ECON 7011, Semester 110.1, Assignment 1, Solutions

1. (a) Let s_i be a pure strategy such that each $s_i' \in \mathcal{S}_i \setminus \{s_i\}$ satisfies $u_i(s_i', s_{-i}) < u_i(s_i, s_{-i})$ for any strategy profile s_{-i} of i's opponents. Any mixed strategy $\sigma_i \in \Delta(\mathcal{S}_i) \setminus \{\delta_{s_i}\}$ satisfies $\sigma_i(s_i) < 1$, hence for any $s_{-i} \in \mathcal{S}_{-i}$, we obtain

$$\begin{aligned} u_i(\sigma_i, s_{-i}) &= \sigma_i(s_i)u_i(s_i) + \sum_{s_i' \in \mathcal{S}_i \setminus \{s_i\}} \sigma_i(s_i')u_i(s_i', s_{-i}) \\ &< \sigma_i(s_i)u_i(s_i) + \sum_{s_i' \in \mathcal{S}_i \setminus \{s_i\}} \sigma_i(s_i')u_i(s_i, s_{-i}) = u_i(s_i, s_{-i}). \end{aligned}$$

Note that for the strict inequality to hold, it was important that $\sigma_i(s_i) < 1$ so that at least one term in the sum has positive weight $\sigma_i(s_i')$.

(b) Suppose towards a contradiction that a mixed strategy σ_i with $\operatorname{supp} \sigma_i = \{s_i^1, s_i^2\}$ is strictly dominant, that is, $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for every pure strategy s_i. Then

$$u_i(\sigma_i, s_{-i}) = \sum_{s_i \in \mathcal{S}_i} \sigma_i(s_i)u_i(s_i, s_{-i}) < \sum_{s_i \in \mathcal{S}_i} \sigma_i(s_i)u_i(\sigma_i, s_{-i}) = u_i(\sigma_i, s_{-i}),$$

where we have used that $\sum_{s_i} \sigma_i(s_i) = 1$ in the last equality. This is a contradiction.

2. (a) Suppose towards a contradiction that B is strictly dominated by $\sigma_3^x = xA + (1-x)C$ for some $x \in [0,1]$. Then Player 3's expected utilities must satisfy

$$2 = u_3(T, L, B) < u_3(T, L, \sigma_3^x) = 3x, \quad 2 = u_3(D, R, B) < u_3(D, R, \sigma_3^x) = 3(1-x).$$

This is system of inequalities is inconsistent as adding the two inequalities shows.

(b) Let us parametrize $\sigma_{-3} = (\sigma_1, \sigma_2)$ by $\sigma_1 = xT + (1-x)D$ and $\sigma_2 = yL + (1-y)R$. By the indifference principle, to find best responses it is sufficient to look at pure-strategy best responses. We compute the expected utility for all of Player 3's pure strategies

$$\begin{aligned} u_3(\sigma_1, \sigma_2, A) &= 3xy, \\ u_3(\sigma_1, \sigma_2, C) &= 3(1-x)(1-y), \\ u_3(\sigma_1, \sigma_2, B) &= 2(xy + (1-x)(1-y)) - 4(x(1-y) + y(1-x)) \\ &= x(6y-4) + (1-x)(2-6y). \end{aligned}$$

If $y \leq \frac{1}{2}$, then $6y - 4 < 0$ and $2 - 6y < 3(1-y)$, hence C is a strictly better response than B for any x. If $y \geq \frac{1}{2}$, then $6y - 4 < 3y$ and $2 - 6y < 0$, hence A is a strictly better response than B for any x. We conclude that B can never be a best response.

(c) To ensure that $\mathcal{R}_3^\infty \neq \Sigma_3^\infty$, we must make sure that neither A nor C are eliminated. Since A is a best response to (T, L) and C is a best response to (D, R), a sufficient condition is that neither strategies of players 1 or 2 are strictly dominated.

3. (a) We begin by showing that the conditions of Theorem 1.11 are met. Each player's actions come from a closed, bounded interval, the utility function is a polynomial, hence continuously differentiable, it is monotone in a_{-i} and strictly concave in a_i as shown by

$$\frac{\partial u_i(a)}{\partial a_{-i}} = (a_i - 2) \leq 0, \qquad \frac{\partial u_i(a)}{\partial a_i} = 1 + a_{-i} - 2a_i \qquad \frac{\partial u_i(a)}{\partial a_i} = -2 < 0.$$

By Theorem 1.11, it is thus sufficient to consider Dirac conjectures $\pi_i = \delta_{a_{-i}}$. By the extreme value theorem, Player i's response to a fixed a_{-i} is attained either at a boundary point or at an interior point $\hat{a}_i$, where the first derivative is 0. Since the utility function is strictly concave, the unique best response on an interval $[\underline{a}, \bar{a}]$ is given by

$$a_i^*(a_{-i}) = \min\Big\{\max\Big\{\tfrac{1+a_{-i}}{2}, \underline{a}\Big\}, \bar{a}\Big\}. \tag{1}$$

A strategy a_i is a best response to some conjecture if there exists $a_{-i} \in [\underline{a}, \bar{a}]$, for which (1) holds. Since (1) is monotonic in a_{-i}, we obtain the extremal best responses for $a_{-i} = \underline{a}$ and $a_{-i} = \bar{a}$, respectively. We start the iterative process at $\mathcal{R}_i^0 = [0, 2]$, that is, we plug in $\underline{a} = 0$, $\bar{a} = 2$ into (1). The extremal best responses are, therefore, $\mathcal{R}_i^1 = [0.5, 1.5]$. The limit $\mathcal{R}_i^\infty$ can be found in one of two ways. Either we observe that in the limit, the lower bound and the upper bound of $\mathcal{R}_i^\infty$ have to be fixed points of (1), or we develop an idea for an explicit form of $\mathcal{R}_i^k$ and prove it by induction. Since we have pursued the former approach in the lecture, let us pursue the second approach here. From (1), we develop a sense that the distance from an end point of the interval to its center be halved in each step of the iteration. That is, we formulate an inductive hypothesis

$$\mathcal{R}_i^k = \left[1 - \frac{1}{2^k}, 1 + \frac{1}{2^k}\right].$$

The base case for $k = 0$ is satisfied since $\mathcal{R}_i^0 = [0, 2]$. We now suppose that the inductive hypothesis holds for k and show that it also holds for $k + 1$. Note that $\underline{a}^k < 1 < \bar{a}^k$ by the inductive hypothesis, hence (1) simplifies to $a_i^*(a_{-i}) = (1 + a_{-i})/2$. The lower end point of the interval $\mathcal{R}_i^{k+1}$ is thus equal to

$$\underline{a}^{k+1} = \frac{1 + \underline{a}^k}{2} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}.$$

In the same way, we verify that the upper end point is indeed $\bar{a}^k = 1 + \frac{1}{2^{k+1}}$. In the limit as we take $k \to \infty$, the unique strategy profile $(1, 1)$ remains.

(b) Common knowledge of rationality is also necessary in this case because the process does not terminate in finitely many rounds. If players had only knowledge of order k, the entire interval $[1 - \frac{1}{2^k}, 1 + \frac{1}{2^k}]$ would remain.