Convex Optimization Midterm, Thursday April 23, 2020. 2:20pm~4:00pm. Exam policy: **Open book**. You can bring any books, handouts, and any kinds of paper-based notes with you. Use of electronic devices (including cellphones, laptops, tablets, etc.), however, is strictly prohibited.

1. (32%) For each of the following functions, prove or disprove if it is a convex function.

   (a) (10%) $f : \mathbf{R}^2 \to \mathbf{R}$, $\mathbf{dom}\ f = \mathbf{R}^2_{++}$ $x_2 > 0, x_1 > 0$

   $$f(x_1, x_2) = \frac{1}{x_1 + \frac{1}{x_2}}.$$

   $\frac{\partial}{\partial x^2} = \frac{2}{y}$ $-\frac{2x}{y^2}$ xy 0 1

   $-\frac{2x}{y^2}$ $\frac{2x^2}{y^3}$ 1 0

   (b) (12%) Let $C \subseteq \mathbf{R}^n$ be a subset of $\mathbf{R}^n$. Define $f_C : \mathbf{R}^n \to \mathbf{R}$ with $\mathbf{dom}\ f_C = \mathbf{R}^n$ as $f_C(x) = \sup_{y \in C} ||x - y||_2$ and $g_C : \mathbf{R}^n \to \mathbf{R}$ with $\mathbf{dom}\ g_C = \mathbf{R}^n$ as $g_C(x) = \inf_{y \in C} ||x - y||_2$. $\frac{x^2}{y}$

   (i) (6%) If C is convex, prove or disprove if f_C and g_C are convex, respectively (3% each). $\frac{\partial}{\partial x} = \frac{2x}{y}$

   (ii) (6%) If C is not convex, prove or disprove if f_C and g_C are convex, respectively (3% each). $\frac{\partial}{\partial y} = -\frac{x^2}{y^2}$

   (c) (10%) $h : \mathbf{R}^n \to \mathbf{R}$, $h(x) = (f(x))^2/g(x)$, with $\mathbf{dom}\ h = \mathbf{dom}\ f \cap \mathbf{dom}\ g$ where $f : \mathbf{R}^n \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$ are both positive and convex within their domains. $\frac{x_1^2}{x_2}$

   $y_1^T x \le 1$ $(1-\theta)y_1^T x + \theta y_2^T x \le 1$

   $y_2^T x \le 1$

2. (10%) Suppose $C \subseteq \mathbf{R}^n$ is nonempty and not convex. Prove or disprove that $S = \{y \in \mathbf{R}^n \mid y^T x \le 1 \text{ for all } x \in C\}$ is convex. $\frac{\partial x_1^2/x_2}{\partial x_1} = 2\frac{x_1}{x_2}$

3. (8%) Let $f(x) = \log x$ with $\mathbf{dom}\ f = \mathbf{R}_{++}$. Find f^*, the conjugate function of f, along with its domain, $\mathbf{dom}\ f^*$. $\frac{\partial}{\partial x_2} = -\frac{x_1^2}{x_2^2}$

   $$\begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & - \end{bmatrix}$$

4. (20%) Determine whether each of the following sets is a **convex set**. You don't have to write down the proofs. The score you get in this section is $s = \max\{0, 5n_c - 10n_w\}$ where n_c and n_w are the numbers of correct answers and wrong answers (not including those left blank).

   $y - \frac{1}{x} = 0$
   $xy - 1 = 0$
   $x = \frac{1}{y}$

   (a) (5%) An ellipsoid, defined as $\{x \mid (x - x_c)^T P^{-1}(x - x_c) \le 1\}$ for any given $x_c \in \mathbf{R}^n$ and $P \in \mathbf{S}^n_{++}$.

   (b) (5%) $\{a \in \mathbf{R}^k \mid p(0) = 1;\ |p(t)| \ge 1 \text{ for } \alpha \le t \le \beta\}$, where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$.

   (c) (5%) $\{a \in \mathbf{R}^k \mid p(0) = 1;\ |p(t)| \le 1 \text{ for } t \le \alpha \text{ or } t \ge \beta\}$, where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$, and $\alpha < \beta$.

   (d) (5%) $\{x \in \mathbf{R}^n \mid ||Ax + b||_2 \le c^T x + d\}$ $A \in R^{m \times n}$, $b \in R^m$, $c \in R^n$, $d \in R$

(More questions on the reverse side)

5. (30%) For the following optimization problems, determine whether each of them is (1) a convex optimization problem[1]; (2) an LP, (3) a QP, (4) a QCQP, (5) a SOCP, (6) a quasi-convex optimization problem. Write your answer as a table of 5 rows and 6 columns, with each entry being T (yes), F (no), or left blank. The score you get in this section is $s = \max\{0, n_c - 2n_w\}$ where n_c and n_w are the numbers of correct answers and wrong answers (not including those left blank).

(a)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x^T P x \leq 1 \end{array}$$

where $P \in \mathbf{S}^n_{++}$.

(b)

$$\begin{array}{ll} \text{minimize} & x_1 \\ \text{subject to} & \sqrt{x_1^2 + 4x_2^2 + 9x_3^2} \leq 2x_1 + x_2 \end{array}$$

(c)

$$\begin{array}{ll} \text{minimize} & (x_1^5 + x_2^5)^{1/5} \\ \text{subject to} & x_1 + x_2 = 1 \\ & x_1 - x_2 \leq 0 \end{array}$$

$\|x\|_r$

(d)

$$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_2^2 + 3x_3^2 \\ \text{subject to} & -x_1 - x_2 - x_3 \leq 1 \end{array}$$

(e)

$$\begin{array}{ll} \text{maximize} & \left(\prod_{k=1}^n x_k\right)^{1/n} \\ \text{subject to} & c^T x \geq 1 \end{array}$$

where $c \in \mathbf{R}^n$.

[1]Note: for an equality constraint $h_1(x) = h_2(x)$, we assume the equality constraint function to be $h(x) = h_1(x) - h_2(x)$; for an inequality constraint $f_1(x) \leq f_2(x)$, we assume the corresponding inequality constraint function to be $f(x) = f_1(x) - f_2(x)$.

Convex Optimization Midterm solution

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1.

(a)

$\mathbb{R}^2_{++}$ is convex for sure. $\Rightarrow$ dom$f$ is convex

(Solution 1)

$$\nabla f(x_1, x_2) = \frac{1}{(x_1 + \frac{1}{x_2})^2} \begin{bmatrix} -1 \\ \frac{1}{x_2^2} \end{bmatrix}$$

$$\nabla^2 f(x_1, x_2) = \frac{2}{(x_1 + \frac{1}{x_2})^3 x_2^3} \begin{bmatrix} x_2^3 & -x_2 \\ -x_2 & -x_1 \end{bmatrix}$$

Take $(x_1, x_2) = (1, 1)$, we have

$$\nabla^2 f(1,1) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \not\succeq 0 \Rightarrow f \text{ is not convex.}$$

(Solution 2) We have a counter example:

Let $\mathbf{x} = (0.5, 0.5), \mathbf{y} = (1.5, 1.5), \theta = 0.5$, we have

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) = f(1,1) = 0.5$$
$$\theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) = 0.5 \times \frac{2}{5} + 0.5 \times \frac{6}{13} = \frac{28}{65}$$
$$\Rightarrow f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) > \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})$$

$\Rightarrow f$ is not convex.

(b)

dom f_C and dom g_C are both $\mathbb{R}^n$, which is convex. (Need not to be discussed)

(i)

Let $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n, \theta \in [0,1]$

$$\begin{aligned}
f_C(\theta\mathbf{v} + (1-\theta)\mathbf{u}) &= \sup_{\mathbf{y}\in C} ||\theta\mathbf{v} + (1-\theta)\mathbf{u} - \mathbf{y}|| \\
&= \sup_{\mathbf{y}\in C} ||\theta(\mathbf{v} - \mathbf{y}) + (1-\theta)(\mathbf{u} - \mathbf{y})|| \\
&\leq \sup_{\mathbf{y}\in C} ||\theta(\mathbf{v} - \mathbf{y})|| + ||(1-\theta)(\mathbf{u} - \mathbf{y})|| \quad (\because \text{triangle ineq.}) \\
&\leq \theta \sup_{\mathbf{y}\in C} ||(\mathbf{v} - \mathbf{y})|| + (1-\theta) \sup_{\mathbf{y}\in C} ||(\mathbf{u} - \mathbf{y})|| \\
&= \theta f_C(\mathbf{v}) + (1-\theta) f_C(\mathbf{u})
\end{aligned}$$

$\Rightarrow f_C$ is convex.

Let $\mathbf{y}_1 \in \mathrm{cl}C$ s.t. $||\mathbf{u} - \mathbf{y}_1||_2 = \inf_{\mathbf{y}\in C} ||\mathbf{u} - \mathbf{y}||_2$,

$\mathbf{y}_2 \in \mathrm{cl}C$ s.t. $||\mathbf{v} - \mathbf{y}_2||_2 = \inf_{\mathbf{y}\in C} ||\mathbf{v} - \mathbf{y}||_2$

Because cl$C$ is convex, $\theta\mathbf{y}_1 + (1-\theta)\mathbf{y}_2 \in \mathrm{cl}C$

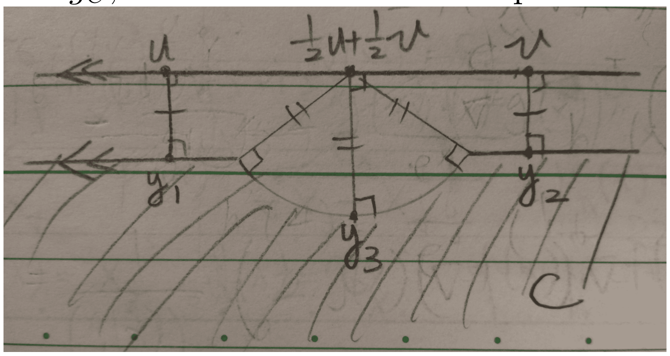
$$\begin{aligned}
\theta g_C(\mathbf{u}) + (1-\theta) g_C(\mathbf{v}) &= \theta \inf_{\mathbf{y}\in C} ||\mathbf{u} - \mathbf{y}||_2 + (1-\theta) \inf_{\mathbf{y}\in C} ||\mathbf{v} - \mathbf{y}||_2 \\
&= \theta ||\mathbf{u} - \mathbf{y}_1||_2 + (1-\theta) ||\mathbf{v} - \mathbf{y}_2||_2 \\
&\geq ||\theta(\mathbf{u} - \mathbf{y}_1) + (1-\theta)(\mathbf{v} - \mathbf{y}_2)||_2 \quad (\because \text{triangle ineq.}) \\
&= ||\theta\mathbf{u} + (1-\theta)\mathbf{v} - (\theta\mathbf{y}_1 + (1-\theta)\mathbf{y}_2)||_2 \\
&\geq \inf_{\mathbf{y}\in C} ||\theta\mathbf{u} + (1-\theta)\mathbf{v} - \mathbf{y}||_2 \quad (\because \theta\mathbf{y}_1 + (1-\theta)\mathbf{y}_2 \in \mathrm{cl}C) \\
&= g_C(\theta\mathbf{u} + (1-\theta)\mathbf{v})
\end{aligned}$$

$\Rightarrow g_C$ is convex.

(ii)

For f_C, we can use the same proof of f_C in (i), therefore we have f_C being convex.

For g_C, we have a counter example on $\mathbb{R}^2$



We have $g_C(\mathbf{u}) = ||\mathbf{u}-\mathbf{y}_1||_2 = g_C(\mathbf{v}) = ||\mathbf{v}-\mathbf{y}_2||_2$, and $g_C(\frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v}) = ||\frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v}-\mathbf{y}_3||_2$

Obviously from the graph, we can find $g_C(\frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v}) > \frac{1}{2}g_C(\mathbf{u}) + \frac{1}{2}g_C(\mathbf{v})$

$\Rightarrow g_C$ is not convex.

(c)

We have a counter example:

Let $f(x) = x^2, g(x) = e^x$, where $x \in \mathbb{R}$, and $\mathrm{dom}h = \mathrm{dom}f = \mathrm{dom}g = \mathbb{R}_{++}$

$$h(x) = \frac{x^4}{e^x} \Rightarrow h''(x) = \frac{x^2(x^2-8x+12)}{e^x}$$

Take $x = 4$

$$h''(4) = \frac{-64}{e^4} < 0 \Rightarrow h'' \not\geq 0 \Rightarrow h \text{ is not convex.}$$

2.

Let $\mathbf{a}, \mathbf{b} \in S$, i.e. $\mathbf{a}^T\mathbf{x} \leq 1, \mathbf{b}^T\mathbf{x} \leq 1, \forall \mathbf{x} \in C$

Let $\theta \in [0,1]$, for all $\mathbf{x} \in C$, we have

$$\begin{aligned}
[\theta\mathbf{a} + (1-\theta)\mathbf{b}]^T\mathbf{x} &= \theta\mathbf{a}^T\mathbf{x} + (1-\theta)\mathbf{b}^T\mathbf{x} \\
&\leq \theta \times 1 + (1-\theta) \times 1 \\
&= 1
\end{aligned}$$

$\Rightarrow \theta\mathbf{a} + (1-\theta)\mathbf{b} \in S \Rightarrow$ is convex.

3.

$$f^*(y) = \sup_{\mathbf{x}\in\mathbb{R}_{++}} (xy - \log x)$$

$$\because \underset{x\to 0^+}{lim} (xy - \log x) = \infty \therefore \sup_{\mathbf{x}\in\mathbb{R}_{++}} (xy - \log x) = \infty, \forall y \in \mathbb{R} \Rightarrow \mathrm{dom} f^* = \emptyset$$

$\Rightarrow$ We don't have the value of f^*

4.

(a) True

Proof. Let $S = \{\mathbf{x} | (\mathbf{x} - \mathbf{x}_c)^T \mathbf{P}^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\}$

Since $\mathbf{P} \in \mathbb{S}^n_{++}$, we have $\mathbf{P}^{-1} \in \mathbb{S}^n_{++}$($\because \exists \mathbf{v}$ s.t. $\mathbf{P}\mathbf{v} = \lambda\mathbf{v} \Rightarrow \mathbf{P}^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$)

$\Rightarrow$ We can let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of the eigenvectors of $\mathbf{P}^{-1}$ corresponding to eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$.

$\exists \mathbf{x}, \mathbf{y} \in S, \theta \in [0,1] \Rightarrow \exists c_i, d_i \in \mathbb{R}, i = 1, 2, \cdots, n$ s.t.

$$\mathbf{x} - \mathbf{x}_c = \sum_{i=1}^n c_i \mathbf{v}_i \text{ and } \mathbf{y} - \mathbf{x}_c = \sum_{i=1}^n d_i \mathbf{v}_i$$

$$\begin{aligned}
&[\theta\mathbf{x} + (1-\theta)\mathbf{y} - \mathbf{x}_c]^T \mathbf{P}^{-1} [\theta\mathbf{x} + (1-\theta)\mathbf{y} - \mathbf{x}_c] \\
&= [\theta(\mathbf{x} - \mathbf{x}_c) + (1-\theta)(\mathbf{y} - \mathbf{x}_c)]^T \mathbf{P}^{-1} [\theta(\mathbf{x} - \mathbf{x}_c) + (1-\theta)(\mathbf{y} - \mathbf{x}_c)] \\
&= [\theta \sum_{i=1}^n c_i \mathbf{v}_i + (1-\theta) \sum_{i=1}^n d_i \mathbf{v}_i]^T \mathbf{P}^{-1} [\theta \sum_{i=1}^n c_i \mathbf{v}_i + (1-\theta) \sum_{i=1}^n d_i \mathbf{v}_i] \\
&= [\sum_{i=1}^n (\theta c_i + (1-\theta) d_i) \mathbf{v}_i]^T \mathbf{P}^{-1} [\sum_{i=1}^n (\theta c_i + (1-\theta) d_i) \mathbf{v}_i] \\
&= [\sum_{i=1}^n (\theta c_i + (1-\theta) d_i) \mathbf{v}_i]^T [\sum_{i=1}^n (\theta c_i + (1-\theta) d_i) \frac{\mathbf{v}_i}{\lambda_i}] \\
&= \sum_{i=1}^n (\theta c_i + (1-\theta) d_i)^2 \frac{1}{\lambda_i} \quad (\because \beta \text{ is an orthonormal basis}) \\
&= \sum_{i=1}^n [\frac{\theta^2 c_i^2}{\lambda_i} + \frac{(1-\theta)^2 d_i^2}{\lambda_i} + 2\theta(1-\theta)\frac{c_i d_i}{\lambda_i}]
\end{aligned}$$

$$
\begin{aligned}
&\sum_{i=1}^{n}[\frac{\theta^2 c_i^2}{\lambda_i} + \frac{(1-\theta)^2 d_i^2}{\lambda_i} + 2\theta(1-\theta)\frac{c_i d_i}{\lambda_i}] \\
&\le \theta^2 + (1-\theta)^2 + 2\theta(1-\theta)\sum_{i=1}^{n}\frac{c_i d_i}{\lambda_i} \\
&\quad (\because \mathbf{x}, \mathbf{y} \in S \therefore \text{we have } (\mathbf{x}-\mathbf{x}_c)^T\mathbf{P}^{-1}(\mathbf{x}-\mathbf{x}_c) = \sum_{i=1}^{n}\frac{c_i^2}{\lambda_i} \le 1, \\
&\quad (\mathbf{y}-\mathbf{x}_c)^T\mathbf{P}^{-1}(\mathbf{y}-\mathbf{x}_c) = \sum_{i=1}^{n}\frac{d_i^2}{\lambda_i} \le 1) \\
&= 1 + 2\theta(1-\theta)(\sum_{i=1}^{n}\frac{c_i}{\sqrt{\lambda_i}}\frac{d_i}{\sqrt{\lambda_i}} - 1) \\
&\le 1 + 2\theta(1-\theta)[(\sum_{i=1}^{n}\frac{c_i^2}{\lambda_i})(\sum_{i=1}^{n}\frac{d_i^2}{\lambda_i}) - 1] \quad (\because \text{Cauchy-Schwarz ineq.})
\end{aligned}
$$

Beacuse $(\mathbf{x}-\mathbf{x}_c)^T\mathbf{P}^{-1}(\mathbf{x}-\mathbf{x}_c) = \sum_{i=1}^{n}\frac{c_i^2}{\lambda_i} \le 1$ and $(\mathbf{y}-\mathbf{x}_c)^T\mathbf{P}^{-1}(\mathbf{y}-\mathbf{x}_c) = \sum_{i=1}^{n}\frac{d_i^2}{\lambda_i} \le 1$, we have

$$
\begin{aligned}
&2\theta(1-\theta)[(\sum_{i=1}^{n}\frac{c_i^2}{\lambda_i})(\sum_{i=1}^{n}\frac{d_i^2}{\lambda_i}) - 1] \le 2\theta(1-\theta)(1\times 1 - 1) = 0 \\
&\Rightarrow 1 + 2\theta(1-\theta)[(\sum_{i=1}^{n}\frac{c_i^2}{\lambda_i})(\sum_{i=1}^{n}\frac{d_i^2}{\lambda_i}) - 1] \le 1 \\
&\Rightarrow \theta\mathbf{x} + (1-\theta)\mathbf{y} \in S \\
&\Rightarrow S \text{ is convex.}
\end{aligned}
$$

□

(b) False

Proof. Let $p_{\mathbf{x}}(t) = \mathbf{x}_1 + \mathbf{x}_2 t + \cdots + \mathbf{x}_k t^{k-1}$

Let $S = \{\mathbf{a} \in \mathbb{R}^k | p_{\mathbf{a}}(0) = 1, |p_{\mathbf{a}}(t)| \ge 1 \text{ for } \alpha \le t \le \beta\}$

$\exists \mathbf{x}, \mathbf{y}, \in S, \theta \in [0,1]$, let $\mathbf{z} = \theta\mathbf{x} + (1-\theta)\mathbf{y}$

$$
p_{\mathbf{z}}(0) = \theta\mathbf{x}_1 + (1-\theta)\mathbf{y}_1 = \theta p_{\mathbf{x}}(0) + (1-\theta)p_{\mathbf{y}}(0) = \theta \times 1 + (1-\theta)\times 1 = 1
$$

$$
\begin{aligned}
\exists t \in [\alpha, \beta], \ \ |p_{\mathbf{z}}(t)| &= |\sum_{i=1}^{k}(\theta \mathbf{x}_i + (1-\theta)\mathbf{y}_i)t^{i-1}| \\
&= |\theta(\sum_{i=1}^{k}\mathbf{x}_i t^{i-1}) + (1-\theta)(\sum_{i=1}^{k}\mathbf{y}_i t^{i-1})| \\
&= |\theta p_{\mathbf{x}}(t) + (1-\theta)p_{\mathbf{y}}(t)|
\end{aligned}
$$

If we take $p_{\mathbf{x}}(t) = -1$ and $p_{\mathbf{y}}(t) = 1$, then we have

$$
|p_{\mathbf{z}}(t)| = |\theta \times (-1) + (1-\theta) \times 1| = |1 - 2\theta|
$$

Take $\theta = 0.5$, we have $|p_{\mathbf{z}}(t)| = 0 \not\geq 1 \Rightarrow S$ is not convex. $\square$

(c) True

Proof. Let $p_{\mathbf{x}}(t) = \mathbf{x}_1 + \mathbf{x}_2 t + \cdots + \mathbf{x}_k t^{k-1}$
Let $S = \{\mathbf{a} \in \mathbb{R}^k | p_{\mathbf{a}}(0) = 1, |p_{\mathbf{a}}(t)| \leq 1 \text{ for } t \leq \alpha \text{ or } t \geq \beta\}$
$\exists \mathbf{x}, \mathbf{y} \in S, \theta \in [0,1]$, let $\mathbf{z} = \theta\mathbf{x} + (1-\theta)\mathbf{y}$

$$
p_{\mathbf{z}}(0) = \theta \mathbf{x}_1 + (1-\theta)\mathbf{y}_1 = \theta p_{\mathbf{x}}(0) + (1-\theta)p_{\mathbf{y}}(0) = \theta \times 1 + (1-\theta) \times 1 = 1 \tag{1}
$$

$$
\begin{aligned}
\text{For } t \leq \alpha \text{ or } t \geq \beta, \ \ |p_{\mathbf{z}}(t)| &= |\sum_{i=1}^{k}(\theta \mathbf{x}_i + (1-\theta)\mathbf{y}_i)t^{i-1}| \\
&= |\theta(\sum_{i=1}^{k}\mathbf{x}_i t^{i-1}) + (1-\theta)(\sum_{i=1}^{k}\mathbf{y}_i t^{i-1})| \\
&= |\theta p_{\mathbf{x}}(t) + (1-\theta)p_{\mathbf{y}}(t)| \qquad (2) \\
&\leq \theta|p_{\mathbf{x}}(t)| + (1-\theta)|p_{\mathbf{y}}(t)| \quad (\because \text{triangle ineq.}) \\
&\leq \theta \times 1 + (1-\theta) \times 1 \\
&= 1
\end{aligned}
$$

By (1)(2), we have $\mathbf{z} = \theta\mathbf{x} + (1-\theta)\mathbf{y} \in S \Rightarrow S$ is convex. $\square$

(d) True

Proof. Let $S = \{\mathbf{x} \in \mathbb{R}^n |\ ||\mathbf{A}\mathbf{x} + \mathbf{b}||_2 \leq \mathbf{c}^T\mathbf{x} + d\}$
$\exists \mathbf{x}, \mathbf{y} \in S, \theta \in [0, 1]$, let $\mathbf{z} = \theta\mathbf{x} + (1 - \theta)\mathbf{y}$

$$\begin{aligned}
||\mathbf{A}\mathbf{z} + \mathbf{b}||_2 &= ||\theta(\mathbf{A}\mathbf{x} + \mathbf{b}) + (1 - \theta)(\mathbf{A}\mathbf{y} + \mathbf{b})||_2 \\
&\leq \theta||\mathbf{A}\mathbf{x} + \mathbf{b}||_2 + (1 - \theta)||\mathbf{A}\mathbf{y} + \mathbf{b}||_2 \quad (\because \text{triangle ineq.}) \\
&\leq \theta(\mathbf{c}^T\mathbf{x} + d) + (1 - \theta)(\mathbf{c}^T\mathbf{y} + d) \\
&= \mathbf{c}^T(\theta\mathbf{x} + (1 - \theta)\mathbf{y}) + d \\
&= \mathbf{c}^T\mathbf{z} + d
\end{aligned}$$

$\Rightarrow \mathbf{z} \in S \Rightarrow S$ is convex.

□

5.

Problem	Convex Prob.	LP	QP	QCQP	SOCP	Quasi-Convex
a	T	F	F	T	T	T
b	T	F	F	F	T	T
c	F	F	F	F	F	F
d	T	F	T	T	T	T
e	F	F	F	F	F	F

Note that if a problem is convex opt. problem. It must also be quasi-convex opt. problem.
Note that for the generality of different problems: SOCP > QCQP > QP > LP. So if a problem is LP, then it is QP, QCQP and SOCP.

For (a), $f_0(x)$ is affine and $f_1(x)$ is a quadratic constraint ($x^T Px$ is a quadratic form, with $P \in S^n_{++}$, $x^T Px \leq 1$ is an ellipsoid). So, (a) is convex opt. problem, QCQP, SOCP and quasi-convex opt. problem.

For (b), $f_0(x)$ is affine and $f_1(x)$ is a second order cone constraint.
To determine if the constraint is equivalent to a convex quadratic constraint, we take the square on both side of the inequality.

$$\begin{aligned}
&\sqrt{x_1^2+4x_2^2+9x_3^2} \le 2x_1+x_2 \\
\Leftrightarrow & x_1^2+4x_2^2+9x_3^2 \le 4x_1^2+x_2^2+4x_1x_2 \\
\Leftrightarrow & -3x_1^2+3x_2^2+9x_3^2-4x_1x_2 \le 0 \\
\Leftrightarrow & \mathbf{x}^T\mathbf{P}\mathbf{x} \le 0, \text{ where } \mathbf{P} = \begin{bmatrix} -3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}
\end{aligned}$$

Since $\mathbf{P} \not\succeq 0$, this constraint is not a convex quadratic constraint and this problem is not a QCQP. So (b) is convex opt. problem, SOCP, quasi-convex.

For (c), note that $f_0(x)$ is not a norm.

- $((|x_1|^5+|x_2|^5)^{1/5}$ is a norm but $(x_1^5+x_2^5)^{1/5}$ is not).

To show $f_0(x)$ is not convex: $a=(2,-1), b=(1,0), (f(a)+f(b))/2-f((a+b)/2) \approx -0.00509 < 0$
To show $f_0(x)$ is not concave: $a=(1,0), b=(0,2), > (f(a)+f(b))/2-f((a+b)/2) \approx 0.4938 > 0$
So, $f_0(x)$ is not convex opt. problem on $dom(f) = \{(x_1,x_2) \mid x_1^5+x_2^5 \ge 0\}$. (If the domain is R_{++}^2 then it's convex.)
To show if $f_0(x)$ is quasi-convex, we can check its sublevel set $S_a = \{(x_1,x_2) \mid 0 \le (x_1^5+x_2^5)^{1/5} \le \alpha\}$ are convex set for all α. For $x,y \in S_a$, we check whether $\theta x+(1-\theta)y$ is in S_a. And we have a counter example
Let $(x_1,x_2)=(1.1,-1), (y_1,y_2)=(1,0)$, we have

$$(x_1^5+x_2^5)^{1/5} \approx 0.906, (y_1^5+y_2^5)^{1/5} = 1$$

Set $\alpha=1, \theta=0.5$, let $\mathbf{z}=\theta\mathbf{x}+(1-\theta)\mathbf{y}$, and then we have

$$(z_1^5+z_2^5)^{1/5} \approx 1.0448 > \alpha$$

$\Rightarrow f_0(x)$ is not quasi-convex.

For (d), it's obvious a QP, whose $f_0(x)$ is quadratic subject to an affine constraint. So (d) is convex opt. problem, QP, QCQP, SOCP and quasi-convex opt. problem.

For (e), $f_0(x)$ is a geometry mean, which is concave function on R^n_{++}, which can be proved by calculating the Hessian matrix and checking the definition of NSD i.e. $v^T D f v \leq 0$.

However, the $dom(f)$ of this problem is not R^n_{++} but $\{x \mid \prod x_i \geq 0\}$. Consider the 2-D case, i.e. $f_0(x) = \sqrt{x_1 x_2}$, the function does not have a convex domain, so it is not convex, nor is quasi-convex (it's sublevel set is not convex).