Macroeconomic Theory: Assignment 4

Exercise 1. (Labor and Growth) Suppose that the household's utility function takes the form

$$\ln c_t + \phi \ln(1 - l_t),$$

where $l_t \in [0, 1]$ is the working hours, and $\phi > 0$. The capital accumulation follows

$$k_{t+1} = Ak_t^{\alpha} l_t^{1-\alpha} - c_t.$$

Then the sequential problem is

$$\begin{aligned} \max_{\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t \left[\ln(c_t) + \gamma \ln(1 - l_t)\right] \\ \text{subject to} \quad & k_{t+1} = Ak_t^{\alpha} l_t^{1-\alpha} - c_t. \end{aligned}$$

The associated functional equation of the value function is

$$\begin{aligned} v(k_t) \quad &= \quad \max_{c_t, k_{t+1}, l_t} \ln(c_t) + \gamma \ln(1 - l_t) + \beta v(k_{t+1}) \\ \text{subject to} \quad & \quad c_t + k_{t+1} = Ak_t^{\alpha} l_t^{1-\alpha}. \end{aligned}$$

We guess that the value function takes the form

$$v(k_t) = E + F \ln k_t,$$

and we assume that the optimal working hour is interior.

1. Use guess and verify to solve for the value function $v(k_t)$ and the policy function $k_{t+1} = g(k_t)$ (Hint: substitute $c_t = Ak_t^{\alpha} l_t^{1-\alpha} - k_{t+1}$ into the functional equation, and then you have two endogenous variables, k_{t+1} and l_t.....)

2. Show that $l_t^* \in [0, 1]$ and l_t^* is irrelevant with k_t

1. $V(k_t) = \max_{l_t, k_{t+1}} \left\{ \ln(Ak_t^{\alpha} l_t^{1-\alpha} - k_{t+1}) + \gamma \ln(1-l_t) + \beta V(k_{t+1}) \right\} \equiv TV(k_t)$

Guess $V(k_t) = E + F \ln k_t$

$$\Rightarrow TV(k_t) = \max_{l_t, k_{t+1}} \left\{ \ln(Ak_t^{\alpha} l_t^{1-\alpha} - k_{t+1}) + \gamma \ln(1-l_t) + \beta[E + F \ln k_{t+1}] \right\}$$

※ 當 $f(k) = Ak^{\alpha}$ 這種形式，做FOC時最好先不要直接展開，而是用 $\alpha \frac{y}{k}$ 來寫，這樣才整理得出來...

FOC: $[l_t]$: $\dfrac{(1-\alpha)Ak_t^{\alpha} l_t^{-\alpha}}{Ak_t^{\alpha} l_t^{1-\alpha} - k_{t+1}} = \dfrac{\gamma}{1-l_t} \Rightarrow ?$

$[k_{t+1}]$: $\dfrac{1}{Ak_t^{\alpha} l_t^{1-\alpha} - k_{t+1}} = \dfrac{\beta F}{k_{t+1}} \Rightarrow k_{t+1} = \beta F [Ak_t^{\alpha} l_t^{1-\alpha} - k_{t+1}]$

$$\Rightarrow k_{t+1} = \frac{\beta F A k_t^{\alpha} l_t^{1-\alpha}}{1+\beta F}$$

$$\Rightarrow TV(k_t) = \ln \frac{k_{t+1}}{\beta F} + \gamma \ln \frac{k_{t+1}}{\beta F} \cdot \frac{\gamma}{(1-\alpha)Ak_t^{\alpha} l_t^{-\alpha}} + \beta[E + F \ln k_{t+1}]$$

$$= (1+\gamma) \ln \frac{k_{t+1}}{\beta F} + \gamma \ln \frac{\gamma}{(1-\alpha)Ak_t^{\alpha} l_t^{-\alpha}} + \beta E + \beta F \ln k_{t+1}$$

$$= (1+\gamma+\beta F) \ln \frac{k_{t+1}}{\beta F} + \gamma \ln \frac{\gamma}{(1-\alpha)Ak_t^{\alpha} l_t^{-\alpha}} + \beta E + \beta F \ln \beta F$$

$$= (1+\gamma+\beta F) \ln \frac{Ak_t^{\alpha} l_t^{1-\alpha}}{1+\beta F} + \gamma \ln \frac{\gamma}{(1-\alpha)Ak_t^{\alpha} l_t^{-\alpha}} + \beta E + \beta F \ln \beta F$$

怎麼處理 l ? → ~~根本不要管~~!

之前 c_{t+1} 是已經給定，才可以不管
這裡擺明要用到 l 求 max，當然不能放掉。

1. ~~$\Rightarrow TV(k_t) = \max_{l_t, k_{t+1}} \left\{ \ln(Ak_t^{\alpha} l_t^{1-\alpha} - k_{t+1}) + \gamma \ln(1-l_t) + \beta[E + F \ln k_{t+1}] \right\}$~~

$[k_{t+1}]$: $\dfrac{1}{Ak_t^{\alpha} l_t^{1-\alpha} - k_{t+1}} = \dfrac{\beta F}{k_{t+1}} \Rightarrow k_{t+1} = \beta F [Ak_t^{\alpha} l_t^{1-\alpha} - k_{t+1}]$

$$\Rightarrow k_{t+1} = \frac{\beta F A k_t^{\alpha} l_t^{1-\alpha}}{1+\beta F}$$

$$\Rightarrow TV(k_t) = \ln \frac{k_{t+1}}{\beta F} + \gamma \ln(1-l_t) + \beta[E + F \ln k_{t+1}]$$

$$= (1+\beta F) \ln k_{t+1} - \ln \beta F + \gamma \ln(1-l_t) + \beta E$$

$$= (1+\beta F) \ln \frac{\beta F A k_t^{\alpha} l_t^{1-\alpha}}{1+\beta F} - \ln \beta F + \gamma \ln(1-l_t) + \beta E$$

$$= (1+\beta F)\alpha \ln k_t + (1+\beta F) \ln \frac{\beta F A \cdot l_t^{1-\alpha}}{1+\beta F} - \ln \beta F + \gamma \ln(1-l_t) + \beta E$$

$$\Rightarrow \quad F = (1 + \beta F)\alpha \quad \Rightarrow \quad F = \alpha + \alpha\beta F \quad \Rightarrow \begin{cases} F = \dfrac{\alpha}{1-\alpha\beta} \\ (1+\beta F) = \dfrac{F}{\alpha} \end{cases}$$

$$\Rightarrow \quad E = \frac{F}{\alpha} \ln \frac{(\frac{F}{\alpha}-1) A\, l_t^{1-\alpha}}{F/\alpha} - \ln\left(\frac{F}{\alpha}-1\right) + \gamma \ln(1-l_t) + \beta E$$

$$\Rightarrow \quad E = \frac{1}{1-\beta}\left[\frac{F}{\alpha} \ln \frac{(\frac{F}{\alpha}-1) A\, l_t^{1-\alpha}}{F/\alpha} - \ln\left(\frac{F}{\alpha}-1\right) + \gamma \ln(1-l_t)\right]$$

Let $y_t = A k_t^{\alpha} l_t^{1-\alpha}$, ⚠ 切記！$\begin{cases} \text{Now 已經從 SP，∴不用怕 } k_t \text{ 到下一期會變 } k_{t+1}\text{，需要微分} \\ y \text{ 裡面有 } k_t \text{ 和 } l_t\text{，FOC 要小心} \end{cases}$

$$Tv(k_t) = \max_{l_t,\, k_{t+1}} \left\{ \ln(y_t - k_{t+1}) + \gamma \ln(1-l_t) + \beta[E + F \ln k_{t+1}] \right\}$$

$$\Rightarrow [k_{t+1}]: \quad \frac{1}{y_t - k_{t+1}} = \frac{\beta F}{k_{t+1}}$$

$$[l_t]: \quad \frac{(1-\alpha)\cancel{A}\cdot\frac{y_t}{l_t}}{y_t - k_{t+1}} = \frac{\gamma}{1-l_t}$$

$$\Rightarrow \quad k_{t+1} = \beta F y_t - \beta F k_{t+1} \quad \Rightarrow \quad k_{t+1} = \frac{\beta F}{1+\beta F} y_t = \frac{\beta F}{1+\beta F}\cdot A \cdot k_t^{\alpha} \cdot l_t^{1-\alpha}$$

$$(1-\alpha)\cancel{A}\,\frac{y_t}{l_t}\cdot(1-l_t) = \gamma y_t - \gamma k_{t+1} \quad \Rightarrow \quad (1-\alpha)\cancel{A}\cdot\frac{1-l_t}{l_t} = \gamma - \gamma \frac{k_{t+1}}{y_t}$$

$$\frac{k_{t+1}}{y_t} = \frac{\beta F}{1+\beta F}$$

這真的很神來一筆…

∴才要 y!!

$$\Rightarrow (1-\alpha)\cancel{A}\cdot(1-l_t) = \left[\gamma - \gamma\cdot\frac{\beta F}{1+\beta F}\right] l_t$$

$$\Rightarrow \quad (1-\alpha)\cancel{A} = \left[\gamma - \gamma\cdot\frac{\beta F}{1+\beta F} + (1-\alpha)\cancel{A}\right] l_t$$

$$\Rightarrow \quad l_t = \frac{(1-\alpha)\cancel{A}}{\gamma - \gamma\cdot\frac{\beta F}{1+\beta F} + (1-\alpha)\cancel{A}} = \frac{(1-\alpha)(1+\beta F)}{\gamma + (1-\alpha)(1+\beta F)} \equiv M$$

$$\Rightarrow \quad Tv(t) = \ln\frac{k_{t+1}}{\beta F} + \gamma \ln(1-M) + \beta[E + F \ln k_{t+1}]$$

$$= \gamma \ln(1-M) + \beta E - \ln \beta F + (1+\beta F)\ln k'$$

$$= \quad〃 \quad + (1+\beta F)\ln \frac{\beta F}{1+\beta F}\cdot A\cdot k_t^{\alpha}\cdot M^{1-\alpha}$$

$$= \gamma \ln(1-M) + \beta E - \ln\beta F + (1+\beta F)\ln\frac{\beta F}{1+\beta F}\cdot A\cdot M^{1-\alpha}$$

$$+ (1+\beta F)\,\alpha \ln k$$

$$\Rightarrow \quad \alpha + \alpha\beta F = F \quad \Rightarrow \quad F = \frac{\alpha}{1-\alpha\beta}$$

2. Show ① $l_t^* \in [0,1]$ and ② irrelevant to k.

①
$$\lim_{r \to 0} l_t^* = 1, \quad \lim_{r \to 1} l_t^* = 0.$$

② $\because F = \frac{\alpha}{1-\alpha\beta}$, $\therefore l_t^*$ does not depend.