Convex Optimization Quiz #3, Thursday May 16, 2019.

1 True or False (84%)

There are 21 questions in this section. For each question, you can choose to write down your answer (T or F) or leave it blank. Suppose the numbers of correct answers, wrong answers are n_c and n_w. Then the score you get from this section is $\max\{4 \cdot n_c - 4 \cdot n_w, 0\}$. Note that $n_c + n_w + n_b = 21$ where n_b is the number of problems left blank.

1. A cone $K \subseteq \mathbf{R}^n$ is called a **proper cone** if it is convex, open, solid, and pointed.
2. A cone K is said to be **pointed** if for any x, $x \in K \setminus \{0\} \Rightarrow -x \notin K$.
3. A cone K is said to be solid if $\mathbf{int}\ K \neq \varnothing$.
4. Let $K \subseteq \mathbf{R}^n$ be a proper cone and consider the generalized inequality "$\preceq_K$". Then, for any $x, y \in \mathbf{R}^n$, at least one of the statements $x \preceq_K y$ and $y \preceq_K x$ must be true.
5. The component-wise inequality between two vectors in $\mathbf{R}^n$ (e.g., $x, y \in \mathbf{R}^n, x \preceq y$) is derived from the proper cone $\mathbf{R}^n_+$ (i.e., $x \preceq y \Leftrightarrow x \preceq_{\mathbf{R}^n_+} y$).
6. Given a generalized inequality on $\mathbf{R}^n$, the **minimum** element of a subset S of $\mathbf{R}^n$ either is unique or does not exist.
7. An element $x \in S$ is a **maximal** element with respect to $\preceq_K$ if for any $y \in S$, $y \succeq_K x$ implies $y = x$.
8. The separating hyperplane theorem states that for any two convex subsets C and D (of $\mathbf{R}^n$) that do not intersect, (i.e., $C \cap D = \varnothing$), there exist $a \in \mathbf{R}^n \setminus \{0\}$ and $b \in \mathbf{R}$ such that the $a^T x + b \geq 0$ for all $x \in C$ and $a^T x + b \leq 0$ for all $x \in D$.
9. Let $C \subseteq \mathbf{R}^n$ and x_0 is a point in the boundary of C (i.e., $x_0 \in \mathbf{bd}\ C$). Then a **supporting hyperplane** at the point x_0 is actually a separating hyperplane of the sets $\{x_0\}$ and C.
10. Let $K \subseteq \mathbf{R}^n$ be a cone. Then the **dual cone** of K is defined as $K^* = \{y \in \mathbf{R}^n \mid x^T y \geq 0 \text{ for all } x \in K\}$.
11. Let $K \in \mathbf{R}^n$ be a proper cone that is its own dual (i.e., $K^* = K$). Then, $K = \mathbf{R}^n_+$.
12. If K is a proper cone, then $K^{**} = K$.
13. A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is said to be K-convex, where $K \subseteq \mathbf{R}^n$ is a proper cone, if for all $x, y \in \mathbf{R}^n$ and $\theta, 0 \leq \theta \leq 1$, $f(\theta x + (1-\theta) y) \preceq \theta f(x) + (1-\theta) f(y)$.
14. A second-order cone K in $\mathbf{R}^{n+1}$ if $K = \left\{(y, t) \in \mathbf{R}^{n+1} \mid y \in \mathbf{R}^n,\ ||y||_2 \leq t\right\}$.
15. Consider a standard-form optimization problem (of the minimization type) with variable $x \in \mathbf{R}^n$, objective function $f_0(x)$, inequality constraint functions $f_i(x), i = 1, ..., m$, and equality constraint functions $h_i(x), i = 1, ..., p$. The **Lagrangian** associated with the problem is defined as $L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$.
16. The **dual function** of a standard-form optimization problem is defined as $g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)$ where $\mathcal{D}$ is the intersection of $\bigcap_{i=0}^m \mathrm{dom} f_i$ and $\bigcap_{i=0}^p \mathrm{dom} h_i$.
17. The dual function of an optimization problem is concave if and only if the primal problem is a convex optimization problem.
18. The dual function $g(\lambda, \nu)$ gives a lower bound of the optimal value of the primal problem as long as $g(\lambda, \nu) > -\infty$.

19. The dual problem of an non-convex optimization problem is still a convex problem.

20. The weak duality ($d^* \leq p^*$ where p^* and d^* are the optimal values of the primal and dual problems, respectively) holds only when the primal problem satisfies the Slater's conditions.

21. The strong duality ($d^* = p^*$) holds only when the primal problem is convex and satisfies the Slater's conditions.

2 Handwriting Problems (16%)

1. (16%) Consider the standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0. \end{array}$$

where $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{p\times n}$ and $b \in \mathbf{R}^p$.

(a) (4%) Find the Lagrangian $L(x, \lambda, \nu)$ associated with the problem where $\lambda \in \mathbf{R}^n$ and $\nu \in \mathbf{R}^p$.

(b) (4%) Find the dual function $g(\lambda, \nu)$ and identify its domain $\mathbf{dom}\, g \subseteq \mathbf{R}^{n+p}$.

(c) (4%) Formulate the dual problem of the problem. In your answer, make the dual constraints explicit.