Review Problems A

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A1

1. If $(x_1, t_1) \succ (x_2, t_2)$, then there exists $\epsilon > 0$ such that $d(x_1, y_1), d(x_2, y_2) < \epsilon \Rightarrow (y_1, t_1) \succ (y_2, t_2)$.
2. We define $u((x,t)) = a$ if $(x,t) \sim (a, 0)$. We now prove that the definition is well-defined.
If $x = 0$, then $(0,t) \sim (0,0)$. If $x \neq 0$, then $(x, 0) \succ (x,t) \succ (0,t) \sim (0,0)$. By continuity, there exists a such that $(x,t) \sim (a,0)$. Moreover, a is unique. Hence, the utility function is well-defined.
3. Trivial. Omit.
4. We gives two ways.
(1) We say $\succ_1$ is more impatient than $\succ_2$ if $\succ_1 \neq \succ_2$ and for $t_1 < t_2$

$$(x_1, t_1) \succ_2 (x_2, t_2) \Rightarrow (x_1, t_1) \succ_1 (x_2, t_2)$$

(2) We say $\succ_1$ is more impatient than $\succ_2$ if $\succ_1 \neq \succ_2$ and for $t > 0$

$$(x,t) \sim_1 (a_1, 0), (x,t) \sim_2 (a_2, 0) \Rightarrow a_1 \leq a_2$$

Easy to see the two definition are equivalent.
5. Notice that the discussion is invalid no matter how we define "impatient". Consider $u_1(x) = x$, $u_2(x) = x^2$,$\delta_1 = \frac{1}{2}$, $\delta_2 = \frac{1}{4}$.
Let $\succsim_1$ represented by $u_1(x)\delta_1^t$ and $\succsim_2$ represented by $u_2(x)\delta_2^t$. Notice that $u_2(x)\delta_2^t = (u_1(x)\delta_1^t)^2$ and $u_1 \geq 0$. Notice that $z \to z^2$ is a strictly increasing function on $[0, \infty)$. Thus, $\succsim_1 = \succsim_2$.
The claim is valid when $u_1 = u_2$. However, that is trivial so we omit the proof.

Comment I want to reclaim that it is not my definition to "impatient" causes the result of 5. But the claim in 5. itself is invalid under any definition.

A2

1. (A1,A2) Y iff number of "S"s> number of "F"s and number of "F"s<2.
(A2,A3) Y no matter what.
(A1,A3) Y iff the first is S.

2. Y iff number of "S"s$>$ number of "F"s.

3. There are three types:

(1) Y only when all S, (2) N only when all F,

(3) There exists number $r > 0$ such that number of "S"s$>$(or $\geq$) $r\times$ number of "F"s.

Suppose D is not of the first two types.

For convenience, we define $S(h)$ =number of "S"s and $F(h)$ =number of "F"s.

We first prove that there exists $k > 0$ such that Y if $S(h) > rF(h)$ and N if $S(h) < rF(h)$.

Suppose not, consider $r = \inf\{\frac{S(h)}{F(h)}|F(h) \neq 0, D(h) = \text{Y}\}$.

First let $r = 0$. Since D is not of second type, there exists h_N with $D(h_N)$ =N with $\frac{S(h_N)}{F(h_N)} > 0$.

Moreover, by $r = 0$, there exists h_Y with $\frac{S(h_Y)}{F(h_Y)} < \frac{S(h_N)}{F(h_N)}$ and $D(h_Y)$ =Y.

Next, let $r > 0$. By the definition of r, we know that $D(h)$ =N if $S(h) < rF(h)$. Hence, there exists h_N such that $\frac{S(h_N)}{F(h_N)} > r$ and $D(h_N)$ =N. By definition of r, there exists h_Y with $\frac{S(h_Y)}{F(h_Y)} < \frac{S(h_N)}{F(h_N)}$ and $D(h_Y)$ =Y.

Therefore, no matter what, we can find h_Y, h_N with $D(h_Y)$ =Y, $D(h_N)$ =N and $\frac{S(h_Y)}{F(h_Y)} < \frac{S(h_N)}{F(h_N)}$.
By A2, we can see $D(h)$ as $D(S(h), F(h))$. And by A3, $D(S(h_Y)S(h_N), F(h_Y)S(h_N))$ =Y and $D(S(h_Y)S(h_N), S(h_Y)N(h_N))$ =N. However, $F(h_Y)S(h_N) > S(h_Y)F(h_N)$, hence we can see $D(S(h_Y)S(h_N), F(h_Y)S(h_N))$ as $D(S(h_Y)S(h_N)+0, S(h_Y)F(h_N)+(F(h_Y)S(h_N)-S(h_Y)F(h_N)))$
Notice that $D(S(h_Y)S(h_N), S(h_Y)N(h_N)) = D(0, F(h_Y)S(h_N) - S(h_Y)F(h_N))$ =N by A1 and so $D(S(h_Y)S(h_N), F(h_Y)S(h_N))$ =N by A3, which contradicts to our assumption.

Now, we shall deal with h with $S(h) = kF(h)$. If k is irrational, then we have nothing to deal with, free to choose $>$ or $\geq$. If k is rational, say $k = \frac{p}{q}$ with $\gcd(p, q) = 1$. Then $(S(h), F(h)) = n(q, p)$ for some n. By A3, $D(S(h), F(h)) = D(q, p)$. Hence, choose $\geq$ when $D(q, p)$ =Y, and $>$ when $D(q, p)$ =N.

A3

1. Let $a \succsim b$ if $a \in C(\{a, b\})$. Completeness and Reflexivity of $\succsim$ is trivial. We shall prove the transitivity.

Let $x \succsim y$ and $y \succsim z$. If $y \in C(\{x, y, z\})$, then $C(\{x, y, z\}) \cap \{x, y\} \neq \emptyset$ and so $x \in C(\{x, y\}) = C(\{x, y, z\}) \cap \{x, y\} \Rightarrow x \in C(\{x, y, z\})$ If $z \in C(\{x, y, z\})$, then $C(\{x, y, z\}) \cap \{y, z\} \neq \emptyset$ and so $y \in C(\{y, z\}) = C(\{x, y, z\}) \cap \{y, z\} \Rightarrow y \in C(\{x, y, z\}) \Rightarrow x \in C(\{x, y, z\})$.

Hence, $x \in C(\{x, y, z\})$ and so $x \in C(\{x, z\})$.

Now, we shall prove that $C(A) = \{x \in A | x \succsim a \text{ for all } a \in A\}$. First suppose $x \succsim a$ for all $a \in A$. Let $y \in C(A)$. If $y \neq x$, by $\{x, y\} \cap C(A) \neq \emptyset$, we have $x \in C(\{x, y\}) = C(A) \cap \{x, y\} \Rightarrow x \in C(A)$.
Next, say $a \succ x$ for some $a \in A$. Then, pick $b \in \{x \in A | x \succsim a \text{ for all } a \in A\} \subseteq C(A)$ (Notice that the set is not empty.). We have $b \succ x$ by transitivity. $\{b\} = C(\{b, x\}) = C(A) \cap \{b, x\}$ since

$b \in C(A)$. Therefore, $x \notin C(A)$, done!
2. No, pick the best two.

A4

1. PI: The best elements in $A \cup B$ is the best among the best in A and the best in B.
E: If x is better than y for all $y \in A$, then x is the best in A.
2. Define $x \succsim y$ iff $x \in C(\{x, y\})$. Complete and reflexive are trivial. Let $x \succ y$ and $y \succ z$.

$$C(\{x,y\}) = C(C(\{x\}) \cup C(\{y,z\})) = C(\{x,y,z\}) = C(C(\{x,y\}) \cup C(\{z\})) = C(\{x,z\})$$

Hence, we have $C(\{x,z\}) = C(\{x,y,z\}) = C(\{x,y\}) = \{x\} \Rightarrow x \succ z$.
We next deal with $C(A) = \{x \in A | \text{for no } y \in A \text{ is } y \succ x\}$.
$\subseteq$: If $y \succ x$ for some $y \in A$, then $C(A) = C(C(\{x,y\}) \cup C(A \setminus \{x,y\}))$. Notice that $x \notin C(\{x,y\})$ and $x \notin A \setminus \{x,y\}$. Hence $x \notin C(A)$.
$\supseteq$: If for no y is $y \succ x$, then $x \succsim y \Rightarrow x \in C(\{x,y\})$ for all $y \in A$. By E, done!
3. PI but no E: the second best, E but no PI: the best two

A5

1. Choose the best two.
2. Choose the best and the worst.
3. Consider X be a set of shoes, which sees left ones and right ones differently. Assign each pair of shoes a number. The set will be looks like $\{\text{left } 1, \text{right } 1, \text{left } 2, \text{right } 2, \cdots\}$. Choose the complete pair with the largest assigned number. If there is no complete pair, then choose shoe with the largest assigned number.
For example, $C(\{\text{left } 1, \text{right } 1, \text{left } 2\}) = \{\text{left } 1, \text{right } 1\}$ and $C(\{\text{left } 1, \text{right } 2, \text{left } 3\}) = \{\text{right } 2, \text{left } 3\}$.
4. We first find a way to order them. Let $C(X) = \{a_1, a_2\}$. (It is fine to assign arbitrary one in the set to be a_1) We can inductively define a_i to be the only element in $C(X \setminus \{a_1, \cdots, a_{i-2}\}) \setminus \{a_{i-1}\}$. It is well defined since $a_{i-1} \in C(X \setminus \{a_1, \cdots, a_{i-2}\})$ by $a_{i-1} \in C(X \setminus \{a_1, \cdots, a_{i-3}\})$ and A1, so there is only one element in $C(X \setminus \{a_1, \cdots, a_{i-2}\}) \setminus \{a_{i-1}\}$.
Define $a_i \succ a_j$ if $i < j$.
We will show that $C(A) = \{a_i, a_j\}$ $(i < j)$ where a_i, a_j being the largest two in $\succ$. We apply induction on $i - j$
1. Let $i - j = 1$, straightforward by definition.
2. Let the proposition holds for $i - j \leq k,\ k \geq 1$. Let $i - j = k + 1$.
$C(A \cup \{a_{j-1}\}) = \{a_i, a_{j-1}\}$ by induction hypothesis. Also, by induction hypothesis $C((A \cup \{a_{j-1}) \setminus \{a_i\}\}) = \{a_{j-1}, a_j\}$. Then $a_j \in C(A)$ by A2 and $a_i \in C(A)$ by A1, done!

A6

1. Let the properties in the list is listed in the following way $P_1, \cdots, P_N$.

Let $a_1 = C(X)$ and $a_i = C(X \setminus \{a_1, \cdots, a_{i-1}\})$ inductively. Define $p(a_i)$ be the maximum of k such that a_i satisfies P_k.

We have $p(a_1) > p(a_i)$ for all $i > 1$ because of the procedure to choose a_1 in X, and $p(a_2) > p(a_i)$ for all $i > 2$ because of the procedure to choose a_2 in X and so on. Hence p is an utility function representing C, done!

2. Suppose that $C = C_{\succsim}$. We know that $\succsim$ is representable, say by u. W.L.O.G, let $a_1 \succ a_2 \succ \cdots \succ a_N$ be the elements in X. Then the properties in the list can be like "if the utility$\geq u(a_N)$, if the utility$\geq u(a_{N-1})$, if the utility$\geq u(a_{N-2})$, $\cdots$ ".

A7

1. Consider (x, y) be the money one will get today and tomorrow. Knowing that money is desirable, and the sooner the better, xDy means that one will prefer x more than y no matter how the preference is explicitly.

D is not a preference relation since it may not be complete.

Let $\succsim_1$ be lexicographical order and $\succsim_2$ be the preference represented by $u(x, y) = x + y$. By checking the definitions, both preferences are in P. However, $(1, 0) \succ_1 (0, 2)$ but $(0, 2) \succ_2 (1, 0)$.

2.

3.

A8

1.

Type 1: There exists $G, B \subseteq X$ be a partition of X. If $A \cap G \neq \emptyset$, then $C(A) = A \cap G$. Otherwise, $C(A) = A$.

Type 2: There exists a set of orderings $\{\succ_k\}_{k=1}^N$. $C(A) = \{x \in A | x \text{ is } \succ_k -\text{maximal in } A \text{ for some } k\}$.

2. Let $G = \{g_1, g_2, \cdots, g_n\}$ and $B = \{b_1, \cdots, b_m\}$.

Let $\succ_{i,j}$ be

$$g_i \succ_{i,j} g_{i+1} \succ_{i,j} \cdots \succ_{i,j} g_n \succ_{i,j} g_1 \succ_{i,j} \cdots \succ_{i,j} g_{i-1} \succ_{i,j} b_j \succ_{i,j} b_{j+1} \succ_{i,j} \cdots \succ_{i,j} b_m \succ_{i,j} b_1 \succ_{i,j} \cdots \succ_{i,j} b_{j-1}$$

Let i, j runs over $1 \sim n, 1 \sim m$, done!

3. Consider $a \succ_1 b \succ_1 c$ and $c \succ_2 a \succ_2 b$.

If it can be described as a decision maker of type 1, then by $C(\{a, b, c\}) = \{a, c\}$, we know that $G = \{a, c\}, B = \{b\}$. However, $C(\{b, c\}) = \{b, c\}$ which should be $\{c\}$ if one is of type 1.