Convex Optimization Midterm, Thursday April 25, 2019.

Exam policy: **Open book**. You can bring any books, handouts, and any kinds of paper-based notes with you. Use of electronic devices (including cellphones, laptops, tablets, etc.), however, is strictly prohibited.

1. (30%) For the following optimization problems, determine whether each of them is (1) a convex optimization problem[1]; (2) an LP, (3) a QP, (4) a QCQP, (5) a SOCP. Write your answer as a table of 4 rows and 5 columns, with each entry being T (yes), F (no), or left blank. The score you get in this section is $s = \max\{0, 1.5n_c - 3n_w\}$ where n_c and n_w are the numbers of correct answers and wrong answers (not including those left blank).

   (a)
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1^2 + x_2^2 + \cdots + x_n^2 \le 1 \end{array}$$
   where $c \in \mathbf{R}^n$.

   (b)
$$\begin{array}{ll} \text{minimize} & 3x_1 + 2x_2 + x_3 \\ \text{subject to} & \sqrt{x_1^2 + 4x_2^2 + 9x_3^2} \le 2x_1 + x_2 \end{array}$$

   (c)
$$\begin{array}{ll} \text{minimize} & (x_1^3 + x_2^3 + x_3^3)^{1/3} \\ \text{subject to} & x_1 - x_2 = 1 \\ & x_1 - x_2 + x_3 \le 0 \end{array}$$

   (d)
$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} & -3x_1 - 4x_2 - 5x_3 \le 1 \end{array}$$

2. (30%) For each of the following sets, prove or disprove if it is **convex**. If it is, write down your proof (i.e., showing that for any x and y in the set and for any $\theta \in [0, 1]$, $\theta x + (1 - \theta)y$ is also in the set). If it is **not** convex, please find x, y and θ that violate the convex property described above.

   (a) $C_1 = \{a \in \mathbf{R}^k \mid p(0) = 1,\ |p(t)| \le 1 \text{ for } -3 \le t \le 5\}$ where
$$p(t) \triangleq a_1 + a_2 t + \cdots + a_k t^{k-1}.$$

   (b) $C_2 = C_1 - 2C_2$ where $C_1 \subseteq \mathbf{R}^k$ and $C_2 \subseteq \mathbf{R}^k$ are both convex sets.

   (c) $C_3 = \{x \in \mathbf{R}^k \mid (|x_1|^{1/2} + |x_2|^{1/2} + \cdots + |x_k|^{1/2})^2 \le 1\}$.

   (d) $C_4 = \left\{x \in \mathbf{R}^k_{++} \mid \prod_{i=1}^k x_i \le 1\right\}$.

   (e) $C_5 = \left\{X \in \mathbf{S}^n \,\middle|\, z^T X z \ge 1,\ \forall z \in \mathbf{R}^n, ||z||_2 = 1\right\}$

(More questions on the reverse side)

[1] Note: for an equality constraint $h_1(x) = h_2(x)$, we assume the equality constraint function to be $h(x) = h_1(x) - h_2(x)$; for an inequality constraint $f_1(x) \le f_2(x)$, we assume the corresponding inequality constraint function to be $f(x) = f_1(x) - f_2(x)$.

3. (22%) True and False. The score you get in this section is $s = \max\{0, 2n_c - 4n_w\}$ where n_c and n_w are the numbers of correct answers and wrong answers (not including those left blank).

   (a) If $C \subseteq \mathbf{R}^n$ is an affine set. Then $0 \in C$.

   (b) A halfspace can be expressed as the intersection of two hyperplanes.

   (c) A polyhedron is the intersection of a finite number of hyperplanes and halfspaces, and therefore is always a convex set.

   (d) An ellipsoid, defined as $\{x \mid (x - x_c)^T P^{-1}(x - x_c) \le 1\}$ for any given $x_c \in \mathbf{R}^n$ and $P \in \mathbf{S}^n_{++}$, is a convex set whose affine dimension is n.

   (e) If a function $f : \mathbf{R}^n \to \mathbf{R}$ is convex and concave at the same time, then it is also an affine function.

   (f) Every norm on $\mathbf{R}^n$ is convex.

   (g) The epigraph of a function $f : \mathbf{R}^n \to \mathbf{R}$, defined as epi $f = \{(x, t) \mid x \in \mathbf{dom}\ f, f(x) \le t\}$, is a convex set if and only if f is a convex function.

   (h) Let $x_1, x_2 \in \mathbf{R}^n$. Then the set $\{x_1, x_2\}$ is a convex set if and only if $x_1 = x_2$.

   (i) Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is twice differentiable. Then, f is strictly convex if and only if its Hessian is always positive definite (i.e., $\nabla^2 f(x) \succ 0, \forall x \in \mathbf{R}^n$).

   (j) The α-sublevel set of a function $f : \mathbf{R}^n \to \mathbf{R}$, defined as $C_\alpha = \{x \in \mathbf{dom}\ f \mid f(x) \le \alpha\}$, is a convex set if and only if f is a convex function.

   (k) A function $f : \mathbf{R}^n \to \mathbf{R}$ is convex if and only if $\forall x \in \mathbf{dom}\ f$, $v \in \mathbf{R}^n$, the function $g(t) = f(x + tv)$ is convex on $\mathbf{dom}\ g = \{t \mid x + tv \in \mathbf{dom}\ f\}$.

4. (18%) Determine whether each of the following sets is a **convex function**, **quasi-convex function**, **concave function**. Write your answer as a table of 6 rows and 3 columns, with each entry being T (yes), F (no), or left blank. You don't have to write down the proofs. The score you get in this section is $s = \max\{0, n_c - 2n_w\}$ where n_c and n_w are the numbers of correct answers and wrong answers (not including those left blank).

   (a) $f_1 : \mathbf{R}^3 \to \mathbf{R}$, $f_1(x) = x^T P x + q^T x + r$ where $P \in \mathbf{S}^3_{++}$.

   (b) $f_2 : \mathbf{R} \to \mathbf{R}$, $f_2(x) = \log x$ with $\mathbf{dom}\ f_2 = \mathbf{R}_{++}$.

   (c) $f_3 : \mathbf{S}^3_{++}$, $f_3(X) = \log(\det(I + X))$.

   (d) $f_4 : \mathbf{R}^2 \to \mathbf{R}$,

   $$f_4(x) = \frac{a^T x + b}{c^T x + d}$$

   where $a = \begin{bmatrix} 6 & 5 \end{bmatrix}^T$, $b = 4$, $c = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$, and $d = 1$, with $\mathbf{dom}\ f_4 = \{x \in \mathbf{R}^2 \mid c^T x + d > 0\}$.

   (e) $f_5 : \mathbf{R}^n_{++} \to \mathbf{R}$, $f_5(x) = -\prod_{k=1}^n x_k$.

   (f) $f_6 : \mathbf{R}^n \to \mathbf{R}$, $f_6(x) = \log(1 + ||x||_2)$.