Convex Optimization – Final Exam, Thursday June 20, 2019.

Exam policy: Open book. You can bring any books, handouts, and any kinds of paper-based notes with you, but use of any electronic devices (including cellphones, laptops, tablets, etc.) is strictly prohibited.

1. (15%) Consider the convex unconstrained optimization problem whose variable is $x \in \mathbf{R}^2$:

$$\text{minimize} \quad f_0(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We will study some types of descent methods in this problem.

(a) (3%) Find ∇f_0, the gradient of f_0 for any $x \in \mathbf{R}^2$.

(b) (4%) Find $\nabla^2 f_0$, the Hessian of f_0 for any $x \in \mathbf{R}^2$.

(c) (3%) Suppose the initial point is chosen to be $x^{(0)} = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$. Find the gradient descent direction Δx_{gd}.

(d) (5%) Again, let the initial point be $x^{(0)} = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$. Find the Newton step Δx_{nt}.

2. (45%) Consider the convex piecewise-linear minimization problem

$$\text{minimize} \quad \max_{i=1,\ldots,m} (a_i^T x + b_i) \tag{1}$$

with variable $x \in \mathbf{R}^n$. Suppose the optimal value is attained and is p^*.

(a) (5%) Find the Lagrange dual function of the problem (1).

(b) (5%) Consider an equivalent problem

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\ldots,m} y_i \\ \text{subject to} & a_i^T x + b_i = y_i, i = 1, \ldots, m, \end{array} \tag{2}$$

with variables $x \in \mathbf{R}^n, y \in \mathbf{R}^m$. Find the Lagrange dual function of problem (2).

(c) (5%) Derive the dual problem for problem (2).

(d) (5%) Formulate the piecewise-linear minimization problem (1) as an equivalent LP. (*Hint: by introducing a slack variable s and putting it in the objective function.*)

(e) (10%) Form the dual problem of the LP you obtained in (d).

(f) (5%) For the LP you obtained in (d) (with variable $\bar{x} = (x, s) \in \mathbf{R}^{n+1}$), and given the barrier method's parameter t, formulate the approximated equality constrained (or unconstrained) problem.

(g) (10%) Write down the KKT conditions of the problem you obtained in (f).

3. (25%) Consider the problem

$$\begin{array}{ll} \text{minimize} & (1/2)x^T x + c^T x \\ \text{subject to} & Ax \preceq b \end{array} \tag{3}$$

where $A \in \mathbf{R}^{m \times n}$. Suppose you are going to apply the barrier method to this problem, whose objective function is denoted f_0 and constraint functions f_i, $i = 1, \ldots, m$. You can denote a_i^T by the ith row of the matrix A.

(a) (10%) Derive $tf_0 + \phi$ as a function of x, where t is any given positive number and ϕ denotes the logarithmic barrier for problem (3).

(b) (15%) For any given strictly feasible point x of (3), derive the Newton step Δx_{nt} at for the "approximated" problem

$$\underset{x}{\text{minimize}} \quad tf_0 + \phi$$

for any given $t > 0$.

4. (15%) For the following pairs of proper cones $K \subseteq \mathbf{R}^q$ and functions $\psi : \mathbf{R}^q \to \mathbf{R}$, determine whether ψ is a **generalized logarithm** for K. Justify your answers.

   (a) (5%) $K = \mathbf{R}^3_+$, $\psi(x) = \log x_1 + 2\log x_2 + 3\log x_3$.

   (b) (5%) $K = \mathbf{R}^3_+$, $\psi(x) = \log(x_1 + x_2 + x_3)$.

   (c) (5%) $K = \mathbf{R}^2_+$, $\psi(x) = \log x_1 - \log x_2$.