11. Repeated with Imperfect Monitoring

ECON 7219 – Games With Incomplete Information

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Imperfect Monitoring

Newly Opened Restaurant

Imperfect monitoring:

- Social beliefs are updated through Google/Yelp reviews.
- Those are based on the individual experiences of the customers, but they may not always accurately reflect the restaurant owner's effort:
    - The customer may not like that kind of food,
    - The customer's mood will have an impact on the review, etc.
- Because the reviews are correlated to the restaurant owner's efforts, we can still use them to update our beliefs.

University Reputation

UNIVERSITÉ DU LUXEMBOURG

University of Luxembourg:

- Founded in 2003 and wishes to build a good reputation.
- Pays professors very well, trying to attract international talents.
- The observable output is the graduating students' success.
- Quality of students and success after graduation are stochastic, but the distribution is correlated with the university's efforts.
- Will the university achieve its intended reputation?

Points We Have to Address

Payoffs:

- If players receive payoffs every period, those payoffs cannot reveal the chosen actions, otherwise it is a game of perfect monitoring.
- Each player's payoffs can depend only on that player's action and the information he/she has available. How much of a complication is this?

Beliefs and reputations:

- Is Bayesian updating significantly harder if actions are not observed?
- Is it harder/easier to build a reputation?
- Are reputations maintained forever as in the perfect monitoring case?
- Does a result like the reputation bound hold?

Public Signal in the Stage Game

Public signal:

- Let $\mathcal{Y}$ be the space of all possible signals with typical element $y \in \mathcal{Y}$.
- The public signal Y is a $\mathcal{Y}$-valued random variable, whose conditional distribution $\pi(\cdot \mid a)$ depends on the *realized* action profile $A = a$.
- Conditional on $A = a$, the distribution of Y does not depend on θ.

Distribution over outcomes:

- The outcome of the game is the random variable (θ, A, Y).
- For a strategy profile σ and common prior beliefs $\mu \in \Delta(\Theta)$, the probability measure P_σ is extended to $\Theta \times \mathcal{A} \times \mathcal{Y}$ by setting

$$P_\sigma(\theta = \vartheta, A = a, Y = y) = \pi(y \mid a)\sigma(\vartheta; a)\mu(\vartheta).$$

Payoffs in the Stage Game

Stage-game payoffs:

- Ex-post payoff $u_i(\vartheta_i, a_i, y)$ of player i depends only on a_i and y.
- Players maximize their ex-ante payoffs

$$u_i(\vartheta_i, a) := \mathbb{E}_a[u_i(\vartheta_i, A_i, Y)].$$

- Ex-ante payoffs of a mixed action profile α is well defined:

$$\begin{aligned} u_i(\vartheta_i, \alpha) := \mathbb{E}_{\vartheta_i,\alpha}[u_i(\vartheta_i, A_i, Y)] &= \sum_{a \in \mathcal{A}} \mathbb{E}_{\vartheta_i,\alpha}[u_i(\vartheta_i, a_i, Y) \mid A = a]\alpha(a) \\ &= \sum_{a \in \mathcal{A}} \sum_{y \in \mathcal{Y}} u_i(\vartheta_i, a_i, y)\pi(y \mid a)\alpha(a) \\ &= \sum_{a \in \mathcal{A}} u_i(\vartheta_i, a)\alpha(a) = \mathbb{E}_{\vartheta_i,\alpha}[u_i(\vartheta_i, A)]. \end{aligned}$$

Ex-Post Payoffs in the Product-Choice Game

	H	L
$\bar{y}$	$\frac{3(1-q)}{p-q}$	$\frac{2p-3q}{p-q}$
$\underline{y}$	$-\frac{3q}{p-q}$	$\frac{p-2q}{p-q}$

Ex-post payoffs

	H	L
H	$2, 3$	$0, 2$
L	$3, 0$	$1, 1$

Ex-ante payoffs

Customer satisfaction:

- Suppose that the customer satisfaction Y can either be high ($\bar{y}$) or low ($\underline{y}$) with conditional distribution given by

$$\pi(\bar{y} \mid a) = \begin{cases} p & \text{if } a_1 = H, \\ q & \text{if } a_1 = L, \end{cases} \qquad \pi(\underline{y} \mid a) = \begin{cases} 1-p & \text{if } a_1 = H, \\ 1-q & \text{if } a_1 = L. \end{cases}$$

  with $1 > p > q > 0$ so that customer satisfaction increases with effort.
- For the above ex-post payoffs, ex-ante payoffs are equal to last week's.

Public Signal in the Repeated Game

Definition 11.1

The public signal $Y = (Y^t)_{t \geq 0}$ is a $\mathcal{Y}$-valued stochastic sequence satisfying:

1. Y^t is independent of Y^s for any $s < t$,
2. Conditional on A^t, Y^t is independent of θ and A^s for any $s < t$,
3. Conditional on $A^t = a$, the distribution of Y^t is $\pi(\,\cdot \mid a)$.

Private histories:

- Player i's private history at time t is $h_i^t = (a_i^0, y^0, \ldots, a_i^{t-1}, y^{t-1})$.
- We denote by $\mathcal{H}_i^t$ the set of i's private histories of length t and by $\mathcal{H}_i = \bigcup_{t=0}^{\infty} \mathcal{H}_i^t$ the set of all of i's private histories.
- As usual, we denote by $H_i^t = (A_i^0, Y^0, \ldots, A_i^{t-1}, Y^{t-1})$ player i's random history of length t and by $H_i = (H_i^t)_{t \geq 0}$ the entire sequence.

Payoffs in the Reputation Game

Player 2's payoff:

- Given history $h^{t+1} = (h^t, a_2^t, y^t)$, player 2 updates their beliefs to

$$\mu(h^{t+1}; \vartheta) = \frac{\sum_{a_1^t \in \mathcal{A}_1} \pi(y^t \mid a^t)\sigma_1(\vartheta, h^t; a_1^t)\mu(h^t; \vartheta)}{\sum_{\vartheta' \in \Theta} \sum_{a_1^t \in \mathcal{A}_1} \pi(y^t \mid a^t)\sigma(\vartheta', h^t; a^t)\mu(h^t; \vartheta')}.$$

- Player 2 maximizes ex-ante stage-game payoff given beliefs $\mu(h^t)$.

Player 1's payoff:

- Writing $u_1(\vartheta_p, a, y) = u_1(a, y)$, the payoff type ϑ_p maximizes

$$U_1(\sigma) := (1-\delta)\sum_{t=0}^{\infty} \delta^t \mathbb{E}_\sigma\left[u_1(A^t, Y^t)\right] = (1-\delta)\sum_{t=0}^{\infty} \delta^t \mathbb{E}_\sigma\left[u_1(A^t)\right],$$

where $u_1(a) = u_1(\vartheta_p, a)$ is defined on Slide 5.

First Implications of Imperfect Monitoring

Imperfect monitoring in theory:

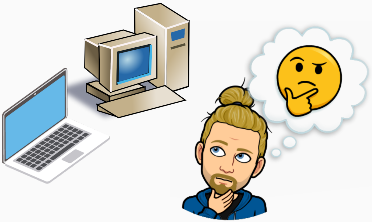
- Updating of beliefs poses no conceptual problem.
- Players' payoffs in the reputation game depend only on ex-ante stage game payoffs, hence we need not specify ex-post payoffs.

Imperfect monitoring in practice:

- Even commitment types cannot prevent a bad outcome.
- This may be an advantage or a disadvantage to player 1:
  - It may be possible gain extra utility by deviating without being noticed.
  - It may take be more difficult to build a reputation.
- How much harder is it to find Nash equilibria in a given game?

Product Choice with Imperfect Monitoring

	H	L
H	$2, 3$	$0, 2$
L	$3, 0$	$1, 1$



Product choice:

- Player 1 is either the commitment type ϑ_H or the payoff type ϑ_p.
- For $1 > p > q > 0$, the customer satisfaction Y is either high ($\bar{y}$) or low ($\underline{y}$) with conditional distribution given by

$$\pi(\bar{y} \mid a) = \begin{cases} p & \text{if } a_1 = H, \\ q & \text{if } a_1 = L, \end{cases} \qquad \pi(\underline{y} \mid a) = \begin{cases} 1-p & \text{if } a_1 = H, \\ 1-q & \text{if } a_1 = L. \end{cases}$$

- What is the Nash equilibrium in the twice-repeated game?

Parametrizing the Strategy

Parametrizing the Firm's strategy:

- In any Nash equilibrium σ, the commitment type must choose H after any history and the payoff type must choose L in the second period.
- The Firm's strategy σ_1 is entirely parametrized by $\sigma_1(\vartheta_p, \emptyset; H) = x$.

Parametrizing the Customers' strategy:

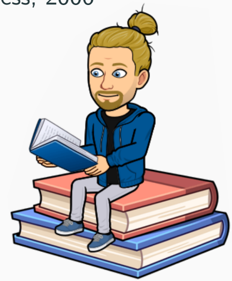
- Let z_0 denote the probability of choosing H in the first period.
- Let $\bar{z}$ and $\underline{z}$ denote the probability of choosing H after $\bar{y}$, and $\underline{y}$.

Comparison to perfect monitoring:

- Similarity: there is one good and one bad outcome.
- Dissimilarity: the bad outcome does not prove that the Firm is type ϑ_p.
- Complete the example in the assignment.

Literature

- D. Fudenberg and J. Tirole: Game Theory, Chapters 5.5–5.6, MIT Press, 1991
- G.J. Mailath and L. Samuelson: Repeated Games and Reputations: Long-Run Relationships, Chapters 7–9, Oxford University Press, 2006
- D. Abreu, D. Pearce and E. Stacchetti: Toward a theory of discounted repeated games with imperfect monitoring, Econometrica, **58** (1990), 1041–1063
- D. Fudenberg and D.K. Levine: Maintaining a Reputation when Strategies are Imperfectly Observed, Review of Economic Studies, **59** (1992), 561–579



Impermanent Reputations

Reputation Effect

Perfect monitoring:

- Ω is the set of all possible outcomes $\omega = (\vartheta, a^0, a^1, \dots)$.
- Player 2 makes no mistakes in the inference on player 1's type.
- Beliefs $\mu(h^t; \vartheta_{\widehat{a}_1})$ are non-decreasing for outcomes in

$$\Omega' = \left\{\omega \in \Omega \;\middle|\; A_1^t(\omega) = \widehat{a}_1 \text{ for all } t\right\}.$$

- We used the correlation between player 1's type and the chosen actions to establish a bound on the frequency, with which $\widehat{a}_1$ is expected.

Imperfect monitoring:

- Beliefs fluctuate because bad signals occur even for commitment types.
- What can we say about the evolution of player 2's beliefs?
- The results need some additional concepts from probability theory.

Belief Process

Naive way to define beliefs:

- Define $\mu(h_2^t) \in \Delta(\Theta)$ by setting $\mu(h_2^t; \vartheta) = P_\sigma(\theta = \vartheta \,|\, H_2^t = h_2^t)$.
- Define the belief process $\mu = (\mu_t)_{t \geq 0}$ by setting $\mu_t = \mu(H_2^t)$.
- This is well-defined only for histories h_2^t with positive measure.

Belief processes:

- Let $\mathcal{F}_t^2$ be the σ-algebra generated by $H^2 = (H_t^2)_{t \geq 0}$.
- Define belief process as $\Delta(\Theta)$-valued stochastic sequence $\mu = (\mu_t)_{t \geq 0}$ by setting $\mu_t(\vartheta) := P_\sigma(\theta = \vartheta \,|\, \mathcal{F}_2^t)$ for every $\vartheta \in \Theta$.
- For a history h_2^t with positive probability, we have

$$\mu_t 1_{\{H_2^t = h_2^t\}} = P_\sigma\left(\theta = \vartheta \,\middle|\, H_2^t = h_2^t\right) 1_{\{H_2^t = h_2^t\}} = \mu(h_2^t) 1_{\{H_2^t = h_2^t\}}.$$

Martingales

Definition 11.2

A sequence of random variables $X = (X_t)_{t \geq 0}$ is a martingale if

1. X is adapted to $\mathbb{F}_2$, that is, X_t is $\mathcal{F}_2^t$-measurable for every $t \geq 0$,
2. X is integrable, i.e., $\mathbb{E}[|X_t|] < \infty$ for every $t \geq 0$,
3. X satisfies the martingale property:

$$\mathbb{E}[X_t \mid \mathcal{F}_2^s] = X_s \qquad \text{for all } s \leq t.$$

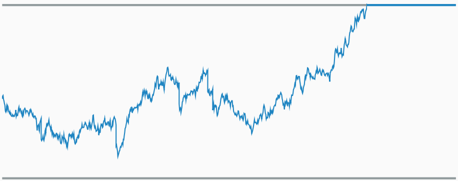
Beliefs are constant on average:

- If μ_t is deterministic, μ_{t+1} is a Bayes-plausible distribution of posteriors.
- If μ_t is a random variable, then μ_{t+1} is a mean-preserving spread of μ_t.
- In general, μ_t is a martingale.

Martingale Convergence Theorem

Theorem 11.3 (Martingale Convergence Theorem)

Any bounded martingale converges P-almost-surely.



Intuition:

- Because a martingale is constant in expectation, expected upward movements are equal to the expected downward movements.
- Once the martingale hits the upper boundary, no upward movements are possible anymore, hence there are no downward movements either.

Identifiability of Deviations

Definition 11.4

Suppose that $\mathcal{Y}$ and $\mathcal{A}_i$ for $i = 1, 2$ are finite. We say that π:

1. has full support if $\pi(y \mid a) > 0$ for every $y \in \mathcal{Y}$ and every $a \in \mathcal{A}$,
2. is linearly independent if, for both players i and every action $a_i \in \mathcal{A}_i$, the vectors $\pi(\,\cdot \mid a_i, a_{-i})$ for $a_{-i} \in \mathcal{A}_{-i}$ are linearly independent.

Remark:

- Linear independence ensures that player i, knowing he/she played a_i, can statistically distinguish $\alpha_{-i} \neq \alpha'_{-i}$ by observing the signal.
- The signal in perfect monitoring games is $Y = A$, hence $\pi(\,\cdot \mid a_i, a_{-i})$ are orthogonal and, in particular, linearly independent.
- Perfect monitoring does not satisfy the full support assumption.

Impermanent Reputations

Proposition 11.5

Suppose that $\mathcal{Y}$ and $\mathcal{A}_i$ for $i = 1, 2$ are finite, that π has full support, and that π is linearly independent. Let $\widehat{\alpha}_1$ be a commitment action with a unique myopic best reply $\widehat{a}_2$ such that $(\widehat{\alpha}_1, \widehat{a}_2)$ is not a stage-game Nash equilibrium. Then in any Nash equilibrium σ of the reputation game,

$$P_\sigma\left(\theta = \vartheta_{\widehat{\alpha}_1} \,\middle|\, \mathcal{F}_2^t\right) \to 0 \qquad \widetilde{P}_\sigma\text{-a.s.},$$

where $\widetilde{P}_\sigma(\cdot) = P_\sigma(\cdot \mid \theta \neq \vartheta_{\widehat{\alpha}_1})$.

Interpretation:

- If the signal is sufficiently informative and player 1 is not the commitment type, then a reputation effects are temporary in any equilibrium.
- This may be surprising since identifiability helps building a reputation.

Idea of Proof

Idea of proof:

- Fix a Nash equilibrium σ. Suppose that there exists a set of outcomes $\Omega'' \subseteq \Omega$ with $\widetilde{P}(\Omega'') = P_\sigma(\Omega'' \,|\, \theta \neq \vartheta_{\widehat{\alpha}_1}) > 0$ such that on Ω'':

$$\lim_{t\to\infty} P_\sigma\left(\theta = \vartheta_p \,\middle|\, \mathcal{F}_2^t\right) > 0, \qquad \lim_{t\to\infty} P_\sigma\left(\theta = \vartheta_{\widehat{\alpha}^1} \,\middle|\, \mathcal{F}_2^t\right) > 0.$$

- On Ω'', player 2 cannot distinguish signal distributions and must believe both types are playing the same strategy on average.
- Linear independence: player 2 will eventually best reply to $\widehat{\alpha}_1$ and player 1 will eventually learn that player 2 is best replying to $\widehat{\alpha}_1$.
- Since $(\widehat{\alpha}_1, \widehat{a}_2)$ is not a Nash equilibrium and beliefs are constant in the limit, player 1 must find it profitable to deviate eventually.
- This contradicts that player 2's beliefs are constant in the limit.

Summary

Imperfect monitoring:

- Building a reputation may take longer than with perfect monitoring.
- Once the reputation is built and player 1 learns that the reputation is effective, he/she exploits the reputation to reap its benefits.
- This causes reputation effects to be temporary.
- We say that reputations are optimally depleted.

Relaxing the full-support assumption:

- Let $\mathcal{Y}(\alpha)$ denote the signals y with $\pi(y \mid \alpha) > 0$.
- The result still holds if player 1 has a profitable deviation $\widetilde{\alpha}_1$ to $(\widehat{\alpha}_1, \widehat{a}_2)$ such that $\mathcal{Y}(\widetilde{\alpha}_1, \widehat{a}_2) \subseteq \mathcal{Y}(\widehat{\alpha}_1, \widehat{a}_2)$.

Reputation Bound with Imperfect Monitoring

Reputation Bound

Reputation bound with perfect monitoring:

- If the payoff type incessantly plays a commitment action $\widehat{a}_1$, player 2 must eventually best reply to $\widehat{a}_1$ by the reputation effect.
- Any Nash equilibrium σ must be robust to the deviation $\widetilde{\sigma}_1$ that mimics $\vartheta_{\widehat{a}_1}$, hence the payoff induced by $\widetilde{\sigma}_1$ is a lower bound for $U_1(\sigma)$.
- Note that this proof does not require that $\widetilde{\sigma}_1$ is a Nash equilibrium.

Reputation effect:

- For player 2 to eventually best respond to $\widehat{a}_1$, we need that:
  - Player 2 can distinguish $\widehat{a}_1$ from other actions,
  - $\mathcal{A}_2$ is finite, so that when player 2 expects to see $\widehat{a}_1$ with probability above some threshold ζ, he/she will play a best response to $\widehat{a}_1$.
- Linear independence should allow us to recover the reputation effect.

Statistical Indistinguishability

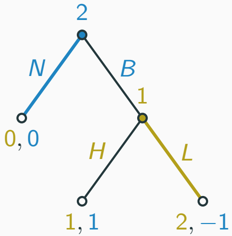
Imperfect monitoring:

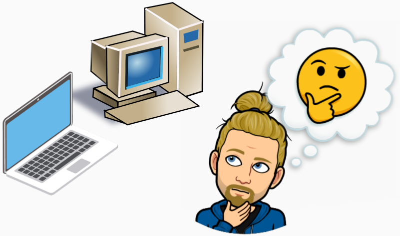
- In a Nash equilibrium, reputations are impermanent because player 1 will eventually find it profitable to deviate.
- Since the reputation bound does not rely on a Nash equilibrium, but the deviation to $\widetilde{\sigma}_1$, maybe weaker requirements on π are sufficient.

Statistical indistinguishability:

- If (α_1, α_2) and $(\widetilde{\alpha}_1, \alpha_2)$ lead to the same signal distribution, player 2 is not able to differentiate between repeated play of α_1 and $\widetilde{\alpha}_1$.
- Payoff type may imitate ϑ_{α_1} but get best responses to $\widetilde{\alpha}_1$.
- This can be very unfortunate for player 1.

Sequential Product Choice





Dynamic stage game:

- Customers choose whether to (B)uy or (N)ot to buy the product.
- If the Customers choose to buy, the Firm decides whether to exert (H)igh or (L)ow effort in producing/delivering the product.
- The terminal node of the stage game is observed: $Y \in \{y_N, y_{BH}, y_{BL}\}$.
- Monitoring is imperfect because no information about the Firm is revealed if Customers choose N.

Signal Distribution in Sequential Product Choice

	H, B	H, N	L, B	L, N
y_N	0	1	0	1
y_{BH}	1	0	0	0
y_{BL}	0	0	1	0

Repeated play of H:

- If the Customers never buy the product, they do not observe whether the Firm chooses H even if $\sigma_1(\vartheta_p, h) = H$ for every history h.
- Since H and L are indistinguishable on the path, the Customers may best reply to what they perceive as L by playing N forever.
- The Firm cannot exploit the reputation effect.
- If $\mu(\vartheta_H) < \frac{1}{2}$, then $(0, 0)$ is an equilibrium payoff even if $\delta > \frac{1}{2}$.

Epsilon-Confirmed Best Responses

Definition 11.6

For any $\varepsilon > 0$, an action $\alpha_2 \in \Delta(\mathcal{A}_2)$ is an ε-confirmed best response to $\alpha_1 \in \Delta(\mathcal{A}_1)$ if there exists $\widetilde{\alpha}_1 \in \Delta(\mathcal{A}_1)$ such that:

1. $\alpha_2 \in \mathcal{B}_2(\widetilde{\alpha}_1)$.
2. $\|\pi(\cdot \mid \alpha_1, \alpha_2) - \pi(\cdot \mid \widetilde{\alpha}_1, \alpha_2)\| \leq \varepsilon$.

Let $\mathcal{B}_2^{\varepsilon}(\alpha_1)$ denote the set of all ε-confirmed best responses to α_1.

Remark:

- This notion measures closeness in signal distribution (hence ε-confirmed best responses), not closeness in utility.
- If strategic player repeatedly plays α_1, for any $\varepsilon > 0$, he/she can expect responses from the set $\mathcal{B}_2^{\varepsilon}(\alpha_1)$ in the long run.

Signal Distribution in Sequential Product Choice

	H, B	H, N	L, B	L, N
y_N	0	1	0	1
y_{BH}	1	0	0	0
y_{BL}	0	0	1	0

0-confirmed best response:

- (H, N) and (L, N) lead to the same signal with probability 1, hence

$$\left\| \pi(\cdot \mid H, N) - \pi(\cdot \mid L, N) \right\| = 0.$$

- N is a best response to L, hence a 0-confirmed best response to H.

Linear Independence

Lemma 11.7

If π is linearly independent, then $\mathcal{B}_2^0(\alpha_1) = \mathcal{B}_2(\alpha_1)$ for any α_1, that is, 0-confirmed best responses are best responses. Moreover, if $\mathcal{Y}$ and $\mathcal{A}_i$ for $i = 1, 2$ are finite, then $\mathcal{B}_2^\varepsilon(\alpha_1) = \mathcal{B}_2(\alpha_1)$ for sufficiently small ε.

Proof:

- By linear independence, $\pi(\,\cdot\,|\,\alpha_2, \alpha_1)$ is injective in α_1 for any α_2.
- If $\mathcal{A}_2$ is finite, then for any α_1, there exists $\varepsilon(\alpha_1) > 0$ such that $\mathcal{B}_2(\widetilde{\alpha}_1) \subseteq \mathcal{B}_2(\alpha_1)$ for any $\widetilde{\alpha}_1$ with $\|\widetilde{\alpha}_1 - \alpha_1\| \leq \varepsilon(\alpha_1)$.
- Since $\pi(\,\cdot\,|\,\alpha_2, \alpha_1)$ is continuous in α_1, there exists $\varepsilon > 0$ such that

$$\|\pi(\,\cdot\,|\,\alpha_1, \alpha_2) - \pi(\,\cdot\,|\,\widetilde{\alpha}_1, \alpha_2)\| \leq \varepsilon$$

  implies $\|\widetilde{\alpha}_1 - \alpha_1\| \leq \varepsilon(\alpha_1)$, hence $\mathcal{B}_2(\widetilde{\alpha}_1) \subseteq \mathcal{B}_2(\alpha_1)$.

Reputation Bound with Imperfect Monitoring

Theorem 11.8

Suppose that $\mu_0(\vartheta_p) > 0$. For any $\varepsilon > 0$ and any $\widehat{\alpha}_1 \in \Delta(\mathcal{A}_1)$ with $\vartheta_{\widehat{\alpha}_1} \in \Theta_c$ and $\mu_0(\vartheta_{\widehat{\alpha}_1}) > 0$, there exists a constant $K(\mu_0) > 0$ such that

$$\inf_{Eq.\ \sigma} U_1(\sigma) \geq (1-\varepsilon)\delta^K \min_{\alpha_2 \in \mathcal{B}_2^{\varepsilon}(\widehat{\alpha}_1)} u_1(\widehat{\alpha}_1, \alpha_2) + (1-(1-\varepsilon)\delta^K) \min_{a \in \mathcal{A}} u_1(a).$$

Interpretation:

- It takes $K(\mu_0)$ periods to build a sufficient reputation to get ε-confirmed best responses to the commitment action $\widehat{\alpha}_1$.
- If π is linearly independent, the long-lived player elicit best responses to $\widehat{\alpha}_1$ instead of ε-confirmed best responses.
- The proof is similar to Theorem 11.8 and can be found in the book.

University Reputation

uni.lu

UNIVERSITÉ DU
LUXEMBOURG

University of Luxembourg:

- If they are the strategic type, they will not fool us asymptotically.
- But: deviating will be in the university's own interest.
- Until the university of Luxembourg finds it beneficial to reveal its type, students actually do get a good education.

Check Your Understanding

True or false:

Scan Me

1. Reputations are impermanent because they are too hard to build with imperfect monitoring.
2. Suppose commitment type $\vartheta_{\hat{a}_1}$ maximizes the long-lived player's payoff in the long-run. Other commitment types do not have an impact on payoffs.
3. When attaining the reputation bound in Theorem 11.8, the long-lived player earns a low payoff during the first $K(\mu_0)$ periods.
4. With imperfect monitoring, beliefs of being a certain commitment type $\vartheta_{\hat{a}_1}$ may decrease even if the payoff type invariably plays $\hat{a}_1$.
5. Suppose all actions are observed. To justify a lower bound of $\underline{v}(\hat{\alpha}_1)$ on equilibrium payoffs for a mixed commitment type $\vartheta_{\hat{\alpha}_1}$ we need the imperfect-monitoring reputation bound.

Literature

- G.J. Mailath and L. Samuelson: Repeated Games and Reputations: Long-Run Relationships, Chapter 15.4–15.6, Oxford University Press, 2006
- M.W. Cripps, G.J. Mailath, and L. Samuelson: Imperfect Monitoring and Impermanent Reputations, Econometrica, **72** (2004), 407–432
- D. Fudenberg and D.K. Levine: Maintaining a Reputation when Strategies are Imperfectly Observed, Review of Economic Studies, **59** (1992), 561–579