(a) $0x + 0y + 0z = 0$

(b) $\begin{aligned} 2x - 3y + z &= 0 \\ 6x - 9y + 3z &= 0 \\ -4x + 6y - 2z &= 0 \end{aligned}$

(c) $\begin{aligned} x - 2y + 7z &= 0 \\ -4x + 8y + 5z &= 0 \\ 2x - 4y + 3z &= 0 \end{aligned}$

(d) $\begin{aligned} x + 4y + 8z &= 0 \\ 2x + 5y + 6z &= 0 \\ 3x + y - 4z &= 0 \end{aligned}$

1 **29.** For what values of s is the solution space of

$$\begin{aligned} x_1 + x_2 + sx_3 &= 0 \\ x_1 + sx_2 + x_3 &= 0 \\ sx_1 + x_2 + x_3 &= 0 \end{aligned}$$

the origin only, a line through the origin, a plane through the origin, or all of R^3?

2 **30.** Which of the following are linear combinations of $\mathbf{u} = (1, -3, 2)$ and $\mathbf{v} = (1, 0, -4)$?

(a) $(0, -3, 6)$ (b) $(3, -9, -2)$

(c) $(0, 0, 0)$ (d) $(1, 6, -16)$

3 **31.** Express the following as linear combinations of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$.

(a) $(-9, -7, -15)$ (b) $(6, 11, 6)$

(c) $(0, 0, 0)$ (d) $(7, 8, 9)$

4 **32.** Which of the following are linear combinations of

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ -2 & 5 \end{bmatrix}?$$

(a) $\begin{bmatrix} 2 & 5 \\ -2 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 5 \\ -2 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 \\ -4 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 9 & 9 \\ -8 & 21 \end{bmatrix}$

5 **33.** In each part express the vector as a linear combination of $\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$.

(a) $6 + 2x + 12x^2$ (b) $2 - x + 6x^2$

(c) 0 (d) $4 + 4x + 6x^2$

6 **34.** In each part, determine whether the given vectors span R^3.

(a) $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (2, 0, 0)$, $\mathbf{v}_3 = (-2, 1, 0)$

(b) $\mathbf{v}_1 = (2, -1, 2)$, $\mathbf{v}_2 = (4, 1, 3)$, $\mathbf{v}_3 = (2, 2, 1)$

(c) $\mathbf{v}_1 = (-1, 5, 2)$, $\mathbf{v}_2 = (3, 1, 1)$, $\mathbf{v}_3 = (2, 0, -2)$, $\mathbf{v}_4 = (4, 1, 0)$

(d) $\mathbf{v}_1 = (3, 2, 4)$, $\mathbf{v}_2 = (-3, -1, 0)$, $\mathbf{v}_3 = (0, 1, 4)$, $\mathbf{v}_4 = (0, 2, 8)$

7 **35.** Suppose that $\mathbf{v}_1 = (2, 1, 3, 0)$, $\mathbf{v}_2 = (3, -1, 2, 5)$, and $\mathbf{v}_3 = (-1, 0, 1, 2)$. Which of the following vectors are in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $(9, 0, 11, 12)$ (b) $(2, 2, 2, 2)$

(c) $(3, 6, 7, -12)$ (d) $(0, 0, 0, 0)$

8 **36.** Determine whether the following polynomials span P_2.

$$\mathbf{p}_1 = 1 + x + 2x^2, \quad \mathbf{p}_2 = 3 + x,$$
$$\mathbf{p}_3 = 5 - x + 4x^2, \quad \mathbf{p}_4 = -2 - 2x + 2x^2$$

9 **37.** Let $\mathbf{f} = \cos^2 x$ and $g = \sin^2 x$. Which of the following lie in the space spanned by $\mathbf{f}$ and $\mathbf{g}$?

(a) $\cos 2x$ (b) 0 (c) 2 (d) $x - 4$ (e) $\sin 2x$

10 **38.** Determine whether the solution space of the system $A\mathbf{x} = \mathbf{0}$ is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation for it. If it is a line, find parametric equations for it.

(a) $A = \begin{bmatrix} 1 & -2 & 6 \\ 3 & -6 & 18 \\ -7 & 14 & -42 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & 9 \\ -2 & 4 & -6 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 9 & -11 & 3 \\ 3 & -4 & 1 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & -4 & 0 \\ -2 & 8 & 1 \\ 4 & -16 & 0 \end{bmatrix}$ (f) $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$

39. (***Calculus required***) Show that the following sets of functions are subspaces of $F(-\infty, \infty)$.

(a) All continuous functions on $(-\infty, \infty)$.

(b) All differentiable functions on $(-\infty, \infty)$.

(c) All differentiable functions on $(-\infty, \infty)$ that satisfy $\mathbf{f}' + 2\mathbf{f} = \mathbf{0}$.

40. (***Calculus required***) Show that the set of continuous functions $\mathbf{f} = f(x)$ on $[a, b]$ such that

$$\int_a^b f(x)\,dx = 0$$

is a subspace of $C[a, b]$.

41. Show that the solution vectors of a consistent nonhomogeneous system of m linear equations in n unknowns do not form a subspace of R^n.

42. Use Theorem 4.2.5 to show that the vectors $\mathbf{v}_1 = (1, 6, 4)$, $\mathbf{v}_2 = (2, 4, -1)$, $\mathbf{v}_3 = (-1, 2, 5)$, and the vectors $\mathbf{w}_1 = (1, -2, -5)$, $\mathbf{w}_2 = (0, 8, 9)$ span the same subspace of R^3.

Linear Independence

11 **43.** Explain why the following are linearly dependent sets of vectors. (Solve this problem by inspection.)

(a) $\mathbf{u}_1 = (3, -1)$ and $\mathbf{u}_2 = (6, -2)$ in R^2

(b) $\mathbf{u}_1 = (-2, 0, 1)$, $\mathbf{u}_2 = (4, -2, 0)$, $\mathbf{u}_3 = (6, -6, 3)$ in R^3

11

(c) $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ -2 & -3 \end{bmatrix}$ in M_{22}

(d) $\mathbf{p}_1 = 2 + x - 3x^2$ and $\mathbf{p}_2 = -4 - 2x + 6x^2$ in P_2

44. Which of the following sets of vectors in R^3 are linearly dependent?

(a) $(4, -1, 2)$, $(-4, 10, 2)$

(b) $(-3, 0, 4)$, $(5, -1, 2)$, $(1, 1, 3)$

(c) $(8, -1, 3)$, $(4, 0, 1)$

(d) $(-2, 0, 1)$, $(3, 2, 5)$, $(6, -1, 1)$, $(7, 0, -2)$

12 45. Which of the following sets of vectors in R^4 are linearly dependent?

(a) $(1, 2, -2, 1)$, $(3, 6, -6, 3)$, $(4, -2, 4, 1)$,

(b) $(5, 2, 0, -1)$, $(0, -3, 0, 1)$, $(1, 0, -1, 2)$, $(3, 1, 0, 1)$

(c) $(2, 1, 1, -4)$, $(2, -8, 9, -2)$, $(0, 3, -1, 5)$, $(0, -1, 2, 4)$

(d) $(1, 0, -6, 3)$, $(0, 1, 3, 0)$, $(0, 2, 7, 0)$, $(0, 2, 0, 1)$

13 46. Which of the following sets of vectors in P_2 are linearly dependent?

(a) $2 - x + 4x^2$, $3 + 6x + 2x^2$, $2 + 10x - 4x^2$

(b) $3 + x + x^2$, $2 - x + 5x^2$, $4 - 3x^2$

(c) $6 - x^2$, $1 + x + 4x^2$

(d) $1 + 3x + 3x^2$, $x + 4x^2$, $5 + 6x + 3x^2$, $7 + 2x - x^2$

16

61. The functions $f_1(x) = \sec x$ and $f_2(x) = \tan x$ are linearly independent in $F(-\frac{\pi}{2}, \frac{\pi}{2})$ because neither function is a scalar multiple of the other. Confirm the linear independence using the Wronskian (the identity $\sec^2 x = 1 + \tan^2 x$ may help).

62. (*Calculus required*) Use the Wronskian to show that the following sets of vectors are linearly independent.

(a) $1, x^2, e^x$ (b) x, x^2, x^3

63. Use Wroński's test to show that the functions $f_1(x) = x$, $f_2(x) = e^x$, and $f_3(x) = e^{-x}$ are linearly independent vectors in $F(-\infty, \infty)$.

64. Use Wroński's test to show that the functions $f_1(x) = \cos x$, $f_2(x) = \sin x$, and $f_3(x) = x \sin x$ are linearly independent vectors in $F(-\infty, \infty)$.

65. (a) In Example 1 of Section 4.3 we showed that the mutually orthogonal vectors **i**, **j**, and **k** form a linearly independent set of vectors in R^3. Do you think that every set of three nonzero mutually orthogonal vectors in R^3 is linearly independent? Justify your conclusion with a geometric argument.

(b) Justify your conclusion with an algebraic argument. [*Hint:* Use dot products.]

Coordinates and Basis

14 66. In words, explain why the following sets of vectors are *not* bases for the indicated vector spaces.

(a) $\mathbf{u}_1 = (3, 2, 1)$, $\mathbf{u}_2 = (-2, 1, 0)$, $\mathbf{u}_3 = (5, 1, 1)$ for R^3

(b) $\mathbf{u}_1 = (1, 1)$, $\mathbf{u}_2 = (3, 5)$, $\mathbf{u}_3 = (4, 2)$ for R^2

(c) $\mathbf{p}_1 = 1 + x$, $\mathbf{p}_2 = 2x - x^2$ for P_2

(d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ -5 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$, $E = \begin{bmatrix} 7 & 1 \\ 2 & 9 \end{bmatrix}$ for M_{22}

15 67. Which of the following sets of vectors are bases for R^2?

(a) $\{(3, 1), (0, 0)\}$ (b) $\{(4, 1), (-7, -8)\}$

(c) $\{(5, 2), (-1, 3)\}$ (d) $\{(3, 9), (-4, -12)\}$

68. Which of the following sets of vectors are bases for R^3?

(a) $\{(1, 0, 0), (2, 2, 0), (3, 3, 3)\}$

(b) $\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$

(c) $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$

(d) $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

16 69. Which of the following form bases for P_2?

(a) $2 - 4x + x^2$, $3 + 2x - x^2$, $1 + 6x - 2x^2$

(b) $3 + 2x - x^2$, $x + 5x^2$, $2 - 4x + x^2$

(c) $1 + x + x^2$, $x + x^2$, x^2

(d) $-4 + x + 3x^2$, $6 + 5x + 2x^2$, $8 + 4x + x^2$

70. Show that the following matrices form a basis for M_{22}.

$$\begin{bmatrix} 3 & 4 \\ 3 & -4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -8 \\ -12 & -2 \end{bmatrix}$$

71. Let W be the space spanned by $\mathbf{f} = \sin x$ and $\mathbf{g} = \cos x$.

(a) Show that for any value of θ, $\mathbf{f}_1 = \sin(x + \theta)$ and $\mathbf{g}_1 = \cos(x + \theta)$ are vectors in W.

(b) Show that $\mathbf{f}_1$ and $\mathbf{g}_1$ form a basis for W.

72. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.

(a) $\mathbf{u}_1 = (0, 1)$; $\mathbf{u}_2 = (1, 0)$; $\mathbf{w} = (5, -3)$

(b) $\mathbf{u}_1 = (3, 8)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (1, 0)$

(c) $\mathbf{u}_1 = (1, 1)$, $\mathbf{u}_2 = (0, 2)$; $\mathbf{w} = (a, b)$

73. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ of R^2.

(a) $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (1, 0)$

(b) $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (0, 1)$

(c) $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (1, 1)$

74. Find the coordinate vector of $\mathbf{v}$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) $\mathbf{v} = (3, 4, 3)$; $\mathbf{v}_1 = (3, 2, 1)$, $\mathbf{v}_2 = (-2, 1, 0)$, $\mathbf{v}_3 = (5, 0, 0)$

(b) $\mathbf{v} = (5, -12, 3)$; $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-4, 5, 6)$, $\mathbf{v}_3 = (7, -8, 9)$

75. Find the coordinate vector of $\mathbf{p}$ relative to the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

(a) $\mathbf{p} = 3 + 2x - 4x^2$, $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x$, $\mathbf{p}_3 = x^2$

(b) $\mathbf{p} = 3 - x - 2x^2$, $\mathbf{p}_1 = 1 + x$, $\mathbf{p}_2 = 1 + x^2$, $\mathbf{p}_3 = x + x^2$

76. Find the coordinate vector of A relative to the basis $S = \{A_1, A_2, A_3, A_4\}$.

$$A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}; \quad A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

In Exercises 77–78, show that $\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, and express A as a linear combination of the basis vectors.

77. $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -5 & 4 \\ 1 & -1 \end{bmatrix}$

78. $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, **90.**

$A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

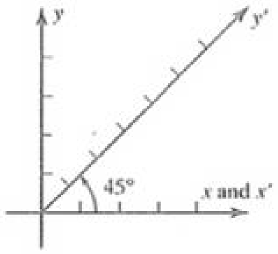
In Exercises **79–80**, show that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for P_2, and express $\mathbf{p}$ as a linear combination of the basis vectors. **91.**

79. $\mathbf{p}_1 = 1 - 2x + x^2, \mathbf{p}_2 = 3 + 7x, \mathbf{p}_3 = 2 + 3x - x^2,$
$\mathbf{p} = 6x$

80. $\mathbf{p}_1 = 1 - x + x^2, \mathbf{p}_2 = x - x^2, \mathbf{p}_3 = x,$
$\mathbf{p} = 7 - x + 2x^2$

81. The accompanying figure shows a rectangular xy-coordinate system and an $x'y'$-coordinate system with skewed axes. Assuming that 1-unit scales are used on all the axes, find the $x'y'$-coordinates of the points whose xy-coordinates are given. **92.**

(a) $(1, 1)$ (b) $(1, 0)$ (c) $(0, 1)$ (d) (a, b)



93.

94.

Figure Ex-81

82. (a) Express $\mathbf{v} = (1, 1)$ as a linear combination of $\mathbf{v}_1 = (1, -1)$, $\mathbf{v}_2 = (3, 0)$, and $\mathbf{v}_3 = (2, 1)$ in two different ways. **95.**

(b) Explain why this does not violate Theorem 4.4.1.

83. Must a basis for P_n contain a polynomial of degree k for each $k = 0, 1, 2, \ldots, n$? Justify your answer. **96.**

Dimension

In Exercises **84–89**, find a basis for the solution space of the homogeneous linear system, and find the dimension of that space. **97.**

17 **84.**
$$\begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ x_1 + x_2 \phantom{{}+x_3} &= 0 \\ -2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

85.
$$\begin{aligned} 3x_1 - x_2 + x_3 - x_4 &= 0 \\ 4x_1 + 2x_2 + x_3 - 2x_4 &= 0 \end{aligned}$$

98.

86.
$$\begin{aligned} 3x_1 - x_2 + 2x_3 + x_4 &= 0 \\ 6x_1 - 2x_2 - 4x_3 \phantom{{}+x_4} &= 0 \end{aligned}$$

99.

18 **87.**
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + 4x_2 - 2x_3 &= 0 \\ -3x_1 - 6x_2 + 3x_3 &= 0 \end{aligned}$$

100.

88.
$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 0 \\ x_1 \phantom{{}+x_2} + 4x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

89. 19
$$\begin{aligned} x + 2y - z &= 0 \\ 2x - y + 2z &= 0 \\ 3x + y + z &= 0 \\ 4x + 3y \phantom{{}+z} &= 0 \end{aligned}$$

126. Let B be a basis for R^n. Prove that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a linearly independent set in R^n if and only if the vectors $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \ldots, [\mathbf{v}_k]_B$ form a linearly independent set in R^n.

127. Let A be an $n \times n$ matrix, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent vectors in R^n expressed as $n \times 1$ matrices. What must be true about A for $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n$ to be linearly independent?

128. Find a formula for the dimension of the vector space of symmetric $n \times n$ matrices.

Row Space, Column Space, and Null Space

129. List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

130. Express the product $A\mathbf{x}$ as a linear combination of the column vectors of A.

(a) $\begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 2 & 6 \\ 1 & -1 & 3 \\ 0 & 1 & 7 \\ 2 & 1 & 3 \\ 4 & -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & 5 \\ 6 & 3 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$

131. Determine whether $\mathbf{b}$ is in the column space of A, and if so, express $\mathbf{b}$ as a linear combination of the column vectors of A.

20

(a) $A = \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1 & 4 \\ 2 & 1 & 1 \\ 2 & 2 & 5 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

(a) Find a vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.

(b) Find a vector form of the general solution of $A\mathbf{x} = \mathbf{b}$.

133. In parts (a)–(d), find the vector form of the general solution of the given linear system $A\mathbf{x} = \mathbf{b}$; then use that result to find the vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.

(a) $\begin{aligned} 3x_1 + x_2 &= 2 \\ 6x_1 + 2x_2 &= 4 \end{aligned}$

(b) $\begin{aligned} x_1 + x_2 + 2x_3 &= 5 \\ x_1 \qquad + x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 3 \end{aligned}$

(c) $\begin{aligned} x_1 - 2x_2 + x_3 + x_4 &= 1 \\ -x_1 + x_2 - 2x_3 + x_4 &= 2 \\ -2x_2 - x_3 - x_4 &= -2 \\ x_1 - 3x_2 \qquad + 3x_4 &= 4 \end{aligned}$

(d) $\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 4 \\ -2x_1 + x_2 + 2x_3 + x_4 &= -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 3 \\ 4x_1 - 7x_2 \qquad - 5x_4 &= -5 \end{aligned}$

134. Find a basis for the null space of A.

(a) $A = \begin{bmatrix} 3 & -1 & 0 \\ 6 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -2 & 10 \\ 2 & -3 & 18 \\ 0 & -7 & 14 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

135. In each part, a matrix in row echelon form is given. By inspection, find bases for the row and column spaces of A.

(a) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 & 6 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$\mathbf{v}_5 = (-7, 18, 2, -8)$

140. Prove: If S is a basis for a vector space V, then for any vectors $\mathbf{u}$ and $\mathbf{v}$ in V and any scalar k, the following relationships hold.

(a) $(\mathbf{u}+\mathbf{v})_S = (\mathbf{u})_S + (\mathbf{v})_S$ (b) $(k\mathbf{u})_S = k(\mathbf{u})_S$

141. Construct a matrix whose null space consists of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 4 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 6 \\ -2 \end{bmatrix}$$

142. (a) Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Show that relative to an xyz-coordinate system in 3-space the null space of A consists of all points on the z-axis and that the column space consists of all points in the xy-plane (see the accompanying figure).

(b) Find a 3×3 matrix whose null space is the x-axis and whose column space is the yz-plane.

ible. Invent and prove a theorem that describes how the row spaces of AB and B are related.

Rank, Nullity, and the Fundamental Matrix Spaces

147. Verify that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

21

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

148. Find the rank and nullity of the matrix; then verify that the values obtained satisfy Formula (4) of Section 4.8 in the Dimension Theorem.

22

(a) $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 4 & 2 & 0 \\ -1 & -3 & 0 & 5 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

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(e) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

149. In each part of Exercise 148, use the results obtained to find the number of leading variables and the number of parameters in the solution of $A\mathbf{x} = \mathbf{0}$ without solving the system.

150. In each part, use the information in the table to find the dimension of the row space of A, column space of A, null space of A, and null space of A^T

157. Suppose that A is a 3×3 matrix whose null space is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

158. Discuss how the rank of A varies with t.

(a) $A = \begin{bmatrix} 1 & -1 & t \\ 1 & t & -1 \\ t^2 & 1 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

159. Are there values of r and s for which

$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

74. Verify that the vectors

$$\mathbf{v}_1 = (1, -1, 2, -1), \quad \mathbf{v}_2 = (-2, 2, 3, 2),$$
$$\mathbf{v}_3 = (1, 2, 0, -1), \quad \mathbf{v}_4 = (1, 0, 0, 1)$$

form an orthogonal basis for R^4 with the Euclidean inner product; then use Theorem 6.3.2*a* to express each of the following as linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$.

(a) $(1, -1, 1, -1)$

(b) $(\sqrt{2}, -3\sqrt{2}, 5\sqrt{2}, -\sqrt{2})$

(c) $(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{4}{3})$

75. (a) Show that the vectors

$$\mathbf{v}_1 = (1, -2, 3, -4), \quad \mathbf{v}_2 = (2, 1, -4, -3),$$
$$\mathbf{v}_3 = (-3, 4, 1, -2), \quad \mathbf{v}_4 = (4, 3, 2, 1)$$

form an orthogonal basis for R^4 with the Euclidean inner product.

(b) Use Theorem 6.3.2*a* to express $\mathbf{u} = (-1\ 2, 3, 7)$ as a linear combination of the vectors in part (a).

In Exercises **76–77**, an orthonormal basis with respect to the Euclidean inner product is given. Use Theorem 6.3.2*b* to find the coordinate vector of $\mathbf{w}$ with respect to that basis.

76. (a) $\mathbf{w} = (3, 7)$; $\mathbf{u}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(b) $\mathbf{w} = (-1, 0, 2)$; $\mathbf{u}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$,

$$\mathbf{u}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \quad \mathbf{u}_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

77. (a) $\mathbf{w} = (2, 0, 5)$; $\mathbf{u}_1 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, $\mathbf{u}_2 = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$,

$$\mathbf{u}_3 = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$$

(b) $\mathbf{w} = (-1, 1, 2)$; $\mathbf{u}_1 = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$,

$$\mathbf{u}_2 = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$

$$\mathbf{u}_3 = \left(-\frac{1}{\sqrt{66}}, -\frac{4}{\sqrt{66}}, \frac{7}{\sqrt{66}}\right)$$

In Exercises **78–79**, the given vectors are orthogonal with respect to the Euclidean inner product. Find $\text{proj}_W \mathbf{x}$, where $\mathbf{x} = (1, 2, 0, -2)$ and W is the subspace of R^4 spanned by the vectors.

In Exercises **80–81**, the given vectors are orthonormal with respect to the Euclidean inner product. Use Theorem 6.3.4*b* to find $\text{proj}_W \mathbf{x}$, where $\mathbf{x} = (1, 2, 0, -1)$ and W is the subspace of R^4 spanned by the vectors.

80. (a) $\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$

(b) $\mathbf{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

81. (a) $\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$,

$$\mathbf{v}_3 = \left(\frac{1}{\sqrt{18}}, 0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}\right)$$

(b) $\mathbf{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$,

$$\mathbf{v}_3 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

82. Find the vectors $\mathbf{w}_1$ in W and $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{x}$ and W are as given in

(a) Exercise 78(a). (b) Exercise 79(a).

83. Find the vectors $\mathbf{w}_1$ in W and $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{x}$ and W are as given in

(a) Exercise 80(a). (b) Exercise 81(a).

84. Let R^2 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ into an orthonormal basis. Draw both sets of basis vectors in the xy-plane.

(a) $\mathbf{u}_1 = (1, -3)$, $\mathbf{u}_2 = (2, 2)$

(b) $\mathbf{u}_1 = (1, 0)$, $\mathbf{u}_2 = (3, -5)$

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85. Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis.

(a) $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, 0, -1)$, $\mathbf{u}_3 = (2, 1, -1)$

(b) $\mathbf{u}_1 = (0, 1, 0)$, $\mathbf{u}_2 = (-7, 4, 2)$, $\mathbf{u}_3 = (-3, 0, -1)$

86. Let R^4 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.

$$\mathbf{u}_1 = (0, -2, 1, 0), \quad \mathbf{u}_2 = (1, 1, 0, 0),$$
$$\mathbf{u}_3 = (-1, -2, 0, -1), \quad \mathbf{u}_4 = (1, 0, 0, -1)$$