Problem 1

1. **Method 1: traditional preferences**

   Consider a tuple of binary relations $\big(P, I, I^*, I'\big)$, defined as follows:

$$\begin{aligned}
&x\,P\,y \quad \text{if} \;\; Q^*(x,y) = \text{“I prefer } x \text{ to } y.\text{”}\\
&y\,P\,x \quad \text{if} \;\; Q^*(x,y) = \text{“I prefer } y \text{ to } x.\text{”}\\
&x\,I^*\,y \quad \text{if} \;\; Q^*(x,y) = \text{“I am indifferent.”}\\
&x\,I'\,y \quad \text{if} \;\; Q^*(x,y) = \text{“I don't know.”}\\
&x\,I\,y \quad \text{if} \;\; x\,I^*\,y \;\; \text{or} \;\; x\,I'\,y.
\end{aligned}$$

   We impose three restrictions on $\big(P, I\big)$:

   (1) A function f assigns to any pair (x, y) of distinct elements in X exactly one of the three "values" $x\,P\,y$, $y\,P\,x$, or I.

   (2) No order effect: $f(x, y) = f(y, x)$.

   (3) Transitivity:
   if $f(x, y) = x\,P\,y$ and $f(y, z) = y\,P\,z$, then $f(x, z) = x\,P\,z$ and
   if $f(x, y) = I$ and $f(y, z) = I$, then $f(x, z) = I$.

   Note that these restrictions are consistent with Definition 1 in Lecture Notes p. 3.

Method 2

Consider a tuple of binary relations $\big(P, I, K\big)$, defined as follows:

$$\begin{aligned}
x\,P\,y &\quad \text{if} \;\; Q^*(x,y) = \text{“I prefer } x \text{ to } y.\text{”} \\
y\,P\,x &\quad \text{if} \;\; Q^*(x,y) = \text{“I prefer } y \text{ to } x.\text{”} \\
x\,I\,y &\quad \text{if} \;\; Q^*(x,y) = \text{“I am indifferent.”} \\
x\,K\,y &\quad \text{if} \;\; Q^*(x,y) = \text{“I don't know.”}.
\end{aligned}$$

We impose three restrictions on $\big(P, I, K\big)$:

(1) A function f assigns to any pair (x, y) of distinct elements in X exactly one of the four "values" $x\,P\,y$, $y\,P\,x$, I, K.

(2) No order effect: $f(x, y) = f(y, x)$.

(3) Transitivity*:
if $f(x, y) = x\,P\,y$ and $f(y, z) = y\,P\,z$, then $f(x, z) = x\,P\,z$ and
if $f(x, y) = I$ or K and $f(y, z) = I$ or K, then $f(x, z) = I$ or K.

2. **Method 1: traditional preferences**

Consider a tuple of binary relations $\left(\succsim, \succsim^*, \succsim' \right)$, defined as follows:

$$\begin{aligned} x &\succsim^* y \quad \text{if} \quad R^*(x,y) = \text{“Yes.”} \\ x &\succsim' y \quad \text{if} \quad R^*(x,y) = \text{“I don't know.”} \\ x &\succsim y \quad \text{if} \quad x \succsim^* y \quad \text{or} \quad x \succsim' y. \end{aligned}$$

We impose three restrictions on $\succsim$:

(1) Reflexivity: For any $x \in X$, $x \succsim x$.

(2) Completeness: For any x, $y \in X$, $x \succsim y$ or $y \succsim x$.

(3) Transitivity: For any x, y, $z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Note that these restrictions are consistent with Definition 2 in Lecture Notes p. 6.

Method 2

Consider a tuple of binary relations $\left(\succsim, ? \right)$, defined as follows:

$$\begin{aligned} x &\succsim y \quad \text{if} \quad R^*(x,y) = \text{“Yes.”} \\ x &\,?\, y \quad \text{if} \quad R^*(x,y) = \text{“I don't know.”} \end{aligned}$$

We could impose some restrictions on $\left(\succsim, ? \right)$:

(1) Reflexivity: For any $x \in X$, $x \succsim x$.

(2) Completeness (optional):

For any x, $y \in X$, at least one of the following four conditions hold:

$$\begin{aligned} &x \succsim y, \\ &y \succsim x, \\ &x \,?\, y, \\ &y \,?\, x. \end{aligned}$$

(3) Complementary Completeness (optional):

For any x, $y \in X$, at least one of the following three conditions hold:

$$x \succsim y,$$
$$y \succsim x,$$
$$x\,?\,y \text{ and } y\,?\,x.$$

(4) ?-Symmetry (optional): For any x, $y \in X$, if $x\,?\,y$, then $y\,?\,x$.

Note that Complementary Completeness does not imply Symmetry.

(5) Mutual Exclusion: For any x, $y \in X$, if $x \succsim y$, then not $x\,?\,y$.

(6) $\succsim$-Transitivity (optional): For any x, y, $z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

(7) ?-Transitivity (optional): For any x, y, $z \in X$, if $x\,?\,y$ and $y\,?\,z$, then $x\,?\,z$.

(8) Transitivity (optional):

For any x, y, $z \in X$, if $[x \succsim y$ or $x\,?\,y]$ and $[y \succsim z$ or $y\,?\,z]$, then $[x \succsim z$ or $x\,?\,z]$.

I think some kind of transitivity should be imposed on $\big(\succsim, ?\big)$, but it needs not be one of the three.

Method 3

We abandon Completeness as an approach to respect the answer "I don't know."

Consider a binary relation $\succsim$, defined as follows:

$$x \succsim y \quad \text{if} \quad R^*(x, y) = \text{"Yes."}$$

We impose two restrictions on $\succsim$:

(1) Reflexivity: For any $x \in X$, $x \succsim x$.

(2) Transitivity: For any x, y, $z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

I don't think this is an appropriate one since in this case, not $x \succsim y$ implies that the answer could be "No" or "I don't know."

3. Before we go into detail about each possible method, we first make an observation. Suppose we impose No Order Effect on $Q^*(x, y)$ but do not impose ?-Symmetry on $R^*(x, y)$. Then we would encounter the following issue: The DM answers "I don't know" to $Q^*(x, y)$ and hence is required to have same answer to $Q^*(y, x)$ by No Order Effect. In contrast, it is possible that she answers "I don't know" to $R^*(x, y)$ while answers "Yes" to $R^*(y, x)$. This issue would lead to an impossibility of finding a correspondence that preserves the same interpretation.

   As a result, if we want to show two definitions are equivalent, then we need to impose No Order Effect (?-Symmetry) on both binary relations induced from Q^* and R^*. Now we would like to use an example to illustrate how to show two definitions are equivalent. The definitions we will use are Method 2 in part 1 (Definition 1) and Method 2 in part 2 with restrictions (1), (3), (4) and (8) (Definition 2). To show they are equivalent, we need to do two things:

   (1) Construct a bijective correspondence.

   (2) Argue this correspondence preserves the interpretation.

   We begin with a table indicating the correspondence for a particular pair (x, y).

$Q^*(x, y) = Q^*(y, x)$	$R^*(x, y)$	$R^*(y, x)$
$x\,P\,y$	Yes ($x \succsim y$)	No
$y\,P\,x$	No	Yes ($y \succsim x$)
I	Yes ($x \succsim y$)	Yes ($y \succsim x$)
K	? ($x\,?\,y$)	? ($y\,?\,x$)

   Note that by Complementary Completeness and ?-Symmetry, four combinations shown in the table are all possible outcomes of R^* for a particular pair (x, y). Denote the correspondence by T. Let f be a preference according to Definition 1. By No Order Effect, the transformation of a particular pair (a, b) is well-defined.

   Step (1)-1: $T(f)$ falls in the desired domain.

   The first step is to show $T(f)$ is indeed a qualified preference under Definition 2, i.e., $T(f)$ falls in the desired codomain. We need to verify restrictions (1), (3), (4) and (8). Since there is no question like (a, a) in Q^*, we just define $a\,T(f)\,a \;=\; a \succsim$

a, and hence Reflexivity holds. By the above table, $T(f)$ satisfies Complementary Completeness and ?-Symmetry. To show Transitivity, assume that the answers $T(f)$ gives to $R^*(a,b)$ and $R^*(b,c)$ are either "Yes" or "I don't know". By the above table, the answer f gives to $Q^*(a,b)$ is $a\,P\,b$, I or K. Similarly, the answer f gives to $Q^*(b,c)$ is $b\,P\,c$, I or K. Assume, for the sake of contradiction, that the answer f gives to $Q^*(a,c)$ is $c\,P\,a$. By Transitivity*, to avoid a direct contradiction, we must have either $\left[Q^*(a,b) = a\,P\,b,\ Q^*(b,c) = I \text{ or } K\right]$ or $\left[Q^*(a,b) = I \text{ or } K,\ Q^*(b,c) = b\,P\,c\right]$. In the former case, by Transitivity*, we have $c\,P\,b$, a contradiction. In the latter case, by Transitivity*, we have $b\,P\,a$, a contradiction. This implies that the answer f gives to $Q^*(a,c)$ is $a\,P\,c$, I or K. Hence, by the above table, the answer $T(f)$ gives to $R^*(a,c)$ is either "Yes" or "I don't know" and thus Transitivity holds.

Step (1)-2: T is one-to-one.

The second step is to show T is one-to-one. Note that for any two different responses f_1 and f_2 to Q^*, there exists a pair (a,b) such that f_1 differs from f_2 in $Q^*(a,b)$. This implies that $T(f_1)$ differs from $T(f_2)$ in either $R^*(a,b)$ or $R^*(b,a)$ (or both). Hence, T is one-to-one.

Step (1)-3: T is onto.

The last step is to show T is onto, i.e., for any $\succsim$, a qualified preference under Definition 2, there exist f, a qualified preference under definition 1, such that $T(f) = \succsim$. A candidate f is given by the table and we need to verify this f is indeed a qualified preference under definition 1. We can see from the table that this f assigns to any pair (x,y) exactly one of the four "values" $x\,P\,y$, $y\,P\,x$, I, K and this f also satisfies No Order Effect. To show Transitivity*, assume first $f(a,b) = a\,P\,b$ and $f(b,c) = b\,P\,c$. Then $a \succsim b$, not $b \succsim a$ and $b \succsim c$, not $c \succsim b$. By Transitivity, $a \succsim c$ and not $c \succsim a$ since if $c \succsim a$, then we would have $c \succsim b$ by transitivity, a contradiction. Thus, $f(a,c) = a\,P\,c$ and this completes the first statement of Transitivity*. Suppose now $f(a,b) = I$ or K and $f(b,c) = I$ or K. If $f(a,b) = I$ and $f(b,c) = I$, then by Transitivity and ?-Symmetry, either $[a \succsim c$ and $c \succsim a]$ or $[a \,?\, c$ and $c \,?\, a]$. Thus, Transitivity* holds in this case. Similarly, Transitivity* holds in the other three cases.

Step (2): Argue this correspondence preserves the interpretation.

It remains to argue the correspondence preserves the same interpretation. The interesting part is $f(x, y) = K$ as the other three cases have been justified in the lecture notes. In general, we do not have $f(x, y)$ corresponds to $x \,?\, y$ and $y \,?\, x$ unless we impose ?-Symmetry on $(\succsim, ?)$ or we weaken No Order Effect. In our case, we choose to impose ?-Symmetry, and thus we avoid the case such as that the DM answers "I don't know" to $R^*(x, y)$ while answers "Yes" to $R^*(y, x)$.

Problem 2

1. **(Method 1)**

Suppose Convexity 1 holds and $x \sim y \sim z$ with $x_1 > y_1 > z_1$. Pick α satisfies

$$\alpha x_1 + (1-\alpha) z_1 = y_1 \Rightarrow \alpha = \frac{y_1 - z_1}{x_1 - z_1}\,.$$

Then by Convexity 1, we have for all $0 < \epsilon < \alpha$,

$$(\alpha - \epsilon)x + (1 - \alpha + \epsilon)z \succsim z \sim y\,.$$

Now, we want to show $\alpha x_2 + (1-\alpha)z_2 \geq y_2$. Suppose instead $\alpha x_2 + (1-\alpha)z_2 < y_2$. Since $x \sim y \sim z$ with $x_1 > y_1 > z_1$, by Monotonicity, we must have $x_2 \leq y_2 \leq z_2$. It must be the case that $x_2 < z_2$ since if $x_2 = z_2$, then $\alpha x_2 + (1-\alpha)z_2 = y_2$. For $\epsilon > 0$ small enough, we still have $(\alpha - \epsilon)x_2 + (1 - \alpha + \epsilon)z_2 < y_2$. Along with $(\alpha - \epsilon)x_1 + (1 - \alpha + \epsilon)z_1 < y_1$, we have by Monotonicity, $y \succ (\alpha - \epsilon)x + (1 - \alpha + \epsilon)z$, a contradiction.

Finally,

$$\begin{aligned}
\alpha x_2 + (1-\alpha)z_2 &\geq y_2 \\
\frac{y_1 - z_1}{x_1 - z_1}\,x_2 + \frac{x_1 - y_1}{x_1 - z_1}\,z_2 &\geq y_2 \\
(y_1 - z_1)\,x_2 &\geq (x_1 - z_1)\,y_2 - (x_1 - y_1)\,z_2\,.
\end{aligned}$$

Subtract $(y_1 - z_1)\,y_2$ from both side and we have

$$\begin{aligned}
(y_1 - z_1)\,x_2 - (y_1 - z_1)\,y_2 &\geq (x_1 - z_1)\,y_2 - (x_1 - y_1)\,z_2 - (y_1 - z_1)\,y_2 \\
(y_1 - z_1)(x_2 - y_2) &\geq (x_1 - y_1)(y_2 - z_2) \\
\frac{x_2 - y_2}{x_1 - y_1} &\geq \frac{y_2 - z_2}{y_1 - z_1}\,.
\end{aligned}$$

□

(Method 2)

Definition (Convexity 3)**.** *The preference relation $\succsim$ satisfies Convexity 3 if for all y the set $AsGoodAs(y) = \{\, z \in X \mid z \succsim y \,\}$ is convex.*

Fact 1. *Convexity 1 and Convexity 3 are equivalent.*

Fact 2. *A function is convex if and only if the set of points lying on or above its graph is convex.*

Fact 3 (Three String Lemma)**.** *If f is a convex function, then for any $a < b < c$,*

$$\frac{f(c)-f(b)}{c-b} \geq \frac{f(c)-f(a)}{c-a} \geq \frac{f(b)-f(a)}{b-a}$$

Fact 4. *A convex real-valued function f defined on (a, b) is continuous on (a, b).*

Now we start working on this problem. Suppose Convexity 3 holds and $x \sim y \sim z$ with $x_1 > y_1 > z_1$. Fix a point $a_1 \in (z_1, x_1)$ and consider $F(a_1) = \{\, (a_1, a_2) \mid (a_1, a_2) \sim y \,\}$. By Monotonicity, $(a_1, z_2) \succsim z \sim y \sim x \succsim (a_1, x_2)$ and thus by Continuity, $F(a_1)$ is non-empty and closed. (It would be easier if here we further assume Strict Monotonicity holds.) Along with the fact that $F(a_1)$ is bounded below by 0, we have $f(a_1) \equiv \min\limits_{a_2} F(a_1)$ exists.

We want to show f is convex and y is in the graph of f. Consider f on the open interval (z_1, x_1). By Monotonicity, every point lying above f is weakly preferred to y and y is strictly preferred to every point lying below f. Hence, by Convexity 3 and Fact 2, f is convex and thus is continuous on (z_1, x_1) by Fact 4.

Suppose y is not in the graph of f, i.e., either $y_2 > f(y_1)$ or $y_2 < f(y_1)$. Assume $y_2 > f(y_1)$. Since f is continuous, there exists $\epsilon > 0$ and a point $b = \big(b_1, f(b_1)\big)$ such that $b_1 < y_1$ and $f(b_1) < f(y_1) + \epsilon < y_2$. By Monotonicity, $y \succ b$, a contradiction. Similarly, assume instead $y_2 < f(y_1)$. Pick $c = \big(c_1, f(c_1)\big)$ such that $c_1 > y_1$ and $f(c_1) > y_2$. Such c exists since f is continuous. By Monotonicity, $c \succ y$, a contradiction.

We have shown $y = \big(y_1, f(y_1)\big)$. By Fact 4 with $x_2 \geq f(x_2)$ and $z_2 \geq f(z_2)$, we have

$$\frac{x_2 - y_2}{x_1 - y_1} \geq \frac{f(x_1) - f(y_1)}{x_1 - y_1} \geq \frac{f(y_1) - f(z_1)}{y_1 - z_1} \geq \frac{y_2 - z_2}{y_1 - z_1}$$

$\square$

2. Suppose Slope-Convexity holds and $x \succsim y$. Assume first $x \sim y$. By Monotonicity, neither $\big[\, x_i > y_i$ for all $i \,\big]$ nor $\big[\, x_i < y_i$ for all $i \,\big]$, so we have four possible cases.

Case 1 $x_1 > y_1$ and $x_2 \leq y_2$

By Monotonicity,

$$\Big(\alpha x_1 + (1-\alpha) y_1,\, y_2\Big) \succsim y \sim x \succsim \Big(\alpha x_1 + (1-\alpha) y_1,\, x_2\Big).$$

By Continuity, there exists $z_2 \in [\, x_2, y_2\,]$ such that $z \equiv \Big(\alpha x_1 + (1-\alpha) y_1,\, z_2\Big) \sim x$. Now we have $x \sim z \sim y$ with $x_1 > z_1 > y_1$. By Slope-Convexity,

$$\frac{z_2 - y_2}{z_1 - y_1} \leq \frac{x_2 - z_2}{x_1 - z_1}\,.$$

Plug in $z_1 = \alpha x_1 + (1-\alpha) y_1$ and we have

$$\begin{aligned} \frac{z_2 - y_2}{\alpha(x_1 - y_1)} &\leq \frac{x_2 - z_2}{(1-\alpha)(x_1 - y_1)} \\ (1-\alpha)(z_2 - y_2) &\leq \alpha(x_2 - z_2) \\ z_2 &\leq \alpha x_2 + (1-\alpha) y_2\,. \end{aligned}$$

Hence, by Monotonicity, $\alpha x + (1-\alpha) y \succsim z \sim y$, as desired.

Case 2 $x_1 = y_1$ and $x_2 \leq y_2$

Suppose not, i.e., there exists α such that $y \succ \alpha x + (1-\alpha) y \equiv z$. Then $x \sim y \succ z$, but by Monotonicity, $z \succsim x$, a contradiction.

Case 3 $x_1 = y_1$ and $x_2 > y_2$

By Monotonicity, $\alpha x + (1-\alpha) y \succsim y$

Case 4 $x_1 < y_1$ and $x_2 \geq y_2$

Case 4 is similar to Case 1.

By Monotonicity,

$$\Big(\alpha x_1 + (1-\alpha)y_1,\, x_2\Big) \succsim x \sim y \succsim \Big(\alpha x_1 + (1-\alpha)y_1,\, y_2\Big).$$

By Continuity, there exists $z_2 \in [\,y_2, x_2\,]$ such that $z \equiv \Big(\alpha x_1 + (1-\alpha)y_1,\, z_2\Big) \sim y$. Now we have $y \sim z \sim x$ with $y_1 > z_1 > x_1$. By Slope-Convexity,

$$\frac{z_2 - x_2}{z_1 - x_1} \leq \frac{y_2 - z_2}{y_1 - z_1}\,.$$

Plug in $z_1 = \alpha x_1 + (1-\alpha)y_1$ and we have

$$\begin{aligned}
\frac{z_2 - x_2}{(1-\alpha)(y_1 - x_1)} &\leq \frac{y_2 - z_2}{\alpha(y_1 - x_1)} \\
\alpha(z_2 - x_2) &\leq (1-\alpha)(y_2 - z_2) \\
z_2 &\leq \alpha x_2 + (1-\alpha)y_2\,.
\end{aligned}$$

Hence, by Monotonicity, $\alpha x + (1-\alpha)y \succsim z \sim y$, as desired.

It remains to show if $x \succ y$, the result still holds. By Monotonicity, $y \succsim (0,\, 0)$. Consider the interval connecting $(0,\, 0)$ and x. By Continuity, there exists $\beta \in (0,\, 1)$ such that $\beta x \sim y$ and thus $\alpha(\beta x) + (1-\alpha)y \succsim y$. Finally, by Monotonicity,

$$\alpha x + (1-\alpha)y \succsim \alpha(\beta x) + (1-\alpha)y \;\Rightarrow\; \alpha x + (1-\alpha)y \succsim y$$

$\square$

Problem 3

Define $x\,?\,y$ if not $x \succsim y$ and not $y \succsim x$. This binary relation has the following properties:

1. $a\,?\,b \Leftrightarrow b\,?\,a$.
2. Suppose $a\,?\,b$ and pick $c \neq a$, b.
   2–1. If $c \succ a$, then either $c\,?\,b$ or $c \succ b$.
   2–2. If $c \sim a$, then $c\,?\,b$.
   2–3. If $a \succ c$, then either $b\,?\,c$ or $b \succ c$.

The first property directly follows from the definition. The second property follows from the fact that $\succsim$ is transitive.

Method 1

For all $a \in X$, define a corresponding function f_a as follows:

$$f_a(x) = \begin{cases} 1 & \text{if } x = a, \\ 1 & \text{if } x \succsim a, \\ 0 & \text{otherwise.} \end{cases}$$

Now we want to show:

$$x \succsim y \quad \Leftrightarrow \quad \text{for all } a \in X,\; f_a(x) \geq f_a(y).$$

(LHS $\Rightarrow$ RHS)

Assume $x \succsim y$. By construction, $f_x(x) = 1 \geq f_x(y)$ and $f_y(x) = 1 \geq f_y(y)$.

Pick $z \neq x$, y.

1. If $y \succsim z$, then by transitivity, $x \succsim z$ and hence $f_z(x) = 1 = f_z(y)$.
2. If $z \succ y$ or $z\,?\,y$, then $f_z(x) \geq 0 = f_z(y)$.

(LHS $\Leftarrow$ RHS)

We prove this by contraposition. Suppose not $x \succsim y$, i.e., either $y \succ x$ or $y\,?\,x$. In both cases, we have $f_y(y) = 1 > 0 = f_y(x)$. □

Method 2

Before proceeding to this problem, we first state two lemmas.

Lemma 1. *Given a transitive but not necessarily complete binary relation $\succsim$ on a finite set X, there exists $y \in X$ such that for all $x \in X$, either $x \succsim y$ or $x \,?\, y$.*

Proof. Suppose not, i.e., for all $y \in X$, there exists $x \in X$ such that $y \succ x$. Then we have $y \succ x_1 \succ x_2 \succ x_3 \succ \cdots$. By transitivity, they are distinct. However, X is finite, a contradiction. □

Lemma 2. *Given a transitive but not necessarily complete binary relation $\succsim$ on a finite set X, we can extend $\succsim$ to a transitive and complete binary relation $\succsim^*$.*

Proof. We prove this by induction.

Suppose $|X| = 2$. If $a \succsim b$ or $b \succsim a$, then we are done. If $a \,?\, b$, then let $a \succsim b$.

Assume as induction hypothesis that when $|X| = n - 1$, the result holds. Suppose $|X| = n$. By previous lemma, there exist $y \in X$ such that for all $x \in X$, either $x \succsim y$ or $x \,?\, y$. Since $|X \setminus \{y\}| = n - 1$, by induction hypothesis, $\succsim$ can be extended to a transitive and complete binary relation $\succsim^*$ on $X \setminus \{y\}$. Now for all $x \in X \setminus \{y\}$, define:

$$x \succ^* y \text{ if } x \succ y \text{ or } x \,?\, y$$

$$x \sim^* y \text{ if } x \sim y.$$

It remains to show $\succsim^*$ is complete and transitive on X. Completeness holds by construction. Transitivity holds by induction hypothesis along with the fact that y is picked as the minimal element. □

Now we start working on this problem. Suppose there are n pairs $Z_i = \{a_i, b_i\}$ such that $a \,?\, b$. Note that $n < \infty$ since X is finite.

Step 1 For i^{th} pair Z_i, construct two binary relations $\succsim'_{2i-1}$ and $\succsim'_{2i}$ as follows.

$$\succsim'_{2i-1} = \succsim \text{ and } a_i \succ'_{2i-1} b_i$$

$$\succsim'_{2i} = \succsim \text{ and } b_i \succ'_{2i} a_i$$

Afterwards, construct their transitive closure $\succsim_{2i-1}$ and $\succsim_{2i}$. By Property 2, which is stated before Method 1, this is doable.

Step 2 By lemma 2, we can extend $\succsim_{2i-1}$ to $\succsim^*_{2i-1}$ and $\succsim_{2i}$ to $\succsim^*_{2i}$ such that $\succsim^*_{2i-1}$ and $\succsim^*_{2i}$ are complete and transitive.

Step 3 Since X is finite, we can construct f_{2i-1} and f_{2i} associated with complete and transitive binary relations $\succsim^*_{2i-1}$ and $\succsim^*_{2i}$ such that

$$x \succsim^*_{2i-1} y \;\Leftrightarrow\; f_{2i-1}(x) \geq f_{2i-1}(y)$$

$$x \succsim^*_{2i} y \;\Leftrightarrow\; f_{2i}(x) \geq f_{2i}(y)$$

Step 4 We need to verify

$$x \succsim y \;\Leftrightarrow\; f_j(x) \geq f_j(y) \;\text{ for all } j = 1, \dots, k$$

($\Rightarrow$)

If $x \succsim y$, then by construction, $x \succsim^*_j y$ for all j and hence $f_j(x) \geq f_j(y)$ for all j.

($\Leftarrow$)

Suppose not, i.e., either (1) $y \succ x$ or (2) $y \,?\, x$.

Case 1

We have $f_j(y) > f_j(x)$ for all j, a contradiction.

Case 2

By construction, there exists i such that

$$\text{either} \quad \begin{pmatrix} f_{2i-1}(x) > f_{2i-1}(y) \\ f_{2i}(x) < f_{2i}(y) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} f_{2i-1}(x) < f_{2i-1}(y) \\ f_{2i}(x) > f_{2i}(y) \end{pmatrix}, \text{ a contradiction.}$$

$\square$

Problem 4

1. **Impatience**

   Suppose for all k, $x_k \leq y_k$ and $x \neq y$. Then we have

$$
\begin{aligned}
& \text{for all } k, \quad \beta^{x_k} \geq \beta^{y_k}, \qquad \beta^{x_j} > \beta^{y_j} \quad \text{for some } j \\
\Rightarrow \quad & \text{for all } k, \quad \pi_k\beta^{x_k} \geq \pi_k\beta^{y_k}, \quad \pi_j\beta^{x_j} > \pi_j\beta^{y_j} \quad \text{for some } j \\
\Rightarrow \quad & \sum_i \pi_i\beta^{x_i} > \sum_i \pi_i\beta^{y_i}.
\end{aligned}
$$

   Hence, $x \succ y$, as desired.

   Stationarity

$$
\begin{aligned}
& x \succsim y \\
\Leftrightarrow \quad & \sum_i \pi_i\beta^{x_i} \geq \sum_i \pi_i\beta^{y_i} \\
\Leftrightarrow \quad & \beta^{\epsilon}\sum_i \pi_i\beta^{x_i} \geq \beta^{\epsilon}\sum_i \pi_i\beta^{y_i} \\
\Leftrightarrow \quad & \sum_i \pi_i\beta^{x_i+\epsilon} \geq \sum_i \pi_i\beta^{y_i+\epsilon} \\
\Leftrightarrow \quad & \big(x_1+\epsilon,\dots,x_n+\epsilon\big) \succsim \big(y_1+\epsilon,\dots,y_n+\epsilon\big)
\end{aligned}
$$

   Independence

$$
\begin{aligned}
& x \succsim y \\
\Leftrightarrow \quad & \sum_{i\in K}\pi_i\beta^{x_i} + \sum_{i\in\overline{K}}\pi_i\beta^{x_i} = \sum_i \pi_i\beta^{x_i} \geq \sum_i \pi_i\beta^{y_i} = \sum_{i\in\overline{K}}\pi_i\beta^{y_i} + \sum_{i\in K}\pi_i\beta^{y_i} \\
\Leftrightarrow \quad & \sum_{i\in K}\pi_i\beta^{x_i} + \sum_{i\in\overline{K}}\pi_i\beta^{x_i} + \sum_{i\in K}\pi_i\beta^{x'_i} \geq \sum_{i\in K}\pi_i\beta^{y'_i} + \sum_{i\in\overline{K}}\pi_i\beta^{y_i} + \sum_{i\in K}\pi_i\beta^{y_i} \\
\Leftrightarrow \quad & \sum_{i\in\overline{K}}\pi_i\beta^{x'_i} + \sum_{i\in K}\pi_i\beta^{x'_i} \geq \sum_{i\in K}\pi_i\beta^{y'_i} + \sum_{i\in\overline{K}}\pi_i\beta^{y'_i} \\
\Leftrightarrow \quad & x' \succsim y'
\end{aligned}
$$

$\square$

2. The central difficulty of this question was to find a path towards the result. Two additional assumptions would be helpful for this. One is additive separability of the utility representation and the other is differentiability of this representation. Additive separability has been discussed in lecture 4 video 19 and it is perfectly fine to impose additional assumptions (Reidemeister condition, Hexagon condition, etc.) or to simply state that independence across three dimensions yields an additive representation. (For precise references, see various results by Debreu (1959) [1], Gorman (1968) [2], Wakker (1989) [3], Vind (1991) [4].) Differentiability makes life easier in deriving the Pexider equation. However, the general purpose of this question was to challenge your ability to break a proof into steps and it is of less importance that each of the steps in the proof is solved.

Step 1

We use the following fact, which is stated in lecture 4 video 19. Please see Debreu (1959) *Topological Methods in Cardinal Utility Theory* for the proof.

Fact 5. *Assume there are at least three commodities. If a preference relation $\succsim$ has a utility representation and satisfies Independence, then $\succsim$ can be represented by an additively separable utility function, i.e., $U(x) = \sum_i v_i(x_i)$.*

Step 2

Fix some $\gamma \in (0,1)$. Let $D = \{\gamma^x \,|\, x \in X\} = (0,1]^n$. Construct a new preference relation R on D, defined as follows:

$$\gamma^x \, R \, \gamma^y \quad \Leftrightarrow \quad x \succsim y \quad \text{where } \gamma^x = \left(\gamma^{x_1}, \gamma^{x_2}, \ldots, \gamma^{x_n}\right).$$

R also satisfies Independence and thus can be represented by $U(\gamma^x) = \sum_i u_i(\gamma^{x_i})$. Besides, R fulfils Monotonicity since $\succsim$ fulfils Impatience.

Step 3

Show R is homothetic.

[1] Debreu (1959) Topological Methods in Cardinal Utility Theory. *Mathematial Methods in the Social Sciences.*

[2] Gorman (1968) The Structure of Utility Functions. *Review of Economic Studies.*

[3] Wakker (1989) *Additive Representations of Preferences: A New Foundation of Decision Analysis.*

[4] Vind (1991) Independent preferences. *Journal of Mathematical Economics.*

Pick $\alpha > 0$ and find ϵ such that $\gamma^\epsilon = \alpha$. By Stationarity, we have

$$\begin{aligned}
& \gamma^x \, R \, \gamma^y \\
\Leftrightarrow \quad & x \succsim y \\
\Leftrightarrow \quad & \big(x_1 + \epsilon, \dots, x_n + \epsilon\big) \succsim \big(y_1 + \epsilon, \dots, y_n + \epsilon\big) \\
\Leftrightarrow \quad & \gamma^{x+\epsilon} \, R \, \gamma^{y+\epsilon} \\
\Leftrightarrow \quad & \gamma^\epsilon \gamma^x \, R \, \gamma^\epsilon \gamma^y \\
\Leftrightarrow \quad & \alpha \gamma^x \, R \, \alpha \gamma^y
\end{aligned}$$

Step 4

R is Continuous, Homothetic, and Monotonic, so R can be represented by a continuous utility function that is homogeneous of degree one. In particular, we have $T\big(U(\gamma^x)\big)$ is homogeneous of degree one, where T is a monotone transformation. For simplicity, let $a = \gamma^x$. Assume T and each $u_i(a_i)$ is differentiable. Then we have

$$\begin{aligned}
& T\big(U(\alpha a)\big) = \alpha T\big(U(a)\big) \\
\Rightarrow \quad & \frac{\partial T\big(U(\alpha a)\big)}{\partial a_i} = \frac{\partial \, \alpha T\big(U(a)\big)}{\partial a_i} \\
\Rightarrow \quad & T'\big(U(\alpha a)\big) \frac{\partial U(\alpha a)}{\partial a_i} = \alpha \, T'\big(U(a)\big) \frac{\partial U(a)}{\partial a_i} \\
\Rightarrow \quad & T'\big(U(\alpha a)\big) \, u_i'(\alpha a_i) = \alpha \, T'\big(U(a)\big) \, u_i'(a_i) \\
\Rightarrow \quad & \frac{u_i'(\alpha a_i)}{u_i'(a_i)} = \frac{T'\big(U(a)\big)}{T'\big(U(\alpha a)\big)} \, .
\end{aligned}$$

Similarly, if we differentiate with respect to a_j, then we have

$$\frac{u_j'(\alpha a_j)}{u_j'(a_j)} = \frac{T'\big(U(a)\big)}{T'\big(U(\alpha a)\big)} = \frac{u_i'(\alpha a_i)}{u_i'(a_i)} \Rightarrow \frac{u_i'(a_i)}{u_j'(a_j)} = \frac{u_i'(\alpha a_i)}{u_j'(\alpha a_j)} \, .$$

If we pick $\alpha = 1/a_j$, then we have

$$\frac{u_i'(a_i)}{u_j'(a_j)} = \frac{u_i'\left(\dfrac{a_i}{a_j}\right)}{u_j'(1)} \equiv f\left(\frac{a_i}{a_j}\right)$$

Now we consider the following change of variables:

$$a_i = e^{\hat{a}_i} \Rightarrow \hat{a}_i = \log a_i \quad \text{and} \quad \frac{1}{a_j} = e^{\hat{a}_j} \Rightarrow \hat{a}_j = -\log a_j\,,$$

and the following functions:

$$\begin{aligned}
\hat{f}(z) &= \log f(e^z) \\
\hat{g}(z) &= \log u_i'(e^z) \\
\hat{h}(z) &= \log \frac{1}{u_j'(e^{-z})} = -\log u_j'(e^{-z})\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
\hat{f}(\hat{a}_i + \hat{a}_j) &= \log f(e^{\hat{a}_i + \hat{a}_j}) = \log f\left(\frac{a_i}{a_j}\right) \\
\hat{g}(\hat{a}_i) &= \log u_i'(e^{\hat{a}_i}) = \log u_i'(a_i) \\
\hat{h}(\hat{a}_j) &= -\log u_j'(e^{-\hat{a}_j}) = -\log u_j'(a_j) \\
\Rightarrow \quad \hat{f}(\hat{a}_i + \hat{a}_j) &= \log f\left(\frac{a_i}{a_j}\right) = \log u_i'(a_i) - \log u_j'(a_j) = \hat{g}(\hat{a}_i) + \hat{h}(\hat{a}_j)\,.
\end{aligned}$$

This is the Pexider equation. The nontrivial and continuous solution of this equation is of the form:

$$\hat{f}(z) = cz + s + t, \quad \hat{g}(z) = cz + s, \quad \hat{h}(z) = cz + t\,.$$

Hence,

$$\begin{aligned}
u_i'(a_i) &= \exp\{\,\hat{g}(\hat{a}_i)\,\} = \exp\{\,c\hat{a}_i + s\,\} = \exp\{\,c\log a_i + s\,\} = e^s a_i^c \equiv w_i a_i^c \\
u_j'(a_j) &= \exp\{\,-\hat{h}(\hat{a}_j)\,\} = \exp\{\,-c\hat{a}_j - t\,\} = \exp\{\,c\log a_j - t\,\} = e^{-t} a_j^c \equiv w_j a_j^c\,.
\end{aligned}$$

This implies that (let $k = c + 1$, $\pi_i = w_i/k$)

$$u_i(a_i) = \pi_i a_i^k \quad \text{for all } i$$

Plug in $a_i = \gamma^{x_i}$ and let $\beta = \gamma^k$. Then we have

$$U(\gamma^x) = \sum_i u_i(\gamma^{x_i}) = \sum_i \pi_i (\gamma^{x_i})^k = \sum_i \pi_i (\gamma^k)^{x_i} = \sum_i \pi_i \beta^{x_i}$$

Step 5

$$\begin{aligned} & x \succsim y \\ \Leftrightarrow \quad & \gamma^x \, R \, \gamma^y \\ \Leftrightarrow \quad & U(\gamma^x) \geq U(\gamma^y) \\ \Leftrightarrow \quad & \sum_i \pi_i \beta^{x_i} \geq \sum_i \pi_i \beta^{y_i} \end{aligned}$$

$\square$