Introduction to Analysis I
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Homework 3

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Problem 1

1.(a)

To prove d_2 is a metric on X, we need to show that

1. $d_2(p,q) > 0$ if $p \neq q$ and $d(p,p) = 0$.
2. $d_2(p,q) = d_2(q,p)$.
3. $d_2(p,q) \leq d_2(p,r) + d_2(r,q)$ $\forall r \in X$.

Also, since d is a metric on X, we have $d(p,q) > 0$ if $p \neq q$ and $d(p,p) = 0$.
To 1., we have

$$\begin{aligned} d_2(p,q) &= \frac{d(p,q)}{1+d(p,q)} &> 0 \\ d_2(p,p) &= \frac{d(p,p)}{1+d(p,p)} &= 0. \end{aligned}$$

To 2., we have

$$d_2(p,q) = \frac{d(p,q)}{1+d(p,q)} = \frac{d(p,q)}{1+d(p,q)} = d_2(q,p).$$

To 3., if $d(p,q) > 0$, then

$$\begin{aligned} d_2(p,q) &= \frac{d(p,q)}{1+d(p,q)} &= 1 - \frac{1}{1+d(p,q)} \\ &\leq 1 - \frac{1}{1+d(p,r)+d(r,q)} &= \frac{d(p,r)+d(r,q)}{1+d(p,r)+d(r,q)} \\ &\leq \frac{d(p,r)}{1+d(p,r)} + \frac{d(r,q)}{1+d(r,q)} &= d_2(p,r) + d_2(r,q). \end{aligned}$$

This property holds when $d(p,q) = 0$ as well.
The significance of d_2 indicates that we can define another metric based on a defined metric.

1.(b)

As 1.(a), we could find ρ is a metric by checking those three properties:

1. $\rho(a,b) > 0$ if $a \neq b$ and $\rho(a,a) = 0$.
2. $\rho(a,b) = \rho(b,q)$.
3. $\rho(a,b) \leq \rho(a,c) + \rho(c,b)$ $\forall c \in \{0,1\}^{\mathbb{N}}$.

To 1., we define $\mathbb{N}_+$ as the set containing positive natural numbers and positive infinity. If $a = b$, then $a_i = b_i \implies \rho(a,b) = 0$, where $i \in \mathbb{N}_+$. If $a \neq b$, then there exists some $i \in \mathbb{N}_+$ is a positive number such that $a_i \neq b_i$, which implies $|a_i - b_i| > 0$. Therefore, $\rho(a,b) > 0$.

To 2., since $|a_i - b_i| = |b_i - a_i|$, we have

$$\rho(a,b) = \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{2^i} = \sum_{i=1}^{\infty} \frac{|b_i - a_i|}{2^i} = \rho(b,a).$$

To 3., since $a_i, b_i \in \mathbb{R}\ \forall i \in \mathbb{N}_+$, and the triangle inequality holds for real numbers, we can say that $\forall c_i \in \{0,1\}^{\mathbb{N}}$, $|a_i + b_i| \le |a_i - c_i| + |c_i - b_i|$ holds. Thus,

$$\begin{aligned}
\rho(a,b) &= \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{2^i} \\
&\le \sum_{i=1}^{\infty} \frac{|a_i - c_i| + |c_i - b_i|}{2^i} \\
&= \sum_{i=1}^{\infty} \frac{|a_i - c_i|}{2^i} + \sum_{i=1}^{\infty} \frac{|c_i - b_i|}{2^i} \\
&= \rho(a,c) + \rho(c,b).
\end{aligned}$$

By checking the definition of metric, we can find that ρ is a metric.

After substituting $\{0,1\}^{\mathbb{N}}$ into the infinite product space $Y = X_1 \times X_2 \times \cdots$, and substituting $|a_i - b_i|$ into $d_i(a_i, b_i)$, to check new ρ is a metric or not, we could still follow the same process as above.

At first, if $a = b$, then $a_i = b_i \forall i \in \mathbb{N}_+$, which implies $d_i(a_i, b_i) = 0 \forall i \in \mathbb{N}_+$. Hence, it's clearly that $\rho(a,b) = \sum_{i=1}^{\infty} \frac{d_i(a_i,b_i)}{2^i} = 0$. If $a \neq b$, then there exists some $i \in \mathbb{N}_+$ is a positive number such that $a_i \neq b_i$, which implies $d_i(a_i, b_i) > 0$ for this i. Therefore, it's clearly that $\rho(a,b) > 0$.

Secondly, since d_i is a metric for $i \in \mathbb{N}_+$, we have

$$\rho(a,b) = \sum_{i=1}^{\infty} \frac{d_i(a_i, b_i)}{2^i} = \sum_{i=1}^{\infty} \frac{d_i(b_i, a_i)}{2^i} = \rho(b,a).$$

Lastly, since d_i is a metric for $i \in \mathbb{N}_+$, the triangle inequality holds. Therefore,

$$\begin{aligned}
\rho(a,b) &= \sum_{i=1}^{\infty} \frac{d_i(a_i - b_i)}{2^i} \\
&\le \sum_{i=1}^{\infty} \frac{d_i(a_i - c_i) + d_i(c_i - b_i)}{2^i} \\
&= \sum_{i=1}^{\infty} \frac{d_i(a_i - c_i)}{2^i} + \sum_{i=1}^{\infty} \frac{d_i(c_i - b_i)}{2^i} \\
&= \rho(a,c) + \rho(c,b).
\end{aligned}$$

By checking the definition of metric, we can find that ρ is a metric.

Problem 2

First, we show that E' is closed.

By **Definition 2.18** in Rudin, it suffices to say every limit point of E' is a limit point of E. Given a limit point $p \in E'$, every neighborhood V of p contains a point p' such that $p' \in E'$. Since p' is a limit point of E, there exists an open neighborhood U of p' contains $q \neq p'$ such that $q \in E$, where

$$U = V \cap B\left(p', \frac{1}{2}d(p,p')\right) \subseteq V.$$

(Note that $B(a,b)$ means the open ball centering at a with radius b.)
Therefore, since $q \neq p$, $q \in U \subseteq V$, and $q \in E$, p is a limit point of E.

Next, we would like to show that $X - E'$ is open, which is equivalent to prove E' is close. Given a point $p \in X - E'$, there exists an open neighborhood V of p contains no points $q \neq p$ such that $q \in E$. Next, to show V is an open neighborhood of p such that $V \subseteq X - E'$, we assume by contradiction that there exists a limit point q of E such that $q \neq p$ and $q \in V$, then here we could contruct a neighborhood U as

$$U = V \cap B\left(q, \frac{1}{2}d(p,q)\right) \subseteq V,$$

where U is an open neighborhood of q containing on points of E, leading to a contrary. Consequently, $V \subseteq X - E'$ is an open neighborhood of $p \in X - E'$ i.e., $X - E'$ is open, equivalent to E' is closed.

Second, we want to prove that E and $\bar{E}$ have the same limit points. It is equivalent to prove that whether $E' = \bar{E}'$.

- ($E' \subseteq \bar{E}'$): Since $E \subseteq \bar{E}$, $E' \subseteq \bar{E}'$ obviously.
- ($E' \subseteq \bar{E}'$): Given a limit point p of $\bar{E} = E \cup E'$. If p is a limit point of E, the proof is done; if p is a limit point of E', since E' is closed, it has to be that $p \in E'$ or p is a limit point of E. In any case, $E' subseteq \bar{E}'$ is true.

Lastly, the problem comes to whether E and E' always have the same limit point. It seems to have an easy yes; however, the answer is false. Consider

$$E = \{\frac{1}{n} : n \in \mathbb{Z}_+\} \subseteq \mathbb{R}.$$

Here $E' = \{0\}$ but $(E')' = \emptyset$.

Problem 3

(a) This problem is equivalent to show that $E^\circ \subseteq (E^\circ)^\circ$. Given any $x \in E^\circ$, there exists $r > 0$ such that $B(x,r) \subseteq E$. Since $B(x,r)$ is an open set, all points in $B(x,r)$ are interior points. That is, $\forall y \in B(x,r)$, $\exists s > 0$ s.t. $B(y,s) \subseteq B(x,r) \subseteq E$. This implies that every $y \in B(x,r)$ is an interior point of E i.e., $B(x,r) \subseteq E^\circ \implies x \in (E^\circ)^\circ \iff E^\circ \subseteq (E^\circ)^\circ$.

(b) By **Definition 2.18** in Rudin, since E is open, every point of E is an interior point of E i.e. $E \subseteq E^\circ$. Also, $E^\circ \subseteq E$ since all points in E° must be in E. Thus, $E = E^\circ$.

(c) $G \subseteq E \implies G^\circ \subseteq E^\circ$, and $G = G^\circ$ by (b). Hence $G = G^\circ \subseteq E^\circ$.

(d)
- ($(E^\circ)^c \subseteq \bar{E}^c$): Given any $x \in (E^\circ)^c$, if $x \notin E$, then $x \in E^c \subseteq \bar{E}^c$; if $x \in E$, then every neighborhood U of x must satisfy $U \cap E^c \neq \emptyset$ i.e., x is a limit point of E^c, which means $x \in (E^c)' \subseteq \bar{E}^c$. For both cases, $x \in \bar{E}^c$ at all.
- ($\bar{E}^c \subseteq (E^\circ)^c$): Given any $x \in \bar{E}^c$, it shows that either $x \in E^c$ or $x \in (E^c)'$. If $x \in E^c$, then $x \neq E \implies x \neq E^\circ$ i.e., $x \in (E^\circ)^c$. On the other hand, if $x \in (E^c)'$, then every neighborhood V of x must satisfy $V \cap E^c \neq \emptyset$. Hence, x is not an interior point of E, i.e., $E \notin E^\circ \implies x \in (E^\circ)^c$.

(e) The answer is **NO**. Consider X is a metric space in $\mathbb{R}$ and $E = \mathbb{Q} \subseteq X$. We have $E^\circ = \emptyset$ since $\bar{\mathbb{Q}}$ is dense in $\mathbb{R}$, and $(\bar{E})^\circ = R$ since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{R}$ is open.

(f) The answer is **NO**. Following by the setting in (e), we have $\bar{E} = \mathbb{R}$ since $\mathbb{Q}$ is dense in $\mathbb{R}$, and $\bar{E^\circ} = \bar{\emptyset} = \emptyset$ since $\bar{\mathbb{Q}}$ is dense in $\mathbb{R}$.

Problem 4

4.(a)

Consider E be the set of points having only rational coordinates. Since $\mathbb{Q}$ is countable, $E = \mathbb{Q}^k$ is also countable.

Now, given any $\mathbf{p} = (p_1, \ldots, p_k) \in \mathbb{R}^k$, we would like to show that $\mathbf{p}$ is a limit point of E.