Macroeconomic Theory - Recursive Methods

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Sequential Problem

Given k_0

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{(SP)}$$

$$\text{subject to} \quad c_t + k_{t+1} = g(k_t)$$

where $g(k_t) = f(k_t) + (1-\delta)k_t$ → just 简化

- (SP) is called the **sequential problem** ↗ 难以证明 $\begin{cases} \text{1. 存在性} \\ \text{2. 唯一性} \\ \text{3. path 的 stable} \end{cases}$
- A special aspect of sequential problem is that the sequential problem is infinite dimensional $k_1, k_2, \ldots k_\infty$
- Dynamic programming turns out to be an ideal tool for dealing with the issues raised by infinity

Infinite Horizon

Let

value func.

$$v_0(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(g(k_t) - k_{t+1})$$

不合理!

start from time 0

initial capital

Let

∵從s期開始，∴沒 β^{t-s} 的話，從當下就會開始折現

$$v_s(k_s) = \max_{\{k_{t+1}\}_{t=s}^{\infty}} \sum_{t=s}^{\infty} \beta^{t-s} u(g(k_t) - k_{t+1})$$

start from time s

We can transform a sequential problem as follows:

把Σ的t=0拿出來

$$\begin{aligned} v_0(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ u(g(k_0) - k_1) + \sum_{t=1}^{\infty} \beta^t u(g(k_t) - k_{t+1}) \right\} \\ &= \max_{k_1} \left\{ u(g(k_0) - k_1) + \beta \boxed{\max_{\{k_{t+1}\}_{t=1}^{\infty}}} \left\{ \underbrace{\sum_{t=1}^{\infty} \beta^{t-1} u(g(k_t) - k_{t+1})}_{= v_1(k_1)} \right\} \right\} \end{aligned}$$

反正不影響，能拿到後面

Then

$$\underbrace{v_0(k_0)}_{\text{value fn for } t_0} = \max_{k_1} \{ \underbrace{u(g(k_0) - k_1)}_{\substack{\text{utility for } t_0 \\ \text{(flow value)}}} + \underbrace{\beta v_1(k_1)}_{\substack{\text{value for } t_1 \\ \text{(stock value)}}} \}$$

Infinite Horizon

從任一期開始都沒差
反正最後都會衝到∞期，∴效用會完全一樣

- We observe that $v_0(k) = v_s(k)$ for all k, and we denote it by $v(k)$, then

v_s,

↳任一期都是 $V(k)$

$$v(k) = \max_{k'} \left\{ u(g(k) - k') + \beta v(k') \right\} \tag{FE}$$

→不需處理∞問題

- We call (FE) the **functional equation** form

the recursive problem

Optimality conditions of the Recursive Problem

↳ How to solve?

- Suppose that we already know the correct form of $v(k)$, we can obtain k' (and c) as a function of k and thus solve for the whole dynamic path of consumption and capital recursively

$$\max_{k'} \underbrace{u(g(k)-k')}_{\because k\downarrow\ \therefore \text{今日效用}\downarrow} + \underbrace{\beta v(k')}_{\text{未来效用高}}$$

↔ → trade off

- Given k, we define $k'^* = h(k)$ by the solution of the functional equation.
- If $v(k)$ is continuous, differentiable, and the solution c and k' are always interior, the first order condition with respect to k' is

$$u'(g(k)-k'^*) = \beta v'(k'^*) \tag{1}$$

↓ 今日减少1单位k的效用 = 明天增加的 benefit

- Then $k'^* = h(k)$ solve (1)

$k_1 = h(k_0) \Rightarrow k_2 = h(k_1) \Rightarrow k_3 = h(k_2)$

$\cdots \Rightarrow$ Dynamic path!

Optimality conditions of the Recursive Problem

- However, we have to know the right form of $v(k)$ before we apply it to find $h(k)$

Ramsey 都符合!

- We have ignored several important questions

1. Can we be sure that a v that satisfies (FE) exists? Yes, 只要符合某個條件
2. How to find the v that solves (FE)? some numerical method!
3. In what circumstance a solution to (FE) is also a solution to (SP)?