Homework 6

Yu-Chieh Kuo B07611039[†]

[†]Department of Information Management, National Taiwan University

Problem 1

To prove that $C + C = [0, 2]$, we show that both directions are correct.

- $(C + C \subseteq [0, 2])$
  If $x \in C$, let $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, where $a_n \in \{0, 2\}$. Therefore, any element of $C + C$ is of the form
  $$\sum_{n=1}^{\infty} \frac{a_n}{3^n} + \sum_{n=1}^{\infty} \frac{b_n}{3_n} = \sum_{n=1}^{\infty} \frac{a_n + b_n}{3^n} = 2\sum_{n=1}^{\infty} \frac{(a_n + b_n)/2}{3^n} = 2\sum_{n=1}^{\infty} \frac{x_n}{3^n},$$
  here $b_n = 0, 2$ as well as a_n, and $x_n = \frac{a_n + b_n}{2}$ for $n \in \mathbb{N}$. It's clear to show that $x_n \in \{0, 1, 2\}$ as $a_n, b_n \in \{0, 2\}$, $n \in \mathbb{N}$. Then, it suffices to say
  $$\sum_{n=1}^{\infty} \frac{x_n}{3^n} \in [0, 1] \implies 2\sum_{n=1}^{\infty} \frac{x_n}{3^n} \in [0, 2].$$

- $([0, 2] \subseteq C + C)$
  Note that $[0, 2] \subseteq C + C$ is equivalent to $[0, 1] \subseteq \frac{C}{2} + \frac{C}{2}$ here, and we show the latter form as below. The reason I prove $[0, 1] \subseteq \frac{C}{2} + \frac{C}{2}$ to show $[0, 2] \subseteq C + C$ is that $[0, 1]$ is more closed to the form we derive in the first direction. Futhermore, we could easily put $x \in [0, 1]$ and $x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$, where $x_n \in \{0, 1, 2\}$. Next, we need to select two element $y, z \in \frac{C}{2}$ such that $x = y + z$. Define $y = \sum_{n=1}^{\infty} \frac{y_n}{3^n}$ and $z = \sum_{n=1}^{\infty} \frac{z_n}{3^n}$, for all $n \in \mathbb{N}$, $y_n = 0$ if $x_n = 0$ and $y_n = 1$ if $x_n = 1, 2$; for all $n \in \mathbb{N}$, $z_n = 0$ if $x_n = 0, 1$ and $z_n = 1$ if $x_n = 2$. Such x, y, z follow $y, z \in \frac{C}{2}$ and $x = y + z$, that is,
  $$x = y + z \in \frac{C}{2} + \frac{C}{2} \implies [0, 1] \subseteq \frac{C}{2} + \frac{C}{2}.$$

Problem 2

(a) If every porper subsequence of $\{x_n\}$ converges, then for the origin sequence $\{x_n\}$, for arbitrary integer N, the subsequence $\{x_1, x_2, \ldots, x_{N_1}, x_{N+1}, \ldots\}$ converges as well. Hence, the origin sequence $\{x_n\}$ converges.

(b) It's the fact that every Cauchy sequence is bounded, and for $\{x_n\}$ is monotonic, $\{x_n\}$ is convergent.

(c) Suppose the subsequence of $\{x_n\}$ converges to x. Let $\varepsilon > 0$, for some large enough $N \in \mathbb{N}$,
$$d(x_m, x_n) < \frac{\varepsilon}{2}, \quad \text{if } m, n > N.$$
Similarly, for some large enough $N' \in \mathbb{N}$, and $N' > N$ such that for $k > N'$, $d(x_{n_k}, x) < \frac{\varepsilon}{2}$. Then, for $n > N'$, we have
$$d(x_n, x) \leq d(x_n, x_{n_{N'}}) + d(x_{n_{N'}}, x) < \varepsilon.$$
Hence $\{x_n\}$ converges to x.

(d) Assume the statement is false, then there exists $\varepsilon > 0$ such that for every k, there exists $n_k > k$ satisfying $d(x_{n_k}, p) \geq \varepsilon$. This implies that no any subsequence converges to p, contrary to the assumption. As a consequence, the statement implies the convergence of $\{x_n\}$.

(e) It's similar to 2.(d), and hence the statement implies the convergence of $\{x_n\}$.

Problem 3

(a) If $\{a_n\}$ is unbounded, then $a_n \to \infty$ as $n \to \infty$. Also,

$$\lim_{n\to\infty} \frac{a_n}{1+a_n} = \lim_{n\to\infty} \frac{1}{\frac{1}{a_n}+1} = 1.$$

As $\dfrac{a_n}{1+a_n}$ does not converge to 0 when $n \to \infty$, by **Theorem 3.23** in Rudin, we claim that $\dfrac{a_n}{1+a_n}$ diverges.

If $\{a_n\}$ is bounded, then there exists $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $a_n \leq M$. It shows that $\dfrac{a_n}{1+a_n} \geq \dfrac{1}{1+M}$, indicating $\dfrac{a_n}{1+a_n}$ diverges as well.

(b) Since for all $n \in \mathbb{N}$ $a_n > 0$ i.e., $s_{N+i} \geq s_{N+j}$ for $i \geq j$, $i, j \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \\
&= \frac{1}{s_{N+k}}(a_{N+1} + \cdots + a_{N+k}) \\
&= \frac{s_{N+k} - s_N}{s_{N+k}} \\
&= 1 - \frac{s_N}{s_{N+k}}.
\end{aligned}$$

It shows that the partial sum of $\sum \dfrac{a_n}{s_n}$ does not in the form of Cauchy sequence, then for every $N \in \mathbb{N}$, we could pick any large enough $k \in \mathbb{N}$ such that $1 - \dfrac{s_N}{s_{N+k}} \geq \dfrac{1}{2}$, deducing the fact that $\sum \dfrac{a_n}{s_n}$ diverges.

(c) Since for all $n \in \mathbb{N}$ $a_n > 0$ i.e., $s_{N+i} \geq s_{N+j}$ for $i \geq j$, $i, j \in \mathbb{N}$, we have

$$\frac{1}{s_n - 1} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1} \cdot s_n} \geq \frac{a_n}{s_n^2}.$$

Futhermore, $\sum \dfrac{1}{s_{n-1}} - \dfrac{1}{s_n}$ converging to $\dfrac{1}{a_1}$ implies that, by **Theorem 3.25** in Rudin, $\sum \dfrac{a_n}{s_n^2}$ converges.

(d) To check the convergence of $\sum \dfrac{a_n}{1+na_n}$, consider two cases as below.

(a) Let $a_n = \dfrac{1}{n}$ and hence $\dfrac{a_n}{1+na_n} = \dfrac{1}{2n}$. It shows that $\sum \dfrac{a_n}{1+na_n} = \dfrac{1}{2} \sum \dfrac{1}{n}$ diverges.

(b) Define a_n as

$$a_n = \begin{cases} 2^k & \text{if } n = 2^k \ \forall k = 0, 1, 2, \ldots \\ 0 & \text{otherwose.} \end{cases}$$

Therefore,

$$\sum \frac{a_n}{1+na_n} = \sum \frac{2^k}{1+2^{2k}} < \sum \frac{1}{2^k},$$

and $\sum \dfrac{1}{2^k}$ diverges by **Theorem 3.26** in Rudin, it indicates that $\sum \dfrac{a_n}{1+na_n}$ converges.

As a result, $\sum \frac{a_n}{1+na_n}$ can either converge or diverge, depending on $\{a_n\}$.

Move on the discussion of $\sum \frac{a_n}{1+n^2a_n}$. Since $a_n > 0$, we have $n^2a_n < 1 + n^2a_n$ and therefore

$$\sum \frac{a_n}{1+n^2a_n} < \sum \frac{1}{n^2}.$$

By **Theorem 3.28** in Rudin, $\sum \frac{a_n}{1+n^2a_n}$ converges.

Problem 4

Define $x_n = \frac{s_n}{n+1}$ to ease note. Here I want to start the derivation from prooving $\limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} a_n$. If $\limsup_{n\to\infty} a_n = +\infty$, it's nothing to do. Next, if $\limsup_{n\to\infty} a_n = a$ where $a \in \mathbb{R}$, for any $\varepsilon > 0$, there exists a $N \in \mathbb{Z}_+$ such that as $n \ge N$, $a_n < a + \varepsilon$. Therefore, we have

$$\begin{aligned} x_n &= \frac{a_1 + \cdots + a_N}{n+1} + \frac{a_{N+1} + \cdots + a_n}{n+1} \\ &\le \frac{a_1 + \cdots + a_N}{n+1} + (a+\varepsilon)\frac{n-N}{n+1}, \end{aligned}$$

implying that $\limsup_{n\to\infty} x_n \le a + \varepsilon \implies \limsup_{n\to\infty} x_n \le a$.

Finally, if $\limsup_{n\to\infty} a_n = -\infty$, it implies $\lim_{n\to\infty} a_n = -\infty$. For any $M > 0$, $M \in \mathbb{R}$, there exists a $N \in \mathbb{Z}_+$ such that as $n \ge N$, $a_n \le -M$. Therefore, we have

$$\begin{aligned} x_n &= \frac{a_1 + \cdots + a_N}{n+1} + \frac{a_{N+1} + \cdots + a_n}{n+1} \\ &\le \frac{a_1 + \cdots + a_N}{n+1} - M\frac{n-N}{n+1}, \end{aligned}$$

implying that $\limsup_{n\to\infty} x_n \le -M \implies \limsup_{n\to\infty} x_n = -\infty$.

Based on the cases we discuss above, it has been proved that $\limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} a_n$. Similarly to show that $\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} x_n$. In the end,

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} a_n.$$

If $a_n \to l$ as $n \to \infty$, $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = l$, and at that case $\liminf_{n\to\infty} \frac{s_n}{n+1} = \limsup_{n\to\infty} \frac{s_n}{n+1} = l$. The converse direction does not hold, and $a_n = (-1)^n$ is a case. It shows that

$$\liminf_{n\to\infty} \frac{s_n}{n+1} = \limsup_{n\to\infty} \frac{s_n}{n+1} = 0.$$

However, a_n diverges.

Problem 5

(a) This problem is not required to do.

(b) Since

$$(1+\frac{1}{n})cosn\pi = \begin{cases} 1 & \text{if } n = 2k \\ -1 & \text{if } n = 2k+1, \end{cases}$$

it's easy to show that

$$\limsup_{n\to\infty}(1+\frac{1}{n})cosn\pi = 1,$$

$$\liminf_{n\to\infty}(1+\frac{1}{n})cosn\pi = -1.$$

(c) Note that

$$nsin(\frac{n\pi}{3}) = \begin{cases} (1+6k)sin\left(\frac{\pi}{3}\right) & \text{if } n = 1+6k \\ -(4+6k)sin\left(\frac{\pi}{3}\right) & \text{if } n = 4+6k. \end{cases}$$

Therefore, we have

$$\limsup_{n\to\infty} nsin\left(\frac{n\pi}{3}\right) = \infty,$$

$$\liminf_{n\to\infty} nsin\left(\frac{n\pi}{3}\right) = -\infty.$$

(d) Since $sin\left(\frac{n\pi}{2}\right) cos\left(\frac{n\pi}{2}\right) = \frac{sin(n\pi)}{2} = 0$, it's easy to say that

$$\limsup_{n\to\infty} sin\left(\frac{n\pi}{2}\right) cos\left(\frac{n\pi}{2}\right) = \liminf_{n\to\infty} sin\left(\frac{n\pi}{2}\right) cos\left(\frac{n\pi}{2}\right) = 0.$$

(e) Since $\lim_{n\to\infty} \frac{(-1)^n n}{(1+n)^n} = 0$, therefore,

$$\limsup_{n\to\infty} \frac{(-1)^n n}{(1+n)^n} = \liminf_{n\to\infty} \frac{(-1)^n n}{(1+n)^n} = 0.$$

(f) Note that

$$\frac{n}{3} - \left\lceil \frac{n}{3} \right\rceil = \begin{cases} \frac{1}{3} & \text{if } n = 3k+1 \\ \frac{2}{3} & \text{if } n = 3k+2 \\ 0 & \text{if } n = 3k, \end{cases}$$

where $k = 0, 1, 2, \ldots$. Therefore, it's easy to say that

$$\limsup_{n\to\infty} \frac{n}{3} - \left\lceil \frac{n}{3} \right\rceil = \frac{2}{3},$$

$$\liminf_{n\to\infty} \frac{n}{3} - \left\lceil \frac{n}{3} \right\rceil = 0.$$

Problem 6

(a) (Not required)

(b) (Not required)

(c) (Not required)

(d) Note that $\frac{1}{n^p - n^q} = \frac{1}{n} \cdot \frac{1}{1 - n^{q-p}}$, for the case $p > 1$, by Limit Comparison Test with $\frac{1}{n^p}$, we have

$$\lim_{n\to\infty} \frac{\frac{1}{n^p - n^q}}{\frac{1}{n^p}} = 1,$$

which means the series converges.

For the case $p \leq 1$, also, by Limit Comparison Test with $\frac{1}{n^p}$, we have

$$\lim_{n\to\infty} \frac{\frac{1}{n^p - n^q}}{\frac{1}{n^p}} = 1,$$

which means the series diverges.

(e) Since $n^{-1-\frac{1}{n}} \geq n^{-1}$ for all $n \in \mathbb{R}$, the series diverges.

(f) Note that $\frac{1}{n^p - n^q} = \frac{1}{n} \cdot \frac{1}{1 - n^{q-p}}$, for the case $p > 1$, by Limit Comparison Test with $\frac{1}{p^n}$, we have

$$\lim_{n\to\infty} \frac{\frac{1}{p^n - q^n}}{\frac{1}{p^n}} = 1,$$

and the series converges. For the case $p \le 1$, also, by Limit Comparison Test with $\frac{1}{p^n}$, we have

$$\lim_{n\to\infty} \frac{\frac{1}{p^n - q^n}}{\frac{1}{p^n}} = 1,$$

and this means the series diverges.

(g) Since $\lim\limits_{n\to\infty} \frac{1}{n \ln\left(1 + \frac{1}{n}\right)} = 1$, the series diverges.

(h) Note that $\log n^{\log n} = n^{\log\log n}$, the series goes to

$$\sum_{n=2}^{\infty} \frac{1}{n^{\log\log n}},$$

and for some large enough $N \in \mathbb{N}$ such that if $n > N$, then $\log\log n > 1$. Hence, the series converges.

(i) By dividing p into three cases here, we could start the proof.

(a) ($p \le 0$):
We derive that

$$\frac{1}{n \log n (\log\log n)} \ge \frac{1}{n \log n}$$

for $n \ge 3$, due to the divergence of $\sum \frac{1}{n \log n}$, it's easy to state that the series diverges.

(b) ($0 < p < 1$):
Pick a large enough $N \in \mathbb{N}$, and by Cauchy Condensation Test, we have

$$\sum_{k=N}^{\infty} \frac{2^k}{2^k \log 2^k (\log\log 2^k)^p} = \frac{1}{\log 2} \sum_{k=N}^{\infty} \frac{1}{i(\log k \log 2)^p} \ge \sum_{k=N}^{\infty} \frac{1}{k(\log k)^p}.$$

To the convergence of $\sum \frac{1}{k(\log k)^p}$, by applying the Cauchy Condensation Test, it shows

$$\sum \frac{1}{k^p (\log 2)^p} > \sum \frac{1}{k^p}$$

which means $\sum \frac{1}{k(\log k)^p}$ diverges. Hence, the origin series diverges as well.

(c) ($p \ge 1$):
Pick a large enough $N \in \mathbb{N}$, and by Cauchy Condensation Test, we have

$$\begin{aligned} \sum_{k=N}^{\infty} \frac{2^k}{2^k \log 2^k (\log\log 2^k)^p} &= \frac{1}{\log 2} \sum_{k=N}^{\infty} \frac{1}{k(\log k \log 2)^p} \\ &\le 2 \sum_{k=N}^{\infty} \frac{1}{k(\log k \log 2)^p} \\ &\le 4 \sum_{k=N}^{\infty} \frac{1}{k(\log k)^p}. \end{aligned}$$

The convergence of $\sum \frac{1}{k(\log k)^p}$ has been discussed above. Consequently, we could say that the origin series diverges since $\sum \frac{1}{k(\log k)^p}$ diverges.

(j) Consider $a_n = \left(\frac{1}{\log\log n}\right)^{\log\log n}$, $b_n = \frac{1}{n}$ for $n \geq 3$ and construct $n = e^{e^x}$ for the future use. Then, by Limit Comparison Test, we have

$$\frac{a_n}{b_n} = \frac{n}{(\log\log n)^{\log\log n}} = \frac{e^{e^x}}{x^x}.$$

As $n \to \infty$, $\frac{e^{e^x}}{x^x} \to \infty$ as well. Therefore, the series diverges.

Problem 7

(a) Since $\sum \frac{1}{n}$ diverges, by Limit Comparison Test, we could clearly state that the series $\sum a_n$ diverges.

(b) The existence of $\lim(n^2 a_n)$ indicates that the series a_n is bounded, therefore, there exists some $M > 0, M \in N$ such that

$$0 < n^2 a_n \leq M \iff 0 < a_n \leq \frac{M}{n^2}.$$

Since $\sum \frac{M}{n^2}$ converges, by comparison test, $\sum a_n$ converges.