Algorithms Midterm

November 11, 2014, 09:10 - 12:10

Please answer the following 5 questions on the answer sheets provided. Be sure to write your name and student ID on all answer sheets you use. No books, notes, or calculators may be used during the exam with the exception of one double-sided, hand-written A4 note. Read all questions first. You may not request for clarification after 10am.

If you want to apply any result or theorem that has been taught in class (including homeworks), you may do so but you must state the result or theorem clearly before using it.

Problem 1 (20%)

Let all functions be positive. Prove or disprove the following three statements. You may only use the definitions of asymptotic notations.

1. (5%) $3n + 5 \in o(n^2)$.
2. (7%) If $f(n)$ and $g(n)$ are monotone increasing functions such that $f(n) \leq g(n) \leq f(n+1)$ for all n, then $f(n) \in \Theta(g(n))$.
3. (8%) If $f_1(n)$ and $f_2(n)$ are both in $o(g(n))$, then $3f_1(n) + 2f_2(n) \in o(g(n))$.

Problem 2 (15%)

Solve the following recurrences. You only need to obtain the asymptotic solution (in $\Theta()$ notation). If you use the master theorem, you must specify all parameters and briefly verity all the conditions.

1. (5%) $T(n) = 4T(\frac{n}{2}) + n^3, T(1) = 1$.
2. (8%) $T(n) = 2T(n-1) - 2T(n-2) + 2, T(0) = 2, T(2) = 4$.
3. (7%) $T(n) = \sum_{i=1}^{n-1} iT(i), T(1) = 3$.

Problem 3 (25%)

Your goal is to split n children $a_1, a_2, \ldots, a_n$ into k equal sized groups $g_1, g_2, \ldots, g_k$ such that every child in group g_i is taller than every child in group g_j for every $i > j$. You are only allowed to directly compare the height of two children. (You may assume that n and k are both powers of 2 and every child has a different height.)

1. (15%) Design an algorithm with running time $O(n \log k)$. You will receive partial credit for an $O(nk)$ algorithm. Briefly justify the running time and the correctness of your algorithm.
2. (10%) Prove that the time complexity of this problem is $\Omega(n \log k)$.

Problem 4 (15%)

The owner of the restaurant "GreedyOne" learned from "DynamicOne" and take customer orders before arranging the seats. One day, many customers arrive. They form m groups of sizes $a_1, a_2, \ldots, a_m$. For each group of customers, you must either seat the whole group or they will leave. There are only n seats in the restaurant. Further assume that these m groups will spend $d_1, d_2, \ldots, d_m$ dollars respectively.

1. (5%) Assume that the groups are already sorted such that $\frac{d_1}{a_1} > \frac{d_2}{a_2} > \cdots > \frac{d_n}{a_n}$. The owner accepts groups $1, 2, \ldots, i$ until group $i+1$ doesn't fit, and rejects groups $i+1, i+2, \ldots, n$. Find an example that the owner doesn't get the maximum profit (assume profit = total amount of money spent).

2. (10%) Following the above question, assume that the owner gets even greedier and seat group $i+1$ on the sidewalk illegally. Prove that the total profit he gained must be larger than his maximum profit in any legal solution (legal solution = solution that only seats customers inside the restaurant).

Problem 5 (20%)

One day, there are n meetings $M_1, M_2, \ldots, M_n$ that Prof. Chen wants to attend. Meeting M_i starts at time s_i, ends at t_i and has an importance factor c_i for every $i = 1, 2, \ldots, n$. He would like to maximize the sum of importance factors of the attended meetings. Further assume that he is not able to any two overlapped meetings.

1. (10%) Design an algorithm that finds the best set of meetings that Prof. Chen can attend. The running time must be polynomial in n. Any polynomial algorithm is good enough for full credit. You must briefly justify the correctness of the algorithm and analyze the running time.

2. (10%) Design a polynomial-time algorithm that maximizes the product of importance factors instead of the sum. (Notice that the importance factors may be negative. If no meeting is chosen, the product is defined to be 0.)

Administrative issues: (read this after exam)

1. The exam score and distribution will be announed on the class website on 11/13(Thu). The answer sheets will be given back on 11/18(Tue) after class. I will go through the problems briefly on that day.

2. If you think you are not performing well in the exam, you may redo this exam as a homework (all homework rules apply) and submit *online* before 11/17(Mon) midnight. This extra work will NOT affect your score unless you are one of the following 3 cases:

a) undergrad student with final score 55-60

b) grad student with final score 65-70

c) phd student doing qualify exam subsitution receiving 75-80 (please email me)

In the above three cases, you will be raised to the lowest passing grade if you perform reasonably well in this extra homework.