Homework 7

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Problem 1

1.(a) $\sin nx$

We start the problem from its partial summation

$$\sum_{k=1}^{n} \frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)}{k} \sin kx.$$

If $x = 2m\pi$ for here $m \in \mathbb{Z}$, it's cleary that the series converges to zero; if $x \neq 2m\pi$, as the series is in the form of the multiplication of two series, we naturally define the following series for further use to attempt to construct a better-to-observe series:

$$a_k = \frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)}{k} \quad \text{and} \quad b_k = \sin kx.$$

Here we have a_k converges as

$$\begin{aligned} a_{k+1} - a_k &= \frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}\right)}{k+1} - \frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)}{k} \\ &= \frac{k\left(1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}\right) - (k+1)\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)}{k(k+1)} \\ &= \frac{\frac{k}{k+1} - \left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)}{k(k+1)} \\ &< 0, \end{aligned}$$

Next we check the convergence of b_k. Recall that

$$-2\sin\frac{x}{2}\sin kx = \cos\left(\frac{(2k+1)x}{2}\right) - \cos\left(\frac{(2k-1)x}{2}\right),$$

and we obtain

$$\begin{aligned}\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \sin kx &= \frac{1}{-2\sin\frac{x}{2}} \sum_{k=1}^{n} -2\sin\frac{x}{2}\sin kx \\ &= \frac{\cos\left(\frac{(2n+1)x}{2}\right) - \cos\frac{x}{2}}{-2\sin\frac{x}{2}} \\ &\vdots \quad \text{(After some trivial derivations.)} \\ &= \frac{\sin\frac{nx}{2}\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}} \\ &\leq \frac{1}{\sin\frac{x}{2}},\end{aligned}$$

which means b_k converges as well.

Therefore, by Dirichlet Test, we know that the series $\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)}{k} \sin kx$ converges.

1.(b) $\cos nx$

As for substituting $\cos nx$ for $\sin nx$ for the origin series, we follow the identical steps to check the convergence. Here, for the series $\sum \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\frac{\cos nx}{n}$, if $x = \frac{1}{2}m\pi$ for $m \in Z$, the series converges to zero clearly; if $x \neq \frac{1}{2}m\pi$, by following **1.(a)**, we construct two series:

$$a_k = \frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)}{k} \quad \text{and} \quad b_k = \cos kx.$$

As a consequence from **1.(a)**, we know that the series a_k converges as $a_{k+1} - a_k < 0$. As for the series b_k, it follows the same procedures to hold its convergence. Recall that

$$2\sin\frac{x}{2}\cos kx = \sin\left(\frac{(2k+1)x}{2}\right) - \sin\left(\frac{(2k-1)x}{2}\right),$$

then it shows

$$\begin{aligned}\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \cos kx &= \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n} 2\sin\frac{x}{2}\sin kx \\ &= \frac{\sin\left(\frac{(2n+1)x}{2}\right) - \sin\frac{x}{2}}{2\sin\frac{x}{2}} \\ &\vdots \quad \text{(After some trivial derivations.)} \\ &= \frac{\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}}{\sin\frac{x}{2}} \\ &\leq \frac{1}{\sin\frac{x}{2}},\end{aligned}$$

which leads to the fact that b_k here converges as well.

Therefore, by Dirichlet Test, we know that the series $\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)}{k} \cos kx$ converges.

Problem 2

2.(a) Rudin Ex. 3.8

Following the concept of **Theorem 3.42** in Rudin, we claim that $\sum a_n b_n$ converges suppose

1. The partial sums A_n of $\sum a_n$ form a bounded sequence,
2. $b_0 \geq b_1 \geq \cdots$ (monotonicity),
3. $\lim_{n\to\infty} b_n = 0$.

Here after we shall attempt to check the above conditions, and there are two possible cases for the series $\{b_n\}$:

1. $\{b_n\}$ increases to b.
   Here we check these three conditions. First we define $\{\beta_n\} = b - b_n$ for the further use.
   (a) The partial sums A_n of $\sum a_n$ form a bounded sequence since $\sum a_n$ converges.
   (b) A series $\{\beta_n\}$ is monotonically increasing.
   (c) $\lim_{n\to\infty} \beta_n = 0$.

   Hence, by the **Theorem 3.42** in Rudin, we have $\sum a_n\beta_n$ converges. In the end, $\sum a_n b_n = -\sum a_n\beta_n + \sum a_n b$ converges as well.
2. $\{b_n\}$ decreases to b.
   Follow the increasing cases, we define $\{\beta_n\} = b_n - b$, then we claim that $\sum a_n\beta_n$ converges as well by following the consequences above. Therefore, we claim that $\sum a_n b_n = \sum a_n\beta_n + \sum a_n b$ converges.

Here $\sum_{n=1}^{\infty} \frac{n^2+1}{n^{100}} \cdot \sum_{k=1}^{n} \frac{1}{k^2}$ is a nontrivial example for this property, where $a_n = \frac{n^2+1}{n^{100}}$ and $\{b_n\} = \sum_{k=1}^{n} \frac{1}{k^2}$.

2.(b) Apostol Ex. 8.27 (a)

We want to prove that $\sum a_n b_n$ converges if $\sum a_n$ converges and if $\sum (b_n - b_{n+1})$ converges absolutely. Next, by **Theorem 3.41** in Rudin, consider a summation by parts that

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k).$$

Here we obtain that $|\sum a_n| = |A_n| \leq M \ \forall n \in \mathbb{N}$ since $\sum a_n$ converges, and M is some real number. In addition, since $\sum (b_n - b_{n+1})$ converges absolutely, we know that $\lim_{n\to\infty} b_n$ exists. Consequently we obtain that $\lim_{n\to\infty} A_n b_{n+1}$ exists and

$$\sum_{k=1}^{n} |A_k (b_{k+1} - b_k)| \leq M \sum_{k=1}^{n} |b_{k+1} - b_k| \leq M \sum_{k=1}^{\infty} |b_{k+1} - b_k|,$$

where the inequality implies that $\sum_{k=1}^{n} A_k (b_{k+1} - b_k)$ converges. In the end, as $\lim_{n\to\infty} A_n b_{n+1}$ exists and $\sum_{k=1}^{n} A_k (b_{k+1} - b_k)$ converges, it shows that $\sum_{k=1}^{n} a_k b_k$ converges.

Here $\sum_{n=1}^{\infty} \frac{n^2+1}{n^{100}} \cdot \frac{1}{n^2}$ is a nontrivial example for this property, where $a_n = \frac{n^2+1}{n^{100}}$ and $b_n = \frac{1}{n^2}$.

2.(c) Apostol Ex. 8.27 (b)

Following **2.(b)**, consider a summation by parts that

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k).$$

Since $b_n \to 0$ as $n \to 0$ and $\sum a_n$ has bounded partial sums, which implies for some $M \in \mathbb{R}$, $|A_n| \leq M \ \forall n$, we obtain that $\lim_{n\to\infty} A_n b_{n+1}$ exists, and

$$\sum_{k=1}^{n} |A_k(b_{k+1} - b_k)| \leq M \sum_{k=1}^{n} |b_{k+1} - b_k| \leq M \sum_{k=1}^{\infty} |b_{k+1} - b_k|,$$

where the inequality implies that $\sum_{k=1}^{n} A_k(b_{k+1} - b_k)$ converges. Consequently, as $\lim_{n\to\infty} A_n b_{n+1}$ exists and $\sum_{k=1}^{n} A_k(b_{k+1} - b_k)$ converges, it shows that $\sum_{k=1}^{n} a_k b_k$ converges.

Here $\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{1}{n}$ is a nontrivial example for this property, where $a_n = \frac{1}{2^n}$ amd $b_n = \frac{1}{n}$.

Problem 3

3.(a) Prove the inequality and show the divergence of $\sum \frac{a_n}{r_n}$

Before the deduction, we observe that r_n has some properties.

1. r_n is positive and finite since $a_n > 0$ and $\sum a_n$ converges.
2. $\{r_n\}$ is monotonically decreasing since $a_n > 0$.
3. $\{r_n\}$ converges to 0 since $\sum a_n$ converges.

Next, we derive the inequality as follows. If $m < n$, the equation is derived as

$$\begin{aligned}
\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} &> \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} && \text{(Since } \{r_n\} \text{ is monotonically decreasing.)} \\
&= \frac{a_m + \cdots + a_n}{r_m} \\
&= \frac{r_m - r_{n+1}}{r_m} \\
&> \frac{r_m - r_n}{r_m} && \text{(Since } r_n > r_{n+1}\text{)} \\
&= 1 - \frac{r_n}{r_m}.
\end{aligned}$$

Next, we would like to conclude that $\sum \frac{a_n}{r_n}$ diverges. Assume by contrary, if $\sum \frac{a_n}{r_n}$ converged, pick $\varepsilon = \frac{1}{2} > 0$, then there exist a $N \in \mathbb{N}$ such that

$$|\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m}| < \varepsilon = \frac{1}{2},$$

where $n \geq m \geq N$. By **Theorem 3.22** in Rudin, taking $m = N$, the inequality becomes

$$1 - \frac{r_n}{r_N} < \frac{1}{2} \iff r_n > \frac{1}{2} r_N$$

for $n \geq N$, contrary to the assumption that $\{r_n\}$ converges to 0.

3.(b) Prove the inequality and show the divergence of $\sum \frac{a_n}{\sqrt{r_n}}$

Note that $a_n = r_n - r_{n+1}$ by definition, then we have

$$\begin{aligned}
\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \quad &\iff \quad \frac{r_n - r_{n+1}}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \\
&\iff \quad \frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n}} < 2 \\
&\iff \quad \sqrt{r_{n+1}} < \sqrt{r_n} \\
&\iff \quad r_{n+1} < r_n.
\end{aligned}$$

This statement holds since $\{r_n\}$ is monotonically decreasing.

Furthermore, to the covergence of $\sum \frac{a_n}{\sqrt{r_n}}$, consider the partial sum that

$$\sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}} < \sum_{k=1}^{n} 2(\sqrt{r_k} - \sqrt{r_{k+1}}) = 2(\sqrt{r_1} - \sqrt{r_{n+1}}) < 2\sqrt{r_1},$$

which implies that $\sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}}$ is bounded by $2\sqrt{r_1}$. In addition, each term of $\sum \frac{a_n}{\sqrt{r_n}}$ is nonnegative since $a_n > 0$ and $\sqrt{r_n} > 0$ for $n \in \mathbb{N}$. Therefore, we conclude that $\sum \frac{a_n}{\sqrt{r_n}}$ converges by **Theorem 3.24** in Rudin.

Problem 4

(a) (No need to write.)

(b) Directly define $\{s_n\}$ by $s_n = (-1)^{n+1}$.

(c) The answer is **yes**. We could construct a weird series as

$$s_n = \begin{cases} \dfrac{1}{n!} + m^{63} & \text{if } n = m^{89} \text{ for some } m \in \mathbb{Z} \\ \dfrac{1}{n!} & \text{otherwise.} \end{cases}$$

It's clear to show that $\limsup s_n = +\infty$. Moreover, for arbitrary $n \in \mathbb{N}$, there always exists a $m \in \mathbb{Z}$ such that $m^{89} \leq n < (m+1)^{89}$. So, the arithmetic means σ_n becomes

$$\begin{aligned}
0 &< \sigma_n \\
&= \frac{1}{n+1}\sum_{k=0}^{n} s_k \\
&\leq \frac{1}{m^{89}+1}\sum_{k=0}^{n} s_k \\
&= \frac{1}{m^{89}+1}\left(\sum_{k=0}^{n}\frac{1}{n!}+\sum_{k=0}^{m}k^{63}\right) \\
&\leq \frac{1}{m^{89}+1}\left(\sum_{k=0}^{\infty}\frac{1}{n!}+\sum_{k=0}^{m}k^{63}\right) \\
&= \frac{e+m\cdot m^{63}}{m^{89}+1}.
\end{aligned}$$

As $n \to \infty$, $m \to \infty$ and thus $\lim \sigma_n = 0$ as a consequence.

(d)

$$\begin{aligned}
\frac{1}{n+1}\sum_{k=1}^{n} ka_k &= \frac{1}{n+1}\sum_{k=1}^{n} k(s_k - s_{k-1}) \\
&= \frac{1}{n+1}\left(\sum_{k=1}^{n} ks_k - \sum_{k=1}^{n} ks_{k-1}\right) \\
&= \frac{1}{n+1}\left(\sum_{k=1}^{n} ks_k - \sum_{k=1}^{n} (k-1)s_{k-1} - \sum_{k=1}^{n} s_{k-1}\right) \\
&= \frac{1}{n+1}\left(ns_n - \sum_{k=1}^{n} s_{k-1}\right) \\
&= \frac{1}{n+1}\left((n+1)s_n - \sum_{k=1}^{n+1} s_{k-1}\right) \\
&= s_n - \sigma_n.
\end{aligned}$$

Here we can re-write s_n as $s_n = \sigma_n + \frac{1}{n+1}\sum_{k=1}^{n} ka_k$ by the consequence above. Since $\lim(na_n) = 0$ and $\{\sigma_n\}$ converges in the statement, we obtain

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n} ka_k = 0,$$

and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\sigma_n + \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n} ka_k = \lim_{n\to\infty}\sigma_n.$$

Thus, we conclude that $\{s_n\}$ converges as well.

(e) The statement follows the outline in Rudin.

(i) If $m < n$, then

$$\begin{aligned}
\sigma_n - \sigma_m &= \frac{1}{n+1}\sum_{k=0}^{n} s_k - \frac{1}{m+1}\sum_{k=0}^{m} s_k \\
&= \frac{1}{n+1}\sum_{k=0}^{n} s_k - \frac{1}{m+1}\sum_{k=0}^{n} s_k + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i \\
&= \frac{m-n}{(m+1)(n+1)}\sum_{k=0}^{n} s_k + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i \\
&= \frac{m-n}{m+1}\sigma_n + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i.
\end{aligned}$$

Multiply $\frac{m+1}{n-m}$ on both sides, we have

$$\begin{aligned}
\frac{m+1}{n-m}(\sigma_n - \sigma_m) &= -\sigma_m + \frac{1}{n-m}\sum_{i=m+1}^{n} s_i \\
&= -\sigma_n - \frac{1}{n-m}\sum_{i=m+1}^{n} (-s_i) \\
&= -\sigma_n - \left(\frac{1}{n-m}\sum_{i=m+1}^{n} (s_n - s_i)\right) + s_n.
\end{aligned}$$

Therefore, by changing terms in the equation, we obtain

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i).$$

(ii) For these i,

$$\begin{aligned}
|s_n - s_i| &= |\sum_{k=i+1}^{n} a_k| \\
&\le \sum_{k=i+1}^{n} |a_k| && \text{(By triangle inequality)} \\
&< \sum_{k=i+1}^{n} \frac{M}{k} && (|ka_k| < M) \\
&\le \sum_{k=i+1}^{n} \frac{M}{i+1} && (k \ge i+1) \\
&= \frac{(n-i)M}{i+1} && \textbf{The term we need to derive.} \\
&= \left(\frac{n-1}{i+1} - 1\right) M \\
&\le \left(\frac{n-1}{m+2} - 1\right) M && (i \ge m+1) \\
&= \frac{(n-m-1)M}{m+2}. && \textbf{The term we need to derive.}
\end{aligned}$$

(iii) Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \le \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Here we know that $m \leq \frac{n-\varepsilon}{1+\varepsilon} < \frac{n}{1+\varepsilon} < n$, thus, by moving terms in this inequality, we obtain

$$\frac{m+1}{n-m} \leq \frac{1}{\varepsilon} \iff \frac{n-m-1}{m+2} < \varepsilon.$$

By the step *(ii)*, $|s_n - s_i| < M\varepsilon$.

(iv) Hence, re-write the equation in *(i)* by adding σ on both side, we have

$$s_n - \sigma = (\sigma_n - \sigma) + \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i),$$

and by triangle inequality, it becomes

$$\begin{aligned} |s_n - \sigma| &\leq |\sigma_n - \sigma| + \frac{m+1}{n-m}|\sigma_n - \sigma_m| + \frac{1}{n-m}\sum_{i=m+1}^{n}|s_n - s_i| \\ &< |\sigma_n - \sigma| + \frac{1}{\varepsilon}|\sigma_n - \sigma_m| + \frac{1}{n-m}\sum_{i=m+1}^{n} M\varepsilon \\ &= |\sigma_n - \sigma| + \frac{1}{\varepsilon}|\sigma_n - \sigma_m| + M\varepsilon. \end{aligned}$$

This inequality holds for m, n satisfying $m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1$. As $\{\sigma_n\}$ converges, there exists $N \in \mathbb{N}$ such that

$$|\sigma_n - \sigma_m| < \varepsilon^2 \quad \text{and} \quad |\sigma_n - \sigma| < \varepsilon$$

for $m, n \leq N$, therefore, $|s_n - \sigma| < (M+2)\varepsilon$ holds for $n \geq 2N+3$. Note that m still satisfies $m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1$. Finally, take $\limsup$ to this inequality and it becomes

$$\limsup_{n\to\infty} |s_n - \sigma| \leq (M+2)\varepsilon.$$

(v) Since ε was arbitrary, $\lim s_n = \sigma$.

Problem 5

5.(a)

First consider the case $e \cdot n!$. Expand it as the form of

$$\begin{aligned} e \cdot n! &= n! \cdot \sum_{k=0}^{\infty} \frac{1}{k!} \\ &= n! \cdot \left(\sum_{k=0}^{n} \frac{1}{k!} + \sum_{k=n+1}^{\infty} \frac{1}{k!} \right), \end{aligned}$$

here the first term goes to 0 as $n \to \infty$. As for the second term, says $b_n = n! \cdot \sum_{k=n+1}^{\infty} \frac{1}{k!}$. Note that $\frac{1}{n+1} < b_n < \frac{1}{n} + \frac{1}{n^2} + \cdots = \frac{1}{n-1}$. Trickly, we can re-write a_n as

$$a_n = n\sin(2e\pi n!) \sim n\sin(2\pi b_n) = n \cdot 2\pi b_n \frac{\sin 2\pi b_n}{2\pi b_n}.$$

Note that the similarity relation holds since as $n \to \infty$, the first term we mentioned previously goes to 0, which implies it does not matter as $n \to \infty$. Thus, we can ignore it naturally. Consequently, take the limit as $n \to \infty$, we conclude $\lim_{n\to\infty} a_n = 2\pi$.

5.(b)

We use the big-O asymptotic notation $O(\cdot)$ to represent the condition as the function goes to the large terms i,e,. when $n \to \infty$. For example, if we say that $f(n) \in O(n^2)$, it indicates that as n increases to very large, $f(n)$ is still be bounded above by cn^2, i.e., $f(n) \le cn^2$ for sufficiently large n, and c here is a constant. Thus, the series a_n can be denoted as $n\sin(2\pi O(\frac{1}{n}))$. Then for the statement,

$$\lim_{n\to\infty} n^k(a_n - 2\pi) = \lim_{n\to\infty}(n^{k+1}\sin(2e\pi n!) - 2\pi n^k) = 2\pi \lim_{n\to\infty} n^k\left(cn\cdot\frac{1}{n} - 1\right),$$

here c is an arbitrary constant corresponding with the big-O notation. Hence, $\lim_{n\to\infty} n^k(a_n - A) = 2\pi(c-1)\lim_{n\to\infty} n^k = L$ exists if $0 \le k < 1$

Problem 6

First we show that $f(E)$ is dense in $f(X)$. It suffices to show that every point $y \in f(X) - f(E)$ is a limit point of $f(E)$. As $y \in f(X) - f(E)$, there exists a point $x \in X - E$ such that $y = f(x)$. In addition, as E is dense in X, there exists a sequence $\{x_n\}$ in E such that as $n \to \infty$, $x_n \to x$. Let $y_n = f(x_n) \in f(E)$, by the continuity of f, it shows $y_n \to y$ as $n \to \infty$, or y is a limit point of $f(E)$.

Next, we show that $g(p) = f(p)$ for all $p \in X$ if $g(p) = f(p)$ for all $p \in E$. It can be proved that $g(p) = f(p)$ for $p \in X - E$. Here, pick any $p \in X - E$, there exists a sequence $\{p_n\}$ in E such that as $n \to \infty$, $p_n \to p$. Note that here $g(p_n) = f(p_n)$, thus, by the continuity of f and g, $g(p) = f(p)$ for $p \in X - E$.

Problem 7

7.(a) Apostol Ex. 4.19

Following the statement, we define $g(x) = \max\{f(k) : k \in [a,x]\}$, and pick any $c \in [a,b]$. Since f is continuous at $x = c$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in (c-\delta, c+\delta)$, $f(c) - \frac{\varepsilon}{2} < f(x) < f(c) + \frac{\varepsilon}{2}$. Since g is defined in the interval $[a,x]$, if we select $x = c+\delta$ and the corresponding function value is $f(j)$, where $j \le c-\delta$, as $x \in (c-\delta, c+\delta)$, we have $g(x) = f(j)$ and $g(c) = f(j) \implies |g(x) - g(c)| = 0$; if we also select $x = c+\delta$, but j here, is larger than $c - \delta$, then the inequality

$$f(c) - \frac{\varepsilon}{2} \le g(x) \le f(c) + \frac{\varepsilon}{2}$$

holds, which also implies that $|g(x) - g(c)| < \varepsilon$. Thus, for any $x \in (c-\delta, c+\delta)$, we have $|g(x) - g(c)| < \varepsilon$, which indicates that g is continuous at $c \in [a,b] \iff g$ is continuous on $[a,b]$.

7.(b) Apostol Ex. 4.20

We prove the statement by mathematical induction. For the case $m = 2$, we have $f = \max(f_1, f_2) = \frac{(f_1+f_2) + |f_1 - f_2|}{2}$, and f is continuous at $x = a$ since by assumption, every f_k is continuous at a of S for

$1 \le k \le m$, and it shows that $f_1 + f_2$ and $f_1 - f_2$ are both continuous at a. By the induction hypothesis, suppose $m = k$ holds, then for $m = k + 1$, we have

$$f = \max(f_1, \dots, f_{k+1}) = \max(\max(f_1, \dots, f_k), f_{k+1}),$$

which is also continuous at $x = a$. Hence, by mathematical induction, f is continuous at $x = a$.

Problem 8

8.(a) Rudin Ex. 4.2

Since f is continuous and $\overline{f(E)}$, $f^{-1}\overline{f(E)}$ are closed, we have

$$E \subseteq f^{-1}(f(E)) \subseteq f^{-1}(\overline{f(E)})$$

(by **Theorem 4.14** in Rudin), and the monotonicity of closure gives that $\overline{E} \subseteq f^{-1}(\overline{f(E)})$, hence,

$$f(\overline{E}) \subseteq f(f^{-1}(\overline{f(E)})) \subseteq \overline{f(E)}.$$

To give an example, let $f : (0, \infty) \to \mathbb{R}$ be a continuous mapping function defined as $f(x) = \dfrac{1}{x}$. Consider $E = \mathbb{Z}_+ \subseteq (0, \infty)$, then $f(E) = \dfrac{1}{n}$ where $n \in \mathbb{Z}_+$. Therefore, we have

$$f(\overline{E}) = \frac{1}{n},\ n \in \mathbb{Z}_+ \quad \text{and} \quad \overline{f(E)} = \frac{1}{n},\ n \in \mathbb{Z}_+ \cap \{0\},$$

which shows that $f(\overline{E})$ is a proper subset of $\overline{f(E)}$.

8.(b) Apostol Ex. 4.30

We prove the statement by proving from both sides.

- $(\Rightarrow)$
  Assume f is continuous on S. Since $f(A) \subseteq cl(f(A))$, it implies $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(cl f(A))$. Note that $f^{-1}(cl(f(A)))$ is closed since the pre-image of a closed set by a continuous function is closed by **Theorem 4.17** in Rudin. Therefore, we have

$$cl(A) \subseteq cl(f^{-1}(cl(f(A)))) \subseteq f^{-1}(cl(f(A))) \implies f(cl(A)) \subseteq f(f^{-1}(cl(f(A)))) \subseteq cl(f(A)).$$

- $(\Leftarrow)$
  Assume that $f(cl(A)) \subseteq cl(f(A))$ for every subset A of S. Consider a closed subset B and defined $f^{-1}(B) = A$, then we have

$$f(cl(f^{-1}(C))) = f(cl(A)) \subseteq cl(f(A)) = cl(f(f^{-1}(C))) \subseteq cl(C),$$

  and as C is closed, $cl(C) = C$ here. Therefore, we have $f(cl(A)) \subseteq C$, and use this relation, we conclude that

$$cl(A) \subseteq f^{-1}(f(cl(A))) \subseteq f^{-1}(C) = A,$$

  which implies $A = f^{-1}(C)$ is a closed set. As a result, f is continuous on S.