Macroeconomic Theory Midterm Examination

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Instructions: You have three hours to complete this two-page examination. Please number each question, **underline your final answers**, and present your work as clearly as possible.

1. Consider the infinite horizon Ak growth model. An infinitely-lived representative household values consumption in each period using the period utility is $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma > 0$, and the subjective discount factor $\beta \in (0,1)$. Production only requires the input of capital and takes the following functional form $y_t = Ak_t$ with $A > 0$. The depreciation rate of capital is $\delta \in (0,1)$ and the initial capital stock is $k_0 > 0$. Assume that $\beta A^{1-\sigma} < 1$, which ensures that utility is bounded. Each of the questions below involves the Social Planner's problem. Further, all questions are to be answered using the utility and production functions listed here.

   (a) State the sequence problem, and derive the Euler equation.

   (b) Formulate a functional (Bellman) equation for this problem, and derive the first-order and functional Euler equation, which involves the Benveniste-Scheinkman conditions, i.e., the envelop conditions.

   (c) Conjecture that the value function takes the form

   $$v(k) = E\frac{(k)^{1-\sigma}}{1-\sigma}, \tag{1}$$

   where $E > 0$ is a function of the parameters of the problem: A, σ and β. Please use the method of undetermined coefficients to find E.

2. Consider the sequential competitive equilibrium of the neoclassical growth model described in question (1). Suppose the household can use k_{t+1} as an instrument of saving and borrowing. The return of the saving in period $t+1$, R_{t+1}, equals the capital rental price, r_{t+1}, plus $1-\delta$.

   (a) Define a sequential competitive equilibrium.

   (b) Derive the household's Euler equation involving c_t, c_{t+1} and the real return to savings, R_{t+1}, in period $t+1$.

   (c) Show that the sequential competitive equilibrium reproduces the sufficient conditions for the Social Planner's problem. Explain the significance of this result.

3. Consider a two period problem. A household consumes and works in both periods. This household faces earning risk in the second period. In particular, in the second

2. (a) Def. $\{c_t, k_{t+1}\}, \{r_{t+1}, R_{t+1}\}$ as sequential competitive eq.

such as : ① $\{c_t, k_t\} = \arg\max_{c_t, k_t} \sum \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$

s.t. $c_t + k_{t+1} = R_t k_t$

② $\{r_t\} = \arg\max_{r_t} A k_t - r_t k_t$

③ Mkt clear : $\begin{cases} c_t + k_{t+1} = A k_t \\ k_t = 0 \end{cases}$

b ② FOC: $A = r_t$

So $R_t = A + (1-\delta) \quad \forall t$

① FOC : $\frac{1}{c_t^{\sigma}} = \beta \cdot \frac{1}{c_{t+1}^{\sigma}} \cdot [R_t]$

$\Rightarrow \quad c_{t+1}^{\sigma} = \beta c_t^{\sigma} \cdot [A + (1-\delta)]$

1. (a) $\max \sum_{t=0}^{T} \beta^t u(c_t)$ s.t. $c_t + k_{t+1} - (1-\delta)k_t = f(k_t)$

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left\{ u(c_t) + \lambda_t \left[f(k_t) + (1-\delta)k_t - k_{t+1} - c_t \right] \right\}$$

FOC: $\begin{cases} u'(c_t) = \lambda_t \\ \lambda_t = \beta \lambda_{t+1} \left[f'(k_t) + (1-\delta) \right] \end{cases}$

Euler: $u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_t) + (1-\delta) \right]$ → 应該要代入!

(b) Let $v(k_t) = \sum_{t=0}^{T} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1})$

$\Rightarrow v(k) = \max_{k'} \{ u(f(k) + (1-\delta)k - k') + \beta v(k') \}$

Let $Tv(k) = \max_{k'} \{ u(f(k) + (1-\delta)k - k') + \beta v(k') \}$

then SP becomes find fixed pt. of T.

FOC: $u'(f(k) + (1-\delta)k - k') = \beta v'(k')$

$$\Rightarrow \frac{1}{[Ak + (1-\delta)k - k']^{\sigma}} = \beta \cdot v'(k').$$

(c) Guess $v(k) = E \frac{(k)^{1-\sigma}}{1-\sigma}$

then $$\frac{1}{[Ak + (1-\delta)k - k']^{\sigma}} = \beta \cdot E \cdot \frac{1}{k'^{\sigma}}$$

$$\Rightarrow k'^{\sigma} = (\beta E)^{\frac{1}{\sigma}} [Ak + (1-\delta)k - k']^{\sigma}$$

$$\Rightarrow k' = (\beta E)^{\frac{1}{\sigma}} (Ak + 1-\delta k) - (\beta E)^{\frac{1}{\sigma}} k'$$

$$\Rightarrow k' = \frac{(\beta E)^{\frac{1}{\sigma}} (Ak + 1-\delta k)}{1 + (\beta E)^{\frac{1}{\sigma}}}$$

$$Tv(k) = \max \left\{ \frac{1}{1-\sigma} \cdot [Ak + (1-\delta)k - k']^{1-\sigma} + \beta \cdot E \frac{(k')^{1-\sigma}}{1-\sigma} \right\}$$

$$= \frac{1}{1-\sigma} \cdot \left[\frac{k'}{(\beta E)^{\frac{1}{\sigma}}} \right]^{1-\sigma} + \beta E \cdot \frac{(k')^{1-\sigma}}{1-\sigma}$$

$$= \left[\frac{1}{1-\sigma} \cdot \frac{1}{(\beta E)^{\frac{1-\sigma}{\sigma}}} + \frac{\beta E}{1-\sigma} \right] (k')^{1-\sigma}$$

$$= \left[(\beta E)^{\frac{\sigma-1}{\sigma}} + \beta E \right] \cdot \frac{1}{1-\sigma} \cdot \left[\frac{(\beta E)^{\frac{1}{\sigma}} (Ak + 1-\delta k)}{1 + (\beta E)^{\frac{1}{\sigma}}} \right]^{1-\sigma}$$

$$= \frac{(\beta E)^{\frac{\sigma-1}{\sigma}} + \beta E}{[1 + (\beta E)^{\frac{1}{\sigma}}]^{1-\sigma}} \cdot \beta E^{\frac{1-\sigma}{\sigma}} (A + 1-\delta) \frac{(k)^{1-\sigma}}{1-\sigma}$$

$$= (1 + (\beta E)^{\frac{1}{\sigma}})^{\sigma} (A + (1-\delta)) \cdot \frac{(k)^{1-\sigma}}{1-\sigma}$$

$$\Rightarrow \left[1+(\beta E)^{\frac{1}{6}}\right]^6\left[A+(1-\delta)\right] = E$$

$$\Rightarrow (\beta E)^{\frac{1}{6}}\left[A+(1-\delta)\right]^{\frac{1}{6}} = E^{\frac{1}{6}}-1$$

$$\Rightarrow \left[A+(1-\delta)\right]^{\frac{1}{6}}+1 = \left[1-\beta^{\frac{1}{6}}\right]E^{\frac{1}{6}}$$

$$\Rightarrow E = \left[\frac{\left[A+(1-\delta)\right]^{\frac{1}{6}}+1}{1-\beta^{\frac{1}{6}}}\right]^6 \quad \#$$

這就是為何定義 $R = r+(1-\delta)$

$\uparrow$

請看為: $c_t + k_{t+1} - (1-\delta)k_t = r_t k_t$

2. A competitive eq is a set of seq $\{c_t\}$, $\{k_{t+1}^*\}$, $\{R_t^*\}$ s.t.

(1) $\{c_t^*, k_{t+1}^*\} = \arg\max \sum \beta u(c_t)$ s.t. $c_t + k_{t+1} = k_t R_t^*$

$c_t \geq 0 \ \forall t,\ k_0 = 0$

No-Ponzi game condition: $\lim k_{t+1}\left(\prod_{s=0}^{\infty} R_{s+1}\right)^{-1} = 0$

(2) $\{k_t^*\} = \arg\max \ Ak_t - r k_t$

$k_t^* = 0$ (asset mkt clearing)

(3) Good Mkt clearing: $c_t^* + k_{t+1}^* = Ak_t^*$

b $\max \ Ak_t - r_t k_t \Rightarrow A = r_t$

So $R_t = r_t + (1-\delta) = A+1-\delta$

⊕ 雖然題目是求家戶的 Euler，
但實際上還是要算到 firm 的 max

$\max \sum \beta^t \frac{c_t^{1-6}}{1-6}$ s.t. $c_t + k_{t+1} \leq R_t k_t$

$$\Rightarrow \begin{cases} \beta^t c_t^{-6} = \tilde{\lambda}_t \\ \tilde{\lambda}_{t+1} R_t = \tilde{\lambda}_t \end{cases}$$

$\Rightarrow$ Euler: $c_t^{-6} = \beta R_t c_{t+1}^{-6}$

c. 略

3. 不考

period there are two possible states of the world $z = \{z_L, z_H\}$. Denote $i = \{L, H\}$. Suppose the probability of state i is $\pi_i = \Pr\{z = z_i\}$ with $\pi_i > 0$ and $\pi_L + \pi_H = 1$. The production function for the first period and second period are $y_0 = zn_0$, $y_{1i} = z_i n_{1i}$, respectively. Furthermore, the household maximizes two periods utility function $u(c_0, n_0) + \beta \sum \pi_i u(c_{1i}, n_{1i})$ where $u(c, n) = \log(c) + \eta \log(1 - n)$ and $n + l = 1$ where $\eta > 0$. This household has access to a complete set of state-contingent bonds.

(a) Denote q_i as the price for state contingent bond i when $z = z_i$. First write down the lifetime budget constraint. Solve for equilibrium consumption, employment, and asset allocations for each date and state. How do these evolve with z_i? Explain the meaning of labor decision you derive.

(b) Derive equilibrium price functions for each Arrow-Security, $q(z_j)$.

(c) Suppose now there are one risk free bond and one state-contingent bond for $z = z_H$ available. And their prices are q, and q_H, respectively. Again, write down the budget constraint for each date and state, and then derive optimal consumption, employment, and asset allocation for each date and state. Is the equilibrium in the current economy different to the one with two state-contingent bond? Explain the economic reason behind this result.

(d) Derive equilibrium price for the discounted risk free bond.

(e) Now suppose there is only risk free bond. How does the equilibrium (consumption, employment, and asset allocations) change? Explain.

$k_{t+1} = x_t + (1-\delta)k_t$

$c = (1-s)y$

$x = sy$

$k_{t+1} = sy + (1-\delta)k_t$

$sy = \delta k^*$

$c = y - sy$
$\quad = y - \delta k^*$

Golden rule def.: S-S that maximize your utility.

∵ 通常 c 愈大, $u(c)$ 就會愈大

∴ 會直接算 $\max_k c$

在 Ramsey model 的所有 S-S: $c^* + \delta k^* = f(k^*)$

$\Rightarrow c^* = f(k^*) - \delta k^*$

So $\max c^* \Leftrightarrow \max f(k^*) - \delta k^*$

FOC: $f'(k^*) = \delta$

↳ 滿足這條的 k, 就是 golden rule capital.

Solow: Assume $\begin{cases} c + x = y \\ c = (1-s)\,y \quad \text{or} \quad x = sy \\ k_{t+1} = x_t + (1-\delta)k_t \end{cases}$

$\Rightarrow$ Law of motion for capital: $k_{t+1} = sy_t + (1-\delta)k_t$

$\Rightarrow$ In S-S, $k^* = sy^* + (1-\delta)k^* \Rightarrow sy^* = \delta k^*$

$\because$ $c^* = (1-s^*)y^* = y^* - sy^* = y^* - \delta k^*$

$\therefore$ $\max c^* \Leftrightarrow \max y^* - \delta k^*$

FOC: $f'(k^*) = \delta k^*$