Optimizing dividends and limited capital injections. A practical solution for the Cramér-Lundberg process, via de Vylder-type approximations

F. Avram, D. Goreac, R. Adenane, U. Solon, S. Zhu

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Abstract

We investigated recently in [AGLW20] the control problem of optimizing dividends in a Cramér-Lundberg model in which capital injections are allowed at a certain cost. We proved there the optimality, with exponential jumps, of bounded buffer $(-a, 0, b)$ policies, which consist in allowing capital injections smaller than a given a^* and declaring bankruptcy at the first time when the size of the overshoot below 0 exceeds a^*, and only pay dividends when the reserve reaches an upper barrier b^*. a^*, b^* are determined via simple formulas expressed in terms os the scale functions W_q, Z_q. Since generalizing to general claims seems rather daunting, we decided to experiment below numerically with applying our formulas (incorrectly) to the general case. While of course, un-optimal, the results show typically improvements of around 33% with respect to to the previous literature (the de Finetti and Shreve, Lehoczky and Gaver solutions).

This approach is very similar in spirit with the de Vylder-type approximations, which consist essentially in replacing the inverse of the exponential rate μ^{-1} in our formulas respectively by m_1, by $\frac{m_2}{2m_1}$, or by $\frac{m_3}{3m_2}$ (more complete descriptions may be obtained by working out some first order Padé approximations). Since we do not have an exact answer for general claims, studying the accuracy of our four approximations is impossible. We can however compare the amount of improvement with respect to the traditional de Finetti and Shreve, Lehoczky and Gaver objectives (which only optimize the dividend barrier b).

Contents

1 Examples of scale computations for Cramér-Lundberg models

In this section, examples along with numerical simulations will be presented. Starting with the case of a Cramér-Lundberg model with exponential mixtures jumps of order three, we plot the graphs of W_q, and also of W_q', W_q'', when they exhibit oscillations, and determine the "winning approximation".

It turns out that the best approximation is ...

In the *example* (2), we try to see what happens in the case of order five of a non-hyperexponential jump density. Finally drawing upon the BUtools package, we deal with a Cramér-Lundberg model with a non phase-type jump density.was that checked?

EXAMPLE 1. *Consider a Cramér-Lundberg process with density function* $\frac{150}{83}e^{-3x} + \frac{42}{83}e^{-2x} + \frac{12}{83}e^{-x}$, *and* $c = 1$, $\lambda = \frac{83}{48}$, $\theta = \frac{263}{235}$, $q = \frac{5}{48}$, $p = \frac{263}{498}$. *The Laplace exponent of this process is* $\kappa(s) = s - \frac{12s}{83(s+1)} - \frac{21s}{83(s+2)} - \frac{50s}{83(s+3)}$ *and from this one can (numerically) invert* $\frac{1}{\kappa(s)-q} = \widehat{W}_q(s)$ *to obtain the scale function*

$$\begin{aligned} W_q(x) &= -0.0813294e^{(-2.60997x)} - 0.179472e^{(-1.68854x)} \\ &\quad -0.373887e^{(-0.779311x)} + 1.63469e^{(0.18198x)}. \end{aligned}$$

	Dominant exponent Φ_q	Percent relative error (Φ_q)	Optimal barrier b^*	Percent relative error (b^*)
Exact	0.18198	0	1.89732	0
Expo	0.184095	1.162215628	2.04608	7.840532962
Dev	0.182011	0.017034839	1.91233	0.79111589
Renyi	0.181708	0.149466974	2.08136	9.699997892
LapDev	0.178939	1.671062754	2.04661	7.868467101

Table 1: Example 1: Values of Φ_q and b^* obtained from the approximations and percent relative error when compared to the exact value

	Dominant exponent Phi_q	Percent relative error (Phi_q)	Optimal barrier b*	Percent relative error (b*)
Exact	0.666084	0	0.538	0
Expo	0.691616	3.833150173	0.506947	5.771933086
Dev	0.670061	0.597071841	0.834488	55.10929368
Renyi	0.650448	2.347451673	0.260532	51.5739777
LapDev	0.587976	11.72644892	0.655954	21.92453532

Table 2: Example 2: Values of Φ_q and b^* obtained from the approximations and percent relative error when compared to the exact value

From here one can obtain the dominant exponent $\Phi_q = 0.18198$*, and since the minimum of* W_q' *is at* $b^* = 1.89732$*, we conclude that this is the optimal barrier that would maximize dividends.*

Continuing, from the parameters of the process, one can obtain the approximations to the scale function W_q *as described in the earlier section. Table 1 gives a summary of the values of* Φ_q *and* b^* *obtained from these approximations, as well as the each one's percent deviation from the exact value. Figure 1 shows the plots of* W_q *as well as its first two derivatives.*

From table 1, we can observe a percent relative error of less than 2% *for each of the approximations'* Φ_q *value, with the DeVylder approximation's* Φ_q *beating the others. Considering the optimal barrier* b^* *obtained from each, we observe only the DeVylder approximation's* b^* *to have a percent relative error of less than* 7%.

EXAMPLE 2. *Consider the Cramér-Lundberg process with density of claims* $f(x) = \frac{5}{2}e^{-5x} + \frac{4}{5}e^{-4x} - \frac{1}{5}e^{-3x} - \frac{1}{5}e^{-2x} + \frac{1}{20}e^{-x}$ *and* $c = \frac{23}{90}$, $q = \frac{1}{10}$.

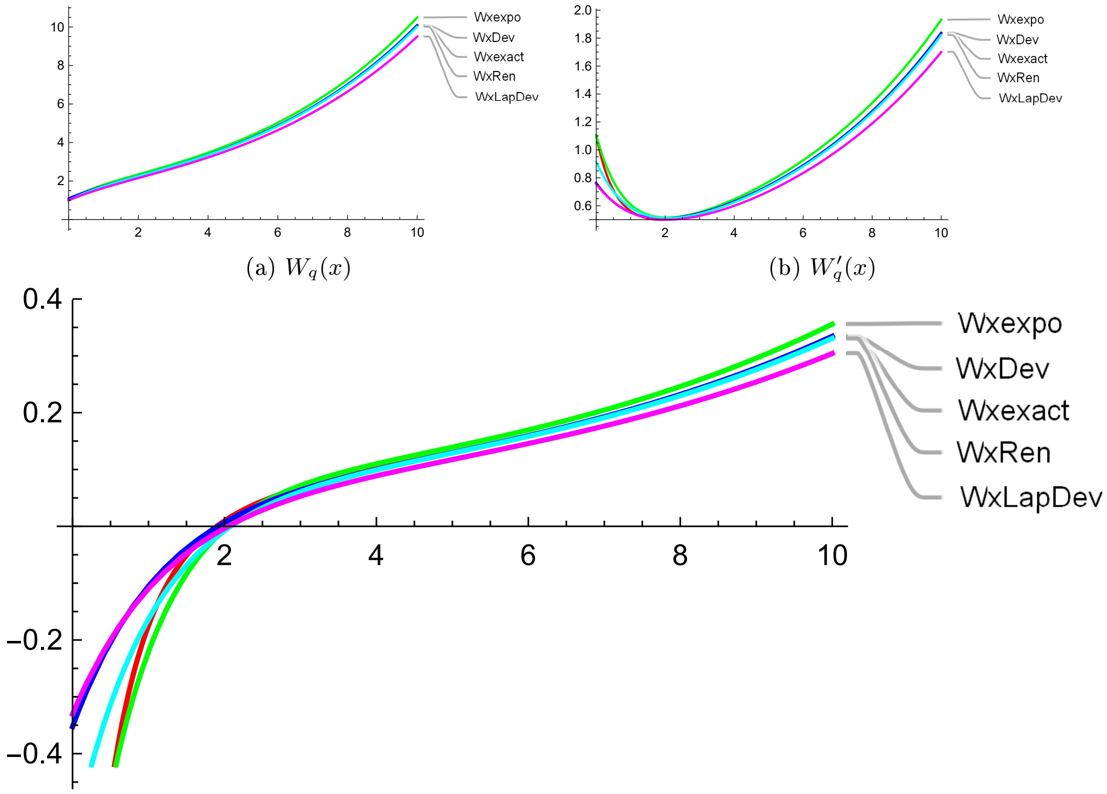
One can solve for the other parameters of the process yielding $\lambda = \frac{7}{12}$, $\theta = 1$, $p = 23/180$, $\rho = \frac{1}{2}$*. The Laplace exponent of this process is* $\kappa(s) = \frac{23s}{90} - \frac{s}{20(s+1)} + \frac{s}{10(s+2)} + \frac{s}{15(s+3)} - \frac{s}{5(s+4)} - \frac{s}{2(s+5)}$ *and from here the scale function is*

$$\begin{aligned} W_q(x) &= -0.0831561\ e^{(-4.35135x)} + 0.684818\ e^{(-2.65126x)} - 0.595164\ e^{(-0.837877x)} + 6.02604\ e^{(0.666084x)} \\ &\quad -2.11949\ e^{(-2.57585x)} \cos[0.811233x] + 2.39748\ e^{(-2.57585x)} \sin[0.811233x] \end{aligned}$$

The approximations for $W_q(x)$ *give again Renyi and classic de Vylder approximationas champions. The dominant exponents of the Renyi and de Vylder approximations are* 0.650448, 0.670061*, one below and one above the real* $\Phi_q = 0.666084$*. This suggests that the average of the two approximations would improve on both of them.*

The exact optimal barrier is $b^* = 0.538$*, the Renyi optimal barrier is* $b_R = 0.260532$*, and the relative error is* 0.515739*. Both* W_q'' *and its approximation are increasing functions.*

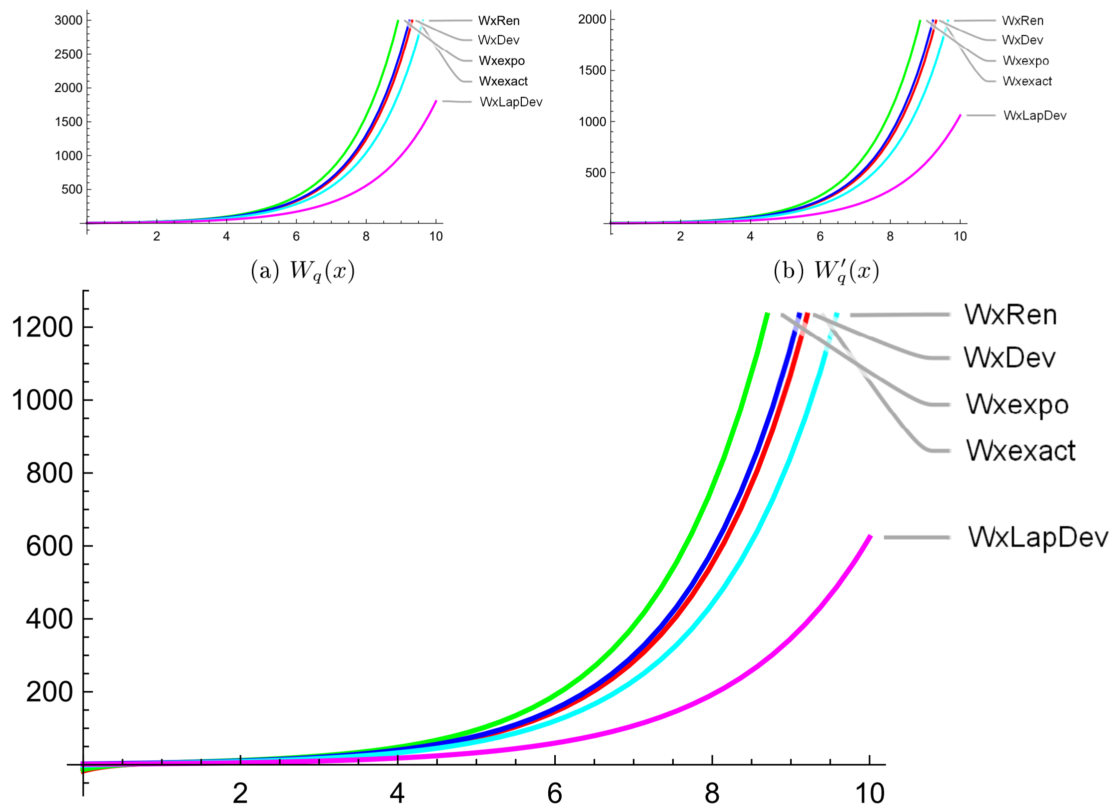
–see [JT89] for more hardest types of approximations; therein the case of mixture of Erlang distributions of sufficiently high common order was investigated.



(a) $W_q(x)$ (b) $W_q'(x)$

(c) $W_q''(x)$

Figure 1: Example 1: Plots of $W_q(x)$, $W_q'(x)$, and $W_q''(x)$ of the exact solution and the approximations



(a) $W_q(x)$

(b) $W_q'(x)$

(c) $W_q''(x)$

Figure 2: Example 2: Plots of $W_q(x)$, $W_q'(x)$, and $W_q''(x)$ of the exact solution and the approximations

	Dominant exponent Phi_q	Percent relative error (Phi_q)	Optimal barrier b*	Percent relative error (b*)
Exact	0.420145	0	0.797999	0
Expo	0.429586	2.247081365	0.857445	7.449382769
Dev	0.420885	0.17612967	0.0686674	91.39505187
Renyi	0.416601	0.843518309	0.794273	0.466917878
LapDev	0.392412	6.600816385	0.271363	65.99456892

Table 3: Example 3: Values of Φ_q and b^* obtained from the approximations and percent relative error when compared to the exact value

EXAMPLE 3. *In The following example, we take a Cramér-Lundberg process with*

$$c = \frac{7}{18}, \lambda = \frac{47}{60}, \theta = 1$$

and non-hyperexponential density of claims $f(x) = \frac{5}{2}e^{-5x} + \frac{4}{5}e^{-4x} + \frac{1}{5}e^{-3x} - \frac{1}{5}e^{-2x} + \frac{1}{20}e^{-x}$.

The scale function is

$$W_q(x) = -0.083013\ 2.71828^{(-4.45457x)} - 0.193651\ 2.71828^{(-3.43626x)} - 0.459603\ 2.71828^{(-0.855781x)}$$
$$+ 4.15668\ 2.71828^{(0.454369x)} - 0.84898\ 2.71828^{(-2.21816x)} \cos[0.513884x] + 1.24626\ 2.71828^{(-2.21816x)} \sin[0.513884x]$$

In this case, the approximations for $W_q(x)$ *give Renyi as winner. The dominant exponents of the Renyi and de Vylder approximations are* 0.44982, 0.455485, *one below and one above the real* $\Phi_q = 0.454369$.

The exact optimal barrier is $b^* = 0.779653$, *the Renyi optimal barrier is* $b_R = 0.732339$, *and the relative error is* 0.060686. *Both* W_q'' *and its approximation are increasing functions.*

EXAMPLE 4. *Next we consider a more challenging example from BUTools, produced by taking a Cramér-Lundberg process with matrix exponential, non phase-type density of claims* $f(x) = \alpha e^{Ax}(-A)\mathbf{1}$, *where* $\alpha = (-2.4, 0.9, 2.5), A = \begin{pmatrix} -6.2, 2, 0 \\ 2, -9, 1 \\ 1, 0, -3 \end{pmatrix}$ *and* $\theta = 1, q = 1/10$.

–see [HWR92] and [?] for other tricky densities.

The scale function is

$$-0.0655864e^{-9.35143x} + 0.149742e^{-6.89805x} - 0.676982e^{-1.26245x} + 1.3476e^{0.142175x}$$

Now $\Phi_q = 0.142175$; *the dominant exponent of the classic de Vylder approximation is very close at* 0.142174, *and the dominant exponent of Renyi,* 0.142238, *is also close.*

The exact optimal barrier is $b^* = 2.61925$, *the Renyi optimal barrier is* $b^* = 2.59638$, *and the relative error is* 0.00873044. *For classic de Vylder approximation the relative error is* 0.00609153.

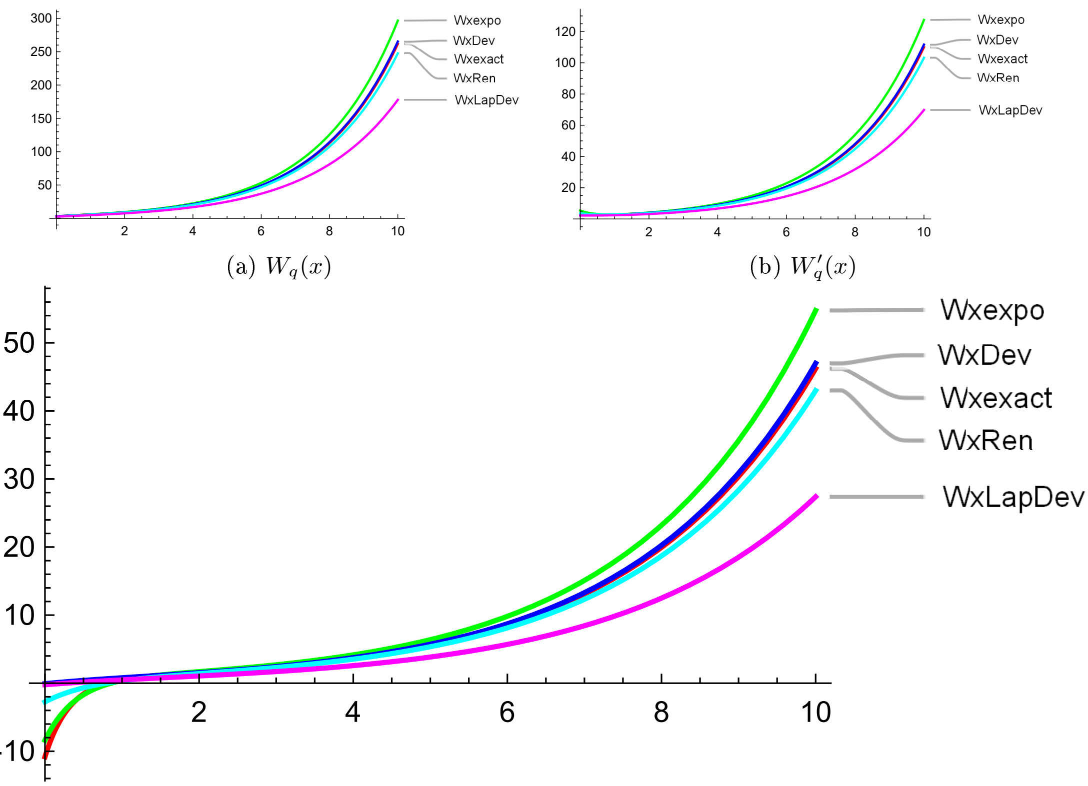
However, we may have a surprise in examples where b^* *is not the unique root of* $W_q''(x)$, *and we may be faced with the Azcue-Muller-Loeffen phenomenon.*

References

[AGLW20] F. Avram, D. Goreac, J Li, and X Wu, *Equity cost induced dichotomy for optimal dividends in the cramér-lundberg model*, IME (2020).

[HWR92] C.M. Harris, G.M. William, and F.B. Robert, *A note on generalized hyperexponential distributions*, Stochastic Models **8** (1992), no. 1, 179–191.

[JT89] M.A. Johnson and M.R. Taaffe, *Matching moments to phase distributions: Mixtures of erlang distributions of common order*, Stochastic Models **5** (1989), no. 4, 711–743.



(a) $W_q(x)$

(b) $W_q'(x)$

(c) $W_q''(x)$

Figure 3: Example 3: Plots of $W_q(x)$, $W_q'(x)$, and $W_q''(x)$ of the exact solution and the approximations