1. Claim : $-E\left[\frac{\partial^2 \log f(y_i|x_i,\theta_0)}{\partial\theta\,\partial\theta'}\right] = E\left[\frac{\partial \log f_i\theta}{\partial\theta}\cdot\frac{\partial \log f_i\theta}{\partial\theta'}\right]$

Pf : $\frac{\partial}{\partial\theta}\log f(y|x,\theta) = \frac{\frac{\partial}{\partial\theta}f(y|x,\theta)}{f(y|x,\theta)}$

$$\frac{\partial^2}{\partial\theta\partial\theta'}\log f(y_i|x_i,\theta_0) = \frac{\frac{\partial}{\partial\theta\partial\theta'}f(y|x,\theta)f(y|x,\theta) - \frac{\partial}{\partial\theta}f(y|x,\theta)\cdot\frac{\partial}{\partial\theta}f(y|x,\theta)}{f(y|x,\theta)^2}$$

$$= \frac{\frac{\partial}{\partial\theta\partial\theta'}f(y|x,\theta)}{f(y|x,\theta)} - \frac{\frac{\partial}{\partial\theta}f(y|x,\theta)\frac{\partial}{\partial\theta}f(y|x,\theta)}{f(y|x,\theta)^2}$$

$$= \frac{\frac{\partial}{\partial\theta\partial\theta'}f(y|x,\theta)}{f(y|x,\theta)} - \frac{\partial}{\partial\theta}\log f(y|x,\theta)\frac{\partial}{\partial\theta}\log f(y|x,\theta)$$

$$= -\frac{\partial}{\partial\theta}\log f(y|x,\theta)\frac{\partial}{\partial\theta}\log f(y|x,\theta)$$

$$\Rightarrow \frac{\frac{\partial}{\partial\theta\partial\theta'}f(y|x,\theta)}{f(y|x,\theta)} =$$

$$E\left[\frac{\frac{\partial}{\partial\theta\partial\theta'}f(y|x,\theta)}{f(y|x,\theta)}\Big| X\right] = \int\frac{\frac{\partial}{\partial\theta\partial\theta'}f(y|x,\theta)}{f(y|x,\theta)}f(y|x,\theta)\,dy$$

$$= \frac{\partial}{\partial\theta\partial\theta'}\int f(y|x,\theta_0)\,dy\,\Big|_{\theta=\theta_0}$$

$$= \frac{\partial}{\partial\theta\partial\theta'}\cdot 1 = 0 \quad ※$$

2. Claim : $V_{\beta_{GMM}} \geq V^*_{\beta_{GMM}}$

Pf : $\sqrt{n}(\hat{\beta}_{GMM}-\beta) = (\frac{1}{n}X'Z\hat{W}\frac{1}{n}ZX)^{-1}(\frac{1}{n}X'Z\hat{W}\frac{1}{\sqrt{n}}Ze)$

$\frac{1}{\sqrt{n}}Z'e = \sqrt{n}\cdot\frac{1}{n}Z'e \xrightarrow{d} N(0, E[(z_ie_i)(z_ie_i)']) \;$ by CLT

$\frac{1}{n}X'Z \xrightarrow{P} E(x_iz_i'),\quad \frac{1}{n}Z'X \xrightarrow{P} E(z_ix_i),\quad \hat{W} \xrightarrow{P} W$

Let $Q' \equiv E[x_iz_i']$.

$\sqrt{n}(\hat{\beta}_{GMM}-\beta) \xrightarrow{d} N(0, (Q'WQ)^{-1}(Q'W\Omega WQ)(Q'WQ^{-1}))$

If $\hat{W} = W_0 = \Omega^{-1}$,

then $V_{\beta_{GMM}} = (Q^{-1}\Omega^{-1}Q)^{-1}(Q'\Omega\Omega^{-1}\Omega Q)(Q^{-1}\Omega^{-1}Q)^{-1} = (Q'\Omega^{-1}Q)^{-1}$

$$V^{-1}_{\beta_{GMM}} - V^{*-1}_{\beta_{GMM}} = Q'(WQ(Q'W\Omega WQ)^{-1}Q^{-1}W - \Omega^{-1})Q$$

$$= Q'\Omega^{-\frac{1}{2}}(\Omega^{\frac{1}{2}}WQ(Q'W\Omega WQ)^{-1}Q^{-1}W\Omega^{-\frac{1}{2}} - I)\Omega^{-\frac{1}{2}}Q$$

Let $P_\Omega = \Omega^{\frac{1}{2}}WQ((\Omega^{\frac{1}{2}}WQ)'(\Omega^{\frac{1}{2}}WQ))^{-1}(\Omega^{\frac{1}{2}}WQ)'$

$$= \Omega^{\frac{1}{2}}WQ(Q'W\Omega WQ)^{-1}Q'W\Omega^{\frac{1}{2}}$$

Let $M = I - P$

then $V^{*-1}_{BGMM} - V^{-1}_{BGMM} = Q'\Omega^{-\frac{1}{2}} M \Omega^{-\frac{1}{2}} Q = Q'\Omega^{-\frac{1}{2}} M M \Omega^{-\frac{1}{2}} Q$

$\Lambda \in \mathbb{R}^2$, $\Lambda'(V^{*-1}_{BGMM} - V^{-1}_{BGMM})\Lambda = \Lambda' Q' \Omega^{-\frac{1}{2}} M M \Omega^{\frac{1}{2}} Q \Lambda \geq 0$

$\Rightarrow V_{BGMM} - V^{*}_{BGMM} \geq 0.$

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