Homework 5

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Problem 1

We can prove the property from both sides respectively. Let S be a topological space.

- ($\Rightarrow$)
  Suppose S is connected. Based on the definition of set, the only two open and closed subsets are $\emptyset$ and the set S. Hence, it's impossible to find a subset T satisfying that T is nonempty, T is a subset of S, and T is an open and closed subset in S.

- ($\Leftarrow$)
  Let S be disconnected, therefore there are two nonempty **open** subsets $T, U \subseteq S$, where $S = T \cup U$ and $T \cap U = \emptyset$. In addition, since $T^c = U$ and $U^c = T$, T and U are open and closed. Hence, if there doesn't exist such subsets as same as $T, U \subseteq S$ satisfying $S = T \cup U$ and $T \cap U = \emptyset$, we claim that S is connected.

Consequently, we can say that a topological space S is connected if and only if the only subsets of S which are both open and closed in S are the empty set $\emptyset$ and S.

Problem 2

Interiors of connected sets are not always connected. Consider $X \subseteq \mathbb{R}^2$ be a metric space with the normal Euclidean metric. Define $B_r(x)$ be an open ball centering at x in radius r. Take

$$E = B_1((2,0)) \cup B_1((100,0)) \cup \{(x,0) \in \mathbb{R}^2 : 2 \leq x \leq 100\}.$$

Here E is connected but

$$E^\circ = B_1((2,0)) \cup B_1((100,0))$$

is disconnected.

Closures of connected sets are always connected. It suffices to show that E is disconnected if $\bar{E}$ is disconnected. Write $\bar{E} = A \cup B$ as an union of two nonempty separated sets, and $A \cap \bar{B} = \emptyset$, $\bar{A} \cap B = \emptyset$. Write E as $E = (A \cap E) \cup (B \cap E)$, and it shows that E is disconnected. Next, we would like to show that $A \cap E$ and $B \cap E$ are separated. In fact,

$$(A \cap E) \cup \overline{B \cap E} \subseteq A \cap \bar{B} = \emptyset,$$

$$\overline{A \cap E} \cap (B \cap E) \subseteq \bar{A} \cap B = \emptyset.$$

If $A \cap E = \emptyset$, therefore

$$E = (A \cap E) \cup (B \cap E) = B \cap E \implies E \subseteq B,$$

and

$$\begin{aligned} A &= (A \cup B) \cap A \\ &= \bar{E} \cap A \\ &\subseteq \bar{B} \cap A \\ &= \emptyset, \end{aligned}$$

contrary to the definition that A is nonempty. Hence, $A \cap E \neq \emptyset$ here, and $B \cap E \neq \emptyset$ similarly. Consequently, E is disconnected if $\bar{E}$ is disconnected, or closures of connected sets are always connected.

Problem 3

3.(a)

Note that $\mathbf{a} \neq \mathbf{b}$ or $|\mathbf{a} - \mathbf{b}| > \mathbf{0}$ since $A \cup B = \emptyset$, and $|\mathbf{p}(t) - \mathbf{p}(s)| = |t - s||\mathbf{a} - \mathbf{b}|$. Hence, $\mathbf{p}(t) = \mathbf{p}(s)$ if and only if $t = s$. Now we want to show that A_0 and B_0 are separated i.e., $A_0 \cap \overline{B_0} = \emptyset$ and $\overline{A_0} \cap B_0 = \emptyset$. If there is $t \in A_0 \cap \overline{B_0}$, then $t \in A_0$ and t is also a limit point of B_0. Since $t \in A_0$, $\mathbf{p}(t) \in A$ as well. In addition, given any $\varepsilon > 0$, there is $s \in B_0$ such that $|t - s| < \frac{\varepsilon}{|\mathbf{a}-\mathbf{b}|}$ for any $s \neq t$, which implies that $\mathbf{p}(t) - \mathbf{p}(s) = |t - s||\mathbf{a} - \mathbf{b}| < \varepsilon$. Here $\mathbf{p}(s) \in B$ and $\mathbf{p}(s) \neq \mathbf{p}(t)$, so $\mathbf{p}(t)$ is a limit point of B. Therefore, $\mathbf{p}(t) \in A \cap \bar{B} = \emptyset$, contrary to the assumption that A and B are separated. It's similar to show that $\overline{A_0} \cap B_0 = \emptyset$. Therefore, we prove that A_0 and B_0 are separated.

3.(b)

Assume for contradiction that for all $t \in (0, 1)$, $\mathbf{p}(t) \in A \cup B$. Since $\mathbf{p}(0) = \mathbf{a} \in A$ and $\mathbf{p}(1) = \mathbf{b} \in B$, it follows that $\mathbf{p}(t) \in A \cup B, \forall t \in [0, 1]$. By definition of A_0, B_0, we have $[0, 1] \subseteq A_0 \cup B_0$. Let $U = [0, 1] \cap A_0$ and $V = [0, 1] \cap B_0$. Since

$$\mathbf{p}(0) = \mathbf{a} \in A \implies 0 \in A_0 \implies 0 \in U,$$

$$\mathbf{p}(1) = \mathbf{b} \in B \implies 1 \in B_0 \implies 1 \in V,$$

U, V are nonempty here. Since A_0, B_0 are separated according to **3.(a)**, it shows that U, V are separated and hence $[0, 1]$ is separated, which leads to a contradiction.

3.(c)

Let S be a subset of $\mathbb{R}^k$. S is convex if $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S, \forall \mathbf{x}, \mathbf{y} \in S$ and $\lambda \in (0, 1)$. Assume for contradiction that S is separated, then there exist nonempty sets X, Y s.t. $X \cup Y = S$ and $X \cap Y = \emptyset$. Pick $\mathbf{x} \in X$ and $\mathbf{y} \in Y$, by **3.(b)**, there exists $\lambda \in (0, 1)$ such that $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \notin S$, which contradicts the assumption that S is a convex set.

Problem 4

Following by the **Definition 3.16** in Rudin, we now put $s^* = \sup E$ and $s_* = \inf E$, where the set E contains all subsequential limits as defined in **Definition 3.5** in Rudin, and plus possibly the number $+\infty, -\infty$. Now we attempt to show that the lim sup and lim inf defined in the statement are the same as that given in Rudin.

- $\limsup_{n\to\infty} x_n = s^*$:
  If (x_n) is not bounded above, $\limsup_{n\to\infty} x_n = +\infty = s^*$.

  If (x_n) is bounded above, for all $\varepsilon > 0$ there exists such $N \in \mathbb{N}$ such that

  $$\limsup_{n\to\infty} x_n - \varepsilon < x_N < \lim_{n\to\infty} \left(\sup_{k\geq n} x_k \right) := \limsup_{n\to\infty} x_n.$$

  Therefore $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \left(\sup_{k\geq n} x_k\right)$ is a limit point of some subsequence of (x_n), and it suffices to say $\limsup_{n\to\infty} x_n \in E$.

  Next, we would like to show that $\limsup_{n\to\infty} x_n$ is bounded. Since $\limsup_{n\to\infty} x_n \in E$, assume that there exists $m > \limsup_{n\to\infty} x_n$, $m \in \mathbb{R}$, such that for some n, $x_n \geq m$ i.e.,

  $$\lim_{n'\to\infty} x_{n'} > \lim_{n\to\infty} \sup_{k\geq n} x_n,$$

  which leads to a contradiction of sup. Therefore, if for some $m > \limsup_{n\to\infty} x_n$, $m \in \mathbb{R}$, $x_n < m$ for $n \geq N$, where $N \in \mathbb{N}$.

  By **Definition 3.17** in Rudin, s^* is the only number with the properties above. Hence, $\limsup_{n\to\infty} x_n$ defined in the statement is the same as s^*.

- $\liminf_{n\to\infty} x_n = s_*$:
  Following the same concept to prove $\limsup_{n\to\infty} x_n = s^*$. If (x_n) is not bounded below, $\liminf_{n\to\infty} x_n = -\infty = s_*$.

  If (x_n) is bounded below, for all $\varepsilon > 0$ there exists such $N \in \mathbb{N}$ such that

  $$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right) < x_N < \liminf_{n\to\infty} x_n + \varepsilon.$$

  Therefore $\liminf_{n\to\infty} x_n := \lim_{n\to\infty}(\inf_{k\geq n} x_k)$ is a limit point of some subsequence of (x_n), and it suffices to say $\liminf_{n\to\infty} x_n \in E$.

  Next, we would like to show that $\liminf_{n\to\infty} x_n$ is bounded. Since $\liminf_{n\to\infty} x_n \in E$, assume that there exists $m > \liminf_{n\to\infty} x_n$, $m \in \mathbb{R}$, such that for some n, $x_n \geq m$ i.e.,

  $$\lim_{n'\to\infty} x_{n'} < \lim_{n\to\infty} \inf_{k\geq n} x_n,$$

  which leads to a contradiction of inf. Therefore, if for some $m < \liminf_{n\to\infty} x_n$, $m \in \mathbb{R}$, $x_n > m$ for $n \geq N$, where $N \in \mathbb{N}$.

  By **Definition 3.17** in Rudin, s_* is the only number with the properties above. Hence, $\liminf_{n\to\infty} x_n$ defined in the statement is the same as s_*.

Problem 5

Following the concept in Rudin for **Theorem 3.37**, we put $\alpha = \liminf_{n\to\infty} \frac{c_{n+1}}{c_n}$. If $\alpha = 0$, it's nothing to prove. If $\alpha > 0$, choose $\beta > \alpha$, and there is an integer N such that $\beta \leq \frac{c_{n+1}}{c_n}$ as $n \geq N$. In particular, for any $p > 0, p \in \mathbb{N}$, we have

$$\beta \cdot c_{N+k} \leq c_{N+k+1} \quad (k = 0, 1, \ldots, p-1).$$

Multiplying these inequalities, we obtain

$$\beta^p c_N \leq c_{N+p} \quad \text{or} \quad c_N \beta^{-N} \cdot \beta^n \leq c_n$$

for $n \geq N$. Hence, it shows

$$\beta \cdot \sqrt[n]{\beta^{-N} c_N} \leq \sqrt[n]{c_n},$$

so that $\liminf_{n\to\infty} \sqrt[n]{c_n} \geq \beta$. Therefore, by **Theorem 3.20** in Rudin, since $\beta \cdot \sqrt[n]{\beta^{-N} c_N} \leq \sqrt[n]{c_n}$ is true for every $\beta > \alpha$, we prove the inequality.