Problem Set 1

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Problem 1
Let $\succsim$ be a preference relation on a set X. Define $I(x)$ to be the set of all $y \in X$ for which $y \sim x$. Show that the set $\{I(x)|x \in X\}$ is a partition of X.

Answer
Suppose that $I(x) \cap I(y) \neq \emptyset$. Hence, there exists $z \in I(x) \cap I(y)$. We have $z \sim x$ and $z \sim y$, so $x \sim z$ and $x \sim y$ by transitivity.
For all $t \in I(x)$, we have $t \sim x$, then by transitivity again, we have, $t \sim y$. And so $t \in I(y)$, which concludes that $I(x) \subseteq I(y)$. Similarly, $I(y) \subseteq I(x)$. Thus, $I(x) = I(y)$.
It remains to prove that for every $x \in X$, there exists $y \in X$ s.t. $x \in I(y)$. However, it is trivial since $x \in I(x)$ by reflexivity.

Problem 2
Kreps introduces another formal definition of preferences. His primitive is a binary relation P interpreted as "stricty preferred". He requires P to satisfy:
Asymmetry: For no x and y do we have both xPy and yPx
Negative Transitivity: For all x, y, and $z \in X$, if xPy, then for any z, either xPz or zPy (or both).
Explain the sense in which Kreps' formalization is equivalent to the traditional definition.

Answer
Let the set of traditional preferences be A, and the set of preferences in Kreps' formalization be denoted as B.
Suppose a function T from B to A be defined as following: For $x, y \in X$, if xPy, then $xT(P)y$ and not $yT(P)x$; if yPx, then $yT(P)x$ and not $xT(P)y$; if neither of xPy and yPx holds, then $xT(P)y$ and $yT(P)x$.
One clearly knows that $T(P)$ is well-defined binary relation by asymmetry. We will prove that $T(P)$ is a traditional preference.

Reflexivity: xPx cannot hold by asymmetry. Hence, $xT(P)x$ by translation rule.
Completeness: True by checking the rules.
Transitivity: Let $xT(P)y$, $yT(P)z$. Then by checking the rules, we have neither of yPx and zPy holds. By Negative Transitivity, zPx must not hold. Thus, by rules, $xT(P)z$.
It remains to prove bijectivity. Injectivity is obvious, since two different preference must act differently on some pair of x, y, and so is their image.
For surjectivity, we construct a function from A to B, which is the inverse function of T, by reversing the rules. It is well-defined by completeness of traditional preferences. Asymmetry is automatically hold by the mapping rule. We shall prove the Negative Transitivity.
Suppose xPy and not xPz, we must show zPy.
We have $xT(P)y$ and $zT(P)x$, and by transitivity, we have $zT(P)y$. However, $xPy \Rightarrow \neg yT(P)x$ and if $yT(P)z$, then $yT(P)x$ by transitivity. Thus, $yT(P)z$ fails and so zPy.
Consider Kreps' formulation as "strongly preferred", that is "more preferred" and traditional as "weakly preferred", that is "more or equally preferred".

Problem 3

Let Z be a finite set and let X be the set of all nonempty subsets of Z. Let $\succsim$ be a preference relation on X. An element $A \in X$ is interpreted as a "menu", that is, "the option to choose an alternative from the set A". Consider the following two properties of preference relations on X:
1. If $A \succsim B$ and C is a set disjoint to both A and B, then $A \cup C \succsim B \cup C$ and if $A \succ B$ and C is a set disjoint to both A and B, then $A \cup C \succ B \cup C$
2. If $x \in Z$ and $\{x\} \succ \{y\}$ for all $y \in A$, then $A \cup \{x\} \succ A$, and if $x \in Z$ and $\{y\} \succ \{x\}$ for all $y \in A$, then $A \succ A \cup \{x\}$.
a. Discuss the plausibility of the properties in the context of interpreting $\succsim$ as the attitude of the individual toward sets from which he will have to make a choice in a "second stage".
b. Provide an example of a preference relation that: (i) satisfies the two properties; (ii) satisfies the first but not the second property; (iii) satisfies the second but not the first property.
c. Show that if there are x, y and $z \in Z$ such that $\{x\} \succ \{y\} \succ \{z\}$, then there is not preference relation satisfying both properties.

Answer a.

1. One can imagine this process as adding page to some restaurant menu. If the two menus are inserted with exactly same page with same content, then the preference between these two menus would not change.
2. If one finds that something he likes very much (more than others on the menu) is added to a menu, then one would prefer such new menu than the first one. However, if something he does not like is added, then he need to take some time to eliminate it or take some risk to choose it accidentally.

Answer b.

(i) Grouping Z into two parts, good part G and bad part W. For $A, B \in X$, $A \succsim B$ if $|A \cap G| - |A \cap W| \geq |B \cap G| - |B \cap W|$.

(ii)-1 Take Z be a finite set of disjoint finite sets. Compare set with the carnality of the union of elements. (Thanks to 羅偉駿 and 黃楚岳)

(ii)-2 Take $Z \subseteq \mathbb{R}$ be finite. Compare $A, B \in X$ in the following way:

First list $A = \{a_1, \cdots, a_m\}$, $B = \{b_1, \cdots, b_n\}$ where $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$.

We say $A \succ B$ if [there exists some positive integer $i \leq \min\{m, n\}$, such that $a_j = b_j$ for all $j < i$ and $a_i > b_i$] or [$m < n$ and for all $i = 1, \cdots, m$, $a_i = b_i$]

(Intuitively, this is the procedure in the following sence:

First compare the greatest. If the greatest are the same, then the second greatest. If still same, then the third and so on. If someone first run out of elements then it is said to be more preferred.)

(iii) Take $Z \subseteq \mathbb{R}$ be a finite set. Compare set with the average of the set.

Answer c.

$\{x\} \succ \{x, y\}$ by 2, and so $\{x, z\} \succ \{x, y, z\}$ by 1.

However, $\{y, z\} \succ \{z\}$ by 2, and so $\{x, y, z\} \succ \{x, z\}$ by 1, which leads to a contradiction.

Problem 4

Let $\succ$ be an asymmetric binary relation on a finite set X that does not have a cycle. Show that $\succ$ can be extended to a complete ordering.

Answer

We apply induction on $|X|$ as what the problem asked us to do. We would like to prove a stronger statement that $|X| \geq 2$. First suppose $|X| = 2$, then the result is trivial.

For $|X| = k > 2$, we first claim that there is a minimal in X, i.e., there is an element x s.t. $x \not\succ y$ for all $y \in X$. If not, then by $|X| < \infty$, there must be a cycle since we can construct a infinite chain in the form of $x_1 \succ x_2 \succ x_3 \succ \cdots$.

Let $X = \{x_1, \cdots, x_k\}$. W.L.O.G. let x_k be a minimal. By induction hypothesis, there is a complete relation $\succ'$ on $X \setminus \{x_k\}$. Again without loss of generality, say $x_1 \succ' x_2 \succ' x_3 \succ' \cdots \succ' x_{k-1}$ (This does not only mean $x_i \succ x_{i+1}$, but also $x_i \succ x_j$ for all $j > i$). Then we claim that

$$x_1 \succ'' x_2 \succ'' x_3 \succ'' \cdots \succ'' x_k$$

is what we want. It suffices to check if the "$\succ$"-relation with x_k is preserved. (relation with other elements are satisfied since the $\succ''$-relation between other elements are inherited by $\succ'$) However, this statement is trivial since x_k is a $\succ$-minimal.

Comment The part that I use minimal to do the argument is not necessary. It is just for mathematical convenience. One can drop such step and still finish the proof by finding a suitable place for x_k to insert.

Problem 5

You have read an article in a "prestigious" journal about a decision maker whose mental attitude towards elements in a finite set X is represented by a binary relation $\succ$, which is asymmetric and transitive but not necessarily complete. The incompleteness in the result of the assumption that a DM is sometimes unable that a DM is sometimes unable to compare between alternatives.

Then, the author states that he is going ot make a stronger assumption: the DM uses the following procedure: he has n criteria in mind, each represented by an ordering $\succ_i (i = 1, \cdots, n)$. He determines that $x \succ y$ if and only if $x \succ_i y$ for every i.

1. Verify that the relation $\succ$ generated by this procedure is asymmetric and transitive. Try to convince a reader of the paper that this attractive assumption by giving a "real life" example in which it is "reasonable" to assume that a DM uses such a procedure in order to compare between alternatives.

It is claimed that the additional assumption is vacuous: given any asymmetric and transitive relation, $\succ$, one can find a set of complete orderings $\succ_1, \cdots, \succ_n$ such that $x \succ y$ iff $x \succ_i y$ for every i.

2. Demonstrate this claim for the binary relation on the set $X = \{a, b, c\}$ according to which only $a \succ b$ and the comparisons between [b and c] and [a and c] are not determined.

3. Prove this claim for the general case.

Answer 1.

Asymmetric: If $x \succ y$, then $x \succ_i y$ for all i. It is impossible that $y \succ_i x$ for all i by asymmetric of $\succ_i$. Hence, $y \not\succ x$.

Transitive: Say $x \succ y$ and $y \succ z$. Then, for all i, $x \succ_i y$ and $y \succ_i z$. By transitivity of $\succ_i$, $x \succ_i z$ for all i. Thus, $x \succ z$.

Example: A college want to choose best researcher in it. Then they have many complete ordering about researchers' devotion on different area. However, the committee will face some difficulty when comparing two researchers from different area.

Answer 2. $c \succ_1 a \succ_1 b$ and $a \succ_2 b \succ_2 c$ can do it.

Answer 3.

We apply induction on $|X|$. For $|X| = 0, 1$, trivial.

Suppose $|X| = k$, and say S is the set of maximals in X.(S must not be empty by a similar argument in answer 4. There is a argument about existence of minimals.)

For each $s \in S$, there exists some $\succ'_{i_s}$s with $i_s \in I_s$ that determines the restriction of $\succ$ on the set $X \setminus \{s\}$. Suppose $\succ_{i_s}$ is defined as $\succ'_{i_s} \cup \{(s,t) | t \in X, t \neq s\}$. (One can imagine it as putting s before the original chain). We claim that $\sqcup_{s \in S}\{\succ_{i_s} | i_s \in I_s\}$ ($\sqcup$ means union of disjoint sets) is what we want.

We shall examine it: Take two distinct $x, y \in X$.

1. $x, y \in S$

Then by the definition of maximal, x, y is not comparable. Moreover, $x \succ_{i_x} y$ and $y \succ_{i_y} x$ for some $i_x \in I_x, i_y \in I_y$.

2. $x \in S$ but $y \notin S$

There are two possibilities:

(1) $x \succ y$: For all $s \in S$ with $s \neq x$, we have $x \succ_{i_s} y$ for all $i_s \in I_s$ by construction of $\succ_{i_s}$ directly. And for $i_x \in I_x$, $x \succ_{i_x} z$ for all $z \neq x$, and so $x \succ_{i_x} y$.

(2) x is not comparable with y: Then we know that x is not the only maximal.(Suppose not, one can find $y_1 \neq x$ with $y_1 \succ x$ and $y_2 \neq x$ with $y_2 \succ y_1 \succ y$ and so on...) Take $s \in S$ with $s \neq x$, by the construction of I_s, there exists some $i_s \in I_s$ with $y \succ_{i_s} x$.

3. $x, y \notin S$ There are three possibilities:

(1) $x \succ y$: For all $s \in S$, we have $x \succ_{i_s} y$ for all $i_s \in I_s$ by construction of $\succ_{i_s}$ directly.

(2) $y \succ x$: same as (1)

(3) x is not comparable with y: For $s \in S$, by construction(x, y are also not comparable in $X - \{s\}$), we have some $i_{s,1}, i_{s_2} \in I_s$ such that $x \succ_{i_{s,1}} y$ but $y \succ_{i_{s,2}} x$.

Comment to Answer 3. The first intuition to this problem for me is to apply induction on $|X|$ and try to construct those complete ordering with those we have in the last step. And soon I found that it is hard for me to insert arbitrary element to X and find such orderings. So I came up with the idea to not insert arbitrary element but maximals, since they have better behavior when we talk about ordering. The rest of work is to write it down properly.

Answer 3. TA version.

For every pair a, b which are incomparable, define

$$\succ_{a,b} = \succ \cup \{(a, b)\}$$

We claim that $\succ_{a,b}$ is asymmetric and does not have any cycle.

Asymmetric: We only need to claim that $(b, a) \notin \succ$. However, this is straightforward by a, b are not comparable.
No cycle: Suppose that there is a cycle. Since there is no cycle in $\succ$, the cycle must contain $a \succ b$. Hence, the cycle must seems like

$$a \succ_{a,b} b \succ_{a,b} \cdots \succ_{a,b} a \succ_{a,b} b$$

Suppose there are no a in $\cdots$ above, then the $\succ_{a,b}$ are actually can be seen as $\succ$. Then, by transitivity of $\succ$, $b \succ a$, contradiction.
By the result of Problem 4, there is a complete ordering extend $\succ_{a,b}$, say $\succ'_{a,b}$.
We claim that $\{\succ'_{a,b} | a, b : \text{incomparable}\}$ is what we want.
(1) If $x \succ y$, then $x \succ_{a,b} y$ for all a, b : incomparable by definition of $\succ_{a,b}$.
(2) If x, y are incomparable, then so are y, x. $x \succ_{x,y} y$ but $y \succ_{y,x} x$, done!

Problem 6 Listen to the illusion called the Shepard Scale. Any economic analogies?

Answer The story of 朝三暮四 maybe one of the possible answer. (Not really sure about it.)