ECON 7011, Semester 110.1, Assignment 6, Solutions

1. (a) Let us parametrize an assessment (σ, μ) by $\sigma_1 = xL + (1-x)R$, $\sigma_2 = y\ell + (1-y)r$, $\sigma_3 = z\lambda + (1-z)\rho$, and by $\mu \in [0,1]$ indicating the probability that Player 3 assigns to the left node in his/her information set h_3. The expected utilities are

$$u_1(\sigma) = 3x(1-z) + (1-x)(1-y+2yz),$$
$$\mathbb{E}_\sigma[u_2(Z) \,|\, h_2] = 3y(1-z) + 2(1-y),$$
$$\mathbb{E}_{\sigma,\mu}[u_3(Z) \,|\, h_3] = 2(\mu z + (1-\mu)(1-z)),$$

with partial derivatives

$$\frac{\partial u_1(\sigma)}{\partial x} = 2 + y - 3z - 2yz, \qquad \frac{\partial u_2(\sigma)}{\partial y} = 1 - 3z, \qquad \frac{\partial u_3(\sigma)}{\partial z} = 2(2\mu - 1).$$

The best-response correspondences are

$$\mathcal{B}_1(y,z) = \begin{cases} x = 1 & \text{if } z < \frac{2+y}{3+2y}, \\ x \in [0,1] & \text{if } z = \frac{2+y}{3+2y}, \\ x = 0 & \text{if } z > \frac{2+y}{3+2y}, \end{cases} \qquad \mathcal{B}_2(z) = \begin{cases} y = 1 & \text{if } z < \frac{1}{3}, \\ y \in [0,1] & \text{if } z = \frac{1}{3}, \\ y = 0 & \text{if } z > \frac{1}{3}, \end{cases}$$

$$\mu(x,y) = \begin{cases} \mu \in [0,1] & \text{if } x = y = 0, \\ \mu = \frac{x}{x+(1-x)y} & \text{otherwise}, \end{cases} \qquad \mathcal{B}_3(\mu) = \begin{cases} z = 1 & \text{if } \mu > \frac{1}{2}, \\ z \in [0,1] & \text{if } \mu = \frac{1}{2}, \\ z = 0 & \text{if } \mu < \frac{1}{2}. \end{cases}$$

We verify consistency by going through the cases of the last player in the tree:

i. If $z \leq \frac{1}{3}$, then $z < \frac{2+y}{3+2y}$ for any $y \in [0,1]$, hence $\mathcal{B}_1$ implies $x = 1$. Therefore, Bayes' rule implies $\mu = 1$, hence $\mathcal{B}_3$ implies $z = 1$, which is inconsistent.

ii. If $z > \frac{1}{3}$, then $\mathcal{B}_2$ implies $y = 0$. We distinguish two cases:
   - If $x > 0$, then Bayes' rule implies $\mu = 1$, hence $\mathcal{B}_3$ implies $z = 1$. This leads to a contradiction because $\mathcal{B}_1$ implies $x = 0$.
   - If $x = 0$, then $\mathcal{B}_1$ implies $z \geq \frac{2}{3}$. Off-path beliefs are either $\mu > \frac{1}{2}$ and $z = 1$, or off-path beliefs are $\mu = \frac{1}{2}$ and $z \in [\frac{2}{3}, 1)$.

All PBE are, therefore, $(R, r, z\lambda + (1-z)\rho)$ for $z \in [\frac{2}{3}, 1)$ and $\mu = \frac{1}{2}$ or $z = 1$ and $\mu \geq \frac{1}{2}$.

(b) We only have to check which off-path beliefs are consistent with strategy profiles of the form $(R, r, z\lambda + (1-z)\rho)$ for $z \geq \frac{2}{3}$. The sequences $x_k = \frac{a}{k}$ and $y_k = \frac{b}{k}$ for constants $a, b > 0$ both approach 0 and Bayes' rule yields

$$\mu_k = \frac{x_k}{x_k + (1-x_k)y_k} = \frac{\frac{a}{k}}{\frac{a}{k} + \frac{k-a}{k}\frac{b}{k}} = \frac{ak}{ak + bk - ab} \to \frac{a}{a+b}.$$

By choosing a and b suitably, these beliefs μ_k can approximate any $\mu \in [\frac{1}{2}, 1)$. For $\mu = 1$, we can use approximating sequences $x_k = \frac{1}{k}$ and $y_k = \frac{1}{k^2}$ as in the lecture notes, for which the beliefs will converge to 1. We conclude that all PBE are sequential equilibria.

2. (a) Let us parametrize a strategy profile $\sigma = (\sigma_1, \sigma_2)$ by

$$\sigma_1(\vartheta_H; H) = \alpha, \qquad \sigma_1(\vartheta_L; H) = \beta, \qquad \sigma_2(h_H; A) = \gamma, \qquad \sigma_2(h_L; A) = \delta.$$

Let $\mu(h_H)$ and $\mu(h_L)$ denote the Insurance Company's posterior beliefs that the Client is of type ϑ_H after observing H and L, respectively. Rejecting a high offer is strictly dominated for the Insurance Company, hence $\mathcal{B}_2(h_H) = \{\gamma = 1\}$. After observing L, the expected utility is $\mathbb{E}_\sigma[u_2(\sigma(\theta)) \,|\, h_L] = (2 - 3\mu(h_L))\delta$. Therefore, rejecting a low offer is a unique best response for the Insurance Company if and only if $\mu(h_L) > \frac{2}{3}$, i.e., the Insurance Company is sufficiently optimistic that the Client is of type ϑ_H. Thus,

$$\mathcal{B}_2(h_L) = \begin{cases} \delta = 1 & \text{if } \mu(h_L) < \frac{2}{3}, \\ \delta \in [0,1] & \text{if } \mu(h_L) = \frac{2}{3}, \\ \delta = 0 & \text{if } \mu(h_L) > \frac{2}{3}. \end{cases}$$

Let us now turn to the Client's best-response correspondence. By the law of total probability, the expected utility of type ϑ_H is equal to

$$\begin{aligned} \mathbb{E}_{\vartheta_H,\sigma}[u_1(A)] &= \mathbb{E}_{\vartheta_H,\sigma}[u_1(A) \,|\, A_1 = H]\alpha + \mathbb{E}_{\vartheta_H,\sigma}[u_1(A) \,|\, A_1 = L](1-\alpha) \\ &= (\gamma - 4(1-\gamma))\alpha + (3\delta - 4(1-\delta))(1-\alpha) \\ &= 5\gamma\alpha + 7\delta(1-\alpha) - 4. \end{aligned}$$

In exactly the same way, we find that the expected payoff of type ϑ_L is given by

$$\mathbb{E}_{\vartheta_L,\sigma}[u_1(A)] = -\gamma\beta + \delta(1-\beta).$$

The partial derivatives with respect to α and β are equal to

$$\frac{\partial \mathbb{E}_{\vartheta_H,\sigma}[u_1(A)]}{\partial \alpha} = 5\gamma - 7\delta, \qquad \frac{\partial \mathbb{E}_{\vartheta_L,\sigma}[u_1(A)]}{\partial \beta} = -\gamma - \delta \le -1,$$

respectively. It follows that $\mathcal{B}_1(\vartheta_L) = \{\beta = 0\}$ is a unique best response for type ϑ_L and that the best-response correspondence of type ϑ_H is

$$\mathcal{B}_1(\vartheta_H) = \begin{cases} \alpha = 1 & \text{if } \delta < \frac{5}{7}, \\ \alpha \in [0,1] & \text{if } \delta = \frac{5}{7}, \\ \alpha = 0 & \text{if } \delta > \frac{5}{7}. \end{cases}$$

Finally, Bayesian updating implies that $\mu(h_H) = 1$ unless $\alpha = \beta = 0$ and

$$\mu(h_L) = \frac{(1-\alpha)\mu_0}{(1-\alpha)\mu_0 + (1-\mu_0)}.$$

To find all the perfect Bayesian equilibria, we distinguish three cases.

Pooling equilibrium. If both types pool on L, that is, $\alpha = \beta = 0$, then $\mu(h_L) = \mu_0$. It follows from $\mathcal{B}_1(\vartheta_H)$ that $\delta \ge \frac{5}{7}$. If $\delta < 1$, the Insurance Company is willing to mix, hence $\mu_0 = \frac{2}{3}$ is implied by $\mathcal{B}_2(h_L)$. Accepting the low offer $\delta = 1$ is a best response for any $\mu_0 \le \frac{2}{3}$. Off-path beliefs are unrestricted since $\gamma = 1$ is strictly dominant.

Separating equilibrium. Note that $\alpha = 1$ implies $\mu(h_L) = 0$, which triggers response $\delta = 1$. However, $\alpha = 0$ is a best response to $\delta = 1$, causing an inconsistency.

Semi-separating equilibria. The last remaining case is $\alpha \in (0,1)$. This implies $\delta = \frac{5}{7}$ by $\mathcal{B}_1(\vartheta_H)$ and hence $\mu(h_L) = \frac{2}{3}$ by $\mathcal{B}_2(h_L)$. Solving the latter identity for α yields $\alpha = 3 - 2/\mu_0$. Such α is within $[0,1]$ if and only if $\mu_0 \ge \frac{2}{3}$.

(b) Since the intuitive criterion refines off-path beliefs, we only need to look at pooling equilibria. Since L is a strict best response for type ϑ_L, deviation H is equilibrium-dominated. For type ϑ_H, deviating to H yields a utility of 1. Since $\delta \ge \frac{5}{7}$ in the pooling equilibrium, the on-path utility is at least $\frac{5}{7} \cdot 3 - 4 \cdot \frac{2}{7} = 1$. Since ϑ_H cannot gain strictly by this deviation, all perfect Bayesian equilibria satisfy the intuitive criterion.

3. (a) Let $s(\vartheta) = e_H$ if and only if $\vartheta \geq \vartheta_*$. Bayesian updating implies that $\mathbb{E}_\sigma[\theta \,|\, e_H] = \frac{\bar{\vartheta}+\vartheta_*}{2}$ and $\mathbb{E}_\sigma[\theta \,|\, e_L] = \frac{\vartheta_*}{2}$. The difference of utilities between choosing e_H and e_L for type ϑ is

$$u_i(\vartheta, w(e_H), e_H) - u_i(\vartheta, w(e_L), e_L) = \frac{\bar{\vartheta}}{2} - \frac{e_H - e_L}{\vartheta}.$$

The cutoff type is indifferent, hence $\vartheta_* = \frac{2}{\bar{\vartheta}}(e_H - e_L)$. The difference of utilities is increasing in ϑ, hence any type $\vartheta < \vartheta_*$ finds e_L a strict best response and any type $\vartheta > \vartheta_*$ finds e_H a strict best response. This shows that the cutoff strategy is indeed a perfect Bayesian equilibrium if $\vartheta_* < \bar{\vartheta}$. This is the case if and only if $\bar{\vartheta}^2 > 2(e_H - e_L)$.

(b) No, because many types choose the same actions.

(c) Yes, trivially, because both actions lie on the path.