

Macro Theory I Part 2 - Assignment 2

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December 13, 2021

Question 1. Human Capital with Externality

(1) Write down a firm's profit maximization problem and derive a set of equations that characterize firm's optimal decision.

$$\max_{h_{1t}^f} (h_{1t}^f)^\alpha (\bar{h}_t)^\eta - w_t h_{1t}^f$$

F.O.C.:

$$[h_{1t}^f] : w_t = \alpha (h_{1t}^f)^{\alpha-1} (\bar{h}_t)^\eta \quad (1)$$

Eq.(1) characterizes firm's optimal decision.

(2) Write down a consumer's utility maximization problem and derive a set of equations that characterize consumer's optimal decision.

$$\begin{aligned} \max_{\{c_t, h_{1t}, h_{2t}, h_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \ln c_t \\ \text{s.t.} & \quad c_t = w_t h_{1t} \\ & \quad h_{t+1} = B h_{2t} \\ & \quad h_t = h_{1t} + h_{2t} \end{aligned}$$

Since

$$\begin{aligned} c_t &= w_t h_{1t} \\ &= w_t (h_t - h_{2t}) \\ &= w_t \left(h_t - \frac{h_{t+1}}{B} \right), \end{aligned}$$

the consumer's utility maximization problem can be written as

$$\begin{aligned} \max_{\{c_t, h_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \ln c_t \\ \text{s.t.} & \quad c_t = w_t \left(h_t - \frac{h_{t+1}}{B} \right) \end{aligned} \quad (2)$$

F.O.C.:

$$\begin{aligned} [c_t] : \frac{1}{c_t} &= \lambda_t \\ [h_{t+1}] : \frac{\lambda_t w_t}{B} &= \beta \lambda_{t+1} w_{t+1} \end{aligned}$$

Combining these two first order conditions gives us

$$\frac{c_{t+1}}{c_t} = \beta B \frac{w_{t+1}}{w_t} \quad (3)$$

Eq.(2) and Eq.(3) characterize the consumer's optimal decision.

(3) Find the growth rates along the balanced growth path.

At the equilibrium, $h_t = \bar{h}_t$ and $h_{1t} = h_{1t}^f$. Thus, from Eq.(3),

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \beta B \frac{w_{t+1}}{w_t} \\ &= \beta B \frac{\alpha(h_{1t+1}^f)^{\alpha-1}(\bar{h}_{t+1})^\eta}{\alpha(h_{1t}^f)^{\alpha-1}(\bar{h}_t)^\eta} \\ &= \beta B \frac{\alpha(h_{1t+1})^{\alpha-1}(h_{t+1})^\eta}{\alpha(h_{1t})^{\alpha-1}(h_t)^\eta} \\ &= \beta B \left(\frac{h_{1t+1}}{h_{1t}} \right)^{\alpha-1} \left(\frac{h_{t+1}}{h_t} \right)^\eta \end{aligned} \quad (4)$$

Look for the growth rate along the balanced growth path. Suppose that $c_t = g_c^t c^*$, $h_t = g_h^t h^*$, $h_{1t} = g_{h1}^t h_1^*$ and $h_{2t} = g_{h2}^t h_2^*$.

1. From $h_t = h_{1t} + h_{2t}$, we have

$$\begin{aligned} g_h^t h^* &= g_{h1}^t h_1^* + g_{h2}^t h_2^* \\ \Rightarrow \quad h^* &= \left(\frac{g_{h1}}{g_h} \right)^t h_1^* + \left(\frac{g_{h2}}{g_h} \right)^t h_2^* \end{aligned}$$

This equation is stationary if

$$g_h = g_{h1} = g_{h2}. \quad (5)$$

2. From Eq.(2),

$$\begin{aligned}
c_t &= w_t \left(h_t - \frac{h_{t+1}}{B} \right) \\
&= \alpha(h_{1t}^f)^{\alpha-1} (\bar{h}_t)^\eta \left(h_t - \frac{h_{t+1}}{B} \right) \\
&= \alpha(h_{1t})^{\alpha-1} (h_t)^\eta \left(h_t - \frac{h_{t+1}}{B} \right) \\
\Rightarrow g_c^t c^* &= \alpha(g_{h1}^t h_1^*)^{\alpha-1} (g_h^t h^*)^\eta \left(g_h^t h^* - \frac{g_h^{t+1} h^*}{B} \right) \\
&= \alpha h_1^{*\alpha-1} h^{*\eta} (g_{h1}^{\alpha-1} g_h^\eta)^t \left(g_h^t h^* - \frac{g_h^{t+1} h^*}{B} \right)
\end{aligned}$$

By Eq.(5), $g_h = g_{h1} = g_{h2}$ implies

$$\begin{aligned}
g_c^t c^* &= \alpha h_1^{*\alpha-1} h^{*\eta} g_h^{(\alpha+\eta-1)t} \left(g_h^t h^* - \frac{g_h^{t+1} h^*}{B} \right) \\
&= \alpha h_1^{*\alpha-1} h^{*\eta+1} g_h^{(\alpha+\eta)t} \left(1 - \frac{g_h}{B} \right) \\
\Rightarrow c^* &= \alpha h_1^{*\alpha-1} h^{*\eta+1} \left(\frac{g_h^{\alpha+\eta}}{g_c} \right)^t \left(1 - \frac{g_h}{B} \right).
\end{aligned}$$

The equation above is stationary if

$$g_c = g_h^{\alpha+\eta}. \quad (6)$$

3. From Eq.(4) and Eq.(5),

$$\begin{aligned}
\frac{c_{t+1}}{c_t} &= \beta B \left(\frac{h_{1t+1}}{h_{1t}} \right)^{\alpha-1} \left(\frac{h_{t+1}}{h_t} \right)^\eta \\
\Rightarrow \frac{g_c^{t+1} \hat{c}}{g_c^t \hat{c}} &= \beta B \left(\frac{g_{h1}^{t+1} \hat{h}_1}{g_{h1}^t \hat{h}_1} \right)^{\alpha-1} \left(\frac{g_h^{t+1} \hat{h}}{g_h^t \hat{h}} \right)^\eta \\
\Rightarrow g_c &= \beta B g_{h1}^{\alpha-1} g_h^\eta \\
&= \beta B g_h^{\alpha+\eta-1}.
\end{aligned}$$

Plugging g_c in Eq.(6) into the above equation yields

$$g_h^{\alpha+\eta} = \beta B g_h^{\alpha+\eta-1}.$$

Thus,

$$g_h = g_{h1} = g_{h2} = \beta B$$

and

$$g_c = (\beta B)^{\alpha+\eta}$$

Question 2. Two Alternative Technologies

(1) Formulate the social planner's dynamic programming problem.

The standard procedure is to transform the economy into a **stationary** one, starting from the resource constraint:

$$N_t c_t + K_{t+1} = \gamma^t K_t^\mu N_t^\phi L_t^{1-\mu-\phi}$$

Remove population growth:

Divide it by the **growth component** N_t , then we have

$$c_t + \frac{N_{t+1}}{N_t} \frac{K_{t+1}}{N_{t+1}} = \gamma^t \left(\frac{K_t}{N_t} \right)^\mu \left(\frac{N_t}{N_t} \right)^\phi \left(\frac{L_t}{N_t} \right)^{1-\mu-\phi}$$

Let $k_t \equiv \frac{K_t}{N_t}$ and $L_t = 1$ for all t , the above equation can be transformed as

$$c_t + \eta k_{t+1} = \gamma^t k_t^\mu \left(\frac{1}{\eta^t N_0} \right)^{1-\mu-\phi}$$

All variables are transformed to be per capita terms. Without loss of generality, I normalize N_0 to unity.

Remove technology growth:

For some variable x , define $\hat{x}_t = \frac{x_t}{g_x^t}$, where g_x is the growth rate of x along the balanced growth path.

The above resource constraint can be rewritten as

$$g_c^t \hat{c}_t + \eta g_k^{t+1} \hat{k}_{t+1} = \left(\frac{\gamma}{\eta^{1-\mu-\phi}} \right)^t \left(g_k^t \hat{k}_t \right)^\mu$$

Divide it by g_c^t :

$$\hat{c}_t + \eta g_k \left(\frac{g_k}{g_c} \right)^t \hat{k}_{t+1} = \left[\left(\frac{\gamma}{\eta^{1-\mu-\phi}} \right) \left(\frac{g_k^\mu}{g_c} \right) \right]^t \hat{k}_t^\mu \quad (7)$$

Eq.(7) is stationary if

$$g_k = g_c$$

and

$$\gamma g_k^\mu = \eta^{1-\mu-\phi} g_c$$

These two equations imply

$$g_c = g_k = \left(\frac{\gamma}{\eta^{1-\mu-\phi}} \right)^{\frac{1}{1-\mu}} \equiv g,$$

where g is common growth rate. Thus, resource constraint, Eq.(7), can be rewritten as

$$\hat{c}_t + \eta g_k \left(\frac{g_k}{g_c} \right)^t \hat{k}_{t+1} = \left[\left(\frac{\gamma}{\eta^{1-\mu-\phi}} \right) \left(\frac{g_k^\mu}{g_c} \right) \right]^t \hat{k}_t^\mu$$

$$\implies \hat{c}_t + \eta g_k \hat{k}_{t+1} = \hat{k}_t^\mu$$

$$\implies \hat{c}_t + (\gamma \eta^\phi)^{\frac{1}{1-\mu}} \hat{k}_{t+1} = \hat{k}_t^\mu$$

We now move to the utility function. Since $N_t = \eta^t N_0 = \eta^t$ and $c_t = \hat{c}_t g^t$, the utility function can be rewritten as

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t N_t \log c_t &= \sum_{t=0}^{\infty} (\beta\eta)^t \log(\hat{c}_t g^t) \\ &= \sum_{t=0}^{\infty} (\beta\eta)^t \log \hat{c}_t + \sum_{t=0}^{\infty} (\beta\eta)^t t \log g \\ &= \sum_{t=0}^{\infty} (\beta\eta)^t \log \hat{c}_t + \frac{\beta\eta}{(1-\beta\eta)^2} \log g, \end{aligned}$$

where $\beta\eta < 1$ and

$$\begin{aligned} \sum_{t=0}^{\infty} (\beta\eta)^t t \log g &= [\beta\eta + 2(\beta\eta)^2 + 3(\beta\eta)^3 + \dots] \log g \\ &= \beta\eta[1 + 2\beta\eta + 3(\beta\eta)^2 + \dots] \log g \\ &= \frac{\beta\eta}{1-\beta\eta} [(1-\beta\eta) + 2\beta\eta(1-\beta\eta) + 3(\beta\eta)^2(1-\beta\eta) + \dots] \log g \\ &= \frac{\beta\eta}{1-\beta\eta} [1 - \beta\eta + 2\beta\eta - 2(\beta\eta)^2 + 3(\beta\eta)^2 - 3(\beta\eta)^3 + \dots] \log g \\ &= \frac{\beta\eta}{1-\beta\eta} [1 + \beta\eta + (\beta\eta)^2 + (\beta\eta)^3 + \dots] \log g \\ &= \frac{\beta\eta}{(1-\beta\eta)^2} \log g. \end{aligned}$$

Finally, the social planner's dynamic programming problem is

$$\begin{aligned} V(\hat{k}) &= \max_{\hat{c}, \hat{k}'} \log \hat{c} + \beta\eta V(\hat{k}') \\ \text{s.t.} \quad &\hat{c} + (\gamma\eta^\phi)^{\frac{1}{1-\mu}} \hat{k}' = \hat{k}^\mu \end{aligned}$$

(2) Characterize the balanced growth path of this economy. Solve explicitly for the growth rate of per capita consumption (c_t) along this path.

Replacing $\hat{c} = \hat{k}^\mu - (\gamma\eta^\phi)^{\frac{1}{1-\mu}} \hat{k}'$, the social planner's dynamic programming problem becomes

$$V(\hat{k}) = \max_{\hat{k}'} \log \left[\hat{k}^\mu - (\gamma\eta^\phi)^{\frac{1}{1-\mu}} \hat{k}' \right] + \beta\eta V(\hat{k}')$$

F.O.C.:

$$[\hat{k}'] : \beta\eta V'(\hat{k}') = \frac{(\gamma\eta^\phi)^{\frac{1}{1-\mu}}}{\hat{k}^\mu - (\gamma\eta^\phi)^{\frac{1}{1-\mu}} \hat{k}'} \quad (8)$$

Envelope condition:

$$\begin{aligned} V'(\hat{k}) &= \frac{\mu \hat{k}^{\mu-1}}{\hat{k}^\mu - (\gamma\eta^\phi)^{\frac{1}{1-\mu}} \hat{k}'} \\ \implies V'(\hat{k}') &= \frac{\mu \hat{k}'^{\mu-1}}{\hat{k}'^\mu - (\gamma\eta^\phi)^{\frac{1}{1-\mu}} \hat{k}''} \quad (9) \end{aligned}$$

Combining Eq.(8) and Eq.(9) gives us Euler equation:

$$\beta\eta\frac{\mu\hat{k}'^{\mu-1}}{\hat{k}'^{\mu} - (\gamma\eta^{\phi})^{\frac{1}{1-\mu}}\hat{k}''} = \frac{(\gamma\eta^{\phi})^{\frac{1}{1-\mu}}}{\hat{k}^{\mu} - (\gamma\eta^{\phi})^{\frac{1}{1-\mu}}\hat{k}}.$$

At the steady state, $\hat{k} = \hat{k}' = \hat{k}'' \equiv \hat{k}^*$ and $\hat{c} = \hat{c}' = \hat{c}'' \equiv \hat{c}^*$. The Euler equation at the steady state is

$$\beta\eta\mu\hat{k}^{*\mu-1} = (\gamma\eta^{\phi})^{\frac{1}{1-\mu}}.$$

It implies that

$$\hat{k}^* = \left[\frac{\beta\mu}{(\gamma\eta^{\phi+\mu-1})^{\frac{1}{1-\mu}}} \right]^{\frac{1}{1-\mu}},$$

and

$$\hat{c}^* = \hat{k}^{*\mu} - (\gamma\eta^{\phi})^{\frac{1}{1-\mu}}\hat{k}^*.$$

(3) Repeat (1) and (2) using the technology instead: $Y_t = \gamma^t K_t^{\theta} N_t^{1-\theta}$.

The standard procedure is to transform the economy into a **stationary** one, starting from the resource constraint:

$$N_t c_t + K_{t+1} = \gamma^t K_t^{\theta} N_t^{1-\theta}$$

Divide it by the **growth component** N_t , then we have

$$c_t + \frac{N_{t+1}}{N_t} \frac{K_{t+1}}{N_{t+1}} = \gamma^t \left(\frac{K_t}{N_t} \right)^{\theta} \left(\frac{N_t}{N_t} \right)^{1-\theta}$$

Let $k_t \equiv \frac{K_t}{N_t}$ for all t , the above equation can be transformed as

$$c_t + \eta k_{t+1} = \gamma^t k_t^{\theta}$$

All variables are transformed to be per capita terms. Without loss of generality, I normalize N_0 to unity. For some variable x , define $\tilde{x}_t = \frac{x_t}{r_x^t}$, where r_x is the growth rate of x along the balanced growth path. The above resource constraint can be rewritten as

$$r_c^t \tilde{c}_t + \eta r_k^{t+1} \tilde{k}_{t+1} = \gamma^t \left(r_k^t \tilde{k}_t \right)^{\theta}$$

Divide it by r_c^t :

$$\tilde{c}_t + \eta r_k \left(\frac{r_k}{r_c} \right)^t \tilde{k}_{t+1} = \left[\frac{\gamma r_k^{\theta}}{r_c} \right]^t \tilde{k}_t^{\theta} \quad (10)$$

Eq.(10) is stationary if

$$r_k = r_c$$

and

$$\gamma r_k^\theta = r_c$$

These two equations imply

$$r_c = r_k = \gamma^{\frac{1}{1-\theta}} \equiv r,$$

where g is common growth rate. Thus, resource constraint, Eq.(10), can be rewritten as

$$\begin{aligned} \tilde{c}_t + \eta r_k \left(\frac{r_k}{r_c} \right)^t \tilde{k}_{t+1} &= \left[\frac{\gamma r_k^\theta}{r_c} \right]^t \tilde{k}_t^\theta \\ \implies \tilde{c}_t + \eta r_k \tilde{k}_{t+1} &= \tilde{k}_t^\theta \\ \implies \tilde{c}_t + \eta \gamma^{\frac{1}{1-\theta}} \tilde{k}_{t+1} &= \tilde{k}_t^\theta \end{aligned}$$

We now move to the utility function. Since $N_t = \eta^t N_0 = \eta^t$ and $c_t = \tilde{c}_t r^t$, the utility function can be rewritten as

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t N_t \log c_t &= \sum_{t=0}^{\infty} (\beta \eta)^t \log(\tilde{c}_t r^t) \\ &= \sum_{t=0}^{\infty} (\beta \eta)^t \log \tilde{c}_t + \sum_{t=0}^{\infty} (\beta \eta)^t t \log r \\ &= \sum_{t=0}^{\infty} (\beta \eta)^t \log \tilde{c}_t + \frac{\beta \eta}{(1 - \beta \eta)^2} \log r, \end{aligned}$$

where $\beta \eta < 1$.

Finally, the social planner's dynamic programming problem is

$$V(\tilde{k}) = \max_{\tilde{k}'} \log \left[\tilde{k}^\theta - \eta \gamma^{\frac{1}{1-\theta}} \tilde{k}' \right] + \beta \eta V(\tilde{k}')$$

F.O.C.:

$$[\tilde{k}'] : \beta \eta V(\tilde{k}') = \frac{\eta \gamma^{\frac{1}{1-\theta}}}{\tilde{k}^\theta - \eta \gamma^{\frac{1}{1-\theta}} \tilde{k}'} \quad (11)$$

Envelope condition:

$$\begin{aligned} V'(\tilde{k}) &= \frac{\theta \tilde{k}^{\theta-1}}{\tilde{k}^\theta - \eta \gamma^{\frac{1}{1-\theta}} \tilde{k}'} \\ \implies V'(\tilde{k}') &= \frac{\theta \tilde{k}'^{\theta-1}}{\tilde{k}'^\theta - \eta \gamma^{\frac{1}{1-\theta}} \tilde{k}''} \quad (12) \end{aligned}$$

Combining Eq.(11) and Eq.(12) gives us Euler equation:

$$\beta \eta \frac{\theta \tilde{k}'^{\theta-1}}{\tilde{k}'^\theta - \eta \gamma^{\frac{1}{1-\theta}} \tilde{k}''} = \frac{\eta \gamma^{\frac{1}{1-\theta}}}{\tilde{k}^\theta - \eta \gamma^{\frac{1}{1-\theta}} \tilde{k}'}.$$

Evaluating Euler equation at steady state yields the result:

$$\tilde{k}^* = \left(\frac{\beta\theta}{\gamma^{\frac{1}{1-\theta}}} \right)^{\frac{1}{1-\theta}},$$

and

$$\tilde{c}^* = \tilde{k}^{*\theta} - \eta\gamma^{\frac{1}{1-\theta}} \tilde{k}^*.$$

(4) Compare how the rate of population growth η affects the rate of per capita growth in the two cases and provide the intuition.

The growth rate in the first case:

$$g = \left(\frac{\gamma}{\eta^{1-\mu-\phi}} \right)^{\frac{1}{1-\mu}}.$$

The growth rate in the second case:

$$r = \gamma^{\frac{1}{1-\theta}}.$$

The population growth rate affects only the growth rate in the first case, g . It is because the production is bounded by a fixed factor of production, land. While the population grows land per capita decreases, slowing down the growth rate.