Macro Theory I Part 2 - Quiz 2

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Question 1

(1) Write down the social planner's problem

Social planner knows how to achieve the most efficient allocation for this economy, so that he/she also knows how many inputs of aggregate human capital to be needed and he/she can directly tells household how many human capitals to be supplied. That is, we can replace $\bar{h}_t$ in the production function with h_t. On the other hand, for social planner's problem, $\bar{h}_t$ can not be taken as given, he/she need to decide the aggregate amount of human capital and thus $\bar{h}_t = h_t$.

Resource constraint:

$$c_t = (l_{1t})^{\alpha} h_t^{\eta}$$

Law of motion for human capital:

$$h_{t+1} = Bh_t (1 - l_{1t})$$

Social planner's problem:

$$\begin{aligned} \max_{\{c_t, h_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \ln c_t \\ \text{s.t.} \quad & c_t = (l_{1t})^{\alpha} h_t^{\eta} \\ & h_{t+1} = Bh_t (1 - l_{1t}) \end{aligned}$$

(2) Find g_c and g_h along the balanced growth path.

Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{\ln c_t + \lambda_t [(l_{1t})^{\alpha} h_t^{\eta} - c_t] + \mu_t [Bh_t (1 - l_{1t}) - h_{t+1}]\}$$

F.O.C.

$$[c_t]: \quad \frac{1}{c_t} = \lambda_t \tag{1}$$

$$[l_{1t}]: \quad \mu_t B h_t = \lambda_t \alpha (l_{1t})^{\alpha-1} h_t^{\eta} \tag{2}$$

$$[h_{t+1}]: \quad \mu_t = \beta \left[\lambda_{t+1} \eta (l_{1t+1})^{\alpha} h_{t+1}^{\eta-1} + \mu_{t+1} \cancel{B} (1 - l_{1t+1})\right] \tag{3}$$

$$\frac{\lambda_{t+1} \alpha (l_{1t+1})^{\alpha-1}}{\cancel{B}\cancel{h_{t+1}}} h_{t+1}^{\eta-1} (1 - l_{1t+1})$$

$\beta \left[\lambda_{t+1} (l_{1t+1})^{\alpha-1} h_{t+1}^{\eta-1} \left[\eta l_{1t+1} + \alpha (1-l_{1t+1}) \right] \right]$

$\alpha + (\eta-\alpha) l_{1t+1}$

Combining Eq.(2) and Eq.(3) yields

$$\frac{\lambda_t \alpha (l_{1t})^{\alpha-1} h_t^{\eta-1}}{B} = \beta \lambda_{t+1} (l_{1t+1})^{\alpha-1} h_{t+1}^{\eta-1} \left[(\eta - \alpha) l_{1t+1} + \alpha \right] \tag{4}$$

Combining Eq.(1) and Eq.(4) yields

$$\frac{c_{t+1}}{c_t} = \frac{\beta B}{\alpha} \left(\frac{l_{1t+1}}{l_{1t}} \right)^{\alpha-1} \left(\frac{h_{t+1}}{h_t} \right)^{\eta-1} \left[(\eta-\alpha) l_{1t+1} + \alpha \right] \tag{5}$$

Look for the growth rate along the balanced growth path. Suppose that $c_t = g_c^t c^*$ and $h_t = g_h^t h^*$. Since time endowment is fixed in a unit, so that the growth of l_{1t} and l_{2t} is not significant. We suppose that $l_{it} = l_{it+1} \equiv l_i^*$, $i = 1, 2$.

1. From resource constraint,

$$\begin{aligned} g_c^t c^* &= (l_1^*)^\alpha \left(g_h^t h^* \right)^\eta \\ \Longrightarrow \qquad c^* &= (l_1^*)^\alpha \left(\frac{g_h^\eta}{g_c} \right)^t (h^*)^\eta \end{aligned} \tag{6}$$

   Eq.(6) is stationary if

$$g_c = g_h^\eta \tag{7}$$

2. From law of motion for human capital, dividing it by h_t gives us

$$\begin{aligned} \frac{h_{t+1}}{h_t} &= B(1 - l_{1t}) \\ \Longrightarrow \quad g_h &= B(1 - l_1^*) \end{aligned}$$

   Thus, we have

$$l_1^* = 1 - \frac{g_h}{B} \tag{8}$$

3. From Eq.(5),

$$\begin{aligned} g_c &= \frac{\beta B}{\alpha} \left(\frac{l_1^*}{l_1^*} \right)^{\alpha-1} \left(\frac{g_h^{t+1} h^*}{g_h^t h^*} \right)^{\eta-1} \left[(\eta-\alpha) l_1^* + \alpha \right] \\ &= \frac{\beta B}{\alpha} g_h^{\eta-1} \left[(\eta-\alpha) l_1^* + \alpha \right] \end{aligned}$$

   Replacing l_1^* by Eq.(8) gives us

$$\begin{aligned} g_c &= \frac{\beta B}{\alpha} g_h^{\eta-1} \left[(\eta-\alpha) \left(1 - \frac{g_h}{B} \right) + \alpha \right] \\ &= \frac{\beta B}{\alpha} g_h^{\eta-1} \left[\eta - \left(\frac{\eta-\alpha}{B} \right) g_h \right] \end{aligned}$$

   Replacing g_c by Eq.(7) gives us

$$\begin{aligned} g_h^\eta &= \frac{\beta B}{\alpha} g_h^{\eta-1} \left[\eta - \left(\frac{\eta-\alpha}{B} \right) g_h \right] \\ \Longrightarrow g_h &= \frac{\beta B}{\alpha} \left[\eta - \left(\frac{\eta-\alpha}{B} \right) g_h \right] \end{aligned}$$

Then, we can solve

$$g_h = \frac{\beta\eta B}{\alpha(1-\beta)+\eta\beta}$$

And thus,

$$g_c = \left[\frac{\beta\eta B}{\alpha(1-\beta)+\eta\beta}\right]^{\eta}$$

Correction. We also can combine all constraints into one constraint and yield result. In the tutorial, the statement that we can not combine them is incorrect.

(3) How the externality affects the growth rate of human capital along the balanced growth rate?

$$\begin{aligned}
\frac{\partial g_h}{\partial \eta} &= \frac{\beta B\left[\alpha(1-\beta)+\eta\beta\right]-\beta^2\eta B}{\left[\alpha(1-\beta)+\eta\beta\right]^2} \\
&= \frac{\beta B\alpha(1-\beta)+\beta^2\eta B-\beta^2\eta B}{\left[\alpha(1-\beta)+\eta\beta\right]^2} \\
&= \frac{\alpha\beta(1-\beta)B}{\left[\alpha(1-\beta)+\eta\beta\right]^2} > 0
\end{aligned}$$