# Homework 1

**Yu-Chieh Kuo B07611039**

**Collaborators statement:** I have a study group with the following memebers:**b06201057 Yu-Chi Hsieh, r08323002 Ze-Wei Chen, b06202004 Han-Wen Chang.** We discuss the problems together but we always do the problem by ourselves first. So, if I specify I have collaborators, then I means I discuss with these group members.
This homework answers the problem set sequentially.

1. (a) **Collaborators: None.**
   Let $c = 0.1,\ n_0 = \frac{5}{2},\ cn^2 \leq 0.5n^2 - n\ \forall n \geq n_0$ holds. Thus, we prove that $0.5n^2 - n \in \Omega(n^2)$.

   (b) **Collaborators: Study Group Members.**
   If $f(n) \in \Omega(n^3)$, then $\exists\ c, n_0 > 0\ s.t.\ cn^3 \leq f(n)\ \forall n \geq n_0$. Let $c', n_0' > 0$, $f(n) \geq cn^3 \geq cn_0' \cdot n^2 \geq c'n^2,\ \forall n \geq n_0'$ holds. Thus, we can say that $\exists\ c', n_0' > 0,\ s.t.\ f(n) \geq cn^2,\ \forall n \geq n_0'\ i.e.\ f(n) \in \omega(n^2)$.

2. **Collaborators: None.**

$$f_1 = (2n)!,\ f_2 = n^n,\ f_3 = n!,\ f_4 = 2^{2n},\ f_5 = (\log_2 n)!,$$

$$f_6 = n^3 + 5n^2,\ f_7 = 8^{\log_2 n},\ f_8 = \sqrt{n} + 3,\ f_9 = n^{0.01},\ f_{10} = \log_2 n,\ f_{11} = \ln n$$

   $f_6, f_7$ and $f_{10}, f_{11}$ are pairs such that $f(n) \in \Theta(g(n))$.

3. By the Master Theorem, we can specify parameters in these two recurrences.

   (a) **Collaborators: None.**
   Let $a = 9,\ b = 3,\ f(n) = n^3,\ n^{log_b a} = n^2$. Since $f(n) = n^3 \in \Omega(n^2 \cdot n^\epsilon),\ \epsilon > 0$, which belongs to the third case in the Master Theorem. As a result, we can say that $T(n) \in \Theta(n^3)$.

   (b) **Collaborators: None.**
   Let $a = 9,\ b = 3,\ f(n) = n^2 + 20n\log n + 3,\ n^{log_b a} = n^2$, and $f(n) = n^2 + 20n\log n + 3$. Given $c_1 = 1,\ c_2 = 10,\ n_0 = 1,\ c_1 n^2 \leq f(n) \leq c_2 n^2$ when $n > n_0$, therefore we have $f(n) \in \Theta(n^2 \cdot (\log n)^0)$, which belongs to the second case in the Master Theorem. As a result, we can say that $T(n) \in \Theta(f(n)\log n) = \Theta(n^2 \log n)$.

4. (a) **Collaborators: Study Group Members.**
   Let $f_i(n) = i \cdot n\ i.e.\ f_1(n) = n,\ f_2(n) = 2n, \cdots, f_n(n) = n^2$, then $g(k) = \sum_{j=1}^{k} f_j(j) = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\ i.e.\ g(n) \in n^3$, which implies the statement is false. Therefore, we can disprove the statement.

(b) **Collaborators: Study Group Members.**
We already know that $\exists\ c, n_0 > 0\ s.t.\ f(n) \le cn\ \forall\ n \ge n_0$. Let $n_0' = n_0 + \sum_{j=1}^{n_0-1} f(j) - cj$, when $n \ge n_0'$,

$$\begin{aligned} g(n) &= \sum_{j=1}^{n_0-1} f(j) + \sum_{j=n_0}^{n_0'} f(j) + \sum_{j=n_0'+1}^{n} f(j) \\ &\le \sum_{j=1}^{n_0-1} cj + \sum_{j=1}^{n_0-1} (f(j) - cj) + \sum_{j=n_0}^{n_0'} cj + \sum_{j=n_0'+1}^{n} cj \\ &\le \sum_{j=1}^{n_0-1} cj + \sum_{j=n_0}^{n_0'} cj + \sum_{j=n_0}^{n_0'} cj + \sum_{j=n_0'+1}^{n} cj \\ &\le \sum_{j=1}^{n} 2cj \\ &\le 2c\frac{n(n+1)}{2} \end{aligned}$$

which means $g(n) \in O(n^2)$. Therefore, we can prove the statement.

5. (a) **Collaborators: Study Group Members.**
First, to prove that $T(n) \in O(2^n)$, we have $T(3) = 6 \le c \cdot 2^3$ and $T(4) = 7 \le c \cdot 2^4$. To prove $T(n) \ge c \cdot 2^n\ \forall\ n > 4$, we have

$$\begin{aligned} T(n) &= T(n-2) + 2T(\lfloor \frac{n}{2} \rfloor) + n \\ &\le c \cdot 2^{n-2} + 2c \cdot 2^{n-2} + 2^{n-2} \\ &\text{(Since } n - 2 \ge \lfloor \frac{n}{2} \rfloor \text{ and } 2^{n-2} \ge n \text{ holds when } n > 4) \\ &= 4c \cdot 2^{n-2} \\ &= c \cdot 2^n \end{aligned}$$

Therefore, we know that there exists $c, n_0 > 0\ s.t\ T(n) \le c \cdot 2^n\ \forall\ n > n_0$, then we prove that $T(n) \in O(2^n)$.

Second, to prove that $T(n) \in \Omega(n^2)$, we have $T(1) = 1 \ge c \cdot 1^2$ when $c \le 1$. By induction, supposing there exists $c, n_0 > 0\ s.t.\ T(n) \ge cn^2$, we have

$$\begin{aligned} T(n) &= T(n-2) + 2T(\lfloor \frac{n}{2} \rfloor) + n \\ &\ge c(n-2)^2 + 2 \cdot 0 + n \\ &= cn^2 - 4cn + 4c + n \\ &\ge cn^2 \end{aligned}$$

Thus, we prove that there exists $c, n_0 > 0\ s.t.\ T(n) \ge cn^2\ i.e.\ T(n) \in \Omega(n^2)$.

(b) **Collaborators: None.**

$$
\begin{aligned}
T(n) &= 3T(\frac{n}{3}) + \frac{n}{2\log_3 n} \\
&= 3[3T\frac{n}{9} + \frac{\frac{n}{3}}{2\log_3 \frac{n}{3}}] + \frac{n}{2\log_3 n} \\
&= 3^2T(\frac{n}{9}) + \frac{n}{2(\log_3 n - 1)} + \frac{n}{2\log_3 n} \\
&\vdots \\
&= 3^{\log_3 n}T(\frac{n}{3^{\log_3 n}}) + \frac{n}{2[\log_3 n - (\log_3 n - 1)]} + \cdots + + \frac{n}{2(\log_3 n - 1)} + \frac{n}{2\log_3 n} \\
&= n \cdot 1 + \frac{n}{2}(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\log_3 n - 1} + \frac{1}{\log_3 n}) \\
&\approx n + \frac{n}{2}(\ln \log_3 n + 0.5772156649) \text{ (by the definition of Harmonic number)} \\
&\leq n + n \ln \log n
\end{aligned}
$$

Therefore, we can say that $T(n) \in \Theta(n)$

6. (a) **Collaborators: None.**
By repeated substitution, we have

$$
\begin{aligned}
T(n) &= 2T(\frac{n}{2}) + f(n) = 2[2T(\frac{n}{4}) + f(\frac{n}{2})] + f(n) \\
&= 2^2T(\frac{n}{2^2}) + 2f(\frac{n}{2}) + f(n) \\
&\vdots \\
&= 2^{\log_2 n}T(\frac{n}{2^{\log_2 n}}) + 2^{\log_2 n - 1}f(\frac{n}{2^{\log_2 n - 1}}) + \cdots + 2f(\frac{n}{2}) + f(n) \\
&= nT(1) + \frac{n}{2}f(2) + \cdots + 2f(\frac{n}{2}) + f(n)
\end{aligned}
$$

Since $f(n) \in \Theta(n^2)$ *i.e.* $\exists\ c_1, c_2, n_0 > 0$ *s.t.* $c_1n^2 \leq f(n) \leq c_2n^2,\ \forall n \geq n_0$ and $T(1) \in \Theta(1)$, therefore, we can easily find another coefficient $c_1', c_2', n_0' > 0$ *s.t.* $c_1'n^2 \leq nT(1) + \frac{n}{2}f(2) + \cdots + 2f(\frac{n}{2}) + f(n) \leq c_2'n^2,\ \forall n \geq n_0'$. As a result, we prove that $T(n) \in \Theta(n^2)$ *i.e.* $T(n) \in \Theta(f(n))$.

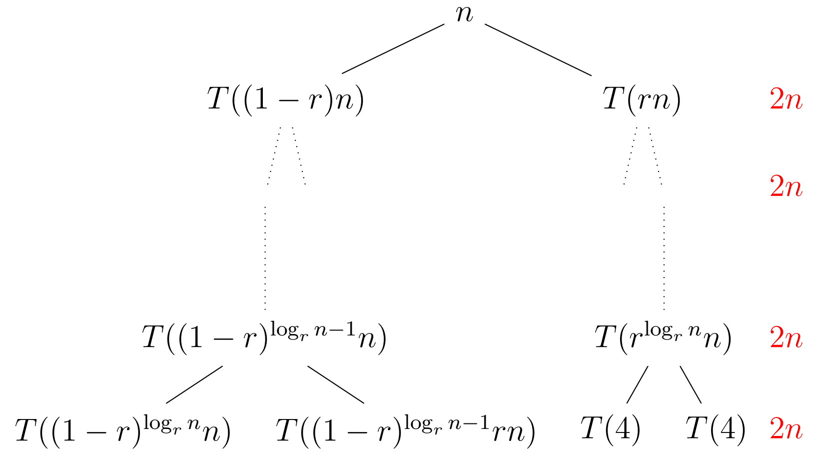
(b) **Collaborators: Study Group Members.**
Let $f(n) = 2^n$ and $T(n) = 2T(\frac{n}{2}) + f(n)$, we have

$$
\begin{aligned}
T(2^k) &= 2T(2^{k-1}) + 2^{2^k} \\
&= 2^kT(1) + \sum_{i=0}^{k} 2^i \cdot 2^{2^{k-i}} \\
&= 2^kT(1) + 2^{2^k}\frac{k+1}{2} \\
&\notin O(2^{2^k})
\end{aligned}
$$

Therefore, we can disprove that $T(n) \in O(f(n))$ for all $n = 2^k$ when $f(n) \in \Omega(n^2)$.

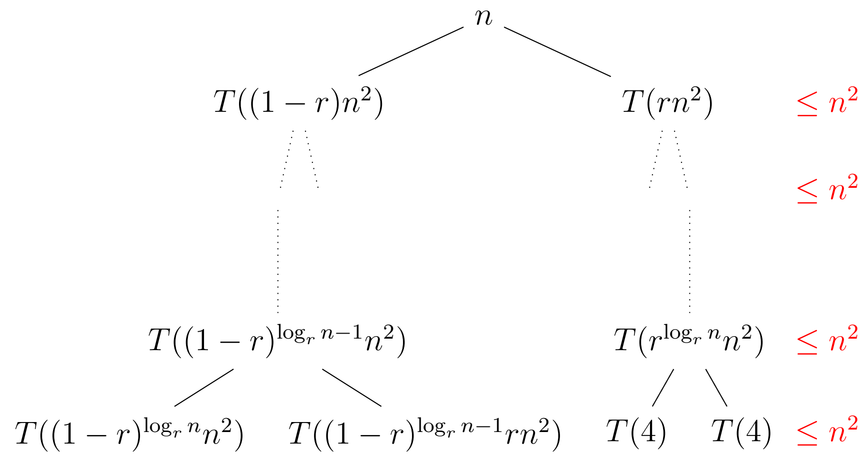
7. (a) **Collaborators: None.** We have already known that $T(n) = T(a) + T(b) + 2n$, where $\frac{1}{4}n \leq a, b \leq \frac{3}{4}n,\ a + b = n$. Let $a = rn, b = (1-r)n$, where $\frac{1}{4} \leq r \leq \frac{1}{2}$.

We can regard this recursive algorithm as a recursive tree, and if $r \neq 0.5$, this recursive will not be balanced at both sides *i.e.* it will have a lower bound and a upper bound. The recursive tree may be as below.




Note that the nodes at left side are still dividable. We deal with the lower bound at first. The height of the lower-bound layer *i.e.* the number of layers from root to the lower-bound leafs is $-\log_r n$, and the time complexity of each layer is $n$. Thus, we can say that $T(n) \in \Omega\ (2^{-\log_r n} + 2n\log_r n) \in \Omega\ (n \log n)$ (since $\sqrt{n} \leq 2^{-\log_r n} \leq n$). As for the upper bound, $T(n) \in O\ (2^{-\log_{1-r} n} + 2n\log_{1-r} n) \in O\ (n\log n)$. Therefore, we obtain that $T(n) \in \Theta(n \log n)$.

(b) **Collaborators: Study Group Members.**
Since in this problem, the splitting and merging procedures requires $n^2$ steps, the time complexity can be presented as $T(n) = T((rn)^2) + T(((1-r)n)^2) + n^2$, where $\frac{1}{4} \leq r \leq \frac{1}{2}$. We can divide this problem into a recursive tree as below.




Where all procedure in $L$-th layer requires $(r^2+(1-r)^2)^L n^2$ steps, which is less than $n^2$. Let $r^2+(1-r)^2 = \theta$, for the lower bound, $T(n) \in \Omega\ (2^{-\log_r n} + n^2(1 + \cdots + \theta^{\log_r n-1})) \in \Omega\ (n^2)$, and the upper bound $T(n) \in O\ (2^{-\log_{1-r} n} + n^2(1 + \cdots + \theta^{\log_{1-r} n-1})) \in O\ (n^2)$ (since $\theta$ is a coefficient less than 1). Therefore, we can say that the asymptotic running time of this problem is $T(n) \in \Theta(n^2)$.