Homework 4

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Notations used in the homework

To avoid notations confusion, we define the common used notations initially.

1. Define $N_r(x)$ as the neighborhood centering at x in radius r. On the other hand, $N(x)$ indicates the neighborhood centering at x without specific radius.
2. Define $B_r(x)$ as the open ball centering at x in radius r.

Problem 1

Following the *hint*, we fix $\delta > 0$ and pick x_1 in X, then choose x_{j+1} s.t. $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Let $E_\delta \subseteq X$ be the set collecting these x_i. Next, we need to show that this process must stop after a finite number of steps and that X can therefore be covered by finite many neighborhoods of radius δ. This statement can be proved by contradiction. Assume by contradiction, the process will not stop after finite number of steps i.e., E_δ is an infinite subset of X, and E_δ has a limit point $p \in X$ since it states that all infinite subsets of X have a limit point. Moreover, based on **Theorem 2.20** in Rudin [1] , we define two points q, r within $N_{\frac{\delta}{4}}(p)$. Note that $q, r \in N_{\frac{\delta}{4}}(p) \subseteq E_\delta$. Therefore, by triangle inequality, we have

$$d(q,r) \leq d_(q,p) + d(p,r) < \frac{\delta}{4} + \frac{\delta}{4} < \delta,$$

which contradicts the construction of E_δ. Consequently, it's proved that X can be covered by finite many neighborhoods of radius δ.

By the definition of *separable*, X is separable if it has a countable dense subset. We derive that it has a countable subset, then we are going to find its dense set. The definition of dense set $D \subseteq X$ is

$$\forall \varepsilon > 0 \, \forall x \in X, \, D \cap B_\varepsilon(x) \neq \emptyset,$$

and, by the Archimedean property, the definition can be re-write to

$$\forall n \in \mathbb{N} \, \forall x \in X, \, D \cap B_{\frac{1}{n}}(x) \neq \emptyset.$$

Use E_δ again. Take $\delta = \frac{1}{n}$, and define the union of all E_δ as

$$E = \bigcap_{n=1}^{\infty} E_{\frac{1}{n}} \subseteq X.$$

Here E is dense in X, since given any $p \in X - E$, there exists $q \in B_r(p)$, where $q \in E$. Pick any $n \in \mathbb{Z}_+$ s.t. $\frac{1}{n} < r$, it suffices to say that $q \in B_r(p)$, that is, E is dense in X.

Finally, since there is a countable dense subset of X, we could say X is separable.

Problem 2

Since metric space K is compact, each of K's open covers has a finite subcover. In other words, for every collection S of open subsets of K, $K \subseteq \bigcup_{x \in S} x$, and there is a finite subset S' of S such that $K \subseteq \bigcup_{x \in S'} x$.

Given any $\delta > 0$, $\delta \in \mathbb{R}$, define an open cover C of K by $C = \{B_\delta(p) : p \in K\}$, and its finite subcover F of K. Here the union of F can be the finite collection of $B_\delta(p) : p \in K$, and $K \subseteq \bigcup_{x \in F} x$. It's equivalent to K is bounded, then, by **Problem 1**, we can say that K is separable.

[1]If p is a limit point of a set E, then every neighborhood of p contains *infinitely* many points of E.

Problem 3

By the results in **Problem 1** and **Ex. 2.23** in Rudin (one of the problems in Homework 3), we have that X is separable and countable. Let $V = \{V_n\}$ be a countable base of X. Give any open cover C of X, for all $V_n \in V$, we take $G_n \in C$ containing V_n and collect all G_n as G, i.e.,

$$G = \{G_n : B_n \subseteq G_n \forall n \in \mathbb{Z}_+\}.$$

Note that G is a countable subset of C and G covers X since V is a countable base of X. Hence, given any open cover C of X, there is a countable subcover G of X. Next, assume by contradiction, if there were no finite subcover of G covering X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\cap F_n$ is empty. Let E be a set containing a point from each F_n, E here is infinite and has a limit point p. Note that $p \in G_n$ for some n since G is an open cover of X. In addition, since G_n is open, there exists an open neighborhood $N(p)$ such that $N(p) \subseteq G_n$. By the construction of F_n, $N(p) \cap F_n = \emptyset$, contrary to the assumption that p is a limit point of E. Consequently, X is compact.

Problem 4

4.(a)

Let $p^{-1} = \alpha, q^{-1} = 1 - \alpha$ and re-write the right-hand-side to

$$\alpha u^p + (1-\alpha)v^q.$$

Take logarithm to this function, it becomes

$$\ln(\alpha u^p + (1-\alpha)v^q),$$

and due to the concavity of logarithm, we have the inequality as

$$\begin{aligned} \ln(\alpha u^p + (1-\alpha)v^q) &\geq \alpha \ln(u^p) + (1-\alpha)\ln(v^q) \\ &= \alpha p \ln(u) + (1-\alpha) q \ln(v) \\ &= \ln(u) + \ln(v). \end{aligned}$$

This inequality is equivalent to $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$. In addition, by the proposition of concavity, the equality holds if and only if $u^p = v^q$.

4.(b)

We can trickily focus on the case $|x|_p = |y|_p = 1$, as all positive numbers are scalars for 1, and multiplying scalars on both sides in the equality is linear i.e.,

$$|\langle ax, by\rangle| \leq |ax|_p |by|_p$$

holds for arbitrary a, b. Starting to our derivation.

$$\begin{aligned} |\langle x, y\rangle| &= |x_1 y_1 + x_2 y_2 + \cdots + x_k y_k| \\ &\leq |x_1||y_1| + |x_2||y_2| + \cdots + |x_k||y_k| \quad \text{(By triangle inequality.)} \\ &\leq \left(\frac{|x_1|^p}{p} + \frac{|y_1|^q}{q} + \cdots + \frac{|x_k|^p}{p} + \frac{|y_k|^q}{q} \right) \quad \text{(By } \mathbf{4(a)}\text{).} \\ &= \frac{1}{p}\sum_{i=1}^{k}|x_i|^p + \frac{1}{q}\sum_{i=1}^{k}|y_i|^p \quad \text{(Note that } (|x|_p)^p = (|y|_p)^p = 1 \text{ as well.)} \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

Here we prove that $|\langle x, y\rangle| \leq |x|_p |y|_p$ for the case $|x|_p = |y|_p = 1$, and the inequality holds for any case as previous discussion. The equality holds if and only if $|x|^p = |y|^q$.

4.(c)

Starting from the case $\sum_{i=1}^{k}|x_i+y_i|^p$, we have

$$\begin{aligned}
\sum_{i=1}^{k}|x_i+y_i|^p &\leq \sum_{i=1}^{k}(|x_i|\cdot|x_i+y_i|^{p-1}+|y_i|\cdot|x_i+y_i|^{p-1}) \quad \text{(By triangle inequality.)} \\
&= \sum_{i=1}^{k}|x_i|\cdot|x_i+y_i|^{p-1}+\sum_{i=1}^{k}|y_i|\cdot|x_i+y_i|^{p-1} \\
&\leq \left(\sum_{i=1}^{k}|x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}(|x_i+y_i|^{p-1})^q\right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{i=1}^{k}|y_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}(|x_i+y_i|^{p-1})^q\right)^{\frac{1}{q}} \quad \text{(By } \mathbf{4(b)}.\text{)} \\
&= |x|_p\left(\sum_{i=1}^{k}(|x_i+y_i|^p)\right)^{\frac{1}{q}}+|y|_p\left(\sum_{i=1}^{k}(|x_i+y_i|^p)\right)^{\frac{1}{q}}.
\end{aligned}$$

Move $\left(\sum_{i=1}^{k}(|x_i+y_i|^p)\right)^{\frac{1}{q}}$ to the left-hand-side, the inequality becomes

$$\begin{aligned}
& \sum_{i=1}^{k}|x_i+y_i|^p\left(\sum_{i=1}^{k}(|x_i+y_i|^p)\right)^{\frac{-1}{q}} \leq |x|_p+|y|_p \\
\Longleftrightarrow\quad & \left(\sum_{i=1}^{k}(|x_i+y_i|^p)\right)^{\frac{1}{p}} \leq |x|_p+|y|_p \\
\Longleftrightarrow\quad & |x+y|_p \leq |x|_p+|y|_p.
\end{aligned}$$

The equality holds if and only if $|x+y|_p=|x|_p+|y|_p$.

4.(d)

Since

$$|x|_p=\left(\sum_{i=1}^{k}|x|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{k}\max_i|x_i|^p\right)^{\frac{1}{p}}=k^{\frac{1}{p}}\max_{1\leq i\leq k}|x_i|,$$

when $p\to\infty$, $\lim_{p\to\infty}|x|_p\leq\max_{1\leq i\leq k}|x_i|$. On the other hand,

$$|x|_p=\left(\sum_{i=1}^{k}|x_i|^p\right)^{\frac{1}{p}} \geq \left(\max_{1\leq i\leq k}|x_i|^p\right)^{\frac{1}{p}}=\max_{1\leq i\leq k}|x_i|.$$

As $p\to\infty$, $\lim_{p\to\infty}|x|_p\geq\max_{1\leq i\leq k}|x_i|$. Hence, we can state that $\lim_{p\to\infty}|x|_p=\max_{1\leq i\leq k}|x_i|$.

To establish inequalities among l-norm for $l=1,p,q,\infty$, $1<p<\infty$, we compare the case $l=1,p,\infty$ initially. It's clearly to say that $|x|_1>|x|_p$, as $(|x|_1)^p\geq(|x|_p)^p$. It's also easy to observe that $|x|_p>|x|_\infty$, since

$$|x|_p=\left(\sum_{i=1}^{k}|x_i|^p\right)^{\frac{1}{p}} \geq \left(\max_{1\leq i\leq k}|x_i|^p\right)^{\frac{1}{p}}=\max_{1\leq i\leq k}|x_i|=|x|_\infty.$$

Last, to compare the relation between p-norm and q-norm, it's a good attempt to differentiate it and then observe the relation. We have

$$\frac{\partial}{\partial p}\left(\sum_{i=1}^{k}|x_i|^p\right)^{\frac{1}{p}}=\left(\frac{-\ln(\sum_{i=1}^{k}x_i^p)}{p^2}+\frac{\sum_{i=1}^{k}x_i^p\ln(x_i)}{p\sum_{i=1}^{k}x_i^p}\right)\left(\sum_{i=1}^{k}x_i^p\right)^{\frac{1}{p}}<0.$$

Hence, as p increases, the value of p-norm decreases. At the end, we have the relation inequality that $|x|_1>|x|_p>|x|_q>|x|_\infty$, where $1<p<q<\infty$.

4.(e)

Yes, they have the same topology.

4.(f)

No. Consider a counter-example $x = (0, 1), y = (1, 0)$ in $\mathbb{R}^2$. For $p \in (0, 1), |x + y|_p = 2^{\frac{1}{p}} > 2 = |x|_p + |y|_p$.

Problem 5

First we check X is compact. Let $O = \{O_i\}_{i \in S}$ be some open cover of X, S be some proper set. Assume that it has no finite subcover, for each finite $F \subseteq S$, there exists some point $p \in X$ s.t. $p \notin \bigcup_{i \in F} O_i$. This statement is equivalent to that for each finite $F \subseteq S$, there exists some point $p \in \bigcap_{i \in S}(X \setminus O_i)$. Hence, the collection $\{X \setminus O_i\}_{i \in S}$ is a collection of closed sets in X having the finite intersection property, which implies there is some point $q \in \bigcap_{i \in S}(X \setminus O_i)$ i.e., $q \notin \bigcap_{i \in S} O_i$, contrary to $\{O_i\}_{i \in S}$ is a cover of X. Hence, we conclude that $\{O_i\}_{i \in S}$ has a finite subcover, i.e., for every collection of closed set in X, it has the finite intersection property. So X is compact.

Next, we check whether the property holds if X is compact. Assume that X is compact and let $\{C_i\}_{i \in S}$ be some family of closed subsets in X having the finite intersection property. Assume by contradiction that $\bigcap_{i \in S} C_i = \emptyset$, then $\{X \setminus C_i\}_{i \in S}$ is an open cover of X, and it has a finite subcover i.e., $\bigcup_{i \in F}(X \setminus C_i) = X$ for some finite subset $F \subseteq S$. However, by asuumption it indicates $\bigcap_{i \in S} C_i = \emptyset$, which contradicts that $\{C_i\}_{i \in S}$ has the finite intersection property. Therefore we conclude that $\bigcap_{i \in S} C_i \neq \emptyset$.