Macro Theory I Part 2 - Quiz 1

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Note and Correction

1. When we consider about the market clearing condition and perceptions in recursive competitive equilibrium (RCE), whether k in the decision rule for household need to be replaced with K or not depends on following to conditions (please be careful on the difference between them):

   - Mass one households:
     - This is the case in Prof. Liao's lecture.
     - Mass one: this economy populated by a large number identical households and they have same preference and same production function.
     - So that we can only focus on representative agent, and we can assume that population equals one ($N_t = 1$).
     - In this case, $k = K$.
     - The decision rule for household can be written in k or K. That is, (use decision rule for working hours as an example)

       $$h(K, k) \text{ or } h(K, K)$$

     - Thus, market clearing condition for labor market is

       $$h^f(K) = h(K, k) \text{ or } h^f(K) = h(K, K)$$

   - Heterogeneous households:
     - For example, there are two types of household with different utility functions.
     - In this case, when we consider about the market clearing conditions, because their utility function are different, k in the decision rule for household can not represent the whole supply side of economy in labor and capital market (whole demand side of economy in consumption good market).
     - To make the decision rule for household can represent the representative agent, we need to impose K "after" we have solved the agent's decision problem.

- The decision rule for household only can be written in K. That is,

  (use decision rule for working hours as an example)

  $$h(K, K)$$

- Thus, market clearing condition for labor market is

  $$h^f(K) = Nh(K, K)$$

- Note that the case of heterogeneous household is more general, that is why I applied decision rule for household in K in TA session.
- Reference: Chapter 7.3, Recursive Macroeconomic Theory, 3rd Edition, Lars Ljungqvist and Thomas J. Sargent.

2. Correction to household's DPP and RCE in Economy B:

   - We also need to be given low of motions for the state variables: C and L.
   - Please check the household's problem for correction.
   - Please also check the sets of equation in RCE for correction.
   - Please also check the perceptions in RCE for correction.

3. Correction to social planner's DPP in Economy C:

   - I corrected the production function for firm 1 in the constraint, please check the suggested solution of this question for more details.

Economy A

Setting

There are a large number of identical households with preferences $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$ in Economy A, where u is continuous, increasing, concave and continuously differentiable in both arguments. Here, c_t and l_t are consumption and leisure in period t, respectively. There is also a single firm that produces output using a constant return to scale (CRTS), $F(k_t, h_t)$, where k_t is capital and h_t is hours worked. Households are endowed with one unit of time that can be allocated to work or leisure. They purchase output from the firm and use it for consumption or as capital in the following period. They are endowed with k_0 units of capital in period 0. Capital is assumed to depreciate at the δ in each period.

Suppose the population for every period is N (i.e., $N_t = N$ for all t). The aggregate consumption, aggregate capital, aggregate leisure and aggregate hours worked at t are $C_t = Nc_t$, $K_t = Nk_t$, $L_t = Nl_t$ and $H_t = Nh_t$, respectively. Define the investment in period t is i_t so that the aggregate investment in period t is $I_t = Ni_t$.

Next, we start from the social planner's sequential problem for Economy A.

$$
\begin{aligned}
\max_{\{c_t, i_t, l_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t N u(c_t, l_t) \\
\text{s.t.} \quad & C_t + I_t = F(K_t, H_t) \\
& K_{t+1} = I_t + (1-\delta) K_t \\
& N = H_t + L_t \\
& k_0 \text{ is given} \\
& (K_0 = Nk_0 \text{ is given.})
\end{aligned}
$$

Combining resource constraint and law of motion for aggregate capital gives us

之所以要整理是∵效用函數內的變數必須是個別HH的c、l

$$
\begin{aligned}
& C_t + K_{t+1} = F(K_t, H_t) + (1-\delta) K_t \\
\Longrightarrow \ & Nc_t + Nk_{t+1} = F(Nk_t, Nh_t) + (1-\delta) Nk_t \\
\Longrightarrow \ & Nc_t + Nk_{t+1} = NF(k_t, h_t) + (1-\delta) Nk_t \\
\Longrightarrow \ & c_t + k_{t+1} = F(k_t, h_t) + (1-\delta) k_t.
\end{aligned}
$$

Also,

$$
\begin{aligned}
N &= H_t + L_t \\
&= Nh_t + Nl_t \\
\implies 1 &= h_t + l_t
\end{aligned}
$$

Thus, the social planner's sequential problem can be written as

$$
\begin{aligned}
&\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N u(c_t, 1 - h_t) \\
&\text{s.t.} \qquad c_t + k_{t+1} = F(k_t, h_t) + (1 - \delta) k_t \\
&\qquad\qquad k_0 \text{ is given.}
\end{aligned}
$$

At $t = 0$, given k_0:

$$
\begin{aligned}
V(k_0) &= \max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N u(c_t, 1 - h_t) \\
&= \max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} N u(c_0, 1 - h_0) + \beta \sum_{t=1}^{\infty} \beta^{t-1} N u(c_t, 1 - h_t) \\
&= \max_{c_0, h_0, k_1} N u(c_0, 1 - h_0) + \beta \max_{\{c_t, h_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} N u(c_t, 1 - h_t) \\
&= \max_{c_0, h_0, k_1} N u(c_0, 1 - h_0) + \beta V(k_1) \\
\implies V(k) &= \max_{c, h, k'} N u(c, l) + \beta V(k')
\end{aligned}
$$

1. Specify the social planner's dynamic programming problem.

Then, the social planner's dynamic programming problem can be written as

雖然是 Social planner，但是我們剛整理完 K 變 k，∴這裡為 k

$$
\begin{aligned}
V(\underline{k}) &= \max_{c, h, k'} N u(c, 1 - h) + \beta V(k') \qquad (1) \\
\text{s.t.} \quad & c + k' = F(k, h) + (1 - \delta) k \\
& k \text{ is given.}
\end{aligned}
$$

2. Define a recursive competitive equilibrium for the economy.

Two state variables

1) k: the capital stock of a household, representing the state/situation/what we have know of the agent.

2) K: the aggregate capital stock, representing the state of the whole economy.

Note:

1. Household need to known K to make decision since his income depends on w and r, and w and r are relative to K.

2. Household only focus on what he has known (k, K) for maximizing life-time utility, but household has no control on the aggregate variable (K).

Household's problem

$$\begin{aligned} V(K,k) = \max_{c,h,k'} \; & u(c, 1-h) + \beta V(K', k') \\ \text{s.t.} \quad & c + k' = wh + rk + (1-\delta)k \\ & K' = G(K) \\ & w = w(K) \\ & r = r(K) \end{aligned} \tag{2}$$

Firm's problem

$$\begin{aligned} \max_{k^f, h^f} \quad & F(k^f, h^f) - wh^f - rk^f \\ \text{s.t.} \quad & w = w(K) \\ & r = r(K) \end{aligned} \tag{3}$$

Recursive Competitive Equilibrium

A *Recursive Competitive Equilibrium* is a set of equations

1. a value function: $V(K, k)$,

2. individual's decision rules: $c(K, k)$, $h(K, k)$ and $k'(K, k)$,

3. decision rules for the firm: $h^f(K)$ and $k^f(K)$,

4. pricing functions: $w(K)$ and $r(K)$,

5. a low of motion for aggregate state variable $K' = G(K)$

such that

1) given $r(K)$, $w(K)$, and $G(K)$,

   $V(K, k)$, $c(K, k)$, $h(K, k)$ and $k'(K, k)$ solve the household's dynamic programming problem (2);

2) given $r(K)$ and $w(K)$,

   the decision rules $h^f(K)$ and $k^f(K)$ solve the firm's problem (3).;

3) market clearing condition:

– labor:

$$h^f(K) = Nh(K, K);$$

– capital:

$$k^f(K) = Nk = K;$$

– consumption goods:

clear by Walras' Law or

$$Nc(K, K) = F(k^f(K), h^f(K));$$

4) perceptions are correct:

$$K' = k'$$

$$\Longrightarrow G(K) = k'(K, K).$$

Note: To make the representative agent representative, we impose $k = K$ on $c(K, k)$, $h(K, k)$ and $k'(K, k)$, but only "after" we have solved the agent's decision problem.

Economy B

Setting

Economy B is same as Economy A, except that utility depends not only on current consumption and leisure, but also on consumption and leisure from the previous period. That is, the period utility function is $u(c_t, c_{t-1}, l_t, l_{t+1})$.

We start from the social planner's sequential problem for Economy B.

$$\begin{aligned} \max_{\{c_t, i_t, l_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t N u(c_t, c_{t-1}, l_t, l_{t-1}) \\ \text{s.t.} \quad & C_t + I_t = F(K_t, H_t) \\ & K_{t+1} = I_t + (1-\delta)K_t \\ & N = H_t + L_t \\ & k_0 \text{ is given} \\ & (K_0 = Nk_0 \text{ is given.}) \end{aligned}$$

$$\max \sum \beta^t N u(C_t, \underline{C_{t-1}}, l_t, \underline{l_{t-1}}) \longrightarrow$$ 实质就是期给定，更为是说 上期已决定!!

$$\text{st} \quad C_t + I_t = f(K_t, H_t)$$

$$L_t + H_t = N$$

$$K_{t+1} = I_t + (1-\delta)K_t$$

$$V(c_{-1}, l_{-1}, k) = \max \{ u(c_{-1}, l_{-1}, c, l) + \beta V(c, l, k') \}$$

$$\text{st.} \quad c + i = f(k, h)$$

$$i = k' - (1-\delta)k$$

$$h + l = 1$$

$$\text{HH} \quad \max \sum \beta^t N u(C_t, C_{t-1}, l_t, l_{t-1})$$

$$\text{st.} \quad \begin{cases} C_t + i_t = w h_t + r k_t \\ h_t + l_t = 1 \\ k_{t+1} = i_t + (1-\delta) k_t \end{cases}$$

$$\text{Firm} \quad \max \; f(k_t^f, h_t^f) - w h_t^f - r k_t^f$$

$$\text{st.} \quad w = w(C_{-1}, L_{-1}, K)$$

$$r =$$

$$\Rightarrow \quad V(c_{-1}, l_{-1}, k, C_{-1}, L_{-1}, K)$$

$$= \max \{ u(1\,2\,3\,4) + \beta V(\text{用}\, k^{-1}) \}$$

$$\text{st.} \quad C_t + i_t = w h_t + r k_t$$

$$h_t + l_t = 1$$

$$k_{t+1} = i_t + (1-\delta) k_t$$

$$w = w(\cdot)$$

$$r = r(\cdot)$$

$$L = g_L(\cdot)$$

$$C = g_C(\cdot)$$

$$K = g_k$$

Then, the social planner's sequential problem can be written as

雖然看起來多了2變數 c_{-1}, l_{-1}
但這都是會給的 state variable,
∴ 根本沒差

$$
\begin{aligned}
\max_{\{c_t, l_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t N u(c_t, c_{t-1}, l_t, l_{t-1}) \\
\text{s.t.} \quad & c_t + k_{t+1} = F(k_t, h_t) + (1-\delta) k_t \\
& 1 = h_t + l_t \\
& k_0 \text{ is given.}
\end{aligned}
$$

k_{t+1} 雖然是另一個要決定的變數，但實際上
k_{t+1} 根本不會影響效用，
∴ 應把它想成決定"c"的一部分
i.e. 用 k_{t+1} 決定如何分配跨期消費

2條限制式分別代入效用函數的2變數

1. Specify the social planner's dynamic programming problem.

The social planner's dynamic programming problem is

$$
\begin{aligned}
V(k, c_{-1}, l_{-1}) = \max_{c,l,h,k'} \quad & N u(c, c_{-1}, l, l_{-1}) + \beta V(k', c, l) \\
\text{s.t.} \quad & c + k' = F(k,h) + (1-\delta)k \\
& h + l = 1 \\
& k \text{ is given.}
\end{aligned} \tag{4}
$$

HH: 直接多加上
總體版變數

2. Define a recursive competitive equilibrium for the economy.

Household's problem

$$
\begin{aligned}
V(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1}) = \max_{c,l,h,k'} \quad & u(c, c_{-1}, l, l_{-1}) + \beta V(K', C, L, k', c, l) \\
\text{s.t.} \quad & c + k' = wh + rk + (1-\delta)k \\
& h + l = 1 \\
& K' = G_K(K, C_{-1}, L_{-1}) \\
& C = G_C(K, C_{-1}, L_{-1}) \\
& L = G_L(K, C_{-1}, L_{-1}) \\
& w = w(K, C_{-1}, L_{-1}) \\
& r = r(K, C_{-1}, L_{-1})
\end{aligned} \tag{5}
$$

總體state
決定價格

個體state
決定效用

除了第0期的東西是給定的
之後每一期都是HH自己要算出來
∴ HH必須知道這些總體變數how變動

We need to know the motion
of these aggregate state variable

So

Aggregate state variable
affect price

Firm's problem

$$
\begin{aligned}
\max_{k^f, h^f} \quad & F(k^f, h^f) - w h^f - r k^f \\
\text{s.t.} \quad & w = w(K, C_{-1}, L_{-1}) \\
& r = r(K, C_{-1}, L_{-1})
\end{aligned} \tag{6}
$$

actually $k^f(K, C_{-1}, L_{-1})$

Recursive Competitive Equilibrium

A *Recursive Competitive Equilibrium* is a set of equations

1. a value function: $V(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$,

2. individual's decision rules: $c(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$, $h(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$, $l(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$ and $k'(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$,

3. decision rules for the firm: $h^f(K, C_{-1}, L_{-1})$ and $k^f(K, C_{-1}, L_{-1})$,

4. pricing functions: $w(K, C_{-1}, L_{-1})$ and $r(K, C_{-1}, L_{-1})$,

5. low of motions for aggregate state variables: $K' = G_K(K, C_{-1}, L_{-1})$, $C = G_C(K, C_{-1}, L_{-1})$ and $L = G_L(K, C_{-1}, L_{-1})$

such that

1) given $r(K, C_{-1}, L_{-1})$, $w(K, C_{-1}, L_{-1})$, $G_K(K, C_{-1}, L_{-1})$, $G_C(K, C_{-1}, L_{-1})$ and $G_L(K, C_{-1}, L_{-1})$, $V(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$, $c(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$, $h(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$, $l(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$ and $k'(K, C_{-1}, L_{-1}, k, c_{-1}, l_{-1})$ solve the household's dynamic programming problem (5);

2) given $r(K, C_{-1}, L_{-1})$ and $w(K, C_{-1}, L_{-1})$,

   the decision rules $h^f(K, C_{-1}, L_{-1})$ and $k^f(K, C_{-1}, L_{-1})$ solve the firm's problem (6);

3) market clearing condition:

   labor:

$$h^f(K, C_{-1}, L_{-1}) = Nh(K, C_{-1}, L_{-1}, K, C_{-1}, L_{-1});$$

   capital:

$$k^f(K, C_{-1}, L_{-1}) = Nk = K;$$

   goods:

   clear by Walras' Law or

$$Nc(K, C_{-1}, L_{-1}, K, C_{-1}, L_{-1}) = F(k^f(K), h^f(K));$$

4) perceptions are correct:

$$K' = k'$$

$$\Longrightarrow G_K(K, C_{-1}, L_{-1}) = k'(K, C_{-1}, L_{-1}, K, C_{-1}, L_{-1}),$$

$$C = c$$

$$\Longrightarrow G_C(K, C_{-1}, L_{-1}) = c(K, C_{-1}, L_{-1}, K, C_{-1}, L_{-1}),$$

$$L = l$$

$$\Longrightarrow G_L(K, C_{-1}, L_{-1}) = l(K, C_{-1}, L_{-1}, K, C_{-1}, L_{-1}).$$

Economy C

Setting

Economy C is the same as Economy A, except that there are two firms that operate constant returns to scale technologies. The first firm produces new capital (i.e., investment goods) using labor and existing capital supplied by the households. The second firm produces consumption goods using the same factor inputs. Denote the technologies operated in sector i by $f^i(k_i, h_i)$, $i = 1, 2$. Households have one unit of time that can be allocated to leisure, working hours in sector 1 (h_1) and working hours in sector 2 (h_2). They also need to allocate his capital to both firms. In addition, investment goods become productive capital in the following period. Productive capital can be allocated to either firm. Note that we have four market in this economy: labor market, capital market, consumption good market and investment good market.

We start from the social planner's sequential problem for Economy C.

s.t.
① $C_t = f^2(K^2, H^2)$ 另一個用增加消費財
$I_t = f^1(K^1, H^1)$ 一個用來增加投資財
③ $K_{t+1} = I_t + (1-\delta)K_t$
$K_t = K^1_t + K^2_t$ 對 Social planner 來說 所有K都有2用途
② $N_t = H^1_t + H^2_t + L_t$

$$\max_{\{c_t, l_t, h_t^1, h_t^2, k_t^1, k_t^2, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N u(c_t, l_t)$$

$$\text{s.t.} \qquad C_t = f^2(K_t^2, H_t^2) \tag{7}$$

廠商1供需

$$K_{t+1} = f^1(K_t^1, H_t^1) + (1-\delta)K_t \tag{8}$$

$$K_t = K_t^1 + K_t^2 \tag{9}$$

$$N = H_{1t} + H_{2t} + L_t \tag{10}$$

2種資源限制式

k_0 is given

使用K^2相當於消費,
使用K^1相當於選擇
跨期消費(投資)

Note that Eq.(8) comes from combinig:

$$\begin{cases} K_{t+1} = I_t + (1-\delta)K_t \\ I_t = f^1(K_t^1, H_t^1) \end{cases}$$

決定資本怎產生

廠商2供需

∵不是同一個商品
∴C, I沒有互相選擇的trade off

From Eq.(9),

⇓
分成2條來看
⇓
去工作 trade off 分配 K_1, K_2 & H_1, H_2

$$K_t = K_t^1 + K_t^2$$
$$\Longrightarrow N k_t = N k_t^1 + N k_t^2$$
$$\Longrightarrow k_t = k_t^1 + k_t^2$$

$u(c, l) \to$ ① c vs l

② 當期 v.s. 跨期

$C_t = f^2(K^2_t, H^2_t) \longrightarrow$ consumption good

$I_t = f^1(K^1_t, H^1_t) \longrightarrow$ investment good

↳ 不要直接想成資本財, 它是專門用來增加資本財的另一種東西

} 這 2 資源卻無法直接互相 trade off, → 當期 C vs 未來 C

∴ 只能看它們的 input: $K^2 H^2$ vs $K^1 H^1$

$H^1_t + H^2_t + L_t = N_t \longrightarrow$ labor

$K^1_t + K^2_t = K_t \longrightarrow$ capital

分給 $H^2, K^2 \longrightarrow$ 增加當期 C 效用

L — 當期 L

H^1, K^1 — 未來 C, L

$K_{t+1} = I_t + (1-\delta)K_t$

↳ law of motion for capital

Q. trade off: 分給 H^1, K^1, 未來能增加多少 C, L?

A: ① ∵ time 只到未來不會增加 (N always 不變)

↳ 假設沒有人口成長

∴ 只需要看 capital

② ∵ capital 增加的方法就是靠 investment good

∴ need one eq connect I_t & K_{t+1}

actually $K^1_t K^2_t$ & $K^1_{t+1} K^2_{t+1}$

跟以前的 i

$C_t + i_t = f(k_t)$

決定存多少就直接可增加 k ↳ $k_{t+1} = i_t + (1-\delta)k_t$

↳ i.e. 雖然 investment good & capital good 不一樣,

但以前直接假設 investment good 能直接無縫轉成 capital,

∴ 只需考慮 c 和 i 如何抉擇. (c_t vs k_{t+1})

But 現在 investment 要必須轉地生產, ∴ 甚至要考慮生產划不划算

HH

$C_t + p i_t = w_t(h^1_t + h^2_t) + r_t(k^1_t + k^2_t) \to$ BC

$h^1_t + h^2_t = 1 \to$ time constrain

$k^1_t + k^2_t = k_t \to$ capital constrain

$i_t = k_{t+1} + (1-\delta)k_t \to$ 先想到 C vs i 的 trade off, 為了衡量跨期決策, need connect k_t & k_{t+1}

$$\boxed{w_t = w(K_t), \quad r_t = r(K_t), \quad K_{t+1} = G(K_t)} \to \text{解 DP 才需要}$$

Firm 1

$\max \; f^1(k^f_{1t}, h^f_{1t}) - r_t k^f_{1t} - w_t h^f_{1t}$

$w_t = w(K_t), \quad r_t = r(K_t)$

Firm 2

$\max \; f^2(k^f_{2t}, h^f_{2t}) - r_t k^f_{2t} - w_t h^f_{2t}$

$w_t = w(K_t), \quad r_t = r(K_t)$

Mkt clear

$N c_t = f^1$

$N i_t = f^2$

$N(h_1 + h_2) = h^f_1 + h^f_2$

$N k_t = k^f_1 + k^f_2$

From Eq.(7),

$$
\begin{aligned}
C_t &= f^2(K_t^2, H_t^2) \\
\implies Nc_t &= f^2(Nk_t^2, Nh_t^2) \\
\implies Nc_t &= Nf^2(k_t^2, h_t^2) \\
\implies c_t &= f^2(k_t^2, h_t^2)
\end{aligned}
$$

From Eq.(8),

$$
\begin{aligned}
K_{t+1} &= f^1(K_t^1, H_t^1) + (1-\delta)K_t \\
\implies Nk_{t+1} &= f^1(Nk_t^1, Nh_t^1) + (1-\delta)Nk_t \\
\implies Nk_{t+1} &= Nf^1(k_t^1, h_t^1) + (1-\delta)Nk_t \\
\implies k_{t+1} &= f^1(k_t^1, h_t^1) + (1-\delta)k_t \\
\implies k_{t+1} &= f^1(k_t - k_t^2, h_t^1) + (1-\delta)k_t
\end{aligned}
$$

Also,

$$
\begin{aligned}
N &= H_t^1 + H_t^2 + L_t \\
&= Nh_t^1 + Nh_t^2 + Nl_t \\
\implies 1 &= h_t^1 + h_t^2 + l_t
\end{aligned}
$$

Thus, the social planner's sequential problem can be written as

$$
\begin{aligned}
\max_{\{c_t, h_t^1, h_t^2, k_t^2, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t N u(c_t, 1 - h_t^1 - h_t^2) \\
\text{s.t.} \quad & c_t = f^2(k_t^2, h_t^2) \\
& k_{t+1} = f^1(k_t - k_t^2, h_t^1) + (1-\delta)k_t \\
& k_0 \text{ is given.}
\end{aligned}
$$

1. Specify the social planner's dynamic programming problem.

The state variable for planner is simply k and there is no need to separately keep k^1 and k^2 around because these two types of capital can be transformed into each other one-for-one.

The social planner's dynamic programming problem is

$$
\begin{aligned}
V(k) = \max_{c, h^1, h^2, k^2, k'} \quad & Nu(c, 1 - h^1 - h^2) + \beta V(k') \qquad (11) \\
\text{s.t.} \quad & c = f^2(k^2, h^2) \\
& k' = f^1(k - k^2, h^1) + (1-\delta)k \\
& k \text{ is given.}
\end{aligned}
$$

2. Define a recursive competitive equilibrium for the economy.

Household's problem

Define p as the relative price of investment goods. And we start from sequential problem for household:

i → 不消費 存下來的錢
⇓
How 存?
⇓
By capital (K) 的形式
存到下一期 ⇒ 下一期能生產更多東西
(可想像成理性人存錢不會單純放家裡，而是會放銀行貸款給其他人，此時貸款的錢如何生更多錢是靠 law of motion for capital.)

$$
\begin{aligned}
\max_{\{c_t, i_t, h_t^1, h_t^2, k_t^1, k_t^2, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t^1 - h_t^2) \\
\text{s.t.} \quad & c_t + p i_t = w(h_t^1 + h_t^2) + r(k_t^1 + k_t^2) \\
& k_{t+1} = i_t + (1-\delta)(k_t^1 + k_t^2) \\
& k_t = k_t^1 + k_t^2 \\
& k_0 \text{ is given.}
\end{aligned}
$$

Since $k_t = k_t^1 + k_t^2$ for all t,

$$
\begin{aligned}
\max_{\{c_t, i_t, h_t^1, h_t^2, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t^1 - h_t^2) \\
\text{s.t.} \quad & c_t + p i_t = w(h_t^1 + h_t^2) + r k_t \\
& k_{t+1} = i_t + (1-\delta) k_t \\
& k_0 \text{ is given.}
\end{aligned}
$$

It implies that there is no difference how we allocate k_t^1 and k_t^2 when we consider this maximization problem since r is same for investing in both firms. Then, combining two constraints yields

$$
\begin{aligned}
\max_{\{c_t, h_t^1, h_t^2, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t^1 - h_t^2) \\
\text{s.t.} \quad & c_t + p k_{t+1} = w(h_t^1 + h_t^2) + [r + p(1-\delta)] k_t \\
& k_0 \text{ is given.}
\end{aligned}
$$

The dynamic programming problem for household is a set of equations

$$
\begin{aligned}
V(K, k) = \max_{c, h^1, h^2, k'} \quad & u(c, 1 - h^1 - h^2) + \beta V(K', k') \\
\text{s.t.} \quad & c + p k' = w(h^1 + h^2) + [r + p(1-\delta)] k \\
& K' = G(K) \\
& w = w(K) \\
& r = r(K)
\end{aligned}
\tag{12}
$$

沒有H
∵ state variable 沒有 h
↓
∴ 預期價格只會受 K 影響
不受 h 影響

Firm I's problem

$$
\begin{aligned}
\max_{k_1^f, h_1^f} \quad & p f^1(k_1^f, h_1^f) - w h_1^f - r k_1^f \\
\text{s.t.} \quad & w = w(K) \\
& r = r(K)
\end{aligned}
\tag{13}
$$

Firm II's problem

$$\max_{k_2^f, h_2^f} \quad f^2(k_2^f, h_2^f) - wh_2^f - rk_2^f \tag{14}$$
$$\text{s.t.} \quad w = w(K)$$
$$r = r(K)$$

Recursive Competitive Equilibrium

A *Recursive Competitive Equilibrium* is

1. a value function: $V(K,k)$,
2. individual's decision rules: $c(K,k)$, $h^1(K,k)$, $h^2(K,k)$ and $k'(K,k)$,
3. decision rules for the firm I: $h_1^f(K)$ and $k_1^f(K)$,
4. decision rules for the firm II: $h_2^f(K)$ and $k_2^f(K)$,
5. pricing functions: $w(K)$ and $r(K)$,
6. a low of motion for aggregate state variable $K' = G(K)$,

such that

1) given $r(K)$, $w(K)$ and $G(K)$,

   $V(K,k)$, $c(K,k)$, $h^1(K,k)$, $h^2(K,k)$ and $k'(K,k)$ solve the household's dynamic programming problem (12).

2) given $r(K)$ and $w(K)$,;

   the decision rules $h_1^f(K)$ and $k_1^f(K)$ solve the firm I's problem (13).;

3) given $r(K)$ and $w(K)$,

   the decision rules $h_2^f(K)$ and $k_2^f(K)$ solve the firm II's problem (14);

4) market clearing condition:

   labor:

$$h_1^f(K) + h_2^f(K) = N\left[h_1(K,K) + h_2(K,K)\right];$$

   capital:

$$k_1^f(K) + k_2^f(K) = Nk = K;$$

consumption goods:

$$Nc(K,K) = f^2(k_2^f(K), h_2^f(K));$$

investment goods:

clear by Walras' Law or

$(1-\delta)$

$$Nk'(K,K) - Nk = f^1(k_1^f(K), h_1^f(K))$$

$$\implies Nk'(K,K) - K = f^1(k_1^f(K), h_1^f(K));$$

$(1-\delta)$

5) perceptions are correct:

$$K' = k'$$

$$\implies G(K) = k'(K,K).$$