Algorithms Midterm

November 5, 2013, 09:10 - 12:10

Please answer the following 8 questions on the answer sheets provided. Be sure to write your name and student ID on all answer sheets you use. No books, notes, or calculators may be used during the exam with the exception of one double-sided, hand-written A4 note. Read all questions first. You may not request for clarification after 10am.

Problem 1 (20%)

Let all functions be positive. Prove or disprove the following three statements. You may only use the definitions of asymptotic notations.

1. (5%) $5n + 10 = O(n^2)$.
2. (7%) If $f(n) = O(g(n))$ and $f(n) \neq o(g(n))$, then $f(n) = \Theta(g(n))$.
3. (8%) If $f(n) = O(g(n))$ and $h(n) = \omega(g(n))$, then $f(n) = o(h(n))$.

Problem 2 (15%)

Solve the following recurrences. You only need to obtain the asymptotic solution (in $\Theta()$ notation).

1. $T(n) = 2T(\frac{n}{2}) + \sqrt{n}, T(1) = 1$.
2. $T(n) = T(n-1) + 2T(n-2) - 4, T(1) = 1, T(2) = 3$.
3. $T(n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} T(n-2i), T(0) = T(1) = 1$.

Problem 3 (15%)

You are given an array of n distinct integers $a_1 < a_2 < \cdots < a_n$. Design an $O(\log n)$ algorithm that determines whether there exists an index i such that $a_i = i$. Briefly justify the correctness and the running time of the algorithm.

Problem 4 (10%)

In randomized quicksort, the pivot is chosen uniformly at random. Suppose we modify the algorithm such that each time five elements are chosen at random and pick the third largest one (in O(1) time) as the pivot. What is the worst case asymptotic running time of this new sorting algorithm? Briefly justify your answer.

Problem 5 (15%)

You are running a restaurant GreedyOne with n seats. One day, many customers arrive. They form groups of different sizes. For each group of customers, you must either seat the whole group or they will leave. You decide to sort the sizes of groups and seat the largest group that can still fit in the restaurant. Repeat until no other groups can fit in.

1. (5%) Show an example that you are not having the maximum number of customers.
2. (10%) If there are at least $\frac{n}{2}$ customers in total and all group sizes are less than n. Prove that you must have at least $\frac{n}{2}$ customers in your restaurant.

Problem 6 (15%)

You are preparing a party for a company. Every person in the company has a non-negative "happiness value" and you would like to maximize the sum of happiness values for the invited guests. To ensure that the party goes smoothly, you may not invite a person and his direct supervisor at the same time. Given that this company's supervisor-employee relationship forms a tree structure, design a polynomial time algorithm that finds the largest total happiness value possible. (Your algorithm only need to determine the largest total happiness value and does not need to actually find the corresponding guest list.) Briefly justify the correctness and analyze the running time.

Problem 7 (10%)

You are now given n different circuits $C_1, C_2, \ldots, C_n$. Each of the circuit has a different purpose and you would like to figure out the function of every one of them. The set of all possible functions $f_1, f_2, \ldots, f_n$ is also given to you in advance. Take any circuit and any subset S of the functions, you may test whether this circuit performs any function in S in one single step. Assume that you do not have other ways to test these circuits, prove that you will need $\Omega(n \log n)$ tests in the worst case.

Problem 8 (10%)

Prof. Chen claimed that he found a new heap structure which supports "Insertion", "Union", "MIN" and "extract-MIN" in amortized $O(1)$ time and other two operations "Decrease" and "Delete" in amortized $O(\log n)$ time. Similar to all other heap structures, his construction only requires comparing pairs of numbers to decided how to change the heap structure. Prove that Prof. Chen must have made a mistake in his analysis.