Macroeconomic Theory

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Linearization

Moti. 我们不只在乎 s-s，也在乎更新我们的下一期、下下期...
∴ 想知道如何求 saddle path (如何收敛到 s-s)

- We would like to know how to identify the convergence path in Ramsey growth model, and how fast capital and consumption converges along the path
- The nonlinear, two dimensional system of Ramsey growth model is complex
- Two methods to solve for the optimal saddle path path
  1. Linearization
  2. Value function iteration

→ linearize non-linear saddle path.

on 离 s-s 太远，會很不准
∵ non lin 太難

- We apply the linearization method in this lecture: We linearize the nonlinear system around the steady state
- To demonstrate how linearization works, we start from a simple one dimensional system: Solow Model

↳ 只有 k 一变数

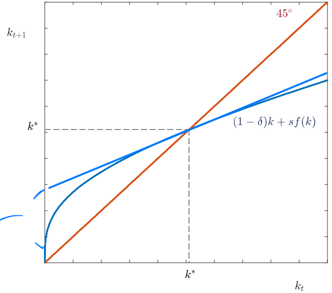
Linearization in Solow Model

- In a Solow model

$$k_{t+1} = sf(k_t) + (1-\delta)k_t$$

- There is a steady state such that

$$k^* = sf(k^*) + (1-\delta)k^*$$



Goal!
s-s 附近和 $(1-\delta)k + sf(k)$
完全一样

太远
误差大

Linearization in Solow Model

- Let $h(k_t) \equiv sf(k_t) + (1-\delta)k_t$

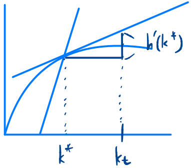
$$k_{t+1} = h(k_t)$$

- Conduct Taylor expansion at the steady state

$\sum_{n=0} \frac{1}{n!} h^n(k^*)(k-k^*)^n$

$$h(k_t) \approx h(k^*) + h'(k^*)(k_t - k^*)$$

At S-S, $k_t = k^*$, then $h(k_t) \approx h(k^*)$



$(k_{t+1} - k^*) = h'(k^*)(k_t - k^*)$

One Dimensional Taylor Expansion

- Let

$$\Phi \equiv h'(k^*) = sf'(k^*) + (1 - \delta)$$

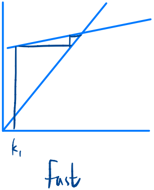
Then,
(Taylor 推来)

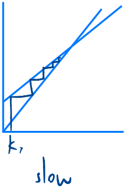
$$k_{t+1} - k^* = \Phi\,(k_t - k^*) \tag{1}$$

o.w. diverge ∵ 愈走愈远

- Note that we must have $\Phi < 1$ if the linear system converges
- Where Φ captures the speed of convergence of capital, and

$$(k_t - k^*) = \Phi^t\,(k_0 - k^*) \tag{2}$$





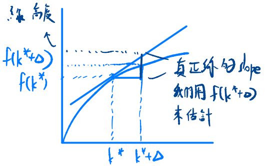
Φ 愈接近 1，收敛愈慢
　　　 0　　　　 快

Linear System

有些函數微分很複雜，需要這樣做

- In practice, we can also apply numerical methods to linearize the system

$$h(k_t) \equiv sf(k_t) + (1-\delta)k_t$$



- Right approximation

$$h'(k^*) \approx \frac{h(k^* + \triangle) - h(k^*)}{\triangle}$$

- Left approximation

$$h'(k^*) \approx \frac{h(k^*) - h(k^* - \triangle)}{\triangle}$$

-

$$\begin{aligned} h'(k^*) &\approx \frac{1}{2}\left[\frac{h(k^* + \triangle) - h(k^*)}{\triangle} + \frac{h(k^*) - h(k^* - \triangle)}{\triangle}\right] \\ &= \frac{1}{2}\left[\frac{h(k^* + \triangle) - h(k^* - \triangle)}{\triangle}\right] \end{aligned}$$

$\Delta = 10^{-8} \sim 10^{-10}$

再小，電腦就沒辦法做

Linear Approximation: Growth Model

The Model

- We assume that the utility function is CRRA:

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \\ \ln(c) & \text{if } \theta = 1 \end{cases}, \theta \geq 0$$

- The dynamic system becomes

$$\begin{aligned} k_{t+1} &= f(k_t) + (1-\delta)k_t - c_t \\ c_{t+1}^{-\theta} &= c_t^{-\theta} \left\{ \beta \left[(1-\delta) + f'(f(k_t) + (1-\delta)k_t - c_t) \right] \right\}^{-1} \end{aligned}$$

$c_{t+1} = u'^{-1}\{u'(c_t)(\quad , \quad)\} \rightarrow$ 蠻難，不適用 CRRA funct.
微分起來很容易

Steady State

ps: 和 L^c, L^k 在刻划完全不同事
不要搞混

$$
\begin{cases}
k_{t+1} = h_k(k_t, c_t) \equiv f(k_t) + (1-\delta)k_t - c_t \\
c_{t+1} = h_c(k_t, c_t) \equiv c_t \left\{\beta\left[(1-\delta) + f'(f(k_t) + (1-\delta)k_t - c_t)\right]\right\}^{\frac{1}{\theta}}
\end{cases}
$$

刻劃 dynamic of capital & consumption

- The steady state

$$
\begin{aligned}
1 &= \beta\left[f'(k^*) + (1-\delta)\right] \\
c^* &= f(k^*) - \delta k^*
\end{aligned}
$$

Linearization

- We conduct Taylor expansion at the steady state (k^*, c^*)

$\begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix}$ || def. $\qquad$ $\begin{pmatrix} k^* \\ c^* \end{pmatrix}$ ||

$$\begin{pmatrix} h_k(k_t, c_t) \\ h_c(k_t, c_t) \end{pmatrix} \approx \begin{pmatrix} h_k(k^*, c^*) \\ h_c(k^*, c^*) \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial k} h_k(c^*, k^*) & \frac{\partial}{\partial c} h_k(c^*, k^*) \\ \frac{\partial}{\partial k} h_c(c^*, k^*) & \frac{\partial}{\partial c} h_c(c^*, k^*) \end{pmatrix} \begin{pmatrix} k_t - k^* \\ c_t - c^* \end{pmatrix}$$

$$\begin{pmatrix} k_{t+1} - k^* \\ c_{t+1} - c^* \end{pmatrix} \approx \begin{pmatrix} \frac{\partial}{\partial k} h_k(k^*, c^*) & \frac{\partial}{\partial c} h_k(k^*, c^*) \\ \frac{\partial}{\partial k} h_c(k^*, c^*) & \frac{\partial}{\partial c} h_c(k^*, c^*) \end{pmatrix} \begin{pmatrix} k_t - k^* \\ c_t - c^* \end{pmatrix} \qquad (3)$$

Linear func! $\qquad$ ... 這些都是 constant

Linearization

- We conduct Taylor expansion at the steady state (k^*, c^*):

$$h_k(k_t, c_t) = f(k_t) + (1-\delta)k_t - c_t$$

$$\begin{aligned}
\frac{\partial}{\partial k} h_k(k^*, c^*) &= f'(k^*) + (1-\delta) = \frac{1}{\beta} \\
\frac{\partial}{\partial c} h_k(k^*, c^*) &= -1
\end{aligned}$$

$\because$ In S-S

Euler: $1 = \beta[f'(k^*) + (1-\delta)]$

Linearization

$c^* \frac{1}{\theta} \left\{ \cancel{\beta} \cdot \cancel{\frac{1}{\beta}} \right\}^{\frac{1}{\theta}-1} \cdot \cancel{\beta} f''(k^*) \cdot \cancel{\frac{1}{\beta}} = \frac{c^*}{\theta} f''(k^*)$

- We conduct Taylor expansion at the steady state:

$$h_c(k_t, c_t) = c_t \left\{ \beta \left[f'(f(k_t) + (1-\delta)k_t - c_t) + (1-\delta) \right] \right\}^{\frac{1}{\theta}}$$

$f'(k^*) + (1-\delta) = \frac{1}{\beta}$

$$\begin{aligned}
\frac{\partial}{\partial k} h_c(k^*, c^*) &= c^* \frac{1}{\theta} \left\{ \beta \left[\underline{f'(f(k^*) + (1-\delta)k^* - c^*) + (1-\delta)} \right] \right\}^{\frac{1}{\theta}-1} \\
&\quad \cdot \beta f''(\underbrace{\boxed{f(k^*) + (1-\delta)k^* - c^*}}_{= k^* \ \because \text{BC}}) \cdot \underbrace{\boxed{\left[f'(k^*) + (1-\delta) \right]}}_{= \frac{1}{\beta} \ \because \text{Euler}} \\
&= \frac{c^*}{\theta} f''(k^*) \\
\frac{\partial}{\partial c} h_c(k^*, c^*) &= \left\{ \beta \left[\underbrace{(1-\delta) + f'(f(k^*) + (1-\delta)k^* - c^*)}_{= \frac{1}{\beta}} \right] \right\}^{\frac{1}{\theta}} \\
&\quad + c^* \frac{1}{\theta} \left\{ \beta \left[\underbrace{(1-\delta) + f'(f(k^*) + (1-\delta)k^* - c^*)}_{= \frac{1}{\beta}} \right] \right\}^{\frac{1}{\theta}-1} \\
&\quad \cdot \beta f''(\underbrace{f(k^*) + (1-\delta)k^* - c^*}_{= k^*})(-1) \\
&= 1 - \frac{c^*}{\theta} \beta f''(k^*)
\end{aligned}$$

$1 + c^* \frac{1}{\theta} \beta f''(k^*)(-1) = 1 - \frac{c^*}{\theta} \beta f''(k^*)$

Linear System

- The linearized system becomes

$$\underbrace{\begin{pmatrix} k_{t+1} - k^* \\ c_{t+1} - c^* \end{pmatrix}}_{= \begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix}} = \underbrace{\begin{bmatrix} \frac{1}{\beta} & -1 \\ \frac{c^*}{\theta} f''(k^*) & 1 - \frac{c^*}{\theta} \beta f''(k^*) \end{bmatrix}}_{= M} \underbrace{\begin{pmatrix} k_t - k^* \\ c_t - c^* \end{pmatrix}}_{= \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}} \tag{4}$$

Let

$$\begin{aligned} \hat{k}_t &= k_t - k^* \\ \hat{c}_t &= c_t - c^* \end{aligned}$$

and

$$\hat{x}_t = \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}$$

We can rewrite (3) as

$$\hat{x}_{t+1} = M \hat{x}_t$$

Linear System

$$\hat{x}_{t+1} = M\hat{x}_t \tag{5}$$

- Our goal is to solve for $\{\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots\}$ that satisfies (5)
- The two-dimensional system of Ramsey model (5) looks similar to the one-dimensional system of Solow model (1)
- However, in Ramsey model, given k_0, there are uncountably many paths satisfying the system, but only one path, the saddle path, is the optimal path and is stable
- We need to identify the saddle path from the two-dimensional system

Discrete Time Dynamic Systems

Linear System

其實是：eigenvalue全部不一樣
也一定可對角化

- We apply diagonalization to solve for the two-dimensional dynamic system

Let 是 assume 嗎? 但不是已經求出來
才這麼假設?

- Let M be a diagonalizable matrix, then we can decompose M as

$$M = V \Lambda V^{-1}$$

- where
  $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is a diagonal matrix, and λ_1 and λ_2 are called the eigenvalues of M
  $V \equiv (v_1, v_2)$, where v_1 and v_2 are the corresponding eigenvectors of λ_1 and λ_2

Households' Problem

- We can thusly rewrite (5) as

  M

  $$\hat{x}_{t+1} = V \Lambda V^{-1} \hat{x}_t$$

  then

  $$\Rightarrow V^{-1} \hat{x}_{t+1} = \Lambda V^{-1} \hat{x}_t$$

- Let $z_t = V^{-1} \hat{x}_t$, then we have

  $$z_{t+1} = \Lambda z_t$$

  $$\Rightarrow z_t = \Lambda^t z_0$$

Households' Problem

z_t

$$V^{-1}\hat{x}_t = \Lambda^t z_0$$

$$\hat{x}_t = V\Lambda^t z_0$$

$$\hat{x}_t = (v_1, v_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^t z_0$$

$$\hat{x}_t = (v_1, v_2)\begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} z_0 \longrightarrow z_0 = \underset{2\times 2}{V^{-1}}\,\underset{2\times 1}{\hat{x}_0} = \begin{bmatrix} z_{1,0} \\ z_{1,0} \end{bmatrix}_{2\times 1}$$

$$\Rightarrow \quad \hat{x}_t = \lambda_1^t v_1 z_{1,0} + \lambda_2^t v_2 z_{2,0}$$

Households' Problem

- $$\overset{\hat{X}_t}{\downarrow}\qquad\qquad \overset{V_1}{\downarrow} \qquad\qquad\qquad \overset{V_2}{\downarrow}$$
  $$\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0} + \lambda_2^t \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0} \tag{6}$$
  where $\underline{z_{1,0} \text{ and } z_{2,0}}$ are undetermined parameters

  ↳ 所有满足 saddle path 的参数

$t=0$: $$\begin{pmatrix} \hat{k}_0 \\ \hat{c}_0 \end{pmatrix} = \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} z_{1,0} + \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix} z_{2,0}$$

↳ start at the diagram → 决定會往哪走
= 决定 $z_{1,0}$ and $z_{2,0}$

Households' Problem

$$\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0} + \lambda_2^t \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0}$$

The dynamic system is a linear combination of two paths

1. let $z_{2,0} = 0, z_{1,0} \neq 0$

$$v_{1,t} \equiv \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0},$$

2. let $z_{1,0} = 0, z_{2,0} \neq 0$

$$v_{2,t} \equiv \lambda_2^t \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0}$$

Households' Problem

$$v_{1,t} \equiv \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0};\; v_{2,t} \equiv \lambda_2^t \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0}$$

- We demonstrate the case that $\lambda_1, \lambda_2 > 0$
- If $\lambda_1, \lambda_2 > 1$: for the paths on the eigenvectors

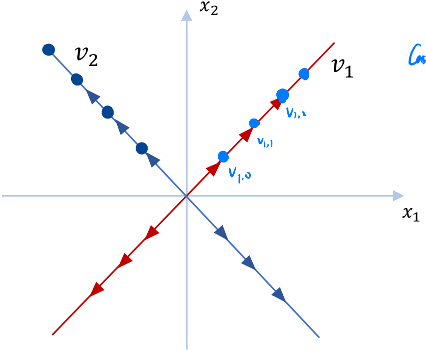
$\lambda_1, \lambda_2 > 1$

Case $v_{12} < 0$ $v_{21} > 0$

$V_{2,0} = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0}$

$V_{2,1} = \lambda_2 V_{2,0}$

$V_{2,2} = \lambda_2^2 V_{2,0}$



Case $v_{11} > 0$ $v_{21} > 0$

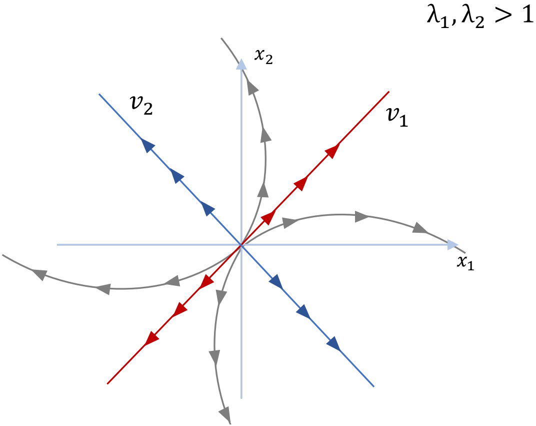
$V_{1,0} = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0}$

$V_{1,1} = \lambda_1 \cdot V_{1,0}$

Households' Problem

$$\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0} + \lambda_2^t \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0}$$

- If $\lambda_1, \lambda_2 > 1$: for the paths outside the eigenvectors



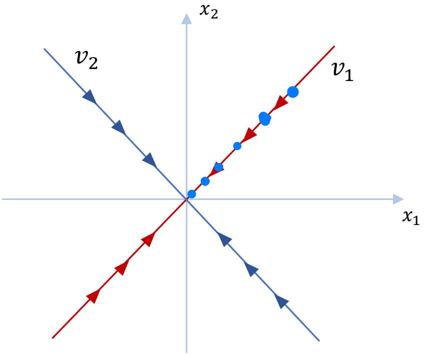
Households' Problem

$$v_{1,t} \equiv \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0};\ v_{2,t} \equiv \lambda_2^t \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0}$$

- If $\lambda_1, \lambda_2 < 1$: for the paths on the eigenvectors

source

$\lambda_1, \lambda_2 < 1$



$v_{1,1} = \lambda_1 v_{1,0}$

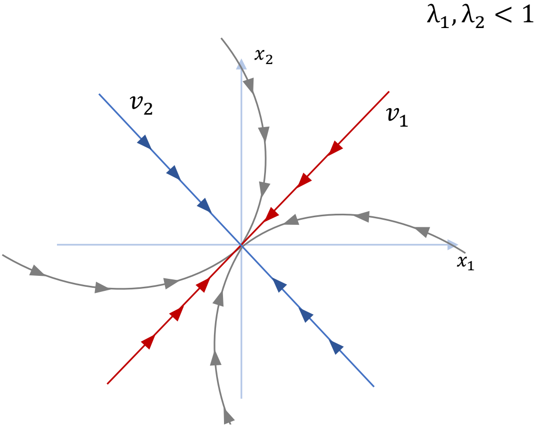
$v_{1,2} = \lambda_1 v_{1,0}$

Households' Problem

$$\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0} + \lambda_2^t \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0}$$

- If $\lambda_1, \lambda_2 < 1$: for the paths outside the eigenvectors

sink



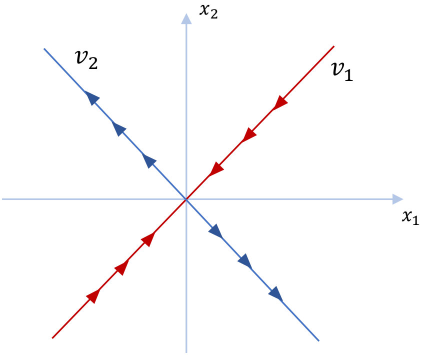
Households' Problem

$$v_{1,t} \equiv \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0};\ v_{2,t} \equiv \lambda_2^t \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0}$$

- If $\lambda_2 > 1 > \lambda_1$: for the paths on the eigenvectors

saddle

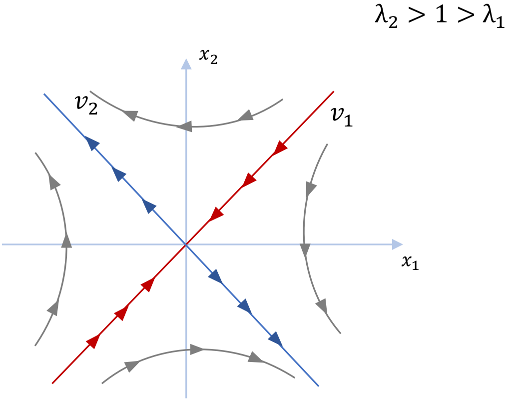
$\lambda_2 > 1 > \lambda_1$



Households' Problem

$$\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0} + \lambda_2^t \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} z_{2,0}$$

- If $\lambda_2 > 1 > \lambda_1$: for the paths outside the eigenvectors



Return to The Growth Model

Linear System

- Our goal is to identify the saddle path of the dynamic system
- We first show that one eigenvalue is greater than one, but another is smaller than one

$$M = \begin{bmatrix} \frac{1}{\beta} & -1 \\ \frac{c^*}{\theta} f''(k^*) & 1 - \frac{c^*}{\theta} \beta f''(k^*) \end{bmatrix}$$

$Ak^{\alpha} \rightarrow \alpha A k^{\alpha-1} \rightarrow \alpha(\alpha-1)A \cdot k^{\alpha-2}$

∴ optimal 一定是saddle

Fact

1. *The sum of eigenvalues* $= tr(M) = \frac{1}{\beta} + 1 - \frac{c^*}{\theta}\beta f''(k^*)$ $\lambda_1 + \lambda_2$
2. *The product of eigenvalues* $= \det(M) = \frac{1}{\beta}$ $\lambda_1 \cdot \lambda_2$

$(x-\lambda_1)(x-\lambda_2) = 0$

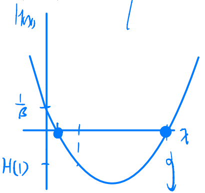
$$H(x) = x^2 - tr(M)x + \det(M)$$

$x^2 - \underbrace{(\lambda_1+\lambda_2)}_{tr(M)} x + \underbrace{\lambda_1\lambda_2}_{\det(M)} = 0$

$$H(0) = \det(M) = \frac{1}{\beta} > 0$$

$$H(1) = 1 - tr(M) + \det \underset{M}{\cancel{A}} = \frac{\beta f''(k^*)c^*}{\theta} < 0$$

∵ $f'' < 0$



∵ $k \to \infty$
$x > 0$
∴ ∃x st. = 0

Linear System

- There are two distinct eigenvalues λ_1 and λ_2; and WLOG, let

$$
\begin{aligned}
\lambda_1 &< 1 \\
\lambda_2 &> 1
\end{aligned}
$$

- Given $\hat{k}_0$, our goal is to find the saddle path $(\hat{k}_t, \hat{c}_t)$
- We set $z_{2,0} = 0$

We know $\hat{k}_0$ find

$$
\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \lambda_1^t \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0} \tag{7}
$$

- $\hat{k}_t$ and $\hat{c}_t$ converges to zero along this path (or this direction $(v_{11} \;\; v_{21})$).

$$
\begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} = \lambda_1^t \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} z_{1,0} + \lambda_2^t \begin{bmatrix} V_{12} \\ V_{12} \end{bmatrix} z_{2,0}
$$

Households' Problem

- To determine $z_{1,0}$, we set $t = 0$: $\hat{k}_0 = v_{11} \cdot z_{1,0}$

$$\begin{pmatrix} \hat{k}_0 \\ \hat{c}_0 \end{pmatrix} = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} z_{1,0} \tag{8}$$

- We know $\hat{k}_0$, and we know (v_{11}, v_{21}).
- Thus, $z_{1,0} = \hat{k}_0 / v_{11}$
- We can apply (8) to solve for the whole path $(\hat{k}_t, \hat{c}_t)$
- Note that (7) and (8) imply

$k_t - k^*$ $\qquad$ $k_0 - k^*$

$$\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \lambda_1^t \begin{pmatrix} \hat{k}_0 \\ \hat{c}_0 \end{pmatrix} \tag{9}$$

$c_t - c^*$ $\qquad$ $c_0 - c^*$

- The eigenvalue λ_1 represents the speed of convergence of the economy

The Dynamics

- Given the system, we consider an economy which is initially at the steady state $(\bar{c}, \bar{k})$ (for $t = -5, -4, -3, -2, -1$). At $t = 0$, there is a productivity shock occurs. We analyze how the capital and consumption change after the shock
- Step 1. Solve for the old steady state $(\bar{c}, \bar{k})$ and the new steady state (c^*, k^*)
- Step 2. linearize the system at the new steady state (c^*, k^*): find the matrix M

The Dynamics

- Step 3. Solve for the eigenvalues and eigenvectors of M
- Step 4. For $t = -5, -4, -3, -2, -1$: we have $k_t = \bar{k}$, $c_t = \bar{c}$
- Step 5. For $t = 0$: Because $k_0 = \bar{k}$ (which is the capital stock at the old steady state), we denote $\hat{k}_0 = \bar{k} - k^*$. We apply (8) to solve for $z_{1,0}$ and $\hat{c}_0$.
- Step 6. For $t = 1, 2, 3, \ldots$: apply (7) or (9) to obtain the whole path of $\hat{k}_t, \hat{c}_t$. Note that the variables we are interested in are $k_t = \hat{k}_t + k^*$ and $c_t = \hat{c}_t + c^*$

The Dynamics

- Step 3. Solve for the eigenvalues and eigenvectors of M
- Step 4. For $t = -5, -4, -3, -2, -1$: we have $k_t = \bar{k}$, $c_t = \bar{c}$
- Step 5. For $t = 0$: Because $k_0 = \bar{k}$ (which is the capital stock at the old steady state), we denote $\hat{k}_0 = \bar{k} - k^*$. We apply (8) to solve for $z_{1,0}$ and $\hat{c}_0$.
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