Convex Optimization Midterm, Thursday April 19, 2018.
Exam policy: Open book. You can bring any books, handouts, and any kinds of paper-based notes with you, but electronic devices (including cellphones, laptops, tablets, etc.) are strictly prohibited.

Problems prepared by Prof. Borching Su. Solution prepared by Tzu-Yu Jeng. Textbook: Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge U. Press, 2004.

1. (44%) Determine whether each of the following sets is a **convex set**, **affine set**, **subspace**, **cone**. Write your answer as a table of 11 rows and 4 columns, with each entry being T (yes), F (no), or left blank. The score you get in this section is $s = \max\{0, n_c - \frac{1}{4}n_w^2\}$ where n_c and n_w are the numbers of correct answers and wrong answers (not including those left blank).

   1. $\mathcal{S}_1 = \left\{ x \in \mathbf{R}^3 \,\middle|\, \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.
      T, T, F, F
      A hyperplane is affine (Lecture 1b, p.25). Being affine, it is convex. Not passing through the origin, it is not a subspace, and not a cone either.
   2. $\mathcal{S}_2 = \left\{ x \in \mathbf{R}^3 \,\middle|\, \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix} x \preceq \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.
      T, F, F, F
      A halfspace is convex, but not affine (Lecture 1b, p.27). Not being affine, it is not a subspace (Lecture 1b, p.24). It is obvious that is not a cone; say, $(0.1, 0, 0) \in S_2$, but $100 \cdot (0.1, 0, 0) = (10, 0, 0)$ is not.
   3. $\mathcal{S}_3 = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \,\middle|\, x \in \mathbf{R}^2 \right\}$.
      T, T, T, T
      It is immediate to observe S_3 is a plane, and thus is a hyperplane. Indeed,
      $$\begin{aligned} a &= (1, 3, 5) \\ b &= (2, 4, 6) \\ c &= (1, 2, 3) \\ n &= a \times b \\ \langle n, x_1 a + x_2 b + c\rangle &= \langle n, c\rangle \end{aligned}$$
      Notice S_3 passes through the origin for $x = (0, -1/2)$. It is affine (Lecture 1b, p.25), and thus convex. It is a cone, since $\theta x_1 a + \theta x_2 b + c \in S_3$ for any $\theta \le 0$, and we saw S_3 is convex.
   4. $\mathcal{S}_4 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \subseteq \mathbf{R}^2$.
      T, T, F, F
      The conditions of convex set, affine set, and subspace all result $(1, 2) = (1, 2)$, and are trivially true, since there is only 1 point. It is not a cone, for $2 \cdot (1, 2) = (2, 4) \notin S_4$.
   5. $\mathcal{S}_5 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} \subseteq \mathbf{R}^2$.
      F, F, F, F
      Obvious the middle point is not in S_5, and it is not convex. It follows it is not affine either, and not a subspace, as both are convex. It is not a cone, since for example $(10, 20) \notin S_5$.
   6. $\mathcal{S}_6 = \left\{ x \in \mathbf{R}^3 \,\middle|\, x_1^2 + x_2^2 \le 1,\ x_3 = 0 \right\}$.
      T, F, F, F
      A ball (here a 2-disk) is convex (Lecture 1b, p.30). It is neither affine, nor (thus) subspace, since (say) in either case it contains points with arbitrarily large norm. For the same reason, considering it passes through the origin, it cannot be a cone either.

7. $\mathcal{S}_7 = -\mathbf{R}^3_+$.
T, F, F, T
Convexity follows from closure law of addition. Not affine because for any x, $-x \notin S_7$. Not being affine, it is not a subspace. Cone follows from closure law of multiplicity.

8. $\mathcal{S}_8 = \left\{ x \in \mathbf{R}^2 \;\middle|\; x_1 x_2 = 0,\ x_1 \geq 0,\ x_2 \geq 0 \right\}$.
F, F, F, T
Not convex, for example because $(0,2)/2 + (2,0)/2 = (1,1) \notin S_8$. Thus not affine, nor subspace. It is a cone since $\theta x_1 \cdot x_2 = 0$ for $x_1 x_2 = 0$, and similarly $x_1 \cdot \theta x_2 = 0$ for $x_1 x_2 = 0$.

9. $\mathcal{S}_9 = \left\{ a \in \mathbf{R}^n \;\middle|\; |1 + a_1 t + a_2 t^2 + \cdots + a_n t^n| \leq 1 \text{ for } \alpha \leq t \leq \beta \right\}$ for any given $\alpha, \beta \in \mathbf{R}$.
T, F, F, F
Notice that for $0 \leq \theta \leq 1$, since norm is convex (Lecture 3, p.38),

$$\begin{aligned}&(1-\theta)|1 + a_1 t + \cdots a_n t^n| + \theta |1 + b_1 t + \cdots b_n t^n| \\ \geq& |1 + ((1-\theta)a_1 + \theta b_1)t + \cdots + ((1-\theta)a_n + \theta b_n)t^n|\end{aligned}$$

So S_9 is convex. But it is not a cone, since we may take $a_1 = -1/\beta$, but $a_i = 0$ for $i = 2, \ldots, n$, and note that $|1 - a_1 t| \to \infty$ as $a_1 \to \infty$. Consequently it is not a subspace. The origin (i.e. $a_i = 0$ for all i's) is in S_9, and the same argument says it is not affine.

10. $\mathcal{S}_{10} = \left\{ X \in \mathbf{S}^n \;\middle|\; z^T X z \geq 1,\ \forall z \in \mathbf{R}^n, ||z||_2 = 1 \right\}$
T, F, F, F
To see it's convex: if

$$\begin{aligned}\langle z, X_1 z\rangle \geq 1 \\ \langle z, X_2 z\rangle \geq 1\end{aligned}$$

Then, for $0 \leq \theta \leq 1$,

$$\begin{aligned}&\langle z, \big(\theta X_1 + (1-\theta)X_2\big)z\rangle \\ =&\theta\langle z, X_1\rangle + (1-\theta)\langle z, X_2\rangle \\ \geq&\theta \cdot 1 + (1-\theta)\cdot 1 \\ =&1.\end{aligned}$$

As required in definition of S_{10}. To see it is not a cone, consider $z = (1, 0, \ldots, 0)$, and $X = I \in \mathbf{S}^n$ (symmetric matrices). Here $\langle z, Xz\rangle = 1$, but $\langle z, 2Iz\rangle = 2$ The reason that it is not affine is the same, by considering $2I = 2 \cdot I + (-1)\cdot O$, the "line" containing O (all-0 matrix) and I. It follows that it is not a subspace.

11. $\mathcal{S}_{11} = \left\{ x \in \mathbf{R}^n \;\middle|\; ||Px + q||_2 \leq c^T x + r \right\}$ given any $P \in \mathbf{R}^{m \times n}$, $q \in \mathbf{R}^m$, $c \in \mathbf{R}^n$, and $r \in \mathbf{R}$.
T, F, F, F
To show convexity, if

$$\begin{aligned}\|Px_1 + q\|_2 - \langle c, x_1\rangle - r \leq 0 \\ \|Px_2 + q\|_2 - \langle c, x_2\rangle - r \leq 0\end{aligned}$$

Then, for $0 \leq \theta \leq 1$,

$$\begin{aligned}&\|P(\theta x_1 + (1-\theta)x_2) + q\|_2 - \langle c, \theta x_1 + (1-\theta)x_2\rangle - r \\ \leq&\theta\|Px_1 + q\|_2 + (1-\theta)\|Px_2 + q\|_2 - \theta\langle c, x_1\rangle - (1-\theta)\langle c, x_2\rangle - \theta r - (1-\theta)r \\ \leq&0 + 0 = 0.\end{aligned}$$

To see it is not a cone, take, for instance,

$$n = 2,\ P = I,\ q = (1,1),\ c = 0,\ r = 0$$

Then $x = (1,1) \in S_{11}$, but $2x$ is not. And as a result, it is not a subspace. $0 \in S_{11}$, and this argument also shows S_{11} is not affine.

2. (18%) Determine whether each of the following sets is a **convex function**, **quasi-convex function**, **concave function**. Write your answer as a table of 6 rows and 3 columns, with each entry being T (yes), F (no), or left blank. The score you get in this section is $s = \max\{0, n_c - \frac{1}{4}n_w^2\}$ where n_c and n_w are the numbers of correct answers and wrong answers (not including those left blank).

(a) $f_1 : \mathbf{R}^3 \to \mathbf{R}$, $f_1(x) = x^T P x + q^T x + r$ where $P \in \mathbf{S}^3_{++}$.
T, T, F
Here P is symmetric and positive definite, and we only need being symmetric. Convexity follows (Lecture 1a, p.22), and thus quasi-convexity (Lecture 6, p.28). In view of the fact that a function both convex and concave is linear, if $P \neq O$, then f_1 is not concave.

(b) $f_2 : \mathbf{R} \to \mathbf{R}$, $f_2(x) = x \log x$ with $\mathbf{dom}\ f_2 = \mathbf{R}_{++}$.
T, T, F
$d^2 f_2 / d^2 x = 1/x > 0$ for all of its domain $x > 0$. It is convex, and not concave, and quasiconvex (Lecture 6, p.28).

(c) $f_3 : \mathbf{S}^3_{++}, f_3(X) = \log(\det(I + X))$.
F, F, T
$\log\det X$ is concave (Lecture 3, p.38), and, not being linear, is not convex. However, it is not quasi-convex. Counterexample:

$$X_1 = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$
$$X_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$
$$\frac{1}{2}\log\det(I + X_1) + \frac{1}{2}\log\det(I + X_2) - \log\det\left(I + \frac{X_1 + X_2}{2}\right) \approx -0.526$$

(d) $f_4 : \mathbf{R}^2 \to \mathbf{R}$,
$$f_4(x) = \frac{a^T x + b}{c^T x + d}$$
where $a = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$, $b = 3$, $c = \begin{bmatrix} 4 & 5 \end{bmatrix}^T$, and $d = 2$, with $\mathbf{dom}\ f_4 = \left\{x \in \mathbf{R}^2 \mid c^T x + d > 0\right\}$.
F, T, F
The function is not convex, nor concave. We take one tangent line on which $f_4(x)$ does not meet the convexity condition, and another on which it does not meet the concavity condition, to justify the claim. For example, let $x = (-6t, 5t)$then $f_4(x) = g(t) := (4t+3)/(t+2)$, but
$$\frac{g(-1)}{2} + \frac{g(1)}{2} - g(0) = -\frac{5}{6},$$
violating convexity condition. On the other hand,
$$\frac{g(-4)}{2} + \frac{g(-8)}{2} - g(-6) = \frac{5}{12},$$
violating concavity condition. To see it is quasi-convex, note that the set
$$\frac{x + 2y + 3}{4x + 5y + 2} \leq \alpha$$
is evidently a plane, and thus convex.

(e) $f_5 : \mathbf{R}^n \to \mathbf{R}$, $f_5(x) = \prod_{k=1}^n x_k$.
F, F, F
f_5 is neither convex nor concave. A violation of the concave condition:

$$x_1 = (1, 1) \tag{1}$$
$$x_2 = (-1, -1) \tag{2}$$
$$\frac{f_5(x_1)}{2} + \frac{f_5(x_2)}{2} - f_5\big(\frac{x_1 + x_2}{2}\big) = 1, \tag{3}$$

A violation of the convex condition:

$$x_1 = (-1, 1) \tag{4}$$
$$x_2 = (1, -1) \tag{5}$$
$$\frac{f_5(x_1)}{2} + \frac{f_5(x_2)}{2} - f_5\big(\frac{x_1 + x_2}{2}\big) = -1 \tag{6}$$

Nor is it quasiconvex. Consider the 2-d case: the set $x_1x_2 \leq 1$ consists of the hyperbola-shaded part of the first and the third quadrant, and all of the second and the fourth quadrant, which is clearly not convex.

(f) $f_6 : \mathbf{R}^n \to \mathbf{R}$, $f_6(x) = \log(1 + ||x||_2)$.
F, T ,F
f_6 is neither convex nor concave. A violation of the concave condition:

$$x_1 = (0, -1) \tag{7}$$
$$x_2 = (0, 1) \tag{8}$$
$$\frac{f_6(x_1)}{2} + \frac{f_6(x_2)}{2} - f_6\big(\frac{x_1 + x_2}{2}\big) \approx 0.693, \tag{9}$$

A violation of the convex condition:

$$x_1 = (0, 0) \tag{10}$$
$$x_2 = (0, 2) \tag{11}$$
$$\frac{f_6(x_1)}{2} + \frac{f_6(x_2)}{2} - f_6\big(\frac{x_1 + x_2}{2}\big) \approx -0.144, \tag{12}$$

Lastly, f_6 is quasiconvex. The sublevel set of $\log(1+\|x\|_2)$ assumes the form $1+\|x\|_2 \leq \text{const}$, or $\|x\|_2 \leq \text{const}$, or a disk, which is convex.

3. (20%) For the following optimization problems, determine whether each of them is (1) a convex optimization problem [1]; (2) an LP, (3) a QP, (4) a QCQP, (5) a SOCP. The score you get in this section is $s = \max\{0, n_c - \frac{1}{4}n_w^2\}$ where n_c and n_w are the numbers of correct answers and wrong answers (not including those left blank).

(a)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \end{array}$$

where $c \in \mathbf{R}^n$. F, F, F, F, F
It is clear that the circle is not convex, and the program is thus none of LP, QP, QCQP, or SOCP.

[1] Note: for an equality constraint $h_1(x) = h_2(x)$, we assume the equality constraint function to be $h(x) = h_1(x) - h_2(x)$; for an inequality constraint $f_1(x) \leq f_2(x)$, we assume the corresponding inequality constraint function to be $f(x) = f_1(x) - f_2(x)$.

(b)

$$\begin{array}{ll} \text{minimize} & 3x_1 + 2x_2 + x_3 \\ \text{subject to} & \sqrt{x_1^2 + 4x_2^2 + 9x_3^2} \le 2(x_1 + x_2 + x_3) \end{array}$$

T, F, F, T/F, T

It is clearly neither an LP nor a QP. However, the problem can be transformed into a QCQP, by noting that the constraint is same as

$$x_1^2 + 4x_2^2 + 9x_3^2 \le 4(x_1 + x_2 + x_3)^2 \tag{13}$$

$$x_1 + x_2 + x_3 \ge 0. \tag{14}$$

The first constraint can be arranged as a quadratic constraint, and the second, linear constraint can be also seens as a trivial quadratic constraint. Considering that the term "equivalent problem" in this course has not been defined formally (compare textbook p.130), we decide to give the credit unconditionally (as of QCQP or not). Lastly, it is an SOCP, by noting lhs is the norm $\|(x_1, 2x_2, 3x_3) - 0\|_2$.

(c)

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} & x_1 + x_2 = 1 \\ & x_1 - x_2 + x_3 \le 0 \end{array}$$

T, F, T, T, T

I think no more explanation is necessary.

(d)

$$\begin{array}{ll} \text{minimize} & -3x_1 - 4x_2 - 5x_3 \\ \text{subject to} & x_1^2 + x_2^2 + x_3^2 \le 1 \end{array}$$

T, F, F, T, T

No more explanation is necessary either.

4. (16%) We consider a robust variation of the linear program:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b. \end{array}$$

We assume that only the vector $c \in \mathbf{R}^n$ is subject to errors, and the other parameters ($A \in \mathbf{R}^{m\times n}, b \in \mathbf{R}^m$) are exactly known. The robust program is defined as

$$\begin{array}{ll} \text{minimize} & \sup_{c\in\mathcal{E}}\{c^T x\} \\ \text{subject to} & Ax \preceq b. \end{array}$$

where $\mathcal{E}$ is the set of possible vectors c. For each of the following sets $\mathcal{E}$, express the robust LP as a tractable convex problem (i.e., any of an LP, QP, QCQP, or SOCP problem)

(a) (8%) A finite set of vectors: $\mathcal{E} = \{c_1, c_2, \cdots, c_K\}$, where $c_i \in \mathbf{R}^n$, $i = 1, \cdots, K$.

Let

$$M = \sup_{c\in\mathcal{E}}(c^T x)$$

Then the original problem can be written as

$$\text{minimize} \quad M$$
$$\text{subject to} \quad \begin{cases} Ax \preceq b \\ c_1^T x \leq M \\ \vdots \\ c_K^T x \leq M \end{cases}$$

where x is a slack variable. It is not difficult to see the form is a LP, if an extended matrix is defined. Namely, for block matrix

$$\begin{bmatrix} A \\ c_1^T \\ \vdots \\ c_K^T \end{bmatrix} x \preceq \begin{bmatrix} b \\ M \\ \vdots \\ M \end{bmatrix}$$

(b) (8%) A set specified by a nominal value $c_0 \in \mathbf{R}^n$ plus a bound on the deviation $c - c_0$:

$$\mathcal{E} = \{c \in \mathbf{R}^n \mid ||c - c_0||_2 \leq \gamma\}$$

where $\gamma \in \mathbf{R}_+$ and $c_0 \in \mathbf{R}^n$.

Hint. Think about epigraph and the technique of introducing slack variables.

Similarly, let M be defined the same way.

$$M = \sup_{c \in \mathcal{E}} (c^T x)$$

Notice that, if we let $u = c - c_0$, we arrive at constraint $\|u\|_2 \leq \gamma$,

$$\begin{aligned} M =& \sup_{c \in \mathcal{E}} (c^T x) \\ =& \sup_{c \in \mathcal{E}} (c_0 + u)^T x \\ =& c_0^T x + \sup_{\|u\|_2 \leq \gamma} u^T x \\ =& c_0^T x + \gamma \|x\|_2 \end{aligned}$$

Then the original problem can be written as

$$\text{minimize} \quad M$$
$$\text{subject to} \quad \begin{cases} \|x\|_2 \leq \dfrac{M}{\gamma} - \dfrac{c_0^T x}{\gamma} \\ \|0x + 0\|_2 \leq -A_{1,1}x_1 - \cdots - A_{1,n}x_n + b_i \\ \vdots \leq \vdots \\ \|0x + 0\|_2 \leq -A_{n,1}x_1 - \cdots - A_{n,n}x_n + b_i \end{cases}$$

where x is a slack variable. It is not difficult to see the form is a SOCP. Few people really wrote the rewritten problem into exactly matrix form, but credit is given once their equations are correct.