PhD Mathematics Progress Report

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April 30, 2020

Outline

Summary - Test

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- Title of thesis (tentative) : Numerical methods for the optimization of dividends, reinsurance, and investments
- Keywords: ruin probability, Pollaczek-Khinchine formula, scale function, optimal dividends, Padé approximations
- Activities for the first year:
  1. Established knowledge of background material
  2. Took and completed necessary trainings (les formations)
  3. Submitted an article to a journal

Ruin Theory for Lévy Processes

Definition

A Lévy process $X = \{X(t)\}_{t \in \mathbb{R}^+}$ is a stochastic process taking values in $\mathbb{R}$ such that X has

1. independent increments: $X(t) - X(s)$ is independent of $\{X(u) : u \leq s\}$ for any $s < t$,
2. stationary increments: $X(s+t) - X(s)$ has the same distribution as $X(t) - X(0)$ for any $s, t > 0$,
3. continuity in probability: $X(s) \to X(t)$ in probability as $s \to t$.

Note: sometimes we write $X(t)$ as X_t

Ruin Theory for Lévy Processes

Remark

Let $X(t)$ be a Levy process. Then

$$\mathbb{E}_0[e^{sX(t)}] = e^{t\kappa(s)}$$

for a function $\kappa(s)$ called the Levy exponent of $X(t)$.

Remark (Levy-Khintchine representation of a Levy process)

Let $X(t)$ be a Levy process. Then

$$\kappa(s) = ps + \frac{\sigma^2}{2}s^2 + \int_0^\infty [e^{-sy} - 1 + sy]\nu(dy)$$

where $p = \mathbb{E}_0[X(1)]$, $\sigma \geq 0$ and ν is the Levy measure of $-X$.

Notes: $\mathbb{E}_x[X(t)] := \mathbb{E}[X(t)|X(0) = x]$
In this presentation we consider $\sigma = 0$

Ruin Theory for Lévy Processes

In the ruin problem, one assumes

1. an initial capital $u > 0$,
2. a constant stream of 'money' $c > 0$ per unit time, and
3. a varying stream of losses represented by $S(t)$ (a Lévy process)

Ruin Theory for Lévy Processes

In the ruin problem, one assumes

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2. a constant stream of 'money' $c > 0$ per unit time, and
3. a varying stream of losses represented by $S(t)$ (a Lévy process)

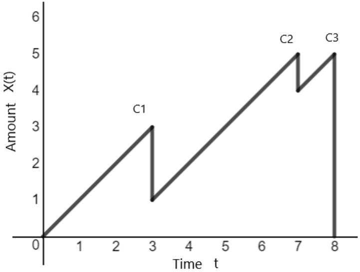
Overall one observes the behaviour of the process

$$X(t) = u + ct - S(t),$$

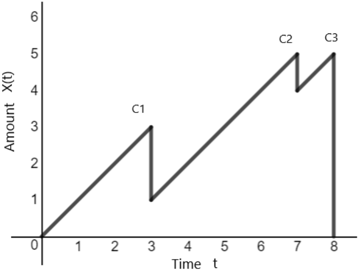
which is also Lévy. Furthermore this process $X(t)$ is said to be spectrally negative, that is, jumps represented by $-S(t)$ can only be negative.

Now, in particular if we assume $S(t) = \sum_{i=1}^{N_\lambda(t)} C_i$, i.e. $S(t)$ is a countable sum of random variables C_i having the same distribution function F, and the number of C_i's being summed, $N_\lambda(t)$, is also assumed to be random variable having a Poisson distribution with parameter λ, we get the Cramer Lundberg process.

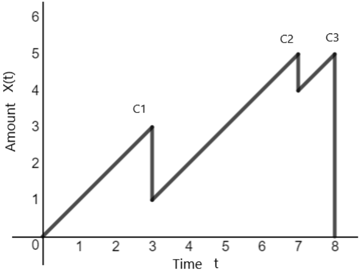
Figure: A sample path of a Cramér Lundberg process

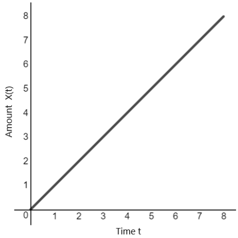


Ruin Theory for Lévy Processes

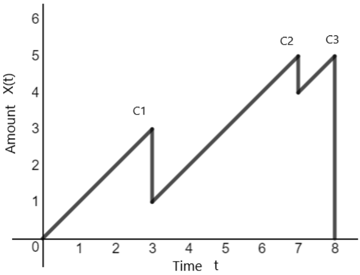


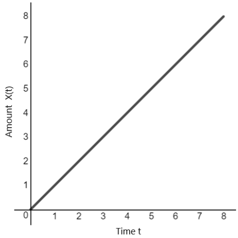
Ruin Theory for Lévy Processes

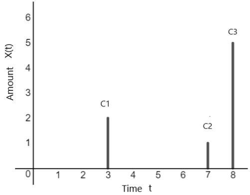




Ruin Theory for Lévy Processes







Define

$$\tau_b^+ = \inf\{t \geq 0, X(t) > b\}$$
$$\tau_b^- = \inf\{t \geq 0, X(t) < b\}$$

the infimum values at which the process $X(t)$ goes above or below a certain value b.
Using this notation, we can denote the time of ruin as $\tau = \tau_0^-$. Moreover, if $X(t) > 0 \ \forall t \geq 0$, then we say $\tau = +\infty$.

Ruin Theory for Lévy Processes

Let $t, u > 0$. The probability of ruin before time t and given initial value u is written as

$$\Psi(t|u) = \mathbb{P}[\tau < t | X(0) = u].$$

We refer to this scenario ruin in finite time.

Ruin Theory for Lévy Processes

Let $t, u > 0$. The probability of ruin before time t and given initial value u is written as

$$\Psi(t|u) = \mathbb{P}[\tau < t|X(0) = u].$$

We refer to this scenario ruin in finite time.
Alternatively, one can study probabilities of survival in finite time by observing the function

$$\bar{\Psi}(t|u) = \mathbb{P}[\tau \geq t|X(0) = u] = 1 - \Psi(t|u).$$

Letting the value of t approach infinity, we get a quantity which can be interpreted as the probability of eventual ruin, or ruin in infinite time. Let $u > 0$. The probability of ruin given initial value u is written as

$$\Psi(u) = \mathbb{P}[\tau < +\infty | X(0) = u].$$

Same as in the last definition, we also can study

$$\bar{\Psi}(u) = \mathbb{P}[\tau = +\infty | X(0) = u] = 1 - \Psi(u).$$

Ruin Theory for Lévy Processes

A formula published by Pollaczek (1930) and used thereafter by contemporaries originally tackling problems in queueing theory found great use in the study of ruin theory:

$$\hat{\Psi}(s) = \int_0^\infty e^{-su}\Psi(u)du = \frac{\lambda(m_1 - \hat{\bar{F}}(s))}{s(c - \lambda\hat{\bar{F}}(s))}$$

where $\hat{\Psi}(s)$ and $\hat{\bar{F}}(s)$ are the Laplace transforms of the ruin function and the survival function of the claim distributions respectively, and m_1 is the first moment of the claims.
Further results concerning the relationship of $\hat{\Psi}(s)$ with other quantities such as the Levy exponent $\kappa(s)$ of the underlying process can be derived from this formula.

Ruin Theory for Lévy Processes

Theorem (The Pollaczek-Khinchine formulas for the ruin function Laplace transform)

Let $X(t) = u + ct - S(t), \quad S(t) = \sum_{i=1}^{N_\lambda(t)} C_i$ be a Cramér Lundberg risk process and Ψ its corresponding ruin function. Then the Laplace transforms of the ruin function and its corresponding survival function can be expressed as

$$\hat{\Psi}(s) = \int_0^\infty e^{-su}\Psi(u)du = \frac{\lambda(m_1 - \hat{\bar{F}}(s))}{s(c - \lambda\hat{\bar{F}}(s))} = \frac{1}{s} - \frac{\kappa'(0)}{\kappa(s)}$$

$$\hat{\overline{\Psi}}(s) = \int_0^\infty e^{-su}\overline{\Psi}(u)du = \frac{1}{s} - \hat{\Psi}(s) = \frac{\kappa'(0)}{\kappa(s)}$$

where $\rho = (\lambda m_1)/c$, $f_e(x) := \bar{F}(x)/m_1$, and $\kappa(s) = s(1 - \rho\hat{f}_e(s))$ the Levy exponent of the Cramer Lundberg process.

The W_q scale function and Discounting

The W_q scale function and Discounting

One may generalize the ruin problem to problems where it is not necessarily the first passage time thru zero that matters.
In the two sided exit problem, we assume $a < b$, and are interested in quantities such as τ_a^+, τ_b^-, $\tau := \min\{\tau_a^+, \tau_b^-\}$ and

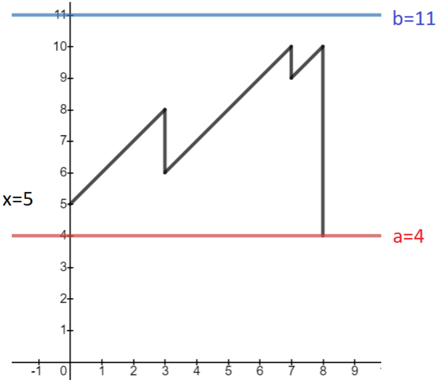
$$\Psi_{a,b}(x) = \mathbb{P}_x[\tau_a^+ < \tau_b^-]$$

$$\overline{\Psi}_{a,b}(x) = \mathbb{P}_x[\tau_a^+ > \tau_b^-]$$

.

The W_q scale function and Discounting

Figure: Sample path for a two sided exit problem



The W_q scale function and Discounting

It can be shown, via the Markov property of the underlying process, that for some function W, we have

$$\overline{\Psi}_{a,b}(x) = \frac{W(x-a)}{W(b-a)}$$

We will refer to this function W as the W scale function of the process. Furthermore upon taking the Laplace transform of the previous equation we get

$$\hat{\overline{\Psi}}_{a,b}(s) = \frac{\hat{W}(s)}{W(b-a)},$$

and in the case of the ruin problem where $\hat{\overline{\Psi}}(s) = \frac{\kappa'(0)}{\kappa(s)}$, we get the idea that

$$\hat{W}(s) = C\frac{1}{\kappa(s)}$$

up to a certain constant C.

The W_q scale function and Discounting

Generalizing further, suppose there exist an ‘interest rate’ $q \geq 0$ such that one considers the ‘discounted value’ of the underlying process with respect to this q when deciding for the first passage time.
In this scenario,

$$\overline{\Psi}_{a,b}(x) = \mathbb{E}_x[e^{-q\tau_b^+} \cdot \mathbb{1}_{\{\tau_a^+ > \tau_b^-\}}]$$

where $\mathbb{1}_A$ is the event that A happens.
Here the probability of survival considers all events where $\tau_a^+ > \tau_b^-$, and gets the expected value of the discounted value of 1 at the time τ_b^+ with respect to q, hence the term $e^{-q\tau_b^+}$.

The W_q scale function and Discounting

Even with discounting, through the same analysis as before one can find a function W_q for which

$$\overline{\Psi}_{a,b}(x) = \frac{W_q(x-a)}{W_q(b-a)}.$$

In fact if one defines $W_q(s) : (0, +\infty) \to [0, +\infty)$ to have a Laplace transform

$$\hat{W}_q(s) = \int_0^\infty e^{-sx} W_q(x)\ dx := \frac{1}{\kappa(s) - q} \quad \forall s > \Phi(q),$$

one would arrive at such a function. As before we call this the W_q scale function of the underlying process.
Note: $\Phi(q) := \sup\{s \geq 0 : \kappa(s) - q = 0\}$

The W_q scale function and Discounting

Remark

The scale function W_q is continuous and increasing on $[0, \infty]$. Bingham (1976), (Bertoin 1998, 49 Thm. VII.8), (Kyprianou 2014, Thm. 8.1)

The W_q scale function and Discounting

Remark (KS07, Lem. 4.3-4.4, KKR13, Lem. 3.2-3.3)

The behavior in the neighborhood of zero of W_q can be obtained from the behavior of its Laplace transform $\hat{W}_q$ at ∞

$$W_q(0^+) = \lim_{s\to\infty} \frac{s}{\kappa(s) - q} = \begin{cases} \frac{1}{c} & \text{if } X \text{ is of bdd variation} \\ 0 & \text{if } X \text{ is of unbdd variation} \end{cases} \tag{1}$$

$$W_q'(0^+) = \lim_{s\to\infty} s\left(\frac{s}{\kappa(s) - q} - W_q(0)\right) = \begin{cases} \frac{q+\nu(u,\infty)}{c^2} & \text{if } X \text{ is of bdd variation} \\ \frac{2}{\sigma^2} & \text{if } X \text{ is of unbdd variation} \end{cases} \tag{2}$$

Note: Perturbation parameter σ

The Dividend Problem

The Dividend Problem

The notion of the problem of dividends is popularized by De Fenetti (1957) and starts by considering a Cramer-Lundberg process $\{X(t)\}_{t\in[0,\infty]}$ (we can of course consider more general cases) and another process $\{L(t)\}_{t\in[0,\infty]}$ which is left continuous, nonnegative, and non decreasing.

The quantity $L(t)$ can be thought of as the cumulative dividends paid up to time t of an insurance company and hence the risk process taking into account the dividends is $\{U(t)\}_{t\in[0,\infty]}$ where

$$U(t) = X(t) - L(t).$$

The general problem is the characterization of L and the maximization of U through the choice of L, often called a dividend strategy.

The Dividend Problem

A dividend strategy is called admissible if at any time before ruin, all dividend payments are smaller than the current value of U, that is

$$L(t^+) - L(t) \leq U(t), \quad \forall t < \tau$$

where τ is the ruin time of U.

There are a number of possible dividend strategies, hence one often associates another parameter to the process L, say L^π.

The Dividend Problem

Denoting all admissible strategies by Π, we write the expected value at the discount rate $q \geq 0$ associated with the dividend strategy $\pi \in \Pi$ with initial surplus $x \geq 0$ as

$$V^\pi(x) = \mathbb{E}_x \left[\int_0^{\tau^\pi} e^{-qt} dL_t^\pi \right].$$

One may sometimes see V^π being referred to as the value function associated with strategy π. The problem is the characterization of

$$V(x) = \sup_{\pi \in \Pi} V^\pi(x).$$

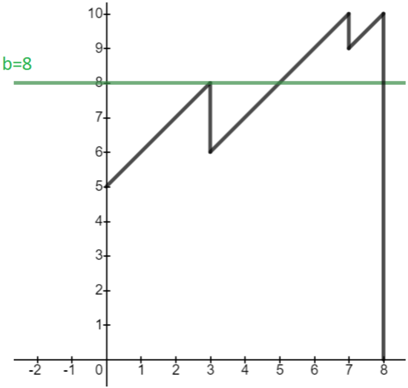
The Dividend Problem

Under the barrier strategy studied by Avram (2007), the process U is never allowed to go above a constant level $b > 0$, by setting

$$L(t) = (X(t) - a)^+ = \max\{0, X(t) - a\}.$$

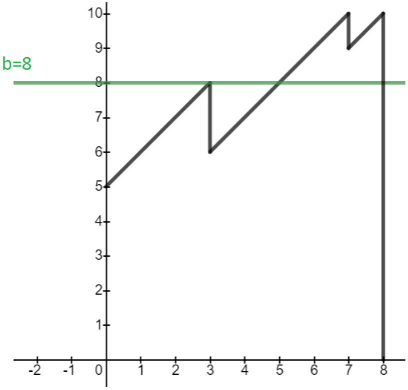
That is, surplus above the barrier b is always paid as dividend.

The Dividend Problem

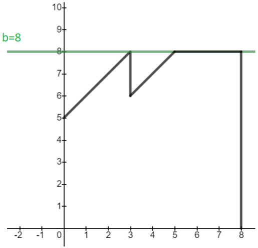


$X(t)$

The Dividend Problem

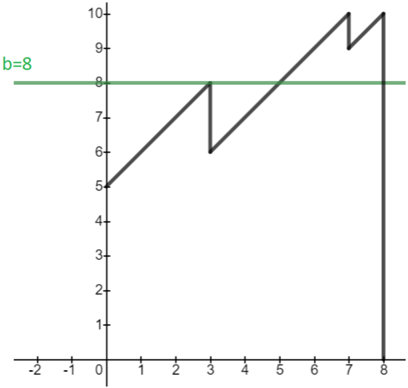


$X(t)$

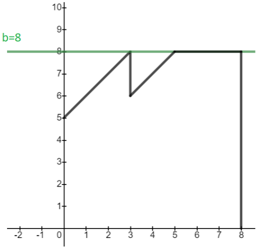


$U(t)$

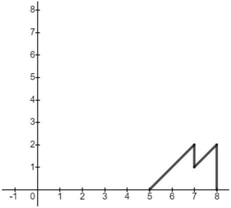
The Dividend Problem



$X(t)$



$U(t)$



$L(t)$

The Dividend Problem

Clearly, this strategy, which we denote by $b] \in \Pi$, is admissible, and referencing Avram (2007), we have

Theorem

Let W_q be the associated scale function to X under the discount rate $q \geq 0$. Under a constant dividend barrier strategy $b] \in \Pi$,

$$V^{b]}(x) = \begin{cases} \frac{W_q(x)}{W_q'(b)} & \text{if } x \leq b \\ x - b + \frac{W_q(b)}{W_q'(b)} & \text{if } x > b \end{cases}$$

The Dividend Problem

Looking at $V^{b]}$ as a function of b instead of x, we can solve for V by looking at where the derivative of $V^{b]}$ is zero. We summarize this result through the use of the 'barrier function' H.

Remark (Avram et al. [2007], Loeffen [2008])

Let $b \geq 0$ and $H(b) = \frac{1}{W_q'(b)}$.
If H is differentiable with $H'(0) > 0$, and has a unique local maximum $b^* > 0$, then this b^* yields the optimal barrier strategy, i.e. $V = V^{b^*]}$

Notes:
$H'(0) > 0 \iff W_q''(0) < 0$
local maximum at $b^* \Rightarrow W_q''(b^*) = 0$
uniqueness of local maximum at $b^* \Rightarrow H(b_1) \geq H(b_2)$, whenever $b* \leq b_1 \leq b_2$

The Dividend Problem

This remark tells us that not only is $V^{b^*]}$ the optimal strategy over the space of the constant barrier strategies, but it is optimal over all strategies in Π.

We will not be tackling other dividend strategies in this presentation, nonetheless, we stress the fact that they do exist (see for example threshold strategies (Gerber and Shiu [2004]), multiband strategies (Azcue and Muler [2005]), etc.)

The Dividend Problem

Example (The Cramer-Lundberg model with exponential jumps)

Consider the Cramer-Lundberg model
$X(t) = u + ct - S(t), \quad S(t) = \sum_{i=1}^{N_\lambda(t)} C_i$
where C_i's are exponentially distributed with $\mathbb{E}C_i = \frac{1}{\mu}$.

Standard computations yield

$$\kappa(s) = s\left(c - \frac{\lambda}{s+\mu}\right), \quad \kappa'(0) = c - \frac{\lambda}{\mu}$$

and solving

$$\kappa(s) - q = 0 \iff cs^2 + (c\mu - \lambda - q)s - q\mu = 0$$

Example (The Cramer-Lundberg model with exponential jumps)

$$\kappa(s) - q = 0 \iff cs^2 + (c\mu - \lambda - q)s - q\mu = 0$$

yields two distinct solutions $\gamma_1 \leq 0 \leq \gamma_2$ given by

$$\gamma_1 = \frac{1}{2c}\left(-(c\mu - \lambda - q) + \sqrt{(c\mu - \lambda - q)^2 + 4cq\mu}\right)$$

$$\gamma_1 = \frac{1}{2c}\left(-(c\mu - \lambda - q) - \sqrt{(c\mu - \lambda - q)^2 + 4cq\mu}\right).$$

The Dividend Problem

Example (The Cramer-Lundberg model with exponential jumps)

In this example the Laplace transform of the W_q scale function is

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - q} = \frac{s + \mu}{cs^2 + (c\mu - \lambda - q)s - q\mu}$$

which, when inverted becomes

$$W_q(x) = \frac{(\gamma_1 + \mu)e^{\gamma_1 x} - (\gamma_2 + \mu)e^{\gamma_2 x}}{c(\gamma_1 - \gamma_2)}$$

Example (The Cramer-Lundberg model with exponential jumps)

Differentiating we get

$$W_q'(x) = \frac{\gamma_1(\gamma_1+\mu)e^{\gamma_1 x} - \gamma_2(\gamma_2+\mu)e^{\gamma_2 x}}{c(\gamma_1-\gamma_2)}$$

$$W_q''(x) = \frac{\gamma_1^2(\gamma_1+\mu)e^{\gamma_1 x} - \gamma_2^2(\gamma_2+\mu)e^{\gamma_2 x}}{c(\gamma_1-\gamma_2)}$$

which implies

$$W_q''(x) = 0 \iff x = \frac{1}{\gamma_1-\gamma_2}\log\frac{\gamma_2(\gamma_2+\mu)}{\gamma_1(\gamma_1+\mu)}.$$

The Dividend Problem

Example (The Cramer-Lundberg model with exponential jumps)

The function $W_q'(x)$ is unimodal with extremum at $x* = \frac{1}{\gamma_1 - \gamma_2} \log \frac{\gamma_2(\gamma_2+\mu)}{\gamma_1(\gamma_1+\mu)}$. Hence if

$$W_q''(0) < 0 \Rightarrow x* \text{ is a global minimum}$$

and

$$W_q''(0) \geq 0 \Rightarrow x* \text{ is a global maximum.}$$

Therefore the optimal boundary $b*$ for the dividend problem is

$$b* = \begin{cases} \frac{1}{\gamma_1 - \gamma_2} \log \frac{\gamma_2(\gamma_2+\mu)}{\gamma_1(\gamma_1+\mu)} & \text{if } W_q''(0) < 0 \\ 0 & \text{if } W_q''(0) \geq 0. \end{cases}$$

Approximations to the W_q scale function

Approximations to the W_q scale function

Consider a spectrally negative Levy risk process $X = u + ct - S(t)$ and associated Levy measure ν with $\kappa(s) = s(c - \hat{\bar{\nu}}(s))$ as its Levy-Khintchine representation.
Letting $\nu_k = \int_0^\infty x^k \nu(dx)$ be the moments of the Levy measure. Then one can obtain a power series representation of the Levy exponent in terms of the moments given by

$$\kappa(s) = s(c - \hat{\bar{\nu}}(s)) = cs + \sum_{k=1}^{\infty} \nu_k \frac{(-s)^k}{k!}.$$

In the Cramer-Lundberg case where the intensity of the claim arrivals are given by the Poisson parameter λ, $\nu_k = \lambda m_k$ where m_k's are the moments of the claims C_i.

One can thus write

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - q} = \frac{1}{cs - \sum_{k=1}^{\infty} \nu_k \frac{(-s)^k}{k!} - q}.$$

We can approximate this expression by a Pade approximation of order $[n-1, n]$

$$\hat{W}_q(s) \approx \frac{P_{n-1}(s)}{Q_n(s)} = \frac{\sum_{i=0}^{n-1} a_i s^i}{cs^n + \sum_{i=0}^{n-1} b_i s^i} \tag{3}$$

for some constants a_i, b_i, $i = 0, \ldots, n-1$.

After the approximation the denominator can be factored as

$$cs^n + \sum_{i=0}^{n-1} b_i s^i = c(s-\gamma_0)\prod_{i=1}^{n}(s+\gamma_i)$$

and after a partial fraction decomposition of (3) we get

$$\hat{W}_q(s) \approx \frac{C_0}{s-\gamma_0} + \sum_{i=1}^{n} \frac{C_i}{s+\gamma_i}$$

$$\Rightarrow W_q(x) \approx C_0 e^{\gamma_0 x} + \sum_{i=1}^{n} C_i e^{-\gamma_i x}$$

Looking back at the properties of the W_q function in the Cramer-Lundberg case, we can force (1) and (2) by setting $a_{n-1} = 1$ and $b_{n-1} = ca_{n-2} - \lambda - q$

$$\hat{W}_q(s) \approx \frac{s^{n-1} + \sum_{i=0}^{n-2} a_i s^i}{cs^n + (ca_{n-2} - \lambda - q)s^{n-1} + \sum_{i=0}^{n-2} b_i s^i}.$$

Since we are fitting two values $W_q(0)$ and $W_q'(0)$ into the Pade approximation, this would be a $[1, 2]$ Pade approximant

$$\hat{W}_q(s) \approx \frac{s + a_0}{cs^2 + (ca_0 - \lambda - q)s + b_0}.$$

Comparing this with the true value of $\hat{W}_q(s)$,

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - q} \approx \frac{s + a_0}{cs^2 + (ca_0 - \lambda - q)s + b_0}$$

$$\Rightarrow cs^2 + (ca_0 - \lambda - q)s + b_0 \approx (s + a_0)(\kappa(s) - q)$$

$$\Rightarrow cs^2 + (ca_0 - \lambda - q)s + b_0 \approx (s + a_0)(cs - \nu_1 s + ... - q)$$

$$\Rightarrow cs^2 + (ca_0 - \lambda - q)s + b_0 \approx (s + a_0)(cs - \lambda m_1 s - q)$$

$$\Rightarrow a_0 = \frac{1}{m_1} \text{ and } b_0 = -\frac{q}{m_1}.$$

That is,

$$\hat{W}_q(s) \approx \frac{s + \frac{1}{m_1}}{cs^2 + (\frac{c}{m_1} - \lambda - q)s - \frac{q}{m_1}}. \tag{4}$$

Approximations to the W_q scale function

We note that the given approximation in (4) is the same approximation if we consider the claims to be exponential with $m_1 = \frac{1}{\mu}$, as in (33).

The optimal boundary $b*$ for this approximation to the W_q function is thus already been solved.

One can also approximate W_q by a higher order Pade approximant, taking into account only (1).
That is, setting $n = 2$ and $a_{n-1} = 1$ we have

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - q} \approx \frac{s + a_0}{cs^2 + b_1 s + b_0}.$$

$$\Rightarrow \hat{W}_q(s) \approx \frac{s + a_0}{cs^2 + b_1 s + b_0}.$$

$$\Rightarrow cs^2 + b_1 s + b_0 \approx (s + a_0)(\kappa(s) - q)$$

$$\Rightarrow cs^2 + b_1 s + b_0 \approx (s + a_0)(cs - \lambda m_1 s + \lambda m_2 \tfrac{s^2}{s} - q)$$

$$\Rightarrow a_0 = \frac{2m_1}{m_2}, b_1 = \frac{2cm_1 - 2\lambda m_1^2 - m_2 q}{m_2}, \text{ and } b_0 = -\frac{2m_1 q}{m_2}.$$

Therefore

$$\hat{W}_q(s) \approx \frac{s + \frac{2m_1}{m_2}}{cs^2 + \left(\frac{2cm_1 - 2\lambda m_1^2 - m_2 q}{m_2}\right) s - \frac{2m_1 q}{m_2}}.$$

This can be rewritten as

$$\hat{W}_q(s) \approx \frac{s + \frac{1}{\tilde{m}_1}}{cs^2 + \left(\frac{c}{\tilde{m}_1} - \lambda \frac{m_1}{\tilde{m}_1} - q\right) s - \frac{q}{\tilde{m}_1}}.$$

where $\tilde{m}_1 = \frac{m_2}{2m_1}$ is the first moment of the equilibrium density $f_e = \frac{\bar{F}}{m_1}$. This approximation is more commonly known as DeVylder's method.

Consider the Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!} \cdot \frac{d^n}{dx^n}[e^{-x}x^n] = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!}$$

for $x \geq 0$, $n = 0, 1, \ldots$. These polynomials are orthogonal with respect to the weight $e^{-\frac{x}{2}}$, and for constant α, $L_n(\alpha x)$ has Laplace transform given by

$$\hat{L}_n(s) = \frac{(s-\alpha)^n}{s^{n+1}}, n = 0, 1, \ldots.$$

Define $W_0^{(\Phi_q)}$ to be the $q = 0$ scale function with respect to the Esscher transformed measure $P^{(\Phi_q)}$ (will not be discussed in detail, see (Albrecher and Asmussen [2010]), (Kyprianou [2014])) given by

$$W_q(x) = e^{x\Phi_q} \cdot W_0^{(\Phi_q)}(x).$$

By dividing by $e^{x\Phi_q}$, one removes the unique positive pole of $\hat{W}_q$

$$\hat{W}_q(s) = \frac{1}{s - \Phi_q} \cdot \widehat{W_0^{(\Phi_q)}}(s)$$

$$\Rightarrow (s - \Phi_q)\hat{W}_q(s) = \widehat{W_0^{(\Phi_q)}}(s).$$

By property of Laplace transforms we also know that

$$\begin{aligned} W_0^{(\Phi_q)}(x) &= e^{-x\Phi_q} \cdot W_q(x) \\ \Rightarrow \widehat{W_0^{(\Phi_q)}}(s) &= \hat{W}_q(s+\Phi_q) = \frac{1}{\kappa(s+\Phi_q)-q} \\ &= \frac{1}{\kappa(s+\Phi_q)-\kappa(\Phi_q)}. \end{aligned}$$

Following (Abate et al. [1998]), one can consider

$$G(x) = W_0^{(\Phi_q)}(\infty) - W_0^{(\Phi_q)}(x) \tag{5}$$

and approximate using Laguerre polynomials

$$G(x) \approx C \sum_{n=0}^{\infty} B_n e^{-\alpha x/2} L_n(\alpha x) \tag{6}$$

for some chosen constants C, α, and coefficients B_n to be solved for. Taking the Laplace transform of (5)

$$\begin{aligned} \widehat{G(x)} &= \widehat{W_0^{(\Phi_q)}(\infty)} - \widehat{W_0^{(\Phi_q)}(x)} \\ \hat{G}(s) &= \frac{W_0^{(\Phi_q)}(\infty)}{s} - \widehat{W_0^{(\Phi_q)}}(s) \\ &= \frac{W_0^{(\Phi_q)}(\infty)}{s} - \frac{1}{\kappa(s+\Phi_q) - \kappa(\Phi_q)}. \end{aligned} \tag{7}$$

Taking the Laplace transform of (6) we have

$$\hat{G}(s) \approx C \sum_{n=0}^{\infty} B_n \cdot \hat{L}_n(s+\alpha/2) = C \sum_{n=0}^{\infty} B_n \frac{(s-\alpha/2)^n}{(s+\alpha/2)^{n+1}}.$$

Some computations yield

$$\hat{H}(s) = (s+\alpha/2)\hat{G}(s) \approx C \sum_{n=0}^{\infty} B_n \frac{(s-\alpha/2)^n}{(s+\alpha/2)^n}$$

$$\Rightarrow \hat{H}\left(\frac{\alpha}{2} \cdot \frac{1+z}{1-z}\right) = \left(\frac{\alpha}{1-z}\right) \hat{G}\left(\frac{\alpha}{2} \cdot \frac{1+z}{1-z}\right) \approx C \sum_{n=0}^{\infty} B_n z^n \qquad (8)$$

upon using the 'collocation' transformation $z = \frac{s-\alpha/2}{s+\alpha/2} \iff s = \frac{\alpha}{2} \cdot \frac{1+z}{1-z}$.

Hence we solve for the coefficients B_n by taking the Taylor expansion of $\hat{H}\left(\frac{\alpha}{2}\frac{1+z}{1-z}\right)$ which is known to us via (8) and (7). The approximation we get for G gives us an approximation for $W_0^{(\Phi_q)}$. An approximation for W_q is then obtained by multiplying by $e^{x\Phi_q}$.

With regards to the choice of C and α, one can do a first order Pade approximation discussed in (1), to obtain an approximation to $W_0^{(\Phi_q)}(x)$ of the form

$$W_0^{(\Phi_q)}(x) \approx W_0^{(\Phi_q)}(\infty) - C\frac{\alpha}{2}e^{-\frac{\alpha}{2}x}.$$

Approximations to the W_q scale function

Example: a Cramer-Lundberg model with exponential mixture jumps of order two

Consider a Cramer-Lundberg process with $c = 1/2$, $\lambda = 29/48$ with claim density given by $f(x) = \frac{8}{29}e^{-x} + \frac{42}{29}e^{-2x}$. To obtain rational values we fix $q = 1/16$.

Computing the Levy exponent, one gets

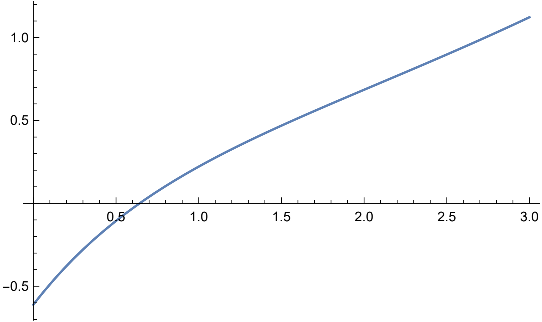
$$\kappa(s) = \frac{1}{2}s + \frac{8}{48}\left(\frac{1}{s+1} - 1\right) + \frac{21}{48}\left(\frac{2}{s+2} - 1\right)$$

which then yields

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - 1/16} = \frac{24(s+1)(s+2)}{(3s-1)(2s+1)(2s+3)}$$

$$\Rightarrow W_q(x) = -\frac{3}{11}e^{-3x/2} - \frac{9}{5}e^{-x/2} + \frac{224}{55}e^{x/3}.$$

Figure: Plot of W_q'', optimal barrier value $b^* = 0.642265$



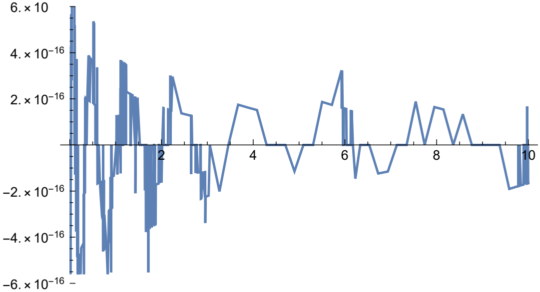
Doing the approximation described in (2), one first computes

$$\widehat{W_0^{(\Phi_q)}(x)} = \frac{8(3s+4)(3s+7)}{s(6s+5)(6s+11)}$$

which then leads to

$$\hat{G}(s) = \frac{72(57s+97)}{55(6s+5)(6s+11)}.$$

Figure: Relative errors of the Laguerre approximation in 3



A [0,1] Pade approximation to $\hat{G}$ gives us $\frac{677448}{55(6177s+5335)}$ implying the Laguerre exponent $\alpha/2 = 5335/6177 = 0.863688$. The resulting error in the Laguerre approximation is plotted in (4), with the largest error being $6 * 10^{-6}$ when 30 terms are considered in the Laguerre expansion.

Trainings completed, Article submitted

Trainings completed, Article submitted

Les Formations	Date	Hours
Gestion du projet de thése	5 Nov 2018	7
Public speaking for scientists	7 Dec 2018	14
Ethique et intégrité scientifique	13 Dec 2018	7
Information Scientifique et technique	28 Jan 2019	7
English language training*	Dec 2018 to April 2019	40**
French language training***	Jan 2019 to May 2019	40**
English for Research Communication	Feb 2019 to April 2019	15**

*resulted in a Cambridge Certificate in Advanced English CEFR level C2

**approximate number of hours

***passed level A2

Trainings completed, Article submitted

Journal name: Risks
Manuscript ID: risks-618671
Type of manuscript: Article
Title: On the Padé and Laguerre-Tricomi-Weeks moments based approximations of the scale function W and of the optimal dividends barrier for spectrally negative Lévy risk processes
Authors: Florin Avram, Andras Horvath, Serge Provost, Ulyses Solon
Received: 30 September 2019

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