# Generalized Inequality

Lecture 8, Convex Optimization

National Taiwan University

April 29, 2021

# Table of contents

# Proper Cone

**Proper Cone**

A cone $K \subseteq \mathbf{R}^n$ is called a **proper cone** if it satisfies the following:

- $K$ is convex.
- $K$ is closed.
- $K$ is **solid**: it has nonempty interior.
- $K$ is **pointed**: it contains no line ($x \in K, -x \in K \implies x = 0$).

# Generalized Inequality – Definitions

- A proper cone $K$ can be used to define a **generalized inequality**, a partial ordering on $\mathbf{R}^n$.
- Specifically, we associate the proper cone $K$ with the partial ordering on $\mathbf{R}^n$ defined by

$$x \preceq_K y \iff y - x \in K.$$

  Also, $x \succeq_K y$ means $y \preceq_K x$.
- Similarly, we define an associated strict partial ordering by

$$x \prec_K y \iff y - x \in \mathbf{int}\ K$$

  and write $x \succ_K y$ for $y \prec_K x$.

# Generalized Inequality – Examples

- When $K = \mathbf{R}_+$, the partial ordering $\preceq_K$ is the usual ordering $\leq$ on $\mathbf{R}$, and the strict partial ordering $\prec_K$ is the same as the usual strict order $<$ on $\mathbf{R}$.
- Let $K = \mathbf{R}^n_+$ (the nonnegative orthant) in $\mathbf{R}^n$. Then $K$ is a proper cone. The associated generalized inequality $\preceq_K$ corresponds to component-wise inequality between vectors:

$$x \preceq_{\mathbf{R}^n_+} y \iff x_i \leq y_i, \;\; i = 1, \ldots, n$$

$$x \prec_{\mathbf{R}^n_+} y \iff x_i < y_i, \;\; i = 1, \ldots, n.$$

- The subscript is usually dropped when the proper cone is chosen as $K = \mathbf{R}^n_+$: $\preceq$ means $\preceq_{\mathbf{R}^n_+}$, and $\prec$ means $\prec_{\mathbf{R}^n_+}$.

# Positive Semidefinite Cones and Matrix Inequalities

- Let $K = \mathbf{S}^n_+$. Then $K$ is a proper cone in $\mathbf{S}^n$.
- The associated generalized inequality $\preceq_K$ is the usual matrix inequality:
  - $X \preceq_K Y$ means $Y - X$ is positive semidefinite.
  - $X \prec_K Y$ means $Y - X$ is positive definite.
- Note that the interior of $\mathbf{S}^n_+$ consists of the positive definite matrices: $\textbf{int}\ \mathbf{S}^n_+ = \mathbf{S}^n_{++}$.
- Similarly, the subscript is usually dropped when the proper cone in $\mathbf{S}^n$ is chosen as $K = \mathbf{S}^n_+$:

$$\preceq \ \text{means} \preceq_{\mathbf{S}^n_+}, \text{and} \ \prec \ \text{means} \ \prec_{\mathbf{S}^n_+} .$$

# Cone of Polynomials Nonnegative on $[0, 1]$

- Let $K \subseteq \mathbf{R}^n$ be defined as

$$K = \left\{ c \in \mathbf{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} \geq 0 \ \forall t \in [0, 1] \right\}.$$

- $K$ is a proper cone. And its interior is:

$$\textbf{int } K = \left\{ c \in \mathbf{R}^n \mid c_1 + c_2 t + \cdots + c_n t^{n-1} > 0 \ \forall t \in [0, 1] \right\}.$$

- For any two vectors $c, d \in \mathbf{R}^n$, we have $c \preceq_K d$ if and only if

$$c_1 + c_2 t + \cdots + c_n t^{n-1} \leq d_1 + d_2 t + \cdots + d_n t^{n-1}$$

for all $t \in [0, 1]$.

# Generalized Inequality – Properties

## Properties of generalized inequalities

1. $\preceq_K$ is **preserved under addition**: if $x \preceq_K y$ and $u \preceq_K v$, then $x + u \preceq_K y + v$.
2. $\preceq_K$ is **transitive**: if $x \preceq_K y$ and $y \preceq_K z$, then $x \preceq_K z$.
3. $\preceq_K$ is **preserved under nonnegative scaling**: if $x \preceq_K y$ and $\alpha \geq 0$, then $\alpha x \preceq_K \alpha y$.
4. $\preceq_K$ is **reflexive**: $x \preceq_K x$.
5. $\preceq_K$ is **antisymmetric**: if $x \preceq_K y$ and $y \preceq_K x$, then $x = y$.
6. $\preceq_K$ is **preserved under limits**: if $x_i \preceq_K y_i$ for $i = 1, 2, \ldots$, and as $i \to \infty$, $x_i \to x$ and $y_i \to y$, then $x \preceq_K y$.

# Generalized Inequality – Properties

## Properties of (strict) generalized inequalities

1. if $x \prec_K y$, then $x \preceq_K y$.
2. if $x \prec_K y$ and $u \preceq_K v$, then $x + u \prec_K y + v$.
3. if $x \prec_K y$ and $\alpha > 0$, then $\alpha x \prec_K \alpha y$.
4. $x \not\prec_K x$.
5. if $x \prec_K y$, then for $u$ and $v$ small enough, $x + u \prec_K y + v$.

# Minimum and Maximum elements

- For the linear ordering "$\leq$" on $\mathbf{R}$, any two points are comparable: either $x \leq y$ or $y \leq x$. This is not always true for a partial ordering like "$\preceq_K$" on $\mathbf{R}^n$.

**Minimum and Maximum Elements**

We say that $x \in S$ is the **minimum** element of $S$ (w.r.t. $\preceq_K$) if $x \preceq_K y$ for all $y \in S$.
Similarly, $x \in S$ is said to be the **maximum** element of $S$ if $x \succeq_K y$ for all $y \in S$.

- Note: If $S$ has a minimum (or maximum) element, then this element is unique.

# Minimal and Maximal Elements

## Minimal and Maximal Elements

We say that $x \in S$ is a **minimal** element of $S$ (w.r.t. $\preceq_K$) if $y \in S$, $y \preceq_K x$ only if $y = x$.
Similarly, $x \in S$ is a **maximal** element of $S$ if $y \in S$, $y \succeq_K x$ only if $y = x$.

- A set can have many different minimal (maximal) elements.

# Minimum and Minimal elements

## Minimum Element

A point $x \in S$ is the minimum element of $S$ if and only if

$$S \subseteq x + K.$$

## Minimal Element

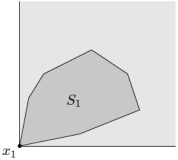
A point $x \in S$ is the minimal element of $S$ if and only if

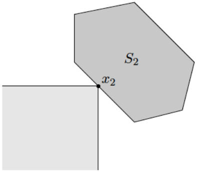
$$(x - K) \cap S = \{x\}.$$

- For $K = \mathbf{R}_+$, the concepts of minimal and minimum are the same.

# Example – Component-wise inequality in $\mathbf{R}^2$

- Consider the cone $\mathbf{R}^2_+$, which induces componentwise inequality in $\mathbf{R}^2$.
- The inequality $x \preceq y$ means $y$ is above and to the right of $x$.
- If we say $x \in S$ is the minimum element of a set $S$, then it means that all other points of $S$ lie above and to the right (e.g., $x_1$ in the figure below).
- If we say $x$ is a minimal element of a set $S$, then it means no other point of $S$ lies to the left and below $x$. (e.g., $x_2$ in the figure below)



# Example – Minimum and Minimal Elements of $\mathbf{S}^n$ (1/2)

- Consider an ellipsoid centered at the origin and associated with $A \in \mathbf{S}^n_{++}$:

$$\mathcal{E}_A = \left\{ x \,\middle|\, x^T A^{-1} x \leq 1 \right\}.$$

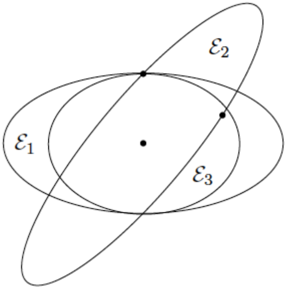
- $A \preceq B$ if and only if $\mathcal{E}_A \subseteq \mathcal{E}_B$.
- Let $v_1, \ldots, v_k \in \mathbf{R}^n$ be given and define the set of ellipsoids that contain points $v_1, \ldots, v_k$:

$$S = \left\{ P \in \mathbf{S}^n_{++} \,\middle|\, v_i^T P^{-1} v_i \leq 1, i = 1, \ldots, k \right\}.$$

- What is the minimum element of $S$?

# Example – Minimum and Minimal Elements of $\mathbf{S}^n$ (2/2)

- The set $S$ does not have a minimum element.
- An ellipsoid is minimal if it contains the points, but no smaller ellipsoid does.




- In the above figure, $\mathcal{E}_1$ is NOT minimal since there exists other smaller ellipsoids (e.g., $\mathcal{E}_3$) that also contain the points. $\mathcal{E}_2$, on the other hand, is minimal since there are no other smaller ellipsoids that contain the points.

# Dual Cones

## Dual Cones

Let $K$ be a cone. The set

$$K^* = \left\{ y \;\middle|\; x^T y \geq 0 \text{ for all } x \in K \right\}$$

is called the **dual cone** of $K$.

## Basic Properties of Dual Cones

- $K^*$ is a cone.
- $K^*$ is convex (even when $K$ is not convex).

# Dual Cones

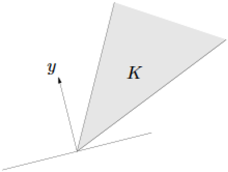
## Dual Cones

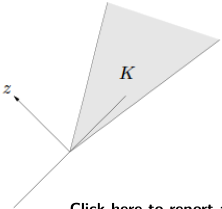
Let $K$ be a cone. The set

$$K^* = \left\{ y \,\middle|\, x^T y \geq 0 \text{ for all } x \in K \right\}$$

is called the **dual cone** of $K$.

- $y \in K^*$ if and only if $-y$ is the normal vector of a hyperplane that supports $K$ at the origin.



# Dual Cones – Examples

## Dual Cones

Let $K$ be a cone. The set

$$K^* = \left\{ y \,\middle|\, x^T y \geq 0 \text{ for all } x \in K \right\}$$

is called the **dual cone** of $K$.

## Subspace

The dual cone of a subspace $V \subseteq \mathbf{R}^n$ (which is a cone) is its orthogonal complement

$$V^\perp = \left\{ y \,\middle|\, y^T v = 0 \quad \forall v \in V \right\}.$$

# Dual Cones – Examples

## Dual Cones

Let $K$ be a cone. The set

$$K^* = \left\{ y \;\middle|\; x^T y \geq 0 \text{ for all } x \in K \right\}$$

is called the **dual cone** of $K$.

## Nonnegative orthant

The cone $\mathbf{R}^n_+$ is its own dual:

$$y^T x \geq 0, \; \forall x \succeq 0 \Leftrightarrow y \succeq 0.$$

We call such a cone **self-dual**.

# Dual of a Positive Semidefinite Cone

## Positive semidefinite cone

The positive semidefinite cone $\mathbf{S}_+^n$ is self-dual, i.e., for $X, Y \in \mathbf{S}^n$,

$$\mathbf{tr}(XY) \geq 0 \text{ for all } X \succeq 0 \Leftrightarrow Y \succeq 0.$$

Here, we used the standard inner product $\mathbf{tr}(XY) = \sum_{i,j=1}^n X_{ij} Y_{ij}$ on the set of symmetric $n \times n$ matrices $\mathbf{S}^n$.

Proof:

- If $Y \notin \mathbf{S}_+^n$, then $\exists q \in \mathbf{R}^n$ s.t. $q^T Y q < 0$. Let $X = qq^T \in \mathbf{S}_+^n$, then $\mathbf{tr}(XY) = \mathbf{tr}(qq^T Y) = \mathbf{tr}(q^T Y q) < 0$.
- If $Y \in \mathbf{S}_+^n$, then for any $X \in \mathbf{S}_+^n$, i.e., $X \succeq 0$, $X$ can be expressed as $X = \sum_{i=1}^n \lambda_i q_i q_i^T$, with $\lambda_i \geq 0$ and $q_i \in \mathbf{R}^n \setminus \{0\}$. Then, $\mathbf{tr}(XY) = \sum_{i=1}^n \lambda_i \mathbf{tr}(q_i q_i^T Y) = \sum_{i=1}^n \lambda_i \mathbf{tr}(q_i^T Y q_i) \geq 0$.

# Properties of Dual Cones

## Properties of Dual Cones

Dual cones satisfy the following properties:

- $K^*$ is closed and convex.
- $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.
- If $K$ has nonempty interior, then $K^*$ is pointed.
- If the closure of $K$ is pointed, then $K^*$ has nonempty interior.
- $K^{**}$ is the closure of the convex hull of $K$.
- If $K$ is convex and closed, then $K^{**} = K$.

Moreover, if $K$ is a proper cone, then

- $K^*$ is also a proper cone.
- $K^{**} = K$.

# Dual Generalized Inequalities

## Dual generalized inequalities

If $K$ is a proper cone which induces a generalized inequality $\preceq_K$. Then we refer to the generalized inequality $\preceq_{K^*}$ as the **dual** of the generalized inequality $\preceq_K$ (note that $K^*$ is also a proper cone).

## Properties of dual generalized inequalities

- $x \preceq_K y$ if and only if $\lambda^T x \leq \lambda^T y$ for all $\lambda \succeq_{K^*} 0$.
- $x \prec_K y$ if and only if $\lambda^T x < \lambda^T y$ for all $\lambda \succeq_{K^*} 0$, $\lambda \neq 0$.

# Theorem of Alternatives for Strict Generalized Inequalities

## Theorem of alternatives for linear strict generalized inequalities

Suppose $K \subseteq \mathbf{R}^m$ is a proper cone. The strict generalized inequality

$$Ax \prec_K b,$$

where $x \in \mathbf{R}^n$, is infeasible iff there exists $\lambda \in \mathbf{R}^m$ s.t.

$$\lambda \neq 0, \quad \lambda \succeq_{K^*} 0, \quad A^T\lambda = 0, \quad \lambda^T b \leq 0.$$

Proof idea:

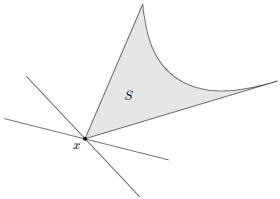
- Apply the separating hyperplane theorem on $\{b - Ax \mid x \in \mathbf{R}^n\}$ and $\mathbf{int}\ K$.
- Converse: can be proved by contradiction.

# Dual Characterization of Minimum Element

**Dual characterization of minimum element**

Consider a set $S \subseteq \mathbf{R}^n$, not necessarily convex. Then, $x$ is the minimum element of $S$, with respect to the generalized inequality $\preceq_K$, if and only if for all $\lambda \succ_{K^*} 0$, $x$ is the unique minimizer of $\lambda^T z$ over $z \in S$.

- This means that for any $\lambda \succ_{K^*} 0$, the hyperplane $\{z \mid \lambda^T(z - x) = 0\}$ is a **strict supporting hyperplane** to $S$ at $x$ (i.e., the hyperplane intersects $S$ only at the point $x$).



# Dual Characterization of Minimum Element

**Dual characterization of minimum element**

Consider a set $S \subseteq \mathbf{R}^n$, not necessarily convex. Then, $x$ is the minimum element of $S$, with respect to the generalized inequality $\preceq_K$, if and only if for all $\lambda \succ_{K^*} 0$, $x$ is the unique minimizer of $\lambda^T z$ over $z \in S$.

Proof:

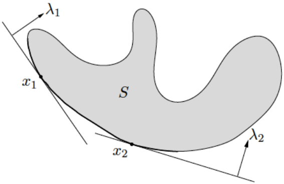
1. If $x$ is the minimum element in $S$:

2. If $x$ is not the minimum element in $S$:

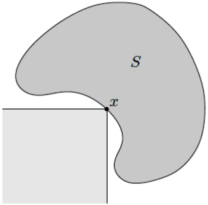
# Dual Characterization of Minimal Elements (1/3)

**Dual Characterization of minimal elements**

If $\exists \lambda \succ_{K^*} 0$ and $x$ minimizes $\lambda^T z$ over $z \in S$, then $x$ is minimal.




- The converse is not true: if $x$ is minimal in $S$, there does not necessarily exist $\lambda \succ_{K^*} 0$ such that $x$ minimizes $\lambda^T z$ over $z \in S$. (Example: if $S$ is not convex).



# Dual Characterization of Minimal Elements (2/3)

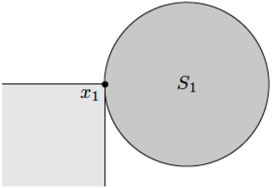
**Dual Characterization of minimal elements in convex sets**

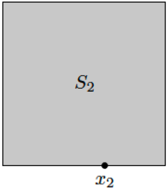
Suppose the set $S \subseteq \mathbf{R}^n$ is convex. Then, for any minimal element $x$ in $S$ there exists a nonzero $\lambda \succeq_{K^*} 0$ such that $x$ minimizes $\lambda^T z$ over $z \in S$.

Proof: Suppose $x$ is minimal: $((x - K) \setminus \{x\}) \cap S = \varnothing$. By applying the separating hyperplane theorem to the convex sets $(x - K) \setminus \{x\}$ and $S$, we have $\exists \lambda \neq 0$ and $\mu$ such that $\lambda^T (x - y) \leq \mu$ for all $y \in K$ and $\lambda^T z \geq \mu$ for all $z \in S$. Note that $\lambda^T x = \mu$ since $x \in S$ and $x \in x - K$. So $\lambda^T y \geq 0$ for all $y \in K$ and therefore $\lambda \in K^*$ (i.e., $\lambda \succeq_{K^*} 0$). Finally, $\lambda^T z \geq \mu = \lambda^T x$ for all $z \in S$ suggests that $x$ minimizes $\lambda^T z$ over $z \in S$.
Question: Must there exist a $\lambda \succ_{K^*} 0$ such that $x$ minimizes $\lambda^T z$ over $z \in S$?

# Dual Characterization of Minimal Elements (3/3)

- A point x can be a minimal point of a convex set $S$, but not a minimizer of $\lambda^T z$ over $z \in S$ for any $\lambda \succ_{K^*} 0$ (left figure).
  - We only need $\lambda \succeq_{K^*} 0$ and $\lambda \neq 0$.




- The converse of the statement in the previous page is not true: not every minimizer of $\lambda^T z$ over $z \in S$, with $\lambda \succeq_{K^*} 0$, is minimal (right figure).

# Monotonicity with respect to a generalized inequality

**Monotonicity with respect to a generalized inequality**

Suppose $K \subseteq \mathbf{R}^n$ is a proper cone with associated generalized inequality $\preceq_K$. A function $f : \mathbf{R}^n \to \mathbf{R}$ is called ***K*-nondecreasing** if

$$x \preceq_K y \implies f(x) \leq f(y),$$

and ***K*-increasing** if

$$x \preceq_K y,\ x \neq y \implies f(x) < f(y).$$

- We define $K$-nonincreasing and $K$-decreasing functions in a similar way.

# Example – Monotone vector functions

- A function $f : \mathbf{R}^n \to \mathbf{R}$ is nondecreasing with respect to $\mathbf{R}^n_+$ if and only if

$$x_1 \leq y_1, \ldots, x_n \leq y_n \Longrightarrow f(x) \leq f(y)$$

  for all $x, y$.
- This is the same as saying that $f$, when restricted to any component $x_i$ (i.e., $x_i$ is considered the variable while $x_j$ for $j \neq i$ are fixed), is nondecreasing.

# Example – Matrix monotone functions

- A function $f : \mathbf{S}^n \to \mathbf{R}$ is called **matrix monotone** (increasing, decreasing) if it is monotone with respect to the positive semidefinite cone.
- Some examples of matrix monotone functions of the variable $X \in \mathbf{S}^n$ :
  1. $\mathbf{tr}(WX)$, where $W \in \mathbf{S}^n$, is matrix nondecreasing if $W \succeq 0$, and matrix increasing if $W \succ 0$ (it is matrix nonincreasing if $W \preceq 0$, and matrix decreasing if $W \prec 0$).
  2. $\mathbf{tr}(X^{-1})$ is matrix decreasing on $\mathbf{S}^n_{++}$.
  3. $\det X$ is matrix increasing on $\mathbf{S}^n_{++}$, and matrix nondecreasing on $\mathbf{S}^n_{+}$.

# Convexity with respect to a generalized inequality

## Convexity with respect to a generalized inequality

Suppose $K \subseteq \mathbf{R}^m$ is a proper cone with associated generalized inequality $\preceq_K$. We say $f : \mathbf{R}^n \to \mathbf{R}^m$ is ***K*-convex** if for all $x, y \in \mathbf{R}^n$, and $0 \leq \theta \leq 1$,

$$f(\theta x + (1-\theta)y) \preceq_K \theta f(x) + (1-\theta)f(y).$$

- Example: A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is convex with respect to componentwise inequality (i.e., the generalized inequality induced by $\mathbf{R}^m_+$) if and only if for all $x, y$ and $0 \leq \theta \leq 1$,

  $$f(\theta x + (1-\theta)y) \preceq \theta f(x) + (1-\theta)f(y),$$

  i.e., each component $f_i$ is a convex function.
- It is strictly convex with respect to componentwise inequality if and only if each component $f_i$ is strictly convex.

# Generalized Inequality Constraints in Convex Problems

- Recall the convex optimization problem in the standard form we studied before:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p. \end{aligned}$$

- One very useful generalization of the above standard form convex optimization problem is obtained by allowing the inequality constraint functions to be vector valued, and using generalized inequalities in the constraints:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

  where $f_0 : \mathbf{R}^n \to \mathbf{R}$, $K_i \subseteq \mathbf{R}^{k_i}$ are proper cones, and $f_i : \mathbf{R}^n \to \mathbf{R}^{k_i}$ are $K_i$-convex.
- We refer to this problem as a (standard form) convex optimization problem with generalized inequality constraints.
- The above convex problem is a special case with $K_i = \mathbf{R}_+, i = 1, \ldots, m$.

# Generalized inequality constraints

- Many of the results for ordinary convex optimization problems hold for problems with generalized inequalities:
  - The feasible set, any sublevel set, and the optimal set are convex.
  - Any point that is locally optimal is globally optimal.
- We will also see that convex optimization problems with generalized inequality constraints can often be solved as easily as ordinary convex optimization problems.

# Conic form problems

- Among the simplest convex optimization problems with generalized inequalities are the conic form problems (or cone programs), which have a linear objective and one inequality constraint function, which is affine (and therefore $K$-convex):

**Conic Form Problems**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b. \end{array}$$

- When $K$ is the nonnegative orthant, the conic form problem reduces to a linear program.
- We can view conic form problems as a generalization of linear programs in which componentwise inequality is replaced with a generalized linear inequality.

# Conic form problems

- Continuing the analogy to linear programming, we refer to the conic form problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & x \succeq_K 0 \\ & Ax = b \end{aligned}$$

  as a conic form problem in standard form.

- Similarly, the problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Fx + g \preceq_K 0 \end{aligned}$$

  is called a conic form problem in inequality form.

# Example – Second-order cone programming

- Recall the second-order cone programming (SOCP) we learned before:

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g. \end{array}$$

- The SOCP can be expressed as a conic form problem

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & -(A_i x + b_i, c_i^T x + d_i) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\ & Fx = g, \end{array}$$

  in which

$$K_i = \left\{(y, t) \in \mathbf{R}^{n_i+1} \mid \|y\|_2 \leq t\right\},$$

  i.e., the second-order cone in $\mathbf{R}^{n_i+1}$.
- This explains the name second-order cone program for the optimization problem.

# Semidefinite programming

- When $K$ is $\mathbf{S}_+^k$, the cone of positive semidefinite $k \times k$ matrices, the associated conic form problem is called a **semidefinite program (SDP)**, and has the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots + x_n F_n + G \preceq 0. \quad ^1 \\ & Ax = b, \end{array}$$

  where $G, F_1, \ldots, F_n \in \mathbf{S}^k$, and $A \in \mathbf{R}^{p \times n}$
- The inequality here is a **linear matrix inequality (LMI)**.
- If the matrices $G, F_1, \ldots, F_n$ are all diagonal, then the LMI is equivalent to a set of $n$ linear inequalities, and the SDP reduces to a linear program.

[1] Note that when $K$ is $\mathbf{S}_+^k$ or $\mathbf{R}_+^k$, the subscript in $\preceq_K$ can be omitted by the notational convention.

# Standard form semidefinite programs

- Following the analogy to LP, a standard form SDP has linear equality constraints, and a (matrix) nonnegativity constraint on the variable $X \in \mathbf{S}^n$:

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \dots, p \\ & X \succeq 0, \end{array}$$

  where $C, A_1, \dots, A_p \in \mathbf{S}^n$. (Recall that $\mathbf{tr}(CX) = \sum_{i,j=1}^n C_{ij} X_{ij}$ is the form of a general real-valued linear function on $\mathbf{S}^n$.)
- This form should be compared to the standard form linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0. \end{array}$$

- In LP and SDP standard forms, we minimize a linear function of the variable, subject to $p$ linear equality constraints on the variable, and a nonnegativity constraint on the variable.

# Inequality form semidefinite programs

- An inequality form SDP, analogous to an inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \end{array}$$ [2]

has no equality constraints, and one LMI:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 A_1 + \cdots + x_n A_n \preceq B, \end{array}$$

with variable $x \in \mathbf{R}^n$, and parameters $B, A_1, \ldots, A_n \in \mathbf{S}^k$, $c \in \mathbf{R}^n$.

[2] Note again that when the subscript in a generalized inequality is omitted, the associated proper cone must be either $\mathbf{S}^k_+$ or $\mathbf{R}^k_+$.

# Multiple LMIs and linear inequalities

- It is common to refer to a problem with linear objective, linear equality and inequality constraints, and several LMI constraints, i.e.,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F^{(i)}(x) = x_1 F_1^{(i)} + \cdots + x_n F_n^{(i)} + G^{(i)} \preceq 0, i = 1, \ldots, K \\ & Gx \preceq h, \\ & Ax = b, \end{array}$$

  as an SDP as well.
- Such problems are readily transformed to an SDP, by forming a large block diagonal LMI from the individual LMIs and linear inequalities:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{diag}\left(Gx - h, F^{(1)}(x), \ldots, F^{(K)}(x)\right) \preceq 0 \\ & Ax = b. \end{array}$$