 0	A2s 0
AKo 1	KK 1	KQs 1	KJs 0	KTs 0	K9s 0	K8s 0	K7s 0	K6s 0	K5s 0	K4s 0	K3s 0	K2s 0
AQo 1	KQo 0	QQ 1	QJs 0	QTs 0	Q9s 0	Q8s 0	Q7s 0	Q6s 0	Q5s 0	Q4s 0	Q3s 0	Q2s 0
AJo 0	KJo 0	QJo 0	JJ 1	JTs 0	J9s 0	J8s 0	J7s 0	J6s 0	J5s 0	J4s 0	J3s 0	J2s 0
ATo 0	KTo 0	QTo 0	JTo 0	TT 1	T9s 0	T8s 0	T7s 0	T6s 0	T5s 0	T4s 0	T3s 0	T2s 0
A9o 0	K9o 0	Q9o 0	J9o 0	T9o 0	99 1	98s 0	97s 0	96s 0	95s 0	94s 0	93s 0	92s 0
A8o 0	K8o 0	Q8o 0	J8o 0	T8o 0	98o 0	88 1	87s 0	86s 0	85s 0	84s 0	83s 0	82s 0
A7o 0	K7o 0	Q7o 0	J7o 0	T7o 0	97o 0	87o 0	77 0.0317	76s 0	75s 0	74s 0	73s 0	72s 0
A6o 0	K6o 0	Q6o 0	J6o 0	T6o 0	96o 0	86o 0	76o 0	66 0	65s 0	64s 0	63s 0	62s 0
A5o 0	K5o 0	Q5o 0	J5o 0	T5o 0	95o 0	85o 0	75o 0	65o 0	55 0	54s 0	53s 0	52s 0
A4o 0	K4o 0	Q4o 0	J4o 0	T4o 0	94o 0	84o 0	74o 0	64o 0	54o 0	44 0	43s 0	42s 0
A3o 0	K3o 0	Q3o 0	J3o 0	T3o 0	93o 0	83o 0	73o 0	63o 0	53o 0	43o 0	33 0	32s 0
A2o 0	K2o 0	Q2o 0	J2o 0	T2o 0	92o 0	82o 0	72o 0	62o 0	52o 0	42o 0	32o 0	22 0

Range: 6.5%

AA 1	AKs 1	AQs 1	AJs 1	ATs 1	A9s 1	A8s 1	A7s 1	A6s 0	A5s 1	A4s 0	A3s 0	A2s 0
AKo 1	KK 1	KQs 1	KJs 1	KTs 1	K9s 0	K8s 0	K7s 0	K6s 0	K5s 0	K4s 0	K3s 0	K2s 0
AQo 1	KQo 1	QQ 1	QJs 1	QTs 1	Q9s 0	Q8s 0	Q7s 0	Q6s 0	Q5s 0	Q4s 0	Q3s 0	Q2s 0
AJo 1	KJo 0	QJo 0	JJ 1	JTs 0	J9s 0	J8s 0	J7s 0	J6s 0	J5s 0	J4s 0	J3s 0	J2s 0
ATo 1	KTo 0	QTo 0	JTo 0	TT 1	T9s 0	T8s 0	T7s 0	T6s 0	T5s 0	T4s 0	T3s 0	T2s 0
A9o 0	K9o 0	Q9o 0	J9o 0	T9o 0	99 1	98s 0	97s 0	96s 0	95s 0	94s 0	93s 0	92s 0
A8o 0	K8o 0	Q8o 0	J8o 0	T8o 0	98o 0	88 1	87s 0	86s 0	85s 0	84s 0	83s 0	82s 0
A7o 0	K7o 0	Q7o 0	J7o 0	T7o 0	97o 0	87o 0	77 1	76s 0	75s 0	74s 0	73s 0	72s 0
A6o 0	K6o 0	Q6o 0	J6o 0	T6o 0	96o 0	86o 0	76o 0	66 1	65s 0	64s 0	63s 0	62s 0
A5o 0	K5o 0	Q5o 0	J5o 0	T5o 0	95o 0	85o 0	75o 0	65o 0	55 1	54s 0	53s 0	52s 0
A4o 0	K4o 0	Q4o 0	J4o 0	T4o 0	94o 0	84o 0	74o 0	64o 0	54o 0	44 0.5053	43s 0	42s 0
A3o 0	K3o 0	Q3o 0	J3o 0	T3o 0	93o 0	83o 0	73o 0	63o 0	53o 0	43o 0	33 0	32s 0
A2o 0	K2o 0	Q2o 0	J2o 0	T2o 0	92o 0	82o 0	72o 0	62o 0	52o 0	42o 0	32o 0	22 0

Range: 13.2%

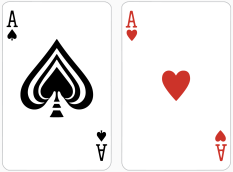
AA 1	AKs 1	AQs 1	AJs 1	ATs 1	A9s 1	A8s 1	A7s 1	A6s 1	A5s 1	A4s 1	A3s 1	A2s 1
AKo 1	KK 1	KQs 1	KJs 1	KTs 1	K9s 1	K8s 0	K7s 0	K6s 0	K5s 0	K4s 0	K3s 0	K2s 0
AQo 1	KQo 1	QQ 1	QJs 1	QTs 1	Q9s 1	Q8s 0	Q7s 0	Q6s 0	Q5s 0	Q4s 0	Q3s 0	Q2s 0
AJo 1	KJo 1	QJo 0	JJ 1	JTs 1	J9s 1	J8s 0	J7s 0	J6s 0	J5s 0	J4s 0	J3s 0	J2s 0
ATo 1	KTo 0	QTo 0	JTo 0	TT 1	T9s 1	T8s 0	T7s 0	T6s 0	T5s 0	T4s 0	T3s 0	T2s 0
A9o 0	K9o 0	Q9o 0	J9o 0	T9o 0	99 1	98s 1	97s 0	96s 0	95s 0	94s 0	93s 0	92s 0
A8o 0	K8o 0	Q8o 0	J8o 0	T8o 0	98o 0	88 1	87s 0	86s 0	85s 0	84s 0	83s 0	82s 0
A7o 0	K7o 0	Q7o 0	J7o 0	T7o 0	97o 0	87o 0	77 1	76s 0	75s 0	74s 0	73s 0	72s 0
A6o 0	K6o 0	Q6o 0	J6o 0	T6o 0	96o 0	86o 0	76o 0	66 1	65s 0	64s 0	63s 0	62s 0
A5o 0	K5o 0	Q5o 0	J5o 0	T5o 0	95o 0	85o 0	75o 0	65o 0	55 1	54s 0	53s 0	52s 0
A4o 0	K4o 0	Q4o 0	J4o 0	T4o 0	94o 0	84o 0	74o 0	64o 0	54o 0	44 1	43s 0	42s 0
A3o 0	K3o 0	Q3o 0	J3o 0	T3o 0	93o 0	83o 0	73o 0	63o 0	53o 0	43o 0	33 0.7837	32s 0
A2o 0	K2o 0	Q2o 0	J2o 0	T2o 0	92o 0	82o 0	72o 0	62o 0	52o 0	42o 0	32o 0	22 0

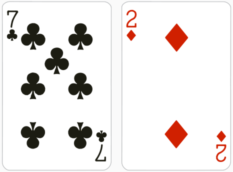
Range: 17.7%

AA 1	AKs 1	AQs 1	AJs 1	ATs 1	A9s 1	A8s 1	A7s 1	A6s 1	A5s 1	A4s 1	A3s 1	A2s 1
AKo 1	KK 1	KQs 1	KJs 1	KTs 1	K9s 1	K8s 1	K7s 1	K6s 0	K5s 0	K4s 0	K3s 0	K2s 0
AQo 1	KQo 1	QQ 1	QJs 1	QTs 1	Q9s 1	Q8s 1	Q7s 0	Q6s 0	Q5s 0	Q4s 0	Q3s 0	Q2s 0
AJo 1	KJo 1	QJo 1	JJ 1	JTs 1	J9s 1	J8s 1	J7s 0	J6s 0	J5s 0	J4s 0	J3s 0	J2s 0
ATo 1	KTo 0.6867	QTo 0	JTo 0	TT 1	T9s 1	T8s 1	T7s 0	T6s 0	T5s 0	T4s 0	T3s 0	T2s 0
A9o 1	K9o 0	Q9o 0	J9o 0	T9o 0	99 1	98s 1	97s 1	96s 0	95s 0	94s 0	93s 0	92s 0
A8o 1	K8o 0	Q8o 0	J8o 0	T8o 0	98o 0	88 1	87s 1	86s 0	85s 0	84s 0	83s 0	82s 0
A7o 0	K7o 0	Q7o 0	J7o 0	T7o 0	97o 0	87o 0	77 1	76s 1	75s 0	74s 0	73s 0	72s 0
A6o 0	K6o 0	Q6o 0	J6o 0	T6o 0	96o 0	86o 0	76o 0	66 1	65s 0	64s 0	63s 0	62s 0
A5o 0	K5o 0	Q5o 0	J5o 0	T5o 0	95o 0	85o 0	75o 0	65o 0	55 1	54s 0	53s 0	52s 0
A4o 0	K4o 0	Q4o 0	J4o 0	T4o 0	94o 0	84o 0	74o 0	64o 0	54o 0	44 1	43s 0	42s 0
A3o 0	K3o 0	Q3o 0	J3o 0	T3o 0	93o 0	83o 0	73o 0	63o 0	53o 0	43o 0	33 1	32s 0
A2o 0	K2o 0	Q2o 0	J2o 0	T2o 0	92o 0	82o 0	72o 0	62o 0	52o 0	42o 0	32o 0	22 1

Range: 24%

Why Blinds are Required in Poker



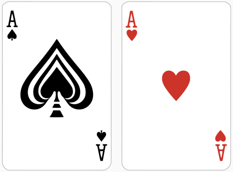


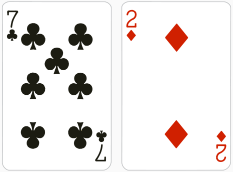
Suppose there are no blinds:

- The best hand pre-flop are two aces, which gives winning chances with 49.2% in a 6-player game.
- It is weakly dominant for aces to bet. Suppose that they do.
- Statistically the worst hand (in all games with 3 players or more) are 7 & 2 off-suit, which is victorious with 8.6% in a 6-player game.
- Since aces bet, the expected value of 7 & 2 off-suit is strictly negative.
- 7 & 2 off-suit should never participate.

Source: http://www.natesholdem.com/pre-flop-odds.php

Why Blinds are Required in Poker





Suppose there are no blinds:

- The next-worst hands are 3 & 2 off-suit in games with 3–6 players or 8 & 2 off-suit in games with 7–10 players.
- Since the only worse hand, 7 & 2 off-suit, does not participate and aces bet, the expectation of the second-worst hand is negative.
- The second-worst hand should never participate.
- ...
- Only pocket aces would bet if there was no dead money to win.

Summary

Signaling games:

- Players reveal information indirectly by taking costly actions.
- In the "standard-signaling model," there is an order of types and actions so that higher types can take higher actions more cheaply/easily.
- These single-crossing property allow types to reveal themselves.

Separating equilibria:

- Separating equilibria are typically a focal point of the analysis because they reveal the most information.
- This is efficient for the receiver, but costly for the sender.
- If doing so is too costly, only pooling equilibria remain.

Check Your Understanding

True or false:

1. No pooling equilibrium is complete without specifying the off-path beliefs.
2. Sequential equilibria satisfy the intuitive criterion.
3. A PBE satisfies the intuitive criterion if off-path beliefs assign beliefs 1 to the type who benefits the most from a deviation.
4. A PBE satisfies the intuitive criterion only if off-path beliefs assign beliefs 1 to the type who benefits the most from a deviation.
5. All sequential equilibria are either pooling equilibria, separating equilibria, or completely mixed equilibria.

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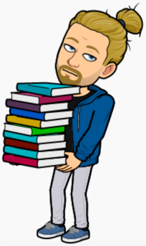


Cheap Talk

Job-Market Signaling

Separating equilibrium:

- High-type worker can successfully distinguish himself/herself from low-type workers because it is too costly for the low-type worker to pool.
- The education signal is credible because it is costly.

Costless signals:

- Suppose workers instead send a non-verifiable costless signal like "I am type H." Can meaningful information be conveyed in such a model?

Cheap-Talk Models

Setup:

- Player 1 has private information about his/her type $\theta \in \Theta$, which he/she can convey to Player 2 via a costless message $m \in \mathcal{M}$.
- Player 2 is initially uninformed with prior beliefs μ_0 on Θ.
- The payoffs of both players depend only on action a_2 taken by Player 2,

$$u_1(m, a_2, \vartheta) = u_1(a_2, \vartheta), \qquad u_2(m, a_2, \vartheta) = u_2(a_2, \vartheta).$$

- An important characteristic is that m is non-verifiable.

Examples:

- Any regular conversation between two individuals, asking somebody for a favor, persuading somebody without verifiable evidence, etc.
- A (possibly biased) expert providing advice to the government.

Cheap-Talk Job Market

	H	M	L
ϑ_H	$3, 1$	$2, 0$	$1, -2$
ϑ_L	$3, -2$	$2, 0$	$1, 1$

Cheap-talk job market:

- The Worker can send either the signal m_H "I am a high-ability worker" or the signal m_L "I am a low-ability worker."
- The Worker's payoff is independent of the signal and his/her type and it depends only on the wage $a_2 \in \{H, M, L\}$ chosen by the Firm.

Perfectly aligned preferences:

- Both types of workers prefer $H \succ M \succ L$.

Cheap-Talk Job-Market

Candidate separating equilibrium:

- If a separating equilibrium exists, we must have $\sigma_1(\vartheta_H) = m_1$ and $\sigma_1(\vartheta_L) = m_2$ for $m_1 \neq m_2$.
- Let us denote by $\mu(m)$ the Firm's posterior beliefs that $\theta = \vartheta_H$. Then,

$$\mu(m_1) = 1, \qquad \mu(m_2) = 0.$$

- A sequentially rational response is thus $\sigma_2(m_1) = H$ and $\sigma_2(m) = L$ for $m \neq m_1$, supported by off-path beliefs $\mu(m) = 0$ for $m \notin \{m_1, m_2\}$.

Incentives to deviate:

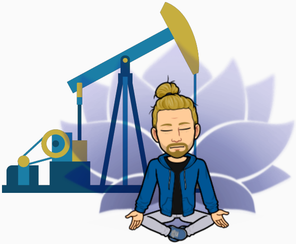
- The low type has an incentive to send signal m_1 instead since

$$u_1(m_1, \sigma_2(m_1), \vartheta_L) = 3 > 1 = u_1(m_2, \sigma_2(m_2), \vartheta_L).$$

- If incentives are perfectly aligned, there is no separating equilibrium.

Improving Company Image

	H	N
ϑ_O	$2, -2$	$0, 0$
ϑ_E	$-2, 2$	$0, 0$



Improving company image:

- In an attempt to improve company image, an Oil Company wants to hire an environmentally-friendly performer for a benefit gala.
- The Oil Company contacts a Performer and asks their type, which can be either ϑ_E "environmentally friendly" or ϑ_O "pro-oil ."

Misaligned preferences between players:

- The Performer accepts a job from an Oil Company only if they are ϑ_O.

Improving Company Image

Candidate separating equilibrium:

- Suppose σ with $\sigma_1(\vartheta_E) = m_1$ and $\sigma_2(\vartheta_O) = m_2$ for $m_1 \neq m_2$ is a PBE.
- Let us denote by $\mu(m)$ the Firm's posterior beliefs that $\theta = \vartheta_E$. Then,

$$\mu(m_1) = 1, \qquad \mu(m_2) = 0.$$

- In any sequentially rational response, $\sigma_2(m_1) = H$ and $\sigma_2(m_2) = N$.

Incentives to deviate:

- Both types have an incentive to change the message they send since

$$u_1(m_2, \sigma_2(m_2), \vartheta_E) = 2 > -2 = u_1(m_1, \sigma_2(m_1), \vartheta_E),$$
$$u_1(m_1, \sigma_2(m_1), \vartheta_O) = 2 > -2 = u_1(m_2, \sigma_2(m_2), \vartheta_O).$$

- If incentives between players are perfectly misaligned, there is no separating equilibrium because every type has an incentive to lie.

Coordination Game

	H	S
ϑ_H	$5, 5$	$4, 4$
ϑ_S	$4, 4$	$5, 5$

Coordination game:

- Suppose two friends want to meet up in New York. Each one of them could go see either Hamilton or Sleep No More.
- Player 1's type is his/her location and he/she can send a message, informing Player 2 of his/her location.
- It is clear that this game has a separating equilibrium, in which Player 1 truthfully reports his/her location.

Babbling Equilibrium

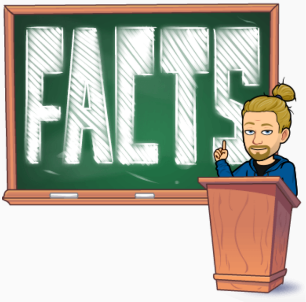
Definition 5.4

A babbling equilibrium is an perfect Bayesian equilibrium σ of a signaling game, in which $\sigma_1(\vartheta)$ is the uniform distribution over $\mathcal{M}$ and player 2 best responds to his/her prior beliefs.

Every cheap talk game has a babbling equilibrium:

- Since σ_1 has full support, there are no off-path messages.
- Given σ_1, player 2's posteriors coincide after any message.
- Since no deviation by player 1 affects player 2's posterior, it will not affect player 2's action. In particular, it is not profitable.

Soliciting Expert Opinions



Soliciting expert opinions:

- The Government solicits expert advice on an issue.
- The Government lacks the expertise to verify the Expert's message.
- While it may be costly to completely misrepresents his/her opinion, it may be costless to report the opinion with some personal bias.

Soliciting Expert Opinions

Setup:

- Suppose that the Government has a continuum of policies $\mathcal{A}_2 = [0, 1]$ available that it could implement.
- The Expert knows the correct policy, i.e., his/her type is $\theta \in \Theta = [0, 1]$.
- The Government's prior μ_0 is the uniform distribution over $[0, 1]$.
- The Government's and the Expert's utility functions are

$$u_1(a_2, \vartheta) = -(a_2 - \vartheta - c)^2, \qquad u_2(a_2, \vartheta) = -(a_2 - \vartheta)^2,$$

  where $c > 0$ is the Expert's bias: he/she prefers a higher action.

Babbling equilibrium:

- The Expert recommends any policy with equal likelihood and the Government implements $a_2 = \frac{1}{2}$, which is sequentially rational, given μ_0.

Two-Message Equilibrium

Cutoff strategy:

- Suppose that a two-message equilibrium σ conveys information so that $\sigma_2(m_1) \neq \sigma_2(m_2)$. Without loss of generality $\sigma_2(m_1) < \sigma_2(m_2)$.
- The difference of payoffs for the Expert is

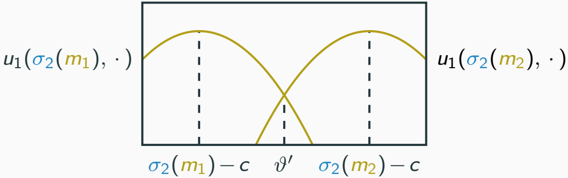
$$\Delta u_1(\vartheta) := (\sigma_2(m_1) - \vartheta - c)^2 - (\sigma_2(m_2) - \vartheta - c)^2.$$

- Since $\Delta u_1(\vartheta)$ is increasing, the Expert must use a cutoff strategy.
- Cutoff ϑ' is where the Expert is indifferent between $\sigma_2(m_1)$ and $\sigma_2(m_2)$.

Bayesian updating:

- After observing m_1, the Government has uniform beliefs on $[0, \vartheta')$.
- After observing m_2, the Government has uniform beliefs on $[\vartheta', 1]$.
- It follows that $\sigma_2(m_1) = \frac{\vartheta'}{2}$ and $\sigma_2(m_2) = \frac{\vartheta'+1}{2}$.

Two-Message Equilibrium



Solving for the cut-off:

- The indifference condition $u_1(\sigma_2(m_1), \vartheta') = u_1(\sigma_2(m_2), \vartheta')$ yields
$$\vartheta' + c - \frac{\vartheta}{2} = \frac{1 + \vartheta'}{2} - \vartheta' - c.$$
- This is equivalent to $\vartheta' = \frac{1}{2} - 2c$, which is positive if and only if $c < \frac{1}{4}$, that is, the Expert is not too biased.
- Note that the equilibrium is asymmetric. Since $\vartheta' < \frac{1}{2}$, message m_1 is more informative than message m_2.

Three-Message Equilibrium

Completely analogously:

- A three-message equilibrium exists if $c < \frac{1}{12}$ with cut-offs $\vartheta_1 = \frac{1}{3} - 4c$ and $\vartheta_2 = \frac{2}{3} - 4c$ such that the Government implements policy

$$\sigma_2(m_1) = \frac{\vartheta_1}{2}, \qquad \sigma_2(m_1) = \frac{\vartheta_1 + \vartheta_2}{2}, \qquad \sigma_2(m_1) = \frac{\vartheta_2 + 1}{2}.$$

- Types ϑ_i for $i = 1, 2$ are indifferent between sending m_i and m_{i+1}
- Observe that the bias must be even smaller to support this equilibrium.

Questions:

- Are all perfect Bayesian equilibria of this form?
- Under what conditions on the model are these the only PBE?

General Model

Setup:

- The informed player (sender)'s type θ is distributed on $\Theta = [0,1]$ with density $f(\vartheta) > 0$ and sends a message m in $\mathcal{M} = [0,1]$.
- Upon receiving the message, the uninformed player (receiver) takes an action $a(m) \in \mathcal{A} \in [0,1]$.
- Utility functions $u_S(a,\vartheta)$ and $u_R(a,\vartheta)$ are both twice differentiable and strictly concave in a for each ϑ. Moreover, for $i = R, S$, we have

$$\frac{\partial^2 u_i(a,\vartheta)}{\partial a\, \partial\vartheta} > 0. \tag{1}$$

Interpretation:

- (1) implies that both sender and receiver prefer higher actions for higher types, that is, preferences are at least partially aligned.
- By concavity, both players have a unique preferred action for each ϑ.

Preferred Actions

Preferred actions:

- Let us denote by $a_S(\vartheta)$ and $a_R(\vartheta)$ the preferred actions of the sender and receiver, respectively, if the true state is ϑ.
- It will be convenient to denote by

$$a_R(\underline{\vartheta}, \bar{\vartheta}) = \max_{a \in \mathcal{A}} \int_{\underline{\vartheta}}^{\bar{\vartheta}} u_R(a, \vartheta) f(\vartheta) \, \mathrm{d}\vartheta$$

  the receiver's best response if he/she learns that $\theta \in [\underline{\vartheta}, \bar{\vartheta}]$.

- Since θ admits density f, a_R is continuous in both arguments.
- Moreover, because $\frac{\partial^2 u_R(a,\vartheta)}{\partial a \, \partial \vartheta} > 0$ and $f(\vartheta) > 0$, it follows that a_R is strictly increasing in both arguments.

Partition Equilibrium

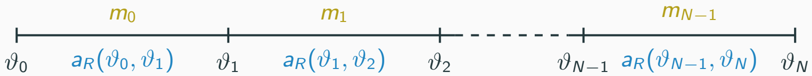
Lemma 5.5

Suppose $a_S(\vartheta) \neq a_R(\vartheta)$ *for every* $\vartheta \in \Theta$. *Consider a partition of* $\Theta = [0,1]$ *into intervals* $[\vartheta_i, \vartheta_{i+1}]$ *for* $0 = \vartheta_0 < \vartheta_1 < \ldots < \vartheta_{N-1} < \vartheta_N = 1$. *The strategy profile* $\sigma = (\sigma_S, \sigma_R)$ *with* $\sigma_S(\vartheta) = m_i$ *for* $\vartheta \in [\vartheta_i, \vartheta_{i+1})$ *and* $\sigma_R(m_i) = a_R(\vartheta_i, \vartheta_{i+1})$ *is a perfect Bayesian equilibrium if and only if*

$$u_S(a_R(\vartheta_i, \vartheta_{i+1}), \vartheta_i) = u_S(a_R(\vartheta_{i-1}, \vartheta_i), \vartheta_i), \qquad \forall\, i = 1, \ldots, N-1. \quad (2)$$

Such an equilibrium is called a partition equilibrium of size N.

Interpretation: Similar types pool together by sending the same message.



Proof of Sufficiency

Sequential rationality:

- Since message m_i is sent only by types in $[\vartheta_i, \vartheta_{i+1})$, Bayesian updating implies that the receiver's beliefs admit conditional density

$$f(\vartheta \mid m_i) = \frac{f(\vartheta)1_{\{\vartheta \in [\vartheta_i, \vartheta_{i+1})\}}}{P(\theta \in [\vartheta_i, \vartheta_{i+1}))}.$$

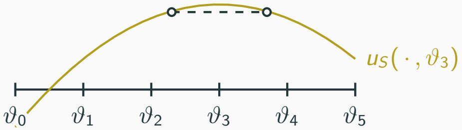
- Therefore, $\sigma(m_i) = a_R(\vartheta_i, \vartheta_{i+1})$ is sequentially rational.

Sender's incentives:

- Without loss of generality, we may assume $\mathcal{M} = \{m_0, \ldots, m_{N-1}\}$.
- It is sufficient to show that $\vartheta \in [\vartheta_i, \vartheta_{i+1})$ satisfies

$$u_S(a_R(\vartheta_i, \vartheta_{i+1}), \vartheta) = \max_j u_S(a_R(\vartheta_j, \vartheta_{j+1}), \vartheta). \qquad (3)$$

Proof of Sufficiency



Boundary types:

- Indifference for two consecutive m_i implies that u_R is maximized at m_i and m_{i+1} by convexity of $u_S(\,\cdot\,, \vartheta_i)$. In particular, (3) holds.

Interior types:

- Since $\frac{\partial^2 u_S(a,\vartheta)}{\partial a\,\partial \vartheta} > 0$, we obtain for any $0 \leq k \leq i$,

$$
\begin{aligned}
&u_S(a_R(\vartheta_i, \vartheta_{i+1}), \vartheta) - u_S(a_R(\vartheta_k, \vartheta_{k+1}), \vartheta) \\
&\qquad \geq u_S(a_R(\vartheta_i, \vartheta_{i+1}), \vartheta_i) - u_S(a_R(\vartheta_k, \vartheta_{k+1}), \vartheta_i) \geq 0.
\end{aligned}
$$

- An analogous argument for $k \geq i$ shows that interior types satisfy (3).

Proof of Necessity

Necessity:

- Suppose that (2) is violated for some ϑ_i, that is,

$$u_S(a_R(\vartheta_i, \vartheta_{i+1}), \vartheta_i) > u_S(a_R(\vartheta_{i-1}, \vartheta_i), \vartheta_i).$$

- By continuity of u_S, there exists a type $\vartheta < \vartheta_i$ such that

$$u_S(a_R(\vartheta_i, \vartheta_{i+1}), \vartheta) > u_S(a_R(\vartheta_{i-1}, \vartheta_i), \vartheta).$$

- Type ϑ can thus benefit by sending signal m_i.

Partition Equilibria Are the Only PBE

Theorem 5.6

Suppose $a_S(\vartheta) \neq a_R(\vartheta)$ for every $\vartheta \in \Theta$. Then there exists N_* such that:

1. *For every $N \leq N_*$, there exists a partition equilibrium of size N.*
2. *All perfect Bayesian equilibria are realization equivalent to a partition equilibrium of size $N \leq N_*$.*

Remark:

- Perfect information transmission is impossible if communication is costless and non-verifiable.
- Expert opinions: the more bias the expert has, the smaller is N_*.
- Note the stark contrast to standard signaling games: even without refinement, there exist only finitely many PBE in cheap-talk games.

Proof of Theorem 5.6

Argument 1:

- Suppose that two actions $a_1 < a_2$ are played in a PBE.
- Thus, there are types ϑ_1, ϑ_2 such that ϑ_i weakly prefers a_i over a_{3-i}.
- By continuity, there exists $\vartheta' \in [\vartheta_1, \vartheta_2]$ such that $u_S(a_1, \vartheta') = u_S(a_2, \vartheta')$.
- Since $u_S(\,\cdot\,, \vartheta')$ is strictly convex, it follows that $a_1 < a_S(\vartheta') < a_2$.
- Moreover, because $\frac{\partial^2 u_S(a,\vartheta)}{\partial a\,\partial\vartheta} > 0$, we must have
  (i) a_1 is not induced by any type $\vartheta > \vartheta'$,
  (ii) a_2 is not induced by any type $\vartheta < \vartheta'$.
- Preferences over actions a_1 and a_2 invert at $\vartheta'(a_1, a_2)$.

Proof of Theorem 5.6

Finitely many actions in any PBE:

- Conditions (i) and (ii) imply via $\frac{\partial^2 u_R(a,\vartheta)}{\partial a\,\partial\vartheta} > 0$ that $a_1 \leq a_R(\vartheta') \leq a_2$.
- Since $a_S(\vartheta) \neq a_R(\vartheta)$ for any ϑ, continuity and compactness of Θ imply that there exists $\varepsilon > 0$ such that $\inf_{\vartheta\in\Theta}|a_S(\vartheta) - a_R(\vartheta)| \geq \varepsilon$.
- Together, these conditions imply that $a_2 - a_1 \geq |a_S(\vartheta') - a_R(\vartheta')| \geq \varepsilon$.
- Because $\frac{\partial^2 u_R(a,\vartheta)}{\partial a\,\partial\vartheta} > 0$, the least and the largest actions $\underline{a}$ and $\bar{a}$ in any perfect Bayesian equilibrium satisfy $a_R(0) \leq \underline{a} \leq \bar{a} \leq a_R(1)$.
- Because $a_R(1) - a_R(0) < \infty$, $a_2 - a_1 > \varepsilon$ implies that only finitely many actions are played in any perfect Bayesian equilibrium.

Proof of Theorem 5.6

Any PBE is a partition equilibrium:

- Order the actions $a_0 < a_1 < \ldots < a_{N-1} < a_N$ that are played in a perfect Bayesian equilibrium and apply Arugment 1 to $a_i < a_{i+1}$.

Existence of such an N_*:

- An upper bound on N_* is $(a_R(1) - a_R(0))/\varepsilon$.
- We will omit the proof that (2) has a solution for every $N \leq N_*$.
- See Crawford and Sobel (1982) and Kono and Kandori (2019).

Comment on the theorem's assumption:

- If $a_R(\vartheta) = a_S(\vartheta)$ for some ϑ, then there is no lower bound on $|a_2 - a_1|$.
- While partition equilibria still exist, they may no longer be finite.

Summary

Cheap-talk games:

- There is always a babbling equilibrium that conveys no information.
- Of more interest are separating equilibria, in which the informed player manages to get some information across despite non-verifiability.

Preference alignments:

- If preferences of types over actions are perfectly aligned, there is no separating equilibrium.
- If preferences between the two players are perfectly misaligned, there is no separating equilibrium.
- If preferences between players are partially aligned, we get partition equilibria, in which some noisy information is transmitted.
- Cheap talk works well in coordination games.

Check Your Understanding

True or false:

1. It is cheap talk if you discuss travel destinations with your friend and mention that Canada is too cold.
2. Conversation during a first date is cheap talk.
3. This class is cheap talk.
4. Academic reference letters are cheap talk.
5. A partition equilibrium of size 1 is a babbling equilibrium.
6. Without loss of generality, we can assume that the sender's message in a partition equilibrium is of the form "you should play action *a*."

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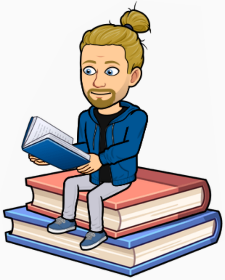


Communication and Mediation

Motivation

Discussing the game:

- The game specifies the rules of the players' interactions, i.e., the available information and payoff-relevant actions.
- With cheap talk, we have a model for costless communication that does not directly affect payoffs.
- In many settings, it seams reasonable that players can engage in non-binding non-verifiable communication.

Questions:

- Can players improve their payoffs if they can talk before the game?
- Can players improve their payoffs if players can talk during the game?

Communication with Perfect Information

	L	R
T	$4, 4$	$1, 5$
B	$5, 1$	$0, 0$

Coordination game:

- Pure-strategy Nash equilibria are (T, R) and (B, L).
- There is a mixed Nash equilibrium, in which players randomize 50-50. The players' expected payoffs are $(2.5, 2.5)$.
- With communication, players can agree on one of the Nash equilibria.
- Week 2: if players have access to a public randomization device, they can attain any convex combination of Nash payoffs.
- Under which conditions can players attain more?

Correlated Equilibrium

Definition 5.7

A correlated equilibrium is a a distribution $\rho \in \Delta(\mathcal{A})$ such that for any realization r of ρ, every player i and every $a_i \in \mathcal{A}_i$,

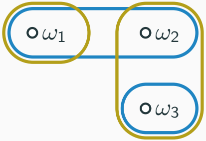
$$\mathbb{E}_\rho[u_i(R) \,|\, R_i = r_i] \geq \mathbb{E}_\rho[u_i(a_i, R_{-i}) \,|\, R_i = r_i].$$

Interpretation:

- There is an impartial mediator who recommends action r_i to player i.
- Players know the joint distribution ρ of recommendations, but not the specific recommendations made to other players.
- Conditional on recommendation r_i, player i has no profitable deviation. This condition is also known as obedience.

Correlated Equilibrium

	L	R
T	$4, 4$	$1, 5$
B	$5, 1$	$0, 0$



Problems:

- Pure-strategy Nash equilibria are (T, R) and (B, L) are asymmetric.
- Mixed-strategy Nash equilibrium assigns positive weight to (B, R).

Mediator:

- Can choose a symmetric distribution that avoids (B, R) by

$$R(\omega_1) = (B, L), \qquad R(\omega_2) = (T, L), \qquad R(\omega_3) = (T, R).$$

- Suppose (B, L) and (T, R) are realized with probability $x \leq \frac{1}{2}$.

Correlated Equilibrium

Obedience by player 1:

- Upon receiving recommendation B, he/she knows the state is ω_1, hence Player 2 received recommendation L.
- Upon receiving recommendation T, he/she updates her beliefs to

$$P_\rho(R_2 = L \,|\, R_2 = T) = \frac{P_\rho(\{\omega_2\})}{P_\rho(\{\omega_2, \omega_3\})} = \frac{1-2x}{1-x}.$$

- A deviation to B is profitable if and only if

$$(1-2x)u_1(B, L) > (1-2x)u_1(T, L) + xu_1(T, R).$$

- The correlated equilibrium is incentive-compatible is $x \in [0.4, 0.5]$.

Expected payoffs:

- The expected payoff is $(4-2x, 4-2x)$, maximized at $x = 0.4$.
- Efficiency is improved by adding a mediator.

Implementation Without Mediator?

Implementation without mediator:

- An impartial mediator may not always be at hand.
- Can the same outcome be implemented without a mediator?

Implementation through cheap talk:

- The outcomes that players can coordinate on through cheap talk must be a subset of the correlated equilibrium outcomes.
- In two-player games, no player can trust the other to send private signals according to the agreed-upon distribution. Therefore, cheap talk does not add anything beyond public randomization.
- In multiplayer games, third players can verify the sampling and reporting procedure. However, communication needs more than one stage.

Mediation with Incomplete Information

Static framework:

- Players have private information that they could share with a mediator.
- Based on the received reports, the mediator suggests an incentive compatible distribution over outcomes.
- This is precisely the topic of mechanism design.

Dynamic framework:

- Players can share information with the mediator in every stage and receive recommendations in every stage.
- This is known as an communication equilibrium.
- We will analyze to what extent communication equilibrium outcomes can be implemented through cheap talk.

Cheap-Talk Extension of a Bayesian Game

Definition 5.8

Let $\Gamma = \big(\mathcal{I}, \Theta, (\mathcal{T}_i), P, (\mathcal{A}_i), (u_i)\big)$ be a finite Bayesian game.

1. A cheap-talk extension of Γ is an extensive-form game Γ', in which players can send private or public messages for finitely many stages before Γ is played in the last stage of Γ'.
2. A communication equilibrium outcome A_* is cheap-talk implementable if there exists a sequential equilibrium σ in Γ' such that the distribution of A_* on $\mathcal{A}$ is equal to P_σ.

Cheap-Talk Implementation

Theorem 5.9

Let $\Gamma = \big(\mathcal{I}, \Theta, (\mathcal{T}_i), P, (\mathcal{A}_i), (u_i)\big)$ be a finite Bayesian game with $|\mathcal{I}| \geq 3$ and $P(\tau) > 0$ for every type profile τ. An outcome A_* of a communication equilibrium is cheap-talk implementable if

1. $P(A_* = a \,|\, T = \tau) \in \mathbb{Q}$ *for each* $a \in \mathcal{A}$ *and* $\tau \in T$.
2. *For each pair of players* (i, j), *there exists a Bayesian Nash equilibrium* σ_{ij} *in* Γ *such that for each* $k \in \{i, j\}$ *and every* $\tau_k \in \mathcal{T}_k$,

$$\mathbb{E}_{\tau_k}[u_k(A_*, \theta)] > \mathbb{E}_{\tau_k, \sigma_{ij}}[u_k(A, \theta)].$$

Interpretation:

- In multiplayer games, cheap-talk communication protocols can be quite powerful, as long as punishment equilibria σ_{ij} exist for each pair (i, j).

Discussion of Assumptions

At least three players:

- To prevent misreporting of information from player i to j, a third player k must be involved to guarantee to j that the information is correct.

Rational weights:

- A technical imposition for the proof in Ben-Porath (2003).

Punishment equilibria:

- Players play σ_{ik} once it becomes common knowledge that players (i, k) have cheated during the communication phase.

Deviation-Proof Lotteries

Joint uniform lottery:

- For any finite set $\mathcal{X}$, let $\Pi(\mathcal{X})$ denote the set of permutations π on $\mathcal{X}$.
- Player i picks a permutation π on $\mathcal{X}$ (uniformly) at random.
- Player j picks a number x from $\mathcal{X}$ (uniformly) at random.
- Both players announce their picks jointly and the outcome is $\pi(x)$.

Unilateral deviations:

- If only player i deviates and chooses a different permutation $\widetilde{\pi}$, then each $\widetilde{\pi}(x)$ is still uniformly distributed on $\{1, \ldots, N\}$.
- If only player i deviates and chooses a different number $\widetilde{x}$, then $\pi(\widetilde{x})$ is still uniformly distributed on $\{1, \ldots, N\}$.

Step 1: Generating the Distribution of A_*

Creating a lottery that implements the distribution of A_*:

- Since $P(A_* = a \mid T = \tau) \in \mathbb{Q}$, there exists a finite set $\mathcal{X}$ such that A_* can be implemented as a lottery on $\mathcal{X}$, followed by a map $f_\tau : \mathcal{X} \to \mathcal{A}$.

Communication protocol that implements the lottery:

1. Player 1 chooses independently and uniformly at random a permutation $\pi_i \in \Pi(\mathcal{T}_i)$ for each $i = 1, \ldots, n-1$ and a permutation $\rho \in \Pi(\mathcal{X})$. Moreover, player 1 reveals these choices to players $2, \ldots, n-1$.
2. Players $i = 1, \ldots, n-1$ send to player n a message $m_i = \pi_i(\tau_i)$.
3. Player n picks at random an element $x \in \mathcal{X}$.

Note: players are allowed to misreport their type τ_i when sending $\pi_i(\tau_i)$.

Step 1: Generating the Distribution of A_*

What have we achieved:

- Denote $\pi(\tau) = (\pi_1(\tau_1), \ldots, \pi_{n-1}(\tau_{n-1}), \tau_n)$.
- The outcome A_* is implemented by $\sigma(m, x) := f_{\pi^{-1}(m)}(\rho(x))$.
- Players $1, \ldots, n-1$ know σ as a function $\sigma : \mathcal{T} \times \mathcal{X} \to \mathcal{A}$.
- Player n knows the inputs (m, x).

Note:

- Nobody has yet learned any information about the other players' types or about the action profile that players are supposed to play.
- Problem: players $1, \ldots, n-1$ cannot compute their action without knowing the inputs to compute everybody else's action.
- We need to encrypt the information transmission from player n to i.

Step 2: Encrypting the Information

Encryption through garblings:

4. Player 1 chooses independently and uniformly at random a permutation $\widehat{\pi}_i \in \Pi(\mathcal{T} \times \mathcal{X})$ for each player $i = 2, \ldots, n-1$. Player 1 sends $\widehat{\pi}_i$ to player n and $\sigma_i \circ \widehat{\pi}_i^{-1}$ to players $i = 2, \ldots, n-1$.
5. Player 2 chooses uniformly at random a permutation $\widehat{\pi}_1 \in \Pi(\mathcal{T} \times \mathcal{X})$, sends $\widehat{\pi}_1$ to player n and $\sigma_1 \circ \widehat{\pi}_1^{-1}$ to player 1.

Steps 4. and 5. is an encryption:

- Player n to encrypts (m, x) intended for player i as $\widehat{\pi}_i(m, x)$, which only player i can decode properly into $\sigma_i\big(\widehat{\pi}_i^{-1}(\widehat{\pi}_i(m, x))\big)$.[1]
- Player n holds the data (m, x) but cannot read it without σ_n.

Problem: Player n could misrepresent the data.

[1] Note that σ_i is not typically invertible since $\mathcal{X}$ may be very large, hence i cannot solve for $\widehat{\pi}_i^{-1}$.

Step 3: Distributing the Information

Distributing the information:

6. Player n chooses independently and uniformly at random a permutation $\beta_i \in \Pi(\mathcal{T} \times \mathcal{X})$ for $i = 1, \ldots n-1$ and sends $\beta_i(\widehat{\pi}_i(m, x))$ to player i.
7. Player n sends β_i^{-1} for $i = 2, \ldots, n-1$ to player 1 and β_1^{-1} to player 2.

What have we achieved:

- Now everybody holds the encrypted data without the key to read it.
- Simultaneously, players publicly reveal each others keys:
  - Players 1 and n reveal the keys β_i^{-1} for players $i = 2, \ldots, n-1$,
  - Players 2 and n reveal the keys β_1^{-1} for player 1,
  - Players 1 and 2 reveal σ_n.
- In case of a report mismatch, we know that one of two players deviated.

Step 4: Verifying Incentives

Unilateral deviations from protocol:

- Do not affect distribution of A_* because of joint lotteries.
- Reveals publicly that one of two players (i, j) deviated. By assumption, there exists a BNE σ_{ij} that punishes those players.
- This incentivizes truthful reporting of the keys.

Misrepresenting types:

- Before players know the realization of the lottery, any communication equilibrium outcome is ex-ante incentive compatible.
- For details, see Ben-Porath (2003).

Communication in Teams

Avalon with special roles:

- One good character, Merlin, knows the identity of every evil player.
- Merlin cannot be too obvious with this information because the evil players get to assassinate one good player at the end of the game.
- If they assassinate Merlin, evil wins.
- Can public communication help the good players identify evil players?

Communication in Teams

Incentives:

- Incentives between good players are perfectly aligned, so this is not actually a cheap talk problem.
- The key is to conceal the information from evil.

Public encryption:

- Each player i chooses a private key π_i^{-1} and releases the corresponding the public key π_i to everybody else.
- Every player j can now send message m to player i encrypted as $\pi_i(m)$.
- Through this encryption, exchange keys π_{ij} and π_{ij}^{-1} that i uses to send messages to j and j uses to decrypt messages from i.
- After this, only keys π_{ij} and π_{ij}^{-1} are used.

Communication in Teams

Communication protocol:

- A good player announces "the following strategy will win for good hence anybody who deviates must be evil."
- Players exchange keys.
- Players send an encrypted message to everyone else, stating either:
  (a) I am not Merlin,
  (b) I am Merlin and I know players i, j, and k are evil.
- Merlin declares (b) to good players and (a) to evil players.
- Every good player declares (a) to everyone.
- Evil players have no choice but to state either (a) or (b).

Communication in Teams

Single Merlin:

- If only one player claims to be Merlin, he must be Merlin, hence good players know the identities of evil.
- This cannot be optimal from evil's perspective.

Every evil player claiming to be Merlin:

- True Merlin will claim that all other Merlins are evil.
- A Merlin, who claims that players are evil that did not themselves claim to be Merlin must be a fake Merlin.
- Thus, evil players must frame each other and one good person as evil.
- Evil players agree on whom to frame in the beginning (encrypted).
- Good players know every good player that is not Merlin, which is often good enough to win the game (unless Merlin is assassinated).

Summary

Communication and mediation:

- The most efficient communication is achieved through a mediator.
- In absence of a mediator, the most efficient communication protocols use encryption to preserve players' privacy.

Features of efficient communication protocols:

- Randomizations are performed through joint lotteries.
- Messages throughout the communication protocol are encrypted.
- In the final step of the protocol, player i's decryption key is revealed by two players j, k simultaneously to detect unilateral deviations.

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