Homework 1

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Problem 1: Definition and properties of exponential functions

1.(a)

Since $b > 1$, the function $t \mapsto b^t$ is increasing. Given any $r, t \in \mathbb{Q},\ t \le r$, since $b^t \le b^r$, we have $\sup B(r) \le b^r$. Conversely, $r \in B(r) \implies b^r \le \sup B(r)$. Therefore, it suffices to say $b^r = \sup B(r)\ \forall r \in \mathbb{Q}$.

1.(b)

Let

$$\begin{aligned} B(x) &= \{b^r | r \in \mathbb{Q},\ r \le x\} \\ B(y) &= \{b^s | s \in \mathbb{Q},\ s \le y\} \\ B(x+y) &= \{b^t | t \in \mathbb{Q},\ t \le x+y\} \\ &= \{b^r b^s | r, s \in \mathbb{Q},\ r \le x,\ s \le y\}, \end{aligned}$$

and

$$\begin{aligned} b^x &= \sup B(x) \\ b^y &= \sup B(y) \\ b^{x+y} &= \sup B(x+y). \end{aligned}$$

Since

$$B(x+y) = b^r b^s \le b^x b^y = \sup B(x) \sup B(y),$$

take supremun, $\sup B(x+y) \le \sup B(x) \sup B(y)$. Moreover, for $b^r b^s = b^{r+s} \le \sup B(x+y)$, we have

$$\begin{aligned} & b^r \le \frac{\sup B(x+y)}{b^s} \\ \implies\ & \sup B(x) \le \frac{\sup B(x+y)}{b^s} \\ & \text{(Here } \frac{\sup B(x+y)}{b^s} \text{ is an upper bound of } b^r\text{.)} \\ \implies\ & \sup B(x) \le \frac{\sup B(x+y)}{\sup B(y)} \\ \implies\ & \sup B(x) \sup B(y) \le \sup B(x+y). \end{aligned}$$

Therefore, we derive $\sup B(x) \sup B(y) = \sup B(x+y)$ i.e., $b^x b^y = b^{x+y}$ for all real x and y.

Problem 2: Logarithm

(a)

$$\begin{aligned} b^n - 1 \quad = b^n - 1^n \quad &= (b-1)\underbrace{(b^{n-1}) + b^{n-2} + \cdots + 1}_{n} \\ &\ge (b-1)\underbrace{(1 + 1 + \cdots + 1)}_{n} \\ &= n(b-1). \end{aligned}$$

(b) Put $b \mapsto b^{\frac{1}{n}}$ in (a) then it's proven.

(c)

$$n > \frac{b-1}{t-1} \implies n(t-1) > b-1 \geq n(b^{\frac{1}{n}}-1),$$

where n, b, t are positive, hence, $t > b^{\frac{1}{n}}$.

(d) Let $t = y \cdot b^{-w} > 1$. By (c), we have

$$\begin{aligned} & b^{\frac{1}{n}} < y \cdot b^{-w} && \text{for } n > \frac{b-1}{y \cdot b^{-w} - 1} \\ \iff & b^{\frac{1}{n}+w} > y && \text{for } n > \frac{b-1}{y \cdot b^{-w} - 1}. \end{aligned}$$

(e) Let $t = y^{-1} \cdot b^{w} > 1$. By (c), we have

$$\begin{aligned} & b^{\frac{1}{n}} < y^{-1} \cdot b^{w} && \text{for } n > \frac{b-1}{y \cdot b^{-w} - 1} \\ \iff & b^{w-\frac{1}{n}} > y && \text{for } n > \frac{b-1}{y \cdot b^{-w} - 1}. \end{aligned}$$

(f) Suppose $b^x < y$, (d) implies that $b^{w+\frac{1}{n}} < y$ for sufficiently large n. Suppose $b^x > y$, (e) implies that $b^{w-\frac{1}{n}} < y$ for sufficiently large n. Both conditions above indicate that x cannot be $\sup A$ satisfying $b^x = y$. Hence, $x = \sup A$ satisfies $b^x = y$.

(g) By contradiction, if there were another $x' \neq x$ such that $b^{x'} = y$, then $x' > x$ or $x' < x$. For the case $x' > x$, $y = b^{x'} > b^x = y$, which is contradicted. Similarly, for the case $x' < x$, $y = b < x' < b^x = y$ is also contradicted. Hence, x should be unique.

Problem 3: Cauchy-Schwarz Inequality

3.(a)

We start deriving from the left side.

$$\begin{aligned} & \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2 \\ = & \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1 \leq k < j \leq n} (a_k^2 b_j^2 + a_j^2 b_k^2 - 2 a_j a_k b_j b_k) - \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k=1}^{n} a_k^2 b_k^2 \\ = & \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{k=1}^{n}\sum_{j=1}^{n} a_k^2 b_j^2 + \sum_{k=1}^{n} a_k b_k \sum_{j=1}^{n} a_k b_j \\ = & \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) + \sum_{k=1}^{n} a_k b_k \sum_{j=1}^{n} a_k b_j \\ = & \left(\sum_{k=1}^{n} a_k b_k\right)^2. \end{aligned}$$

3.(b)

$$
\begin{aligned}
|\langle \mathbf{a}, \mathbf{b}\rangle|^2 &= \left|\sum_{k=1}^{n} a_k \bar{b}_k\right|^2 \\
&= \sum_{k=1}^{n} a_k \bar{b}_k \cdot \sum_{k=1}^{n} \bar{a}_k b_k \\
&= \sum_{1\le j,k\le n} a_k \bar{a}_j b_j \bar{b}_k \\
&= \sum_{k=1}^{n} |a_k \bar{b}_k|^2 + \sum_{1\le k<j\le n} a_k \bar{a}_j b_j \bar{b}_k + a_j \bar{a}_k b_k \bar{b}_j + \sum_{1\le j,k\le n, j\ne k} |a_k \bar{b}_j|^2 - \sum_{1\le j,k\le n, j\ne k} |a_k \bar{b}_j|^2 \\
&= \sum_{k=1}^{n} |a_k|^2 \sum_{j=1}^{n} |\bar{b}_j|^2 - \left(\sum_{1\le k<j\le n} |a_k \bar{b}_j|^2 + |a_j \bar{b}_k|^2 + a_k \bar{a}_j b_j \bar{b}_k + a_j \bar{a}_k b_k \bar{b}_j\right) \\
&= \sum_{k=1}^{n} |a_k|^2 \sum_{j=1}^{n} |\bar{b}_j|^2 - \sum_{1\le k<j\le n} |a_k \bar{b}_j - a_j \bar{b}_k|^2.
\end{aligned}
$$

As a result, the Lagrange identity of complex number is

$$
\left|\sum_{k=1}^{n} a_k \bar{b}_k\right|^2 = \sum_{k=1}^{n} |a_k|^2 \sum_{j=1}^{n} |\bar{b}_j|^2 - \sum_{1\le k<j\le n} |a_k \bar{b}_j - a_j \bar{b}_k|^2.
$$

3.(c)

Observe the results in 3(a) and 3(b) above, we may derive that

$$
\begin{aligned}
\left(\sum_{k=1}^{n} a_k b_k\right)^2 &\le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \\
\left|\sum_{k=1}^{n} a_k \bar{b}_k\right|^2 &\le \sum_{k=1}^{n} |a_k|^2 \sum_{j=1}^{n} |\bar{b}_j|^2,
\end{aligned}
$$

and these two inequalities representing the Cauchy-Schwarz inequalities in respective real and complex number. Under the condition that, when $\sum_{1\le k<j\le n}(a_k b_j - a_j b_k)^2$ and $\sum_{1\le k<j\le n}|a_k \bar{b}_j - a_j \bar{b}_k|^2$ hold, the inequalities hold respectively.

Problem 4: Equivalent Properties

We classify that group A contains (a), (e), and group B contains (b), (c), (d).

(Group A)

1. **(a) $\implies$ (e)**

   Suppose order field F holds the least-upper-bound property (In the following, we abbreviate least-upper-bound property as LUB property), then for two non-empty bounded below subsets $A, B \subset F$, with the property that $x \le y$ for every $x \in A$ and every $y \in B$, by **Theorem 1.11** in Rudin, we claim that $\inf B$ exists, and $\sup A$ exists as it's bounded by B.

   Next, by the relation between x and y, we derive

$$
\begin{aligned}
& x \le y \ \forall x \in A,\ \forall y \in B \\
\iff\ & y \text{ is an upper bound of } A,\ \forall y \in B \\
\iff\ & y \ge \sup A,\ \forall y \in B \\
\iff\ & \sup A \text{ is a lower bound of } B \\
\iff\ & \sup A \le \inf B.
\end{aligned}
$$

Last, since $x \leq \sup A \leq \inf B \leq y,\ \forall x \in A,\ \forall y \in B$, defince c as $\sup A (\in F)$, then we prove that there exists $c \in F$ such that $x \leq c \leq y$ for all $x \in A$ and $y \in B$.

2. **(e) $\implies$ (a)**
   Let A, B be nomempty subsets of F with the property that $x \leq y$ for every $x \in A$ and every $y \in B$, and there exists $c \in F$ such that $x \leq c \leq y$ for all $x \in A$ and $y \in B$. Since c is an upper bound of A and $c \leq y$ i.e., c is bounded by any other upper bound. To expand to any set, consider B' as the set of every upper bound, and A' as the set of those which are not upper bound. For every element $x' \in A'$ and $y' \in B'$, we obtain the relation for $x' \leq y'$. Since A' is bounded above by B', we could find $c' \in B'$ such that $x' \leq c' \leq y'$ for all $x' \in A'$ and $y' \in B'$. Hence, c' is the least upper bound, and we derive that if there exists the upper bound of the set, then the LUB property is true.

As above, e prove that properties in the group A are equivalent to each other.

(Group B)

1. **(b) $\implies$ (c)**
   Let $x, y \in F$ and $0 < y - x$. By the archimedean property, there exists $n \in \mathbb{N}$ s.t. $n(y-x) > 1$. Moreover, there exists $m \in N$ s.t. $nx < m < ny \implies x < \frac{m}{n}$ and $y > \frac{m}{n}$. Therefore, we can conclude that $\mathbb{Q}$ is in dense in F.

2. **(c) $\implies$ (d)**
   As $\mathbb{Q}$ is in dense in F, by (c), therefore, for any $x \in F$, $x > 0$, there exists $m, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \frac{m}{n} < x$

3. **(d) $\implies$ (b)**
   Let $x, y, z \in F,\ x, y, z > 0$ and $z = \frac{x}{y}$, by (d), there exists $n \in N$ such that

$$0 < \frac{1}{n} < z = \frac{x}{y} \implies nx > y.$$

   That is, (b) could be implied by (d).

As above, we prove that properties in group B are equivalent to each other.

Next, to prove that every property in group A implies any properties in group B but not the converse, we take property (a) and property (b) as an example. Since the properties in the same group are equivalent, it suffices to prove the condition between two groups by merely handling the relation between property (a) and (b).

1. **(a) $\implies$ (b)**
   We use the fact that the set of positive integers $\mathbb{N}$ is not bounded above, to prove this deduction.

   Lemma 1. *The set of positive integers $\mathbb{N}$ is not bounded above.*

   Proof. Since $\mathbb{N} \subset \mathbb{Q} \subset F$ and F holds the LUB property i.e., F has a least upper bound, by contradiction, assume $\mathbb{N}$ has a least upper bound $a \in F$, where $n \leq a$ for all $n \in \mathbb{N}$. Consequently, $a - 1$ is not an upper bound for $\mathbb{N}$ as if it were, $a - 1 < a$ implies that a is not the *least* upper bound. Therefore, there exists some integers b such that $a - 1 < b \implies a < b + 1$. However, it contradicts that a is an upper bound of $\mathbb{N}$. ☐

   Let $x, y \in F,\ x > 0$ and $n \in \mathbb{N}$. Assume by contradiction that there is no $n \in \mathbb{N}$ such that $nx > y$ i.e., the archimedean property is satisfied, then it indicates $nx \leq y \iff n \leq \frac{y}{x}$ for all $n \in N$. This statement is equivalent that the set of positive integers $\mathbb{N}$ has an upper bound, and it contradicts the lemma. Therefore, we prove that the LUB property implies the archimedean property.

2. **(b) $\not\Longrightarrow$ (a)**
   Let $x \in \mathbb{Q}$ and $x > 0$, we can write x as $\frac{n}{m}$ with $n \in \mathbb{N}$ and $m \neq 0,\ m \in \mathbb{N}$. Since $m \geq 1$, then we have $n = mx \geq x$ which means $n + 1 > x$. So the rational number has the archimedean property. However, $\mathbb{Q}$ does not have the LUB property. For example, consider $\{y \in \mathbb{Q}\ |y^2 = 2\}$, which has an upper bound but no least upper bound. Hence, we find a counter-example to prove that property (b) cannot imply property (a).

Problem 5: Ordered Field

5.(a)

To indicate that $\prec$ is an order making $Q[x]$ an ordered field, we need to prove

1. $x+y \prec x+z$ if $x,y,z \in Q[x]$ and $y \prec z$.
2. $0 \prec xy$ if $x,y \in Q[x]$, $0 \prec x$ and $0 \prec y$.

Let $x,y,z \in Q[x]$. For 1., assume that $x \prec y$, then

$$y - x \in K \implies y + z - (x+z) \in K \implies x + z \prec y + z.$$

For 2., assume that $0 \prec x$ and $0 \prec y$, it implies

$$x - 0 \in K,\ y - 0 \in K \implies x, y \in K.$$

Since $x, y \in K$, their corresponding uniqle polynomials, denote as p^x, q^x, p^y, q^y, satisfy that $p_n^x > 0,\ p_n^y > 0$ and $p_n^x \cdot p_n^y > 0$ i.e., $xy \in K$. Hence, $0 \prec xy$ holds.

5.(b)

If $Q[x]$ have the archimedean property, then for $x, y \in Q[x]$ and $0 \prec x$, there exists a possitive number n such that $y \prec nx \iff nx - y \in K$. However, when $m > n$, we could always find y with the positive leading coefficient such that $nx - y \notin K$. Hence, $Q[x]$ does not have the archimedean property.

Let $f \in Q[x]$ with the positive leading coefficient and $f = (n+1)x$ where $n \in \mathbb{N}$, then $\forall n \in \mathbb{N},\ \forall x > 0$,

$$nx < f = nx + n \implies f \text{ is an upper bound of } nx.$$

Similarly, let $g \in Q[x]$ with the positive leading coefficient and $g = (n-1)x$ where $n \in \mathbb{N}$, then $\forall n \in \mathbb{N},\ \forall x > 0$,

$$nx > g = nx - x \implies g \text{ is a lower bound of } nx.$$

Hence, it suffices to say that $\mathbb{N}$ is bounded in $Q[x]$.

Problem 6: Nested Interval Property

6.(a)

Since $I_n = [a_n, b_n]$ is a nested sequence of closed intervals, we obtain that

$$\forall n, m \in \mathbb{N},\ a_n \le a_{\max\{n,m\}} \le b_{\max\{n,m\}} \le b_n.$$

In other words, for every $m \in \mathbb{N}$, b_m is a upper bound of a_n $(n \in \mathbb{N})$. Let $c = \lim\limits_{n\to\infty} a_n$, and b_m $(m \in \mathbb{N})$ is an upper bound of a_n $(n \in \mathbb{N})$, we have $c \le b_m\ \forall m \in \mathbb{N}$; also, $a_n \le c\ \forall n \in \mathbb{N}$. Therefore, we find $c \in \cap_{n=1}^{\infty} I_n$ i.e., $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

6.(b)

Let $A = \{a_n\}(n \in \mathbb{N})$ and $B = \{b_n\}(n \in N)$, a_n is a strictly increasing sequence, b_n is a strictly decreasing sequence, and $b_n(n \in \mathbb{N})$ is an upper bound of A. Since a_n and b_n are bounded, a_n and b_n converge to some limit points α, β respectively, and $\alpha \le \beta$. For any arbitrarily small $\varepsilon > 0$ and $|I_k| = b_k - a_k$, we have

$$\lim_{k\to\infty} |I_k| = \lim_{k\to\infty} b_k - a_k = \lim_{k\to\infty} b_k - \lim_{k\to\infty} a_k = \beta - \alpha = 0 \implies \alpha = \beta.$$

Therefore, the point common to all the intervals $\cap_{n=1}^{\infty} I_n$ is unique.

6.(c)

If the nested sets I_n are open intervals, their intersection may sometimes be an emptyset. For example, let the nested set $I_n = (0, \frac{1}{n})$, the intersection of I_n is an emptyset.