Dept. IM. Name 郭宇杰 Student ID B07611039 Grade ______________

1. [R] Ex. 1.6 (c) and (d) (definition and properties of exponential functions)
Ans. 可使用中文。於截止時限前上傳 pdf 檔至 NTU COOL 完成繳交作業，其中 ID 改為自己的學號。 □

2. [R] Ex. 1.7 (logarithm of $y > 0$ to the base $b > 1$).
Ans.

3. [A] Ex. 1.23 & 1.48, [R] Ex. 15 (real and complex Lagrange identities)

   (a) Prove Lagrange identity for real numbers:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2 . \tag{1}$$

   In fact, the left hand side of (1) is the inner product $\langle \mathbf{a}, \mathbf{b}\rangle$ of $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$.

   (b) Define the inner product of $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbb{C}^n$ by

$$\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{k=1}^{n} a_k \bar{b}_k .$$

   Derive a Lagrange identity of complex numbers by starting with $|\langle \mathbf{a}, \mathbf{b}\rangle|^2$.

   (c) Derive the real and complex Cauchy-Schwarz inequalities from the respective real and complex Lagrange identities. Under what conditions do equalities hold, respectively?

Ans.

   (a) We start deriving from the left side.

$$\begin{aligned}
&\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2 \\
=& \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\le k<j\le n} (a_k^2 b_j^2 + a_j^2 b_k^2 - 2a_j a_k b_j b_k) - \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k=1}^{n} a_k^2 b_k^2 \\
=& \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{k=1}^{n}\sum_{j=1}^{n} a_k^2 b_k^2 + \sum_{k=1}^{n} a_k b_k \sum_{j=1}^{n} a_k b_j \\
=& \left(\sum_{k=1}^{n} a_k b_k\right)^2
\end{aligned}$$

□

4. Let F be an ordered field containing $\mathbb{Q}$. Classify the following properties of F to group A and group B such that (1) properties in the same group are equivalent to each other and (2) every property in group A implies any property in group B, but not the converse.

   (a) The least-upper-bound property.

   (b) The archimedean property.

   (c) $\mathbb{Q}$ is dense in F.

(d) For any $x \in F, x > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.

(e) If A and B are nonempty subsets of F having the property that $x \leq y$ for every $x \in A$ and every $y \in B$, then there exists $c \in F$ such that $x \leq c \leq y$ for all $x \in A$ and $y \in B$.

Ans. 這裡展示，例如，$(a) \Rightarrow (b)$ 的寫法。 $\square$

5. Let $Q[x]$ be the field of rational functions with real coefficients. For each $f \in Q[x]$ there exist unique polynomials $p = \sum_{j=0}^{n} p_j x^j, q = \sum_{j=0}^{m} q_j x^j$ such that $p_n \neq 0, q_m = 1$ and $f = p/q$ is in lowest terms (that is, p and q have no nonconstant factors in common). With this notation, let $K = \{f \in Q[x] : p_n > 0\}$ and define $f \prec g$ if $g - f \in K$.

   (a) Show that $\prec$ is an order which makes $Q[x]$ an ordered field.

   (b) Show that $Q[x]$ does not have the archimedean property. Give an upper bound and a lower bound to show that $\mathbb{N}$ is bounded in $Q[x]$.

   Ans.

6. (Nested Interval Property) A sequence of subsets S_n is called *nested* if $S_{n+1} \subseteq S_n, n \in \mathbb{N}$. Let $I_n = [a_n, b_n] \in \mathbb{R}$ be a nested sequence of closed intervals.

   (a) Prove that the nested sequence has a nonempty intersection; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

   (b) Moreover, if it is known that for every $\varepsilon > 0$ there is an interval I_k whose length $|I_k| < \varepsilon$, then the point common to all the intervals is unique.

   (c) What can be said if the nested sets I_n are open intervals $I_n = (a_n, b_n), n \in \mathbb{N}$? Give a proof or an example of your answer.

   Ans.