# 1. Knowledge and Equilibrium

ECON 7219 – Games With Incomplete Information

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# Selling Farmland

Two of Taiwan's most valuable crops are tea and rice.

**Annual average yield:**

- Tea: 5.35m NTD/km$^2$.
- Rice: 4.2m NTD/km$^2$.

A Rice Farmer considers selling his/her land to a Tea Farmer.

- Suppose high/low-quality soil yields 33% above/below average.
- The Rice Farmer knows the quality, but the Tea Farmer does not.
- Can trade between the two farmers occur? If so, at which price?

Source: https://www.visualcapitalist.com/the-worlds-most-valuable-cash-crop/

# Modeling the Interaction as a Game

**Uncertainty:**

- Quality of soil $\vartheta$ can either be low, medium, or high: $\vartheta \in \{L, M, H\}$.
- Suppose the value of the land is a constant $c$ times the annual yield

$$y_T(L) = 3.6, \qquad y_T(M) = 5.35, \qquad y_T(H) = 7.1,$$
$$y_R(L) = 2.8, \qquad y_R(M) = 4.2, \qquad y_R(H) = 5.6.$$

**Players and actions:**

- Tea Farmer can offer a price $p \geq 0$.
- Rice Farmer chooses $a \in \{\text{sell}, \text{not sell}\}$.

**Preferences over outcomes as utility functions:**

- $u_T(p, a, \vartheta) = (c y_T(\vartheta) - p) 1_{\{a = \text{sell}\}}$,
- $u_R(p, a, \vartheta) = p \cdot 1_{\{a = \text{sell}\}} + c y_R(\vartheta) 1_{\{a = \text{not sell}\}}$.

# Solving the Model

**Players maximize expected utility:**

- We must describe the players' beliefs over the unknown state.
- Players choose action that maximizes their expected utility, given their beliefs about the state and the action of opponent.
- We must describe the players' higher-order beliefs.

**Solution concept:**

- Complete information: Nash equilibrium,
- Incomplete information: Bayesian Nash equilibrium.

# Nash Equilibrium

# Static Game

### Definition 1.1

A static game $\mathcal{G} = \big(\mathcal{I}, (\mathcal{A}_i), (u_i)\big)$ consists of:

1. A finite set of players $\mathcal{I} = \{1, \dots, n\}$,
2. A set $\mathcal{A}_i$ of pure actions available to player $i$ for each $i \in \mathcal{I}$,
3. A payoff function $u_i : \mathcal{A} \to \mathbb{R}$ for each player $i \in \mathcal{I}$, where

$$\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$$

   is the set of pure action profiles $a = (a_1, \dots, a_n)$.

**Remark:**

- $(\mathcal{A}_i)$ and $(u_i)$ are short for $(\mathcal{A}_i)_{i \in \mathcal{I}}$ and $(u_i)_{i \in \mathcal{I}}$, indicating that it is a pure action set and a payoff function for each player $i \in \mathcal{I}$.
- $u : \mathcal{A} \to \mathbb{R}^n$ defined by $u = (u_1, \dots, u_n)$ is the players' payoff vector.

# Assumptions About the Players

**We impose:**

1. Rationality: each player maximizes his/her expected utility.
2. Awareness: each player knows $\mathcal{G}$, that is, who is playing, the available actions of each player, and each player's preferences over outcomes.
3. It is common knowledge that players are rational and aware.

**Common knowledge:**

- An event $Y$ is common knowledge if every player knows $Y$, every player knows that every player knows $Y$, etc.
- Imposition 3 above is a very strong assumption, but we need it for almost everything in game theory.
- It allows us to analyze situations which require higher-order reasoning.

# Rock, Paper, Scissors

| | $R$ | $P$ | $S$ |
|---|---|---|---|
| $R$ | $0, 0$ | $-1, 1$ | $1, -1$ |
| $P$ | $1, -1$ | $0, 0$ | $-1, 1$ |
| $S$ | $-1, 1$ | $1, -1$ | $0, 0$ |

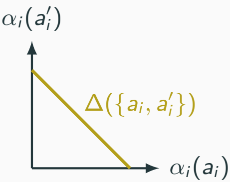
**Rock paper scissors:**

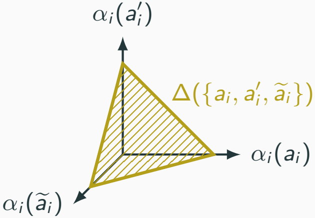
- The players $\mathcal{I} = \{1, 2\}$ are the literal players of the game.
- The available pure actions are $\mathcal{A}_1 = \mathcal{A}_2 = \{$Rock, Paper, and Scissors$\}$.
- Payoffs for player 1 and 2 are given in the payoff matrix.

**Need to randomize:**

- Players decide with which frequency to choose each of the pure actions.

# Mixed Actions




**Randomizing actions:**

- A mixed action of player $i$ is a distribution $\alpha_i \in \Delta(\mathcal{A}_i)$.
- If $\mathcal{A}_i$ is finite, a distribution over $\mathcal{A}_i$ is an $|\mathcal{A}_i|$-dimensional vector with

$$\alpha_i(a_i) \in [0,1] \quad \forall\, a_i \in \mathcal{A}_i, \qquad \textstyle\sum_{a_i \in \mathcal{A}_i} \alpha_i(a_i) = 1,$$

  where $\alpha_i(a_i)$ indicates the probability with which $a_i \in \mathcal{A}_i$ is selected.[1]
- If $\mathcal{A}_i$ is an interval, we can represent $\alpha_i$ by a distribution function.

[1] The set of $n$-dimensional vectors with non-negative entries that sum up to 1 is called the $(n-1)$-simplex.

# Outcome

**What do players observe?**

- In rock, paper, scissors, your opponent sees only one of these three outcomes, but not how frequently you choose them.
- Players do not observe the chosen distribution, only a realization of it, that is, players observe a random variable $A_i$ with distribution $\alpha_i$.
- We say that such a random variable $A_i$ implements the $\alpha_i$.

**Outcome:**

- A mixed action profile $\alpha = (\alpha^1, \dots, \alpha^n)$ is implemented independently, that is, the random variables $A^i$ and $A^j$ are independent for any $j \neq i$.
- The outcome of the game is $A = (A^1, \dots, A^n)$.
- If players choose a pure action profile, the outcome is deterministic.

# Notation

**Subscript vs. superscript:**

- Subscript $i$ of a vector $x$ refers to component that pertains to player $i$.

  *Example: $a_1$ is the first player's action in action profile $a$.*

- Negative subscript $-i$ refers to components of all players except $i$.

  *Example: $\alpha_{-i}$ is the action profile $(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$.*

- Superscripts distinguish different entities, possibly indexed by time.

  *Example: action profiles $a^1$ and $a^2$, played at times $t = 1, 2$.*

**Small, capital, Greek, and calligraphic letters:**

- Small letter $x$ refers to a fundamental, deterministic element.
- Calligraphic letter $\mathcal{X}$ refers to set of fundamental elements $x$.
- Capital letter $X$ refers to a random variable, taking values in $\mathcal{X}$.
- Greek letter $\xi$ refers to the distribution of $X$, itself an element of $\Delta(\mathcal{X})$.

# Payoffs in a Mixed Action Profile

**Ex-post payoff:**

- Player $i$ receives the random payoff $u_i(A)$.
- Whether that is $u_i(a)$ or $u_i(\widetilde{a})$ depends on the realization of $A$.

**Ex-ante expected payoff:**

- Risk-neutral players aim to maximize their expected payoff under the probability measure $P_\alpha$ induced by play of $\alpha$, defined by

$$P_\alpha(A_i = a_i) = \alpha_i(a_i).$$

- $P_\alpha$ is the probability measure under which players observe the game.
- Expected payoff of mixed action profile $\alpha = (\alpha_1, \dots, \alpha_n)$ is

$$u_i(\alpha) := \mathbb{E}_\alpha[u_i(A)] = \sum_{a \in \mathcal{A}} \prod_{j=1}^{n} \alpha_j(a_j) u_i(a).$$

# Nash Equilibrium

### Definition 1.2

A Nash equilibrium of a static game $\mathcal{G} = \big(\mathcal{I}, (\mathcal{A}_i), (u_i)\big)$ is a (possibly mixed) action profile $\alpha$ such that for every $i \in \mathcal{I}$ and every $a_i \in \mathcal{A}_i$,

$$\mathbb{E}_\alpha[u_i(A)] \geq \mathbb{E}_{(a_i,\alpha_{-i})}[u_i(A)],$$

where we denote $\alpha_{-i} = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$.

**Interpretation:**

1. Each player $i$ predicts the behavior $\alpha_{-i}$ of his/her opponents correctly.
2. Each player $i$ best replies to his/her conjecture with $\alpha_i$.

**Note:** Requirement 1. is a rather strong requirement.

# Nash Equilibrium as a Solution Concept

**Why study Nash equilibria?**

- If we allow incorrect beliefs about opponent's play, then almost any outcome can be rationalized.
- Correct predictions of opponents' play gives us a workable theory.
- As a theoretical solution concept, it is extremely useful.

**Existence of Nash equilibria:**

- If $\mathcal{A}$ is finite, existence in mixed actions follows from Nash (1950).
- If $\mathcal{A}$ is closed and $u$ is bounded and upper semi-continuous, existence in mixed actions follows from Dasgupta and Maskin (1986).
- If $\mathcal{A}$ is convex and compact and $u$ is continuous and quasi-concave, existence in pure actions follows from Glicksberg (1952).

# Strategies

**Actions vs. Strategies:**

- An action is one possible way to influence the game in a given instant.
- A strategy is a contingent plan of action choices, made by the player for the entire game.
- Formally, a strategy is a function that maps the available information to some available action whenever player $i$ is called upon to play.

**Static game:**

- A strategy is simply the choice of an action: $\mathcal{S}_i = \mathcal{A}_i$.
- The definitions in this section are equally valid for strategies.
- We will use actions/strategies interchangeably in this section.

# Tipping

| | $T$ | $N$ |
|---|---|---|
| $G$ | $2, 2$ | $-1, 3$ |
| $B$ | $3, -1$ | $0, 0$ |


**Available actions:**

- Client chooses to (T)ip or (N)ot tip.
- Waiter provides (G)ood or (B)ad service.

**Strictly dominated actions:**

- Regardless of the quality of service, the Client is better off not tipping.
- Regardless of tips, it is better for the Waiter to provide bad service.
- The unique Nash equilibrium is $(B, N)$.

# Strictly Dominated Strategies

**Lemma 1.3**

*Action $a_i \in \mathcal{A}_i$ is strictly dominated by $\widetilde{a}_i \in \mathcal{A}_i$ if $u_i(a_i, a_{-i}) < u_i(\widetilde{a}_i, a_{-i})$ for all $a_{-i} \in \mathcal{A}_{-i}$. Strictly dominated actions cannot be played in any Nash equilibrium $\alpha$, that is, $\alpha_i(a_i) = 0$ for any strictly dominated action $a_i$.*

**More generally:**

- Nash equilibrium assigns positive weight only to actions $a_i$ that survive iterated deletion of strictly dominated strategies.
- Remove in round $k$ those strategies that are strictly dominated when opponents play strategies that survive round $k$ of elimination.

**Note:**

- Nash equilibria can assign positive weight to weakly dominated actions.

# Indifference Principle

**Lemma 1.4**

*In a mixed Nash equilibrium $\alpha$, every player $i$ is indifferent between the pure actions in the support of $\alpha_i$, that is,*

$$u_i(a_i, \alpha_{-i}) = u_i(\alpha)$$

*for every $a_i \in supp(\alpha_i)$, where $supp(\alpha_i) := \{a_i \in \mathcal{A}_i \mid \alpha_i(a_i) > 0\}$.*

**Remark:**

- The indifference principle is useful to find completely mixed equilibria, which are equilibria that assign positive weight to every pure action.
- The indifference principle is of no help if we do not know $\mathrm{supp}(\alpha_i)$.

# Finding Completely Mixed Equilibria

| | $R$ | $P$ | $S$ |
|---|---|---|---|
| $R$ | $0, 0$ | $-1, 1$ | $1, -1$ |
| $P$ | $1, -1$ | $0, 0$ | $-1, 1$ |
| $S$ | $-1, 1$ | $1, -1$ | $0, 0$ |

**Need to randomize:**

- Parametrize mixed action $\alpha_i$ for $i = 1, 2$ by $(r_i, p_i, 1 - r_i - p_i)$.
- Player 1's choice must make Player 2 indifferent between $R$, $P$, and $S$:

$$u_2(\alpha_1, P) = 2r_1 + p_1 - 1 \;=\; u_2(\alpha_1, S) = p_1 - r_1 \quad \text{yields} \quad r_1 = \tfrac{1}{3},$$

$$u_2(\alpha_1, R) = 1 - r_1 - 2p_1 \;=\; u_2(\alpha_1, S) = p_1 - r_1 \quad \text{yields} \quad p_1 = \tfrac{1}{3}.$$

- By symmetry, player 2 also chooses $R$, $P$, and $S$ with probability $\frac{1}{3}$ each.

# Best-Response Correspondences

## Definition 1.5

Player $i$'s best-response correspondence is

$$\mathcal{B}_i(\alpha_{-i}) = \left\{ \alpha_i \in \Delta(\mathcal{A}_i) \;\middle|\; u_i(\alpha_i, \alpha_{-i}) = \max_{a_i \in \mathcal{A}_i} u_i(a_i, \alpha_i) \right\}.$$

**Remark:**

- If $\mathcal{A}_i$ is finite, then $u_i(\alpha)$ is affine in $\alpha_i(a_i)$. Finding the best-response correspondence corresponds to maximizing an affine function:
  - If the slope is 0, any $\alpha_i(a_i) \in [0, 1]$ attains the maximum.
  - Otherwise, the maximum is attained at either 0 or 1.
- Contrary to the indifference principle, we can find mixed Nash equilibria that are not completely mixed using best-response correspondences.

# Crossing an Intersection

| | $C$ | $W$ |
|---|---|---|
| $C$ | $-4, -4$ | $1, 0$ |
| $W$ | $0, 1$ | $0, 0$ |

**Crossing an intersection:**

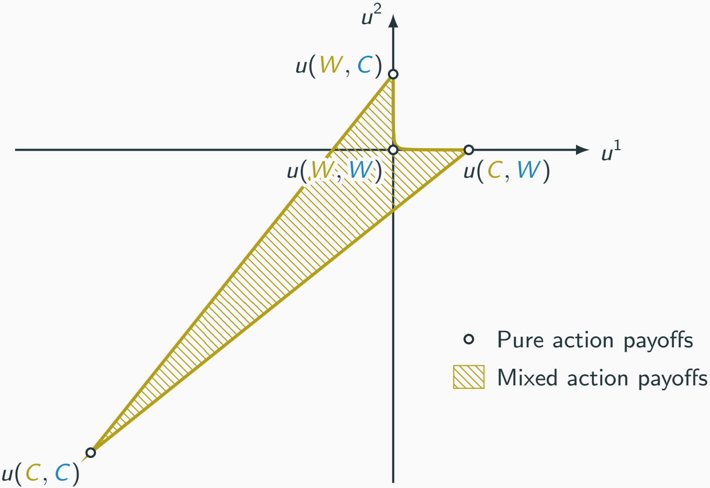
- Both Pedestrian and Driver can either (C)ross or (W)ait.
- Parametrize $\alpha$ by $\alpha_1(C) = x$ and $\alpha_2(C) = y$. The expected utilities are

$$u_1(\alpha) = -4xy + x(1-y), \qquad u_2(\alpha) = -4xy + y(1-x).$$

- The partial derivatives are $\frac{u_1(\alpha)}{\partial x} = 1 - 5y$ and $\frac{\partial u_2(\alpha)}{\partial y} = 1 - 5x$.
- There are three mutual best responses $(x, y) \in \{(0,1), (1,0), (0.2, 0.2)\}$.

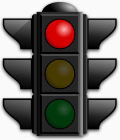
Image source: https://www.taiwannews.com.tw/en/news/3506399

# Crossing an Intersection: Payoffs



# Crossing an Intersection: Public Correlation

| | $C$ | $W$ |
|---|---|---|
| $C$ | $-4, -4$ | $1, 0$ |
| $W$ | $0, 1$ | $0, 0$ |



**Public correlation:**

- Players can use a public signal to correlate their actions:
  - Choose $(C, W)$ when the traffic light is red,
  - Choose $(W, C)$ when the traffic light is green.
- Realization of signal is only a recommendation, it is not binding.
- Neither player has an incentive to deviate after observing the signal.
- Attains $\frac{1}{2}u(W, C) + \frac{1}{2}u(C, W)$ if red and green are equally likely.

# Public Correlation

**Definition 1.6**

1. A public correlation device is an $\mathcal{M}$-valued random variable $M$, independent of $A$, where $\mathcal{M}$ is the space of possible signals $m$.
2. A publicly correlated action profile $\alpha = (\alpha_1, \ldots, \alpha_n)$ consists of a map $\alpha_i : \mathcal{M} \to \Delta(\mathcal{A}_i)$ for each player $i \in \mathcal{I}$.

**Remark:**

- By independence of $A$, distribution of $M$ is not affected by mixing.
- Correlation device is observed before players choose their actions.
- Consequence: a Nash equilibrium with public correlation is robust to unilateral deviations ex post.
- This is *not* a correlated equilibrium in the sense of Aumann (1974).

# Crossing an Intersection: Public Correlation

**Formally:**

- Traffic light $M$ takes values in $\mathcal{M} = \{m_G, m_R\}$.
- Equilibrium and its outcome with public correlation are

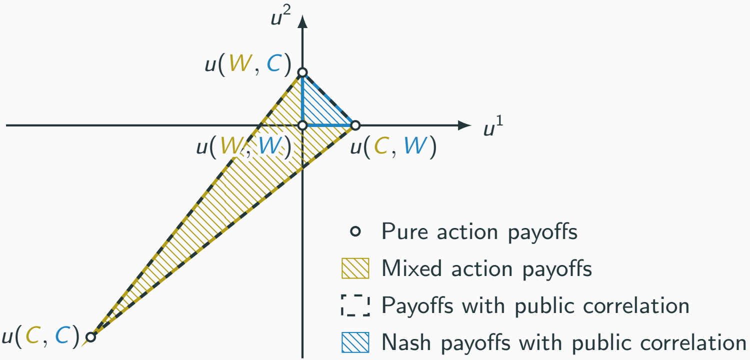
$$\alpha(m) = \begin{cases} (C, W) & \text{if } m = m_R, \\ (W, C) & \text{if } m = m_G, \end{cases} \qquad A = \begin{cases} (C, W) & \text{if } M = m_R, \\ (W, C) & \text{if } M = m_G. \end{cases}$$

- Compute expected payoff by conditioning on correlation device:

$$\begin{aligned} \mathbb{E}_\alpha[u(A)] &= \mathbb{E}_\alpha[u(C, W) \mid M = m_R] P_\alpha(M = m_R) \\ &\quad + \mathbb{E}_\alpha[u(W, C) \mid M = m_G] P_\alpha(M = m_G). \end{aligned}$$

- If red and green are equally likely, the equilibrium with public correlation attains payoff pair $\frac{1}{2}u(W, C) + \frac{1}{2}u(C, W) = (\frac{1}{2}, \frac{1}{2})$.

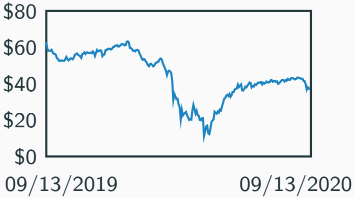
# Crossing an Intersection: Payoffs




**Use of public correlation:**

- With public correlation, we can attain any payoffs in the convex hull.
- Whether its use is appropriate depends on the application.

# Price Wars





**Cournot duopoly:**

- Russian and OPEC oil producers had a price war in April 2020.
- Each firm $i = 1, 2$ can produce quantity $q_i \in [0, 90]$ at cost $c(q_i) = 10q_i$.
- Suppose the market price is $p(q) = 100 - q_1 - q_2$ so that $i$'s payoff is

$$u_i(q) = (p(q) - 10)q_i = 90q_i - q_i^2 - q_1 q_2.$$

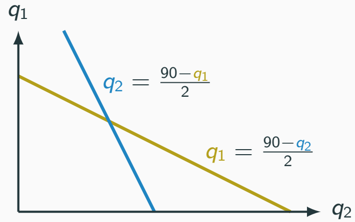
- What is the Nash equilibrium prediction?

Source: https://www.macrotrends.net/2516/wti-crude-oil-prices-10-year-daily-chart

# Price Wars


Source: https://www.bbc.com/news/business-52350082

# Price Wars





**Best responses:**

- If Firm 1 knew $q_2$, its payoff would be maximized where $\frac{\partial u_1(q)}{\partial q_1} = 0$.
- We derive Firm 1's best response correspondence by solving:

$$0 = \frac{\partial u_1(q)}{\partial q_1} = 90 - 2q_1 - q_2, \quad \implies \quad q_1 = \frac{90 - q_2}{2}.$$

- By symmetry, Firm 2's best-response correspondence is $q_2 = \frac{90-q_1}{2}$.
- The unique Nash equilibrium is $(30, 30)$.

# Correctly Predicting Behavior

**By design:**

- There is explicit communication on which equilibrium will be played.
- Example: the law informs people how they are expected to behave.
- Example: when countries agree to sign a treaty, they design it with specific behavior in mind that they wish to enforce.

**Evolutionary process:**

- Players continually observe opponents' strategies and they adapt by best responding iteratively until the process converges.
- Example: at a poker table, you may update your beliefs about the frequency, with which opponents bluff and adapt your strategy.
- Example: social norms are upheld because we are all following them.

# What Is a Nash Equilibrium?

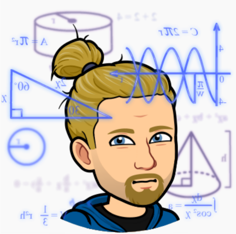
**Consistent outcomes:**

- Nash equilibria are the set of outcomes of a game that are consistent with mutually rational behavior.
- No player would want to deviate if they knew their opponents' strategy.

**Predictive power:**

- Nash equilibria are useful to determine which outcomes can be sustained through self-enforcing agreements.
- A Nash equilibrium is not necessarily a prediction of rational behavior in a single interaction without any means for communication.
- If the equilibrium is not unique, players may miscoordinate.

# Guessing Two Thirds of the Average



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**Setup of game:**

- Guess a number between 0 and 100, that is, choose $a_i \in [0, 100]$.
- The goal is that your guess is as close as possible to two thirds of the average of all guesses, i.e.,

$$u_i(a) = \left| a_i - \frac{2}{3n} \sum_{j=1}^{n} a_j \right|.$$

# Common Knowledge of Rationality

**Common knowledge of rationality:**

- Predicting a unique Nash equilibrium requires common knowledge that every player will conduct infinite-order reasoning.
- It is not enough that everybody is rational.
- It is not enough that everybody knows everybody else is rational.

**Knowing everybody's rationality:**

- If I believe that some players believe others will guess a positive number, it is a best response for everybody to guess a positive number.
- Nevertheless, somebody's beliefs will be incorrect.

# Check Your Understanding

**True or false:**

1. In a mixed Nash equilibrium, players choose their mixing probabilities independently from each other.
2. In every outcome of a mixed Nash equilibrium, every player best responds to the others' outcome.
3. The outcome of a correlation device is binding in a Nash equilibrium with public correlation.
4. Everybody knowing everybody's rationality is enough to deduce the Nash equilibrium in the tipping game.

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**Short-answer question:**

5. Find all Nash equilibria in the static game displayed to the right.

| | $L$ | $R$ |
|---|---|---|
| $T$ | $1, 2$ | $1, 2$ |
| $B$ | $3, 1$ | $0, 0$ |

# Literature

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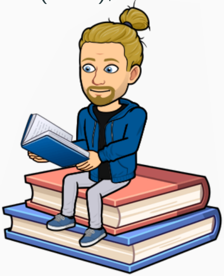

# Information and Knowledge

# Selling Farmland

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**Annual average yield:**

- Tea: 5.35m NTD/km$^2$.
- Rice: 4.2m NTD/km$^2$.

A Rice Farmer considers selling his/her land to a Tea Farmer.

- Suppose high/low-quality soil yields 33% above/below average.
- The Rice Farmer knows the quality, but the Tea Farmer does not.
- Can trade between the two farmers occur? If so, at which price?

Source: https://www.visualcapitalist.com/the-worlds-most-valuable-cash-crop/

# Information Sets

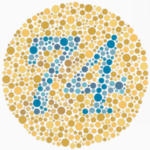

**Modeling knowledge:**

- Quantity of interest $\vartheta$ is called the state of nature.
- We model knowledge via information sets. An information set $\tau_i$ of player $i$ is a set of states of nature that player $i$ cannot distinguish.

**Example:** Quality of the soil $\vartheta$ lies in $\Theta = \{L, M, H\}$.

- Tea Farmer does not know the quality of the soil.
- Tea Farmer has only one information set $\tau_T = \{L, M, H\}$.
- Rice Farmer knows $\theta$ from experience.
- Rice Farmer has information sets $\tau_R^L = \{L\}$, $\tau_R^M = \{M\}$, and $\tau_R^H = \{H\}$.

# Color Blindness

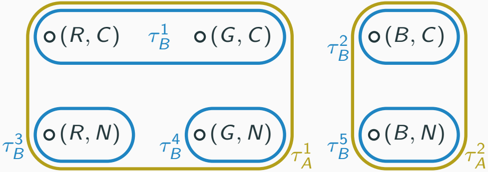

Alan and Ben are looking at a hat, wondering what color it is.

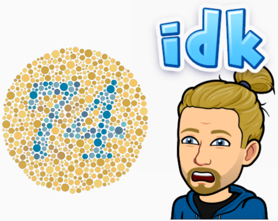
- The hat can have one of three colors $\vartheta \in \{R, G, B\}$.
- Alan is color blind and cannot distinguish $R$ and $G$.
- Ben is not color blind and can distinguish all three colors.

Very intuitive, but it does not allow us to model higher-order knowledge:

Does Ben know whether Alan knows the color of the hat?

# Color Blindness




**Extend the model:**

- Add 2nd dimension to the state: Ben is (C)olor blind, (N)ot color blind.
- Set of all states is $\{(R, C), (R, N), (G, C), (R, N), (B, C), (B, N)\}$.

**Can we describe:**

- Everyone knows Alan is color blind and cannot distinguish $R$ and $G$.
- Nobody except Ben knows whether Ben is color blind.

# Aumann Model of Incomplete Information

**Definition 1.7**

Let $\Theta$ be a set of finitely many states of nature. An Aumann model of incomplete information over $\Theta$ consists of:

1. A finite set of players $\mathcal{I} = \{1, \ldots, n\}$.
2. A finite set $\Omega$ of possible states of the world $\omega$.
3. A partition $\mathcal{T}_i = \left\{\tau_i^1, \ldots, \tau_i^{m_i}\right\}$ of $\Omega$ for each player $i \in \mathcal{I}$.
4. A function $\theta : \Omega \to \Theta$, indicating the state of nature $\theta(\omega)$ in each $\omega$.

**Remarks:**

- We are interested in the players' knowledge of the states of nature.
- States of the world are needed to describe higher-order knowledge.
- An element $\tau_i \in \mathcal{T}_i$ is an information set of player $i$.
- Also works if $\Theta$ or $\Omega$ are uncountable, but requires measure theory.

# Knowledge

**Definition 1.8**

For any state $\omega \in \Omega$, denote by $T_i(\omega)$ the partition element of $\mathcal{T}_i$ that contains $\omega$, i.e., $T_i(\omega)$ is the set of all states that $i$ deems possible in $\omega$.

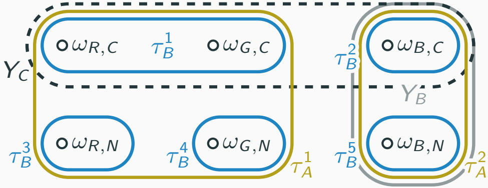
1. An event $Y$ is a subset of $\Omega$.
2. Player $i$ knows event $Y$ in state $\omega$ if $T_i(\omega) \subseteq Y$.
3. Define player $i$'s knowledge operator $K_i : 2^\Omega \to 2^\Omega$ as

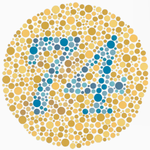
$$K_i(Y) := \{\omega \in \Omega \mid T_i(\omega) \subseteq Y\}.$$

**Remarks:**

- $2^\Omega$ is the set of all subsets of $\Omega$, i.e., the set of all events.
- Whether a state of nature $\vartheta$ obtains is the event $\{\omega \in \Omega \mid \theta(\omega) = \vartheta\}$.
- $K_i Y = K_i(Y)$ is the event that player $i$ knows $Y$.

# Color Blindness



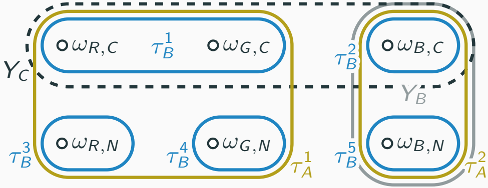

**Apply the formalism:**

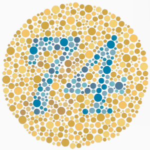
- States of nature are $\Theta = \{R, G, B\}$.
- States of the world are $\Omega = \{\omega_{R,C}, \omega_{R,N}, \omega_{G,C}, \omega_{G,N}, \omega_{B,C}, \omega_{B,N}\}$.
- Associated states of nature are $\theta(\omega_{col,blind}) = col$.

**Can define several events:**

- "The hat is green" is $Y_B = \{\omega \mid \theta(\omega_{col,blind}) = B\} = \{\omega_{B,C}, \omega_{B,N}\}$.
- "Ben is color blind" $Y_C = \{\omega_{R,C}, \omega_{G,C}, \omega_{B,C}\}$.

# Color Blindness

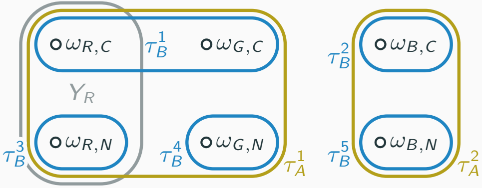



**Who knows what in state $\omega_{B,C}$?**

- Ben's information set is $\tau_B(\omega_{B,C}) = \{\omega_{B,C}\}$.
- Since $\tau_B(\omega_{B,C}) \subseteq Y_B$, Ben knows $Y_B$.
- The states, in which Ben knows $Y_B$ is $K_B(Y_B) = \{\omega_{B,N}, \omega_{B,C}\}$.
- Since $\tau_A(\omega_{B,C}) = \{\omega_{B,N}, \omega_{B,C}\} \subseteq Y_B$, Alan knows $Y_B$ as well.
- Alan does not know whether Ben is color blind since $\tau_A(\omega_{B,C}) \not\subseteq Y_C$.
- Still, Alan knows Ben knows $Y_B$ because $\tau_A(\omega_{B,C}) \subseteq K_B(Y_B)$.

# Check Your Understanding

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**True or false:**

1. If $X \subseteq Y$ for two events $X$ and $Y$, then $Y$ is more informative than $X$.
2. For any player $i$, any state $\omega$ must lie in some information set $\tau_i \in \mathcal{T}_i$.

**Short-answer questions:**

3. What does Alan know about the color of the hat in state $\omega_{R,N}$?
4. What does Ben know about the color of the hat in state $\omega_{R,N}$?
5. Compute $K_A(K_B(Y_R))$. What does it mean?

# Characterization of Knowledge Operators

**Proposition 1.9**

*An operator $K_i : 2^\Omega \to 2^\Omega$ is a knowledge operator if and only if it satisfies the following properties:*

(K1) *Axiom of awareness:* $K_i\Omega = \Omega$,

(K2) *Axiom of knowledge:* $K_i Y \subseteq Y$ *for any event* $Y$,

(K3) *Distribution axiom:* $K_i(X \cap Y) = K_i X \cap K_i Y$ *for any events* $X, Y$,

(K4) *Axiom of introspection:* $K_i\big(K_i Y\big) = K_i Y$ *for any event* $Y$,

(K5) *Axiom of negative introspection:* $(K_i Y)^c = K_i\big((K_i Y)^c\big)$ *for any* $Y$.

**Notes:**

- (K2) implies that you cannot know something that is not true.
- (K3) is sometimes stated in a different, but equivalent form.

# Equivalent Representation of Knowledge Operator

**Lemma 1.10**

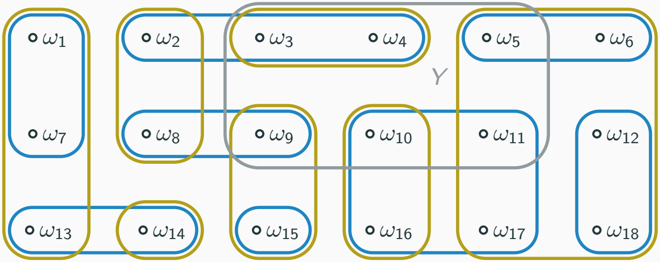
*Fix an information partition $\mathcal{T}_i$ of $\Omega$. For any event $Y$, the event $K_i(Y)$ is the union of all of $i$'s information sets that are fully contained in $Y$:*

$$K_i(Y) = \bigcup_{\tau_i \in \mathcal{T}_i : \tau_i \subseteq Y} \tau_i.$$

**Proof:**

- If $i$ knows $Y$ in state $\omega$, that is, $T_i(\omega) \subseteq Y$, then $i$ knows $Y$ in any state $\omega'$ with $T_i(\omega') = T_i(\omega)$ since then $T_i(\omega') \subseteq Y$.
- Thus, any information set with one state in $K_i(Y)$ must fully lie in $Y$.
- On the other hand, $K_i(Y)$ cannot contain any states of an information set $\tau_i$ that are not fully contained in $Y$ by definition of $K_i(Y)$.

# Example




**Example:**

- $K_1(Y) = \{\omega_3, \omega_4\}$: Player 1 knows event $Y$ in states $\omega_3$ and $\omega_4$.
- $K_2(Y) = \emptyset$: Player 2 cannot know event $Y$.
- Player 1 is aware that Player 2 does not know $Y$: $K_1(K_2^c(Y)) = \Omega$.

# Proof of Proposition 1.9

**Consequence of Lemma 1.10:**

- Lemma 1.10 immediately implies $K_i(Y) \subseteq Y$ (K2).
- If $Y$ is the union of some of $i$'s information sets, then $K_i(Y) = Y$.
- This readily implies $K_i(K_i Y) = K_i(Y)$ (K4) and $K_i(\Omega) = \Omega$ (K1).
- Since $\mathcal{T}_i$ is a partition of $\Omega$ and $K_i$ is a union of information sets, then so is $K_i^c(Y)$. In particular $K_i(K_i^c Y) = K_i^c(Y)$ (K5).

**Distribution axiom (K3):**

- $K_i(X)/K_i(Y)$ are the union of information sets in $X/Y$.
- Thus, $K_i(X) \cap K_i(Y)$ is the union of information sets that lie in both.
- By Lemma 1.10, this set coincides with $K_i(X \cap Y)$.

See Bacharach (1985) for proof of converse.

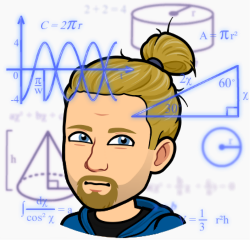
# How Strong Is This Concept of Knowledge?

**Corollary 1.11**

*If $X \subseteq Y$ and $\omega \in K_i(X)$, then $\omega \in K_i(Y)$.*

**Implications:**

- If you know an event $X$, then you know everything that $X$ implies.
- If you know the 9 ZFC axioms, then for every mathematical statement you know whether it is true, false, or undecidable!




**Unbounded rationality:**

- In game theory and most of economics, we assume that players can make infinitely many logical deductions infinitesimally quickly.
- Behavioral game theory and bounded rationality relax this assumption.

# Common Knowledge

# Higher-Order Knowledge

# Knowledge Hierarchies

**Proposition 1.12**

*Let $\Theta$ be a set of states of nature. An Aumann model of incomplete information $(\mathcal{I}, \Omega, (\mathcal{T}_i)_{i \in \mathcal{I}}, \theta)$ over $\Theta$ uniquely defines a knowledge hierarchy $K_{i_1} K_{i_2} \dots K_{i_k}$ for any finite sequence of players $i_1, i_2, \dots, i_k$.*

**Proof:**

- For any event $Y$, the event $K_i Y$ is a well-defined subset of $\Omega$.
- By induction, $K_{i_1} K_{i_2} \dots K_{i_k} Y$ is well defined for any $i_1, i_2, \dots, i_k$.

**Application to Friends:**

- $Y = \{\text{Monica and Chandler are dating}\}$.
- Chandler, Monica, Rachel, and Phoebe could keep pranking each other until arbitrarily high order of knowledge about $Y$.
- If they talk about it in the same room, $Y$ becomes common knowledge.

# Common Knowledge

**Definition 1.13**

An event $Y \subseteq \Omega$ is common knowledge in state $\omega$ if for every finite sequence of players $i_1, \ldots, i_k$,

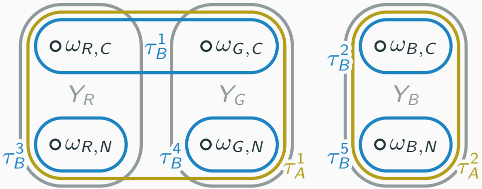
$$\omega \in K_{i_1} K_{i_2} \ldots K_{i_k} Y. \tag{1}$$

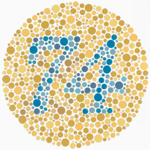
- In particular, equation (1) must hold for arbitrarily large $k$.
- Common knowledge = everybody knows $Y$, they all know that they know $Y$, they all know that they know $Y$, and so on.

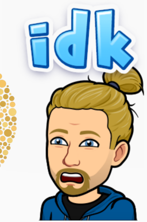
**Games with complete information:**

- Set of players, available actions, and payoffs are common knowledge.
- Every player is rational and rationality of players is common knowledge.

# Color Blindness






**What events are common knowledge?**

- Since $K_A(Y_B) = K_B(Y_B) = Y_B$, it follows that $K_{i_1} K_{i_2} \ldots K_{i_k} Y_B = Y_B$.
- Therefore, $Y_B$ is common knowledge in $\omega_{B,N}$ and $\omega_{B,C}$.
- Similarly, $Y_R \cup Y_G$ is common knowledge in $\omega_{R,C}$, $\omega_{R,N}$, $\omega_{G,C}$, and $\omega_{G,N}$.
- Are there other events that are common knowledge?

# How to Find Events of Common Knowledge

**Lemma 1.14**

*An event $X$ is called self-evident if $K_i X = X$ for every player $i$. An event $Y$ is common knowledge in $\omega$ if and only if there exists a self-evident event $X \subseteq Y$ with $\omega \in X$.*

**Showing equivalence between (A) and (B):**

- Show (A) implies (B) or, equivalently, not (B) implies not (A).
- Show (B) implies (A) or, equivalently, not (A) implies not (B).

We will show (B) implies (A) and not (B) implies not (A).

# Proof of Lemma 1.14

**Preparation:**

- Corollary 1.11 is equivalently stated as $K_i X \subseteq K_i Y$ for any $X \subseteq Y$.
- We say that $K_i$ is monotone for any player $i$.

**Suppose such self-evident X exists:**

- Monotonicity implies that for any sequence $i_1, i_2, \ldots$

$$X = K_{i_1} X \subseteq K_{i_1} Y, \qquad X = K_{i_2} K_{i_1} X \subseteq K_{i_2} K_{i_1} Y, \qquad \ldots$$

**Suppose no suchself-evident event exists:**

- Then for every $X \subseteq Y$, there exists a player $i$ with $K_i X \subsetneq X$.
- There exists a sequence $i_1, i_2, \ldots$ such that

$$Y \supsetneq K_{i_1} Y \supsetneq K_{i_2} K_{i_1} Y \supsetneq \ldots$$

- Since $Y$ is finite, the sequence converges to the empty set.

# Common Knowledge Component

**Definition 1.15**

For any state $\omega \in \Omega$, the common knowledge component $C(\omega)$ in $\omega$ is the smallest set self-evident event $X$ with $\omega \in X$.

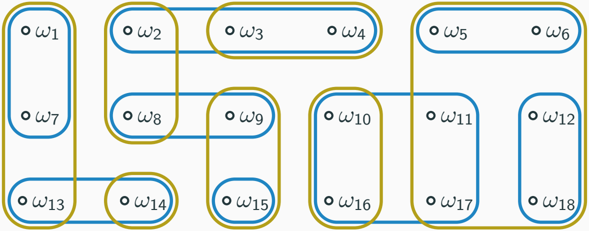
**Notes:**

- $C(\omega)$ is the most informative event that is common knowledge in $\omega$.
- An event $Y$ is common knowledge in $\omega$ if and only if $C(\omega) \subseteq Y$.

**Finding $C(\omega)$:**

1. Start with $C_0(\omega) = \{\omega\}$.
2. For $k \geq 1$, define $C_k(\omega)$ as the union over all information sets $\tau_i$ for all players $i$ with $\tau_i \cap C_{k-1}(\omega) \neq \emptyset$.
3. If $C_k(\omega) = C_{k-1}(\omega)$, then $C_k(\omega) = C(\omega)$.

# Finding Common Knowledge Component Graphically




**Common knowledge in $\omega_9$:**

- Start with $C_0(\omega_9) = \{\omega_9\}$.
- $C_1(\omega_9) = \{\omega_8, \omega_9, \omega_{15}\}$ contains all information sets, which contain $\omega_9$.
- We continue with $C_2(\omega_9) = \{\omega_2, \omega_8, \omega_9, \omega_{15}\}$ and

$$C(\omega_9) = C_3(\omega_9) = \{\omega_2, \omega_3, \omega_4, \omega_8, \omega_9, \omega_{15}\}.$$

Common knowledge components = connected components of the graph.

# Crossing Xinhai Road


Rowan and Layla are waiting to cross Xinhai road. They stand at an angle, the countdown for the green light is partially obstructed:

- Rowan sees only the right-most edge of the digits,
- Layla sees only the left-most edge of the digits.

**Questions:**

- What are the information partitions of Rowan and Layla?
- Which digits do they know when they see it?
- What numbers are common knowledge?

# Learning

# Learning New Information

**Definition 1.16**

If player $i$ with information partition $\mathcal{T}_i$ learns information in $\mathcal{T}_*$, then his/her information partition with the new information is simply $\mathcal{T}_i \cap \mathcal{T}_*$ with information sets $\tau_i \cap \tau_*$ for any $\tau_i \in \mathcal{T}_i$ and $\tau_* \in \mathcal{T}_*$.

**Note:**

- Since player retains information in $\mathcal{T}_i$, information partition gets finer.

**Examples:**

- Players are allowed to communicate, then $\mathcal{T}_* = \mathcal{T}_j$ for $j \neq i$.
- Player observes event $Y$, then $\mathcal{T}_* = \{Y, Y^c\}$.

# Crossing Xinhai Road


**Sharing information:**

- What numbers do Rowan and Layla know if they can communicate?
- Suppose Layla steps onto the road whenever the countdown could be zero and Rowan observes this. What is Rowan's information set now?

**Gathering information individually:**

- Suppose they observe the countdown for two consecutive seconds.
- What are Rowan's and Layla's information sets?
- What numbers are common knowledge now?

# Releasing Prisoners

**Three prisoners are to be released:**

- Each of the three has a mark on their forehead, which others can see but they themselves cannot.
- A prisoner with a mark can simply walk out.
- A prisoner without a mark that attempts to flee will be stopped and sentenced to a life in jail.

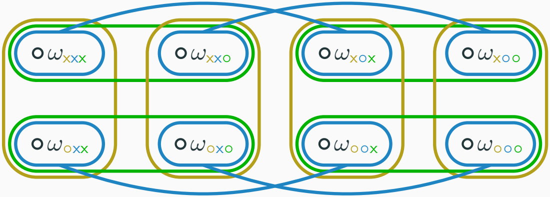
**Available information:**

- There are no reflective surfaces and communication is not allowed.
- All prisoners see each other once a day in the cafeteria for lunch.

**Questions:**

1. Will any prisoners leave? If so, after how many days?
2. If, additionally, the warden announces in the cafeteria that at least one prisoner is marked. Will anybody leave? If so, after how many days?

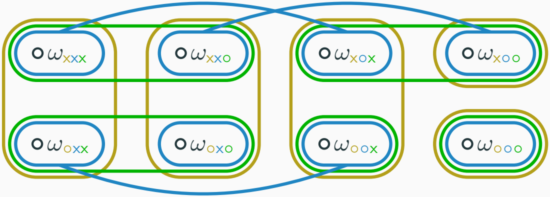
# Releasing Prisoners




**Reducing the scope of the problem:**

- Because of the strong punishment, prisoners will try to leave only if they know they carry the mark.
- By axiom of knowledge (K2), nobody without a mark will try to leave.
- It is enough to consider the three marked prisoners: there are 8 states.
- State $\omega_{xox}$ indicates that prisoners 1 and 3 are marked.

## Releasing Prisoners

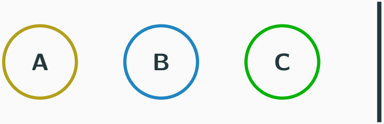



**Learning by observing others:**

- Let $Y_i$ denote the event that $i$ is marked for freedom.
- If $i$ does not leave, other prisoners learn $K_i^c Y_i = (K_i Y_i)^c$, that is, they receive the information contained in partition $\{K_i Y_i, (K_i Y_i)^c\}$.
- Without the warden's announcement $K_i^c Y_i = \emptyset$ for any prisoner $i$.
- Warden's announcement contains the information $\{\omega_{\text{ooo}}, \omega^c_{\text{ooo}}\}$. Now:

$$K_1 Y_1 = \{\omega_{\text{xoo}}\}, \qquad K_2 Y_2 = \{\omega_{\text{oxo}}\}, \qquad K_3 Y_3 = \{\omega_{\text{oox}}\}.$$

# Releasing More Prisoners




**There is no room for Arya, Brienne, and Cersei:**

- Prison guard tells them there are three blue hats and two white hats and he will put one on each of their heads.
- If anybody guesses the color of their hat correctly, they are released.
- If anybody guesses the color of their hat incorrectly, they are executed.

**Questions:**

1. How many states are required to describe this situation?
2. If they are, in fact, all wearing a blue hat. Who announces first that they know the color of their hat?

# Summary

**Aumann model of incomplete information:**

- Formal model to express knowledge over a set of possible states.
- The concept of knowledge is understood in a very strong sense:
  - Players are unboundedly rational: if they know $Y$, they know everything that is deducible from $Y$.
  - This is in line with equilibrium concepts used in game theory.
- It is easy to verify graphically whether an event is common knowledge.

**Applications:**

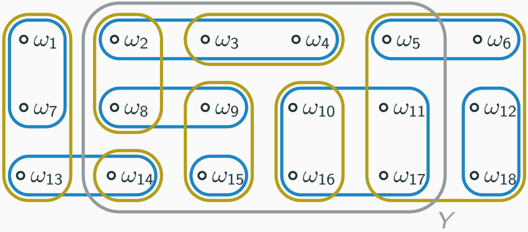
- Solving games/riddles that involve higher-order or common knowledge.
- Especially useful if we enhance the model with beliefs over uncertainty.

# Social Deduction Games


**Social deduction game:**

- Some players are “evil” and hidden from the rest.
- Some actions provide hard evidence that a player is evil.
- Aumann’s model of incomplete information can be used.

# Check Your Understanding

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**True or false:**

1. Event $Y$ is common knowledge in state $\omega_9$.
2. Suppose two players share all the information they have. Then any known event is also commonly known.
3. Suppose prisoner $i$ leaves the prison in the setting of Slide 59. Does this carry the same information for other prisoners as if $i$ stays?

# Literature


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- M. Bacharach: Some Extensions of a Claim of Aumann in an Axiomatic Model of Knowledge, Journal of Economic Theory, **37** (1985), 167–190.