ECON 7011, Semester 110.1, Assignment 2, Solutions

1. (a) In the game to the left, no pure strategy is strictly dominated. To ensure that we find all Nash equilibria, we find the players' best response correspondences and check consistency. To do so, we parametrize the strategy profile $\sigma = (\sigma_1, \sigma_2)$ by $\sigma_1 = xT + (1-x)D$ and $\sigma_2 = yL + zC + (1-y-z)R$. The players' expected utilities are

$$u_1(\sigma) = x(y+2z) + (1-x)(1+y-z),$$
$$u_2(\sigma) = x(2+y-2z) + (1-x)(1-y+z).$$

The partial derivatives are

$$\frac{\partial u_1(\sigma)}{\partial x} = 3z - 1, \qquad \frac{\partial u_2(\sigma)}{\partial y} = 2x - 1, \qquad \frac{\partial u_2(\sigma)}{\partial z} = 1 - 3x.$$

If both the derivatives with respect to y and z are negative, Player 2's best response is $y = z = 0$, i.e., choosing R with certainty. The best-response correspondences are

$$\mathcal{B}_1(y,z) = \begin{cases} x = 1 & \text{if } z > \frac{1}{3}, \\ x \in [0,1] & \text{if } z = \frac{1}{3}, \\ x = 0 & \text{if } z < \frac{1}{3}, \end{cases} \qquad \mathcal{B}_2(x) = \begin{cases} y = 1, z = 0 & \text{if } x > \frac{1}{2}, \\ z = 0, y \in [0,1] & \text{if } x = \frac{1}{2}, \\ y = z = 0 & \text{if } x \in (\frac{1}{3}, \frac{1}{2}), \\ y = 0, z \in [0,1] & \text{if } x = \frac{1}{3}, \\ z = 1, y = 0 & \text{if } x < \frac{1}{3}. \end{cases}$$

We verify consistency:

i. If $z > \frac{1}{3}$, then $\mathcal{B}_1$ implies $x = 1$, hence $\mathcal{B}_2$ implies $z = 0$, a contradiction.
ii. If $z = \frac{1}{3}$, then $\mathcal{B}_2$ implies $x = \frac{1}{3}$ and $y = 0$. This is consistent.
iii. If $z < \frac{1}{3}$, then $\mathcal{B}_1$ implies $x = 0$, hence $\mathcal{B}_2$ implies $z = 1$, a contradiction.

We conclude that the unique Nash equilibrium is $\left(\frac{1}{3}T + \frac{2}{3}D, \frac{1}{3}C + \frac{2}{3}R\right)$.

In the game to the right, we note that L is strictly dominated by $yC + (1-y)R$ for $y \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Anticipating that no rational Player 2 will play L, Player 1 now has a strictly dominant response D. Finally, given that Player 1 will play D, Player 2 strictly prefers R. We conclude that the unique Nash equilibrium is (D, R).

(b) We are more confident in the predictions of game 2 since common knowledge of rationality is sufficient for players to deduce the Nash equilibrium. In game 1, any strategy profile is rationalizable, hence we need the stronger assumption of conjectures being correct.

2. Let us parametrize Firm 2's strategy by $\sigma_2 = 20x + 40y$, understood as choosing 20 with probability x and choosing 40 with probability y. The firms' expected utilities are

$$u_1(q_1, \sigma_2) = xq_1(70 - q_1) + yq_1(50 - q_1) + (1-x-y)q_1(90 - q_1),$$
$$u_2(q_1, \sigma_2) = 20x(70 - q_1) + 40y(50 - q_1).$$

The partial derivatives with respect to q_1, x, and y are

$$\frac{\partial u_1(q_1, \sigma_2)}{\partial q_1} = 70x + 50y + 90(1-x-y) - 2q_1 = 90 - 20x - 40y - 2q_1,$$
$$\frac{\partial^2 u_1(q_1, \sigma_2)}{\partial q_1^2} = -2, \qquad \frac{\partial u_2(q_1, \sigma_2)}{\partial x} = 20(70 - q_1), \qquad \frac{\partial u_2(q_1, \sigma_2)}{\partial y} = 40(50 - q_1).$$

Similarly as in the first problem, producing $q_2 = 0$ is a best response only if both derivatives are non-positive, i.e., only if $q_1 \geq 70$. For $q_1 < 50$, an increase in the probability of choosing $q_2 = 20$ and $q_2 = 40$ both have a positive impact on Firm 2's utility. The utility is maximized for whichever the marginal impact is larger, i.e., $y = 1$ is a best response if $40(50 - q_1) \geq 20(70 - q_1)$ or, equivalently, $q_1 \leq 30$. We conclude that the best-response correspondences are

$$\mathcal{B}_1(x, y) = \frac{(90 - 20x - 40y)^+}{2}, \qquad \mathcal{B}_2(q_1) = \begin{cases} y = 1 & \text{if } q_1 < 30, \\ x + y = 1 & \text{if } q_1 = 30, \\ x = 1 & \text{if } q_1 \in (30, 70), \\ x \in [0, 1], y = 0 & \text{if } q_1 = 70, \\ x = y = 0 & \text{if } q_1 > 70, \end{cases}$$

where we have used that u_1 is strictly concave. Next, we verify consistency. Note that any best response by firm 1 satisfies $q_1 \leq 45$. We distinguish the three remaining cases:

i. $q_1 < 30$ implies via $\mathcal{B}_2$ that $y = 1$, hence $\mathcal{B}_1$ implies that $q_1 = 25$.✓
ii. $q_1 = 30$ implies via $\mathcal{B}_1$ that $20x + 40y = 30$, hence $x = y = \frac{1}{2}$.✓
iii. $q_1 > 30$ implies via $\mathcal{B}_2$ that $x = 1$, hence $\mathcal{B}_1$ implies that $q_1 = 35$.✓

All of the above are consistent, hence there are three Nash equilibria in this game.

3. (a) Observe first that snowboarder i's utility is continuous only if $p_{-i} = 1$, in which case $u_i(p_i, 1) = 1 - p_i$ has best response is $p_i = 0$. For any $p_{-i} < 1$, snowboarder i's utility is decreasing in p_i except at p_{-i}, where there are two upward discontinuities. In particular, $p_i = p_{-i}$ is not a best response because $p_i' = p_{-i} + \varepsilon$ for $\varepsilon > 0$ is a better response. Moreover, no $p_i > p_{-i}$ is a best response because $p_i' = (p_i + p_{-i})/2$ is a better response. Since $u_i(p)$ is decreasing on $[0, p_{-i})$, only a safe run $p_i = 0$ can be a best response. It is a best response if and only if the right limit at p_{-i} yields a lower utility, i.e., if

$$p_{-i} = u_i(0, p_{-i}) \geq \lim_{x \searrow p_{-i}} u_i(p) = 1 - p_{-i}.$$

In conclusion,

$$\mathcal{B}_i(p_{-i}) = \begin{cases} \emptyset & \text{if } p_{-i} < \frac{1}{2}, \\ 0 & \text{if } p_{-i} \geq \frac{1}{2}. \end{cases}$$

(b) If $p_{-i} < \frac{1}{2}$, then snowboarder i has no best response. If $p_{-i} \geq \frac{1}{2}$, snowboarder i best responds with $p_i = 0$, to which snowboarder $-i$ has no best response.

(c) Suppose the two snowboarders mix according to distribution function F with density f. The indifference principle implies that the snowboarders gain a constant utility on supp F. Thus, there exists a constant $k > 0$ such that for any $p_i \leq \bar{p}$, we have

$$u_i(p_i, F) = (1 - p_i)F(p_i) + (1 - p_i)\int_{p_i}^{\bar{p}} x f(x)\, \mathrm{d}x = k. \tag{1}$$

Since $F(\bar{p}) = 1$, it follows that $1 - \bar{p} = k$. This implies that $\bar{p} < 1$ because a deviation to $p_i = 0$ would otherwise be profitable. Since we only have to find one Nash equilibrium, let us make our life simpler by assuming that F is differentiable on $(0, \bar{p})$. Dividing (1) by $1 - p_i$ (which is possible because $\bar{p} < 1$) and taking the derivative on both sides yields

$$f(p_i) - p_i f(p_i) = \frac{k}{(1 - p_i)^2}.$$

This equation is solved by $f(p_i) = k/(1-p_i)^3$. We obtain the cdf by integrating

$$F(p_i) = \int_0^{p_i} f(x)\,\mathrm{d}x = \frac{k}{2(1-p_i)^2} - \frac{k}{2} = \frac{kp_i(2-p_i)}{2(1-p_i)^2},$$

hence $F(\bar{p}) = 1$ implies $k = 2(1-\bar{p})^2/(\bar{p}(2-\bar{p}))$. Equating with $k = 1-\bar{p}$ yields the quadratic equation $\bar{p}^2 - 4\bar{p} + 2 = 0$ with solutions $2 \pm \sqrt{2}$, of which only the smaller one lies in $(0,1)$. In conclusion, there is a candidate equilibrium with $\bar{p} = 2-\sqrt{2}$ and

$$F(p_i) = \frac{p_i(2-p_i)}{\bar{p}(2-\bar{p})}\frac{(1-\bar{p})^2}{(1-p_i)^2}.$$

It remains to shows that choosing p_i outside the support of F is not a better response. Indeed, any $p_i > \bar{p}$ yields $u_i(p_i, F) = 1 - p_i < 1 - \bar{p} = k$.

(d) Only a safe run can be a best response to a pure strategy, hence we include it in the support. Moreover, atoms cause problems with discontinuities, hence we exclude them.