Homework 3

Yu-Chieh Kuo B07611039

This homework answers the problem set sequentially.

1. To avoid the notation confusion, we replace the coefficient c and a in the theorem with x and y. Thefore, we can rewrite the theorem as below:

   For all constants $x > 0$ and $y > 1$, and for all monotonically increasing functions $f(n)$, we have $(f(n))^x = o(y^{f(n)})$.

   Let $f(n) = \log_2 n$, $x = a$, $y = 2^b$, $a, b > 0$, by using the above theorem, we have

   $$(\log_2 n)^a = o(2^{b(\log_2 n)}) = o(n^b)$$

   Thus, we prove the function by using the above theorem.

2. (a) Let $f(n) = (\log n)^{\log n}$, $g(n) = \frac{n}{\log n}$, we can find the limit of $\frac{g(n)}{f(n)}$ when $n \to \infty$ to justify whether $f(n) \in O(g(n))$.

   $$\begin{aligned}
   \lim_{n\to\infty} \frac{g(n)}{f(n)} &= \lim_{n\to\infty} \frac{\frac{n}{\log n}}{\log n^{\log n}} \\
   &= \lim_{n\to\infty} \frac{n}{\log n \log n^{\log n}} \\
   &= \lim_{n\to\infty} \frac{n}{\log n^{\log n+1}} \\
   &= \lim_{n\to\infty} \frac{1}{\log n \cdot \frac{1}{n} \log e} \\
   &= 0
   \end{aligned}$$

   Hence, we can obviously say that $f(n)$ is larger than $g(n)$ (as an inaccurate statement), so $f(n) \notin O(g(n))$.
   To justify whether $f(n) \in \Omega(g(n))$, we can prove by Ω's definition:

   $$\Omega(g(n)) = \{f(n) \mid \exists\ c,\ n_0 > 0\ s.t.\ cg(n) \le f(n),\ \forall n \ge n_0\}$$

   By definition, we have:

   $$c(\frac{n}{\log n}) \le (\log n)^{\log n}$$

   Substitute $\log n$ by x and than take the $\log_2$ at both sides, we have:

   $$x + \log c - \log x \le x \log x$$

   When $c = 1$, $x > 1$ *i.e.* $n > 2$, the inequality holds. Thus, by definition, we prove that $f(n) \in \Omega(g(n))$.

(b) If $f(n) \in O(g(n))$, then there exists $c, n_0 > 0$ $s.t. f(n) \leq cg(n)$. We have

$$n^2 2^n \leq c3^n$$

Divide by 2^n and take the log at both sides, we have:

$$2\log n \leq n(\log c + \log 3 - \log 2)$$

Let $c = 1$, when $n \geq 13$, the inequality holds. Therefore, we can say that $f(n) \in O(g(n))$.

As for whether $f(n) \in \Omega(g(n))$, we can find the limit of $\frac{f(n)}{g(n)}$ when $n \to \infty$ to justify it.

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{n^2 2^n}{3^n} \\
&= \lim_{n\to\infty} \frac{n^2}{(\frac{3}{2})^n} \\
&= \lim_{n\to\infty} \frac{2n}{\ln\frac{3}{2}(\frac{3}{2})^n} \\
&= \lim_{n\to\infty} \frac{2}{(\ln\frac{3}{2})^2(\frac{3}{2})^n} \\
&= 0
\end{aligned}$$

Hence, we can obviously say that $g(n)$ is larger than $f(n)$ (as an inaccurate statement), so $f(n) \notin \Omega(g(n))$.

3. Since $f(n) \in O(g(n))$, there exists $c, n_0 > 0$ $s.t.$ $f(n) \leq cg(n)$ $\forall n > n_0$. Taking the log at both sides, we have

$$\log f(n) \leq \log c + \log g(n),\ \forall n > n_0$$

If the statement is true, then we can find a coefficient c' such that

$$\log f(n) \leq \log c + \log g(n) \leq c' \log g(n),\ \forall n > n_0$$

If $c' \geq \frac{\log c + \log g(n)}{\log g(n)}$, then the inequality holds. Fortunately there exists c', when $c' \geq \frac{\log c + \log g(n_0)}{\log g(n_0)}$, to hold this inequality. Therefore, we can say that it is true that $\log f(n) \in O(\log g(n))$.

As for whether it is true for $2^{f(n)} \in O(2^{g(n)})$, we can prove by contradiction easily. Let $f(n) = 2\log n$ and $g(n) = log(n)$, we have $2^{f(n)} = n^2$ and $2^{g(n)} = n$, which is obviously false for $2^{f(n)} \in O(2^{g(n)})$.

At the end, if $f(n)$ is a constant, the assumption for $f(n)$ is a strictly increasing function is false, which implies this proposition is false, as a result, we cannot justify the property. So deflated QQ.

4. By the recurrence relation, we have

$$\begin{aligned}
T(n-1) &= (n-1) + [T(n-2) + T(n-3) + \cdots + T(2) + T(1)] \\
T(n) &= n + [T(n-1) + T(n-2) + \cdots + T(2) + T(1)]
\end{aligned}$$

Decrease these two equations, we have:

$$T(n) - T(n-1) = n - (n-1) + T(n-1)$$

which is equal to

$$T(n) = 2T(n-1) + 1$$

Use the recurrence relation again, we can substitute $T(n-1)$ by $2T(n-2)+1$ and so on. Therefore we have a new recurrence:

$$\begin{aligned} T(n) &= 2T(n-1) + 1 \\ &= 2^i T(n-1) + (2^{i-1} + 2^{i-2} + \cdots + 2^2 + 2 + 1) \end{aligned}$$

Since $T(1) = 1$, by substituting i with $n-1$, we have:

$$\begin{aligned} T(n) &= 2^{n-1}T(1) + (2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1) \\ &= 2^{n-1} + \frac{1 \cdot (2^{n-1})}{2-1} \\ &= 2^n \end{aligned}$$

5. The attempt fails since we cannot eliminate the constant in the original recurrence relation, but if we modify the guess inequality from $T(n) \le cn$ into $T(n) \le cn - 1$, it can deal with the problem. In accuracy, we will have

$$\begin{aligned} T(n) &= 2T(\frac{n}{2}) + 1 \\ &\le 2(\frac{cn}{2} - 1) + 1 \\ &= cn - 1 \\ &\le cn \end{aligned}$$

By proving $T(n) \le cn$, we have $T(n) \in O(n)$.