Optimizing dividends and limited capital injections. Practical approximations for the Cramér-Lundberg process

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Abstract

The recent papers [?, AGLW20] investigated the control problem of optimizing dividends when capital injections and bankruptcy are allowed as well. The first paper works under the spectrally negative Lévy model; the second works under the Cramér-Lundberg model with exponential jumps, where the results are considerably more explicit. The current paper illustrates the fact that quite reasonable approximations of the first problem may be obtained using the exponential particular case. We start by experimenting here with de Vylder type approximations, which amount essentially to replacing our process by one with exponential jumps and cleverly crafted parameters based on the first three moments of the claims. The winner however is a new approximation specific to our problem, which consists in plugging into the exponential objective function (21) the exact values of both the scale functions and the survival and mean functions of our non-exponential examples.

Contents

1 Introduction

Motivation. The recent papers [?, AGLW20] investigated the important control problem of optimizing dividends and capital injections for processes with jumps, when bankruptcy is allowed as well. The first paper works under the spectrally negative Lévy model; the second works under the Cramér-Lundberg model with exponential jumps. The results are considerably more explicit in the latter case, and our paper shows that they provide quite reasonable approximations to the case of matrix exponential jumps and general jumps (as the de Vylder approximation provides for the ruin problem). We focus here on the case of matrix exponential jumps jumps despite the fact that non matrix exponential jumps jumps yield similar numeric results, for two reasons. One is in order to highlight several equations which are similar to their exponential versions, and which may at their turn be used to produce more accurate approximations, and also since this class is known to be dense in the class of general nonnegative jumps (even error bounds are available for completely monotone jumps [?]).

The results of [?, AGLW20] may be divided in four parts:

1. Compute the value of "bounded buffer policies", which consist in allowing capital injections smaller than a given a and declaring bankruptcy at the first time when the size of the overshoot below 0 exceeds a, and pay dividends when the reserve reaches an upper barrier b. These will briefly be described as $(-a, 0, b)$ policies from now on.

   The first step is carried out for the spectrally negative Lévy processesmodel only in [?]; in [AGLW20] it is only carried out for exponential claims. However, as usual, this can be easily extended to the matrix exponential case, and this extension is spelled out below.

2. Equations determining candidates for the optimal a^*, b^* are obtained by differentiating the objective (which is expressed in terms of the scale functions W_q, Z_q).

3. The optimal pair (a^*, b^*) is determined using second order derivatives.

4. an HJB equation associated to the stochastic control problem is formulated and optimality of the $(-a^*, 0, b^*)$ policy is established.

Note that the last three steps are quite non-trivial and are achieved by different methods in [?] and [AGLW20]).

The object of this paper is to investigate experimentally the accuracy of exponential approximations in the case of general claims.

REMARK 1. *The objective may be optimized numerically, for each instance of the parameters, using the first step only. To facilitate this, we offer below in (??) a simplification of [?]'s formula, valid for Cramér-Lundberg models with matrix exponential jumps. Interestingly, this formula may be derived via steps analogous to the exponential case, after introducing a vector Z_q scale function.*

Beyond matrix exponential jumps, one could resort to approximation by processes with matrix exponential jumps – see for example [?].

To make life even easier, one can approximate by processes with exponential claims, and resort directly to formulas in [AGLW20]. The results below show that the (un-optimal) $(\widetilde{a}, \widetilde{b})$ obtained this way lead to small relative errors with respect to the (exact) numeric optimization of the objective.

It turns out that the value functions of exponential approximations show considerable improvements with respect to the previous exact results for the de Finetti and Shreve, Lehoczky and Gaver solutions.

Our conclusion is that from a practical point of view, exponential approximations are typically sufficient in this problem.

Our exponential approximation is very similar in spirit with the de Vylder-type approximations, which consist essentially in replacing the inverse μ^{-1} of an exponential rate μ in the problem considered respectively by m_1, by $\frac{m_2}{2m_1}$, or by $\frac{m_3}{3m_2}$ (a more complete description of these formulas and proofs are included in section ??).

The model. [AGLW20] work under the Cramér-Lundberg model

$$X_t = x + ct - \sum_{i=1}^{N_t} C_i \; c \geq 0, \tag{1}$$

where $(C_i)_{i \geq 1}$ is a family of i.i.d.r.v. whose distribution, density and moments are denoted respectively by $F, f, m_k, k \leq K \geq 1$, and N is an independent Poisson process of intensity $\lambda > 0$. The space is then endowed with the natural right-continuous, completed filtration $\mathbb{F}$ satisfying the usual assumptions of right-continuity and completeness.

For further details on the formulation of the dividends and capital injections problem see Section ??.

Computing the value function was considerably simplified by the use of the first passage recipes available for spectrally negative Lévy processes [?, ?, ?], which are built around two ingredients: the W_q and Z_q scale functions, defined respectively for $x \geq 0, q \geq 0$ as:

1. the inverse Laplace transform of $\frac{1}{\kappa(s)-q}$, where $\kappa(s)$ is the Laplace exponent (which characterizes a Lévy process) and

2. $Z_q(x) = 1 + q\int_0^x W_q(y)dy$

– see the papers [?, ?, ?] for the first appearance of these functions.

A further important role in the results below will be played by the **convolution function**

$$C_q(x) = \lambda \int_0^x W_q(x-y)\overline{F}(dy) = cW_q(x) - Z_q(x) \tag{2}$$

where the equality holds for an arbitrary survival function of claims $\overline{F}$ by the harmonicity of Z_q, and by the $Z_{q,1}$ function, defined by $Z_{q,1}(x) = \int_0^x Z_q(y)dy - (c-\lambda m_1)\int_0^x W_q(y)dy$, which intervenes in capital injections problems.

We highlight now in figure 1 the fact that for exponential jumps, the limited capital injections objective function J_0 given by (**??**) for arbitrary b but optimal $a = s(b)$ (via a complicated formula) improves the value function with respect to de Finetti and Shreve, Lehoczky and Gaver, for any b.

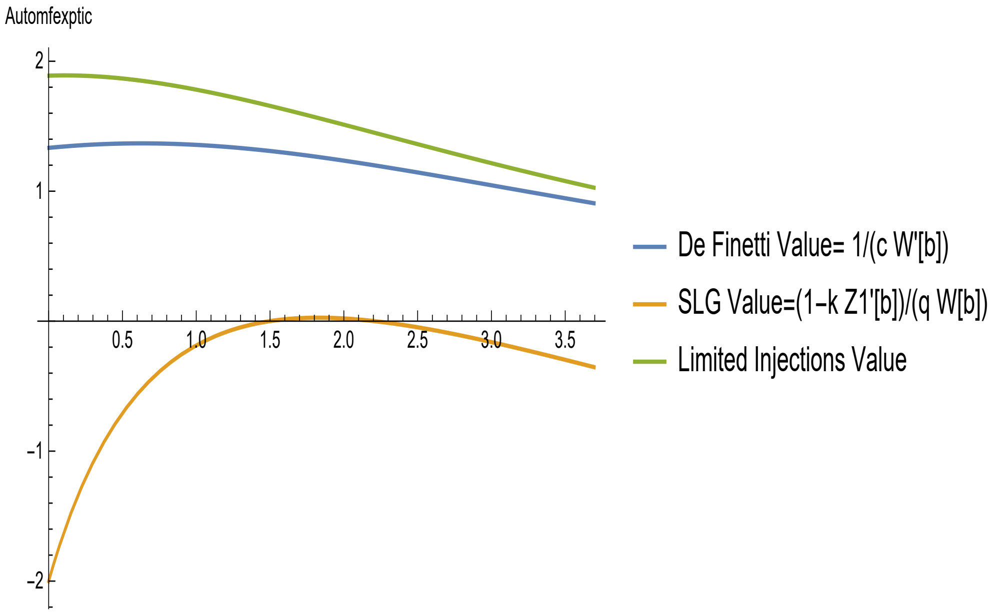


Figure 1: The value function J_0 given by (**??**), for arbitrary b but optimal a. The inequality observed is a consequence of the properties of the Lambert function. The improvement with respect to de Finetti is considerable, of 0.382292% (the SLG approach is not competitive in this case). Note also that the optimal barrier $b = 0.109023$ is smaller than the de Finetti and SLG optima of $0.626672, 1.82726$.

As the formulas in [AGLW20] are entirely expressed in terms of the scale functions, we may apply them directly to non-exponential cases, as ad-hoc approximations; this is clearly in the spirit, if not in the letter of the de Vylder approximation.

Recall that the philosophy of the de Vylder approximation is to approximate a Cramèr-Lundberg process by a simpler process with exponential jumps. The efficiency of the de Vylder approximation for approximating ruin probabilities is well documented [?]. The natural question of whether this type of techniques may work for other objectives, like for example for optimizing dividends and/or reinsurance was already discussed in [?, ?, ?, ?, ?]. In this paper, following on previous works [?, ?, ?], we draw first the attention to the fact that we have not one, but many types of de Vylder-type approximations, and we compare some of them numerically on simple applications like determining the optimal dividends barrier, which requires approximating the scale function $W_q(x)$. The best approximation in our experiments turn out to be the classic de Vylder approximation, as well as that obtained by a Padé approximation of the Laplace transform

$\widehat{W}_q(s)$, while fixing the value $W_q(0) = \frac{1}{c}$, which we will call Renyi approximation. These two approximation yield quite reasonable answers for completely monotone claims. In the opposite case however– see for example Figures ??, ??, our completely monotone approximation cannot fully reproduce functions like $W_q'(x), W_q''(x)$, when they exhibits oscillations. In such cases, higher order generalizations should be used, and an investigation of these will be the object of a future paper.

Contents and contributions. Section 2 reviews, for completeness, to the de Vylder approximation-type approximations. Section 2.1 recalls, for warm-up, some of the oldest exponential approximations for ruin probabilities. Section 2.2 recalls in Proposition 1, following [?, ?] three approximations of the scale function $W_q(x)$ [§] , obtained by approximating its Laplace transform. Section 3 examines numerically the performance of de Vylder approximation-type approximations on some chosen characteristics (Φ_q, and the optimal de Finetti barrier), on some cases with matrix-exponential claims. Our observation here is that the classic de Vylder approximation typically wins, just like in the ruin problem, but there are also exceptions where Renyi wins.

Note that the previous two sections do not involve capital injections; they were included to offer the reader a glimpse into the historical roots of the exponential approximation idea.

Section 4 revisits the Equity Cost Induced Dichotomy of [AGLW20], while taking also advantage of properties of the Lambert-W function, which were not exploited in that paper. The final result is summarized in Section ??, but we offer more details (which may be skipped) in Section 4.1. Section 4.1 is useful however for understanding Section ??, where we provide a new (straightforward) extension to the matrix exponential jumps case.

Section ?? examines numerically, on the same matrix exponential examples considered in section 3, the performance of our exponential approximations with respect to the exact optimum, and also the improvement with respect to the value of the de Finetti and SLG approaches. Using matrix-exponential claims is practical both since here we know the exact solution, and since the InverseLaplaceTransform command in symbolic algebra systems provides the scale functions as functions on $\mathbb{R}_+$.

Section ?? gives some idea of the programs we used, which are available upon request

Finally, section ?? recalls the derivation of some de Vylder-type approximations, including the original derivation of the Renyi and de Vylder approximations using process cumulants, in section ??.

To prevent this paper form becoming too long, we decided to postpone for the future the investigation of the performance of the approximation (21) on some non-matrix exponential favorites of the statistical modeling like gamma (including χ^2), Pareto, Weibull, Mittag-Leffler, and beta. In that case, our approximation must be tested against the exact formula of [?].

2 Which exponential approximation?

2.1 Some de Vylder-type approximations for the ruin probability

In the simplest case of exponential jumps and $\sigma = 0$, the formula for the ruin probability is

$$\Psi(x) = P_x[\exists t \geq 0 : X_t < 0] = \frac{1}{1+\theta}\ exp\left(-\frac{x\theta m_1^{-1}}{1+\theta}\right), \tag{3}$$

where $\theta = \frac{c-\lambda m_1}{\lambda m_1}$ is the loading coefficient. This formula may serve as the **simplest approximation** for processes with general claims, simply by plugging the mean of the claims.

The Renyi and De Vylder's approximations. More sophisticated is the Renyi exponential approximation

$$\Psi_R(x) = \frac{1}{1+\theta}\ exp\left(-\frac{x\theta m_R^{-1}}{1+\theta}\right), m_R = \frac{m_2}{2m_1};$$

This formula can be obtained as a two point Padé approximation of the Laplace transform, which conserves also the value $\Psi(0) = (1+\theta)^{-1}$ [?]. It may be also viewed heuristically as an exponential

[§] essentially, this is the "dividend function with fixed barrier", which had been also extensively studied in previous literature before the introduction of $W_q(x)$

approximation of the **excess/severity density**

$$f_e(x) = \frac{\overline{F}(x)}{m_1} = \frac{1-F(x)}{m_1}.$$

Heuristically, it makes more sense to make $f_e(x)$ exponential instead of the original density $f(x)$, since $f_e(x)$ is a monotone function (and also an important component of the Pollaczek-Khinchine formula for the Laplace transform $\widehat{\Psi}(s) = \int_0^\infty e^{-sx} F(dx)$) – see [?, ?].

More moments may be put to work in the de Vylder approximation

$$\Psi_{DV}(x) = \frac{1}{1+\widetilde{\theta}}\, exp\left(-\frac{x\widetilde{\theta}\widetilde{m}^{-1}}{1+\widetilde{\theta}}\right),\ \widetilde{m} = \frac{m_3}{3m_2} = \widehat{m}_3,\ \widetilde{\theta} = \frac{2m_1m_3}{3m_2^2}\theta = \frac{\widehat{m}_3}{\widehat{m}_2}\theta. \tag{4}$$

Interestingly, the result may be expressed in terms of the so-called "normalized moments"

$$\widehat{m}_i = \frac{m_i}{i\ m_{i-1}} \tag{5}$$

introduced in [?].

The de Vylder approximation parameters above may be obtained either from

1. a Padé approximation of the Laplace transform of the ruin probability [?], or
2. by equating the first cumulants of our process to those of a process with exponentially distributed claim sizes of mean $\widetilde{m}$, and modified λ, c [?]; (however $p = c - \lambda m_1$ must be conserved, since this is the first cumulant). This results – see (**??**) – in the modified parameters

   $$\widetilde{m} = \widehat{m}_3,\ \widetilde{\lambda} = \frac{9m_2^3}{2m_3^2}\lambda = \frac{m_2/2}{\widehat{m}_3^2}\lambda,\ \ \widetilde{c} = c - \lambda m_1 + \widetilde{\lambda}\widetilde{m} = c - \lambda m_1 + \lambda\frac{m_2^2}{2m_3}$$

   where we gave also expressions involving the less standard normalized moments $\widehat{m}_k$ of [?].

The first derivation via Padé shows that higher order approximations may be easily obtained as well (but they might not be admissible, due to negative values).

The second derivation of De Vylder is a **process approximation** (i.e., independent of the problem considered); as such, it may be applied to other functionals of interest besides ruin probabilities($W_q(x)$, dividend barriers, etc), simply by plugging the modified parameters in the exact formula for the ruin probability of the simpler process.

2.2 Padé approximations of the scale functions $W_q, Z_q, q > 0$

Our goal in this section is to investigate whether the approximations described previously are precise enough to yield reasonable estimates for quantities important in control like $W_q''(0)$ and the global minimum of $W_q'(\cdot)$, which yields (typically) the de Finetti optimal dividends barrier b_{DeF}. Note that other performance measures like the dual optimal dividends barrier, and the reflected optimal dividends barrier could be investigated as well.

Most of our approximations may be obtained by plugging appropriate values in the exact formula (6) for W_q, for the Cramèr-Lundberg process with exponential claims. We will call them **process approximations**, and we have encountered already three:

1. the naive,
2. the Renyi, and
3. the de Vylder exponential approximating processes.

They yield each an approximation, simply by plugging in the exact exponential formula (**??**) with $\sigma = 0$ the appropriate parameters.

EXAMPLE 1. The Cramér-Lundberg model with exponential jumps *Consider the Cramér-Lundberg model with exponential jump sizes with mean $1/\mu$, jump rate λ, premium rate $c > 0$, and Laplace exponent $\kappa(s) = s\left(c - \frac{\lambda}{\mu+s}\right)$, assuming $\kappa'(0) = c - \frac{\lambda}{\mu} > 0$. Let $\gamma = \mu - \lambda/c$ denote the adjustment coefficient, and let $r = \frac{\lambda}{c\mu}$. Solving $\kappa(s) - q = 0 \Leftrightarrow cs^2 + s(c\mu - \lambda - q) - q\mu = 0$ for s yields two distinct solutions $\gamma_2 \le 0 \le \gamma_1 = \Phi_q$ given by*

$$\begin{aligned}\gamma_1 =& \frac{1}{2c}\left(-(\mu c - \lambda - q) + \sqrt{(\mu c - \lambda - q)^2 + 4\mu q c}\right),\\ \gamma_2 =& \frac{1}{2c}\left(-(\mu c - \lambda - q) - \sqrt{(\mu c - \lambda - q)^2 + 4\mu q c}\right).\end{aligned}$$

The W scale function is:

$$W_q(x) = \frac{A_1 e^{\gamma_1 x} - A_2 e^{\gamma_2 x}}{c(\gamma_1 - \gamma_2)} \Leftrightarrow \widehat{W}_q(s) = \frac{s+\mu}{cs^2 + s(c\mu - \lambda - q) - q\mu}, \tag{6}$$

where $A_1 = \mu + \gamma_1, A_2 = \mu + \gamma_2$.

Furthermore, it is well-known and easy to check that the function $W_q'(x)$ is in this case unimodal with global minimum at

$$b_{DeF} = \frac{1}{\gamma_1 - \gamma_2}\begin{cases}\log\frac{(\gamma_2)^2 A_2}{(\gamma_1)^2 A_1} = \log\frac{(\gamma_2)^2(\mu+\gamma_2)}{(\gamma_1)^2(\mu+\gamma_1)} & \text{if } W_q''(0) < 0 \Leftrightarrow (q+\lambda)^2 - c\lambda\mu < 0\\ 0 & \text{if } W_q''(0) \ge 0 \Leftrightarrow (q+\lambda)^2 - c\lambda\mu \ge 0\end{cases}, \tag{7}$$

since $W_q''(0) = \frac{(\gamma_1)^2(\mu+\gamma_1) - (\gamma_2)^2(\mu+\gamma_2)}{c(\gamma_1-\gamma_2)} = \frac{(q+\lambda)^2 - c\lambda\mu}{c^3}$ and that the optimal strategy for the de Finetti problem is the barrier strategy at level b_{DeF} [?].

We may also attempt to use Padé approximations, or two-point Padé approximations of the Laplace transform of W_q, which incorporate into the Padé approximation the following initial values (these can be derived easily via the initial value theorem, from the Pollaczek-Khinchine Laplace transform):

$$W_q(0) = \lim_{s\to\infty} s\widehat{W}_q(s) = \frac{1}{c}, \tag{8}$$

$$W_q'(0) = \lim_{s\to\infty} s\left(\frac{s}{\kappa(s) - q} - W_q(0)\right) = \frac{q+\lambda}{c^2}. \tag{9}$$

Furthermore, when the jump distribution has a density f, it holds that : [‖]

$$W_q''(0_+) = \lim_{s\to\infty} s\left(s\left(\frac{s}{\kappa(s)-q} - W_q(0)\right) - W_q'(0_+)\right) = \frac{1}{c}\Big(\big(\frac{\lambda+q}{c}\big)^2 - \frac{\lambda}{c}f(0)\Big). \tag{10}$$

Further derivatives at 0 could be computed, but we stop at order 2, since $W_q''(0)$ already requires estimating $f_C(0)$, which is a rather delicate task starting from real data.

We recall below three types of two-point Padé approximations [?, Prop. 1], and particularize them to the case when the denominator degree is $n = 2$ (which are studied further below). Note that the first two are of the type we saw in Example 1.

PROPOSITION 1. *[?, Prop. 1]* **Three matrix exponential approximations for the scale function.**

1. *To secure both the values of $W_q(0)$ and $W_q'(0)$, take into account (8) and (9), i.e. use the Padé approximation*

$$\widehat{W}_q(s) \sim \frac{\sum_{i=0}^{n-1} a_i s^i}{cs^n + \sum_{i=0}^{n-1} b_i s^i}, a_{n-1} = 1, b_{n-1} = ca_{n-2} - \lambda - q.$$

This yields the naive approximation $\mu \to \frac{1}{m_1}$

$$\widehat{W}_q(s) \sim \frac{\frac{1}{m_1} + s}{cs^2 + s\left(\frac{c}{m_1} - \lambda - q\right) - \frac{q}{m_1}}. \tag{11}$$

[‖]This equation is important in establishing the nonnegativity of the optimal dividends barrier.

2. *To ensure $W_q(0) = \frac{1}{c}$, we must only impose the behavior specified in* (8)*, i.e. use the Padé approximation*

$$\widehat{W}_q(s) \sim \frac{\sum_{i=0}^{n-1} a_i s^i}{cs^n + \sum_{i=0}^{n-1} b_i s^i}, a_{n-1} = 1.$$

For $n = 2$, this yields the Renyi approximation $\mu \to \frac{1}{m_R}, \lambda- > \lambda \frac{m_1}{m_R}$,

$$\widehat{W}_q(s) \sim \frac{\frac{2m_1}{m_2} + s}{cs^2 + \frac{s\left(2cm_1 - 2\lambda m_1^2 - m_2 q\right)}{m_2} - \frac{2m_1 q}{m_2}} = \frac{\frac{1}{m_R} + s}{cs^2 + s\left(\frac{c}{m_R} - \lambda\frac{m_1}{m_R} - q\right) - \frac{q}{m_R}}, \tag{12}$$

where $m_R = \frac{m_2}{2m_1}$ is the first moment of the excess density $f_e(x)$. This is called DeVylder B) method in [?, (5.6-5.7)], and is the well-known result of fitting the first two cumulants of the risk process. Note that it equals the scale function of a process with exponential claims of rate m_R and with λ modified to $\lambda_R = \lambda \frac{m_1}{m_R}$, and that, since c is unchanged, the latter equation is equivalent to the conservation of $\rho = \frac{\lambda m_1}{c}$, and to the conservation of θ.

3. *The pure Padé approximation*

$$\widehat{W}_q(s) \sim \frac{s + \frac{3m_2}{m_3}}{s^2\left(c - \lambda m_1 + \lambda\frac{3m_2^2}{2m_3}\right) + s\left(c\frac{3m_2}{m_3} - \frac{3m_1 m_2}{m_3}\lambda - q\right) - \frac{3m_2}{m_3}q}$$
$$= \frac{s + \frac{1}{\widehat{m}_3}}{s^2\left(c - \lambda m_1 + \widetilde{\lambda}_L \widehat{m}_2\right) + s\left(c\frac{1}{\widehat{m}_3} - \tilde{\lambda}_L - q\right) - \frac{1}{\widehat{m}_3}q}, \; \tilde{\lambda}_L = \frac{3m_1 m_2}{m_3}\lambda = \frac{m_1}{\widehat{m}_3}\lambda. \tag{13}$$

Note that the coefficient of s^2 in the denominator coincides with the one in the classic de Vylder approximation, but the coefficient of s doesn't. Also, $\widetilde{\lambda}_L$ is different from the classic de Vylder approximation parameter $\widetilde{\lambda}$ (also called DeVylder (A) in [?, (5.2-5.4)]).

Finally, comparing with (6) *we see that this cannot be viewed as an "exponential process" approximation. Indeed, if it were, the first two parameters should be*

$$\widetilde{m} = \widehat{m}_3, \tilde{\lambda}_L = \frac{3m_1 m_2}{m_3}\lambda = \frac{m_1}{\widehat{m}_3}\lambda.$$

Now the coefficient of s imposes preserving c, but the coefficient of s^2 contradicts this (unless $\lambda m_1 = \tilde{\lambda}_L \widehat{m}_2 \Leftrightarrow 2m_1 m_3 = 3m_2^2$.

REMARK 2. *In the case of exponential claims, these three approximations are exact. Indeed, it suffices to check that for exponential claims all the normalized moments are equal to μ^{-1}.*

We have included a derivation of the less-known "De Vylder-Laplace" third approximation in Section ??.

3 Examples of scale approximations for Cramèr-Lundberg models

In this section, examples along with numerical simulations will be presented. We plot the graphs of W_q, and also of W_q', W_q'', when they exhibit oscillations, and determine the "winning approximation" with respect to the exact solution. We consider the Cramèr-Lundberg model with exponential mixtures of jumps (in this case, the computation of W_q, Z_q is fast and error-less with symbolic algebra systems, since it belongs to the realm of rational computations).

It turns out that the de Vylder approximation of Φ_q is always the best, but Renyi may also win occasionally when approximating b_{DeF} – see ...

3.1 A Cramér-Lundberg process with hyperexponential claims of order 3

Consider a Cramér-Lundberg process with density function $f(x) = \frac{12}{83}e^{-x} + \frac{42}{83}e^{-2x} + \frac{150}{83}e^{-3x}$, and $c = 1$, $\lambda = \frac{83}{48}$, $\theta = \frac{263}{235}$, $p = \frac{263}{498}$, $q = \frac{5}{48}$.

The Laplace exponent of this process is $\kappa(s) = s - \frac{12s}{83(s+1)} - \frac{21s}{83(s+2)} - \frac{50s}{83(s+3)}$ and from this one can invert $\frac{1}{\kappa(s)-q} = \widehat{W}_q(s)$ to obtain the scale function [1]

$$W_q(x) \quad = -0.0813294e^{(-2.60997x)} - 0.179472e^{(-1.68854x)} - 0.373887e^{(-0.779311x)} + 1.63469e^{(0.18198x)}.$$

From this, we see that the dominant exponent is $\Phi_q = 0.18198$. The unique minimum of W_q' is at $b_{DeF} = 1.89732$, and we conclude that this is the optimal barrier that maximizes dividends.

Figure 2 shows the plots of W_q as well as its first two derivatives. The plots of the exact W_q and its derivatives are labelled Wxexact, and coloured as the darkest. The plot of W_q' an exhibits the noticeable minimum around $x = 1.9$.

Continuing, from the parameters of the process, one can obtain the approximations to the scale function W_q as described in the earlier section. Table 1 gives a summary of the values of Φ_q and b_{DeF} obtained from these approximations, as well as the relative errors from the exact value. [2] We can observe a relative error of less than 2% for each of the approximations' Φ_q value, with the DeVylder approximation's Φ_q beating the others.

We also tried varying the safety loading factor θ, keeping the density f and the discount rate q to be the same. Over certain values of $\theta \in [0, 263/235]$ we observe the trend of the DeVylder approximation exhibiting the least relative error in approximating Φ_q. Though the DeVylder approximations displayed increasing errors for decreasing values of θ, the errors never went above 2%. A table summarizing the results can be found at table 2. Unlike this observation for Φ_q, we are unable to conclude anything when looking at the approximate values for b_{DeF}. For $\theta = 43/235, 23/235, 3/235$, all of the approximations yielded $b_{DeF} = 0$ as the optimal barrier. This turned out to be true for $\theta = 3/235$.

REMARK 3. *Note (see for example [?, Sec. 3]) that the necessary, and in our case sufficient condition for the nonnegativity of the optimal dividends barrier is $W_q''(0_+) < 0 \Leftrightarrow (\frac{\lambda+q}{c})^2 < \frac{\lambda}{c}f(0)$. Here, this condition is satisfied for the exact when $\theta \geq \ldots$. For the approximations, it is satisfied at ...*

	Dominant exponent Φ_q	Percent relative error (Φ_q)	Optimal barrier b_{DeF}	Percent relative error (b_{DeF})
Exact	0.18198	0	1.89732	0
Expo	0.184095	1.162215628	2.04608	7.840532962
Dev	0.182011	0.017034839	1.91233	0.79111589
Renyi	0.181708	0.149466974	2.08136	9.699997892
LapDev	0.178939	1.671062754	2.04661	7.868467101

Table 1: Exact and approximate values of Φ_q and b_{DeF} for $f(x) = \frac{12}{83}e^{-x} + \frac{42}{83}e^{-2x} + \frac{150}{83}e^{-3x}$, $c = 1$, $q = \frac{5}{48}$. The DeVylder approximation displayed the least relative error among the four approximations considered.

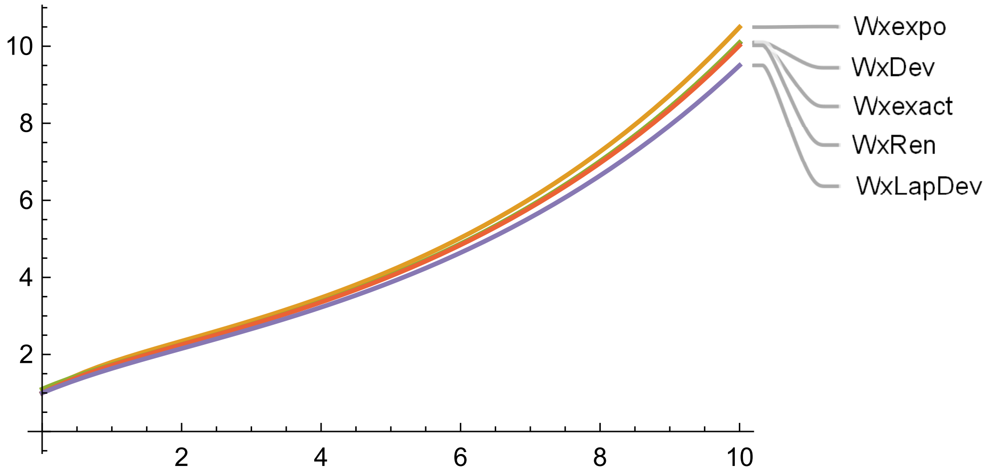
3.2 A Cramér-Lundberg process with non-hyperexponential claims of order 5

Consider the Cramér-Lundberg process with density of claims

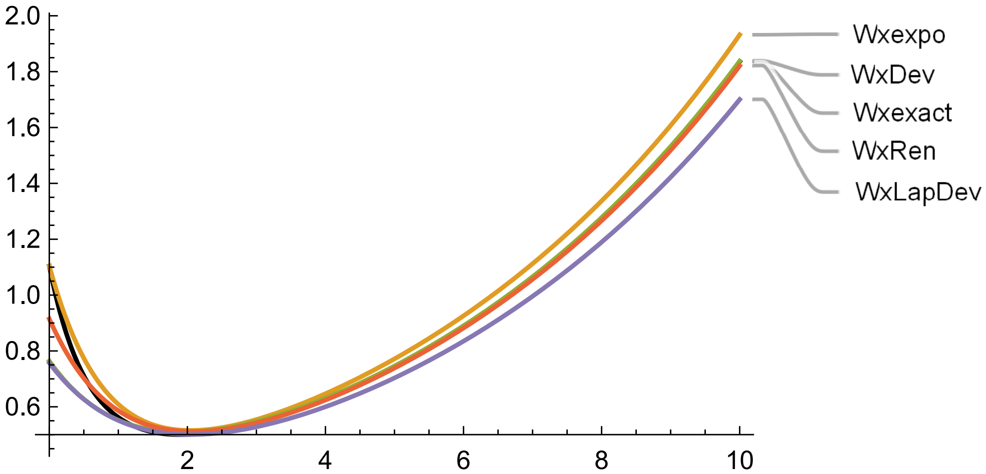
$$f(x) = \frac{5}{2}e^{-5x} + \frac{4}{5}e^{-4x} - \frac{1}{5}e^{-3x} - \frac{1}{5}e^{-2x} + \frac{1}{20}e^{-x}$$

[1] Laplace inversion done via Mathematica; coefficients and exponents are decimal approximations of the real values.

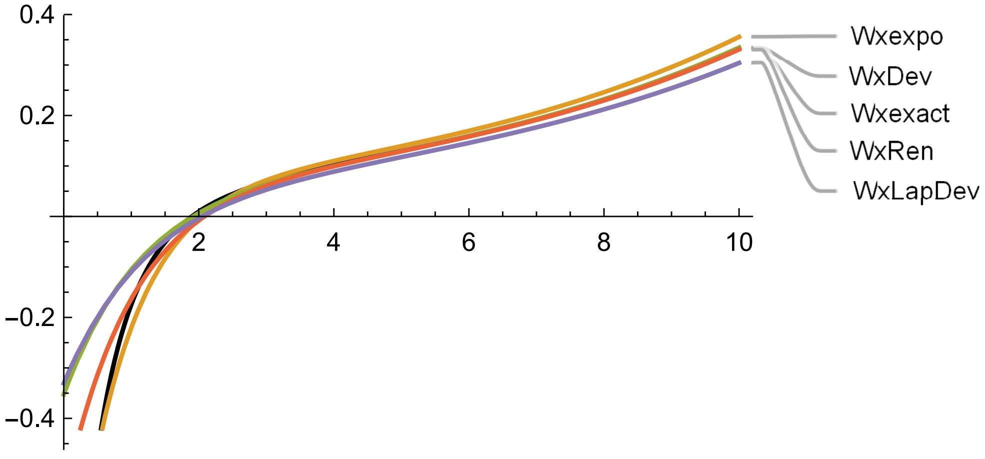
[2] Percent relative error is computed as the absolute value of the difference between the approximation and the exact, divided by the exact value, times 100. Differences in values when computed from the table and the displayed value may be explained by rounding off errors.



(a) $W_q(x)$ (in black)



(b) $W'_q(x)$



(c) $W''_q(x)$

Figure 2: Plots of $W_q(x)$, $W'_q(x)$, and $W''_q(x)$ of the exact solution and the approximations for $f(x) = \frac{12}{83}e^{-x} + \frac{42}{83}e^{-2x} + \frac{150}{83}e^{-3x}$, $c = 1$, $q = \frac{5}{48}$.

and $c = \frac{23}{90}$, $\lambda = \frac{7}{12}$, $\theta = 1$, $p = 23/180$, $\rho = \frac{1}{2}$, $q = \frac{1}{10}$.

The Laplace exponent of this process is $\kappa(s) = \frac{23s}{90} - \frac{s}{20(s+1)} + \frac{s}{10(s+2)} + \frac{s}{15(s+3)} - \frac{s}{5(s+4)} - \frac{s}{2(s+5)}$ and

θ	Closest approximation	Φ_q exact	Φ_q approximation	% error Φ_q
263/235	Dev	0.18198	0.182011	0.0168217
243/235	Dev	0.194712	0.194754	0.0213671
223/235	Dev	0.209221	0.209279	0.0274827
203/235	Dev	0.225876	0.225957	0.0358309
183/235	Dev	0.245146	0.245262	0.0474032
163/235	Dev	0.267635	0.267806	0.0637063
143/235	Dev	0.294126	0.294382	0.0870647
123/235	Dev	0.325643	0.326038	0.121115
103/235	Dev	0.363539	0.364163	0.171618
83/235	Dev	0.40961	0.410625	0.247788
63/235	Dev	0.466261	0.46796	0.364457
43/235	Dev	0.536719	0.539647	0.545532
23/235	Dev	0.62533	0.630516	0.829419
3/235	Dev	0.737962	0.747389	1.27736

Table 2: Exact and approximate values of Φ_q for $f(x) = \frac{12}{83}e^{-x} + \frac{42}{83}e^{-2x} + \frac{150}{83}e^{-3x}$, varying the value of θ. The DeVylder approximation displayed the least relative error among the four approximations considered.

θ	Closest approximation	Barrier exact	Barrier approx	% error Barrier
263/235	Dev	1.89732	1.91233	0.791183
243/235	Dev	1.79954	1.78002	1.08482
223/235	LapDev	1.69334	1.74547	3.07875
203/235	LapDev	1.57785	1.56951	0.528553
183/235	Ren	1.45224	1.52484	4.9989
163/235	Ren	1.31579	1.33691	1.60463
143/235	Ren	1.16804	1.12368	3.79796
123/235	Expo	1.00898	1.04123	3.19653
103/235	Expo	0.839228	0.794964	5.27444
83/235	Expo	0.660338	0.513179	22.2854
63/235	Expo	0.474896	0.196234	58.6785
43/235	Expo	0.286563	0	100
23/235	Expo	0.0998863	0	100
3/235	Exact	0	0	0

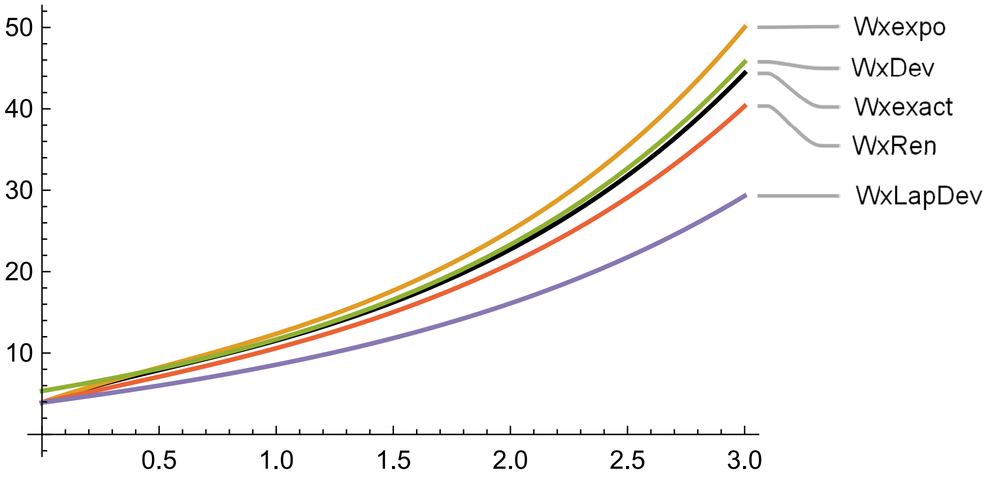
Table 3: Exact and approximate values of b_{DeF} for $f(x) = \frac{12}{83}e^{-x} + \frac{42}{83}e^{-2x} + \frac{150}{83}e^{-3x}$, varying the value of θ. No single approximation displayed a noticeable advantage over the others.

from here the scale function is

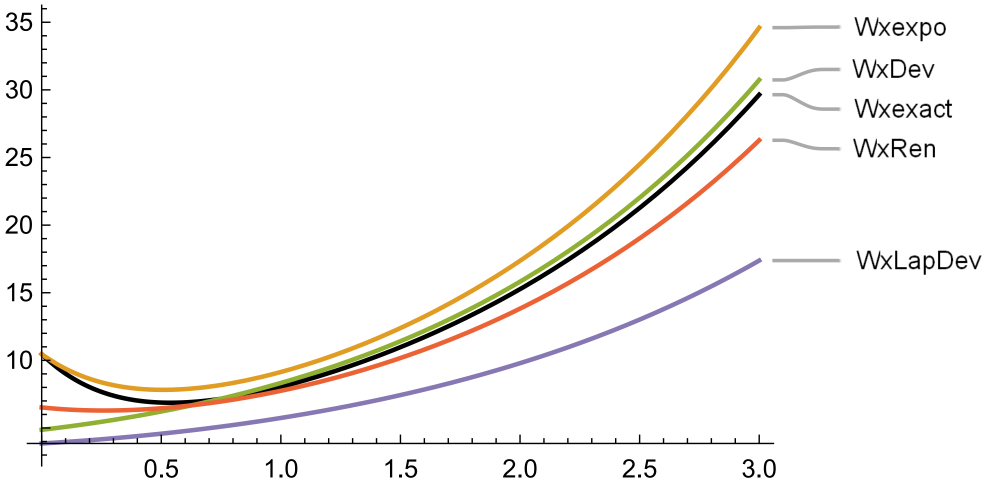
$$\begin{aligned} W_q(x) \;&= -0.0831561\; e^{(-4.35135x)} + 0.684818\; e^{(-2.65126x)} - 0.595164\; e^{(-0.837877x)} + 6.02604\; e^{(0.666084x)} \\ &\quad -2.11949\; e^{(-2.57585x)} \cos[0.811233x] + 2.39748\; e^{(-2.57585x)} \sin[0.811233x] \end{aligned}$$

As seen in table 4, the approximations for W_q give again the DeVylder as the closest approximation when looking at the Φ_q value. However, W_q' for the DeVylder attains its minimum at $x = 0$, which is relatively far from the exact barrier at $b_{DeF} = 0.538$. The exponential approximation shows the best barrier approximate at $x = 0.506947$.

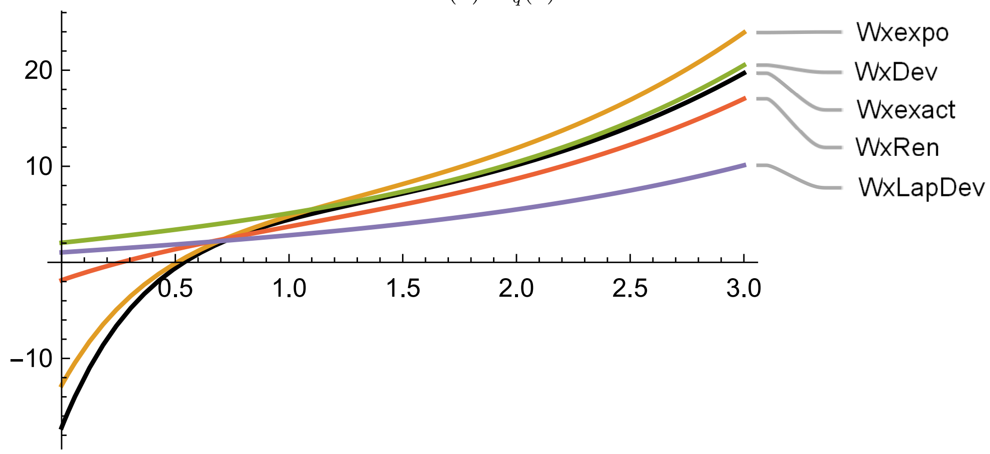
The lack of a minimum of W_q' for the DeVylder and the Laplace DeVylder approximations inside the interval $x \in (0, 3)$ can be seen clearly in the plots in figure 3b. The lack of zero of the corresponding W_q'' plots in figure 3c supports this idea. The exponential approximation has the closest root to the exact at



(a) $W_q(x)$ (in black)



(b) $W_q'(x)$



(c) $W_q''(x)$

Figure 3: Plots of $W_q(x)$, $W_q'(x)$, and $W_q''(x)$ of the exact solution and the approximations, for $f(x) = \frac{5}{2}e^{-5x} + \frac{4}{5}e^{-4x} - \frac{1}{5}e^{-3x} - \frac{1}{5}e^{-2x} + \frac{1}{20}e^{-x}$, $\theta = 1$, $q = 1/10$

around $x = 0.5$.

	Dominant exponent Φ_q	Percent relative error (Φ_q)	Optimal barrier b_{DeF}	Percent relative error (b_{DeF})
Exact	0.666084	0	0.538	0
Expo	0.691616	3.8331	0.506947	5.77181
Dev	0.670061	0.596931	0	100
Renyi	0.650448	2.34749	0.260532	51.5739
LapDev	0.587976	11.7265	0	100

Table 4: Exact and approximate values of Φ_q and b_{DeF} for $f(x) = \frac{5}{2}e^{-5x} + \frac{4}{5}e^{-4x} - \frac{1}{5}e^{-3x} - \frac{1}{5}e^{-2x} + \frac{1}{20}e^{-x}$, $\theta = 1$, $q = 1/10$. The DeVylder approximation displayed the least percent relative error in approximating Φ_q, but struggled in approximating b_{DeF}.

θ	Closest approximation	Φ_q exact	Φ_q approximation	% error Φ_q
1	Dev	0.666084	0.670061	0.596931
0.9	Dev	0.721302	0.726797	0.761704
0.8	Dev	0.785584	0.793322	0.985015
0.7	Dev	0.861148	0.872279	1.29261
0.6	Dev	0.950932	0.967325	1.72389
0.5	Dev	1.05887	1.08366	2.34057
0.4	Dev	1.19032	1.22891	3.24173
0.3	Dev	1.35264	1.41474	4.59125
0.2	Dev	1.55609	1.65988	6.6701
0.1	Expo	1.81514	1.95652	7.78877

Table 5: Exact and approximate values of Φ_q for $f(x) = \frac{5}{2}e^{-5x} + \frac{4}{5}e^{-4x} - \frac{1}{5}e^{-3x} - \frac{1}{5}e^{-2x} + \frac{1}{20}e^{-x}$, varying the value of θ. The DeVylder approximation displayed the least percent relative error among the four approximations considered.

3.3 A Cramér-Lundberg process with a matrix exponential density

Next we consider an example produced by taking a Cramér-Lundberg process with matrix exponential density of claims $f(x) = \alpha e^{Ax}(-A)\mathbf{1}$, where $\alpha = (-2.4, 0.9, 2.5)$, $A = \begin{pmatrix} -6.2, 2, 0 \\ 2, -9, 1 \\ 1, 0, -3 \end{pmatrix}$ and $\lambda = 1$, $\theta = 1$, $q = \frac{1}{10}$.

The Laplace exponent of this process is $\kappa(s) = 1.3249s - \frac{3.34053s}{s+2.87761} + \frac{3.03143s}{s+5.34047} - \frac{0.690902s}{s+9.98192}$ and the scale function is

$$W_q(x) \quad = -0.0655864e^{-9.35143x} + 0.149742e^{-6.89805x} - 0.676982e^{-1.26245x} + 1.3476e^{0.142175x}.$$

θ	Closest approximation	Barrier exact	Barrier approx	% error Barrier
1	Expo	0.538	0.506947	5.77181
0.9	Expo	0.496513	0.448391	9.69186
0.8	Expo	0.448937	0.382485	14.8021
0.7	Expo	0.394055	0.308318	21.7575
0.6	Expo	0.330415	0.225085	31.8783
0.5	Expo	0.256331	0.13227	48.3988
0.4	Expo	0.169933	0.0299613	82.3687
0.3	Expo	0.0693293	0	100
0.2	Exact	0	0	0
0.1	Exact	0	0	0

Table 6: Exact and approximate values of Φ_q for $f(x) = \frac{5}{2}e^{-5x} + \frac{4}{5}e^{-4x} - \frac{1}{5}e^{-3x} - \frac{1}{5}e^{-2x} + \frac{1}{20}e^{-x}$, varying the value of θ. The exponential approximation displayed the least percent relative error among the four approximations considered.

	Dominant exponent Φ_q	Percent relative error (Φ_q)	Optimal barrier b_{DeF}	Percent relative error (b_{DeF})
Exact	0.142175	0	2.61925	0
Expo	0.139202	2.09118	2.79162	6.58074
Dev	0.142174	0.00104491	2.60329	0.609153
Renyi	0.142238	0.0439717	2.59638	0.873044
LapDev	0.143232	0.743272	2.48327	5.19155

Table 7: Exact and approximate values of Φ_q and b_{DeF} for $f(x) = \alpha e^{Ax}(-A)\mathbf{1}$, $\theta = 1$, $q = \frac{1}{10}$. The DeVylder approximation displayed the least percent relative error among the four approximations considered.

θ	Closest approximation	Φ_q exact	Φ_q approximation	% error Φ_q
1	Dev	0.142175	0.142174	0.00104491
0.9	Dev	0.156039	0.156036	0.00148235
0.8	Dev	0.172688	0.172684	0.00216274
0.7	Dev	0.192979	0.192973	0.00325715
0.6	Dev	0.218117	0.218106	0.0050835
0.5	Dev	0.249825	0.249804	0.0082542
0.4	Dev	0.290594	0.290553	0.013988
0.3	Dev	0.344047	0.343962	0.0247701
0.2	Dev	0.415404	0.415214	0.0457098
0.1	Dev	0.512	0.511554	0.0871333

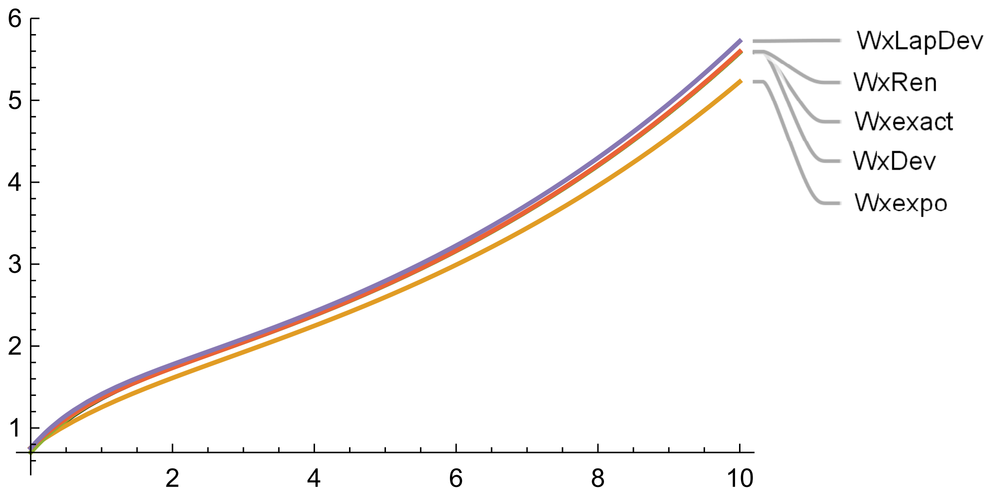
Table 8: Exact and approximate values of Φ_q for $f(x) = \alpha e^{Ax}(-A)\mathbf{1}$, varying the value of θ. The DeVylder approximation displayed the least percent relative error among the four approximations considered.

3.4 A Cramér-Lundberg process with an Erlang density

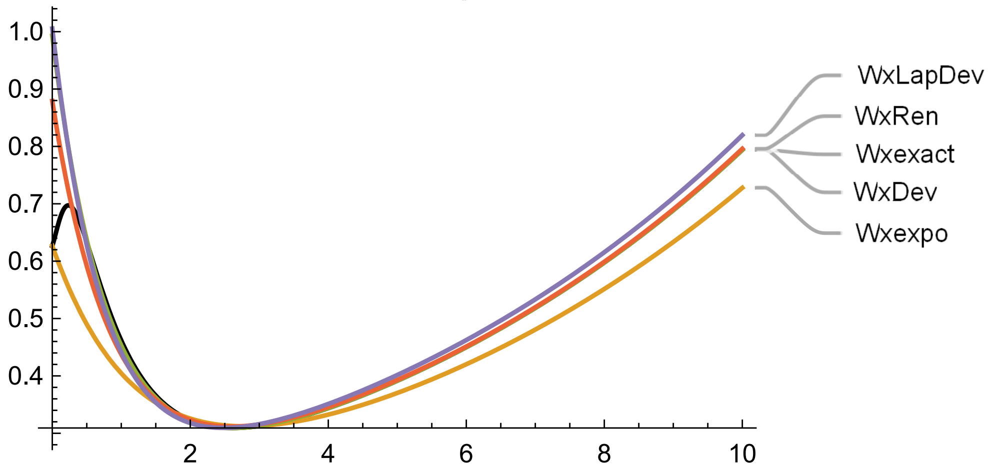
In the following example, we study a Cramèr-Lundberg model with $Erlang(2,1)$ density of claims $f(x) = 10\,(xe^{-x})$ and $\lambda = 10, \theta = \frac{7}{100},\ c = \frac{107}{5},\ q = \frac{1}{10}$.

The Laplace exponent of this process is $\kappa(s) = 21.4s - \frac{10s}{s+1} - \frac{10s}{(s+1)^2}$ and the scale function is

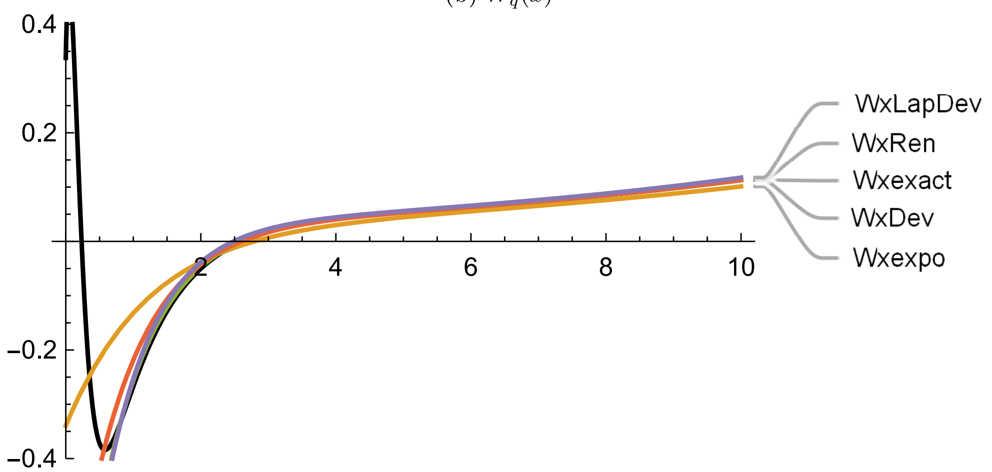
$$W_q(x) \quad = 0.00517511e^{-1.48825x} - 0.23639e^{-0.0793553x} + 0.277943e^{0.0395672x}.$$



(a) $W_q(x)$ (in black)



(b) $W_q'(x)$



(c) $W_q''(x)$

Figure 4: Plots of $W_q(x)$, $W_q'(x)$, and $W_q''(x)$ of the exact solution and the approximations for $f(x) = \alpha e^{Ax}(-A)\mathbf{1}$, $\theta = 1$, $q = \frac{1}{10}$.

θ	Closest approximation	Barrier exact	Barrier approx	% error Barrier
1	Dev	2.61925	2.60329	0.609154
0.9	Dev	2.52243	2.50437	0.716072
0.8	Expo	2.39997	2.39528	0.195427
0.7	Dev	2.24344	2.22233	0.941111
0.6	Dev	2.04159	2.01986	1.06465
0.5	Dev	1.77996	1.75855	1.20296
0.4	Dev	0	1.4216	-
0.3	Expo	0	0.518511	-
0.2	Exact	0	0	-
0.1	Exact	0	0	-

Table 9: Exact and approximate values of b_{DeF} for $f(x) = \alpha e^{Ax}(-A)\mathbf{1}$, varying the value of θ. The DeVylder approximation displayed the least percent relative error among the four approximations considered.

	Dominant exponent Φ_q	Percent relative error (Φ_q)	Optimal barrier b_{DeF}	Percent relative error (b_{DeF})
Exact	0.142175	0	2.61925	0
Expo	0.139202	2.09118	2.79162	6.58074
Dev	0.142174	0.00104491	2.60329	0.609153
Renyi	0.142238	0.0439717	2.59638	0.873044
LapDev	0.143232	0.743272	2.48327	5.19155

Table 10: Exact and approximate values of Φ_q and b_{DeF} for $f(x) = 10\,(xe^{-x})$, $\theta = \frac{7}{100}$, $q = \frac{1}{10}$. The DeVylder approximation displayed the least percent relative error among the four approximations considered.

3.5 A Cramér-Lundberg process with a matrix exponential density

In the following example, we study a Cramèr-Lundberg model with density of claims given by

$$\begin{aligned} f(x) =& ue^{-ax} 2\cos^2\left(\frac{\omega x+\phi}{2}\right) = ue^{-ax}(1+\cos(\omega x+\phi)) = \\ =& e^{-ax}(u + u\cos(\phi)\cos(\omega x) - u\sin(\phi)\sin(\omega x)) \end{aligned}$$

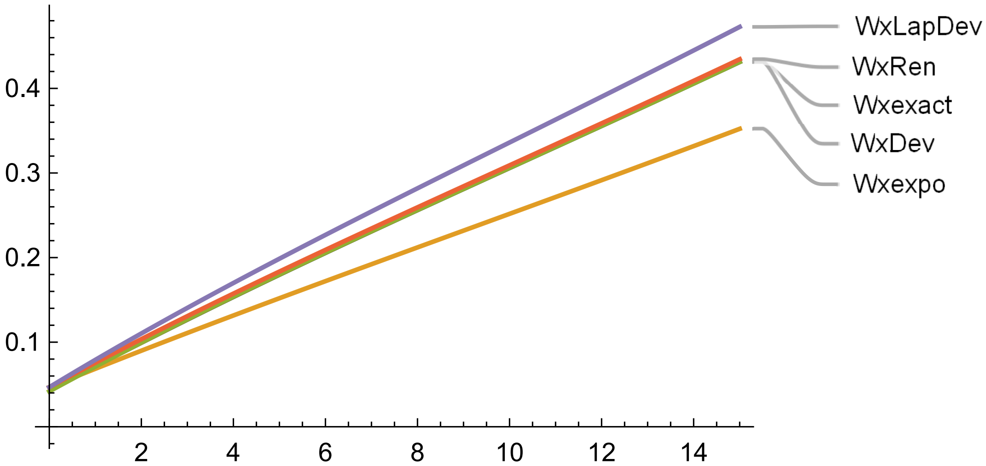
where

$$u = \frac{a\left(a^2+\omega^2\right)}{a^2+\omega^2+a^2\cos(\phi)-a\omega\sin(\phi)}.$$

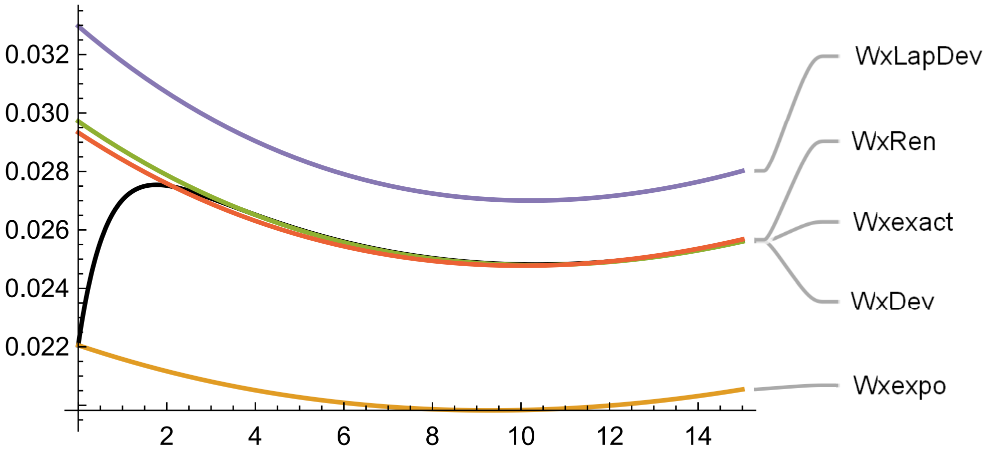
One can check that $\int f(x)dx = 1$ with such value of u.

Assuming further that $a = 1$, $\phi = 2$, $\omega = 20$, and that $\theta = 1$, $q = 1/10$, the Laplace exponent for this process is $\kappa(s) = \frac{s\left(2.09898s^3+5.29695s^2+843.502s+420.846\right)}{(s+1.)(s^2+2.s+401.)}$ and the scale function is

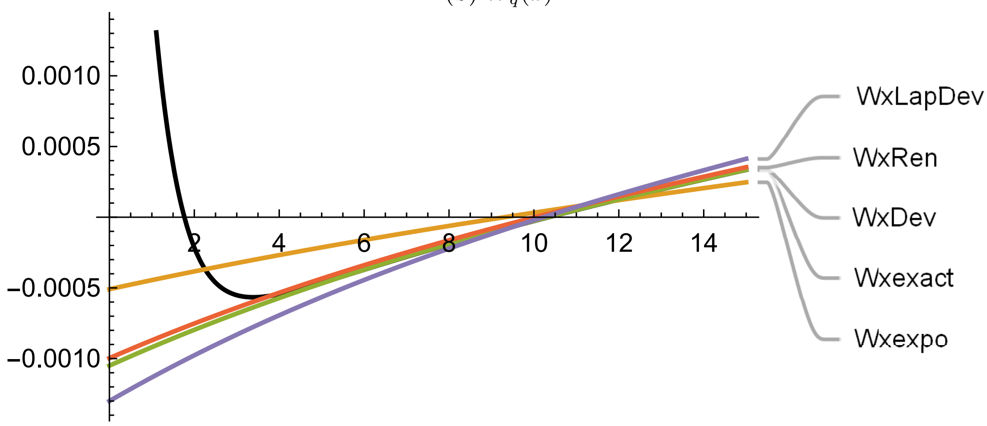
$$\begin{aligned} W_q(x) =& 0.824723e^{0.0881484x} - 0.348141e^{-0.540677x} \\ &+ e^{-1.0117x}\cos(19.9957x)\Big(-(0.000285494+0.0000804151i)\sin(39.9914x) \\ &- (0.0000804151+0.000285494i) + (-0.0000804151+0.000285494i)\cos(39.9914x)\Big) \\ &+ e^{-1.0117x}\sin(19.9957x)\Big(-(0.0000804151-0.000285494i)\sin(39.9914x) \\ &+ (0.000285494+0.0000804151i)\cos(39.9914x) - (0.000285494-0.0000804151i)\Big). \end{aligned}$$



(a) $W_q(x)$ (in black)



(b) $W_q'(x)$



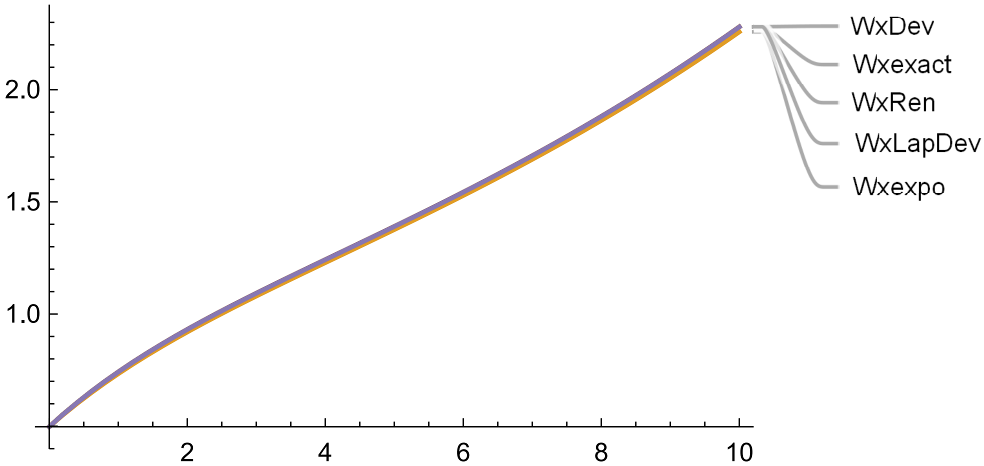
(c) $W_q''(x)$

Figure 5: Plots of $W_q(x)$, $W_q'(x)$, and $W_q''(x)$ of the exact solution and the approximations for $f(x) = 10\,(xe^{-x})$, $\theta = \frac{7}{100}$, $q = \frac{1}{10}$.

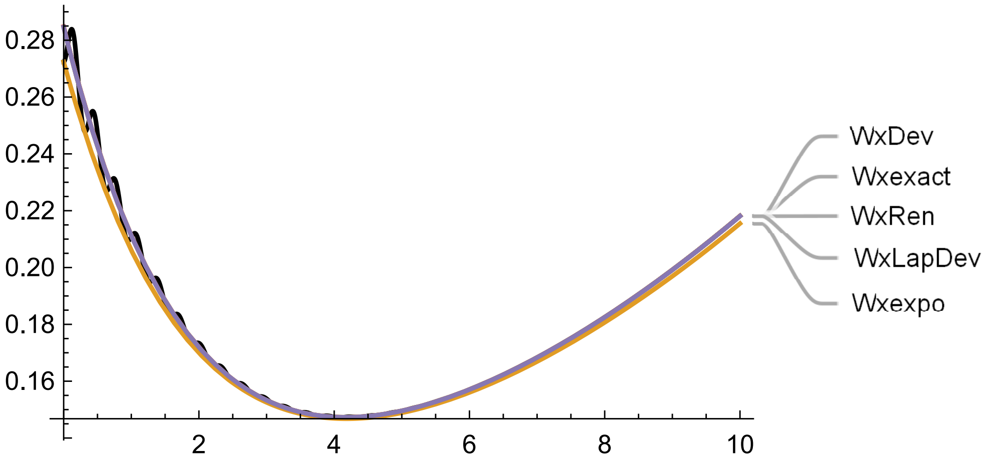
4 Optimizing dividends and capital injections[?, AGLW20]

We refer to [?, AGLW20] for the formulation of this stochastic control problem.

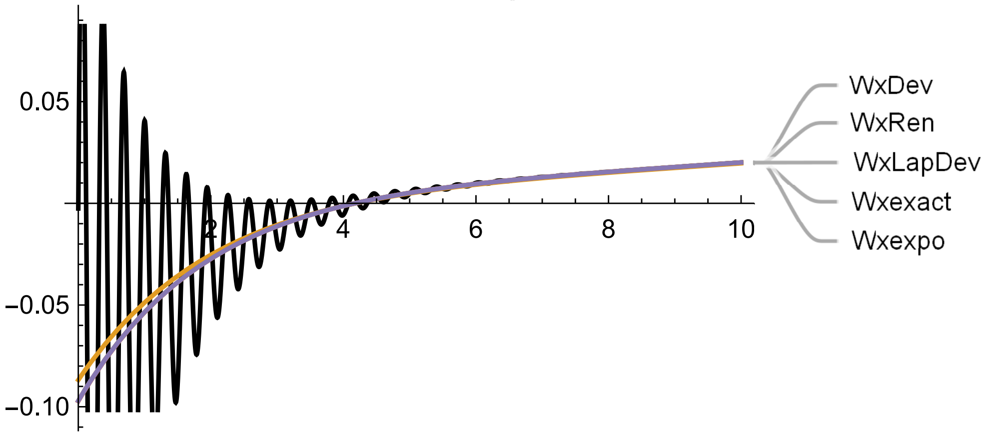
In this section we revisit the problem of optimizing numerically the value of "bounded buffer $(-a, 0, b)$



(a) $W_q(x)$ (in black)



(b) $W_q'(x)$



(c) $W_q''(x)$

Figure 6: Plots of $W_q(x)$, $W_q'(x)$, and $W_q''(x)$ of the exact solution and the approximations for $f(x) = ue^{-ax}2\cos^2\left(\frac{\omega x+\phi}{2}\right)$, $\theta = 1$, $q = \frac{1}{10}$.

policies", which consist in allowing capital injections smaller than a given a and declaring bankruptcy at the first time when the size of the overshoot below 0 exceeds a, and pay dividends when the reserve reaches an upper barrier b.

	Dominant exponent Φ_q	Percent relative error (Φ_q)	Optimal barrier b_{DeF}	Percent relative error (b_{DeF})
Exact	0.0881484	0	4.38201	0
Expo	0.0878658	0.32053	4.42263	0.927122
Dev	0.0881484	6.11743*10^-6	4.39745	0.352331
Renyi	0.0881481	0.000314617	4.39788	0.362284
LapDev	0.0881449	0.00395543	4.3982	0.369586

Table 11: Exact and approximate values of Φ_q and b_{DeF} for $f(x) = ue^{-ax}2\cos^2\left(\frac{\omega x+\phi}{2}\right)$, $\theta = 1$, $q = \frac{1}{10}$. The DeVylder approximation displayed the least percent relative error among the four approximations considered.

4.1 The cost function and its optimization

We present here a synthesis of the results of [?, AGLW20] (in order to relate the results, one needs to replace γ in the objective of [?] by $1/k$). They are expressed in terms of the functions

$$\begin{cases} R_a(x) = \lambda \int_0^x W_q(x-y)\ \overline{F}(y+a)\ dy \\ G_a(x) = \lambda \int_0^x W_q(x-y)\ m_y(a)\ \ dy, \quad m_y(a) = \int_0^a zf(y+z)dz \\ S_a(x) = Z_q(x) + R_a(x) \end{cases} \tag{14}$$

([?] use s_c, r_c, instead of $G_a(x), R_a(x)$, respectively).

REMARK 4. *The relation* (21) *below earns the function $S(x,a)$ the name of scale function for our problem, which basically means that it will appear in several different problems involving a reflecting barrier at b and a limited reflection buffer $(-a, 0]$. See [AGLW20, Rem. 8] for some other examples, and see the equation [?, (3)] for an additional example.*

REMARK 5. *Cf. [?, Lem. A.4], these functions satisfy*

$$G_a(x) + aR_a(x) = \int_0^a R_y(x)dy = \lambda \int_0^x W_q(x-y) \int_0^a \overline{F}(y+z)dzdy \Longrightarrow \begin{cases} G_{a'}(x) = -aR_{a'}(x) \\ G'_a(x) + aR'_a(x) = \int_0^a R'_y(x)dy \end{cases}, \tag{15}$$

where $R_{a'}(x), G_{a'}(x)$ denote derivatives with respect to the subscript, and where we added a missing minus in the last statement of [?, Lem. A.4].

EXAMPLE 2. *In the particular case of exponential jumps, the functions* (14) *become [AGLW20]*

$$\begin{cases} G_a(x) = C_q(x)m(a),\ R_a(x) = C_q(x)e^{-\mu a}, \\ C_q(x) = cW_q(x) - Z_q(x),\ m(a) = \int_0^a y\ F(\mathrm{d}y) = \frac{1-e^{-\mu a}(\mu a+1)}{\mu} \end{cases},$$

as follows from the identities $\overline{F}(y+a) = e^{-\mu a}\overline{F}(y), m_y(a) = e^{-\mu y}m(a)$.

Note also the formulas

$$\begin{cases} R_a(x) = C_q(x)(1-F(a)) \\ G_a(x) = C_q(x)\int_0^a y\ F(\mathrm{d}y) \end{cases}, \tag{16}$$

which will be used below as a heuristic approximation in non-exponential cases.

EXAMPLE 3. *Consider now the more general case when the claims are distributed according to a matrix exponential density generated by a row vector $\vec{\beta}$ and by an invertible matrix B of order n, which are such that the vector $\vec{\beta}e^{xB}$ is decreasing componentwise to 0, and $\vec{\beta}.\mathbf{1} \neq 0$, with $\mathbf{1}$ a column vector. As customary, we restrict w.l.o.g. to the case when $\vec{\beta}$ is a probability vector, and $\vec{\beta}.\mathbf{1} = 1$, so that*

$$\overline{F}(x) = \vec{\beta}e^{xB}\mathbf{1}$$

is a valid survival function.

The matrix versions of our functions are:

$$\begin{cases} R_a(x) = \lambda \int_0^x W_q(x-y)\, \overline{F}(y+a)\, dy = \lambda \vec{\beta} \int_0^x W_q(x-y)\, e^{yB}\, dy\, e^{aB}\mathbf{1} = \vec{C}_q(x) e^{aB}\mathbf{1} \\ m_y(a) = \int_0^a z f(y+z) dz = \vec{\beta}\, e^{yB}\, \int_0^a z\, e^{zB}(-B)\, dz\, \mathbf{1} = \vec{\beta}\, e^{yB} M(a)\mathbf{1} \\ G_a(x) = \lambda \int_0^x W_q(x-y)\, m_y(a)\ \ dy = \vec{C}_q(x) M(a)\mathbf{1} \end{cases}, \tag{17}$$

where

$$\begin{cases} C_q(x) = \lambda \int_0^x W_q(x-y)\, e^{yB}\, dy \\ \vec{C}_q(x) = \lambda \vec{\beta} \int_0^x W_q(x-y)\, e^{yB}\, dy \end{cases}. \tag{18}$$

The product formulas (17) *may also be established directly in the phase-type case, using the conditional independence of the ruin probability of the overshoot size -see section* **??**.

PROPOSITION 2. *[**?**, Thm. 4]* **Cost function for** (a, b) **policies**

For a Cramèr-Lundberg process (compound Poisson) with exponential jumps, let

$$J_x = J^{a,b}(x) := \mathbb{E}_x \left[\int_0^{T_{a,-}} e^{-qt} \left(\mathrm{d}D_t - k\, \mathrm{d}C_t\right) \right]$$

denote the expected discounted dividends minus capital injections associated to policies consisting in paying capital injections with proportional cost $k \geq 1$, provided that the severity of ruin is smaller than $a > 0$, and paying dividends as soon as the process reaches some upper level b.

Then,

1. $$J_x = \begin{cases} kG_a(x) + J_0^{(a,b)} S_a(x) = kG_a(x) + \frac{1-kG_a'(b)}{S_a'(b)} S_a(x), & x \in [0, b] \\ kx + J_0^{(a,b)} & x \in [-a, 0] \\ 0 & x \leq -a \end{cases}. \tag{19}$$

2. *For fixed $b \geq 0$, the optimality equation $\frac{\partial}{\partial a} J_0^{a,b} = 0$ may be written as*

$$ka = J_0^{a,b} \Leftrightarrow J_{-a}^{a,b} = 0. \tag{20}$$

Proof: The first statement is [**?**, Thm. 4], and the second is a consequence of (15). □

In the exponential case, further simplification is possible. In particular, we will take advantage of properties of the Lambert-W function, which were not exploited in [AGLW20].

COROLLARY 3. Cost function and optimality conditions in the exponential case

1. $$J_0^{(a,b)} = \frac{1 - k\, m(a) C_q'(b)}{(1 - F(a)) C_q'(b) + q W_q(b)} = \frac{\gamma(b) - k\, m(a)}{1 - F(a) + q\theta(b)}, \tag{21}$$

   where we put

$$\gamma(b) = \frac{1}{C_q'(b)}, \theta(b) = \frac{W_q(b)}{C_q'(b)}.$$

2. *For fixed $a \geq 0$, the optimality equation $\frac{\partial}{\partial b} J_0^{a,b} = 0$ may be written as*

$$J_0^{a,b} = j(b),\ j(b) := \frac{\gamma'(b)}{q\theta'(b)}. \tag{22}$$

   At a critical point $(a^*, b^*), a^* > 0, b^* > 0$, we must have $J_0^{a^*,b^*} = j(b^*) = ka^* \Longrightarrow$

$$a^* = s(b^*), s(b) := \frac{j(b)}{k}. \tag{23}$$

   In conclusion, b^* for such critical points may be computed solving

$$\eta(b) := \frac{\gamma(b)}{\theta(b)} - qj(b) - \frac{k}{\mu\theta(b)} F\left(\frac{j(b)}{k}\right) = 0. \tag{24}$$

REMARK 6. *a) The important equation* (23) *identifies the optimal buffer associated with a dividends barrier b via the explicit function s(b). In the general framework of [?], s(b) is only defined implicitly as solution of [?, (6)].*

b) For fixed $b \geq 0$*, the optimality equation*

$$\frac{\partial}{\partial a} J_0^{a,b} = 0 \Leftrightarrow J_0^{(a,b)} = ka = \frac{\gamma(b) - km(a)}{e^{-\mu a} + q\theta(b)}$$

may be written as also as $ka = \frac{\gamma(b) - k\frac{1-e^{-\mu a}}{\mu}}{q\theta(b)}$.

3. *In the special case* $b^* = 0$*, the optimality equation* (20) *implies that* $a^* = a_k := a_{k,0}$ *satisfies the simpler equation*

$$\delta(k, a) := c - k\left(aq + \frac{\lambda}{\mu}F(a)\right) = 0. \tag{25}$$

Rewriting the equation (25) *as* $ze^z = \frac{\lambda}{q}e^g, z = \mu a + g)$ *implies that the solution is*

$$\mu a = -g + L_W\left(\frac{\lambda}{q}e^g\right), g = \frac{\lambda}{q} - \frac{\mu c}{kq},$$

where L_W *denotes the real Lambert-W function [?, ?, ?, ?] (this observation is missing in [AGLW20]).*

Proof: 2. From (21), the optimality equation $\frac{\partial}{\partial b} J_0^{a,b} = 0$ simplifies to

$$J_0^{(a,b)} = \frac{\gamma'(b)}{q\theta'(b)} = j(b)(= -\frac{C_q''(b)}{q\left(W_q'(b)C_q'(b) - C_q''(b)W_q(b)\right)}),$$

and recalling $J_0^{(a,b)} = ka$ yields the result. □

REMARK 7. *Without switching to* γ, θ*, the previous computation is more complicated*

$$J_0^{(a,b)} = \frac{-k\ m(a)C_q''(b)}{(1 - F(a))C_q''(b) + qW_q'(b)}$$

We have now a further look at the functions introduced in Proposition 2. It may be checked that γ is increasing-decreasing (from $\frac{c}{\lambda}$ to 0), with a maximum at the root of $C_q''(x) = 0$, which is

$$\bar{b} := \frac{1}{\Phi_q - \rho_-}\log\left(\frac{\rho_-^2}{\Phi_q^2}\right), \tag{26}$$

and θ increases from $\theta(0) = \frac{1/c}{\lambda/c} = \frac{1}{\lambda}$ to $\theta(\infty) = \frac{1}{c\Phi_q - q}$.

The following result is to be found, albeit with somewhat different notations, in [AGLW20, Proof of Theorem 11, A2].

Lemma 4. *The function* $j(b) = \frac{\gamma'(b)}{q\theta'(b)}$ *is decreasing, with* $j(0) = \frac{\lambda}{\mu q}\frac{-C_q''(0)}{(C_q'(0))^2} = \frac{c\mu - (q+\lambda)}{\mu q}$*. If* $c\mu - (q + \lambda) > 0$*, then* $\bar{b} > 0$ *defined in* (26) *is the unique positive root of* $j(b)$.

REMARK 8. *Introducing*

$$\eta(b, a) := \frac{\gamma(b)}{\theta(b)} - k\left(qa + \frac{1}{\mu\theta(b)}F(a)\right) = \frac{1}{W_q(b)} - k\left(qa + \frac{1}{\mu\theta(b)}F(a)\right),$$

we note by using $C_q'(0) = \frac{\lambda}{c}, W_q(0) = \frac{1}{c}$ *that*

$$\eta(0, a) := c - k\left(qa + \frac{\lambda}{\mu}F(a)\right) = \delta(k, a).$$

The continuous function $\left[0,\bar{b}\right] \ni b \mapsto \eta(b)$,

$$\eta(b) := \eta(b,s(b)) = \frac{1}{W_q(b)} - qj(b) - \frac{k}{\mu\theta(b)}F\left(s(b)\right),\ s(b) := \frac{j(b)}{k}, \tag{27}$$

already defined in (24)*, will play an important role below.*

Note that

$$\eta(0) = \frac{c}{\lambda} - \frac{1}{\lambda}\left(c - \frac{q+\lambda}{\mu}\right) - \frac{k}{\mu}F\left(\frac{j(0)}{k}\right) = \frac{1}{\lambda\mu}\left(\lambda + q - \lambda k F\left(\frac{j(0)}{k}\right)\right) = \frac{1}{\lambda}\delta(k) < 0.$$

and $\eta\left(\bar{b}\right) = \frac{1}{W_q(\bar{b})} > 0$*. Therefore,* $\eta(b) = 0$ *has at least one solution of in* $[0,\bar{b}]$*; the first such solution will be denoted by* b^**.*

REMARK 9. *Putting* $h = \frac{1}{q\theta(b)} - \frac{\mu}{k}\frac{\gamma(b)}{q\theta(b)}$*, the structure equation* (24) *may be rewritten as:*

$$(\frac{\mu}{k}j(b) + h)e^{\frac{\mu}{k}j(b)+h} = \frac{e^h}{q\theta(b)} \Longrightarrow \frac{\mu}{k}j(b) + h = L_w(\frac{e^h}{q\theta(b)}).$$

The following (new!) result relates the dichotomy domains to the Lambert-W function.

Lemma 5. *The function*

$$\delta(k) := \delta(k, j(0)/k) = \delta(k, \frac{c\mu - (q+\lambda)}{k\mu q}) = \mu^{-1}\left(\lambda + q - \lambda k\left(1 - e^{-\frac{c\mu-\lambda-q}{qk}}\right)\right) \tag{28}$$

has a unique nonnegative root k^* *iff*

$$c\mu > \lambda^{-1}\left(\lambda+q\right)^2 > \left(\lambda+q\right). \tag{29}$$

Explicitly,

$$k^* = \frac{q+\lambda}{\lambda}\frac{f}{f + L_W\left(-fe^{-f}\right)},\ f = \frac{\lambda}{q+\lambda}\frac{c\mu - (\lambda+q)}{q}. \tag{30}$$

Proof: The equation to be solved is similar to (25), but this time the unknown is k.

Putting $d = \frac{c\mu-(\lambda+q)}{q}$ reduces the equation $\delta(k) = 0$ to

$$ke^{-\frac{d}{k}} = k - \frac{q+\lambda}{\lambda} \Leftrightarrow e^{-\frac{d}{k}} = 1 - \frac{q+\lambda}{\lambda k} \Leftrightarrow 1 = e^{\frac{d}{k}}\left(1 - \frac{(q+\lambda)d}{d\lambda k}\right) := e^z\left(1 - z/f\right), f = \frac{d\lambda}{q+\lambda}.$$

Rewriting the latter as $-f = e^z\left(z - f\right)$ we recognize, by putting $z = y + f$, an equation reducible to $ye^y = -fe^{-f}$, whose real solution is

$$y = L_W\left(-fe^{-f}\right),$$

where L_W denotes the real Lambert-W function. The final solution is (30).

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