# A generalization of De Vylder's formula and an application

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## 1 Abstract

In 1978 *De Vylder* proposed a method of approximating the ultimate ruin probability in the classical Cramèr-Lundberg risk model, by replacing it via a simpler process with exponential jumps. In this paper we consider a generalization : we approximate the underlying process by a Brownian motion

perturbed process with exponential jumps. We show numerically that, as expected, the obtained approximation works typically better, and illustrate this with important applications including the optimization of dividends and the computation of ruin probabilities under reinsurance.

# 2 Introduction

We recall first the Cramér-Lundberg risk model with an added Brownian perturbation [?, ?]

$$X_t = x + ct + \sigma B(t) - S_t, S_t = \sum_{i=1}^{N_\lambda(t)} C_i. \tag{1}$$

Here $x \geq 0$ is the initial surplus, $c \geq 0$ is the linear premium rate. The $C_i$'s, $i = 1, 2, ...$ are independent identically distributed (i.i.d) random variables with distribution $F(z) = F_C(z)$ representing nonnegative jumps arriving after independent exponentially distributed times with mean $1/\lambda$, and $N_\lambda(t)$ denotes the associated Poisson process counting the arrivals of claims on the interval $[0, t]$. Finally, $\sigma B(t), \sigma > 0$ is an independent Brownian perturbation. Ruin happens when, for the first time, a jump takes $X_t$ below 0.

**First passage theory** concerns the first passage times above and below, and the hitting time of a level $b$. For any process $X_t, t \geq 0$, these are defined by

$$\begin{aligned} T_{b,+} &= T^X_{b,+} = \inf\{t \geq 0 : X_t > b\}, \\ T_{a,-} &= T^X_{a,-} = \inf\{t \geq 0 : X_t < a\}, \\ T_{\{b\}} &= T^X_{\{b\}} = \inf\{t \geq 0 : X_t = b\}, \end{aligned} \tag{2}$$

with $\inf \emptyset = +\infty$. The upper script $X$ will be typically omitted, as well as the signs $+, -$, when they are clear from the context.

Risk theory revolved initially around evaluating and minimizing the probability of ruin. Insurance companies are also interested in maximizing company value. This lead to the study of optimal dividend policies. As suggested by De Finetti in the 1950's [?] – see also [?] – an interesting objective is that of maximizing the expected value of the sum of discounted future dividend payments until the time of ruin.

The most important class of dividend policies is that of a constant barrier at $b$, which modifies the surplus only when $X_t > b$, by a lump payment bringing the surplus at $b$, and then keep it there by Skorokhod reflection, until the next negative jump. In financial terms, in the absence of a Brownian component, this amounts to paying out all the income while at $b$. In case of Brownian perturbation, Skorokhod reflection means keeping the process above the barrier by minimal capital injections (whenever necessary), or below a barrier, by taking out dividends (if necessary) [?].

In presence of the barrier at $b$, the de Finetti objective (the expected value of the sum of discounted future dividend payments until ruin) has a simple expression [?] in terms of the so called "scale function" $W$ introduced by [?, ?] :

$$V^{b]}(x) = E_x^{b]}\left[\int_{[0,T_0^{b]}]} e^{-qt} d\Delta_t^b\right] = \begin{cases} \frac{W_q(x)}{W_q'(b)}, & x \le b \\ x - b + \frac{W_q(b)}{W_q'(b)}, & x > b \end{cases}, \tag{3}$$

where $T_0^{b]}$ is the time of ruin, $q$ denotes the discount rate, $\Delta_t^b$ the total local time at $b$ before time $t$, and $E^{b]}$ the law of the process reflected from above at $b$ and absorbed at 0 and below.

The scale function $W_q(x) : \mathbf{R} \to [0, \infty), q \ge 0$ is defined on the positive half-line by the Laplace transform

$$\widehat{W}_q(s) := \int_0^\infty \mathrm{e}^{-sx} W_q(x) dx = \frac{1}{\kappa(s) - q}, \quad \forall s > \Phi_q, \tag{4}$$

where the "symbol" $\kappa(s)$ (also called cumulant generating function) is defined in (9) in Section 3 where we provide the necessary background information, and $\Phi_q$ is the unique nonnegative root of the Cramér-Lundberg equation

$$\Phi_q := \sup\{s \ge 0 : \kappa(s) - q = 0\}, \quad q \ge 0. \tag{5}$$

The scale function $W_q(x)$ is continuous and increasing on $[0, \infty)$ [?], [?, Thm. VII.8], [?, Thm. 8.1]. It may have however many inflection points (such an example is depicted in Figure ??), and these play an important role in the optimization of dividends [?, ?, ?]. For convenience, $W_q(x)$ is extended to be 0 on $\mathbf{R}_-$. An important fact that will be exploited is that the Laplace transform of our function has a unique non-negative pole $\Phi_q$, see (4) and (5).

This paper aims at computing/approximating the scale function $W_q(x)$, using its moments. The techniques being used are classic : Padé approximation and Laguerre expansions. The order $(m, n)$ Padé approximation of a function $g(x)$ is a rational function in the form

$$R(x) = \frac{a_0 + a_1 x + a_2 x^2 + ... + a_m x^m}{1 + b_1 x + b_2 x^2 + ... + b_n x^n}$$

for which $R(0) = g(0), R'(0) = g'(0), R''(0) = g''(0), ..., R^{(m+n)}(0) = g^{(m+n)}(0)$. In the context of probability distributions, given a density function $f(x)$ and its Laplace transform $\widehat{f}(s)$, the inverse Laplace transform of the order $(m, m)$ Padé approximant of $\widehat{f}(s)$ provides a matrix exponential approximation of $f(x)$ that matches the first $2m$ moments of $f(x)$ (including $m_0$). In [**?**] this approach was used to approximate ruin probabilities. In this paper we develop the same approach to approximate the scale function $W_q(x)$ (Section **??**). An extension of the above idea is the so-called two-point Padé approximation which allows to match not only the moments of $W_q(x)$ but also the behavior of the function at 0, i.e., to match $W_q(0)$, $W_q'(0), ...$ (Section 5). For more details of this extension see [**?**] where ruin probabilities are approximated.

Let us draw attention now to several numeric challenges which were absent in the ruin probability problem.

1. **Optimizing dividends** starts by optimizing the so called "barrier function"

$$H_D(b) := \frac{1}{W_q'(b)}, \; b \geq 0, \tag{6}$$

and the optimal dividend policy is often simply a barrier strategy at its maximum. This is the case in particular when the barrier function $H_D(b)$ is differentiable with

$$H_D'(0) > 0 E q W_q''(0) < 0 \tag{7}$$

and has a unique local maximum $b^* > 0 \Longrightarrow W_q''(b^*) = 0$; then this $b^*$ yields the optimal dividend policy, and the optimal barrier function,

$$V(x) := sup_{b \geq 0} V^{b]}(x) = V^{b^*]}(x), \tag{8}$$

turns out to be the largest concave minorant of $W_q(x)$. [§]

§. Even when barrier strategies do not achieve the optimum, and multi-band policies must be used instead, constructing the solution must start by determining the global maximum of the barrier function [**?**, **?**, **?**].

2. **The challenge of multiple inflection points**. In the presence of several inflection points, however, the optimal policy is multiband [?, ?, ?, ?]. The first numerical examples of multiband policies were produced in [?, ?], with Erlang claims $Erl_{2,1}$. However, it was shown in [?] that multibands cannot occur when $W_q'(x)$ is increasing after its last global minimum $b^*$ (i.e., when no local minima are allowed after the global minimum).

   [?] further made the interesting observation that for Erlang claims $ER_{2,1}$ (which are non-monotone), multiband policies may occur for volatility parameters $\sigma$ smaller than a threshold value, but barrier policies (with non-concave value function!) will occur when $\sigma$ is large enough.

   Figure **??** displays the first derivative $W_q'(x)$, for $\sigma^2/2 \in \{\frac{1}{2}, 1, \frac{3}{2}, 2\}$. The last two values yield barrier policies with non-concave value function, due to the presence of an inflection point in the interior of the interval $[0, b^*]$.

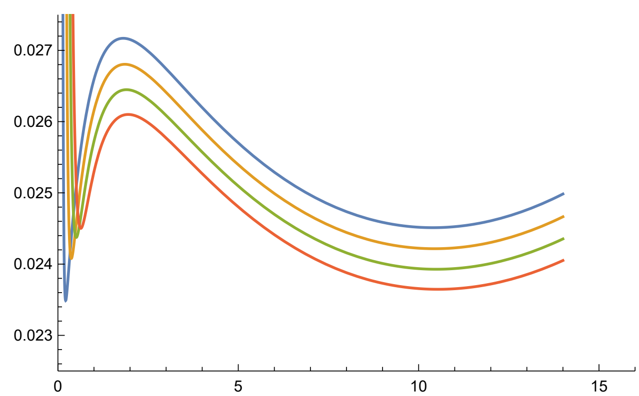


FIGURE 1 – Graphs of the Loeffen example for $\kappa(s) = \frac{\sigma^2 s^2}{2} + c\ s + \lambda\left(\frac{1}{(s+1)^2} - 1\right), c = \frac{107}{5}, \lambda = 10, q = \frac{1}{10}$, $\sigma^2/2 \in \{1/2, 1, 3/2, 2\}$.

Below we will investigate whether our approximations are precise enough to yield reasonable approximations for $W_q''(0)$ and the root(s) of $W_q''(\cdot)$.

**Special features**. While our methods consist essentially of Padé approximation and Laguerre-Tricomi-WeeksLaplace transform inversion, we found that exploiting the special features of our problem is useful. These are :

1. including known values of $W_q(0), W_q'(0)$ (using thus two-point Padé approximations).
2. shifting the approximations around $\Phi_q$ specified in (5), which transforms $W_q(x)$ into a survival probability. As a consequence, we end up using a certain judicious choice of the Laguerre exponential decay parameter (**??**), which is usually left to be tuned by the user in the Laguerre-Tricomi-Weeks method [**?**].

**Contents**. We briefly review classical ruin theory in Section 3. Padé approximations are provided in Section **??**, where we also spell out the simplest algorithm for the computation of the scale function. In Section 5, we derive low-order Padé and two-point Padé approximations of $W_q(x)$, reminiscent of the de Vylder approximation of the ruin probability. Some of these approximations appeared already in [**?**], where however the Padé method and the fact that they can be easily extended to higher orders is not mentioned. Section **??** offers our personal strategy for inverting Laplace transforms of interest in probability, in the presence of uncertainty. Subsection **??** implements the Laguerre-Tricomi-Weeks Laplace transform inversion with a certain judicious choice of the exponential decay parameter (**??**), which is believed to be new. Section **??** presents numeric experiments with mixed exponential claims. Section **??** presents experiments with Pareto claims ; since these have heavy tail and, consequently, finitely many moments, we apply "shifted" Padé approximation of the claim distribution. Section **??** includes a computer program required to obtain test cases with exact rational answers, using the Wiener-Hopf factorization ; of course, this is quite convenient for the initial testing of the precision of our algorithms. Finally, Section **??** reviews a more general version of the Laguerre-Tricomi-Weeks Laplace inversion method, which may be of interest for further experiments.

# 3 A short review of classical ruin theory

The process defined in (1) is a particular example of spectrally negative Lévy processes, with finite mean, which are defined by assuming instead of (1) that $S_t$ is a subordinator with $\sigma$-finite Lévy measure $\nu(dx)$ that integrates $x$, but having possibly infinite activity near the origin $\nu(0,\infty) = \infty$ [**?**] (for (1), the Lévy measure is given by $\nu(dx) = \lambda F(dx) \Longrightarrow \nu(0,\infty) = \lambda$). A spectrally negative Lévy process is characterized by its Lévy-Khintchine/Laplace exponent/cumulant generating function/symbol $\kappa(s)$ defined by $E_0\left[e^{sX(t)}\right] =$

$e^{t\kappa(s)}$, with $\kappa(s)$ of the particular form

$$\kappa(s) = cs + \int_0^\infty (e^{-sx} - 1)\nu(dx) + \frac{\sigma^2 s^2}{2} := s\left(c - \widehat{\overline{\nu}}(s) + \frac{\sigma^2}{2}s\right). \tag{9}$$

Some concepts of interest in classical risk theory are :

— *First passage times* below and above a level $a$ :

$$T_{a,-(+)} := \inf\{t > 0 : X(t) < (>)a\}.$$

— The first first passage quantity to be studied historically was the *eventual ruin probability* :

$$\Psi(x) := P[T_0 < \infty | X(0) = x]. \tag{10}$$

In order that the eventual ruin probability not be identically 1, the parameter

$$p := c - \lambda m_1 = \kappa'(0), \text{ where } m_1 = \int_0^\infty zF(dz),$$

which is called drift or profit rate, must be assumed positive.
The Laplace transform of the ruin probability is explicit, given by the so called Pollaczek-Khinchine formula, which states that the Laplace transform of $\overline{\Psi}(x) = 1 - \Psi(x)$ is :

$$\widehat{\overline{\Psi}}(s) = \int_0^\infty e^{-sx}\overline{\Psi}(x)dx = \frac{c - \lambda m_1}{\kappa(s)} = \frac{\kappa'(0)}{\kappa(s)}. \tag{11}$$

The roots with negative real part of the Cramér Lundberg equation

$$\kappa(s) = q, q \geq 0 \tag{12}$$

are important, when such roots exist. They will be denoted by $-\gamma_1, -\gamma_2, \cdots, -\gamma_N, N \geq 0$, and ordered by their absolute values $|\gamma_1| \leq |\gamma_2| \leq ..., \leq |\gamma_N|$. $\gamma_1 > 0$ is called the adjustment coefficient, and furnishes the Cramér-Lundberg asymptotic approximation

$$\Psi(u) \sim \frac{\kappa'(0)}{-\kappa'(-\gamma_1)} e^{-\gamma_1 u}.$$

# 4 De Vylder approximation

An approximation of the ruin probability, in the absence of the Gaussian component,is given explicitely in this section by the *De Vylder* approximation which is a Padé approximation of the *Pollaczek-Khinchin* transform for the classical **Cramèr-Lundberg** model.
In the case of exponentially distributed claim size, the main idea of De Vylder is based on replacing the risk reserve process $X(t)$ by another risk process $\tilde{X}(t)$ such that for some $n \geq 1$ the moments up to order $n$ of certain characteristics of $X(t)$ and $\tilde{X}(t)$ coincide.

$$X(t) - u = pt - \sum_{i=1}^{N_t}(C_i - EC_i) = \theta\lambda m_1 t - \sum_{i=1}^{N_t}(C_i - EC_i)$$

Where $p := c - \lambda m_1 = \lambda m_1 \theta > 0$ is the profit rate.

Furthermore, $\tilde{X}(t)$ is chosen in such a way that the ruin probability $\tilde{\Psi}(u)$ of $\tilde{U}(t)$ is easier to determine than $\Psi(u)$.

Let us designate by $\mathcal{K}_k(Y)$ the k-order cumulant of a random variable $Y$ , i.e. the coefficients of the Taylor expansion :

$$log(\mathbb{E}e^{uY}) = \sum_{k=0}^{\infty} \mathcal{K}_k(Y)\frac{u^k}{k!}$$

## 4.1 Proposition :

$a)$The k-order cumulant of a compound Poisson process $L(t)$ is of the form :

$$\mathcal{K}_k(L(t)) = \int_0^t \mathcal{K}_k[X(ds)]ds = \lambda t \int x^k f(x)dx, \ \forall k \geq 1.$$

$b)$

$$\mathcal{K}_1(X(t)) = pt, \ \mathcal{K}_2(X(t)) = t(\sigma^2 + \lambda \int x^k f(x)dx), \ \mathcal{K}_k(X(t)) = (-1)^k \lambda t \int x^k f(x)dx, \ \forall k \geq 3.$$

## 4.2 DeVylder Approximation :

The idea of the *De Vylder* approximation is to replace $X(t)$ by $\tilde{X}(t)$, where $\tilde{U}(t)$ has exponentially distributed claim size and :

$$\mathcal{K}_k(X(t)) = \mathcal{K}_k(\tilde{X}(t)) \tag{13}$$

.
Let $\tilde{\lambda}$, $\tilde{p}$, $\tilde{m}$ et $\tilde{\theta}$ the new parameters of $\tilde{U}(t)$, representing respectively the average rate of arrival claim, the profit rate, the loading security coefficient, and the average claim reimbursement amounts.
From *proposition 4.2* ,this implies that (10) holds if :

$$\begin{cases} p = \theta\lambda m_1 = \tilde{p} = \tilde{\theta}\tilde{\lambda}\tilde{m}_1 \\ \lambda m_2 = 2\tilde{\lambda}\tilde{m}^2 \\ \lambda m_3 = 6\tilde{m}^3 \end{cases} \tag{14}$$

A computation leads to :

$$\lambda = \frac{2\tilde{\lambda}\tilde{m}^2}{m_2}$$

$$\tilde{\lambda} = \frac{\lambda m_3}{6\tilde{m}^3} = \frac{\lambda m_3}{6(\frac{m_3}{3m_2})^3} = \frac{9m_2^3}{2m_3^2}\lambda$$

And

$$\tilde{\theta} = \frac{\lambda m_1}{\tilde{\lambda}\tilde{m}}\theta = \frac{2m_1 m_3}{3m_2^2}\theta$$

Now we use the first $2n$ ruin moments $\lambda_1 = \frac{\tilde{m}_1}{\theta},\ \lambda_2 = \frac{\tilde{m}_2}{2\theta} + \ (\frac{\tilde{m}^2}{\theta})^2, ..., \lambda_{2n-1}$. For $n = 1$, one may set up a system for the Padé $(0, 1)$ approximation in 0 , and from the Laplace transform of ruin probability in (4), we obtain :

$$\widehat{\Psi}_{DV}(s) = \frac{1}{s} - \ \frac{p}{s(p + \lambda m_2 s/2 - \lambda m_3 s^{2/6} + ...)}$$

$$= \frac{\lambda m_2/2 - \lambda m_3 s/6 + ...}{p + \lambda m_2 s/2 - \lambda m_3 s^{2/6} + ...} \approx \frac{a}{s+b}$$

$$\Leftrightarrow as(p + \lambda m_2 s/2 - \lambda m_3 s^{2/6} + ...)$$

$$\approx (s + b)(\lambda m_2 s/2 - \lambda m_3 s^2/6 + ...)$$

$$\Leftrightarrow ap = b\lambda m_2/2,\ \ am_2/2 = m_2/2 - \ bm_3/6$$

$$\Leftrightarrow a = \frac{3\lambda m_2^2}{3\lambda m_2^2 + 2pm_3},\ \ b = \frac{6pm_2}{3\lambda m_2^2 + 2pm_3}.$$

This leads to the **DeVylder**'s famous approximation.

$$\Psi_{DV}(u) \approx ae^{-bu} \tag{15}$$

Hence

$$\Psi_{DV}(u) = \frac{1}{1+\tilde{\theta}}\, exp\left(-\frac{u\tilde{\theta}\mu}{1+\tilde{\theta}}\right) \tag{16}$$

Where $\mu = \frac{3m_2}{m_3}$, $\tilde{\theta} = \frac{2pm_3}{3\lambda m_2^2} = \frac{2m_1m_3}{3m_2^2}\theta$ represent the exponential reimbursement rate and the relative safety factor in the associated process $\tilde{X}(t)$, respectively.

# 5 Two-point Padé approximations, with low order examples

One may obtain better results by incorporating into the Padé approximation the following initial values, that can be derived easily from the Laplace transform :

$$W_q(0) = \lim_{s\to\infty} s\widehat{W}_q(s) = \begin{cases} \frac{1}{c}, & \text{if } X \text{ is of bounded variation/Cramér-Lundberg} \\ 0, & \text{if } X \text{ is of unbounded variation} \end{cases} \tag{17}$$

$$W_q'(0) = \lim_{s\to\infty} s\left(\frac{s}{\kappa(s)-q} - W_q(0)\right) = \begin{cases} \frac{q+\nu(0,\infty)}{c^2}, & \text{if } X \text{ is of bounded variation} \\ \frac{2}{\sigma^2}, & \text{if } \sigma > 0, \\ \infty, & \text{if } \sigma = 0 \text{ and } \nu(0,\infty) = \infty \end{cases} \tag{18}$$

Furthermore, when the jump distribution has a density $f$, it holds that : ‖

$$\begin{aligned} W_q''(0_+) &= \lim_{s\to\infty} s\left(s\left(\frac{s}{\kappa(s)-q} - W_q(0)\right) - W_q'(0_+)\right) \\ &= \begin{cases} \frac{1}{c}\left((\frac{\lambda+q}{c})^2 - \frac{\lambda}{c}f(0)\right), & \text{if } X \text{ is of bounded variation} \\ -c(\frac{2}{\sigma^2})^2, & \text{if } \sigma > 0 \end{cases} . \end{aligned} \tag{19}$$

Further derivatives at 0 could be computed, but we stop at order 2, since $W_q''(0)$ already requires estimating $f_C(0)$, which is a rather delicate task starting from real data.

We will provide in Proposition 1 below a couple of two-point Padé approximations, when $n = 2$. Before that, it is worth recalling the case of exponential claims.

‖. This equation is important in establishing the nonnegativity of the optimal dividends barrier.

**Example 1.** ***The Cramér-Lundberg model with exponential jumps*** *Consider the Cramér-Lundberg model with exponential jump sizes with mean* $1/\mu$*, jump rate* $\lambda$*, premium rate* $c > 0$*, and Laplace exponent* $\kappa(s) = s\left(c - \frac{\lambda}{\mu+s}\right)$*, assuming* $\kappa'(0) = c - \frac{\lambda}{\mu} > 0$*. Let* $\gamma = \mu - \lambda/c$ *denote the adjustment coefficient, and let* $\rho = \frac{\lambda}{c\mu}$*. Solving* $\kappa(s) - q = 0 \Leftrightarrow cs^2 + s(c\mu - \lambda - q) - q\mu = 0$ *for* $s$ *yields two distinct solutions* $\gamma_2 \le 0 \le \gamma_1 = \Phi_q$ *given by*

$$\gamma_1 = \frac{1}{2c}\left(-(\mu c - \lambda - q) + \sqrt{(\mu c - \lambda - q)^2 + 4\mu q c}\right),$$
$$\gamma_2 = \frac{1}{2c}\left(-(\mu c - \lambda - q) - \sqrt{(\mu c - \lambda - q)^2 + 4\mu q c}\right).$$

*The* $W$ *scale function is :*

$$W_q(x) = \frac{A_1 e^{\gamma_1 x} - A_2 e^{\gamma_2 x}}{c(\gamma_1 - \gamma_2)} \Leftrightarrow \widehat{W}_q(s) = \frac{s + \mu}{cs^2 + s(c\mu - \lambda - q) - q\mu} \tag{20}$$

*where* $A_1 = \mu + \gamma_1, A_2 = \mu + \gamma_2$*.*

*Furthermore, it is well-known and easy to check that the function* $W_q'(x) = H_D(x)^{-1}$ *is in this case unimodal with global minimum at*

$$b^* = \frac{1}{\gamma_1 - \gamma_2}\begin{cases}\log\frac{(\gamma_2)^2 A_2}{(\gamma_1)^2 A_1} = \log\frac{(\gamma_2)^2(\mu+\gamma_2)}{(\gamma_1)^2(\mu+\gamma_1)} & \text{if } W_q''(0) < 0 \Leftrightarrow (q+\lambda)^2 - c\lambda\mu < 0 \\ 0 & \text{if } W_q''(0) \ge 0 \Leftrightarrow (q+\lambda)^2 - c\lambda\mu \ge 0\end{cases}, \tag{21}$$

*since* $W_q''(0) = \frac{(\gamma_1)^2(\mu+\gamma_1) - (\gamma_2)^2(\mu+\gamma_2)}{c(\gamma_1-\gamma_2)} \sim (q+\lambda)^2 - c\lambda\mu$ *and that the optimal strategy for the de Finetti problem is the barrier strategy at level* $b^*$ *[**?**].*

**Proposition 1.** *1. To secure both the values of* $W_q(0)$ *and* $W_q'(0)$*, take into account* (17) *and* (18)*, i.e. use the Padé approximation*

$$\widehat{W}_q(s) \sim \frac{\sum_{i=0}^{n-1} a_i s^i}{cs^n + \sum_{i=0}^{n-1} b_i s^i}, a_{n-1} = 1, b_{n-1} = ca_{n-2} - \lambda - q.$$

*This yields*

$$\widehat{W}_q(s) \sim \frac{\frac{1}{m_1} + s}{cs^2 + s\left(\frac{c}{m_1} - \lambda - q\right) - \frac{q}{m_1}}. \tag{22}$$

*In view of* (20)*, this yields the same result as approximating the claims by exponential claims, with* $\mu = \frac{1}{m_1}$*.*

2. *To ensure $W_q(0) = \frac{1}{c}$, we must only impose the behavior specified in (17), i.e. use the Padé approximation*

$$\widehat{W}_q(s) \sim \frac{\sum_{i=0}^{n-1} a_i s^i}{cs^n + \sum_{i=0}^{n-1} b_i s^i}, a_{n-1} = 1.$$

*For $n = 2$, this yields*

$$\widehat{W}_q(s) \sim \frac{\frac{2m_1}{m_2} + s}{cs^2 + \frac{s\left(2cm_1 - 2\lambda m_1^2 - m_2 q\right)}{m_2} - \frac{2m_1 q}{m_2}} = \frac{\frac{1}{\widetilde{m}_1} + s}{cs^2 + s\left(\frac{c}{\widetilde{m}_1} - \lambda\frac{m_1}{\widetilde{m}_1} - q\right) - \frac{q}{\widetilde{m}_1}}, \tag{23}$$

*where we denoted by $\widetilde{m}_1 = \frac{m_2}{2m_1}$ the first moment of the excess density $f_e(x)$. For exponential claims this coincides with (22) (since $f_e(x) = f(x)$). This is the DeVylder B) method [?, (5.6-5.7)], derived therein by assuming exponential claims, with $\mu = \frac{2m_1}{m_2}$, and simultaneously modifying $\lambda$ to fit the first two moments of the risk process.*

3. *When the pure Padé approximation is applied, the first step yields*

$$\widehat{W}_q(s) \sim \frac{s + \frac{3m_2}{m_3}}{s^2\left(c + \lambda(\frac{3m_2^2}{2m_3} - m_1)\right) + s\left(c\frac{3m_2}{m_3} - \frac{3m_1 m_2}{m_3}\lambda - q\right) - \frac{3m_2}{m_3}q} \tag{24}$$

$$= \frac{s + \frac{1}{\widehat{m}_3}}{s^2\left(c + \lambda(\frac{\widehat{m}_2}{\widehat{m}_3} - 1)\right) + s\left(c\frac{1}{\widehat{m}_3} - \frac{m_1}{\widehat{m}_3}\lambda - q\right) - \frac{1}{\widehat{m}_3}q}, \tag{25}$$

*where $\widehat{m}_i = \frac{m_i}{i\, m_{i-1}}$ is a so-called "normalized moment" [?].*
*This is the DeVylder A) method [?, (5.2-5.4)], derived therein by assuming exponential claims, with $\mu = \frac{3m_2}{m_3}$, and simultaneously modifying both $\lambda$ and $c$ to fit the first three moments of the risk process.*

**Lemma 1.** *In the case of exponential claims, the three approximations given above are exact.*

**Proof:** It suffices to check that for exponential claims all the normalized moments are equal to $m_1 = \mu^{-1}$. □

In particular, the optimal barrier for exponential claims obtained by the explicit formula (21) is the same. For example, $\mu = 2/5, \lambda = 9/10, c = 1, q = 1/10$ yields

$$W_q(x) \sim 0.6529892.71828^{0.0659646x} - 0.1529892.71828^{-1.51596x}$$

and $b^* = 3.04576$.

The Laplace transform of $\psi(x)$ and $\overline{\psi}(u)$ are respectively given by the so called **Pollaczek-Khinchine** :

$$\widehat{\psi}(s) = \int_0^\infty e^{-sx}\psi(x)dx = \frac{1}{s} - \frac{1-\rho}{s - \frac{\lambda}{c}(1-\hat{f}(s))} = \frac{1}{s}\left(1 - \frac{1-\rho}{1-\rho\hat{f}_e(s)}\right). = \rho\frac{\widehat{\overline{F}}_e(s)}{1-\rho\hat{f}_e(s)}$$

$$\Leftrightarrow \widehat{\overline{\psi}}(s) = \int_0^\infty e^{-sx}\overline{\psi}(x)dx = \frac{c(1-\rho)}{k(s)} = \frac{1-\rho}{s(1-\rho\hat{f}_e(s))} \tag{26}$$

Where $\hat{f}(s) = \int_0^\infty e^{-sx}dF(x)$, $\overline{F}(x) = 1 - F(x)$
$f_e(x) = \overline{F}(x)/m_1$ : *which denotes the stationary excess density of the claims ;*
*(*$\int_0^\infty \overline{F}(x)dx = m_1$*)*
$\hat{f}_e(s) = \int_0^\infty e^{-sx}f_e(x)dx = (1-\hat{f}(s))/(m_1 s)$, $\widehat{\overline{F}}_e(s) = (1-\hat{f}_e(s))/s$.

## Proof :

Using the Laplace transform of the ruin probability that satisfy the integro-differential equation of [IfM3, eq.(13.9)] which gives :

$$\hat{\psi}(s) = \frac{c\psi(0) - \lambda\widehat{\overline{F}}(s)}{cs(1-\lambda\widehat{\overline{F}}(s))} \tag{27}$$

Now by substituting $s = 0$ in the denominator we can clearly see that the denominator explodes into zero which means that we have a singularity at this point. We assume that our probability of ruin declines fast enough to have a finite integral .Then in order to have $\widehat{\psi}(0) = \int_0^\infty \psi(x)dx < \infty$ it is necessary to check that the numerator admits also a singularity in zero.
To achieve this we'll need a results that's easy to demonstrate based on integration by parts, we obtain the Laplace transformation of the survival function in terms of the density transformation mentioned above, we get :

$$\widehat{\overline{F}}(s) = \frac{1 - \hat{f}(s)}{s}, \qquad \widehat{\overline{F}}(0) = \int_0^\infty \overline{F}(x)dx = m_1 \tag{28}$$

As the denominator becomes 0 for $s = 0$ and $\widehat{\psi}(0) < \infty$ then :

$$\psi(0) = \frac{\lambda \int_0^\infty \overline{F}(x)dx}{c} = \frac{\lambda m_1}{c} := \rho. \tag{29}$$

For the survival probability, we use $\overline{\psi}(x) = 1 - \psi(x)$. Thus we obtain

$$\widehat{\overline{\psi}}(s) = \frac{1}{s} - \widehat{\psi}(s)., \qquad \overline{\psi}(0) = 1 - \frac{\lambda m_1}{c} = 1 - \rho. \tag{30}$$

Hence the result.

## 5.1 Corollary :

The ruin probability $\psi(u)$ is 1 for all u when $\rho \geq 1$ , and $< 1$ for all u when $\rho < 1$.

# 6 The Pollaczek-Khinchine formula

We shall here exploit the decomposition of the claims process as sum of ladder heights. Since our model is a *compound Poisson process*, then the ladder heights are i.i.d.
Let us recall the Laplace transform defined in (4).
We have $\frac{1}{1-\rho\hat{f}_e(s)}$ is expandable in a geometric series, and by inverting the Laplace transform we get :

$$\overline{\psi}(u) = \sum_{n=0}^{\infty}(1-\rho)\ \rho^n\ F_e(u)^{\star n} \ \Leftrightarrow \psi(u) = \sum_{n=1}^{\infty}(1-\rho)\ \rho^n\ \overline{F}_e(u)^{\star n} \tag{31}$$

Reciprocally, from (9), we calculate its Laplace transform, the convolution becomes product, and easily we find (4).
To intepret this formula, we would need the following definition.

## 6.1 Definition :

The variable $Y = X_\tau \mid \{\tau < \infty,\ X_0 = u\} > 0$ is the severity of ruin.

Let $S_{\tau_-}$ denotes the first (weak) descending ladder heights past u (defined on $\{\tau_- < \infty\}$ only).

## 6.2 Geometric Interpretation :

Let's take the graph of **Cramèr-Lundberg** model with initial capital u.
Let's assume that on our kth descending ladder heights ruin occurs.
The ruin severity density is denoted by $f_e$ as defined above.
We try to decompose the initial reserve of the Cramèr-Lundberg model as well as the severity of ruin as the sum of the kth ladder heights.
We will call a walk without jumps strictly greater than 1 up/down continuous up/down. So when we take a walk that goes up, the walk will have a finite number of ladder heights down, so the number of ladder heights down is finite, we will denote it by **N**. We assume that **N** is a geometric varible with parameter $\rho$.

Thus, for $N = 0$

$$\overline{\psi}(u) = 1 - \rho.$$

and for $N = 1$

$$\overline{\psi}(u) = (1 - \rho)\ \rho\ \mathbb{P}(S_{\tau_-} < \ u)$$

Further, The law of the ladder heights is the law of equilibrium $F_e$. Therefore,

$$\overline{\psi}(u) = (1 - \rho)\ \rho^n\ F_e(u)^{\star u}$$

So the ruin occurs if and only if the total sum of the heights exceeds u.