Convex Functions (II)

Lecture 4, Convex Optimization

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Basic Operations that Preserve Convexity

- If f is convex and $\alpha \geq 0$, then αf is also convex.
- If both f_1 and f_2 are convex, then $f_1 + f_2$ is also convex.
- More generally, if $f_1, \ldots, f_n$ are convex functions, then any of their "conic combinations",

$$f = w_1 f_1 + \cdots + w_n f_n,$$

  is also convex (with $w_1, \ldots, w_n \geq 0$). This is also called the **nonnegative weighted sum**.
- Extension: if $f(x, y)$ is convex in x for any $y \in \mathcal{A}$, and $w(y) \geq 0$ for any $y \in \mathcal{A}$, then the function

$$g(x) = \int_{\mathcal{A}} w(y) f(x, y) dy$$

  is convex in x.

Basic Operations that Preserve Convexity

- Suppose $f : \mathbf{R}^n \to \mathbf{R}$, $A \in \mathbf{R}^{n\times m}$, and $b \in \mathbf{R}^n$. Define $g : \mathbf{R}^m \to \mathbf{R}$ by
$$g(x) = f(Ax + b)$$
with $\mathbf{dom}\ g = \left\{ x \;\middle|\; Ax + b \in \mathbf{dom}\ f \right\}$.
- If f is convex, then g is also convex.
- If f is concave, so is g.

Pointwise maximum

- If f_1 and f_2 are convex functions then their **pointwise maximum** f, defined as
$$f(x) = \max\{f_1(x), f_2(x)\},$$
with $\mathbf{dom}\ f = \mathbf{dom}\ f_1 \cap \mathbf{dom}\ f_2$, is also convex.
- Proof:
$$\begin{aligned} f(\theta x + (1-\theta)y) &= \\ &\leq \\ &\leq \\ &= \theta f(x) + (1-\theta) f(y). \end{aligned}$$
- It can be easily extended: if $f_1, \ldots, f_m$ are convex, then their pointwise maximum
$$f(x) = \max\{f_1(x), \ldots, f_m(x)\},$$
is also convex.

Pointwise maximum – Examples

Piecewise-linear functions

A **piecewise-linear** function $f(x) = \max\left\{a_1^T x + b_1, \ldots, a_L^T x + b_L\right\}$ is convex, since the affine functions $a_i^T x + b_i$ are all convex.

Sum of r largest components

For $x \in \mathbf{R}^n$, we denote by $x_{[i]}$ the ith largest component of x, i.e.,

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$$

are the components of x sorted in nonincreasing order $\left(\left\{x_{[1]}, \ldots, x_{[n]}\right\} = \{x_1, \ldots, x_n\}\right)$. Then the function $f(x) = \sum_{i=1}^r x_{[i]}$ is convex.

- Note that, as a generalization, the function $f(x) = \sum_{i=1}^r w_i x_{[i]}$ is also convex as long as $w_1 \geq w_2 \geq \ldots \geq w_r \geq 0$.

Pointwise supremum

- If for each $y \in \mathcal{A}$, $f(x, y)$ is convex in x, then the function g, defined as

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex in x. Here

$$\textbf{dom } g = \left\{ x \;\middle|\; (x, y) \in \textbf{dom } f \; \forall y \in \mathcal{A}, \;\; \sup_{y \in \mathcal{A}} f(x, y) < \infty \right\}.$$

- Similarly, the pointwise infimum of a set of concave functions is a concave function.

Recall: the **supremum** and **infimum** of a set $\mathcal{A}$ are defined as

$$\sup \mathcal{A} = \min \{ y \mid y \geq x, \forall x \in \mathcal{A} \} \text{ (i.e., the minimum upper bound of } \mathcal{A})$$

and

$$\inf \mathcal{A} = \max \{ y \mid y \leq x, \forall x \in \mathcal{A} \} \text{ (i.e., the maximum lower bound of } \mathcal{A}),$$

respectively.

Pointwise supremum

- In terms of epigraphs, the pointwise supremum of functions corresponds to the intersection of epigraphs: if

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y),$$

then we have

$$\textbf{epi } g = \bigcap_{y \in \mathcal{A}} \textbf{epi } f(\cdot, y).$$

- Thus, the result follows from the fact that the intersection of a family of convex sets is convex.

Pointwise supremum – Examples

Support function of a set

Let $C \subseteq \mathbf{R}^n$ with $C \neq \varnothing$. The **support function** S_C associated with the set C, defined as

$$S_C(x) = \sup \left\{ x^T y \mid y \in C \right\},$$

with $\mathbf{dom}\ S_C = \left\{ x \mid \sup_{y \in C} x^T y < \infty \right\}$, is convex.

Distance to farthest point of a set

Let $C \subseteq \mathbf{R}^n$. The distance (in any norm) to the farthest point of C,

$$f(x) = \sup_{y \in C} ||x - y||,$$

is convex.

Pointwise supremum – Examples

Maximum eigenvalue of a symmetric matrix

The function $f(X) = \lambda_{max}(X)$, with **dom** $f = \mathbf{S}^m$, is convex.

Proof:

$$f(X) = \sup \left\{ y^T X y \mid ||y||_2 = 1 \right\}.$$

Norm of a matrix

The function $f(X) = ||X||_2$ with **dom** $f = \mathbf{R}^{p \times q}$, where $|| \cdot ||_2$ denotes the spectral norm or maximum singular value, is convex.

Proof:

$$f(X) = \sup \left\{ u^T X v \mid ||u||_2 = 1, ||v||_2 = 1 \right\},$$

is the pointwise supremum of a family of linear functions of X.

Convexity of composition of functions

Convexity of composition of functions

Let $h : \mathbf{R} \to \mathbf{R}$, and $g : \mathbf{R} \to \mathbf{R}$ and $f = h \circ g : \mathbf{R} \to \mathbf{R}$, $f(x) = h(g(x))$. Let $\mathbf{dom}\ f = \mathbf{dom}\ g = \mathbf{dom}\ h = \mathbf{R}$ and f, g, h be differentiable. Then,

- f is convex if h is convex and nondecreasing, and g is convex,
- f is convex if h is convex and nonincreasing, and g is concave,
- f is concave if h is concave and nondecreasing, and g is concave,
- f is concave if h is concave and nonincreasing, and g is convex.

Proof (for the case where h and g are both twice differentiable):

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x).$$

Proof for the case where h and g are not twice differentiable

- If h and g are convex, then $g(y) \geq g(x) + g'(x)(y-x)$ and $h(y) \geq h(x) + h'(x)(y-x)$.
- If h is further nondecreasing, then $h'(x) \geq 0$.
- Now, we have

$$\begin{aligned} f(y) = h(g(y)) &\geq h(g(x) + g'(x)(y-x)) \quad (\because h \text{ is n.d. } \& \ g \text{ is convex}) \\ &\geq h(g(x)) + h'(g(x)) \cdot g'(x)(y-x) \\ &= f(x) + f'(x)(y-x). \end{aligned}$$

Examples – Convexity of composition of functions

- If g is convex then $\exp g(x)$ is convex.
- If g is concave and positive, then $\log g(x)$ is concave.
- If g is concave and positive, then $1/g(x)$ is convex.
- If g is convex and nonnegative and $p \geq 1$, then $g(x)^p$ is convex.
- If g is convex then $-\log(-g(x))$ is convex on $\{x \mid g(x) < 0\}$.

A generalization

Convexity of composition of functions

Let $h : \mathbf{R} \to \mathbf{R}$, and $g : \mathbf{R}^n \to \mathbf{R}$ and $f = h \circ g : \mathbf{R}^n \to \mathbf{R}$, $f(x) = h(g(x))$. Let $\textbf{dom } f = \textbf{dom } g = \mathbf{R}^n, \textbf{dom } h = \mathbf{R}$, and f, g, h be differentiable. Then,

- f is convex if h is convex and nondecreasing, and g is convex,
- f is convex if h is convex and nonincreasing, and g is concave,
- f is concave if h is concave and nondecreasing, and g is concave,
- f is concave if h is concave and nonincreasing, and g is convex.

Proof idea: convexity is determined by the behavior of a function on arbitrary lines that intersect its domain.

Vector composition – A further generalization

Vector Composition

Suppose $f(x) = h(g(x)) = h(g_1(x), ..., g_k(x))$, with $h : \mathbf{R}^k \to \mathbf{R}$, $g_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, ..., k$. Then,

- f is convex if h is convex, h is n.d. in each argument, and g_i are convex,
- f is convex if h is convex, h is n.i. in each argument, and g_i are concave,
- f is concave if h is concave, h is n.d. in each argument, and g_i are concave.
- f is concave if h is concave, h is n.i. in each argument, and g_i are convex.

Proof: W.l.o.g., we can assume $n = 1$.

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x),$$

Vector composition examples

- Let $h(z) = z_{[1]} + \ldots + z_{[r]}$, the sum of the r largest components of $z \in \mathbf{R}^k$. Then h is convex and nondecreasing in each argument.
- Suppose $g_1, \ldots, g_k$ are convex functions on $\mathbf{R}^n$. Then the composition function $f = h \circ g$, i.e., the pointwise sum of the r largest g_i's, is convex.
- The function $h(z) = \log(\sum_{i=1}^k e^{z_i})$ is convex and nondecreasing in each argument, so $\log(\sum_{i=1}^k e^{g_i})$ is convex whenever g_i are.
- For $0 < p \leq 1$, the function $h(z) = (\sum_{i=1}^k z_i^p)^{1/p}$ on $\mathbf{R}_+^k$ is concave, and its extension (which has the value $-\infty$ for $z \not\succeq 0$) is nondecreasing in each component. So if g_i are concave and nonnegative, we conclude that $f(x) = (\sum_{i=1}^k g_i(x)^p)^{1/p}$ is concave.

Vector composition examples

- Suppose $p \geq 1$, and $g_1, \ldots, g_k$ are convex and nonnegative. Then the function $(\sum_{i=1}^{k} g_i(x)^p)^{1/p}$ is convex.
  - Proof idea: The ℓ_p-norm is convex, and is nondecreasing in each argument if the considered domain is $\mathbf{dom} \; ||\cdot||_p = \mathbf{R}_+^k$.
- The geometric mean $h(z) = (\prod_{i=1}^{k} z_i)^{1/k}$ on $\mathbf{R}_+^k$ is concave and its extension is nondecreasing in each argument. It follows that if $g_1, \ldots, g_k$ are nonnegative concave functions, then so is their geometric mean,

$$\left(\prod_{i=1}^{k} g_i\right)^{1/k}.$$

Minimization

Minimization and convexity

If f is convex in (x, y), and C is a convex nonempty set, then

$\forall y$, f(x) is convex
x f(y)

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex in x, provided $g(x) > -\infty$ for some x (which implies $g(x) > -\infty$ for all x), with

$$\mathbf{dom}\ g = \{x \mid (x, y) \in \mathbf{dom}\ f,\ \exists y \in C\}\,.$$

- Proof: For $x_1, x_2 \in \mathbf{dom}\ g$. Let $\epsilon > 0$. Then $\exists y_1, y_2 \in C$ such that $f(x_i, y_i) \leq g(x_i) + \epsilon$ for $i = 1, 2$. For any $\theta, 0 \leq \theta \leq 1$, we have

$$\begin{aligned} g(\theta x_1 + (1-\theta)x_2) &= \inf_{y \in C} f(\theta x_1 + (1-\theta)x_2, y) \\ &\leq f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \\ &\leq \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2) \\ &\leq \theta g(x_1) + (1-\theta) g(x_2) + \epsilon \end{aligned}$$

why?

convex

Minimization

Minimization and convexity

If f is convex in (x, y), and C is a convex nonempty set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex in x, provided $g(x) > -\infty$ for some x (which implies $g(x) > -\infty$ for all x), with

$$\mathbf{dom}\ g = \{x \mid (x, y) \in \mathbf{dom}\ f,\ \exists y \in C\}.$$

- Alternative proof (based on epigraph): Since

$$g(x) = \inf_{y \in C} f(x, y),$$

we have

$$\mathbf{epi}\ g = \{(x, t) \mid (x, y, t) \in \mathbf{epi}\ f,\ \ \exists y \in C\}.$$

A challenge of the previous proof and a potential correction of the proof

- I received from a student that if we define $f : \mathbf{R}^2 \to \mathbf{R}$, $\mathbf{dom}\ f = \mathbf{R} \times \mathbf{R}_{++}$, with

$$f(x, y) = x + \frac{1}{y},$$

and letting $C = \mathbf{R}_{++}$, then we have

$$g(x) = \inf_{y \in C} f(x, y) = x.$$

So, it is obvious that $(0, 0) \in \mathbf{epi}\ g$.

- However, $(0, 0) \notin \{(x, t) \mid (x, y, t) \in \mathbf{epi}\ f, \quad \exists y \in C\}$, since $\forall y \in C$, $f(x, y) > x$. This example has essentially invalidate the proof in the previous page.
- A possible correction of this proof is to instead argue that

$$\mathbf{epi}\ g = \mathbf{cl}\ \{(x, t) \mid (x, y, t) \in \mathbf{epi}\ f, \quad \exists y \in C\},$$

noting that the epigraph of any function is a closed set, and that the closure operation preserves convexity of a set.

Example – Distance to a set

convex set

- The distance of a point x to a set $S \subseteq \mathbf{R}^n$, in the norm $||\cdot||$, is defined as

$$\mathbf{dist}\ (x, S) = \inf_{y \in S} ||x - y||.$$

is norm ⇒ convex

- The function $||x - y||$ is convex in (x, y), so if the set S is convex, the distance function $\mathbf{dist}\ (x, S)$ is a convex function of x.

取 sup ⇒ 不管 S 是 convex 也可

Example

- Suppose h is convex. Then the function g defined as

$$g(x) = \inf \{h(y) \mid Ay = x\}$$ [1]

is convex.

- Proof: We define f by [2]

$$f(x,y) = \begin{cases} h(y) & \text{if } Ay = x \\ \infty & \text{otherwise} \end{cases}$$

which is convex in (x, y). Then g is the minimum of f over y, and hence is convex. (It is not hard to show directly that g is convex.)

[1] In fact, it can be shown that $g(x) = \min \{h(y) \mid Ay = x\}$.
[2] Note that $\mathbf{dom}\, f = \{(x, y) \mid Ay = x\}$ is convex.