ECON 7011, Semester 109.1, Final Exam

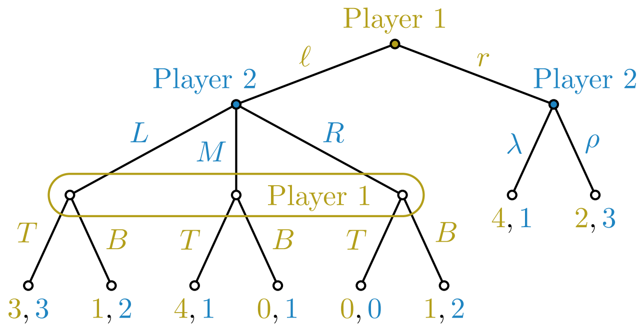
Time: 2.5 hours. Please provide a justification for all your answers.

Problem 4 is on the back side. Total points: 60. Good luck!

Points required for full score: 44.

1. Consider the following simultaneous-move and extensive-form two-player games.

	L	M	R
T	$3, 3$	$4, 1$	$0, 0$
B	$1, 2$	$0, 1$	$1, 2$



   (a) [8 points] Find all Nash equilibria in the simultaneous-move game to the left.

   (b) [2 points] Is common knowledge of rationality sufficient for players to coordinate on one specific Nash equilibrium? Briefly justify your answer.

   (c) [4 points] Find all subgame-perfect equilibria in the game to the right.
   Note: the subgame starting at the left node of Player 2 is the game to the left.

2. An entrepreneur (agent) approaches a venture capitalist (principal) to obtain funding q for their business idea of quality θ in exchange for payments p in the next period. Suppose that the idea is either good ($\vartheta_H = 2$) or bad ($\vartheta_L = 1$) with common prior $\mu = P(\theta = \vartheta_H) = \frac{1}{5}$. Payments in the next period are discounted by $\delta = \frac{4}{5}$ so that the principal's utility is $u_1(q, p) = \delta p - q$ and the agent's utility is $u_2(q, p, \vartheta) = \vartheta\sqrt{q} - \delta p$.

   (a) [6 points] What is the optimal ex-ante contract?

   (b) [8 points] What is the optimal limited-liability contract if the limit is $\ell = \frac{2}{5}\mu = \frac{2}{25}$?

   Note: Theorem 5.9 characterizing the optimal limited-liability contract is on the back side.

3. Consider a Stackelberg competition with demand uncertainty, in which the inverse demand function is given by $p(q, \theta) = \theta - q_1 - q_2$, where θ takes two possible values $\{70, 90\}$ with equal probability. The per-unit cost of both firms is 10 so that ex-post utilities are $u_i(q, \vartheta) = p(q, \vartheta)q_i - 10q_i$. The Stackelberg leader (firm 1) observes θ before choosing quantity $q_1 \geq 0$. The Stackelberg follower (firm 2) observes q_1 (but not θ) before choosing $q_2 \geq 0$.

   (a) [1 point] How many types does each firm have?

   Find all separating perfect Bayesian equilibria in pure strategies with the following steps:

   (b) [3 points] Find those off-path beliefs that make deviations the least attractive.

   (c) [10 points] Find all separating PBE for the off-path beliefs in (b).

   (d) [2 points] How do the equilibria compare to the Stackelberg equilibrium with complete information? Explain why they are similar or dissimilar.

4. Consider a first-price auction among two bidders, in which the two bidders' valuations are uniformly distributed on $\{\vartheta_L, \vartheta_H\}$. If the bids are a tie, the item is awarded to both players with equal probability, that is, ex-post payoffs for bids $b = (b_1, b_2)$ are given by

$$u_i(b, \vartheta_i) = \begin{cases} \vartheta_i - b_i & \text{if } b_i > b_{-i}, \\ \frac{1}{2}(\vartheta_i - b_i) & \text{if } b_i = b_{-i}, \\ 0 & \text{if } b_i < b_{-i}. \end{cases}$$

Find the unique symmetric Bayesian Nash equilibrium with the following steps:

(a) [4 points] Show that in any equilibrium σ, we must have $\sigma_i(\vartheta_L) = \vartheta_L$.

(b) [3 points] Show that no pure-strategy Bayesian Nash equilibrium exists.

Let $\sigma_i(\vartheta_H)$ be parametrized by distribution function F. One can show that F admits a density function f on an interval with end points $\underline{b} < \bar{b}$ (you do not have to show this).

(c) [3 points] Show that, in equilibrium, the lower end of the support of F is ϑ_L.

(d) [1 points] Show that ϑ_L itself is not in the support of F.

Together, these steps show that $\operatorname{supp} F = (\vartheta_L, \bar{b}]$ *Note: $\bar{b}$ can be included by continuity.*

(e) [3 points] Use the indifference principle to find $\bar{b}$ and F.

(f) [2 points] Verify that the derived strategy profile is a Bayesian Nash equilibrium.

Hint: You can solve parts (b)–(e) even if you did not solve earlier parts. In particular, I encourage you to start with the simpler parts (b), (e), and (f). These may then give you an idea how to solve (c) and (d). To prevent a circular argument, your answers to (c) and (d) should not depend on (e)–(f) and your answer to (a) should not depend on (b)–(f).

Theorem 5.9. Under limited liability, the optimal contract is of the form:

1. If $\ell \geq \mu \Delta v(q_L^*)$, the optimal contract is $\{(q_H^*, p_H^0), (q_L^*, p_L^0)\}$.

2. If $\mu \Delta v(\hat{q}_L) < \ell < \mu \Delta v(q_L^*)$, then the optimal contract solves

$$\Delta v(q_L^\ell) = \frac{\ell}{\mu}, \qquad p_L^\ell = v(q_L^\ell, \vartheta_L) + \ell$$

   and satisfies $q_H^\ell = q_H^*$ and $p_H^\ell = p_L^\ell + v(q_H^\ell, \vartheta_H) - v(q_L^\ell, \vartheta_H)$.

3. If $\ell \leq \mu \Delta v(\hat{q}_L)$, the optimal contract is $\{(q_H^*, \hat{p}_H + \ell), (\hat{q}_L, \hat{p}_L + \ell)\}$.

q_L^*, q_H^* are the quantities in the first-best contract, $\hat{q}_L$, $\hat{q}_H$ are the quantities and $\hat{p}_L$, $\hat{p}_H$ are the payments in the second-best contract, and p_L^0, p_H^0 are the payments in the ex-ante contract

$$p_L^0 = \mu v(q_L^*, \vartheta_H) + (1 - \mu) v(q_L^*, \vartheta_L),$$

$$p_H^0 = p_L^0 + v(q_H^*, \vartheta_H) - v(q_L^*, \vartheta_H).$$

and $\Delta v(q) = v(q, \vartheta_H) - v(q, \vartheta_L)$.