NTU Macroeconomic Theory I

Homework 1

- The deadline is 9:00AM, November 29, 2021.
- Please submit to NTU COOL.
- Each student has to submit his/her own version.
- You DO NOT need to include the Matlab code in your homework.

1 Value Function Iteration

Consider the following more complicated problem:

$$\max \sum_{t=0}^{\infty} \beta^t (\log c_t - Bh_t), \quad 0 < \beta < 1$$

$$\text{s.t. } c_t + k_{t+1} \leq A k_t^{\theta} h_t^{1-\theta}, \quad 0 < \theta < 1$$

$$k_0 \text{ given}$$

where h_t is hours worked in period t; A and B are constant.

(1) Write down the Bellman equation. Use value function iteration to find the optimal law of motion, i.e., expressing k_{t+1} as a function of the state variable(s) and the optimal decision rule; expressing h_t as a function of the state variable(s).

(2) Derive the first order condition and envelope condition associated with the dynamic programming problem in (1). Use these to find the steady state capital stock and hours worked.

(3) Matlab exercise

a. Assume that $\beta = 0.9$, $A = 5$, $\theta = 0.33$, and $B = 1$. Grid $k \in [0.02, 10]$ with interval 0.1. Use Matlab to solve for the value function and optimal policy function. Provide the graphs for the value function and policy function in your homework. What is the value of the steady-state capital stock?

b. Repeat a using $k \in [0.02, 10]$ with interval 0.01. Compare your results to a.

c. Repeat a using $k \in [0.02, 2.4]$ with interval 0.1. Compare your results to a. What do you find?

(1) $$V(k) = \max_{k',h} \left\{ \ln\left(Ak^{\theta}h^{1-\theta} - k'\right) - Bh + \beta V(k') \right\}$$

Guess $v(k) = E + F\ln k$

then $$v(k) = \max_{k',h} \left\{ \ln\left(Ak^{\theta}h^{1-\theta} - k'\right) - Bh + \beta\left(E + F\ln k'\right) \right\}$$

FOC to k': $$\frac{1}{Ak^{\theta}h^{1-\theta} - k'} = \beta F \frac{1}{k'} \Rightarrow k' = \beta F Ak^{\theta}h^{1-\theta} - \beta F k'$$

$$\Rightarrow k' = \frac{\beta F}{1+\beta F} \cdot Ak^{\theta} \cdot h^{1-\theta}$$

FOC to h: $$\frac{(1-\theta)Ak^{\theta}h^{-\theta}}{Ak^{\theta}h^{1-\theta} - k'} = B \Rightarrow (1-\theta)Ak^{\theta}h^{-\theta} = BAk^{\theta}h^{1-\theta} - Bk'$$

$$\because \quad v(k) = \ln\left(\frac{1+\beta F}{\beta F}k' - k'\right) - Bh + \beta\left(E + F\ln k'\right)$$

$$= (1+\beta F)\ln k' - \ln \beta F - Bh + \beta E$$

$$= (1+\beta F)\theta \ln k + C$$, where C is a constant.

$$\therefore \quad F = (1+\beta F)\theta \Rightarrow F = \frac{\theta}{1-\beta\theta} > 0$$

[h]: $$(1-\theta)\cancel{Ak^{\theta}}h^{-\theta} = BAk^{\theta}h^{1-\theta} - B\frac{\beta F}{1+\beta F} \cdot Ak^{\theta} \cdot h^{1-\theta}$$

$$= B\left(\frac{1}{1+\beta F}\right)\cancel{Ak^{\theta}}h^{1-\theta}$$

$$\Rightarrow h = \frac{(1-\theta)(1+\beta F)}{B} = \frac{(1-\theta)F}{B\theta} = \frac{(1-\theta)\theta}{B\theta(1-\beta\theta)} = \frac{1-\theta}{B(1-\beta\theta)}$$ ※

[k']: $$k' = \frac{\beta F}{1+\beta F} \cdot Ak^{\theta} \cdot h^{1-\theta} = \frac{\theta}{\cancel{F}} \cdot \beta\cancel{F} Ak^{\theta} \cdot \left(\frac{1-\theta}{B(1-\beta\theta)}\right)^{1-\theta}$$

$$= A\beta\theta \cdot \left(\frac{1-\theta}{B(1-\beta\theta)}\right)^{1-\theta} k^{\theta}$$ ※

(2) $$V(k) = \max_{k',h} \left\{ \ln\left(Ak^{\theta}h^{1-\theta} - k'\right) - Bh + \beta V(k') \right\}$$

FOC to k': $$\frac{1}{Ak^{\theta}h^{1-\theta} - k'} = \beta V'(k')$$

to h: $$\frac{(1-\theta)Ak^{\theta}h^{-\theta}}{Ak^{\theta}h^{1-\theta} - k'} = B$$

Envelope condition: $$\cancel{V'(k') = \frac{\theta Ak^{\theta-1}h^{1-\theta}}{Ak^{\theta}h^{1-\theta} - k'}}$$

$$V'(k) \overset{\text{envelope thm}}{=} \frac{\partial U(k)}{\partial k} = \frac{\theta Ak^{\theta-1}h^{1-\theta}}{Ak^{\theta}h^{1-\theta} - k'}$$

$$\Rightarrow V'(k') = \frac{\theta Ak'^{\theta-1}h'^{1-\theta}}{Ak'^{\theta}h'^{1-\theta} - k''}$$

$\Rightarrow$ Euler equation: $$\frac{1}{Ak^{\theta}h^{1-\theta} - k'} = \beta \frac{\theta Ak^{\theta-1}h^{1-\theta}}{Ak^{\theta}h^{1-\theta} - k'}$$ 代入

In S-S, let $k = k' = k^*$, $h = h^*$,

then
$$\frac{1}{A k^{*\theta} h^{*1-\theta} - k^*} = \beta \frac{\theta A k^{*\theta-1} h^{*1-\theta}}{A k^{*\theta} h^{*1-\theta} - k^*} \Rightarrow 1 = \beta \theta A k^{*\theta-1} \cdot h^{*1-\theta} \quad \cdots (☆1)$$

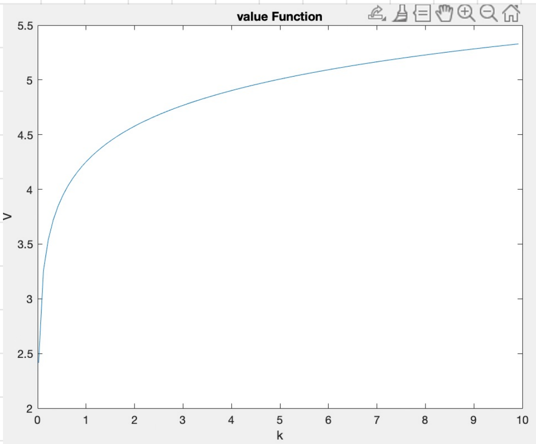
$$\frac{(1-\theta) A k^{*\theta} h^{*-\theta}}{A k^{*\theta} h^{*1-\theta} - k^*} = B \quad \cdots (☆2)$$

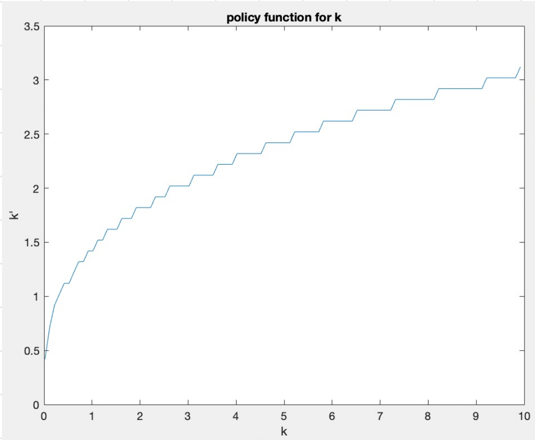
$$(☆1) + (☆2) \Rightarrow \frac{\frac{1}{\beta\theta} k^* \frac{1}{h^*}(1-\theta)}{\frac{1}{\beta\theta} \cdot k^* - k^*} = B \Rightarrow \frac{\frac{1}{\cancel{\beta\theta}} \cancel{k^*} \frac{1}{h^*}(1-\theta)}{\frac{1-\beta\theta}{\cancel{\beta\theta}} \cdot \cancel{k^*}} = B$$

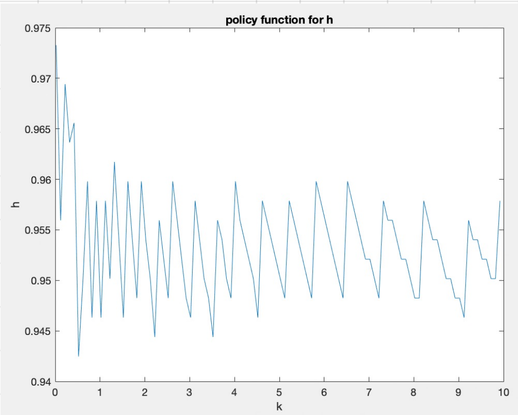
$$\Rightarrow h^* = \frac{1-\theta}{B(1-\beta\theta)} \quad \cdots (☆3)$$

$$(☆1) + (☆3) \Rightarrow k^* = \left[(A\beta\theta)^{-1} \cdot h^{*\theta-1}\right]^{\frac{1}{\theta-1}} = (A\beta\theta)^{\frac{1}{1-\theta}} \cdot \frac{1-\theta}{B(1-\beta\theta)}$$

(3) (a)



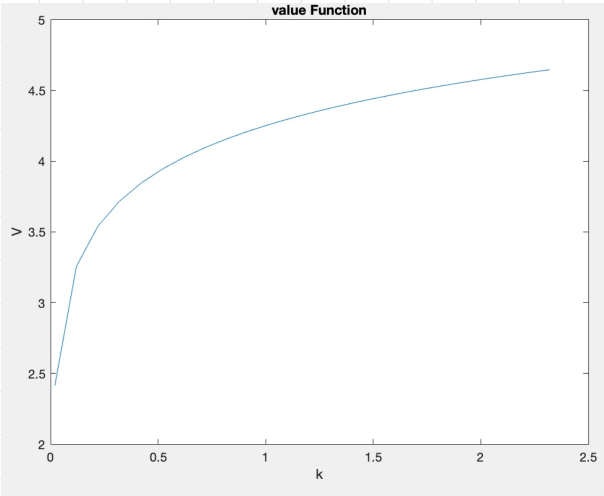




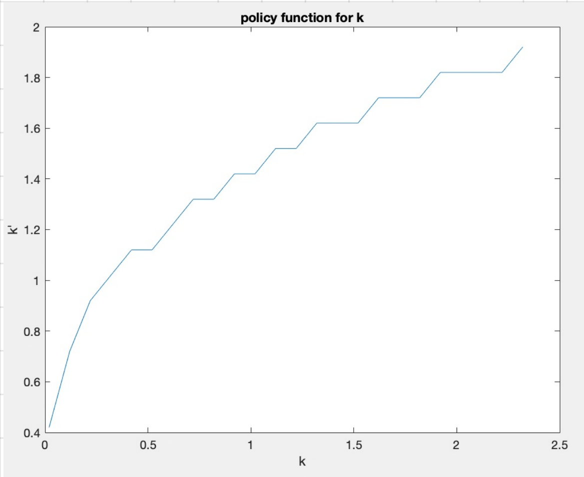
$k^* = 1.72$

(b) My computer ran code for twenty minute, and it cannot get the result.

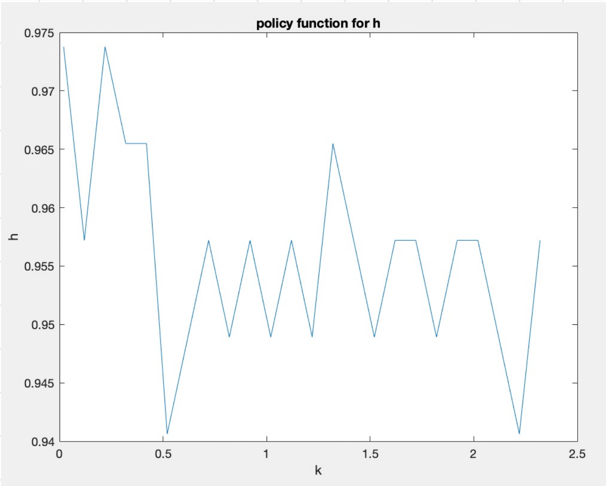
(c)



value Function
V
5
4.5
4
3.5
3
2.5
2
0
0.5
1
1.5
2
2.5
k



policy function for k
k'
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0
0.5
1
1.5
2
2.5
k



policy function for h
h
0.975
0.97
0.965
0.96
0.955
0.95
0.945
0.94
0
0.5
1
1.5
2
2.5
k