Numerical methods for the optimization of dividends, reinsurance, and investments

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Abstract

We review concepts related to the ruin problem in the case of spectrally negative Lévy processes, the W_q scale function, and the De Finetti dividend problem. We then present some methods of approximation for the W_q scale function and some examples.

I. Ruin Theory for Lévy Processes

Definition. A Lévy process $X = \{X(t)\}_{t\in\mathbb{R}^+}$ is a stochastic process taking values in $\mathbb{R}$ such that X has

1. independent increments: $X(t) - X(s)$ is independent of $\{X(u) : u \leq s\}$ for any $s < t$,
2. stationary increments: $X(s+t) - X(s)$ has the same distribution as $X(t) - X(0)$ for any $s, t > 0$,
3. continuity in probability: $X(s) \to X(t)$ in probability as $s \to t$.

Remark. Let $X(t)$ be a Levy process. Then

$$\mathbb{E}_0[e^{sX(t)}] = e^{t\kappa(s)}$$

for a function $\kappa(s)$ called the Levy exponent of $X(t)$.

Remark (Levy-Khintchine representation of a Levy process)**.** Let $X(t)$ be a Levy process. Then

$$\kappa(s) = ps + \frac{\sigma^2}{2}s^2 + \int_0^\infty [e^{-sy} - 1 + sy]\nu(dy)$$

where $p = \mathbb{E}_0[X(1)]$, $\sigma \geq 0$ and ν is the Levy measure of $-X$, which must satisfy $\int_0^\infty (y \wedge y^2)\nu(dy) < \infty$.

We further note that we will be using the notation $\mathbb{E}_x[X(t)] := \mathbb{E}[X(t)|X(0) = x]$, and in this article we mostly consider cases where the perturbation parameter $\sigma = 0$.

In the ruin problem, one assumes

1. an initial capital $u > 0$,
2. a constant stream of 'money' $c > 0$ per unit time, and
3. a varying stream of losses represented by a spectrally positive Lévy process $S(t)$ (i.e. without negative jumps).

Overall one observes the behaviour of the process

$$X(t) = u + ct - S(t),$$

which is also Lévy, but spectrally negative, that is its jumps represented can only be negative.

Now, in particular if we assume $S(t) = \sum_{i=1}^{N_\lambda(t)} C_i$, i.e. $S(t)$ is a Poisson sum of random variables C_i having the same distribution function F (with the number of C_i's being summed, $N_\lambda(t)$, being assumed to be an

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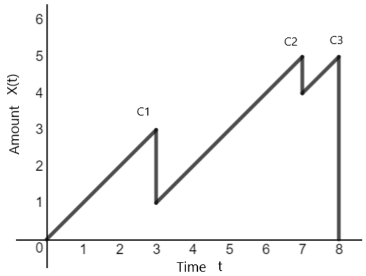
Figure 1: *A sample path of a Cramér-Lundberg process*



independent random variable having a Poisson distribution with parameter λ), we get the Cramer-Lundberg process.

We define the first passage times

$$\tau_b^+ = \inf\{t \geq 0, X(t) > b\}$$

$$\tau_b^- = \inf\{t \geq 0, X(t) < b\}$$

as the infimum values at which the process $X(t)$ goes above or below a certain value b. Using this notation, we can denote the time of ruin as $\tau = \tau_0^-$. Moreover, if $X(t) > 0 \ \forall t \geq 0$, then we say $\tau = +\infty$.

Let $t, u > 0$. The probability of ruin before time t and given initial value u is written as

$$\Psi(t|u) = \mathbb{P}[\tau < t|X(0) = u].$$

We refer to this scenario ruin in finite time.

Alternatively, one can study probabilities of survival in finite time by observing the function

$$\bar{\Psi}(t|u) = \mathbb{P}[\tau \geq t|X(0) = u] = 1 - \Psi(t|u).$$

Letting the value of t approach infinity, we get a quantity which can be interpreted as the probability of eventual ruin, or ruin in infinite time.

Let $u > 0$. The probability of ruin given initial value u is written as

$$\Psi(u) = \mathbb{P}[\tau < +\infty|X(0) = u].$$

Same as in the last definition, we also can study

$$\bar{\Psi}(u) = \mathbb{P}[\tau = +\infty|X(0) = u] = 1 - \Psi(u).$$

The ruin probability is a measure of the risk associated to a financial company.

For the case Cramer-Lundberg case, the following formula which may be inferred form the work of Pollaczek, Khinchine (Rolski et al. [2009]) was found great of great use in ruin theory (and in queueing theory) :

$$\hat{\Psi}(s) = \int_0^\infty e^{-su}\Psi(u)du = \frac{\lambda(m_1 - \hat{\bar{F}}(s))}{s(c - \lambda\hat{\bar{F}}(s))}$$

where $\hat{\Psi}(s)$ and $\hat{\bar{F}}(s)$ are the Laplace transforms of the ruin function and the survival function of the claim distributions respectively, and m_1 is the first moment of the claims, which will be assumed from now on to be finite.

We state now several forms of this result, emphasizing the relationship of $\hat{\Psi}(s)$ to the crucial Levy exponent $\kappa(s)$ of the underlying process. This is important, since results expressed in terms of $\kappa(s)$ hold typically for all spectrally negative Lévy processes.

Theorem (The Pollaczek-Khinchine formulas for the ruin function Laplace transform)**.** Let $X(t) = u + ct - S(t), \quad S(t) = \sum_{i=1}^{N_\lambda(t)} C_i$ be a Cramér Lundberg risk process and Ψ its corresponding ruin function. Then the Laplace transforms of the ruin function and its corresponding survival function can be expressed as

$$\hat{\Psi}(s) = \int_0^\infty e^{-su}\Psi(u)du = \frac{\lambda(m_1 - \hat{\bar{F}}(s))}{s(c - \lambda\hat{\bar{F}}(s))} = \frac{1}{s} - \frac{\kappa'(0)}{\kappa(s)}$$

$$\hat{\overline{\Psi}}(s) = \int_0^\infty e^{-su}\overline{\Psi}(u)du = \frac{1}{s} - \hat{\Psi}(s) = \frac{\kappa'(0)}{\kappa(s)}.$$

It is also possible to express this result in terms of the equilibrium density $f_e(x) := \bar{F}(x)/m_1$ and of $\rho = (\lambda m_1)/c$, since for Cramer Lundberg processes $\kappa(s) = s(1 - \rho\hat{f}_e(s))$.

II. The Dividend Problem

The notion of the problem of dividends is popularized by De Fenetti (de Finetti [1957]) and starts by considering a Cramer-Lundberg process $\{X(t)\}_{t\in[0,\infty]}$ (we can of course consider more general cases) and another process

$\{L(t)\}_{t\in[0,\infty]}$ which is left continuous, nonnegative, and non decreasing.
The quantity $L(t)$ can be thought of as the cumulative dividends paid up to time t of an insurance company and hence the risk process taking into account the dividends is $\{U(t)\}_{t\in[0,\infty]}$ where

$$U(t) = X(t) - L(t).$$

The general problem is the characterization of L and the maximization of U through the choice of L, often called a dividend strategy.

A dividend strategy is called admissible if at any time before ruin, all dividend payments are smaller than the current value of U, that is

$$L(t^+) - L(t) \leq U(t), \quad \forall t < \tau$$

where τ is the ruin time of U.
There are a number of possible dividend strategies, hence one often associates another parameter to the process L, say L^π.

Denoting all admissible strategies by Π, we write the expected value at the discount rate $q \geq 0$ associated with the dividend strategy $\pi \in \Pi$ with initial surplus $x \geq 0$ as

$$V^\pi(x) = \mathbb{E}_x \left[\int_0^{\tau^\pi} e^{-qt} dL_t^\pi \right].$$

One may sometimes see V^π being referred to as the value function associated with strategy π. The problem is the characterization of

$$V(x) = \sup_{\pi\in\Pi} V^\pi(x).$$

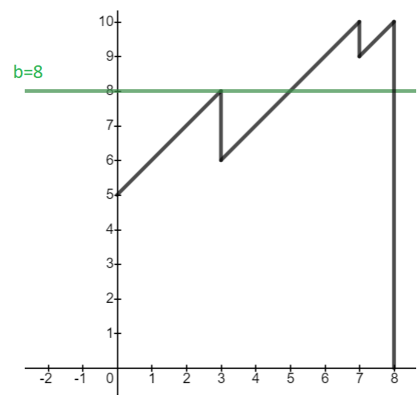
Under the barrier strategy studied by Avram (Avram et al. [2007]), the process U is never allowed to go above a constant level $b > 0$, by setting

$$L(t) = (X(t) - a)^+ = \max\{0, X(t) - a\}.$$

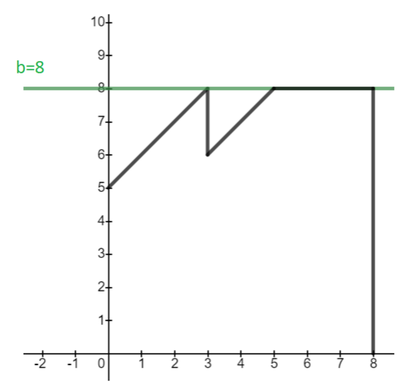
That is, surplus above the barrier b is always paid as dividend.

Clearly, this strategy, which we denote by $b] \in \Pi$, is admissible, and referencing Avram (2007), we have the following result.

Figure 2: *Sample path for a risk process and under a dividend barrier b*



(a) $X(t)$



(b) $U(t)$

Theorem. Let W_q be the associated scale function to X under the discount rate $q \geq 0$. Under a constant dividend barrier strategy $b] \in \Pi$,

$$V^{b]}(x) = \begin{cases} \frac{W_q(x)}{W_q'(b)} & \text{if } x \leq b \\ x - b + \frac{W_q(b)}{W_q'(b)} & \text{if } x > b \end{cases}$$

Looking at $V^{b]}$ as a function of b instead of x, we can solve for V by looking at where the derivative of $V^{b]}$ is zero. We summarize this result through the use of the 'barrier function' H.

Remark (Avram et al. [2007], Loeffen [2008])**.** Let $b \geq 0$ and $H(b) = \frac{1}{W_q'(b)}$.
If H is differentiable with $H'(0) > 0$, and has a unique local maximum $b^* > 0$, then this b^* yields the optimal barrier strategy, i.e. $V = V^{b^*]}$.
Furthermore, note that

- $H'(0) > 0 \iff W_q''(0) < 0$
- local maximum at $b^* \Rightarrow W_q''(b^*) = 0$
- uniqueness of local maximum at $b^* \Rightarrow H(b_1) \geq H(b_2)$, whenever $b* \leq b_1 \leq b_2$

This remark tells us that not only is $V^{b^*]}$ the optimal strategy over the space of the constant barrier strategies, but it is optimal over all strategies in Π.
We will not be tackling other dividend strategies in this presentation, nonetheless, we stress the fact that they do exist (see for example threshold strategies (Gerber and Shiu [2004]), multiband strategies (Azcue and Muler [2005]), etc.)

i. The Cramer-Lundberg model with exponential jumps

Consider the Cramer-Lundberg model $X(t) = u + ct - S(t), \quad S(t) = \sum_{i=1}^{N_\lambda(t)} C_i$
where C_i's are exponentially distributed with $\mathbb{E}C_i = \frac{1}{\mu}$.
Standard computations yield

$$\kappa(s) = s\left(c - \frac{\lambda}{s+\mu}\right), \quad \kappa'(0) = c - \frac{\lambda}{\mu}$$

and solving

$$\kappa(s) - q = 0 \iff cs^2 + (c\mu - \lambda - q)s - q\mu = 0$$

yields two distinct solutions $\gamma_1 \leq 0 \leq \gamma_2$ given by

$$\gamma_1 = \frac{1}{2c}\left(-(c\mu - \lambda - q) + \sqrt{(c\mu - \lambda - q)^2 + 4cq\mu}\right)$$

$$\gamma_1 = \frac{1}{2c}\left(-(c\mu - \lambda - q) - \sqrt{(c\mu - \lambda - q)^2 + 4cq\mu}\right)$$

In this example the Laplace transform of the W_q scale function is

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - q} = \frac{s+\mu}{cs^2 + (c\mu - \lambda - q)s - q\mu}$$

which, when inverted becomes

$$W_q(x) = \frac{(\gamma_1 + \mu)e^{\gamma_1 x} - (\gamma_2 + \mu)e^{\gamma_2 x}}{c(\gamma_1 - \gamma_2)}$$

Differentiating we get

$$W_q'(x) = \frac{\gamma_1(\gamma_1 + \mu)e^{\gamma_1 x} - \gamma_2(\gamma_2 + \mu)e^{\gamma_2 x}}{c(\gamma_1 - \gamma_2)}$$

$$W_q''(x) = \frac{\gamma_1^2(\gamma_1 + \mu)e^{\gamma_1 x} - \gamma_2^2(\gamma_2 + \mu)e^{\gamma_2 x}}{c(\gamma_1 - \gamma_2)}$$

which implies

$$W_q''(x) = 0 \iff x = \frac{1}{\gamma_1 - \gamma_2} \log \frac{\gamma_2(\gamma_2 + \mu)}{\gamma_1(\gamma_1 + \mu)}.$$

The function $W_q'(x)$ is unimodal with extremum at $x* = \frac{1}{\gamma_1 - \gamma_2} \log \frac{\gamma_2(\gamma_2+\mu)}{\gamma_1(\gamma_1+\mu)}$. Hence if

$$W_q''(0) < 0 \Rightarrow x* \text{ is a global minimum}$$

and

$$W_q''(0) \geq 0 \Rightarrow x* \text{ is a global maximum.}$$

Therefore the optimal boundary $b*$ for the dividend problem is

$$b* = \begin{cases} \frac{1}{\gamma_1 - \gamma_2} \log \frac{\gamma_2(\gamma_2+\mu)}{\gamma_1(\gamma_1+\mu)} & \text{if } W_q''(0) < 0 \\ 0 & \text{if } W_q''(0) \geq 0. \end{cases}$$

III. Approximations to the W_q scale function

Consider a spectrally negative Levy risk process $X = u + ct - S(t)$ and associated Levy measure ν with $\kappa(s) = s(c - \hat{\nu}(s))$ as its Levy-Khintchine representation.

Letting $\nu_k = \int_0^\infty x^k \nu(dx)$ be the moments of the Levy measure. Then one can obtain a power series representation of the Levy exponent in terms of the moments given by

$$\kappa(s) = s(c - \hat{\nu}(s)) = cs + \sum_{k=1}^{\infty} \nu_k \frac{(-s)^k}{k!}.$$

In the Cramer-Lundberg case where the intensity of the claim arrivals are given by the Poisson parameter λ, we have $\nu_k = \lambda m_k$ where m_k's are the moments of the claims C_i.

One can thus write

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - q} = \frac{1}{cs - \sum_{k=1}^{\infty} \nu_k \frac{(-s)^k}{k!} - q}. \tag{1}$$

i. Pade Approximation

We can approximate (1) by a Pade approximation of order $[n-1, n]$

$$\hat{W}_q(s) \approx \frac{P_{n-1}(s)}{Q_n(s)} = \frac{\sum_{i=0}^{n-1} a_i s^i}{cs^n + \sum_{i=0}^{n-1} b_i s^i} \tag{2}$$

for some polynomials P_{n-1}, Q_n, and some constants a_i, b_i, $i = 0, \ldots, n-1$.

After the approximation the denominator can be factored as

$$cs^n + \sum_{i=0}^{n-1} b_i s^i = c(s - \gamma_0) \prod_{i=1}^{n} (s + \gamma_i)$$

and after a partial fraction decomposition of (2) we get

$$\hat{W}_q(s) \approx \frac{C_0}{s - \gamma_0} + \sum_{i=1}^{n} \frac{C_i}{s + \gamma_i}$$

$$\Rightarrow W_q(x) \approx C_0 e^{\gamma_0 x} + \sum_{i=1}^{n} C_i e^{-\gamma_i x}$$

Looking back at the properties of the W_q function in the Cramer-Lundberg case, we can force (8) and (9) by setting $a_{n-1} = 1$ and $b_{n-1} = ca_{n-2} - \lambda - q$

$$\hat{W}_q(s) \approx \frac{s^{n-1} + \sum_{i=0}^{n-2} a_i s^i}{cs^n + (ca_{n-2} - \lambda - q)s^{n-1} + \sum_{i=0}^{n-2} b_i s^i}.$$

Since we are fitting two values $W_q(0)$ and $W_q'(0)$ into the Pade approximation, this would be a $[1, 2]$ Pade approximant

$$\hat{W}_q(s) \approx \frac{s + a_0}{cs^2 + (ca_0 - \lambda - q)s + b_0}.$$

Comparing this with the true value of $\hat{W}_q(s)$,

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - q} \approx \frac{s + a_0}{cs^2 + (ca_0 - \lambda - q)s + b_0}$$

$$\Rightarrow cs^2 + (ca_0 - \lambda - q)s + b_0 \approx (s + a_0)(\kappa(s) - q)$$

$$\Rightarrow cs^2 + (ca_0 - \lambda - q)s + b_0 \approx (s + a_0)(cs - \nu_1 s + \ldots - q)$$

$$\Rightarrow cs^2 + (ca_0 - \lambda - q)s + b_0 \approx (s + a_0)(cs - \lambda m_1 s - q)$$

$$\Rightarrow a_0 = \frac{1}{m_1} \text{ and } b_0 = -\frac{q}{m_1}.$$

That is,

$$\hat{W}_q(s) \approx \frac{s + \frac{1}{m_1}}{cs^2 + (\frac{c}{m_1} - \lambda - q)s - \frac{q}{m_1}}. \tag{3}$$

We note that the given approximation in (3) is the same approximation if we consider the claims to be exponential with $m_1 = \frac{1}{\mu}$, as in (i). The optimal boundary $b*$ for this approximation to the W_q function is thus already been solved.

One can also approximate W_q by a higher order Pade approximant, taking into account only (8).

That is, setting $n = 2$ and $a_{n-1} = 1$ we have

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - q} \approx \frac{s + a_0}{cs^2 + b_1 s + b_0}.$$

$$\Rightarrow \hat{W}_q(s) \approx \frac{s + a_0}{cs^2 + b_1 s + b_0}.$$

$$\Rightarrow cs^2 + b_1 s + b_0 \approx (s + a_0)(\kappa(s) - q)$$

$$\Rightarrow cs^2+b_1s+b_0 \approx (s+a_0)(cs-\lambda m_1 s+\lambda m_2 \tfrac{s^2}{s}-q)$$

$$\Rightarrow a_0 = \frac{2m_1}{m_2}, b_1 = \frac{2cm_1-2\lambda m_1^2-m_2q}{m_2},$$

$$\text{and } b_0 = -\frac{2m_1q}{m_2}$$

Therefore

$$\hat{W}_q(s) \approx \frac{s+\frac{2m_1}{m_2}}{cs^2+\left(\frac{2cm_1-2\lambda m_1^2-m_2q}{m_2}\right)s-\frac{2m_1q}{m_2}}.$$

This can be rewritten as

$$\hat{W}_q(s) \approx \frac{s+\frac{1}{\tilde{m}_1}}{cs^2+\left(\frac{c}{\tilde{m}_1}-\lambda\frac{m_1}{\tilde{m}_1}-q\right)s-\frac{q}{\tilde{m}_1}}.$$

where $\tilde{m}_1 = \frac{m_2}{2m_1}$ is the first moment of the equilibrium density $f_e = \frac{\bar{F}}{m_1}$. This approximation is more commonly known as DeVylder's method (Gerber et al. [2008]).

ii. Laguerre Approximation

Consider the Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!}\cdot\frac{d^n}{dx^n}[e^{-x}x^n] = \sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k}{k!}$$

for $x \geq 0$, $n = 0, 1, \ldots$. These polynomials are orthogonal with respect to the weight $e^{-\frac{x}{2}}$, and for constant α, $L_n(\alpha x)$ has Laplace transform given by

$$\hat{L}_n(s) = \frac{(s-\alpha)^n}{s^{n+1}}, n = 0, 1, \ldots.$$

Now, define $W_0^{(\Phi_q)}$ to be the $q = 0$ scale function with respect to the Esscher transformed measure $P^{(\Phi_q)}$ (will not be discussed in detail, see (Albrecher and Asmussen [2010]), (Kyprianou [2014])) given by

$$W_q(x) = e^{x\Phi_q}\cdot W_0^{(\Phi_q)}(x).$$

By dividing by $e^{x\Phi_q}$, one removes the unique positive pole of $\hat{W}_q$

$$\hat{W}_q(s) = \frac{1}{s-\Phi_q}\cdot\widehat{W_0^{(\Phi_q)}}(s)$$

$$\Rightarrow (s-\Phi_q)\hat{W}_q(s) = \widehat{W_0^{(\Phi_q)}}(s).$$

By property of Laplace transforms we also know that

$$W_0^{(\Phi_q)}(x) = e^{-x\Phi_q}\cdot W_q(x)$$

$$\Rightarrow \widehat{W_0^{(\Phi_q)}}(s) = \hat{W}_q(s+\Phi_q) = \frac{1}{\kappa(s+\Phi_q)-q}$$

$$= \frac{1}{\kappa(s+\Phi_q)-\kappa(\Phi_q)}.$$

Following (Abate et al. [1998]), one can consider

$$G(x) = W_0^{(\Phi_q)}(\infty) - W_0^{(\Phi_q)}(x) \tag{4}$$

and approximate using Laguerre polynomials

$$G(x) \approx C\sum_{n=0}^{\infty} B_n e^{-\alpha x/2} L_n(\alpha x) \tag{5}$$

for some chosen constants C, α, and coefficients B_n to be solved for. Taking the Laplace transform of (4)

$$\widehat{G(x)} = \widehat{W_0^{(\Phi_q)}(\infty)} - \widehat{W_0^{(\Phi_q)}(x)}$$

$$\hat{G}(s) = \frac{W_0^{(\Phi_q)}(\infty)}{s} - \widehat{W_0^{(\Phi_q)}}(s)$$

$$= \frac{W_0^{(\Phi_q)}(\infty)}{s} - \frac{1}{\kappa(s+\Phi_q)-\kappa(\Phi_q)}. \tag{6}$$

Taking the Laplace transform of (5) we have

$$\hat{G}(s) \approx C\sum_{n=0}^{\infty} B_n\cdot\hat{L}_n(s+\alpha/2)$$

$$\hat{G}(s) \approx C\sum_{n=0}^{\infty} B_n\frac{(s-\alpha/2)^n}{(s+\alpha/2)^{n+1}}.$$

Some computations yield

$$\hat{H}(s) = (s+\alpha/2)\hat{G}(s)$$

$$\approx C\sum_{n=0}^{\infty} B_n\frac{(s-\alpha/2)^n}{(s+\alpha/2)^n}$$

$$\Rightarrow \hat{H}\left(\frac{\alpha}{2}\cdot\frac{1+z}{1-z}\right) = \left(\frac{\alpha}{1-z}\right)\hat{G}\left(\frac{\alpha}{2}\cdot\frac{1+z}{1-z}\right) \tag{7}$$

$$\approx C\sum_{n=0}^{\infty} B_n z^n$$

upon using the 'collocation' transformation $z = \frac{s-\alpha/2}{s+\alpha/2} \iff s = \frac{\alpha}{2} \cdot \frac{1+z}{1-z}$.

Hence we solve for the coefficients B_n by taking the Taylor expansion of $\hat{H}\left(\frac{\alpha}{2}\frac{1+z}{1-z}\right)$ which is known to us via (7) and (6). The approximation we get for G gives us an approximation for $W_0^{(\Phi_q)}$. An approximation for W_q is then obtained by multiplying by $e^{x\Phi_q}$.

With regards to the choice of C and α, one can do a first order Pade approximation discussed in (i), to obtain an approximation to $W_0^{(\Phi_q)}(x)$ of the form

$$W_0^{(\Phi_q)}(x) \approx W_0^{(\Phi_q)}(\infty) - C\frac{\alpha}{2}e^{-\frac{\alpha}{2}x}.$$

iii. Example: a Cramer-Lundberg model with exponential mixture jumps of order two

Consider a Cramer-Lundberg process with $c = 1/2$, $\lambda = 29/48$ with claim density given by $f(x) = \frac{8}{29}e^{-x} + \frac{42}{29}e^{-2x}$. To obtain rational values we fix $q = 1/16$.

Computing the Levy exponent, one gets

$$\kappa(s) = \frac{1}{2}s + \frac{8}{48}\left(\frac{1}{s+1} - 1\right) + \frac{21}{48}\left(\frac{2}{s+2} - 1\right)$$

which then yields

$$\hat{W}_q(s) = \frac{1}{\kappa(s) - 1/16} = \frac{24(s+1)(s+2)}{(3s-1)(2s+1)(2s+3)}$$

$$= -\frac{6}{11} \cdot \frac{1}{2s+3} - \frac{18}{5} \cdot \frac{1}{2s+1} + \frac{672}{55} \cdot \frac{1}{3s-1}$$

$$\Rightarrow W_q(x) = -\frac{3}{11}e^{-3x/2} - \frac{9}{5}e^{-x/2} + \frac{224}{55}e^{x/3}.$$

A plot of this scale function is given in (3), and upon doing some calculations one gets the optimal barrier value $b^* = 0.642265$. Furthermore this reveals that the unique positive root $\Phi_q = 1/3$

Doing a Pade [2,3] approximation of $\hat{W}_q$ described in i would yield the exact result (in fact, this is true for all Cramer Lundberg processes with mixed exponential claims of order 2).

Figure 3: *Plot of W_q'' in iii*

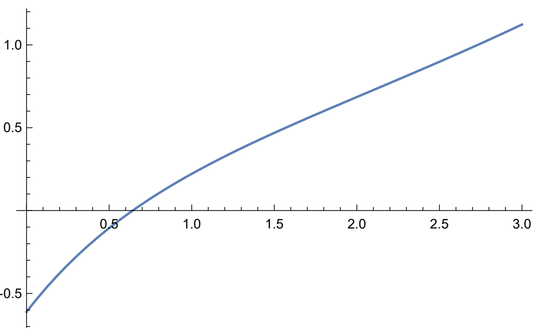
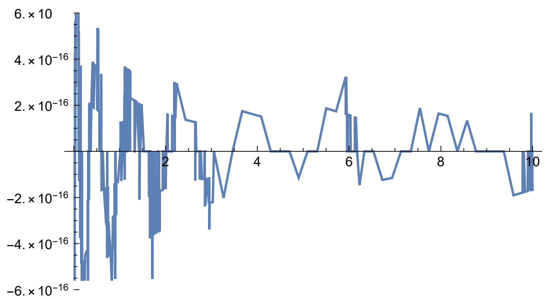


Figure 4: *Relative errors of the Laguerre approximation in iii*



Meanwhile, doing the approximation described in (ii), one first computes

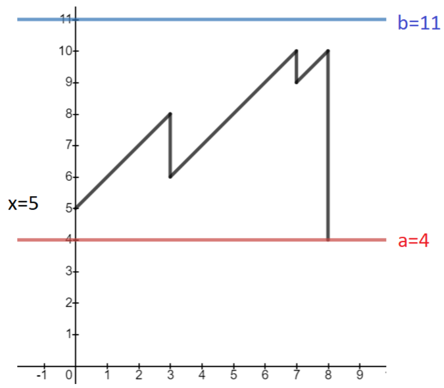
$$\widehat{W_0^{(\Phi_q)}}(x) = \frac{8(3s+4)(3s+7)}{s(6s+5)(6s+11)}$$

which then leads to

$$\hat{G}(s) = \frac{72(57s+97)}{55(6s+5)(6s+11)}.$$

A [0,1] Pade approximation to $\hat{G}$ gives us $\frac{677448}{55(6177s+5335)}$ implying the Laguerre exponent $\alpha/2 = 5335/6177 = 0.863688$. The resulting error in the Laguerre approximation is plotted in (4), with the largest error being $6 * 10^{-6}$ when 30 terms are considered in the Laguerre expansion.

Figure 5: *Sample path for a two sided exit problem*



IV. The W_q scale function and Discounting

The underlying structure of ruin theory appears more clearly after generalizing the ruin problem to the two sided exit problem. Assume $a < b$, and consider quantities such as $\tau_a^+, \tau_b^-, \tau := \min\{\tau_a^+, \tau_b^-\}$ and

$$\Psi_{a,b}(x) = \mathbb{P}_x[\tau_a^+ < \tau_b^-]$$

$$\overline{\Psi}_{a,b}(x) = \mathbb{P}_x[\tau_a^+ > \tau_b^-].$$

It can be shown, via the Markov property of the underlying process, that for some function W, we have

$$\overline{\Psi}_{a,b}(x) = \frac{W(x-a)}{W(b-a)}$$

We will refer to this function W (determined up to a proportionality constant) as the W scale function of the process. Furthermore upon taking the Laplace transform of the previous equation we get

$$\hat{\overline{\Psi}}_{a,b}(s) = \frac{\hat{W}(s)}{W(b-a)},$$

and in the case of the ruin problem where $\hat{\overline{\Psi}}(s) = \frac{\kappa'(0)}{\kappa(s)}$, we get the idea that

$$\hat{W}(s) = C\frac{1}{\kappa(s)}$$

up to a certain constant C.

Generalizing further, suppose there exist an 'interest rate' $q \geq 0$ such that one considers the 'discounted value' of the underlying process with respect to this q when deciding for the first passage time.
In this scenario,

$$\overline{\Psi}_{a,b}(x) = \mathbb{E}_x[e^{-q\tau_b^+} \cdot \mathbb{1}_{\{\tau_a^+ > \tau_b^-\}}]$$

where $\mathbb{1}_A$ is the event that A happens.
Here the probability of survival considers all events where $\tau_a^+ > \tau_b^-$, and gets the expected value of the discounted value of 1 at the time τ_b^+ with respect to q, hence the term $e^{-q\tau_b^+}$.

Even with discounting, through the same analysis as before one can find a function W_q for which

$$\overline{\Psi}_{a,b}(x) = \frac{W_q(x-a)}{W_q(b-a)}.$$

In fact if one defines $W_q(s) : (0, +\infty) \to [0, +\infty)$ to have a Laplace transform

$$\hat{W}_q(s) = \int_0^\infty e^{-sx} W_q(x)\, dx := \frac{1}{\kappa(s) - q} \quad \forall s > \Phi(q),$$

where $\Phi(q) := \sup\{s \geq 0 : \kappa(s) - q = 0\}$, one would arrive at such a function. As before we call this the W_q scale function of the underlying process.

Remark (Bingham [1976], Bertoin [1998], Kyprianou [2014])**.** The scale function W_q is continuous and increasing on $[0, \infty]$.

Remark (Kyprianou and Surya [2007],Kuznetsov et al. [2012])**.** The behavior in the neighborhood of zero of W_q can be obtained from the behavior of its Laplace transform $\hat{W}_q$ at ∞

$$W_q(0^+) = \lim_{s\to\infty} \frac{s}{\kappa(s) - q} = \begin{cases} \frac{1}{c} & \text{if } X \text{ is of bdd variation} \\ 0 & \text{if } X \text{ is of unbdd variation} \end{cases} \tag{8}$$

$$\begin{aligned} W_q'(0^+) &= \lim_{s\to\infty} s\left(\frac{s}{\kappa(s) - q} - W_q(0)\right) \\ &= \begin{cases} \frac{q+\nu(u,\infty)}{c^2} & \text{if } X \text{ is of bdd variation} \\ \frac{2}{\sigma^2} & \text{if } X \text{ is of unbdd variation} \end{cases} \end{aligned} \tag{9}$$

V. Minimizing the ruin probability via reinsurance

Of course, insurance (and reinsurance) are not for free; they influence the premium income of the insurers via so called premium principles.

Premium principles are specified by a deterministic premium function $\pi : L_1(\Omega, P) \to [0, \infty)$. Some of the most popular choices are

a) the *expectation principle*, with

$$\pi(Z) = (1+\eta)E[Z],$$

where η denotes a safety loading

b) the *variance principle*

$$\pi(Z) = E[Z] + \eta Var[Z]$$

c) the *exponential principle*

$$\pi(Z) = \frac{E[e^{\eta Z}] - 1}{\eta}.$$

Under the assumption of the expected value principle, it holds that :

$$c = (1+\theta)E \sum_{i=1}^{N_\lambda(1)} C_i = (1+\theta)\lambda m_1,$$

where $\theta > 0$ is the safety loading for the insurer.

Suppose now that the insurer attempts to reduce his risk exposure by purchasing a proportional reinsurance with a retention level of $\alpha \in [0, 1]$. Specifically, for each claim of size C_i, the insurer covers αC_i and the reinsurer covers the rest $(1-\alpha)C_i$. Suppose that the reinsurer also uses the expected premium principle, but with a larger safety loading $\eta > \theta$ (the reinsurance is then called "non-cheap"). The premium rate for reinsurance is then

$$c_\alpha = (1+\eta)(1-\alpha)\lambda m_1.$$

Then, the surplus process with reinsurance can be expressed as

$$\begin{aligned} \widetilde{X}_t^{(\alpha)} &= x + (c - c_\alpha)t - \alpha \sum_{i=1}^{N_\lambda(t)} C_i \\ &= x + \lambda m_1 \Big(\theta - \eta + \alpha(1+\eta)\Big)t - \alpha \sum_{i=1}^{N_\lambda(t)} C_i. \end{aligned}$$

The underlying question is how to chose α in order to minimize the probability of ruin.

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